Uniform Bounds for Maximal Flat Periods on SL(n,R)

by

Phillip Lee Harris

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The dissertation is approved by the following members of the Final Oral Committee:
Simon Marshall, Professor, Mathematics
Sergey Denisov, Professor, Mathematics
Betsy Stovall, Professor, Mathematics
Andreas Seeger, Professor, Mathematics

Abstract

Let X be a locally symmetric space associated to $SL(n, \mathbb{R})$ and $f_i \in C^\infty(X)$ an orthonormal basis of Maass forms, with associated spectral parameters ν_i. Let $Y \subset X$ be a flat submanifold of maximal dimension. We prove a bound for f_i integrated against a smooth cutoff function on Y, uniform in the spectral parameter.

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Contents

Chapter 1

Introduction

Let G be a noncompact semisimple Lie group with Iwasawa decomposition $G = NAK$ and Lie algebras $\mathfrak{g} = \mathfrak{n} + \mathfrak{a} + \mathfrak{k}$. Let G/K be the associated symmetric space and $X = \Gamma\backslash G/K$ a compact quotient. Let $(f_i)_i \in L^2(X)$ be an orthonormal basis of Maass forms with spectral parameters $\nu_i \in \mathfrak{a}^*_{\mathbb{C}}$. Let $Y \subset X$ be a maximal flat submanifold (not necessarily closed) and $b \in C_c^\infty(Y)$ a smooth cutoff function. We are interested in the growth of the *flat periods*

$$\mathcal{P}_i = \int_Y f_i(x) b(x) dx$$

with respect to ν_i.

There is a general bound for Laplace eigenfunctions on compact manifolds due to Zelditch [1] which in our case gives

$$|\mathcal{P}_i| \ll \|\nu_i\|^{(\dim X - \dim Y - 1)/2} = \|\nu_i\|^{(n^2-n-2)/4}.$$

Michels [2] was the first to study flat periods on higher rank locally symmetric spaces. His work implies[1] an averaged estimate for flat periods, as follows. Let X be a compact locally symmetric space of noncompact type, of rank r. Let $\Sigma \subset \mathfrak{a}^*$ be the restricted roots of X and Σ^+ be the positive roots. Define the set of "generic points" $(\mathfrak{a}^*)^{\text{gen}} \subset \mathfrak{a}^*$ to be

[1] This result for Maass forms follows from applying the trace formula argument in Section 2.2 to Michels' bound for spherical functions.

the set of points that are regular and that do not lie in any proper subspace spanned by roots. Fix a closed cone $D \subset (\mathfrak{a}^*)^{\text{gen}}$. Let $\beta : \mathfrak{a}^* \to \mathbb{R}$ be the Plancherel density. Then there exists $C > 0$ such that uniformly for $\lambda \in D$ we have

$$\sum_{\|\operatorname{Re}\nu_i - \lambda\| \le C} |\mathcal{P}_i|^2 \ll \beta(\lambda)(1 + \|\lambda\|)^{-r}.$$

In other words, consider all periods $\mathcal{P}_i$ such that $\operatorname{Re}\nu_i$ lies in a ball of fixed radius around λ, then their average norm squared is $\ll (1 + \|\lambda\|)^{-r}$. For certain choices of X and Y enumerated in [2, Theorem 1.3] Michels proves a lower bound (replacing $\ll$ with $\asymp$).

For X associated to $SL(n, \mathbb{R})$, we give an estimate uniform in λ, elucidating the behavior on the non-generic set. Define $\beta_0(\lambda) = \prod_{\alpha \in \Sigma^+} 1 + |\langle \lambda, \alpha \rangle|$, which is essentially the maximum of β taken over a ball around λ.

Theorem 1. *Uniformly for* $\lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_n) \in \mathfrak{a}^*$ *we have*

$$\sum_{\|\operatorname{Re}\nu_i - \lambda\| \le 1} |\mathcal{P}_i|^2 \ll \beta_0(\lambda)(1 + \|\lambda\|)^{-n+1} L_n(\lambda)$$

with the implied constant depending only on n *and the test function* b*. The function* $L_n(\lambda)$ *is Weyl invariant, and in the Weyl chamber* [2] *with* $\lambda_1 \le \cdots \le \lambda_n$*, we define it as follows. Define* $\log' x = \log(2 + x)$*. For* $n \ne 4$*,*

$$L_n(\lambda) := \left(\log' \frac{\|\lambda\|}{1 + |\lambda_2| + |\lambda_{n-1}|} \right)^{n-2},$$

and for $n = 4$*,*

$$L_4(\lambda) := \left(\log' \frac{\|\lambda\|}{1 + |\lambda_2| + |\lambda_3|} \right)^2 \log' \frac{\|\lambda\|}{1 + |\lambda_1 - \lambda_2| + |\lambda_3 - \lambda_4|}.$$

Since $\beta_0(\lambda) \ll \|\lambda\|^{n(n-1)/2}$ we obtain $|\mathcal{P}_i| \ll \|\nu_i\|^{(n^2-3n+2)/4} L_n(\nu_i)^{1/2}$, an improvement on the bound for general manifolds. The factor $L_n(\lambda)$ can be given a geometric interpre-

[2] This is opposite from the canonical positive chamber.

tation, namely, it grows when λ is collinear with a root. In the $n = 4$ case, it grows when λ is collinear with a root or a sum of two orthogonal roots.

1.1 Outline of the proof

Using a standard pre-trace formula argument, we reduce the period estimate to an estimate for spherical functions: let $A \subset SL(n, \mathbb{R})$ be the diagonal subgroup and $b \in C_c^\infty(A)$, then

$$\int_A \varphi_\lambda(g) b(g) dg \ll_b (1 + \|\lambda\|)^{-n+1} L_n(\lambda). \tag{1.1}$$

We use a theorem of Duistermaat to linearize the problem, replacing the spherical function φ_λ on the Lie group with a "Euclidean" approximation on the Lie algebra. The problem then reduces to the following lemma: Let $\lambda \in \mathfrak{a}^*$ be regular. For a real $n \times n$ matrix A, let $d(A) = \sum_{1 \leq i \leq n} A_{ii}^2$ be the size of the diagonal. Choose $k \in SO(n)$ at random according to the Haar measure, then

$$\mathrm{Prob}[d(k\lambda k^{-1}) < 1] \asymp (1 + \|\lambda\|)^{-n+1} L_n(\lambda).$$

While we were unable to prove a lower bound in Theorem 1, this lemma and Michels' lower bounds are at least suggestive that our upper bound is sharp.

Chapter 2

Proof of theorem 1

2.1 Preliminaries and Notation

The notation $A \ll B$ means there exists $C > 0$ such that $A \le CB$, and $A \asymp B$ means $A \ll B \ll A$. The implied constant C may always depend on the dimension n and bump function b; any other dependency is indicated with a subscript. When A is a matrix, $\|A\|$ will denote the Frobenius norm $\|A\|^2 = \sum_{ij} \left|A_{ij}\right|^2$. Indicator functions are denoted $\mathbf{1}_S(x)$ where S is a set or $\mathbf{1}[P(x)]$ where P is a predicate.

Let $G = SL(n, \mathbb{R})$. We write the Iwasawa decomposition $G = NAK$, where N is the upper triangular subgroup with 1's on the diagonal, A is the diagonal subgroup and K is $SO(n)$. We have the corresponding Lie algebra decomposition $\mathfrak{g} = \mathfrak{n} + \mathfrak{a} + \mathfrak{k}$ and the Cartan decomposition $\mathfrak{g} = \mathfrak{p} + \mathfrak{k}$. Then $X = \Gamma\backslash G/K$ for some cocompact lattice $\Gamma \subset G$, and Y lies in the image of gA for some $g \in G$. Using a partition of unity we may assume the support of b is small, say, having diameter $< R/100$ where R is the injectivity radius of X, and lift b to $C_c^\infty(gA)$.

2.2 Spherical Functions and the Trace Formula

Let $H : G \to \mathfrak{a}$ be the smooth map satisfying $g \in Ne^{H(g)}K$ for all $g \in G$ (sometimes called the "Iwasawa projection"). Let ρ be the half sum of the positive roots and W be the Weyl group. Recall the spherical functions

$$\varphi_\lambda(g) = \int_K e^{(i\lambda+\rho)H(kg)} dk,$$

the Harish-Chandra transform

$$\widehat{f}(\lambda) = \int_G f(x)\varphi_{-\lambda}(x)dx \qquad f \in C_c^\infty(K\backslash G/K),$$

and its inverse

$$f(x) = \frac{1}{|W|}\int_{\mathfrak{a}^*} \varphi_\lambda(x)\widehat{f}(\lambda)\beta(\lambda)d\lambda.$$

In order to prove Theorem 1 we use the pre-trace formula for $SL(n, \mathbb{R})$. Let $k \in C_c^\infty(K\backslash G/K)$ be a bi-K-invariant test function and

$$K(x, y) = \sum_{\gamma\in\Gamma} k(x^{-1}\gamma y)$$

be the corresponding automorphic kernel on X. Then the pre-trace formula [3] states

$$\sum_{\gamma\in\Gamma} k(x^{-1}\gamma y) = \sum_i \widehat{k}(-\nu_i)f_i(x)\overline{f_i(y)}$$

where $\widehat{k}$ is the Harish-Chandra transform of k. Integrating against $b(x)b(y) \in C_c^\infty(gA\times gA)$ we obtain

$$\int_{gA}\int_{gA}\sum_{\gamma\in\Gamma} k(x^{-1}\gamma y)b(x)b(y)dxdy = \sum_i \widehat{k}(-\nu_i)|\mathcal{P}_i|^2 . \tag{2.1}$$

Next we construct a k such that $\widehat{k}$ concentrates around a given λ_0. Recall that the spectral parameters of Maass forms satisfy $W\nu_i = W\overline{\nu_i}$ and $\|\mathrm{Im}\,\nu_i\| \le \|\rho\|$. Define $\Omega = \{\lambda \in \mathfrak{a}_{\mathbb{C}}^* : W\lambda = W\overline{\lambda}, \|\mathrm{Im}\,\lambda\| \le \|\rho\|\}$. Using the Paley-Wiener theorem for the Harish-Chandra transform we construct k with the following properties:

1. k is supported in a ball of radius $R/100$, where R is the injectivity radius of X.

2. $\widehat{k}(\lambda) \ge 0$ for $\lambda \in \Omega$.

3. $\widehat{k}(\lambda) \geq 1$ for $\lambda \in \Omega$ with $\|\mathrm{Re}\ \lambda - \lambda_0\| \leq 1$.

4. $\widehat{k}(\lambda) \ll_N (1 + \|\lambda - \lambda_0\|)^{-N}$ for $\nu \in \Omega$ uniformly in λ_0.

The details of this construction may be found in Section 4.1 of [4]. By property (1), all terms except $\gamma = e$ on the geometric side drop out. By properties (2) and (3) we have

$$\int_{gA}\int_{gA} k(x^{-1}y)b(x)b(y)dxdy \geq \sum_{\|\mathrm{Re}\,\nu_i - \lambda_0\| \leq 1} |\mathcal{P}_i|^2 .$$

Note that $x^{-1}y \in A$. Defining $b_1(z) = \int_{gA} b(x)b(xz)dx \in C_c^\infty(A)$ and changing variables $z = x^{-1}y$ the left hand side becomes

$$\begin{aligned}\int_{gA}\int_{gA} k(x^{-1}y)b(x)b(y)dxdy &= \int_A k(z)b_1(z)\,dz \\ &= \int_A \left(\frac{1}{|W|}\int_{\mathfrak{a}^*} \varphi_\lambda(z)\widehat{k}(\lambda)\beta(\lambda)\,d\mu\right) b_1(z)dz \\ &= \frac{1}{|W|}\int_{\mathfrak{a}^*} \widehat{k}(\lambda)\beta(\lambda)\left(\int_A \varphi_\lambda(z)b_1(z)dz\right) d\lambda.\end{aligned}$$

The task is now to bound the integral of a spherical function φ_λ against a smooth cutoff function on A. In other words, we need the following bound: for $b \in C_c^\infty(A)$,

$$\int_A \varphi_\lambda(z)b(z)dz \ll_b (1 + \|\lambda\|)^{-n+1} L_n(\lambda).$$

With this bound, Theorem 1 follows from the rapid decay of $\widehat{k}$ away from $-\lambda_0$ (property 4) and the polynomial growth of β.

2.3 Euclidean Approximation of Spherical Functions

Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition. For $X \in \mathfrak{p}, k \in K$ write $k.X = kXk^{-1}$ for the adjoint action of K on $\mathfrak{p}$. Let $\pi : \mathfrak{p} \to \mathfrak{a}$ be the orthogonal projection with respect to the Killing form. By a theorem of Duistermaat [5, Equation 1.11], there exists a nonnegative analytic function $a \in C^\infty(\mathfrak{p})$ such that

$$\varphi_\lambda(e^X) = \int_K e^{\langle i\lambda, \pi(k.X)\rangle} a(k.X) dk$$

where dk is the Haar measure. Changing variables $z = e^X$ and using Duistermaat's formula gives

$$\begin{aligned}(*) := \int_A \varphi_\lambda(z) b(z) dz &= \int_{\mathfrak{a}} \int_K e^{\langle i\lambda, \pi(k.X)\rangle} a(k.X) b(e^X) dX dk \\ &= \int_K \left(\int_{\mathfrak{a}} e^{\langle ik^{-1}.\lambda, X\rangle} a(k.X) b(e^X) dX \right) dk,\end{aligned}$$

where in the second line we extend $\lambda \in \mathfrak{a}^*$ to $\mathfrak{p}^*$ by pulling back along π, and replace the adjoint action with the coadjoint action. Let $c(k, X) := a(k.X)b(e^X)$ and note that $c \in C_c^\infty(K \times \mathfrak{a})$. Let $\widehat{c}(k, \xi) \in C^\infty(K \times \mathfrak{a}^*)$ be the Fourier transform of c in the second variable. Then the inner integral is just $\widehat{c}$ evaluated at the frequency $\xi = -\pi^*(k^{-1}.\lambda)$, where $\pi^* : \mathfrak{p}^* \to \mathfrak{a}^*$ is restriction:

$$(*) = \int_K \widehat{c}(k, -\pi^*(k^{-1}.\lambda)) dk.$$

Since $\widehat{c}(k, \xi)$ has rapid decay in ξ for all k and K is compact, $\sup_k \widehat{c}(k, \xi)$ also has rapid decay in ξ. Let B be the indicator function of the unit ball on $\mathfrak{a}^*$, pulled back along π^* to $\mathfrak{p}^*$. Taking the supremum in the first variable and upper bounding in the second variable by balls gives

$$\left|(*)\right| \leq \sum_{r=1}^\infty d_r \int_K B(r^{-1}(k^{-1}.\lambda)) dk$$

where the sequence d_r has rapid decay. Essentially, we want to know how much the coadjoint orbit $K.\lambda$ can concentrate near $(\mathfrak{a}^*)^\perp$. It suffices to show the following lemma:

Lemma 1. *Let $Sym_n(\mathbb{R})$ be the set of real symmetric $n \times n$ matrices. For $X \in Sym_n(\mathbb{R})$ let $\pi(X)$ be the diagonal part of X and $B(X) = \mathbf{1}[\|\pi(X)\| < 1]$. Let $\lambda = diag(\lambda_1, \ldots, \lambda_n)$*

with $\lambda_1 < \lambda_2 < \cdots < \lambda_n$. *Define*

$$I_n(\lambda) := \int_{SO(n)} B(k.\lambda)\, dk$$

and

$$A_n(\lambda) := (1 + \|\lambda\|)^{-n+1} L_n(\lambda).$$

Then if $\operatorname{Tr}\lambda = 0$ *we have* $I_n(\lambda) \asymp A_n(\lambda)$.

Finally, since $(*)$ is continuous in λ, we may perturb λ to be regular (i.e. $\lambda_i < \lambda_{i+1}$ rather than $\leq$), then

$$\begin{aligned} \bigl|(*)\bigr| &\leq \sum_{r=1}^{\infty} d_r (1 + r^{-1}\|\lambda\|)^{-n+1} L_n(r^{-1}\lambda) \\ &\ll (1 + \|\lambda\|)^{-n+1} L_n(\lambda). \end{aligned}$$

Chapter 3

The coadjoint orbit $K.\lambda$

The remainder of the paper will be proving Lemma 1.

3.1 Preliminaries

We note some properties of $I_n(\lambda)$ and $A_n(\lambda)$.

- $I_n(\lambda)$ is monotone under scaling: if $t \geq 1$ then $I_n(t\lambda) \leq I_n(\lambda)$.
- For a fixed $C \geq 0$, we have $A_n(C\lambda) \asymp_C A_n(\lambda)$, and $\|\lambda - \lambda'\| \leq C$ implies $A_n(\lambda) \asymp_C A_n(\lambda')$.
- An estimate for $I_n(\lambda)$ when $\operatorname{Tr}\lambda \neq 0$ easily follows from the tracefree case. Write $\lambda = \lambda_0 + (\operatorname{Tr}\lambda/n)I$. Then

$$\begin{aligned}\|\pi(k.\lambda)\|^2 &= \|\pi(k.\lambda_0)\|^2 + 2\langle \pi(k.\lambda_0), (\operatorname{Tr}\lambda/n)I\rangle + \|\pi((\operatorname{Tr}\lambda/n)I)\|^2 \\ &= \|\pi(k.\lambda_0)\|^2 + (\operatorname{Tr}\lambda)^2/n,\end{aligned}$$

so $\|\pi(k.\lambda)\| < 1$ if and only if $\|\pi(k.\lambda_0)\| < \sqrt{1-(\operatorname{Tr}\lambda)^2/n}$. Thus

$$I_n(\lambda) = \begin{cases} I_n(\lambda_0/\sqrt{1-(\operatorname{Tr}\lambda)^2/n}) & |\operatorname{Tr}\lambda| \leq \sqrt{n} \\ 0 & |\operatorname{Tr}\lambda| > \sqrt{n} \end{cases}. \tag{3.1}$$

We also note the following soft lower bound for $I_n(\lambda)$.

Lemma 2. $I_n(\lambda) \gg (1+\|\lambda\|)^{-\dim SO(n)}$.

Proof. An easy inductive argument shows that for all λ with $\operatorname{Tr}\lambda = 0$, there exists some $k_0 \in SO(n)$ such that $\pi(k_0.\lambda) = 0$. Indeed, suppose $\lambda_1 < 0 < \lambda_2$, and let $R_\theta \in SO(2)$ act on the first two coordinates, then

$$R_{\pi/2}.\begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \end{bmatrix} = \begin{bmatrix} \lambda_2 & & \\ & \lambda_1 & \\ & & \ddots \end{bmatrix},$$

so by continuity there exists θ such that $R_\theta.\lambda$ has a 0 in the upper left corner, etc.

Choose $X \in \mathfrak{so}(n)$ with $\|X\| < 1/100$. Taylor expansion gives

$$(\exp X)k_0.\lambda = k_0.\lambda + X(k_0.\lambda) - (k_0.\lambda)X + \frac{1}{2}\left(X^2(k_0.\lambda) - 2X(k_0.\lambda)X + (k_0.\lambda)X^2\right) + \dots$$

Thus $\left\|\pi((\exp X)k_0.\lambda)\right\| \ll \|X\|\|\lambda\|$, and $\left\|\pi(k.\lambda)\right\| < 1$ on a ball of radius $\gg \min(1/100, \|\lambda\|^{-1})$ around k_0. □

Since $(1+\|\lambda\|)^{-\dim SO(n)} \ll I_n(\lambda) \leq 1$, in the rest of the proof we may assume that $\|\lambda\|$ is greater than some constant depending only on n and prove $I_n(\lambda) \asymp \|\lambda\|^{-n+1} L_n(\lambda)$ instead of $I_n(\lambda) \asymp (1+\|\lambda\|)^{-n+1} L_n(\lambda)$.

3.2 Inductive step

For $n=2$, we have $\lambda = \begin{bmatrix} -a & 0 \\ 0 & a \end{bmatrix}$ and the integral can be evaluated with basic trigonometry:

$$\begin{aligned}\int_{SO(2)} B(k.\lambda)dk &= \frac{1}{2\pi}\int_0^{2\pi} B\left(\begin{bmatrix} -a\cos 2\theta & a\sin 2\theta \\ a\sin 2\theta & a\cos 2\theta \end{bmatrix}\right) d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \mathbf{1}[2a^2\cos^2(2\theta) < 1] d\theta \\ &\asymp (1+|a|)^{-1},\end{aligned}$$

so assume $n \geq 3$.

Let $e_1, \ldots, e_n$ be basis vectors for $\mathbb{R}^n$ and let $SO(n-1) \subset SO(n)$ be the subgroup fixing e_n. Let $\Pi : \mathrm{Sym}_n(\mathbb{R}) \to \mathrm{Sym}_{n-1}(\mathbb{R})$ be the projection to the upper left $n-1 \times n-1$ submatrix; we have $\Pi(k'.X) = k'.\Pi(X)$ for $k' \in SO(n-1)$. Let π' and B' be the $n-1$-dimensional versions of π and B. Let $X = k.\lambda$. We have $\left\|\pi(X)\right\|^2 = \left\|\pi'(\Pi(X))\right\|^2 + X_{nn}^2$, so $B(X) \leq B'(\Pi(X))$ and

$$I_n(\lambda) \leq \int_{SO(n)} B'(\Pi(k.\lambda))\, dk. \tag{3.2}$$

On the other hand, since $\operatorname{Tr} X = 0$ we have $X_{nn} = -\sum_{i=1}^{n-1} X_{ii}$ and by Cauchy-Schwarz $X_{nn}^2 \leq (n-1)\sum_{i=1}^{n-1} X_{ii}^2 = (n-1)\left\|\pi'(\Pi(X))\right\|^2$, so $\left\|\pi(X)\right\| \leq \sqrt{n}\left\|\pi'(\Pi(X))\right\|$ and

$$I_n(\lambda) \geq \int_{SO(n)} B'(\sqrt{n}\Pi(k.\lambda))\, dk.$$

To do the induction we will convert the integral over $SO(n)$ to an integral over $SO(n-1)$:

Lemma 3. *Define*

$$\mathcal{M} := \left\{ \begin{bmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_{n-1} \end{bmatrix} : \lambda_i < \mu_i < \lambda_{i+1} \right\}$$

and

$$J(\mu) := \frac{\prod_{1 \le i < j \le n-1} |\mu_i - \mu_j|}{\prod_{\substack{1 \le i \le n \\ 1 \le j \le n-1}} |\lambda_i - \mu_j|^{1/2}}.$$

For any non-negative $f : Sym_{n-1}(\mathbb{R}) \to \mathbb{R}$, we have

$$\int_{SO(n)} f(\Pi(k.\lambda))\, dk = c_n \int_{\mathcal{M}} \int_{SO(n-1)} f(k'.\mu)\, dk'\, J(\mu)\, d\mu. \tag{3.3}$$

for some absolute constant $c_n > 0$.

Proof. Let $\mathcal{A} \subset \mathrm{Sym}_{n-1}(\mathbb{R})$ be the set of matrices conjugate to some $\mu \in \mathcal{M}$, equipped with the metric inherited from $\mathbb{R}^{(n-1)^2}$. We will show both sides are proportional to

$$\int_{\mathcal{A}} f(X) \prod_{\substack{1 \le i \le n \\ 1 \le j \le n-1}} |\lambda_i - \mu_j(X)|^{-1/2}\, dX \tag{3.4}$$

where $\mu_j(X)$ is the j-th highest eigenvalue of X, viewed as a function on $\mathcal{A}$. Define a map

$$F : SO(n) \to \mathrm{Sym}_{n-1}(\mathbb{R})$$
$$F(k) = \Pi(k.\lambda).$$

We begin by pushing the left hand side of (3.3) forward along F to get (3.4). First we exclude the degenerate case where $\Pi(k.\lambda)$ shares an eigenvalue with λ.

Lemma 4. *λ_i is an eigenvalue of $\Pi(k.\lambda)$ if and only if $\langle e_i, k^{-1} e_n \rangle = 0$.*

Proof. By replacing λ with $\lambda - \lambda_i I$, we may assume without loss of generality that $\lambda_i = 0$. Let $P = I - e_n e_n^T$, the projection killing the n-th coordinate. Then for any $A \in \mathrm{Sym}_n(\mathbb{R})$,

$$PAP = \begin{bmatrix} \Pi(A) & 0 \\ 0 & 0 \end{bmatrix},$$

so the eigenvalues of PAP are the eigenvalues of $\Pi(A)$, with an extra multiplicity at 0.

Suppose λ_i is an eigenvalue of $\Pi(k.\lambda)$, then $Pk\lambda k^{-1}Pv = 0$ for some v with $Pv \neq 0$. If $\lambda k^{-1}Pv = 0$, then $k^{-1}Pv = ce_i$ for some $c \neq 0$, and $\langle e_i, k^{-1}e_n\rangle = c^{-1}\langle k^{-1}Pv, k^{-1}e_n\rangle = c^{-1}\langle Pv, e_n\rangle = c^{-1}\langle v, Pe_n\rangle = 0$. Otherwise, if $Pk\lambda k^{-1}Pv = 0$ and $\lambda k^{-1}Pv \neq 0$ then $k\lambda k^{-1}Pv = ce_n$ for some $c \neq 0$ and $\langle e_i, k^{-1}e_n\rangle = c^{-1}\langle e_i, \lambda k^{-1}Pv\rangle = c^{-1}\langle \lambda e_i, k^{-1}Pv\rangle = 0$.

Conversely, if $\langle e_i, k^{-1}e_n\rangle = 0$ then $Pke_i = ke_i$ and $Pk\lambda k^{-1}Pke_i = Pk\lambda e_i = 0$, so ke_i is the desired eigenvector for $\Pi(k.\lambda)$. □

Let $\mathcal{S} = \{k \in SO(n) : \langle e_i, k^{-1}e_n\rangle = 0 \text{ for some } i\ \}$. Note that the quotient $SO(n-1)\backslash\mathcal{S}$ is the intersection of the sphere $S^n \simeq SO(n-1)\backslash SO(n)$ with the coordinate planes, so $\mathcal{S}$ is negligible. The following calculation of Fan and Pall [6, Theorem 1 and pp.300-301] shows that $F(SO(n) - \mathcal{S}) = \mathcal{A}$.

Lemma 5 (Fan-Pall)**.** *Let $\mu_1 \leq \cdots \leq \mu_{n-1}$ and $z_1, \ldots, z_n \in \mathbb{R}$. Suppose the matrix*

$$\begin{bmatrix} \mu_1 & & & z_1 \\ & \ddots & & \vdots \\ & & \mu_{n-1} & z_{n-1} \\ z_1 & \cdots & z_{n-1} & z_n \end{bmatrix} \tag{3.5}$$

has eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$. Then $\lambda_1 \leq \mu_1 \leq \lambda_2 \leq \cdots \leq \mu_{n-1} \leq \lambda_n$. Furthermore, if $\lambda_i \neq \mu_j$ for all i, j then

$$z_i^2 = \frac{\prod_{1\leq j\leq n}|\lambda_j - \mu_i|}{\prod_{\substack{1\leq j\leq n-1 \\ i\neq j}}|\mu_i - \mu_j|}$$

for $1 \leq i \leq n-1$, and taking traces gives $z_n = \sum_i \lambda_i - \sum_j \mu_j$.

Conversely, for any μ and λ satisfying $\lambda_i \neq \mu_j$ the above choice of z_i gives a matrix with eigenvalues λ.

We check that F restricted to $SO(n) - \mathcal{S}$ is a smooth covering and compute its differential. Let $Y \in \mathcal{A}$. Conjugating by $SO(n-1)$ we may assume Y is diagonal. There are 2^{n-1} choices of $X \in \mathrm{Sym}_n(\mathbb{R})$ conjugate to λ such that $\Pi(X) = Y$, corresponding to choices of sign for the z_i in (3.5). Furthermore, since the centralizer of λ in $SO(n)$ consists of reflections through an even number of coordinate axes, for each X there are 2^{n-1} choices of $k \in SO(n) - \mathcal{S}$ such that $k.\lambda = X$.

Let $\mathbf{E_{ij}}$ be the $n \times n$ matrix with a 1 at position (i,j), a -1 at (j,i) and 0 elsewhere. Then $\{\mathbf{E_{ij}}k : 1 \leq i < j \leq n\}$ is an orthonormal basis for $T_k SO(n)$ (up to some constant). Let $\mathbf{F_{ij}}$ be the $n-1 \times n-1$ matrix with a 1 at (i,j) and (j,i) (or a single 1 if $i = j$). Then $\{\mathbf{F_{ij}} : 1 \leq i \leq j \leq n-1\}$ is a orthonormal basis for $T_{F(k)}\mathcal{A}$.

We calculate

$$
\begin{aligned}
D_k F(\mathbf{E_{ij}}k) &= \lim_{t \to 0} \frac{1}{t} \left[\Pi((\exp t\mathbf{E_{ij}})k.\lambda) - \Pi(k.\lambda) \right] \\
&= \Pi([\mathbf{E_{ij}}, k.\lambda]) \\
&= \begin{cases} (\mu_j - \mu_i)\mathbf{F}_{ij} & j < n \\ \sum_{h=1}^{n-1} z_h (1 + \delta_{ih}) \mathbf{F}_{ih} & j = n \end{cases} .
\end{aligned}
$$

The Jacobian determinant has one nonzero term (the one that associates $\mathbf{E_{ij}}$ to $\mathbf{F_{ij}}$ for $j < n$ and $\mathbf{F_{ii}}$ for $j = n$). Hence

$$
\det D_k F = \prod_{1 \leq i < j \leq n-1} (\mu_j - \mu_i) \prod_{1 \leq i \leq n-1} 2z_i = \pm 2^{n-1} \prod_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n-1}} \left| \lambda_i - \mu_j \right|^{1/2},
$$

which is nonvanishing everywhere. Thus F defines a smooth 2^{2n-2}-fold covering, and pushing forward along F we see that the left hand side of (3.3) is proportional to (3.4).

On the other hand, define

$$G : SO(n-1) \times \mathcal{M} \to \mathcal{A}$$
$$G(k', \mu) = k'.\mu.$$

It suffices to compute the differential at $k' = 1$. Take $\{\mathbf{E}'_{\mathbf{ij}} : 1 \le i < j \le n-1\}$ as a basis for $T_1 SO(n-1)$ and let $(e'_i)_{1 \le i \le n-1}$ be the obvious basis for $T_\mu \mathcal{M}$. Then

$$D_{(1,\mu)} G(\mathbf{E}'_{\mathbf{ij}}, 0) = [\mathbf{E}'_{\mathbf{ij}}, \mu] = (\mu_j - \mu_i)\mathbf{F}'_{\mathbf{ij}}$$
$$D_{(1,\mu)} G(0, e'_i) = \mathbf{F}'_{\mathbf{ii}}$$

and the Jacobian determinant is $\pm \prod_{1 \le i < j \le n-1} |\mu_i - \mu_j|$, so the right hand side of (3.3) pushes forward to (3.4).

□

Define

$$J_n(\lambda) := \int_{\mathcal{M}} A_{n-1}(\mu) J(\mu) \, d\mu.$$

The next step is to show $I_n(\sqrt{n-1}\lambda) \ll J_n(\lambda) \ll I_n(\lambda/\sqrt{n})$. Then it will suffice to prove $J_n(\lambda) \asymp \|\lambda\|^{-n+1} L_n(\lambda)$. Applying Lemma 3 to Equation (3.2) we get

$$I_n(\lambda) \ll \int_{\mathcal{M}} \int_{SO(n-1)} B'(k'.\mu) \, dk' J(\mu) \, d\mu$$
$$= \int_{\mathcal{M}} I_{n-1}(\mu) J(\mu) \, d\mu.$$

By Equation (3.1) we may restrict to the region where $|\operatorname{Tr} \mu| < \sqrt{n-1}$, in which we have $I_{n-1}(\mu) = I_{n-1}(\mu_0 / \sqrt{1 - (\operatorname{Tr} \mu)^2/(n-1)})$, where μ_0 is the tracefree part of μ. By monotonicity, this is $\le I_{n-1}(\mu_0)$, by induction $I_{n-1}(\mu_0) \asymp A_{n-1}(\mu_0)$ and since $\|\mu_0 - \mu\|$ is bounded, $A_{n-1}(\mu_0) \asymp A_{n-1}(\mu)$. The upper bound now reads

$$I_n(\lambda) \ll \int_{\substack{\mu \in \mathcal{M} \\ |\operatorname{Tr} \mu| < \sqrt{n-1}}} A_{n-1}(\mu) J(\mu) d\mu.$$

Scaling λ and μ by $\sqrt{n-1}$ this is equivalent to

$$I_n(\sqrt{n-1}\lambda) \ll \int_{\substack{\mu\in\mathcal{M}\\ |\operatorname{Tr}\mu|<1}} A_{n-1}(\sqrt{n-1}\mu)J(\mu)d\mu,$$

and since $A_{n-1}(\sqrt{n-1}\mu) \asymp A_{n-1}(\mu)$ the RHS is $\asymp J_n(\lambda)$, so $I_n(\sqrt{n-1}\lambda) \ll J_n(\lambda)$.

On the other hand, for the lower bound we have

$$\begin{aligned} I_n(\lambda) &\gg \int_{\mathcal{M}}\int_{SO(n-1)} B'(\sqrt{n}k'.\mu)\,dk'\,J(\mu)\,d\mu \\ &= \int_{\mathcal{M}} I_{n-1}(\sqrt{n}\mu)J(\mu)\,d\mu, \end{aligned}$$

and we may restrict the integral to the region where $\operatorname{Tr}\mu < 1/\sqrt{n}$, in which

$$\begin{aligned} I_{n-1}(\sqrt{n}\mu) &= I_{n-1}(\sqrt{n}\mu_0/\sqrt{1-(\operatorname{Tr}\sqrt{n}\mu)^2/(n-1)}) \\ &\geq I_{n-1}(\sqrt{n}\mu_0/\sqrt{(n-2)/(n-1)}) \\ &\asymp A_{n-1}(\mu_0) \\ &\asymp A_{n-1}(\mu). \end{aligned}$$

So

$$I_n(\lambda) \gg \int_{\substack{\mu\in\mathcal{M}\\ |\operatorname{Tr}\mu|<1/\sqrt{n}}} A_{n-1}(\mu)J(\mu)\,d\mu,$$

and scaling λ and μ by $1/\sqrt{n}$ we get $I_n(\lambda/\sqrt{n}) \gg J_n(\lambda)$.

Our strategy for showing $J_n(\lambda) \asymp \|\lambda\|^{-n+1} L_n(\lambda)$ will be to simplify $J_n(\lambda)$ to a convolution. Define

$$\begin{aligned} \mathcal{H} &:= \{\mu\in\mathcal{M} : \left|\textstyle\sum_i \mu_i\right| < 1\} \\ F_i(x) &:= \mathbf{1}_{[\lambda_i,\lambda_{i+1}]}|\lambda_i - x|^{-1/2}|\lambda_{i+1}-x|^{-1/2} \\ \kappa_{i,j} &:= \left|\mu_i-\mu_j\right|\left|\lambda_i-\mu_j\right|^{-1/2}\left|\lambda_{j+1}-\mu_i\right|^{-1/2}. \end{aligned}$$

Then

$$J_n(\lambda) = \int_{\mathcal{H}} (1+\|\mu\|)^{-n+2} L_{n-1}(\mu) \prod_{1\le i<j\le n-1} \kappa_{i,j} \prod_{1\le i\le n-1} F_i(\mu_i)\, d\mu. \tag{3.6}$$

Note that

$$F_i(x) \asymp \mathbf{1}_{[\lambda_i,\lambda_{i+1}]} |\lambda_i - \lambda_{i+1}|^{-1/2} \left(|\lambda_i - x|^{-1/2} + |\lambda_{i+1} - x|^{-1/2} \right),$$

implying $\int F_i(x)dx \asymp 1$. Also, $\kappa_{i,j} \le 1$ on $\mathcal{H}$. If we can eliminate the factors $(1+\|\mu\|)^{-n+2}$, $L_{n-1}(\mu)$, and $\kappa_{i,j}$ then the integral becomes simply a convolution evaluated at 0:

$$\begin{aligned}\int_{\mathcal{H}} \prod_{1\le i\le n-1} F_i(\mu_i)\, d\mu &= \int \cdots \int \mathbf{1}_{[-1,1]} \left(-\textstyle\sum_i \mu_i\right) \prod_{1\le i\le n-1} F_i(\mu_i)\, d\mu_{n-1} \dots d\mu_1 \\ &= \left(\mathbf{1}_{[-1,1]} * \left(\mathop{\ast}_{1\le i\le n-1} F_i \right) \right)(0).\end{aligned}$$

Let $d = \lambda_n - \lambda_1$ and let $d_i = \lambda_{i+1} - \lambda_i$ be the "spectral gaps." Then Lemma 1 follows from the following cases:

First, by Lemma 6, we have unconditionally that $J_n(\lambda) \gg \|\lambda\|^{-n+1}$.

- Let I be the index of the largest gap. Suppose $d_I > (1 - 1/100n)d$. Then:
  - If $(n, I) \ne (4, 2)$, then $L_n(\lambda) \asymp 1$ and it remains to show $J_n(\lambda) \ll \|\lambda\|^{-n+1}$. See Lemma 9.
  - If $(n, I) = (4, 2)$, then the exceptional term term in $L_n(\lambda)$ is large. See Lemma 10.
- Otherwise, let $I < J$ be the two largest gaps, breaking ties arbitrarily. Then $(1 - 1/100n)d \ge d_I, d_J \ge d/100n^2$.
  - If $(I, J) \ne (1, n-1)$ then $L_n(\lambda) \asymp 1$ and it remains to show $J_n(\lambda) \ll \|\lambda\|^{-n+1}$. See Lemma 11.
  - If $(I, J) = (1, n-1)$ then the main term in $L_n(\lambda)$ is large. See Lemma 13.

Chapter 4

Convolution Bounds

4.1 The general lower bound

Lemma 6. *For all λ with $\lambda_1 < \lambda_2 < \cdots < \lambda_n$, we have $J_n(\lambda) \gg \|\lambda\|^{-n+1}$.*

Proof. Applying the inequalities $L_n(\mu) \gg 1$ and $(1+\|\mu\|)^{-n+2} \gg \|\lambda\|^{-n+2}$ to the definition of $J_n(\lambda)$ gives

$$J_n(\lambda) \gg \|\lambda\|^{-n+2} \int_{\mathcal{H}} J(\mu) d\mu.$$

We define a subset of $\mathcal{H}$ on which the $\kappa_{i,j}$ are bounded away from 0. Let $\theta = -\lambda_1/d$. From $(n-1)\lambda_1 + \lambda_n \leq \lambda_1 + \lambda_2 + \cdots + \lambda_n = 0$ we deduce $\lambda_1 \leq -d/n$ and similarly $\lambda_1 + (n-1)\lambda_n \geq \lambda_1 + \lambda_2 + \cdots + \lambda_n = 0 \implies (n-1)d \geq -n\lambda_1$. Thus $1/n \leq \theta \leq 1 - 1/n$. Let $m_i = \theta\lambda_i + (1-\theta)\lambda_{i+1}$. Then $\sum_{1 \leq i \leq n-1} m_i = 0$. Define intervals $M_i \subset [\lambda_i, \lambda_{i+1}]$ centered on the m_i, given by

$$M_i = [m_i - d_i/2n, m_i + d_i/2n]$$

and restrict the integral to the set

$$\mathcal{H}' = \{\mu : \mu_i \in M_i, \left|\textstyle\sum_i \mu_i\right| < 1\} \subset \mathcal{H}.$$

Then for $\mu \in \mathcal{H}'$ we have

$$\frac{|\mu_i - \mu_j|}{|\mu_i - \lambda_{j+1}|} = \frac{|\mu_i - \lambda_j| + |\lambda_j - \mu_j|}{|\mu_i - \lambda_j| + |\lambda_j - \lambda_{j+1}|} \geq \frac{|\lambda_j - \mu_j|}{|\lambda_j - \lambda_{j+1}|} \geq \frac{1}{2n},$$

and similarly $|\mu_i - \mu_j| / |\lambda_i - \mu_j| \geq 1/2n$, so $\kappa_{ij} \geq 1/2n$. Also, for $\mu_i \in M_i$ we have

$$F_i(\mu_i) \asymp |\lambda_i - \lambda_{i+1}|^{-1/2} \left(|\lambda_i - \mu_i|^{-1/2} + |\lambda_{i+1} - \mu_i|^{-1/2} \right) \gg d_i^{-1}.$$

Applying the lower bounds for $\kappa_{i,j}$ and F_i gives

$$J_n(\lambda) \gg \|\lambda\|^{-n+2} \int_{\mathcal{H}'} \prod_{1 \leq i \leq n-1} d_i^{-1} \mathbf{1}_{M_i}(\mu_i) d\mu_i,$$

and this is equivalent to a convolution

$$\begin{aligned} J_n(\lambda) &\gg \|\lambda\|^{-n+2} G(0) \\ G &= \mathbf{1}_{[-1,1]} * \left(\mathop{\ast}_{1 \leq i \leq n-1} d_i^{-1} \mathbf{1}_{M_i} \right). \end{aligned}$$

Since $\sum_i m_i = 0$ we can translate each factor to be centered at 0,

$$G = \mathbf{1}_{[-1,1]} * \left(\mathop{\ast}_{1 \leq i \leq n-1} d_i^{-1} \mathbf{1}_{[-d_i/2n, d_i/2n]} \right).$$

Then by the following Lemma, $G(x)$ is maximized at $x = 0$.

Lemma 7. *For a sequence of reals $a_i \geq 0$ let*

$$f_n(x) = \mathop{\ast}_{1 \leq i \leq n} \mathbf{1}_{[-a_i, a_i]}(x)$$

then f_n is maximized at 0 for all n.

Proof. The f_n are obviously even. We prove a stronger statement, that f_n is nonincreasing on $[0,\infty)$. The case $n=1$ is trivial. For $n>1$ and $x>0$ we have

$$\begin{aligned} f_n(x) &= \int \mathbf{1}_{[-a_n,a_n]}(x-y) f_{n-1}(y)dy \\ &= \int_{x-a_n}^{x+a_n} f_{n-1}(y)dy \end{aligned}$$

implying, for $0<x<x'$

$$\begin{aligned} f_n(x)-f_n(x') &= \int_{x-a_n}^{x+a_n} f_{n-1}(y)dy - \int_{x'-a_n}^{x'+a_n} f_{n-1}(y)dy \\ &= \int_{x-a_n}^{x'-a_n} f_{n-1}(y)dy - \int_{x+a_n}^{x'+a_n} f_{n-1}(y)dy \\ &= \int_{x}^{x'} f_{n-1}(y-a_n) - f_{n-1}(y+a_n)dy, \end{aligned}$$

and by induction the integrand is ≥ 0. □

Finally, note that $\int G(x)dx = 2\prod_i \left\| d_i^{-1}\mathbf{1}_{M_i} \right\|_1 \gg 1$ and $G(x)$ is supported in an interval of length $\ll \|\lambda\|$. Thus, $G(0) \gg \|\lambda\|^{-1}$ and we are done. □

4.2 Monotone rearrangement

In order to upper bound various convolutions we introduce the following convenient tool. For $f:\mathbb{R}\to[0,\infty)$ define the level set

$$\{f>t\} = \{x\in\mathbb{R} : f(x)>t\},$$

and the "layer cake representation" of f,

$$f(x) = \int \mathbf{1}_{\{f>t\}}(x)\,dt.$$

For a set $S \subset \mathbb{R}$ define the *rearranged* set $S^* = [0, \mu(S))$. Finally define the *monotone rearrangement*

$$f^*(x) = \int \mathbf{1}_{\{f>t\}^*}(x)\, dt.$$

For example, the monotone rearrangement of $\mathbf{1}_{[a,b]}$ is $\mathbf{1}_{[0,b-a]}$, and the monotone rearrangement of

$$f(x) = \mathbf{1}_{[a,b]}|a-x|^{-1/2}|b-x|^{-1/2}$$

is

$$\begin{aligned} f^*(x) &= \mathbf{1}_{[0,b-a]}(x/2)^{-1/2}((b-a)-x/2)^{-1/2} \\ &\ll (b-a)^{-1/2}x^{-1/2}. \end{aligned}$$

We note basic properties such as $f \le g \implies f^* \le g^*$ and $(af)^* = af^*$. The real workhorse is the n-ary Hardy-Littlewood Rearrangement inequality:

Lemma 8. *For $f_1, \ldots, f_n : \mathbb{R} \to [0, \infty)$ we have*

$$\int \prod_{i=1}^{n} f_i(x) dx \le \int \prod_{i=1}^{n} f_i^*(x)\, dx.$$

Proof.

$$\begin{aligned} \int_{\mathbb{R}} f_1(x) \ldots f_n(x)\, dx &= \int_{\mathbb{R}} \int_0^\infty \cdots \int_0^\infty \mathbf{1}_{\{f_1>t_1\}}(x) \ldots \mathbf{1}_{\{f_n>t_n\}}(x)\, dt_n \ldots dt_1\, dx \\ &= \int_0^\infty \cdots \int_0^\infty \mu(\{f_1 > t_1\} \cap \cdots \cap \{f_n > t_n\})\, dt_n \ldots dt_1 \\ &\le \int_0^\infty \cdots \int_0^\infty \mu(\{f_1^* > t_1\} \cap \cdots \cap \{f_n^* > t_n\})\, dt_n \ldots dt_1 \\ &= \int_{\mathbb{R}} f_1^*(x) \ldots f_n^*(x)\, dx. \end{aligned}$$

□

For $n = 2$ this implies $\|f * g\|_\infty \le \langle f^*, g^* \rangle$, where $f * g$ is the convolution of f and g.

We also note the following inequalities for $0 < a < b$:

$$\int_a^b x^{-1/2} \left(\log' \frac{T}{x}\right)^k dx \ll_k b^{1/2} \left(\log' \frac{T}{b}\right)^k \tag{4.1}$$

$$\int_a^b x^{-1} \left(\log' \frac{T}{x}\right)^k dx \ll \log \frac{b}{a} \left(\log' \frac{T}{a}\right)^k, \tag{4.2}$$

the first of which can be verified by applying integration by parts k times.

4.3 One Large Gap

In this section we assume there exists I such that $d_I > (1 - 1/(100n))d$.

Let $d' = d - d_I = |\lambda_1 - \lambda_I| + |\lambda_{I+1} - \lambda_n| \leq d/100n$. We show that n and I determine the positions of λ_i and μ_i up to an $O(d')$ error. Write $\pm C$ for an error of absolute value $\leq C$. Then $\lambda_i = \lambda_1 \pm d'$ for $i \leq I$ and $\lambda_i = \lambda_n \pm d'$ for $i > I$. Writing

$$\begin{aligned} 0 &= \sum_i \lambda_i \\ &= I\lambda_1 + (n - I)\lambda_n \pm nd' \end{aligned}$$

we get $\lambda_n = Id/n \pm d'$ and $\lambda_1 = (I - n)d/n \pm d'$. Then

$$\lambda_i = \begin{cases} \dfrac{(I-n)d}{n} \pm 2d' & i \leq I \\ \dfrac{Id}{n} \pm 2d' & i > I \end{cases} \tag{4.3}$$

and (using $\left|\sum_i \mu_i\right| \leq 1$ to find μ_I)

$$\mu_i = \begin{cases} \dfrac{(I-n)d}{n} \pm 2d' & i < I \\ \dfrac{(I-2n)d}{n} \pm (2nd' + 1) & i = I \\ \dfrac{Id}{n} \pm 2d' & i > I \end{cases} \tag{4.4}$$

4.3.1 The upper bound when $(n, I) \neq (4, 2)$

Lemma 9. *Assume there exists I such that $d_I > (1 - 1/(100n))d$ and $(n, I) \neq (4, 2)$. Then $L_n(\lambda) \asymp 1$ and $J_n(\lambda) \asymp \|\lambda\|^{-n+1}$.*

By Equation 4.3 we have $|\lambda_2| \gg d$ so

$$\log' \frac{\|\lambda\|}{1 + |\lambda_2| + |\lambda_{n-1}|} \ll 1.$$

Also, if $n = 4$ but $I \neq 2$ then at least one of $|\lambda_1 - \lambda_2|$ and $|\lambda_3 - \lambda_4|$ is $\gg d$ so

$$\log' \frac{\|\lambda\|}{1 + |\lambda_1 - \lambda_2| + |\lambda_3 - \lambda_4|} \ll 1.$$

Thus $L_n(\lambda) \asymp 1$, and by Lemma 6, $J_n(\lambda) \gg \|\lambda\|^{-n+1}$, so it remains to show $J_n(\lambda) \ll \|\lambda\|^{-n+1}$.

If $I \geq 2$ then $|\mu_1| \gg d$, otherwise, if $I = 1$ then $|\mu_{n-1}| \gg d$. Thus $(1 + \|\mu\|)^{-n+2} \asymp \|\lambda\|^{-n+2}$.

If $n = 4$ and $I \neq 2$ then $|\mu_2| \gg d$ and $L_{n-1}(\mu) \asymp 1$.

If $n \geq 5$, then at least one of $|\mu_2|$ and $|\mu_{n-2}|$ must be $\gg d$, so

$$\log' \frac{\|\mu\|}{1 + |\mu_2| + |\mu_{n-2}|} \asymp 1.$$

If $n = 5$ then the exceptional factor in $L_{n-1}(\mu)$ is also $\asymp 1$ since at least one of $|\mu_1 - \mu_2|$ and $|\mu_3 - \mu_4|$ must be $\gg d$. Thus $L_{n-1}(\mu) \asymp 1$ for $n \geq 5$.

Bounding $\kappa_{i,j} \leq 1$ for all i, j we get

$$J_n(\lambda) \ll \|\lambda\|^{-n+2} \int_{\mathcal{H}} \prod_i F_i(\mu_i)\, d\mu.$$

By Equation (4.4) we have $\mu_I = (I - 2n)d/n \pm (2nd' + 1)$. Then (taking d sufficiently large) $|\lambda_I - \mu_I|$ and $|\lambda_{I+1} - \mu_I|$ are $\gg d$ so $F_I(t) \asymp \|\lambda\|^{-1}$. Finally

$$\begin{aligned} J_n(\lambda) &\ll \|\lambda\|^{-n+1} \int_{\mathcal{H}} \prod_{i \neq I} F_i(\mu_i)\, d\mu \\ &= \|\lambda\|^{-n+1} \left(\mathbf{1}_{[-1,1]} * \left(\mathop{\ast}_{i \neq I} F_i \right) \right) (0) \\ &\ll \|\lambda\|^{-n+1} \prod_{i \neq I} \|F_i\|_1 \\ &\ll \|\lambda\|^{-n+1} \end{aligned}$$

and we are done.

4.3.2 The case $(n, I) = (4, 2)$

Lemma 10. *Assume there exists I such that $d_I > (1 - 1/(100n))d$ and $(n, I) = (4, 2)$. Then*

$$L_n(\lambda) \asymp \log' \frac{\|\lambda\|}{1 + |\lambda_1 - \lambda_2| + |\lambda_3 - \lambda_4|}$$

, and $J_n(\lambda) \asymp \|\lambda\|^{-n_1} L_n(\lambda)$.

The proof proceeds similarly to Lemma 9, except that we cannot eliminate $L_{n-1}(\mu) = \log' \|\lambda\| / (1 + |\mu_2|)$. We write instead

$$J_n(\lambda) \ll \|\lambda\|^{-n+2} \int_{-\infty}^{\infty} L(\mu_2) G(\mu_2)\, d\mu_2.$$

where

$$\begin{aligned} L(t) &= \log' \frac{\|\lambda\|}{1 + |t|} \\ G &= \mathbf{1}_{[-1,1]} * F_1 * F_3. \end{aligned}$$

Using Young's inequality and the fact that $\|F_i\|_1 \asymp 1$ we have

$$\begin{aligned} J_n(\lambda) &\ll \|\lambda\|^{-n+1} \left\| L * \mathbf{1}_{[-1,1]} * F_1 * F_3 \right\|_\infty \\ &\leq \|\lambda\|^{-n+1} \|L * F_1\|_\infty \left\| \mathbf{1}_{[-1,1]} \right\|_1 \|F_3\|_1 \\ &\ll \|\lambda\|^{-n+1} \|L * F_1\|_\infty . \end{aligned}$$

Then using the Hardy-Littlewood rearrangement inequality and equation (4.1):

$$\begin{aligned} \|L * F_1\|_\infty &\leq \langle L^*, F_1^* \rangle \\ &\ll \int_0^{d_1} \log' \frac{\|\lambda\|}{1 + |x|} d_1^{-1/2} x^{-1/2} \, dx \\ &\ll \log' \frac{\|\lambda\|}{1 + d_1}, \end{aligned}$$

so $J_n(\lambda) \ll \|\lambda\|^{-n+1} \frac{\|\lambda\|}{1+d_1}$ and similarly $J_n(\lambda) \ll \|\lambda\|^{-n+1} \log' \frac{\|\lambda\|}{1+d_3}$. Together these imply

$$J_n(\lambda) \ll \|\lambda\|^{-n+1} \log' \frac{\|\lambda\|}{1 + |\lambda_1 - \lambda_2| + |\lambda_3 - \lambda_4|}$$

as desired. For the lower bound, we restrict to $\mathcal{H}'$, then since $\kappa_{i,j} \gg 1$ for all $i < j$, and $(1 + \|\mu\|^{-2} \gg \|\lambda\|^{-2}$ we have

$$J_n(\lambda) \gg \|\lambda\|^{-2} \int_{\mathcal{H}'} \log' \frac{\|\lambda\|}{1 + |\mu_2|} \prod_{1 \leq i \leq 3} F_i(\mu_i) \, d\mu.$$

By Equation (4.4) we have $|\mu_2| \leq 1 + 4d'$ for $\mu \in \mathcal{H}'$, so

$$J_n(\lambda) \gg \|\lambda\|^{-2} \log' \frac{\|\lambda\|}{2 + 4d'} \int_{\mathcal{H}'} \prod_{1 \leq i \leq 3} F_i(\mu_i) \, d\mu$$

and using the same argument as in Lemma 6 the integral is $\gg \|\lambda\|^{-1}$. Finally since $d' = |\lambda_1 - \lambda_2| + |\lambda_3 - \lambda_4|$ we have

$$J_n(\lambda) \gg \|\lambda\|^{-3} \log' \frac{\|\lambda\|}{1 + |\lambda_1 - \lambda_2| + |\lambda_3 - \lambda_4|}.$$

4.4 Two Large Gaps

In this section we assume no index I satisfies $d_I > (1-1/100n)d$. This implies there exist indices $I < J$ such that $d_I, d_J \geq d/100n^2$.

4.4.1 The case $(I,J) \neq (1, n-1)$

Lemma 11. *Suppose there exist $I < J$ with $(I,J) \neq (1,n-1)$ such that $d_I, d_J \geq d/100n^2$. Then $L_n(\lambda) \asymp 1$ and $J_n(\lambda) \asymp \|\lambda\|^{-n+1}$.*

Proof. If $|\lambda_2| + |\lambda_{n-1}| \leq d/100n^2$, then $d_i < d/100n^2$ for $2 \leq i \leq n-2$, forcing $(I,J) = (1, n-1)$. Hence $|\lambda_2| + |\lambda_{n-1}| \gg \|\lambda\|$ and the main term in $L_n(\lambda)$ is $\ll 1$. The exceptional term of $L_n(\lambda)$ is also $\ll 1$ since one of $|\lambda_1 - \lambda_2|, |\lambda_3 - \lambda_4|$ must be d_I or d_J. It remains to prove $J_n(\lambda) \ll \|\lambda\|^{-n+1}$.

If $n = 3$ then $(I,J) = (1,2)$ is forced, so we must have $n \geq 4$. By symmetry (replacing $\lambda_1, \ldots, \lambda_n$ with $-\lambda_n, \ldots, -\lambda_1$) we may assume $J \leq n-2$.

First we rewrite $L_{n-1}(\mu)$ to depend solely on μ_J:

Lemma 12. *Under the assumptions of Lemma 11,*

$$L_{n-1}(\mu) \ll \mathcal{L}(\mu_J) := \left(\log' \frac{\|\lambda\|}{|\lambda_J - \mu_J|}\right) \left(\log' \frac{\|\lambda\|}{|\lambda_{J+1} - \mu_J|}\right)^{n-2} \left(\log' \frac{\|\lambda\|}{|\mu_J|}\right)^{n-3}.$$

Proof. If $J = n-2$ then bound

$$\left(\log' \frac{\|\mu\|}{1 + |\mu_2| + |\mu_{n-2}|}\right)^{n-3} \ll \left(\log' \frac{\|\lambda\|}{|\mu_J|}\right)^{n-3}.$$

Otherwise, suppose $J \leq n-3$. Since $1 \leq I < J$ we must have $n \geq 5$. If $(I,J) = (1,2)$ then $|\mu_2| + |\mu_{n-2}| \geq |\mu_2 - \mu_3| \geq |\mu_2 - \lambda_3| = |\lambda_{J+1} - \mu_J|$ and

$$\left(\log' \frac{\|\mu\|}{1 + |\mu_2| + |\mu_{n-2}|}\right)^{n-3} \ll \log' \left(\frac{\|\lambda\|}{|\lambda_{J+1} - \mu_J|}\right)^{n-3}.$$

Otherwise, if $(I,J) \neq (1,2)$ then $n \geq 6$ is forced, and $|\mu_2| + |\mu_{n-2}| \geq |\mu_2 - \mu_{n-2}| \geq$

$|\lambda_3 - \lambda_{n-2}| \ge d_J \gg \|\lambda\|$, so

$$\left(\log' \frac{\|\mu\|}{1 + |\mu_2| + |\mu_{n-2}|}\right)^{n-3} \ll 1.$$

As for the exceptional factor in $L_{n-1}(\mu)$ that appears when $n = 5$, if $J = 3$ we bound

$$\log' \frac{\|\mu\|}{1 + |\mu_1 - \mu_2| + |\mu_3 - \mu_4|} \ll \log' \frac{\|\lambda\|}{|\lambda_{J+1} - \mu_J|},$$

and if $J = 2$ we bound

$$\ll \frac{\|\lambda\|}{|\lambda_J - \mu_J|}.$$

□

Since $\|\mu\| \gg |\mu_1 - \mu_{n-1}| \gg d_J \gg \|\lambda\|$, we have $(1 + \|\mu\|)^{-n+2} \ll \|\lambda\|^{-n+2}$.

Since $|\lambda_{J+1} - \mu_I| \ge d_J \gg \|\lambda\|$ we have

$$\begin{aligned} \kappa_{I,J} &= |\lambda_I - \mu_J|^{-1/2} |\lambda_{J+1} - \mu_I|^{-1/2} |\mu_I - \mu_J| \\ &\ll \|\lambda\|^{-1/2} |\mu_I - \mu_J|^{1/2}, \end{aligned}$$

and since $|\lambda_J - \mu_{n-1}| \ge d_J \gg \|\lambda\|$ we have

$$\begin{aligned} \kappa_{J,n-1} &= |\lambda_J - \mu_{n-1}|^{-1/2} |\lambda_n - \mu_J|^{-1/2} |\mu_J - \mu_{n-1}| \\ &\ll \|\lambda\|^{-1/2} |\mu_J - \mu_{n-1}|^{1/2} \\ &\ll \|\lambda\|^{-1/2} |\lambda_n - \mu_J|^{1/2}. \end{aligned}$$

Applying all the above inequalities, and $\kappa_{i,j} \le 1$ for all other pairs i, j, to Equation (3.6) and converting to a convolution we have

$$J_n(\lambda) \ll \|\lambda\|^{-n+1} \int F(t) G(-t)\, dt$$

where

$$F(t) = \int_{\mu_I+\mu_J=t} F_I(\mu_I)F_J(\mu_J)|\mu_I-\mu_J|^{1/2}|\mu_J-\lambda_n|^{1/2}\,\mathcal{L}(\mu_J)\,d\mu_J$$

$$G(s) = \left(\mathbf{1}_{[-1,1]} * \left(\mathop{\ast}_{i\neq I,J} F_i\right)\right)(s).$$

The estimation of $F(t)$ is contained in Lemma 14, with

$$A=\lambda_I \quad B=\lambda_{I+1} \quad C=\lambda_J \quad D=\lambda_{J+1} \quad E=\lambda_n \quad T=d, \quad k=2n-4$$

giving

$$F(t) \ll \left(\log' \frac{|\lambda_J-\lambda_{I+1}|}{|\lambda_{I+1}+\lambda_J-t|}\right)^{k+1} + \left(\log' \frac{|\lambda_n-\lambda_{J+1}|}{|\lambda_I+\lambda_{J+1}-t|}\right)^{k+1} + \left(\log' \frac{d}{|\lambda_I+\lambda_J-t|}\right)^{k} + \left(\log' \frac{d}{|\lambda_{I+1}+\lambda_{J+1}-t|}\right)^{k}.$$

Let $d' = d - d_I - d_J$. Let $H_1(t)$ be the sum of the first two log terms and $H_2(t)$ be the sum of the second two log terms. We have $|\lambda_J-\lambda_{I+1}|$,$|\lambda_n-\lambda_{J+1}| < d'$ so $H_1^*(t) \ll (\log' \frac{d'}{t})^{k+1}$. Let d_K be the next largest gap after d_I and d_J, we have $d_K \geq d'/(n-3)$. Then

$$\begin{aligned}
\int H_1(t)G(-t)\,dt &\ll \|H_1 * F_K\|_\infty \prod_{i\neq I,J,K} \|F_i\|_1 \\
&\ll \langle H_1^*, F_K^*\rangle \\
&\ll \int_0^{d_K} \left(\log' \frac{d'}{x}\right)^{k+1} d_K^{-1/2} x^{-1/2}\,dx \\
&\ll \left(\log' \frac{d'}{d_K}\right)^{k+1} \\
&\ll 1,
\end{aligned}$$

as desired. To bound $\int H_2(t)G(-t)dt$ there are two cases. First suppose $d' \geq d/(100n)$.

Then

$$\begin{aligned}\int H_2(t)G(-t)\,dt &\ll \langle H_2^*, F_K^*\rangle \\ &\ll \int_0^{d_K} \left(\log' \frac{d}{x}\right)^k d_K^{-1/2} x^{-1/2}\,dx \\ &\ll \left(\log' \frac{d}{d_K}\right)^k \\ &\ll 1,\end{aligned}$$

since $d_K \gg d$. Otherwise $d' < d/(100n)$. If $G(-t) \neq 0$ then

$$\begin{aligned} t &\in -\operatorname{supp} G \\ &\subset [-1,1] + \sum_{\substack{1\le i\le n-1 \\ i\neq I,J}} [-\lambda_{i+1}, -\lambda_i] \\ &= [-1+\lambda_1+\lambda_{I+1}+\lambda_{J+1},\ 1+\lambda_n+\lambda_I+\lambda_J].\end{aligned}$$

Recall that $\lambda_1 < -d/n$. Subtracting $\lambda_I + \lambda_J$ from the lower bound for t we get

$$\begin{aligned} t-\lambda_I-\lambda_J &\ge -1+\lambda_1+d_I+d_J \\ &\ge -1-d/n+d_I+d_J \\ &\ge -1+(1-1/100n-1/n)d\end{aligned}$$

so for sufficiently large λ, $|t-\lambda_I-\lambda_J| \gg \|\lambda\|$, and similarly $|t-\lambda_{I+1}+\lambda_{J+1}| \gg \|\lambda\|$. Hence $H_2(t) \ll 1$ on the support of $G(-t)$ and $\int H_2(t)G(-t)\,dt \ll 1$ as desired. □

4.4.2 The case $(I,J) = (1, n-1)$

Lemma 13. *Let $(I,J) = (1,n-1)$ and suppose $d_I, d_J \ge d/100n^2$. Then*

$$L_n(\lambda) \asymp \left(\frac{\|\lambda\|}{1+|\lambda_2|+|\lambda_{n-1}|}\right)^{n-2},$$

and $J_n(\lambda) \asymp \|\lambda\|^{-n+1} L_n(\lambda)$.

Proof. Clearly the exceptional term in $L_4(\lambda)$ is trivial when $d_1, d_3 \gg \|\lambda\|$. We split into three cases: the upper bound for $n = 3$, the upper bound for $n \geq 4$, and the lower bound.

The upper bound when $n = 3$

Since $|\lambda_1 - \lambda_2|, |\lambda_2 - \lambda_3| \gg \|\lambda\|$ we have

$$\begin{aligned}(1 + \|\mu\|)^{-1}\kappa_{1,2} &\ll |\lambda_1 - \mu_2|^{-1/2} |\lambda_3 - \mu_1|^{-1/2} \\ &\ll |\lambda_1 - \lambda_2|^{-1/2} |\lambda_3 - \lambda_2|^{-1/2} \\ &\ll \|\lambda\|^{-1}.\end{aligned}$$

Then

$$J_n(\lambda) \ll \|\lambda\|^{-1} \int_{-1}^{1} \int_{\substack{\mu_1+\mu_2=t \\ \lambda_1 \leq \mu_1 \leq \lambda_2 \leq \mu_2 \leq \lambda_3}} |\lambda_1 - \mu_1|^{-1/2} |\lambda_2 - \mu_1|^{-1/2} |\lambda_2 - \mu_2|^{-1/2} |\lambda_3 - \mu_2|^{-1/2} \, d\mu_1 \, dt.$$

By Lemma 15 the inner integral is

$$\ll \|\lambda\|^{-1} \left(\log' \frac{\|\lambda\|}{|2\lambda_2 - t|} + \log' \frac{\|\lambda\|}{|\lambda_1 + \lambda_3 - t|} \right) = \|\lambda\|^{-1} \left(\log' \frac{\|\lambda\|}{|2\lambda_2 - t|} + \log' \frac{\|\lambda\|}{|\lambda_2 + t|} \right),$$

and

$$\begin{aligned}J_n(\lambda) &\ll \|\lambda\|^{-2} \int_{-1}^{1} \log' \frac{\|\lambda\|}{|\lambda_2 + t|} + \log' \frac{\|\lambda\|}{|2\lambda_2 - t|} \, dt \\ &\ll \|\lambda\|^{-2} \log' \frac{\|\lambda\|}{1 + |\lambda_2|}\end{aligned}$$

as desired.

For $2 \le i \le n-2$ we have

$$\begin{aligned}\kappa_{1,i} =& |\mu_1 - \mu_i||\lambda_{i+1} - \mu_1|^{-1/2}|\lambda_1 - \mu_i|^{-1/2} \\ \ll& |\mu_1 - \mu_i||\lambda_{i+1} - \mu_1|^{-1/2}\|\lambda\|^{-1/2} \\ \ll& |\mu_1 - \mu_i|^{1/2}\|\lambda\|^{-1/2}\end{aligned}$$

and similarly $\kappa_{i,n} \ll |\mu_i - \mu_n|^{1/2}\|\lambda\|^{-1/2}$ so by AM-GM, $\kappa_{1,i}\kappa_{i,n} \ll |\mu_1 - \mu_{n-1}|\|\lambda\|^{-1}$. Also, $\kappa_{1,n} \ll |\mu_1 - \mu_{n-1}|\|\lambda\|^{-1}$. We can use these factors to cancel $(1+\|\mu\|)^{-n+2}$, and the extra factor in $L_{n-1}(\mu)$ if $n = 5$:

$$\begin{aligned}&\left(\log' \frac{\|\mu\|}{1 + |\mu_1 - \mu_2| + |\mu_3 - \mu_4|}\right)(1+\|\mu\|)^{-n+2}\kappa_{1,n-1}\prod_{2\le i\le n-2}\kappa_{1,i}\kappa_{i,n-1} \\ &\qquad \ll \log' \frac{\|\mu\|}{|\mu_1 - \mu_2|}\|\mu\|^{-1/2}|\mu_1 - \mu_2|^{1/2}\|\lambda\|^{-n+2} \\ &\qquad \ll \|\lambda\|^{-n+2},\end{aligned}$$

where in the last line we use the fact that $x\log(1/x) \le 1$ for $x \le 1$. Bound $\kappa_{i,j} \le 1$ for all remaining i, j.

Bound

$$\left(\log' \frac{\|\mu\|}{1 + |\mu_2| + |\mu_{n-2}|}\right)^{n-3} \ll \left(\log' \frac{\|\mu\|}{1 + |\mu_{n-2}|}\right)^{n-3}.$$

Write

$$J_n(\lambda) \ll \|\lambda\|^{-n+2}\int F(t)G(-t)\,dt$$

where

$$F(t) = \int_{\substack{\mu_1+\mu_{n-1}=t \\ \lambda_1 \le \mu_1 \le \lambda_2 \\ \lambda_{n-1} \le \mu_{n-1} \le \lambda_n}} F_1(\mu_1) F_{n-1}(\mu_{n-1})\, d\mu_1$$

$$G = \mathbf{1}_{[-1,1]} * F_{n-2}^L * \left(\mathop{\ast}_{i \ne 1, n-2, n-1} F_i \right)$$

$$F_{n-2}^L(x) = F_{n-2}(x) \left(\log' \frac{\|\mu\|}{1+|x|} \right)^{n-3}.$$

Applying Lemma 15 and using $d_1, d_{n-1} \gg \|\lambda\|$ gives

$$F(t) \ll \|\lambda\|^{-1} \left(\log' \frac{\|\lambda\|}{|\lambda_1 + \lambda_n - t|} + \log' \frac{\|\lambda\|}{|\lambda_2 + \lambda_{n-1} - t|} \right).$$

By symmetry (replacing $\lambda_1, \ldots, \lambda_n$ with $-\lambda_n, \ldots, -\lambda_1$) we may assume $|\lambda_{n-1}| \ge |\lambda_2|$, which also implies $\lambda_{n-1} \ge 0$. Now we split into three cases:

- If $\lambda_{n-1} \le 100$ then bound

  $$F_{n-2}^L(x) \ll (\log' \|\lambda\|)^{n-3} F_{n-2}(x)$$

  and

  $$\begin{aligned} \int F(t) G(-t)\, dt &\ll (\log' \|\lambda\|)^{n-3} \left\| F * \mathbf{1}_{[-1,1]} \right\|_\infty \prod_{i \ne 1, n-1} \|F_i\|_1 \\ &\ll \|\lambda\|^{-1} (\log' \|\lambda\|)^{n-3} \int_0^2 \log' \frac{\|\lambda\|}{x}\, dx \\ &\ll \|\lambda\|^{-1} (\log' \|\lambda\|)^{n-2} \\ &\asymp \|\lambda\|^{-1} \left(\log' \frac{\|\lambda\|}{1+|\lambda_2|+|\lambda_{n-1}|} \right)^{n-2} \end{aligned}$$

  and we are done.

- Suppose $\lambda_{n-1} > 100$ and $|\lambda_2 - \lambda_{n-1}| \le |\lambda_{n-1}|/100$. Then $\lambda_2, \ldots, \lambda_{n-1} > 0$ and

$\lambda_2 > 10$. Since F^L_{n-2} is supported on $[\lambda_{n-2}, \lambda_{n-1}]$ we can bound

$$\begin{aligned} F^L_{n-2}(x) &\ll \left(\log' \frac{\|\lambda\|}{|\lambda_{n-2}|}\right)^{n-3} F_{n-2}(x) \\ &\ll \left(\log' \frac{\|\lambda\|}{1 + |\lambda_2| + |\lambda_{n-1}|}\right)^{n-3} F_{n-2}(x). \end{aligned}$$

For t satisfying $G(-t) \neq 0$ we have

$$\begin{aligned} t &\in -\operatorname{supp} G \\ &\subset [-1 - \sum_{2 \le i \le n-2} \lambda_{i+1}, 1 - \sum_{2 \le i \le n-2} \lambda_i] \\ &= [-1 + \lambda_1 + \lambda_2 + \lambda_n, 1 + \lambda_1 + \lambda_{n-1} + \lambda_n] \end{aligned}$$

so $t - \lambda_1 - \lambda_n \ge \lambda_2 - 1 \gg |\lambda_2| + |\lambda_{n-1}|$. On the other hand,

$$\begin{aligned} \lambda_2 + \lambda_{n-1} - t &\ge \lambda_2 + \lambda_{n-1} - (1 + \lambda_1 + \lambda_{n-1} + \lambda_n) \\ &\ge \lambda_2 - 1 - \lambda_1 - \lambda_n \\ &\ge \lambda_2 - 1 + (\lambda_2 + \cdots + \lambda_{n-1}) \\ &\ge (n-1)\lambda_2 - 1 \\ &\gg 1 + |\lambda_2| + |\lambda_{n-1}|. \end{aligned}$$

Thus

$$F(t) \ll \|\lambda\|^{-1} \log' \frac{\|\lambda\|}{1 + |\lambda_2| + |\lambda_{n-1}|},$$

and the result follows.

- Finally suppose $|\lambda_{n-1}| > 100$ and $|\lambda_2 - \lambda_{n-1}| \ge |\lambda_{n-1}|/100$. Then $|\lambda_2 - \lambda_{n-1}| \asymp 1 + |\lambda_2| + |\lambda_{n-1}|$. If $\lambda_{n-2} < \lambda_{n-1}/2$, then $d_{n-2} > |\lambda_2 - \lambda_{n-1}|/4 \gg |\lambda_2| + |\lambda_{n-2}|$ and by monotone rearrangement,

$$\begin{aligned}\int F(t)G(-t)\,dt &\ll \|\lambda\|^{-1}\left\|F * F_{n-2}^{L}\right\|_{\infty}\prod_{i\neq 1,n-2,n-1}\|F_i\|_1\\ &\ll \|\lambda\|^{-1}\int_0^{d_{n-2}} d_{n-2}^{-1/2}x^{-1/2}\left(\log'\frac{\|\lambda\|}{x}\right)^{n-2}\,dx\\ &\ll \|\lambda\|^{-1}\left(\log'\frac{\|\lambda\|}{d_{n-2}}\right)^{n-2}\\ &\ll \|\lambda\|^{-1}\left(\log'\frac{\|\lambda\|}{1+|\lambda_2|+|\lambda_{n-2}|}\right)^{n-2}\end{aligned}$$

and we are done. Otherwise if $\lambda_{n-2} > \lambda_{n-1}/2$ then

$$\begin{aligned}F_{n-2}^{L}(x) &\ll \left(\log'\frac{\|\lambda\|}{|\lambda_{n-2}|}\right)^{n-3} F_{n-2}(x)\\ &\ll \left(\log'\frac{\|\lambda\|}{1+|\lambda_2|+|\lambda_{n-1}|}\right)^{n-3} F_{n-2}(x).\end{aligned}$$

Choose $2 \le K \le n-2$ such that d_K is the next largest after d_J, then

$$\begin{aligned}d_K &> (d-d_I-d_J)/(n-3)\\ &= |\lambda_2-\lambda_{n-1}|/(n-3)\\ &\gg 1+|\lambda_2|+|\lambda_{n-2}|\end{aligned}$$

and

$$\begin{aligned}\int F(t)G(-t)\,dt &\ll \|\lambda\|^{-1}\left(\log'\frac{\|\lambda\|}{1+|\lambda_2|+|\lambda_{n-2}|}\right)^{n-3}\|F*K\|_{\infty}\prod_{i\neq 1,K,n-1}\|F_i\|_1\\ &\ll \|\lambda\|^{-1}\left(\log'\frac{\|\lambda\|}{1+|\lambda_2|+|\lambda_{n-2}|}\right)^{n-3}\int_0^{d_K} d_K^{-1/2}x^{-1/2}\log'\frac{\|\lambda\|}{x}\,dx\\ &\ll \|\lambda\|^{-1}\left(\log'\frac{\|\lambda\|}{1+|\lambda_2|+|\lambda_{n-2}|}\right)^{n-3}\left(\log'\frac{\|\lambda\|}{d_K}\right)\\ &\ll \|\lambda\|^{-1}\left(\log'\frac{\|\lambda\|}{1+|\lambda_2|+|\lambda_{n-2}|}\right)^{n-2}.\end{aligned}$$

Lower bound when $n \geq 3$ and $(I, J) = (1, n-1)$

By assuming $L_n(\lambda)$ is larger than some constant, we may assume $1 + |\lambda_2| + |\lambda_{n-1}| < d/(100n)$. This further implies $|\lambda_1 + \lambda_n| = |\lambda_2 + \cdots + \lambda_{n-1}| \leq d/100$, and since $d = \lambda_n - \lambda_1$, we have $\lambda_1 = (\lambda_1 + \lambda_n - d)/2 \leq -d/2 + d/200 \leq -d/3$ and $\lambda_n \geq d/3$.

Similarly to Section 4.1, we restrict the integral to the region

$$\mathcal{H}'' = \{\mu \in \mathcal{H} : \mu_i \in M_i \text{ for } 2 \leq i \leq n-2,\ \mu_1 < -d/4,\ \mu_{n-1} > d/4\},$$

on which we have $\kappa_{i,j} \geq 1/4$ for all $i < j$, and $F_i \gg d_i^{-1}\mathbf{1}_{M_i}$ for $2 \leq i \leq n-2$. Since $\lambda_2 < \mu_2 < \mu_{n-2} < \lambda_{n-1}$ we have

$$L_{n-1}(\mu) \gg \left(\log' \frac{\|\lambda\|}{1 + |\lambda_2| + |\lambda_{n-1}|}\right)^{n-3}.$$

Bound $|\lambda_2 - \mu_1|^{-1/2}|\lambda_{n-1} - \mu_{n-1}|^{-1/2} \gg \|\lambda\|^{-1}$ and $(1 + \|\mu\|)^{-n+2} \gg \|\lambda\|^{-n+2}$. Using all the above inequalities gives

$$\begin{aligned} J_n(\lambda) \gg & \|\lambda\|^{-n+1} \left(\log' \frac{\|\lambda\|}{1 + |\lambda_2| + |\lambda_{n-1}|}\right)^{n-3} \times \\ & \int_{\mathcal{H}''} |\lambda_1 - \mu_1|^{-1/2}|\lambda_n - \mu_{n-1}|^{-1/2} \prod_{2 \leq i \leq n-2} F_i(\mu_i) J(\mu) d\mu, \end{aligned}$$

which we rewrite as a convolution

$$\begin{aligned} J_n(\lambda) &\gg \|\lambda\|^{-n+1} \int F(t)G(-t)\, dt \\ F(t) &= \int_{\substack{\mu_1 + \mu_{n-1} = t \\ \lambda_1 \leq \mu_1 \leq -d/4 \\ d/4 \leq \mu_{n-1} \leq \lambda_n}} |\lambda_1 - \mu_1|^{-1/2}|\lambda_n - \mu_{n-1}|^{-1/2}\, d\mu \\ G &= \mathbf{1}_{[-1,1]} * \left(\mathop{\ast}_{2 \leq i \leq n-2} d_i^{-1}\mathbf{1}_{M_i} \right). \end{aligned}$$

Since $M_i \subset [\lambda_2, \lambda_{n-1}]$ for $2 \leq i \leq n-2$, we have $G(-t) \neq 0 \implies |t| \leq d/100$. If

$t \leq \lambda_1 + \lambda_n$, then substituting $y = \mu_1 - \lambda_1$ in the equation for F gives

$$F(t) = \int_0^{t-d/4-\lambda_1} y^{-1/2}(\lambda_n - (t - (y + \lambda_1)))^{-1/2}\, dy$$
$$= \int_0^{t-d/4-\lambda_1} y^{-1/2}(y + \lambda_1 + \lambda_n - t)^{-1/2}\, dy.$$

Apply Lemma 17 with $a = \lambda_1 + \lambda_n - t$ and $T = t - d/4 - \lambda_1$. The preceding inequalities for $|t|, \lambda_1$, and $|\lambda_1 + \lambda_n|$ ensure $a \leq T$. This gives

$$F(t) \gg \log' \frac{\|\lambda\|}{\lambda_1 + \lambda_n - t}.$$

Likewise if $t \geq \lambda_1 + \lambda_n$ then

$$F(t) \gg \log' \frac{\|\lambda\|}{\lambda_1 + \lambda_n - t},$$

thus

$$F(t) \gg \log' \frac{\|\lambda\|}{\left|t - (\lambda_1 + \lambda_n)\right|}.$$

Since $|\lambda_1 + \lambda_n| \ll |\lambda_2| + |\lambda_{n-1}|$ we have $\left|t - (\lambda_1 + \lambda_n)\right| \ll |t| + |\lambda_2| + |\lambda_{n-1}|$ and

$$F(t) \gg \log' \frac{\|\lambda\|}{|t| + |\lambda_2| + |\lambda_{n-1}|}.$$

Meanwhile, we have $\int G(t)\, dt \gg 1$, and the support of G is contained in an interval of radius $\ll 1 + |\lambda_2| + |\lambda_{n-1}|$ around 0, so

$$\int F(t)G(-t)\, dt \gg \log' \frac{\|\lambda\|}{1 + |\lambda_1| + |\lambda_n|}$$

as desired. □

4.5 Lemmas

Here we have separated out the main calculation for the two large gaps case.

Lemma 14. *Let* $A \le B \le C \le D \le E$, T, *and* $(L_i)_{1\le i\le k}$ *be given with* $|A-E| \le T$. *Then for* $A+C \le t \le B+D$,

$$(*) = \int_{\substack{x+y=t\\ A\le x\le B\\ C\le y\le D}} \big(|x-A||x-B||y-C||y-D|\big)^{-1/2}|x-y|^{1/2}|y-E|^{1/2} \prod_{1\le i\le k} \log' \frac{T}{|L_i-y|}\,dx$$

is bounded above by

$$\ll_k T|A-B|^{-1/2}|C-D|^{-1/2} \times$$
$$\left[\left(\log' \frac{|B-C|}{|B+C-t|}\right)^{k+1} + \left(\log' \frac{|D-E|}{|A+D-t|}\right)^{k+1} + \left(\log' \frac{T}{|A+C-t|}\right)^{k} + \left(\log' \frac{T}{|B+D-t|}\right)^{k}\right].$$

Proof. For $x \in [P,Q]$ we have

$$|x-P|^{-1/2}|x-Q|^{-1/2} \ll |P-Q|^{-1/2}\left(|x-P|^{-1/2} + |x-Q|^{-1/2}\right). \tag{4.5}$$

Using this identity twice with $(P,Q) = (A,B)$ and $(P,Q) = (C,D)$ and substituting $y = t-x$ gives

$$(*) \ll |A-B|^{-1/2}|C-D|^{-1/2} \sum_{\substack{X\in\{A,B\}\\ Y\in\{C,D\}}} I(X,Y)$$

where $I(X,Y) =$

$$\int_{\max(A,t-D)}^{\min(B,t-C)} |x-X|^{-1/2}|x-t+Y|^{-1/2}|2x-t|^{1/2}|x-t+E|^{1/2} \prod_{1\le i\le k} \log' \frac{T}{|L_i-t+x|}dx.$$

First consider $I(A,C)$. Extend the bounds of the integral to $[A, t-C]$. Bound

$|2x-t|^{1/2}|x-t+E|^{1/2} \ll T$. Then

$$I(A,C) \ll T \int_A^{t-C} |x-A|^{-1/2} \big|x-(t-C)\big|^{-1/2} \prod_{1\le i\le k} \log' \frac{T}{|L_i - t + x|}\, dx$$

and using Equation (4.5) with $(P,Q) = (A, t-C)$ and then using monotone rearrangement on each factor we obtain

$$I(A,C) \ll T|t-C-A|^{-1/2} \int_0^{t-C-A} x^{-1/2} \left(\log' \frac{T}{x}\right)^k dx \ll_k T \left(\log' \frac{T}{|A+C-t|}\right)^k.$$

The calculation for $I(B,D)$ is similar to $I(A,C)$.

Now consider $I(A,D)$. Bound $|2x-t|^{1/2} \ll T^{1/2}$. Suppose $t \le A+D$. Substitute $x' = x - A$. The integral now runs from 0 to $B-A$, extend it to $[0,T]$. The integral now reads

$$T^{1/2} \int_0^T \big|x'\big|^{-1/2} \big|x' + A + D - t\big|^{-1/2} \big|x' + A + E - t\big|^{1/2} \prod_{1\le i\le k} \log' \frac{T}{|L_i - t + x' + A|}\, dx.$$

Applying Lemma 16 with $a = A+D-t$ and $b = A+E-t$ gives

$$\begin{aligned} I(A,D) &\ll_k T \left(\log' \frac{A+E-t}{A+D-t}\right)^{k+1} \\ &= T\left(\log'\left(1 + \frac{E-D}{A+D-t}\right)\right)^{k+1} \\ &\ll T\left(\log' \frac{|D-E|}{|A+D-t|}\right)^{k+1}. \end{aligned}$$

If $t \ge A+D$, then substitute $x' = x - t + D$, extend the bounds of the integral to $[0,T]$, and apply Lemma 16 with $a = t-A-D, b = E-D$. The calculation for $I(B,C)$ is similar. □

Lemma 15. *Let $A \le B \le C \le D$ be given with $|A - D| \le T$. Then for $A+B \le t \le C+D$,*

$$\int_{\substack{x+y=t\\ A\le x\le B\\ C\le y\le D}} \left(|x-A||x-B||y-C||y-D|\right)^{-1/2}\, dx$$

is bounded above by

$$\ll |A-B|^{-1/2}|C-D|^{-1/2}\left(\log' \frac{T}{|B+C-t|} + \log' \frac{T}{|A+D-t|}\right).$$

Proof. Substitute $A, B, C+T, D+T, D+2T, 3T, t+T$ for A, B, C, D, E, T, t in Lemma 14. We obtain that

$$\int_{\substack{x+y=t+T\\ A\le x\le B\\ C+T\le y\le D+T}} \left(|x-A||x-B||y-C-T||y-D-T|\right)^{-1/2}|x-y|^{1/2}|y-E-2T|^{1/2}\, dx$$

is $\ll$ than

$$T|A-B|^{-1/2}|C-D|^{-1/2}\left(\log' \frac{T+|B-C|}{|B+C-t|} + \log' \frac{T+|D-E|}{|A+D-t|}\right).$$

Note that $|x-y|, |y-E-2T|$ in the integrand and $T+|B-C|, T+|D-E|$ on the right hand side are all $\asymp T$, so dividing both sides by T gives the desired inequality. □

Lemma 16. *For $0 < a, b < T$ and all $L_1, \ldots, L_k$ we have*

$$\int_0^T x^{-1/2}(x+a)^{-1/2}(x+b)^{1/2} \prod_{1\le i\le k} \log' \frac{T}{|L_i + x|}\, dx \ll_k T^{1/2}\left(\log' \frac{b}{a}\right)^{k+1}.$$

Proof. Since $x^{-1/2}(x+a)^{-1/2}(x+b)^{1/2}$ is decreasing on $[0, T]$, we may use monotone rearrangement and assume $L_i = 0$ for all i. If $0 < a < b < T$ then the integral is

$$\begin{aligned}
&\ll \left[a^{-1/2}b^{1/2}\int_0^a x^{-1/2} + b^{1/2}\int_a^b x^{-1} + \int_b^T x^{-1/2}\right]\left(\log' \frac{T}{x}\right)^k dx \\
&\ll_k b^{1/2}\left(\log' \frac{T}{a}\right)^k + b^{1/2}\left(\log' \frac{b}{a}\right)\left(\log' \frac{T}{a}\right)^k + T^{1/2} \\
&\ll_k T^{1/2}\frac{b^{1/2}}{T^{1/2}}\left(\log' \frac{T}{b} + \log' \frac{b}{a}\right)^k + T^{1/2}\frac{b^{1/2}}{T^{1/2}}\left(\log \frac{b}{a}\right)\left(\log' \frac{T}{b} + \log' \frac{b}{a}\right)^k + T^{1/2} \\
&\ll_k T^{1/2}\left(\log' \frac{b}{a}\right)^{k+1}
\end{aligned}$$

where on the third line, we use the fact that $(b/T)^{1/2}(\log'(T/b))^k \ll_k 1$. Otherwise if $0 < b < a < T$ then $x^{-1/2}(x+a)^{-1/2}(x+b)^{1/2} \leq x^{-1/2}$ and the integral is

$$\leq \int_0^T x^{-1/2}\left(\log' \frac{T}{x}\right)^k dx \ll_k T^{1/2}.$$

□

Lemma 17. *For* $0 < a \leq T$,

$$\int_0^T x^{-1/2}(x+a)^{-1/2}\, dx \asymp \log'(T/a).$$

Proof.

$$\begin{aligned}
\int_0^T x^{-1/2}(x+a)^{-1/2}\, dx &= \int_0^a x^{-1/2}(x+a)^{-1/2}\, dx + \int_a^T x^{-1/2}(x+a)^{-1/2}\, dx \\
&\asymp a^{-1/2}\int_0^a x^{-1/2}\, dx + \int_a^T x^{-1}\, dx \\
&\asymp 1 + \log(T/a).
\end{aligned}$$

□

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