L^p Regularity Estimates for Radon-like Operators with Fibered Folding Canonical Relations

By

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Abstract

Local Radon-like transforms are examples of Fourier integral operators which appear in many areas of harmonic analysis and integral geometry. These transforms integrate a function along a family of submanifolds of $\mathbb{R}^d$, and as such we expect that they exhibit some smoothing. The (local) L^p-Sobolev regularity of a local Radon-like transform is in part determined by the geometry of its canonical relation. In almost all cases excluding averages over hypersurfaces the canonical relation always projects with singularities, meaning the calculus of Fourier integral operators due to Hörmander does not apply.

In this work we investigate the (local) L^p-Sobolev regularity of local Radon-like transforms with one-sided folds, specifically transforms which integrate over families of curves in $\mathbb{R}^3$. We prove L^p-Sobolev estimates for a class of these local Radon-like transforms associated to fibered folding canonical relations which are optimal except possibly for endpoints. The proof of this main result relies on L^2 estimates for frequency-localized oscillatory integral operators, which we prove in all dimensions, and decoupling inequalities by Wolff and Bourgain-Demeter for plate decompositions of thin neighborhoods of cones.

We investigate applications of these results to two model cases, restricted X-ray transforms, and Heisenberg convolutions with compactly supported measures on curves in the Heisenberg group. We also construct a Sobolev space adapted to translations on the Heisenberg group which permits a global extension of our main result for this second model case.

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List of Figures

Notation and Symbols

We use x, w, y, z to represent physical variables in $\mathbb{R}^d$. Meanwhile ξ, η and τ will typically denote frequency variables. In this work we consider families of curves in $\mathbb{R}^3$, and use local coordinates with the last coordinate distinguished, hence we will write $x = (x', x_3)$, $w = (w', w_3)$, $y = (y', y_3)$, $z = (z', z_3)$, $\xi = (\xi', \xi_3)$, and $\eta = (\eta', \eta_3)$. For ease of reading the dot $\cdot$ will be reserved for the inner product on $\mathbb{R}^2$, and $\langle\ ,\ \rangle$ for inner product on $\mathbb{R}^3$. In cases where this choice affects readability we default to $\langle\ ,\ \rangle$, but these instances should be clear from context. In this work we also consider local coordinates on dimension n submanifolds of $\mathbb{R}^d$, in which case we use local coordinates with the last n coordinates distinguished, i.e. $x = (x', x_{d-n+1}, ..., x_d) \in \mathbb{R}^d$. In these circumstances the dot $\cdot$ will be used to denote the inner product on $\mathbb{R}^{d-n}$ while $\langle\ ,\ \rangle$ will be reserved for the inner product on $\mathbb{R}^d$. For $i = 1, ..., d$, e_i represent the standard unit basis vectors in $\mathbb{R}^d$. The Euclidean ball of radius r centered at $x \in \mathbb{R}^d$ is denoted $B_r(x)$ (where the dimension is implied by the context), and the volume of $B_r(x)$ is given by $V(d)r^d$ so that $V(d)$ is the volume of the unit ball in $\mathbb{R}^d$.

In this work C and c will represent positive arbitrary constants. The values of these constants may change from line to line. If a constant C depends on a parameter ε, we write C_ε to reflect this dependence. Additionally, for non-negative quantities X and Y we will write $X \lesssim Y$ to denote the existence of a positive constant C such that $X \leq CY$. If this constant depends on a parameter such as ε we write $X \lesssim_\varepsilon Y$. If $X \lesssim Y$ and $Y \lesssim X$ then we write $X \simeq Y$.

Generally we denote smooth compactly supported functions by χ, ψ, ζ, and η, whereas the indicator function of a set E is denoted by $\mathbb{1}_E$. For a real number x the functions $\lfloor x \rfloor$ and $\lceil x \rceil$ denote respectively the largest integer less than x and the smallest integer greater than x, while $\lceil x \rfloor$ denotes the closest integer to x (note if $x = n + \frac{1}{2}$ for some integer n then we define $\lceil x \rfloor = \lfloor x \rfloor = n$).

We write $\partial_{x_i} f$ to mean the partial derivative of f with respect to x_i. A multiindex α is a tuple $\alpha = (\alpha_1, \alpha_2, ..., \alpha_d)$ with $\alpha_i \in \mathbb{N}$ for all $i = 1, ..., d$. The length (or order) of a multiindex is given by $|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_d$. We write

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_d^{\alpha_d}$$

$$\partial_x^\alpha f = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \cdots \partial_{x_d}^{\alpha_d} f$$

for any multiindex α. The dimension of the multiindex may vary within different contexts.

The Fourier transform of a function f is denoted

$$\mathfrak{F}[f](\xi) = \int e^{2\pi i \langle x, \xi \rangle} f(x)\, dx$$

or more simply $\hat{f}(\xi)$. A partial Fourier transform in only some variables will be denoted by subscripts; for example given $f \in L^2(\mathbb{R}^3)$, the partial Fourier transform of f in the first two variables is given by

$$\mathfrak{F}_{1,2}[f(\cdot, x_3)](\xi') = \int e^{2\pi i x' \cdot \xi'} f(x', x_3)\, dx'.$$

The inverse Fourier transform of a function $f(\xi)$ is denoted $\mathfrak{F}^{-1}[f](x)$ or $\check{f}(x)$.

Below is a non-exhaustive list of spaces used throughout this work.

- $\mathbb{N}$ - the natural numbers, $0, 1, 2, ...$

- $\mathbb{Z}$ - the integers
- $\mathbb{R}^d$ - The Euclidean space of dimension d
- $\mathbb{C}$ - The complex plane
- $\mathbb{S}^{d-1}$ - the $(d-1)$-dimensional sphere.
- $\mathbb{H}$ - The (first) Heisenberg group, see Section 4.2
- $\mathbb{H}_{\mathbb{Z}}$ - The discrete Heisenberg group, see Chapter 6
- $\mathbb{M}_{d,n}$ - The space of affine n-planes in $\mathbb{R}^d$, see Section 4.1
- $\mathbb{G}_{d,n}(\mathbb{R}^d)$ - The Grassmannian; the space of all n-planes through the origin in $\mathbb{R}^d$, see Section 4.1.
- $\ell^p = L^p(\mathbb{Z})$ - The discrete Lebesgue spaces; for $0 < p \leq \infty$, the space of sequences $a : \mathbb{Z} \to \mathbb{C}$ such that
$$\|a\|_{\ell^p} = \Big(\sum_{n \in \mathbb{Z}} |a(n)|^p \Big)^{1/p} < \infty.$$
- $L^p, L^p(\mathbb{R}^d)$ - the Euclidean Lebesgue spaces; for $1 \leq p \leq \infty$, the space of functions $f : \mathbb{R}^d \to \mathbb{C}$ such that
$$\|f\|_p = \Big(\int |f|^p \Big)^{1/p} < \infty.$$
- $L^p_{\text{comp}}(\mathbb{R}^d)$ - The space of compactly supported functions $f \in L^p(\mathbb{R}^d)$
- $L^p_{\text{loc}}(\mathbb{R}^d)$ - The space of functions f such that $f|_K \in L^p(\mathbb{R}^d)$ for any compact set $K \subset \mathbb{R}^d$.
- $C^\infty(\mathbb{R}^d)$ - The space of functions $f : \mathbb{R}^d \to \mathbb{C}$ which are infinitely differentiable

- $C_0^\infty(\mathbb{R}^d)$ - The space of functions $f : \mathbb{R}^d \to \mathbb{C}$ which are infinitely differentiable such that $\lim_{|x|\to\infty} f(x) = 0$.

- $C_c^\infty(\mathbb{R}^d)$ - The space of compactly supported functions $f : \mathbb{R}^d \to \mathbb{C}$ which are infinitely differentiable

- $\mathcal{S}(\mathbb{R}^d)$ - The Schwartz space; the space of functions $f : \mathbb{R}^d \to \mathbb{C}$ such that for any multiindices α, β
$$\sup_{x\in\mathbb{R}^d} |x^\alpha \partial_x^\beta f| < \infty.$$

- $\mathcal{H}^p, \mathcal{H}^p(\mathbb{R}^d)$ - The Hardy spaces; for $0 < p \leq \infty$ and $\Phi \in \mathcal{S}(\mathbb{R}^d)$, the space of tempered distributions f such that the maximal function
$$M_\Phi f(x) = \sup_{t>0} \big| f * \Phi_t(x) \big|$$
is in $L^p(\mathbb{R}^d)$, where $\Phi_t(x) = t^{-d}\Phi(x/t)$.

- $L_s^p, L_s^p(\mathbb{R}^d)$ - The Sobolev or Bessel potential spaces; for $1 \leq p \leq \infty$ and $s \in \mathbb{R}$, the space of functions $f : \mathbb{R}^d \to \mathbb{C}$ such that
$$(I - \Delta)^{s/2} f \in L^p(\mathbb{R}^d).$$

- $L_s^p(\mathbb{H})$ - The Heisenberg Sobolev spaces, see Definition 6.1

- $F_s^{p,q}(\mathbb{R}^d) = F_s^{p,q}$ - The Triebel-Lizorkin spaces, see Definition 8.7

- $B_s^{p,q}(\mathbb{R}^d) = B_s^{p,q}$ - The Besov spaces, see Definition 8.7.

Contents

Chapter 1

Introduction

The classical Radon transform maps a function defined on the plane to a function defined on the space of lines in the plane by taking its integral over each line. More specifically, given a smooth compactly supported function $f : \mathbb{R}^2 \to \mathbb{C}$, the Radon transform of f is defined for each line $\ell \subset \mathbb{R}^2$ by integration of f over ℓ, i.e.

$$Rf(\ell) = \int_\ell f.$$

The Radon transform was introduced over 100 years ago by Johann Radon [47], who was interested in recovering f from its Radon transform. See [34] for a more detailed introduction to the classical Radon transform in $\mathbb{R}^d$.

Any line ℓ in the plane can be defined as the solution set of an equation $y \cdot \theta = s$, where $\theta \in \mathbb{S}^1$ and $s \in \mathbb{R}$. Note that the pairs (θ, s) and $(-\theta, -s)$ are associated to the same line ℓ, hence the map $\vartheta : (\theta, s) \to \ell$ is a double covering of $\mathbb{S}^1 \times \mathbb{R}$ onto the space of lines. We can furnish the space of lines with a canonical manifold structure with respect to which ϑ is differentiable and regular [34]. Indeed, we may identify $Rf(\ell)$ with an even function on $\mathbb{S}^1 \times \mathbb{R}$ given by

$$J(\theta, s) = \int_{y\cdot\theta\ =\ s} f(y)\, d\sigma(y),$$

where σ is Lebesgue measure on the line $y \cdot \theta = s$. If $f(x) = O(|x|^{-N})$ for $N > 2$ then we can recover f from Rf by the formula

$$f(x) = 2(-\Delta)^{1/2} \int_{\ell \ni x} Rf(\ell)\, d\mu_x,$$

where μ_x is the unique probability measure on the compact set $\{\ell \ : \ x \in \ell\}$ invariant under rotations about x [47]. This inversion formula has a more concrete presentation as

$$f(x) = (2\pi)^{-1}(-\Delta)^{1/2} \int_{\mathbb{S}^1} J(\theta, \langle \theta, x \rangle)\, d\theta \tag{1.1}$$

where $d\theta$ is the 1 dimensional Hausdorff measure on the unit circle $\mathbb{S}^1$ [34].

Using a coordinate patch on the space of lines, we can view the Radon transform locally as an operator acting on smooth functions compactly supported in $\mathbb{R}^2$, which integrates over a certain family of curves parametrized by $x \in \mathbb{R}^2$. For example, by parametrizing the circle $\mathbb{S}^1$ by the map $x_1 \mapsto (\cos(x_1), \sin(x_1))$ and utilizing the projection ϑ, we see that for x_1 near 0 we have

$$Rf(\vartheta(x_1, x_2)) = \int f(x_2 \sec(x_1) - y_2 \tan(x_1), y_2)\, dy_2.$$

thus we see that in a certain coordinate patch on the space of lines, the Radon transform integrates over the family of non-horizontal lines $\{y \in \mathbb{R}^2 \ : \ y_1 = x_2 \sec(x_1) - y_2 \tan(x_1)\}$ parametrized by $x \in \mathbb{R}^2$ near the origin.

This perspective leads to a natural and well-studied local variant: given open sets $\Omega_L, \Omega_R \subset \mathbb{R}^d$, suppose we have a family of n-dimensional submanifolds $\mathcal{M}_x \subset \Omega_R$ parametrized by and smoothly varying with $x \in \Omega_L$. A ***local Radon-like transform*** $\mathcal{R} : C_c^\infty(\Omega_R) \to C^\infty(\Omega_L)$ associated to this family of manifolds is defined for $f \in C_c^\infty(\Omega_R)$ by

$$\mathcal{R}f(x) = \int_{\mathcal{M}_x} f(y)\chi(x, y) d\sigma_x(y), \ x \in \Omega_L, \tag{1.2}$$

where $\chi \in C_c^\infty(\Omega_L \times \Omega_R)$ and $d\sigma_x$ is the restriction of Lebesgue measure onto $\mathcal{M}_x$. For convenience, let $n' = d - n$, the codimension of the manifolds $\mathcal{M}_x$.

Local Radon-like transforms appear in many areas of harmonic analysis and integral geometry; we will see a few examples in Chapters 3 and 4. Due to their appearance in many different contexts, local Radon-like transforms have been studied from a variety of perspectives. In integral geometry questions of inversion are common, and have been investigated for restricted X-ray transforms and the classical Radon transform in $\mathbb{R}^d$, to name a few examples (see [27] and also [34, Ch. II]). The L^p-improving properties of local Radon-like transforms are related to a well-studied problem in harmonic analysis, the L^p-improving properties of convolutions with measures supported on curves (see [12, 54, 13]). More generally, the L^p-improving properties of local Radon-like transforms have been studied by Gressman (see for example [30, 31, 32]), Greenleaf and Seeger [23], and many others in more specific contexts.

In this work we focus on analyzing the regularity properties of $\mathcal{R}$ - how does the smoothness of $\mathcal{R}f$ compare to the smoothness of f? To measure the smoothness of f and $\mathcal{R}f$ we use L^p-Sobolev norms; however, to define Sobolev norms on an open set like Ω_L using Bessel potentials would require careful consideration of the boundary. The choices of the particular open sets Ω_L and Ω_R are arbitrary, so to avoid assumptions about the regularity of the boundaries of Ω_L and Ω_R and to emphasize the local nature of our analysis we will investigate conditions under which $\mathcal{R}$ extends to a continuous operator

$$\mathcal{R} \; : \; L^p_{\text{comp}}(\Omega_R) \to L^p_{s,\text{loc}}(\Omega_L)$$

for certain $1 < p < \infty$ and $s \in \mathbb{R}$. An estimate of this form means that for any C^∞

function v_0 compactly supported in Ω_L and for any compact set $K \subset \Omega_R$ we have for all L^p functions f supported in K,

$$\|v_0 \mathcal{R} f\|_{L^p_s(\mathbb{R}^d)} \leq C_p(v_0, K)\|f\|_{L^p(\mathbb{R}^d)}.$$

Here $L^p_s(\mathbb{R}^d)$ is the standard Sobolev space consisting of tempered distributions g on $\mathbb{R}^d$ such that $(I - \Delta)^{s/2} g \in L^p(\mathbb{R}^d)$.

In this work we will often use local changes of variables to transform general operators into a model case locally. Thus it is helpful to note that changes of variables leave the local Sobolev spaces defined above invariant. We present this result below for s between 0 and 1, but applying Leibniz rule and chain rule we can prove the same result for local changes variables for Sobolev spaces of integer order, then apply interpolation to obtain the equivalence for all s.

Lemma 1.1. *If $\psi : \Omega_L \to \Omega'_L$ is a C^∞ diffeomorphism then for $0 \leq s \leq 1$ $f \in L^p_{s,\mathrm{loc}}(\Omega'_L)$ if and only if $f \circ \psi \in L^p_{s,\mathrm{loc}}(\Omega_L)$, and $\|f\|_{L^p_{s,\mathrm{loc}}(\Omega_L)} \simeq \|f \circ \psi\|_{L^p_{s,\mathrm{loc}}(\Omega'_L)}$ with constants only depending on p, s and $|\det J_\psi|$.*

Proof. Let J_ψ denote the Jacobian matrix of ψ. Suppose $f \in L^p_{0,\mathrm{loc}}(\Omega'_L)$. Then by a change of variables $f \circ \psi \in L^p_{0,\mathrm{loc}}(\Omega_L)$ since for any compactly supported $v \in C^\infty(\Omega'_L)$

$$\Big(\int |f(x)v(x)|^p \, dx\Big)^{1/p} = \Big(\int |(f \circ \psi)(y)(v \circ \psi)(y)|^p |\det J_\psi(y)| \, dy\Big)^{1/p},$$

$|\det(J_\psi)|^{1/p}$ is bounded above and below on Ω_L, and $v \circ \psi$ is a compactly supported function in $C^\infty(\Omega_L)$. Note that this implies

$$\inf_{y \in \Omega_L} |\det(J_\psi(y))|^{1/p} \|f \circ \psi\|_{L^p_{0,\mathrm{loc}}(\Omega_L)} \leq \|f\|_{L^p_{0,\mathrm{loc}}(\Omega'_L)} \leq \sup_{y \in \Omega_L} |\det(J_\psi(y))|^{1/p} \|f \circ \psi\|_{L^p_{0,\mathrm{loc}}(\Omega_L)}$$

Next, suppose that $f \in L^p_{1,\mathrm{loc}}(\Omega'_L)$. By chain rule $\nabla(f\circ\psi) = ((\nabla f)\circ\psi)^\intercal J_\psi$. Thus for any compactly supported $v \in C^\infty(\Omega'_L)$

$$\begin{aligned}\Big(\int |(v\circ\psi)\nabla(f\circ\psi)|^p |\det J_\psi|^{1-p}\Big)^{1/p} &= \Big(\int \big|(v\circ\psi)\big((\nabla f)\circ\psi\big)^\intercal J_\psi\big|^p |\det J_\psi|^{1-p}\Big)^{1/p}\\ &\le \Big(\int |(v\circ\psi)|^p |(\nabla f)\circ\psi|^p |\det J_\psi|\Big)^{1/p}\\ &= \Big(\int |v\nabla f|^p\Big)^{1/p}.\end{aligned}$$

Repeating the same argument with ψ^{-1} and using the uniform bounds on $|\det J_\psi|$, we see that

$$\begin{aligned}\inf_{y\in\Omega_L} |\det J_\psi(y)|^{\frac{1}{p}-1}\Big(\int |(v\nabla f)\circ\psi|^p\Big)^{1/p} &\le \Big(\int |vf|^p\Big)^{1/p}\\ \Big(\int |vf|^p\Big)^{1/p} &\le \sup_{y\in\Omega_L} |\det J_\psi(y)|^{\frac{1}{p}-1}\Big(\int |(v\nabla f)\circ\psi|^p\Big)^{1/p}.\end{aligned}$$

Interpolation between $L^p_0(\mathbb{R}^d) = L^p(\mathbb{R}^d)$ and $L^p_1(\mathbb{R}^d)$ implies the desired equivalence with a constant depending on v_0. $\square$

Suppose T is an integral operator of the form $Tf(x) = \int K(x,y)f(y)\,dy$ such that T is bounded from $L^{p_0}_{s_0,\mathrm{loc}}(\Omega_R) \to L^{p_1}_{s_1,\mathrm{loc}}(\Omega_L)$ where $0 \le s_0, s_1 \le 1$ and $1 \le p_0, p_1 \le \infty$. If $\sigma : \Omega'_R \to \Omega_R$ and $\eta : \Omega'_L \to \Omega_L$ are C^∞ diffeomorphisms, by the above lemma the operator $\tilde{T}$ with Schwartz kernel $K(\eta(x),\sigma(y))$ is also bounded from $L^{p_0}_{s_0,\mathrm{loc}}(\Omega'_R) \to L^{p_1}_{s_1,\mathrm{loc}}(\Omega'_L)$ with operator norm $C\|T\|$, where C depends only on p_0, p_1, s_0, s_1, $|\det J_\sigma|$, and $|\det J_\eta|$. Thus we may freely apply local changes of variables in x and y separately to integral operators throughout this work. Because we typically are free to change our open sets Ω_L and Ω_R freely we may even suppose that σ and η are only local diffeomorphisms.

1.1 Outline

In Chapter 2 we review the microlocal analysis of local Radon-like transforms and more general Fourier integral operators. We begin with a review of the work of Hörmander, who developed a calculus of Fourier integral operators associated to so-called "nondegenerate" canonical relations. In the context of local Radon-like transforms this condition is equivalent to the notion of nonvanishing rotational curvature, due to Phong and Stein. As we will see, local Radon-like transforms cannot satisfy the nonvanishing rotational curvature condition unless the dimension of the ambient space and codimension of the manifolds $\mathcal{M}_x$ satisfy a very restrictive number theoretic relation; in particular, rotational curvature must vanish for local Radon-like transforms over families of curves in $\mathbb{R}^3$. From here we introduce the definitions of folding canonical relations and fibered folding canonical relations.

In Chapter 3 we give a brief history of L^p-Sobolev regularity results to date, beginning with results on L^2. Results on L^2 are the most well-studied, and depend on the associated microlocal geometry. Sharp L^p-Sobolev regularity was not established for $p \neq 2$ until 2007, with a result by Pramanik and Seeger [45] about convolutions with measures supported on curves in $\mathbb{R}^2$. The novel method in the proof of sharp L^p-Sobolev regularity for large p is the use of Bourgain-Demeter-Wolff decoupling of thin plates associated to a general cone with one non-vanishing principal curvature. Pramanik and Seeger were later able to generalize this method of proof to a class of local Radon-like transforms associated to folding canonical relations, which we present as an introduction to our main results, Theorems 3.14 and 3.15, which further generalize this method of proof to include sharp L^p-Sobolev estimates for all p for a class of local Radon-like transforms associated to fibered folding canonical relations.

In Chapter 4 we introduce two examples of local Radon-like transforms associated to families of curves in $\mathbb{R}^3$, and analyze the microlocal geometry associated to each operator. First, we introduce restricted X-ray transform, which have been studied in integral geometry for decades to model problems in tomography. Greenleaf and Uhlmann proved nonlocal inversion formulas for a class of restricted X-ray transforms in [27], and we show that transforms in this class are associated to fibered folding canonical relations, and generically satisfy the conditions of the main theorem. We also introduce a noncommutative version of a convolution with a measure supported on a curve, in this case set on the Heisenberg group. We characterize the microlocal structure of this operator, and use it as an example to show the sharpness of the main theorem in Chapter 5. We also can use the geometric structure of the "Heisenberg convolution" to extend the the local L^p-Sobolev regularity to L^p regularity on an analogue of the global Sobolev space which is adapted translations on the Heisenberg group, which we introduce in Chapter 6.

In Chapter 7 we begin the proof of Theorem 3.15 by proving using Hardy space estimates to interpolate L^p-Sobolev regularity for small values of p. This result is general, relying only on local L^1 boundedness and some L^2 regularity.

We begin the proof of Theorem 3.14 in Chapter 8. Here, we decompose our local Radon-like transform using Littlewood-Paley theory and the techniques of Phong and Stein in order to formulate an oscillatory integral estimate which is the essential estimate needed to prove Theorem 3.14. We also outline the structure of the proof of Theorem 3.14, which involves three main parts, constituting the next four chapters. In Chapter 9 we prove an L^2-Sobolev estimate using a similar argument to the proof of the Calderón-Vaillancourt Theorem for a general class of oscillatory integral operators.

The heart of the proof lies in Chapters 10 and 11, where we relate L^p regularity to decoupling inequalities via a microlocal analysis. A model case is introduced in detail in Chapter 10 and families of changes of variables are introduced in Chapter 11 to reduce the study of the general case to the model case. Finally, in Chapter 12 we use a Calderón-Zygmund type estimate to relate the oscillatory integral estimates in Chapter 8 to the L^p-Sobolev estimates in Theorem 3.14, finishing the proof of Theorems 3.14 and 3.15.

This work is in part based on results from preprints of the author [6, 7]. In particular, Section 4.2 and Chapters 5-7 draw from [6] while Section 4.1 and Chapters 9-11 draw from [7].

Chapter 2

The Microlocal Picture for Local Radon-like Transforms

The formula (1.1) for inverting the Radon transform involves two dual forms of integration, an integration over all the points x in a given line ℓ, and an integral over all lines ℓ containing a given point x. These dual integrals suggest two operators which integrate over dual fibers of the same manifold; this construction is known as the ***double fibration formalism*** [22]. Let's assume that $(x, \mathcal{M}_x)$ are fibers of a manifold known as an ***incidence relation***; more specifically, assume that

$$\mathcal{M}_x = \{y \in \Omega_R \ : \ (x,y) \in \mathcal{M}\},$$

where $\mathcal{M} \subset \Omega_L \times \Omega_R$ has codimension $d-n$, and the natural projections

$$\begin{tikzcd} & \mathcal{M} \arrow[dl, "\rho_L"'] \arrow[dr, "\rho_R"] & \\ \Omega_L & & \Omega_R \end{tikzcd} \tag{2.1}$$

are submersions. By shrinking Ω_L, Ω_R we can additionally assume that ρ_L, ρ_R are surjective. We can define a local Radon-like transform directly from this manifold $\mathcal{M}$. Since we assume that the natural projections $\rho_L : \mathcal{M} \to \Omega_L$ and $\rho_R : \mathcal{M} \to \Omega_R$ are surjective submersions, by an application of the implicit function theorem we have that for each

$x \in \Omega_L$

$$\mathcal{M}_x = \rho_R \rho_L^{-1}(\{x\}) = \{y \in \Omega_R \ : \ (x, y) \in \mathcal{M}\}$$

is a smooth immersed n-dimensional submanifold of Ω_R, depending smoothly on x. Indeed, since $d\rho_L$ has rank d near x, locally we can pick coordinates in Ω_R such that

$$\rho_L^{-1}(\{x\}) = \{(x, y_1, ..., y_n, g(y_1, ..., y_n)), (y_1, ..., y_n) \in U\}$$

for some smooth function $g : \mathbb{R}^n \to \mathbb{R}^{d-n}$. In these coordinates

$$\rho_R \rho_L^{-1}(\{x\}) = \{(y_1, ..., y_n, g(y_1, ..., y_n)), (y_1, ..., y_n) \in U\},$$

hence $\mathcal{M}_x$ is locally a n-dimensional manifold in Ω_R.

Helgason [34] observed that we can construct a dual operator associated to $\mathcal{M}$ by the same argument; $\mathcal{M}^y = \rho_L \rho_R^{-1}(\{y\})$ are also smooth immersed n-dimensional manifolds smoothly depending on $y \in \Omega_R$. This perspective gives rise to two operators,

$$\mathcal{R}f(x) = \int_{\mathcal{M}_x} f$$

and its adjoint

$$\mathcal{R}^* g(y) = \int_{\mathcal{M}^y} g. \tag{2.2}$$

These two operators, integrating over dual fibers of the same incidence relation, allowed Helgason to develop inversion formulas for more general Radon-like operators over homogeneous spaces which satisfy this double fibration condition (see for example [34]). For our purposes, the fact that $\mathcal{R}$ and $\mathcal{R}^*$ share the same incidence relation will be important in relating the microlocal behaviors of $\mathcal{R}$ and $\mathcal{R}^*$, a relationship observed by Hormander in [35].

A local Radon-like transform $\mathcal{R}$ can be related to an oscillatory integral by writing the Schwartz kernel of $\mathcal{R}$ as an oscillatory integral distribution. Since $\mathcal{M}$ is embedded in $\Omega_L \times \Omega_R \subset \mathbb{R}^{2d}$, in a neighborhood of some reference point $P^\circ = (x^\circ, y^\circ) \in \mathcal{M}$ we can use the implicit function theorem to find a smooth $\mathbb{R}^{d-n}$-valued function Φ such that near P°

$$\mathcal{M} = \{(x,y) \in \Omega_L \times \Omega_R \ : \ \Phi(x,y) = 0\}.$$

The Schwartz kernel of $\mathcal{R}$ is then given by the measure $\chi\delta \circ \Phi$, where δ is the Dirac measure on $\mathbb{R}^{d-n}$, and χ is C^∞ and compactly supported near $P^\circ \in \Omega_L \times \Omega_R$, which we can take to be the origin in $\mathbb{R}^d \times \mathbb{R}^d$. Thus by the Fourier inversion formula in $\mathbb{R}^{d-n}$

$$\begin{aligned}\mathcal{R}f(x) &= \int_{\{y \in \mathbb{R}^d \ : \ \Phi(x,y)=0\}} f(y)\chi(x,y)\,dy \\ &= \int \chi(x,y)\delta \circ \Phi(x,y) f(y)\,dy \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^{d-n}} e^{2\pi i \tau \cdot \Phi(x,y)}\chi(x,y)f(y)\,d\tau\,dy, \qquad (2.3)\end{aligned}$$

where $\tau \cdot \Phi(x,y) = \sum_{i=1}^{d-n} \tau_i \Phi^i(x,y)$. The formula (2.3) reveals that $\mathcal{R}$ is an example of a Fourier integral operator (FIO), the theory of which we will discuss in the next section. First, we note that the assumption that ρ_L, ρ_R are submersions implies that $\mathcal{R}$ is locally a bounded operator on $L^1(\mathbb{R}^d)$ and $L^\infty(\mathbb{R}^d)$, and hence all $L^p(\mathbb{R}^d)$, $1 \le p \le \infty$.

Lemma 2.1 ([25, p. 4])**.** *Suppose that $\mathcal{R}$ and $\mathcal{M}$ are defined as above and the natural projections $\rho_L : \mathcal{M} \to \Omega_L$, $\rho_R : \mathcal{M} \to \Omega_R$ are submersions. Then $\mathcal{R}$ extends to continuous operator*

$$\mathcal{R} \ : \ L^p_{\text{comp}}(\Omega_R) \to L^p_{\text{loc}}(\Omega_L), \ 1 \le p \le \infty.$$

Proof. Let $v_0 \in C_c^\infty(\Omega_L)$ and $K \subset \Omega_R$ be a compact set; let $H = \operatorname{supp} v_0$. Let $\eta \in C_c^\infty(\mathbb{R}^{d-n})$ such that $0 \le \eta \le 2V(d-n)^{-1}$, $\int \eta = 1$, and $\operatorname{supp} \eta \subset B_1(0)$, where $V(d-n)$

is the volume of the ball of radius 1 in $\mathbb{R}^{d-n}$. Then $\varepsilon^{-(d-n)}\eta\big(\frac{x}{\varepsilon}\big)$ converges in the sense of distributions to the Dirac delta function on $\mathbb{R}^{d-n}$ as $\varepsilon \to 0^+$. Thus for $f \in C_c^\infty(\Omega_R)$ supported in K,

$$\mathcal{R}f(x) = \lim_{\varepsilon\to 0^+} \int \chi(x,y)\varepsilon^{-(d-n)}\eta\big(\tfrac{\Phi(x,y)}{\varepsilon}\big) f(y)\, dy.$$

To prove that $\mathcal{R}$ is bounded on L^1, we estimate

$$\int \Big| \lim_{\varepsilon\to 0^+} \int v_0(x)\chi(x,y)\varepsilon^{-(d-n)}\eta\big(\tfrac{\Phi(x,y)}{\varepsilon}\big) f(y)\, dy \Big|\, dx.$$

By Fatou's Lemma this is bounded by

$$\liminf_{\varepsilon\to 0^+} \int\int \big|v_0(x)\chi(x,y)\varepsilon^{-(d-n)}\eta\big(\tfrac{\Phi(x,y)}{\varepsilon}\big) f(y)\big|\, dy\, dx$$

Interchanging the order of integration and applying Hölder's inequality to pull out $\|f\|_{L^1}$, it suffices to estimate the expression

$$\sup_{y\in K} \int \big|v_0(x)\chi(x,y)\varepsilon^{-(d-n)}\eta\big(\tfrac{\Phi(x,y)}{\varepsilon}\big)\big|\, dx.$$

uniformly in ε. As η is supported in the unit ball, for fixed $y \in K$ the function $v_0(x)\chi(x,y)\eta\big(\frac{\Phi(x,y)}{\varepsilon}\big)$ is supported in the set

$$E^y = \{x \in H \ : \ |\Phi(x,y)| < \varepsilon\}.$$

Since ρ_R is a submersion, E^y is the ε-neighborhood of the immersed n-dimensional manifold $\mathcal{M}^y$. Applying the implicit function theorem we can represent $\mathcal{M}^y$ locally as the graph of a $\mathbb{R}^{d-n}$-valued C^1 function in a neighborhood $\mathcal{U} = U \times V \subset \Omega_L \times \Omega_R$ of a fixed point $(x_0, y_0) \in \mathcal{M}$. On this set we see that $|E^y \cap U| \le C\varepsilon^{d-n}$ uniformly in y. As $H \times K \subset \Omega_L \times \Omega_R$ is compact, we can cover $H \times K$ by finitely many such neighborhoods $\mathcal{U}$; thus by a partition of unity we see that

$$\sup_{y\in K} \int \big|v_0(x)\chi(x,y)\varepsilon^{-(d-n)}\eta\big(\tfrac{\Phi(x,y)}{\varepsilon}\big)\big|\, dx \le C_{d,n}\|v_0\chi\|_{L^1_x L^\infty_y}.$$

Thus

$$\liminf_{\varepsilon\to 0^+} \int\int \left|v_0(x)\chi(x,y)\varepsilon^{-(d-n)}\eta\left(\tfrac{\Phi(x,y)}{\varepsilon}\right)f(y)\right| dy\,dx \le C\|\chi\|_{L^1(H)L^\infty(K)}\,\|f\|_{L^1(\Omega_R)}$$

Next we prove the estimate on L^∞. For fixed $x \in H$, $v_0(x)\chi(x,y)\eta\left(\frac{\Phi(x,y)}{\varepsilon}\right)$ is supported in the set

$$E_x = \{y \in K \ : \ |\Phi(x,y)| \le \varepsilon\}.$$

Again, since ρ_L is a submersion, E_x is the ε-neighborhood of the immersed n-dimensional manifold $\mathcal{M}_x$. Applying the implicit function theorem we can represent $\mathcal{M}_x$ locally as the graph of a $\mathbb{R}^{d-n}$-valued C^1 function in a neighborhood $\mathcal{U} = U \times V \subset \Omega_L \times \Omega_R$ of a fixed point $(x_0, y_0) \in \mathcal{M}$. On this neighborhood we see that $|E_x \cap V| \le C\varepsilon^{d-n}$ uniformly in x. Applying a partition of unity on the finite cover of $H \times K$ obtained by the same argument as in the L^1 case, we see that

$$\left|\lim_{\varepsilon\to 0^+} \int v_0(x)\chi(x,y)\varepsilon^{-(d-n)}\eta\left(\tfrac{\Phi(x,y)}{\varepsilon}\right)f(y)\,dy\right| \le C_{d,n}\|f\|_{L^\infty(\Omega_R)}\|\chi\|_{L^\infty(H\times K)}.$$

Interpolating between L^1 and L^∞ yields the desired L^p estimates for $1 \le p \le \infty$. □

2.1 Fourier Integral Operators

Let X, Y be open sets of $\mathbb{R}^d$. A ***Fourier integral operator*** $\mathcal{F}$ is locally given by a sum of oscillatory integral operators of the form

$$Ff(x) = \int\int e^{2\pi i\phi(x,y,\theta)}a(x,y,\theta)f(y)\,dy\,d\theta,$$

where $\theta \in \mathbb{R}^N$ for some N, $x \in X$, $y \in Y$, $\phi, a \in C^\infty(X \times Y \times \mathbb{R}^N)$, $\nabla\phi_{\theta_i}$ are independent at $\{\phi_\theta = 0\}$, and ϕ satisfies a homogeneity condition $\phi(x,y,t\theta) = t\phi(x,y,\theta)$ for $|\theta| = 1$

and $t \gg 1$. We say $\mathcal{F}$ is a FIO of order μ if $a \in S^{\mu+(d-N)/2}$, the standard symbol class of order $\mu + (d-N)/2$, for each F. The ***canonical relation*** associated to F is locally given by

$$\mathfrak{C} = \{(x, \phi_x, y, -\phi_y) \ : \ \phi_\theta = 0\},$$

and we assume that $\mathfrak{C} \subset T^*X \setminus 0 \times T^*Y \setminus 0$, where $T^*X \setminus 0 = \{(x,\xi) \in T^*X \ : \ \xi \neq 0\}$. Staying away from the zero-sections in T^*X and T^*Y implies that

$$|\phi_x(x,y,\theta)| \approx |\theta| \approx |\phi_y(x,y,\theta)|$$

when ϕ_θ is small, hence $\mathfrak{C}$ is conic (in the (ξ,η)-variables) [25]. If σ_X and σ_Y are the canonical 2-forms on T^*X and T^*Y respectively, then $\mathfrak{C}$ is Lagrangian with respect to the symplectic form $\sigma_X - \sigma_Y$ [20, § 3.6]. As explored in the work of Hörmander [35] and the general theory of FIOs, the L^2-Sobolev regularity of a Fourier integral operator F depends on the geometry of its canonical relation, more specifically the geometry of the natural projections

$$\begin{tikzcd} & \mathfrak{C} \arrow[dl, "\pi_L"'] \arrow[dr, "\pi_R"] & \\ T^*\Omega_L & & T^*\Omega_R \end{tikzcd} \tag{2.4}$$

If $\mathfrak{C}$ is locally the graph of a canonical transformation, meaning that π_L and π_R are locally diffeomorphisms, then we have the following theorem, due to Hörmander [35] as a consequence of his work developing a calculus for FIOs.

Theorem 2.2 ([35], cf. [25, p. 4]). *Suppose F is a Fourier integral operator of order μ associated to a canonical relation which is locally the graph of a canonical transformation. Then for all $s \in \mathbb{R}$, F extends to a continuous operator from $L^2_{s,\mathrm{comp}}(Y)$ into $L^2_{s-\mu,\mathrm{loc}}(X)$.*

If the canonical relation associated to F is locally the graph of a canonical transformation, we say that F is associated to a local canonical graph.

2.2 Applications to Local Radon-like Transforms

Given this context, $\mathcal{R}$, as described in (2.3), is a FIO of order $-n/2$ (as $\chi(x,y)$ is a symbol of order 0), and the canonical relation associated to $\mathcal{R}$ is given by

$$(N^*\mathcal{M})' = \big\{(x,(\tau\cdot\Phi)_x, y, -(\tau\cdot\Phi)_y),\ :\ \Phi(x,y)=0\big\}. \tag{2.5}$$

The set $(N^*\mathcal{M})'$ is related to the conormal bundle of the incidence relation $\mathcal{M}$ (hence the notational similarity) by

$$(N^*\mathcal{M})' = \big\{(x,\xi,y,-\eta)\ :\ (x,y,\xi,\eta)\in N^*\mathcal{M}\big\},$$

hence we refer to $(N^*\mathcal{M})'$ as the ***twisted conormal bundle*** of $\mathcal{M}$. Recall the conormal bundle of a manifold $\mathcal{M}$ is given by

$$N^*\mathcal{M} = \{(x,y,\xi,\eta)\in T^*(\Omega_L\times\Omega_R)\setminus\{0\}\ :\ (\xi,\eta)\perp T_{(x,y)}\mathcal{M}\}$$

Note that the diagram (2.4) corresponds to a refinement to the cotangent spaces of the double fibration formalism (2.1). Note that since the adjoint (2.2) shares an incidence relation with $\mathcal{R}$, the canonical relation associated to $\mathcal{R}^*$ is the inverse image of $(N^*\mathcal{M})'$ under the map $T^*\Omega_R\times T^*\Omega_L\to T^*\Omega_L\times T^*\Omega_R$, interchanging the two factors. This in turn interchanges the projections π_L and π_R between $\mathcal{R}$ and $\mathcal{R}^*$. This symmetry allows us to state theorems with assumptions on π_L without loss of generality, as we can treat the adjoint to interchange to the projections.

Since the projections in (2.1) are submersions, we can choose local coordinates to parametrize $\mathcal{M}$ as a graph so that

$$\Phi(x,y) = S(x,y'') - y'$$

with $y' = (y_1, ..., y_{n'})$, $y'' = (y_{n'+1}, ..., y_d)$, and $S = (S^1, ..., S^{n'})$. Recall that $n' = d - n$ is the codimension of $\mathcal{M}_x \subset \Omega_L$. This process is described in detail in [46] in the case $d = 3, n = 1$, and is presented below for general d and n.

Indeed, since ρ_L is a submersion the $n' \times d$ matrix Φ_y has rank n', so by a linear change of variables in y we can find coordinates y' (defined above) so that $\det(\nabla_{y'}\Phi) \neq 0$ near a reference point y°. Then by the implicit function theorem, we can choose (x, y'') as local coordinates on $\mathcal{M}$ so that the equation $\Phi(x, y) = 0$ is equivalent to

$$y_i = S^i(x, y''),\ \ i = 1, ..., n' \tag{2.6}$$

near y°. Thus we can write

$$\Phi(x, y) = \sum_{i=1}^{n'} (S^i(x, y'') - y_i) B_i(x, y), \tag{2.7}$$

where

$$B_i(x, y) = -\int_0^1 \Phi_{y_i}\Big(x, S(x, y'') + s(y' - S(x, y'')), y''\Big)\, ds.$$

Since Φ_y has rank n', Φ_{y_i} are linearly independent on $\mathcal{M}$, hence if we choose χ supported sufficiently close to $\mathcal{M}$ we can ensure that B_i are linearly independent as well. Thus we can rewrite (2.7) as

$$\Phi(x, y) = B(x, y)\Big(S(x, y'') - y'\Big),$$

where $B(x, y)$ is a $n' \times n'$ invertible matrix whose column vectors are B_i. Additionally, since ρ_R is a submersion the gradients $\{S^i_x(x, y'')\}_{i=1,..,n'}$ are linearly independent as well, so through a change of variables we can rewrite (2.3) as

$$\int \chi(x, y)\delta \circ \Phi(x, y) f(y)\, dy = \int \frac{\chi(x, y)}{|\det(B(x, y))|} \int e^{2\pi i \tau \cdot (S(x, y'') - y')}\, d\tau f(y)\, dy. \tag{2.8}$$

By redefining χ we have parametrized $\mathcal{M}$ as the graph $y' = S(x, y'')$.

Given these changes of variables, the twisted conormal bundle associated to $\mathcal{R}$ is given by

$$(N^*\mathcal{M})' = \Big\{(x,\xi,y,\eta)\ :\ y' = S(x,y''),\ \xi = \sum_{i=1}^{n'} \tau_i S^i(x,y''),$$
$$\eta = \Big(\tau, -\sum_{i=1}^{n'} \tau_i S^i_{y''}(x,y'')\Big)\Big\}.$$

Thus parametrizing $(N^*\mathcal{M})'$ by the coordinates (x,τ,y''), the projection π_L is identified with the map

$$\tilde{\pi}_L : (x,\tau,y'') \mapsto \Big(x, \sum_{i=1}^{n'} \tau_i S^i_x(x,y'')\Big) \tag{2.9}$$

and the projection π_R is identified with the map

$$\tilde{\pi}_R : (x,\tau,y'') \mapsto \Big(S(x,y''), y'', \tau, -\sum_{i=1}^{n'} \tau_i S^i_{y''}(x,y'')\Big) \tag{2.10}$$

From these identifications we can see that the rank of the differentials of π_L and π_R must be equal; this is a more general consequence of the canonical relation being Lagrangian (see [35]). Since we can identify the differential of π_L and π_R with the Jacobians of $\tilde{\pi}_L$ and $\tilde{\pi}_R$ respectively, we see that

$$\begin{aligned} \operatorname{corank} d\pi_L &= \operatorname{corank} \begin{pmatrix} I_{d\times d} & 0_{d\times n'} & 0_{d\times n} \\ (\tau\cdot S)_{xx} & S_x & (\tau\cdot S)_{xy''} \end{pmatrix} \\ &= \operatorname{corank} \begin{pmatrix} S_x & (\tau\cdot S)_{xy''} \end{pmatrix} \end{aligned} \tag{2.11}$$

and

$$\operatorname{corank} d\pi_R = \operatorname{corank}\begin{pmatrix} (S_x)^{\intercal} & 0_{n'\times n'} & (S_{y''})^{\intercal} \\ 0_{n\times d} & 0_{n\times n'} & I_{n\times n} \\ 0_{n'\times d} & I_{n'\times n'} & 0_{n'\times n} \\ (\tau\cdot S)_{y''x} & S_{y''} & (\tau\cdot S)_{y''y''} \end{pmatrix}$$
$$= \operatorname{corank}\begin{pmatrix} (S_x)^{\intercal} \\ (\tau\cdot S)_{y''x} \end{pmatrix}. \tag{2.12}$$

This implies in particular that π_L is a local diffeomorphism if and only if π_R is a local diffeomorphism.

2.3 Inherent Singularities

In the case of local Radon-like transforms over families of hypersurfaces the local canonical graph condition of Theorem 2.2 coincides the notion of ***nonvanishing rotational curvature*** [53], given by the invertibility of the matrix

$$\begin{pmatrix} \tau\cdot\Phi_{xy} & \Phi_y \\ \Phi_x^{\intercal} & 0 \end{pmatrix}.$$

However, if the codimension of $\mathcal{M}_x$ exceeds 1 then the local canonical graph condition does not generically hold. In fact, as noted by Gressman in [32] (using the language of rotational curvature), the projections π_L and π_R must be singular unless d and n satisfy a strict number-theoretic relation. We reproduce his result below.

Lemma 2.3 ([32, Theorem 3]). *Suppose that $\mathcal{R}, \mathcal{M}$, and the projections π_L, π_R are defined as above. Suppose that n (the dimension of the manifolds $\mathcal{M}_x$) factors into the* *form* $2^{4q+r}s$ *for integers* q, r, s*, such that* s *is odd and* $0 \leq r \leq 3$*. Then if* $d-n$ *(the codimension of the manifolds* $\mathcal{M}_x$*) exceeds* $8q+2^r$*, then* $\mathcal{R}$ *is not associated to a local canonical graph. Specifically, for every point* $(x,y) \in \mathcal{M}$ *there is a* $P = (x, \xi, y, \eta) \in (N^*\mathcal{M})'$ *such that* $(d\pi_L)_P$ *and* $(d\pi_R)_P$ *are singular.*

Proof. Fix $(x,y) \in \Omega_L \times \Omega_R$. As discussed above, it suffices to check whether $(d\pi_L)$ is invertible at a point $P = (x, \xi, y, \eta)$. As noted above, we can identify π_L with the map (2.9), so by (2.11), $d\pi_L$ is invertible if and only if the matrix

$$\begin{pmatrix} S^1_x(x,y'') & \cdots & S^{n'}_x(x,y'') & \sum_i \tau_i S^i_{xy_{n'+1}}(x,y'') & \cdots & \sum_i \tau_i S^i_{xy_d}(x,y'') \end{pmatrix}$$

is invertible. Since $\{S^i_x\}_i$ are linearly independent, $d\pi_L$ is invertible if and only if

$$\operatorname{rank} \begin{pmatrix} \sum_i \tau_i S^i_{xy_{n'+1}}(x,y') & \cdots & \sum_i \tau_i S^i_{xy_d}(x,y'') \end{pmatrix} = n.$$

By renaming and possibly reordering coordinates, this is equivalent to invertibility of

$$\sum_i \tau_i \begin{pmatrix} S^i_{x_{n'+1}y_{n'+1}}(x,y'') & \cdots & S^i_{x_{n'+1}y_d}(x,y'') \\ \vdots & \ddots & \vdots \\ S^i_{x_d y_{n'+1}}(x,y'') & \cdots & S^i_{x_d y_d}(x,y'') \end{pmatrix}.$$

Since τ may be any nonzero vector in $\mathbb{R}^{d-n}$, for fixed x, y' we are asked to find a family of $d-n$ real matrices of dimension $n \times n$ such that every linear combination of them is invertible. If such a family of matrices exists then it is easy to construct $S(x,y'')$, linear in x and y'', for which the associated projections π_L and π_R are local diffeomorphisms; if no such family exists then for every choice of $S(x,y'')$ and every (x,y'') we can find some $\tau \neq 0$ such that π_L (and hence π_R) is singular at (x, y'', τ).

The existence of such families of matrices has been completely characterized for some time, originally due to Adams, Lax, and Phillips in [1, 2] (see also a minor correction in

[3]). In particular, the maximal number of $n \times n$ real matrices for which every nontrivial linear combination is invertible is given by the Radon-Hurwitz function $\rho(n) = 8q + 2^r$, where q and r are defined as in the statement of this lemma.

$\square$

The possible values of d and n' which admit examples of $\mathcal{R}$ associated to a local canonical graph are few and far between. A table of such pairs (n', d) is shown in Figure 1.

$n' \backslash d$	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
2	×		✓		✓		✓		✓		✓		✓		✓
3	×	×				✓				✓				✓	
4	×	×	×				✓				✓				✓
5	×	×	×	×								✓			
6	×	×	×	×	×								✓		
7	×	×	×	×	×	×								✓	
8	×	×	×	×	×	×	×								✓

Figure 1: Pairs (n', d) which admit local Radon-like transforms $\mathcal{R}$ satisfying the conditions of Lemma 2.2

For example, if n is odd then for the local canonical graph condition to hold n' must equal 1 (This phenomenon is essentially the observation made by Christ in [10] in the setting of Fourier restriction). In general, one can only expect to find examples of local Radon-like transforms over families of manifolds of large codimension which satisfy the conditions of Theorem 2.2 if n is divisible by a large power of 2, and even then the possible codimension n' only grows logarithmically with n. This severely restricts the application of Theorem 2.2; in particular, local Radon-like transforms over families of curves in $\mathbb{R}^d$ must have singular projections π_L and π_R unless $d = 2$.

We can observe this degeneracy directly for averages over curves in $\mathbb{R}^d$, $d \geq 3$ by parametrizing the projections π_L and π_R. Recalling our definition of Φ from (2.8), after a change of coordinates we can parametrize $\mathcal{M}$ as a graph

$$\Phi(x, y) = S(x, y_d) - y'$$

with $y' = (y_1, ..., y_{d-1}) \in \mathbb{R}^{d-1}$ and $S = (S^1, ..., S^{d-1})$. Then the condition that π_L is locally diffeomorphic is equivalent to the nonvanishing of the determinant

$$\det \begin{pmatrix} \tau \cdot \Phi_{xy} & \Phi_x \\ {}^{\intercal}\Phi_y & 0 \end{pmatrix} = (-1)^n \det \begin{pmatrix} \tau \cdot S_{xy_d} & S_x \end{pmatrix}$$

for all $\tau \in \mathbb{S}^{d-2}$. This determinant is a linear functional in τ, and thus for each fixed (x, y) vanishes for all τ in a hyperplane.

2.4 Classification of Singularities

In light of Lemma 2.3, it is unsurprising that many local Radon-like transforms encountered in the literature are associated to canonical relations with singular projections. Recall that in view of the symplectic structure of $T^*\Omega_L \times T^*\Omega_R$ we always have

$$\operatorname{rank} d\pi_L = \operatorname{rank} d\pi_R,$$

but the behavior of the singularities of π_L and π_R may differ. In this section we explore different notions of singularities which can and do occur in the projections π_L and π_R.

Here we focus on the case corank $d\pi_{L(R)} \leq 1$, which is the situation when the manifolds $\mathcal{M}_x$ are curves.

Let X, Y be smooth d-dimensional manifolds, and $\pi : X \to Y$ a smooth map between them. We call a smooth vector field V on a neighborhood U of $P \in X$ a ***kernel field*** of π if V is not identically zero, is smooth on U, and if there exists a smooth vector field W on $\pi(U)$ so that $d\pi_P V = \det(d\pi_P) W_{\pi(P)}$ for all $P \in U$. Note that this definition implies $V_P \in \ker d\pi_P$ for every $P \in X$ such that $\det d\pi_P = 0$, hence why these vector fields are called kernel fields. As shown by Greenleaf and Seeger in [26], on the set

$$\mathcal{L} = \{P \in X \ : \ \det(d\pi)_P = 0\} \tag{2.13}$$

kernel fields are unique up to scaling by smooth functions.

Lemma 2.4 ([26, pp. 2-3])**.** *If* corank $(d\pi)_P \leq 1$ *then there is a kernel field of* π *defined in a neighborhood of* P*. Moreover, if* V *and* $\tilde{V}$ *are both kernel fields on* U*, then* $\tilde{V} = \alpha V - \det(d\pi) W$ *for some smooth function* α *and smooth vector field* W*.*

Proof. Following [26], assume corank $(d\pi)_P \leq 1$, and $P \in X$. Then we can pick coordinates (x', x_d) on X and (y', y_d) on Y vanishing at P and $\pi(P)$ respectively so that $(d\pi)_P = \left(\begin{smallmatrix} A & p \\ q^\intercal & r \end{smallmatrix}\right)$, where A is an invertible $(d-1) \times (d-1)$ matrix, p and q are vectors in $\mathbb{R}^{d-1}$, $r \in \mathbb{R}$, and $A, p, q,$ and r depend smoothly on x. Define a vector field $V = \partial_{x_d} - \langle A^{-1}p, \partial_{x'}\rangle$. We see that $d\pi(V) = (r - q^\intercal A^{-1} p)\partial_{y_d}$, and $\det d\pi = (r - q^\intercal A^{-1}p) \det A$; thus V is a kernel field.

Furthermore, assume that $\tilde{V} = \langle \beta', \partial_{x'}\rangle + \beta_d \partial_{x_d}$ is also a kernel field of π near P, i.e. $d\pi(\tilde{V}) = \det(d\pi)\tilde{W}$, where $\tilde{W} = \langle \sigma', \partial_{y'}\rangle + \sigma_d \partial_{y_d}$, and $\beta = (\beta', \beta_d)$ and $\sigma = (\sigma', \sigma_d)$ are smooth functions of x and y respectively. Then, at any x, $A\beta' + p\beta_d = \det(d\pi)\sigma'$; since

A is invertible this implies that $\beta' = \det(d\pi)A^{-1}\sigma' - A^{-1}p\beta_d$, and therefore

$$\tilde{V} = \beta_d V + \det(d\pi)\langle A^{-1}\sigma', \partial_{x'}\rangle.$$

$\square$

We say π ***drops rank simply*** at a point P if $\operatorname{rank}(d\pi)_P = d-1$ and $d(\det d\pi)_P \neq 0$. By an application of the implicit function theorem we see that if π drops rank simply at P, then $\mathcal{L}$ is locally a hypersurface near P. If we assume π drops ranks simply whenever it is singular then we can classify many types of singularities that π may exhibit using kernel fields. The first type is called a Whitney fold, introduced by Whitney in [57].

Definition 2.5. *We say π has a* ***Whitney fold*** *at $P \in X$ if π drops rank simply at P and $V\det(d\pi)_P \neq 0$ for any (and therefore every) kernel field V.*

The prototypical example of a map with a Whitney fold is $f : (x, y) \mapsto (x, y^2)$. This is also in some sense the only example of a Whitney fold, since in local coordinates every Whitney fold can be expressed as the graph of a parabola in the final two coordinates. We can see the "fold" more clearly if we consider f as a composition of the maps

$$(x, y) \mapsto (x, y, y^2) \mapsto (x, y^2),$$

illustrated in Figure 2. The map f "folds" the lower half plane onto the upper half plane, and the crease of this fold is the line $y = 0$. Unsurprisingly, this crease is also the set on which f exhibits a Whitney fold. Indeed, the differential of f at (x, y) is the 2×2 matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 2y \end{smallmatrix}\right)$, which drops rank by 1 when $y = 0$, and the determinant of the differential of f vanishes to order 1 in the dy direction, which is also the direction of the kernel of df, along $\{y = 0\}$.

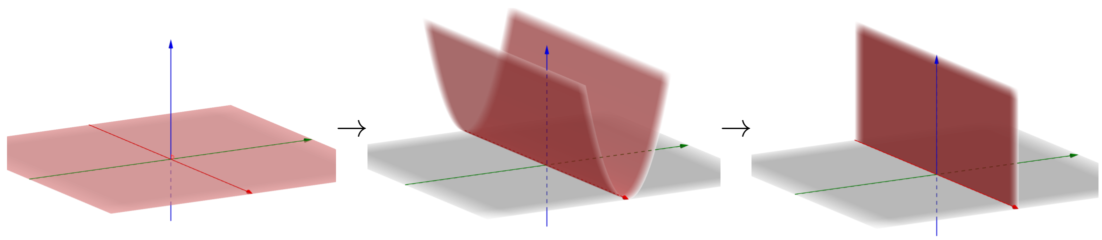

Figure 2: Prototypical Whitney Fold

In this work we consider local Radon-like transforms for which at least one of the projections π_L, π_R has at most fold singularities. For the purpose of simplicity we shall usually assume that this map is π_L, although analogous results can be derived easily from the mapping properties of the adjoint (2.2).

The essential characteristic of a Whitney fold is the order 1 vanishing of the determinant of $d\pi$ in the direction of the kernel of $d\pi$. A natural generalization of this condition is due to Comech, who proposed the following classification of singularities of finite type.

Definition 2.6 ([15, p. 3])**.** *We say π is of **type k** at P if π drops rank simply at P and for all $j < k$ we have $V^j \det(d\pi)_P = 0$, but $V^k \det(d\pi)_P \neq 0$.*

*We say π has **maximal type k** if at every point in its domain, π is either nonsingular or is type j with $j \leq k$.*

From this definition we see that a Whitney fold is equivalent to a type 1 singularity. The Morin singularities (cusps, swallowtails, etc.) are examples of finite type singularities which are stable under perturbations, but we will not discuss them further here (see [33]). An example of a map with a type k singularity is $g_k : (x, y, z) \mapsto$

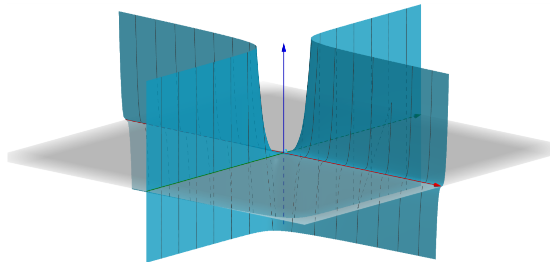

Figure 3: Surface $\mathcal{L}$ associated to g_3

$(x, y, xz + \frac{1}{k+1} yz^{k+1})$. The differential of g_k is given by

$$dg_k\big|_{(x,y,z)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ z & z^{k+1} & x + yz^k \end{pmatrix}$$

and g_k drops rank simply along the surface $\mathcal{L}$ defined by $x = -yz^k$, which contains the y- and z-axes. This surface is illustrated in Figure 3 in the case $k = 3$. A kernel field for g_k along this surface is ∂_z, and $\partial_z \det(dg) = kyz^{k-1}$. This quantity is nonzero on almost all of $\mathcal{L}$, implying that g_k has Whitney folds on almost all of $\mathcal{L}$. However, g_k is of type k along the y- and z- axes, where $kyz^{k-1} = 0$.

While there are many exotic singularities which do not fall into the classes described above, there is arguably one "worst" case, when π is maximally degenerate.

Definition 2.7 (cf. [25, p. 5]). *We say π is a **blowdown** on $\mathcal{L}$ if π drops rank simply on $\mathcal{L}$, but every kernel field V of π, when restricted to $\mathcal{L}$, is everywhere tangential to $\mathcal{L}$.*

Note that the blowdown condition implies that $V^k(\det dg)\big|_P = 0$ for all $k \in \mathbb{N}$ and all $P \in \mathcal{L}$. An example of a blowdown is the map $h : (x, y, z) \mapsto (x, y, yz)$. The differential of h is

$$dh_{(x,y,z)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & z & y \end{pmatrix}$$

which is singular along the plane $y = 0$. A kernel field for h along $\{y = 0\}$ is ∂_z, which lies tangent to the plane $y = 0$ everywhere.

Chapter 3

History of Results

3.1 History of Results on L^2

The L^2-Sobolev regularity of $\mathcal{R}$ is well-studied when one of π_L, π_R projects with only fold singularities. The first results in this case are due to Melrose and Taylor [37] in the context of scattering of plane waves. They proved L^2-Sobolev results in the case when both π_L and π_R project with only fold singularities; a canonical relation with this property is called a ***folding canonical relation***. Phong and Stein, motivated by the earlier work of [52] and [17], were instrumental in unifying the subject by introducing a dyadic frequency decomposition relative to $\mathcal{L}$ that became crucial to proving L^2-Sobolev estimates. We will use a modified version of their argument in Chapter 8. We state three results that will be crucial for this work. First, we describe the L^2-Sobolev boundedness of $\mathcal{R}$ associated to a folding canonical relation.

Theorem 3.1 ([37, 40])**.** *Suppose that both π_L and π_R project with only fold singularities. Then $\mathcal{R}$ extends to a continuous operator*

$$\mathcal{R} \ : \ L^2_{s,\mathrm{comp}}(\Omega_R) \to L^2_{s+\frac{n}{2}-\frac{1}{6},\mathrm{loc}}(\Omega_L).$$

Greenleaf and Seeger proved a uniform estimate of L^2 regularity under the assumption that one of the projections π_L, π_R has only fold singularities, with no assumption on the other projection.

Theorem 3.2 ([23, Theorem 1.1])**.** *Suppose that one of the projections* π_L, π_R *has maximal type* 1*. Then for* $s \in \mathbb{R}$*,* $\mathcal{R}$ *extends to a continuous operator*

$$\mathcal{R} \ : \ L^2_{s,\mathrm{comp}}(\Omega_R) \to L^2_{s+\frac{n}{2}-\frac{1}{4},\mathrm{loc}}(\Omega_L).$$

As was shown in [28, 29], this L^2 regularity estimate is sharp for local Radon-like transforms associated to canonical relations where one projection has at most fold singularities and the other has a blowdown singularity; a canonical relation with such a configuration of projections is called a ***fibered folding canonical relation***.

However, when one of the projections has only fold singularities and the other is less degenerate than a blowdown one might expect better L^2 regularity. This was proven in the finite type case by Comech, who obtained a sharp loss of $s(k) = (4+\frac{2}{k})^{-1}$ derivatives if one of π_L and π_R has only fold singularities, and the other has maximal type k.

Theorem 3.3 ([15, Theorem 1.1])**.** *Suppose that one of the projections* π_L, π_R *has maximal type* 1 *and the other has maximal type* k*. Then for* $s \in \mathbb{R}$*,* $\mathcal{R}$ *extends to a continuous operator*

$$\mathcal{R} \ : \ L^2_{s,\mathrm{comp}}(\Omega_R) \to L^2_{s+\frac{n}{2}-s(k),\mathrm{loc}}(\Omega_L).$$

Note that for curves in $\mathbb{R}^3$ the quantity $\frac{n}{2} - s(k)$ ranges between $\frac{1}{3}$ for folding canonical relations and $\frac{1}{4}$ for fibered folding canonical relations, interpolating between the regularity results in Theorems 3.1 and 3.2.

L^2-Sobolev estimates are also known for larger classes of singularities, such as two-sided and one-sided cusps ([16, 24]), and higher order singularities ([25, 26, 18]). While much progress has been made in L^2, sharp L^p-Sobolev regularity has largely been out of reach for local Radon-like transforms $\mathcal{R}$ for arbitrary dimensions d and n.

As a first attempt at L^p-Sobolev regularity we can interpolate the estimates of these three theorems with the L^p estimates of Lemma 2.1. However, we cannot interpolate L^1 or L^∞ with a Sobolev space to obtain L^p-Sobolev estimates; instead we adapt an analytic interpolation method due to Fefferman and Stein [21] involving Hardy space estimates.

Proposition 3.4. *Let $\mathcal{R}$ be a local Radon-like transform. Assume there exists $\alpha > 0$ such that $\mathcal{R}$ extends to a bounded operator*

$$\mathcal{R} \ : \ L^2_{\mathrm{comp}}(\Omega_R) \to L^2_{\alpha,\mathrm{loc}}(\Omega_L). \tag{3.1}$$

Then for $1 < p < 2$, $\mathcal{R}$ is bounded from $L^p_{\mathrm{comp}}(\Omega_R)$ to $L^p_{\alpha(p),\mathrm{loc}}(\Omega_L)$ where $\alpha(p) = (2\alpha - \frac{2\alpha}{p})$. Note that $\alpha(2) = \alpha$.

We will prove Proposition 3.4 in Chapter 7. Applying the result to $\mathcal{R}$ and $\mathcal{R}^*$ we obtain the following L^p regularity estimates.

Theorem 3.5. *Suppose that $\mathcal{R}$ is a local Radon-like transform such that π_L projects with folds. Let k be the maximal type of π_R, with $k = \infty$ and $s(\infty) = \lim_{k\to\infty} s(k)$ if π_R does not have maximal type. Then $\mathcal{R}$ extends to a bounded operator from $L^p_{\mathrm{comp}}(\Omega_R)$ to $L^p_{s,\mathrm{loc}}(\Omega_L)$, where $(1/p, s)$ lies within the shaded region of Figure 4.*

These L^p-Sobolev estimates are not sharp. Results exist for improvements to this range which are sharp in the plane [50, 51]. More recently, sharp results on L^p have been proven for local Radon-like transforms over families of curves in $\mathbb{R}^3$, which will be the focus of this work.

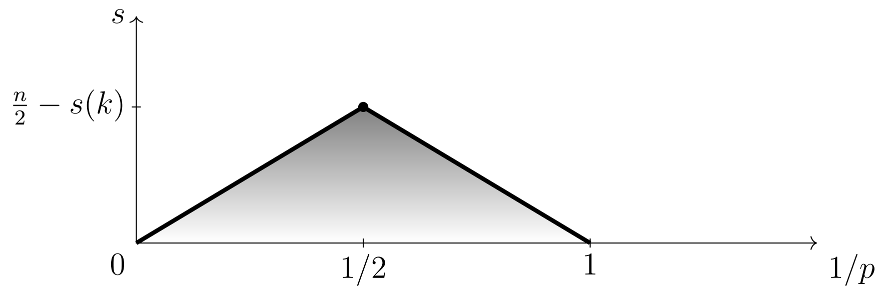

Figure 4: A priori $L^p \to L^p_s$ mapping for $\mathcal{R}$ with one-sided fold singularities

3.2 An Example of Sharp L^p Regularity

We introduce the discussion of sharp L^p regularity with an example. Let $\gamma : [0,1] \to \mathbb{R}^3$ be smooth and regular (i.e. γ is C^∞ and $\gamma' \neq 0$), and let χ be a smooth nonnegative function supported on $[0,1]$. We can define a measure μ supported on γ given by $\langle f, \mu \rangle = \int f(\gamma(t))\chi(t)\,dt$. Then the convolution operator

$$\mathcal{A}_{\mathbb{R}} f(x) = f * \mu(x) = \int_0^1 f(x - \gamma(t))\chi(t)\,dt \tag{3.2}$$

is an example of a local Radon-like transform associated to the family of curves $\mathcal{M}_x = \{x - \gamma(t) \ : \ t \in [0,1]\}$ in $\mathbb{R}^3$. In [45], Pramanik and Seeger proved that $\mathcal{A}_{\mathbb{R}}$ satisfies sharp L^p-Sobolev estimates for sufficiently large p provided $\mathcal{A}_{\mathbb{R}}$ is associated to folding canonical relations.

In this section we will investigate which class of curves γ are associated with folding canonical relations for $\mathcal{A}_{\mathbb{R}}$ and examine the proof of Pramanik and Seeger's result, as the techniques introduced in [45] provide the foundation for later sharp L^p-Sobolev regularity results.

Since $\gamma' \neq 0$ we may choose coordinates so that locally $\gamma(t) = (\gamma_1(t), \gamma_2(t), t)$. Then the incidence relation associated to $\mathcal{A}_{\mathbb{R}}$ is given by

$$\mathcal{M} = \{(x,y) \ : \ \Phi(x,y) = 0\}$$

where $\Phi = (\Phi^1, \Phi^2)^\intercal$, and $\Phi^i(x,y) = x_i - y_i - \gamma_i(x_3 - y_3)$ for $i = 1, 2$. The twisted conormal bundle of $\mathcal{M}$ is then

$$(N^*\mathcal{M})' = \big\{(x,\xi,y,\eta) \ : \ y_i = x_i - \gamma_i(x_3,y_3), \ i = 1,2, \\ \xi = \eta = \big(\tau_1, \tau_2, -\tau_1\gamma_1'(x_3 - y_3) - \tau_2\gamma_2'(x_3 - y_3)\big)\big\}$$

In the coordinates induced by Φ we can identify the differentials of π_L and π_R with the Jacobians of the maps

$$\tilde{\pi}_L : (x, y_3, \tau) \mapsto (x, \tau_1, \tau_2, -\tau_1\gamma_1'(x_3 - y_3) - \tau_2\gamma_2'(x_3 - y_3))$$

$$\tilde{\pi}_R : (x, y_3, \tau) \mapsto \big(x_1 - \gamma_1(x_3 - y_3), x_2 - \gamma_2(x_3 - y_3), y_3 \\ \tau_1, \tau_2, -\tau_1\gamma_1'(x_3 - y_3) - \tau_2\gamma_2'(x_3 - y_3)\big)$$

respectively. We obtain $(d\pi_L)\big|_{(x,y_3,\tau)} = \left(\begin{smallmatrix} I & 0 \\ A & B \end{smallmatrix}\right)$, where

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\gamma_1'(x_3 - y_3) & -\gamma_2'(x_3 - y_3) & \tau_1\gamma_1''(x_3 - y_3) + \tau_2\gamma_2''(x_3 - y_3) \end{pmatrix},$$

and $(d\pi_R)\big|_{(x,\tau,y_3)}$ equals

$$\begin{pmatrix} 1 & 0 & -\gamma_1'(x_3 - y_3) & 0 & 0 & \gamma_1'(x_3 - y_3) \\ 0 & 1 & -\gamma_2'(x_3 - y_3) & 0 & 0 & \gamma_2'(x_3 - y_3) \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -\tau \cdot \gamma''(x_3 - y_3) & -\gamma_1'(x_3 - y_3) & -\gamma_2'(x_3 - y_3) & \tau \cdot \gamma''(x_3 - y_3) \end{pmatrix}.$$

Calculating the determinant of both of these matrices, we see that

$$\det(d\pi_L)\Big|_{(x,y_3,\tau)} = -\det(d\pi_R)\Big|_{(x,y_3,\tau)} = \tau_1\gamma_1''(x_3-y_3) + \tau_2\gamma_2''(x_3-y_3).$$

Thus both π_L and π_R are singular on the set $\tau\cdot\gamma''(x_3-y_3)=0$; if we assume that γ has nonvanishing curvature (i.e. $\gamma''(t)\neq 0$) then this condition is only satisfied for a 1-parameter family of $\tau\in\mathbb{R}^2$, specifically $(\tau_1,\tau_2)=\rho(-\gamma_2''(x_3-y_3),\gamma_1''(x_3-y_3))$ for $\rho\in\mathbb{R}$. Thus

$$\begin{aligned}\mathcal{L} &= \{P\in(N^*\mathcal{M})' \;:\; \det(d\pi_L)|_P=0\}\\ &= \Big\{(x,\xi,y,\eta) \;:\; y_i = x_i-\gamma_i(x_3-y_3),\ i=1,2\\ &\qquad\qquad \xi=\eta=\big(\tau_1,\tau_2,-\tau_1\gamma_1(x_3-y_3)-\tau_2\gamma_2(x_3-y_3)\big)\\ &\qquad\qquad\quad (\tau_1,\tau_2)=\rho(-\gamma_2''(x_3-y_3),\gamma_1''(x_3-y_3)),\ \rho\in\mathbb{R}\Big\}. \end{aligned}\tag{3.3}$$

Kernel fields for π_L and π_R are given by $V_L=\partial_{y_3}$ and $V_R=\gamma_1'(x_3-y_3)\partial_{x_1}+\gamma_2'(x_3-y_3)\partial_{x_2}+\partial_{x_3}$ respectively. Since $\det(d\pi_L)$ and $\det(d\pi_R)$ are identical (up to a minus sign) and are functions of τ and x_3-y_3 (in particular they are constant in x_1,x_2), the types of π_L and π_R at a particular point in $(N^*\mathcal{M})'$ will always be identical. Indeed, for any $k\in\mathbb{N}$

$$\begin{aligned} V_L^k\det(d\pi_L)\Big|_{\tau\cdot\gamma''(x_3-y_3)=0} &= (-1)^k\tau\cdot\gamma^{(k+2)}(x_3-y_3)\Big|_{\tau=\rho\left(-\gamma_2''(x_3-y_3),\gamma_1''(x_3-y_3)\right)}\\ &= (-1)^k\rho\det\begin{pmatrix}\gamma_1''(x_3-y_3) & \gamma_2''(x_3-y_3)\\ \gamma_1^{(k+2)}(x_3-y_3) & \gamma_2^{(k+2)}(x_3-y_3)\end{pmatrix} \end{aligned}\tag{3.4}$$

$$\begin{aligned} V_R^k\det(d\pi_R)\Big|_{\tau\cdot\gamma''(x_3-y_3)=0} &= (-1)^{k+1}\tau\cdot\gamma^{(k+2)}(x_3-y_3)\Big|_{\tau=\rho\left(-\gamma_2'(x_3-y_3),\gamma_1''(x_3-y_3)\right)}\\ &= (-1)^{k+1}\rho\det\begin{pmatrix}\gamma_1''(x_3-y_3) & \gamma_2''(x_3-y_3)\\ \gamma_1^{(k+2)}(x_3-y_3) & \gamma_2^{(k+2)}(x_3-y_3)\end{pmatrix}. \end{aligned}\tag{3.5}$$

Since the absolute value of these two expressions are equal both must vanish on the same set. Thus we can summarize the conditions on which π_L and π_R have maximal type k.

Proposition 3.6. *Let I be a compact interval, and suppose $\gamma : I \to \mathbb{R}^3$ is a smooth compactly supported curve such that $\gamma'(t), \gamma''(t) \neq 0$ for all $t \in I$. For each $t \in I$ let $k_t \geq 1$ be the smallest integer such that*

$$\det\left(\gamma'(t)\, \gamma''(t)\, \gamma^{(k_t+2)}(t)\right) \neq 0. \tag{3.6}$$

Suppose that $\max_{t\in I} k_t = K$. Then π_L and π_R both have maximal type K.

Proof. As above we may assume that $\gamma_1(t) = t$ by a change of variables. This applied to (3.6) immediately yields the expression (3.4) (and (3.5)) at any point $P = P(x, y_3, \tau) \in (N^*\mathcal{M})'$ such that $x_3 - y_3 = t$ and $\tau \cdot \gamma''(x_3 - y_3) = 0$. Thus all that remains is to check that π_L and π_R drop rank simply at P. This is not hard to see, as $\nabla_\tau \det(d\pi_L) = -\nabla_\tau \det(d\pi_R) = \gamma''(t) \neq 0$. Hence π_L and π_R are type k_t at points such that $x_3 - y_3 = t$, and have maximal type K if the maximum exists.

$\square$

In particular, we see that π_L and π_R have only fold singularities if and only if

$$\det\begin{pmatrix} \gamma_1''(x_3-y_3) & \gamma_2''(x_3-y_3) \\ \gamma_1'''(x_3-y_3) & \gamma_2'''(x_3-y_3) \end{pmatrix} \neq 0.$$

Note that $\gamma(t)$ has nonvanishing curvature and torsion if and only if $\gamma'(t), \gamma''(t), \gamma'''(t)$ are linearly independent, or equivalently if

$$\det\begin{pmatrix} \gamma'(t) & \gamma''(t) & \gamma'''(t) \end{pmatrix} = \det\begin{pmatrix} 1 & 0 & 0 \\ \gamma_2'(t) & \gamma_2''(t) & \gamma_2'''(t) \\ \gamma_3'(t) & \gamma_3''(t) & \gamma_3'''(t) \end{pmatrix} \neq 0.$$

Thus the condition that the only singularities of π_L (and thus π_R) are folds is equivalent to the condition that γ has nonvanishing curvature and torsion. Under this condition, Pramanik and Seeger used the fact that $\mathcal{A}_{\mathbb{R}}$ is a Fourier multiplier to prove the following sharp result.

Theorem 3.7 ([45, Theorem 1.1])**.** *If γ is a smooth regular curve with nonvanishing curvature and torsion then $\mathcal{A}_{\mathbb{R}}$ is bounded from $L^p(\mathbb{R}^3) \to L^p_{1/p}(\mathbb{R}^3)$ for $p > 4$.*

The proof of Theorem 3.7 relies on an observation about the non-isotropic decay of the Fourier transform of the measure μ. Let $\hat{\mu}$ be the Fourier transform of μ, given by

$$\hat{\mu}(\xi) = \int e^{-2\pi i \langle \xi, \gamma(t) \rangle} \chi(t)\, dt. \tag{3.7}$$

If γ has nonvanishing curvature and torsion then $\gamma'(t), \gamma''(t), \gamma'''(t)$ span $\mathbb{R}^3$ for each $t \in [0,1]$; thus there is a constant $c > 0$ such that for every $\xi \in \mathbb{R}^3 \setminus 0$ at least one of the inequalities

$$|\langle \xi, \gamma'(t) \rangle| \geq c|\xi| > 0$$

$$|\langle \xi, \gamma''(t) \rangle| \geq c|\xi| > 0$$

$$|\langle \xi, \gamma'''(t) \rangle| \geq c|\xi| > 0$$

must hold. Applying the method of nonstationary phase and Van der Corput's Lemma to (3.7) shows that $\hat{\mu}$ decays at a uniform rate

$$|\hat{\mu}(\xi)| \leq C|\xi|^{-1/3}. \tag{3.8}$$

Since $\hat{\mu}(\xi)$ is also bounded by $\|\chi\|_1$ we conclude

$$(1 + |\xi|^{2/3})^{1/2} \hat{\mu}(\xi)$$

is a Fourier multiplier on $L^2(\mathbb{R}^3)$, hence by Plancherel's theorem $\mathcal{A}_{\mathbb{R}}$ maps $L^2(\mathbb{R}^3) \to L^2_{1/3}(\mathbb{R}^3)$ boundedly, matching the regularity of Theorem 3.3 in the case of folding canonical relations.

However, there are many directions in which $\hat{\mu}$ decays faster than the estimate (3.8). Suppose $\theta \in C_c^\infty(\mathbb{R}^3)$ is a smooth cutoff function such that for $\xi \in \operatorname{supp}\theta$, we have

$$|\langle \xi, \gamma'(t)\rangle| + |\langle \xi, \gamma''(t)\rangle| \geq c|\xi|$$

for some $c > 0$ uniformly for all $t \in [0,1]$. Then by Van der Corput's Lemma

$$|\theta(\xi)\hat{\mu}(\xi)| \leq C|\xi|^{-1/2}.$$

By the same argument as above this implies that $(1+|\xi|)^{1/2}\theta\hat{\mu}$ is a Fourier multiplier on $L^2(\mathbb{R}^3)$, meaning that the operator $f * \mathfrak{F}^{-1}[\theta\hat{\mu}]$ is bounded from $L^2(\mathbb{R}^3) \to L^2_{1/2}(\mathbb{R}^3)$, the same regularity as Theorem 2.2, when the canonical relation is a local graph. The set of ξ which do not lie in the support of θ is a neighborhood of the conic set of directions binormal to γ,

$$\mathfrak{B} = \{\rho\gamma'(t) \wedge \gamma''(t) \;:\; \rho \in \mathbb{R},\; t \in I\}. \tag{3.9}$$

If γ has nonvanishing curvature and torsion $\mathfrak{B}$ is a cone with one nonvanishing principal curvature. For example, if γ is the moment curve $\gamma(t) = (t, t^2, t^3)$, the set $\mathfrak{B}$ is given by

$$\mathfrak{B} = \{\rho(3t^2, -3t, 1) \;:\; \rho \in \mathbb{R},\; t \in I\}.$$

An important observation made by Pramanik and Seeger is that $\mathfrak{B}$ coincides exactly with the fibers of $\pi_L(\mathcal{L})$ [45]. Indeed, for each x we define the fibers of $\pi_L(\mathcal{L})$ to be

$$\Sigma_x = \{\xi \;:\; (x,\xi) \in \pi_L(\mathcal{L})\}. \tag{3.10}$$

Then given the parametrization of $\mathcal{L}$ from (3.3) we see that

$$\Sigma_x = \left\{\rho\begin{pmatrix} \gamma_2''(x_1-y_1)\gamma_3'(x_1-y_1) - \gamma_3''(x_1-y_1)\gamma_2'(x_1-y_1) \\ \gamma_3''(x_1-y_1) \\ -\gamma_2''(x_1-y_1)\end{pmatrix} \;:\; \rho \in \mathbb{R},\; x_1 - y_1 \in I\right\},$$

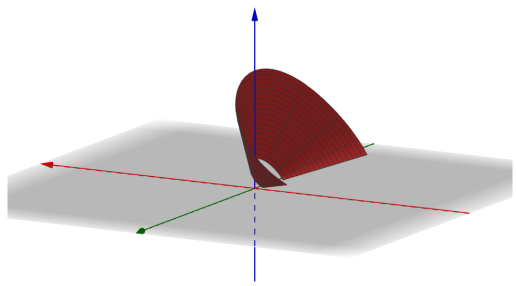

Figure 5: Section of Binormal Cone $\mathfrak{B}$ for the Moment Curve $\gamma(t) = (t, t^2, t^3)$

which is exactly the cone $\mathfrak{B}$ for every x.

By applying a dyadic decomposition of the support of $\hat{\mu}$ in the distance away from this cone we obtain a sum of functions whose Fourier transforms are supported in a neighborhood of a curved cone. For $\ell \in \mathbb{N}$ let $\theta_\ell(\xi)$ be a smooth cutoff function supported where $\operatorname{dist}(\xi, \mathfrak{B}) \simeq 2^{-\ell}|\xi|$ such that

$$(1-\theta)\hat{\mu}(\xi) = \sum_{\ell \geq 0} \theta_\ell(\xi)\hat{\mu}(\xi).$$

Then by an argument in [45, Lemma 3.3] involving Van der Corput's Lemma and an almost orthogonal decomposition we see that the decay of $\hat{\mu}$ improves quantitatively as the distance from $\mathfrak{B}$ increases. In particular,

$$|\theta_\ell(\xi)\hat{\mu}(\xi)| \lesssim 2^{\ell/2}|\xi|^{-1/2}.$$

When $\ell = 0$, $\hat{\mu}$ is supported far away from $\mathfrak{B}$, and $|\hat{\mu}(\xi)| \lesssim |\xi|^{-1/2}$, the optimal bound. On the other hand, once if $2^{-\ell}$ is smaller than $|\xi|^{-1/3}$ the estimate is no better than the uniform decay of $\hat{\mu}$.

3.3 ℓ^p Decoupling for the Cone

To estimate the L^p boundedness of $\mathcal{A}_{\mathbb{R}}$ Pramanik and Seeger interpolated the "quantitative" L^2 estimates with ***ℓ^p-decoupling estimates***, first proven by Wolff [58] for large p and subsequently extended by Bourgain and Demeter [8] to the optimal range $p > 6$. We present the extension of their results for a general cone in $\mathbb{R}^3$ with one nonvanishing principal curvature (cf. [45, Proposition 2.1] and [8, Theorem 1.2]).

Let $I \subset [-1, 1]$ be a closed interval and let $g : I \to \mathbb{R}^2$ define a C^3 curve in the plane. Suppose there are constants $c_0, c_1, c_2 > 0$ such that

$$\begin{aligned} \|g\|_{C^3} &\leq c_0, \\ |g'(b)| &\geq c_1, \\ \left| \det \left(\begin{smallmatrix} g_1'(b) & g_1''(b) \\ g_2'(b) & g_2''(b) \end{smallmatrix} \right) \right| &\geq c_2, \end{aligned}$$

for all $b \in I$. Then

$$\mathcal{C}_g = \{\xi \in \mathbb{R}^3 \ : \ \xi = \lambda(g_1(b), g_2(b), 1), \ b \in I, \ \lambda > 0\} \tag{3.11}$$

is a cone in $\mathbb{R}^3$ with one nonvanishing principal curvature. A basis for the tangent space of $\mathcal{C}_g$ at $\lambda(g(b), 1)$ is given by

$$\begin{aligned} u_1(b) &= (g(b), 1) \qquad (3.12) \\ \tilde{u}_2(b) &= (g'(b), 0), \end{aligned}$$

and a vector normal to $\mathcal{C}_g$ at $\lambda(g(b), 1)$ is given by

$$u_3(b) = u_1(b) \wedge \tilde{u}_2(b) = \big(-g_2'(b), g_1'(b), g_1(b)g_2'(b) - g_2 g_1'(b) \big). \tag{3.13}$$

Thus $T_{\lambda(g(b),1)}\mathcal{C}_g$ has an orthogonal basis given by $u_1(b)$ and

$$u_2(b) := u_1(b) \wedge u_3(b). \tag{3.14}$$

Given this basis, we can define thin ***plates*** adapted to the cone $\mathcal{C}_g$ at the point $(g(b),1)$.

Definition 3.8. *Let $A > 1$, $0 < \delta < 1$, and $b \in I$. Given pairwise orthogonal vectors $u_1(b), u_2(b), u_3(b) \in \mathbb{R}^3$, let $\Pi_{A,b}(\delta)$ be the set of $\xi \in \mathbb{R}^3$ defined by the inequalities*

$$A^{-1} \leq \left|\left\langle \tfrac{u_1(b)}{|u_1(b)|}, \xi \right\rangle\right| \leq A$$

$$\left|\left\langle \tfrac{u_2(b)}{|u_2(b)|}, \xi \right\rangle\right| \leq A\delta \tag{3.15}$$

$$\left|\left\langle \tfrac{u_3(b)}{|u_3(b)|}, \xi \right\rangle\right| \leq A\delta^2. \tag{3.16}$$

The sets $\Pi_{A,b}(\delta)$ are unions of $A \times A\delta \times A\delta^2$-boxes with long, middle, and short sides parallel to $u_1(b), u_2(b)$, and $u_3(b)$ respectively. Decoupling inequalities allow one to efficiently estimate the L^p norm of a sum of functions whose Fourier transforms are supported on a family of plates $\Pi_{A,b_\nu}(\delta)$ for a set of separated points $b_\nu \in I$. We formulate this theorem in terms of ℓ^p decoupling for small p for the purposes of our later proof, but they are equivalent to the typical presentation of ℓ^2 decoupling inequalities for large p.

Theorem 3.9 (cf. [8, Theorem 1.2],[58, Theorem 1], see also [45, Proposition 2.1])**.** *Let $\varepsilon > 0$ and $A > 1$. There exists a constant $C(\varepsilon, A)$ depending on c_0, c_1, c_2 such that the following holds for any choice of $0 < \delta_0 < \delta_1 < 1$. Let $M > 1$ and let $B = \{b_\nu\}_{\nu=1}^M$ be a set of points in an interval $J \subset I$ of length δ_0 such that $|b_\nu - b'_\nu| \geq \delta_1$ for $\nu \neq \nu'$. Let $2 \leq p \leq 6$. Let $f_\nu \in L^p(\mathbb{R}^3)$ such that $\hat{f}_\nu$ is supported in $\Pi_{A,b_\nu}(\delta_1)$ for each $\nu = 1, 2, ..., M$. Then*

$$\Big\| \sum_{\nu=1}^M f_\nu \Big\|_p \leq C(\varepsilon, A)(\delta_0/\delta_1)^{\frac{1}{2}-\frac{1}{p}+\varepsilon} \Big(\sum_{\nu=1}^M \|f_\nu\|_p^p \Big)^{1/p}.$$

To apply this method of decoupling to a local Radon-like transform over curves in $\mathbb{R}^3$ there are two obstructions to circumvent. First, the fibers associated to $\mathcal{A}_{\mathbb{R}}$ are fixed only because $\mathcal{A}_{\mathbb{R}}$ is a convolution operator and is thus translation invariant; the fibers of $\mathcal{L}$ associated to a generic local Radon-like transform a priori vary with x. We will deal with this obstruction using iterations of changes of variables, which we will introduce in Chapter 10.

Second, the fibers of $\pi_L(\mathcal{L})$ may not in general be curved cones. To ensure that they are we need an assumption, first formulated in the context of FIOs in [23] and later characterized for local Radon-like transforms over families of curves in $\mathbb{R}^3$ in [46].

Lemma 3.10 ([46, § 3])**.** *Let $\varpi : (N^*\mathcal{M})' \to \mathcal{M}$ be the natural projection. If the restriction of ϖ to $\mathcal{L}$*

$$\varpi|_{\mathcal{L}} : \mathcal{L} \to \mathcal{M}$$

is a submersion, then the fibers of $\pi_L(\mathcal{L})$ are curved cones in $T_x^*\Omega_L$ for each x.

3.4 The Surjectivity Condition on ϖ

In this section we examine the assumption on ϖ in Lemma 3.10, and how it impacts the geometry of the conormal bundle of $\mathcal{M}$. In this section Ω_L, Ω_R are 3-dimensional manifolds, and $\mathcal{M} \subset \Omega_L \times \Omega_R$ is a 4-dimensional submanifold such that the projections ρ_L, ρ_R defined in (2.1) are submersions. As discussed in §2.2, in a neighborhood of a reference point $P \in \mathcal{M}$,

$$\mathcal{R}f(x) = \int\int e^{2\pi i\left(\tau_1(S^1(x,y_3)-y_1)+\tau_2(S^2(x,y_3)-y_2)\right)}\chi(x,y)f(y)\,d\tau\,dy.$$

Recall from §2.2 that the canonical relation associated to $\mathcal{R}$ is the twisted conormal bundle of $\mathcal{M}$, given by

$$(N^*\mathcal{M})' = \big\{(x,\xi,y,\eta) \;:\; y_i = S^i(x,y_3),\ i=1,2 \quad \xi = \sum_{i=1}^{2} \tau_i S^i_x(x,y_3)$$

$$\eta_i = \tau_i,\ i=1,2 \quad \eta_3 = \sum_{i=1}^{2} \tau_i S^i_{y_3}(x,y_3)\big\},$$

and π_L, π_R are defined as in (2.4). From (2.11) and (2.12) we see that

$$\det(d\pi_L) = \det\Big(S^1_x(x,y_3)\, S^2_x(x,y_3)\, \sum_{i=1}^{2} \tau_i S^i_{x,y_3}(x,y_3)\Big).$$

To see that the fibers of $\pi_L(\mathcal{L})$ are conic, let

$$\Delta^i(x,y_3) = \det(S^1_x\, S^2_x\, S^i_{xy_3})\Big|_{x,y_3},\ i=1,2.$$

Then we can rewrite $\det(d\pi_L) = \tau_1\Delta^1(x,y_3) + \tau_2\Delta^2(x,y_3)$. Given the parametrization above, $\mathcal{L}$ (see (2.13)) is the subset of $(N^*\mathcal{M})'$ such that

$$\tau_1\Delta^1(x,y_3) + \tau_2\Delta^2(x,y_3) = 0.$$

Let ϖ be the projection defined in Lemma 3.10. Then we have the following result from Pramanik and Seeger.

Lemma 3.11 ([46, Lemma 3.1])**.** *If $\varpi|_{\mathcal{L}} : \mathcal{L} \to \mathcal{M}$ is a submersion and π_L is a fold then*

$$|\Delta^1(x,y_3)| + |\Delta^2(x,y_3)| \neq 0$$

for (x,y) near P.

This lemma implies that for any (x,y_3) we can find $\tilde{\tau}$ such that $(x,y_3,\tilde{\tau})$ parametrizes a point in $\mathcal{L}$; in particular

$$\tilde{\tau} = \pm\rho\big(-\Delta^2(x,y_3), \Delta^1(x,y_3)\big),$$

for some $\rho > 0$. Although [46] deals with the case of folding canonical relations, Lemmas 3.11 and 3.10 only require a fold condition on one of the projections.

Next we examine the fibers of $\pi_L(\mathcal{L})$, defined by

$$\Sigma_x = \{\xi \ : \ (x,\xi) \in \pi_L(\mathcal{L})\}.$$

Given the parametrization of $(N^*\mathcal{M})'$ above we see that

$$\Sigma_x = \{(\tau \cdot S)_x(x,y_3) \ : \ \tau \cdot \Delta(x,y_3) = 0\} = \{\pm\rho\Xi(x,y_3) \ : \ \rho > 0\} \tag{3.17}$$

where

$$\Xi(x,y_3) = -\Delta^2(x,y_3)S^1_x(x,y_3) + \Delta^1(x,y_3)S^2_x(x,y_3).$$

Thus Σ_x is conic, and we can construct a basis for its tangent and normal spaces. Let $a \in \Omega_L$ be fixed. We begin with an observation. By an identity for vectors in $\mathbb{R}^3$, for $i = 1,2$

$$\Delta^i(x,y_3) = \langle S^1_x(x,y_3) \wedge S^2_x(x,y_3), S^i_{x,y_3}(x,y_3)\rangle.$$

This implies that

$$\begin{aligned}\langle S^1_x \wedge S^2_x, \Delta^1 S^2_{xy_3} - \Delta^2 S^1_{xy_3}\rangle &= \Delta^1\langle S^1_x \wedge S^2_x, S^2_{xy_3}\rangle - \Delta^2\langle S^1_x \wedge S^2_x, S^1_{xy_3}\rangle \\ &= \Delta^1\Delta^2 - \Delta^2\Delta^1 \\ &= 0,\end{aligned}$$

implying that $-\Delta^2 S^1_{xy_3} + \Delta^1 S^2_{xy_3} \in \mathrm{Span}(S^1_x, S^2_x)$ for fixed (a, y_3). The tangent space of Σ_a at a point parametrized by $(y_3, \pm\rho)$ is spanned by

$$\begin{aligned} T_1(a,y_3) &= \Xi(a,y_3) \qquad (3.18)\\ \tilde{T}_2(a,y_3) &= \Xi_{y_3}(a,y_3),\end{aligned}$$

so a normal vector at this point is given by

$$
\begin{aligned}
T_1 \wedge \tilde{T}_2 &= \Xi \wedge \Xi_{y_3} \\
&= (\Delta^1 \Delta^2_{y_3} - \Delta^2 \Delta^1_{y_3})(S^1_x \wedge S^2_x) \\
&\quad + (\Delta^1 S^2_x - \Delta^2 S^1_x) \wedge (\Delta^1 S^2_{xy_3} - \Delta^2 S^1_{xy_3}).
\end{aligned}
$$

Since $-\Delta^2 S^1_{xy_3} + \Delta^1 S^2_{xy_3} \in \mathrm{Span}(S^1_x, S^2_x)$ the expression in the final line of the calculation of $T_1 \wedge \tilde{T}_2$ is either 0 or a scalar multiple of the vector $S^1_x \wedge S^2_x$, meaning that a normal vector to Σ_x at a point parametrized by $(y_3, \pm\rho)$ is given by

$$
N(a, y_3) := S^1_x(a, y_3) \wedge S^2_x(a, y_3). \tag{3.19}
$$

Finally, to construct an orthogonal vector in the tangent space to Σ_a we define

$$
T_2(a, y_3) := T_1(a, y_3) \wedge N(a, y_3). \tag{3.20}
$$

In the proof of Lemma 3.10, Pramanik and Seeger proved in particular that Σ_x is a two-dimensional cone that has one non-vanishing principal curvature given by

$$
\rho \langle \Xi_{y_3, y_3}, N \rangle.
$$

It is useful to construct explicit kernel fields of π_L and π_R in conic neighborhoods of $\mathcal{L}$. Note that $\mathcal{L}$ splits as a disjoint union of two cones,

$$
\mathcal{L}^\pm = \big\{ \big(x, \pm\rho(-\Delta^2 S^1_x + \Delta^1 S^2_x), S^1, S^2_{xy_3}, y_3, \tau, \pm\rho(\Delta^2 S^1_{y_3} - \Delta^1 S^2_{y_3})\big) \ : \ \rho > 0 \big\}.
$$

Lemma 3.12. *A kernel field for π_R near $\mathcal{L}^\pm$ is given by*

$$
V_R = \langle N(x, y_3), \nabla_x \rangle.
$$

Define $\Gamma_i(x, y_3)$, $i = 1, 2$ *by*

$$\Gamma_1(x, y_3) = \det(S^1_x\, S^2_{xy_3}\, S^1_{xy_3}) \tag{3.21}$$

$$\Gamma_2(x, y_3) = \det(S^1_{xy_3}\, S^2_x\, S^2_{xy_3}). \tag{3.22}$$

Then

$$V_L^{\pm} = \frac{\pm|\tau|}{|\Delta(x, y_3)|}\Big(\Gamma_2(x, y_3)\partial_{\tau_1} - \Gamma_1(x, y_3)\partial_{\tau_2}\Big) + \partial_{y_3}$$

is a kernel field for π_L *near* $\mathcal{L}^{\pm}$.

Proof. We begin with V_R. Applying the local representation of $d\pi_R$ in (2.12) (which map coordinates $(x, \tau, y_3) \to (w_1, ..., w_6)$) to $V_R = S^1_x(x, y_3) \wedge S^2_x(x, y_3)$ we obtain

$$\begin{aligned} d\pi_R V_R\big|_{(x,\tau,y_3)} &= \begin{pmatrix} \langle S^1_x(x, y_3) \wedge S^2_x(x, y_3), S^1_x(x, y_3)\rangle \\ \langle S^1_x(x, y_3) \wedge S^2_x(x, y_3), S^2_x(x, y_3)\rangle \\ 0 \\ 0 \\ 0 \\ \langle S^1_x(x, y_3) \wedge S^2_x(x, y_3), \tau_1 S^1_{xy_3}(x, y_3) + \tau^2 S^2_{x,y_3}(x, y_3)\rangle \end{pmatrix} \\ &= \big(\tau_1\Delta^1(x, y_3) + \tau_2\Delta^2(x, y_3)\big)\partial_{w_6} \\ &= \det(d\pi_R)\big|_{\pi_L(x,y_3)}\partial_{w_6}. \end{aligned}$$

Clearly V_R is a kernel field for π_R. Note this implies that $-\Delta^2 S^1_{xy_3} + \Delta^1 S^2_{xy_3} \in \mathrm{Span}(S^1_x, S^2_x)$.

To show that $V_L^{\pm}$ is a kernel field for π_L near $\mathcal{L}^{\pm}$ we follow the argument in the proof of [46, Lemma 3.2]. Applying the local representation of $d\pi_L$ in (2.11) (mapping coordinates $(x,\tau,y_3) \to (w_1,...,w_6)$) to the definition of $V_L^{\pm}$ above we obtain

$$d\pi_L V_L^{\pm}\big|_{(x,\tau,y_3)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \left(\frac{\pm|\tau|}{|\Delta|}\left(S_x^1\Gamma_2 - S_x^2\Gamma_1\right) + \left(\tau_1 S^1_{xy_3} + \tau^2 S^2_{xy_3}\right)\right)\Big|_{(x,y_3)} \end{pmatrix}.$$

Evaluating at $\tau = \pm\rho(-\Delta^2(x,y_3),\Delta^1(x,y_3))$ it suffices show that

$$S_x^1\Gamma_2 - S_x^2\Gamma_1 - \Delta^1 S^1_{xy_3} + \Delta^2 S^2_{xy_3}\Big|_{(x,y_3)} = 0. \tag{3.23}$$

Let W equal the left hand side of the above equation. To prove (3.23) we note that since

$$\det(S^1\ S_x^2\ \Delta^1 S^1_{xy_3} + \Delta^2 S^2_{xy_3})\Big|_{(x,y_3)} = |\Delta(x,y_3)|^2,$$

which is nonvanishing by Lemma 3.11, the vectors $S_x^1(x,y_3)$, $S_x^2(x,y_3)$, and $\Delta^1 S^1_{xy_3} + \Delta^2 S^2_{xy_3}\big|_{(x,y_3)}$ form a basis on $\mathbb{R}^3$. This implies that $S_x^1 \wedge S_x^2\big|_{(x,y_3)}$, $S_x^1 \wedge \Delta^1 S^1_{xy_3} + \Delta^2 S^2_{xy_3}\big|_{(x,y_3)}$, and $S_x^2 \wedge \Delta^1 S^1_{xy_3} + \Delta^2 S^2_{xy_3}\big|_{(x,y_3)}$ also form a basis of $\mathbb{R}^3$. We can show that $W = 0$ by testing it against these three basis vectors.

First, we test W against $S_x^1 \wedge S_x^2$ to obtain

$$\begin{aligned}\langle W, S_x^1 \wedge S_x^2\rangle &= \langle S_x^1 \wedge S_x^2, -\Delta^2 S^1_{xy_3}\rangle + \langle S_x^1 \wedge S_x^2, \Delta^1 S^2_{x,y_3}\rangle \\ &= -\Delta^2\Delta^1 + \Delta^1\Delta^2 \\ &= 0.\end{aligned}$$

Next, we test W against $S_x^1 \wedge (\Delta^1 S^1_{xy_3} + \Delta^2 S^2_{xy_3})$ and obtain

$$\begin{aligned}\langle W, S_x^1 \wedge (\Delta^1 S^1_{xy_3} + \Delta^2 S^2_{xy_3})\rangle &= -\Gamma_1\Delta^1\langle S_x^2, S_x^1 \wedge S^1_{xy_3}\rangle - \Gamma_1\Delta^2\langle S_x^2, S_x^1 \wedge S^2_{xy_3}\rangle \\ &\quad - (\Delta^2)^2\langle S^1_{xy_3}, S_x^1 \wedge S^2_{xy_3}\rangle + (\Delta^1)^2\langle S^2_{xy_3}, S_x^1 \wedge S^1_{xy_3}\rangle.\end{aligned}$$

By the definition of Γ_1, Γ_2 this vanishes. An analogous argument can be made to show $\langle W, S^2_x \wedge (\Delta^1 S^1_{xy_3} + \Delta^2 S^2_{xy_3})\rangle = 0$.

$\square$

We can give an explicit condition that π_L is a fold along $\mathcal{L}$ by computing for $\rho > 0$

$$V_L^{\pm}\big(\tau_1 \Delta^2(x, y_3) + \tau_2 \Delta^2(x, y_3)\big)\Big|_{\tau=\pm\rho(-\Delta^2,\Delta^1)} = \rho\kappa(x, y_3)$$

where

$$\kappa(x, y_3) = \Gamma_2\Delta^1 - \Gamma_1\Delta^2 + \Delta^1\Delta^2_{y_3} - \Delta^2\Delta^1_{y_3}\Big|_{(x,y_3)}. \tag{3.24}$$

Since we assume π_L has fold singularities along $\mathcal{L}$, κ must be nonzero.

3.5 Sharp L^p-Sobolev Estimates

Under the assumption of Lemma 3.10, Pramanik and Seeger were able to prove that the same $1/p$ gain in regularity holds on L^p for $p > 4$ for a large class of local Radon-like transforms over families of curves in $\mathbb{R}^3$ associated to folding canonical relations. Since $\mathcal{R}$ has folding canonical relations if and only if $\mathcal{R}^*$ has folding canonical relations, the same result can be applied to $\mathcal{R}$ and $\mathcal{R}^*$. Interpolating their result with the L^2 estimate from Theorem 3.3 we obtain the following characterization of the L^p-Sobolev regularity of $\mathcal{R}$ with folding canonical relations.

Theorem 3.13 ([46, Theorem 1.1])**.** *Let $\Omega_L, \Omega_R \subset \mathbb{R}^3$ be open sets, and $\mathcal{M} \subset \Omega_L \times \Omega_R$ be a four-dimensional manifold such that the projections $\mathcal{M} \to \Omega_L$ and $\mathcal{M} \to \Omega_R$ are submersions. Let $\mathcal{R}$ be the local Radon-like transform associated to $\mathcal{M}$. Assume that the only singularities on $\pi_L : (N^*\mathcal{M})' \to T^*\Omega_L$ and $\pi_R : (N^*\mathcal{M})' \to T^*\Omega_R$ are Whitney* *folds. Let $\mathcal{L}$ be the conic submanifold on which $d\pi_L$ and $d\pi_R$ drop rank by one, and let ϖ be the projection of $(N^*\mathcal{M})'$ onto the base $\mathcal{M}$. Further, suppose that $\varpi|_{\mathcal{L}} : \mathcal{L} \to \mathcal{M}$ is a submersion. Then $\mathcal{R}$ extends to a continuous operator*

$$\mathcal{R} \ : \ L^p_{\text{comp}}(\Omega_R) \to L^p_{s,\text{loc}}(\Omega_L)$$

for $(1/p, s)$ within the shaded region of Figure 6.

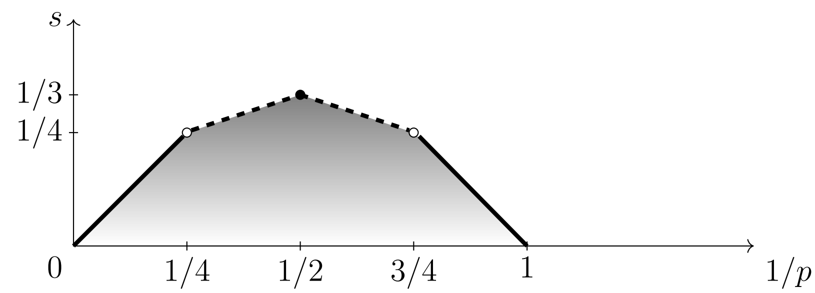

Figure 6: Sharp $L^p \to L^p_s$ mapping for $\mathcal{R}$ with folding canonical relations

As the curvature assumption on the fibers of $\mathcal{L}$ only requires that π_L be a Whitney fold, it is conjectured that this method of proof could be extended to all local Radon-like transforms over families of curves in $\mathbb{R}^3$ with one-sided folds satisfying the submersion condition of Lemma 3.10. In this work our main result is to prove one case of this conjecture, that $\mathcal{R}$ still gains $1/p$ derivatives on L^p for $p > 4$ when π_R is a blowdown.

Theorem 3.14. *Let Ω_L, Ω_R be open sets and let $\mathcal{M} \subset \Omega_L \times \Omega_R$ be a four-dimensional manifold such that the projections $\rho_L : \mathcal{M} \to \Omega_L$ and $\rho_R : \mathcal{M} \to \Omega_R$ are submersions. Let $\mathcal{R}$ be the local Radon-like transform associated to $\mathcal{M}$. Let $\mathcal{L}$ be the conic submanifold on which $d\pi_L$ and $d\pi_R$ drop rank by one. Assume that the only singularities on $\pi_L : (N^*\mathcal{M})' \to T^*\Omega_L$ are Whitney folds, and that $\pi_R : (N^*\mathcal{M})' \to T^*\Omega_R$ is a blowdown* *on $\mathcal{L}$. Let ϖ be the projection of $(N^*\mathcal{M})'$ onto the base $\mathcal{M}$. Further, suppose that $\varpi|_{\mathcal{L}} : \mathcal{L} \to \mathcal{M}$ is a submersion. Then $\mathcal{R}$ extends to a continuous operator*

$$\mathcal{R} \ : \ L^p_{\text{comp}}(\Omega_R) \to L^p_{1/p,\text{loc}}(\Omega_L), \ 4 < p < \infty.$$

Theorem 3.14 generalizes the results of [6] and [44]. Interpolating the results of Theorem 3.14 with the L^2-Sobolev estimate in Theorem 3.3 we obtain L^p-Sobolev estimates for $2 \leq p \leq 4$. To obtain estimates for $p < 2$, we apply Proposition 3.4. We combine the estimates for all p together in the following result.

Theorem 3.15. *If $\mathcal{R}$ satisfies the conditions of Theorem 3.14 then $\mathcal{R}$ maps boundedly from $L^p_{\text{comp}}(\mathbb{R}^3)$ into $L^p_{s,\text{loc}}(\mathbb{R}^3)$, where $(1/p, s)$ lies within the shaded region of Figure 7.*

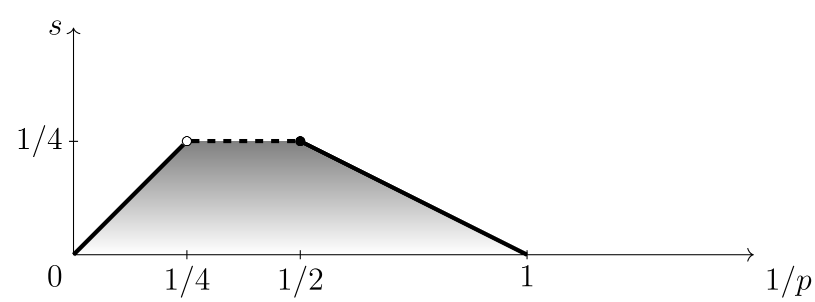

Figure 7: Sharp $L^p \to L^p_s$ mapping for $\mathcal{R}$ with fibered folding canonical relations

Examples in [44, 6] show that the L^p-Sobolev estimates of Theorem 3.15 cannot be expanded beyond the boundary of the trapezoidal region of Figure 7, although it may be possible to extend to the endpoints (i.e. the closure of the trapezoidal region). We will introduce these examples in Chapter 4 and prove related sharpness results in Chapter 5.

The proof of Theorem 3.14 follows the same basic structure of the proofs of Theorem

3.7 and of Theorem 3.13, first showing "quantitative" improvements in L^2-Sobolev estimates moving away from $\mathcal{L}$, then applying decoupling to cones deriving from the fibers of $\pi_L(\mathcal{L})$. However, the maximal degeneracy on π_R introduces difficulties not present in the case of folding canonical relations.

First, the dyadic decomposition away from $\mathcal{L}$ is halted once the the "quantitative" estimate is no better than the uniform L^2-Sobolev estimate. In the case of fibered folds, the decomposition continues much closer to $\mathcal{L}$ because the uniform L^2-Sobolev estimate of Theorem 3.2 is worse than in the case of folding canonical relations (Theorem 3.1). Because the support of the decomposed pieces is much closer to the singularities in $\mathcal{L}$, proving estimates there requires greater care in the analysis.

Second, since π_R is a blowdown on $\mathcal{L}$, V_R is parallel to $\mathcal{L}$, which implies $V_R^k\tau \cdot \Delta(x, y_3) = 0$ on $\mathcal{L}$ for all $k \geq 0$. Moreover, since $V_R = \langle N(x, y_3), \nabla_x\rangle$ we also have by definition that $V_R\tau \cdot S_{y_3}(x, y_3) = 0$ on $\mathcal{L}$. In other words, the blowdown imposes a flatness condition in the V_R direction which does not permit almost orthogonal decompositions in the V_R direction which were possible in the case of a fold. These difficulties will be discussed in more detail in Chapters 8, 9, and 10 when we introduce the dyadic frequency decomposition in distance from $\mathcal{L}$.

Chapter 4

Examples of Local Radon-like Transforms

In this chapter we explore and characterize examples arising from problems in harmonic analysis and integral geometry which motivate our investigations into L^p-Sobolev regularity. In the study of local Radon-like transforms, model cases are key as the general picture can usually be seen as a perturbation of one (or more) examples. The notation in each section is self-contained.

4.1 n-plane transforms and Restricted X-ray transforms

Let $\mathbb{M}_{d,n}$ be the bundle of affine n-planes in $\mathbb{R}^d$, and define the n-plane transform

$$\mathcal{P}_{d,n}f(\pi) = \int_\pi f(x)d\mu_\pi(x), \pi \in \mathbb{M}_{d,n}.$$

where $d\mu_\pi$ is Lebesgue measure on the n-plane π. The dimension of $\mathbb{M}_{d,n}$ is $(n+1)(d-n)$, and can be illustrated by relating $\mathbb{M}_{d,n}$ to the Grassmannian $\mathbb{G}_n(\mathbb{R}^d)$, the space of all n-planes through the origin in $\mathbb{R}^d$. The Grassmannian can be realized as a homogeneous space by the identity

$$\mathbb{G}_n(\mathbb{R}^d) \simeq \frac{O(d)}{O(n) \times O(d-n)}$$

which also implies that $\dim(\mathbb{G}_n(\mathbb{R}^d)) = n(d-n)$ [38]. We can parametrize $\mathbb{M}_{d,n}$ by (θ, y), where $\theta \in \mathbb{G}_n(\mathbb{R}^d)$, and $y \in \theta^\perp$, the $(d-n)$-dimensional subspace of $\mathbb{R}^d$ orthogonal to θ. We identify (θ, y) with the n-plane parallel to θ containing the point y in $\mathbb{M}_{d,n}$ [11]. This parametrization is one to one, and shows that the dimension of $\mathbb{M}_{d,n}$ is indeed $(n+1)(d-n)$.

In the case of hypersurfaces, $\mathcal{P}_{d,d-1}$ is the higher-dimensional analogue of the Radon transform introduced at the beginning of this work. At the other extreme, 1-plane transforms, more commonly referred to as X-ray transforms, have served as a model for X-ray tomography, where an important problem involves reconstituting f from $\mathcal{P}_{d,1}f$. Hence the possibility of inversion is an important question regarding n-plane transforms. When $n < d-1$ the dimension of $\mathbb{M}_{d,n}$ is strictly larger than $\mathbb{R}^d$, so the problem of finding f from $\mathcal{P}_{d,n}f$ is overdetermined. Thus it is natural to restrict the domain of $\mathcal{P}_{d,n}f$ to an d-dimensional submanifold $\mathcal{F} \subset \mathbb{M}_{d,n}$ (called an n-plane complex), and ask for which n-plane complexes $\mathcal{F}$ can the associated restricted n-plane transform $\mathcal{P}_{d,n}f|_{\mathcal{F}}$ be inverted (see e.g. [22, 27, 48]).

Local versions of restricted X-ray transforms have served as model examples for both folding canonical relations [46, § 4.2] and fibered folding canonical relations [27, 44]. We introduce a concrete example of the latter from [44]. Let I be a compact interval and suppose that $\gamma : I \to \mathbb{R}^2$ is a smooth regular curve with nonvanishing curvature (i.e. $\gamma'(s), \gamma''(s) \neq 0$ for $s \in I$). For a Schwartz function $f \in \mathcal{S}(\mathbb{R}^3)$ and $\alpha \in I$ define

$$\mathcal{X}f(x', \alpha) = \int_1^2 f(x' + s\gamma(\alpha), s)\chi_1(s)\chi_2(\alpha)\, ds, \tag{4.1}$$

where χ_1 and χ_2 are smooth real-valued functions supported in the interior of $[1,2]$ and I respectively. The family of curves associated to this local Radon-like transform is parametrized by $(x',\alpha) \in \mathbb{R}^3$; for each (x',α) the restricted X-ray transform integrates f over the unique line through the point $(x',0)$ pointing in the direction $(\gamma(\alpha),1)$.

The operator $\mathcal{X}$ belongs to a class of restricted X-ray transforms initially formulated in the complex setting by Gelfand and Graev [22] to give an essentially complete characterization of when inversion of the X-ray transform is possible.

Definition 4.1 (Gelfand Admissibility)**.** *Given a three-dimensional line complex $\mathcal{F} \subset \mathbb{M}_{1,3}$, let Γ_P be the conic set generated by lines in $\mathcal{F}$ through the point P. We say that $\mathcal{F}$ is* ***Gelfand-admissible*** *if Γ_P is a two-dimensional cone for each P, and Γ_P is tangent to Γ_Q along the line between the points Q and P for every Q in the cone Γ_P. Additionally, let $\mathcal{X}_{\mathcal{F}}$ be the restricted X-ray transforms associated to $\mathcal{F}$. We say that $\mathcal{X}_{\mathcal{F}}$ is Gelfand-admissible if $\mathcal{F}$ is a Gelfand-admissible complex.*

Examples of Gelfand-admissible line complexes include the set of light rays in $\mathbb{R}^3$ (i.e. all lines which make an angle of $\pi/4$ radians with the horizontal plane, see for example [41]), and the complex of lines (called a Chow variety, see [27]) which intersect a curve which intersects almost every affine hyperplane ($\mathcal{X}$ is the Chow variety of γ and thus an example). Gelfand-admissible restricted X-ray transforms have been studied by many authors, including Greenleaf and Uhlmann who, in [27], showed that Gelfand admissibility, along with the assumption that Γ_P is curved for each P, is sufficient for the inversion of $\mathcal{X}_{\mathcal{F}}$, extending the results of Gelfand-Graev to the real setting. Moreover, generic restricted X-ray transforms which are Gelfand-admissible (including (4.1) if γ has nonvanishing curvature) satisfy the conditions of Theorem 3.14.

Proposition 4.2. *Suppose that $\mathcal{F}$ is Gelfand-admissible and that for each point Q the cone Γ_Q of lines in $\mathcal{F}$ through Q is curved. Then $(\mathcal{X}_\mathcal{F})^*$ satisfies the assumptions of Theorem 3.14.*

4.1.1 Jacobi Fields and the Canonical Relation

We begin by showing that the assumption on ϖ from Theorem 3.14 holds under a basic nondegeneracy condition on the sets Γ_P. The presentation of the canonical relation associated to $\mathcal{X}_\mathcal{F}$ in this section is due to Phong in the survey paper [41]; see also [27].

Lemma 4.3. *Let ϖ and $\mathcal{L}$ be defined as in Theorem 3.14. Suppose that for each $P \in \mathbb{R}^3$ the set Γ_P is a locally a conic submanifold of dimension 2 away from P. Then $\varpi|_\mathcal{L}$ is a submersion.*

As in the case of the classical Radon transform in the introduction, we can locally identify each line l in $\mathcal{F} \subset \mathbb{M}_{1,3}$ with a point $P \in \mathbb{R}^3$ and a direction $\gamma \in \mathbb{R}^3$ with $|\gamma| = 1$ via the map $(P, \gamma) \mapsto \{P + s\gamma \ : \ s \in \mathbb{R}\} = l$. As a consequence we can view $\mathbb{M}_{1,3}$ locally as a submanifold of $T\mathbb{R}^3$. For each line $l \in \mathcal{F}$ $\mathcal{X}_\mathcal{F}$ integrates over all points $Q \in l$, hence the incidence relation for $\mathcal{X}_\mathcal{F}$ is given by

$$Z = \{((P, \gamma), Q) \ : \ (P, \gamma) \in \mathcal{F}, \ Q \in l\} = \{((P, \gamma), Q) \ : \ (Q - P) \wedge \gamma = 0\} \subset \mathcal{F} \times \mathbb{R}^3.$$

Note that $(Q - P) \wedge \gamma = 0$ if and only if Q lies on the line parametrized by (P, γ). As $Q \in l$, there is some $t \in \mathbb{R}$ such that $Q - P = t\gamma$.

At this point we use Jacobi fields (see [19, Ch. 5]) to make a more concrete characterization of $T_l\mathcal{F}$ and $T_l\mathbb{M}_{1,3}$. We omit some details in the construction of Jacobi fields as we only focus on Jacobi fields for Euclidean spaces, which avoids much of the necessary Riemannian geometry. We refer to [19] and [27] as general references for Jacobi fields and their connection to X-ray transforms respectively.

Definition 4.4 ([19])**.** *Let (M, g) be a Riemannian manifold with curvature tensor R defined in [19, Ch. 4]. Let $l(s)$ be a geodesic on M parametrized by s. Note that $l'(s)$ is then a vector field along l. A vector field $J(s)$ along $l(s)$ is called a* ***Jacobi field*** *if it satisfies*

$$\frac{D^2}{ds^2}J(s) + R(J(s), l'(s))l'(s) = 0, \tag{4.2}$$

where $\frac{D}{ds}$ is the covariant derivative along $l(s)$ arising from the Levi-Civita connection (see [19, pp.50-56]).

Jacobi fields describe the difference between a given geodesic and infinitesimally close geodesics, meaning that we can use Jacobi fields to form a basis for the tangent space at a geodesic l in the space of geodesics on (M, g) (cf. [27, § 2]). In the case of Euclidean spaces the geodesics are lines, the curvature tensor R is uniformly 0, and $\frac{D}{ds}$ coincides with the usual derivative with respect to s. Thus (4.2) implies that Jacobi fields $J(s)$ on a given line $l \subset \mathbb{R}^d$ are precisely the vector fields along l that are linear in s.

Returning to the X-ray transform restricted to $\mathcal{F} \subset \mathbb{M}_{1,3}$, we fix $l_0 = (P_0, \gamma_0) \in \mathcal{F}$. Letting $u_0 = \gamma_0$ we can pick u_1, u_2 such that u_0, u_1, u_2 form an orthonormal basis of vectors on $\mathbb{R}^3$. Then the set of $\mathcal{J}$ of solutions to (4.2) for the line $l_0 \in \mathcal{F}$ is a 6-dimensional vector space which splits as $\mathcal{J}^\intercal \oplus \mathcal{J}^\perp$, where $\mathcal{J}^\intercal$ is spanned by u_0, su_0 and $\mathcal{J}^\perp$ is spanned by u_1, u_2, su_1, su_2. Informally, we see that $T_l\mathbb{M}_{1,3}$ can be identified with $\mathcal{J}^\perp$ by considering perturbations of l_0. The line l_0 can be deformed to another line in

$\mathbb{M}_{1,3}$ by

$$P_0 + s\gamma_0 \mapsto P_0 + s\gamma_0 + (a_1 s + b_1)u_1 + (a_2 s + b_2)u_2,$$

where a_i, b_i are any constants, giving a basis for $T_{l_0}\mathbb{M}_{1,3}$ in terms of the Jacobi fields u_1, u_2, su_1, su_2. The formal proof of the statement that $T_{l_0}\mathbb{M}_{1,3}$ is canonically isomorphic to $\mathcal{J}^\perp$ is given in [27, p. 209]. Given a Jacobi field $X(s) = (a_1 s + b_1)u_1 + (a_2 s + b_2)u_2$, we can view the deformation above using the identification $l = (P, \gamma) \in T\mathbb{R}^3$ as

$$(P_0, \gamma_0) \mapsto (P_0 + X(0), \gamma_0 + X').$$

Thus a tangent vector in $T_{l_0}\mathbb{M}_{1,3}$ can be identified as a pair $(X(0), X')$ lying in $T^*(T\mathbb{R}^3)$ where $X(0), X' \in \operatorname{span}(u_1, u_2)$.

Since we identify $\mathbb{M}_{1,3}$ as a subset of $T(\mathbb{R}^3)$ we can identify $((P,\gamma), Q) \mapsto (Q - P) \wedge \gamma$ with a defining function $\Phi : (\mathbb{R}^3 \times \mathbb{R}^3) \times \mathbb{R}^3 \to \mathbb{R}^3$ where we let P, γ, Q each vary in $\mathbb{R}^3$; we then restrict the domain of Φ to $((P,\gamma), Q) \in \mathcal{F} \times \mathbb{R}^3$, and the output of Φ will be thus restricted to the two-dimensional space orthogonal to γ. Using this scheme the restriction $(Q - P) \wedge \gamma = 0$ is equivalent to $\Phi = 0$ for $(P, \gamma) \in \mathcal{F}$. Let γ', P' be the projections of the variables γ, P to the plane spanned by u_1, u_2; these are the directions of $T_{l_0}\mathbb{M}_{1,3}$ coming from the Jacobi fields in $\mathcal{J}^\perp$. Then covectors (Γ, ξ) in $N^*_{(P_0,\gamma_0)}Z$ are given by the restriction of

$$\begin{aligned}(\Gamma, \xi) &= \Big(\big(\nabla_{P'}(\tau \cdot \Phi), \nabla_{\gamma'}(\tau \cdot \Phi)\big)\big|_{T_{l_0}\mathcal{F}}, \nabla_Q(\tau \cdot \Phi)\Big)\Big|_{((P,\gamma),Q) = ((P_0,\gamma_0), P_0 + t\gamma_0)} \\ &= \Big(\big(\tau_2 u_1 - \tau_1 u_2, t(\tau_1 u_2 - \tau_2 u_1)\big)\big|_{T_{l_0}\mathcal{F}}, \tau \wedge \gamma|_{\gamma=\gamma_0}\Big).\end{aligned}$$

Thus we can use (2.5) to define the twisted conormal bundle $N^*_{(l_0,Q)}Z'$

$$\Big\{ \Big((P_0, \gamma_0), \big(\tau_2 u_1 - \tau_1 u_2, t(\tau_1 u_2 - \tau_2 u_1)\big)\big|_{T_{l_0}\mathcal{F}}, P_0 + t\gamma_0, \tau \wedge \gamma|_{\gamma=\gamma_0}\Big) \ : \ t \in \mathbb{R}, \tau \in \mathbb{R}^2 \setminus 0 \Big\} \tag{4.3}$$

4.1.2 The Proof of Lemma 4.3

Suppose that for each $Q \in \mathbb{R}^3$ the set Γ_Q is a cone. We can pick u_1, u_2 such that for some constants $a, b_1, b_2 \in \mathbb{R}$ the Jacobi field $X_4(s) = au_1 + s(b_1u_1 + b_2u_2) \in T_{(P_0,\gamma_0)}\mathbb{M}_{1,3}$ is normal to $\mathcal{F}$ at (P_0, γ_0). We remark that if $b_1 = b_2 = 0$ the set Γ_{P_0} contains all lines in $\mathbb{M}_{1,3}$ through P_0 for γ in a neighborhood of γ_0, contradicting our assumption that Γ_{P_0} forms a cone. Thus we may assume that $b_1^2 + b_2^2 \neq 0$. Applying the Gram Schmidt process, the Jacobi fields

$$
\begin{aligned}
X_1(s) &= u_2 \\
X_2(s) &= s(-b_2u_1 + b_1u_2) \\
X_3(s) &= (b_1^2 + b_2^2)u_1 - sa(b_1u_1 + b_2u_2)
\end{aligned}
$$

form an orthogonal basis for $T_{(P_0,\gamma_0)}\mathcal{F}$. Let Ψ_i be the dual basis to X_i in $T^*_{(P_0,\gamma_0)}\mathcal{F}$, i.e. let $\Psi_i \in T^*_{(P_0,\gamma_0)}\mathcal{F}$ such that the pairing $\langle \Psi_i, X_j \rangle = \delta^i_j$ for each $i, j = 1, 2, 3$. Then projecting our expression in (4.3) to $T^*_{(P_0,\gamma_0)}\mathcal{F}$ we obtain

$$
\begin{aligned}
N^*_{(l_0,Q)}Z' = \Big\{ \big((P_0, \gamma_0); (-\tau_1\Psi_1 + t(b_2\tau_2 + b_1\tau_1)\Psi_2 + (at(b_2\tau_1 - b_1\tau_2) + \tau_2(b_1^2 + b_2^2))\Psi_3); \\
P_0 + t\gamma_0; \tau \wedge \gamma|_{\gamma=\gamma_0} \big) \Big\}.
\end{aligned}
$$

Let $\pi_L : N^*Z' \to T^*\mathcal{F}$ be the natural projection as in (2.4). The differential $(d\pi_L)_{((P_0,\gamma_0),t,\tau)}$ is given in the local coordinates induced by $(\{X_i\}_{i=1}^3, t, \tau)$ by the matrix $\left(\begin{smallmatrix} I & 0 \\ A & B \end{smallmatrix}\right)$, where

$$
B = \begin{pmatrix} 0 & -1 & 0 \\ b_1\tau_1 + b_2\tau_2 & b_1t & b_2t \\ ab_2\tau_1 - ab_1\tau_2 & tab_2 & (b_1^2 + b_2^2) - ab_1t \end{pmatrix}. \tag{4.4}
$$

The determinant of this matrix is $(b_1^2 + b_2^2)(\tau_1(b_1 - at) + \tau_2 b_2)$. Since $b_1^2 + b_2^2 \neq 0$ for sufficiently small t (or equivalently for Q sufficiently near P) the set $\mathcal{L}_{l_0} = \{(\Gamma, \xi) \in$

$N^*_{l_0}Z' \ : \ \det d\pi_L = 0\}$ is the subset of $N^*_{l_0}Z'$ such that $(\tau_1, \tau_2) \perp (b_1 - at, b_2)$.

Our analysis of $T_{l_0}\mathcal{F}$ can be repeated for each l in a neighborhood of l_0, leading to the definition of smooth functions $u_i(P, \gamma)$, $i = 0, 1, 2$, such that $u_i(P_0, \gamma_0) = u_i$ for $i = 0, 1, 2$, $\{u_i(P, \gamma)\}_{i=0,1,2}$ is an orthonormal basis for $\mathbb{R}^3$ for each (P, γ), $u_0(P, \gamma) = \gamma$, and $u_1(P, \gamma)$ is normal to the cone Γ_P at the line $l = \{P + s\gamma\}$. Using these smooth functions we obtain for each line l local coordinates $\Psi_i(P, \gamma)$ of $T^*_{(P,\gamma)}\mathcal{F}$ which depend smoothly on (P, γ) near (P_0, γ_0). The normal Jacobi field $X_4(s) =: X_4^{(P,\gamma)}(s)$ also varies with (P, γ), hence the parameters a, b_1, b_2 in (4.4) also vary smoothly with (P, γ). Repeating the argument above, for each l in a sufficiently small neighborhood of l_0 and Q in a neighborhood of P, $\mathcal{L} \cap N^*_l Z$ is the subset of $N^*_l Z'$ such that $(\tau_1, \tau_2) \perp (b_1 - at, b_2) \neq 0$, i.e. $(\tau_1, \tau_2) = \pm\rho(b_2, at - b_1)$ for any $\rho > 0$.

The projection $\varpi : N^*Z' \to Z$ from Theorem 3.14 maps

$$\begin{aligned}
&((P, \gamma), (-\tau_2\Psi_1(P, \gamma) + t(b_2\tau_1 - b_1\tau_2)\Psi_2(P, \gamma) + (at(b_1\tau_1 + b_2\tau_2) - \tau_1(b_1^2 + b_2^2)))\Psi_3(P, \gamma), \\
&\qquad P + t\gamma, \tau \wedge \gamma) \mapsto ((P, \gamma); P + t\gamma),
\end{aligned}$$

where a, b_1, b_2 smoothly depend on (P, γ). By plugging in the restriction in τ, the restriction of ϖ to $\mathcal{L}$ near l_0 and Q near P is then defined for $\rho > 0$

$$\begin{aligned}
&\big((P, \gamma), \pm\rho((b_1 - at)\Psi_1(P, \gamma) + t(b_2^2 + b_1^2 - ab_1t)\Psi_2(P, \gamma) + b_2((at)^2 - (b_1^2 + b_2^2))\Psi_3(P, \gamma), \\
&\qquad P + t\gamma, \pm\rho\big((at - b_1)u_1(P, \gamma) - b_2u_2(P, \gamma)\big) \mapsto ((P, \gamma); P + t\gamma).
\end{aligned}$$

Thus we see that $\varpi|_{\mathcal{L}}$ is still a projection in the first and last coordinates to Z near l_0 and is thus a submersion.

4.1.3 The Proof of Proposition 4.2

The implication that the canonical relations of Gelfand admissible restricted X-ray transforms are fibered folds is shown in [41, § II.3]; we present the proof below giving more detail. Pramanik and Seeger also gave an explicit parametrization of the fibered folding canonical relation for a model restricted X-ray transform in [46, § 4.2]. In addition to implying the conditions of Lemma 4.3, the Gelfand admissibility condition (Definition 4.1) allows us additional control over the Jacobi field $X_4(s)$. It states that along the line l_0, the normal space to $\mathcal{F}$ is proportional to a fixed vector in $\mathbb{R}^3$. Given our previous choice of u_1, u_2 the Gelfand admissibility condition implies that $X_4(s) = au_1 + b_1 s u_1$ with $b_1 \neq 0$, and therefore by rescaling

$$X_1(s) = u_2$$

$$X_2(s) = su_2$$

$$X_3(s) = b_1 u_1 - sau_1$$

$$X_4(s) = au_1 + sb_1 u_1$$

form an orthogonal basis for $T_{(P_0,\gamma_0)}\mathbb{M}_{1,3}$ while $X_1(s), X_2(s), X_3(s)$ form an orthogonal basis for $T_{(P_0,\gamma_0)}\mathcal{F}$. Repeating the argument above and applying the additional constraint $b_2 = 0$ to (4.4) we see that $\det d\pi_L|_{((P_0,\gamma_0),t,\tau)} = \tau_1(b_1 - at)$, which implies that for (P, γ) near (P_0, γ_0) and Q near P, $\mathcal{L}$ is the subset of N^*Z' such that $\tau_1 = 0$.

We also see from (4.4) that the kernel field for π_L at $((P_0, \gamma_0), P_0 + t\gamma_0)$ is given by $V_L|_{((P_0,\gamma_0),P_0+t\gamma_0)} = \tau_1 \partial_{\tau_2}$. Since $\mathcal{L}$ has no restriction on τ_2, V_L is tangent to $\mathcal{L}$ at $((P_0, \gamma_0), P_0+t\gamma_0)$. Repeating this same argument for each l we see that π_L is a blowdown along $\mathcal{L}$.

Next we examine the projection π_R. Note that given our discussion above we may rewrite $\pi_R(N^*Z) = (P+t\gamma, \tau_2 u_1(P,\gamma) - \tau_1 u_2(P,\gamma)$. We use $u_0 = \gamma_0, u_1 = u_1(P_0,\gamma_0), u_2 = u_2(P_0,\gamma_0)$ as a basis for $\mathbb{R}^3$ near P_0 and for $T^*_{P_0}\mathbb{R}^3$. Then we can identify the differential of π_R at $((P_0,\gamma_0),t,\tau)$ in the local coordinates induced by $(\{X_i(s)\}_{i=1}^3, t, \tau)$ as

$$d\pi_R\big|_{((P_0,\gamma_0),t,\tau)} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & b_1 - at & 0 & 0 & 0 \\ 1 & t & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ & \mathrm{A} & & 0 & 0 & 1 \\ & & & 0 & 1 & 0 \end{pmatrix}$$

where A describes the derivatives of $\tau_2 u_1(P,\gamma) - \tau_1 u_2(P,\gamma)$ with respect to the variations of (P,γ) within $\mathcal{F}$ (i.e. the Jacobi fields $X_1(s), X_2(s), X_3(s)$). For an element $\sum_{i=0}^2 \beta_i \partial_{X_i(s)} + c\partial_t + d_1\partial_{\tau_1} + d_2\partial_{\tau_2}$ to lie in the kernel of $d\pi_R|_{((P_0,\gamma_0),t,\tau)}$ it must be that $\beta_3 = c = 0$ and $\beta_2 = -t\beta_1$. We know that $\tau_1 = 0$ on $\mathcal{L}$, so finding the kernel of $d\pi_R|_{((P_0,\gamma_0),t,\tau)}$ on $\mathcal{L}$ amounts to determining the variation of $u_1(P,\gamma)$ with respect to $X_2(s)$, i.e. varying γ in the u_2 direction leaving P fixed at P_0. To make this more concrete we give a parametrization of Γ_{P_0} and relate the variation of u_1 to a projection of a derivative of a curve on the cone. Recall that u_1 is normal to the cone Γ_{P_0} at the line l_0 and u_2 is tangent to Γ_{P_0} at the line l_0, and that Γ_{P_0} has nonvanishing curvature. Thus for some $\epsilon > 0$ we can find a smooth map $g : (-\epsilon,\epsilon) \to \mathbb{R}^2$ such that $|g|^2 = g_1^2 + g_2^2 = 1$, g is parametrized by arc length, $\gamma_0 = \frac{\sqrt{2}}{2}(g(0),1)$, and

$$\Gamma_{P_0} = \{P_0 + s(g(\alpha),1) \ : \ s \in \mathbb{R}, \ |u| < \epsilon\}.$$

Then if (P_0,γ) is is sufficiently near (P_0,γ_0) we can find $\alpha \in (-\epsilon,\epsilon)$ such that $\gamma =$

$\frac{\sqrt{2}}{2}(g(\alpha),1) =: u_0(\alpha)$, $u_2(P_0,\gamma) = (g'(\alpha),0) =: u_2(\alpha)$, and

$$u_1(P_0,\gamma) = \gamma \wedge u_2(P_0,\gamma) = \tfrac{\sqrt{2}}{2}\big(-g_2'(\alpha), g_1'(\alpha), g_1(\alpha)g_2'(\alpha) - g_2(\alpha)g_1'(\alpha)\big) =: u_1(\alpha). \quad (4.5)$$

Thus $\{u_i(\alpha)\}$ forms an orthonormal basis of $\mathbb{R}^3$ adapted to the cone Γ_{P_0} so that $u_1(\alpha)$ is normal to the cone at $P_0 + s(g(\alpha),1)$, and $u_2(\alpha)$ is the horizontal tangent vector at the same point. Then the variation of $u_1(P,\gamma)$ in the $X_2(s)$ direction at (P_0,γ_0) is equivalent to the projection

$$\begin{aligned}\langle u_1'(0), u_2(0)\rangle &= \big\langle \tfrac{\sqrt{2}}{2}\big(-g_2''(0), g_1''(0), g_1(0)g_2''(0) - g_2(0)g_1''(0)\big), (g_1'(0), g_2'(0), 0)\big\rangle \\ &= \frac{\sqrt{2}}{2}(g_1''(0)g_2'(0) - g_1'(0)g_2''(0)),\end{aligned}$$

which is proportional to the principal curvature of Γ_{P_0} along $u_2(0) = u_2(P_0,\gamma_0)$. This implies that elements of $\ker d\pi_R|_{((P_0,\gamma_0),t,\tau)}$ must have $d_1 \neq 0$, implying that V_R is transversal to $\mathcal{L}$. Additionally, since $\det(d\pi_R)|_{((P_0,\gamma_0),t,\tau)}$ is linear in τ_1 we conclude that π_R has a fold singularity at $((P_0,\gamma_0),t,0,\tau_2)$.

4.2 Convolution-type Operators on the Heisenberg Group

The Heisenberg group $\mathbb{H}$ can be defined as $\mathbb{R}^3$ with the group law

$$(x_1,x_2,u) \odot (y_1,y_2,v) = \big(x_1+y_1, x_2+y_2, u+v+\frac{1}{2}(x_1y_2 - x_2y_1)\big).$$

Note that with this presentation the group inverse of an element is its negative (i.e. $x^{-1} = -x$), and the center of $\mathbb{H}$ is the x_3-axis.

Let $\gamma : [0,1] \to \mathbb{R}^3$ be a smooth regular curve (e.g. C^∞ and $\gamma' \neq 0$) and let μ be a smooth measure supported on $\gamma([0,1])$. Then we can define a family of curves by translating γ (or rather γ^{-1}) by $x \in \mathbb{H}$ via the group law, $\mathcal{M}_x = \{\gamma(t)^{-1} \odot x \in \mathbb{H} \ : \ t \in [0,1]\}$. The local Radon-like transform associated to this family of curves is

$$\mathcal{A}_{\mathbb{H}} f(x) = \int f(\gamma(t)^{-1} \odot x)\chi(t)\, dt. \tag{4.6}$$

The definition of $\mathcal{A}_{\mathbb{H}}$ is very similar to the definition of $\mathcal{A}_{\mathbb{R}}$ in §3.2, as both operators average over families of curves generated by group translation, one by Euclidean translation and one by Heisenberg translation. In other words, both operators are convolutions in their respective groups with measures supported on curves. This perspective in part suggests that the behavior of both operators may share some similarities. In particular, we may expect that curves in $\mathbb{H}$ obeying some group-invariant notion of nonvanishing curvature and torsion would correspond to $\mathcal{A}_{\mathbb{H}}$ being associated to a two-sided fold. Secco, in [49] provided such a notion of curvature and torsion. The derivative $\gamma'(t)$ is an element of $T_{\gamma(t)}\mathbb{H}$, so it is natural to compare higher derivatives of γ by first mapping $\gamma'(t)$ to the tangent space at the origin.

Let $R_{\gamma(t)} : \mathbb{H} \to \mathbb{H}$ be right translation by $\gamma(t)$ (i.e. $x \mapsto x \odot \gamma(t)$). Then the pullback $dR_{\gamma(t)}$ is a map from $T_{\gamma(t)}\mathbb{H} \to T_0\mathbb{H}$. Similarly, let $L_{\gamma(t)} : \mathbb{H} \to \mathbb{H}$ be left translation by $\gamma(t)$ (i.e. $x \mapsto \gamma(t) \odot x$). Then the pullback $dL_{\gamma(t)}$ is also a map from $T_{\gamma(t)}\mathbb{H} \to T_0\mathbb{H}$. Since $\mathbb{H}$ is noncommutative this choice of pullbacks results in two distinct notions of nonvanishing curvature and torsion.

Definition 4.5 (Secco '99)**.** *Given the definitions of $dL_{\gamma(t)}$ and $dR_{\gamma(t)}$ above, we define* *the left-invariant derivatives and right-invariant derivatives of* γ

$$\begin{aligned} \gamma_L'(t) &= dL^{-1}_{\gamma(t)}\gamma'(t) & \qquad \gamma_R'(t) &= dR^{-1}_{\gamma(t)}\gamma'(t) \\ \gamma_L''(t) &= \tfrac{d}{dt}\gamma_L'(t) & \qquad \gamma_R''(t) &= \tfrac{d}{dt}\gamma_R'(t) \\ \gamma_L'''(t) &= \tfrac{d}{dt}\gamma_L''(t) & \qquad \gamma_R'''(t) &= \tfrac{d}{dt}\gamma_R''(t). \end{aligned}$$

We say that γ *has* ***left- (resp. right-) invariant nonvanishing curvature and torsion*** *at* t *if* $\gamma_L'(t), \gamma_L''(t), \gamma_L'''(t)$ *(resp.* $\gamma_R'(t), \gamma_R''(t), \gamma_R'''(t)$*) are linearly independent.*

Since we assume $\gamma' \neq 0$, we may change variables so that either $\gamma_1'(t) = t$, $\gamma_2'(t) = t$, or $\gamma_3'(t) = t$. However, because the center of $\mathbb{H}$ is the x_3-axis, we split our analysis into two cases: first when γ is nowhere vertical and second when γ is near vertical. In the second case we see that the notions of left and right invariant derivatives of γ coincide since γ is parallel to the center of $\mathbb{H}$. Indeed, suppose that $\gamma_2'(t_0) = \gamma_3'(t_0) = 0$ and $\gamma_3(t_0)' = c > 0$. Via Definition 4.5 we see that the nonvanishing left- and right-invariant curvature and torsion is equivalent to the nonvanishing of the determinants

$$\begin{aligned} \det\left(\gamma_L' \ \gamma_L'' \ \gamma_L'''\right)|_{t=t_0} &= \begin{vmatrix} 0 & \gamma_1''(t_0) & \gamma_1'''(t_0) \\ 0 & \gamma_2''(t_0) & \gamma_2'''(t_0) \\ c & \frac{1}{2}\Big(\gamma_1''(t_0)\gamma_2(t_0) - \gamma_2''(t_0)\gamma_1(t_0)\Big) + \gamma_3''(t_0) & \frac{1}{2}\Big(\gamma_1'''(t_0)\gamma_2(t_0) - \gamma_2'''(t_0)\gamma_1(t_0)\Big) + \gamma_3'''(t_0) \end{vmatrix} \\ &= c\Big(\gamma_1''(t_0)\gamma_2'''(t) - \gamma_2''(t_0)\gamma_1'''(t_0)\Big) \end{aligned}$$

and

$$\begin{aligned} \det\left(\gamma_R' \ \gamma_R'' \ \gamma_R'''\right)|_{t=t_0} &= \begin{vmatrix} 0 & \gamma_1''(t_0) & \gamma_1'''(t_0) \\ 0 & \gamma_2''(t_0) & \gamma_2'''(t_0) \\ 1 & -\frac{1}{2}\Big(\gamma_1''(t_0)\gamma_2(t_0) - \gamma_2''(t_0)\gamma_1(t_0)\Big) + \gamma_3''(t_0) & -\frac{1}{2}\Big(\gamma_1'''(t_0)\gamma_2(t_0) - \gamma_2'''(t_0)\gamma_1(t_0)\Big) + \gamma_3''(t_0) \end{vmatrix} \\ &= c\Big(\gamma_1''(t_0)\gamma_2'''(t) - \gamma_2''(t_0)\gamma_1'''(t_0)\Big) \end{aligned}$$

respectively. Since these quantities are identical when γ is vertical, γ must either have non-vanishing left- and right-invariant curvature and torsion in a neighborhood of t_0 or both must vanish at t_0. Therefore we do not expect to see asymmetric behavior of the projections π_L and π_R when γ is near vertical.

Thus we assume that γ is nowhere vertical, i.e. $(\gamma_1', \gamma_2') \neq 0$. Then without loss of generality (though possibly with a reordering of the first two coordinates) we can write $\gamma(t) = (t, \gamma_2(t), \gamma_3(t))$, where $\gamma_2, \gamma_3 \in C^\infty(\mathbb{R})$. Given this parametrization of γ, we can again relate nonvanishing left- and right-invariant curvature and torsion of γ to the nonvanishing of the determinants

$$\begin{aligned}\det\left(\gamma_L' \ \gamma_L'' \ \gamma_L'''\right) &= \begin{vmatrix} 1 & 0 & 0 \\ \gamma_2'(t) & \gamma_2''(t) & \gamma_2'''(t) \\ \gamma_3'(t) - \frac{1}{2}(t\gamma_2'(t) - \gamma_2(t)) & \gamma_3''(t) - \frac{1}{2}\gamma_2''(t) & \gamma_3'''(t) - \frac{1}{2}(\gamma_2''(t) + t\gamma_2'''(t)) \end{vmatrix} \\ &= \det\begin{pmatrix} \gamma_2''(t) & \gamma_3''(t) \\ \gamma_2'''(t) & \gamma_3'''(t) \end{pmatrix} + \tfrac{1}{2}(\gamma_2''(t))^2 \end{aligned} \tag{4.7}$$

and

$$\begin{aligned}\det\left(\gamma_R' \ \gamma_R'' \ \gamma_R'''\right) &= \begin{vmatrix} 1 & 0 & 0 \\ \gamma_2'(t) & \gamma_2''(t) & \gamma_2'''(t) \\ \gamma_3'(t) + \frac{1}{2}(t\gamma_2'(t) - \gamma_2(t)) & \gamma_3''(t) + \frac{1}{2}\gamma_2''(t) & \gamma_3'''(t) + \frac{1}{2}(\gamma_2''(t) + t\gamma_2'''(t)) \end{vmatrix} \\ &= \det\begin{pmatrix} \gamma_2''(t) & \gamma_3''(t) \\ \gamma_2'''(t) & \gamma_3'''(t) \end{pmatrix} - \tfrac{1}{2}(\gamma_2''(t))^2 \end{aligned} \tag{4.8}$$

respectively. If $\gamma_2'' \neq 0$ then these quantities differ, and therefore at least one of them must be nonzero. When we compare these notions of torsion and curvature of γ to the microlocal behavior of $\mathcal{A}_{\mathbb{H}}$ we see very similar behavior to the Euclidean operator of §3.2. Indeed, when γ exhibits both left- and right-invariant nonvanishing curvature and torsion then both π_L and π_R have at most fold singularities. However, if γ only satisfies one of these conditions then only one of the projections has folds. Thus we can find examples of curves in $\mathbb{H}$ where the associated projections π_L and π_R have asymmetric behavior, unlike in the Euclidean case.

4.2.1 One-Sided Fold Conditions for $\mathcal{A}_{\mathbb{H}}$

First we give a characterization of the curves for which $\mathcal{A}_{\mathbb{H}}$ satisfies the conditions of Theorem 3.14. Specifically, this case occurs when (4.7) is nonzero, but (4.8) is uniformly zero, or equivalently, if γ_2'' never vanishes, but (4.8) is uniformly zero.

Proposition 4.6. *Suppose that $\gamma(t) = (t, \gamma_2(t), \gamma_3(t))$ is a smooth regular curve in $\mathbb{H}$ for $t \in [0,1]$. If $\gamma_2''(t) \neq 0$ and*

$$\det \begin{pmatrix} \gamma_2''(t) & \gamma_3''(t) \\ \gamma_2'''(t) & \gamma_3'''(t) \end{pmatrix} = \tfrac{1}{2}(\gamma_2''(t))^2$$

for all $t \in [0,1]$ then $\mathcal{A}_{\mathbb{H}}$ satisfies the conditions of Theorem 3.14. On the other hand, if $\gamma_2''(t) \neq 0$ and

$$\det \begin{pmatrix} \gamma_2''(t) & \gamma_3''(t) \\ \gamma_2'''(t) & \gamma_3'''(t) \end{pmatrix} = -\tfrac{1}{2}(\gamma_2''(t))^2$$

for all $t \in [0,1]$ then $\mathcal{A}_{\mathbb{H}}^*$ satisfies the conditions of Theorem 3.14.

The conditions of Proposition 4.6 restrict the class of admissible curves γ quite significantly, as we can rewrite the conditions

$$\det \begin{pmatrix} \gamma_2'' & \gamma_3'' \\ \gamma_2''' & \gamma_3''' \end{pmatrix} = \pm\tfrac{1}{2}(\gamma_2'')^2$$

as $(\gamma_3''/\gamma_2'')' = \pm\frac{1}{2}$. This implies the existence of constants $C_1, C_2, C_3 \in \mathbb{R}$ such that

$$\begin{aligned} \gamma_3''(t) &= (\pm\tfrac{1}{2}t + C_1)\gamma_2''(t) \\ \gamma_3'(t) &= (\pm\tfrac{1}{2}t + C_1)\gamma_2'(t) \mp \tfrac{1}{2}\gamma_2(t) + C_2 \\ \gamma_3(t) &= (\pm\tfrac{1}{2}t + C_1)\gamma_2(t) \mp \Gamma(t) + C_2 t + C_3, \end{aligned} \tag{4.9}$$

where

$$\Gamma(t) = \int_0^t \gamma_2(s)\, ds. \tag{4.10}$$

A concrete example which satisfies each of the two conditions is given by $\gamma(t) = (t, t^2, \pm\frac{1}{6}t^3)$.

We can additionally characterize the curves γ for which $\mathcal{A}_{\mathbb{H}}$ is associated to fold and finite type conditions.

Proposition 4.7. *Suppose that $\gamma_2''(t) \neq 0$ for $t \in [0,1]$. Let*

$$h_k^L(t) = \det\begin{pmatrix} \gamma_2''(t) & \gamma_3''(t) \\ \gamma_2^{(k+2)}(t) & \gamma_3^{(k+2)}(t) \end{pmatrix} + \tfrac{k}{2}\gamma_2''(t)\gamma_2^{(k+1)}(t), \qquad k = 1, 2, ...$$

$$h_k^R(t) = \det\begin{pmatrix} \gamma_2''(t) & \gamma_3''(t) \\ \gamma_2^{(k+2)}(t) & \gamma_3^{(k+2)}(t) \end{pmatrix} - \tfrac{k}{2}\gamma_2''(t)\gamma_2^{(k+1)}(t), \qquad k = 1, 2, ...$$

Then for each $t \in [0,1]$ at least one of $h_1^L(t), h_1^R(t)$ is nonzero. Further:

1. *Suppose $h_1^L(t) \neq 0$ and $h_1^R(t) \neq 0$ for all $t \in [0,1]$. Then the only singularities of π_L and π_R are Whitney folds.*

2. *Suppose $h_1^L(t) \neq 0$ for all $t \in [0,1]$, but that $h_1^R(t)$ vanishes at finitely many isolated points t_j, $j = 1, ..., N$. For each point t_j, suppose that there is $k_j > 1$ such that $h_{k_j}^R(t_j) \neq 0$, and $h_i^R(t_0) = 0$ for all $1 \leq i < k_j$. Let $k = \max\{k_j \ : \ j = 1, ..., N\}$. Then π_L has at most Whitney folds and π_R has maximal type k.*

3. *Suppose $h_1^R(t) \neq 0$ for all $t \in [0,1]$, but that $h_1^R(t)$ vanishes at finitely many isolated points t_j, $j = 1, ..., N$. For each point t_j, suppose that there is $k_j > 1$ such that $h_{k_j}^L(t_j) \neq 0$, and $h_i^L(t_0) = 0$ for all $1 \leq i < k_j$. Let $k = \max\{k_j \ : \ j = 1, ..., N\}$. Then π_R has at most Whitney folds and π_L has maximal type k.*

Note that h_1^L and h_1^R are equal to (4.7) and (4.8). An example of a curve that satisfies condition (1) is $\gamma(t) = (t, t^2, \alpha t^3)$, where $\alpha \neq \pm\frac{1}{6}$. An example of a curve that satisfies condition (2) is $\gamma(t) = (t, t^2, \frac{1}{6}t^3 + t^{k+2})$. An example of a curve that satisfies condition (3) is $\gamma(t) = (t, t^2, -\frac{1}{6}t^3 + t^{k+2})$.

4.2.2 The Proof of Propositions 4.6 and 4.7

The incidence relation $\mathcal{M}$ associated to $\mathcal{A}_{\mathbb{H}}$ is the zero locus of $\Phi = (\Phi^2, \Phi^3)^\intercal$, defined by

$$\begin{aligned}
\Phi^2(x,y) &= x_2 - y_2 - \gamma_2(x_1 - y_1) \\
\Phi^3(x,y) &= x_3 - y_3 - \gamma_3(x_1 - y_1) + \tfrac{1}{2}(x_1\gamma_2(x_1 - y_1) - x_2(x_1 - y_1)).
\end{aligned} \tag{4.11}$$

We change variables to rewrite Φ^3 so that it no longer depends on x_2. First, we rearrange

$$\Phi^3(x,y) = (x_3 - \tfrac{1}{2}x_2x_1) - y_3 - \gamma_3(x_1 - y_1) + \tfrac{1}{2}x_1\gamma_2(x_1 - y_1) + \tfrac{1}{2}x_2y_1.$$

Next, on $\mathcal{M} = \{\Phi(x,y) = 0\}$ we may substitute $x_2 = y_2 + \gamma_2(x_1 - y_1)$ and rearrange to obtain

$$\begin{aligned}
\Phi^3(x,y) &= (x_3 - \tfrac{1}{2}x_2x_1) - y_3 - \gamma_3(x_1 - y_1) + \tfrac{1}{2}x_1\gamma_2(x_1 - y_1) - \tfrac{1}{2}y_1(y_2 + \gamma_2(x_1 - y_1)) \\
&= (x_3 - \tfrac{1}{2}x_2x_1) - (y_3 - \tfrac{1}{2}y_1y_2) - \gamma_3(x_1 - y_1) + \tfrac{1}{2}(x_1 + y_1)\gamma_2(x_1 - y_1).
\end{aligned}$$

By a smooth change of variables $\tilde{x}_3 = x_3 - \frac{1}{2}x_1x_2$ and $\tilde{y}_3 = y_3 - \frac{1}{2}y_1y_2$ (and abusing notation slightly by rewriting $\tilde{x}_3$ and $\tilde{y}_3$ as x_3 and y_3 respectively) we can define $\mathcal{M}$ as the zero locus of $\tilde{\Phi}(x,y) = (x' - y' - S(x_1, y_1))$, where

$$S^2(x_1, y_1) = \gamma_2(x_1 - y_1) \tag{4.12}$$

$$S^3(x_1, y_1) = \gamma_3(x_1 - y_1) - \tfrac{1}{2}(x_1 + y_1)\gamma_2(x_1 - y_1). \tag{4.13}$$

In the coordinates induced by the defining function $\tilde{\Phi}$, the twisted conormal bundle is given by

$$\mathfrak{C} := \left\{ \big(x, (\tau \cdot \tilde{\Phi})_x, y, -(\tau \cdot \tilde{\Phi})_y\big) \,:\, \tilde{\Phi}(x, y) = 0 \right\}.$$

We define $\mathcal{L} \subset \mathfrak{C}$ as in (2.13).

Note that $\tilde{\Phi}$ can be expressed as the graph of an $\mathbb{R}^2$-valued function $F(x, y_1) = x' - S(x_1, y_1)$, hence the differentials of the projections π_L and π_R at a point $P(x, y_1, \tau) \in \mathfrak{C}$ can be expressed as the Jacobians of the functions $\pi_L^{\tilde{\Phi}} : (x, y_1, \tau) \mapsto (x, -\tau \cdot S_{x_1}(x_1, y_1), \tau)$ and $\pi_R^{\tilde{\Phi}} : (x, y_1, \tau) \mapsto (y_1, x' - S(x_1, y_1), \tau \cdot S_{y_1}(x_1, y_1), \tau)$, respectively. Thus $d\pi_L|_{P(x,y_1,\tau)} = \left(\begin{smallmatrix} I_{3\times 3} & 0 \\ (\tau\cdot\tilde{\Phi})_{x_i x_j} & B \end{smallmatrix}\right)$ where

$$B = \begin{pmatrix} -\tau \cdot S_{x_1 y_1}(x_1, y_1) & -S^2_{x_1}(x_1, y_1) & -S^3_{x_1}(x_1, y_1) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus we see that $\det(d\pi_L) = -\tau \cdot S_{x_1 y_1}(x_1, y_1)$. Then in the coordinates induced by $\tilde{\Phi}$,

$$\begin{aligned} \mathcal{L} = \{P(x, y_1, \tau) \in \mathfrak{C} \,:\, \tau_2 &= \pm\rho(\gamma_3''(x_1 - y_1) - \tfrac{1}{2}(x_1 + y_1)\gamma_2''(x_1 - y_1)) \\ \tau_3 &= \mp\rho\gamma_2''(x_1 - y_1),\ \rho > 0\}. \end{aligned} \tag{4.14}$$

Let the restriction of τ in $\mathcal{L}$ be denoted $\tau = \pm\rho\tilde{\tau}(x_1, y_1)$. In these coordinates the restriction of the projection $\varpi|_{\mathcal{L}} : \mathcal{L} \to \mathcal{M}$ defined in Theorem 3.14 maps

$$\begin{aligned} &\big(x, \mp \rho\tilde{\tau}(x_1, y_1) \cdot S_{x_1}(x_1, y_1), \pm\rho\tilde{\tau}(x_1, y_1), y_1, x' - S(x_1, y_1), \\ &\qquad \pm \rho\tilde{\tau}(x_1, y_1) \cdot S_{y_1}(x_1, y_1), \pm\tilde{\tau}(x_1, y_1)\big) \mapsto \big(x, y_1, x' - S(x_1, y_1)\big). \end{aligned}$$

Clearly the restriction of ϖ to $\mathcal{L}$ is still a projection and therefore a submersion.

We also see that π_L and π_R drop rank simply on $\mathcal{L}$. Indeed, since $\gamma_2''(t) \neq 0$ for all t,

$$\partial_{\tau_2} \det(d\pi_{L(R)}) = \gamma_2''(x_1 - y_1) \neq 0.$$

Next we characterize the various finite type conditions possible for π_L and π_R. Following the procedure in Lemma 2.4, we see that a kernel field for π_L is given by $V_L = \partial_{y_1}$; thus for any point $P(x, y_1, \tau) \in \mathcal{L}$ and any $k \geq 1$

$$\begin{aligned} V_L^k \det d\pi_L\Big|_P &= -\tau \cdot S_{x_1 y_1^{k+1}}(x_1, y_1)\Big|_{\tau \cdot S_{x_1 y_1}(x_1,y_1)=0} \\ &= \rho(-1)^k \Big(\det \left(\begin{smallmatrix} \gamma_2''(x_1-y_1) & \gamma_3''(x_1-y_1) \\ \gamma_2^{(k+2)}(x_1-y_1) & \gamma_3^{(k+2)}(x_1-y_1) \end{smallmatrix} \right) + \tfrac{k}{2}\gamma_2''(x_1-y_1)\gamma_2^{(k+1)}(x_1-y_1) \Big) \end{aligned} \tag{4.15}$$

A similar calculation yields that $\det(d\pi_R)|_P = -\det(d\pi_L)|_P$. Again following the procedure of Lemma 2.4 we can construct a kernel field for π_R, given by

$$V_R = \partial_{x_1} + S_{x_1}^2(x_1, y_1)\partial_{x_2} + S_{x_1}^3(x_1, y_1)\partial_{x_3}.$$

As above, for $P(x, y_1, \tau) \in \mathcal{L}$ and any $k \geq 1$ we have

$$\begin{aligned} V_R^k \det d\pi_R\Big|_P &= \tau \cdot S_{x_1^{k+1} y_1}(x_1, y_1)\Big|_{\tau \cdot S_{x_1 y_1}(x_1,y_1)=0} \\ &= \rho(-1)^k \Big(\det \left(\begin{smallmatrix} \gamma_2''(x_1-y_1) & \gamma_3''(x_1-y_1) \\ \gamma_2^{(k+2)}(x_1-y_1) & \gamma_3^{(k+2)}(x_1-y_1) \end{smallmatrix} \right) - \tfrac{k}{2}\gamma_2''(x_1-y_1)\gamma_2^{(k+1)}(x_1-y_1) \Big). \end{aligned} \tag{4.16}$$

Note that (4.15) and (4.16) correspond to the definitions of h_k^L and h_k^R respectively. In particular, if $h_j^L(t_0) = 0$ for $1 \leq j < k$ but $h_k^L(t_0) \neq 0$, then we see that π_L is type k at points in $\mathcal{L}$ where $x_1 - y_1 = t_0$, and we can make an analogous statement for π_R. Thus we have proven Proposition 4.7.

Finally we prove Proposition 4.6. Suppose $\gamma_2''(t) \neq 0$ and $h_1^R(t) = 0$ for all t. Then $h_1^L(t) \neq 0$ for all $t \in [0, 1]$ and π_L is Whitney fold. As in (4.9) there are constants

$C_1, C_2, C_3 \in \mathbb{R}$ such that

$$\gamma_3(t) = (\tfrac{1}{2}t + C_1)\gamma_2(t) - \Gamma(t) + C_2 t + C_3,$$

where $\Gamma(t)$ is given by (4.10). This implies that

$$S^3_{x_1 y_1}(x_1, y_1) = -(\tfrac{1}{2}(x_1 - y_1) + C_1)\gamma_2''(t) + \tfrac{1}{2}(x_1 + y_1)\gamma_2''(t) = (y_1 - C_1)\gamma_2''(x_1 - y_1).$$

This in turn implies that $\det(d\pi_L) = 0$ when

$$(\tau_2, \tau_3) = \rho(y_1 - C_1, 1)\gamma_2''(t)$$

for any $\rho \in \mathbb{R}$. Since $\gamma_2''(t) \neq 0$ this implies that $\mathcal{L}$ is given by the set

$$\mathcal{L} = \{P(x, y_1, \tau) \in \mathfrak{C} \ : \ (\tau_2, \tau_3) = \rho(y_1 - C_1, 1), \ \rho \in \mathbb{R}\}$$

Since V_R lies in the span of $\{\partial_{x_i}\}_{i=1,2,3}$, V_R is clearly tangent to $\mathcal{L}$ everywhere along $\mathcal{L}$, implying that π_R is a blowdown. On the other hand, if $\gamma_2''(t) \neq 0$ and $h_1^L(t) = 0$ for all t, an almost identical argument shows that π_R is a fold and π_L is a blowdown.

Chapter 5

Sharpness Results on the Heisenberg Group

In this chapter we prove that Theorem 3.15 is sharp for examples, meaning that there exist examples of local Radon-like transforms $\mathcal{R}$ which satisfy the conditions of Theorem 3.15 but for which L^p-Sobolev estimates cannot be improved past the boundary of the shaded region in Figure 7. Since Theorem 3.14 is implied by Theorem 3.15 we will consequently establish the sharpness of Theorem 3.14. The sharpness of the region in Figure 7 was observed for the restricted X-ray transform (4.1) in [44], where Pramanik and Seeger proved the following proposition.

Proposition 5.1 ([44, Proposition 1.1])**.** *Suppose that $\mathcal{X}$ is defined as in* (4.1) *such that γ has nonvanishing curvature. Suppose that $\mathcal{X}^* : L^p_{\mathrm{comp}} \to L^p_{s,\mathrm{loc}}$. Then* $s \leq \min\left\{\frac{1}{p}, \frac{1}{2}\left(1 - \frac{1}{p}\right), \frac{1}{4}\right\}$.

The proof of the necessity of $s \leq \frac{1}{4}$ is closely related to the sharpness of decoupling inequalities. A general correspondence between L^p-Sobolev estimates for averages over curves and decoupling inequalities is established in [5] in the case of translation invariant operators. Unfortunately that link is less transparent in the variable coefficient setting, where the fibers of $\pi_L(\mathcal{L})$ may vary with x (recall our discussion of the fibers of $\pi_L(\mathcal{L})$

and their relation to decoupling in §3.3).

However, we can adapt techniques from [44] to establish the first two of these necessary conditions also hold in the case of Heisenberg convolutions with measures supported on curves satisfying the conditions of Proposition 4.6 (see [6]). In addition, we establish the third necessary condition ($s \leq 1/4$) for the case of the moment curve $\gamma(t) = (t, t^2, \frac{1}{6}t^3)$ through a change of variables which renders the associated fibers of $\pi_L(\mathcal{L})$ fixed in x.

Proposition 5.2. *Let $\mathcal{A}_{\mathbb{H}}$ be defined as in* (4.6)*. If γ satisfies the conditions of Proposition 4.6 and $\mathcal{A}_{\mathbb{H}}$ extends to a continuous operator from $L^p_{\mathrm{comp}} \to L^p_{s,\mathrm{loc}}$ then*

$$s \leq \min\left\{\tfrac{1}{p}, \tfrac{1}{2}\left(1 - \tfrac{1}{p}\right)\right\}.$$

Additionally, if $\gamma(t) = (t, t^2, \frac{1}{6}t^3)$ then we may also conclude $s \leq \frac{1}{4}$.

To prove this proposition we recall the oscillatory integral representation of $\mathcal{A}_{\mathbb{H}}$, where after a change of variables

$$\mathcal{A}_{\mathbb{H}} f(x) = \int\int e^{2\pi i\tau\cdot(x'-y'-S(x_1,y_1))}\chi(x_1, y_1) f(y)\, d\tau\, dy,$$

where $S(x_1, y_1)$ is given by (4.12),(4.13) and χ is smooth and compactly supported. Then the adjoint of $\mathcal{A}_{\mathbb{H}}$ is given by

$$\mathcal{A}^*_{\mathbb{H}} g(y) = \int e^{-i\tau\cdot(x'-y'-S(x_1,y_1))}\chi(x_1, y_1) g(x)\, dx.$$

For simplicity, the sharpness examples will be proven on the adjoint of $\mathcal{A}_{\mathbb{H}}$.

Consider a Fourier multiplier m_k in $\mathbb{R}^2$ of order 0 which vanishes for $|\xi'| \leq c2^k$. Then identifying $\xi' = (\xi_2, \xi_3)$ when $\xi = (\xi_1, \xi') \in \mathbb{R}^3$ we can let m_k act on functions on $\mathbb{R}^3$ via

$$m_k(D')f(x) = \mathfrak{F}^{-1}[m_k(\xi')\hat{f}(\xi)](x).$$

Observe that $\mathcal{A}_{\mathbb{H}}^*$ (and $\mathcal{A}_{\mathbb{H}}$) commutes with $m_k(D')$ since $\mathcal{A}_{\mathbb{H}}^*$ is translation invariant in y', i.e.

$$\mathcal{A}_{\mathbb{H}}^* g(y_1, y' - \tilde{y'}) = \mathcal{A}_{\mathbb{H}}[g(\cdot, \cdot - \tilde{y'})](y).$$

This implies that $m_k(D')$ commutes with $\mathcal{A}_{\mathbb{H}}^*$. Indeed, applying Fourier inversion,

$$\begin{aligned}
m_k(D')\mathcal{A}_{\mathbb{H}}^* g(y) &= \int e^{2\pi i y'\cdot\xi'} m_k(\xi') \int e^{-2\pi i z'\cdot\xi'} \mathcal{A}_{\mathbb{H}}^* g(y_1, z')\, dz'\, d\xi' \\
&= \int e^{2\pi i\tau\cdot(S(x_1,y_1)-x')} \left(\int\int e^{-2\pi i z'\cdot(\xi'-\tau)} e^{2\pi i y'\cdot\xi'} m_k(\xi') d\xi'\, dz' \right) \\
&\qquad\qquad \times \chi(x_1, y_1) g(x)\, dx\, d\tau \\
&= \int e^{2\pi i\tau\cdot(S(x_1,y_1)-x')} e^{2\pi i y'\cdot\tau} m_k(\tau)\chi(x_1, y_1) g(x)\, dx\, d\tau \\
&= \int e^{2\pi i\tau\cdot(S(x_1,y_1)-x'+y')} m_k(\tau)\chi(x_1, y_1) g(x)\, dx\, d\tau.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\mathcal{A}_{\mathbb{H}}^*[m_k(D')g](y) &= \mathcal{A}_{\mathbb{H}}^*\Big[\int e^{2\pi i x'\cdot\xi'} m_k(\xi') \int e^{-2\pi i\xi'\cdot w'} g(x_1, w')\, dw'\, d\xi' \Big](y) \\
&= \int e^{2\pi i\tau\cdot(S(x_1,y_1)+y')} \left(\int\int e^{2\pi i x'(\xi'-\tau)} e^{-2\pi i\xi'\cdot w'} m_k(\xi')\, d\xi'\, dx' \right) \\
&\qquad\qquad \times \chi(x_1, y_1) g(x_1, w')\, dx_1\, dw'\, d\tau \\
&= \int e^{2\pi i\tau\cdot(S(x_1,y_1)-w'+y')} m_k(\tau)\chi(x_1, y_1) g(x_1, w')\, dx_1\, dw'\, d\tau.
\end{aligned}$$

If $\mathcal{A}_{\mathbb{H}} : L^p_{\mathrm{comp}} \to L^p_s$ for some $p \in (1, \infty)$ it follows that $\mathcal{A}_{\mathbb{H}}^* : L^p_{-s,\mathrm{comp}} \to L^{p'}$, and thus that

$$\|m_k(D')\mathcal{A}_{\mathbb{H}}^* g\|_{p'} = \|\mathcal{A}_{\mathbb{H}}^*[m_k(D')g]\|_{p'} \leq C_{p'} \|m_k(D')g\|_{L^{p'}_{-s}} \leq C_{p'} 2^{-ks} \|g\|_{p'}$$

for any compactly supported $g \in L^{p'}(\mathbb{R}^3)$.

5.1 The Necessity of $s \le 1/p$

Let ζ_1 be supported in $\{\xi' : 1/2 \le |\xi'| \le 2\}$ with $\widehat{\zeta_1}(0) = 1$. Let m_k be the Fourier multiplier acting on functions in $\mathbb{R}^3$ by $m_k(D')f(x) = \mathfrak{F}^{-1}[\zeta_1(2^{-k}\xi')\hat{f}(\xi)]$. Then

$$m_k(D')\mathcal{A}_{\mathbb{H}}^* g(y_1, y') = \int 2^{2k}\widehat{\zeta_1}(2^k(x' - y' - S(x_1, y_1)))g(x)\chi(x_1, y_1)\, dx. \tag{5.1}$$

Let y_1 be fixed for now and let $x_0 \in [-1/2, 1/2]$ such that $\chi(x_0, y_1) > 0$. Choose k large enough that $\chi(x_1, y_1) > c > 0$ for $|x_1 - x_0| \le 2^{-k}$. Let g_k be the indicator function of a ball of radius 2^{-k} centered at $(x_0, 0, 0)$, and let c_{y_1} be the curve $\{S(x_1, y_1) : x_1 \in \operatorname{supp} \chi(\cdot, y_1)\} \subset \mathbb{R}^2$. For small $\varepsilon > 0$ let E_{y_1} be the set of all y' such that $\operatorname{dist}(y', c_{y_1}) \le \varepsilon 2^{-k}$. Since $\gamma_2'' \neq 0$ on $[-1, 1]$ we can conclude that $S(\cdot, y_1)$ is a regular curve in $\mathbb{R}^2$ on a neighborhood of (x_0, y_1) that has diameter at least $1/2$, hence we estimate $|E_{y_1}| \approx 2^{-2k}$ for each fixed y_1. As $\widehat{\zeta_1}$ is positive near the origin we see that the integrand in (5.1) is bounded below by $c2^{2k}$ if $y' \in E_{y_1}$, whence we can bound the integral (5.1) below by 2^{-k}. After integrating in y' over the size of E_{y_1} and in y_1 over a fixed compact set, we see that $\|m_k(D')\mathcal{A}_{\mathbb{H}}^* g_k\|_{p'} \gtrsim 2^{-k}2^{-2k/p'}$. On the other hand, $\|g_k\|_{p'} \lesssim 2^{-3k/p'}$, hence by a scaling argument we must have $s \le 1 - 1/p' = 1/p$.

5.2 The Necessity of $s \le \frac{1}{2}(1 - \frac{1}{p})$

Notice that since $\gamma'' \neq 0$ the direction of the vector $S_{x_1 y_1}(x_1, y_1) = \gamma_2''(x_1 - y_1)(1, -y_1)$ does not depend on x_1. Let $T(y_1) = (1, -y_1)$ and let $N(y_1) = (y_1, 1)$. Let $\zeta_2 \in \mathcal{S}(\mathbb{R})$ be such that $\widehat{\zeta_2}$ is non-negative everywhere and is positive in $[-1/2, 1/2]$. Let ζ_3 be supported in $\{1/2 \le |t| \le 2\}$ with $\widehat{\zeta_3} \ge 1/2$ on $[-C, C]$. Pick b such that $\chi(b) > 0$ and define the Fourier multiplier m_k by

$$m_k(\tau_2, \tau_3) = \zeta_2(2^{-k/2}\langle \tau, T(b)\rangle)\zeta_3(2^{-k}\langle \tau, N(b)\rangle).$$

Again, m_k acts on functions in $\mathbb{R}^3$ as

$$m_k(D')f(x) = \mathfrak{F}^{-1}\Big[m_k(\tau_2, \tau_3)\widehat{f}(\xi_1, \tau_2, \tau_3)\Big].$$

Since $m_k(\tau)$ vanishes for $|\tau| \leq c2^k$ we have $\|m_k(D')\mathcal{A}_{\mathbb{H}}^* g\|_{p'} \leq 2^{-ks}\|g\|_{p'}$, and that

$$\begin{aligned} m_k(D')\mathcal{A}_{\mathbb{H}}^* g(y) = \int & 2^{3k/2}\widehat{\zeta}_2(2^{k/2}\langle x' - y' - S(x_1, y_1), T(b)\rangle) \\ & \times \widehat{\zeta}_3(2^k\langle x' - y' - S(x_1, y_1), N(b)\rangle)g(x)\chi(x_1, y_1)\, dx. \end{aligned} \tag{5.2}$$

Let $g_k(x)$ be the indicator function of the set defined by the equations

$$\begin{aligned} |\langle x' - S(x_1, b), T(b)\rangle| &\leq 2^{-k/2} \\ |\langle x' - S(x_1, b), N(b)\rangle| &\leq 2^{-k} \\ |x_1| &\leq 1/2. \end{aligned}$$

Let P_k be the set of y such that $|\langle y', T(b)\rangle| \leq 2^{-k/2}$, $|\langle y', N(b)\rangle| \leq 2^{-k}$, and $|y_1 - b| \leq 2^{-k/2}$. For $x \in \operatorname{supp} g_k$ and $y \in P_k$ we see that since $|y_1 - b| \leq 2^{-k/2}$,

$$|\langle x' - y' - S(x_1, y_1), T(b)\rangle| \leq C2^{-k/2}.$$

However, we have better decay in the $N(b)$ direction, as $S(x_1, \cdot)$ vanishes to second order in the $N(b)$ direction. Indeed, a Taylor expansion reveals

$$|\langle S(x_1, y_1) - S(x_1, b), N(b)\rangle| = |(y_1 - b)^2(-\gamma_2'(x_1 - b)) + |y_1 - b|^2 R_1(x_1, y_1)| \leq C2^{-k},$$

where $R_1(x_1, y_1)$ is smooth and uniformly bounded. Thus

$$|\langle y' - x' - S(x_1, y_1), N(b)\rangle| \leq C2^{-k},$$

implying by the conditions on $\widehat{\zeta}_2$ and $\widehat{\zeta}_3$ that the integrand in (5.2) is greater than $c2^{3k/2}$, implying that $m_k(D')\mathcal{A}_{\mathbb{H}}^* g_k(y)$ is bounded below by a positive constant for all $y \in P_k$. Thus $\|m_k(D')\mathcal{A}_{\mathbb{H}}^* g_k\|_{p'} \geq c2^{-2k/p'}$. On the other hand, $\|g_k\|_{p'} \leq 2^{-3k/2p'}$, implying that $s \leq \frac{1}{2p'} = \frac{1}{2}(1 - \frac{1}{p})$.

5.3 The Necessity of $s \leq 1/4$ for the Moment Curve

In the case $\gamma(t) = (t, t^2, \frac{1}{6}t^3)$, we can make a change of variables to transform $\mathcal{A}_{\mathbb{H}}$ into the restricted X-ray transform (4.1). This allows us to give another example of a local Radon-like transform which cannot map $L^p \to L^p_s$ locally unless $s \leq 1/4$. Let $\eta(y) = (y_2 + y_1^2, y_3 - \frac{2}{3}y_1^3 - \frac{1}{2}y_1 y_2, y_1)$. Note η is a smooth function whose Jacobian always has determinant 1. We apply the operator $\mathcal{A}_{\mathbb{H}}$ to $f \circ \eta$ to obtain

$$\mathcal{A}_{\mathbb{H}}(f \circ \eta)(x) = \int f(x_2 + x_1^2 - 2x_1 t, x_3 - \tfrac{2}{3}x_1^3 - \tfrac{1}{2}x_1 x_2 + 2x_1^2 t - x_1 t, x_1 - t)\chi(t)dt$$

Next, we change variables $(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) = (x_1, x_2 - x_1^2, x_3 - \frac{1}{2}x_1 x_2 + \frac{1}{3}x_1^3)$ to get

$$\mathcal{A}_{\mathbb{H}}(f \circ \eta)(\tilde{x}) = \int f(\tilde{x}_2 + 2\tilde{x}_1(\tilde{x}_1 - t), \tilde{x}_3 - \tilde{x}_1(\tilde{x}_1 - t)^2, \tilde{x}_1 - t)\chi(t)dt.$$

The map $x \to \tilde{x}$ is also smooth with Jacobian always equal to 1. Finally, letting $y_3 := \tilde{x}_1 - t$ we see that our operator has been transformed into the adjoint of (4.1), associated to the curve $y_3 \mapsto (-2y_3, y_3^2)$.

Given this transformation, we can directly apply the third sharpness constraint of Proposition 5.1, that $s \leq 1/4$.

Chapter 6

An Extension to the Heisenberg-Sobolev Space

Convolution operators like (3.2) commute with translation on $\mathbb{R}^3$; thus we can bootstrap local results like Theorem 3.13 to global results for convolution operators by translating and stitching together compactly supported functions into a global function. Suppose that $\mathcal{R}$ is a local Radon-like transform that is also a convolution operator such that $\mathcal{R}f$ is supported in $B_C(0)$ whenever f is supported in $B_1(0)$ for some uniform constant $C \geq 1$. Suppose also that $\mathcal{R}$ extends to a continuous operator

$$\mathcal{R} \;:\; L^p_{\text{comp}}(\Omega_R) \to L^p_{\alpha,\text{loc}}(\Omega_L) \tag{6.1}$$

for some $\alpha \in \mathbb{R}$. Then let $V \subset \mathbb{R}^3$ be a countable collection of points such that $\{B_1(\nu)\}_{\nu\in V}$ covers $\mathbb{R}^3$ and $\{B_C(\nu)\}_{\nu\in V}$ is finitely overlapping. Using a partition of unity we split $f = \sum_{\nu\in V} f_\nu(x)$, where $f_\nu(x)$ are supported in $B_1(\nu)$ for each ν. Since $\mathcal{R}$ is translation invariant, $\mathcal{R}f_\nu(x-\nu) = \mathcal{R}[f_\nu(\cdot-\nu)](x)$, and therefore $\mathcal{R}f_\nu(x)$ is supported in $B_C(\nu)$ for each $\nu \in V$. Thus $\{\mathcal{R}f_\nu(x)\}_{\nu\in V}$ is a collection of functions with bounded overlap. By applying (6.1) to each $\mathcal{R}f_\nu$ and using this observation of bounded overlap we see that

$$\|\mathcal{R}f\|_{L^p_\alpha} \leq \sum_\nu \|\mathcal{R}f_\nu\|_{L^p_\alpha} \leq \sum_\nu C\|f_\nu\|_p \leq C\|\sum_\nu f_\nu\|_p = C\|f\|_p.$$

This argument can be adapted to other operators which are not invariant under Euclidean translation. For example the averaging operator $\mathcal{A}_{\mathbb{H}}$ from Chapter 4 can be viewed as a Heisenberg convolution with a measure supported on the curve γ; thus $\mathcal{A}_{\mathbb{H}}$ commutes with the action of the Heisenberg group on itself (which we call Heisenberg translation). Hence we can prove via an adaptation of the above argument that $\mathcal{A}_{\mathbb{H}}$ is bounded from $L^p(\mathbb{R}^3)$ to an analogue of the space $L^p_{1/p}(\mathbb{R}^3)$ adapted to translations on the Heisenberg group, which we now introduce.

Define the ***discrete Heisenberg group*** $\mathbb{H}_{\mathbb{Z}} := \{(x_1, x_2, x_3 + \frac{1}{2}x_1x_2) : x_j \in \mathbb{Z}\} \subset \mathbb{H}$. As in §4.2, let R_λ denote right (Heisenberg) translation by $\lambda \in \mathbb{H}_{\mathbb{Z}}$ and R^*_λ the associated translation operator defined by $R^*_\lambda f(x) = f(x \odot \lambda^{-1})$. The discrete Heisenberg group acts as a discrete approximation of $\mathbb{H}$, as $\mathbb{Z}^3$ is a discrete approximation of $\mathbb{R}^3$, and we can use its integer-like properties to construct a partition of unity adapted to $\mathbb{H}_{\mathbb{Z}}$ (this construction is an example of a ***uniform partition of unity*** on a locally compact group, see [36]). $\mathbb{H}_{\mathbb{Z}}$ is a ***uniform lattice*** on $\mathbb{H}$, meaning that we can find a compact set $C \subset \mathbb{H}$ such that $\mathbb{H} = \bigcup_{\lambda \in \mathbb{H}_{\mathbb{Z}}} C \odot \lambda$. This condition is satisfied by $C = \{(a, b, c + \frac{1}{2}ab) : -\frac{1}{2} \le a, b, c \le \frac{1}{2}\} \subset B_1(0)$. Indeed, given $(x, y, z) \in \mathbb{H}$ we can find a discrete translate of C that contains (x, y, z). Let $\lceil \cdot \rfloor$ denote the nearest integer function, so that $x = \lceil x \rfloor + a$, $y = \lceil y \rfloor + b$, and $z = \lceil z \rfloor + c$, where $|a|, |b|, |c| \le \frac{1}{2}$. Then defining

$$\lambda = \big(\lceil x \rfloor, \lceil y \rfloor, \lceil z - \tfrac{1}{2}xy + b\lceil y \rfloor \rfloor + \tfrac{1}{2}\lceil x \rfloor \lceil y \rfloor\big) \in \mathbb{H}_{\mathbb{Z}},$$

we see that $(x, y, z) = (a, b, c) \odot \lambda$, implying that $(x, y, z) \in C \odot \lambda$. Also note that the interiors of $C \odot \lambda$ for every $\lambda \in \mathbb{H}_{\mathbb{Z}}$ are mutually disjoint. Moreover, $B_2(0) \supset B_1(0) \supset C$ contains finitely many elements of $\mathbb{H}_{\mathbb{Z}}$. Thus we can pick $\psi \in C^\infty_c(B_2(0))$ with uniformly bounded derivatives such that $0 \le \phi \le 1$, and define translates $\phi_\lambda(x) = \phi(x \odot \lambda^{-1}) =$

$R_\lambda^*\phi$ so that $\sum_{\lambda\in\mathbb{H}_\mathbb{Z}}\phi_\lambda \simeq 1$ with finitely overlapping support. Then since the sum $\sum_{\lambda\in\mathbb{H}_\mathbb{Z}}\phi_\lambda$ is locally finite the functions

$$\psi_\lambda = \frac{\phi_\lambda}{\sum_{\lambda'\in\mathbb{H}_\mathbb{Z}}\phi_{\lambda'}}$$

form a C^∞ partition of unity. Moreover, the family of functions is still generated by the group action of $\mathbb{H}_\mathbb{Z}$ as

$$\psi_\lambda(x) = \frac{\phi(x\odot\lambda^{-1})}{\sum_{\lambda'\odot\lambda\in\mathbb{H}_\mathbb{Z}}\phi\big((x\odot(\lambda')^{-1}\odot\lambda^{-1})\odot\lambda\big)} = R_\lambda\frac{\phi(x)}{\sum_{\lambda'\in\mathbb{H}_\mathbb{Z}}\phi_{\lambda\odot\lambda'}},$$

implying $\psi_\lambda = R_\lambda^*\psi_{(0,0,0)} =: R_\lambda^*\psi$. Given this partition of unity, we define the following norm.

Definition 6.1. *Let $\psi \in C_c^\infty(B_2(0))$ such that $0 \le \psi \le 1$ with uniformly bounded derivatives and $\sum_{\lambda\in\mathbb{H}_\mathbb{Z}} R_\lambda^*\psi \equiv 1$ with finitely overlapping supports. We define $L_s^p(\mathbb{H})$ to be the space of functions in $L^p(\mathbb{R}^3)$ such that the norm*

$$\|f\|_{L_s^p(\mathbb{H})} := \Big\| \sum_{\lambda\in\mathbb{H}_\mathbb{Z}} R_\lambda^*(I-\Delta)^{s/2}\psi R_{\lambda^{-1}}^* f\Big\|_{L^p(\mathbb{R}^3)}$$

is finite.

By an adaptation of the argument above, Theorem 3.14 and Proposition 4.6 imply the following.

Theorem 6.2. *If γ satisfies the same conditions as in Proposition 4.6, then $\mathcal{A}_\mathbb{H}$ is bounded from $L^p(\mathbb{R}^3)$ to $L_{1/p}^p(\mathbb{H})$ for $p > 4$.*

In this chapter we will prove Theorem 6.2 and examine some aspects of the norm $\|\cdot\|_{L_s^p(\mathbb{H})}$. This norm is a natural choice for a Sobolev space on $\mathbb{H}$ for three reasons. First, the standard (Euclidean) Sobolev norm and the Heisenberg-Sobolev norm are comparable for functions supported near the origin. Given a compact set K containing the origin there are only finitely many ψ_λ that are nonzero on K, hence for functions supported on K we can apply finitely many changes of variables and use the finitely overlapping support of $\{\psi_\lambda\}$ to conclude

$$\|f\|_{L^p_s(\mathbb{R}^3)} \simeq_K \|f\|_{L^p_s(\mathbb{H})}.$$

Second, if we replace Heisenberg translations over $\mathbb{H}_\mathbb{Z}$ with Euclidean translations over the integers (denote these translations τ_n) we see that

$$\begin{aligned}\Big\|\sum_{n\in\mathbb{Z}} \tau_n (I-\Delta)^{s/2}\psi\tau_{-n} f\Big\|_p &= \Big\|\sum_{n\in\mathbb{Z}} \tau_n (I-\Delta)^{s/2}\tau_{-n}\psi_n f\Big\|_p \\ &= \Big\|(I-\Delta)^{s/2}\sum_{n\in\mathbb{Z}}\psi_n f\Big\|_p \\ &= \|(I-\Delta)^{s/2} f\|_p,\end{aligned}$$

assuming that $\sum_{n\in\mathbb{Z}}\psi_n \equiv 1$. So the main obstruction between this space and the standard (Euclidean) Sobolev space is the fact that $(I-\Delta)^{s/2}$ does not commute with Heisenberg translations, making it a natural analogue of the Sobolev space in a non-commutative setting. Third, this norm is independent of our choice of smooth cutoff function ψ.

Proposition 6.3. *The choice of a different ψ in the definition of the Heisenberg-Sobolev norm results in an equivalent norm.*

We will prove this proposition in Section 6.2. First, we prove Theorem 6.2.

6.1 The Proof of Theorem 6.2

We use finitely overlapping support of ψ_λ and the fact that $\mathcal{A}_{\mathbb{H}}$ commutes with Heisenberg translation to show

$$\|\mathcal{A}_{\mathbb{H}} f\|_{L^p_{1/p}(\mathbb{H})} \lesssim \Big(\sum_{\lambda\in\Lambda} \big\|R^*_\lambda (I-\Delta)^{\frac{1}{2p}} \psi \mathcal{A}_{\mathbb{H}} R^*_{\lambda^{-1}} f\big\|_p^p\Big)^{1/p}.$$

We first remove the right translation by λ by an affine change of variables. We observe that for $\mathcal{F}$ a fixed dilate of the support of ψ we have $\psi\mathcal{A}_{\mathbb{H}} R^*_{\lambda^{-1}} f = \psi\mathcal{A}_{\mathbb{H}} \mathbb{1}_{\mathcal{F}} R^*_{\lambda^{-1}} f$. This combined with Theorem 3.14 gives

$$\begin{aligned}\Big(\sum_{\lambda\in\Lambda} \big\|(I-\Delta)^{\frac{1}{2p}} \psi \mathcal{A}_{\mathbb{H}} R^*_{\lambda^{-1}} f\big\|_p^p\Big)^{1/p} &\lesssim \Big(\sum_{\lambda\in\Lambda} \|\mathbb{1}_{\mathcal{F}} R^*_{\lambda^{-1}} f\|_p^p\Big)^{1/p} \\ &\lesssim \|f\|_p,\end{aligned}$$

finishing the proof.

6.2 Independence from ψ

We now prove Proposition 6.3. Suppose $\{\tilde{\psi}_\lambda\}_{\lambda\in\mathbb{H}_{\mathbb{Z}}}$ is another partition of unity satisfying the conditions in Definition 6.1. Observe that there is a finite set $\mathcal{B} \subset \mathbb{H}_{\mathbb{Z}}$ contained in the Euclidean ball $B_4(0)$ (independent of $\tilde{\psi}$ and ψ) such that

$$\psi = \psi\Big(\sum_{\sigma\in\mathcal{B}} \tilde{\psi}_\sigma\Big).$$

Next, for each $\sigma \in \mathcal{B}$ and $\lambda \in \mathbb{H}_{\mathbb{Z}}$ we have

$$\begin{aligned}\psi\tilde{\psi}_\sigma R^*_{\lambda^{-1}} f &= \psi R^*_\sigma \tilde{\psi} R^*_{\sigma^{-1}} R^*_{\lambda^{-1}} f \\ &= R^*_\sigma \psi_{\sigma^{-1}} \tilde{\psi} R^*_{(\sigma\lambda)^{-1}} f.\end{aligned}$$

Since the supports of ψ_λ are finitely overlapping and $\mathcal{B}$ is finite, we obtain

$$\begin{aligned}\Big\| \sum_{\lambda\in\mathbb{H}_\mathbb{Z}} R^*_\lambda (I-\Delta)^{s/2}\psi R^*_{\lambda^{-1}} f\Big\|_p &= \Big\| \sum_{\lambda\in\mathbb{H}_\mathbb{Z}} R^*_\lambda (I-\Delta)^{s/2} \sum_{\sigma\in\mathcal{B}} \psi\tilde{\psi}_\sigma R^*_{\lambda^{-1}} f\Big\|_p \\ &= \Big\| \sum_{\lambda\in\mathbb{H}_\mathbb{Z}} \sum_{\sigma\in\mathcal{B}} R^*_\lambda (I-\Delta)^{s/2} R^*_\sigma \psi_{\sigma^{-1}}\tilde{\psi} R^*_{(\sigma\lambda)^{-1}} f\Big\|_p \\ &\simeq \Big(\sum_{\lambda\in\mathbb{H}_\mathbb{Z}} \sum_{\sigma\in\mathcal{B}} \| R^*_\lambda (I-\Delta)^{s/2} R^*_\sigma \psi_{\sigma^{-1}}\tilde{\psi} R^*_{(\sigma\lambda)^{-1}} f\|_p^p\Big)^{\frac{1}{p}}. \quad (6.2)\end{aligned}$$

Let $g_{\lambda,\sigma} = \psi_{\sigma^{-1}}\tilde{\psi} R^*_{(\sigma\lambda)^{-1}} f$. We prove that $\|(I-\Delta)^{s/2} R^*_\sigma g_{\lambda,\sigma}\|_p \simeq \|R^*_\sigma (I-\Delta)^{s/2} g_{\lambda,\sigma}\|_p$ uniformly in σ and λ. To show this we need some technical details from the definition of Triebel-Lizorkin spaces (cf. [55, 56]).

Definition 6.4. *Let Ω be the collection of all sequences $\{\omega_j\}_{j=0}^\infty \subset \mathcal{S}(\mathbb{R}^3)$ with the properties*

1. *there exist positive constants A, B, C such that*

$$\operatorname{supp}\omega_0 \subset \{\xi \,:\, |\xi| \le A\}$$
$$\operatorname{supp}\omega_j \subset \{\xi \,:\, B2^{j-1} \le |\xi| \le C2^{j+1}\}, \qquad j = 1, 2, 3, ...$$

2. *for every multi-index α there exists $c_\alpha > 0$ such that*

$$\sup_{x\in\mathbb{R}^3} \sup_{j\in\mathbb{N}} 2^{j|\alpha|} |\partial^\alpha \omega_j(\xi)| \le c_\alpha,$$

3. *for every $\xi \in \mathbb{R}^3$*

$$\sum_{j=0}^\infty \omega_j(\xi) = 1.$$

For a sequence $\{\omega_j\} \in \Omega$ we define the Triebel-Lizorkin norm

$$\|f\|_{F^{p,q}_s} = \Big\| \Big(\sum_{j=0}^\infty |2^{js} \check{\omega}_j * f|^q \Big)^{1/q} \Big\|_{L^p}.$$

We remark that a different choice of $\{\omega_j\}$ results in an equivalent norm.

Let $\{\omega_j\} \in \Omega$ with associated constants A, B, C, c_α. Recall that $\|R^*_\sigma g_{\lambda,\sigma}\|_{L^p_s} \simeq \|R^*_\sigma g_{\lambda,\sigma}\|_{F^{p,2}_s}$. A direct calculation reveals that

$$\widehat{R^*_\sigma g}(\xi) = e^{-2\pi i\langle\sigma,\xi\rangle}\widehat{g}(\xi_1 + \tfrac{\sigma_2}{2}\xi_3, \xi - \tfrac{\sigma_1}{2}\xi_3, \xi_3)$$

We define $\vartheta(\eta) = (\eta_1 - \frac{\sigma_2}{2}\eta_3, \eta_2 + \frac{\sigma_1}{2}\eta_3, \eta_3)$. Then by a linear change of variables

$$\begin{aligned}
\check{\omega}_j * R^*_\sigma g_{\lambda,\sigma} &= \int e^{2\pi i\langle x,\xi\rangle}\omega_j(\xi)e^{-2\pi i\langle\sigma,\xi\rangle}\widehat{g}_{\lambda,\sigma}(\eta(\xi))\, d\xi \\
&\int e^{2\pi i(\langle x\odot\sigma^{-1}),\eta\rangle}\omega_j(\vartheta(\eta))\widehat{g}_{\lambda,\sigma}(\eta)\, d\eta \\
&= R^*_\sigma\Big[\widecheck{\omega_j \circ \vartheta} * g_{\lambda,\sigma}\Big].
\end{aligned}$$

The smooth cutoff $\omega_j \circ \vartheta$, $j = 1, 2, 3...$ is supported where

$$B2^{j-1} \le |\vartheta(\eta)| \le C2^{j+1}.$$

Since $|\sigma_j| \le 4$ for all $\sigma \in \mathcal{B}$ these inequalities imply that

$$\operatorname{supp}\omega_j(\vartheta(\eta)) \subset \big\{\eta\,:\, \tfrac{B}{5}2^{j-1} \le |\eta| \le 5C2^{j+1}\big\}.$$

The same argument also implies that $\operatorname{supp}\omega_0(\vartheta(\eta)) \subset \{\eta\,:\,|\eta| \le 5A\}$. Next, since $\vartheta(\eta)$ is linear and $|\sigma_j| \le 4$ for $j = 1, 2, 3$ we can conclude that for any multi-index α

$$\sup_{\eta\in\mathbb{R}^3}\sup_{j\in\mathbb{N}} 2^{j|\alpha|}|\partial^\alpha\omega_j(\vartheta(\eta))| \le 3^{|\alpha|}c_\alpha.$$

Since clearly $\sum_{j=0}^\infty \omega_j(\vartheta(\eta)) = 1$ for every η we conclude that $\{\tilde{\omega}_j\}_{j=0}^\infty = \{\omega_j \circ \vartheta\}_{j=0}^\infty \in \Omega$, hence

$$\begin{aligned}
\|(I-\Delta)^{s/2}R^*_\sigma g_{\lambda,\sigma}\|_p &\simeq \Big\|\Big(\sum_{j=0}^\infty |2^{js}\check{\omega}_j * R^*_\sigma g_{\lambda,\sigma}|^2\Big)^{1/2}\Big\|_p \\
&= \Big\|R^*_\sigma\Big(\sum_{j=0}^\infty |2^{js}\widecheck{\omega_j \circ \vartheta} * g_{\lambda,\sigma}|^2\Big)^{1/2}\Big\|_p \simeq \|R^*_\sigma(I-\Delta)^{s/2}g_{\lambda,\sigma}\|_p.
\end{aligned}$$

Plugging this into (6.2) we obtain

$$\begin{aligned}\Big\|\sum_{\lambda\in\mathbb{H}_{\mathbb{Z}}} R^*_\lambda(I-\Delta)^{s/2}\psi R^*_{\lambda^{-1}}f\Big\|_p &\simeq C\Big(\sum_{\lambda\in\mathbb{H}_{\mathbb{Z}}}\sum_{\sigma\in\mathcal{B}}\|R^*_{\sigma\lambda}(I-\Delta)^{s/2}\psi_{\sigma^{-1}}\tilde{\psi}_0 R^*_{(\sigma\lambda)^{-1}}f\|_p^p\Big)^{\frac{1}{p}}\\ &\leq C\Big(\sum_{\tilde{\lambda}\in\mathbb{H}_{\mathbb{Z}}}\sum_{\sigma\in\mathcal{B}}\|R^*_{\tilde{\lambda}}(I-\Delta)^{s/2}\psi_{\sigma^{-1}}\tilde{\psi} R^*_{\tilde{\lambda}^{-1}}f\|_p^p\Big)^{\frac{1}{p}}\\ &\simeq \Big\|\sum_{\tilde{\lambda}\in\mathbb{H}_{\mathbb{Z}}} R^*_{\tilde{\lambda}}(I-\Delta)^{s/2}\tilde{\psi}\Big(\sum_{\sigma\in\mathcal{B}}\psi_{\sigma^{-1}}\Big)R^*_{\tilde{\lambda}^{-1}}f\Big\|_p\\ &= C\Big\|\sum_{\tilde{\lambda}\in\mathbb{H}_{\mathbb{Z}}} R^*_{\tilde{\lambda}}(I-\Delta)^{s/2}\tilde{\psi} R^*_{\tilde{\lambda}^{-1}}f\Big\|_p,\end{aligned}$$

proving that the Heisenberg-Sobolev norm is equivalent for different choices of cutoff function.

Chapter 7

The Proof of Proposition 3.4

In this chapter we present an analytic interpolation argument for local Radon-like transforms of arbitrary dimensions d and n with a given L^2-Sobolev estimate. Let Ω_L, Ω_R be n-dimensional manifolds, and let $\mathcal{M} \subset \Omega_L \times \Omega_R$ be a $(d+n)$-dimensional submanifold such that the projections ρ_L, ρ_R defined in (2.1) are submersions. Then as in (2.3) we can express $\mathcal{M}$ locally as the zero locus of a smooth $\mathbb{R}^{d-n}$-valued function Φ, and we can write

$$\mathcal{R}f(x) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^{d-n}} e^{2\pi i \tau \cdot \Phi(x,y)} \chi(x,y) f(y)\, d\tau\, dy.$$

Recall that $\nabla_x \Phi^j(x,y)$ are linearly independent, as are $\nabla_y \Phi^j(x,y)$ for $j = 1, 2, ..., d-n$. By the implicit function theorem we can find $C_0 > 0$ such that for $(x,y) \in \operatorname{supp} \chi$

$$4C_0^{-1}|\tau| \leq |(\tau \cdot \Phi)_x| \leq C_0/4|\tau| \tag{7.1}$$

$$4C_0^{-1}|\tau| \leq |(\tau \cdot \Phi)_y| \leq C_0/4|\tau|. \tag{7.2}$$

We now introduce a dyadic partition of unity which we will use many times throughout the remainder of this work. Let $\chi_0 \in C_c^\infty(\mathbb{R})$ be nonnegative such that $\chi_0 = 1$ on $[-1,1]$ and is supported on $[-2,2]$. For $k \geq 1$ define $\tilde{\chi}_k(x) = \chi_0(2^{-k}x) - \chi_0(2^{1-k}x)$. Thus $\chi_1 \in C_c^\infty(\mathbb{R})$ is supported where $1 \leq |x| \leq 4$, $\chi_k(\cdot) = \chi_1(2^{1-k}\cdot)$ is supported where $2^{k-1} \leq |x| \leq 2^{k+1}$, and $\sum_{k \geq 0} \chi_k \equiv 1$. We can use this dyadic partition of unity to dyadically decompose $\mathcal{R}$ as follows. For $k \geq 0$ and $f \in \mathcal{S}(\mathbb{R}^d)$ define

$$\mathcal{R}_k f(x) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^{d-n}} e^{2\pi i \tau \cdot \Phi(x,y)} \chi(x,y) \chi_k(|\tau|) f(y) \, d\tau \, dy.$$

For $k \geq 0$ let P_k and $\tilde{P}_k$ be any standard Littlewood-Paley multipliers such that $\mathfrak{F}[P_k f]$ and $\mathfrak{F}[\tilde{P}_k f]$ are supported where $|\xi| \simeq 2^k$ for $k \geq 1$ and $\mathfrak{F}[P_0 f]$ is supported where $|\xi| \lesssim 1$. We introduce a lemma about Littlewood-Paley decompositions of local Radon-like transforms.

Lemma 7.1. *Suppose $C_0 > 0$ is such that* (7.1) *and* (7.2) *hold. For each $k \in \mathbb{N}$ let*

$$\mathcal{D}_k = \{(k', k'') \in \mathbb{N}^2 \ : \ |k - k'| > C_1\} \cup \{(k', k'') \in \mathbb{N}^2 \ : \ |k - k''| > C_1\}, \tag{7.3}$$

where C_1 depends on C_0. Let $v_0 \in C_c^\infty(\Omega_L)$ and $v_1 \in C_c^\infty(\Omega_R)$. Then for any $k \in \mathbb{N}$ and any $(k', k'') \in \mathcal{D}_k$

$$\|P_k v_0 \mathcal{R}_{k'} v_1 \tilde{P}_{k''}\|_{L^p \to L^p} \leq C \min\{2^{-kN}, 2^{-k'N}, 2^{-k''N}\}.$$

Proof of Lemma 7.1. This integration by parts argument is essentially due to Hörmander [35], based on the fact that the canonical relation stays away from zero sections (cf. [50, Lemma 2.1]). Note that the Schwartz kernel of the operator $P_k v_0 \mathcal{R}_{k'} v_1 P_{k''}$ is given by

$$\int\int\int\int\int e^{2\pi i [\langle x-w, \eta\rangle + \tau \cdot \Phi(w,z) + \langle z-y, \xi\rangle]} \chi_k(|\eta|) \chi_{k'}(|\tau|) \chi_{k''}(|\xi|)$$
$$\times \chi(x,y) v_0(x) v_1(y) \, dw \, dz \, d\tau \, d\eta \, d\xi.$$

Our assumption on Φ implies that if $\max\{|k - k'|, |k' - k''|\} > C_1$ we have

$$\nabla_{(z,w)} \left[\langle x - w, \eta\rangle + \tau \cdot \Phi(w,z) + \langle z - y, \xi\rangle\right] \geq c \max\{2^k, 2^{k'}, 2^{k''}\}.$$

We integrate by parts many times in the (w,z) variables and use the compact support of the kernel and Minkowski's integral inequality to obtain the desired bound on L^p.

□

Suppose there exists $\alpha \in \mathbb{R}$ such that $\mathcal{R}$ extends to a bounded operator from $L^2_{\mathrm{comp}}(\Omega_R) \to L^2_{\alpha,\mathrm{loc}}(\Omega_L)$, and define $\alpha(p)$ as in Lemma 3.1. We first construct an analytic family of operators. As before, let $v_0 \in C_c^\infty(\Omega_L)$, $v_1 \in C_c^\infty(\Omega_R)$, and define $T_z f = v_0 \mathcal{R}[v_1 (I-\Delta)^{\frac{z}{2}} f]$. Then we prove

$$\|T_z f\|_2 \leq C\|f\|_2, \qquad \operatorname{Re} z = \alpha \tag{7.4}$$

$$\|T_z f\|_{L^1} \leq C_y \|f\|_{\mathcal{H}^1}, \qquad \operatorname{Re} z = 0, \tag{7.5}$$

where C_y depends at most polynomially on y. Here $\mathcal{H}^1 = \mathcal{H}^1(\mathbb{R}^d)$ refers to the Hardy space on $\mathbb{R}^d$. Since $(I-\Delta)^{iy/2}$ is a Calderon-Zygmund operator, it is bounded from $L^2 \to L^2$ and $\mathcal{H}^1 \to L^1$ with constants depending at most polynomially on y. Thus the estimate (7.4) follows from the assumption (3.1) and the L^2 boundedness of $(I-\Delta)^{iy}$; note that (3.1) still holds for $\mathcal{R}$ with $\chi(x,y)$ replaced by $v_0(x)\chi(x,y)v_1(y)$. On the other hand, (7.5) follows from the local L^1-boundedness of $\mathcal{R}$ and the $\mathcal{H}^1 \to L^1$ boundedness of $(I-\Delta)^{iy}$. Thus we can use an analytic interpolation theorem of Stein found in [21, § 5] to deduce that the operator $T_{\alpha(p)}$ extends to a bounded operator

$$T_{\alpha(p)} : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$$

for $1 < p < 2$. Since $\mathcal{R}$ is not translation-invariant we cannot directly commute $\mathcal{R}$ and $(I-\Delta)^{\alpha(p)/2}$ to conclude that $\mathcal{R}$ is bounded from $L^p_{\mathrm{comp}}(\Omega_R) \to L^p_{\alpha(p),\mathrm{loc}}(\Omega_L)$. However, we can use Littlewood-Paley theory and Lemma 7.1 to achieve the same result. Suppose that $v_1 \equiv 1$ in a neighborhood of the origin. For $f \in L^p$ supported in the set $\{v_1 \equiv 1\}$

we can write

$$\|v_0\mathcal{R}f\|_{L^p_{\alpha(p)}} = \Big\|\Big(\sum_{k\geq 0}|2^{k\alpha(p)}P_k v_0\mathcal{R}v_1 f|^2\Big)^{1/2}\Big\|_{L^p}.$$

Note that since $P_k\big(2^{-k\alpha(p)}(I-\Delta)\big)^{\alpha(p)/2}$ is also a Littlewood-Paley multiplier of order k, for each $k\geq 0$

$$\begin{aligned}
P_k v_0\mathcal{R}v_1 f &= P_k v_0\bigg(\sum_{(k',k'')\notin\mathcal{D}_k}\mathcal{R}'_k v_1 P_{k''}\big(2^{-k''\alpha(p)}(I-\Delta)^{\alpha(p)/2}\big)f \\
&\qquad + \sum_{(k',k'')\in\mathcal{D}_k}\mathcal{R}'_k v_1 P_{k''}\big(2^{-k''\alpha(p)}(I-\Delta)^{\alpha(p)/2}\big)f\bigg) \\
&= P_k v_0\bigg(\sum_{|s_1|,|s_2|\leq C_1}\mathcal{R}_{k+s_1}v_1 P_{k+s_2}\big(2^{-k+s_2\alpha(p)}(I-\Delta)^{\alpha(p)/2}\big)f \\
&\qquad + \sum_{(k',k'')\in\mathcal{D}_k}\mathcal{R}'_k v_1 P_{k''}\big(2^{-k''\alpha(p)}(I-\Delta)^{\alpha(p)/2}\big)f\bigg).
\end{aligned}$$

Thus by the triangle inequality and an application of Lemma 7.1, we can estimate

$$\begin{aligned}
\big\|v_0\mathcal{R}f\big\|_{L^p_{\alpha(p)}} &\leq \bigg\|\bigg(\sum_{k\geq 0}\Big|\sum_{|s_1|,|s_2|\leq C_1}2^{-s_2\alpha(p)}P_k v_0\mathcal{R}_{k+s_1}v_1 P_{k+s_2}(I-\Delta)^{\alpha(p)/2}f\Big|^2\bigg)^{1/2}\bigg\|_p \\
&\qquad + C\|f\|_p \\
&\leq 2^{C_1\alpha(p)}\bigg\|\bigg(\sum_{k\geq 0}\Big|\sum_{|s_1|,|s_2|\leq C_1}P_k v_0\mathcal{R}_{k+s_1}v_1 P_{k+s_2}(I-\Delta)^{\alpha(p)/2}f\Big|^2\bigg)^{1/2}\bigg\|_p \\
&\qquad + C\|f\|_p.
\end{aligned}$$

Note that by the triangle inequality

$$\begin{aligned}
\Big|\sum_{|s_1|,|s_2|\leq C_1}\mathcal{R}_{k+s_1}v_1 P_{k+s_2}\Big| &= \Big|\sum_{k',k''\geq 0}\mathcal{R}_{k'}v_1 P_{k''} - \sum_{(k',k'')\in\mathcal{D}_k}\mathcal{R}_{k'}v_1 P_{k''}\Big| \\
&\leq \Big|\sum_{k',k''\geq 0}\mathcal{R}_{k'}v_1 P_{k''}\Big| + \Big|\sum_{(k',k'')\in\mathcal{D}_k}\mathcal{R}_{k'}v_1 P_{k''}\Big| \\
&= |\mathcal{R}v_1| + \Big|\sum_{(k',k'')\in\mathcal{D}_k}\mathcal{R}_{k'}v_1 P_{k''}\Big|.
\end{aligned}$$

Thus by another application of Lemma 7.1

$$\begin{aligned}\Big\|\Big(\sum_{k\geq 0}\Big|\sum_{|s_1|,|s_2|\leq C_1} P_k v_0 \mathcal{R}_{k+s_1} v_1 P_{k+s_2}(I-\Delta)^{\alpha(p)/2} f\Big|^2\Big)^{1/2}\Big\|_p &\leq C\Big\|\Big(\sum_{k\geq 0}|P_k T_{\alpha(p)} f|^2\Big)^{1/2}\Big\|_p \\ &\quad + C\|f\|_p \\ &\leq \|T_{\alpha(p)} f\|_p + C\|f\|_p \\ &\leq C\|f\|_p\end{aligned}$$

finishing the proof that $\mathcal{R}$ extends to a bounded operator from $L^p_{\mathrm{comp}}(\Omega_R) \to L^p_{\alpha(p),\mathrm{loc}}(\Omega_L)$ for $1 < p < 2$.

Chapter 8

The Structure of the Proof of Theorem 3.14

In this chapter we begin the proof of Theorem 3.14 by relating L^p-Sobolev estimates to estimates on oscillatory integral operators. Recall the definitions of χ_k for $k \geq 0$ from Chapter 7. We decompose dyadically in $|\tau|$ as in §7, then dyadically in the size of $|\det(d\pi_L)|$, following the ideas of Phong and Stein in [42]. Since π_L is a fold the decomposition in $|\det(d\pi_L)|$ also decomposes dyadically in the distance away from $\mathcal{L}$, locally in the direction of V_L. As the Schwartz kernel of $\mathcal{R}$ is compactly supported in (x, y), we have a uniform bound $|\Delta(x, y_3)| \leq 2^{-C_2}$ for some $C_2 \in \mathbb{R}$. Let $\varepsilon > 0$ be a small constant to be determined and let $\ell_\circ = \ell_\circ(k, \varepsilon) = \left\lfloor \frac{k}{2+\varepsilon} \right\rfloor$. Then for $C_2 \leq \ell \leq \ell_\circ$ let

$$
\begin{aligned}
a_{k,\ell,\pm}(x, y_3, \tau) &= \chi_1(2^{\ell+1-k}(\pm\tau \cdot \Delta(x, y_3))) \qquad\qquad C_2 \leq \ell < \ell_\circ \\
a_{k,\ell_\circ}(x, y_3, \tau) &= \chi_0(2^{\ell_\circ+1-k}|\tau \cdot \Delta(x, y_3)|).
\end{aligned}
$$

For $C_2 \leq \ell \leq \ell_\circ$ we then define

$$
\mathcal{R}_{k,\ell,\pm} f(x) = \int e^{2\pi i\tau\cdot(S(x,y_3)-y')} \chi(x, y) f(y) \chi_k(|\tau|) a_{k,\ell,\pm}(x, y_3, \tau)\, dy\, d\tau. \tag{8.1}
$$

By our assumption on Δ and the definitions of χ_k, $\sum_\pm \sum_{k\geq 0} \sum_{C_2 \leq \ell \leq \ell_\circ(k,\varepsilon)} \mathcal{R}_{k,\ell,\pm} = \mathcal{R}$. We will suppress the dependence of $\mathcal{R}$ on $\pm$ as we deal with $\mathcal{R}_{k,\ell,+}$ and $\mathcal{R}_{k,\ell,-}$ identically, and we will suppress the dependence of $\ell_\circ$ on k and ε when clear from context.

8.1 Oscillatory Integral Estimates

The main estimate we prove for our decomposed operator is the following.

Proposition 8.1. *For $p > 4$ there exists $\varepsilon_0(p) > 0$ such that for all $C_2 \le \ell \le \ell_\circ$,*

$$\|\mathcal{R}_{k,\ell}\|_{L^p \to L^p} \le C_p 2^{-(k+\ell\varepsilon_0)/p}.$$

This proposition follows by interpolation with L^2 estimates, L^∞ estimates, and a decoupling inequality. Let $\mathcal{I}$ be a collection of intervals of length $2^{-\ell}$ with disjoint interiors intersecting a small neighborhood of 0. Then for a function $f : \mathbb{R}^3 \to \mathbb{C}$ supported in small enough neighborhood of the origin and any $I \in \mathcal{I}$, let $f_I(y) := f(y)\mathbb{1}_I(y_3)$, so that $f = \sum_{I\in\mathcal{I}} f_I$ almost everywhere, with almost disjoint supports in y_3. We also define the operator

$$\mathcal{R}^I_{k,\ell} f(x) = \mathcal{R}_{k,\ell}[\mathbb{1}_I f](x)$$

for each $I \in \mathcal{I}$. Note that $\mathcal{R}_{k,\ell} f_I(x) = \mathcal{R}^I_{k,\ell} f_I(x)$.

The decomposition in distance from $\mathcal{L}$ allows us to quantitatively estimate the improvement in L^2 estimates as we move away from $\mathcal{L}$. This observation is analogous to the nonisotropic Fourier decay of measures supported on curves in $\mathbb{R}^3$ away from the binormal cone, illustrated in §3.2.

Proposition 8.2. *For every $k > 0$, every $\varepsilon > 0$ and $C_2 \le \ell \le \ell_\circ$*

$$\|\mathcal{R}_{k,\ell}\|_{L^2 \to L^2} \lesssim 2^{\frac{\ell-k}{2}+\ell\varepsilon}. \tag{8.2}$$

Moreover, by almost disjoint supports of the functions f_I,

$$\Big(\sum_{I\in\mathcal{I}} \|\mathcal{R}^I_{k,\ell} f_I\|^2_{L^2}\Big)^{1/2} \lesssim 2^{\frac{\ell-k}{2}+\ell\varepsilon}\Big(\sum_{I\in\mathcal{I}} \|f_I\|^2_{L^2}\Big)^{1/2}. \tag{8.3}$$

We will prove a more general version of Proposition 8.2 in Chapter 9, following methods of almost-orthogonality found in the proof of the Calderón-Vaillancourt theorem (see [39], § 9.2). The foundation of this method was originally introduced into this context by Phong and Stein [42], Cuccagna [18], and Comech [14]. The general version of Proposition 8.2 (Theorem 9.1) drops the assumption on ϖ and on dimension, and applies to all FIOs associated to fibered folding canonical relations.

The main estimate in the proof of Theorem 3.14 is the decoupling inequality.

Proposition 8.3. *If $\ell \le \ell_\circ$, for every $\varepsilon > 0$*

$$\Big\|\sum_{I\in\mathcal{I}} \mathcal{R}_{k,\ell} f_I\Big\|_{L^p} \lesssim_\varepsilon 2^{\ell(\frac{1}{2}-\frac{1}{p}+\varepsilon)}\Big(\sum_{I\in\mathcal{I}} \|\mathcal{R}_{k,\ell} f_I\|_{L^p}^p\Big)^{1/p} + 2^{-10k}\|f\|_{L^p}$$

for $2 \le p \le 6$.

Following a similar approach to [4] and [46], we prove Proposition 8.3 using an inductive argument, at each step combining l^p decoupling with suitable changes of variables. We first prove one step in this inductive argument for a model case in Chapter 10, then reduce the general case to the model case and perform the induction in Chapter 11.

To show that Propositions 8.2 and 8.3 imply Proposition 8.1 we interpolate with an easy L^∞ estimate.

Proposition 8.4. *For every $k \ge 0$ and $\ell \le \ell_\circ$*

$$\sup_{I\in\mathcal{I}} \|\mathcal{R}^I_{k,\ell} f_I\|_\infty \lesssim 2^{-\ell} \sup_{I\in\mathcal{I}} \|f_I\|_\infty \tag{8.4}$$

$$\|\mathcal{R}_{k,\ell} f\|_\infty \lesssim \|f\|_\infty. \tag{8.5}$$

Interpolating the estimates (8.4) with (8.3) for the vector-valued operator $\{\mathcal{R}^I_{k,\ell}\}_{I\in\mathcal{I}}$

applied to $\{f_I\}$ we obtain

$$\Big(\sum_{I\in\mathcal{I}}\|\mathcal{R}_{k,\ell}f_I\|_p^p\Big)^{1/p}\lesssim_\varepsilon 2^{\ell(\frac{3}{p}-1+\varepsilon)}2^{-k/p}\Big(\sum_{I\in\mathcal{I}}\|f_I\|_p^p\Big)^{1/p}, \qquad 2\le p\le\infty. \tag{8.6}$$

Combining this estimate with Proposition 8.3 we obtain

$$\|\mathcal{R}_{k,\ell}f\|_p\lesssim_\varepsilon 2^{\ell(\varepsilon+\frac{2}{p}-\frac{1}{2})}2^{-k/p}\Big(\sum_{I\in\mathcal{I}}\|f_I\|_p^p\Big)^{1/p}+2^{-10k}\|f\|_p, \qquad 2\le p\le 6. \tag{8.7}$$

Note that the power of 2^ℓ in (8.7) is negative if $4<p\le 6$ and ε is sufficiently small. A further interpolation with the L^∞ estimate (8.5) yields Proposition 8.1 for $p>4$.

8.2 Integration by Parts and Nonstationary Phase Arguments

Each of these propositions relies on integration by parts estimates, with careful consideration of the derivatives applied to the various symbols. We begin by stating a general integration by parts estimate, which we will apply many times throughout the next four chapters.

Lemma 8.5. *Suppose $\phi\in C^\infty(\mathbb{R}^d)$, and define the differential operator $L=\langle\frac{\nabla\phi}{|\nabla\phi|^2},\nabla\cdot\rangle$. Suppose $g\in C_c^\infty(\mathbb{R}^d)$ and there exists $D>0$ such that for every derivative ∂^j of order $j\in\mathbb{N}$, $|\partial^j g|\le D^j$. Assume that there exists some $E>0$ such that $|\nabla\phi|\ge E$ and $|\partial^j\phi|\le C_jD^{j-1}E$ for $j\ge 2$. Then for every $N>0$,*

$$|(L^*)^N(g)|\lesssim_{N,d}\left(\frac{D}{E}\right)^N. \tag{8.8}$$

Proof. This lemma is a special case of [4, Lemma A.2], which describes the structure of $(L^*)^N g$.

Definition 8.6 ([4, Definition A.1]). *1. The term g is of type $(A, 0)$. A term is of type (A, j) for some $j \geq 1$ if it is $\partial^j g/|\nabla\phi|^j$ where ∂^j is a derivative of order j.*

2. A term is of type $(B, 0)$ if it is equal to 1. A term is of type (B, j) for some $j \geq 1$ if it is of the form $\partial^{j+1}\phi/|\nabla\phi|^{j+1}$ where ∂^{j+1} is a derivative of order $j + 1$.

Let $N = 0, 1, 2, ...$ Then per [4, Lemma A.2], we can write

$$(L^*)^N g = \sum_{\nu=1}^{K(N,d)} c_{N,\nu} g_{N,\nu}.$$

Each $g_{N,\nu}$ is of the form

$$P\big(\tfrac{\nabla\phi}{|\nabla\phi|}\big)\alpha_A \prod_{m=1}^{M} \beta_m$$

where P is a polynomial of d variables (independent of g and ϕ), α_A is of type (A, j_A) for some $j_A \in \{0, ..., N\}$ and the terms β_m are of type (B, κ_m) so that $j_A + \sum_{m=1}^{M} \kappa_m = N$.

Our assumptions on the derivatives of ϕ and g imply that terms of type (A, j) and terms of type (B, j) are both bounded by $C_j D^j/E^j$. Then we have for each N, ν

$$|g_{N,\nu}| \leq \|P\|_{L^\infty(B_1(0))}\Big(\frac{D}{E}\Big)^N,$$

implying that

$$|(L^*)^N g| \leq C_{N,d}\Big(\frac{D}{E}\Big)^N.$$

□

8.2.1 The Proof of Proposition 8.4

To prove (8.4) we estimate the Schwartz kernel of $\mathcal{R}_{k,\ell}$,

$$R_{k,\ell}(x, y) = \chi(x, y) \int e^{2\pi i\tau\cdot(S(x,y_3)-y')}\chi_k(|\tau|)a_{k,\ell,\pm}(x, y_3, \tau)\, d\tau$$

by integrating by parts in the τ variables. For fixed x, y we integrate by parts in the distinguished directions $(\Delta^1(x,y_3), \Delta^2(x,y_3))$ and $(-\Delta^2(x,y_3), \Delta^1(x,y_3))$. Here our submersion assumption of ϖ comes into play, implying via Lemma 3.11 that both (Δ^1, Δ^2) and $(-\Delta^2, \Delta^1)$ are nonzero. Since $\tau \cdot (S(x,y_3) - y')$ is linear in τ,

$$\Big|\big(\Delta^1(x,y_3)\partial_{\tau_1} + \Delta^2(x,y_3)\partial_{\tau_2}\big)^j[\chi_k(|\tau|)]\Big| \leq C_j 2^{-kj}$$
$$\Big|\big(\Delta^1(x,y_3)\partial_{\tau_1} + \Delta^2(x,y_3)\partial_{\tau_2}\big)^j[a_{k,\ell,\pm}(x,y_3,\tau)]\Big| \leq C_j 2^{(\ell-k)j},$$

and

$$\Big|\big(-\Delta^2(x,y_3)\partial_{\tau_1} + \Delta^1(x,y_3)\partial_{\tau_2}\big)^j[\chi_k(|\tau|)]\Big| \leq C_j 2^{-kj}$$
$$\Big|\big(-\Delta^2(x,y_3)\partial_{\tau_1} + \Delta^1(x,y_3)\partial_{\tau_2}\big)^j[a_{k,\ell,\pm}(x,y_3,\tau)]\Big| = 0,$$

for any $j \geq 1$, we can apply Lemma 8.5 in the (Δ^1, Δ^2) direction and then the $(-\Delta^2, \Delta^1)$ direction to obtain

$$|R_{k,\ell}(x,y)| \leq C_N \big(2^{k-\ell}|\Delta^1(y_1 - S^1) + \Delta^2(y_2 - S^2)|\big)^{-N} \big(2^k|-\Delta^2(y_1 - S^1) + \Delta^2(y_2 - S^2)|\big)^{-N}.$$

On the other hand, for fixed x, y the symbol of $R_{k,\ell}(x,y)$ is supported in a rectangle which has length $2^{k-\ell}$ in the (Δ^1, Δ^2) direction and length 2^k in the $(-\Delta^2, \Delta^1)$ direction. This shows that $|R_{k,\ell}(x,y)| \leq C_N U_1(x,y) U_2(x,y)$, where

$$U_1(x,y) = \frac{2^{k-\ell}}{(1 + 2^{k-\ell}|\Delta^1(y_1 - S^1) + \Delta^2(y_2 - S^2)|)^N}$$
$$U_2(x,y) = \frac{2^k}{(1 + 2^k| - \Delta^2(y_1 - S^1) + \Delta^2(y_2 - S^2)|)^N}$$

We integrate the kernel in y' first, then over $y_3 \in I$, which is an interval of length $2^{-\ell}$. To prove (8.5) we apply the same argument, but integrate over a larger interval in y_3.

8.3 Recombining into L^p-Sobolev estimates

As in [44, 45, 46, 6], we prove Theorem 3.14 from Proposition 8.1 by estimating the Triebel-Lizorkin norm of

$$\mathcal{R}_\ell = \sum_{k \geq (2+\varepsilon)\ell} \mathcal{R}_{k,\ell}$$

First, we introduce the definitions of global and local Triebel-Lizorkin and Besov norms.

Definition 8.7 ([55], cf. [46, pp.33-34])**.** *For $k \in \mathbb{N}$ let P_k be standard Littlewood-Paley multipliers on $\mathbb{R}^d$. For $0 < p, q < \infty$ and $s \in \mathbb{R}$ the Triebel-Lizorkin norm $\|\cdot\|_{F^{p,q}_s(\mathbb{R}^d)}$ is given by*

$$\|f\|_{F^{p,q}_s(\mathbb{R}^d)} = \Big\| \Big(\sum_k |2^{ks} P_k f|^q \Big)^{1/p} \Big\|_{L^p(\mathbb{R}^d)}$$

and the Besov norm $\|\cdot\|_{B^{p,q}_s(\mathbb{R}^d)}$ is given by

$$\|f\|_{B^{p,q}_s(\mathbb{R}^d)} = \Big(\sum_k \|2^{ks} P_k f\|^q_{L^p(\mathbb{R}^d)} \Big)^{1/q}.$$

Given open sets $\Omega_L, \Omega_R \subset \mathbb{R}^d$ we say a linear operator T is bounded from $\big(B^{p_0,q_0}_{s_0}\big)_{\mathrm{comp}}(\Omega_R)$ to $\big(F^{p_1,q_1}_{s_1}\big)_{\mathrm{loc}}(\Omega_L)$ if for any $v_0 \in C^\infty_c(\Omega_L)$ we have for all $f \in B^{p_0,q_0}_{s_0}(\mathbb{R}^d)$ which are supported in a compact set $K \subset \Omega_R$

$$\|v_0 T f\|_{F^{p_1,q_1}_{s_1}(\mathbb{R}^d)} \leq C_p(v_0, K) \|f\|_{B^{p_0,q_0}_{s_0}(\mathbb{R}^d)}.$$

Proposition 8.1 implies the following local estimate on Triebel-Lizorkin and Besov spaces.

Proposition 8.8. *For $f \in C^\infty_c(\Omega_R)$ and $v_0 \in C^\infty_c(\Omega_L)$*

$$\|v_0 \mathcal{R}_\ell f\|_{F^{p,q}_{1/p}} \leq 2^{-\ell\varepsilon(p)} \|f\|_{B^{p,p}_0}, \qquad 0 < q \leq 2 < 4 < p < \infty. \tag{8.9}$$

The decay in ℓ allows us to sum in ℓ with $q \geq 1$, and conclude that

$$\mathcal{R} : \left(B_0^{p,p}\right)_{\text{comp}}(\Omega_R) \to \left(F_{1/p}^{p,q}\right)_{\text{loc}}(\Omega_L), \qquad q \leq 2 < 4 < p < \infty.$$

Since $L_s^p = F_s^{p,2} \hookrightarrow B_s^{p,p}$ for $p > 2$ and $F_{1/p}^{p,q} \hookrightarrow F_{1/p}^{p,2} = L_{1/p}^p$ for $q \leq 2$, this implies the asserted L^p-Sobolev bounds for $\mathcal{R}$.

We will prove Proposition 8.8 in Chapter 12 by applying [43, Theorem 1.1], a now-standard argument previously used in [44, 45, 46] and [6].

Chapter 9

L^2 Estimates for Oscillatory Integral Operators

In this chapter we prove an L^2-Sobolev estimate for a general class of oscillatory integral operators which implies Proposition 8.2. These oscillatory integral operators are related to FIOs associated to fibered folding canonical relations (see [23, § 2]), in particular the local Radon-like transforms considered in Theorem 3.14. As in §2.1, let X, Y be open sets in $\mathbb{R}^d$, and define for $k \in \mathbb{N}$

$$\mathcal{A}_k f(x) = \int e^{2\pi i 2^k \Phi(x,y)} f(y)\sigma(x,y)dy, \tag{9.1}$$

where $x \in X$, $y \in Y$, $\Phi \in C^\infty(X \times Y)$, and $\sigma \in C_0^\infty(X \times Y)$.

We define the canonical relation associated to an oscillatory integral operator of the form (9.1) to be

$$\mathfrak{C}_{\mathcal{A}} = \{(x, \Phi_x, y, -\Phi_y) \ : \ x \in X, \ y \in Y\} \subset T^*X \times T^*Y.$$

We will see this set is directly related to the canonical relation for related FIOs [25]; in particular $\mathfrak{C}_{\mathcal{A}}$ again has natural projections $\pi_L : \mathfrak{C}_{\mathcal{A}} \to T^*X$ and $\pi_R : \mathfrak{C}_{\mathcal{A}} \to T^*Y$ which we associate with the maps $(x,y) \mapsto (x, \Phi_x)$ and $(x,y) \mapsto (y, -\Phi_y)$ respectively. Let $h(x,y) = \det \Phi_{xy}$ and let $\mathcal{L}$ be the subset of $\mathfrak{C}_{\mathcal{A}}$ on which $h(x,y)$ vanishes. We assume that the only singularities of π_L are folds.

9.1 Reductions

By a partition of unity we may choose the support of σ small enough such that we can choose coordinate $x = (x', x_d)$, $y = (y', y_d)$ in $\mathbb{R}^{d-1} \times \mathbb{R}$ vanishing at a reference point $P^\circ = (x^\circ, y^\circ) \in X \times Y$ such that π_L is a fold at P°, and in these new coordinates

$$\Phi_{x'y'}(0,0) = I_{d-1} \tag{9.2}$$

$$\Phi_{x_d y'}(0,0) = 0 \tag{9.3}$$

$$\Phi_{x' y_d}(0,0) = 0 \tag{9.4}$$

$$\Phi_{x_d y_d}(0,0) = 0 \tag{9.5}$$

and for (x, y) in a small neighborhood of the origin

$$\max\{|\Phi_{x'y_d}(x,y)|, |\Phi_{x_d y'}(x,y)|\} < \varepsilon. \tag{9.6}$$

We present the proof of this statement from [23, § 2]. Let $e_1, ..., e_d$ denote the standard orthonormal basis vectors in $\mathbb{R}^d$. First, suppose that $0 \neq a \in \mathrm{coker}\Phi_{xy}(x^\circ, y^\circ)$ and that $0 \neq b \in \ker \Phi_{xy}(x^\circ, y^\circ)$. Set $\phi(x,y) = \Phi(x^\circ + B_1 x, y^\circ + B_2 y)$ where $B_1, B_2 \in GL(d, \mathbb{R})$ have the properties

$$B_1 e_d = a$$

$$B_2 e_d = b$$

$$B_2 e_j \perp \partial_y^2 \langle a, \Phi_x \rangle b, \qquad j = 1, ..., d-1.$$

The fold condition on π_L at P° implies that the quadratic form $\eta \mapsto \left\langle \partial_y^2 \langle a, \Phi_x \rangle \eta, \eta \right\rangle$ is nondegenerate on $\ker d\pi_L$, which in turn implies that B_2 can be made invertible. Clearly $e_d \in \mathrm{coker}\Phi_{xy}(0,0)$ and $e_d \in \ker \Phi_{xy}(0,0)$; this implies (9.3) and (9.4), which in turn implies (9.6) if we shrink the support of σ accordingly. The fold condition on π_L implies (9.2) and (9.5). Applying a linear change of variables we can thus assume our phase is $\phi(x, y)$ and σ is supported in a small neighborhood of the origin.

Let $\phi^{x'y'} = \phi^{-1}_{x'y'}$. Then using the construction from Lemma 2.4 we can define the kernel fields

$$V_R = \partial_{x_d} - \phi_{x_d y'}(\phi^{x'y'})^{\intercal}\partial_{x'}$$
$$V_L = \partial_{y_d} - \phi_{x' y_d}\phi^{x'y'}\partial_{y'}$$

for π_R and π_L respectively. The assumption on π_L implies that that there is a fixed constant $c_L > 0$ such that

$$h(x, y) = 0 \implies |V_L h(x, y)| \geq c_L > 0.$$

Note that if π_R is a blowdown V_R is tangent to the singularity surface $\mathcal{L}$, implying

$$h(x, y) = 0 \implies |V_R^j h(x, y)| = 0 \ \forall j \geq 0.$$

Note that (9.6) additionally implies

$$|(V_L - \partial_{y_d})h(x, y)| \leq \varepsilon \|\phi\|_{C^3}$$

for (x, y) in the support of σ.

Through the loss of a constant we may assume that $|h(x, y)| \leq 1$. Then we decompose dyadically in the size of $h(x, y)$, which in view of π_L having only fold singularities, is also a decomposition in the distance to $\mathcal{L}$. For $0 \leq \ell < \ell_\circ = \lfloor \frac{k}{2+\varepsilon} \rfloor$ let

$$\mathcal{A}_{k,\ell} f(x) := \int e^{2\pi i 2^k \phi(x,y)} f(y)\sigma(x, y)\chi_1(2^\ell h(x, y))dy,$$

and for $\ell = \ell_\circ$ define

$$\mathcal{A}_{k,\ell_\circ} f(x) := \int e^{2\pi i 2^k \phi(x,y)} f(y)\sigma(x,y)\chi_0(2^{\ell_\circ} h(x,y))dy.$$

We prove the following decay estimate.

Theorem 9.1. *Suppose that π_R is a blowdown on the set $\{h(x,y) = 0\}$. Then for all $k \geq 0$, all $\varepsilon > 0$ and all $0 \leq \ell \leq \ell_\circ$*

$$\|\mathcal{A}_{k,\ell} f\|_2 \leq C_\varepsilon 2^{\frac{\ell - dk}{2} + \ell\varepsilon} \|f\|_2. \tag{9.7}$$

9.2 Connections to Local Radon-like Transforms

For local Radon-like transforms we can derive L^2-Sobolev estimates directly from the rate of decay of associated oscillatory integral operators via the Fourier transform. Indeed, let $\mathcal{R}_{k,\ell}$ be defined as in (8.1)

$$\mathcal{R}_{k,\ell} f(x) = \int\int e^{2\pi i \tau \cdot (S(x,y_3) - y')} \chi(x,y) f(y)\, dy\, d\tau,$$

where $S \in C^\infty$ and $\chi \in C_c^\infty$. As seen in §3.4, the canonical relation associated to $\mathcal{R}_{k,\ell}$ is given by

$$\mathfrak{C} = \big\{ \big(x, \tau \cdot S_x(x,y_3), S(x,y_3), y_3, \tau, -\tau \cdot S_{y_3}(x,y_3)\big) \ : \ (x,y_3) \in \operatorname{supp}\chi,\ \tau \in \mathbb{R}^2 \big\},$$

We will assume for now that $\chi(x,y) = \tilde{\chi}(x,y_3)\eta_2(y')$. Then if we apply a partial Fourier transform to f in the y' variables we see that

$$\begin{aligned}
\mathcal{R}_{k,\ell}f(x) &= \int\int e^{2\pi i\tau\cdot(S(x,y_3)-y')}\tilde{\chi}(x,y_3)\eta_2(y')f(y)\chi_k(|\tau|)a_{k,\ell}(x,y_3,\tau)f(y)\,dy\,d\tau \\
&= \int\int e^{2\pi i\tau\cdot S(x,y_3)}\tilde{\chi}(x,y_3)\chi_k(|\tau|)a_{k,\ell}(x,y_3,\tau)\int e^{-2\pi i\tau\cdot y'}f(y',y_3)\eta_2(y')\,dy'\,dy_3\,d\tau \\
&= \int\int e^{2\pi i\tau\cdot S(x,y_3)}\tilde{\chi}(x,y_3)\chi_1(2^{1-k}|\tau|)a_{1,\ell}(x,y_3,2^{1-k}\tau)\mathfrak{F}_{y'}[f\eta_2](\tau,y_3)\,dy_3\,d\tau \\
&= 2^{2k}\int\int 2^{2\pi i 2^k\mu\cdot S(x,y_3)}\tilde{\chi}(x,y_3)\chi_1(2|\mu|)a_{1,\ell}(x,y_3,2\mu)\mathfrak{F}_{y'}[f\eta_2](\mu,y_3)\,dy_3\,d\mu,
\end{aligned}$$

with obvious modifications via the definition of $\mathcal{R}_{k,\ell}$ (8.1) if $k=0$ or $\ell = \ell_\circ = \lfloor \frac{k}{2+\varepsilon}\rfloor$. Thus $\mathcal{R}_{k,\ell}$ is directly related to the oscillatory integral operator $\mathcal{A}_{k,\ell}$ with $\phi(x,y) = y'\cdot S(x,y_3)$, $h(x,y) = y'\cdot\Delta(x,y_3)$, and $\sigma(x,y) = \tilde{\chi}(x,y_3)\chi_1(2|y'|)$. In addition, the canonical relation associated to $\mathcal{A}_{k,\ell}$ is given by

$$\big\{\big(x, y'\cdot S_x(x,y_3), y, -S(x,y_3), -y'\cdot S_{y_3}(x,y_3)\big) \;:\; (x,y_3)\in\operatorname{supp}\tilde{\chi},\; 2|y'|\in\chi_1\big\},$$

which by rearranging the coordinates is the subset of the canonical relation associated to $\mathcal{R}_{k,\ell}$ such that $|\tau|\simeq 1$. In particular, if $\mathcal{R}_{k,\ell}$ is associated to a fibered folding canonical relation, then so is the canonical relation associated to $\mathcal{A}_{k,\ell}$. Additionally, since $\|\mathfrak{F}_{y'}[f\eta_2]\|_{L^2} = \|f\eta_2\|_{L^2} \le C\|f\|_{L^2}$ by Plancherel we can prove Proposition 8.2 from Theorem 9.1 by this same argument.

To reduce to the case where $\chi(x,y) = \tilde{\chi}(x,y_3)\eta_2(y')$ we first assume by scaling and translating to the origin that χ is supported in $[\frac{1}{4},\frac{3}{4}]^6$. Then applying the Fourier inversion formula on the unit interval six times there exist constants $c_{r,s}$ for $r,s\in\mathbb{Z}^3$ such that for $x,y\in[0,1]^3$

$$\chi(x,y) = \sum_{r\in\mathbb{Z}^3}\sum_{s\in\mathbb{Z}^3} c_{r,s}e^{2\pi i\langle r,x\rangle}e^{2\pi i\langle s,y\rangle}$$

and for any $N > 0$

$$|c_{r,s}| \leq C_N(1 + |r| + |s|)^{-N}.$$

Let $\eta \in C_c^\infty(\mathbb{R}^4)$, and $\eta_1 \in C_c^\infty(\mathbb{R}^2)$ such that $\eta(x, y_3)\eta_1(y') = 1$ on the support of χ and is supported in the unit cube. Then $\chi(x, y) = \chi(x, y)\eta(x, y_3)\eta_1(y')$, and thus

$$\chi(x, y) = \sum_{r \in \mathbb{Z}^3} \sum_{s \in \mathbb{Z}^3} c_{r,s} e^{2\pi i(\langle r,x\rangle + s_3 y_3)} \eta(x, y_3) e^{2\pi i s' \cdot y'} \eta_1(y').$$

Fix $r, s \in \mathbb{Z}^3$ and let

$$\tilde{\chi}(x, y_3) = \tfrac{2}{3} c_{r,s} e^{2\pi i(\langle r,x\rangle + s_3 y_3} \eta(x, y_3)$$
$$\eta_2(y') = \tfrac{1}{3} c_{r,s} e^{2\pi i s' \cdot y'} \eta_1(y')$$

Each of these functions is smooth and compactly supported. Since the coefficients $c_{r,s}$ are rapidly decaying we can apply Theorem 9.1 for each choice of $r, s \in \mathbb{Z}^3$ then sum in r and s.

9.2.1 Connections to L^2-Sobolev Estimates of FIOs

The L^2-Sobolev estimates introduced in §3.1 are also typically deduced from estimates on oscillatory integral operators such as $\mathcal{A}_k$. In particular, [23, Theorem 2.1] states that if π_L has at most fold singularities

$$\|\mathcal{A}_k\|_{L^2 \to L^2} \lesssim 2^{\frac{k}{4} - \frac{dk}{2}}, \tag{9.8}$$

while [15, Theorem 1.2] states that if both π_L and π_R have at most fold singularities

$$\|\mathcal{A}_k\|_{L^2 \to L^2} \lesssim 2^{\frac{k}{6} - \frac{dk}{2}}. \tag{9.9}$$

However, as seen in Theorem 9.1, if the kernel of $\mathcal{A}_k$ is supported away from $\mathcal{L}$ the estimates on L^2 improve quantitatively, analogous to the nonisotropic Fourier decay of measures supported on curves with nonvanishing curvature and torsion, as in §3.2. Indeed, the estimate (9.7) is better than (9.8) until $\ell = \ell_\circ = \left\lfloor \frac{k}{2+\varepsilon} \right\rfloor$. The estimate (9.7) is also proven in [15] in the case of folding canonical relations, but for a reduced range of ℓ. The reason for this reduction is that (9.7) matches the uniform bound (9.9) when $\ell = k/3$ in the case of folding canonical relations, whereas (9.7) doesn't equal the uniform bound until $\ell = \ell_\circ$. This increased range in ℓ makes the estimates harder to prove than in the case of folding canonical relations. Additionally, our proof of (9.7) relies on the uniform bound (9.8) for the case $\ell = \ell_\circ$, so it cannot be used to reprove (9.8).

9.3 The Proof of Theorem 9.1

Let $\varepsilon > 0$. We note that by global estimates of $\mathcal{A}_k$ proven in [23], if Theorem 9.1 holds for $\ell < \ell_\circ$ then it also holds for $\ell = \ell_\circ$. Indeed, since $\mathcal{A}_k = \sum_{C_2 \leq \ell \leq \ell_\circ} \mathcal{A}_{k,\ell} = \sum_{C_2 \leq \ell < \frac{k}{2+\varepsilon}} \mathcal{A}_{k,\ell} + \mathcal{A}_{k,\ell_\circ}$, we can estimate by (9.7)

$$\begin{aligned} \left\| \mathcal{A}_{k,\ell_\circ} \right\|_{L^2 \to L^2} &\leq \|\mathcal{A}_k\|_{L^2 \to L^2} + \sum_{C_2 \leq \ell < \ell_\circ} \|\mathcal{A}_{k,\ell}\|_{L^2 \to L^2} \\ &\lesssim 2^{\frac{k}{4} - \frac{dk}{2}} + C_\varepsilon \sum_{C_2 \leq \ell < \ell_\circ} 2^{\frac{\ell - dk}{2} + \ell\varepsilon} \\ &\leq C_\varepsilon \big(2^{\frac{k}{4} - \frac{dk}{2}} + 2^{\frac{\ell_\circ - dk}{2} + \varepsilon \ell_\circ} \big). \end{aligned}$$

Since $k \leq (2+\varepsilon)\ell_\circ$, we see $\frac{k}{4} \leq \frac{\ell_\circ}{2} + \varepsilon \frac{\ell_\circ}{2}$, hence

$$\|\mathcal{A}_{k,\ell_\circ} f\|_2 \lesssim 2^{\frac{\ell_\circ - dk}{2} + \ell_\circ \varepsilon} \|f\|_2,$$

proving Theorem 9.1 for this case.

In the rest of this section we assume that $\ell < \ell_\circ = \lfloor \frac{k}{2+\varepsilon} \rfloor$. In this range we decompose the support of $\sigma(x,y)$ using methods of the proof of the Calderon-Vaillancourt theorem on the L^2 boundedness of pseudodifferential operators [9], following the ideas of Phong and Stein [42], Comech [14], and Cuccagna [18].

Let $m \in \mathbb{Z}^d$, $n_d \in \mathbb{Z}$, and let $\psi \in C_c^\infty(\mathbb{R})$ supported in $[-2,2]$ such that $0 \le \psi \le 1$ and $\sum_{k\in\mathbb{Z}} \psi(\cdot - k) = 1$. We decompose the support of $\sigma(x,y)$ along $2^{-\ell}$ diameter boxes in y-space, by way of smooth cutoffs

$$\psi_m(y) = \prod_{j=1}^{d} \psi(2^\ell y_j - m_j)).$$

We also decompose the support of $\sigma(x,y)$ in x_d into much larger $2^{-\ell\varepsilon}$ length intervals with smooth cutoffs $\tilde{\psi}_{n_d}(x_d) = \psi(2^{\ell\varepsilon} x_d - n_d)$. Because of the flatness in the x_d direction introduced by the blowdown condition on π_R, we will not show orthogonality in the x_d decomposition, instead summing in n_d loss of a large (but controlled) constant depending on ε. This loss (as well as our restricted range of $\ell < \ell_\circ$) introduce constants $2^{\ell\varepsilon}$ in our estimates which are too large to prove endpoint L^2-Sobolev estimates with this method of proof.

We fix k, ℓ for now and let $\mathcal{A}_{k,\ell}^{m,n_d} := \mathcal{A}_{k,\ell}[\psi_m(y)\tilde{\psi}_{n_d}(x_d)\cdot]$. Then $\mathcal{A}_{k,\ell}^{m,n_d}\big(\mathcal{A}_{k,\ell}^{\tilde{m},n_d}\big)^*$ has Schwartz kernel

$$K^{\mathcal{A}\mathcal{A}^*}_{m,\tilde{m},n_d}(x,w) = \int e^{2\pi i 2^k(\phi(x,y)-\phi(w,y))} \sigma_{m,\tilde{m},n_d}(x,w,y)\, dy,$$

where

$$\begin{aligned}\sigma_{m,\tilde{m},n_d}(x,w,y) &= \sigma(x,y)\chi_1(2^\ell h(x,y))\psi_m(y)\tilde{\psi}_{n_d}(x_d) \\ &\qquad \times \overline{\sigma(w,y)\chi_1(2^\ell h(w,y))\psi_{\tilde{m}}(y)\tilde{\psi}_{n_d}(w_d)}.\end{aligned}$$

Similarly, the Schwartz kernel for $\left(\mathcal{A}_{k,\ell}^{m,n}\right)^*\mathcal{A}_{k,\ell}^{\tilde{m},n}$ is given by

$$K_{m,\tilde{m},n}^{\mathcal{A}^*\mathcal{A}}(y,z) = \int e^{2\pi i 2^k(\phi(x,y)-\phi(x,z))}\tilde{\sigma}_{m,\tilde{m},n}(x,y,z)\,dx,$$

where

$$\begin{aligned}\tilde{\sigma}_{m,\tilde{m},n}(x,y,z) = \sigma(x,y)\chi_1(2^\ell h(x,y))\psi_m(y)|\tilde{\psi}_{n_d}(x_d)|^2 \\ \times \overline{\chi_1(2^\ell h(x,z)\sigma(x,z)\psi_{\tilde{m}}(z)}.\end{aligned}$$

By splitting $\mathcal{A}_{k,\ell}$ into a finite number of collections of $\{\mathcal{A}_{k,\ell}^{m,n_d}\}$ we may assume that if $m_j \neq \tilde{m}_j$ then $|m_j - \tilde{m}_j| > \max\{\frac{15}{c_L}, 2\sqrt{d}\}$. We first prove two lemmas.

Lemma 9.2. *There exists a constant $C > 0$ such that for every $m \in \mathbb{Z}^d, n_d \in \mathbb{Z}$*

$$\|\mathcal{A}_{k,\ell}^{m,n_d}\|_{L^2\to L^2} \leq C2^{\frac{\ell-dk}{2}}.$$

Lemma 9.3. *For every $N > 0$ and every $n_d \in \mathbb{Z}$ the following estimates hold.*

(a) If $m \neq \tilde{m}$ then

$$\|\mathcal{A}_{k,\ell}^{m,n_d}\left(\mathcal{A}_{k,\ell}^{\tilde{m},n_d}\right)^*\| = 0.$$

(b) If $m \neq \tilde{m}$ and $|m' - \tilde{m}'| \leq \frac{c_L}{10\|\phi\|_{C^3}}|m_d - \tilde{m}_d|$ then

$$\|\left(\mathcal{A}_{k,\ell}^{m,n_d}\right)^*\mathcal{A}_{k,\ell}^{\tilde{m},n_d}\| = 0.$$

(c) If $m \neq \tilde{m}$ and $|m' - \tilde{m}'| \geq \frac{c_L}{10\|\phi\|_{C^3}}|m_d - \tilde{m}_d|$ then

$$\|\left(\mathcal{A}_{k,\ell}^{m,n}\right)^*\mathcal{A}_{k,\ell}^{\tilde{m},n}\| \lesssim_N 2^{\ell-dk}\left(2^{k-2\ell}|m-\tilde{m}|\right)^{-N}.$$

We state a few remarks. First, the estimates in Lemma 9.3 rely on neither the decomposition of the support in x_d nor the blowdown assumption; they essentially reprove results of Comech in [14], albeit through different approaches. Second, the separation of ℓ from $k/2$ is only necessary for part (c) of Lemma 9.3; in particular, it is not needed to prove Lemma 9.2.

We prove Theorem 9.1 using the Cotlar-Stein Lemma.

Lemma 9.4 (Cotlar-Stein Lemma). *Consider a family of operators T_j, $j \in \mathbb{N}$, such that T_j is a bounded linear operator on $L^2(\mathbb{R}^d)$. Define*

$$a_{j,k} = \|T_j T_k^*\|_2, \qquad b_{j,k} = \|T_j^* T_k\|_2.$$

*We say that the family of operators $\{T_j\}_{j\in\mathbb{N}}$ is **almost orthogonal** if*

$$A = \sup_j \sum_k \sqrt{a_{j,k}} < \infty \qquad B = \sup_j \sum_k \sqrt{b_{j,k}} < \infty.$$

If T_j are almost orthogonal then

$$\Big\| \sum_j T_j \Big\|_2 \leq \sqrt{AB}.$$

Lemmas 9.2 and 9.3 prove that for each n_d the family of operators $\{\mathcal{A}_{k,\ell}^{m,n_d}\}_{m\in\mathbb{Z}^d}$ are almost orthogonal, and by choosing N large enough (depending on ε) we can ensure that $A \leq C2^{\frac{\ell-dk}{2}}$ and $B \leq C_\varepsilon 2^{\frac{\ell-dk}{2}+\ell\varepsilon}$ uniformly in n_d. Additionally, since σ is compactly supported inside the unit ball, we see that $\mathcal{A}_{k,\ell}^{m,n_d} f(x) = 0$ for $|n_d| \geq C2^{\ell\varepsilon}$. Thus

$$\|\mathcal{A}_{k,\ell}\|_{L^2\to L^2} \leq C2^{\ell\varepsilon} \sup_{|n_d|\leq C2^{\ell\varepsilon}} \Big\| \sum_{m\in\mathbb{Z}^d} \mathcal{A}_{k,\ell}^{m,n_d} \Big\|_{L^2\to L^2},$$

which by an application of the Cotlar-Stein Lemma, gives the desired result.

9.3.1 The Proof of Lemma 9.2: Individual Box Estimates

We prove this lemma by noting that $\|\mathcal{A}_{k,\ell}^{m,n_d}\|_{L^2\to L^2}^2 = \|\mathcal{A}_{k,\ell}^{m,n_d}(\mathcal{A}_{k,\ell}^{m,n_d})^*\|_{L^2\to L^2}$ and estimating the Schwartz kernel $K_{m,m,n_d}^{\mathcal{A},\mathcal{A}^*}(x,w)$ uniformly in m and n_d. We wish to apply the differential operator

$$L_y = \Big\langle \frac{\nabla_y\big(\phi(x,y)-\phi(w,y)\big)}{\big|\nabla_y\big(\phi(x,y)-\phi(w,y)\big)\big|^2}, \nabla_y \cdot \Big\rangle$$

using Lemma 8.5 to integrate by parts many times in the y variables, which requires us to find a lower bound for $\nabla_y(\phi(x,y)-\phi(w,y))$. We investigate estimates for $\nabla_{y'}(\phi(x,y)-\phi(w,y))$ and $\partial_{y_d}(\phi(x,y)-\phi(w,y))$ separately.

Since $|\phi_{x'y'}|>c>0$ the set of equations $\nabla_{y'}(\phi(x,y)-\phi(w,y))=0$ is solved uniquely by $x'=\mathfrak{x}'(w,x_d,y)$. By the implicit function theorem we see that

$$\frac{1}{4}|x'-\mathfrak{x}'(w,x_d,y)| \le |\phi_{y'}(x,y)-\phi_{y'}(w,y)| \le 4|x'-\mathfrak{x}'(w,x_d,y)|.$$

This implicit function also helps us to find a lower bound for $\phi_{y_d}(x,y)-\phi_{y_d}(w,y)$. Note that

$$\mathfrak{x}'(w,w_d,y)=w', \tag{9.10}$$

and by the implicit function theorem,

$$\begin{aligned}\partial_{x_d}\mathfrak{x}'(w,x_d,y) &= -(\phi^{x'y'})^\intercal(\mathfrak{x}'(w,x_d,y),x_d,y)(\phi_{x_dy'})^\intercal(\mathfrak{x}'(w,x_d,y),x_d,y)\\ &= -\big(\phi_{x_dy'}\phi^{x'y'}\big)^\intercal(\mathfrak{x}'(w,x_d,y),x_d,y).\end{aligned} \tag{9.11}$$

These two statements imply that

$$\begin{aligned}\partial_{x_d}[\phi_{y_d}(\mathfrak{x}'(w,x_d,y),x_d,y)]\big|_{x_d=w_d} &= \phi_{x_dy_d}(w,y)-\phi_{y_dx'}(w,y)\big(\phi_{x_dy'}\phi^{x'y'}\big)^\intercal(w,y)\\ &= \phi_{x_dy_d}(w,y)-\phi_{x_dy'}\phi^{x'y'}\phi_{x'y_d}(w,y)\\ &= h(w,y)\det\phi^{x'y'}(w,y).\end{aligned} \tag{9.12}$$

Furthermore, the definition of $\mathfrak{x}'$ allows us to exchange ∂_{x_d} for V_R.

Lemma 9.5. *Suppose $g \in C^1(\mathbb{R}^d \times \mathbb{R}^d)$. Then we have the identity*

$$\partial_{x_d}[g(\mathfrak{x}'(w, x_d, y), x_d, y)]\Big|_{x_d = w_d} = V_R[g](w, y).$$

Proof. Applying chain rule,

$$\partial_{x_d}[g(\mathfrak{x}'(w, x_d, y), x_d, y)] = g_{x_d}(\mathfrak{x}'(w, x_d, y), x_d, y) + \langle g_{x'}(\mathfrak{x}'(w, x_d, y), x_d, y), \partial_{x_d}\mathfrak{x}'(w, x_d, y)\rangle.$$

Applying (9.11) and evaluating at $x_d = w_d$ we obtain

$$\begin{aligned} \partial_{x_d}[g(\mathfrak{x}'(w, x_d, y), x_d, y)]\big|_{x_d = w_d} &= g_{x_d}(w, y) - \big\langle \big(\phi_{x_d y'}\phi^{x'y'}(w, y)\big), g_{x'}(w, y)\big\rangle \\ &= V_R[g](w, y). \end{aligned}$$

□

Using Lemma 9.5 along with (9.10) and (9.12), we can apply a Taylor expansion about $x_d = w_d$ to $\phi_{y_d}(\mathfrak{x}'(w, x_d, y), x_d, y) - \phi_{y_d}(w, y)$ to obtain

$$\begin{aligned} \phi_{y_d}(\mathfrak{x}'(w, x_d, y), x_d, y) - \phi_{y_d}(w, y) = \sum_{j=0}^{N} V_R^j[h \det \phi^{x'y'}](w, y)\frac{(x_d - w_d)^{j+1}}{(j+1)!} \\ + E(w, x_d, y)(x_d - w_d)^{N+2}, \end{aligned}$$

where $E \in C^\infty$ has bounded derivatives independent of k and ℓ, and $N \simeq \frac{1}{\varepsilon}$ is large enough that $|x_d - w_d|^{N+1} \leq C2^{-\ell}$.

Since π_L is a fold and $V_L\big|_{(0,0)} = \partial_{y_d}$, we see that $h(w, y) = 0$ is solved uniquely by $y_d = \mathfrak{y}_d(w, y')$ near 0. Again from the implicit function theorem,

$$\frac{1}{4}|y_d - \mathfrak{y}_d(w, y')| \leq |h(w, y)| \leq 4|y_d - \mathfrak{y}_d(w, y')|.$$

Because π_R is a blowdown and $|h(w,y)| \simeq 2^{-\ell}$ on the support of σ_{m,m,n_d} we see by an application of Taylor's theorem that

$$|V_R^j h(w,y)| = |V_R^j h(w,y',\mathfrak{y}_d(w,y')) + (y_d - \mathfrak{y}_d(w,y'))\partial_{y_d} V_R^j h(w,y',z_d)| \leq C2^{-\ell}$$

implying by the properties of differentiation of products

$$|\phi_{y_d}(\mathfrak{x}'(w,x_d,y),x_d,y) - \phi_{y_d}(w,y)| \geq c2^{-\ell}|x_d - w_d|.$$

On the other hand, we know that

$$|\phi_{y_d}(x,y) - \phi_{y_d}(\mathfrak{x}'(w,x_d,y),x_d,y)| \leq \|\phi_{x'y_d}\|\big|x' - \mathfrak{x}'(w,x_d,y)\big| \lesssim \varepsilon|x' - \mathfrak{x}'(w,x_d,y)|.$$

Thus by the reverse triangle inequality

$$\begin{aligned}|\phi_{y_d}(x,y) - \phi_{y_d}(w,y)| \geq \Big||\phi_{y_d}(x,y) &- \phi_{y_d}(\mathfrak{x}'(w,x_d,y),x_d,y)| \\ &- |\phi_{y_d}(w,y) - \phi_{y_d}(\mathfrak{x}'(w,x_d,y),x_d,y)|\Big|,\end{aligned}$$

and therefore,

$$\big|\nabla_y\big(\phi(x,y) - \phi(w,y)\big)\big| \geq C\max\big\{2^{-\ell}|x_d - w_d|, |x' - \mathfrak{x}'(w,x_d,y)|\big\}.$$

With these estimates in place we apply Lemma 8.5 to L_y, noting that for any multi-index α

$$|\partial_y^\alpha \sigma_{m,m,n_d}| \leq C_{|\alpha|}2^{\ell|\alpha|}$$

and that for $|\alpha| > 1$,

$$|\partial_y^\alpha \phi| \leq C_{|\alpha|}|x - w|.$$

As $|\sigma_{m,m,n_d}| \lesssim 1$ and is supported (in y) on the set $\{y \ : \ |2^\ell y - m| \leq 1\}$ and noting that $(1 + \max\{A,B\})^2 \geq (1+A)(1+B)$ for any $A, B \geq 0$ we obtain

$$|K_{m,m,n_d}^{\mathcal{A}\mathcal{A}^*}(x,w)| \lesssim_N \int_{|2^\ell y - m| \leq 1} \frac{1}{(1 + 2^{k-\ell}|x' - \mathfrak{x}'(w,x_d,y)|)^N} \frac{1}{(1 + 2^{k-2\ell}|x_d - w_d|)^N}\, dy.$$

Integrating in x we see that

$$\begin{aligned}\int |K^{\mathcal{A}\mathcal{A}^*}_{m,m,n_d}(x,w)|\ dx &\lesssim_{N,\varepsilon} \int \frac{1}{(1+2^{k-2\ell}|x_d-w_d|)^N}\ dx_d \\ &\qquad \times \sup_{x_d,y} 2^{-d\ell} \int \frac{1}{(1+2^{k-\ell}|x'-\mathfrak{x}'(w,x_d,y)|)^N}\, dx' \\ &\lesssim 2^{2\ell-k}2^{-d\ell}2^{(d-1)(\ell-k)} \\ &\lesssim 2^{\ell-dk}.\end{aligned}$$

Repeating the entire argument switching the roles of x and w yields the same estimate for $\int |K^{\mathcal{A}\mathcal{A}^*}_{m,m,n_d}(x,w)|\, dw$ uniformly in x. Thus the lemma follows by Schur's test.

9.3.2 The Proof of Lemma 9.3: Almost Orthogonality Estimates

Part (a) follows immediately since the supports of $\psi_m(y)$ and $\psi_{\tilde{m}}(y)$ are disjoint when m and $\tilde{m}$ are sufficiently separated.

The kernel $K^{\mathcal{A}^*\mathcal{A}}_{m,\tilde{m},n_d}(y,z)$ vanishes under the assumption in (b) because π_L has a fold singularity on $\mathcal{L}$. To see why, note that since $|h(x,y)|$ and $|h(x,z)|$ are both bounded above by $2^{-\ell+1}$, their sum is bounded by $2^{-\ell+3}$. Expanding the difference about $y=z$

we see

$$\begin{aligned}
h(x,y) - h(x,z) &= (y_d - z_d)\partial_{y_d} h(x,z) \\
&\qquad + (y' - z') \cdot \nabla_{y'} h(x,z) + R_1(x,y,z)|y-z|^2 \\
&= (y_d - z_d)[\partial_{y_d} - V_L] h(x,z) \\
&\qquad + (y_d - z_d) V_L h(x,z) \\
&\qquad + (y' - z') \cdot \nabla_{y'} h(x,z) + R_2(x,y,z)|y-z|^2 \\
|h(x,y) - h(x,z)| &\geq \frac{c_L}{3} |y_d - z_d| \\
&\geq 5(2^{-\ell}),
\end{aligned}$$

where R_1, R_2 are C^∞ with derivatives uniformly bounded independently of k and ℓ. We see there are no y, z that satisfy these conditions, hence

$$\chi_1(2^\ell h(x,y))\overline{\chi_1(2^\ell h(x,z))} = 0.$$

To prove (c) we will apply Lemma 8.5 with the differential operator

$$L_{x'} = \Big\langle \frac{\nabla_{x'}\big(\phi(x,y) - \phi(x,z)\big)}{\big|\nabla_{x'}\big(\phi(x,y) - \phi(x,z)\big)\big|^2}, \nabla_{x'} \Big\rangle.$$

To do so requires (among other things) a lower bound on $\nabla_{x'}\big(\phi(x,y) - \phi(w,y)\big)$. Using a Taylor approximation we see

$$\nabla_{x'}[\phi(x,y) - \phi(x,z)] = \phi_{x'y_d}(x,z)(y_d - z_d) + \phi_{x'y'}(y' - z') + R_3(x,y,z)|y-z|^2, \quad (9.13)$$

where $R_3 \in C^\infty$ has uniformly bounded derivatives independent of k and ℓ. Since $|\det \phi_{x'y'}| \geq c > 0$ we see that $|\phi_{x'y'}(x,z)(y'-z')| \geq C_d|y'-z'|$, and $|\phi_{x'y_d}(x,z)(y_d-z_d)| \lesssim \varepsilon|y_d - z_d|$. By assumption

$$|y' - z'| \geq \tfrac{c_L}{3\|\phi\|_{C^3}} |y_d - z_d| \geq \tfrac{10\varepsilon}{C_d} |y_d - z_d|.$$

Thus

$$|\nabla_{x'}[\phi(x,y)-\phi(x,z)]| \geq c|y-z|$$

for some small constant $c>0$ independent of k,ℓ, or ε. We also have for each multi-index α

$$|\partial_{x'}^{j}\tilde{\sigma}_{m,\tilde{m},n_d}(x,y,z)| \leq C_j 2^{\ell j}$$

and

$$|\partial_{x'}^{j}\big(\phi(x,y)-\phi(x,z)\big)| \leq C_j|y-z|$$

Thus we can apply Lemma 8.5 with $L_{x'}$ to obtain

$$|K_{m,\tilde{m},n_d}^{\mathcal{A}^*\mathcal{A}}(y,z)| \leq C_N \int \frac{1}{(2^{k-\ell}|y-z|)^N} \mathbb{1}_{\operatorname{supp}\tilde{\sigma}_{m,\tilde{m},n_d}}\, dx.$$

Since $k>2\ell$ and $|y-z|\simeq 2^{-\ell}|m-\tilde{m}|\geq 2^{-\ell}$ on the support on $\tilde{\sigma}$,

$$\frac{1}{2^{k-\ell}|y-z|} \leq \min\left\{\frac{C}{1+2^{k-\ell}|y-z|},\ \frac{C}{2^{k-2\ell}|m-\tilde{m}|}\right\}.$$

Integrating in y (or z)

$$\begin{aligned}\int |K_{m,\tilde{m},n_d}^{\mathcal{A}^*\mathcal{A}}(y,z)|\, dy &\leq C_{N,d}\int \frac{1}{(1+2^{k-\ell}|y-z|)^{d+1}}\frac{1}{(2^{k-2\ell}|m-\tilde{m}|)^N}\, dy \\ &\leq C_{N,d}2^{d(\ell-k)}(2^{k-2\ell}|m-\tilde{m}|)^{-N}.\end{aligned}$$

Since $k-2\ell>\ell\varepsilon$, if we let $N=\frac{d-1}{\varepsilon}$ then by Schur's test

$$\|\mathcal{A}_{m,n_d}^{*}\mathcal{A}_{\tilde{m},n_d}\|_{2\to 2} \leq C(\varepsilon,d)2^{(\ell-dk)}|m-\tilde{m}|^{-N},$$

proving part (c) of Lemma 9.3.

Chapter 10

Decoupling in a Model Case

In this chapter we begin the proof of Proposition 8.3. The decoupling inequality is based on the idea that the Fourier support of $\mathcal{R}_{k,\ell}$ should be concentrated in a neighborhood of a cone related to the fibers of $\pi_L(\mathcal{L})$. In the case of translation invariant local Radon-like transforms, such as $\mathcal{A}_{\mathbb{R}}$ and $\mathcal{X}$ (see (3.2) and (4.1) respectively), this concentration actually occurs because the fibers of $\pi_L(\mathcal{L})$ are fixed for each x [44, 45]. In general the fibers vary with x and this argument is no longer possible. To explain how to work around this obstruction we return to our model case on the Heisenberg group.

10.1 A First Example

Let us consider $\mathcal{A}_{\mathbb{H}}$ (see §4.2) with $\gamma(t) = (t, t^2, \frac{1}{6}t^3)$ the moment curve. We additionally assume that $\chi(x_1, y_1) = \chi(x_1)\tilde{\chi}(y_1)$ as an example. Then

$$\left(\mathcal{A}_{\mathbb{H}}\right)_{k,\ell} f(x) = \chi(x_1) \int\int e^{2\pi i\tau\cdot(x'-y'+S(x_1,y_1))}\tilde{\chi}(y_1)\chi_k(|\tau|)a_{k,\ell}(x_1,y_1,\tau)f(y)\,dy\,d\tau,$$

where as in (4.12) and (4.13),

$$S^1(x_1,y_1) = -(x_1-y_1)^2 = -x_1^2 + 2x_1y_1 - y_1^2$$

$$S^2(x_1,y_1) = y_1(x_1-y_1)^2 + \tfrac{1}{3}(x_1-y_1)^3 = \tfrac{1}{3}x_1^3 + \tfrac{2}{3}y_1^3 - x_1y_1^2.$$

Recall from (3.10) the fibers of $\pi_L(\mathcal{L})$ associated to $\mathcal{A}_{\mathbb{H}}$ are given by

$$\Big\{\rho\begin{pmatrix} (x_1-y_1)^2 \\ y_1 \\ 1 \end{pmatrix} \;:\; \rho\in\mathbb{R}, y_1\in \operatorname{supp}\chi\Big\},$$

which vary with x_1. However, through a nonlinear change of variables

$$\sigma(x) = \big(\sigma_1(x),\sigma_2(x),\sigma_3(x)\big) = \big(x_1, x_2+x_1^2, x_3-\tfrac{1}{3}x_1^3\big)$$

and

$$\eta(y) = \big(\eta_1(y),\eta_2(y),\eta_3(y),\big) = \big(y_1, y_2+y_1^2, y_3+\tfrac{2}{3}y_1^3\big)$$

we are able to freeze the fibers so they no longer vary with x. Note that the Jacobians of each of these maps is 1. Incorporating the above changes of variables

$$\big(\mathcal{A}_{\mathbb{H}}\big)_{k,\ell}(f\circ\eta)(\sigma(x)) = \chi(x_1)\int\!\!\int\!\!\int e^{2\pi i\tau\cdot(x'-y'-\tilde{S}(x_1,y_1))}\chi_k(|\tau|)a_{k,\ell}(y_1,\tau)\tilde{\chi}(y_1)f(y)\,dy\,d\tau,$$

where $\tilde{S}(x_1,y_1) = (\tilde{S}^2,\tilde{S}^3) = (-2x_1y_1, x_1y_1^2)$. After these changes of variables, $(N^*\mathcal{M})'$ is given by

$$\big\{(x,\xi,y,-\eta) \;:\; \xi_1 = \tau\cdot\tilde{S}_{x_1},\ \eta_1 = \tau\cdot\tilde{S}_{y_1},\ y' = x'-\tilde{S}(x_1,y_1),\ \xi'=\eta'=\tau\big\},$$

and $\mathcal{L}$ consists of points in $(N^*\mathcal{M})'$ such that $\tau\cdot\tilde{S}_{x_1y_1}(x_1,y_1) = -2\tau_2+2y_1\tau_3 = 0$. The fibers of $\pi_L(\mathcal{L})$, defined in (3.10), are thus given by

$$\tilde{\Sigma}_x = \{\xi\in\mathbb{R}^3 \;:\; \xi = \rho(y_1^2, y_1, 1), \rho\in\mathbb{R}\},$$

which no longer varies with x. As such, we can prove that the Fourier transform of $\big(\mathcal{A}_{\mathbb{H}}\big)_{k,\ell}f(x)$ essentially lies in a $2^{k-2\ell}$ neighborhood of Σ_x. Thus we can apply Theorem 3.9 directly to $\big(\mathcal{A}_{\mathbb{H}}\big)_{k,\ell}f(x)$ to decouple down to plates adapted to that $2^{k-2\ell}$-neighborhood.

Indeed, applying a Fourier transform we can rearrange the order of integration

$$\begin{aligned}\mathfrak{F}\Big[\big(\mathcal{A}_{\mathbb{H}}\big)_{k,\ell}f\Big](\xi) &= \int e^{-2\pi i\langle x,\xi\rangle}\chi(x_1)\int\int e^{2\pi i\tau\cdot(x'-y'-\tilde{S}(x_1,y_1))}f(y)\\ &\qquad\times \chi_k(|\tau|)\chi_1(2^{\ell-k+2}|\tau_3 y_1-\tau_2|)\tilde{\chi}(y_1)\,dy\,d\tau\,dx\\ &= \int e^{-2\pi i x_1\xi_1}\chi(x_1)\int \tilde{\chi}(y_1)f(y)\Big(\int\int e^{-2\pi i x'\cdot(\xi'-\tau)}\\ &\qquad\times e^{-2\pi i\tau\cdot(y'+\tilde{S}(x_1,y_1))}\chi_k(|\tau|)\chi_1(2^{\ell-k+2}|\tau_3 y_1-\tau_2|)\,d\tau\,dx'\Big)\,dy\,dx_1.\end{aligned}$$

Evaluating the innermost two integrals applies Fourier and inverse Fourier transforms in x' and τ respectively to obtain

$$\begin{aligned}\mathfrak{F}\Big[\big(\mathcal{A}_{\mathbb{H}}\big)_{k,\ell}f\Big](\xi_1,\xi') &= \int e^{-2\pi i x_1\xi_1}\chi(x_1)\int e^{-2\pi i\xi'\cdot(y'+\tilde{S}(x_1,y_1))}\tilde{\chi}(y_1)f(y)\\ &\qquad\times \chi_k(|\xi'|)\chi_1(2^{\ell-k+2}|\xi_3 y_1-\xi_2|)\,dy\,dx_1.\end{aligned}$$

Evaluating the Fourier transform in y' we obtain

$$\begin{aligned}\mathfrak{F}\Big[\big(\mathcal{A}_{\mathbb{H}}\big)_{k,\ell}f\Big](\xi_1,\xi') &= \int e^{-2\pi i x_1\xi_1}\chi(x_1)\int e^{-2\pi i\xi'\cdot\tilde{S}(x_1,y_1)}\tilde{\chi}(y_1)\chi_k(|\xi'|)\chi_1(2^{\ell-k+2}|\xi_3 y_1-\xi_2|)\\ &\qquad\times\int e^{-2\pi i\xi'\cdot y'}f(y)\,dy'\,dy_1\,dx_1\\ &= \int e^{-2\pi i x_1\xi_1}\chi(x_1)\int e^{-2\pi i\xi'\cdot\tilde{S}(x_1,y_1)}\tilde{\chi}(y_1)\chi_k(|\xi'|)\chi_1(2^{\ell-k+2}|\xi_3 y_1-\xi_2|)\\ &\qquad\times\mathfrak{F}_{2,3}\big[f(y_1,\cdot)\big](\xi')\,dy_1\,dx_1.\end{aligned}$$

Note that $\tilde{S}(x_1,y_1)$ is linear is x_1, implying that

$$e^{-2\pi i\xi'\cdot\tilde{S}(x_1,y_1)} = e^{2\pi i x_1(\xi_3 y_1^2-2\xi_2 y_1)}.$$

Finally, interchanging the order of integration and evaluating the integral in x_1 we obtain

$$\begin{aligned}\mathfrak{F}\Big[\big(\mathcal{A}_{\mathbb{H}}\big)_{k,\ell}f\Big](\xi_1,\xi') &= \int \tilde{\chi}(y_1)\chi_k(|\xi'|)\chi_1(2^{\ell-k+2}|\xi_3 y_1-\xi_2|)\mathfrak{F}_{2,3}\big[f(y_1,\cdot)\big](\xi') \\ &\qquad \times \Big(\int e^{-2\pi i x_1(\xi_1+\xi_3 y_1^2-2\xi_2 y_1)}\chi(x_1)\,dx_1\Big)\,dy_1 \\ &= \int \hat{\chi}(\xi_1-2\xi_2 y_1+\xi_3 y_1^2)\tilde{\chi}(y_1)\chi_k(|\xi'|)\chi_1(2^{\ell-k+2}|\xi_3 y_1-\xi_2|) \\ &\qquad \times \mathfrak{F}_{2,3}\big[f(y_1,\cdot)\big](\xi')\,dy_1. \end{aligned} \tag{10.1}$$

In view of the smoothness of χ, $\hat{\chi}(\xi_1-2\xi_2y_1+\xi_3y_1^2)$ decays rapidly off of the set $\{|\xi_1 - 2\xi_2y_1+\xi_3y_1^2| \simeq 1\}$. By an error term argument [44, p. 11] we may replace $\hat{\chi}(\cdot)$ with $\chi_0(2^{k-2\ell(1-\varepsilon)}|\cdot|)\hat{\chi}(\cdot)$ so that integrand of (10.1) vanishes unless

$$\begin{aligned} |\xi| &\simeq 2^k \\ \Big|\Big\langle \tfrac{\xi}{|\xi|}, \Big(\begin{smallmatrix}0\\1\\-y_1\end{smallmatrix}\Big)\Big\rangle\Big| &\lesssim 2^{-\ell} \\ \Big|\Big\langle \tfrac{\xi}{|\xi|}, \Big(\begin{smallmatrix}1\\-2y_1\\y_1^2\end{smallmatrix}\Big)\Big\rangle\Big| &\lesssim 2^{-2\ell(1-\varepsilon)}. \end{aligned}$$

Let $g(t) = (t^2, t)$ and recall the definition of u_1, u_2, u_3, and $\mathcal{C}_g$ from (3.12),(3.14),(3.13), and (3.11) respectively. These inequalities imply that for each y_1 the integrand of (10.1) vanishes outside the the set $\Pi_{C2^k,y_1}(2^{-\ell})$ (recall Definition 3.8) associated to the orthogonal vectors $u_1(y_1), u_2(y_1), u_3(y_1)$ (note that $(0,1,-y_1)$ is orthogonal to $u_1(y_1) = (g(y_1),1)$, so the second inequality implies the analogous result for $u_2(y_1)$). Thus $\mathfrak{F}\big[\big(\mathcal{A}_{\mathbb{H}}\big)_{k,\ell}f\big](\xi)$ vanishes outside a $C2^{-2\ell}$ neighborhood of the cone $\mathcal{C}_g$. Because of this support restriction, decoupling for the cone (i.e. Theorem 3.9) can be applied immediately to $\big(\mathcal{A}_{\mathbb{H}}\big)_{k,\ell}f(x)$, proving Proposition 8.3 for $\mathcal{A}_{\mathbb{H}}$ when γ is the moment curve.

In general it cannot be hoped that we can apply just one nonlinear change of variables to fix Σ_x in x. However, if we cut up the support in x into small boxes and apply local changes of variables on each box to partially "freeze" the variation of the cone Σ_x with x, we can apply decoupling to each decomposed piece separately. We cannot expect to be able to decouple down to the $2^{-\ell}$ scale immediately because of the variation in Σ_x even over this small box of x, but by decoupling many times to smaller and smaller boxes, changing variables at each step to further "freeze" the cone Σ_x depending on the decoupling step, we can recover the same estimate as in the model case above, with a large constant depending on ε.

In this chapter we show this iterative method works in a model case, then in Chapter 11 we reduce the general case to the model case by families of changes of variables. In the model case, the functions S^i are replaced by $\mathfrak{S}^i$ satisfying simplifying assumptions at the origin. The fold and blowdown conditions imply additional assumptions near the origin.

10.2 The Model Case Setup

Because the decoupling inequality involves a constant which depends on $\varepsilon > 0$, and is only proven for ℓ in the range $C_2 \leq \ell \leq \ell_\circ(k, \varepsilon) = \frac{k}{2+\varepsilon}$, we may choose a large enough constant C_ε such that the decoupling inequality holds for small k, and also small ℓ. Hence we may assume that $2^k \gg 1 \gg 2^{-\ell} > 0$.

Let $w = (w', w_3) \in \mathbb{R}^3$, $z = (z', z_3) \in \mathbb{R}^3$, and $\kappa_0 > 0$ be a constant (κ_0 will stand in for our fold assumption, i.e. the nonvanishing of $\kappa(x, y_3)$ in §3.4, where κ is defined as in (3.24)). Consider C^∞ maps $(w, z_3) \mapsto \mathfrak{S}^i(w, z_3)$ defined on a neighborhood of $[-r, r]^4$

for some $r \in (0,1)$. For $n \in \mathbb{N}$ define $M_n > 0$ such that

$$M_n \geq 2 + \|\mathfrak{S}^1\|_{C^{n+5}([-r,r]^4)} + \|\mathfrak{S}^2\|_{C^{n+5}([-r,r]^4)}, \tag{10.2}$$

where the C^n norm is the supremum of all derivatives orders 0 to n. We assume that for $w \in [-r,r]^3$,

$$(\mathfrak{S}^1, \mathfrak{S}^2, \mathfrak{S}^1_{z_3})\Big|_{(w,0)} = (w_1, w_2, w_3); \tag{10.3}$$

we also assume

$$\mathfrak{S}^2_{w,z_3}(0,0) = 0, \tag{10.4}$$

and

$$\mathfrak{S}^2_{w_3 z_3^2}(0,0) = \kappa_0. \tag{10.5}$$

As the functions $\mathfrak{S}^1, \mathfrak{S}^2$ play the part of S^1, S^2 in our model case, we can analyze the geometry of the conormal bundle associated to $\mathfrak{S}^1, \mathfrak{S}^2$, given by

$$\mathfrak{C}_{\mathfrak{S}} = \{(w, \xi, z, -\eta) \ : \ z' = \mathfrak{S}(w, z_3), \ \xi = \mu \cdot \mathfrak{S}_w(w, z_3), \ \eta = (\mu_1, \mu_2, \mu \cdot \mathfrak{S}_{z_3}(w, z_3)\}.$$

The projections π_L, π_R defined in (2.4) will be identified with the maps

$$\begin{aligned} \tilde{\pi_L} \ &: (w, \mu, z_3) \mapsto \big(w, \mu_1 \mathfrak{S}^1_w(w, z_3) + \mu_2 \mathfrak{S}^2_w(w, z_3)\big) \\ \tilde{\pi}_R \ &: (w, \mu, z_3) \mapsto \big(\mathfrak{S}(w, z_3), z_3, \mu, -(\mu_1 \mathfrak{S}^1_{z_3}(w, z_3) + \mu_2 \mathfrak{S}^2_{z_3}(w, z_3))\big). \end{aligned}$$

Define $\Delta^i_{\mathfrak{S}} = \det(\mathfrak{S}^1_w \ \mathfrak{S}^2_w \ \mathfrak{S}^i_{wz_3})$ for $i = 1, 2$. Then the submanifold $\mathcal{L}_{\mathfrak{S}} \subset \mathfrak{C}_{\mathfrak{S}}$ on which π_L and π_R are singular is given by the restriction $\mu_1 \Delta^1_{\mathfrak{S}}(w, z_3) + \mu_2 \Delta^2_{\mathfrak{S}}(w, z_3) = 0$. Note that (10.3) and (10.4) imply that $\mathcal{L}_{\mathfrak{S}}$ contains the point P parametrized by $(w, z_3) = (0, 0)$, and (10.5) implies that π_L has a fold at that point.

A kernel field for π_R at a point P parametrized by (w, μ, z_3) is given by

$$V_R(w, z_3) = \langle \mathfrak{S}^1_w(w, z_3) \wedge \mathfrak{S}^2_w(w, z_3), \nabla_w \rangle.$$

Our final assumption on $\mathfrak{S}^1, \mathfrak{S}^2$ is that π_R has a blowdown singularity along $\mathcal{L}_{\mathfrak{S}}$, i.e. that $V_R(w, z_3)$ lies in the tangent space of $\mathcal{L}_{\mathfrak{S}}$ whenever (w, μ, z_3) parametrizes a point in $\mathcal{L}_{\mathfrak{S}}$. This blowdown assumption implies that

$$V_R^N[\mu_1 \Delta_{\mathfrak{S}}^1 + \mu_2 \Delta_{\mathfrak{S}}^2]\Big|_{\mu \perp \Delta_{\mathfrak{S}}(w,z_3)} = 0$$

for all $N \geq 0$. Since $\mathfrak{S}_w^1(w, 0) = e_1$ and $\mathfrak{S}_w^2(w, 0) = e_2$, we see that $V_R(w, 0) = \partial_{w_3}$. The above conditions imply that

$$\partial_{w_3}^N \mathfrak{S}_{w_3 z_3}^2(w, 0) = 0, \ \forall N \geq 1 \tag{10.6}$$

$$\partial_{w_3}^N \Delta_{\mathfrak{S}}^2(w, 0) = 0, \ \forall N \geq 1. \tag{10.7}$$

Now that we have introduced the assumptions on our model functions $\mathfrak{S}^1, \mathfrak{S}^2$ we can define our model Radon-like operators. Let $(w, z_3) \mapsto \alpha(w, z_3)$ be a C^∞ function satisfying for $|(w, z_3)|_\infty < r$ and any multi-index β,

$$M_0^{-1} \leq |\alpha(w, z_3)| \leq M_0 \tag{10.8}$$

$$|\partial_w^\beta \alpha(w, z_3)| \leq M_{|\beta|} \tag{10.9}$$

Let $(w, z, \mu) \mapsto \zeta(w, z, \mu)$ belong to a bounded family of C^∞ functions supported where $|(w, z)|_\infty \leq r$ and $1/4 \leq |\mu| \leq 4$. For $k \gg 1$ and $1 < \ell < \ell_\circ(k, \varepsilon)$ let $\mathcal{T}_{k,\ell,\pm}$ be an operator with Schwartz kernel

$$2^{2k} \int e^{2\pi i 2^k \mu \cdot (\mathfrak{S}(w,z_3) - z')} \chi_1\big(\pm 2^\ell \alpha(w, z_3) \mu \cdot \Delta_{\mathfrak{S}}(w, z_3)\big) \zeta(w, z, \mu)\, d\mu, \tag{10.10}$$

and let $\mathcal{T}_{k,\ell_\circ}$ be an operator with Schwartz kernel

$$2^{2k} \int e^{2\pi i 2^k \mu \cdot (\mathfrak{S}(w,z_3) - z')} \chi_0\big(2^{\ell_\circ} \big|\alpha(w, z_3) \mu \cdot \Delta_{\mathfrak{S}}(w, z_3)\big|\big) \zeta(w, z, \mu)\, d\mu. \tag{10.11}$$

The operators $\mathcal{T}_{k,\ell,\pm}, \mathcal{T}_{k,\ell_\circ}$ will play the role of $\mathcal{R}_{k,\ell}$ after a nonlinear change of variables, while $\alpha(w, z_3)$ is introduced in the localization as a byproduct of those changes of variables. As in our analysis of $\mathcal{R}_{k,\ell,\pm}$, we will drop the dependence on $\pm$ as the same techniques apply for $\mathcal{T}_{k,\ell,+}$ and $\mathcal{T}_{k,\ell,-}$. The inductive step in our iterated decoupling method is the following estimate.

Proposition 10.1. *Let* $0 < \varepsilon \leq 1$, $k \gg 1$, $0 \leq \ell \leq \ell_0$,

$$\delta_0 \in [2^{-\ell(1-\varepsilon)}, 2^{-\ell\varepsilon}],$$

and

$$\delta_1 \in [\max\{2^{-\ell(1-\varepsilon/2)}, \delta_0 2^{-\ell\varepsilon/4}\}, \delta_0).$$

Define $\varepsilon_1 = (\delta_1/\delta_0)^2$. *Let* J *be an interval of length* δ_0 *containing* 0, *and* $\mathcal{I}_J$ *be a collection of intervals of length* δ_1 *with disjoint interior and whose interiors all intersect* J. *Let* $\sigma \in C_c^\infty(\mathbb{R}^3)$ *be supported* $(-1,1)^3$ *and define* $\sigma_{\ell,\varepsilon_1}(w) = \sigma(2^\ell w_1, 2^\ell w_2, \varepsilon_1^{-1} w_3)$. *Let* $2 \leq p \leq 6$, *let* $g \in L^p(\mathbb{R}^3)$ *and define* $g_I(y) = g(y)\mathbb{1}_I(y_3)$. *Then*

$$\Big\|\sigma_{\ell,\varepsilon_1} \sum_{I \in \mathcal{I}_J} \mathcal{T}_{k,\ell} g_I\Big\|_p \lesssim_\varepsilon (\delta_0/\delta_1)^{\frac{1}{2}-\frac{1}{p}+\varepsilon} \left(\sum_{I \in \mathcal{I}_J} \Big\|\sigma_{\ell,\varepsilon_1} \mathcal{T}_{k,\ell} g_I\Big\|_p^p\right)^{1/p} + C(\varepsilon) 2^{-10k} 2^{-2\ell} \varepsilon_1 \|g\|_p.$$

The idea behind this proposition is to show that the Fourier transforms of $\sigma_{\ell,\varepsilon_1}\mathcal{T}_{k,\ell} g_I$ are concentrated on thin plates in the neighborhood the plates $\Pi_{A,b_I}(\delta_1)$ for some $b_I \in I$ and some large enough $A > 1$, and thus decoupling applies.

Recall from (3.10) that the fibers of $\pi_L(\mathcal{L}_{\mathfrak{S}})$ are given for fixed w by

$$\begin{aligned}\tilde{\Sigma}_w &= \{\mu_1 \mathfrak{S}_w^1(w, z_3) + \mu_2 \mathfrak{S}_w^2(w, z_3) \ : \ \mu_1 \Delta_{\mathfrak{S}}^1(w, z_3) + \Delta_{\mathfrak{S}}^2(w, z_3) = 0\} \\ &= \{\pm \rho \Xi_{\mathfrak{S}}(w, z_3) \ : \ \rho > 0, \ |z_3| \leq r\},\end{aligned}$$

where $\Xi_{\mathfrak{S}}(w, z_3) = -\mathfrak{S}^1_w(w, z_3)\Delta^2_{\mathfrak{S}}(w, z_3) + \mathfrak{S}^1_w(w, z_3)\Delta^1_{\mathfrak{S}}(w, z_3)$. Narrowing our perspective to the origin, $\tilde{\Sigma}_0$ is a cone parametrized by (ρ, z_3) given by

$$\tilde{\Sigma}_0 = \{\pm\rho\Xi_{\mathfrak{S}}(0, z_3) \ : \ \rho > 0, \ |z_3| \leq r\}.$$

Analogous to (3.19), we can define $N(z_3) := \mathfrak{S}^1_w \wedge \mathfrak{S}^2_w(0, z_3)$, the vector normal to $\tilde{\Sigma}_0$ at the point P parametrized by (ρ, b). Analogous to (3.18) and (3.20), the tangent space of $\tilde{\Sigma}_0$ at a point parametrized by (ρ, z_3) is given by the vectors

$$T_1(b) = \Xi_{\mathfrak{S}}(0, b) \tag{10.12}$$

$$T_2(b) = T_1(b) \wedge N(b). \tag{10.13}$$

Given these three pairwise orthogonal vectors, we consider the plates $\Pi_{A,b}(\delta)$ for $A > 1$ and $0 < \delta < 1$ associated to $T_1(b), T_2(b), N(b)$ from Definition 3.8. These plates cover a δ^2-neighborhood of $\tilde{\Sigma}_0$, and because $\tilde{\Sigma}_0$ is curved we can apply Theorem 3.9 to sums of functions whose Fourier transforms are supported on these plates $\Pi_{A,b}(\delta)$.

10.3 Derivatives of $\mathfrak{S}$ and Δ

Before we proceed with the proof of Proposition 10.1, we write some approximations of $\mathfrak{S}$ and $\Delta^i_{\mathfrak{S}}$ derived from the simplifying assumptions at the origin. For the rest of Chapter 10.2 we omit the subscript dependence on $\mathfrak{S}$. Because of (10.3) we may conclude that for any multi-index β of length at least 1,

$$\partial^\beta_w \mathfrak{S}^1_w\big|_{(w,0)} = 0 \tag{10.14}$$

$$\partial^\beta_w \mathfrak{S}^2_w\big|_{(w,0)} = 0 \tag{10.15}$$

$$\partial^\beta_w \mathfrak{S}^1_{wz_3}\big|_{(w,0)} = 0. \tag{10.16}$$

For $w \in [-r,r]^3$,

$$\Delta^1(w,0) = 1 \tag{10.17}$$

$$\Delta^2(w,0) = \mathfrak{S}^2_{w_3z_3}(w,0) \tag{10.18}$$

$$\Delta^1_{z_3}(0,0) = \mathfrak{S}^1_{w_3z_3^2}(0,0) \tag{10.19}$$

$$\Delta^2_{z_3}(0,0) = \mathfrak{S}^2_{w_3z_3^2}(0,0) = \kappa_0 \tag{10.20}$$

and by applying these identities

$$\Xi(w,0) = -\Delta^2(w,0)\mathfrak{S}^1_w(w,0) + \Delta^1(w,0)\mathfrak{S}^2_w(w,0) = e_2 - \mathfrak{S}^2_{w_3z_3}(w,0)e_1 \tag{10.21}$$

$$\Xi_{w_3^n}(0,0) = -\mathfrak{S}^2_{w_3^{n+1}z_3}(0,0)e_1 = 0, \qquad n \geq 1 \tag{10.22}$$

$$\Xi_{z_3}(0,0) = -\kappa_0 e_1 + \mathfrak{S}^1_{w_3z_3^2}(0,0)e_2. \tag{10.23}$$

Let e_1, e_2, e_3 be the standard basis vectors in $\mathbb{R}^3$. We use Taylor expansions with appropriate remainders. Therefore, as in Chapter 9, for any $i \in \mathbb{N}$ let $R_i(w, z_3)$ denote C^∞ functions which are bounded by a uniform constant. Using the above identities on Ξ and $\mathfrak{S}$ and appropriate Taylor expansions, we can approximate

$$\begin{aligned} T_1(b) &= \Xi(0,b) && (10.24)\\ &= \Xi(0,0) + b\Xi_{z_3}(0,0) + b^2R_1(0,b) \\ &= -\kappa_0 be_1 + (1 + b\mathfrak{S}^1_{w_3z_3^2}(0,0))e_2 + b^2R_2(0,b) && (10.25) \end{aligned}$$

and

$$\begin{aligned} N(b) &= \mathfrak{S}^1_w(0,b) \wedge \mathfrak{S}^2_w(0,b) && (10.26)\\ &= (e_1 + be_3 + b^2R_3(0,b)) \wedge (e_2 + b^2R_4(0,b)) \\ &= -be_1 + e_3 + b^2R_5(0,b). && (10.27) \end{aligned}$$

Applying these approximations to the definition of $T_2(b)$ we see

$$T_2(b) = T_1(b) \wedge N(b) = (1 + b\mathfrak{S}^1_{w_3 z_3^2}(0,b))e_1 + \kappa_0 b e_2 + b e_3 + b^2 R_6(0,b). \tag{10.28}$$

Let $\beta = (\beta_{w_1}, \beta_{w_2}, \beta_{w_3}, \beta_{z_3})$ be a multi-index and let $\partial^\beta_{(w,z_3)}$ denote a derivative of order $|\beta| = \beta_{w_1} + \beta_{w_2} + \beta_{w_3} + \beta_{z_3}$ in the variables w, z_3. By using the upper bounds M_n, trilinearity of determinants, and differentiation rules for products we can additionally estimate

$$|\partial^\beta_{(w,z_3)} \Delta^i| \leq 3^{|\beta|} M^3_{|\beta|}. \tag{10.29}$$

Similarly, by differentiating products,

$$|\partial^\beta_{(w,z_3)} \Xi| \leq 4^{|\beta|} M^4_{|\beta|}. \tag{10.30}$$

10.4 Plate Localization

We show in this section that under certain assumptions the w-gradient of the phase of $\mathcal{T}_{k,\ell}$ is contained in the plate $\Pi_{A,b}(\delta_1)$ given a large enough A. This will later be used to apply an integration by parts argument in Section 10.5.

Lemma 10.2. *Let $\varepsilon > 0$, and $\delta_0, \delta_1, \varepsilon_1$ be as in Proposition 10.1. Let $\ell \leq \ell_\circ$ such that $2^{-\ell} \ll r$, $M_0 2^{-\ell} \leq 2^{-10}$, and let $\frac{1}{4} \leq |\mu| \leq 4$, $|w'| \leq 2^{-\ell}$, $|w_3| \leq \varepsilon_1$, $|b| \leq \delta_0$, and $|z_3 - b| \leq \delta_1$.*

If

$$|\mu_1 \Delta^1(w, z_3) + \mu_2 \Delta^2(w, z_3)| \leq M_0 2^{-\ell}, \tag{10.31}$$

then there exists $A(\varepsilon) > 1$ such that

$$\mu_1 \mathfrak{S}^1_w(w, z_3) + \mu_2 \mathfrak{S}^2_w(w, z_3) \in \Pi_{A(\varepsilon),b}(\delta_1).$$

More specifically,

$$A(\varepsilon)^{-1}|T_1(b|) \leq |\langle T_1(b), \mu_1 \mathfrak{S}^1_w(w, z_3) + \mu_2 \mathfrak{S}^2_w(w, z_3)\rangle| \leq A(\varepsilon)|T_1(b)| \tag{10.32}$$

$$|\langle T_2(b), \mu_1 \mathfrak{S}^1_w(w, z_3) + \mu_2 \mathfrak{S}^2_w(w, z_3)\rangle| \leq A(\varepsilon)|T_2(b)|\delta_1. \tag{10.33}$$

$$|\langle N(b), \mu_1 \mathfrak{S}^1_w(w, z_3) + \mu_2 \mathfrak{S}^2_w(w, z_3)\rangle| \leq A(\varepsilon)|N(b)|\delta_1^2. \tag{10.34}$$

Note that the constant $A(\varepsilon)$ does not depend on δ_0, δ_1.

Proof. The estimate in (10.32) is clearly true for some $A > 1$ independent of ε. Throughout the remainder of this proof we use Taylor expansions. Because we will be taking large numbers of derivatives to prove estimates we must be careful to track the appropriate Taylor remainders. Therefore, as in the previous section for any $i \in \mathbb{N}$ the function $R_i(w, \mu, z_3)$ is C^∞ and uniformly bounded by 1.

10.4.1 The Normal Direction

We start with the proof of (10.34). Let $G = \lceil 3\varepsilon^{-1} \rceil$. We employ an order G Taylor expansion of $\langle N(b), \mu \cdot \mathfrak{S}_w(w, z_3)\rangle$ about $(w, z_3) = (0, b)$. Since $|z_3 - b|^2 \leq \delta_1^2$, $|w'|^2 \leq 2^{-2\ell} < \delta_1^2$, and $|w_3|^G \leq \varepsilon_1^G \leq \delta_1^2$, we can estimate the remainder

$$\langle N(b), \mu \cdot \mathfrak{S}_w(w, z_3)\rangle = \sum_{n=0}^{G} \sum_{|\alpha|=0}^{G-n} C_{n,\alpha}(z_3 - b)^n w^\alpha \langle N(b), \nabla_w \Big((\partial_{z_3})^n (\partial_w)^\alpha [\mu \cdot \mathfrak{S}]\Big)(0, b)\rangle$$
$$+ M_G \delta_1^2 R_1(w, \mu, z_3).$$

We can rearrange the terms of the Taylor expansion to see that

$$\begin{aligned}\langle N(b), \mu\cdot\mathfrak{S}_w(w,z_3)\rangle &= \langle N(b), \mu_1\mathfrak{S}^1_w + \mu_2\mathfrak{S}^2_w(0,b)\rangle \\ &\quad + (z_3-b)\langle N(b), \mu_1\mathfrak{S}^1_{wz_3} + \mu_2\mathfrak{S}^2_{wz_3}(0,b)\rangle \\ &\quad + \sum_{i=1}^{2} w_i\langle N(b), \mu_1\mathfrak{S}^1_{ww_i} + \mu_2\mathfrak{S}^2_{ww_i}(0,b)\rangle \\ &\quad + I + II + III + M_G\delta_1^2 R_2(w,\mu,z_3), \end{aligned} \tag{10.35}$$

where

$$\begin{aligned} I &= \sum_{n=1}^{G}\frac{w_3^n}{n!}\langle N(b), \mu_1\mathfrak{S}^1_{ww_3^n} + \mu_2\mathfrak{S}^2_{ww_3^n}(0,b)\rangle \\ II &= \sum_{n=2}^{G}\frac{w_3^{n-1}(z_3-b)}{n!}\langle N(b), \mu_1\mathfrak{S}^1_{ww_3^{n-1}z_3} + \mu_2\mathfrak{S}^2_{ww_3^{n-1}z_3}(0,b)\rangle \\ III &= \sum_{n=2}^{G}\sum_{i=1}^{2}\frac{w_3^{n-1}w_i}{n!}\langle N(b), \mu_1\mathfrak{S}^1_{ww_3^{n-1}w_i} + \mu_2\mathfrak{S}^2_{ww_3^{n-1}w_i}(0,b)\rangle. \end{aligned}$$

The first term in (10.35) vanishes by the definition of $N(b)$ (see (10.26)). The second term in (10.35) is

$$(z_3-b)\langle N(b), \mu_1\mathfrak{S}^1_{wz_3} + \mu_2\mathfrak{S}^2_{wz_3}(0,b)\rangle = (z_3-b)(\mu_1\Delta^1(0,b) + \mu_2\Delta^2(0,b)).$$

Now, since $|w'|, |z_3-b| \le \delta_1$, and $|w_3|^G \le \delta_1^2$, we can apply a Taylor expansion about (w,z_3). Using trilinearity of determinants and differentiation of products we get

$$\begin{aligned}\mu_1\Delta^1(0,b) + \mu_2\Delta^2(0,b) &= \sum_{n=0}^{G}\frac{w_3^n}{n!}\Big(\mu_1\Delta^1_{w_3^n}(w,z_3) + \mu_2\Delta^2_{w_3^n}(w,z_3)\Big) \\ &\quad + 3^G M_G^3\delta_1 R_3(w,\mu,z_3).\end{aligned}$$

By (10.31) the first term is bounded by $M_0 2^{-\ell}$. For each $1 \le n \le G$, from (10.7) and (10.17) we have $\Delta^i_{w_3^n}(w,0) = 0$ for $i = 1,2$, and so by trilinearity of determinants, and differentiation of products, expanding about $z_3 = 0$ we get

$$|\Delta^i_{w_3^n}(w, z_3)| \leq \left|\Delta^i_{w_3^n}(w, 0) + 3^n M_n^3 z_3\right| \leq 3^n M_n^3 \delta_0.$$

Thus

$$|\mu_1 \Delta^1(0, b) + \mu_2 \Delta^2(0, b)| \leq M_0 2^{-\ell} + 3^G M_G^3 \varepsilon_1 \delta_0 + 3^G M_G^3 \delta_1 \leq 3^{G+1} M_G^3 \delta_1,$$

and the second term in (10.35) is bounded by $3^{G+1} M_G^3 \delta_1^2$.

Next we deal with the first order w'-derivatives in (10.35). We approximate about $z_3 = 0$. For $i = 1, 2$, using the estimates (10.14) and (10.15), we get

$$\begin{aligned}
|w_i \langle \mathfrak{S}^1_w(0, b) \wedge \mathfrak{S}^2_w(0, b), \mu \cdot \mathfrak{S}_{ww_i}(0, b)\rangle| &\leq |w_i| \Big[\left|\left\langle \mathfrak{S}^1_w(0, 0) \wedge \mathfrak{S}^2_w(0, 0), \mu \cdot \mathfrak{S}_{ww_i}(0, 0)\right\rangle\right| \\
&\qquad + 3M_0^3 b R_4(0, \mu, b)\Big] \\
&\leq 2^{-\ell}(0 + 3M_0^3 \delta_0).
\end{aligned}$$

Note that the condition $\delta_1 \geq \max\{M_0^2 2^{20-\ell(1-\varepsilon/2}, 2^{-\ell\varepsilon/4}\delta_0\}$ from Proposition 10.1 implies that $2^{-\ell}\delta_0 \leq \delta_1^2$.

Finally, we estimate I, II, and III. The estimates of all three sums rely on the blowdown condition at the origin. We begin with the estimate of I. For all $n \geq 1$, we employ a Taylor expansion about $z_3 = 0$ to obtain

$$\begin{aligned}
&\langle \mathfrak{S}^1_w(0,b) \wedge \mathfrak{S}^2_w(0,b), \mu_1 \mathfrak{S}^1_{ww_3^n}(0,b) + \mu_2 \mathfrak{S}^2_{ww_3^n}(0,b)\rangle \\
&\qquad = \langle \mathfrak{S}^1_w(0,0) \wedge \mathfrak{S}^2_w(0,0), \mu_1 \mathfrak{S}^1_{ww_3^n}(0,0) + \mu_2 \mathfrak{S}^2_{ww_3^n}(0,0)\rangle \\
&\qquad\quad + b\Big[\det(\mathfrak{S}^1_{wz_3}\ \mathfrak{S}^2_w\ \mu_1 \mathfrak{S}^1_{ww_3^n} + \mu_2 \mathfrak{S}^2_{ww_3^n})\Big|_{(0,0)} \\
&\qquad\quad + \det(\mathfrak{S}^1_w\ \mathfrak{S}^2_{wz_3}\ \mu_1 \mathfrak{S}^1_{ww_3^n} + \mu_2 \mathfrak{S}^2_{ww_3^n})\Big|_{(0,0)} \\
&\qquad\quad + \det(\mathfrak{S}^1_w\ \mathfrak{S}^2_w\ \mu_1 \mathfrak{S}^1_{ww_3^n z_3} + \mu_2 \mathfrak{S}^2_{ww_3^n z_3})\Big|_{(0,0)}\Big] \\
&\qquad\quad + 3^2 M_n^3 b^2 R_5(0,\mu,b)
\end{aligned}$$

Using the estimates (10.14), (10.15), (10.26), (10.16), and (10.6), we observe

$$\begin{aligned}
\langle \mathfrak{S}^1_w(0,0) \wedge \mathfrak{S}^2_w(0,0), \mu_1 \mathfrak{S}^1_{ww_3^n}(0,0) + \mu_2 \mathfrak{S}^2_{ww_3^n}(0,0)\rangle &= 0 \\
\det(\mathfrak{S}^1_{wz_3}\ \mathfrak{S}^2_w\ \mu_1 \mathfrak{S}^1_{ww_3^n} + \mu_2 \mathfrak{S}^2_{ww_3^n})\Big|_{(0,0)} &= 0 \\
\det(\mathfrak{S}^1_w\ \mathfrak{S}^2_{wz_3}\ \mu_1 \mathfrak{S}^1_{ww_3^n} + \mu_2 \mathfrak{S}^2_{ww_3^n})\Big|_{(0,0)} &= 0 \\
\det(\mathfrak{S}^1_w\ \mathfrak{S}^2_w\ \mu_1 \mathfrak{S}^1_{ww_3^n z_3} + \mu_2 \mathfrak{S}^2_{ww_3^n z_3})\Big|_{(0,0)} = \mu_1 \mathfrak{S}^1_{w_3^{n+1} z_3}(0,0) + \mu_2 \mathfrak{S}^1_{w_3^{n+1} z_3}(0,0) &= 0.
\end{aligned}$$

This implies

$$|I| \le 3^2 M_G^3 \sum_{n=1}^{G} \frac{\varepsilon_1^n \delta_0^2}{n!} \le 3^3 M_G^3 \varepsilon_1 \delta_0^2 \le 3^3 M_G^3 \delta_1^2.$$

Next we estimate II. For $n \ge 2$, we expand about $z_3 = 0$ again to obtain

$$\begin{aligned}
\langle \mathfrak{S}^1_w \wedge \mathfrak{S}^2_w, \mu \cdot \mathfrak{S}_{ww_3^{n-1} z_3}\rangle\big|_{(0,b)} &= \det(\mathfrak{S}^1_w\ \mathfrak{S}^2_w\ \mu \cdot \mathfrak{S}_{ww_3^{n-1} z_3})\big|_{(0,0)} \\
&\quad + 3M_G^3 b R_6(0,\mu,b).
\end{aligned}$$

Thus by the calculation from the estimate on I, this determinant vanishes, and

$$|II| \le 3M_G^3 \sum_{n=2}^{G} \frac{\varepsilon_1^{n-1} \delta_1 \delta_0}{n!} \le 3M_G^3 \varepsilon_1 \delta_1 \delta_0 \le 3M_G^3 \delta_1^2.$$

Finally we estimate III. Again using the calculations from I, for $n \geq 2$ and $i = 1, 2$

$$\begin{aligned}\langle \mathfrak{S}^1_w \wedge \mathfrak{S}^2_w, \mu \cdot \mathfrak{S}_{ww_3^{n-1}w_i} \rangle\big|_{(0,b)} &= \langle \mathfrak{S}^1_w(0,0) \wedge \mathfrak{S}^2_w(0,0), \mu \cdot \mathfrak{S}_{ww_3^{n-1}w_i}(0,0) \rangle \\ &\quad + 3M_G^3 b R_7(0, \mu, b) \\ &= \mu \cdot \mathfrak{S}_{w_i w_3^n}(0,0) + 3M_G^3 b R_7(0,\mu,b) \\ &= 3M_G^3 b R_7(0,\mu,b).\end{aligned}$$

This implies that

$$|III| \leq 3M_G^3 \sum_{n=2}^{G} \frac{\varepsilon_1^{n-1} 2^{-\ell}\delta_0}{n!} \leq 3M_G^3 \varepsilon_1 2^{-\ell}\delta_0 \leq 3M_G^3 \delta_1^2.$$

Since $|N(b)| \geq 1/2$ this proves (10.34) with any $A(\varepsilon) \geq 3^{\lceil \frac{3}{\varepsilon} \rceil + 2} M^3_{\lceil \frac{3}{\varepsilon} \rceil}$.

10.4.2 The Tangential Estimate

Having proven (10.34), we prove (10.33). Using (10.28), define

$$T_2^*(b) = (1 + b\mathfrak{S}^1_{w_3 z_3^2}(0,0))e_1 + \kappa_0 b e_2 + b e_3$$

and note that $|T_2(b) - T_2^*(b)| \leq M_0 \delta_0^2$. Next, we will approximate μ by the projection of $\mu_1 \Delta^1(w, z_3) + \mu_2 \Delta^2(w, z_3)$ onto $\mathcal{L}_{\mathfrak{S}}$. In particular, let

$$\mu^\circ = \pm \tfrac{|\mu|}{|\Delta(w,z_3)|}(-\Delta^2(w,z_3), \Delta^1(w,z_3)),$$

so that $\mu_1^\circ \Delta^1(w, z_3) + \mu_2^\circ \Delta^2(w, z_3) = 0$, $|\mu| = |\mu^\circ|$, and where the sign is picked so that

$$|\mu - \mu^\circ| \leq 2|\mu| M_0 2^{-\ell}.$$

This is possible since $|\mu_1 \Delta^1 + \mu_2 \Delta^2| \leq M_0 2^{-\ell}$ and $|\Delta(w, z_3)| \neq 0$. Then

$$\mu_1^\circ \mathfrak{S}^1_w(w, z_3) + \mu_2^\circ \mathfrak{S}^2_w(w, z_3) = \tfrac{|\mu|}{|\Delta(w,z_3)|}\Xi(w, z_3),$$

and thus

$$\left|\mu_1\mathfrak{S}^1_w(w,z_3)+\mu_2\mathfrak{S}^2_w(w,z_3)-\frac{|\mu|}{|\Delta(w,z_3)|}\Xi(w,z_3)\right|\le|\mu-\mu^\circ||\mathfrak{S}_w|\le 8M_0^2 2^{-\ell}.$$

Thus we have reduced proving (10.28) to proving the estimate

$$|\langle T_2^*(b),\Xi(w,z_3)\rangle|\le A(\varepsilon)|T_2(b)|\delta_1.$$

We approximate $\langle T_2^*(b),\Xi(w,z_3)\rangle$ by an order G Taylor expansion about $(w,z_3)=(0,0)$. Since $|z_3|^2\le\delta_0^2\le\delta_1$ $|w_3||z_3|\le\varepsilon_1\delta_0\le\delta_1$, $|w'|\le 2^{-\ell}\le\delta_1$, and $|w_3|^G\le\varepsilon_1^G\le\delta_1^2\le\delta_1$, we can estimate the remainder of this Taylor expansion by $4^G M_G^4\delta_1 R_8(w,z_3)$ using differentiation of products. Reorganizing, we obtain

$$\begin{aligned}\langle T_2^*(b),\Xi(w,z_3)\rangle&=\sum_{n=0}^{G}\sum_{|\alpha|=0}^{G-n}C_{n,\alpha}w^\alpha z_3^n\langle T_2^*(b),(\partial_{z_3})^n(\partial_w)^\alpha\Xi\rangle\Big|_{(0,0)}+4^G M_G^4\delta_1 R_8(w,\mu,z_3)\\&=\langle T_2^*(b),\Xi(0,0)\rangle+z_3\langle T_2^*(b),\Xi_{z_3}(0,0)\rangle\\&\quad+\sum_{n=1}^{G}\frac{w_3^n}{n!}\langle T^*(b),\Xi_{w_3^n}(0,0)\rangle+4^G M_G^4\delta_1 R_9(w,\mu,z_3).\end{aligned}$$

Using (10.21), (10.22), and (10.23)

$$\begin{aligned}\langle T_2^*(b),\Xi(0,0)\rangle&=\kappa_0 b\\\langle T_2^*(b),\Xi_{z_3}(0,0)\rangle&=-\kappa_0(1+b(\mathfrak{S}^1_{w_3z_3^2}(0,0)-\mathfrak{S}^1_{w_3z_3^2}(0,b)))\\\langle T_2^*(b),\Xi_{w_3^n}(0,0)\rangle&=0,\qquad\qquad n\ge 1.\end{aligned}$$

Thus using the assumption $b\le\delta_0$,

$$|\langle T_2^*(b),\Xi(w,z_3)\rangle|\le\kappa_0\delta_1+\kappa_0M_0\delta_0^2+4^G M_G^4\delta_1$$

and therefore we can estimate

$$\begin{aligned}|\langle T_2(b),\mu_1\mathfrak{S}^1_w(w,z_3)+\mu_2\mathfrak{S}^2_w(w,z_3)\rangle|&\le M_0\delta_0^2+8M_0^2 2^{-\ell}+\kappa_0\delta_1+\kappa_0M_0\delta_0^2+4^G M_G^4\delta_1\\&\le\kappa_0(1+4^{G+2}M_G^4)\delta_1.\end{aligned}$$

Thus picking

$$A(\varepsilon) \geq \max\{3^{\lceil \frac{3}{\varepsilon} \rceil + 2} M^3_{\lceil \frac{3}{\varepsilon} \rceil}, \kappa_0(1 + 4^{\lceil \frac{3}{\varepsilon} \rceil + 2} M^4_{\lceil \frac{3}{\varepsilon} \rceil})\} \tag{10.36}$$

the lemma is proven. $\square$

10.5 Proof of Proposition 10.1

Now that we have shown that the phase function of the Schwartz kernel of $\mathcal{T}_{k,\ell}$ lies in $\Pi_{A(\varepsilon),b}(\delta_1)$ whenever $|z_3 - b| \leq \delta_1$ we can begin the proof of the decoupling step. For each $I \in \mathcal{I}_J$ pick $b_I \in I$. Note that since $b_I \in J$ and J si an interval of length δ_0 containing the origin, $|b_I| \leq \delta_0$. Let m_{A,b_I,δ_1} be a multiplier equal to 1 on $\Pi_{2A,b_I}(\delta_1)$ which vanishes on $\Pi_{3A,b_I}(\delta_1)$. Let

$$\widehat{P_{k,A,b_I,\delta_1} f}(\xi) = m_{A,b_I,\delta_1}(2^k \xi) \hat{f}(\xi).$$

Then by Bourgain-Demeter decoupling on the cone (Theorem 3.9),

$$\Big\| \sigma_{\ell,\varepsilon_1} \sum_I P_{k,A(\varepsilon),b_I,\delta_1} \mathcal{T}_{k,\ell} g_I \Big\|_p \leq C(\varepsilon, A(\varepsilon))(\delta_0/\delta_1)^{\frac{1}{2} - \frac{1}{p} + \varepsilon} \Big(\sum_I \big\| \sigma_{\ell,\varepsilon_1} \mathcal{T}_{k,\ell} g_I \big\|_p^p \Big)^{1/p},$$

for $2 \leq p \leq 6$ and any sufficiently small $\varepsilon > 0$. Thus it suffices to estimate the remainder

$$\big\| \sigma_{\ell,\varepsilon_1} \sum_I \big(\mathrm{Id} - P_{k,A(\varepsilon),b_I,\delta_1}\big) \mathcal{T}_{k,\ell} g_I \big\|_p \leq C 2^{-10k} 2^{-2\ell} \varepsilon_1 \|g\|_p.$$

Note that since $\ell \leq \frac{k}{2+\varepsilon}$ and $\varepsilon_1 \geq 2^{-\ell\varepsilon/2}$, $2^{-2\ell}\varepsilon_1 \geq 2^{-k} 2^{3\ell\varepsilon/2} \geq 2^{-k}$, so it suffices to prove the above estimate with 2^{-10k} replaced by 2^{-11k}.

For each $I \in \mathcal{I}_J$ the Schwartz kernel of the operator $f \mapsto \big(\mathrm{Id} - P_{k,A(\varepsilon),b_I,\delta_1}\big) \mathcal{T}_{k,\ell} f$ is given by a sum of kernels $\sum_{n=0}^\infty K_{n,k,\ell,b_I}(w,z)$, where

$$K_{n,k,\ell,b_I}(w,z) = 2^{2k} \int\int\int e^{2\pi i \Psi(w,v,z,\mu,\xi)} \sigma_1(v,z,\mu) \sigma_{n,2}(\xi)\, dv\, d\xi d\mu,$$

the phase function Ψ is given by

$$\Psi(w,v,z,\mu,\xi) = \langle w-v,\xi\rangle + 2^k\mu\cdot(\mathfrak{S}(v,z_3)-z')$$

and the symbols $\sigma_1, \sigma_{n,2}$ are given by

$$\sigma_1(v,z,\mu) = \sigma_{\ell,\varepsilon_1}(v)\chi_1\big(2^\ell\alpha(v,z_3)\mu\cdot\Delta(v,z_3)\big)\zeta(v,z,\mu),$$

$$\sigma_{n,2}(\xi) = \big(1-m_{A(\varepsilon),b_I,\delta_1}(2^k\xi)\big)\chi_n(|\xi|)$$

with $\chi_1(2^\ell\alpha(v,z_3)\mu\cdot\Delta(v,z_3))$ replaced by $\chi_0(2^\ell\alpha(v,z_3)\mu\cdot\Delta(v,z_3))$ if $\ell=\ell_\circ$. Note that the symbol of K_{n,k,ℓ,b_I} is supported where $|\xi|\simeq 2^n$ for $n\geq 1$ (and $|\xi|\leq 4$ if $n=0$), $|\mu|\simeq 1$, $|v|+|z|\leq r$, and a priori unbounded w. We prove the following lemma to reduce to the case when $|\xi|\simeq 2^k$.

Lemma 10.3. *Let $C_1>0$ be the necessary constant from Lemma 7.1 applied to $\mathcal{T}_{k,\ell}$. Suppose that $|k-n|>C_1$. Then for $N>1$*

$$|K_{n,k,\ell,b_I}(w,z)| \leq C_{N,\varepsilon}\frac{1}{(1+|w|)^4}2^{-N(k+n)}\mathbb{1}_{[-r,r]}(|z|). \tag{10.37}$$

If $|n-k|<C_1$ we can apply integration by parts using the fact that $2^{-k}\xi$ is bounded away from the plate $\Pi_{A(\varepsilon),b_I}(\delta_1)$ while $\mu\cdot\mathfrak{S}_w$ lies in $\Pi_{A(\varepsilon),b_I}(\delta_1)$ to obtain lower bounds on $|\Psi_v|$. In particular, we prove the following estimate.

Lemma 10.4. *If $|n-k|<C_1$ then*

$$|K_{n,k,\ell,b_I}(w,z)| \leq C_\varepsilon 2^{-11k}\frac{1}{(1+|w|)^4}\mathbb{1}_{[-r,r]}(z).$$

Together the estimates in Lemmas 10.3 and 10.4 along with the compact support of $K_{n,k,\ell,b_I}(w,z)$ in z imply

$$\sup_z\int\big|K_{n,k,\ell,b_I}(w,z)\big|\,dw + \sup_w\int\big|K_{n,k,\ell,b_I}(w,z)\big|\,dz \leq C_\varepsilon 2^{-11k-n}.$$

Thus

$$\Big\| \sum_{I\in\mathcal{I}_J} \big(\mathrm{Id} - P_{k,A(\varepsilon),b_I,\delta_1}\big)\big[\sigma_{\ell,\varepsilon_1}\mathcal{T}_{k,\ell} g_I\big]\Big\|_p = \sum_{I\in\mathcal{I}_J}\sum_{n\geq 0}\Big\| \int K_{n,k,\ell,b_I}(\cdot,z) g_I(z)\,dz\Big\|_p$$

and applying Young's inequality and the almost disjoint support of $\{g_I\}_{I\in\mathcal{I}_J}$

$$\begin{aligned}\sum_{I\in\mathcal{I}_J}\sum_{n\geq 0}\Big\| \int K_{n,k,\ell,b_I}(\cdot,z) g_I(z)\,dz\Big\|_p &\leq \sum_{I\in\mathcal{I}_J}\sum_{n\geq 0} C_\varepsilon 2^{-11k-n}\|g_I\|_p \\ &\leq C_\varepsilon 2^{-11k}\|g\|_p.\end{aligned}$$

This will complete the proof of Proposition 10.1.

10.5.1 The Proof of Lemma 10.3: Large and Small ξ

First, we introduce the details of integration by parts in the ξ variables with the differential operator

$$L_\xi = \langle \tfrac{w-v}{|w-v|^2}, \nabla_\xi \cdot \rangle$$

which will give the desired decay in w. Note that $\nabla_\xi \Psi = w - v$, so $\partial_\xi^\beta \Psi = 0$ for any multi-index β with $|\beta| \geq 2$, and

$$|\partial_\xi^\beta \sigma_{n,2}| \leq C_{|\beta|}\min\{A(\varepsilon)^{-1}\delta_1^2 2^k, 2^n\}^{-|\beta|} \leq C_{|\beta|} A(\varepsilon)^{|\beta|}$$

for any multi-index β with $|\beta|\geq 1$. Applying Lemma 8.5 gives a bound of

$$\big|(L_\xi^*)^N \sigma_{n,2}(\xi)\big| \leq \frac{C_N A(\varepsilon)^N}{|w-v|^N}$$

for any $N > 0$. Since $\sigma_{n,2}$ is bounded and supported where $|\xi| \simeq 2^n$, we obtain an estimate

$$\Big|\int e^{2\pi i \Psi(w,v,z,\mu,\xi)}\sigma_{n,2}(\xi)\,d\xi\Big| \leq C_N \frac{2^{3n}}{(1 + A(\varepsilon)^{-1}|w-v|)^N}, \tag{10.38}$$

allowing us to later integrate in w.

Suppose that $|n-k| > C_1$. Then since $\mathfrak{S}^i_w(w,z_3)$ are linearly independent for $i=1,2$

$$|\nabla_v \Psi| = \big| -\xi + 2^k \nabla_v(\mu\cdot \mathfrak{S}(v,z_3))\big| \geq \big||\xi| - |2^k\nabla_v(\mu\cdot\mathfrak{S}(v,z_3))|\big| \geq C_0 \max\{2^k, 2^n\}$$

by the implicit function theorem. We also see that $|\partial_v^\beta \Psi| \leq A(\varepsilon)2^k$ for any multi-index β with $|\beta|\geq 2$, and $|\partial_v^\beta \sigma| \leq C_{|\beta|}2^{\ell|\beta|}$ for any multi-index β with $|\beta|\geq 1$. Since $\ell < k/2$, integrating by parts in the v variables with the differential operator $L_v = \langle \frac{\nabla_v\Psi}{|\nabla_v\Psi|^2}, \nabla_v \cdot\rangle$ and Lemma 8.5 gives the estimate

$$\big|(L_v^*)^N \sigma_1(v,z,\mu)\big| \leq C_N \frac{A(\varepsilon)2^\ell}{C_0\max\{2^k,2^n\}} \leq C_N A(\varepsilon)\max\{2^{k/2},2^{n/2}\}^{-N}.$$

Combining this estimate with (10.38), we obtain

$$\begin{aligned}|K_{n,k,\ell,b_I}(w,z)| &\leq \int\int\Big|\int e^{2\pi i \Psi(w,v,z,\mu,\xi)}\sigma_{n,2}(\xi)\,d\xi\Big|\big|(L_v^*)^{2N}\sigma_1(v,z,\mu)\big|\,dv\,d\mu \\ &\leq C_N \int\int \frac{A(\varepsilon)}{\max\{2^k,2^n\}^N}\frac{1}{(1+A(\varepsilon)^{-1}|w-v|)^N}\,dv\,d\mu.\end{aligned}$$

As $\sigma_1(v,z,\mu)$ is supported where $|v|+|z|+|\mu|\leq 6$ by loss of a constant depending on ε we can integrate in v and μ to obtain (10.37).

10.5.2 The Proof of Lemma 10.4: Off Plate Estimates

In view of the support of $(1-M_{A(\varepsilon),b_I,\delta_1}(2^k\xi))$, one of four inequalities

$$\big|\big\langle \tfrac{T_1(b_I)}{|T_1(b_I)|},\xi\big\rangle\big| \geq 3A(\varepsilon)2^k \tag{10.39}$$

$$\big|\big\langle \tfrac{T_1(b_I)}{|T_1(b_I)|},\xi\big\rangle\big| \leq \tfrac{1}{3}A(\varepsilon)^{-1}2^k \tag{10.40}$$

$$\big|\big\langle \tfrac{T_2(b_I)}{|T_2(b_I)|},\xi\big\rangle\big| \geq 3A(\varepsilon)2^k\delta_1 \tag{10.41}$$

$$\big|\big\langle \tfrac{N(b_I)}{|N(b_I)|},\xi\big\rangle\big| \geq 3A(\varepsilon)2^k\delta_1^2 \tag{10.42}$$

must hold. Since we know that $|\xi| \simeq 2^n$ and $|n-k| < C_1$ we may assume (if necessary by making $A(\varepsilon)$ larger) that (10.41) or (10.42) must hold. Thus we apply integration by parts in the tangential and normal directions, as in the proof of Lemma 10.2. Also similar to the proof of Lemma 10.2, the normal direction will require much more careful estimates.

Suppose that (10.41) holds. Define the one-dimensional differential operator $\partial_{T_2(b_I)} = \langle T_2(b_I), \nabla_v \cdot \rangle$. Then by Lemma 10.2 (specifically (10.33))

$$|\partial_{T_2(b_I)}\Psi| \geq 2A(\varepsilon)2^k\delta_1.$$

We can also estimate for $j \geq 1$

$$|\partial^j_{T_2(b_I)}\sigma_1| \leq C_j A(\varepsilon)2^{\ell j},$$

and for $j \geq 2$

$$|\partial^j_{T_2(b_I)}\Psi| \leq C_j A(\varepsilon)2^k \leq C_j A(\varepsilon)2^{\ell(j-1)}2^k\delta_1.$$

Thus applying Lemma 8.5 to the operator $L_{T_2(b_I)} = \frac{1}{\partial_{T_2(b_I)}\Psi}\partial_{T_2(b_I)}$ and applying the estimate (10.38), we obtain

$$\begin{aligned}|K_{n,k,\ell,b_I}(w,z)| &\leq \int\int\Big|\int e^{2\pi i\Psi(w,v,z,\mu,\xi)}\sigma_{n,2}(\xi)\,d\xi\Big|\big|(L^*_{T_2(b_I)})^N\sigma_1(v,z,\mu)\big|\,dv\,d\mu \\ &\leq C_N\int\int\frac{2^{3k}}{(1+A(\varepsilon)^{-1}|w-v|)^4}\frac{1}{(2^{k-\ell}\delta_1)^N}\,dv\,d\mu.\end{aligned}$$

Since $2^{k-\ell}\delta_1 \geq 2^{k\varepsilon/2}$, integrating by parts in the $T_2(b_I)$ direction $\simeq 10/\varepsilon$ times and integrating over the compact support of σ_1 in v, μ gives the required estimate.

Next we assume that $|\langle N(b_I), \xi\rangle| \geq 3A(\varepsilon)2^k\delta_1^2$. Define $\partial_{N(b_I)} = \langle N(b_I), \nabla_v\rangle$. We will apply Lemma 8.5 to the one-dimensional differential operator $L_{N(b_I)} = \frac{1}{\partial_{N(b_I)}\Psi}\partial_{N(b_I)}$.

First, (10.34) implies

$$|\partial_{N(b_I)}\Psi| \geq 2A(\varepsilon)2^k\delta_1^2. \tag{10.43}$$

We claim that

$$|\partial^j_{N(b_I)}\sigma_1(v,z,\mu)| \leq C_jA(\varepsilon)\max\{2^\ell\delta_0,\varepsilon_1^{-1}\}^j \tag{10.44}$$

for every $j \geq 1$. To see this, we use the approximation $N(b_I) = -b_Ie_1 + e_3 + C(b_I)b_I^2$, where $|C(b_I)| < M_0$, from (10.27). From the definition of σ_1 we see for every $j \geq 1$ and every multi-index β with $|\beta| \leq j$ that

$$\begin{aligned}\left|(b_I\partial_{v_1})^{j-|\beta|}C(b_I)^{|\beta|}b_I^{2|\beta|}\partial_v^\beta\sigma_1(v,z,\mu)\right| &\leq C_j\left(2^\ell\delta_0\right)^{j-|\beta|}\left(2^\ell\delta_0^2\right)^{|\beta|}\\ &\leq C_j\left(2^\ell\delta_0\right)^j.\end{aligned} \tag{10.45}$$

Thus it suffices to check that (10.44) holds for mixed derivatives of the form

$$|b_I|^{|\beta|}\partial_{v_3}^{j-|\beta|}\partial_{v'}^\beta\sigma_1(v,z,\mu),$$

where $v' = (v_1, v_2)$, and β is a 2-dimensional multi-index such that $|\beta| < j$. Note that

$$\begin{aligned}\left||b_I|^{|\beta|}\partial_{v_3}^{j-|\beta|}\partial_{v'}^\beta\sigma_{\ell,\varepsilon_1}(v)\right| &\leq C_j(2^\ell\delta_0)^{|\beta|}\varepsilon_1^{|\beta|-j}\\ \left||b_I|^{|\beta|}\partial_{v_3}^{j-|\beta|}\partial_{v'}^\beta\zeta(v,z,\mu)\right| &\leq C_j\delta_0^{|\beta|},\end{aligned}$$

so it suffices to estimate

$$|b_I|^{|\beta|}\partial_{v_3}^{j-|\beta|}\partial_{v'}^\beta\chi_1(2^\ell\alpha(v,z_3)\mu\cdot\Delta(v,z_3)),$$

with $\chi_1(\cdot)$ replaced with $\chi_0(\cdot)$ if $\ell = \ell_\circ$. Note that terms for which no derivative hits $\mu\cdot\Delta(v,z_3)$ will be negligible since $|\mu\cdot\Delta(v,z_3)| \simeq 2^{-\ell}$. Using (10.17), (10.7), and a Taylor expansion about $z_3 = 0$ we see that

$$\begin{aligned}\left||b_I|^{|\beta|}\partial_{v_3}^{j-|\beta|}\partial_{v'}^\beta\mu\cdot\Delta(v,z_3)\right| &\leq \delta_0^{|\beta|}\partial_{v'}^\beta\mu\cdot\Delta_{v_3^j}(v,0) + A(\varepsilon)\delta_0^{|\beta|+1}\\ &= A(\varepsilon)\delta_0^{|\beta|+1}.\end{aligned}$$

Thus we see by differentiation of compositions and products

$$\left||b_I|^{|\beta|}\partial_{v_3}^{j-|\beta|}\partial_{v'}^{\beta}\sigma_1(v,z,\mu)\right| \leq C_j C_j A(\varepsilon)\max\{2^\ell\delta_0, \varepsilon_1^{-1}\}^j, \tag{10.46}$$

and by combining (10.45) and (10.46) the claim (10.44) is proven.

To apply Lemma 8.5 we also need to show that for $j \geq 2$

$$|\partial_{N(b_I)}^j \Psi| \leq C_j A(\varepsilon) 2^k \max\{2^\ell\delta_0, \varepsilon_1^{-1}\}^{j-1}\delta_1^2.$$

In fact, we claim that

$$|\partial_{N(b_I)}^j \Psi| \leq C_j A(\varepsilon) 2^k \delta_0^2 \tag{10.47}$$

for $j \geq 2$. We use (10.27) again to see that

$$\partial_{N(b_I)}^j \Psi = \langle b_I e_1 + e_3 + C(b_I) b_I^2, \nabla_v\rangle^j \Psi$$

where again $|C(b_I)| \leq M_0$. Rearranging terms using the fact that $|\partial_v^\beta \Psi| \leq A(\varepsilon)2^k$ for any multi-index β with $|\beta| \geq 2$ we obtain

$$\partial_{N(b_I)}^j \Psi = b_I \Psi_{v_1 v_3^{j-1}} + \Psi_{v_3^j} + A(\varepsilon) 2^k b_I^2 R_{10}(v, z_3).$$

Next, we estimate via a Taylor expansion about $z_3 = 0$,

$$\begin{aligned}\Psi_{v_1 v_3^{j-1}}(v, z_3) &= \mu \cdot \mathfrak{S}_{v_1 v_3^{j-1}}(v, 0) + 2M_j z_3 R_{11}(v, z_3)\\ \Psi_{v_3^j}(v, z_3) &= \mu \cdot \mathfrak{S}_{v_3^j}(v, 0) + z_3 \mu \cdot \mathfrak{S}_{v_3^j z_3}(v, 0) + 2M_j z_3^2 R_{12}(v, z_3).\end{aligned}$$

From (10.14), (10.15), and (10.16) we see that

$$\begin{aligned}\mu \cdot \mathfrak{S}_{v_1 v_3^{j-1}}(v, 0) &= 0,\\ \mu \cdot \mathfrak{S}_{v_3^j}(v, 0) &= 0,\\ \mathfrak{S}^1_{v_3^j z_3}(v, 0) &= 0.\end{aligned}$$

Moreover (10.6) ensures that

$$\mathfrak{S}^2_{v_3^j z_3}(v,0) = 0,\ \ j \geq 2.$$

Hence for $j \geq 2$

$$|\partial^j_{N(b_I)}\Psi| \leq C_j A(\varepsilon)\delta_0^2 \leq C_j A(\varepsilon)\varepsilon_1^{-j+1}\delta_1^2,$$

satisfying the claim (10.47).

Now that we have verified the conditions (10.43), (10.44), and (10.47), we can apply Lemma 8.5 with $L_{N(b_I)} = \frac{1}{\partial_{N(b_I)}\Psi}\partial_{N(b_I)}$ to obtain for every $M > 0$

$$|(L^*_{N(b_I)})^M \leq C_M \min\{2^{k-\ell}\delta_1^2/\delta_0, 2^k\delta_1^2\varepsilon_1\}^{-M}. \tag{10.48}$$

Combining (10.48) with (10.38) we can estimate

$$\begin{aligned}|K_{n,k,\ell,b_I}(w,z)| &\leq C_{M,\varepsilon}\int\int\left|\int e^{2\pi i\Psi(w,v,z,\mu,\xi)}\sigma_{n,2}(\xi)\,d\xi\right|\left|(L^*_{N(b_I)})^M\sigma_1(v,z,\mu)\right|dv\,d\mu \\ &\leq \Big(\frac{1}{\min\{2^{k-\ell}\delta_1^2/\delta_0, 2^k\delta_1^2\varepsilon_1\}}\Big)^M\int\int\frac{1}{(1+A(\varepsilon)^{-1}|w-v|)4}dv\,d\mu.\end{aligned}$$

Since $\delta_1 \geq 2^{-\ell(1-\varepsilon/2)}$ and $\delta_1 \geq 2^{-\ell\varepsilon/4}\delta_0$, we have

$$\begin{aligned}2^{k-\ell}\delta_1^2/\delta_0 \geq 2^{k-2\ell+\ell\varepsilon/4} &\geq 2^{k\varepsilon/8} \\ 2^k\delta_1^2\varepsilon_1 \geq 2^{k-2\ell+\ell\varepsilon/2} &\geq 2^{k\varepsilon/4}.\end{aligned}$$

So if $M \simeq 50/\varepsilon$ and we integrate over the compact support of σ_1 in v and μ we obtain the desired estimate.

Chapter 11

Decoupling in the General Case

Let $P^\circ = (a^\circ, y^\circ) \in \mathcal{M}$, with $y^\circ = (S^1(a^\circ, b^\circ), S^2(a^\circ, b^\circ), b^\circ)$. For $r > 0$ let

$$Q(r) = \{(x_1, x_2, x_3) \ : \ |x - a^\circ| \le r\}$$

and

$$I(r) = \{y_3 \ : \ |y_3 - b^\circ| \le r\}.$$

For $i = 1, 2$, let S^i be smooth functions in a neighborhood of $Q(2r_0) \times I(2r_0)$, for some $r_0 > 0$. After possibly permuting the variables y_1, y_2 we may assume in light of Lemma 3.11 that $\Delta^1(x, y_3) = \det(S^1_x, S^2_x, S^1_{xy_3}) \ne 0$ on $Q(2r_0) \times I(2r_0)$. Choose $M > 0$ so that

$$M > 2 + \|S\|_{C^5(Q(2r_0)\times I(2r_0))} + \max_{(x,y_3)\in Q(2r_0)\times I(2r_0)} |\Delta^1(x, y_3)|^{-1}.$$

We now will consider (a, b) close to (a°, b°) and construct changes of variables so that in the new coordinates the model case decoupling theorem in Proposition 10.1 can be applied at the suitable scale. These changes of variables were constructed in [46] to prove variable coefficient decoupling theorems in the case of folding canonical relations. However, since the canonical relation is invariant under changes of variables, we can use the same changes of variables with the additional assumption of a blowdown on π_R for our work.

11.1 Families of Changes of Variables

Let Γ_1, Γ_2 be defined as in (3.21),(3.22), and for $a \in Q(2r_0)$, $b \in I(2r_0)$ let $\rho(a,b) \in \mathbb{R}^3$ be defined by

$$(\rho_1, \rho_2, \rho_3) := \frac{1}{\Delta^1(a,b)}(-\Gamma_2(a,b), \Gamma_1(a,b), \Delta^2(a,b)).$$

For $x, a \in Q(r_0)$ and $b \in I(2r_0)$, consider the map

$$(x,a,b) \mapsto \mathfrak{w}(x,a,b) \in \mathbb{R}^3$$

given by

$$\begin{aligned}
\mathfrak{w}_1(x,a,b) &= S^1(x,b) - S^1(a,b)\\
\mathfrak{w}_2(x,a,b) &= S^2(x,b) - \rho_3(a,b)S^1(x,b) - S^2(a,b) + \rho_3(a,b)S^1(a,b)\\
\mathfrak{w}_3(x,a,b) &= S^1_{y_3}(x,b) - S^1_{y_3}(a,b).
\end{aligned}$$

Then

$$\det(D\mathfrak{w}/Dx) = \det\left(S^1_x(x,b),\ S^2_x - \rho_3(a,b)S^1_x(x,b),\ S^1_{xy_3}(x,b)\right) = \Delta^1(x,b) \neq 0.$$

By the implicit function theorem, there exists $r_1 \in (0, r_0)$ such that for $|w|_\infty < 2r_1$, $a \in Q(2r_1)$, and $b \in I(2r_1)$ the equation $\mathfrak{w}(x,a,b) = w$ is solved by a unique C^∞ function $x = \mathfrak{x}(w,a,b)$. Note that

$$|\rho_i(a,b)| \leq 6M^4,\ a \in Q(2r_0),\ b \in I(2r_0),\ i = 1,2,3.$$

By the definition of $\mathfrak{w}$ and the mean value theorem this implies $|\mathfrak{w}(x,a,b)|_\infty \leq 3M(1+6M^4)|x-a|_\infty$ for $x, a \in Q(2r_0)$ and $b \in I(2r_0)$. Hence for any $r_2 < r_1$ if $|x-a|_\infty < r_2$ and $|a - a^\circ| < r_2$ then $|\mathfrak{w}(x,a,b)|_\infty \leq 42M^5 r_2$. If we define

$$r_2 = (50M^5)^{-1} r_1$$

then we get $|\mathfrak{w}(x,a,b)| < r_1$ for $x, a \in Q(r_2),\ b \in I(2r_1)$. Thus by the uniqueness of the function $\mathfrak{x}$ we therefore have $\mathfrak{x}(\mathfrak{w}(x,a,b),a,b) = x$ for $x, a \in Q(r_2)$ and $b \in I(2r_1)$.

Note that $\mathfrak{w}(a,a,b) = 0$, implying $\mathfrak{x}(0,a,b) = \mathfrak{x}(\mathfrak{w}(a,a,b)) = a$. We also see that $\det(D\mathfrak{x}(w,ab)/Dw) = \frac{1}{\Delta^1(\mathfrak{x}(w,a,b),b)}$.

We also change variables in y. Define $\mathfrak{z} : \mathbb{R}^2 \times I(2r_0) \times Q(2r_0) \times I(2r_0) \to \mathbb{R}^3$ by

$$\mathfrak{z}_1(y,a,b) = y_1 - S^1(a,y_3)$$

$$\mathfrak{z}_2(y,a,b) = y_2 - S^2(a,y_3) - \rho_3(a,b)(y_1 - S^1(a,y_3)) - (y_3 - b)\sum_{i=1}^{2} \rho_i(a,b)(y_i - S^i(a,y_3))$$

$$\mathfrak{z}_3(y,a,b) = y_3 - b.$$

The Jacobian of this map is given by

$$\det(D\mathfrak{z}/Dy) = (1 - \rho_2(a,b)(y_3 - b)), \tag{11.1}$$

which by the bound on $|\rho_i|$ on lies between $(1/2, 3/2)$ if $z_3, b \in I(r_3)$, $a \in Q(2r_0)$, where $r_3 < \min\{r_1, (24M^4)^{-1}\}$. For $|z_3| \leq r_3$, $b \in I(r_3)$, and $a \in Q(2r_0)$ we can define the inverse $z \mapsto \mathfrak{y}(z,a,b)$ explicitly by

$$\begin{aligned}
\mathfrak{y}_1(z,a,b) &= z_1 + S^1(a, b + z_3) \\
\mathfrak{y}_2(z,a,b) &= \frac{z_2 + z_1(\rho_3(a,b) + \rho_1(a,b)z_3) + (1 - z_3)S^2(a, b + z_3)}{1 - \rho_2(a,b)z_3} \\
\mathfrak{y}_3(z,a,b) &= b + z_3.
\end{aligned}$$

Notice that $\mathfrak{y}(0,a,b) = (S^1(a,b), S^2(a,b), b)$. Other properties of these changes variables are contained in the following lemma, proven in [46].

Lemma 11.1 ([46, Lemma 7.1])**.** *The function $\mathfrak{x}, \mathfrak{y}$ defined above have the following properties.*

1. *Let $\rho = \rho(a,b)$, and let*

$$B(z_3,a,b) = \begin{pmatrix} 1 & 0 \\ -\rho_3 - \rho_1 z_3 & 1-\rho_2 z_3 \end{pmatrix}.$$

Then for $|z_3| \le r_3$, $a \in Q(r_2)$, $|w| \le r_2$

$$B(z_3,a,b)\begin{pmatrix} S^1(\mathfrak{x}(w,a,b),b+z_3) - \mathfrak{y}_1(z,a,b) \\ S^2(\mathfrak{x}(w,a,b),b+z_3) - \mathfrak{y}_2(z,a,b) \end{pmatrix} = \begin{pmatrix} \mathfrak{S}^1(w,z_3,a,b) - z_1 \\ \mathfrak{S}^2(w,z_3,a,b) - z_2 \end{pmatrix},$$

where $\mathfrak{S}^i$ are C^∞ satisfying

$$(\mathfrak{S}^1, \mathfrak{S}^2, \mathfrak{S}^1_{z_3})\big|_{(w,0,a,b)} = w$$

and $\mathfrak{S}^2_{wz_3}(0,0,a,b) = 0$.

2. *Let*

$$\Delta^i_S(x,y_3) = \det(S^1_x, S^2_x, S^i_{xy_3})\big|_{(x,y_3)}$$

$$\Delta^i_{\mathfrak{S}}(x,y_3,a,b) = \det(\mathfrak{S}^1_w, \mathfrak{S}^2_w, \mathfrak{S}^i_{wz_3})\big|_{(w,z_3,a,b)}.$$

Then for $(\tau_1,\tau_2) = (\mu_1,\mu_2)B(z_3,a,b)$,

$$\sum_{i=1}^{2} \tau_i \Delta^i_S(\mathfrak{x}(w,a,b), b+z_3) = \frac{\Delta^1_S(\mathfrak{x}(w,a,b),b)}{1-\rho_2(a,b)z_3} \sum_{i=1}^{2} \mu_i \Delta^i_{\mathfrak{S}}(w,z_3,a,b).$$

3. *Let $\kappa(x,y_3)$ be defined as in (3.24). Then*

$$\mathfrak{S}^2_{w_3 z_3 z_3}(0,0,a,b) = \frac{\kappa(a,b)}{(\Delta^1_S(a,b))^2}.$$

11.2 Decoupling in the General Case

We begin by proving one step of the induction in the general case by using the above changes of variables to reduce to the model case in Proposition 10.1. After applying a partition of unity in x, y_3 to the symbol of $\mathcal{R}_{k,\ell}$ we may assume that the support of $\chi(x,y)$ in (x,y_3) lies within $Q(r_2)\times I(r_3)$, which will allow us to apply Lemma 11.1.

Proposition 11.2. *Let* $0<\varepsilon<\frac{1}{10}$, $k\gg 1$, $1<\ell\le \ell_\circ(k,\varepsilon)$. *Let* $\delta_0\in(2^{-\ell(1-\varepsilon)},2^{-\ell\varepsilon}]$, *and let* $\delta_1\in(0,\delta_0)$ *such that*

$$\max\{2^{-\ell(1-\varepsilon/2)},\delta_0 2^{-\ell\varepsilon/4}\}<\delta_1<\delta_0.$$

Define $\varepsilon_1=(\delta_1/\delta_0)^2$. *Let* J *be an interval of length* δ_0 *such that* $\operatorname{dist}(b,b^\circ)\le r_3$ *for any* $b\in J$, *and let* $\mathcal{I}_J$ *be a collection of intervals* I *of length* δ_1 *which have disjoint interior and intersect* J. *For each* $I\in\mathcal{I}_J$, *define* $f_I(y)=f(y)\mathbb{1}_I(y_3)$. *Then given a compactly supported function* $v_0\in C_c^\infty(\Omega_L)$, *for* $2\le p\le 6$

$$\Big\|\sum_{I\in\mathcal{I}_J} v_0\mathcal{R}_{k,\ell}f_I\Big\|_p\le C_\varepsilon(\delta_0/\delta_1)^{\frac12-\frac1p+\varepsilon}\Big(\sum_{I\in\mathcal{I}_J}\|v_0\mathcal{R}_{k,\ell}f_I\|_p^p\Big)^{1/p}+C_\varepsilon 2^{-10k}\|f\|_p.$$

Proof. Fix $b\in J$. For each $a\in\operatorname{supp}v_0\cap Q(r_2)$ define the connected open set

$$\mathcal{U}_{a,b}=\Big\{x\in\mathbb{R}^3\ :\ |\mathfrak{w}_1(x,a,b)|\le 2^{-\ell},\ |\mathfrak{w}_2(x,a,b)|\le 2^{-\ell},\ |\mathfrak{w}_3(x,a,b)|\le\varepsilon_1\Big\}.$$

Because $\mathcal{U}_{a,b}$ are open sets which contain a neighborhood of a, $\{\mathcal{U}_{a,b}\}_{a\in\operatorname{supp}v_0\cap Q(r_2)}$ is an open cover of the compact set $\operatorname{supp}v_0\cap Q(r_2)$, which thus admits a finite subcover $\{\mathcal{U}_{a_\lambda,b}\}_{\lambda\in\Lambda}$. Moreover, because $\det(D\mathfrak{w}/dx)=\Delta^1(a,b)\ne 0$, each set $\mathcal{U}_{a,b}$ contains a, has measure $\simeq 2^{-2\ell}\varepsilon_1$, and is the image of a $2^{-\ell}\times 2^{-\ell}\times\varepsilon_1$ rectangle under a C^∞ map smoothly varying with a; thus we may further assume $|\Lambda|\lesssim 2^{2\ell}\varepsilon_1^{-1}r_2^3$ and each point x is covered by a uniformly bounded number of sets $\mathcal{U}_{a_\lambda,b}$ by the support of χ in x.

Let $\{\varsigma_{\ell,\varepsilon_1,a_\lambda,b}(x)\}_{\lambda\in\Lambda}$ be a smooth partition of unity adapted to $\mathcal{U}_{a_\lambda,b}$. Then since $|x|,|a_\lambda|<r_2/2$ and $|b|<r_1$ we have for each $\lambda\in\Lambda$ that

$$\sigma_{\ell,\varepsilon_1,a_\lambda,b}(w):=v_0(\mathfrak{x}(w,a,b))\varsigma_{\ell,\varepsilon_1,a_\lambda,b}(\mathfrak{x}(w,a_\lambda,b))$$

is a smooth function supported on the set $\{w \ : \ |w'| < 2^{-\ell}, \ |w_3| < \varepsilon_1\}$. By Hölder's inequality

$$\Big\| \sum_{I\in\mathcal{I}_J} v_0 \mathcal{R}_{k,\ell} f_I \Big\|_p = \Big\| \sum_{I\in\mathcal{I}_J}\sum_{\lambda\in\Lambda} v_0 \varsigma_{\ell,\varepsilon_1,a_\lambda,b}\mathcal{R}_{k,\ell} f_I \|_p$$
$$\leq C_p \Big(\sum_{\lambda\in\Lambda} \Big\| \sum_{I\in\mathcal{I}_J} v_0 \varsigma_{\ell,\varepsilon_1,a_\lambda,b}\mathcal{R}_{k,\ell} f_I \Big\|_p^p \Big)^{1/p}.$$

Fix $a \in \{a_\lambda\}_{\lambda\in\Lambda}$. Write $g(z,a,b) = f(\mathfrak{y}(z,a,b))$, noting that $f_I(\mathfrak{y}(z,a,b)) = g_{-b+I}(z,a,b)$ since $\mathfrak{y}_3(z,a,b) = b + z_3$. Next, we apply changes of variables $y = \mathfrak{y}(z,a,b)$ and $\tau = 2^k B^{\intercal -1}(z_3,a,b)\mu$, noting from (11.1) and the definition of B that

$$\det(D\mathfrak{y}/dz)\det B = 1,$$

so that

$$\mathcal{R}_{k,\ell} f_I(x) = 2^{2k} \int\int e^{2\pi i 2^k \mu\cdot(\mathfrak{S}(\mathfrak{w}(x,a,b))-z')} \tilde{\chi}_{k,\ell}(x,z,\mu,a,b) g_{-b+I}(z,a,b)\, d\mu dz,$$

where

$$\tilde{\chi}_{k,\ell}(x,z,\mu,a,b) = \chi_1\Big(2^\ell \tfrac{\Delta^1(x)}{1-\rho_3(a,b)z_3}\big(\mu_1 \Delta^1_{\mathfrak{S}}(\mathfrak{w}(x,a,b),z_3,a,b) + \mu_2\Delta^2_{\mathfrak{S}}(\mathfrak{w}(x,a,b),z_3,a,b)\big)\Big)$$
$$\times\, \chi(x,\mathfrak{y}(z,a,b))\chi_1(|B^{\intercal -1}(z_3,a,b)\mu|)$$

Thus applying the change of variables $x = \mathfrak{x}(w,a,b)$ we see that

$$v_0(\mathfrak{x}(w,a,b))\varsigma_{\ell,\varepsilon_1,a,b}(\mathfrak{x}(w,a,b)) \sum_{I\in\mathcal{I}_J} \mathcal{R}_{k,\ell} f_I(\mathfrak{x}(w,a,b)) = \sigma_{\ell,\varepsilon_1,a,b}(w)\sum_{I\in\mathcal{I}_J}\mathcal{T}_{k,\ell,a,b} g_{-b+I}(w),$$

where $\mathcal{T}_{k,\ell,a,b} \equiv \mathcal{T}_{k,\ell}$ from (10.10) in the model case. Define

$$M_n(a,b) \geq 2 + ||\mathfrak{S}^1(\cdot,a,b)||_{C^{n+5}([-r_0,r_0]^4)} + ||\mathfrak{S}^2(\cdot,a,b)||_{C^{n+5}([-r_0,r_0]^4)}$$
$$\tilde{A}(\varepsilon) = \sup_{a,b\in[-r_0,r_0]^4} \max\{3^{\lceil\frac{3}{\varepsilon}\rceil+2} M_{\lceil\frac{3}{\varepsilon}\rceil}(a,b), \kappa_0(a,b)(1+4^{\lceil\frac{3}{\varepsilon}\rceil+2}M^4_{\lceil\frac{3}{\varepsilon}\rceil}(a,b))\};$$

these are the uniform versions of (10.2) and (10.36) respectively. We can then write

$$\begin{aligned}\Big\|v_0\varsigma_{\ell,\varepsilon_1,a,b}\sum_{I\in\mathcal{I}_J}\mathcal{R}_{k,\ell}f_I\Big\|_p &= \Big(\int\Big|v_0\varsigma_{\ell,\varepsilon_1,a,b}\sum_{I\in\mathcal{I}_J}\mathcal{R}_{k,\ell}f_I\Big|_{x=\mathfrak{r}(w,a,b)}\Big|^p|\det(\tfrac{D\mathfrak{r}}{Dw})|\,dw\Big)^{1/p}\\ &\leq C_p\Big\|\sigma_{\ell,\varepsilon_1,a,b}\sum_{I\in\mathcal{I}_J}\mathcal{T}_{k,\ell,a,b}g_{-b+I}\Big\|_p\end{aligned}$$

by the uniform upper bound on $|\det(\frac{D\mathfrak{r}}{Dw})|$. This inequality allows us to apply Proposition 10.1 with $A(\varepsilon)=\tilde{A}(\varepsilon)$ to get

$$\Big\|\sigma_{\ell,\varepsilon_1,a,b}\sum_{I\in\mathcal{I}_J}\mathcal{T}_{k,\ell}g_I\Big\|_p\leq C_\varepsilon(\delta_0/\delta_1)^{\frac{1}{2}-\frac{1}{p}+\varepsilon}\Big(\sum_{I\in\mathcal{I}_J}\big\|\sigma_{\ell,\varepsilon_1,a,b}\mathcal{T}_{k,\ell}g_I\big\|_p^p\Big)^{1/p}+C_\varepsilon 2^{-10k}2^{-3\ell}\|g\|_p.$$

Undoing the changes of variables above (and again applying the uniform lower bounds on $|\det(\frac{D\mathfrak{r}}{Dw})|$) we may bound this by

$$C'_\varepsilon(\delta_0/\delta_1)^{\frac{1}{2}-\frac{1}{p}+\varepsilon}\Big(\sum_{I\in\mathcal{I}_J}\big\|\varsigma_{\ell,\varepsilon_1,a,b}\mathcal{R}_{k,\ell}f_I\big\|_p^p\Big)^{1/p}+C_\varepsilon 2^{-10k}2^{-3\ell}\|f\|_p.$$

Finally, we recombine our partition of unity in x using the fact that $|\Lambda|\lesssim 2^{2\ell}\varepsilon^{-1}r_3^3$ to get

$$\begin{aligned}\Big\|\sum_{I\in\mathcal{I}_J}v_0\mathcal{R}_{k,\ell}f_I\Big\|_p &\leq C_p\Big(\sum_{\lambda\in\Lambda}\Big\|\sum_{I\in\mathcal{I}_J}v_0\varsigma_{\ell,\varepsilon_1,a_\lambda,b}\mathcal{R}_{k,\ell}f_I\Big\|_p^p\Big)^{1/p}\\ &\leq C_pC_\varepsilon(\delta_0/\delta_1)^{\frac{1}{2}-\frac{1}{p}+\varepsilon}\Big(\sum_{\lambda\in\Lambda}\sum_{I\in\mathcal{I}_J}\big\|v_0\varsigma_{\ell,\varepsilon_1,a_\lambda,b}\mathcal{R}_{k,\ell}f_I\big\|_p^p\Big)^{1/p}\\ &\quad+\sum_{\lambda\in\Lambda}C_\varepsilon 2^{-3\ell}2^{-10k}\|f\|_p\\ &\leq C_{p,\varepsilon}(\delta_0/\delta_1)^{\frac{1}{2}-\frac{1}{p}+\varepsilon}\Big(\sum_{I\in\mathcal{I}_J}\|v_0\mathcal{R}_{k,\ell}f_I\|_p^p\Big)^{1/p}+C_\varepsilon 2^{-10k}\|f\|_p.\end{aligned}$$

$\square$

11.3 Iteration of the Decoupling Step

Let $v_0 \in C_c^\infty(\Omega_L)$ and let $f \in L^p(\Omega_R)$ be compactly supported. Let $\delta_0 = 2^{-\ell\varepsilon}$, and define $\delta_j = \delta_{j-1} 2^{-\ell\varepsilon/4}$ for $j = 1, 2, ...$ Note that this implies $\varepsilon_1 = (\delta_1/\delta_0)^2 = 2^{-\ell\varepsilon/2}$. We will iterate the estimate in Proposition 11.2 until $\delta_j \leq 2^{-\ell(1-\varepsilon)}$. Let j^* be the smallest j such that $\delta_j < 2^{-\ell(1-\varepsilon)}$. Clearly $j^* \leq 4/\varepsilon$ and $2^{-\ell(1-\varepsilon/2)} \leq \delta_{j^*} \leq 2^{-\ell(1-\varepsilon)}$.

To iterate the decoupling argument we construct a nested family of intervals which at each level have disjoint interior. Let $J = [b^\circ - r_3, b^\circ + r_3]$ and for each $j = 0, 1, 2, ...$ tile J by a family of intervals $\mathcal{I}_{J,j}$ such that each $I_j \in \mathcal{I}_{J,j}$ intersects J and has length δ_j, and such that all intervals in $\mathcal{I}_{J,j}$ have mutually disjoint interiors. For an interval $I_j \in \mathcal{I}_{J,j}$, let $\mathcal{I}_{I_j}$ denote the collection of intervals $I_{j+1} \in \mathcal{I}_{J,j+1}$ which intersect I_j. Then since $r_3 < 1$ and $\delta_0 = 2^{-\ell\varepsilon}$, using Hölder's and Minkowski's inequalities we have

$$\|v_0 \mathcal{R}_{k,\ell} f\|_p \lesssim 2^{\ell\varepsilon/p'} \Big(\sum_{I_0 \in \mathcal{I}_{J,0}} \Big\| v_0 \mathcal{R}_{k,\ell} f_{I_0} \Big\|_p^p \Big)^{1/p}. \tag{11.2}$$

The function and operator $\mathcal{R}_{k,\ell} f_{I_0}$ now satisfy the conditions of Proposition 11.2. We claim that for each $0 \leq j \leq j^*$,

$$\begin{aligned} \|v_0 \mathcal{R}_{k,\ell} f\|_p \lesssim C(\varepsilon)^j 2^{\ell\varepsilon/p'} (\delta_0/\delta_j)^{\frac{1}{2}-\frac{1}{p}+\varepsilon} \Big(\sum_{I_j \in \mathcal{I}_{J,j}} \|v_0 \mathcal{R}_{k,\ell} f_{I_j}\|_p^p \Big)^{1/p} \\ + j 2^{2\ell} C(\varepsilon)^j 2^{-10k} \|f\|_p. \end{aligned} \tag{11.3}$$

The case $j = 0$ follows immediately from (11.2). Assume (11.3) holds for some j. Then by applying Proposition 11.2 we get

$$\begin{aligned}\Big(\sum_{I_j\in\mathcal{I}_{J,j}}\|v_0\mathcal{R}_{k,\ell}f_{I_j}\|_p^p\Big)^{1/p} &\le \Big(\sum_{I_j\in\mathcal{I}_{J,j}}\Big[C(\varepsilon)\big(\tfrac{\delta_j}{\delta_{j+1}}\big)^{\frac12-\frac1p+\varepsilon}\Big(\sum_{I_{j+1}\in I_{I_j}}\|v_0\mathcal{R}_{k,\ell}f_{I_{j+1}}\|_p^p\Big)^{1/p}\\ &\qquad + C(\varepsilon)2^{-10k}\|f\|_p^p\Big]^p\Big)^{1/p}\\ &\le C(\varepsilon)\big(\tfrac{\delta_j}{\delta_{j+1}}\big)^{\frac12-\frac1p+\varepsilon}\Big(\sum_{I_{j+1}\in\mathcal{I}_{J,j+1}}\|v_0\mathcal{R}_{k,\ell}f_{I_{j+1}}\|_p\Big)^{1/p}\\ &\qquad + C(\varepsilon)\delta_j^{-1/p}2^{-10k}\|f\|_p. \end{aligned}\tag{11.4}$$

Plugging the above estimate into (11.3) gives us

$$\begin{aligned}\|\mathcal{R}_{k,\ell}f\|_p \le{}& C(\varepsilon)^{j+1}2^{\ell\varepsilon/p'}\big(\tfrac{\delta_0}{\delta_{j+1}}\big)^{\frac12-\frac1p+\varepsilon}\Big(\sum_{I_{j+1}\in\mathcal{I}_{J,j+1}}\|v_0\mathcal{R}_{k,\ell}f_{I_{j+1}}\|_p^p\Big)^{1/p}\\ &+ C(\varepsilon)^j2^{\ell\varepsilon/p'}\big(\tfrac{\delta_0}{\delta_j}\big)^{\frac12-\frac1p+\varepsilon}C(\varepsilon)\delta_j^{-1/p}2^{-10k}\|f\|_p\\ &+ j2^{2\ell}C(\varepsilon)^j2^{-10k}\|f\|_p.\end{aligned}$$

Using the fact that $\delta_0 = 2^{-\ell\varepsilon}$, $\delta_j \ge 2^{\ell(1-\varepsilon/2)}$ for $j \le j^*$, and $2 \le p \le 6$, the last two terms of the above inequality are bounded by

$$(j+1)C(\varepsilon)^{j+1}2^{2\ell}2^{-10k}\|f\|_p,$$

proving the claim.

We apply (11.3) for $j = j^*$ and use the fact that $j^* \le 4/\varepsilon$ as well as the assertion

$$\frac{\varepsilon}{p'} - \frac{\varepsilon}{2} + \frac{\varepsilon}{p} - \varepsilon^2 - \frac{\varepsilon}{4} + \frac{\varepsilon}{2p} + \frac{\varepsilon^2}{2} \le \varepsilon$$

to deduce

$$\begin{aligned}\|v_0\mathcal{R}_{k,\ell}f\|_p &\le C(\varepsilon)^{4/\varepsilon}2^{\ell\varepsilon/p'}2^{-\ell\varepsilon\left(\frac{1}{2}-\frac{1}{p}+\varepsilon\right)}2^{\ell(1-\varepsilon/2)\left(\frac{1}{2}-\frac{1}{p}+\varepsilon\right)}\Big(\sum_{I_{j^*}\in\mathcal{I}_{J,j^*}}\|v_0\mathcal{R}_{k,\ell}f_{I_{j^*}}\|_p^p\Big)^{1/p}\\ &\qquad + \tfrac{4}{\varepsilon}C(\varepsilon)^{4/\varepsilon}2^{-10k+2\ell}\|f\|_p\\ &\lesssim_\varepsilon 2^{\ell\left(\frac{1}{2}-\frac{1}{p}+2\varepsilon\right)}\Big(\sum_{I_{j^*}\in\mathcal{I}_{J,j^*}}\|v_0\mathcal{R}_{k,\ell}f_{I_{j^*}}\|_p^p\Big)^{1/p}+C(\varepsilon)2^{-9k}\|f\|_p. \end{aligned}\tag{11.5}$$

Picking $\varepsilon' = 2\varepsilon$ completes the proof.

Chapter 12

The Proof of Proposition 8.8

In this chapter we prove Proposition 8.8 using Littlewood-Paley theory and a Calderón-Zygmund estimate, the main result of [43]. Let $v_0 \in C_c^\infty(\Omega_L)$ and let $f \in L^p$ be supported in a compact set $K \subset \Omega_R$. Let $0 < q < 2 < 4 < p < \infty$. By an application of Lemma 7.1 we can reduce the proof of (8.9) to the estimate

$$\Big\|\Big(\sum_{k+C_1 \geq (2+\varepsilon)\ell} \big|2^{k/p} P_k v_0 \mathcal{R}_{k+s_1,\ell} v_1 P_{k+s_2} f\big|^q\Big)^{1/q}\Big\|_{L^p} \leq C 2^{-\ell\varepsilon(p)} \Big\|\Big(\sum_{k>0} |P_{k+s_2} f|^p\Big)^{1/p}\Big\|_{L^p} \tag{12.1}$$

where $v_1 \in C_c^\infty(\Omega_R)$ is equal to 1 on the support of f, and $|s_1|, |s_2| \leq C_1$, where C_1 is the necessary constant from Lemma 7.1. Indeed, by expanding the definition of $\mathcal{R}_\ell$ and applying a similar argument to the proof of Proposition 3.4, we see for every $k \geq 0$

$$\begin{aligned} P_k v_0 \mathcal{R}_\ell f &= P_k \Big(\sum_{k' \geq (2+\varepsilon)\ell} v_0 \mathcal{R}_{k',\ell} \Big(\sum_{k''} v_1 P_{k''} f\Big)\Big) \\ &= P_k \Big(\sum_{|s_1|, |s_2| \leq C_1} \mathcal{R}_{k+s_1,\ell} v_1 P_{k+s_2} f + \sum_{(k',k") \in \mathcal{D}_k} \mathcal{R}_{k',\ell} v_1 P_{k''} f\Big). \end{aligned}$$

Note that terms in the first sum vanish if $k + s_1 < (2+\varepsilon)\ell$. Applying Hölder's inequality

$$\begin{aligned} \|v_0 \mathcal{R}_\ell f\|_{F^{p,q}_{1/p}} \leq 2^{\frac{1}{q}-1} &\Big\|\Big(\sum_{k+C_1 \geq (2+\varepsilon)\ell} \Big|2^{k/p} P_k \Big(\sum_{|s_1|, |s_2| \leq C_1} v_0 \mathcal{R}_{k+s_1,\ell} v_1 P_{k+s_2} f\Big)\Big|^q\Big)^{1/q}\Big\|_{L^p} \\ &+ 2^{\frac{1}{q}-1} \Big\|\Big(\sum_k \Big|\sum_{(k',k'') \in \mathcal{D}_k} 2^{k/p} P_k v_0 \mathcal{R}_{k',\ell} v_1 P_{k''} f\Big|^q\Big)^{1/q}\Big\|_{L^p}, \end{aligned}$$

where $\mathcal{D}_k$ is defined as in (7.3). Applying Hölder's inequality a second time we can break $\|v_0\mathcal{R}_\ell f\|_{F^{p,q}_{1/p}}$ into pieces for which (12.1) apply, and a remainder

$$\mathcal{E}_{p,q}f = \Big(\sum_{k\geq 0}\Big|\sum_{(k',k'')\in\mathcal{D}_k} 2^{k/p}P_k v_0\mathcal{R}_{k',\ell}v_1P_{k''}f\Big|^q\Big)^{1/q}.$$

Indeed,

$$\begin{aligned}\|v_0\mathcal{R}_\ell f\|_{F^{p,q}_{1/p}} \leq (8C_1^2)^{\frac{1}{q}-1}\sum_{|s_1|,|s_2|\leq C_1}\Big\|\Big(\sum_{k+C_1\geq(2+\varepsilon)\ell}\big|2^{k/p}P_kv_0\mathcal{R}_{k+s_1,\ell}v_1P_{k+s_2}f\big|^q\Big)^{1/q}\Big\|_{L^p}\\ + 2^{\frac{1}{q}-1}\|\mathcal{E}_{p,q}f\|_{L^p}.\end{aligned}$$

We next show that the remainder $\mathcal{E}_{p,q}$ is bounded from $B^{p,p}_0 \to L^p$. This holds due to Lemma 7.1 after several applications of Hölder's inequality, which we present below. Note that since $\ell \leq \frac{k+s_1}{2}$ Lemma 7.1 still applies even though the symbol of the kernel of $\mathcal{R}_{k+s_1,\ell}$ depends on ℓ. For any $\varepsilon > 0$ by applying Hölder's inequality several times we obtain

$$\begin{aligned}\|\mathcal{E}_{p,q}f\|_{L^p} &\leq \sum_{k',k''\geq 0}\Big\|\Big(\sum_{\substack{k\geq 0\\(k',k'')\in\mathcal{D}_k}} 2^{-k\varepsilon q}\big|2^{k\varepsilon+k/p}P_kv_0\mathcal{R}_{k',\ell}v_1P_{k''}f\big|^q\Big)^{1/q}\Big\|_p\\
&\leq \sum_{k',k''\geq 0}\Big(\sum_{k\geq 0}2^{-k\varepsilon q}\Big)^{1/q}\sup_{\substack{k\geq 0\\(k',k'')\in\mathcal{D}_k}} 2^{k\varepsilon+k/p}\big\|P_kv_0\mathcal{R}_{k',\ell}v_1P_{k''}f\big\|_p\\
&\leq C_{q,\varepsilon}\sum_{k',k''\geq 0}2^{-k'\varepsilon}2^{k'\varepsilon}\sup_{\substack{k\geq 0\\(k',k'')\in\mathcal{D}_k}} 2^{k\varepsilon+k/p}\big\|P_kv_0\mathcal{R}_{k',\ell}v_1P_{k''}f\big\|_p\\
&\leq C_{q,\varepsilon}\sum_{k''\geq 0}\Big(\sum_{k'\geq 0}2^{-k\varepsilon}\Big)\sup_{\substack{k,k'\geq 0\\(k',k'')\in\mathcal{D}_k}} 2^{(k+k')\varepsilon+k/p}\big\|P_kv_0\mathcal{R}_{k',\ell}v_1P_{k''}f\big\|_p\\
&\leq C_{q,\varepsilon}\sum_{k''\geq 0}2^{-k''\varepsilon}2^{k''\varepsilon}\sup_{\substack{k,k'\geq 0\\(k',k'')\in\mathcal{D}_k}} 2^{(k+k')\varepsilon+k/p}\big\|P_kv_0\mathcal{R}_{k',\ell}v_1P_{k''}f\big\|_p\\
&\leq C_{q,\varepsilon}\Big(\sum_{k''\geq 0}2^{-k\varepsilon p'}\Big)^{1/p'}\Big(\sum_{k''\geq 0}\sup_{\substack{k,k'\geq 0\\(k',k'')\in\mathcal{D}_k}} 2^{(k+k'+k'')\varepsilon p+k}\big\|P_kv_0\mathcal{R}_{k',\ell}v_1P_{k''}f\big\|_p^p\Big)^{1/p}.\end{aligned}$$

Since $P_{k''}$ is a Littlewood-Paley multiplier $P_{k''}f = \sum_{|s|\lesssim 1} P_{k''}P_{k''+s}f$. By Lemma 7.1 we can thus estimate for each $k'' \geq 0$

$$\sup_{\substack{k,k'\geq 0\\ (k',k'')\in\mathcal{D}_k}} 2^{(k+k'+k'')\varepsilon+k/p}\big\|P_k v_0 \mathcal{R}_{k',\ell} v_1 P_{k''} f\big\|_p \leq C_p \Big\|\sum_{|s|\lesssim 1} P_{k''+s} f\Big\|_p,$$

where C_p does not depend on k''. The claim holds by one last application of the triangle inequality and rearrangement of the sum.

To prove (12.1) we apply the main result from [43].

Theorem 12.1 ([43, Theorem 1.1])**.** *Let T_k be a family of operators on Schwartz functions by*

$$T_k f(x) = \int K_k(x,y) f(y)\, dy.$$

Let $\phi \in \mathcal{S}(\mathbb{R}^3)$, $\phi_k = 2^{3k}\phi(2^k\cdot)$, and $\Pi_k f = \phi_k * f$. Let $\varepsilon > 0$ and $1 < p_0 < p < \infty$. Assume T_k satisfies

$$\sup_{k>0} 2^{k/p}\|T_k\|_{L^p\to L^p} \leq A \tag{12.2}$$

$$\sup_{k>0} 2^{k/p_0}\|T_k\|_{L^{p_0}\to L^{p_0}} \leq B_0. \tag{12.3}$$

Further let $A_0 \geq 1$, and assume that for each cube Q there is a measurable set E_Q such that

$$|E_Q| \leq A_0 \max\{|Q|^{2/3}, |Q|\}, \tag{12.4}$$

and for every $k \in \mathbb{N}$ and every cube Q with $2^k\mathrm{diam}(Q) \geq 1$,

$$\sup_{x\in Q}\int_{\mathbb{R}^d\setminus E_Q} |K_k(x,y)|\, dy \leq B_1 \max\left\{\left(2^k\mathrm{diam}(Q)\right)^{-\varepsilon}, 2^{-k\varepsilon}\right\}. \tag{12.5}$$

Let

$$\mathcal{B} = B_0^{q/p}(AA_0^{1/p} + B_1)^{1-q/p}.$$

Then for any $q > 0$ there is a C depending on ε, p, p_0, q such that

$$\Big\|\Big(\sum_k 2^{kq/p}|\Pi_k T_k f_k|^q\Big)^{1/q}\Big\|_p \le CA\left[\log\left(3+\tfrac{\mathcal{B}}{A}\right)\right]^{\frac{1}{q}-\frac{1}{p}}\Big(\sum_k \|f_k\|_p^p\Big)^{1/p}. \tag{12.6}$$

We apply this theorem on the family of operators $T_k := \mathcal{R}_{k,\ell}$ for $k \ge (2+\varepsilon)\ell$ (here ℓ is fixed). By Proposition 8.1 the assumptions (12.2) and (12.3) are satisfied with $A \lesssim 2^{-\ell\varepsilon(p)}$ and $B_0 \lesssim 2^{-\ell\varepsilon(p_0)}$. We next check assumptions (12.4) and (12.5). For a given cube Q with center x_Q let

$$E_Q = \{y \,:\, |S(x^Q, y_3) - y'| \le C2^\ell \mathrm{diam}(Q)\}$$

if $\mathrm{diam}(Q) < 1$, and a cube centered at x^Q of diameter $C2^\ell\mathrm{diam}(Q)$ if $|Q| \ge 1$. By an integration by parts argument in the τ variables we derive the bound

$$|K_k(x,y)| \lesssim_N \frac{2^{2k}}{(1+2^{k-\ell}|S(x^Q,y_3)-y'|)^N}.$$

Then clearly assumptions (12.4) and (12.5) are satisfied with $A_0 \lesssim 2^{3\ell}$ and $B_1 \lesssim 2^{2\ell}$ respectively. Theorem 12.1 then implies (12.1) with $\Pi_k = P_{k+s_1}$ and $f_k = P_{k+s_2}f$, finishing the proof of Proposition 8.8.

Bibliography

[1] J. F. Adams, *Vector fields on spheres*, Ann. of Math. (2), 75 (1962), pp. 603–632.

[2] J. F. Adams, P. D. Lax, and R. S. Phillips, *On matrices whose real linear combinations are non-singular*, Proc. Amer. Math. Soc., 16 (1965), pp. 318–322.

[3] ——, *Correction to "On matrices whose real linear combinations are nonsingular"*, Proc. Amer. Math. Soc., 17 (1966), pp. 945–947.

[4] T. C. Anderson, L. Cladek, M. Pramanik, and A. Seeger, *Spherical means on the Heisenberg group: Stability of a maximal function estimate*, To appear in J. Anal. Math., preprint: arXiv:1801.06981, (2018).

[5] D. Beltran, S. Guo, J. Hickman, and A. Seeger, *Sobolev improving for averages over curves in* $\mathbb{R}^4$, preprint: arXiv:2102.08806, (2021).

[6] G. Bentsen, *L^p regularity for a class of averaging operators on the Heisenberg group*, To appear in Indiana Univ. Math. J., preprint arXiv:2002.01917, (2020).

[7] ——, *L^p regularity estimates for a class of integral operators with fold blowdown singularities*, preprint: arXiv:2101.08233, (2021).

[8] J. Bourgain and C. Demeter, *The proof of the l^2 decoupling conjecture*, Ann. of Math. (2), 182 (2015), pp. 351–389.

[9] A. Calderón and R. Vaillancourt, *On the boundedness of pseudo-differential operators*, J. Math. Soc. Japan, 23 (1971), pp. 374–378.

[10] M. Christ, *Restriction of the Fourier transform to submanfiolds of low codimension*, ProQuest LLC, Ann Arbor, MI, 1982. Thesis (Ph.D.)–The University of Chicago.

[11] ——, *Estimates for the k-plane transform*, Indiana Univ. Math. J., 33 (1984), pp. 891–910.

[12] ——, *Convolution, curvature, and combinatorics: a case study*, Internat. Math. Res. Notices, (1998), pp. 1033–1048.

[13] M. Christ, S. Dendrinos, B. Stovall, and B. Street, *Endpoint Lebesgue estimates for weighted averages on polynomial curves*, Amer. J. Math., 142 (2020), pp. 1661–1731.

[14] A. Comech, *Integral operators with singular canonical relations*, in Spectral theory, microlocal analysis, singular manifolds, vol. 14 of Math. Top., Akademie Verlag, Berlin, 1997, pp. 200–248.

[15] ——, *Optimal regularity of Fourier integral operators with one-sided folds*, Comm. Partial Differential Equations, 24 (1999), pp. 1263–1281.

[16] A. Comech and S. Cuccagna, *Integral operators with two-sided cusp singularities*, Internat. Math. Res. Notices, 23 (2000), pp. 1225–1242.

[17] M. Cowling and G. Mauceri, *Inequalities for some maximal functions. I*, Trans. Amer. Math. Soc., 287 (1985), pp. 431–455.

[18] S. Cuccagna, *L^2 estimates for averaging operators along curves with two-sided k-fold singularities*, Duke Math. J., 89 (1997), pp. 203–216.

[19] M. P. a. do Carmo, *Riemannian geometry*, Mathematics: Theory & Applications, Birkhäuser Boston, Inc., Boston, MA, 1992. Translated from the second Portuguese edition by Francis Flaherty.

[20] J. J. Duistermaat, *Fourier integral operators*, vol. 130 of Progress in Mathematics, Birkhäuser Boston, Inc., Boston, MA, 1996.

[21] C. Fefferman and E. Stein, *H^p spaces of several variables*, Acta Math., 129 (1972), pp. 137–193.

[22] I. M. Gelfand and M. I. Graev, *Line complexes in the space C^n*, Funkcional. Anal. i Priložen., 2 (1968), pp. 39–52.

[23] A. Greenleaf and A. Seeger, *Fourier integral operators with fold singularities*, J. Reine Angew. Math., 455 (1994), pp. 35–56.

[24] ——, *Fourier integral operators with cusp singularities*, Amer. J. Math., 120 (1998), pp. 1077–1119.

[25] ——, *Oscillatory and Fourier integral operators with degenerate canonical relations*, in Proceedings of the 6th International Conference on Harmonic Analysis and Partial Differential Equations (El Escorial, 2000), 2002, pp. 93–141.

[26] ——, *Oscillatory integral operators with low-order degeneracies*, Duke Math. J., 112 (2002), pp. 397–420.

[27] A. Greenleaf and G. Uhlmann, *Nonlocal inversion formulas for the X-ray transform*, Duke Math. J., 58 (1989), pp. 205–240.

[28] A. Greenleaf and G. Uhlmann, *Composition of some singular Fourier integral operators and estimates for restricted X-ray transforms*, Ann. Inst. Fourier (Grenoble), 40 (1990), pp. 443–466.

[29] A. Greenleaf and G. Uhlmann, *Composition of some singular Fourier integral operators and estimates for restricted X-ray transforms. II*, Duke Math. J., 64 (1991), pp. 415–444.

[30] P. T. Gressman, *Lp-Lq estimates for Radon-like operators*, ProQuest LLC, Ann Arbor, MI, 2005. Thesis (Ph.D.)–Princeton University.

[31] ——, *L^p-improving properties of averages on polynomial curves and related integral estimates*, Math. Res. Lett., 16 (2009), pp. 971–989.

[32] ——, *L^p-nondegenerate Radon-like operators with vanishing rotational curvature*, Proc. Amer. Math. Soc., 143 (2015), pp. 1595–1604.

[33] V. Guillemin and S. Sternberg, *Geometric asymptotics*, American Mathematical Society, Providence, R.I., 1977. Mathematical Surveys, No. 14.

[34] S. Helgason, *The Radon transform*, vol. 5 of Progress in Mathematics, Birkhäuser Boston, Inc., Boston, MA, second ed., 1999.

[35] L. Hörmander, *Fourier integral operators. I*, Acta Math., 127 (1971), pp. 79–183.

[36] H. Leptin and D. Müller, *Uniform partitions of unity on locally compact groups*, Adv. Math., 90 (1991), pp. 1–14.

[37] R. B. Melrose and M. E. Taylor, *Near peak scattering and the corrected*

Kirchhoff approximation for a convex obstacle, Adv. in Math., 55 (1985), pp. 242–315.

[38] J. W. Milnor and J. D. Stasheff, *Characteristic classes*, Princeton University Press, Princeton, N. J.; University of Tokyo Press, Tokyo, 1974. Annals of Mathematics Studies, No. 76.

[39] C. Muscalu and W. Schlag, *Classical and multilinear harmonic analysis. Vol. I*, vol. 137 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 2013.

[40] Y. Pan and C. D. Sogge, *Oscillatory integrals associated to folding canonical relations*, Colloq. Math., 60/61 (1990), pp. 413–419.

[41] D. H. Phong, *Singular integrals and Fourier integral operators*, in Essays on Fourier analysis in honor of Elias M. Stein (Princeton, NJ, 1991), vol. 42 of Princeton Math. Ser., Princeton Univ. Press, Princeton, NJ, 1995, pp. 286–320.

[42] D. H. Phong and E. M. Stein, *Radon transforms and torsion*, Internat. Math. Res. Notices, 4 (1991), pp. 49–60.

[43] M. Pramanik, K. M. Rogers, and A. Seeger, *A Calderón-Zygmund estimate with applications to generalized Radon transforms and Fourier integral operators*, Studia Math., 202 (2011), pp. 1–15.

[44] M. Pramanik and A. Seeger, *L^p Sobolev regularity of a restricted X-ray transform in $\mathbb{R}^3$*, in Harmonic analysis and its applications, Yokohama Publ., Yokohama, 2006, pp. 47–64.

[45] ——, *L^p regularity of averages over curves and bounds for associated maximal operators*, Amer. J. Math., 129 (2007), pp. 61–103.

[46] ——, *L^p Sobolev estimates for a class of integral operators with folding canonical relations*, To appear in J. Geom. Anal., preprint: arXiv: 1909.04173, (2019).

[47] J. Radon, *Über die Bestimmung von Funktionen durch ihre Integralwerte längs gewisser Mannigfaltigkeiten*, in 75 years of Radon transform (Vienna, 1992), Conf. Proc. Lecture Notes Math. Phys., IV, Int. Press, Cambridge, MA, 1994, pp. 324–339.

[48] B. Rubin, *Overdetermined transforms in integral geometry*, in Complex analysis and dynamical systems VI. Part 1, vol. 653 of Contemp. Math., Amer. Math. Soc., Providence, RI, 2015, pp. 291–313.

[49] S. Secco, *L^p-improving properties of measures supported on curves on the Heisenberg group*, Studia Math., 132 (1999), pp. 179–201.

[50] A. Seeger, *Degenerate Fourier integral operators in the plane*, Duke Math. J., 71 (1993), pp. 685–745.

[51] ——, *Radon transforms and finite type conditions*, J. Amer. Math. Soc., 11 (1998), pp. 869–897.

[52] C. D. Sogge and E. M. Stein, *Averages of functions over hypersurfaces in R^n*, Invent. Math., 82 (1985), pp. 543–556.

[53] C. D. Sogge and E. M. Stein, *Averages over hypersurfaces. Smoothness of generalized Radon transforms*, J. Analyse Math., 54 (1990), pp. 165–188.

[54] B. Stovall, *Endpoint $L^p \to L^q$ bounds for integration along certain polynomial curves*, J. Funct. Anal., 259 (2010), pp. 3205–3229.

[55] H. Triebel, *Theory of function spaces*, vol. 78 of Monographs in Mathematics, Birkhäuser Verlag, Basel, 1983.

[56] ——, *Theory of function spaces. II*, vol. 84 of Monographs in Mathematics, Birkhäuser Verlag, Basel, 1992.

[57] H. Whitney, *On singularities of mappings of euclidean spaces. I. Mappings of the plane into the plane*, Ann. of Math. (2), 62 (1955), pp. 374–410.

[58] T. Wolff, *Local smoothing type estimates on L^p for large p*, Geom. Funct. Anal., 10 (2000), pp. 1237–1288.