OPERATOR LEVEL LIMITS OF β-ENSEMBLES

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Abstract

Beta ensembles (β-ensembles) can be viewed as one parameter generalizations of the eigenvalues of random matrix ensembles. Under the appropriate scaling, these finite point processes converge in distribution to certain limiting point processes as the number of the points grows. This is known as the local scaling limit. The limiting point processes can be characterized as the spectra of certain random differential operators. This thesis mainly includes two types of results: the local scaling limits of certain β-ensembles in the appropriate operator level sense, and properties of the limiting objects.

We first consider the hard-to-soft edge transition for β-ensembles. The soft and hard edge scaling limits of β-ensembles can be characterized as the spectra of certain random Sturm-Liouville operators [51, 48]. By tuning the parameter of the hard edge process one can obtain the soft edge process as a scaling limit [6, 48, 50]. We prove that this limit can be realized on the level of the corresponding random operators. More precisely, the random operators can be coupled in a way so that the scaled versions of the hard edge operators converge to the soft edge operator a.s. in the norm resolvent sense. This part is based on joint work with Laure Dumaz and Benedek Valkó [16].

Next, we prove an operator level limit for the circular Jacobi β-ensemble. As a result, we characterize the counting function of the limit point process via coupled systems of stochastic differential equations. This diffusion description allows us to derive several properties of the limit point process (e.g. large gap probability and a process level transition). We show that the normalized characteristic polynomials converge to a random analytic function, which we characterize via the joint distribution of its Taylor coefficients at zero and as the solution of a stochastic differential equation system. We also provide analogous results for the real orthogonal β-ensemble. This is based on joint work with Benedek Valkó [37].

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Chapter 1

Introduction and Background

Random matrices were first emerged from the study of sample covariance matrices in statistics [71], and later were used to model the energy levels of nuclei of heavy atoms in nuclear physics [69, 68]. A central problem of random matrix theory is to understand the structure of the *spectra of large random matrices.* Over the years, many tools have been developed from various branches of mathematics (including combinatorics, analysis, representation theory, etc.) to study the eigenvalue statistics of random matrix ensembles on different scales. To understand the global picture, one could study the empirical spectral measure, i.e. a random probability measure supported on the spectrum so that each eigenvalue has equal weight (up to multiplicity). Under the appropriate scaling, the empirical spectral measures of a wide class of ensembles converge in distribution to a deterministic limit as the size of the matrix grows to infinity. For example, the Wigner semicircle law arises for random symmetric matrices with independent entries, and the Marchenko–Pastur law for the sample covariance matrices, see Section 1.1 for the precise statements. These classical results indicate certain universal asymptotic properties of the spectrum, and are essentially Law of Large Numbers type results.

Another natural object to investigate is the *local scaling limit*, which describes the limit of the spectrum in the scaling regime where the spacings between the eigenvalues remain of constant order. In contrary to the global picture, the local limit will describe the asymptotic behavior near a certain value (reference point), and the limit process could depend on whether the reference point is in the bulk or at the edge of the spectrum. For several classical ensembles, the bulk and edge scaling limits were derived by Dyson, Gaudin, and Mehta in the 1960s utilizing the algebraic structures present in the joint eigenvalue densities. They showed that these algebraic structures are preserved in the limit, and the joint intensity functions of the limit point processes can be fully described.

By generalizing the eigenvalue distributions of the classical ensembles of random matrix theory, we get the so-called β-ensembles, which can be viewed as one-parameter families of particle systems, see Section 1.2 below for more details. In this thesis, we study local scaling limits of certain β-ensembles and prove various results related to the limiting objects.

The outline of the remaining of the Chapter is as follows. In Section 1.1 we will introduce some of the classical random matrix ensembles and review the known results. In Sections 1.2 and 1.3, we will give a brief introduction to β-ensembles and the random operator approach that we used in their study. Section 1.4 provides a brief overview of the main results of this thesis.

1.1 Classical random matrix ensembles

We start this section with introducing the classical matrix ensembles that are most related to our work. Then we will state the results regarding the global and local scaling limits of the eigenvalues.

The study of random matrices could be traced back the work of Wishart [71] in 1920s. The models considered by Wishart are matrices of the form $MM^\dagger$ where M is an $n\times(n+$

a) matrix with i.i.d. standard real/complex/quaternion Gaussian entries. The models are called the Laguerre (or Wishart) ensembles. Noticing that the Laguerre ensembles are invariant under certain group conjugations, the joint eigenvalue densities can be computed explicitly. For a size n Laguerre ensemble (indexed by n, a) the eigenvalues have a joint density function given by

$$p_{n,\beta}(\lambda_1, \cdots, \lambda_n) = \frac{1}{Z_{n,\beta}} \prod_{j<k} |\lambda_k - \lambda_k|^\beta \prod_{k=1}^{n} f(\lambda_k) \tag{1.1}$$

on $\mathbb{R}_+^n$, where the reference measure $f(x) = x^{\frac{\beta}{2}(a+1)-1} e^{-\frac{\beta}{2}x}$ is proportional to the Gamma density, and the parameter $\beta = 1, 2, 4$ corresponds to the real/complex/quaternion entries, respectively. Here $Z_{n,\beta}$ is an explicitly computable normalizing constant. Notice that the matrix $m^{-1}MM^\dagger$ is the correlation matrix of n independent individuals whose m characteristics are i.i.d. standard Gaussians.

Fix $\beta = 1, 2$ or 4, denote by $\Lambda_{n,\beta,a} = (\lambda_{1,n}, \ldots, \lambda_{n,n})$ a size n Laguerre ensemble with parameter a. One way of understanding the global picture of the spectrum is to study the rescaled empirical spectral measure, that is a random probability measure defined by $\nu_n := \frac{1}{n} \sum_{k=1}^{n} \delta_{\lambda_{k,n}/n}$. When $n + a$ is of the same order as n, the macroscopic behavior of this ensemble is described by the famous Marchenko-Pastur limit law. Let $a_n > -1, n \geq 1$ be a sequence such that $\lim_{n\to\infty} \frac{n}{n+a_n} = \gamma \in (0, 1]$ exists. The Marchenko-Pastur theorem [39] states that the sequence of random probability measures $\nu_n, n \geq 1$ converges in distribution a.s. to a deterministic measure with density given by

$$\sigma_\gamma(x) = \frac{\sqrt{(x - b_-)(b_+ - x)}}{2\pi x} \mathbf{1}_{[b_-,b_+]}(x), \qquad b_\pm = b_\pm(\gamma) = (1 \pm \sqrt{\gamma})^2. \tag{1.2}$$

See Figure 1 for a plot of Marchenko-Pastur law for various values of γ. Note that in the case $\gamma = 1$, the density becomes $\frac{\sqrt{x(4-x)}}{2\pi x} \mathbf{1}_{[0,4]}(x)$.

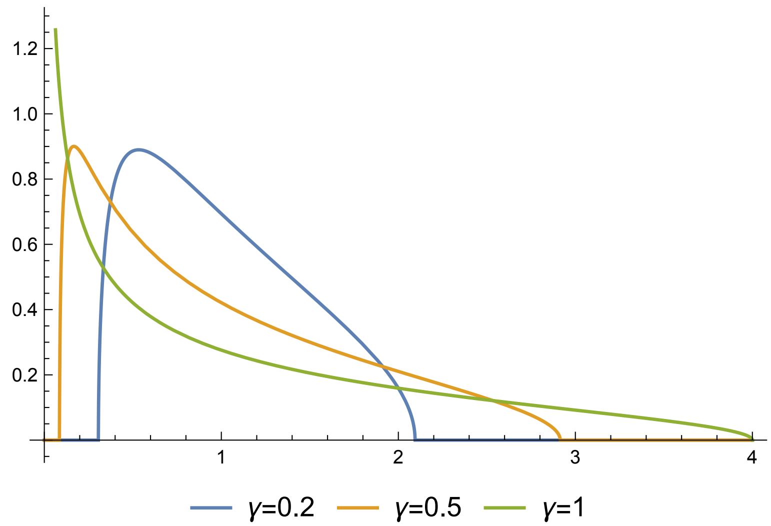

Figure 1: Plot of the Marchenko-Pastur law for different values of γ.

Next we turn to Gaussian ensembles, probably the most famous model of random matrices. In 1950s Wigner [70] used random matrices to model the energy levels of nuclei of heavy atoms in nuclear physics. The idea was to approximate a self-adjoint operator (Hamiltonians with certain symmetries) using a large symmetric or Hermitian matrix with i.i.d. real or complex standard normals. The resulting models are called the Gaussian orthogonal ensemble (GOE) or Gaussian unitary ensemble (GUE), which are classified by the group over which they are invariant. It turns out the joint eigenvalue densities of these Gaussian ensembles have the same structure as equation (1.1) with the Gaussian reference measure $f(x) = e^{-\frac{\beta}{4}x^2}$, and $\beta = 1, 2$ for GOE or GUE, respectively.

Let $(\lambda_{1,n}, \lambda_{2,n}, \ldots, \lambda_{n,n})$ be the eigenvalues of a size n Gaussian ensemble. Consider the rescaled empirical spectral measure of size n Gaussian ensembles defined by $\mu_n := \frac{1}{n}\sum_{k=1}^{n} \delta_{\lambda_{k,n}/\sqrt{n}}$. It was proved by Wigner that the sequence of random probability measures $\mu_n, n \geq 1$ converges in distribution a.s. to the deterministic Wigner semicircle

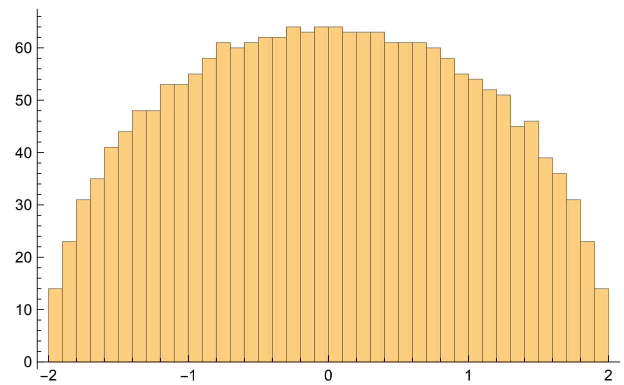

Figure 2: Histogram of rescaled eigenvalues of a 2000×2000 GUE

law with density given by

$$\rho_{sc}(x) = \frac{1}{2\pi}\sqrt{4-x^2}\mathbf{1}_{|x|\leq 2}.$$

Figure 2 above shows a simulation of the histogram of the rescaled eigenvalues of a GUE matrix of dimension 2000×2000.

Another well-studied ensemble in random matrix theory is the circular unitary ensemble (CUE), which describes the eigenvalue distribution of finite Haar unitary matrices. The model was introduced by Dyson [18] as a generalization of GUE on the unit circle. For a size n CUE, the eigenangles have a joint density proportional to $\prod_{j<k\leq n}|e^{i\theta_j}-e^{i\theta_k}|^{\beta}$ with $\beta = 2$, which can be viewed as the density (1.1) evaluated on the unit circle with $f = 1$. The cases when $\beta = 1, 4$ correspond to symmetric/self-dual unitary matrices.

Other commonly studied random matrix ensembles include Ginibre ensemble, Jacobi/MANOVA (multivariate analysis of variance) ensemble, etc. One common feature of these ensembles is that the joint eigenvalue functions satisfy (1.1) with different reference measures $f(x)$. We summarize in the Table 1 below the matrix representations and the associated joint density functions. Here we assume the entries of the random matrices $A \in \mathbb{C}^{n\times n}$, $X \in \mathbb{C}^{n\times n_1}$, $Y \in \mathbb{C}^{n\times n_2}$ are i.i.d. complex standard normal random variables $\mathcal{CN}(0,1)$, and denote the difference of the dimensions by $a := n_1 - n \geq 0$, $b := n_2 - n \geq 0$.

Ensembles	**Reference measure** f	**Matrix Models** $(\beta = 2)$
Gaussian/Hermite	$e^{-x^2/2}$	$M = \frac{A+A^\dagger}{\sqrt{2}}$
Wishart/Laguerre	$x^a e^{-x}\mathbf{1}_{x>0}$	$M = XX^\dagger$
Circular	$\mathbf{1}_{x=e^{\mathrm{i}\theta}}$	uniformly chosen $M \in \mathbb{U}(n)$
Ginibre	$e^{-\lvert z\rvert^2/2}\mathbf{1}_{z\in\mathbb{C}}$	$M = A$
MANOVA/Jacobi	$(1-x)^a x^b \mathbf{1}_{x\in(0,1)}$	$M = XX^\dagger(XX^\dagger + YY^\dagger)^{-1}$

Table 1: Classical random matrix ensembles

We now turn to the local scaling limit. The microscopic behavior of the spectrum $\Lambda_n = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ can be described by the large n limit of the point process $c_n(\Lambda_n - d_n)$ where d_n is the centering point and c_n is the appropriate scaling parameter. In order to get a meaningful point process limit, the scaling parameter c_n would need to be chosen so that it is roughly the inverse of the average spacing between the particles near d_n.

In the remaining of the section, we will take the GUE and LUE as examples to introduce the bulk and the edge limit of random matrices. It follows from the Wigner Semicircle law that asymptotically the eigenvalues of a size n GUE lie on an interval of size $4\sqrt{n}$ (see also [25] for a stronger statement regarding the largest/smallest eigenvalues of random Hermitian matrices). This suggests us to rescale the eigenvalues up by $\sqrt{n}$ to see what happens on a local scale in the bulk of the spectrum. Indeed, using the algebraic structures of GUE and analyzing the asymptotic of Hermite polynomials, it was proved by Gaudin-Mehta (see e.g. [42]) that for any $E \in (-2, 2)$,

$$\rho_{sc}(E)\sqrt{n}(\Lambda_n - E\sqrt{n}) \Rightarrow \text{Sine}_2 \qquad \text{as } n \to \infty,$$

where $\rho_{sc}(E) = \frac{1}{2\pi}\sqrt{4 - E^2}$ is the Wigner semicircle density, and where the Sine_2 process is the determinantal point process with the sine kernel $K(x, y) = \frac{\sin(\pi(x-y))}{\pi(x-y)}$ (see e.g. [29] for more on determinantal point process). Note that the convergence is universal in the sense that the limiting point process is independent of reference point E.

A similar type of result can be found at the edge of the spectrum $E \in \{-2, 2\}$ with a more careful saddle point analysis of the Hermite polynomials. By symmetry, it suffices to look at the points near the maximum eigenvalue. If the Wigner semicircle law holds on small scales, the expected number of points in the interval $[(2-\varepsilon)\sqrt{n}, 2\sqrt{n}]$ is of order $n\int_{2-\varepsilon}^{2} \rho_{sc}(x)dx \sim n\varepsilon^{3/2}$. We can expect the spacing of rescaled eigenvalue at the edge is of order $n^{-2/3}$. Then by rescaling the spectrum up by $n^{1/6}$, we get $n^{1/6}(\Lambda_n - 2\sqrt{n}) \Rightarrow \text{Airy}_2$ as $n \to \infty$, see [42] and [21]. Here the Airy_2 process is the determinantal point process with kernel constructed from the Airy function of the first kind. This is called the soft edge behavior in the sense that the rescaled largest/smallest can fluctuate around the endpoint of the limiting spectral measure. Indeed, there is positive probability of the event that some of the rescaled eigenvalues are located outside of the support of the Wigner Semicircle law.

In contrast, consider a finite Laguerre unitary ensemble, the eigenvalues have to be positive since the random matrix is a.s. nonnegative definite. In the case when $a_n = o(n)$, the Marchenko-Pastur states that the density of the limiting measure becomes $\frac{\sqrt{x(4-x)}}{2\pi x}\mathbf{1}_{[0,4]}(x)$. This indicates that the smallest eigenvalues of the Laguerre ensembles are pushed to 0, and locally the spectrum will feel the hard constraint at the origin.

Therefore, we expect to see a different limiting point process supported on the positive half line. It turns out that in the case when $a_n \equiv a > -1$, the smallest k eigenvalues of LUE (under the appropriate rescaling) converge in distribution to a point process indexed by a, see [61]. The limiting process, which we will denote as $\text{Bessel}_{2,a}$, is a determinantal point process with kernel constructed from the Bessel functions of the first kind. This is known as a hard edge behavior.

1.2 β-ensembles

In this section, we define the β-ensembles, which can be viewed as a one-parameter generalization of the eigenvalues of classical matrix models. First observe that the measure defined by (1.1) makes sense for all $\beta > 0$ for a broad class of reference measures f under certain regularity assumptions. Hence the classical Laguerre/Gaussian/circular ensembles (which correspond to Dyson's threefold way $\beta = 1, 2, 4$) can be naturally generalized to Laguerre β-ensemble, Gaussian β-ensemble, and circular β-ensemble for all positive β. More precisely, the size n Laguerre β-ensemble is a two-parameter family of distributions on $\mathbb{R}_+^n$ with density function

$$p^L_{n,\beta,a}(\lambda_1, \dots, \lambda_n) = \frac{1}{Z^L_{n,\beta,a}} \prod_{j<k} |\lambda_j - \lambda_k|^\beta \prod_{k=1}^n \lambda_k^{\frac{\beta}{2}(a+1)-1} e^{-\frac{\beta}{2}\lambda_k} . \tag{1.3}$$

The parameters satisfy $\beta > 0$ and $a > -1$. This density corresponds to the Gibbs measure of n positively charged particles living on the positive half-line with a log-Gamma potential.

Similarly, we define the size n Gaussian (Hermitian) β-ensemble as a family of distributions on $\mathbb{R}^n$ with density function

$$p^G_{n,\beta}(\lambda_1,\ldots,\lambda_n) = \frac{1}{Z^G_{n,\beta}} \prod_{j<k} |\lambda_j - \lambda_k|^\beta \prod_{k=1}^n e^{-\frac{\beta}{4}\lambda_k^2}, \qquad \beta > 0.$$

Denote by $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $\partial\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}$. The size n circular β-ensemble is defined as the joint distribution of n distinct points $\{e^{i\theta_1}, \ldots, e^{i\theta_n}\}$ on $\partial\mathbb{D}$ with $\theta_j \in [-\pi, \pi)$, where the joint density function of the angles θ_j is given by

$$p^c_{n,\beta}(\theta_1,\ldots,\theta_n) = \frac{1}{Z^c_{n,\beta}} \prod_{j<k\le n} \left|e^{i\theta_j} - e^{i\theta_k}\right|^\beta. \tag{1.4}$$

We will refer to the points obeying (1.1) as the associated β-ensembles, and we will sometimes call these points eigenvalues/eigenangles. This slight abuse of terminology will be made rigorous in Section 1.3 as there are sparse tridiagonal matrix models whose eigenvalues are distributed according to (1.1) for positive β for certain reference measures.

Table 2 below records the β-generalizations of classical matrix ensembles that are considered in this thesis.

We note that the real Jacobi β-ensemble can be connected to another named ensemble on the unit circle via a change of variables. Suppose that $(x_1, \cdots, x_n)$ is a size n real Jacobi β-ensemble with parameters $a, b > -1$. Introduce $\theta_j = \arccos(1 - 2x_j)$, then $\{\theta_1, \ldots, \theta_n\} \in (0, \pi)^n$ has joint density function

$$\begin{aligned} p^o_{n,\beta,a,b}(\theta_1,\ldots,\theta_n) = \frac{1}{Z^o_{n,\beta,a,b}} &\prod_{j<k\le n} |\cos(\theta_j) - \cos(\theta_k)|^\beta \\ &\times \prod_{k=1}^n |1-\cos(\theta_k)|^{\frac{\beta}{2}(a+1)-1/2} |1+\cos(\theta_k)|^{\frac{\beta}{2}(b+1)-1/2}. \end{aligned} \tag{1.5}$$

β-ensembles	Parameters	Support	Reference measure f
Gaussian/Hermite	$n \in \mathbb{Z}_+$	$\mathbb{R}^n$	$e^{-\frac{\beta}{4}\lambda^2}$
Wishart/Laguerre	$n \in \mathbb{Z}_+, a > -1$	$\mathbb{R}_+^n$	$\lambda^{\frac{\beta}{2}(a+1)-1}e^{-\frac{\beta}{2}\lambda}$
MANOVA/Jacobi	$n \in \mathbb{Z}_+, a, b > -1$	$[0,1]^n$	$\lambda^{\frac{\beta}{2}(a+1)-1}(1-\lambda)^{\frac{\beta}{2}(b+1)-1}$
Circular	$n \in \mathbb{Z}_+$	$(\partial\mathbb{D})^n$	1
Circular Jacobi	$n \in \mathbb{Z}_+, \delta \in \mathbb{C} : \Re\delta > -\frac{1}{2}$	$(\partial\mathbb{D})^n$	$(1-e^{\mathrm{i}\theta})^{\bar{\delta}}(1-e^{-\mathrm{i}\theta})^{\delta}$

Table 2: β-generalization of the classical models

The symmetrized version of $(\theta_1, \ldots, \theta_n)$ is called the *real orthogonal β-ensemble* (ROβE). Precisely, ROβE is a family of distributions describing an even number of points on the unit circle in a reflection symmetric configuration. If we parametrize the points as $\{\pm e^{i\theta_1}, \ldots, \pm e^{i\theta_n}\}$ with $\theta_j \in (0, \pi)$ then the joint density for $(\theta_1, \ldots, \theta_n)$ is given by (1.5).

For β-ensembles, it turns out that the global scaling limits for general $\beta > 0$ remain the same as in the classical $\beta = 1, 2, 4$ cases, e.g. the empirical measure of Laguerre β-ensembles and Gaussian β-ensembles are still governed by the Marchenko-Pastur law and the Wigner semicircle law, respectively (see e.g. [21]). In the next section, we will discuss the local scaling limits of β-ensembles, and in particular the random operator approach that are used in the study of these local behaviors.

1.3 The random operator approach

In the classical $\beta = 1, 2, 4$ cases, the ensembles are integrable (exactly solvable) in the sense that the local behavior (e.g. the joint intensity functions) of the eigenvalues can be explicitly described. However, the tools developed by Dyson, Gaudin and Mehta can not be adapted for general $\beta > 0$. This is due to the lack of the determinantal structure presented in the classical case. One particularly fruitful approach to study the asymptotic behavior of β-ensembles (especially on a microscopic level) is via random differential operators.

The random operator approach can be traced back to the work of Dumitriu and Edelman [17], where the authors constructed two families of tridiagonal matrix models whose eigenvalues obey Gaussian β-ensemble and Laguerre β-ensemble, respectively. Take the Gaussian β-ensemble as an example, the matrix models introduced in [17] is as follows:

$$H_{n,\beta} = \frac{1}{\sqrt{\beta}} \begin{pmatrix} a_1 & b_1 & & & \\ b_1 & a_2 & b_2 & & \\ & \ddots & \ddots & \ddots & \\ & & b_{n-2} & a_{n-1} & b_{n-1} \\ & & & b_{n-1} & a_n \end{pmatrix}, \tag{1.6}$$

where the entries $\{a_i, b_j, 1 \le i \le n, 1 \le j \le n-1\}$ are independent, $a_i \sim \mathcal{N}(0, 2)$ has normal distribution with mean 0 and variance 2, and $b_i \sim \chi_{\beta(n-i)}$ has chi distribution with $\beta(n-i)$ degrees of freedom. Sparse matrix models for circular β-ensemble and Jacobi β-ensemble are studied and constructed in [33] using Verblunsky coefficients, a main ingredient of understanding the orthogonal polynomials on the unit circle.

To study the local scaling limit of the spectrum, that is, to find the asymptotic limits of the rescaled (and possibly recentered) spectrum, Edelman and Sutton [19] presented heuristics that the scaled tridiagonal matrices can be treated as the discrete versions of certain random second order differential operators, and that they should converge to these operators. They set up conjectures for the form of the limiting operators.

The arguments of [19] have been made rigorous by Ramírez, Rider, and Virág [51], and Ramírez and Rider [48] for the soft edge limit and the hard edge limit, respectively. In the soft edge case, if Λ_n denotes the set of eigenvalues of the tridiagonal matrix model $H_{n,\beta}$ as defined in (1.6), then it was proved in [51] that $n^{1/6}(2\sqrt{n} - \Lambda_n)$ converges in distribution to the stochastic Airy process, which we will denote as Airy_β. In the hard edge case, Ramírez and Rider [48] considered the bottom of the spectrum of Laguerre β-ensemble with parameters $a_n \equiv a$. Then under appropriate rescaling, the smallest eigenvalues of Laguerre β-ensemble converge in distribution to the stochastic Bessel process indexed by $\beta > 0, a > -1$, denoted as $\text{Bessel}_{\beta,a}$. It has also been shown in [51] and [48] that both processes Airy_β and $\text{Bessel}_{\beta,a}$ can be characterized as spectra of certain random differential operators. Note that though the k-point correlation functions of the Airy_β and $\text{Bessel}_{\beta,a}$ processes were unknown (unlike the classical $\beta = 1, 2, 4$ cases), the limit processes can be described by their counting functions, which tells the distribution of the number of points in every interval. We further remark that the limiting random differential operators constructed in [51] and [48] are second order Sturm-Liouville operators acting on certain subsets of L^2 functions, see Section 2.1 and Section 3.1 below for more details.

Centering in the bulk of the spectrum, the local scaling limit was derived by Valkó and Virág [62] for the Gaussian β-ensemble. Killip and Stoiciu [34] provided a related but different description of the point process limit for the circular β-ensemble. In the Gaussian case, if Λ_n denotes the size n Gaussian β-ensemble (set of eigenvalues of $H_{n,\beta}$

defined in (1.6)), then it was proved in [62] that for all $E \in (-2, 2)$ we have

$$\rho_{sc}(E)\sqrt{n}(\Lambda_n - E\sqrt{n}) \Rightarrow \mathrm{Sine}_\beta,$$

where the limiting point process Sine_β dependents on β and generalizes the Sine_2 process (when $\beta = 2$). In [62], the authors provided descriptions of the Sine_β process via its counting function (which are related to a coupled system of stochastic differential equations), and also via the so-called Brownian carousel. In the case of the circular β-ensemble, the limiting process was characterized by its counting function via a different coupled system of stochastic differential equations [34]. Later it was proved in [44, 64] that the two descriptions are equivalent, and the Sine_β process can be characterized as the spectrum of a first order random Dirac operator acting on two-dimensional vector valued functions. We will give the precise definition of the considered Dirac operators in Sections 2.2.1 and 4.1 below. For now, we would like to remark that the authors in [64] also provided similar operator representations for a variation of the $\mathrm{Bessel}_{\beta,a}$ process, finite unitary or orthogonal ensembles and their limits. These operators constructed in [64] are parametrized by a path in the upper half plane $\mathbb{H} = \{z : \Im z > 0\}$ and two points on the boundary $\partial\mathbb{H}$.

Specifically, for finitely supported probability measures on the unit circle, [64] showed that the Szegő recursion of the normalized orthogonal polynomials (the analogue of the three-term recursion for real orthogonal polynomials) can be translated into an eigenvalue problem for a Dirac operator with piecewise constant path. The path of the associated Dirac operator is built from the Verblunsky coefficients, the main ingredient of the Szegő recursion, see e.g. [57]. The spectrum of the constructed Dirac operator is the periodic lifting of the angles corresponding to the support of the probability measure.

Let us also mention that the point process scaling limits for β-ensembles have been shown to be universal for a wide class of β-ensembles, see [9, 8, 35, 54]. Moreover, the last two results also show universality on the level of random operators near the edge. To end the introductory part of β-ensembles, we note that the random differential operator descriptions are novel even at the classical $\beta = 1, 2, 4$ cases. Using oscillation theory from differential equations, one can describe the limit point processes via their counting functions. These counting functions are related to coupled systems of stochastic differential equations. This representation provides access to various properties of the limit objects (e.g. large gap probability, tail asymptotics) simultaneously for all values of β using techniques from stochastic analysis.

1.4 Summary of the main results

The main results of this thesis can be divided into three parts. Chapter 3 proves the operator level hard-to-soft edge transition for β-ensembles. In Chapter 4 we study the point process limits of the circular Jacobi β-ensembles and provide several descriptions of the limiting objects. In Chapter 5, we prove various properties of the limiting objects proved in Chapter 4. Chapter 3 relies on the joint work with Laure Dumaz and Benedek Valkó [16]. Chapter 4 is a modified version of a submitted paper [37], which is joint with Benedek Valkó. The results of Chapter 5 are announced without proofs in [37].

Consider the bottom of the spectra of a sequence of Laguerre β-ensemble (as defined in (1.3)). Depending on whether the parameter $a_n \equiv a$ or $\liminf_{n\to\infty} a_n/n > 0$ it has been proved in [48, 51] that we get the convergence of (rescaled and possibly recentered) Laguerre β-ensemble to the $\text{Bessel}_{\beta,a}$ process and the Airy_β process, respectively. It is expected that the regime when $a_n = o(n)$ and $\lim_n a_n = \infty$ should also fall into the case of the soft edge (which will give the soft edge limit near the hard edge), but this statement has not been fully proved yet.

However, if working directly with the limiting point processes without considering the finite ensembles, one can obtain the soft edge process as a scaling limit by tuning the parameter of the hard edge process [6, 48, 50]. In Chapter 3, we prove that this limit can be realized on the level of the corresponding random operators. More precisely, the random operators can be coupled in a way so that the scaled versions of the hard edge operators converge to the soft edge operator a.s. in the norm resolvent sense. See Section 2.1 and Chapter 3 below for more details.

In Chapter 4, we study various limits of the circular Jacobi β-ensemble. The finite n ensemble can be viewed as a one parameter generalization of circular β-ensemble, see Table 2 for the precise definition. Using the Dirac differential operator framework introduced in [64] and [65], we prove an operator level limit for the circular Jacobi β-ensemble. The convergence is strong enough so that the convergence of the eigenangles follows. More precisely, if $\Lambda_{n,\beta,\delta}$ denotes the size n circular Jacobi β-ensemble indexed by δ, then $n\Lambda_{n,\beta,\delta}$ converges in distribution to the limiting point process $\mathrm{HP}_{\beta,\delta}$ indexed by β, δ. The limiting point process $\mathrm{HP}_{\beta,\delta}$ can be characterized as the spectrum of a Dirac differential operator, and can be viewed as a one parameter generalization of the Sine_β process (when $\delta = 0$). We also show in Chapter 4 that the normalized characteristic polynomials converge to a random analytic function. Moreover, we provided several equivalent descriptions of the limiting objects. Using the same operator theoretic framework, we provide analogous results for the real orthogonal β-ensemble. See Section 2.2 and Chapter 4 for more details.

In Chapter 5, we prove several results on the $\mathrm{HP}_{\beta,\delta}$ process. Specifically, we prove the asymptotics of large gap probabilities (probabilities of having no eigenvalues in large intervals) of the $\mathrm{HP}_{\beta,\delta}$ process, a process level transition from the $\mathrm{HP}_{\beta,\delta}$ process to the Sine_β process, and a Central Limit Theorem for the counting function of the $\mathrm{HP}_{\beta,\delta}$ process. We refer to Section 2.3 for the precise statements and Chapter 5 for the proofs.

Chapter 2

Results

2.1 Operator level hard-to-soft transition for β-ensembles

We start this section with reviewing the local scaling limits of β-ensembles. In particular, in Section 2.1.1 we will use the Laguerre β-ensemble as an example to illustrate the two different type of behaviors at the edge of the spectrum of certain β-ensembles: the soft edge behavior and the hard edge behavior.

In Section 2.1.2 we will define precisely the hard-to-soft edge transition for β-ensembles and state our main result.

2.1.1 Limits of β-ensembles at the edge

Recall that the size n Laguerre β-ensemble is a two-parameter family of distributions on $\mathbb{R}_+^n$ with density function (1.3). The global picture of Laguerre β-ensemble is still governed by the Marchenko-Pastur limit law in the case when $n+a_n$ is of the same order of n. More precisely, fix $\beta > 0$ and let $a_n > -1$ be a sequence such that $\lim_{n\to\infty} \frac{n}{n+a_n} = \gamma \in (0,1]$ exists. Denote by $\Lambda_{\beta,a_n,n} = (\lambda_{1,n}, \lambda_{2,n}, \ldots, \lambda_{n,n})$ a size n Laguerre β-ensemble with parameter a_n. Then the rescaled empirical spectral measure $\nu_n := \frac{1}{n}\sum_{k=1}^{n} \delta_{\lambda_{k,n}/n}$

converges in distribution a.s. to the Marchenko-Pastur law (1.2), see e.g. [39, 21].

The microscopic behavior of the Laguerre ensemble can be described by the large n limit of the point process $c_n(\Lambda_{n,\beta,a_n} - d_n)$ where d_n is the centering point and c_n is the appropriate scaling parameter. From now on, we will focus on the *lower edge behavior* i.e. the case $d_n := b_-$. (See [31] and [51] for the bulk and upper edge behavior.)

The distribution of the limiting point process depends on the asymptotic behavior of the sequence a_n. If $a_n = a > -1$ does not depend on n, then Ramírez and Rider [48] showed that the scaling limit of $n\Lambda_{n,\beta,a}$ exists, and gave an explicit description of the limiting point process. This is called the hard edge scaling limit.

Theorem 2.1 (Hard edge limit of the Laguerre ensemble, [48]). *Fix $\beta > 0$ and $a > -1$, and let $\Lambda_{n,\beta,a}$ be a size n Laguerre β-ensemble with parameter a. Then the sequence $n\Lambda_{n,\beta,a}$ converges in distribution to a point process* $\mathrm{Bessel}_{\beta,a}$ *as $n \to \infty$. The* $\mathrm{Bessel}_{\beta,a}$ *process has the same distribution as the a.s. discrete spectrum of the random differential operator*

$$\mathfrak{G}_{\beta,a} = -\frac{1}{m(x)}\frac{d}{dx}\left(\frac{1}{s(x)}\frac{d}{dx}\,\cdot\right), \tag{2.1}$$

$$m(x) = m_a(x) = e^{-(a+1)x-\frac{2}{\sqrt{\beta}}B_a(x)}, \qquad s(x) = s_a(x) = e^{ax+\frac{2}{\sqrt{\beta}}B_a(x)}. \tag{2.2}$$

Here B_a is a standard Brownian motion, and the operator $\mathfrak{G}_{\beta,a}$ is defined on a subset of $L^2(\mathbb{R}_+, m)$ with Dirichlet boundary condition at 0 and Neumann boundary condition at infinity.

We will come back to the precise definition of $\mathfrak{G}_{\beta,a}$ in Section 3.1. Let us just mention that since the functions s, m are a.s. continuous, this differential operator fits into the framework of classical Sturm-Liouville operators.

If the sequence $a_n, n \geq 1$ goes to infinity with at least a constant speed then the Marchenko-Pastur theorem and the expression of the limiting measure (1.2) suggest a different scaling than the one seen in the hard edge case. This is called the soft edge scaling limit. The description of the limiting point process follows from the work of Ramírez, Rider, and Virág [51].

Theorem 2.2 (Soft edge limit, [51])**.** *Fix $\beta > 0$ and suppose that the sequence $a_n, n \geq 1$ satisfies* $\liminf_{n\to\infty} a_n/n > 0$*. Then there is a point process* Airy_β *so that the following limit in distribution holds as $n \to \infty$:*

$$\frac{((n+a_n)n)^{1/6}}{(\sqrt{n+a_n}-\sqrt{n})^{4/3}}(\Lambda_{n,\beta,a_n} - (\sqrt{n+a_n}-\sqrt{n})^2) \Rightarrow \mathrm{Airy}_\beta\,.$$

The point process Airy_β *has the same distribution as the a.s. discrete spectrum of the random differential operator*

$$\mathsf{A}_\beta = -\frac{d^2}{dx^2} + x + \frac{2}{\sqrt{\beta}}B' \tag{2.3}$$

defined on a subset of $L^2(\mathbb{R}_+)$ with Dirichlet boundary conditions at 0. Here B' is the standard white noise on $\mathbb{R}_+$.

The precise definition of the operator A_β will be discussed in Section 3.1. Note that a priori it is not even clear that the operator A_β is well-defined, due to the irregularity of the white noise term in the potential.

Remark 2.3. *In [51], the authors actually proved the soft edge scaling limit near the upper edge $d_n := b_+$ for the Laguerre β-ensemble, but the proof can be easily adapted to the lower edge $d_n := b_+$ under the assumption that* $\liminf a_n/n > 0$*. They also proved the soft edge limit for the Gaussian β-ensemble. In particular, if $\Lambda_{n,\beta}$ denotes a size n*

Gaussian β-ensemble, then

$$n^{1/6}(2\sqrt{n} - \Lambda_{n,\beta}) \Rightarrow \mathrm{Airy}_\beta\,.$$

2.1.2 Hard-to soft transition

It is natural to conjecture that the condition $\liminf_{n\to\infty} a_n/n > 0$ in Theorem 2.2 could be relaxed to $\lim_{n\to\infty} a_n = \infty$, but the tools developed in [51] do not seem to be sufficient to prove this. (See however [13] for the treatment of the case $\beta = 2$, $a_n = c\sqrt{n}$, where the appropriate limit is proved using the determinantal structure present at $\beta = 2$.) This conjecture, together with a diagonal argument, would imply the following point process level transition from the $\mathrm{Bessel}_{\beta,a}$ process to Airy_β:

$$a^{-4/3}(\mathrm{Bessel}_{\beta,2a} - a^2) \Rightarrow \mathrm{Airy}_\beta, \qquad \text{as } a \to \infty. \tag{2.4}$$

See [62] for a similar diagonal argument for the transition between the soft edge and the bulk limiting processes.

The process level limit (2.4) is called hard to soft edge transition. It can be analyzed without considering the finite n ensembles, working directly with the limiting point processes appearing in the statement. This transition was first proved in [6] for $\beta = 2$ using again the determinantal structure present in this case. For general $\beta > 0$, Ramírez and Rider [48] proved the scaling limit for the first point of the respective point processes. This result was extended in [50] to a full process level limit.

In light of Theorems 2.1 and 2.2, the statement of (2.4) can be rewritten using the operators $\mathfrak{G}_{\beta,2a}$ and A_β as

$$a^{-4/3}(\mathrm{spec}(\mathfrak{G}_{\beta,2a}) - a^2) \Rightarrow \mathrm{spec}(\mathsf{A}_\beta),$$

where spec(Q) denotes the spectrum of the operator Q. It is natural to ask whether it is possible to prove the corresponding limit on the level of the operators. This is the main result of Chapter 3. Theorem 2.4 below shows that one can realize the operator level limit as an a.s. limit with an appropriate coupling between the Brownian motion B_a of the Bessel operator (2.1) and the white noise B' of the Airy operator (2.3).

To describe our coupling, we introduce a simple transformation of $\mathfrak{G}_{\beta,2a}$. For $a > 0$ let θ_a be the 'stretching' transformation defined via

$$(\theta_a f)(x) = f(a^{2/3}x), \tag{2.5}$$

and define the following transform of the hard-edge operator corresponding to $2a$:

$$\mathsf{G}_{\beta,2a} = \theta_a^{-1}\left(m_{2a}^{1/2}\mathfrak{G}_{\beta,2a}m_{2a}^{-1/2}\right)\theta_a, \tag{2.6}$$

where $m(\cdot)$ is defined in (2.2). As we will see in Section 3.1, $\mathsf{G}_{\beta,2a}$ is a self-adjoint operator with the same spectrum as $\mathfrak{G}_{\beta,2a}$, and the operators A_β^{-1} and $(\mathsf{G}_{\beta,2a} - a^2)^{-1}$ are Hilbert-Schmidt integral operators acting on the same space of $L^2(\mathbb{R}_+)$ functions. Our main result in Chapter 3 is the following.

Theorem 2.4 (Operator level hard-to-soft transition)**.** *Let B' be white noise on $\mathbb{R}_+$ and let B be a Brownian motion defined as $B(x) := \int_0^x B'(y)dy$. Set $B_{2a}(x) = a^{-1/3}B(a^{2/3}x)$ for $a > 0$. Consider A_β defined as (2.3) using the white noise B', and $\mathsf{G}_{\beta,2a}$ defined with the Brownian motion B_{2a} via (2.1) and (2.6) for $a > 0$. Then $a^{4/3}(\mathsf{G}_{\beta,2a} - a^2)^{-1} \to \mathsf{A}_\beta^{-1}$ a.s. in Hilbert-Schmidt norm as $a \to \infty$.*

We expect that with a more careful application of our methods one could also get estimates on the speed of convergence in our coupling. See Remark 3.20 in Section 3.5.

The theorem implies that $a^{-4/3}(\mathsf{G}_{\beta,2a} - a^2) \to \mathsf{A}_\beta$ a.s. in norm resolvent sense from which the process level transition $a^{-4/3}(\mathrm{spec}(\mathfrak{G}_{\beta,2a}) - a^2) \Rightarrow \mathrm{spec}(\mathsf{A}_\beta)$, and therefore the limit (2.4) follows. The coupling of the operators produces a coupling of the point processes in a way that almost surely the points in the scaled hard edge processes converge to the points in the soft edge point process. More precisely, a version of the Hoffman-Wielandt inequality (see e.g. [3]) shows that if we denote the ordered points in the scaled hard edge process $a^{-4/3}(\mathrm{Bessel}_{\beta,2a} - a^2)$ by $\lambda_{k,2a}, k \geq 0$, and the ones in the soft edge process Airy_β by $\lambda_k, k \geq 0$, then in the coupling of Theorem 2.4 we have a.s.

$$\lim_{a\to\infty} \sum_{k=0}^{\infty} \left|\lambda_k^{-1} - \lambda_{k,2a}^{-1}\right|^2 = 0 \, .$$

Moreover, as the spectrum of the operators are discrete, and each eigenvalue has multiplicity 1, the a.s. norm resolvent convergence also implies the a.s. convergence of the respective normalized eigenfunctions in L^2.

The proof of Theorem 2.4 will be given in Chapter 3.

2.2 Limits of the circular Jacobi β-ensemble

In this section, we will first review the bulk scaling limit of β-ensembles, and introduce the random Dirac differential operators. Then we will explain the point process convergence of the circular Jacobi β-ensemble under the Dirac operator framework.

2.2.1 Limits of β-ensembles in the bulk

The bulk limit refers to the case when the reference point d_n is chosen inside of the spectrum. Recall that in the GUE case, after appropriate centering and rescaling, we obtain the Sine_2 process in the limit. Under the same scaling, the point process limit of Gaussian β-ensemble was derived by Valkó and Virág [62] for the general $\beta > 0$ case. Due to the lack of the determinantal algebraic structure, the Sine_β process is not characterized by its k-point correlation function, but by its counting function instead. Roughly speaking, the counting function of Sine_β tells the number of points in the limiting process for all intervals, and can be analyzed by solving a coupled system of stochastic differential equations.

In Killip and Stoiciu [34], the authors studied the scaling limits of circular β-ensemble. If Λ_n is the finite point process with joint distribution (1.4), then $n\Lambda_n$ converges in distribution to a limiting point process, whose counting function can be described with a related but different stochastic differential equation system. Later it was proved in [44, 64] that the two descriptions are equivalent. Moreover, the Sine_β process can be characterized as the spectrum of a first order differential operator acting on functions $f : [0, 1) \mapsto \mathbb{R}^2$.

Theorem 2.5. *Consider the differential operator of the form*

$$\mathtt{Sine}_\beta : f \to 2R_t^{-1} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} f'(t), \quad f : [0, 1) \to \mathbb{R}^2,$$

where $R_t, t \in [0, 1)$ is a positive definite matrix built from a hyperbolic Brownian motion (see (2.7) below for the construction). Then the operator $\mathtt{Sine}_\beta$ is self-adjoint on an appropriately defined domain, and its spectrum is given by the Sine_β process.

The operators of this form are often called Dirac operators, see Section 4.1 below for more details. [64] also showed that a number of random matrix models (and their limits) can be represented using random Dirac differential operators. The ingredients to define a Dirac operator are a generating path $x + iy : [0, 1) \mapsto \mathbb{H} = \{z \in \mathbb{C} : \Im z > 0\}$, and two non-zero, non-parallel vectors $u_1, u_2 \in \mathbb{R}^2$. Then we consider differential operators of the form

$$\tau : f \to R^{-1}(t) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} f', \qquad f : [0, 1) \to \mathbb{R}^2, \qquad R = \frac{1}{2y} \begin{pmatrix} 1 & -x \\ -x & x^2 + y^2 \end{pmatrix}. \quad (2.7)$$

In Theorem 2.5, the generating path of the $\texttt{Sine}_\beta$ operator is a time-changed hyperbolic Brownian motion in $\mathbb{H}$ satisfying $(x+iy)(t) = (\widetilde{x}+i\widetilde{y})(-\frac{4}{\beta}\log(1-t))$ for $t \in [0, 1)$, where $\widetilde{x} + i\widetilde{y}$ is the solution of the SDE

$$dx = ydB_1, \quad dy = ydB_2, \qquad x(0) = 0, y(0) = 1,$$

where B_1, B_2 are independent Brownian motions. It has also been shown in [64] that under some mild conditions on the triple $(x + iy, u_1, u_2)$, the associated Dirac operator is self-adjoint with pure point spectrum, and its inverse is a Hilbert-Schmidt integral operator with explicit kernel.

Using the theory of orthogonal polynomials on the unit circle (see e.g. [57]), the authors in [64] provided Dirac operator representations for finite unitary matrices. The idea was that for finitely supported probability measures on the unit circle, the Szegő recursion of the normalized orthogonal polynomials can be translated into the eigenvalue equation for a Dirac operator with piecewise constant path. The path can be constructed from the so-called modified Verblunsky coefficients appearing in the Szegő recursion.

The main goal of Chapter 4 is to understand the local scaling limit of circular Jacobi β-ensemble, using the Dirac operator theoretic framework. The main results in Chapter 4 are summarized in the next section.

2.2.2 Limits of the circular Jacobi β-ensemble

For a given integer $n \geq 1$, $\beta > 0$, and $\delta \in \mathbb{C}$ with $\Re\delta > -1/2$ the size n *circular Jacobi β-ensemble* (CJβE) with parameters β, δ is the joint distribution of n distinct points $\{e^{i\theta_1}, \ldots, e^{i\theta_n}\}$ with $\theta_j \in [-\pi, \pi)$, where the joint density function of the angles θ_j is given by

$$p^{cj}_{n,\beta,\delta}(\theta_1, \ldots, \theta_n) = \frac{1}{Z^{cj}_{n,\beta,\delta}} \prod_{j<k\leq n} \left|e^{i\theta_j} - e^{i\theta_k}\right|^\beta \prod_{k=1}^{n} (1 - e^{-i\theta_k})^\delta (1 - e^{i\theta_k})^{\bar{\delta}}, \qquad \theta_j \in [-\pi, \pi). \tag{2.8}$$

Here $Z^{cj}_{n,\beta,\delta}$ is an explicitly computable normalizing constant (see e.g. Section 4.1 of [21]).

We write $\Lambda_n \sim \mathrm{CJ}_{n,\beta,\delta}$ to denote that the random set $\Lambda_n = \{\theta_1, \ldots, \theta_n\}$ has joint density given by (2.8). This family of distributions extends several other named ensembles. For $\beta = 2$ the distribution was studied by Hua [30] and Pickrell [45], and is known as the Hua-Pickrell measure in the literature. The local scaling limit of the angles was derived in [23] using the determinantal structure present in this case. For $\delta = 0$ the distribution recovers CβE, where the scaling limit was given by the Sine_β process [34],[64], see the discussions in Section 2.2.1 above. When $\beta = 2$ and $\delta = 0$ we get the circular unitary ensemble, which gives the joint eigenvalue distribution of an $n \times n$ Haar unitary matrix. For $k \in \mathbb{Z}_+$ with $\delta = \frac{\beta k}{2}$ the measure given by (2.8) can also be realized as a conditioned version of the size $n+k$ circular β-ensemble, conditioned to have k points at 1 (i.e. $\theta = 0$). See [7], [23], Section 3.12 of [21], and the references within for additional information about the ensemble.

In [10] the authors constructed matrix models for CJβE and described explicitly the distribution of the modified Verblunsky coefficients. These constructions lead to the random Dirac operators $\mathtt{CJ}_{n,\beta,\delta}$, whose spectrum is the periodic extension of $\mathrm{CJ}_{n,\beta,\delta}$

with an extra magnification by n. Under the appropriate scaling, the piecewise constant paths associated to the random operators $\mathtt{CJ}_{n,\beta,\delta}$ converge to the time-changed hyperbolic Brownian motion with drift. As shown in [64], one can construct random differential operators in terms of the limiting diffusion. Denote the limiting operator and its spectrum by $\mathtt{HP}_{\beta,\delta}$ and $\mathrm{HP}_{\beta,\delta}$, respectively (note that $\mathtt{HP}_{\beta,0} = \mathtt{Sine}_\beta$ and $\mathrm{HP}_{\beta,0} = \mathrm{Sine}_\beta$). Our main result in Chapter 4 is the following operator level convergence.

Theorem 2.6. *Fix $\beta > 0$ and $\Re\delta > -1/2$. Then there is a coupling of the random operators $\mathtt{CJ}_{n,\beta,\delta}, n \geq 1$ and $\mathtt{HP}_{\beta,\delta}$ so that $\mathtt{CJ}_{n,\beta,\delta}$ converges to $\mathtt{HP}_{\beta,\delta}$ a.s. in norm resolvent sense as $n \to \infty$. In particular, if $\Lambda_{n,\beta,\delta} \sim \mathrm{CJ}\beta\mathrm{E}_n$ with parameter δ then $n\Lambda_{n,\beta,\delta} \Rightarrow \mathrm{HP}_{\beta,\delta}$. In this coupling the normalized characteristic polynomial of $\Lambda_{n,\beta,\delta}$ converges a.s. uniformly on compacts to a random analytic function constructed from the $\mathtt{HP}_{\beta,\delta}$ operator, which we denote as $\zeta^{\mathtt{HP}}_{\beta,\delta}$.*

The limit objects are interesting on their own. In Chapter 4, we also characterize the $\mathrm{HP}_{\beta,\delta}$ process via its counting function, and derive various characterizations of the limiting random analytic function.

Theorem 2.7. *Fix $\beta > 0$ and $\Re\delta > -1/2$, and let Z be a standard complex Brownian motion. Then the counting function $N(\cdot)$ of the $\mathrm{HP}_{\beta,\delta}$ process has the same distribution as the right continuous version of the function $\lambda \mapsto \lim_{t\to\infty} \frac{\alpha_\lambda(t)}{2\pi}$, where α_λ solves the coupled system of SDE*

$$d\alpha_\lambda = \lambda\frac{\beta}{4}e^{-\frac{\beta}{4}t}dt + \Re\left[(e^{-i\alpha_\lambda} - 1)(dZ - i\delta dt)\right], \qquad \alpha_\lambda(0) = 0. \tag{2.9}$$

The limiting analytic function $\zeta^{\mathtt{HP}}_{\beta,\delta}$ can be characterized through its Taylor coefficients at 0 given in terms of certain stochastic differential equation, and as the solution of a stochastic differential equation system.

Using the same strategy, we also study the point process limit of *real orthogonal β-ensemble* (ROβE), which is a finite point processes on the unit circle, see the discussion around Table 2. The ensemble was introduced in [33] and [32] as a generalization of the joint eigenvalue distributions of some of the classical ensembles on the orthogonal and special orthogonal group of matrices. E.g. with $\beta = 2$, $a = b = \frac{1}{\beta} - 1$, we get the joint eigenvalue distribution of a $2n \times 2n$ special orthogonal matrix chosen according to Haar measure on $\mathbb{SO}(2n)$. We write $\Lambda_{2n} \sim \mathrm{RO}_{2n,\beta,a,b}$ to denote that the random set $\Lambda_{2n} = \{\pm\theta_1, \dots, \pm\theta_n\}$ has a distribution determined by the joint density given by (1.5).

The distribution of the modified Verblunsky coefficients of ROβE were described explicitly in [33, 32]. These coefficients can be used to construct the random Dirac operators $\mathtt{RO}_{2n,\beta,a,b}$, whose spectrum is the periodic extension of $\mathrm{RO}_{2n,\beta,a,b}$ with an extra magnification by $2n$. Under the appropriate scaling, the piecewise constant driving paths associated to the random operators $\mathtt{RO}_{2n,\beta,a,b}$ converge to the exponential Brownian motion on the imaginary axis with drift, which can be used to construct the limiting operator [64]. Denote the limiting operator and its spectrum by $\mathtt{Bess}_{\beta,a}$ and $\mathrm{Bess}_{\beta,a}$. Then we have the following operator level convergence.

Theorem 2.8. *Fix $\beta > 0$ and $a, b > -1$. Then there is a coupling of the random operators $\mathtt{RO}_{2n,\beta,a,b}, n \geq 1$ and $\mathtt{Bess}_{\beta,a}$ so that $\mathtt{RO}_{2n,\beta,a,b}$ converges to $\mathtt{Bess}_{\beta,a}$ a.s. in norm resolvent sense as $n \to \infty$. In particular, if $\Lambda_{2n,\beta,a,b} \sim \mathrm{RO}_{2n,\beta,a,b}$ then $2n\Lambda_{2n,\beta,a,b} \Rightarrow \mathrm{Bess}_{\beta,a}$.*

In this coupling the normalized characteristic polynomial of $\Lambda_{2n,\beta,a,b}$ converges a.s. uniformly on compacts to $\zeta^{\mathtt{B}}_{\beta,a}$, a random analytic function constructed from the $\mathtt{Bess}_{\beta,a}$ operator. The function $\zeta^{\mathtt{B}}_{\beta,a}$ can be described through its Taylor expansion at 0, and as the solution of a stochastic differential equation system.

The proofs of Theorems 2.6, 2.7, and 2.8 will be provided in Chapter 4.

2.3 Additional results on the $\mathrm{HP}_{\beta,\delta}$ process

The third part of this thesis consists of several results on the limiting process $\mathrm{HP}_{\beta,\delta}$ introduced in Theorem 2.6. The starting point is the diffusion description given in Theorem 2.7. This representation provides access to various properties of the limit process (e.g. large gap probability, a process level transition, and a Central Limit Theorem) using techniques from stochastic analysis.

We start with the large gap probability. For $\beta > 0, \delta \in \mathbb{C}$ with $\Re\delta > -1/2$, denote the gap probability

$$GAP_\lambda = P(\mathrm{HP}_{\beta,\delta} \cap [0,\lambda] = \emptyset), \qquad \lambda > 0.$$

Then Theorem 2.7 allows us to write the gap probability as

$$GAP_\lambda = P\left(\lim_{t\to\infty} \alpha_\lambda(t) = 0\right).$$

The problem boils down to understand the asymptotic expansion of the probability of the process α_λ converges to 0. Following the work of Valkó and Virág [63] on the large gap probability of the Sine_β process, the argument involves estimating the Cameron-Martin-Girsanov term (the Radon-Nikodym derivative) of a suitably chosen change of variable. In Section 5.1 we will present the construction of such a change of measure, and prove the following asymptotic expansion of GAP_λ as $\lambda \to \infty$.

Theorem 2.9. *As $\lambda \to \infty$, we have*

$$GAP_\lambda = (\kappa_{\beta,\delta} + o(1))\lambda^{\gamma_{\beta,\delta}} \exp\left(-\frac{\beta}{64}\lambda^2 + \left(\frac{\beta}{8} - \frac{1}{4} + \frac{1}{2}\Im\delta\right)\lambda\right),$$

where

$$\gamma_{\beta,\delta} = \frac{1}{4}\Big(\frac{\beta}{2} + \frac{2}{\beta} - 3\Big) - \Re\delta + \frac{2}{\beta}\Re(\delta + \delta^2).$$

Using the diffusion description (2.9), we are also able to show a process level transition from the $\mathrm{HP}_{\beta,\delta}$ process to the Sine_β process, and a Central Limit Theorem of the counting function of the $\mathrm{HP}_{\beta,\delta}$ process. The proofs rely on the techniques introduced in Holcomb [26], where the author considered similar results for the square root of the hard edge process (constant multiple of the $\mathrm{Bess}_{\beta,a}$ process).

Theorem 2.10. *Fix $\beta > 0$ and $\delta \in \mathbb{C}$ with $\Re\delta > -1/2$. Then as $\lambda \to \infty$, we have*

$$(\mathrm{HP}_{\beta,\delta} - \lambda) \Rightarrow \mathrm{Sine}_\beta\,.$$

Let $N(\cdot)$ be the counting function of the $\mathrm{HP}_{\beta,\delta}$ process, as $\lambda \to \infty$ we have

$$\tfrac{1}{\sqrt{\log\lambda}}(N(\lambda) - \tfrac{\lambda}{2\pi}) \Rightarrow \mathcal{N}(0, \tfrac{2}{\beta\pi^2}),$$

where $\mathcal{N}(\mu, \sigma^2)$ is the mean μ, variance σ^2 normal distribution.

The proofs of Theorems 2.9 and 2.10 will be presented in Chapter 5.

Chapter 3

Operator level hard-to-soft transition for β-ensembles

The content of this chapter is joint work with Laure Dumaz and Benedek Valkó and is a modified version of an published article [16].

The structure of the rest of the chapter is as follows. In Section 3.1 we show how one can describe the appearing differential operators using the generalized Sturm-Liouville theory, show that A_β^{-1} and $(\mathsf{G}_{\beta,2a} - a^2)^{-1}$ are Hilbert-Schmidt integral operators, and describe their kernels in terms of certain diffusions. Section 3.2 outlines the main steps of the proof of the main Theorem 2.4. Our proof uses the approximation of the integral operators by their truncated version. We state the convergences of the truncated operators towards their full operator as well as the convergence of the truncated hard edge integral operators to the truncated soft edge integral operator in several lemmas whose proofs are postponed to later sections. Section 3.3 estimates the truncation error of the soft edge integral operator. Section 3.4 shows that the truncated hard edge integral operators converge to the truncated soft edge integral operator by proving that the integral kernels converge uniformly on compacts with probability one. Section 3.5 describes the asymptotic behavior of the diffusions connected to the operator $\mathsf{G}_{\beta,2a}$ and provides the results needed to estimate the truncation error for the hard edge integral operators. Finally, the final section gathers the proof of some technical lemmas needed for the results of Sections 3.3 and 3.5.

3.1 The operators A_β and $\mathfrak{G}_{\beta,2a}$ as generalized Sturm-Liouville operators

This section briefly introduces the background for the differential operators appearing in this work, and shows how it can be used to describe the random differential operators $\mathfrak{G}_{\beta,2a}, \mathsf{G}_{\beta,2a}, \mathsf{A}_\beta$ and their inverses. We use the classical theory discussed in [67] and Chapter 9 of [60].

3.1.1 Generalized Sturm-Liouville operators

We consider generalized Sturm-Liouville (S-L) operators of the form

$$\tau u(x) = \frac{1}{r(x)} \left(-(p_1(x)u'(x) - q_0(x)u(x))' - q_0(x)u'(x) + p_0(x)u(x) \right), \tag{3.1}$$

where u is a real valued function on $[0, L]$ for some $L > 0$ or on $\mathbb{R}_+$ (which we consider to be the $L = \infty$ case in the following). We assume that the real functions p_0, p_1, q_0, r are continuous on $[0, \infty)$ and $r(x), p_1(x) > 0$ for $x \geq 0$.

The operation τu is well-defined if both u and $p_1 u' - q_0 u$ are absolutely continuous on $[0, L]$. From the standard theory of differential equations we have that for any $\lambda \in \mathbb{C}$ the differential equation $\tau u = \lambda u$ has a unique differentiable solution on $[0, L]$ with initial conditions $u(0) = c_0, u'(0) = c_1$. We note that if f_1, f_2 are both solutions of $\tau f = \lambda f$ then integration by parts shows that the Wronskian $p_1(f_1 f_2' - f_1' f_2)$ is constant on $\mathbb{R}_+$.

We consider differential operators satisfying the following three assumptions:

(A1) The solution $u_{\mathbf{d}}$ of the equation $\tau u_{\mathbf{d}} = 0$ with Dirichlet initial condition $u_{\mathbf{d}}(0) = 0$, $u'_{\mathbf{d}}(0) = 1$ is not in $L^2(\mathbb{R}_+, r)$, i.e. $\int_0^\infty u_{\mathbf{d}}^2(x) r(x) dx = \infty$.

(A2) There is a unique solution u_∞ of the equation $\tau u_\infty = 0$, with initial condition $u_\infty(0) = 1$ that is in $L^2(\mathbb{R}_+, r)$.

(A3) With $u_{\mathbf{d}}, u_\infty$ defined from (A1), (A2), we have $\int_0^\infty \int_0^x u_\infty(x)^2 u_{\mathbf{d}}(y)^2 r(x) r(y)\, dy dx < \infty$.

Under these assumptions, the operator τ can be made self-adjoint on an appropriate subset of $L^2([0, L], r)$ or $L^2(\mathbb{R}_+, r)$. We introduce

$$\mathcal{D}_L = \left\{ u \in L^2([0, L], r) : \tau u \in L^2([0, L], r),\ u, p_1 u' - q_0 u \in \mathrm{AC}([0, L]) \right\},$$

and we drop the subscript L for $L = \infty$. Here $\mathrm{AC}([0, L])$ is the set of absolutely continuous real functions on $[0, L]$.

The continuity of the functions p_0, p_1, q_0 and r implies that the operator τ is regular at 0 and at any finite L and therefore is limit circle at those points. The condition (A1) implies that the operator τ is limit point at $+\infty$ thanks to the Weyl's alternative theorem. Conditions (A2) and (A3) ensure that the inverse and the resolvent are Hilbert Schmidt operators.

The following propositions summarize the basic properties of generalized Sturm-Liouville differential operators satisfying conditions (A1)-(A3).

Proposition 3.1 (Self-adjoint version of τ)**.** *Assume that τ is of the form (3.1) and that it satisfies the condition (A1-A3), and let $L \in (0, \infty]$. Then there is a self-adjoint version of the operator on $[0, L]$ with Dirichlet boundary conditions on the domain*

$$\mathcal{D}_{L,0} = \mathcal{D}_L \cap \{u : u(0) = 0, u(L) = 0\},$$

where the end condition $u(L) = 0$ is dropped in the case $L = \infty$. We denote this self-adjoint operator by τ_L.

Proposition 3.2 (Inverse as an integral operator). *Consider the operator τ_L from Proposition 3.1. If L is finite then assume that $u_{\mathbf{d}}(L) \neq 0$ (i.e. that 0 is not an eigenvalue of τ_L). Then the inverse τ_L^{-1} is an integral operator of the form $\tau_L^{-1} f(x) = \int_0^L K^{(L)}(x,y) f(y) r(y) dy$ on $L^2([0,L], r)$ with*

$$K^{(L)}(x,y) = \frac{1}{p_1(0)} \left(u_L(x) u_{\mathbf{d}}(y) \mathbf{1}(x \geq y) + u_{\mathbf{d}}(x) u_L(y) \mathbf{1}(x < y) \right). \tag{3.2}$$

Here $u_{\mathbf{d}}$ is defined in (A1). If $L = \infty$ then u_L is u_∞ from (A2), and in the case $L < \infty$ the function u_L is defined as the solution of $\tau u_L = 0$ with $u_L(0) = 1, u_L(L) = 0$. The inverse operator τ_L^{-1} is a Hilbert-Schmidt operator in $L^2([0,L], r)$, and it has a bounded pure point spectrum.

Proposition 3.3 (Resolvent as an integral operator). *Consider τ_L from Proposition 3.1, and assume that a given $\lambda \in \mathbb{R}$ is not an eigenvalue of τ_L. Then the resolvent $(\tau_L - \lambda)^{-1}$ is a Hilbert-Schmidt integral operator of the same form as $K^{(L)}$ from (3.2), where now $u_{\mathbf{d}}, u_L$ are the appropriate solutions of $\tau u = \lambda u$ with the respective boundary conditions. For $L = \infty$ the function $u_L = u_\infty$ is the unique solution of $\tau u_\infty = \lambda u_\infty$ with $u_\infty(0) = 1$ and $u_\infty \in L^2(\mathbb{R}_+, r)$.*

The proofs of these propositions follow from the theory of Sturm-Liouville operators. Again, we refer to the monograph [67]. Note that the classical theory (when $q_0 = 0$) is treated in a self-contained way in Chapter 9 of [60] (see in particular Theorems 9.6 and 9.7).

3.1.2 Bessel and Airy operators as generalized S-L operators

The operators $\mathfrak{G}_{\beta,a}$, $\mathsf{G}_{\beta,2a}$, and A_β can be represented as a generalized Sturm-Liouville operators for which Assumptions (A1-A3) are satisfied, and hence the appropriate resolvents are a.s. Hilbert-Schmidt integral operators. We summarize the relevant results in the propositions below.

Proposition 3.4 ($\mathfrak{G}_{\beta,2a}$ as a Sturm-Liouville operator)**.** *The operator $\mathfrak{G}_{\beta,2a}$ is a Sturm-Liouville operator of the form (3.1) with $r = m_{2a}, p_1 = s_{2a}^{-1}$, $p_0 = q_0 = 0$. The operator satisfies the conditions (A1-A3) with probability one if $a > 1/2$.*

If ϕ solves the equation $\mathfrak{G}_{\beta,2a}\phi = \lambda\phi$ with deterministic initial conditions $\phi(0) = c_0$, $\phi'(0) = c_1$ then (ϕ, ϕ') is the unique strong solution of the stochastic differential equation system

$$d\phi(x) = \phi'(x)dx, \quad d\phi'(x) = \tfrac{2}{\sqrt{\beta}}\phi'(x)dB_{2a}(x) + \Big((2a + \tfrac{2}{\beta})\phi'(x) - \lambda e^{-x}\phi(x)\Big)\,dx, \tag{3.3}$$

with the corresponding initial conditions.

Proof. The fact that $\mathfrak{G}_{\beta,2a}$ is a Sturm-Liouville operator is contained in the statement of Theorem 2.1, the statement about the solution of the eigenvalue equation can be checked with Itô's formula (see [48]). As explained in [49], the Neumann boundary condition for $\mathfrak{G}_{\beta,2a}$ at ∞ for $a > 0$ can be dropped. The SDE (3.3) satisfies the usual conditions for existence and uniqueness, so (ϕ, ϕ') is a well-defined process for all times.

We only need to check that the conditions (A1-A3) are satisfied for $a > 1/2$. This can be done directly using the a.s. sublinear growth of the Brownian motion by noting that $u_{\mathbf{d}}(x) = \int_0^x s_{2a}(y)dy$ and $u_\infty(x) = 1$. $\square$

Proposition 3.5 (Integral kernel for $(\mathsf{G}_{\beta,2a}-a^2)^{-1}$)**.** *For a given $a > 1/2$, let $\phi_{\mathbf{d}}^{(2a)}$ be the unique strong solution of (3.3) with $\lambda = a^2$ and initial conditions $\phi(0) = 0$, $\phi'(0) = 1$. Let $\mathcal{E}_a$ be the event that a^2 is not an eigenvalue of $\mathfrak{G}_{\beta,2a}$. Denote by $\phi_\infty^{(2a)}$ the unique solution of $\mathfrak{G}_{\beta,a}\phi_\infty^{(2a)} = a^2\phi_\infty^{(2a)}$ with $\phi_\infty^{(2a)}(0) = 1$ and $\phi_\infty^{(2a)} \in L^2(\mathbb{R}_+, m_{2a})$, this exists on $\mathcal{E}_a$. Then on the event $\mathcal{E}_a$ the operator $a^{4/3}(\mathsf{G}_{\beta,2a} - a^2)^{-1}$ is a Hilbert-Schmidt integral operator in $L^2(\mathbb{R}_+)$ with integral kernel*

$$K_{\mathsf{G},2a}(x,y) = \tilde\phi_\infty(x)\tilde\phi_{\mathbf{d}}(y)\mathbf{1}(x \ge y) + \tilde\phi_{\mathbf{d}}(x)\tilde\phi_\infty(y)\mathbf{1}(x < y),$$

where

$$\tilde\phi_{\mathbf{d}}(x) = a^{2/3}m_{2a}^{1/2}(a^{-2/3}x)\phi_{\mathbf{d}}^{(2a)}(a^{-2/3}x), \qquad \tilde\phi_\infty(x) = m_{2a}^{1/2}(a^{-2/3}x)\phi_\infty^{(2a)}(a^{-2/3}x). \quad (3.4)$$

On the event $\mathcal{E}_a$ the operator $a^{4/3}(\mathsf{G}_{\beta,2a} - a^2)^{-1}$ has a bounded pure point spectrum that is the same as the spectrum of $a^{4/3}(\mathfrak{G}_{\beta,2a} - a^2)^{-1}$.

Proof. By Proposition 3.3, the function $\phi_\infty^{(2a)}$ is well-defined on $\mathcal{E}_a$, and the operator $(\mathfrak{G}_{\beta,2a} - a^2)^{-1}$ is Hilbert-Schmidt on $L^2(\mathbb{R}_+, m_{2a})$ with integral kernel

$$K_{\mathfrak{G},2a}(x,y) = \phi_\infty^{(2a)}(x)\phi_{\mathbf{d}}^{(2a)}(y)\mathbf{1}(x \ge y) + \phi_{\mathbf{d}}^{(2a)}(x)\phi_\infty^{(2a)}(y)\mathbf{1}(x < y).$$

Recalling the definition of $\mathsf{G}_{\beta,2a}$ from (2.6) we get that $a^{4/3}(\mathsf{G}_{\beta,2a} - a^2)^{-1}$ is a Hilbert-Schmidt integral operator on $L^2(\mathbb{R}_+)$ with kernel

$$K_{\mathsf{G},2a}(x,y) = a^{2/3}m_{2a}^{1/2}(a^{-2/3}x)\, K_{\mathfrak{G},2a}(a^{-2/3}x, a^{-2/3}y)\, m_{2a}^{1/2}(a^{-2/3}y),$$

from which the proposition follows. □

Note that for any fixed $a > 1/2$, the event $\mathcal{E}_a$ has a probability 1, see Remark 3.25. Later, in Corollary 3.18 in Section 3.5 we show that *in our coupling* if a is large enough then a^2 is not an eigenvalue for $\mathfrak{G}_{\beta,2a}$.

Proposition 3.6 (The operator A_β as a generalized S-L operator)**.** *The operator A_β is a generalized Sturm-Liouville operator of the form (3.1) with $r(x) = p_1(x) = 1$, $q_0(x) = \frac{2}{\sqrt{\beta}} B(x)$, $p_0(x) = x$. The operator satisfies the conditions (A1-A3) with probability one.*

If ψ solves the equation $\mathsf{A}_\beta \psi = 0$ with deterministic initial conditions $\psi(0) = c_0$, $\psi'(0) = c_1$, $(c_0, c_1) \neq (0,0)$, then (ψ, ψ') is the strong solution of the SDE system

$$d\psi(x) = \psi'(x)dx, \qquad d\psi'(x) = \psi(x)\left(\tfrac{2}{\sqrt{\beta}} dB + x dx\right), \tag{3.5}$$

which is well defined for all times, and satisfies

$$\frac{\psi'(x)}{\psi(x)\sqrt{x}} \to 1 \quad \textit{a.s. as } x \to \infty. \tag{3.6}$$

A.s. 0 is not an eigenvalue of A_β, and the operator A_β^{-1} is a Hilbert-Schmidt integral operator with kernel

$$K_{\mathsf{A}}(x,y) = \psi_\infty(x)\psi_{\mathbf{d}}(y)\mathbf{1}(x \geq y) + \psi_{\mathbf{d}}(x)\psi_\infty(y)\mathbf{1}(x < y). \tag{3.7}$$

Here $\psi_{\mathbf{d}}$ is the solution of $\mathsf{A}_\beta\psi = 0$ with initial condition $\psi_{\mathbf{d}}(0) = 0$, $\psi'_{\mathbf{d}}(0) = 1$, and $\psi_\infty \in L^2(\mathbb{R}_+)$ is the unique function satisfying $\mathsf{A}_\beta \psi_\infty = 0$, $\psi_\infty(0) = 1$ (see Figure 3).

Proof. The fact that the soft-edge operator Airy_β can be represented as a generalized Sturm-Liouville operator of the form (3.1) with the listed coefficients was shown in [5] (see also [43]). The SDE representation of the solutions of $\mathsf{A}_\beta\psi = 0$ with a deterministic initial condition is shown in [51]. Since the SDE (3.5) satisfies the usual conditions of existence and uniqueness for SDEs, the solution is well defined for all times. The asymptotics (3.6) was stated without proof in [51], we include a proof of this statement in Proposition 3.11 in Section 3.6.1 below for completeness.

To check that the conditions (A1)-(A3) are satisfied we first observe that if $\psi_{\mathbf{d}}$ is the solution of $\mathsf{A}_\beta\psi = 0$ with Dirichlet initial condition then by (3.6) for any fixed $\varepsilon > 0$ we have

$$e^{(2/3-\varepsilon)x^{3/2}} \le \psi_{\mathbf{d}}(x) \le e^{(2/3+\varepsilon)x^{3/2}} \quad \text{for } x \text{ large enough,} \tag{3.8}$$

hence $\psi_{\mathbf{d}}$ is not in $L^2(\mathbb{R}_+)$. This means that a.s. there can be at most one $L^2(\mathbb{R}_+)$ solution of $\mathsf{A}_\beta\psi = 0$ with initial condition $\psi(0) = 1$. We will construct such a function using $\psi_{\mathbf{d}}$.

Denote by z_0 the largest zero of $\psi_{\mathbf{d}}$ on $\mathbb{R}_+$, and let $z_0 = 0$ if such a zero does not exists. Motivated by the Wronskian identity we introduce the function

$$\psi_\infty(x) = \psi_{\mathbf{d}}(x) \int_x^\infty \psi_{\mathbf{d}}(y)^{-2} dy \tag{3.9}$$

which is well defined for $x > z_0$. One can check that ψ_∞ satisfies $\mathsf{A}_\beta\psi_\infty = 0$ and the Wronskian identity

$$\psi'_\infty(x)\psi_{\mathbf{d}}(x) - \psi_\infty(x)\psi'_{\mathbf{d}}(x) = -1 \tag{3.10}$$

for $x > z_0$. Then, the function ψ_∞ can be uniquely extended to $\mathbb{R}_+$ as a solution of $\mathsf{A}_\beta\psi = 0$. This function satisfies (3.10) on $\mathbb{R}_+$, hence it will satisfy $\psi_\infty(0) = 1$.

Using (3.9) we see that for $x > z_0$ we have

$$\psi_{\mathbf{d}}(x)\psi_\infty(x) = \int_x^\infty \frac{\psi_{\mathbf{d}}(x)^2}{\psi_{\mathbf{d}}(y)^2} dy = \int_x^\infty \exp\left(-2\int_x^y \frac{\psi'_{\mathbf{d}}(z)}{\psi_{\mathbf{d}}(z)} dz\right) dy,$$

and from (3.6) we get the bounds

$$\sqrt{y}\int_0^y \psi_{\mathbf{d}}(x)^2\psi_{\mathbf{d}}(y)^{-2} dx \le C, \qquad \sqrt{y}\int_y^\infty \psi_{\mathbf{d}}(x)^{-2}\psi_{\mathbf{d}}(y)^2 dx \le C, \tag{3.11}$$

for some random $C < \infty$. Together with the bound (3.8) this is now sufficient to show that ψ_∞ is in $L^2(\mathbb{R}_+)$, and that

$$\int_0^\infty \int_0^x \psi_\infty(x)^2\psi_{\mathbf{d}}(y)^2 dy\, dx < \infty.$$

By Propositions 3.2 and 3.6 it follows immediately that A_β^{-1} is almost surely a Hilbert-Schmidt integral operator with kernel given in (3.7). $\square$

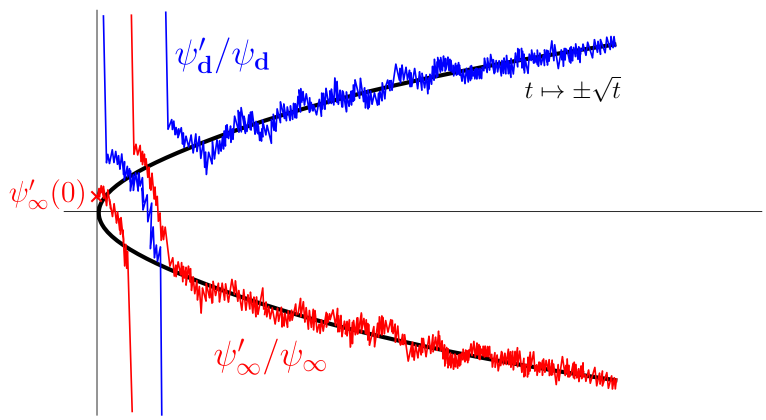

Figure 3: Representation of the log-derivatives of $\psi_{\mathbf{d}}$ and ψ_∞.

Remark 3.7. *Using the identity (3.9) and the limit (3.6) one can show that* $\psi'_\infty(x)/\psi_\infty(x) \to -\sqrt{x}$ *a.s. as* $x \to \infty$*, and that* $\psi_\infty(x) \le e^{-(2/3-\varepsilon)x^{3/2}}$ *for* x *large enough. This behavior was also noted in [51]. See Figure 3 for an illustration for the behavior of* $\psi_{\mathbf{d}}, \psi_\infty$*.*

We record here the Wronskian identities for the appropriate operators:

$$\psi_{\mathbf{d}}(x)\psi'_\infty(x) - \psi'_{\mathbf{d}}(x)\psi_\infty(x) = -1, \qquad \phi_{\mathbf{d}}(x)\phi'_\infty(x) - \phi'_{\mathbf{d}}(x)\phi_\infty(x) = -s_{2a}(x). \quad (3.12)$$

where we dropped the a-dependence in $\phi_{\mathbf{d}}^{(2a)}, \phi_\infty^{(2a)}$ to alleviate the notation. From the second equation of (3.12) one can obtain the following analogue of the identity (3.9) for the hard edge diffusions:

$$\phi_\infty(x) = \phi_{\mathbf{d}}(x) \int_x^\infty \phi_{\mathbf{d}}(y)^{-2} s_{2a}(y)dy, \quad (3.13)$$

if x is larger than the largest zero of $\phi_{\mathbf{d}}$.

Note that the functions $\psi_{\mathbf{d}}, \phi_{\mathbf{d}}$ are diffusions with respect to the natural filtrations of the Brownian motions B, B_{2a}. This is not the case for the functions ψ_∞ and ϕ_∞, as the starting values of these processes depend on the σ-field generated by the whole Brownian motion $B(t), t \geq 0$. In particular, those functions are not Markovian.

3.2 Proof of Theorem 2.4

Proof of Theorem 2.4. In order to prove the theorem, we first need to show that in our coupling with probability one a^2 is not an eigenvalue of the operator $\mathsf{G}_{\beta,2a}$ if a is large enough. This will be the content of Corollary 3.18 in Section 3.5: we will show that there is an a.s. finite random variable C_{ev} such that the operator $\mathsf{G}_{\beta,2a} - a^2$ is invertible for all $a > C_{\mathrm{ev}}$. In particular, this means that on the event $\{a > C_{\mathrm{ev}}\}$ the operator $(\mathsf{G}_{\beta,2a} - a^2)^{-1}$ is a well-defined integral operator with kernel given in Proposition 3.5.

By the results of Section 3.1, to prove Theorem 2.4 we need to show that we have

$$\lim_{a\to\infty} \int_0^\infty \int_0^\infty |K_{\mathsf{A}}(x,y) - K_{\mathsf{G},2a}(x,y)|^2 \, dx \, dy = 0 \qquad a.s. \tag{3.14}$$

We do this by approximating K_{A} and $K_{\mathsf{G},2a}$ with the resolvent kernels of the appropriate differential operators restricted to $[0, L]$, with $L > 0$. We denote these operators by $K_{\mathsf{A}}^{(L)}$ and $K_{\mathsf{G},2a}^{(L)}$. More specifically, set

$$K_{\mathsf{A}}^{(L)}(x,y) = \psi_L(x)\psi_{\mathbf{d}}(y)\mathbf{1}(y \leq x \leq L) + \psi_{\mathbf{d}}(x)\psi_L(y)\mathbf{1}(x < y \leq L), \tag{3.15}$$

where ψ_L which solves $\mathsf{A}_\beta \psi = 0$ with boundary conditions $\psi_L(0) = 1$, $\psi_L(L) = 0$. The function ψ_L is well-defined if $\psi_{\mathbf{d}}(L) \neq 0$.

Moreover, set

$$K_{\mathsf{G},2a}^{(L)}(x,y) = \tilde{\phi}_L(x)\tilde{\phi}_{\mathbf{d}}(y)\mathbf{1}(y \leq x \leq L) + \tilde{\phi}_{\mathbf{d}}(x)\tilde{\phi}_L(y)\mathbf{1}(x < y \leq L) \tag{3.16}$$

where

$$\tilde{\phi}_L(x) = m_{2a}^{1/2}(a^{-2/3}x)\phi_{a^{-2/3}L}(a^{-2/3}x),$$

and $\phi_{a^{-2/3}L}$ solves the equation $\mathfrak{G}_{\beta,2a}\phi = a^2\phi$ with $\phi_{a^{-2/3}L}(0) = 1$, $\phi_{a^{-2/3}L}(a^{-2/3}L) = 0$. The function $\tilde{\phi}_L$ is well-defined if $\phi_{\mathbf{d}}(a^{-2/3}L) \neq 0$. (Note that ϕ and $\tilde{\phi}$ depend on a as well, which we do not denote.)

By the triangle inequality we have

$$\|K_{\mathsf{A}} - K_{\mathsf{G},2a}\|_2 \leq \|K_{\mathsf{A}} - K_{\mathsf{A}}^{(L)}\|_2 + \|K_{\mathsf{A}}^{(L)} - K_{\mathsf{G},2a}^{(L)}\|_2 + \|K_{\mathsf{G},2a} - K_{\mathsf{G},2a}^{(L)}\|_2.$$

We will show that all three terms on the right will vanish in the limit if we let $a \to \infty$ and then $L \to \infty$ along a particular sequence, this is the content of the Lemmas 3.8, 3.9 and 3.10 below. From these three lemmas, we deduce the convergence (3.14), and hence Theorem 2.4 follows. □

More precisely, we will prove the following three lemmas.

Lemma 3.8 (Truncation of the Airy operator)**.** $\|K_{\mathsf{A}} - K_{\mathsf{A}}^{(L)}\|_2^2 \to 0$ *a.s. as* $L \to \infty$*.*

Lemma 3.9 (Convergence of the truncated operators)**.** *For any fixed* $L > 0$ *we have*

$$\|K_{\mathsf{A}}^{(L)} - K_{\mathsf{G},2a}^{(L)}\|_2^2 \to 0 \text{ a.s. as } a \to \infty\,.$$

Lemma 3.10 (Truncation of the Bessel operator)**.** *With probability* 1*, we have,*

$$\lim_{L\to\infty} \limsup_{a\to\infty} \|K_{\mathsf{G},2a} - K_{\mathsf{G},2a}^{(L)}\|_2^2 = 0\,.$$

We prove Lemma 3.8 in Section 3.3 using the the asymptotics (3.8). The proof of Lemma 3.9 is given in Section 3.4, we will show that for a fixed $L < \infty$ the kernel $K_{\mathsf{G},2a}^{(L)}$ converges uniformly to $K_{\mathsf{A}}^{(L)}$ on $[0, L]^2$ as $a \to \infty$. Finally, the proof of Lemma 3.10 will be given in Section 3.5, and it will rely on a careful analysis of the asymptotic behavior of $\phi_{\mathbf{d}}^{(2a)}$.

3.3 Truncation of the Airy operator

We analyze the solutions of the SDE (3.5) via the Riccati transform $\frac{\psi'(t)}{\psi(t)}$. Suppose that ψ, ψ' is the strong solution of the SDE (3.5) with deterministic initial conditions $\psi(0) = c_0$, $\psi'(0) = c_1$, $(c_0, c_1) \neq (0,0)$. Set $X(t) = \frac{\psi'(t)}{\psi(t)}$, by Itô's formula X satisfies the SDE

$$dX(t) = (t - X(t)^2)dt + \tfrac{2}{\sqrt{\beta}}dB(t), \tag{3.17}$$

with initial condition $X(0) = c_1/c_0$. The initial condition is ∞ if $c_0 = 0$, $c_1 \neq 0$. Note that the diffusion blows up to $-\infty$ at the zeros of ψ, and it restarts at ∞ instantaneously whenever this happens.

The drift in (3.17) vanishes on the parabola $x^2 = t$, it is positive for $|x| < \sqrt{t}$, and negative for $|x| > \sqrt{t}$. This suggests that the asymptotic behavior of $X(t)$ should be $\sqrt{t}$ (since the branch $x = -\sqrt{t}$ is unstable), as stated in (3.6). The proposition below proves this statement by providing quantitative bounds on $|X(t) - \sqrt{t}|$. See Figure 4 for an illustration of the asymptotic behavior of X. Note that less precise asymptotic bounds on X were also proved in [15] for the study of the small β limit.

Proposition 3.11. *Let ψ, ψ' be the strong solution of (3.5) with deterministic initial conditions $\psi(0) = c_0$, $\psi'(0) = c_1$, $(c_0, c_1) \neq (0,0)$. Let $X(t) = \frac{\psi'(t)}{\psi(t)}$. Then there is an a.s. finite random time T such that*

$$|X(t) - \sqrt{t}| \leq t^{-1/4} \ln t, \qquad \textit{for all } t \geq T. \tag{3.18}$$

Our upper bound in (3.18) is not optimal. In fact by evaluating the error terms in the proof given below it can be shown that $t^{-1/4} \ln t$ can be replaced with $t^{-1/4}\sqrt{\ln t}\, g(t)$ for any positive function $g(t)$ satisfying $\lim_{t\to\infty} g(t) = \infty$.

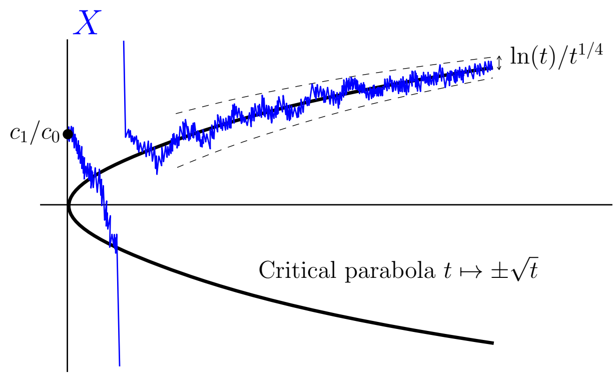

Figure 4: Schematic illustration of the asymptotic behavior of the diffusion X

The proof of Proposition 3.11 relies on the following two technical lemmas, whose proofs are postponed to Section 3.6.1.

Lemma 3.12. *Let X be a strong solution of the SDE (3.17). For a given $s \geq 10$ set*

$$\sigma_s = \inf\left\{t \geq s \ : \ |X(t) - \sqrt{t}| \leq \tfrac{1}{2}t^{-1/4}\ln t\right\}. \tag{3.19}$$

Then σ_s is a.s. finite.

Lemma 3.13. *For a given $t_0 > 0, x_0 \in \mathbb{R}$ consider the solution X of the SDE (3.17) on $[t_0, \infty)$ with initial condition $X(t_0) = x_0$, and denote by P_{t_0,x_0} its distribution. Then*

$$\lim_{t_0\to\infty} \inf_{|x_0-\sqrt{t_0}|\leq\frac{1}{2}t_0^{-1/4}\ln t_0} P_{t_0,x_0}\left(|X(t) - \sqrt{t}| \leq t^{-1/4}\ln t,\ \textit{for all } t \geq t_0\right) = 1. \tag{3.20}$$

Lemma 3.12 shows that for any solution X of the SDE (3.17) and any $s \geq 10$ the process $X(t) - \sqrt{t}$ will get close enough to 0 after time s. Lemma 3.13 shows that if $X(t) - \sqrt{t}$ is close to 0 for a given large $t = t_0$ then with a high probability it will stay close to 0 for all $t \geq t_0$.

Proof of Proposition 3.11. Let $f(t) = t^{-1/4}\ln t$. By Lemma 3.12 for any fixed $s \geq 10$ there is an a.s. finite stopping time σ_s with $\sigma_s \geq s$ so that $|X(\sigma_s) - \sqrt{\sigma_s}| \leq \frac{1}{2}f(\sigma_s)$ with probability one. Lemma 3.13 shows that if the diffusion is close to $\sqrt{t}$ then with a high probability it will stay close forever.

More precisely, for a given $\varepsilon > 0$ one can choose $s \geq 10$ so that

$$\inf_{\substack{t_0 \geq s \\ |x_0 - \sqrt{t_0}| \leq \frac{1}{2} f(t_0)}} P_{t_0, x_0}\left(|X(t) - \sqrt{t}| \leq f(t), \text{ for all } t \geq t_0 \right) \geq 1 - \varepsilon.$$

The strong Markov property and Lemma 3.12 now imply that the inequality (3.18) holds with $T = \sigma_s$ with probability at least $1 - \varepsilon$. This shows that the random time

$$T_0 = \inf \left\{ s \geq 10 : |X(t) - \sqrt{t}| \leq f(t) \text{ for all } t \geq s \right\}$$

is finite with probability at least $1 - \varepsilon$, hence it is a.s. finite. Therefore (3.18) holds with probability one with $T = T_0$. □

We can now prove Lemma 3.8.

Proof of Lemma 3.8. By Proposition 3.6 with probability one the operator A_β^{-1} is a Hilbert-Schmidt integral operator with kernel K_A. From (3.6) and the estimate (3.8) it follows that $\psi_\mathbf{d}$ has a largest zero (if it has one), hence if L is larger than that, the linearity of the equation $\mathsf{A}_\beta \psi = 0$ implies that

$$\psi_L(y) = \psi_\infty(y) - \frac{\psi_\infty(L)}{\psi_\mathbf{d}(L)} \psi_\mathbf{d}(y). \tag{3.21}$$

Hence the truncated operator $K_\mathsf{A}^{(L)}$ is well-defined in this case. From the definition of $K_\mathsf{A}^{(L)}$ we get

$$\|K_\mathsf{A} - K_\mathsf{A}^{(L)}\|_2^2 = \iint_{[0,L]^2} \left| K_\mathsf{A}(x,y) - K_\mathsf{A}^{(L)}(x,y) \right|^2 dx\, dy + \iint_{\mathbb{R}_+^2 \setminus [0,L]^2} |K_\mathsf{A}(x,y)|^2\, dx\, dy. \tag{3.22}$$

By Proposition 3.6, with probability one we have $\|K_{\mathsf{A}}\|_2^2 < \infty$. This implies that the term $\iint_{\mathbb{R}_+^2 \setminus [0,L]^2} |K_{\mathsf{A}}(x,y)|^2 \, dx \, dy$ converges to 0 a.s. as $L \to \infty$. In fact, by the arguments described in the proof of Proposition 3.6 it follows that $\iint_{\mathbb{R}_+^2 \setminus [0,L]^2} |K_{\mathsf{A}}(x,y)|^2 \, dx \, dy$ can be bounded by $CL^{-1/2}$ with a random constant C.

We now estimate the first term on the right hand side of (3.22). By symmetry we have

$$\iint_{[0,L]^2} \left| K_{\mathsf{A}}(x,y) - K_{\mathsf{A}}^{(L)}(x,y) \right|^2 dx \, dy = 2 \int_0^L \int_0^y \left| K_{\mathsf{A}}(x,y) - K_{\mathsf{A}}^{(L)}(x,y) \right|^2 dx \, dy.$$

From (3.21), for L large enough, and $0 \le x \le y \le L$, we get

$$K_{\mathsf{A}}(x,y) - K_{\mathsf{A}}^{(L)}(x,y) = (\psi_\infty(y) - \psi_L(y))\psi_{\mathbf{d}}(x) = \psi_{\mathbf{d}}(x)\psi_{\mathbf{d}}(y) \int_L^\infty \psi_{\mathbf{d}}(z)^{-2} dz,$$

and

$$\int_0^L \int_0^y \left| K_{\mathsf{A}}(x,y) - K_{\mathsf{A}}^{(L)}(x,y) \right|^2 dx \, dy = \frac{1}{2} \left(\int_0^L \frac{\psi_{\mathbf{d}}(x)^2}{\psi_{\mathbf{d}}(L)^2} dx \right)^2 \left(\int_L^\infty \frac{\psi_{\mathbf{d}}(L)^2}{\psi_{\mathbf{d}}(z)^2} dz \right)^2. \tag{3.23}$$

From the bounds of (3.11) we get that the expression in (3.23) is bounded by a random constant times L^{-2}, and thus it converges to zero a.s. as $L \to \infty$. This concludes the proof of Lemma 3.8. □

3.4 Convergence of the truncated operators

Recall the definition of $\tilde{\phi}_L, \psi_L$ from Section 3.2. Lemma 3.9 will follow from the following statement:

Lemma 3.14. *For any fixed $L > 0$ we have $\tilde{\phi}_{\mathbf{d}} \to \psi_{\mathbf{d}}$ and $\tilde{\phi}_L \to \psi_L$ uniformly on $[0, L]$ with probability one as $a \to \infty$.*

Proof of Lemma 3.9. From (3.15), (3.16), and Lemma 3.14 it follows that if $L > 0$ is fixed then $K^{(L)}_{\mathsf{G},2a}(x,y) \to K^{(L)}_{\mathsf{A}}(x,y)$ uniformly on $[0,L]^2$ with probability one. From this Lemma 3.9 follows. □

The proof of Lemma 3.14 relies on the following proposition:

Proposition 3.15. *Let B' be standard white noise on $\mathbb{R}_+$, and B the corresponding Brownian motion. Define $\mathfrak{G}_{\beta,2a}$ using $B_{2a}(x) = a^{-1/3}B(a^{2/3}x)$, and A_β with B' as in Theorem 2.4. Let η_0, η_1 be fixed real numbers. Suppose that the processes $u_a, a \geq 1$ satisfy the following conditions:*

(a) $\mathfrak{G}_{\beta,2a}u_a = a^2 u_a$,

(b) $u_a(0), u_a'(0)$ *are deterministic, depend continuously on a, and satisfy*

$$(a^{2/3}u_a(0), u_a'(0) - au_a(0)) \to (\eta_0, \eta_1)$$

as $a \to \infty$.

Let $\hat{u}_a(x) = a^{2/3}e^{-a^{1/3}x}u_a(a^{-2/3}x)$. Then for any $L > 0$ we have $(\hat{u}_a, \hat{u}_a') \to (\psi, \psi')$ a.s. uniformly on $[0,L]$ where ψ, ψ' is the unique solution of $\mathsf{A}_\beta\psi = 0$ with initial conditions $\psi(0) = \eta_0$, $\psi'(0) = \eta_1$.

Proof. To ease notation, we drop the dependence on a in $u_a, \hat{u}_a$. By Proposition 3.4 the process $(u(t), u'(t))$ satisfies the SDE

$$du(x) = u'(x)dx, \quad du'(x) = \tfrac{2}{\sqrt{\beta}}u'(x)dB_{2a}(x) + \left((2a + \tfrac{2}{\beta})u'(x) - a^2e^{-x}u(x)\right)dx. \tag{3.24}$$

The initial conditions for $\hat{u}$ are

$$\hat{u}(0) = a^{2/3}u(0), \qquad \hat{u}'(0) = u'(0) - au(0),$$

hence by the conditions of the proposition we see that $(\hat{u}(0), \hat{u}'(0)) \to (\eta_0, \eta_1)$. Note that $\hat{u}'(x) = -a^{1/3}\hat{u}(x) + e^{-a^{1/3}x}u'(a^{-2/3}x)$, by Itô's formula and (3.24) we have that

$$d\hat{u}' = \tfrac{2}{\sqrt{\beta}}(a^{-1/3}\hat{u}' + \hat{u})dB(x) + \left(a^{2/3}(1 - e^{-a^{-2/3}x})\hat{u} + \tfrac{2}{\beta}a^{-1/3}\hat{u} + \tfrac{2}{\beta}a^{-2/3}\hat{u}'\right) dx.$$

This means that $\hat{u}, \hat{u}'$ satisfies

$$\begin{aligned} d\hat{u}(x) &= \hat{u}'(x)dx, \\ d\hat{u}'(x) &= \hat{u}(x)(\tfrac{2}{\sqrt{\beta}}dB(x) + xdx) + F_1(\varepsilon, x, \hat{u}(x), \hat{u}'(x))dx + F_2(\varepsilon, x, \hat{u}(x), \hat{u}'(x))dB, \end{aligned} \tag{3.25}$$

where $\varepsilon = a^{-1/3}$ and

$$F_1(\varepsilon, x, p, q) = (\varepsilon^{-2}(1 - e^{-\varepsilon^2 x}) - x)p + \tfrac{2}{\beta}\varepsilon p + \tfrac{2}{\beta}\varepsilon^2 q, \qquad F_2(\varepsilon, x, p, q) = \tfrac{2}{\sqrt{\beta}}\varepsilon q. \tag{3.26}$$

With a bit of abuse of notation we will use $\hat{u}_\varepsilon, \hat{u}'_\varepsilon$ to denote the dependence on $\varepsilon \in (0, 1]$.

The functions F_1, F_2 can be continuously extended to $\varepsilon = 0$ by setting $F_i(0, x, p, q) = 0$. Define $(\hat{u}_0, \hat{u}'_0)$ to be the solution of (3.25) with $\varepsilon = 0$ and initial conditions (η_0, η_1). This is exactly the solution (ψ, ψ') of $\mathsf{A}_\beta \psi = 0$ and $\psi(0) = \eta_0$, $\psi'(0) = \eta_1$.

Note that for $x \in [0, L]$, $\varepsilon \in [0, 1]$ the functions F_1, F_2 are globally Lipschitz in p and q, and $(\hat{u}_\varepsilon, \hat{u}'_\varepsilon), \varepsilon \in [0, 1]$ gives a stochastic flow where the deterministic initial conditions are continuous for $\varepsilon \in [0, 1]$. Standard theory of stochastic flows (see e.g. Theorem 37 in Chapter 7 of [46]) shows that there is a unique one-parameter family of strong solutions for the SDE (3.25) for $\varepsilon \in [0, 1]$ which is a.s. uniformly continuous in ε for $x \in [0, L]$. But this implies that $(\hat{u}_\varepsilon, \hat{u}'_\varepsilon) \to (\hat{u}_0, \hat{u}'_0)$ a.s. uniformly on $[0, L]$ as $\varepsilon \to 0$, proving the statement of the lemma. $\square$

Proof of Lemma 3.14. Consider $u_a(x) = \phi_{\mathbf{d}}(x)$. These functions satisfy the conditions of Proposition 3.15 with $\eta_0 = 0, \eta_1 = 1$. Thus $\hat{u}(x) = a^{2/3}e^{-a^{1/3}x}\phi_{\mathbf{d}}(a^{-2/3}x)$ converges to

$\psi_{\mathbf{d}}$ a.s. uniformly on $[0, L]$ as $a \to \infty$. Then the same is true for

$$\tilde{\phi}_{\mathbf{d}}(x) = a^{2/3} m_{2a}^{1/2}(a^{-2/3}x)\phi_{\mathbf{d}}(a^{-2/3}x) = \hat{u}(x)e^{-\frac{a^{-2/3}}{2}x - \frac{a^{-1/3}}{\sqrt{\beta}}B(x)}.$$

To show the convergence of $\tilde{\phi}_L$ we first consider ϕ_*, the solution of $\mathfrak{G}_{\beta,2a}\phi_* = a^2\phi_*$ with initial conditions $\phi_*(0) = a^{-2/3}$, $\phi_*'(0) = a^{1/3}$. Then $v_a(x) = \phi_*(x)$ satisfies the conditions of Proposition 3.15 with $\eta_0 = 1, \eta_1 = 0$. This means that $\hat{v}(x) = a^{2/3}e^{-a^{1/3}x}\phi_*(a^{-2/3}x)$ converges uniformly to $\psi_*(x)$ where $\mathsf{A}_\beta\psi_* = 0$ and $\psi_*(0) = 1$, $\psi_*'(0) = 0$ (i.e. the solution with Neumann initial conditions).

By linearity $\psi_L(x) = \psi_*(x) - \frac{\psi_*(L)}{\psi_{\mathbf{d}}(L)}\psi_{\mathbf{d}}(x)$. Note that $\psi_{\mathbf{d}}(L) \neq 0$ with probability one for a fixed L, so ψ_L is a.s. well-defined. This also implies that for a fixed L the random variable $\tilde{\phi}_{\mathbf{d}}(L)$ is not zero if a is larger than a random constant, and in this case $\tilde{\phi}_L$ is also well-defined.

The function ψ_L satisfies $\mathsf{A}_\beta\psi_L = 0$ with $\psi_L(0) = 1$, $\psi_L(L) = 0$. By our previous arguments we have $\hat{v}(x) - \frac{\hat{v}(L)}{\hat{u}(L)}\hat{u}(x) \to \psi_L(x)$ a.s. uniformly for $x \in [0, L]$, as $a \to \infty$. We have

$$\begin{aligned}
\hat{v}(x) - \frac{\hat{v}(L)}{\hat{u}(L)}\hat{u}(x) &= a^{2/3}e^{-a^{1/3}x}\phi_*(a^{-2/3}x) - \frac{a^{2/3}e^{-a^{1/3}L}\phi_*(a^{-2/3}L)}{a^{2/3}e^{-a^{1/3}L}\phi_{\mathbf{d}}(a^{-2/3}L)}a^{2/3}e^{-a^{1/3}x}\phi_{\mathbf{d}}(a^{-2/3}x) \\
&= a^{2/3}e^{-a^{1/3}x}\left(\phi_*(a^{-2/3}x) - \frac{\phi_*(a^{-2/3}L)}{\phi_{\mathbf{d}}(a^{-2/3}L)}\phi_{\mathbf{d}}(a^{-2/3}x)\right),
\end{aligned}$$

and we can check (by plugging in $x = 0$ and $x = L$) that

$$\hat{v}(x) - \frac{\hat{v}(L)}{\hat{u}(L)}\hat{u}(x) = e^{-a^{1/3}x}\phi_{a^{-2/3}L}(a^{-2/3}x) = \tilde{\phi}_L(x)e^{\frac{a^{-2/3}}{2}x + \frac{a^{-1/3}}{\sqrt{\beta}}B(x)}.$$

But this now implies that $\tilde{\phi}_L \to \psi_L$ uniformly on $[0, L]$ with probability one, completing the proof. $\square$

3.5 Truncation of the Bessel operator

In order to control $\|K_{\mathsf{G},2a} - K^{(L)}_{\mathsf{G},2a}\|_2^2$ and prove Lemma 3.10, we need to understand the asymptotic behavior of $\phi_{\mathbf{d}}(t) = \phi^{(2a)}_{\mathbf{d}}(t)$ uniformly in a. As before, we turn to the Riccati transform $p = p^{(2a)}(t) = \frac{\phi'_{\mathbf{d}}(t)}{\phi_{\mathbf{d}}(t)}$. Itô's formula together with (3.3) implies that $p(t)$ satisfies the diffusion

$$dp(t) = \frac{2}{\sqrt{\beta}} p(t) dB_{2a}(t) + \Big((2a + \tfrac{2}{\beta}) p(t) - p(t)^2 - a^2 e^{-t} \Big)\, dt \tag{3.27}$$

with initial condition $p(0) = \infty$. The diffusion could reach $-\infty$ at a finite time, in which case it restarts at $+\infty$ instantaneously.

Our next proposition describes the behavior of p in the region $[a^{-2/3}L, \infty)$ uniformly in a. In words the asymptotic behavior of p can be explained as follows: on a microscopic $a^{-2/3}$ time scale the scaled version of p (that is $a^{-2/3}(p(a^{-2/3}t) - a)$) will mimic $\frac{\psi'_{\mathbf{d}}(t)}{\psi_{\mathbf{d}}(t)}$ by Proposition 3.15, and this behavior can be extended up to a small macroscopic time of order $a^{2/3}$. For large macroscopic times the diffusion $p(t)/a$ will behave like a time-stationary diffusion supported on $\mathbb{R}_+$, which yields logarithmic bounds on $\ln p(t) - \ln a$.

For the rest of this section we set $t_0 := 1/8$. Recall that for $a > 0$ we have $B_{2a}(t) = a^{-1/3} B(a^{2/3} t)$.

Proposition 3.16 (Behavior of the Bessel diffusion). *Let $d_1, d_2 > 0$. For a given $L > 0$ and $a_1 \geq 1$, define $\mathcal{C}_{L,a_1}$ to be the event where the following inequalities hold for all $a \geq a_1$:*

$$p^{(2a)}(t) \geq a(1 + d_1\sqrt{t}), \qquad \textit{for all } t \in [a^{-2/3}L, t_0], \tag{3.28}$$

$$\exp(-a^{-1/6}\ln t) \leq p^{(2a)}(t)/a \leq \exp(d_2 + a^{-1/6}\ln t), \qquad \textit{for all } t \geq t_0, \tag{3.29}$$

$$\tfrac{2}{\sqrt{\beta}} |B_{2a}(t) - B_{2a}(s)| \leq a^{1/2}(t - s) + a^{-1/6}\ln(a^{2/3}s), \qquad \textit{for all } t \geq s \geq a^{-2/3}L. \tag{3.30}$$

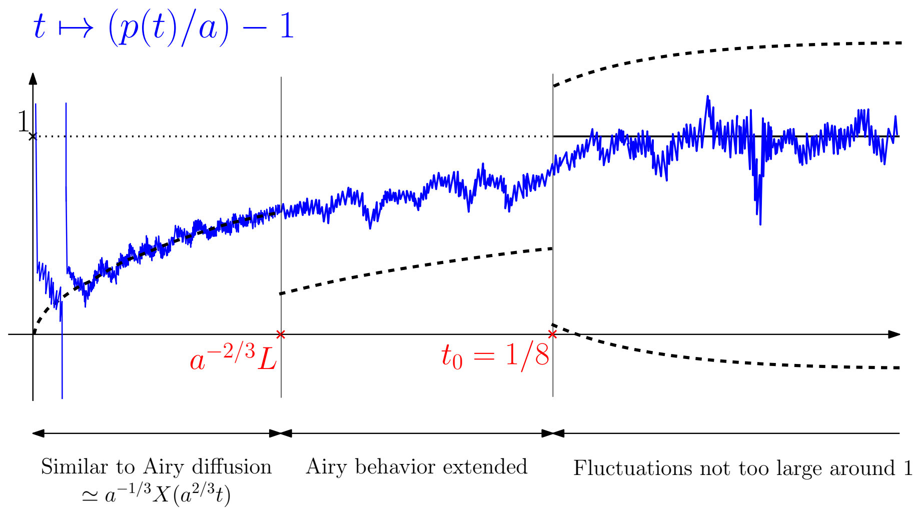

Figure 5: Schematic representation of the behavior of the diffusion $t \mapsto (p(t)/a) - 1$

Then we can choose deterministic constants $d_1, d_2 > 0$ so that

$$\lim_{L\to\infty} \lim_{a_1\to\infty} P\big(\mathcal{C}_{L,a_1}\big) = 1\,. \tag{3.31}$$

See Figure 5 for an schematic illustration of the behavior of the Bessel diffusion. The proof of Proposition 3.16 is postponed to Section 3.6.2. Using this proposition we can control the products $\tilde\phi_{\mathbf{d}}(x)\,\tilde\phi_\infty(x)$ and $\tilde\phi_{\mathbf{d}}(y)^{-2}\tilde\phi_{\mathbf{d}}(x)^2$ when $y \ge x \ge L$. This will be key to estimate $\|K_{\mathsf{G},2a} - K^{(L)}_{\mathsf{G},2a}\|_2^2$.

In the rest of this section, we assume $L \ge 10$ and set $c_L = (10L)^{3/2} \vee 4(1-e^{-t_0})^{-2}$.

Proposition 3.17. *Define*

$$\mathcal{I}(s,t) := -2\int_s^t (p(z)-a)dz + \frac{2}{\sqrt{\beta}}(B_{2a}(t) - B_{2a}(s))\,. \tag{3.32}$$

There are absolute constants c, c' so that for all $a_1 \ge c_L$, the following inequalities hold *on the event $\mathcal{C}_{L,a_1}$ (as defined in Proposition 3.16): for all $a \geq a_1$,*

$$\mathcal{I}(s,t) \leq \begin{cases} -c\,a\sqrt{s}(t-s)+c' & t \geq s, \quad t_0 \geq s \geq a^{-2/3}L, \\ -c\,a(t-s)+5a^{-1/6}\ln s + c' & t \geq s \geq t_0. \end{cases} \tag{3.33}$$

Proof. We first prove the case when $t \geq s \geq t_0$ in (3.33). From this point on we will work on the event $\mathcal{C}_{L,a_1}$ with $a_1 \geq c_L$, allowing us to assume the inequalities (3.28)–(3.30). Let us define

$$q(t) := q^{(2a)}(t) := \ln p^{(2a)}(t) - \ln a.$$

On the event $\mathcal{C}_{L,a_1}$, and for $t \geq t_0$, $q(t)$ is well defined as $p(t) > 0$. By Itô's formula the process q satisfies the following differential equation:

$$dq(t) = \tfrac{2}{\sqrt{\beta}}dB_{2a}(t) + a(2 - e^{q(t)} - e^{-t-q(t)})dt,$$

with the initial condition $q(t_0) = \ln(p(t_0)/a) > 0$. Note that the drift of the diffusion q will be close to $a(2-e^q)$ for large t. The corresponding diffusion

$$d\tilde{q} = \tfrac{2}{\sqrt{\beta}}dB_{2a}(t) + a(2 - e^{\tilde{q}(t)})dt,$$

converges to a stationary distribution supported on $\mathbb{R}$ (which can be computed explicitly). This suggests that q behaves like the stationary solution of $\tilde{q}$, and hence we cannot expect to get a uniform constant bound on $a(e^{q(t)}-1) = p(t) - a$ in (3.32). Because of this we instead look for a bound on the integral term in (3.32).

We start with the following identity: for all $t \geq s \geq t_0$, we have

$$a\int_s^t (e^{q(z)}-1)dz = a(t-s) + \tfrac{2}{\sqrt{\beta}}(B_{2a}(t) - B_{2a}(s)) - (q(t)-q(s)) - a\int_s^t e^{-q(z)-z}dz\,. \tag{3.34}$$

Using the lower bound from (3.29) and the fact that $-a^{-1/6}\ln t \geq -t + t_0$ for all $t \geq t_0$, we get

$$a\int_s^t (e^{q(z)} - 1)dz \geq a(1 - e^{-t_0})(t - s) + \tfrac{2}{\sqrt{\beta}}(B_{2a}(t) - B_{2a}(s)) - (q(t) - q(s)). \quad (3.35)$$

and thus

$$\mathcal{I}(s,t) \leq -2a(1 - e^{-t_0})(t - s) - \tfrac{2}{\sqrt{\beta}}(B_{2a}(t) - B_{2a}(s)) + 2q(t) - 2q(s).$$

Using the inequality $\ln t \leq \ln s + t_0^{-1}(t - s)$ for $t \geq s \geq t_0$, the bounds (3.29), (3.30), and by our choice of c_L, we get that there exist positive constants c_1, c_1' such that for all $t \geq s \geq t_0$, we have

$$\mathcal{I}(s,t) \leq -c_1\, a\,(t - s) + 5a^{-1/6}\ln s + c_1'\,.$$

This completes the proof of (3.33) in the case $t \geq s \geq t_0$.

Let us consider now the case $a^{-2/3}L \leq s < t_0$. From (3.28) we have for all $a^{-2/3}L \leq s \leq t \leq t_0$,

$$\int_s^t (p(z) - a)dz \geq \frac{2}{3}\,a\,d_1(t^{3/2} - s^{3/2}) \geq \frac{2}{3}\,a\,d_1\sqrt{s}(t - s)\,.$$

Using the lower bound from (3.30) we deduce that for all $a^{-2/3}L \leq s \leq t \leq t_0$,

$$\mathcal{I}(s,t) \leq -\frac{4}{3}\,a\,d_1\sqrt{s}(t - s) + a^{1/2}(t - s) + a^{-1/6}\ln(a^{2/3}s)\,.$$

As $a^{-2/3}L \leq s \leq t_0$ and $a \geq a_1 \geq c_L$, we get that there exists a constant c_I such that:

$$\mathcal{I}(s,t) \leq -d_1\,a\,\sqrt{s}(t - s) + c_I\,.$$

For $t \geq t_0 \geq s \geq a^{-2/3}L$, note that $\mathcal{I}(s,t) = \mathcal{I}(s,t_0) + \mathcal{I}(t_0,t)$. Therefore, we get

$$\begin{aligned}\mathcal{I}(s,t) &\leq -d_1\,a\,\sqrt{s}(t_0 - s) + c_I - c_1\,a\,(t - t_0) + 5a^{-1/6}\ln t_0 + c_1'\,,\\ &\leq -c_2\,a\,\sqrt{s}(t - s) + c_I',\end{aligned}$$

where $c_2 = \min\{d_1, c_1 t_0^{-1/2}\}$. We choose $c = \min\{c_1, c_2\}$ and $c' = \max\{c_1', c_I, c_I'\}$ to conclude the proof of (3.33). □

As a consequence of Proposition 3.16, we can also show that a^2 is not an eigenvalue of $\mathfrak{G}_{\beta,2a}$ if a is large enough.

Corollary 3.18. *Let $a_1 \geq c_L$. On the event $\mathcal{C}_{L,a_1}$ defined in Proposition 3.16, a^2 is not an eigenvalue of $\mathfrak{G}_{\beta,2a}$ for all $a \geq a_1$. As a consequence, there exists an a.s. finite random variable $C_{ev} > 0$ such that a^2 is not an eigenvalue of $\mathfrak{G}_{\beta,2a}$ on the event $\{a \geq C_{ev}\}$.*

Proof. The value a^2 is not an eigenvalue of $\mathfrak{G}_{\beta,2a}$ exactly if the function $\phi_{\mathbf{d}}^{(2a)}$ is not in $L^2(\mathbb{R}_+, m_{2a})$. On $\mathcal{C}_{L,a_1}$ and for $a \geq a_1$, using the identity (3.34) and the bound (3.35) in the proof of Proposition 3.17, we get

$$a \int_{t_0}^{t} e^{q(z)} dz \geq a(2 - e^{-t_0})(t - t_0) + \tfrac{2}{\sqrt{\beta}}(B_{2a}(t) - B_{2a}(t_0)) - q(t) + q(t_0).$$

Recall that $a e^{q(t)} = p(t) = \frac{\phi_{\mathbf{d}}'(t)}{\phi_{\mathbf{d}}(t)}$. Using the above lower bound on the integral of $a e^{q(t)}$, and the bounds (3.29) and (3.30), we get

$$\begin{aligned} \phi_{\mathbf{d}}(t)^2 m_{2a}(t) &= \phi_{\mathbf{d}}(t_0)^2 \exp\left(2 \int_{t_0}^{t} p(z) dz\right) \exp(-(2a+1)t - \tfrac{2}{\sqrt{\beta}} B_{2a}(t)) \\ &\geq c(t_0) \exp\left(2a(1 - e^{-t_0})t - t - a^{1/2} t - 2a^{-1/6} \ln t\right), \end{aligned}$$

where $c(t_0)$ is an a.s. finite random constant. Choosing $a \geq a_1 \geq c_L \geq (1 - e^{-t_0})^{-2}$, we get that $\int_0^\infty \phi_{\mathbf{d}}(t)^2 m_{2a}(t) dt$ is infinite, proving the statement.

Now set

$$C_{\text{ev}} = 1 + \inf_{a_1 \geq c_L, L \geq 10} a_1 \cdot \mathbf{1}_{\mathcal{C}_{L,a_1}}.$$

If $a \geq C_{\text{ev}}$ then a^2 is not an eigenvalue of $\mathfrak{G}_{\beta,2a}$. By the limit (3.31), the random variable C_{ev} is a.s. finite, which completes the proof. □

Proposition 3.19. *Recall the definition of the event $\mathcal{C}_{L,a_1}$ from Proposition 3.16. On this event a^2 is not an eigenvalue of $\mathfrak{G}_{\beta,2a}$ (or $\mathsf{G}_{\beta,2a}$) if $a \geq a_1 \geq c_L$ by Corollary 3.18, hence $\tilde{\phi}_\infty$ is well-defined. There exist deterministic constants $c_1, c > 0$ such that for all $L \geq 10$ and $a_1 \geq c_L$, the following inequalities hold on $\mathcal{C}_{L,a_1}$: for all $a \geq a_1$,*

$$\tilde{\phi}_{\mathbf{d}}(x)\tilde{\phi}_\infty(x) \leq \begin{cases} c\, x^{-1/2} & L \leq x < a^{2/3}t_0, \\ c\, a^{-1/3}e^{-a^{-2/3}x/2} & x \geq a^{2/3}t_0, \end{cases} \tag{3.36}$$

and

$$\tilde{\phi}_{\mathbf{d}}(y)^{-2}\tilde{\phi}_{\mathbf{d}}(x)^2 \leq \begin{cases} \exp(-c_1\sqrt{x}(y-x)+c) & y \geq x, \quad a^{2/3}t_0 \geq x \geq L \\ \exp(-c_1\, a^{1/3}\,(y-x) + 5\, a^{-1/6}\ln x + c) & y \geq x \geq a^{2/3}t_0\,. \end{cases} \tag{3.37}$$

Moreover, under the same conditions, we also get the following inequality for all $y \geq x \geq L$:

$$\tilde{\phi}_{\mathbf{d}}(y)^{-2}\tilde{\phi}_{\mathbf{d}}(x)^2 \leq \exp\left(-c_1\sqrt{L}(y-x) + 5a^{-1/6}\ln x + c\right). \tag{3.38}$$

Proof. Recall the definition of $\tilde{\phi}_{\mathbf{d}}, \tilde{\phi}_\infty$ from (3.4). On $\mathcal{C}_{L,a_1}$, the diffusion $p(t)$ does not explode on $[a^{-2/3}L, \infty)$, which also implies the largest zero of $\phi_{\mathbf{d}}^{(2a)}$ is smaller than $a^{-2/3}L$. By the Wronskian identity (3.13), for all $x \geq L$ we have

$$\begin{aligned} \tilde{\phi}_\infty(x)\tilde{\phi}_{\mathbf{d}}(x) &= a^{2/3}s(a^{-2/3}x)m_{2a}(a^{-2/3}x)\int_{a^{-2/3}x}^{\infty} \phi_{\mathbf{d}}(a^{-2/3}x)^2\phi_{\mathbf{d}}(y)^{-2}\frac{s(y)}{s(a^{-2/3}x)}dy \\ &= e^{-a^{-2/3}x}\int_x^\infty \exp\left(\mathcal{I}(a^{-2/3}x, a^{-2/3}y)\right)dy\,, \end{aligned} \tag{3.39}$$

where

$$\mathcal{I}(s,t) := -2\int_s^t (p(z)-a)dz + \frac{2}{\sqrt{\beta}}(B_{2a}(t) - B_{2a}(s))\,.$$

For the product $\tilde{\phi}_d(y)^{-2}\tilde{\phi}_d(x)^2$ for $y \geq x \geq L$, we have

$$\tilde{\phi}_{\mathbf{d}}(y)^{-2}\tilde{\phi}_{\mathbf{d}}(x)^2 = \exp\left(a^{-2/3}(y-x) + \mathcal{I}(a^{-2/3}x, a^{-2/3}y)\right).$$

For $a_1 \geq c_L$, (3.37) follows from (3.33) directly. Integrating the exponential of (3.32) and using the upper bounds (3.33), we get (3.36) and the statement of the proposition. The inequality (3.38) follows by comparing the upper bounds in (3.37). □

We now turn to the proof of Lemma 3.10. We will use the following identity, that follows from the linearity of the equation $\mathfrak{G}_{\beta,2a}\phi = a^2\phi$:

$$\tilde{\phi}_\infty(x) - \tilde{\phi}_L(x) = m_{2a}^{1/2}(a^{-2/3}x)\frac{\phi_\infty(a^{-2/3}L)}{\phi_{\mathbf{d}}(a^{-2/3}L)}\phi_{\mathbf{d}}(a^{-2/3}x) = \frac{\tilde{\phi}_\infty(L)}{\tilde{\phi}_{\mathbf{d}}(L)}\tilde{\phi}_{\mathbf{d}}(x). \tag{3.40}$$

By Propositions 3.16 and 3.17, we have that $\tilde{\phi}_{\mathbf{d}}(L) \neq 0$ and $\tilde{\phi}_\infty$ is well-defined for all $a \geq a_1$ on the event $\mathcal{C}_{L,a_1}$.

Proof of Lemma 3.10. For $L \geq 10$ define the event

$$\mathcal{C}_L^{(1)} = \left\{\psi_{\mathbf{d}}(K)^{-2}\int_0^K \psi_{\mathbf{d}}(x)^2 dx \leq 2K^{-1/2}, \quad \text{for all } K \geq L\right\} \cap \{\psi_{\mathbf{d}}(t) > 0, \quad \forall t \geq L\}.$$

The family of events $\mathcal{C}_L^{(1)}, L \geq 10$ is non-decreasing in L and $\lim_{L\to\infty} P(\mathcal{C}_L^{(1)}) = 1$, by Proposition 3.11. Define the events

$$\mathcal{C}_{L,a_1}^{(2)} = \mathcal{C}_{L,a_1} \cap \mathcal{C}_L^{(1)} \cap \left\{\tilde{\phi}_{\mathbf{d}}^{(2a)}(L)^{-2}\int_0^L \tilde{\phi}_{\mathbf{d}}^{(2a)}(x)^2 dx \leq 3L^{-1/2}, \quad \forall a \geq a_1\right\}.$$

The family $\mathcal{C}_{L,a_1}^{(2)}$ is non-decreasing in a_1 for fixed L and the events $\cup_{a_1}\mathcal{C}_{L,a_1}^{(2)}$ are non-decreasing in L. By the uniform convergence of $(\tilde{\phi}_{\mathbf{d}}, \tilde{\phi}'_{\mathbf{d}}) \to (\psi, \psi')$ on $[0, L]$, we have

$$\lim_{L\to\infty}\lim_{a_1\to\infty} P(\mathcal{C}_{L,a_1}^{(2)}) = 1.$$

We now prove inequalities on the event $\mathcal{C}^{(2)}_{L,a_1}$ for all $a_1 \geq c_L$. In the following, c' is a constant that may change from line to line. We start with the following identity:

$$\|K_{\mathsf{G},2a} - K^{(L)}_{\mathsf{G},2a}\|_2^2 = \iint_{[0,L]^2} \left|K_{\mathsf{G},2a}(x,y) - K^{(L)}_{\mathsf{G},2a}(x,y)\right|^2 dx\,dy + \iint_{\mathbb{R}^2_+ \setminus [0,L]^2} |K_{\mathsf{G},2a}(x,y)|^2\,dx\,dy.$$

On $[0,L]^2$ we have

$$\begin{aligned}
\iint_{[0,L]^2} \left|K_{\mathsf{G},2a}(x,y) - K^{(L)}_{\mathsf{G},2a}(x,y)\right|^2 dx\,dy &= 2\int_0^L \int_0^y \tilde{\phi}_{\mathbf{d}}(x)^2 (\tilde{\phi}_\infty(y) - \tilde{\phi}_L(y))^2 dxdy\,, \\
&= \left(\tilde{\phi}_{\mathbf{d}}(L)^{-2} \int_0^L \tilde{\phi}_{\mathbf{d}}(x)^2 dx\right)^2 \tilde{\phi}_\infty(L)^2 \tilde{\phi}_{\mathbf{d}}(L)^2\,, \\
&\leq (3L^{-1/2})^2 (cL^{-1/2})^2\,,
\end{aligned}$$

using identity (3.40) for the second line and the bound (3.36) for $x = L$ for the third line. Thus this term is bounded by $c'L^{-2}$ uniformly in a.

We further split the region $\mathbb{R}^2_+ \setminus [0,L]^2$ into the union of $\mathcal{R}_1 = [L,\infty) \times [0,L] \cup [0,L] \times [L,\infty)$ and $\mathcal{R}_2 = [L,\infty)^2$. On $\mathcal{R}_1$ we have:

$$\iint_{\mathcal{R}_1} |K_{\mathsf{G},2a}(x,y)|^2\,dx\,dy = \left(2\tilde{\phi}_{\mathbf{d}}(L)^{-2} \int_0^L \tilde{\phi}_{\mathbf{d}}(x)^2 dx\right) \tilde{\phi}_{\mathbf{d}}(L)^2 \int_L^\infty \tilde{\phi}_\infty(y)^2 dy\,.$$

The first term $2\,\tilde{\phi}_{\mathbf{d}}(L)^{-2} \int_0^L \tilde{\phi}_{\mathbf{d}}(x)^2 dx$ is bounded from above by $6\,L^{-1/2}$. For the second term, we split the integral, and apply Proposition 3.19 to get the following upper bound:

$$\begin{aligned}
&\tilde{\phi}_{\mathbf{d}}(L)^2 \int_L^\infty \tilde{\phi}_\infty(y)^2 dy \\
&\quad = \int_L^{a^{2/3}t_0} \tilde{\phi}_\infty(y)^2 \tilde{\phi}_{\mathbf{d}}(y)^2 \tilde{\phi}_{\mathbf{d}}(y)^{-2} \tilde{\phi}_{\mathbf{d}}(L)^2 dy + \int_{a^{2/3}t_0}^\infty \tilde{\phi}_\infty(y)^2 \tilde{\phi}_{\mathbf{d}}(y)^2 \tilde{\phi}_{\mathbf{d}}(y)^{-2} \tilde{\phi}_{\mathbf{d}}(L)^2 dy \\
&\quad \leq \int_L^{a^{2/3}t_0} c^2 y^{-1} e^{-c_1\sqrt{L}(y-L)+c} dy + \int_{a^{2/3}t_0}^\infty c^2 a^{-2/3} e^{-a^{-2/3}y} e^{-c_1\sqrt{L}(y-L)+c} dy \\
&\quad \leq c'(L^{-3/2} + L^{-1/2}a^{-2/3}).
\end{aligned}$$

At last, on $\mathcal{R}_2$ we have

$$\iint_{\mathcal{R}_2} |K_{\mathsf{G},2a}(x,y)|^2\,dxdy = 2\int_L^{a^{2/3}t_0}\int_L^y \tilde{\phi}_{\mathbf{d}}(x)^2\tilde{\phi}_\infty(y)^2 dxdy + 2\int_{a^{2/3}t_0}^\infty \int_L^y \tilde{\phi}_{\mathbf{d}}(x)^2\tilde{\phi}_\infty(y)^2 dxdy.$$

We use (3.36) and (3.37) to bound the first integral,

$$\begin{aligned}
\int_L^{a^{2/3}t_0}\int_L^y \tilde{\phi}_{\mathbf{d}}(x)^2\tilde{\phi}_\infty(y)^2 dxdy &= \int_L^{a^{2/3}t_0} \tilde{\phi}_{\mathbf{d}}(y)^2\tilde{\phi}_\infty(y)^2 \int_L^y \tilde{\phi}_{\mathbf{d}}(y)^{-2}\tilde{\phi}_{\mathbf{d}}(x)^2 dxdy \\
&\leq \int_L^{a^{2/3}t_0} c^2 y^{-1}\int_L^y e^{-c_1\sqrt{x}(y-x)+c} dxdy \\
&\leq \int_L^{a^{2/3}t_0} c' y^{-3/2} dy \\
&\leq c' L^{-1/2}\,.
\end{aligned}$$

For the second integral, we use (3.36) and (3.38),

$$\begin{aligned}
\int_{a^{2/3}t_0}^\infty \int_L^y \tilde{\phi}_\infty(y)^2\tilde{\phi}_{\mathbf{d}}(x)^2 dxdy &= \int_{a^{2/3}t_0}^\infty \tilde{\phi}_\infty(y)^2\tilde{\phi}_{\mathbf{d}}(y)^2 \int_L^y \tilde{\phi}_{\mathbf{d}}(y)^{-2}\tilde{\phi}_{\mathbf{d}}(x)^2 dxdy \\
&\leq \int_{a^{2/3}t_0}^\infty c^2 a^{-2/3} e^{-a^{-2/3}y}\int_L^y e^{-c_1\sqrt{L}(y-x)+5a^{-1/6}\ln y + c} dxdy \\
&\leq \int_{a^{2/3}t_0}^\infty c' L^{-1/2} a^{-2/3} e^{-a^{-2/3}y+5a^{-1/6}\ln y} dy \\
&\leq c' L^{-1/2}\,.
\end{aligned}$$

Recall that the family of events $\mathcal{C}^{(2)}_{L,a_1}$ is non-decreasing in a_1 for fixed L, and the events $\mathcal{C}^{(2)}_L := \cup_{a_1}\mathcal{C}^{(2)}_{L,a_1}$ satisfy $\mathcal{C}^{(2)}_L \uparrow \Omega$ as $L \to \infty$ with $P(\Omega) = 1$. On the event Ω we have

$$\lim_{L\to\infty}\limsup_{a\to\infty} \|K_{\mathsf{G},2a} - K^{(L)}_{\mathsf{G},2a}\|_2^2 = 0,$$

which completes the proof. □

Remark 3.20. *Note that our estimates give an upper bound of the order $O(L^{-1/2})$ on the squared Hilbert-Schmidt norm difference of $K_{\mathsf{G},2a}$ and $K^{(L)}_{\mathsf{G},2a}$. A bound of the same order was shown on the truncation error for K_{A}.*

By choosing $L = L_a$ to be dependent on a with $L_a \to \infty$ at some rate, one could potentially obtain a bound on the rate of convergence in (3.14). This would require the extension of the result of Lemma 3.14 to increasing intervals $[0, L_a]$. We do not explore this path in this paper, but we want to present a hand-waving argument to show that our methods are not expected to give better than logarithmic convergence.

In the proof of Proposition 3.15, we viewed the process $(\hat{u}, \hat{u}')$ as a stochastic flow depending on two variables $\varepsilon = a^{-1/3}$ and x. It is reasonable to expect that if the statement of Lemma 3.14 holds on the interval $[0, L_a]$ then $\sup_{x \le L_a} |\hat{u}_\varepsilon(x) - \hat{u}_0(x)|$ should vanish as $a \to \infty$. This quantity should be of the same order as $\varepsilon \sup_{x \le L_a} |v(x)|$ where $v(x) = \partial_\varepsilon \hat{u}_\varepsilon(x)|_{\varepsilon=0}$. One can check that v satisfies the stochastic differential equation,

$$dv = v' dx, \quad dv' = v(\tfrac{2}{\sqrt{\beta}} dB + x dx) + \tfrac{2}{\beta} \hat{u}_0(x) dx + \tfrac{2}{\sqrt{\beta}} \hat{u}_0'(x) dB$$

with initial values $v(0) = 0$ and $v'(0) = 0$. If we assume that v' grows at least as fast as the contribution of the $\frac{2}{\beta} \hat{u}_0(x) dx$ term then we would get that v grows at least as fast as $e^{\frac{1}{2} x^{3/2}}$. This would lead to the requirement $a^{-1/3} e^{\frac{1}{2} L_a^{3/2}} \to 0$, and $L_a \ll (\ln a)^{2/3}$. Hence the speed of convergence could not be faster than $(\ln a)^{-1/3}$.

3.6 Bounds on the soft and hard edge diffusions

3.6.1 Asymptotic properties of the soft edge diffusion $\psi_{\mathbf{d}}$

This section contains the proofs of Lemma 3.12 and 3.13, which were used for the asymptotic analysis of the diffusion X in (3.17). In this section we set $f(t) = t^{-1/4} \ln t$.

Proof of Lemma 3.12. We will prove that

$$\lim_{t_0\to\infty} P\left(|X(t)-\sqrt{t}| \le \tfrac{1}{2}f(t) \text{ for some } t\in[t_0, t_0+\tfrac{1}{\sqrt{t_0}}\ln^3(t_0)]\right) = 1. \tag{3.41}$$

This means that with higher and higher probability we will hit the region $|X(t)-\sqrt{t}| \le \frac{1}{2}f(t)$ within a small time interval, which implies that $\sigma_s < \infty$ with probability one.

To prove (3.41) we consider X with initial condition $X(t_0) = x_0$ with $t_0 \ge 10$, $x_0 \in \mathbb{R}$, and give a bound on the probability in (3.41) in each of the following cases (see Figure 6):

$$\begin{aligned}
&\text{Case I:} && x_0 > \sqrt{t_0} + f(t_0)/2\\
&\text{Case II:} && x_0 < -\sqrt{t_0} - f(t_0)\\
&\text{Case III:} && -\sqrt{t_0} + f(t_0) < x_0 < \sqrt{t_0} - \frac{1}{2}f(t_0)\\
&\text{Case IV:} && -\sqrt{t_0} - f(t_0) \le x_0 \le -\sqrt{t_0} + f(t_0).
\end{aligned}$$

In each one of these cases we will compare the diffusion to a time-homogeneous version of itself. Then in Cases I-III we use the idea that as long as we control the maximal value of the Brownian motion B, the diffusion will stay close to the deterministic path solving the ODE $x(t)' = t - x(t)^2$ which is what we get if we remove the noise from the SDE of X. In Case IV we will use explicit computations about hitting times of diffusions.

Let $g(x) = x + \frac{1}{\sqrt{x}}\ln(x)$. We consider Case I, when $x_0 > \sqrt{t_0} + f(t_0)/2$. We set $t_1 = g(t_0)$ and assume that t_0 is large enough. Let the time-homogeneous diffusion X_+ on $[t_0, t_1]$ be given by the strong solution of

$$dX_+(t) = (t_1 - X_+(t)^2)dt + \tfrac{2}{\sqrt{\beta}}dB(t), \qquad X_+(t_0) = +\infty.$$

Comparing the drifts of X_+ and X we see that on the event $\{X(t) > \sqrt{t}, t \in [t_0, t_1]\}$ we have $X_+(t) \ge X(t)$ for $t \in [t_0, t_1]$.

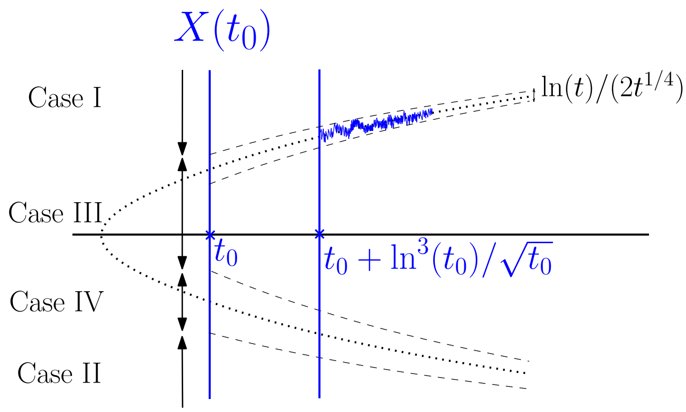

Figure 6: Representation of the four different cases for the position of $X(t_0)$

The process $Z(t) := X_+(t) - \frac{2}{\sqrt{\beta}}\tilde{B}(t)$ with $\tilde{B}(t) = B(t) - B(t_0)$ satisfies the ODE

$$Z'(t) = t_1 - Z(t)^2(1 + \frac{\frac{2}{\sqrt{\beta}}\tilde{B}(t)}{Z_t})^2, \qquad Z(t_0) = \infty$$

for all time $t \geq t_0$ smaller than the first hitting time of 0 for Z. We set

$$M := \tfrac{1}{10}f(t_0) = \tfrac{1}{10}t_0^{-1/4}\ln t_0,$$

and introduce the event

$$\mathcal{A} = \mathcal{A}_{t_0} := \{\sup_{t\in[t_0,t_1]} |B(t) - B(t_0)| \leq \tfrac{\sqrt{\beta}}{2}M\}.$$

Note that

$$P(\mathcal{A}) = P\left(\sup_{s\in[0,1]} |B(s)| \leq \tfrac{\sqrt{\beta}}{20}\sqrt{\ln t_0}\right)$$

which shows that $P(\mathcal{A}_{t_0}) \to 1$ as $t_0 \to \infty$.

On the event $\mathcal{A}$, if $Z(s) = \sqrt{t_0}$ for an $s \in [t_0, t_1]$ then this would imply

$$X(s) \leq \sqrt{t_0} + M \leq \sqrt{s} + f(s)/2.$$

On $\widetilde{\mathcal{A}} = \mathcal{A} \cap \{Z(t) > \sqrt{t_0}, t \in [t_0, t_1]\}$, Z is bounded from above by the deterministic solution of

$$F'(t) = t_1 - F(t)^2(1 - 2M/\sqrt{t_0}), \qquad F(t_0) = \infty,$$

which is given by

$$F(t) = \sqrt{t_1/D}\coth(\sqrt{t_1 D}(t - t_0)), \qquad D = 1 - 2M/\sqrt{t_0}.$$

Using Taylor-expansion, we get that for t_0 large enough we have $F(t_1) \leq \sqrt{t_0} + 2M$ which implies that on $\widetilde{\mathcal{A}}$ we must have $X(t_1) \leq \sqrt{t_0} + 3M \leq \sqrt{t_1} + \frac{1}{2}f(t_1)$. This shows that

$$\mathcal{A} \subset \{|X(t) - \sqrt{t}| \leq \tfrac{1}{2}f(t) \text{ for some } t \in [t_0, t_1]\},$$

which implies

$$\lim_{t_0 \to \infty} \inf_{x_0 > \sqrt{t_0} + \frac{1}{2}f(t_0)} P_{x_0, t_0}\left(|X(t) - \sqrt{t}| \leq \tfrac{1}{2}f(t) \text{ for some } t \in [t_0, t_0 + \tfrac{1}{\sqrt{t_0}}\ln^3(t_0)]\right) = 1.$$

Next we consider the case $x_0 < -\sqrt{t_0} - f(t_0)$ (this is Case II). Similar arguments used as in Case I show that for t_0 large enough X explodes to $-\infty$ before time $t_1 = g(t_0)$ on the event $\mathcal{A}$. Since X restarts at $+\infty$ at the explosion, we are back in Case I, and by the arguments presented there we get that $|X(t) - \sqrt{t}| \leq \frac{1}{2}f(t)$ must hold before time $g(t_1)$ with high probability. Since $g(t_1) \leq t_0 + \ln^3 t_0/\sqrt{t_0}$ for t_0 large, we get

$$\lim_{t_0 \to \infty} \inf_{x_0 < -\sqrt{t_0} - \frac{1}{2}f(t_0)} P_{x_0, t_0}\left(|X(t) - \sqrt{t}| \leq \tfrac{1}{2}f(t) \text{ for some } t \in [t_0, t_0 + \tfrac{1}{\sqrt{t_0}}\ln^3(t_0)]\right) = 1.$$

Now consider Case III, when $x_0 \in (-\sqrt{t_0} + f(t_0), \sqrt{t_0} - f(t_0)/2)$. We show that X reaches $\sqrt{t_0} - f(t_0)/2$ before time t_1 with probability going to 1. For this we can just assume that $x_0 = -\sqrt{t_0} + f(t_0)$, since the other cases stochastically dominate this one by a simple coupling. Let us examine again $Z = X - \frac{2}{\sqrt{\beta}}\tilde{B}$. The process $Z(t)$ satisfies the ODE

$$Z'(t) = t - (Z(t) + \tfrac{2}{\sqrt{\beta}}\tilde{B}(t))^2, \quad Z(t_0) = -\sqrt{t_0} + f(t_0).$$

On the event $\mathcal{A}$, the process Z is increasing when $-\sqrt{t} + M \leq Z(t) \leq \sqrt{t} - M$, in particular $Z'(t_0) > 0$. Before Z hits $\sqrt{t_0}$, we can bound Z from below by $G(t)$ where

$$G'(t) = (\sqrt{t_0} - \tfrac{3}{2}M)^2 - G^2(t), \quad G(t_0) = -\sqrt{t_0} + f(t_0).$$

Solving the above initial value problem, we get $G(t) = (\sqrt{t_0} - \frac{3}{2}M)\tanh((\sqrt{t_0} - \frac{3}{2}M)(t - t_0) + c)$ where $c < 0$ is chosen such that $G(t_0) = -\sqrt{t_0} + f(t_0)$. Here $c \sim -\frac{3}{8}\ln t_0$ if t_0 is large. Using Taylor-expansion again, we get $G(t_1) \geq \sqrt{t_0} - 2M$ which implies that $X(t) \geq \sqrt{t} - f(t)/2$ somewhere in $[t_0, t_1]$.

For the last case IV when $x_0 \in [-\sqrt{t_0} - f(t_0), -\sqrt{t_0} + f(t_0)]$, denote by τ the exit time of $X(t)$ from the interval $[q^-, q^+] := [-\sqrt{t_1} - f(t_1), -\sqrt{t_1} + f(t_1)]$. We use the time-homogeneous diffusion $\tilde{X}(t)$ satisfying the SDE

$$d\tilde{X}(t) = (t_0 - \tilde{X}(t)^2)dt + \tfrac{2}{\sqrt{\beta}}dB(t), \quad \tilde{X}(t_0) = x_0.$$

Let us denote by $\tilde{\tau}$ the first exit time for $\tilde{X}$ after time t_0 from (q^-, q^+). By the Cameron-Martin-Girsanov formula, the Radon-Nikodym derivative of X with respect to $\tilde{X}$ on the time interval $[t_0, t_1]$ can be expressed as $e^{G(\tilde{X})}$ where

$$G(\tilde{X}) = \frac{1}{(\frac{2}{\sqrt{\beta}})^2}\Big(\int_{t_0}^{t_1}(\tilde{X}(t_1) - \tilde{X}(t))dt - \frac{(t_1 - t_0)^3}{6} - \int_{t_0}^{t_1} t\,(t_0 - \tilde{X}(t)^2)dt\Big).$$

On the event $\{\tilde{X}(t) \in [q^-, q^+], t \in [t_0, t_1]\}$ one can bound $G(\tilde{X})$ by a constant. This means that $P(\tau > t_1)$ can be bounded by a constant times $P(\tilde{\tau} > t_1)$.

We can explicitly compute $E[\tilde{\tau}]$ in terms of the scale function and speed measure of $\tilde{X}$. The scale function sc and speed measure sp for $\tilde{X}(t)$ are given by

$$sc(x) = \int_{-\infty}^{x} \exp(-2t_0 y + \frac{2}{3}y^3)dy, \quad sp(dx) = \frac{2}{sc'(x)}dx.$$

From this we can express the first moment of $\tilde{\tau}$ as

$$\begin{aligned} E[\tilde{\tau} - t_0] = &\int_{q^-}^{x_0} \frac{(sc(y) - sc(q^-))(sc(q^+) - sc(x_0))}{sc(q^+) - sc(q^-)} sp(dy) \\ &+ \int_{x_0}^{q^+} \frac{(sc(x_0) - sc(q^-))(sc(q^+) - sc(y))}{sc(q^+) - sc(q^-)} sp(dy). \end{aligned}$$

(See for example Theorem VII.3.6 [52].) By analyzing the above integrals as $t_0 \to \infty$, one can bound $E[\tilde{\tau} - t_0]$ by $c\frac{\ln \ln t_0}{\sqrt{t_0}}$ with an absolute constant c for all t_0 large enough and all $x_0 \in [-\sqrt{t_0} - f(t_0), -\sqrt{t_0} + f(t_0)]$. (We refer to Lemma 5.7. of [14] for additional details for this argument.) By Markov's inequality, we get

$$P_{x_0,t_0}[\tau > t_1] = E[\exp(G(\tilde{X}))\mathbf{1}_{\{\tilde{\tau} > t_1\}}] \le c' \frac{\ln \ln t_0}{(t_1 - t_0)\sqrt{t_0}} = c' \frac{\ln \ln t_0}{\ln t_0},$$

with an absolute constant c'. Therefore X exits the region (q^-, q^+) before time t_1 with probability tending to 1 as $t_0 \to \infty$. Once $X(t)$ exits this region, we get to Case II or III, and repeating the arguments there we can show that

$$\lim_{t_0 \to \infty} \inf_{x_0 : |x_0 + \sqrt{t_0}| \le f(t_0)} P_{x_0,t_0} \left(|X(t) - \sqrt{t}| \le \tfrac{1}{2} f(t) \text{ for some } t \in [t_0, t_0 + \tfrac{1}{\sqrt{t_0}} \ln^3(t_0)] \right) = 1.$$

This completes the proof of (3.41) and hence the statement of the lemma. $\square$

Proof of Lemma 3.13. Introduce $Y(t) := X(t) - \sqrt{t}$, then $Y(t)$ satisfies the stochastic differential equation

$$dY(t) = (-Y(t)^2 - 2\sqrt{t}Y(t) - \frac{1}{2\sqrt{t}})dt + \tfrac{2}{\sqrt{\beta}} dB(t),$$

with initial condition $y_0 = x_0 - \sqrt{t_0}$.

With the same driven noise dB, we define two families of diffusions $Y_1(t) = Y_1^{y_0,t_0}(t)$, $Y_2(t) = Y_2^{y_0,t_0}(t)$ on $[t_0, \infty)$ with initial condition y_0 as follows:

$$dY_1(t) = -2\sqrt{t}Y_1(t)dt + \tfrac{2}{\sqrt{\beta}}dB(t), \quad Y_1(t_0) = y_0,$$
$$dY_2(t) = (-2\sqrt{t}Y_2(t) - 2f(t)^2)dt + \tfrac{2}{\sqrt{\beta}}dB(t), \quad Y_2(t_0) = y_0.$$

By comparing the drift terms in Y, Y_1, Y_2 we see that if for a given t_0 we start Y_1, Y_2 from $y_0 = Y(t_0)$ at time t_0 then the coupling $Y_2(t) \le Y(t) \le Y_1(t)$ holds for all $t \ge t_0$ on the event

$$\mathcal{D}_{t_0,y_0} := \{-f(t) \le Y_2(t), Y_1(t) \le f(t) \text{ for all } t \ge t_0\}. \tag{3.42}$$

Consequently, this shows that

$$\mathcal{D}_{t_0,y_0} \subset \{|Y(t)| \le f(t), \forall t \ge t_0\}, \tag{3.43}$$

and thus it is enough to prove

$$\lim_{t_0\to\infty} \inf_{|y_0|\le\frac{1}{2}f(t_0)} P(\mathcal{D}_{t_0,y_0}) = 1. \tag{3.44}$$

Using the integrating factor trick, both Y_1 and Y_2 can be solved explicitly:

$$Y_1(t) = \exp(-\tfrac{4}{3}(t^{3/2} - t_0^{3/2}))y_0 + \tfrac{2}{\sqrt{\beta}}e^{-\frac{4}{3}t^{3/2}}\int_{t_0}^t e^{\frac{4}{3}s^{3/2}}dB_s,$$
$$Y_2(t) = \exp(-\tfrac{4}{3}(t^{3/2} - t_0^{3/2}))y_0 - 2\,e^{-\frac{4}{3}t^{3/2}}\int_{t_0}^t f^2(s)e^{\frac{4}{3}s^{3/2}}ds + \tfrac{2}{\sqrt{\beta}}e^{-\frac{4}{3}t^{3/2}}\int_{t_0}^t e^{\frac{4}{3}s^{3/2}}dB_s.$$

Let $\xi(t) = \int_1^t e^{\frac{8}{3}s^{3/2}}ds$. There exists a Brownian motion W such that we have the following distributional identity:

$$\left(\int_1^t e^{\frac{4}{3}s^{3/2}}dB_s,\, t \ge 1\right) \stackrel{d}{=} (W(\xi(t)),\, t \ge 1).$$

By the Law of Iterated Logarithm, there exist finite random constant C such that

$$|W(u)| \leq C\sqrt{u \ln\ln u}, \qquad \text{for all } u \geq 20.$$

Note that $\xi(t) \leq \frac{1}{2} e^{\frac{8}{3}t^{3/2}} t^{-1/2}$ for all $t \geq 1$. We may assume $t_0 \geq \max(10, \xi^{-1}(20))$, then for $t \geq t_0$ we get

$$\begin{aligned} Y_1(t) &\leq \tfrac{1}{2} e^{-\frac{4}{3}(t^{3/2} - t_0^{3/2})} f(t_0) + \tfrac{2}{\sqrt{\beta}}\, C\, e^{-\frac{4}{3}t^{3/2}} \big(\sqrt{\xi(t)\ln\ln\xi(t)} + \sqrt{\xi(t_0)\ln\ln\xi(t_0)}\big) \\ &\leq e^{-\frac{4}{3}(t^{3/2} - t_0^{3/2})} \big(\tfrac{1}{2} f(t_0) + \tfrac{2}{\sqrt{\beta}} C t_0^{-1/4} \sqrt{\ln t_0}\big) + \tfrac{2}{\sqrt{\beta}} C t^{-1/4} \sqrt{\ln t}. \end{aligned}$$

Integration by parts yields the bound

$$\int_{t_0}^{t} f(s)^2 e^{\frac{4}{3}s^{3/2}} ds \leq \frac{1}{\sqrt{t}}\, f(t)^2 e^{\frac{4}{3}t^{3/2}}.$$

Therefore, we obtain that

$$Y_2(t) \geq -e^{-\frac{4}{3}(t^{3/2} - t_0^{3/2})} \big(\tfrac{1}{2} f(t_0) + \tfrac{2}{\sqrt{\beta}} C t_0^{-1/4} \sqrt{\ln t_0}\big) - 2t^{-1/2} f(t)^2 - \tfrac{2}{\sqrt{\beta}} C t^{-1/4} \sqrt{\ln t}\,.$$

For a large enough deterministic c_0, we have $-f(t) \leq Y_2(t) \leq Y_1(t) \leq f(t)$ for all $t \geq t_0 \geq c_0$ on the event $\{C < \frac{\sqrt{\beta}}{20}\sqrt{\ln t_0}\}$. Hence if $t_0 \geq c_0$ then

$$\inf_{|y_0| \leq \frac{1}{2} f(t_0)} P(\mathcal{D}_{y_0, t_0}) \geq P(C < \tfrac{\sqrt{\beta}}{20}\sqrt{\ln t_0})$$

which completes the proof of (3.44). □

3.6.2 Bounds for the hard edge diffusion

We start this section with a lemma controlling the fluctuations of Brownian motion. Although the bounds in the lemma are not optimal they are sufficient for our purposes.

Lemma 3.21. *Let B be a standard Brownian motion. Then there is a random finite positive C so that a.s. we have the following inequality:*

$$|B(s+h)-B(s)| \le C\sqrt{h\ln(2+\tfrac{s}{h}+|\ln h|)}, \qquad \textit{for all } h>0,\ s>0. \tag{3.45}$$

This implies in particular the following simple bounds:

$$|B(s+h)-B(s)| \le C_1(h+\ln s), \qquad \textit{for all} \quad h>0,\ s\ge 10, \tag{3.46}$$

with a random constant C_1.

Proof. First set $h=2^n$, $s=m2^n$, for $n\in\mathbb{Z}$ and $m\in\mathbb{N}$. We have

$$\begin{aligned} P(\max_{x\le h}|B(s)-B(s+x)| \ge 4\cdot 2^{n/2}\sqrt{\ln(2+|n|+m)}) &\le 2P(|B(1)|\ge 4\sqrt{\ln(2+|n|+m)}) \\ &\le 2e^{-8\ln(2+|n|+m)} = \frac{2}{(2+|n|+m)^8}, \end{aligned}$$

which is summable for $n\in\mathbb{Z}, m\in\mathbb{N}$. Hence by the Borel-Cantelli Lemma, there is a random $\tilde{C}$ so that

$$\max_{x\le h}|B(s)-B(s+x)| \le \tilde{C}\sqrt{h}\sqrt{\ln(2+|\ln h|+\tfrac{s}{h})} \tag{3.47}$$

for all $s=m2^n$, $h=2^n$. For general $s>0, h>0$, there exist $n\in\mathbb{Z}, m\in\mathbb{N}$ such that $2^n<h\le 2^{n+1}$ and $m2^n<s\le(m+1)2^n$. Using (3.47) and the triangle inequality, we get

$$|B(s+h)-B(s)| \le 8\tilde{C}\sqrt{h\ln(2+|\ln h|+\tfrac{s}{h})},$$

which proves the first part of the lemma with $C=8\tilde{C}$.

For $s\ge 10$ we have

$$\sqrt{h\ln(2+\tfrac{s}{h}+|\ln h|)} \le \sqrt{h\ln(2+\tfrac{1}{h}+|\ln h|)+h\ln(1+s)}.$$

For $h \geq \ln s$, we have

$$\ln(2 + \tfrac{1}{h} + |\ln h|) < h, \quad \ln(1+s) < \ln(2s) \leq 2h,$$

which implies $\sqrt{h \ln(2 + \frac{s}{h} + |\ln h|)} \leq 2h$ in this case.

Now assume $h < \ln s$. We have $h \ln(2 + \frac{2}{h} + \ln h) \leq 2$ for $h \in [0,1]$, which yields $h \ln(2 + \frac{1}{h} + |\ln h|) \leq 2\ln(s) \ln\ln(s)$ for $h < \ln s$, $s \geq 10$. We also have $h \ln(1+s) \leq (3/2)(\ln s)^2$ under the same conditions, which yields $\sqrt{h \ln(2 + \frac{s}{h} + |\ln h|)} \leq 2 \ln s$. The bound (3.46) now follows from (3.45). $\square$

The next lemma gives estimates on the diffusion $p^{(2a)}(t)$ at time $t = a^{-2/3}L$ using the convergence result of Proposition 3.15.

Lemma 3.22. *For all positive L and a_1, let $\mathcal{A}^{(1)}_{L,a_1}$ be the event that*

$$a\big(1 + \tfrac{4}{5} a^{-1/3}\sqrt{L}\big) \leq p(a^{-2/3}L) \leq a\big(1 + \tfrac{6}{5} a^{-1/3}\sqrt{L}\big), \text{ for all } a \geq a_1.$$

Then $\lim_{L\to\infty} \lim_{a_1 \to \infty} P(\mathcal{A}^{(1)}_{L,a_1}) = 1$.

Proof. The uniform convergence of Proposition 3.15 implies that almost surely,

$$p(a^{-2/3}L)a^{-2/3} - a^{1/3} \to X(L)\,, \quad \text{as } a \to \infty\,. \tag{3.48}$$

Indeed

$$(a^{2/3}\phi(a^{-2/3}t)e^{-a^{1/3}t}, \phi'(a^{-2/3}t)e^{-a^{1/3}t} - a\phi(a^{-2/3}t)e^{-a^{1/3}t}) \to (\psi(t), \psi'(t))\,,$$

uniformly on $[0, L]$ and $p(t) = \phi'(t)/\phi(t)$ and $X(t) = \psi'(t)/\psi(t)$.

Fix L large and define the event:

$$\mathcal{A}_L := \{\tfrac{9}{10}\sqrt{t} \leq X(t) \leq \tfrac{11}{10}\sqrt{t}, \quad \forall t \geq L\}.$$

Note that the family $\mathcal{A}_L$ is non-decreasing in L. From Proposition 3.11 it follows that $\lim_{L\to\infty} P(\mathcal{A}_L) = 1$. For all L and a_1, define

$$\mathcal{A}_{L,a_1} = \mathcal{A}_L \cap \{a(1+\tfrac{4}{5}a^{-1/3}\sqrt{L}) \leq p(a^{-2/3}L) \leq a(1+\tfrac{6}{5}a^{-1/3}\sqrt{L}), \quad \forall a \geq a_1\}.$$

By (3.48) and the condition $\frac{9}{10}\sqrt{L} \leq X(L) \leq \frac{11}{10}\sqrt{L}$ on $\mathcal{A}_L$, we have $\mathcal{A}_{L,a_1} \uparrow \mathcal{A}_L$ as $a_1 \to \infty$ which concludes the proof. $\square$

Let us introduce $q = q^{(2a)} = \ln p^{(2a)} - \ln a$. By Lemma 3.22, the diffusion q is well-defined at time $a^{-2/3}L$ on the event $\mathcal{A}^{(1)}_{L,a_1}$. By Itô's formula, for $t \geq a^{-2/3}L$ we have

$$dq(t) = \tfrac{2}{\sqrt{\beta}} dB_{2a}(t) + a(2 - e^{q(t)} - e^{-t-q(t)})dt\,. \tag{3.49}$$

The diffusion q blows-up when p reaches 0, so q may not be well-defined on the whole interval $[a^{-2/3}L, +\infty)$.

The next proposition controls the growth of q from small times starting at $a^{-2/3}L$ until a positive deterministic time. In this time-interval, q is small and therefore p is close to $a(1+q)$. Analyzing the drift of the q diffusion for small t and q, we see that one can compare the behavior of q with the diffusion X defined in (3.17). This allows us to bound q with constant multiples of the square root function with large probability.

Proposition 3.23. *Fix $t_0 := 1/8$. For all positive L and a_1 with $a_1^{-2/3}L \leq t_0$, we define $\mathcal{A}^{(2)}_{L,a_1}$ to be the event that*

$$\tfrac{2}{5}\sqrt{t} \leq q^{(2a)}(t) \leq \tfrac{7}{5}\sqrt{t}, \quad \forall t \in [a^{-2/3}L, t_0] \quad \textit{for all } a \geq a_1. \tag{3.50}$$

Then $\lim_{L\to\infty} \lim_{a_1\to\infty} P\big(\mathcal{A}^{(2)}_{L,a_1}\big) = 1$.

Note that the inequality (3.50) implies

$$p^{(2a)}(t) \geq a(1+\tfrac{2}{5}\sqrt{t}), \quad \forall t \in [a^{-2/3}L, t_0] \quad \text{for all } a \geq a_1.$$

Proof. If $a_1 > (8L)^{3/2}$ then on the event $\mathcal{A}^{(1)}_{L,a_1}$ of Lemma 3.22, we have

$$\tfrac{3}{5}\sqrt{L} \le a^{1/3} q(a^{-2/3}L) \le \tfrac{6}{5}\sqrt{L}, \qquad \text{for all } a \ge a_1.$$

For $0 \le q \le 1/2, t \le t_0$ we have the following inequalities:

$$-q^2 + t \ge 2 - e^q - e^{-t-q} = 2 - e^q - e^{-q} + e^{-q}(1 - e^{-t}) \ge -2q^2 + \tfrac{1}{2}t\,.$$

Let $q_1 = q_1^{(2a)}$ and $q_2 = q_2^{(2a)}$ be the diffusions on $[a^{-2/3}L, t_0]$ so that

$$dq_1(t) = \tfrac{2}{\sqrt{\beta}} dB_{2a}(t) + a(\tfrac{1}{2}t - 2q_1(t)^2)dt, \quad dq_2(t) = \tfrac{2}{\sqrt{\beta}} dB_{2a}(t) + a(t - q_2(t)^2)dt\,,$$

with $q_1(a^{-2/3}L) = q_2(a^{-2/3}L) = q(a^{-2/3}L)$. Then the coupling $\{q_1(t) \le q(t) \le q_2(t)\}$ holds on the event $\{0 \le q_1(t) \le q_2(t) \le 1/2, \forall t \in [a^{-2/3}L, t_0]\}$.

Recall that $B_{2a}(t) = a^{-1/3}B(a^{2/3}t)$. Setting $y_1(t) = 2\,a^{1/3}q_1(a^{-2/3}t)$ and $y_2(t) = a^{1/3}q_2(a^{-2/3}t)$, we get

$$dy_1(t) = \tfrac{4}{\sqrt{\beta}} dB(t) + (t - y_1(t)^2)dt, \quad dy_2(t) = \tfrac{2}{\sqrt{\beta}} dB(t) + (t - y_2(t)^2)dt,$$

with $\frac{6}{5}\sqrt{L} \le y_1(L) \le \frac{12}{5}\sqrt{L}$ and $\frac{3}{5}\sqrt{L} \le y_2(L) \le \frac{6}{5}\sqrt{L}$. Thanks to Proposition 3.11, we know that the event

$$\Big\{\forall t \ge L, \quad y_1(t) \in [\tfrac{4}{5}\sqrt{t}, \tfrac{13}{5}\sqrt{t}], \quad y_2(t) \in [\tfrac{1}{2}\sqrt{t}, \tfrac{7}{5}\sqrt{t}]\Big\} \tag{3.51}$$

has probability going to 1 when $L \to \infty$. On the event (3.51) we have

$$0 \le \tfrac{2}{5}\sqrt{t} \le q_1(t) \le q(t) \le q_2(t) \le \tfrac{7}{5}\sqrt{t} \le \tfrac{1}{2}, \quad \forall t \in [a^{-2/3}L, t_0],$$

implying that $p(t) \ge a(1 + \frac{2}{5}\sqrt{t})$ on $[a^{-2/3}L, t_0]$. $\square$

Next we estimate the growth of $q(t)$ in the time interval $t \in [t_0, \infty)$. As we will see, q will have a different behavior for large times: it oscillates near the value $\ln 2$ with possibly making large excursions away from this value. We will prove bounds on those fluctuations using a comparison with a non-exploding, stationary version of the diffusion q.

Proposition 3.24. *Recall the definition of* $\mathcal{A}^{(2)}_{L,a_1}$ *from Proposition 3.23. Define*

$$\mathcal{A}^{(3)}_{L,a_1} = \mathcal{A}^{(2)}_{L,a_1} \cap \{-a^{-1/6}\ln t \le q^{(2a)}(t) \le c + a^{-1/6}\ln t, \forall t \ge t_0, \forall a \ge a_1\}.$$

Then, there exists a constant $c > 0$ *such that* $\lim_{L\to\infty}\lim_{a_1\to\infty} P\big(\mathcal{A}^{(3)}_{L,a_1}\big) = 1$.

Proof. For each a, we bound $q(t)$ using two stationary diffusions $q_1(t) = q_1^{(2a)}(t)$ and $q_2(t) = q_2^{(2a)}(t)$, and we show that the growth of q_1, q_2 is at most logarithmic with a large probability. Let q_1 and q_2 be the following diffusions:

$$dq_1(t) = \tfrac{2}{\sqrt{\beta}}dB_{2a}(t) + a(c_1 - e^{q_1(t)})dt, \quad dq_2(t) = \tfrac{2}{\sqrt{\beta}}dB_{2a}(t) + a(c_2 - e^{q_2(t)})dt,$$

with $c_1 = 2 - e^{-t_0}, c_2 = 2$, and $q_1(t_0) = q_2(t_0) = q(t_0)$. Comparing the drift terms of q, q_1, q_2 we see that the event $\{q_1(t) \ge -t + t_0, \forall t \ge t_0\}$ implies the event $\{q_1(t) \le q(t) \le q_2(t), \forall t \ge t_0\}$.

Notice that the SDEs for q_i for $i = 1, 2$ can be solved. We get that for $t \ge t_0$, $i = 1, 2$,

$$\begin{aligned}\exp(-q_i(t)) &= \exp(-q_i(t_0))\exp\Big(a\,c_i(t_0 - t) + \tfrac{2}{\sqrt{\beta}}(B_{2a}(t_0) - B_{2a}(t))\Big)\\ &\quad + a\int_{t_0}^t \exp\Big(a\,c_i(s - t) + \tfrac{2}{\sqrt{\beta}}(B_{2a}(s) - B_{2a}(t))\Big)ds.\end{aligned}$$

Recall that $B_{2a}(t) = a^{-1/3}B(a^{2/3}t)$. Applying the bound (3.46) of Lemma 3.21 on the event $\{C_1 < a_1^{1/6}\}$ for the Brownian motion B, we have the following inequality for $x \ge a^{-2/3}L$, $L \ge 10$ and for all $a \ge a_1$:

$$\tfrac{2}{\sqrt{\beta}}|B_{2a}(x+h) - B_{2a}(x)| \le C_1 a^{-1/3}(a^{2/3}h + \ln(a^{2/3}x)) \le a^{1/2}h + a^{-1/6}\ln(a^{2/3}x). \tag{3.52}$$

Note that this is exactly inequality (3.30) of Proposition 3.16.

Moreover, on $\mathcal{A}^{(2)}_{L,a_1}$, for $a \geq a_1$, we have $\exp(q(t_0)) \geq \exp(2\sqrt{t_0}/5) > c_1$. We get that there is an absolute constant $c_3 > 0$ so that for all $a \geq a_1 \geq c_3$ we have

$$\begin{aligned}
e^{-q_1(t)} &\leq \exp\big(-q(t_0) + (ac_1 - a^{1/2})(t_0 - t) + a^{-1/6}\ln(a^{2/3}t_0)\big) \\
&\quad + \exp\big(a^{-1/6}\ln(a^{2/3}t)\big)(c_1 - a^{-1/2})^{-1}\Big(1 - \exp\big((ac_1 - a^{1/2})(t_0 - t)\big)\Big) \\
&\leq e^{a^{-1/6}\ln(a^{2/3}t)}\Big((c_1 - a^{-1/2})^{-1} + e^{(ac_1 - a^{1/2})(t_0 - t)}(e^{-q(t_0)} - (c_1 - a^{-1/2})^{-1})\Big) \\
&\leq t^{a^{-1/6}}.
\end{aligned}$$

We conclude that for all $a \geq a_1 \geq c_3$ we have

$$q_1(t) \geq -a^{-1/6}\ln t \geq -t + t_0, \quad \forall t \geq t_0,$$

which also implies that the coupling $q_2(t) \geq q(t) \geq q_1(t)$ holds on $\{C_1 < a_1^{1/6}\} \cap \mathcal{A}^{(2)}_{L,a_1}$.

For the upper bound, first note that $\exp(q(t_0)) < e^{1/2} < c_2 = 2$ on $\mathcal{A}^{(2)}_{L,a_1}$. Then there is an absolute constant $c_4 > 0$, so that for all $a \geq a_1 \geq c_4$ and $t \geq t_0$, we have

$$\begin{aligned}
e^{-q_2(t)} &\geq e^{-a^{-1/6}\ln(a^{2/3}t)}\Big((c_2 + a^{-1/2})^{-1} + e^{(ac_2 + a^{1/2})(t_0 - t)}(e^{-q(t_0)} - (c_2 + a^{-1/2})^{-1})\Big) \\
&\geq e^{-a^{-1/6}\ln a^{2/3} - q(t_0)}t^{-a^{-1/6}}.
\end{aligned}$$

Therefore, we deduce

$$-a^{-1/6}\ln t \leq q(t) \leq a^{-1/6}\ln t + 1, \quad \forall t \geq t_0$$

on the event $\{C_1 < a_1^{1/6}\} \cap \mathcal{A}^{(2)}_{L,a_1}$ for all $a \geq a_1 \geq c_5$ with a fixed $c_5 > 0$, which completes the proof of the proposition. $\square$

Now we are ready to complete the proof of Proposition 3.16.

Proof of Proposition 3.16. The statement follows from Propositions 3.23 and 3.24, and the inequality (3.52). □

Remark 3.25. *A more careful analysis of the diffusion $\phi_{\mathbf{d}}^{(2a)}$ (using ideas described in the proofs of Lemma 3.20 and Lemma 3.23) can provide a logarithmic bound on the diffusion q for a fixed $a > 0$. More precisely, it can be shown that for a fixed $a > 1/2$ with probability one the diffusion q satisfies $|q(t)| \leq \frac{2(32)^2}{\beta\, a} \ln t$ for all large t. In particular, this result implies that $\phi_{\mathbf{d}} := \phi_{\mathbf{d}}^{(2a)}$ is a.s. not in $L^2(\mathbb{R}_+, m_{2a})$ for $a > 1/2$ thanks to the identities (3.34) and*

$$\phi_{\mathbf{d}}(t)^2 m_{2a}(t) = \phi_{\mathbf{d}}(t_0)^2 \exp(2\,a \int_{t_0}^{t} e^{q(s)} ds) \exp(-(2a+1)t - \frac{2}{\sqrt{\beta}} B_{2a}(t)).$$

Chapter 4

Operator level limits of circular Jacobi β-ensembles

The content of this chapter is joint work with Benedek Valkó and is a modified version of a submitted paper [37].

We study point process limits of the circular Jacobi β-ensemble (CJβE) and the real orthogonal β-ensemble (ROβE), together with the scaling limits of some related objects, in particular the limits of the normalized characteristic polynomials. Our approach follows the framework introduced in [64] and [65]. This framework, together with a high level description of our main results is summarized in the following outline.

1. **Differential operators from probability measures.** [64] describes how the spectral information (the modified Verblunsky coefficients) of a finitely supported probability measure on the unit circle can be used to construct a differential operator (a Dirac operator) with a pure point real spectrum. The spectrum of the constructed differential operator is the periodic lifting of the angles corresponding to the support of the probability measure, see Proposition 4.3 for the precise statement. We summarize the background and the relevant results in Section 4.1.

2. **Random Dirac operators.** [10] and [33] provide constructions for random probability measures on the unit circle where the support of the measure is given by the CJβE and ROβE, respectively, and the distribution of the modified Verblunsky coefficients can be explicitly described, see Theorems 4.6 and 4.7. These constructions together with Proposition 4.3 lead to the construction of the random differential operators $\mathtt{CJ}_{n,\beta,\delta}$ and $\mathtt{RO}_{2n,\beta,a,b}$ with pure point spectrum. The spectrum of $\mathtt{CJ}_{n,\beta,\delta}$ is distributed as $n\Lambda_n + 2\pi n\mathbb{Z}$ with $\Lambda_n \sim \mathrm{CJ}_{n,\beta,\delta}$, and the spectrum of $\mathtt{RO}_{2n,\beta,a,b}$ is distributed as $2n\Lambda_{2n} + 4\pi n\mathbb{Z}$ where $\Lambda_{2n} \sim \mathrm{RO}_{2n,\beta,a,b}$, see Section 4.2.1. The inverses of these differential operators (after a change of basis) are denoted by $\mathtt{r\,CJ}_{n,\beta,\delta}$ and $\mathtt{r\,RO}_{2n,\beta,a,b}$, these are random Hilbert-Schmidt integral operators acting on L^2 functions of the form $[0,1) \to \mathbb{R}^2$.

3. **Operator level convergence.** The operators $\mathtt{CJ}_{n,\beta,\delta}$ and $\mathtt{RO}_{2n,\beta,a,b}$ and their inverses can be parameterized in terms of certain random walks in the hyperbolic plane. Under the appropriate scaling these random walks converge to diffusions in the hyperbolic plane. As shown in [64], one can construct random differential operators in terms of these diffusions, these will be called $\mathtt{HP}_{\beta,\delta}$ and $\mathtt{Bess}_{\beta,a}$, respectively. (See Section 4.2.2.) Both of these random differential operators have pure point spectra, the distribution of the point processes of eigenvalues are denoted by $\mathrm{HP}_{\beta,\delta}$ and $\mathrm{Bess}_{\beta,a}$, respectively. The process $\mathrm{HP}_{\beta,\delta}$ for $\delta = 0$ is the process Sine_β introduced in [62] as the bulk scaling limit of the Gaussian β-ensemble. The process $\mathrm{Bess}_{\beta,a}$ is just a symmetrized (and scaled) version of the square root of the hard edge process $\mathrm{Bessel}_{\beta,a}$ introduced in [48].

We will prove that in appropriate couplings we have the operator level convergence

$$\|\mathtt{r}\,\mathtt{CJ}_{n,\beta,\delta} - \mathtt{r}\,\mathtt{HP}_{\beta,\delta}\|_{HS} \to 0 \text{ almost surely as } n \to \infty, \tag{4.1}$$

$$\|\mathtt{r}\,\mathtt{RO}_{2n,\beta,a,b} - \mathtt{r}\,\mathtt{Bess}_{\beta,a}\|_{HS} \to 0 \text{ almost surely as } n \to \infty. \tag{4.2}$$

The precise version of these results are stated in Theorems 4.14 and 4.16 in Section 4.3.1. These results identify the point process scaling limits of the ensembles CJβE and ROβE as the point processes $\mathrm{HP}_{\beta,\delta}$ and $\mathrm{Bess}_{\beta,a}$. (See Corollaries 4.15 and 4.17.) The distribution of the point process $\mathrm{HP}_{\beta,\delta}$ can be characterized via its counting function using coupled systems of stochastic differential equations. Two equivalent characterizations are given in Theorems 4.18 and 4.19 in Section 4.3.2.

4. **Convergence of characteristic polynomials.** [65] introduced the secular function for a Dirac operator τ which is an entire function with zero set given by the spectrum of τ. This is a generalization of the normalized characteristic polynomial of a unitary matrix. We review the definition in Section 4.1. [65] also showed that results of the form of (4.1) and (4.2) (together with similar statements on the so-called integral trace) imply that the scaled and normalized characteristic polynomials of CJβE and ROβE converge to the secular functions of the operators $\mathtt{HP}_{\beta,\delta}$ and $\mathtt{Bess}_{\beta,a}$. These results are stated as part of Corollaries 4.15 and 4.17. Theorems 4.20 and 4.21 provide two separate characterizations of the limiting random entire functions: by describing the joint distribution of the Taylor coefficients, and a characterization using entire function valued stochastic differential equations.

For the circular Jacobi β-ensemble the operators $\mathtt{CJ}_{n,\beta,\delta}$ and $\mathtt{HP}_{\beta,\delta}$ were introduced in [64], and the convergence (4.1) was stated as a conjecture. (More precisely: as a statement to be proved in a future paper.) In [1] Assiotis and Najnudel showed the existence of the point process limit of the circular Jacobi β-ensemble by providing a coupling of the scaled finite ensembles so that they posses an a.s. point process limit. However their result does not provide an explicit characterization for the limiting point process.

Our main new contributions for the study of the scaling limits of CJβE are the operator level convergence of Theorem 4.14, the various characterizations of the limit point process $\mathrm{HP}_{\beta,\delta}$ (Theorems 4.18 and 4.19), and the description and characterization of the limit of the normalized characteristic polynomials (Corollary 4.15 and Theorem 4.20). Some of our results are extensions and generalizations of corresponding results for the circular β-ensemble and the Sine_β process proved in [34], [64], [66], [65].

In the $\beta = 1, 2, 4$ cases the limiting point processes have been described via their n-point correlation functions in [23]. In [38] the limiting correlation functions were derived in the case when β is an even integer, together with exact formulas for expectations of products of characteristic polynomials. (Note that the normalization for the characteristic polynomials in [38] is different from ours.) [22] provides corrections to these results in the case when β is an even integer or equal to 1. Scaling limits of characteristic polynomials of classical random matrix ensembles were also studied in [12] and [11].

A version of the first three steps of the outline above was carried out by Holcomb and Moreno Flores in [27] for the real Jacobi β-ensemble. Using the change of variables of $x_j = \frac{1}{2}(1 - \cos\theta_j)$, their results also imply the point process level convergence of ROβE. The proof in [27] relies on a tridiagonal representation of the real Jacobi β-ensemble together with the operator convergence approach for studying the hard edge limit, introduced in [48] for the Laguerre β-ensemble. [64] provided a representation of the hard edge limit operator as a random Dirac operator. [26] provides various descriptions and properties of the limiting (hard edge) point process. Our main new contributions for the study of ROβE are the existence and description of the limit of the normalized characteristic polynomials (Corollary 4.17 and Theorem 4.21), and a new approach to prove the point process limit via operator convergence (Theorem 4.16).

Outline of the reminder of the chapter

In Section 4.1 we outline the used operator theoretic framework, the presentation will mostly follow that of [64] and [65]. In Section 4.2 we introduce the random differential operators corresponding to the finite ensembles and their limits. Section 4.3 states our precise results, including the description of the limiting point processes and random analytic functions. Sections 4.4, 4.5, and 4.6 provide the proofs for the operator convergence results, while Section 4.7 contains the proofs of the statements of the properties and characterizations of the limiting objects.

4.1 The operator theoretic framework

This section collects all the deterministic operator theoretic ingredients. We describe the type of differential and integral operators we consider, the definition of the secular function, and how these objects can be used to study finitely supported probability measures on the unit circle.

4.1.1 Dirac operators

We start by collecting some basic facts about differential operators of the form

$$\tau : f \to R^{-1}(t)Jf', \qquad f : [0,1) \to \mathbb{R}^2, \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{4.3}$$

where $R(t)$ is a positive definite real symmetric 2×2 matrix valued function on $[0,1)$. These differential operators are called Dirac operators, for more details see [67] or [64].

We consider differential operators of the form (4.3) where the matrix valued function $R(t)$ is defined from a locally bounded measurable function $x + iy : [0,1) \to \mathbb{H} = \{z \in \mathbb{C} : \Im z > 0\}$ as follows:

$$R = \frac{1}{2}X^t X, \quad X = \frac{1}{\sqrt{y}} \begin{pmatrix} 1 & -x \\ 0 & y \end{pmatrix}. \tag{4.4}$$

We call R the *weight function*, and $x + iy$ the *generating path* of τ.

The boundary conditions for τ at 0 and 1 are given by nonzero, non-parallel $\mathbb{R}^2$ vectors $\mathfrak{u}_0, \mathfrak{u}_1$. We will assume that these vectors are normalized so that they satisfy the condition

$$\mathfrak{u}_0^t J \mathfrak{u}_1 = 1. \tag{4.5}$$

We will also have the following integrability assumption for the boundary conditions:

Assumption 4.1.

$$\int_0^1 \|R(s)\mathfrak{u}_1\| ds < \infty \quad \text{and} \quad \int_0^1 \int_0^t \mathfrak{u}_0^t R(s) \mathfrak{u}_0 \, \mathfrak{u}_1^t R(t) \mathfrak{u}_1 ds dt < \infty. \tag{4.6}$$

Under these conditions τ will be self-adjoint on the following domain:

$$\operatorname{dom}(\tau) = \{v \in L^2_R \cap \mathrm{AC} : \tau v \in L^2_R, \lim_{s \to 0} v(s)^t J \mathfrak{u}_0 = 0, \lim_{s \to 1} v(s)^t J \mathfrak{u}_1 = 0\}. \tag{4.7}$$

Here L^2_R is the L^2 space of functions $f:[0,1)\to\mathbb{R}^2$ with the L^2 norm $\|f\|^2_R=\int_0^1 f^tRf\,ds$, and AC([0,1)) is the set of absolutely continuous real functions on $[0,1)$. We will use the notations $\mathtt{Dir}(R,\mathfrak{u}_0,\mathfrak{u}_1)$ or $\mathtt{Dir}(x+iy,\mathfrak{u}_0,\mathfrak{u}_1)$ for the the operator τ defined via (2.7) and (4.4) with boundary conditions $\mathfrak{u}_0,\mathfrak{u}_1$ on the domain (4.7). We sometimes replace the $\mathbb{R}^2$ vector by the element in $\mathbb{R}\cup\{\infty\}$ corresponding to the ratio of its two coordinates: $[a,b]^t$ corresponds to a/b if $b\neq 0$ and ∞ if $b=0$.

The inverse of $\tau=\mathtt{Dir}(x+iy,\mathfrak{u}_0,\mathfrak{u}_1)$ is a Hilbert-Schmidt integral operator on L^2_R with kernel given by

$$K_{\tau^{-1}}(s,t)=\big(\mathfrak{u}_0\mathfrak{u}_1^t 1_{s<t}+\mathfrak{u}_1\mathfrak{u}_0^t 1_{s\geq t}\big)R(t). \tag{4.8}$$

This means that if $f\in\mathrm{dom}(\tau)$ and $g=\tau f$ then $f(s)=\int_0^1 K_{\tau^{-1}}(s,t)g(t)dt$. The fact that the integral operator is Hilbert-Schmidt follows from the second inequality of (4.6), and implies that τ has a discrete pure point spectrum with nonzero real eigenvalues $\lambda_k, k\in\mathbb{Z}$ that satisfy $\sum_k \lambda_k^{-2}<\infty$. We label the eigenvalues so that they are in an increasing order with $\lambda_{-1}<0<\lambda_0$.

After the change of variables $\hat\tau=X\tau X^{-1}$, the inverse $\mathtt{r}\,\tau:=\hat\tau^{-1}$ is an integral operator on the L^2 space of functions $f:[0,1)\to\mathbb{R}^2$ with norm $\|f\|^2=\int_0^1 f^tf\,ds$, and its kernel is given by

$$K_{\mathtt{r}\,\tau}(s,t)=\tfrac12\big(a(s)c(t)^t1_{s<t}+c(s)a(t)^t1_{s\geq t}\big),\quad a(s)=X(s)\mathfrak{u}_0,\quad c(s)=X(s)\mathfrak{u}_1. \tag{4.9}$$

We define the *integral trace* of $\mathtt{r}\,\tau$ as the integral of the trace of the kernel $K_{\mathtt{r}\,\tau}$, and denote it by $\mathfrak{t}_\tau$:

$$\mathfrak{t}_\tau=\int_0^1 \mathrm{tr}\,K_{\mathtt{r}\,\tau}(s,s)ds=\tfrac12\int_0^1 a(s)^tc(s)ds=\int_0^1\mathfrak{u}_0^tR(s)\mathfrak{u}_1ds. \tag{4.10}$$

By the first inequality of (4.6) the integral trace is finite.

We define the *secular function* of τ with the expression

$$\zeta_\tau(z) = e^{-z\mathfrak{t}_\tau}\det{}_2(I - z\,\mathtt{r}\,\tau) = e^{-\frac{z}{2}\int_0^1 a(s)^t c(s)ds} \prod_k (1 - z/\,\lambda_k)e^{z/\lambda_k}. \tag{4.11}$$

Here $\det_2$ is the second regularized determinant, see [58]. The secular function ζ_τ is an entire function with zero set given by $\lambda_k, k \in \mathbb{Z}$, it is an analogue of the normalized characteristic polynomial of a square matrix. (See [65] for details.)

The next statement provides comparisons for the spectra and secular functions of two Dirac operators.

Proposition 4.2. *Let τ_1, τ_2 be two Dirac operators on $[0,1)$ satisfying assumptions (4.5) and (4.6). Denote by $\lambda_{k,i}, \zeta_i, \mathtt{r}\,_i, \mathfrak{t}_i$ the eigenvalues, secular function, resolvent and integral trace of τ_i. Let $\|\cdot\|$ denote the Hilbert-Schmidt norm. Then*

$$\sum_k \left|\lambda_{k,1}^{-1} - \lambda_{k,2}^{-1}\right|^2 \leq \|\mathtt{r}\,_1 - \mathtt{r}\,_2\|^2, \tag{4.12}$$

and there is a universal constant $a > 1$ so that for all $z \in \mathbb{C}$

$$|\zeta_1(z) - \zeta_2(z)| \leq \Big(e^{|z||\mathfrak{t}_1 - \mathfrak{t}_2|} - 1 + |z|\big\|\mathtt{r}\,_1 - \mathtt{r}\,_2\big\|\Big)a^{|z|^2(\|\mathtt{r}\,_1\|^2 + \|\mathtt{r}\,_2\|^2) + |z|(|\mathfrak{t}_1| + |\mathfrak{t}_2|) + 1} \tag{4.13}$$

The inequality (4.12) is just the Hoffman-Wielandt inequality in infinite dimensions (see e.g. [3]), the bound (4.13) follows from standard properties of the regularized determinant [58] (see Proposition 21 in [65] for additional details). Proposition 4.2 shows that the Hilbert-Schmidt convergence of Dirac operators implies the convergence of the spectra, and if the integral traces converge as well then we have uniform on compacts convergence of the secular functions.

The end points of a Dirac operator can be classified as limit circle or limit point based on the integrability of the solutions of $(\tau - \lambda)u = 0$ near that end point. By the

Weyl's alternative theorem (e.g. Theorem 5.6 in [67]) the integrability of the solutions does not depend on λ. Hence one can choose $\lambda = 0$, and just check the integrability of the constant vectors. Since $R(t)$ is locally bounded near 0, the left endpoint of the interval $[0, 1)$ is *limit circle* with respect to the weight function R: for any $v \in \mathbb{R}^2$ the function $v^t R v$ is integrable near 0. Assumption (4.6) shows that vRv is integrable near 1 if $v \parallel \mathfrak{u}_1$, but that might not be the case if $v \nparallel \mathfrak{u}_1$. This shows that the right endpoint could be limit circle or *limit point*.

For certain applications of the limiting objects, it is more convenient to consider operators that have 0 as the endpoint that could possibly be limit point. In this case the domain of the operator is $(0, 1]$, and we have to modify our setup and assumptions. This *reversed framework* will be introduced in Section 4.7.1, where we also discuss other transformations of Dirac operators.

4.1.2 Dirac operators for finitely supported probability measures on the unit circle

We review the construction given Section 3 of [65] that shows how a finitely supported probability measure on the unit circle can be represented using a Dirac operator of the form (2.7). (See also Section 5 of [64].)

Let μ be a probability measure whose support is a set of n distinct points $e^{i\lambda_j}, 1 \leq j \leq n$ on the unit circle, and assume $\mu(\{1\}) = 0$. The characteristic polynomial of μ, normalized at 1, is defined as

$$p_\mu(z) = \prod_{j=1}^{n} \frac{z - e^{i\lambda j}}{1 - e^{i\lambda j}}. \tag{4.14}$$

For $0 \le k \le n$, the kth orthogonal polynomial $\Phi_k(z)$ is defined as the unique polynomial with main coefficient 1 of degree k that is orthogonal to $1, \dots, z^{k-1}$ in $L^2(\mu)$. We denote by $\Psi_k(z) = \frac{\Phi_k(z)}{\Phi_k(1)}$ the normalized orthogonal polynomials. Note that we have $\Phi_0 = \Psi_0 = 1$ and $p_\mu = \Psi_n$. For $0 \le k \le n$ we define Φ_k^*, Ψ_k^* as the reversed polynomials

$$\Phi_k^*(z) = z^k \overline{\Phi_k(1/\bar{z})}, \qquad \Psi_k^*(z) = z^k \overline{\Psi_k(1/\bar{z})}.$$

The vector $\binom{\Phi_k}{\Phi_k^*}$ satisfies the Szegő recursion [57]:

$$\begin{pmatrix} \Phi_{k+1}(z) \\ \Phi_{k+1}^*(z) \end{pmatrix} = A_k \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \Phi_k(z) \\ \Phi_k^*(z) \end{pmatrix}, \qquad 0 \le k \le n-1. \tag{4.15}$$

Here $A_k = \begin{pmatrix} 1 & -\bar{\alpha}_k \\ -\alpha_k & 1 \end{pmatrix}$, the complex numbers $\alpha_0, \dots, \alpha_{n-1}$ are called the Verblunsky coefficients. They satisfy $|\alpha_k| < 1$ for $0 \le k \le n-1$ and $|\alpha_{n-1}| = 1$. The normalized orthogonal polynomials Ψ_k, Ψ_k^* satisfy a similar recursion as (4.15), with the matrix

$$\widetilde{A}_k = \begin{pmatrix} \frac{1}{1-\gamma_k} & -\frac{\gamma_k}{1-\gamma_k} \\ -\frac{\bar{\gamma}_k}{1-\bar{\gamma}_k} & \frac{1}{1-\bar{\gamma}_k} \end{pmatrix}$$

in place of A_k. The complex numbers γ_k, $0 \le k \le n-1$ are called the modified or deformed Verblunsky coefficients (see [10]). They satisfy

$$\gamma_k = \bar{\alpha}_k \prod_{j=0}^{k-1} \frac{1-\bar{\gamma}_j}{1-\gamma_j}, \qquad 0 \le k \le n-1, \tag{4.16}$$

from which it follows that $|\gamma_k| = |\alpha_k|$.

Define $w_k, v_k \in \mathbb{R}$ with

$$\frac{2\gamma_k}{1-\gamma_k} = w_k - i v_k. \tag{4.17}$$

Set $x_0 = 0$, $y_0 = 1$, and define recursively

$$x_{k+1} = x_k + v_k y_k, \qquad y_{k+1} = y_k(1 + w_k), \qquad 0 \le k \le n-1. \tag{4.18}$$

Note that $|\gamma| \le 1$ implies $\Re \frac{2\gamma}{1-\gamma} \ge -1$, and we have equality if and only if $|\gamma| = 1$, $\gamma \ne 1$. Hence $y_k > 0$ for $1 \le k \le n-1$ and $y_n = 0$. The following proposition was proved in [65].

Proposition 4.3 ([65]). *Set $x(t) + iy(t) = x_{\lfloor nt \rfloor} + iy_{\lfloor nt \rfloor}$ for $t \in [0, 1]$. Let*

$$\tau = R^{-1} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \frac{d}{dt}, \qquad R = \frac{X^t X}{2 \det X}, \qquad X = \begin{pmatrix} 1 & -x \\ 0 & y \end{pmatrix}, \tag{4.19}$$

with boundary conditions $\mathfrak{u}_0 = [1, 0]^t$, $\mathfrak{u}_1 = [-x(1), -1]^t$.

Then τ satisfies our assumptions, the spectrum of τ is given by the set

$$\operatorname{spec} \tau = \{n\lambda_k + 2\pi n j : 1 \le k \le n, j \in \mathbb{Z}\},$$

and the secular function of τ satisfies

$$\zeta_\tau(z) = p_\mu(e^{iz/n}) e^{-iz/2} = \prod_{j=1}^{n} \frac{\sin(\lambda_j/2 - z/(2n))}{\sin(\lambda_j/2)}. \tag{4.20}$$

4.2 Random Dirac operators

This section introduces the random Dirac operators corresponding to the finite ensembles and to their limits.

4.2.1 Operators for the finite ensembles

The results of this section provide descriptions of random probability measures with support given by the CJβE and ROβE, respectively, where the joint distribution of the modified Verblunsky coefficients can be described explicitly.

Definition 4.4. *For $a > 0$ and $\Re\delta > -1/2$ we denote by $\Theta(a+1,\delta)$ the distribution on $\{|z| < 1\}$ that has probability density function*

$$c_{a,\delta}(1-|z|^2)^{a/2-1}(1-z)^{\bar{\delta}}(1-\bar{z})^{\delta}, \tag{4.21}$$

where $c_{a,\delta} = \frac{\Gamma(a/2+1+\delta)\Gamma(a/2+1+\bar{\delta})}{\pi\Gamma(a/2)\Gamma(a/2+1+\delta+\bar{\delta})}$.

We extend the definition for the $a = 0$, $\Re\delta > -1/2$ case as follows: $\Theta(1,\delta)$ is the distribution on $\{|z| = 1\}$ with probability density function

$$\tfrac{\Gamma(1+\delta)\Gamma(1+\bar{\delta})}{\Gamma(1+\delta+\bar{\delta})}(1-z)^{\bar{\delta}}(1-\bar{z})^{\delta}. \tag{4.22}$$

Definition 4.5. *For $s, t > 0$ let $\widetilde{\mathrm{B}}(s,t)$ denote the scaled (and flipped) beta distribution on $(-1,1)$ that has probability density function*

$$\tfrac{2^{1-s-t}\Gamma(s+t)}{\Gamma(s)\Gamma(t)}(1-x)^{s-1}(1+x)^{t-1}.$$

Theorem 4.6 (Theorems 3.2 and 3.3 of [10])**.** *For given $\beta > 0$, $\Re\delta > -1/2$ and $n \geq 1$ let $\mu = \mu^{\mathrm{cj}}_{n,\beta,\delta}$ be the random probability measure $\mu = \sum_{k=1}^n r_k \delta_{e^{i\theta_k}}$ on the unit circle where $(\theta_1,\dots,\theta_n)$ and $(r_1,\dots,r_n)$ are independent, the joint density of $\theta_k, 1 \leq k \leq n$ is given by (2.8), and the joint density of $r_k, 1 \leq k \leq n-1$ is given by $\frac{1}{C_{n,\beta}}\prod_{k=1}^n r_k^{\beta/2-1}$. In other words, μ is a probability measure where the support has distribution CJβE, and the weights are Dirichlet($\beta/2,\dots,\beta/2$) distributed, independently of the support.*

Then the modified Verblunsky coefficients $\gamma_0,\dots,\gamma_{n-1}$ of μ are independent, and γ_k has distribution $\Theta(\beta(n-k-1)+1,\delta)$ for $0 \leq k \leq n-1$.

Theorem 4.7 (Theorem 2 of [33], Proposition 4.5 in [32])**.** *For given $\beta > 0$, $a, b > -1$ and $n \geq 1$ let $\mu = \mu^{\mathrm{o}}_{2n,\beta,a,b}$ be the random probability measure $\mu = \sum_{k=1}^n \frac{1}{2} r_k(\delta_{e^{i\theta_k}} + \delta_{e^{-i\theta_k}})$ on the unit circle where $(\theta_1,\dots,\theta_n)$ and $(r_1,\dots,r_n)$ are independent, the joint density* *of* $\theta_k, 1 \leq k \leq n$ *is given by* (1.5)*, and the joint density of* $r_k, 1 \leq k \leq n-1$ *is given by* $\frac{1}{C_{n,\beta}} \prod_{k=1}^{n} r_k^{\beta/2-1}$.

Then the Verblunsky coefficients $\alpha_0, \ldots, \alpha_{2n-1}$ *corresponding to* μ *are real, independent of each other. We have* $\alpha_{2n-1} = -1$*, and the distribution of* $\alpha_k, 0 \leq k \leq 2n-2$ *is given by*

$$\alpha_k \sim \begin{cases} \widetilde{\mathrm{B}}\big(\frac{\beta}{4}(2n-k+2a), \frac{\beta}{4}(2n-k+2b)\big), & \text{if } k \text{ is even,} \\ \widetilde{\mathrm{B}}\big(\frac{\beta}{4}(2n-k+2a+2b+1), \frac{\beta}{4}(2n-k-1)\big), & \text{if } k \text{ is odd.} \end{cases}$$

Since all the Verblunsky coefficients are real, we have $\gamma_k = \alpha_k$ *for all* $0 \leq k \leq 2n-1$.

Theorems 4.6 and 4.7 together with Proposition 4.3 provide random Dirac operator representations for the CJβE and ROβE.

Definition 4.8. *We denote by* $\mathtt{CJ}_{n,\beta,\delta}$ *the random Dirac operator constructed from the random probability measure* $\mu^{\mathrm{cj}}_{n,\beta,\delta}$ *of Theorem 4.6 using Proposition 4.3. We denote by* $\mathtt{RO}_{2n,\beta,a,b}$ *the random Dirac operator constructed from the random probability measure* $\mu^{\mathrm{o}}_{2n,\beta,a,b}$ *of Theorem 4.7 using Proposition 4.3 .*

The modified Verblunsky coefficients are independent for both $\mu^{\mathrm{cj}}_{n,\beta,\delta}$ and $\mu^{\mathrm{o}}_{2n,\beta,a,b}$. Hence the sequence $x_k + iy_k$ defined by the recursion (4.18) is a Markov chain for both of these random measures. The generating paths of the $\mathtt{CJ}_{n,\beta,\delta}$ and $\mathtt{RO}_{2n,\beta,a,b}$ operators are just these Markov chains embedded into continuous time.

4.2.2 The limiting operators

As we show below, the generating paths of both $\mathtt{CJ}_{n,\beta,\delta}$ and $\mathtt{RO}_{2n,\beta,a,b}$ approximate certain diffusions in $\mathbb{H}$, and the operators themselves approximate the Dirac operators built from these diffusions. In this section we introduce the two limiting operators.

For the rest of the paper, we set

$$\upsilon_\beta(t) = -\tfrac{4}{\beta}\log(1-t). \tag{4.23}$$

Hua-Pickrell operator

Fix $\beta > 0$ and $\delta \in \mathbb{C}$ with $\Re\delta > -1/2$. Let B_1, B_2 be independent standard Brownian motion, and let $x_t + iy_t, t \geq 0$ be the strong solution of the SDE

$$dy = (-\Re\delta dt + dB_1)\, y, \quad dx = (\Im\delta dt + dB_2)\, y, \quad y(0) = 1, x(0) = 0. \tag{4.24}$$

Proposition 4.9 (Proposition 31 of [64]). *Let $x(t) + iy(t)$ be defined via (4.24). The limit $q = \lim_{t\to\infty} x(t)$ exists, and it is non-zero with probability one. Define $\widetilde{x}(t) = x(\upsilon_\beta(t))$,$\widetilde{y}(t) = y(\upsilon_\beta(t))$, and set $\mathfrak{u}_0 = [1,0]^t$, $\mathfrak{u}_1 = [-q,-1]^t$. Then the random Dirac operator $\mathtt{HP}_{\beta,\delta} = \mathtt{Dir}(\widetilde{x} + i\widetilde{y}, \mathfrak{u}_0, \mathfrak{u}_1)$ satisfies the assumptions of Section 4.1.1.*

We record the following estimates for $\widetilde{x}, \widetilde{y}$ from the proof of Proposition 31 of [64]. For any $\varepsilon > 0$ small there exists a random finite $C = C(\varepsilon)$ such that

$$C^{-1}(1-t)^{\frac{4}{\beta}(\Re\delta+\frac{1}{2}+\varepsilon)} \leq \widetilde{y}(t) \leq C(1-t)^{\frac{4}{\beta}(\Re\delta+\frac{1}{2}-\varepsilon)}, \quad |q - \widetilde{x}(t)| \leq C(1-t)^{\frac{4}{\beta}(\Re\delta+\frac{1}{2}-\varepsilon)}. \tag{4.25}$$

The distribution of $q = \lim_{t\to\infty} x(t)$ was identified in [2].

Definition 4.10. *For $m > 1/2$ and $\mu \in \mathbb{R}$ we denote by $P_{IV}(m,\mu)$ the distribution of the (unscaled) Pearson type IV distribution on $\mathbb{R}$ that has density function*

$$\frac{2^{2m-2}|\Gamma(m+\frac{\mu}{2}i)|^2}{\pi\,\Gamma(2m-1)}(1+x^2)^{-m}e^{-\mu\arctan x}. \tag{4.26}$$

Theorem 4.11 ([2]). *The random variable q in Proposition 4.9 has $P_{IV}(\Re\delta+1, -2\Im\delta)$ distribution.*

There is an interesting connection between the distributions P_{IV} and Θ: the map $z(e^{i\theta}) = -\cot(\theta/2)$ transforms $\Theta(1,\delta)$ into $P_{IV}(\Re\delta + 1, -2\Im\delta)$. The map z can be extended to the conformal map $w \to i\frac{w+1}{-w+1}$ from $\{|w| \le 1\}$ to $\{\Im z > 0\}$, which provides an isometry between the unit disk and half-plane representations of the hyperbolic plane. In other words, $\Theta(1,\delta)$ and $P_{IV}(\Re\delta + 1, -2\Im\delta)$ are different representations of the same distribution on the boundary of the hyperbolic plane.

Hard edge operator

The point process scaling limit of the Laguerre β-ensemble near the hard edge was identified by Ramírez and Rider in [48], see Theorem 2.1. [64] provided a Dirac operator representation for $\mathfrak{G}_{\beta,a}$, we summarize the result below.

Proposition 4.12 (Theorem 30 of [64])**.** *Fix $\beta > 0, a > -1$, and let B be a standard Brownian motion. We set $y(t) = e^{-\frac{\beta}{4}(2a+1)t - B(2t)}$, $\widetilde{y}(t) = y(\upsilon_\beta(t))$, $\mathfrak{u}_0 = [1,0]^t$, and $\mathfrak{u}_1 = [0,-1]^t$. Then the operator $\mathtt{Bess}_{\beta,a} := \mathtt{Dir}(i\widetilde{y}, \mathfrak{u}_0, \mathfrak{u}_1)$ satisfies the assumptions of Section 4.1.1, and its spectrum is symmetric about 0: $\lambda_{-k} = -\lambda_{k-1}, k \ge 1$.*

Moreover, the set $\{\frac{1}{16}\lambda_0^2, \frac{1}{16}\lambda_1^2, \dots\}$ has the same distribution as the spectrum of the hard edge operator $\mathfrak{G}_{\beta,a}$ defined in (2.1)*.*

Remark 4.13. *Theorem 30 of [64] is stated in a slightly different (but equivalent) way. With the notations of Proposition 4.12 the statement of that theorem is about the operator $\mathsf{G}_{\beta,2a} = \mathtt{Dir}(i\widetilde{y}^{-1}, \mathfrak{u}_1, \mathfrak{u}_0)$. Note however that conjugating $\mathtt{Bess}_{\beta,a}$ with the permutation matrix transposing the first and second coordinate in $\mathbb{R}^2$ gives $-\mathsf{G}_{\beta,2a}$, and since the spectra of $\mathtt{Bess}_{\beta,a}$ and $\mathsf{G}_{\beta,2a}$ are symmetric about 0, the statement of the proposition follows.*

4.3 Precise results

We are ready to state our results in a precise form.

4.3.1 Convergence of random operators and normalized characteristic polynomials

Theorem 4.14. *Fix $\beta > 0$ and $\Re\delta > -1/2$. Then there is a coupling of the random operators $\mathtt{CJ}_{n,\beta,\delta}, n \geq 1$ and $\mathtt{HP}_{\beta,\delta}$ so that $\|\mathtt{r}\,\mathtt{CJ}_{n,\beta,\delta} - \mathtt{r}\,\mathtt{HP}_{\beta,\delta}\|_{HS}$ and $\mathtt{t}_{\mathtt{CJ}_{n,\beta,\delta}} - \mathtt{t}_{\mathtt{HP}_{\beta,\delta}}$ both converge to 0 almost surely as $n \to \infty$.*

From Theorem 4.14 and Proposition 4.2 we immediately get the following corollary.

Corollary 4.15. *Consider the coupling of Theorem 4.14. Denote by Λ_n the eigenangles of $\mathtt{CJ}_{n,\beta,\delta}$ inside $(-\pi, \pi]$, and let $\lambda_{k,n}, k \in \mathbb{Z}$ be the sequence of ordered elements of the set $n\Lambda_n + 2\pi n\mathbb{Z}$ with $\lambda_{-1,n} < 0 < \lambda_{0,n}$. Let $p_n(z)$ be the normalized characteristic polynomial of Λ_n defined via (4.14). Denote by $\mathrm{HP}_{\beta,\delta} = \{\lambda_{k,\mathtt{HP}}, k \in \mathbb{Z}\}$ the ordered spectrum of the operator $\mathtt{HP}_{\beta,\delta}$, and by $\zeta^{\mathtt{HP}}_{\beta,\delta}$ the secular function of $\mathtt{HP}_{\beta,\delta}$. Then*

$$\sum_k |\lambda_{k,n}^{-1} - \lambda_{k,\mathtt{HP}}^{-1}|^2 \to 0 \qquad \textit{almost surely as } n \to \infty, \tag{4.27}$$

$$|p_n(e^{iz/n})e^{-iz/2} - \zeta^{\mathtt{HP}}_{\beta,\delta}(z)| \to 0 \qquad \textit{almost surely, uniformly on compacts as } n \to \infty. \tag{4.28}$$

In particular, if $\Lambda_n \sim \mathrm{CJ}_{n,\beta,\delta}$ then $n\Lambda_n \Rightarrow \mathrm{HP}_{\beta,\delta}$.

The following theorem and its corollary state the corresponding result for the real orthogonal ensemble.

Theorem 4.16. *Fix $\beta > 0$ and $a, b > -1$. Then there is a coupling of the random operators $\mathtt{RO}_{2n,\beta,a,b}, n \geq 1$ and $\mathtt{Bess}_{\beta,a}$ so that $\|\mathtt{r}\,\mathtt{RO}_{2n,\beta,a,b} - \mathtt{r}\,\mathtt{Bess}_{\beta,a}\|_{HS}$ converges to 0 almost surely as $n \to \infty$.*

Note that since the driving paths are purely imaginary, we have $\mathtt{t}_{\mathtt{RO}_{2n,\beta,a,b}} = \mathtt{t}_{\mathtt{Bess}_{\beta,a}} = 0$.

Corollary 4.17. *Consider the coupling of Theorem 4.16. Denote by Λ_{2n} the eigenangles of $\mathtt{RO}_{2n,\beta,a,b}$ inside $(-\pi, \pi]$, and let $\lambda_{k,2n}, k \in \mathbb{Z}$ be the ordered elements of the set $2n\Lambda_{2n} + 4\pi n\mathbb{Z}$ with $\lambda_{-1,2n} < 0 < \lambda_{0,2n}$. Let $p_{2n}(z)$ be the normalized characteristic polynomial of Λ_{2n} defined via (4.14). Denote by $\mathrm{Bess}_{\beta,a} = \{\lambda_{k,\mathtt{B}}, k \in \mathbb{Z}\}$ the ordered spectrum of the operator $\mathtt{Bess}_{\beta,a}$, and by $\zeta^{\mathtt{B}}_{\beta,a}$ the secular function of $\mathtt{Bess}_{\beta,a}$. Then*

$$\sum_k |\lambda_{k,2n}^{-1} - \lambda_{k,\mathtt{B}}^{-1}|^2 \to 0 \qquad \textit{almost surely as } n \to \infty, \tag{4.29}$$

$$|p_{2n}(e^{iz/(2n)})e^{-iz/2} - \zeta^{\mathtt{B}}_{\beta,a}(z)| \to 0 \qquad \textit{almost surely, uniformly on compacts as } n \to \infty. \tag{4.30}$$

Moreover, if $\Lambda_{2n} \sim \mathrm{RO}_{2n,\beta,a,b}$ then $2n\Lambda_{2n} \Rightarrow \mathrm{Bess}_{\beta,a}$.

4.3.2 Characterization of the limiting point processes

The point process $\mathrm{HP}_{\beta,\delta}$ is a generalization of the Sine_β process: $\mathrm{HP}_{\beta,0} = \mathrm{Sine}_\beta$. The Sine_β process has various descriptions via its counting function using stochastic differential equations, we will show that these descriptions can be extended to the process $\mathrm{HP}_{\beta,\delta}$ as well.

Theorem 4.18. *Let $\beta > 0$, $\delta \in \mathbb{C}$ with $\Re\delta > -1/2$. Let $Z = B_1 + iB_2$ be a standard complex Brownian motion, and let $\theta \in (-\pi, \pi]$ be a random variable independent of Z so that $e^{i\theta}$ has distribution $\Theta(1, \delta)$.*

There is a unique process $\psi_\lambda(t)$ with $t \in (0,1]$, $\lambda \in \mathbb{R}$ that is continuous in both variables, and for each $\lambda \in \mathbb{R}$ the process $t \to \psi_\lambda(t)$ is a strong solution of

$$d\psi_\lambda = \lambda dt + \Re[(e^{-i\psi_\lambda} - 1)(\tfrac{2}{\sqrt{\beta t}}dZ - i\delta\tfrac{4}{\beta t}dt)], \qquad \lim_{t\to 0}\psi_\lambda(t) = 0. \tag{4.31}$$

The point process $\mathrm{HP}_{\beta,\delta}$ has the same distribution as the random set

$$\Xi = \{\lambda \in \mathbb{R} : \psi_\lambda(1) \in \theta + 2\pi\mathbb{Z}\}. \tag{4.32}$$

Note that this is an extension of the Killip-Stoiciu characterization of the Sine_β process, see [34], [64]. The following theorem provides another, equivalent characterization of $\mathrm{HP}_{\beta,\delta}$, which is an extension of the description of Sine_β given in Proposition 4 of [62].

Theorem 4.19. *Let $\beta > 0$, $\delta \in \mathbb{C}$ with $\Re\delta > -1/2$. Let $Z = B_1 + iB_2$ be a standard complex Brownian motion. Then the following SDE system has a unique strong solution on $t \in [0,\infty)$, $\lambda \in \mathbb{R}$*

$$d\alpha_\lambda = \lambda\tfrac{\beta}{4}e^{-\frac{\beta}{4}t}dt + \Re[(e^{-i\alpha_\lambda} - 1)(dZ - i\delta dt)], \qquad \alpha_\lambda(0) = 0. \tag{4.33}$$

With probability one the process $\lambda \to \alpha_\lambda(t)$ is increasing for all $t > 0$. For each $\lambda \in \mathbb{R}$ the limit $\operatorname{sgn}(\lambda)\cdot\lim_{t\to\infty}\frac{1}{2\pi}\alpha_\lambda(t)$ exists almost surely, and it has the same distribution as the number of points of $\mathrm{HP}_{\beta,\delta}$ in $[0,\lambda]$ for $\lambda \ge 0$ (and in $[\lambda,0]$ for $\lambda < 0$). Moreover, if $N(\lambda)$ is the right-continuous version of the function $\lambda \to \lim_{t\to\infty}\frac{1}{2\pi}\alpha_\lambda(t)$, then $N(\cdot)$ has the same distribution as the counting function of the $\mathrm{HP}_{\beta,\delta}$ process.

The diffusion description given in Theorem 4.19 allows us to study various properties of the counting function of the $\mathrm{HP}_{\beta,\delta}$ process via the SDE (4.33). See Chapter 5 for results on the large gap asymptotics of the point process $\mathrm{HP}_{\beta,\delta}$, a central limit theorem on the counting function of $\mathrm{HP}_{\beta,\delta}$, and a process level transition from $\mathrm{HP}_{\beta,\delta}$ to the Sine_β process (see Theorems 5.1 and 5.2 for the precise statements).

4.3.3 Characterization of the limiting random analytic functions

Theorem 4.20 (Characterization of $\zeta^{\mathtt{HP}}_{\beta,\delta}$). *Fix $\beta > 0$ and $\delta \in \mathbb{C}$ with $\Re\delta > -1/2$. Let B_1, B_2 independent copies of two-sided Brownian motion, and let q be an independent random variable with distribution $P_{IV}(\Re\delta + 1, -2\Im\delta)$. Denote by $\mathrm{HP}_{\beta,\delta}$ the spectrum of the operator $\mathtt{HP}_{\beta,\delta}$, and by $\zeta^{\mathtt{HP}}_{\beta,\delta}$ its secular function. Then $\zeta^{\mathtt{HP}}_{\beta,\delta}$ has the same distribution as the random analytic function $[1, -q]\mathcal{H}_0$ where $\mathcal{H}_u(z)$ is the unique analytic solution of the system of stochastic differential equations*

$$d\mathcal{H} = \begin{pmatrix} 0 & -dB_1 \\ 0 & dB_2 \end{pmatrix} \mathcal{H} + \begin{pmatrix} 0 & -\Im\delta du \\ 0 & -\Re\delta du \end{pmatrix} \mathcal{H} - z\frac{\beta}{8} e^{\beta u/4} J\mathcal{H} du, \qquad u \in \mathbb{R} \tag{4.34}$$

with the boundary condition $\lim_{u\to-\infty} \sup_{|z|<1} \left|\mathcal{H}_u(z) - \binom{1}{0}\right| = 0$. Moreover, $\zeta^{\mathtt{HP}}_{\beta,\delta}(z)$ has the same distribution as the random power series $\sum_{n=0}^\infty (\mathcal{A}_0^{(n)} - q\mathcal{B}_0^{(n)}) z^n$ where $\mathcal{A}_u^{(n)}, \mathcal{B}_u^{(n)}$ are processes satisfying the recursion

$$\mathcal{B}_u^{(n)} = -e^{B_2(u) - (\frac{1}{2} + \Re\delta)u} \int_{-\infty}^u \tfrac{\beta}{8}\, e^{-B_2(s) + (\frac{\beta}{4} + \frac{1}{2} + \Re\delta)s}\, \mathcal{A}_s^{(n-1)}\, ds, \tag{4.35}$$

$$\mathcal{A}_u^{(n)} = \int_{-\infty}^u \left(\tfrac{\beta}{8}\, e^{\beta s/4}\, \mathcal{B}_s^{(n-1)} - \Im\delta \mathcal{B}_s^{(n)}\right) ds - \int_{-\infty}^u \mathcal{B}_s^{(n)}\, dB_1. \tag{4.36}$$

with $\mathcal{A}^{(0)} \equiv 1, \mathcal{B}^{(0)} \equiv 0$.

Theorem 4.21 (Characterization of $\zeta^{\mathtt{B}}_{\beta,a}$). *Fix $\beta > 0$, $a > -1$. Let B be a two-sided Brownian motion on $\mathbb{R}$, $y(t) = \exp(-\frac{\beta}{4}(2a+1)t + B(2t))$ and $\hat{y}(t) = y(\frac{4}{\beta}\log t)$. Denote by $\mathrm{Bess}_{\beta,a}$ the spectrum of the operator $\mathtt{Bess}_{\beta,a}$, and by $\zeta^{\mathtt{B}} = \zeta^{\mathtt{B}}_{\beta,a}$ its secular function. Then $\zeta^{\mathtt{B}}$ has the same distribution as $1 + \sum_{k=1}^\infty r_k z^{2k}$, where*

$$r_k = (-1)^k 2^{-2k} \iiint\limits_{0<s_1<s_2<\cdots<s_{2k}\le 1} \frac{\hat{y}(s_2)\hat{y}(s_4)\cdots\hat{y}(s_{2k})}{\hat{y}(s_1)\hat{y}(s_3)\cdots\hat{y}(s_{2k-1})} ds_1 \cdots ds_{2k}. \tag{4.37}$$

Moreover, $\zeta^{\mathtt{B}}(z)$ has the same distribution as $[1,0]\mathcal{H}_0(z)$, where $\mathcal{H}_u(z)$ is the unique strong solution of the SDE

$$d\mathcal{H} = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{2}dB + (1 - \frac{\beta}{4}(2a+1))du \end{pmatrix} \mathcal{H} - z\frac{\beta}{8}e^{\beta u/4} J\mathcal{H} du \tag{4.38}$$

with boundary conditions $\lim_{u\to-\infty} \sup_{|z|<1} \left|\mathcal{H}_u(z) - \binom{1}{0}\right| = 0.$

Remark 4.22. *The random analytic function $\zeta^{\mathtt{B}}_{\beta,a}$ can also be represented in a product form as follows:*

$$\zeta^{\mathtt{B}}_{\beta,a}(z) = \prod_{0<\lambda\in \mathrm{Bess}_{\beta,a}} \left(1 - \frac{z^2}{\lambda^2}\right). \tag{4.39}$$

To see this we use definition (4.11)*, and note that the integral trace of the operator* $\mathtt{r\,Bess}_{\beta,a}$ *is zero. Moreover, by Proposition 4.12 the point process* $\mathrm{Bess}_{\beta,a}$ *is symmetric about 0, and* $\mathtt{r\,Bess}_{\beta,a}$ *is Hilbert-Schmidt, which gives us*

$$\prod_{\lambda\in \mathrm{Bess}_{\beta,a}} (1 - z/\lambda)e^{z/\lambda} = \prod_{0<\lambda\in \mathrm{Bess}_{\beta,a}} \left(1 - \frac{z^2}{\lambda^2}\right).$$

The random analytic function $\zeta^{\mathtt{HP}}_{\beta,\delta}(z)$ should also have a similar representation in terms of its zeros, it should be equal to the principal value product

$$\lim_{r\to\infty} \prod_{\substack{\lambda\in \mathrm{HP}_{\beta,\delta},\\ |\lambda|<r}} \left(1 - \frac{z}{\lambda}\right). \tag{4.40}$$

For $\delta = 0$ this statement was proved in [65]. Using the results of the current paper one should be able to extend the proof in [65] for the general δ case.

4.4 Convergence of discrete Dirac operators

This section collects some of the tools that will be used to prove Theorems 4.14 and 4.16. We first prove a general convergence result for the resolvents and integral traces of Dirac operators where the driving paths converge pointwise and are also 'regular' in a certain sense. Then we review some probabilistic tools: a standard result on the convergence of Markov chains to diffusions, and an iterated logarithm type result for products of independent random variables.

4.4.1 Convergence of resolvents and secular functions of Dirac operators

The following proposition gives a sufficient condition for the convergence of the resolvents and integral traces of deterministic Dirac operators.

Proposition 4.23. *Suppose that the Dirac operators $\tau^{(n)}, n \in \mathbb{Z}_+ \cup \{\infty\}$ are parametrized by paths $x^{(n)} + iy^{(n)}$ and boundary conditions $\mathfrak{u}_0 = [1,0]^t$, $\mathfrak{u}_1^{(n)} = [-q^{(n)}, -1]^t$. Introduce the notation $\lfloor t \rfloor_n = \lfloor nt \rfloor / n$ with the understanding that $\lfloor t \rfloor_\infty = t$. Assume that there are constants $c_1, c_2 > -1$, $c_3 > 0$, and $\kappa > 0$ so that the following bounds hold for all $0 \le t < 1$,*

$$\kappa^{-1}(1 - \lfloor t \rfloor_n)^{c_2} \le y^{(n)}(t) \le \kappa(1 - \lfloor t \rfloor_n)^{c_1}, \qquad |q^{(n)} - x^{(n)}(t)| \le \kappa(1 - \lfloor t \rfloor_n)^{c_3} \tag{4.41}$$

uniformly in $n \in \mathbb{Z}_+ \cup \{\infty\}$ with

$$c_3 > c_2 - 1, \qquad c_1 > c_2 - 2. \tag{4.42}$$

Assume that $x^{(n)} + iy^{(n)} \to x^{(\infty)} + iy^{(\infty)}$ point-wise on $[0, 1)$.

Then

$$\lim_{n\to\infty} \|\mathtt{r}\,\tau^{(n)} - \mathtt{r}\,\tau^{(\infty)}\|_{HS} = 0, \quad \textit{and} \quad \lim_{n\to\infty} |\mathfrak{t}_{\tau^{(n)}} - \mathfrak{t}_{\tau^{(\infty)}}| = 0. \tag{4.43}$$

Proof. From the second inequality of (4.41) and the triangle inequality we have $q^{(n)} \to q^{(\infty)}$.

Denote by $R^{(n)}$ the weight function of $\tau^{(n)}$, and by $X^{(n)}$ the 2×2 matrix defined in (4.4). Recall that $\mathtt{r}\,\tau^{(n)}, n \in \mathbb{Z}_+ \cup \{\infty\}$ is an integral operator with kernel given by (4.9). From $q^{(n)} \to q^{(\infty)}$ and the pointwise convergence of $x^{(n)} + iy^{(n)}$ we get the pointwise convergence of the integral kernels of $\mathtt{r}\,\tau^{(n)}$ on $[0,1)^2$.

The bounds (4.41) and the conditions on the constants c_1, c_2, c_3 provide integrable upper bounds for the functions

$$\begin{aligned} \operatorname{tr} K_{\mathtt{r}\,\tau^{(n)}}(s,s) &= \mathfrak{u}_0^t R^{(n)}(s)\mathfrak{u}_1^{(n)} = \frac{x^{(n)}(s) - q^{(n)}}{2y^{(n)}(s)}, \\ \operatorname{tr} K_{\mathtt{r}\,\tau^{(n)}}(s,t)^t K_{\mathtt{r}\,\tau^{(n)}}(s,t) &= \frac{1}{4}\|X^{(n)}(s \vee t)\mathfrak{u}_1^{(n)}\|^2 \|X^{(n)}(s \wedge t)\mathfrak{u}_0\|^2 \\ &= \frac{1}{4}\left(\frac{|q^{(n)} - x^{(n)}(s \vee t)|^2}{y^{(n)}(s \vee t)^2} + 1\right)\frac{y^{(n)}(s \vee t)}{y^{(n)}(s \wedge t)}, \end{aligned}$$

on $[0,1)$ and $[0,1)^2$, respectively. This shows that condition (4.6) is satisfied for $\tau^{(n)}$ for each $n \in \mathbb{Z}_+ \cup \{\infty\}$. Moreover, the General Dominated Convergence Theorem (see e.g. Theorem 1.4.19 in [56]) and the point-wise convergence of the kernels lead to (4.43). $\square$

As an immediate consequence we have the following corollary for random Dirac operators.

Corollary 4.24. *Suppose that $\tau^{(n)}, n \in \mathbb{Z}_+ \cup \{\infty\}$ are random Dirac operators built from the processes $x^{(n)} + iy^{(n)}$, and boundary conditions $\mathfrak{u}_0 = [1,0]^t$ and $\mathfrak{u}_1^{(n)} = [-q^{(n)}, -1]$, with random variables $q^{(n)}$. Assume that the following conditions are satisfied:*

1. *$x^{(n)} + iy^{(n)} \to x^{(\infty)} + iy^{(\infty)}$ in distribution on $[0,1)$ with respect to the Skorohod topology.*

2. *There exists constants* $c_1, c_2 > -1$, $c_3 > 0$ *satisfying* (4.42), *and a sequence of tight positive random variables* $\kappa^{(n)}, n \in \mathbb{Z}_+ \cup \{\infty\}$ *so that for* $0 \le t < 1$

$$(\kappa^{(n)})^{-1}(1 - \lfloor t \rfloor_n)^{c_2} \le y^{(n)}(t) \le \kappa^{(n)}(1 - \lfloor t \rfloor_n)^{c_1}, \tag{4.44}$$

$$|q^{(n)} - x^{(n)}(t)| \le \kappa^{(n)}(1 - \lfloor t \rfloor_n)^{c_3}. \tag{4.45}$$

Then there is a coupling of $\tau^{(n)}, n \in \mathbb{Z}_+\{\infty\}$ *so that almost surely both* $\|\mathtt{r}\,\tau^{(n)} - \mathtt{r}\,\tau^{(\infty)}\|_{HS}$ *and* $|\mathfrak{t}_{\tau^{(n)}} - \mathfrak{t}_{\tau^{(\infty)}}|$ *converge to* 0 *as* $n \to \infty$.

Proof. We will show that the quadruple $(x^{(n)} + iy^{(n)}, q^{(n)}, \mathtt{r}\,\tau^{(n)}, \mathfrak{t}_{\tau^{(n)}})$ converges jointly in distribution to $(x^{(\infty)} + iy^{(\infty)}, q^{(\infty)}, \mathtt{r}\,\tau^{(\infty)}, \mathfrak{t}_{\tau^{(\infty)}})$ in the appropriate product space. Since both the space of cadlag functions on $[0, 1)$ under the Skorohod topology and the space of L^2 bounded integral operators on $\mathbb{R}^2$ are separable, the statement follows by Skorohod's representation theorem (see e.g. Theorem 1.6.7 in [4]).

We have to show that for any subsequence $n_j, j \in \mathbb{Z}_+$ we can choose a further subsequence $n_{j(m)}$ along which the appropriate convergence in distribution holds. By the tightness of $\kappa^{(n)}, n \in \mathbb{Z}_+$ we may choose $n_{j(m)}$ so that $(x^{(n_{j(m)})} + iy^{(n_{j(m)})}, \kappa^{(n_{j(m)})}) \Rightarrow (x^{(\infty)} + iy^{(\infty)}, \kappa^{(\infty)})$ with an a.s. finite $\kappa^{(\infty)}$. Using Skorohod's representation theorem there is a coupling where this convergence in distribution holds in the a.s. sense with $x+iy$ converging pointwise on $[0, 1)$. We can now use Proposition 4.23 to conclude that in this coupling the quadruple $(x^{(n_{j(m)})} + iy^{(n_{j(m)})}, q^{(n_{j(m)})}, \mathtt{r}\,\tau^{(n_{j(m)})}, \mathfrak{t}_{\tau^{(n_{j(m)})}})$ converges a.s. to $(x^{(\infty)} + iy^{(\infty)}, q^{(\infty)}, \mathtt{r}\,\tau^{(\infty)}, \mathfrak{t}_{\tau^{(\infty)}})$ in the appropriate product metric. This also implies convergence in distribution along the subsubsequence $n_{j(m)}$, finishing the proof. $\square$

4.4.2 Probabilistic tools

The following two results will allow us to check the conditions in Corollary 4.24. The first is a special case of a classical result about the diffusion limit of discrete time Markov chains due to Ethier and Kurtz.

Proposition 4.25. *Suppose that for each $n \in \mathbb{Z}_+$ the the sequence of pairs of random variables $Z_k^{(n)} = (v_k^{(n)}, w_k^{(n)})$, $0 \le k \le n-1$ are independent. For a given n let $(x_k^{(n)}, y_k^{(n)}), 0 \le k \le n$ be the solution of the recursion (4.18) built from $(v_k^{(n)}, w_k^{(n)})$, and introduce the notation $(x^{(n)}(t), y^{(n)}(t)) := (x_{\lfloor nt \rfloor}^{(n)}, y_{\lfloor nt \rfloor}^{(n)})$.*

Assume that there exist continuous functions $a_1, a_2, \sigma_1^2, \sigma_2^2$ on $[0,1)$ such that

$$n\mathbb{E}(Z_k^{(n)}) = \begin{pmatrix} a_1(\frac{k}{n}) & a_2(\frac{k}{n}) \end{pmatrix} + \mathrm{err}_1(k,n), \tag{4.46}$$

$$n\mathrm{Cov}(Z_k^{(n)}, Z_k^{(n)}) = \begin{pmatrix} \sigma_1^2(\frac{k}{n}) & 0 \\ 0 & \sigma_2^2(\frac{k}{n}) \end{pmatrix} + \mathrm{err}_2(k,n), \tag{4.47}$$

and

$$n\mathbb{E}(|v_k^{(n)}|^4 + |w_k^{(n)}|^4) = \mathrm{err}_3(k,n), \tag{4.48}$$

where the error terms satisfy

$$\limsup_{n\to\infty} \max_{k/n \le 1-\delta} |\mathrm{err}_j(k,n)| = 0$$

for any $\delta \in (0,1)$, $1 \le j \le 3$.

Then $x^{(n)} + iy^{(n)}$ converges in distribution to $x + iy$, the solution of the stochastic differential equation

$$dx = (a_1(t)dt + \sigma_1(t)dB_1)y, \quad dy = (a_2(t)dt + \sigma_2(t)dB_2)y, \quad x(0) = 0, y(0) = 1, \tag{4.49}$$

on $[0,1)$ with respect to the Skorohod topology. Here B_1 and B_2 are independent standard Brownian motion.

Proof. The proposition follows from Theorem 7.4.1 and Corollary 7.4.2 of [20] (see Section 11.2 in [59] as well). $\square$

Our next statement provides a sufficient condition to check the inequality (4.44) for our models. The proposition is a straightforward extension of Lemma 5 of [48], we do not present the proof here. (See (2.4)-(2.5) of Lemma 5 and also Claim 10 in [48].)

Proposition 4.26. *Let $\xi_k^{(n)}, 0 \le k \le n-1, 1 \le n$ be a positive triangular array with independent entries for any given n. Define $y_j^{(n)} = \prod_{k=0}^{j-1} \xi_k^{(n)}$. Assume that there are constants $\lambda_0 > 0$, $c_1 \in \mathbb{R}$ and $c_2, c_3 > 0$, so that for $|\lambda| < \lambda_0$ and $0 \le j \le n-1$ we have*

$$\log \mathbb{E}[\exp(\lambda \log y_j^{(n)})] = c_1 \lambda \log(1 - \tfrac{j}{n}) - c_2 \lambda^2 \log(1 - \tfrac{j}{n}) + \mathrm{err}_n(j), \tag{4.50}$$

where $|\mathrm{err}_n(j)| \le c_3$ for all j, n. Then for any $\varepsilon > 0$ small, there exists a sequence of tight positive random variables $\kappa^{(n)} = \kappa^{(n)}(\varepsilon)$ such that for all $0 \le k \le n-1$ we have

$$(\kappa^{(n)})^{-1}(1 - \tfrac{k}{n})^{c_1+\varepsilon} \le y_k^{(n)} \le \kappa^{(n)}(1 - \tfrac{k}{n})^{c_1-\varepsilon}.$$

4.5 Path convergence for the discrete models

In this section, we prove that the driving paths of the operators $\mathtt{CJ}_{n,\beta,\delta}$ and $\mathtt{RO}_{2n,\beta,a,b}$ converge in distribution to the driving paths of the operators $\mathtt{HP}_{\beta,\delta}$ and $\mathtt{Bess}_{\beta,a}$, respectively. For this we will check that the discrete models satisfy the conditions in Proposition 4.25.

4.5.1 Circular Jacobi ensemble

Recall the definition of the distributions $\Theta(a+1, \delta)$ and $P_{IV}(m, \mu)$ from Definitions 4.4 and 4.10. We also introduce an additional distribution.

Definition 4.27. *For* $s,t>0$ *let* $\mathrm{B}'(s,t)$ *denote the 'beta prime' distribution on* $(0,\infty)$ *that has the probability density function*

$$\tfrac{\Gamma(s+t)}{\Gamma(s)\Gamma(t)}y^{s-1}(1+y)^{-s-t}.$$

Note that if $X_i, i=1,2$ are independent Gamma distributed random variables with density $\Gamma(\alpha_i)^{-1}x^{\alpha_i-1}e^{-x}$ on $(0,\infty)$ then $\frac{X_1}{X_2}$ has $\mathrm{B}'(\alpha_1,\alpha_2)$ distribution, and $\frac{X_2-X_1}{X_1+X_2}$ has $\widetilde{\mathrm{B}}(\alpha_1,\alpha_2)$ distribution.

The following statement follows by a change of variables.

Fact 4.28. *Suppose that* $\gamma\in\mathbb{C}$ *is distributed as* $\Theta(a+1,\delta)$ *with* $a\ge 0$ *and* $\Re\delta>-1/2$. *Define* $w,v\in\mathbb{R}$ *with* $\frac{2\gamma}{1-\gamma}=w-iv$. *Then the random variables* w *and* $\frac{v}{2+w}$ *are* *independent, and*

$$1+w\sim \mathrm{B}'(\tfrac{a}{2},\tfrac{a}{2}+2\Re\delta+1),\qquad \tfrac{v}{2+w}\sim P_{IV}(\tfrac{a}{2}+\Re\delta+1,-2\Im\delta).$$

In the $a=0$ *case* w *degenerates to* -1, *and hence* $\frac{v}{2+w}=v$.

We record here the following facts of the beta prime and Pearson type IV distributions.

Fact 4.29. *Let* $s,t>0$, *and* $Y\sim\mathrm{B}'(s,t)$. *Then for any* $-s<k<t$,

$$\mathbb{E}[Y^k]=\tfrac{\Gamma(s+k)\Gamma(t-k)}{\Gamma(s)\Gamma(t)}.$$

Let $m>5/2$, $\mu\in\mathbb{R}$, *and* $Z\sim P_{IV}(m,\mu)$. *Then we have*

$$\mathbb{E}[Z]=-\tfrac{\mu}{2m-2},\qquad \mathbb{E}[Z^2]=\tfrac{2m-2+\mu^2}{(2m-2)(2m-3)},\qquad \mathbb{E}[Z^4]=\tfrac{12(m+(\mu^2-3)/2)^2-2\mu^4-2\mu^2-3}{(2m-5)(2m-4)(2m-3)(2m-2)}.$$

We are now ready to prove that the driving paths of the operators $\mathtt{CJ}_{n,\beta,\delta}$ converge to the driving path of the operator $\mathtt{HP}_{\beta,\delta}$.

Proposition 4.30. *Fix $\beta > 0$ and $\delta \in \mathbb{C}$ with $\Re\delta > -1/2$. Let $\{\gamma_k^{(n)}, 0 \le k \le n-1\}$ be random variables that are independent for a fixed n, and have distributions $\gamma_k^{(n)} \sim \Theta(\beta(n-k-1)+1, \delta)$. Define $v_k^{(n)}, w_k^{(n)} \in \mathbb{R}$ via (4.17) using $\gamma_k = \gamma_k^{(n)}$, and let $x_k^{(n)}, y_k^{(n)}, 0 \le k \le n$ be the solution of the recursion (4.18) using $v_k = v_k^{(n)}$, $w_k = w_k^{(n)}$. Set $(x^{(n)}(t), y^{(n)}(t)) := (x_{\lfloor nt \rfloor}^{(n)}, y_{\lfloor nt \rfloor}^{(n)})$. Let $\widetilde{x} + \widetilde{y}$ be the process defined in Proposition 4.9. Then $x^{(n)} + iy^{(n)}$ converges in distribution to $\widetilde{x} + i\widetilde{y}$ on $[0,1)$ with respect to the Skorohod topology.*

Proof. Let $N_\delta = \lceil \frac{2}{\beta}(2 - \Re\delta) \rceil \vee 0$.

Set $z_k^{(n)} = v_k^{(n)}/(2 + w_k^{(n)})$. By Fact 4.28 we have that $1 + w_k^{(n)}$ and $z_k^{(n)}$ are independent with distributions

$$1 + w_k^{(n)} \sim \mathrm{B}'(\tfrac{\beta}{2}(n-k-1), \tfrac{\beta}{2}(n-k-1) + 2\Re\delta + 1), \tag{4.51}$$

$$z_k^{(n)} \sim P_{IV}(\tfrac{\beta}{2}(n-k-1) + \Re\delta + 1, -2\Im\delta). \tag{4.52}$$

From Fact 4.29, we get that for $0 \le k \le n - N_\delta - 1$

$$\mathbb{E}[w_k^{(n)}] = \tfrac{-4\Re\delta}{\beta(n-k-1)+4\Re\delta}, \qquad \mathbb{E}[(w_k^{(n)})^2] = \tfrac{4\beta(n-k-1)-8\Re\delta+16(\Re\delta)^2}{(\beta(n-k-1)+4\Re\delta-2)(\beta(n-k-1)+4\Re\delta)}, \tag{4.53}$$

$$\mathbb{E}[v_k^{(n)}] = \tfrac{4\Im\delta}{\beta(n-k-1)+4\Re\delta}, \qquad \mathbb{E}[(v_k^{(n)})^2] = \tfrac{4\beta(n-k-1)+8\Re\delta+16(\Im\delta)^2}{(\beta(n-k-1)+4\Re\delta-2)(\beta(n-k-1)+4\Re\delta)}. \tag{4.54}$$

Moreover, there exists a constant $c > 0$ so that for $0 \le k \le n - N_\delta - 1$ we have

$$\left|\mathbb{E}[v_k^{(n)} w_k^{(n)}]\right| + \mathbb{E}[(v_k^{(n)})^4] + \mathbb{E}[(w_k^{(n)})^4] \le c(n-k)^{-2}.$$

This means that the conditions (4.46) and (4.48) of Proposition 4.25 are satisfied with the functions $a_1(t) = \Im\delta \upsilon_\beta'(t)$, $a_2(t) = -\Re\delta \upsilon_\beta'(t)$, $\sigma_1^2(t) = \sigma_2^2(t) = \upsilon_\beta'(t)$, with $\upsilon_\beta(t) = -\frac{4}{\beta}\log(1-t)$. Hence the processes $x^{(n)}(t) + iy^{(n)}(t)$ converge in distribution to the solution of the sde

$$dx = \left(\Im\delta \upsilon_\beta'(t)dt + \sqrt{\upsilon_\beta'(t)}dB_1\right)y, \quad dy = \left(-\Re\delta \upsilon_\beta'(t)dt + \sqrt{\upsilon_\beta'(t)}dB_2\right)y \tag{4.55}$$

with independent Brownian motions B_1, B_2 and initial values $x(0) = 0, y(0) = 1$. The distribution of the process in (4.55) is the same as that of the SDE (4.24) with the time change $t \to v_\beta(t)$, which is completes the proof of the proposition. $\square$

4.5.2 Real orthogonal ensemble

Now we turn to the path convergence of the real orthogonal ensemble. By Theorem 4.7, the modified Verblunsky coefficients of the real orthogonal ensemble are all real. Hence (4.17) and (4.18) imply that $v_k = x_k = 0$, $1 + w_k = \frac{1+\gamma_k}{1-\gamma_k}$, and $y_k = \prod_{j=0}^{k-1} \frac{1+\gamma_k}{1-\gamma_k}$.

Proposition 4.31. *Fix $a, b > -1, \beta > 0$. Let $\{\gamma_k^{(2n)}, 0 \leq k \leq 2n-1\}$ be random variables that are independent for a fixed n with the following distributions: $\gamma_{2n-1}^{(2n)} = -1$, and for $0 \leq k \leq 2n-2$*

$$\gamma_k^{(2n)} \sim \begin{cases} \widetilde{\mathrm{B}}\big(\frac{\beta}{4}(2n-k+2a), \frac{\beta}{4}(2n-k+2b)\big), & \text{if } k \text{ is even,} \\ \widetilde{\mathrm{B}}\big(\frac{\beta}{4}(2n-k+2a+2b+1), \frac{\beta}{4}(2n-k-1)\big), & \text{if } k \text{ is odd.} \end{cases} \tag{4.56}$$

Define $y^{(2n)}(t) = \prod_{k=0}^{\lfloor 2nt \rfloor - 1} \frac{1+\gamma_k^{(2n)}}{1-\gamma_k^{(2n)}}$ for all $0 \leq t < 1$. Let $\widetilde{y}$ be the process defined in Proposition 4.12. Then $y^{(2n)}$ converges in distribution to $\widetilde{y}$ on $[0, 1)$ with respect to the Skorohod topology.

Proof. We first consider the multiplicative random walk with step size 2 and define $y_1^{(2n)}(t) := \prod_{k=0}^{2\lfloor nt \rfloor - 1} \frac{1+\gamma_k^{(2n)}}{1-\gamma_k^{(2n)}}$. We will check the conditions in Proposition 4.25 for $y_1^{(2n)}(t)$ (with $x_1^{(2n)} = 0$).

If $\gamma \sim \widetilde{\mathrm{B}}(s_1, s_2)$ then $\frac{1+\gamma}{1-\gamma} \sim \mathrm{B}'(s_2, s_1)$. Using the moment formulas of Fact 4.29 one readily checks that with

$$v_k^{(2n)} = 0, \qquad w_k^{(2n)} = \frac{1+\gamma_{2k}^{(2n)}}{1-\gamma_{2k}^{(2n)}} \cdot \frac{1+\gamma_{2k+1}^{(2n)}}{1-\gamma_{2k+1}^{(2n)}} - 1$$

the conditions (4.46) and (4.48) of Proposition 4.25 are satisfied with $a_1 = \sigma_1^2 = 0$, $a_2(t) = \frac{4/\beta-(2a+1)}{(1-t)}$ and $\sigma_2^2(t) = \frac{8}{\beta(1-t)}$. Hence the limit in distribution of $y_1^{(2n)}(\cdot)$ exist and it has the distribution of the strong solution of the diffusion

$$d\widetilde{y} = \tfrac{4/\beta-(2a+1)}{(1-t)}\widetilde{y}dt + \sqrt{\tfrac{8}{\beta(1-t)}}\widetilde{y}dB, \qquad \widetilde{y}(0) = 1,$$

where B is a standard Brownian motion.

The solution of this SDE has the same distribution as the process $\widetilde{y}$ in Proposition 4.12. Using the the fourth moment bounds of Fact 4.29 one can show that $|y_1^{(2n)}/y^{(2n)}-1|$ converges to 0 in the sup-norm in probability on any compact subset of $[0,1)$. From this it follows that that $y^{(2n)}$ converges to $\widetilde{y}$ in distribution as well, proving the proposition. $\square$

4.6 Proofs of the operator limit theorems

We are ready to prove Theorem 4.14. We will do that by applying Corollary 4.24 to the processes described in Propositions 4.30, for this we need to prove the path bounds (4.44) and (4.45). This is the content of Propositions 4.32 and 4.33 below.

Proposition 4.32. *Fix $\beta > 0$, $\delta \in \mathbb{C}$ with $\Re\delta > -1/2$. Let $x_k^{(n)} + iy_k^{(n)}, 0 \le k \le n$ be defined as in Proposition 4.30. Then for any $0 < \varepsilon < c_\delta = \frac{4}{\beta}(\Re\delta + \frac{1}{2})$, there exists a sequence of tight random variables $\kappa^{(n)} = \kappa^{(n)}(\varepsilon)$ such that for all $0 \le k \le n-1$,*

$$(\kappa^{(n)})^{-1}(1-\tfrac{k}{n})^{c_\delta+\varepsilon} \le y_k^{(n)} \le \kappa^{(n)}(1-\tfrac{k}{n})^{c_\delta-\varepsilon}. \tag{4.57}$$

Proof. Using the definition of $y_k^{(n)}$ together with Fact 4.28 we get that

$$y_k^{(n)} = \prod_{j=0}^{k-1}(1+w_k^{(n)}),$$

where for a fixed n the random variables $w_k^{(n)}, 0 \le k \le n-1$ are independent with distribution given in (4.51). By Fact 4.29, for $|\lambda| < \Re\delta + 1/2$ and $0 \le k \le n-1$ we have

$$\log E[(y_k^{(n)})^\lambda] = \sum_{j=0}^{k-1} \log\left(\frac{\Gamma(s_j^{(n)}+\lambda)\Gamma(t_j^{(n)}-\lambda)}{\Gamma(s_j^{(n)})\Gamma(t_j^{(n)})}\right),$$

where $s_j^{(n)} = \frac{\beta}{2}(n-j-1)$, $t_j^{(n)} = \frac{\beta}{2}(n-j-1) + 2\Re\delta + 1$. By the asymptotics of the Gamma function for any $r > 0$ there is a $c_r > 0$ so that

$$\left|\log\Gamma(x) - \left((x-\tfrac{1}{2})\log x + x - \tfrac{\log 2\pi}{2} - \tfrac{1}{12}x^{-1}\right)\right| \le c_r x^{-2} \quad \text{for } x \ge r.$$

From this (and some basic Taylor expansion estimates) it follows that $y_k^{(n)}$ satisfies condition (4.50) of Proposition 4.26 with $c_1 = c_\delta$ and $c_2 = \frac{2}{\beta}$, and the statement follows by Proposition 4.26. $\square$

Proposition 4.33. *Fix $\beta > 0$, $\delta \in \mathbb{C}$ with $\Re\delta > -1/2$. Let $x_k^{(n)} + iy_k^{(n)}, 0 \le k \le n$ be defined as in Proposition 4.30. Then for any $0 < c' < c_\delta = \frac{4}{\beta}(\Re\delta + \frac{1}{2})$, there exist tight random constants $\kappa_1^{(n)} > 0$ such that*

$$|x_n^{(n)} - x_j^{(n)}| \le \kappa_1^{(n)}(1 - \frac{j}{n})^{c'} \quad \textit{for all } 0 \le j \le n-1. \tag{4.58}$$

Proof. Fix $\varepsilon > 0$ so that $c' + 2\varepsilon < c_\delta$. By Proposition 4.32 there is a sequence of tight random variables $\kappa^{(n)}$ so that (4.57) holds, and the sequence $\kappa^{(n)}$ is measurable with respect to the sigma-field generated by the random variables $y_k^{(n)}, 0 \le k \le n-1$.

Set $z_k^{(n)} = v_k^{(n)}/(2 + w_k^{(n)})$. Then from (4.18) we get

$$x_{k+1}^{(n)} = x_k^{(n)} + z_k^{(n)}\big(2 + w_k^{(n)}\big)y_k^{(n)} = x_k^{(n)} + z_k^{(n)}\big(y_{k+1}^{(n)} + y_k^{(n)}\big),$$

and

$$x_n^{(n)} - x_j^{(n)} = \sum_{k=j}^{n-1} z_k^{(n)}\big(y_k^{(n)} + y_{k+1}^{(n)}\big).$$

Introduce

$$A^{(n)} := \max_{0\le j\le n-1}\left|\sum_{k=j}^{n-1} z_k^{(n)}\big(y_k^{(n)}+y_{k+1}^{(n)}\big)\right|\left(1-\frac{j}{n}\right)^{-c'},$$

the statement will follow once we show that the sequence $A^{(n)}, n \ge 1$ is tight. We will do that by first separating finitely many terms in the maximum, and then splitting the sum using centered versions of $z_k^{(n)}$.

Set $N_\delta = \lceil \frac{2}{\beta}(4-\Re\delta)\rceil \vee 0$ and $\widetilde{n} = n - N_\delta - 1$. Note that by Fact 4.29, the fourth moment of $z_k^{(n)}$ is finite for $j \le \widetilde{n}$. By (4.52) the distribution of $z_k^{(n)}$ only depends on $n-k$, hence the path bounds (4.57) on $y_k^{(n)}$ (together with $c_\delta - 2\varepsilon - c' > 0$) imply that the following sequence of random variables is tight:

$$A_0^{(n)} := \max_{\widetilde{n}+1\le j\le n-1}\left|\sum_{k=j}^{n-1} z_k^{(n)}\big(y_k^{(n)}+y_{k+1}^{(n)}\big)\right|\left(1-\frac{j}{n}\right)^{-c'}. \tag{4.59}$$

Since the sequence $A_0^{(n)}, n\ge 1$ is tight, it suffices to show the tightness of the following sequence:

$$\widetilde{A}^{(n)} := \max_{0\le j\le \widetilde{n}}\left|\sum_{k=j}^{\widetilde{n}} z_k^{(n)}\big(y_k^{(n)}+y_{k+1}^{(n)}\big)\right|\left(1-\frac{j}{n}\right)^{-c'}. \tag{4.60}$$

We introduce

$$A_1^{(n)} = \max_{0\le j\le \widetilde{n}}\left|\sum_{k=j}^{\widetilde{n}} \mathbb{E}\big[z_k^{(n)}\big]\big(y_k^{(n)}+y_{k+1}^{(n)}\big)\right|\left(1-\frac{j}{n}\right)^{-c'},$$

$$A_2^{(n)} = \max_{0\le j\le \widetilde{n}}\left|\sum_{k=j}^{\widetilde{n}} \bar{z}_k^{(n)}\big(y_k^{(n)}+y_{k+1}^{(n)}\big)\right|\left(1-\frac{j}{n}\right)^{-c'},$$

where $\overline{X} = X - \mathbb{E}[X]$. Note that $\widetilde{A}^{(n)} \le A_1^{(n)} + A_2^{(n)}$.

By (4.52) and Fact 4.29 we have

$$\mathbb{E}\big[z_k^{(n)}\big] = \frac{2\Im\delta}{\beta(n-k-1)+2\Re\delta}.$$

Using the bounds in (4.57) with $\varepsilon < c_\delta - c'$ we get

$$A_1^{(n)} \leq \max_{0\leq j\leq \widetilde{n}} \left\{ \left(1-\frac{j}{n}\right)^{-c'} \left(\sum_{k=j}^{\widetilde{n}} \frac{4\kappa^{(n)}|\Im\delta|}{\beta(n-k-1)+2\Re\delta} \left(1-\frac{k}{n}\right)^{c_\delta-\varepsilon} \right) \right\} \leq c\kappa^{(n)}, \quad (4.61)$$

with a deterministic constant c that only depends on δ and β. This shows that the sequence $A_1^{(n)}, n \geq 1$ is tight.

Next we turn to the tightness of the sequence $A_2^{(n)}$. Choose $1 < \theta < (c_\delta - \frac{3}{2}\varepsilon)/c'$. Define

$$m = m^{(n)} = \inf\{i \in \mathbb{Z}^+ : \theta^i \geq \log\left(\tfrac{n}{N_\delta+1}\right)\},$$

$$\sigma_0 = \sigma_0^{(n)} = 0, \qquad \sigma_i = \sigma_i^{(n)} = \min\left(\lfloor n\big(1-e^{-\theta^i}\big)\rfloor, \widetilde{n}\right) \quad \text{for } 1 \leq i \leq m.$$

Note that $\sigma_0 = 0 \leq \sigma_1 \leq \cdots \leq \sigma_m = \widetilde{n}$. In order to bound the tail of $A_2^{(n)}$ we will split the index set of the sums into blocks $\{\sigma_i, \sigma_i+1, \cdots, \sigma_{i+1}\}$ to control the term $(1-j/n)^{-c'}$, and then control the fluctuations within each block. Fix $K > 0$, then we have

$$P(A_2^{(n)} \geq K) \leq \sum_{i=0}^{m-1} P\left(\max_{\sigma_i\leq j\leq\sigma_{i+1}} \left| \sum_{k=j}^{\widetilde{n}} \bar{z}_k^{(n)} \big(y_k^{(n)} + y_{k+1}^{(n)}\big) \right| \left(1-\frac{j}{n}\right)^{-c'} \geq K, \kappa^{(n)} \leq \sqrt{K} \right) \quad (4.62)$$

$$+ P\big(\kappa^{(n)} > \sqrt{K}\big).$$

Since $\kappa^{(n)}$ are tight, we have

$$\lim_{K\to\infty} \limsup_{n\to\infty} P\big(\kappa^{(n)} > \sqrt{K}\big) = 0.$$

We now estimate the terms in the sum in (4.62) for each $0 \le i \le m-1$. We have

$$
\begin{aligned}
P&\left(\max_{\sigma_i \le j \le \sigma_{i+1}} \Big| \sum_{k=j}^{\widetilde{n}} \bar{z}_k^{(n)}\big(y_k^{(n)} + y_{k+1}^{(n)}\big)\Big| \Big(1 - \frac{j}{n}\Big)^{-c'} \ge K, \kappa^{(n)} \le \sqrt{K}\right) \\
&\le P\left(\Big| \sum_{k=\sigma_i}^{\widetilde{n}} \bar{z}_n^{(n)}\big(y_k^{(n)} + y_{k+1}^{(n)}\big)\Big| \ge \frac{K}{2}\Big(1 - \frac{\sigma_{i+1}}{n}\Big)^{c'}, \kappa^{(n)} \le \sqrt{K}\right) \\
&\quad + P\left(\max_{\sigma_i \le j \le \sigma_{i+1}} \Big| \sum_{k=\sigma_i}^{j} \bar{z}_k^{(n)}\big(y_k^{(n)} + y_{k+1}^{(n)}\big)\Big| \ge \frac{K}{2}\Big(1 - \frac{\sigma_{i+1}}{n}\Big)^{c'}, \kappa^{(n)} \le \sqrt{K}\right).
\end{aligned}
$$

Note that the sequence $\kappa^{(n)}$ is measurable with respect to $y_k^{(n)}, 0 \le k \le n$ and $\bar{z}_k^{(n)}$ are independent of $y_k^{(n)}$. Hence by conditioning on $y_k^{(n)}, 0 \le k \le n$, using Doob's maximal inequality, and the path bound (4.57) we get

$$
\begin{aligned}
P&\left(\max_{\sigma_i \le j \le \sigma_{i+1}} \Big| \sum_{k=\sigma_i}^{j} \bar{z}_k^{(n)}\big(y_k^{(n)} + y_{k+1}^{(n)}\big)\Big| \ge \frac{K}{2}\Big(1 - \frac{\sigma_{i+1}}{n}\Big)^{c'}, \kappa^{(n)} \le \sqrt{K}\right) \\
&\le \mathbb{E}\left[\mathbf{1}\big(\kappa^{(n)} \le \sqrt{K}\big)\left(\sum_{k=\sigma_i}^{\sigma_{i+1}} 4\mathbb{E}\big[(\bar{z}_k^{(n)})^2\big]\big(y_k^{(n)} + y_{k+1}^{(n)}\big)^2 K^{-2}\Big(1 - \frac{\sigma_{i+1}}{n}\Big)^{-2c'}\right)\right] \\
&\le \mathbb{E}\left[\mathbf{1}\big(\kappa^{(n)} \le \sqrt{K}\big)\left(\sum_{k=\sigma_i}^{\sigma_{i+1}} 16(\kappa^{(n)})^2\mathbb{E}\big[(\bar{z}_k^{(n)})^2\big]\Big(1 - \frac{k}{n}\Big)^{2(c_\delta - \varepsilon)} K^{-2}\Big(1 - \frac{\sigma_{i+1}}{n}\Big)^{-2c'}\right)\right] \\
&\le K^{-1}\sum_{k=\sigma_i}^{\sigma_{i+1}} 16\mathbb{E}\big[(\bar{z}_k^{(n)})^2\big]\Big(1 - \frac{k}{n}\Big)^{2(c_\delta - \varepsilon)}\Big(1 - \frac{\sigma_{i+1}}{n}\Big)^{-2c'}. \qquad (4.63)
\end{aligned}
$$

Using (4.52) and Fact 4.29 one can show that there exists an absolute constant c such that

$$\text{R.H.S. of (4.63)} \le cK^{-1}(1 - \frac{\sigma_i}{n})^{2(c_\delta - \varepsilon)}(1 - \frac{\sigma_{i+1}}{n})^{-2c'} \le cK^{-1}e^{-2\theta^i(c_\delta - \varepsilon - c'\theta)} \le cK^{-1}e^{-\varepsilon\theta^i}.$$

Similarly, Chebishev's inequality, conditioning, and the path bound (4.57) give the upper bound

$$P\left(\Big| \sum_{k=\sigma_i}^{\widetilde{n}} \bar{z}_k^{(n)}\big(y_k^{(n)} + y_{k+1}^{(n)}\big)\Big| \ge \frac{K}{2}\Big(1 - \frac{\sigma_{i+1}}{n}\Big)^{c'}, \kappa^{(n)} \le \sqrt{K}\right) \le cK^{-1}e^{-\varepsilon\theta^i}.$$

This shows that the sum on the right of (4.62) can be bounded from above by

$$2\sum_{i=0}^{m} cK^{-1}e^{-\varepsilon\theta^i} \le c_1K^{-1}$$

with an absolute constant c_1. This proves the tightness of the sequence $A_2^{(n)}, n \ge 1$, and completes the proof of the proposition. $\square$

Now we have all the pieces for the proof of Theorem 4.14.

Proof of Theorem 4.14. Consider the random variables $x_k^{(n)} + iy_k^{(n)}, 0 \le k \le n$ defined in Proposition 4.30, and define $(x^{(n)}(t), y^{(n)}(t)) := (x_{\lfloor nt \rfloor}^{(n)}, y_{\lfloor nt \rfloor}^{(n)})$. Let $\widetilde{x} + i\widetilde{y}$ be the process defined in Proposition 4.9. Set $q^{(n)} = x_n^{(n)}$ and $q = \lim_{t\to 1} \widetilde{x}(t)$. Define $\tau^{(n)}, n \in \mathbb{Z}_+$ using $(x^{(n)} + iy^{(n)}, q^{(n)})$, and $\tau^{(\infty)}$ using $(\widetilde{x} + i\widetilde{y}, q)$. Then $\tau^{(n)} \sim \mathtt{CJ}_{n,\beta,\delta}$ and $\tau^{(\infty)} \sim \mathtt{HP}_{\beta,\delta}$.

By Propositions 4.32 and 4.33 there exists a tight sequence $\kappa^{(n)}, n \in \mathbb{Z}_+$ so that the inequalities (4.44) and (4.45) are satisfied for $n \in \mathbb{Z}_+$ with $c_1 = c_\delta - \varepsilon, c_2 = c_\delta + \varepsilon$, $c_3 = c_\delta - \varepsilon$. Here $c_\delta = \frac{4}{\beta}(\Re\delta + 1/2)$ and $\varepsilon \in (0, \min(c_\delta, \frac{1}{2}))$ is arbitrary. By (4.25) there is a finite random variable $\kappa^{(\infty)}$ so that (4.44) and (4.45) are satisfied for $\widetilde{x} + i\widetilde{y}$ with the just defined c_1, c_2, c_3. Together with Proposition 4.30 this means that the conditions of Corollary 4.24 are satisfied, and hence the statement of the theorem follows. $\square$

The proof of Theorem 4.16 follows along the same line.

Proposition 4.34. *Fix $\beta > 0$, $a, b > -1$. Let $y_k^{(2n)}, 0 \le k \le 2n$ be defined as in Proposition 4.31. Then for any $\varepsilon > 0$ small, there exists a sequence of tight random variables $\kappa^{(2n)} = \kappa^{(2n)}(\varepsilon)$ such that for all $0 \le k \le 2n - 1$,*

$$(\kappa^{(2n)})^{-1}(1 - \tfrac{k}{2n})^{2a+1+\varepsilon} \le y_k^{(2n)} \le \kappa^{(2n)}(1 - \tfrac{k}{2n})^{2a+1-\varepsilon}.$$

Proof. One can just mimic the steps of the proof of Proposition 4.32 using the parameters

$$(s_k^{(2n)}, t_k^{(2n)}) = \begin{cases} \left(\frac{\beta}{4}(2n-k+2a), \frac{\beta}{4}(2n-k+2b)\right) & \text{if } k \text{ is even,} \\ \left(\frac{\beta}{4}(2n-k+2a+2b+1), \frac{\beta}{4}(2n-k-1)\right) & \text{if } k \text{ is odd,} \end{cases}$$

and $c_1 = 2a+1$, $c_2 = \frac{4}{\beta}$. □

Proof of Theorem 4.16. Consider the random variables $y_k^{(2n)}, 0 \le k \le n$ defined in Proposition 4.31, and define $(x^{(2n)}(t), y^{(2n)}(t)) := (0, y_{\lfloor 2nt \rfloor}^{(2n)})$. Let $\widetilde{y}$ be the process defined in Proposition 4.12 and set $\widetilde{x} = 0$. Set $q^{(2n)} = q = 0$, and define $\tau^{(2n)}, n \in \mathbb{Z}_+$ using $(x^{(2n)}+iy^{(2n)}, q^{(2n)})$, and $\tau^{(\infty)}$ using $(\widetilde{x}+i\widetilde{y}, q)$. Then $\tau^{(2n)} \sim \mathtt{RO}_{2n,\beta,a,b}$ and $\tau^{(\infty)} \sim \mathtt{Bess}_{\beta,a}$.

By Propositions 4.34 there exists a tight sequence $\kappa^{(2n)}, n \in \mathbb{Z}_+$ so that the inequalities (4.44) and (4.45) are satisfied for $n \in \mathbb{Z}_+$ with $c_1 = 2a+1-\varepsilon, c_2 = 2a+1+\varepsilon$, $c_3 = \max(c_1, 1)$. (Note that since $x^{(2n)} = q^{(2n)} = 0$ the inequality (4.45) holds for any positive c_3.) Here $\varepsilon \in (0, \frac{1}{2})$ is chosen so that $c_1 > -1$. By the sublinearity of Brownian motion there is a finite random variable $\kappa^{(\infty)}$ so that (4.44) and (4.45) are satisfied for $\widetilde{x} + i\widetilde{y}$ with the just defined c_1, c_2, c_3. Together with Proposition 4.31 this means that the conditions of Corollary 4.24 are satisfied, and hence the statement of the theorem follows. □

4.7 Proofs of the theorems related to the limiting operators

In this section we provide the proofs for our results on the properties and characterizations of the limiting point processes and random analytic functions arising from the circular Jacobi β-ensemble and the real orthogonal β-ensemble (Theorems 4.18, 4.19, 4.20 and 4.21).

4.7.1 Simple transformations of Dirac operators

For some of our results it will be more convenient to consider Dirac operators that live on $(0,1]$, with a potential limit point at 0. (In fact this is the framework used in [65].) In order to do this, the framework introduced in Section 4.1.1 has to be extended to also include the following setup (we call this the *reversed framework*):

a) Both the generating path $x+iy$ and the weight function R (defined via (4.4)) are defined on $(0,1]$. The operator τ in (2.7) acts on $(0,1] \to \mathbb{R}^2$ functions.

b) In Assumption 4.1 the first integral condition is replaced with $\int_0^1 \|R(s)\mathfrak{u}_0\| ds < \infty$.

Otherwise we have the same assumptions: $x+iy$ is measurable and locally bounded on its domain, the boundary conditions $\mathfrak{u}_0, \mathfrak{u}_1$ satisfy (4.5). Then τ is self-adjoint on the domain $\mathrm{dom}(\tau)$ given by (4.7), its inverse is a Hilbert-Schmidt integral operator with the kernel given in (4.8). The operator $\mathtt{r}\,\tau$, the integral trace $\mathfrak{t}_\tau$, and the secular function ζ_τ can be defined the same way as before (see Section 4.1.1).

There is a simple way to move between the two frameworks. Introduce the time reversal operator $\rho f(t) := f(1-t)$ acting on functions defined on $[0,1)$ or $(0,1]$. Let $\iota : \mathbb{H} \to \mathbb{H}$ be defined as the reflection $x+iy \to -x+iy$, and set

$$S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

If a weight function R is generated by the path $z = x+iy$, then SRS is the weight function corresponding to the path ιz.

The statements of the following two lemmas are contained in Lemma 36 of [65].

Lemma 4.35 ([65]). *Assume that the Dirac operator* $\tau = \mathtt{Dir}(R, \mathfrak{u}_0, \mathfrak{u}_1)$ *satisfies the assumptions* (4.5) *and* (4.6) *with boundary conditions* $\mathfrak{u}_0, \mathfrak{u}_1$, *weight function* R, *and generating path* $z = x + iy$. *Then the operator* $\rho^{-1} S\tau S\rho$ *satisfies the assumptions of the reversed framework with boundary conditions* $-\mathfrak{u}_1, -\mathfrak{u}_0$, *weight function* ρSRS, *and generating path* $\iota\rho z$. *The operators* τ *and* $\rho^{-1}S\tau S\rho$ *are orthogonally equivalent in the respective* L^2 *spaces, they have the same integral traces and secular functions.*

Lemma 4.36 ([65]). *Let* Q *be a* 2×2 *orthogonal matrix with determinant 1. Let* $\mathcal{Q} : \bar{\mathbb{H}} \to \bar{\mathbb{H}}$ *be the corresponding linear isometry of* $\bar{\mathbb{H}}$ *mapping* $z \in \bar{\mathbb{H}}$ *to the ratio of the entries of* $Q[z, 1]^t$. *Suppose that the Dirac operator* τ *satisfies the assumptions* (4.5) *and* (4.6) *with boundary conditions* $\mathfrak{u}_0, \mathfrak{u}_1$ *and generating path* $x + iy$. *Then the operator* $Q\tau Q^{-1}$ *also satisfies the same assumptions, with boundary conditions* $\mathcal{Q}\mathfrak{u}_0, \mathcal{Q}\mathfrak{u}_1$ *and generating path* $\mathcal{Q}(x+iy)$. *The two operators are orthogonally equivalent, they have the same integral traces and secular functions. The same statement holds if* τ *satisfies the assumptions of the reversed framework.*

4.7.2 Proofs of the theorems related to $\mathtt{HP}_{\beta,\delta}$

Our first step is to produce a unitary equivalent form of the operator $\mathtt{HP}_{\beta,\delta}$ where the driving path is independent of the boundary conditions. In order to do that, we use the following factorization lemma for the diffusion (4.24). This is a generalization of Proposition X.3.1 in [24] which treats the $\delta = 0$ case, i.e. the hyperbolic Brownian motion.

We recall that in the Poincaré half plane model of the hyperbolic plane the isometries are of the form $z \to \frac{az+b}{cz+d}$ with $a, b, c, d \in \mathbb{R}$ and $ad - bc \neq 0$. For $r \in \mathbb{R}$ we set

$$T_r(z) = \frac{rz+1}{r-z}. \tag{4.64}$$

T_r is the hyperbolic rotation about the point i taking r to ∞ and ∞ to $-r$.

Theorem 4.37. *Fix $\delta \in \mathbb{C}$ with $\Re\delta > -1/2$. Consider the diffusion $w = x + iy$ defined in (4.24), and denote by w_∞ the a.s. limit as $t \to \infty$. Then the process $\widetilde{w}_t = T_{w_\infty} w_t$ satisfies the diffusion*

$$d\widetilde{w} = \Im\widetilde{w}(d\widetilde{Z} + i(1+\bar{\delta})dt), \qquad \widetilde{w}_0 = i. \tag{4.65}$$

where $\widetilde{Z}$ is standard complex Brownian motion.

Moreover, if a process $\widetilde{w}$ satisfies the SDE (4.65), and q is a random variable with distribution $P_{IV}(\Re\delta+1, -2\Im\delta)$ then the process $x_t+iy_t = T_q^{-1}\widetilde{w}_t$ satisfies the SDE (4.24) with B_1, B_2 being independent copies of standard Brownian motion.

Proof. By Theorem 4.11 the distribution of w_∞ is given by $P_{IV}(\Re\delta+1, -2\Im\delta)$. The SDE (4.24) is invariant under affine transformations of the form $z \to a+bz$ with $a \in \mathbb{R}, b > 0$. Hence for $a \in \mathbb{R}, b > 0$ the solution of (4.24) with initial condition $a + ib$ will converge in distribution to $a + bw_\infty$ where $w_\infty \sim P_{IV}(\Re\delta + 1, -2\Im\delta)$. Now using either Doob's h-transform or the technique of enlargement of filtrations (c.f. [55], or [40]) one can show that for a given $r \in \mathbb{R} \cup \{\infty\}$ the process w conditioned on the event $\{w_\infty = r\}$ satisfies the diffusion

$$dz^{(r)} = \Im z^{(r)} \left(dZ + i(1+\bar{\delta}) \frac{z^{(r)} - r}{\overline{z^{(r)}} - r} dt \right), \qquad z^{(r)}(0) = i. \tag{4.66}$$

Here Z is a standard complex Brownian motion, and in the $r = \infty$ case the $\frac{z^{(r)}-r}{\bar{z}^{(r)}-r}$ term in the drift is replaced by the constant one. In particular, $z^{(\infty)}$ has the same distribution as the process $\widetilde{w}$ from (4.65), and it hits ∞ with probability one. Using Ito's formula one can readily check that for $r \in \mathbb{R}$ the rotated process $\widetilde{w}^{(r)} = T_r(z^{(r)}) = \frac{rz^{(r)}+1}{r-z^{(r)}}$ satisfies the SDE (4.65), in particular, its distribution does not depend on r. This shows that the rotated process $t \to T_{w_\infty} w_t$ has the same distribution as $\widetilde{w}$ from (4.65), and that it is independent of w_∞. Using $w_\infty \sim P_{IV}(\Re\delta + 1, -2\Im\delta)$ the second half of the theorem follows as well. $\square$

We will now construct a reversed and transformed version of $\mathtt{HP}_{\beta,\delta}$. Let B_1, B_2 be independent two-sided real Brownian motion. Consider the two-sided version of $x + iy$ from (4.24) defined using B_1, B_2, i.e.,

$$y_s = e^{B_2(s) - (\Re\delta + \frac{1}{2})s}, \quad x_s = \begin{cases} -\int_s^0 y(t)dB_1 - \Im\delta \int_s^0 y(t)dt & s \le 0, \\ \int_0^s y(t)dB_1 + \Im\delta \int_0^s y(t)dt & s \ge 0. \end{cases} \tag{4.67}$$

We also introduce the time change

$$u_\beta(t) = -\upsilon_\beta(1-t) = \frac{4}{\beta}\log t.$$

Definition 4.38. *Let q be a random variable with distribution $P_{IV}(1 + \Re\delta, -2\Im\delta)$ independent of B_1, B_2. Set $\hat{x}(t) + i\hat{y}(t) = x(u_\beta(t)) + iy(u_\beta(t))$ for $t \in (0,1]$. Define the reversed and transformed version of the $\mathtt{HP}_{\beta,\delta}$ operator as*

$$\tau_{\beta,\delta}^{\mathtt{HP}} = \mathtt{Dir}(\hat{x} + i\hat{y}, \mathfrak{u}_0, \mathfrak{u}_1),$$

where $\mathfrak{u}_0 = [1,0]^t, \mathfrak{u}_1 = [-q,-1]^t$.

In this section we will use the simplified notation $\tau_{\beta,\delta}$ for $\tau_{\beta,\delta}^{\mathtt{HP}}$, and denote the secular function of $\tau_{\beta,\delta}$ by $\zeta_{\beta,\delta}$.

Lemma 4.39. *The operator $\tau_{\beta,\delta}$ is orthogonal equivalent to an operator which has the same distribution as the $\mathtt{HP}_{\beta,\delta}$ operator. In particular, the random analytic function $\zeta_{\beta,\delta}$ has the same distribution as $\zeta^{\mathtt{HP}}_{\beta,\delta}$.*

Proof. Recall the transformations ι, S and ρ defined in and around Lemma 4.35. Let T_q be the hyperbolic rotation defined in (4.64). Consider the Dirac operator

$$\widetilde{\tau} = \rho^{-1} S \,\mathtt{Dir}(T_q(\hat{x}+i\hat{y}), T_q\mathfrak{u}_0, T_q\mathfrak{u}_1)\, S\rho = \mathtt{Dir}(\rho\iota T_q(\hat{x}+i\hat{y}), -T_q\mathfrak{u}_1, -T_q\mathfrak{u}_0).$$

Here we identify the boundary condition $\mathfrak{u} = [a,b]^t$ with its projection a/b onto the real axis so that $T_q\mathfrak{u}_0$, $T_q\mathfrak{u}_1$ are well defined:

$$-T_q\mathfrak{u}_1 = \infty, \qquad -T_q\mathfrak{u}_0 = q.$$

By Lemmas 4.35 and 4.36 the operator $\widetilde{\tau}$ is orthogonal equivalent to $\tau_{\beta,\delta}$, hence we just have to show that $\widetilde{\tau}$ has the same distribution as $\mathtt{HP}_{\beta,\delta}$.

Note that $T_q = T_{-q}^{-1}$ and $-q \sim P_{IV}(\Re\delta+1, 2\Im\delta)$. From the definition (4.67) it follows that the reversed process $(x_{-s}+iy_{-s}), s \geq 0$ satisfies the SDE (4.65) with drift $i(1+\delta)$ in place of $i(1+\bar{\delta})$. Hence by Theorem 4.37, the process $T_q(x_{-s}+iy_{-s}), s \geq 0$ satisfies the SDE

$$dw = \Im w(dZ - i\bar{\delta}ds), \qquad w(0) = i,$$

with standard complex Brownian motion Z, and the path converges to $T_q\infty = -q$ as $s \to \infty$. From this it follows that

$$\rho\iota T_q(x_{u(\cdot)}+iy_{u(\cdot)}) \stackrel{d}{=} \rho(x_{-u(\cdot)}+iy_{-u(\cdot)}) = (x_{\upsilon_\beta(\cdot)}+iy_{\upsilon_\beta(\cdot)}),$$

with $\lim_{t\to 1} \rho\iota T_q(x_{u(t)}+iy_{u(t)}) = q$. This shows that the driving path and boundary conditions of $\widetilde{\tau}$ match up (in distribution) with the corresponding ingredients of the $\mathtt{HP}_{\beta,\delta}$ operator, proving the statement of the lemma. $\square$

The independence of the boundary point and the driving path in the reversed operator $\tau_{\beta,\delta}$ allows us to prove Theorem 4.20. Our proof follows the proof of Theorem 1 of [65], which can be considered the $\delta = 0$ case of our theorem.

Proof of Theorem 4.20. By Lemma 4.39 the random analytic function $\zeta_{\beta,\delta}$ has the same distribution as $\zeta^{\mathtt{HP}}_{\beta,\delta}$. Hence we can work with the reversed operator $\tau_{\beta,\delta}$, and prove the statements of the theorem for $\zeta_{\beta,\delta}$.

By Proposition 13 in [65] the secular function of $\tau_{\beta,\delta}$ can be characterized as follows. Let $R(t)$ be the weight function built from the driving path of the reversed $\tau_{\beta,\delta}$ operator according to (4.4). Then there exists a unique function $H : (0,1] \times \mathbb{C} \mapsto \mathbb{C}^2$ so that for every $z \in \mathbb{C}$ the function $H(\cdot, z)$ solves the ODE

$$J \frac{d}{dt} H(t,z) = zR(t)H(t,z), \qquad \lim_{t\to 0} H(t,z) = \mathfrak{u}_0 = [1,0]^t. \tag{4.68}$$

The secular function $\zeta_{\beta,\delta}$ can be obtained from H using the formula $\zeta_{\beta,\delta}(z) = [1,-q]H(1,z)$.

Consider the process $X_u = \begin{pmatrix} 1 & -x_u \\ 0 & y_u \end{pmatrix}$, $u \le 0$, where $x_u + iy_u$ is defined in (4.67). Define $\mathcal{H}_u(z) = X_u H(t(u), z)$ with $t(u) = e^{\frac{\beta}{4}u}$ being the inverse of $u(t) = \frac{4}{\beta}\log t$. Since $X_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, we have $\zeta_{\beta,\delta}(z) = [1,-q]\mathcal{H}_0(z)$. A direct computation using Itô's formula shows that $\mathcal{H}_u$ solves the SDE (4.34). To be precise, one first has to consider approximations of $\mathcal{H}_u$ that are defined on $[\varepsilon, 1]$, for this one has to use the approximation method introduced in Propositions 20 and 43 in [65]. A simple extension of those arguments also shows the characterization of $\mathcal{H}_u(z)$ as the unique solution of (4.34) with the conditions given.

Now write $\mathcal{H}_u = [\mathcal{A}_u, \mathcal{B}_u]^t$. The functions $\mathcal{A}_u, \mathcal{B}_u$ are entire functions on $\mathbb{C}$, we denote their Taylor coefficients at 0 by $\mathcal{A}_u^{(n)}, \mathcal{B}_u^{(n)}$. Since the SDE system (4.34) depends analytically on its parameter z, Itô's formula can be applied to get SDEs for derivatives in this parameter as well, see e.g. Section V.7 of [46]. Differentiating (4.34) n times in z and considering $z = 0$ shows that the Taylor coefficients $\mathcal{A}^{(n)}, \mathcal{B}^{(n)}$ satisfy the following system of SDEs

$$\begin{aligned} d\mathcal{B}^{(n)} &= \mathcal{B}^{(n)} dB_2 - \Re\delta\mathcal{B}^{(n)} du - \frac{\beta}{8} e^{\beta u/4} \mathcal{A}^{(n-1)} du, \\ d\mathcal{A}^{(n)} &= -\mathcal{B}^{(n)} dB_1 - \Im\delta\mathcal{B}^{(n)} du + \frac{\beta}{8} e^{\beta u/4} \mathcal{B}^{(n-1)} du, \end{aligned}$$

with initial conditions $\mathcal{B}^{(0)} \equiv 0$, $\mathcal{A}^{(0)} \equiv 1$. Mimicking the proof of Propositions 45 and 47 in [65] one can prove that the solution of the above system exist, and it is given by equations (4.35), (4.36). $\square$

Using the SDE characterization of $\zeta^{\mathtt{HP}}_{\beta,\delta}$ given in Theorem 4.20 we are able to prove Theorem 4.18.

Proof of Theorem 4.18. As in the proof of Theorem 4.20, we work with the operator $\tau_{\beta,\delta}$. The spectrum of this operator has the same distribution as the $\mathrm{HP}_{\beta,\delta}$ process.

Consider the random analytic function valued processes $\mathcal{A}_u, \mathcal{B}_u$ introduced in the proof of Theorem 4.20. Recall that $\zeta_{\beta,\delta} = [1, -q]\mathcal{H}_0 = \mathcal{A}_0 - q\mathcal{B}_0$, with q given in the definition of $\tau_{\beta,\delta}$, see Definition 4.38.

We introduce the structure function $\mathcal{E}(u, z) = \mathcal{A}_u(z) - i\mathcal{B}_u(z)$, note that this can also be expressed as $[1, -i]\mathcal{H}(u, z)$ with $\mathcal{H}_u$ defined in the proof of Theorem 4.20. For $\lambda \in \mathbb{R}$ we define $2 \log \mathcal{E}(u, \lambda) = \mathcal{L}_\lambda(u) + i\alpha_\lambda(u)$ with $\mathcal{L}_\lambda, \alpha_\lambda \in \mathbb{R}$, where for each $u \in \mathbb{R}$ the function is chosen so that it is continuous in λ and $\alpha_0(u) = 0$. (This is possible because $\mathcal{H}_u(z)$ is continuous in z and it is never equal to $[0, 0]^t$.) By (4.34) and Itô's formula we get

$$d\alpha_\lambda = \lambda \tfrac{\beta}{4} e^{\frac{\beta}{4}u} du + \Re[(e^{-i\alpha_\lambda} - 1)(dZ - i\delta du)], \qquad \alpha_\lambda(-\infty) = 0. \tag{4.69}$$

The process $\psi_\lambda(t) = \alpha_\lambda(u(t))$ with $u(t) = \frac{4}{\beta}\log t$ satisfies the SDE (4.31), and simple coupling arguments show that it is the unique solution of (4.31) with the conditions given in Theorem 4.18. (See e.g [34] for more details in the $\delta = 0$ case.)

Set $\theta = -2\,\mathrm{arccot}\, q$. By the comment following Theorem 4.11 we have $e^{i\theta} \sim \Theta(1, \delta)$, and θ is independent of the complex Brownian motion Z in (4.69). The eigenvalues of $\tau_{\beta,\delta}$ are given by the zeros of $\zeta_{\beta,\delta}$. By definition we have $\zeta_{\beta,\delta}(\lambda) = 0$ if and only if $\mathcal{E}(0, \lambda)$ is a real multiple of $q - i$, or equivalently $\alpha_\lambda(0) = \psi_\lambda(1) = 2\log(q - i) = \theta \mod 2\pi$. Using $\mathrm{spec}(\tau_{\beta,\delta}) \stackrel{d}{=} \mathrm{HP}_{\beta,\delta}$ finishes the proof. $\square$

Now we turn to the proof of Theorem 4.19. We first isolate the statements regarding the SDE (4.33) in a separate lemma.

Lemma 4.40. *The SDE system* (4.33) *has a unique strong solution on* $t \in [0, \infty)$, $\lambda \in \mathbb{R}$. *With probability one the process* $\lambda \to \alpha_\lambda(t)$ *is increasing for all* $t > 0$. *For each* $\lambda \in \mathbb{R}$ *the limit* $\lim_{t\to\infty} \frac{1}{2\pi}\alpha_\lambda(t)$ *exists almost surely and it is an integer. Moreover, if* $\beta \le 4(\Re\delta + \frac{1}{2})$ *and* $\lambda > 0$ *then a.s.* $\frac{1}{2\pi}\alpha_\lambda(t)$ *converges to an integer from above.*

Note that for $\delta = 0$ these statements were proved in Theorem 7 and Proposition 9 of [62].

Proof. The fact that the system (4.33) has a unique strong solution follows from standard theory, the monotonicity property is a consequence of the monotone dependence of the drift function of the parameter λ.

For a fixed $\lambda \in \mathbb{R}$ the process α_λ solves the SDE

$$d\alpha_\lambda = \lambda \tfrac{\beta}{4} e^{-\frac{\beta}{4}t} dt + (\Im\delta(\cos\alpha_\lambda - 1) - \Re\delta \sin\alpha_\lambda)dt + 2\sin(\tfrac{\alpha_\lambda}{2})dW, \quad \alpha_\lambda(0) = 0, \quad (4.70)$$

where W is a standard real Brownian motion depending on λ.

For $\lambda = 0$ we have $\alpha_\lambda(t) = 0$. It is sufficient to show the statement for $\lambda > 0$, since $-\alpha_{-\lambda}$ solves the same SDE as α_λ with $\bar{\delta}$. From the monotonicity in λ it follows that for $\lambda > 0$ we have $\alpha_\lambda(t) > 0$ for $t > 0$ almost surely, and if $t_0 > 0$, $m \in \mathbb{Z}$ then on the event $\alpha_\lambda(t_0) > 2m\pi$ one has $\alpha_\lambda(t) > 2m\pi$ for all $t > t_0$ with probability one. (See Proposition 9 in [62] for the proof of these statements in the $\delta = 0$ case.)

Fix $\lambda > 0$, and introduce the diffusion

$$X(t) = \begin{cases} \log(\tan(\alpha_\lambda(t)/4)), & \text{if } \alpha_\lambda(t) \in [4k\pi, (4k+2)\pi), \\ -\log(-\tan(\alpha_\lambda(t)/4)), & \text{if } \alpha_\lambda(t) \in [(4k+2)\pi, (4k+4)\pi). \end{cases}$$

By Itô's formula, this diffusion satisfies the SDE

$$dX = \tfrac{\lambda\beta}{8} e^{-\beta t/4} \cosh X dt + (\Re\delta + \tfrac{1}{2}) \tanh X_i dt - \Im\delta \operatorname{sech} X dt + dW, \;\; X(0) = -\infty, \qquad (4.71)$$

with a W standard Brownian motion that is a simple transformation of the W from (4.70). Note that the diffusion might blow up to ∞ in finite time, in which case it restarts immediately from $-\infty$. To prove the convergence statement for $\frac{1}{2\pi}\alpha_\lambda(t)$ we need to show that with probability one $\lim\limits_{t\to\infty} X(t)$ exists and it is an element of $\{-\infty, \infty\}$. This can be proved with fairly straightforward coupling arguments, we will only give a sketch of the proof.

For given $t_0 > 0, x \in \mathbb{R}$ we can consider the solution of (4.71) on $[t_0, \infty)$ with $X(t_0) = x$. We denote the distribution of the process by $P_{t_0,x}$.

Denote the drift term in the SDE (4.71) by

$$R(x,t) = \tfrac{\lambda\beta}{8} e^{-\beta t/4} \cosh x + (\Re\delta + \tfrac{1}{2}) \tanh x - \Im\delta \operatorname{sech} x.$$

Note that when $|x| \le 2M$, the function $|R(x,t)|$ could be bounded from above by a constant $c = c(M, \delta, \beta, \lambda)$ that is independent of t. By coupling R with a Brownian motion with drift c, it follows that for any fixed $M > 0$ there is an $\varepsilon \in (0,1)$ so that

$$\sup_{t_0 > 0, |x| \le M} P_{t_0,x} \left(|X(t)| \le M \text{ for all } t \in [t_0, t_0 + 1] \right) \le 1 - \varepsilon.$$

Using the strong Markov property it now follows that for any $t_0 > 0$, $x \in [-M, M]$ we have

$$P_{t_0,x} \left(|X(t)| \le M \text{ for all } t \ge t_0 \right) = 0. \tag{4.72}$$

We will show that there is a positive constant c_1, so that

$$\lim_{M \to \infty} \inf_{\substack{t_0 \ge c_1 M \\ |x| \ge M}} P_{t_0,x} (\lim_{t \to \infty} X(t) \in \{-\infty, \infty\}) = 1. \tag{4.73}$$

This statement together with (4.72) implies that with probability one $\lim_{t\to\infty} X(t) \in \{-\infty, \infty\}$.

Fix $x \ge M$, $t_0 > 0$. For any fixed $0 < c_+ < \Re\delta + \frac{1}{2}$, we could choose M large so that $R(x,t) \ge c_+$ for all $x \ge M/2, t \ge 0$. Under the distribution $P_{t_0,x}$, the coupling

$$X(t) - M \ge W_{c_+}(t_0, t) := W(t) - W(t_0) + c_+(t - t_0)$$

holds on $[t_0, \sigma]$ where

$$\sigma := \inf_{t \ge t_0} \{ X(t-) = \infty \text{ or } W_{c_+}(t_0, t) \le -M/2 \}.$$

Since $c_+ > 0$, the random variable $-\inf_{t \geq t_0} W_{c_+}(t_0, t)$ is distributed as an exponential random variable with parameter $2c_+$ (see e.g. [41]). Thus,

$$P_{t_0,x}(W_{c_+}(t_0,t) > -\tfrac{M}{2}, \forall t \geq t_0) = 1 - e^{-c_+ M}.$$

Using the sublinearity of Brownian motion we get that

$$\inf_{\substack{t_0 > 0 \\ x \geq M}} P_{t_0,x}(\lim_{t\to\infty} X(t) = \infty \text{ or } X(t) \text{ blows up in finite time}) \geq 1 - e^{-c_+ M}. \tag{4.74}$$

Next we fix the constants c_-, c_2 with $0 < c_- < c_2 < \min\{\Re\delta + \frac{1}{2}, \frac{\beta}{4}\}$, and fix $t_0 \geq 2c_2^{-1} M$, $x_0 \leq -M$. The bound $R(x,t) \leq -c_-$ holds in the region

$$\mathcal{R} := \{(t,x) : -M/2 \geq x \geq -c_2 t, t \geq t_0\},$$

if M is larger than a fixed constant that only depends on λ, δ and β. Thus under P_{t_0,x_0} we can couple $X(t) - x_0$ on $[t_0, \infty)$ from above with the process

$$W_{-c_-}(t_0,t) := W(t) - W(t_0) - c_-(t - t_0),$$

on the event that $(t, -M + W_{-c_-}(t_0,t))$ stays in the region $\mathcal{R}$. Note that by our assumption $(t_0, -M + W_{-c_-}(t_0,t_0)) \in \mathcal{R}$. Note that both

$$\sup_{t \geq t_0} W_{-c_-}(t_0,t) \quad \text{and} \quad -\inf_{t\geq t_0} W_{-c_-}(t_0,t) + c_2(t - t_0)$$

are exponentially distributed, with parameters $2c_-$ and $2(c_2 - c_-)$, respectively. Hence the probability of $(t, -M + W_{-c_-}(t_0,t))$ not staying in the region $\mathcal{R}$ is exponentially small in M. Since $-M + W_{-c_-}(t_0,t)$ converges to $-\infty$ as $t \to \infty$, we get

$$\lim_{M\to\infty} \inf_{\substack{t_0 > 2c_2^{-1}M \\ x_0 \leq -M}} P_{t_0,x_0}(\lim_{t\to\infty} X(t) = -\infty) = 1. \tag{4.75}$$

From (4.74) and (4.75) we get (4.73), which implies that a.s. X converges to either ∞ or $-\infty$.

In the case $\beta \le 4(\Re\delta + \frac{1}{2})$, the $\mathtt{HP}_{\beta,\delta}$ operator is limit point at $t = 1$. In this case, for $\lambda > 0$ one can show that the limit of $X(t)$ has to be $-\infty$. This generalizes Theorem 7 of [62] which proves the statement for $\delta = 0$. The idea is that for any fixed δ with $\Re\delta + 1/2 > 0$ one can choose M large so that the term $-\Im\delta \operatorname{sech} x$ in $R(x,t)$ is negligible on the event $\{X(t) \ge M \text{ for } t \ge t_0\}$. After dropping that term, one can just mimic the proof of the $\delta = 0$ case from Theorem 7 of [62]. This proves that a.s. X converges to $-\infty$ when $\beta \le 4(\Re\delta + \frac{1}{2})$ and hence a.s. α_λ converges from above for any fixed $\lambda > 0$. $\square$

We now have all the ingredients to prove Theorem 4.19.

Proof of Theorem 4.19. The statements about the SDE (4.33) are proved in Lemma 4.40. The rest of the proof will follow along the lines of the proof of Theorem 26 in [64], where the $\delta = 0$ case is handled.

Consider the operator $\mathtt{HP}_{\beta,\delta}$ defined in Proposition 4.9. Let $v = v_\lambda = [v_1, v_2]^t$ be the solution of the differential equation $\mathtt{HP}_{\beta,\delta} v = \lambda v$ with $v(0) = [1,0]^t$. Then the ratio of the two components $r_\lambda(t) = \frac{v_1(\lambda,t)}{v_2(\lambda,t)}$ satisfies the ODE

$$r'_\lambda = \lambda \frac{\widetilde{y}^2 + (\widetilde{x} - r_\lambda)^2}{2\widetilde{y}}, \tag{4.76}$$

with initial condition $r_\lambda(0) = \infty$. Consider the hyperbolic angle $\widetilde{\alpha}_\lambda = \widetilde{\alpha}_{\lambda,\delta}$ between the points $\infty, \widetilde{x} + i\widetilde{y}, r_\lambda$, this is given by $\widetilde{\alpha}_\lambda = 2\operatorname{arccot}(\frac{\widetilde{x} - r_\lambda}{\widetilde{y}})$. More precisely, we can define a "lifted" version of this function on $\mathbb{R}$ that is continuous in λ and t, satisfies $\widetilde{\alpha}_\lambda(0) = 0$ and $\cot(\widetilde{\alpha}_\lambda/2) = \frac{\widetilde{x} - r_\lambda}{\widetilde{y}}$.

By Itô's formula, together with a change of variable $\alpha_\lambda(t) = \widetilde{\alpha}_{\lambda,\delta}(e^{-\beta t/4})$, we get the

SDE system

$$d\alpha_\lambda = \lambda \tfrac{\beta}{4} e^{-\frac{\beta}{4}t} dt + \Re[(e^{-\mathrm{i}\alpha_\lambda} - 1)(dZ - \mathrm{i}\delta dt)], \qquad \alpha_\lambda(0) = 0.$$

Let $N(\lambda)$ be the right-continuous version of the limit $\lim_{t\to\infty} \frac{1}{2\pi}\alpha_\lambda(t)$. It remains to prove that $N(\cdot)$ has the same distribution as the counting function of the spectrum of the $\mathtt{HP}_{\beta,\delta}$ operator. The proof relies on the oscillation theory of Dirac operators, see Section 4 in [64], and it can be done exactly the same way as in Theorem 26 in [64]. The only ingredients that are needed to cover the general $\Re\delta + 1/2 > 0$ case are the following: (1) the right endpoint of the $\mathtt{HP}_{\beta,\delta}$ operator is limit point if $\beta \le 4(\Re\delta + 1/2)$ and limit circle otherwise (see Proposition 31 in [64]), and (2) for $\beta \le 4(\Re\delta + 1/2)$ in the $\lambda > 0$ case $\alpha_\lambda(t)$ converges to its limit from above a.s. by Lemma 4.40. □

4.7.3 Proofs of the theorems related to $\mathtt{Bess}_{\beta,a}$

Proof of Theorem 4.21. It will be more convenient to work with a time reversed version of the operator $\mathtt{Bess}_{\beta,a}$. Let $y(u) = \exp(-\frac{\beta}{4}(2a+1)u + B(2u))$ and $\hat{y}(t) = y(u_\beta(t))$ with $u_\beta(t) = \frac{4}{\beta}\log t$. We consider the reversed Dirac operator

$$\tau^{\mathtt{B}}_{\beta,a} = \mathtt{Dir}(i\hat{y}(t), \mathfrak{u}_0, \mathfrak{u}_1), \qquad t \in (0,1],$$

where $\mathfrak{u}_0 = [1,0]^t, \mathfrak{u}_1 = [0,-1]^t$. Within this proof we use the simplified notation $\tau_{\beta,a}$ for $\tau^{\mathtt{B}}_{\beta,a}$, and denote the secular function of $\tau_{\beta,a}$ by $\zeta_{\beta,a}$. By the symmetry of $\mathtt{Bess}_{\beta,a}$, Lemmas 4.35 and 4.36, we have

$$\rho J \tau_{\beta,a} J \rho^{-1} \stackrel{d}{=} \mathtt{Bess}_{\beta,a}.$$

Hence $\tau_{\beta,a}$ is orthogonal equivalent to $\mathtt{Bess}_{\beta,a}$, its eigenvalues have the same law of the $\mathrm{Bess}_{\beta,a}$ process, and $\zeta^{\mathtt{B}}_{\beta,a}$ has the same distribution as $\zeta_{\beta,a}$.

The statement about the Taylor expansion of $\zeta_{\beta,a}$ follows from Proposition 9 in [65], which shows that the nth Taylor coefficient of $\zeta_{\beta,a}$ can be evaluated using the multiple integral

$$-\iiint\limits_{0<s_1<s_2<\cdots<s_n\le 1} \mathfrak{u}_0^t R(s_1)JR(s_2)J\cdots R(s_n)\mathfrak{u}_1 ds_1\cdots ds_n, \quad R(s)=\frac{1}{2}\begin{pmatrix}\hat{y}(s)^{-1} & 0\\ 0 & \hat{y}(s)\end{pmatrix}.$$

Noting that the multiple integral is 0 when n is odd, the statement about the Taylor expansion of $\zeta_{\beta,a}$ follows.

The SDE representation of $\zeta_{\beta,a}$ can be shown similarly as the analogue statement for $\zeta^{\mathtt{HP}}_{\beta,\delta}$. By Proposition 13 in [65], we have $\zeta_{\beta,a}(z)=[1,0]H(1,z)$, where $H:(0,1]\times\mathbb{C}\mapsto\mathbb{C}^2$ is the unique function that solves the ODE

$$J\frac{d}{dt}H(t,z)=zR(t)H(t,z), \qquad \lim_{t\to 0}H(t,z)=\mathfrak{u}_0.$$

Introduce $X_u=\begin{pmatrix}1 & 0\\ 0 & y(u)\end{pmatrix}$, $u\le 0$. Then we have $\zeta_{\beta,a}(z)=[1,0]\mathcal{H}_0(z)$ where $\mathcal{H}_u(z)=X_uH(e^{\frac{\beta}{4}u},z)$. The fact that $\mathcal{H}$ satisfies the SDE (4.38) can be checked using Itô's formula and an adaptation of the approximating scheme described in Propositions 20 and 43 in [65].

Note that the Taylor coefficients of $\zeta_{\beta,a}$ can also be expressed by differentiating the SDE (4.38) and solving the resulting system of SDEs. This gives another way to derive (4.37). $\square$

Chapter 5

Additional results related to the $\mathrm{HP}_{\beta,\delta}$ process

The diffusion description given in Theorem 4.19 allows us to study various properties of the counting function of the $\mathrm{HP}_{\beta,\delta}$ process via the SDE (4.33). For a given $\lambda \in \mathbb{R}$ the process α_λ given by (4.33) has the same distribution as the unique strong solution of

$$d\alpha_\lambda = \lambda \tfrac{\beta}{4} e^{-\frac{\beta}{4}t} dt + (\Im\delta(\cos\alpha_\lambda - 1) - \Re\delta \sin\alpha_\lambda)dt + 2\sin(\tfrac{\alpha_\lambda}{2})dW, \quad \alpha_\lambda(0) = 0. \tag{5.1}$$

Here W is a standard Brownian motion (which also depends on λ).

As an application of Theorem 4.19, one can study the asymptotics of large gap probabilities of the $\mathrm{HP}_{\beta,\delta}$ process. For $\beta > 0, \Re\delta > -1/2$ let

$$GAP_\lambda = P(\mathrm{HP}_{\beta,\delta} \cap [0,\lambda] = \emptyset), \qquad \lambda > 0, \tag{5.2}$$

be the probability of $\mathrm{HP}_{\beta,\delta}$ having no points in the interval $[0,\lambda]$. The asymptotics of GAP_λ as $\lambda \to \infty$ can be studied with a change of measure argument, by comparing α_λ to a similar diffusion which converges to 0 a.s. This approach was carried out in [63] for the Sine_β process, and the proof in [63] can be extended to cover the $\mathrm{HP}_{\beta,\delta}$ process. Our main result is the following asymptotic expansion.

Theorem 5.1. *Fix $\beta > 0$ and $\delta \in \mathbb{C}$ with $\Re\delta > -1/2$. Then as $\lambda \to \infty$ we have*

$$GAP_\lambda = (\kappa_{\beta,\delta} + o(1))\lambda^{\gamma_{\beta,\delta}} \exp\left(-\frac{\beta}{64}\lambda^2 + \left(\frac{\beta}{8} - \frac{1}{4} + \frac{1}{2}\Im\delta\right)\lambda\right),$$

where

$$\gamma_{\beta,\delta} = \frac{1}{4}\Big(\frac{\beta}{2} + \frac{2}{\beta} - 3\Big) - \Re\delta + \frac{2}{\beta}\Re\Big(\delta + \delta^2\Big).$$

A similar type of result was proved in [47] for the asymptotic gap probability of the hard edge process (spectrum of the operator $\mathfrak{G}_{\beta,a}$ given in (2.1)). For the square root of the hard edge process (which is a constant multiple of the $\mathrm{Bess}_{\beta,a}$ process, see Proposition 4.12 and Remark 4.13), Holcomb [26] proved a similar stochastic differential equation description of its counting function, building on the results of [48]. Let $M_{a,\beta}(\lambda)$ be the counting function of the $\mathrm{Bess}_{\beta,a}$ process and B a standard Brownian motion. Then, by Theorem 1.4 of [26], the function $\lambda \to M_{a,\beta}(\lambda)$ has the same distribution as the right continuous version of the function $\lambda \to \lim_{t\to\infty} \lfloor \frac{1}{4\pi}\varphi_{a,\lambda}(t) \rfloor$, where $\varphi_{a,\lambda}$ solves the SDE

$$d\varphi_{a,\lambda} = \frac{\beta}{2}\Big(a + \frac{1}{2}\Big)\sin\Big(\frac{\varphi_{a,\lambda}}{2}\Big)dt + \lambda\frac{\beta}{4}e^{-\beta t/8}dt + \frac{\varphi_{a,\lambda}}{2}dt + 2\sin\Big(\frac{\varphi_{a,\lambda}}{2}\Big)dB, \tag{5.3}$$

with initial conditions $\varphi_{a,\lambda}(0) = 2\pi$. By analyzing the coupled system of SDE (5.3), Holcomb [26] also proved various properties (for example a transition to Sine_β process and a Central Limit Theorem) for the square root of the hard edge process. Using the techniques introduced in [26], we get similar results for the $\mathrm{HP}_{\beta,\delta}$ process.

Theorem 5.2. *Fix $\beta > 0$ and $\delta \in \mathbb{C}$ with $\Re\delta > -1/2$. Then as $\lambda \to \infty$, we have*

$$(\mathrm{HP}_{\beta,\delta} - \lambda) \Rightarrow \mathrm{Sine}_\beta\,.$$

Let $N(\cdot)$ be the counting function of the $\mathrm{HP}_{\beta,\delta}$ process, as $\lambda \to \infty$ we have

$$\frac{1}{\sqrt{\log\lambda}}\Big(N(\lambda) - \frac{\lambda}{2\pi}\Big) \Rightarrow \mathcal{N}\Big(0, \frac{2}{\beta\pi^2}\Big),$$

where $\mathcal{N}(\mu,\sigma^2)$ is the mean μ, variance σ^2 normal distribution.

Section 5.1 provides the proof of Theorem 5.1, the asymptotics of the gap probability. In Section 5.2, we will prove Theorem 5.2 as two separate propositions.

5.1 Large gap probability

Fix $\beta > 0, \delta \in \mathbb{C}$ with $\Re\delta > -1/2$, the gap probability for $\lambda > 0$ is defined as in (5.2). Using the counting function description of the $\mathrm{HP}_{\beta,\delta}$ process proved in Theorem 4.19, we could rewrite the gap probability as

$$GAP_\lambda = P\Big(\lim_{t\to\infty} \alpha_\lambda(t) = 0\Big), \tag{5.4}$$

where α_λ is the unique strong solution of (5.1).

To analyze the diffusion α_λ, it is more convenient to remove the space dependence from the diffusion coefficient. Recall also the change of variable used in the proof of Theorem 4.19,

$$X(t) = X_\lambda(t) = \begin{cases} \log(\tan(\alpha_\lambda(t)/4)), & \text{if } \alpha_\lambda(t) \in [4k\pi, (4k+2)\pi), \\ -\log(-\tan(\alpha_\lambda(t)/4)), & \text{if } \alpha_\lambda(t) \in [(4k+2)\pi, (4k+4)\pi). \end{cases} \tag{5.5}$$

By Itô's formula, this diffusion satisfies the SDE

$$dX = \tfrac{\lambda\beta}{8} e^{-\frac{\beta}{4}t} \cosh X dt + (\Re\delta + \tfrac{1}{2}) \tanh X dt - \Im\delta \operatorname{sech} X dt + dW, \;\; X(0) = -\infty, \tag{5.6}$$

with a W standard Brownian motion depending on λ. Note that the diffusion might blow up to ∞ in finite time, in which case it restarts immediately from $-\infty$. By analyzing the drift term in (5.1) when α_λ crosses $2\pi\mathbb{Z}$, we get the function $\lfloor \frac{\alpha_\lambda(t)}{2\pi} \rfloor$ is non-decreasing in t. Together with Theorem 4.19, we see that the event $\{\lim_{t\to\infty} \alpha_\lambda(t) = 0\}$ implies $X(t) < \infty$ for all time $t \geq 0$. This shows that

$$GAP_\lambda = P\left(\{X(t) < \infty, \forall t < \infty\} \cap \{\lim_{t\to\infty} X(t) = -\infty\}\right).$$

It would be useful later on to consider the diffusion (5.6) with a general initial condition $X(0) = x \in [-\infty, \infty)$. Introduce the passage probability

$$p_\lambda(x) := P_x \left(X(t) \text{ is finite for all } t \geq 0 \text{ and does not converge to } \infty \text{ as } t \to \infty \right),$$

where P_x denotes the distribution of $X(t)$ under the condition $X(0) = x$. We drop the initial condition when $x = -\infty$ and write $p_\lambda = p_\lambda(-\infty) = GAP_\lambda$ for all fixed $\lambda > 0$. The proof of Theorem 5.1 will be postponed to the end of this section.

Remark 5.3. *In this work, we will describe the constant term $\kappa_{\beta,\delta}$ as the expectation of a functional of a certain diffusion, but we do not attempt to identify the exact value of $\kappa_{\beta,\delta}$. In the case when $\delta = 0$, the constant term was known for $\beta = 1, 2, 4$, we refer to [63] and the references therein for more details.*

First observe that a time shift of equation (5.6) only changes the parameter λ and the initial condition. More precisely, the process $\hat{\alpha}_\lambda(t) := \alpha_\lambda(t + T)$ where $T = \frac{4}{\beta} \log \lambda$ satisfies (5.6) with $\lambda = 1$ and a random initial condition $\hat{\alpha}_\lambda(0) = \alpha_\lambda(T)$. Together with the Markov property of $X(t)$ we get

$$p_\lambda(x) = \mathbb{E}_x \left[\mathbf{1}(X(t) \text{ is finite for all } t \leq T) p_1(X(T)) \right],$$

where $\mathbb{E}_x$ denotes the expectation under the distribution P_x. Following the work of Valkó and Virág [63] on the large gap probability of the Sine_β process (note that $\text{Sine}_\beta = \text{HP}_{\beta,0}$), the idea is to find a new diffusion Y which approximates the conditional distribution of the diffusion X under the event that it does not blow up. Our main tool is the following version of the Cameron-Martin-Girsanov formula.

Proposition 5.4 ([63]). *Let $B, \widetilde{B}$ be standard real Brownian motions. Consider the stochastic differential equations*

$$dX = g(t,X)dt + dB, \qquad \lim_{t\to 0} X(t) = \infty, \tag{5.7}$$

$$dY = h(t,Y)dt + d\widetilde{B}, \qquad \lim_{t\to 0} Y(t) = \infty \tag{5.8}$$

on the interval $(0,T]$. Assume that the equation (5.7) has a unique solution $X : (0,T] \mapsto (-\infty,\infty]$. Let

$$G_s = G_s(X) = \int_0^s (h(t,X) - g(t,X))dX - \frac{1}{2}\int_0^s (h(t,X)^2 - g(t,X)^2)dt, \tag{5.9}$$

and assume that

(a). $g^2 - h^2$ and $g - h$ are bounded when x is bounded from above.

(b). G_s is bounded from above by a deterministic constant.

(c). If X hits $+\infty$ at time τ then $G_s \to -\infty$ when $s \uparrow \tau$. In this case, we define $G_s := -\infty$ for $s \geq \tau$.

Consider the process $\widetilde{Y}$ whose density with respect to the distribution of X is given by e^{G_T}. then $\widetilde{Y}$ satisfies the SDE (5.8) and never blows up to ∞ almost surely. Moreover, for any nonnegative function ϕ of the path of X that vanishes when X blows up we have

$$\mathbb{E}[\phi(X)] = \mathbb{E}[\phi(Y)e^{-G_T(Y)}].$$

From now on, we focus on the construction of the Y diffusion. Precisely, we will construct an a.s. finite diffusion Y which solves

$$dY = h(t,Y)dt + dB_t, \qquad Y_0 = -\infty, \tag{5.10}$$

and such that the Radon-Nikodym derivative e^{G_T} with G_T defined in (5.9) is close to the asymptotic expansion of p_λ with the desired logarithmic correction exponent $\gamma_{\beta,\delta}$.

Lemma 5.5 (Construction of Y)**.** *Consider the solution of (5.6) and set $T = \frac{4}{\beta}\log\lambda$. There exists a function $h(t,x)$ so that assumptions a-c of Proposition 5.4 hold, and G_T satisfies*

$$-G_T(X) = -\frac{\beta}{64}\lambda^2 + \Big(\frac{\beta}{8} - \frac{1}{4} + \frac{1}{2}\Im\delta\Big)\lambda + \bigg(\frac{1}{4}\Big(\frac{\beta}{2} + \frac{2}{\beta} - 3\Big) - \Re\delta + \frac{2}{\beta}\Re(\delta + \delta^2)\bigg)\log\lambda$$
$$+ \frac{\beta}{8}e^{X(T)} + \Big(2 - \frac{\beta}{2} + 2\Re\delta\Big)X(T)^+ + \omega(X(T)) + \int_0^T \phi(T-t, X(t))dt.$$

Here the function ω is bounded and continuous, ϕ is continuous and bounded by a function $\widetilde{\phi}(t)$ which has a finite integral on $[0,\infty)$.

Moreover, the function $h(t,x)$ has the following form

$$h(t,x) = -\frac{\lambda}{2}f(t)\sinh(x) + h_0(t,x), \tag{5.11}$$

where $f(t) = \frac{\beta}{4}e^{-\frac{\beta}{4}t}$ and $|h_0(t,x)| < c$ for all $t \in [0,T]$.

Proof. Following the work of Valkó-Virág [63], we would like to present the process of how one can find the appropriate drift function, rather than check directly that G_T satisfies the statement under the choice of the given drift function $h(t,x)$.

Recall that the diffusion X satisfies the SDE (5.6). We further decompose the drift term of (5.6) as $g = g_1 + g_2 + g_3$, where

$$g_1(t,x) = \frac{\lambda}{2}f\cosh(x), \qquad g_2(x) = (\Re\delta + \frac{1}{2})\tanh x, \qquad g_3(x) = -\Im\delta\,\mathrm{sech}\,x.$$

Our goal is to find h such that the diffusion Y will approximate the conditional distribution of X under the event that it does not blow up in the interval $[0,T]$. If such an approximation exists, by Proposition 5.4 we have

$$-G_s(X) = \int_0^s (g(t,X) - h(t,X))dX + \frac{1}{2}\int_0^s (h^2(t,X) - g^2(t,X))dt.$$

We will start with the highest order. To that end, we write $h = h_1 + h_2 + h_3 + h_4$. Define $h_1(t,x) = -\frac{\lambda}{2} f \sinh x$, then we have

$$\frac{1}{2}\int_0^s h_1^2 - g_1^2 dt = -\frac{\lambda^2}{8}\int_0^s f(t)^2 dt,$$

which will give the leading term in the asymptotic. On the other hand, the stochastic integral of g_1, h_1 with respect to dX is given by

$$\begin{aligned}\int_0^s (g_1(t,X) - h_1(t,X))dX &= \frac{\lambda}{2}\int_0^s e^X f(t) dX \\ &= \frac{\lambda}{2} f(s) e^{X(s)} + \frac{\lambda}{2}\Big(\frac{\beta}{4} - \frac{1}{2}\Big)\int_0^s e^X f(t)dt,\end{aligned}$$

where in the second equality we have used $f'(t) = -\frac{\beta}{4} f(t)$ and the following version of Itô's formula

$$a(t)b'(X)dX = d(a(t)b(X)) - a'(t)b(X)dt - \frac{1}{2}a(t)b''(X)dt.$$

Next, we choose $h_2 = (\frac{\beta}{4} - \frac{1}{2})(1 + \tanh(x/2))$ so that

$$\int_0^s h_1 h_2 dt = \frac{\lambda}{2}\Big(\frac{\beta}{4} - \frac{1}{2}\Big)\int_0^s (1 - e^X) f(t) dt,$$

and gives the cancellation

$$\int_0^s (g_1 - h_1)dX + \int_0^s h_1 h_2 dt = \frac{\lambda}{2} f(s) e^{X(s)} + \frac{\lambda}{2}(\frac{\beta}{4} - \frac{1}{2})\int_0^s f(t)dt.$$

Together with

$$-\int_0^s g_1 g_3 dt = \Im\delta\frac{\lambda}{2}\int_0^s f(t)dt,$$

we will get the coefficient of the linear term in the asymptotic expansion.

We will choose the next term h_3 so that the cross term $\int h_1 h_3 dt$ cancels the cross term $-\int g_1 g_2 dt$. In particular, with $h_3(t,x) = -(\Re\delta + \frac{1}{2})$ we have

$$h_1 h_3 = g_1 g_2 = \frac{\lambda}{2}\left(\Re\delta + \frac{1}{2}\right)\sinh(x),$$

which implies

$$\int_0^s (h_1 h_3 - g_1 g_2) dt = 0.$$

Now consider the stochastic integral $\int_0^s u(X)dX$ where

$$\begin{aligned} u(x) &= g_2 + g_3 - h_2 - h_3 \\ &= 1 - \frac{\beta}{4} + \Re\delta + \left(\Re\delta + \frac{1}{2}\right)\tanh x - \Im\delta \operatorname{sech} x - \left(\frac{\beta}{4} - \frac{1}{2}\right)\tanh(x/2). \end{aligned}$$

Denote by

$$\begin{aligned} \widetilde{u}(x) = &\left(1 - \frac{\beta}{4} + \Re\delta\right)x + \left(\Re\delta + \frac{1}{2}\right)\log\cosh x \\ &- 2\Im\delta \arctan\big(\tanh(x/2)\big) + \left(1 - \frac{\beta}{2}\right)\log\cosh(x/2) \end{aligned} \tag{5.12}$$

the anti-derivative of $u(x)$. Note that $\lim_{t\to 0}\widetilde{u}(X(t)) = c_1$ is well defined. By Itô's formula, $\int_0^s u(X)dX - \widetilde{u}(X(s)) + \widetilde{u}(X(0))$ is given by

$$\begin{aligned} &-\frac{1}{2}\int_0^s u'(X)dt \\ &= -\frac{1}{2}\int_0^s \left(\left(\Re\delta + \frac{1}{2}\right)\operatorname{sech}^2(X) + \Im\delta \operatorname{sech} X \tanh X - \frac{\beta - 2}{8}\operatorname{sech}^2(X/2)\right)dt \\ &= \left(\frac{\beta - 6}{16} - \frac{1}{2}\Re\delta\right)s \\ &\quad + \frac{1}{2}\int_0^s \left(\left(\Re\delta + \frac{1}{2}\right)\tanh^2(X) + \frac{2-\beta}{8}\tanh^2(X/2) - \Im\delta \operatorname{sech} X \tanh X\right)dt. \end{aligned} \tag{5.13}$$

Now we evaluate $\frac{1}{2}\int_0^s (h_2+h_3)^2 - (g_2+g_3)^2 dt$, we get

$$
\begin{aligned}
\frac{1}{2}\big((h_2+h_3)^2 - (g_2+g_3)^2\big) = \frac{1}{2}&\left(\left(\frac{\beta}{4} - \Re\delta - 1\right)^2 - (\Im\delta)^2\right) \\
&+ \frac{(\beta - 4\Re\delta - 4)(\beta - 2)}{16}\tanh(x/2) \\
&+ \frac{\beta^2 - 4\beta + 4}{32}\tanh^2(x/2) \\
&+ \frac{1}{2}\left((\Im\delta)^2 - \left(\Re\delta + \frac{1}{2}\right)^2\right)\tanh^2(x) \\
&+ \Im\delta\left(\Re\delta + \frac{1}{2}\right)\tanh x \operatorname{sech} x
\end{aligned}
\tag{5.14}
$$

Collecting this computation, and expanding $(h_2+h_3+h_4)^2$ in the integral of $\int h^2(t)dt$ we get

$$
\begin{aligned}
-G_s(X) = &-\frac{\lambda^2}{8}\int_0^s f^2(t) \\
&+ \frac{\lambda}{2}\Big(\frac{\beta}{4} - \frac{1}{2} + \Im\delta\Big)\int_0^s f(t)dt \\
&+ \left(\frac{\beta^2 - 6\beta + 4}{32} - \frac{\beta}{4}\Re\delta + \frac{(\Re\delta)^2 - (\Im\delta)^2 + \Re\delta}{2}\right)s \\
&+ \frac{\lambda}{2}f(s)e^{X(s)} + \widetilde{u}(X(s)) - c_1 \\
&+ \int_0^s h_1 h_4 dt + \frac{1}{2}h_4^2 + h_4(h_2+h_3)dt \\
&- \int_0^s h_4 dX + \int_0^s \eta(X)dt,
\end{aligned}
\tag{5.15}
$$

where

$$
\begin{aligned}
\eta(x) = &\frac{(\beta - 4(1+\Re\delta))(\beta-2)}{16}\tanh(x/2) + \frac{\beta^2 - 6\beta + 8}{32}\tanh^2(x/2) \\
&+ \frac{4(\Im\delta)^2 - 4(\Re\delta)^2 + 1}{8}\tanh^2(x) + \Im\delta\Re\delta\tanh x \operatorname{sech} x.
\end{aligned}
$$

Here the coefficient of s in (5.15) comes from the linear term on the right hand side of (5.13), and the constant term of (5.14). The function $\eta(x)$ collects the rest terms from (5.13), (5.14) and $-g_2^2/2$. The function $\eta(x)$ contributes to an uniformly bounded error term. Note that the function $\eta(x)/\sinh(x)$ is bounded by a constant. We define $h_4(t,x) = -\eta(x)/h_1(t,x)$ so that

$$\int_0^s h_1(t,X)h_4(t,X)dt + \int_0^s \eta(X(t))dt = 0.$$

Let $\widetilde{h}_4(t,x) = \int_0^x h_4(t,y)dy$, we obtain that

$$-\int_0^s h_4(t,X)dX = -\widetilde{h}_4(s,X(s)) + \frac{\beta}{4}\int_0^s \widetilde{h}_4(t,X(t))dt + \frac{1}{2}\int_0^s \partial_x h_4(t,X(t))dt.$$

Substituting this into (5.15), we end up with

$$\begin{aligned}
-G_s(X) = &-\frac{\lambda^2}{8}\int_0^s f^2(t)dt + \frac{\lambda}{2}\Big(\frac{\beta}{4} - \frac{1}{2} + \Im\delta\Big)\int_0^s f(t)dt \\
&+ \left(\frac{\beta^2 - 6\beta + 4}{32} - \frac{\beta}{4}\Re\delta + \frac{(\Re\delta)^2 - (\Im\delta)^2 + \Re\delta}{2}\right)s \\
&+ \frac{\lambda}{2}f(s)e^{X(s)} + \widetilde{u}(X(s)) - c_1 - \widetilde{h}_4(s,X(s)) \\
&+ \int_0^s \frac{1}{2}h_4^2 + h_4(h_2+h_3) + \frac{\beta}{4}\widetilde{h}_4 + \frac{1}{2}\partial_x h_4 dt.
\end{aligned} \tag{5.16}$$

Since h_2, h_3 are uniformly bounded and only depend on x, the functions $h_4, \widetilde{h}_4, \partial_x h_4$ are all bounded by a constant times $1/(\lambda f(t)) = \frac{16}{\beta}f(T-t) \le c$ for $0 \le t \le T$, we can rewrite the integrand in (5.16) as

$$\int_0^s \phi(T-t, X(t))dt$$

where ϕ is continuous, independent of λ, and satisfies $|\phi(t,x)| \le \varphi(t)$ with $\int_0^\infty \varphi(t)dt < \infty$. Using (5.12) and the fact that $(\log\cosh x - |x|)$ is bounded, the terms in the third line of (5.16) can be written as

$$\frac{\lambda}{2}f(s)e^{X(s)} + \Big(2 - \frac{\beta}{2} + 2\Re\delta\Big)X(s)^+ + \omega_0(X(s)) - \widetilde{h}_4(s,X(s))$$

with a bounded and continuous ω_0. Plug $s = T = \frac{4}{\beta}\log\lambda$ into (5.16), then the first two lines give

$$-\frac{\lambda^2\beta}{64}(1-\lambda^{-2}) + \frac{\lambda}{2}\Big(\frac{\beta}{4} - \frac{1}{2} + \Im\delta\Big)(1-\lambda^{-1}) \\ + \left(\frac{\beta^2 - 6\beta + 4}{8\beta} - \Re\delta + \frac{2}{\beta}(\Re\delta^2 + \Re\delta)\right)\log\lambda$$

and the third line gives

$$\frac{\beta}{8}e^{X(T)} + \Big(2 - \frac{\beta}{2} + 2\Re\delta\Big)X(T)^+ + \omega_0(X(T)) - \widetilde{h}_4(T, X(T)).$$

This proves that $-G_T$ has the desired form in the statement of Lemma 5.5 . It remains to check that h satisfies assumptions $(a)-(c)$ of Proposition 5.4.

For assumption (a), we note that as $x \to -\infty$ we have $g(t,x) = \frac{1}{4}\lambda f(t)e^{-x} - (\Re\delta + \frac{1}{2}) + \hat{g}(t,x)$, and $h(t,x) = \frac{1}{4}\lambda f(t)e^{-x} - (\Re\delta + \frac{1}{2}) + \hat{h}(t,x)$, where $\max\{|\hat{g}|, |\hat{h}|\} \le ce^x$ with constant that only depends on β if $0 \le t \le T$. This proves that both $g - h$ and $g^2 - h^2$ are bounded if x is bounded from above. For assumption (b), comparing with the construction of h in [63], the only difference is that we need to show

$$\Big(2 - \frac{\beta}{2} + 2\Re\delta\Big)X(s)^+ + \frac{\lambda}{2}f(t)e^{X(s)}$$

is bounded from below. Since $s \le \frac{4}{\beta}\log\lambda$, we have $(2 - \frac{\beta}{2} + 2\Re\delta)X(s)^+ + \frac{\lambda}{2}f(t)e^{X(s)} \le (2 - \frac{\beta}{2} + 2\Re\delta)X(s)^+ + \frac{\beta}{8}e^{X(s)}$, which is bounded below by a constant depending only on β. Lastly for assumption (c), we have $(2 - \frac{\beta}{2} + 2\Re\delta)X(s)^+ + \frac{\lambda}{2}f(t)e^{X(s)}$ converges to ∞ as s converges to the hitting time of ∞, which implies $G_s \to -\infty$ as desired. □

We also need the following preliminary estimation of the gap probability.

Proposition 5.6. *Consider the solution of* (5.6) *with* $\lambda = 1$ *and* $X(0) = x$. *Recall that* $p_1(x)$ *denotes the probability that* X *does not blow up in finite time and does not* *converge to* ∞ *as* $t \to \infty$*. Then we have*

$$0 < p_1(x) < c_\beta e^{-\frac{\beta}{60}e^x}.$$

Proof. The proof is a minor modification of the proof of Lemma 5 in [63]. It is enough to consider the case when x is large since otherwise the upper bound for x can be obtained by making the constant $c_{\beta,\delta}$ large. For fixed $\beta > 0$ and $\delta \in \mathbb{C}$ with $\Re\delta > -1/2$, we define $M = M_\delta = \sinh^{-1}(|\Im\delta|/(\Re\delta + \frac{1}{2}))$. Then for $x > M + 4$, we have

$$g(x,t) := \frac{\beta}{8}e^{-\frac{\beta}{4}t}\cosh x + (\Re\delta + \frac{1}{2})\tanh x - \Im\delta \operatorname{sech} x > \frac{\beta}{16}e^{x-\frac{\beta}{4}t}.$$

Consider the diffusion

$$dR = \frac{\beta}{16}e^{R-\frac{\beta}{4}t}dt + dB, \qquad R(0) = x$$

which has the same noise term as X. Then we have $R \leq X$ in this coupling while $R \geq M$. This implies that for every $t \geq 0$ we have

$$p_1(x) \leq P\left(\min_{0\leq s\leq t} R(s) < M \text{ or } R \text{ does not blow up before time t}\right).$$

The difference $U = R - B$ satisfies the ODE

$$e^{-U}dU = \frac{\beta}{16}e^{B-\frac{\beta}{4}t}dt, \qquad U(0) = x,$$

which gives

$$e^{-x} - e^{-U(t)} = \frac{\beta}{16}\int_0^t e^{B(s)-\frac{\beta}{4}s}ds.$$

This implies $U(t) \geq x$, and the event

$$\{\min_{0\leq s\leq t} R(s) < M\} \subset \{\min_{0\leq s\leq t} B(s) < M - x\}.$$

Moreover, if $\min_{0\le s\le t} B(s)$ is not sufficiently small (for example $\min_{0\le s\le t} B_s > -b$ such that $e^{-b}(1-e^{-\beta t/4}) > 4e^{-x}$), then we have

$$e^{-x} < \frac{\beta}{16}\int_0^t e^{B(s)-\frac{\beta}{4}s}ds.$$

This shows that U blows up before time t, and if in particular $b < x - M$ also holds, then

$$\begin{aligned} P\left(\min_{0\le s\le t} R(s) < M \text{ or } R \text{ does not blow up before time t}\right) &\le P(\min_{0\le s\le t} B_s < -b) \\ &\le \frac{\sqrt{t}}{b}e^{-\frac{b^2}{2t}}. \end{aligned}$$

Set $t = 16e^{2-x}/\beta, b = 4e/\sqrt{30}$, then both $b < x$ and $\frac{e^{-b}}{4}(1-e^{-\beta t/4}) > e^{-x}$ are satisfied. This gives the upper bound

$$p_1(x) \le \frac{\sqrt{t}}{b}e^{-\frac{b^2}{2t}} < c_\beta e^{-\frac{\beta}{60}e^x}.$$

For the lower bound, (4.75) shows that $p_1(x) > 0$ for $x < -M$, where M is a fixed constant that only depends on β, δ. On the other hand, if $X(0) = x > 0$, we set $c := \frac{\beta}{8}e^{2x} + \Re\delta + \frac{1}{2} + |\Im\delta|$,and then we couple the process X with $\widetilde{B}(t) := B(t) + ct + x$, the Brownian motion with drift c starting at x. Note that $|g(y,t)| \le c$ for all $|y| \le 2x$. The coupling $X(t) \le \widetilde{B}(t)$ holds when $|\widetilde{B}(t)| \le 2x$. Since

$$P\left(\max_{0\le s\le 1} \widetilde{B}(s) < 2x, \min_{0\le s\le 1} \widetilde{B}(s) < -2x\right) > 0,$$

there exists a positive constant c_1 so that $p_1(x) > c_1 p_1(-x)$. This proves that $p_1(x) > 0$ for $x > M$. Using the monotonicity of $p_1(x)$ in x completes the proof. □

We are now ready to prove Theorem 5.1.

Proof of Theorem 5.1. Set $T = \frac{4}{\beta}\log\lambda$, the time-shifted diffusion $t \mapsto X(t+T)$ satisfies (5.6) with $\lambda = 1$ with initial condition $X(-T) = -\infty$. Using the Markov property of X, we get that

$$p_\lambda = \mathbb{E}\left[\mathbf{1}(X(t) \text{ is finite for all } 0 \le t \le T) p_1(X(T))\right].$$

Consider the diffusion Y satisfying the SDE (5.10) with the drift function $h(t,x)$ constructed in Lemma 5.5. By Proposition 5.4 and Lemma 5.5, we have

$$p_\lambda = \lambda^{\gamma_{\beta,\delta}} \exp\Big(-\frac{\beta}{64}\lambda^2 + \big(\frac{\beta}{8} - \frac{1}{4} + \frac{1}{2}\Im\delta\big)\lambda\Big)\mathbb{E}\left[p_1(Y(T))\exp\{\psi(Y)\}\right],$$

where

$$\gamma_{\beta,\delta} = \frac{1}{4}\Big(\frac{\beta}{2} + \frac{2}{\beta} - 3\Big) - \Re\delta + \frac{2}{\beta}\Big(\delta^2 + \Re\delta\Big).$$

and where

$$\psi(Y) = \Big(2 - \frac{\beta}{2} + 2\Re\delta\Big)Y(T)^+ + \frac{\beta}{8}e^{Y(T)} + \omega(Y(T)) + \int_0^T \phi(T-t, Y(t))dt.$$

It suffices to show that the limit $\lim_{\lambda\to\infty}\mathbb{E}[p_1(Y(T))\exp\{\psi(Y)\}]$ exists, and is finite and positive. This limit would equal the constant $\kappa_{\beta,\delta}$ of the asymptotics, but we would not attempt to find the exact value, see Remark 5.3.

Following the proof of Theorem 1 in [63], the idea is to run the process $Y_\lambda(t)$ with a shifted time $\tau = t - T$ and consider $\widetilde{Y}_T(\tau) := Y_\lambda(\tau + T)$. Now the diffusions $\widetilde{Y}_T(\lambda)$ for different λ satisfy the same SDE on nested time intervals

$$d\widetilde{Y}_T(\tau) = \widetilde{h}(\tau, \widetilde{Y})d\tau + dB, \qquad \tau > -T, \qquad \widetilde{Y}_T(-T) = -\infty,$$

where the drift term is given by

$$\widetilde{h}(\tau, y) = h(T+\tau, y) = -\frac{\beta}{8}e^{-\frac{\beta}{4}\tau}\sinh(y) + h_0(T+\tau, y).$$

Here the processes $\widetilde{Y}_T$ are driven by the same Brownian motion, and h_0 is the bounded function constructed in (5.11). Then for $T_1 > T_2$ we have $\widetilde{Y}_{T_1}(\tau) > \widetilde{Y}_{T_2}(\tau)$ for $\tau \geq -T_2$ and the domination is preserved. It now suffices to prove

$$\lim_{T\to\infty} \mathbb{E}\left[p_1(\widetilde{Y}_T(0)) \exp\{\widetilde{\psi}(\widetilde{Y}_T)\}\right] \tag{5.17}$$

is positive and finite, where

$$\widetilde{\psi}(\widetilde{Y}) = \left(2 - \frac{\beta}{2} + 2\Re\delta\right)\widetilde{Y}(0)^+ + \frac{\beta}{8}e^{\widetilde{Y}(0)} + \omega(\widetilde{Y}(0)) + \int_0^T \phi(t, \widetilde{Y}(-t))dt. \tag{5.18}$$

Consider a nonnegative diffusion $Z(t)$ satisfying the SDE

$$dZ = r(Z)dt + dB$$

which is reflected at 0 and the drift term

$$r(y) = -\frac{\beta}{16}e^y + c_1, \tag{5.19}$$

where the constant c_1 is chosen so that

$$r(z) \geq \sup_{\tau<0, 0\leq y\leq z} h(\tau, y).$$

We will use the stationary version of Z to control the diffusion $\widetilde{Y}_T$. Since Z and $\widetilde{Y}$ are driven by the same Brownian motion, then if $Z, \widetilde{Y} > 0$ we have

$$d(Z - \widetilde{Y}) = (r(Z) - f(t, Y))dt.$$

This implies that if $Z(\tau_0) \geq \widetilde{Y}(\tau_0)$ for a negative time τ_0, then this ordering is preserved until time 0.

Consider the process Z in its stationary distribution, since $Z(-T) > \widetilde{Y}_T(-T)$ we have $Z > \widetilde{Y}_T$ on $[-T, 0]$. Since $\widetilde{Y}_T(\tau)$ is increasing in T and bounded by $Z(\tau)$ we have

$\widetilde{Y}_\infty(\tau) = \lim_{T\to\infty} \widetilde{Y}_T(\tau)$ exists and is dominated by $Z(\tau)$. By (5.18) we have

$$\widetilde{\psi}(\widetilde{Y}_T) = \omega(\widetilde{Y}_T(0)) + \int_0^T \phi(t, \widetilde{Y}(-t))dt,$$

where ω is continuous and $\phi(t,y) \le \widetilde{\phi}(y)$ such that $\int_0^\infty \widetilde{\phi}(t)dt < \infty$ (see Lemma 5.5 for the construction). This implies that $\widetilde{\psi}(\widetilde{Y}_T) \to \widetilde{\psi}(\widetilde{Y}_\infty)$ and

$$q_T := \exp\{\widetilde{\psi}(\widetilde{Y}_T)\}p_1(\widetilde{Y}_T) \to q_\infty := \exp\{\widetilde{\psi}(\widetilde{Y}_\infty)\}p_1(\widetilde{Y}_\infty)$$

as $T \to \infty$. By Proposition 5.6, we have

$$q_T \le c\exp\left\{\left(2 - \frac{\beta}{2} + 2\Re\delta\right)\widetilde{Y}_T(0)^+ + \frac{\beta}{8}e^{\widetilde{Y}_T(0)} - \frac{\beta}{60}e^{\widetilde{Y}_T(0)}\right\} \le c'\chi(\widetilde{Y}_T(0)),$$

with $\chi(y) = \exp\{(\frac{\beta}{8} - \frac{\beta}{61}e^y)\}$. Note that the stationary density of Z is given by

$$g(z) = c\exp\left(-\frac{\beta}{8}e^z + 2c_1 z\right),$$

see e.g. Chapter VII of [53]. This implies that

$$\mathbb{E}[\chi(Z(0))] \le \int_0^\infty \chi(z)g(z)dz < \infty,$$

hence the dominated convergence theorem gives that

$$\mathbb{E}[q_T] \to \mathbb{E}[q_\infty] < \infty.$$

This proves the finiteness (and existence) of the limit (5.17). Finally, we have

$$q_\infty \ge cp_1(\widetilde{Y}_\infty(0))\exp\left\{\left(2 - \frac{\beta}{2} + 2\Re\delta\right)Y_\infty(0)^+\right\}.$$

The positivity of the limit (5.17) follows from $\widetilde{Y}_\infty(0) < \infty$ a.s. and the positivity of $p_1(\cdot)$ proved in Proposition 5.6. This completes the proof. $\square$

5.2 Transition and a CLT

The goal of this section is to prove Theorem 5.2, which states the transition from the $\mathrm{HP}_{\beta,\delta}$ process to the Sine_β process and a Central Limit Theorem of the counting function of the $\mathrm{HP}_{\beta,\delta}$ process. This will be the content of Propositions 5.10 and 5.12 below.

By Theorem 4.19, the counting function $N(\cdot) = N_{\beta,\delta}(\cdot)$ of the $\mathrm{HP}_{\beta,\delta}$ process has the same distribution as the right-continuous version of the function $\lambda \mapsto \lim_{t\to\infty} \frac{1}{2\pi}\alpha_\lambda(t)$, where the processes $(\alpha_\lambda(t), \lambda \in \mathbb{R})$ solves the coupled system of SDE (4.33). For a fixed λ the process α_λ satisfies the SDE (5.1) where W is a standard Brownian motion depending on λ. For large $\lambda > 0$, observing that the process α_λ will be rapidly increasing until time on the order of $\log\lambda$ since $\lambda\frac{\beta}{4}e^{-\frac{\beta}{4}t}$ would be the dominating term in (5.1) on this time regime. Moreover, the trigonometric terms of α_λ would be rapidly oscillating on this regime and hence vanish in the $\lambda \to \infty$ limit.

Following the work of Holcomb [26] on similar results for the $\mathrm{Bess}_{\beta,a}$ process, the key estimation for proving Propositions 5.10 and 5.12 is the following control of the oscillatory integrals.

Proposition 5.7 ([28], [26]). *For each $\lambda \in \mathbb{R}$, suppose that $A_{\lambda,t}$ is an adapted finite variation process so that $|A_{\lambda,t}| \le \xi$ is uniformly bounded for all t a.s., and suppose that $X_{\lambda,t}$ is a martingale satisfying $d[X_\lambda]_t \le 2$. Let $u_{\lambda,t}$ be the process satisfying*

$$du_{\lambda,t} = \lambda f(t)dt + A_{\lambda,t}dt + dX_{\lambda,t} \qquad u_{\lambda,0} = 0,$$

where $f(t) = f_\beta(t) = \frac{\beta}{4}e^{-\frac{\beta}{4}t}$.

Then for each fixed $\beta > 0$, there exists constants R and η uniform in T and $\lambda, a \in \mathbb{R}$

such that

$$\mathbb{E}\left[\sup_{0\le t\le T}\left|\int_0^t e^{iau_{\lambda,s}}ds\right|\right]\le \frac{R(1+C)}{|a\lambda|f(T)},$$

and for all $C>0$,

$$P\left(\sup_{0\le t\le T}\left|\int_0^t e^{iau_{\lambda,s}}ds\right|-\frac{R(1+\xi)}{|a\lambda|f(T)}\ge C\right)\le \exp\left(-\eta(Ca\lambda f(T))^2\right).$$

We also need the following standard results on characterization of Brownian motions and weak convergence of diffusions.

Proposition 5.8 (Theorem 7.1.4 of [20])**.** *Let $\{M^{(n)}\}$ be a sequence of $\mathbb{R}^d$-valued martingales. Suppose that*

$$\lim_{n\to\infty}\mathbb{E}\Big[\sup_{s\le t}|M^{(n)}(s)-M^{(n)}(s-)|\Big]=0,$$

and the quadratic variation $[M_i^{(n)},M_j^{(n)}]:=C_{ij}^{(n)}\to c_{i,j}(t)$ in probability for all $t\ge 0$, where $M_i^{(n)}$ denotes the i-th component of $M^{(n)}$, and where $C=[c_{ij}]_{i,j\le n}$ is a continuous, symmetric matrix valued function on $[0,\infty)$ with $C(0)=0$ and $C(t)-C(s)\ge 0$ for $t>s\ge 0$. Then $M^{(n)}\Rightarrow M$, where M is a Gaussian process with independent increments and $\mathbb{E}[M(t)M(t)^T]=C(t)$.

Consider a sequence of diffusions X_n satisfying the stochastic integral equations

$$X_n(t)=X_n(0)+\int_0^t\sigma(X_n,s-)dM_n(s)+\int_0^t b(X_n,s)dV_n(s),$$

where M_n is a d-dimensional martingale such that $\mathbb{E}[[M_n]_t]<\infty$ for every $t\ge 0$, and where V_n is a $\mathbb{R}^{d\times d}$ valued process with uniformly bounded finite variation. The following convergence result is a special case of Theorem 5.4 of [36].

Proposition 5.9. *Let W be a standard Brownian motion, and $V(t) = tI$. Assume $(M_n, V_n) \Rightarrow (W, V)$, and the diffusion X satisfies*

$$X_n(t) = X_n(0) + \int_0^t \sigma(X,s)dW(s) + \int_0^t b(X,s)dV(s),$$

then $X_n \Rightarrow X$.

Using the convergence of $\alpha_\lambda(t)$ to $2\pi\mathbb{Z}$ as $t \to \infty$, together with Propositions 5.7, 5.8 and 5.9, we are now ready to prove the following transition result.

Proposition 5.10 (Transition)**.** *Fix $\beta > 0$ and $\delta \in \mathbb{C}$ with $\Re\delta > -1/2$. Then as $\lambda \to \infty$ we have*

$$(\mathrm{HP}_{\beta,\delta} - \lambda) \Rightarrow \mathrm{Sine}_\beta\,.$$

Proof. Fix $\beta > 0$ and $\delta \in \mathbb{C}$ with $\Re\delta > -1/2$. Let $N(\cdot) := N_{\delta,\beta}(\cdot)$ be the counting function of the $\mathrm{HP}_{\beta,\delta}$ process. Theorem 4.19 shows that $N(\cdot)$ has the same distribution as the right-continuous version of the function $\lambda \mapsto \lim_{t\to\infty} \frac{1}{2\pi}\alpha_\lambda(t)$, where the processes $(\alpha_\lambda(t), \lambda \in \mathbb{R})$ solves the coupled system of SDE (4.33). Denote by $M(\cdot) = M_\beta(\cdot)$ the counting function of the Sine_β process, note that $M(\cdot) = N_{0,\beta}(\cdot)$.

It is enough to show that the convergence of the finite dimensional marginals of the counting function. More precisely, we will prove that for any finite collection $\{x_1, x_2, \cdots, x_k\}$ we have

$$\{N(\lambda + x_i) - N(\lambda)\}_{1\le i\le k} \Rightarrow \{M(x_i)\}_{1\le i\le k}$$

joint in distribution as $\lambda \to \infty$.

To that end, we will use the SDE characterization of the difference $N(\lambda + x) - N(\lambda)$.

Let $\psi_{\lambda,x}(t) := \alpha_{\lambda+x}(t) - \alpha_\lambda(t)$, a direct calculation shows that

$$d\psi_{\lambda,x}(t) = x\frac{\beta}{4}e^{-\frac{\beta}{4}t}dt + \Re[(e^{-i\psi_{\lambda,x}} - 1)e^{-i\alpha_\lambda(t)}dZ] \\ + \Im\delta\Re[(e^{-i\psi_{\lambda,x}} - 1)e^{-i\alpha_\lambda(t)}]dt + \Re\delta\Im[(e^{-i\psi_{\lambda,x}} - 1)e^{-i\alpha_\lambda(t)}]dt, \tag{5.20}$$

with initial condition $\psi_{\lambda,x}(0) = 0$.

By the triangle inequality,

$$\left|\sup_{0\le s\le T}\int_0^s \Re[(e^{-\mathrm{i}\psi_{\lambda,x}} - 1)e^{-\mathrm{i}\alpha_\lambda(t)}]dt\right| \le \left|\sup_{0\le s\le T}\int_0^s e^{-i\alpha_{\lambda+x}}dt\right| + \left|\sup_{0\le s\le T}\int_0^s e^{-i\alpha_\lambda}dt\right|.$$

A similar estimate also holds for the term $\Im[(e^{-i\psi_{\lambda,x}} - 1)e^{-i\alpha_\lambda(t)}]dt$. We can then use Proposition 5.7 to conclude that the two drift terms on the second line of equation 5.20 vanish as $\lambda \to \infty$.

Write $dZ = dB_1 + idB_2$ where B_1, B_2 are independent standard real Brownian motion. For a fixed x, we have

$$\Re[(e^{-i\psi_{\lambda,x}} - 1)e^{-i\alpha_\lambda(t)}dZ] = \sin\psi_{\lambda,x}(\cos\alpha_\lambda dB_2 - \sin\alpha_\lambda dB_1) \\ + (\cos\psi_{x,\lambda} - 1)(\cos\alpha_\lambda dB_1 + \sin\alpha_\lambda dB_2).$$

Define

$$W_{\lambda,1}(t) = \int_0^t (\cos\alpha_\lambda dB_1 + \sin\alpha_\lambda dB_2), \qquad W_{\lambda,2}(t) = \int_0^t (\cos\alpha_\lambda dB_2 - \sin\alpha_\lambda dB_1).$$

It follows from the independence of B_1 and B_2 that $[W_{\lambda,1}]_t = t$, $[W_{\lambda,2}]_t = t$, and $[W_{\lambda,1}, W_{\lambda,2}]_t = 0$. By Proposition 5.8, this implies that $(W_{\lambda,1}, W_{\lambda_2}) \Rightarrow (W_1, W_2)$ as $\lambda \to \infty$, where W_1, W_2 are independent real Brownian motions.

Therefore, for a fixed x, the limiting diffusion of $\psi_{\lambda,x}$ should satisfy the SDE

$$d\hat{\psi}_x(t) = x\frac{\beta}{4}e^{-\frac{\beta}{4}t}dt + \Re[(e^{-\hat{\psi}_x(t)} - 1)dZ],$$

where Z is a standard complex Brownian motion $Z = W_1 + iW_2$. Note that this is exactly the stochastic sine equation ((4.33) when $\delta = 0$) and the unique strong solution is given by the phase function of the Sine$_\beta$ process. Note also that the limit $(W_{\lambda,1}, W_{\lambda,2}) \Rightarrow (W_1, W_2)$ is independent of x. So the limiting diffusions $\hat{\psi}_{x_i}, 1 \le i \le k$ are driven by the same Brownian motion Z. Therefore, for any fixed T and $k \in \mathbb{Z}_+$ we have

$$(\psi_{\lambda,x_1}(T), \ldots, \psi_{\lambda,x_k}(T)) \Rightarrow (\hat{\psi}_{x_1}(T), \ldots, \hat{\psi}_{x_k}(T)) \tag{5.21}$$

as $\lambda \to \infty$, which implies that for any $l_1, \ldots, l_k \in \mathbb{Z}_+$

$$\lim_{\lambda\to\infty} P\left(\lfloor \psi_{\lambda,x_i}(T) + \pi \rfloor_{2\pi} = 2\pi l_i, i = 1, \ldots, k\right) = P\left(\lfloor \hat{\psi}_{x_i}(T) + \pi \rfloor_{2\pi} = 2\pi l_i, i = 1, \ldots, k\right), \tag{5.22}$$

where $\lfloor x \rfloor_{2\pi} := 2\pi \lfloor \frac{x}{2\pi} \rfloor$.

By Theorem 4.19, we have that $M(\cdot)$ has the same distribution as the right-continuous version of the function $x \mapsto \frac{1}{2\pi}\hat{\psi}_x(\infty)$, where evaluation at ∞ should be understood as a limit. In particular,

$$P\left(\frac{1}{2\pi}\hat{\psi}_{x_i}(\infty) = M(x_i), i = 1, \ldots, k\right) = 1. \tag{5.23}$$

This implies that $\hat{\psi}_x(T)$ should be close to $2\pi\mathbb{Z}$ for large enough T. By Theorem 4.19 again we have $\psi_{\lambda,x_i}(\infty) \in 2\pi\mathbb{Z}_{\ge 0}$ for any $\lambda > 0$, and

$$P\left(\frac{1}{2\pi}\psi_{\lambda,x_i}(\infty) = N(\lambda + x_i) - N(\lambda), i = 1, \ldots, k\right) = 1, \tag{5.24}$$

For any finite collection $\{x_1, \ldots, x_k\}$ and $0 < \eta < \pi$, (5.23) and (5.24) imply that for any $\varepsilon > 0$ we may choose T large enough so that

$$P\left(|\hat{\psi}_{x_i}(\infty) - \hat{\psi}_{x_i}(T)| < \eta, i = 1, \ldots, k\right) > 1 - \varepsilon. \tag{5.25}$$

Combining (5.25), (5.22), (5.23) and (5.24), it suffices to prove for any $\varepsilon > 0$ we can choose λ and T sufficiently large so that

$$P\left(\psi_{\lambda,x_i}(\infty) = \lfloor \psi_{\lambda,x_i}(T) + \pi \rfloor_{2\pi}, i = 1, \ldots, k\right) > 1 - \varepsilon. \tag{5.26}$$

Mimicking the proof of Lemma 3.5 in [26], we now show that for $|\eta| < \eta_0 < 1/16$ there exists λ_0 and A uniform in x so that for $\lambda \geq \lambda_0$ and $T \geq -\frac{4}{\beta}\log \rho_0$,

$$P\left(\psi_{\lambda,x}(\infty) \neq \lfloor \psi_{\lambda,x}(T) + \pi \rfloor_{2\pi} \middle| \psi_{\lambda,x}(T) - \lfloor \psi_{\lambda,x}(T) + \pi \rfloor_{2\pi} = \eta\right) \leq (x + A)\sqrt{\eta_0}. \tag{5.27}$$

Set $T_{\lambda,\eta_0} := \frac{4}{\beta}\log(\lambda\eta_0)$. Without loss of generality, we may assume $\eta > 0$, in which case $\lfloor \psi_{\lambda,x}(T) \rfloor_{2\pi} = \lfloor \psi_{\lambda,x}(T) + \pi \rfloor_{2\pi}$. The $\eta < 0$ case can be treated similarly with a reflection argument by considering the diffusion $\psi_{\lambda,-x}$. On the time regime $[T, T_{\lambda,\eta_0}]$, we have

$$\psi_{\lambda,x}(T_{\lambda,\eta_0}) - \lfloor \psi_{\lambda,x}(T) \rfloor_{2\pi} = \int_T^{T_{\lambda,\eta_0}} d\psi_{\lambda,x}(t) + \psi_{\lambda,x}(T) - \lfloor \psi_{\lambda,x}(T) \rfloor_{2\pi}. \tag{5.28}$$

Taking expectation of the right hand side of (5.28) and then applying Proposition 5.7, we can find R so that the oscillatory integrals are bounded by $R\delta$. Thus,

$$\mathbb{E}\left[\psi_{\lambda,x}(T_{\lambda,\eta_0}) - \lfloor \psi_{\lambda,x}(T) \rfloor_{2\pi} \middle| \psi_{\lambda,x}(T) - \lfloor \psi_{\lambda,x}(T) + \pi \rfloor_{2\pi} = \eta\right] \leq x e^{-\frac{\beta}{4}T} + R\delta.$$

By Markov's equality, for $T \geq -\frac{4}{\beta}\log \eta_0$ we have

$$P\left(\psi_{\lambda,x}(T_{\lambda,\eta_0}) - \lfloor \psi_{\lambda,x}(T) \rfloor_{2\pi} > \sqrt{\delta} \,\middle|\, \psi_{\lambda,x}(T) - \lfloor \psi_{\lambda,x}(T) + \pi \rfloor_{2\pi} = \eta\right) \leq (x + R)\sqrt{\delta}. \tag{5.29}$$

On the region $[T_{\lambda,\eta_0}, \infty)$, we have $(\lambda + x)e^{-\frac{\beta}{4}T_{\lambda,\eta_0}} = \frac{1}{\eta_0} + \frac{x}{\lambda\eta_0}$. To show that $\psi_{\lambda,x}(\infty) = \lfloor \psi_{\lambda,x}(T_{\lambda,\eta_0}) \rfloor_{2\pi}$ with high probability, it is equivalent to study the original diffusions $\alpha_{\lambda+x}$ and α_λ restarted at T_{λ,η_0}. Consider $\widetilde{\alpha}_{\frac{1}{\eta_0}+\frac{x}{\lambda\eta_0}}$ and $\widetilde{\alpha}_{\frac{1}{\eta_0}}$ satisfying (4.33) with initial conditions $\widetilde{\alpha}_{\frac{1}{\eta_0}+\frac{x}{\lambda\eta_0}}(0) = \alpha_{\lambda+x}(T_{\lambda,\eta_0})$ and $\widetilde{\alpha}_{\frac{1}{\eta_0}}(0) = \alpha_\lambda(T_{\lambda,\eta_0})$. Then it follows from the property that $\lfloor\widetilde{\alpha}(t)\rfloor_{2\pi}$ is non-decreasing with an almost surely finite limit that for large enough S,

$$P\left(\lfloor\widetilde{\alpha}_{\frac{1}{\eta_0}+\frac{x}{\lambda\eta_0}}(S)\rfloor_{2\pi}=\lfloor\widetilde{\alpha}_{\frac{1}{\eta_0}+\frac{x}{\lambda\eta_0}}(\infty)\rfloor_{2\pi},\lfloor\widetilde{\alpha}_{\frac{1}{\eta_0}}(S)\rfloor_{2\pi}=\lfloor\widetilde{\alpha}_{\frac{1}{\eta_0}}(\infty)\rfloor_{2\pi}\right)>1-\varepsilon/2. \quad (5.30)$$

Then by using continuous dependence on parameters and initial conditions $\widetilde{\alpha}_{\frac{1}{\eta_0}+\frac{x}{\lambda\eta_0}}(0)-\widetilde{\alpha}_{\frac{1}{\eta_0}}(0)=\eta$, there exist η' and λ' so that for $\eta_0<\eta'$ and $\lambda>\lambda'$ that

$$P\left(\left|\widetilde{\alpha}_{\frac{1}{\eta_0}+\frac{x}{\lambda\eta_0}}(S)-\widetilde{\alpha}_{\frac{1}{\eta_0}}(S)\right|<2\pi\right)>1-\varepsilon/2. \quad (5.31)$$

Taking the intersection of the two events in (5.30) and (5.31) we get the event $\psi_{\lambda,x}(\infty)=\lfloor\psi_{\lambda,x}(T_{\lambda,\eta_0})\rfloor_{2\pi}$ happens with probability at least $1-\varepsilon$. Combining with the estimate (5.29), we conclude the proof of (5.27).

We are now ready to finish the proof of Proposition 5.10. Observe that (5.27) implies

$$P\left(\psi_{\lambda,x}(\infty)\neq\lfloor\psi_{\lambda,x}(T)+\pi\rfloor_{2\pi},\left|\psi_{\lambda,x}(T)-\lfloor\psi_{\lambda,x}(T)+\pi\rfloor_{2\pi}\right|<\eta_0\right)\leq(x+A)\sqrt{\eta_0}.$$

By (5.23), $\hat{\psi}_x(T)$ will be close to a multiple of 2π with high probability for large T. By (5.22), this implies that for any $\varepsilon>0$ we may choose T and λ large enough so that $P(|\psi_{\lambda,x}(T)-\lfloor\psi_{\lambda,x}(T)+\pi\rfloor_{2\pi}|<\eta_0)>1-\varepsilon$. These bounds imply that

$$P(\psi_{\lambda,x}(\infty)\neq\lfloor\psi_{\lambda,x}(T)+\pi\rfloor_{2\pi}|)<\varepsilon+(x+A)\sqrt{\eta_0}.$$

Choosing ε and δ small enough proves (5.26) and completes the proof. □

Remark 5.11. *Recall that the* $\mathrm{HP}_{\beta,\delta}$ *process was obtained as the local scaling limit ($n\to\infty$ limit) of the circular Jacobi β-ensemble near the origin (or equivalent, the point* 1 *on the unit circle). Heuristically speaking, if the $n\to\infty$ and $\lambda\to\infty$ limits can be exchanged, then Proposition 5.10 indicates that if we chooses a sequence of points on* *the unit circle moving away from the point* 1*, then we expect to get the* Sine_β *process in the* $n \to \infty$ *limit.*

Proposition 5.12 (CLT)**.** *Fix* $\beta > 0$ *and* $\delta \in \mathbb{C}$ *with* $\Re\delta > -1/2$*. Let* $N(\cdot) = N_{\delta,\beta}(\cdot)$ *be the counting function of the* $\mathrm{HP}_{\beta,\delta}$ *process, then as* $\lambda \to \infty$ *we have*

$$\frac{1}{\sqrt{\log \lambda}}\Big(N(\lambda) - \frac{\lambda}{2\pi}\Big) \Rightarrow \mathcal{N}\Big(0, \frac{2}{\beta\pi^2}\Big),$$

where $\mathcal{N}(0, \frac{2}{\beta\pi^2})$ *has normal distribution with mean* 0 *and variance* $\frac{2}{\beta\pi^2}$*.*

Proof. Let $T = \frac{4}{\beta}\log\lambda$. Notice that the time-shifted process $\hat\alpha_\lambda(t) := \alpha_\lambda(T+t)$ satiesfies the same SDE (5.1) with $\lambda = 1$ with (random) initial condition $\hat\alpha_\lambda(0) = \alpha_\lambda(T)$. Since the equation (5.1) is 2π invariant, the difference with of $\hat\alpha_\lambda(t)$ with its limits $\hat\alpha_\lambda(\infty)$ is stochastically bounded by $\alpha_1(\infty) + 1$, this implies that

$$\frac{\alpha_\lambda(\infty) - \alpha_\lambda(T)}{\sqrt{\log\lambda}} \to 0 \qquad \text{in distribution.} \tag{5.32}$$

On the other hand, solving the SDE (5.1) gives

$$\alpha_\lambda(T) = \lambda - 1 + \int_0^T (\Im\delta(\cos\alpha_\lambda - 1) - \Re\delta\sin\alpha_\lambda)dt + 2\int_0^T \sin(\tfrac{\alpha_\lambda}{2})dW.$$

Proposition 5.7 shows that the expected value of the first integral is finite for all λ. Therefore, after dividing by $\sqrt{\log\lambda}$, the first integral vanishes in the limit as $\lambda \to \infty$. For the remaining term, there exists a Brownian motion $\hat B$ such that we have the following distributional identity

$$\frac{2}{\sqrt{\log\lambda}}\int_0^T \sin\Big(\frac{\alpha_\lambda}{2}\Big)dW = \hat B\left(\frac{4}{\log\lambda}\int_0^T \sin^2\Big(\frac{\alpha_\lambda}{2}\Big)dt\right) = \hat B\left(\frac{\beta}{8} - \frac{2}{\log\lambda}\int_0^T \cos\alpha_\lambda dt\right).$$

By Proposition 5.7 again, we have the integral $\frac{2}{\log\lambda}\int_0^T \cos\alpha_\lambda dt$ converges to 0 in probability, hence

$$\frac{\alpha_\lambda(T) - \lambda}{\sqrt{\log\lambda}} \;\Rightarrow\; \mathcal{N}\Big(0, \frac{8}{\beta}\Big). \tag{5.33}$$

Combining (5.32) and (5.33) and then dividing by 2π finishes the proof. $\square$

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