A Local Model of the Compactified Jacobian

by

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Abstract

The compactified Jacobian of a singular curve X is a moduli space that parameterizes the torsion-free sheaves of generic rank one over X. In general, the topology of the compactified Jacobian is interesting to analyze; for example, it is known that it is irreducible if and only if the embedding dimension of all the singularities is two. However, it is not known how to compute the number of components for a given X with singularities with larger embedding dimension, except in a small number of particular cases.

In this thesis, we start with a unibranch curve singularity rather than a whole curve, and construct a parameter space based on torsion-free modules of generic rank one over the singularity. The space we construct turns out to be homeomorphic to a certain compactified Jacobian. Our construction, however, allows for analysis of its topology using methods distinct from those used to study the compactified Jacobian. We give direct proofs that our space is reducible if the embedding dimension of the singularity is greater than two, as well as computations of the number of components for certain specific singularities. Because of the homeomorphism, this gives new proofs of the results for the corresponding compactified Jacobians.

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Contents

Chapter 1

Introduction

We fix an algebraically closed field k. Let C be a (projective, irreducible) curve defined over k. Associated to C are its **Picard scheme** Pic_C, which is the moduli space of line bundles on C, and its **Jacobian** Pic^0_C, which is the moduli space of degree-0 line bundles on C, or equivalently the connected component of the trivial bundle in Pic_C. When C is nonsingular, Pic^0_C is an abelian variety; in particular it is complete. However, when C is singular, Pic^0_C fails to be complete. Equivalently, on a singular curve, there are flat families of line bundles that fail to converge to line bundles. For example, the ideal sheaf of a smooth point is a line bundle, but as the point degenerates to a singular point, the sheaf degenerates to the ideal sheaf of the singular point which is no longer invertible. Furthermore, the Jacobian of a singular curve fits into a short exact sequence of algebraic groups as follows:

$$0 \to \bigoplus_{x \in C^{\mathrm{sing}}} P_x \to \mathrm{Pic}^0_C \to \mathrm{Pic}^0_{C^\nu} \to 0$$

where C^{sing} is the set of singular points of C, P_x is an affine group depending only on the completed local ring at x, and C^ν is the normalization of C.

Given a noncomplete moduli space such as Pic^0_C for C singular, it is natural to ask how to compactify it, i.e. find a complete moduli space in which the original space embeds. In our case, this is asking whether there is a class of sheaves on a singular curve that contains all line bundles, and such that the corresponding moduli space exists and is complete. Such a class turns out to exist: the class of torsion-free sheaves of generic rank one. The corresponding moduli spaces are the **Compactified Picard Scheme** $\overline{\mathrm{Pic}}_C$ and the **Compactified Jacobian** $\overline{\mathrm{Pic}}^0_C$, the connected component of the trivial bundle. See [6] for an introduction to the Compactified Jacobian.

The Jacobian of a smooth curve is typically studied using the Abel map, which is a surjective morphism $C^{(n)} \to \mathrm{Pic}^0_C$ for sufficiently large n (here $C^{(n)}$ is the n^{th} symmetric power of C), whose fibers are projective spaces–the complete linear systems of divisors associated to line bundles. There is an Abel map for $\overline{\mathrm{Pic}}^0_C$ as well, whose source is $\mathrm{Quot}^n\,\omega$, where ω is the dualizing sheaf of C (note that when C is smooth, ω is a line bundle, so $\mathrm{Quot}^n\,\omega = \mathrm{Hilb}^n C = C^{(n)}$). Again the fibers are projective spaces, and this follows from the development of a divisor theory for torsion-free sheaves. Thus many properties of $\overline{\mathrm{Pic}}^0_C$ can be deduced from study of this Quot scheme. For example, the following is a fundamental result about the compactified Jacobian (see [1] and [7]):

Fact 1.0.1. $\overline{\mathrm{Pic}}^0_C$ *is reducible if and only if all the singularities of* C *are planar.*

This is shown by proving the same property for $\mathrm{Quot}^n\,\omega$ for large enough n, from which the result follows immediately. The problem of determining the number of components of the compactified Jacobian of a curve is largely open, with exact counts known only in certain special cases.

There is another possible perspective on the compactified Jacobian: rather than proceeding in an analogous manner to the case of a smooth curve, one can analyze the individual contributions of each singularity. The short exact sequence above gives an example of this perspective, as it shows how each singularity of C contributes to the failure of completeness of Pic^0_C. We wish to extend this idea to study the compactified Jacobian. The completed stalk of a torsion-free sheaf of generic rank one at a singularity is a torsion-free module of generic rank one over the completed local ring of a singularity. Therefore, in order to see how a single singularity contributes to compactified Jacobian, we wish to parameterize the class of such modules. However, parameterizing the isomorphism type of the module alone does not give a suitable parameter space, but rather we must also keep track of how such a module can attach to a line bundle away from the singularity; in other words we must also keep track of local descent data associated to the module.

In [4], Greuel and Pfister construct a space parameter space along these lines, but do not connect their construction to the compactified Jacobian. In [2], Beauville restricts to unibranch singularities, and notes that the constructed space is homeomorphic to the compactified Jacobian. In this thesis, we elaborate on these ideas, using new methods to carefully construct a suitable parameter space for torsion-free modules of generic rank one over a unibranch curve singularity, and show that these parameter spaces are homeomorphic to certain compactified Jacobians. Furthermore, we show that known topological properties of compactified Jacobians can be proven directly about our parameter spaces.

In Sections 2 and 3, we introduce the main tools for our analysis, lattices and spaces of lattices. In Section 4, we construct our parameter space and prove that it has some properties that are analogous to those of the compactified Jacobian. In Section 5, we prove that the space we construct is homeomorphic to a certain compactified Jacobian. Finally, in Section 6, we give some explicit examples of our construction and use the theory we built up to analyze some individual cases.

Chapter 2

Lattices and Lattice Spaces

2.1 Basics of Lattices

Our primary tool will be a detailed study of *lattices*, which are special linear subspaces of $k((t))$. This terminology comes from the Tate linear algebra, introduced in [3], section 2.6.9.

Definition 2.1.1.

1. A vector subspace $L \subset k((t))$ is a **lattice** if there exist $n, m \in \mathbb{Z}$ such that $t^n k[\![t]\!] \subset L \subset t^m k[\![t]\!]$. A subspace of the form $t^n k[\![t]\!]$ is a **standard lattice**. If $R \subset k((t))$ is a subring, an R**-lattice** is a lattice which is closed under the action of R on $k((t))$.

2. Given two lattices $L_1, L_2 \subset k((t))$, the **relative dimension of** L_2 **in** L_1, denoted $\dim(L_1 : L_2)$, is $\dim(L_1/(L_1 \cap L_2)) - \dim(L_2/(L_1 \cap L_2)) \in \mathbb{Z}$.

3. Given a lattice L such that $t^n k[\![t]\!] \subset L \subset t^m k[\![t]\!]$, we say that the (m, n) is a *bound* of L. If n is minimal and m maximal, define the **standard bounds** of L to be the pair (m, n), with m the **standard lower bound** and n the **standard upper bound**. The **width** of L to be $w(L) := n - m$, and the **size** of L to be $s(L) := \dim(L/t^n k[\![t]\!])$. If $t^n k[\![t]\!] \subset L \subset t^m k[\![t]\!]$ with (m, n) the standard bounds of L, we will also say $t^n k[\![t]\!] \subset L \subset t^m k[\![t]\!]$ **tightly**.

4. A **lattice ring** is lattice which is a subring of $k((t))$. Note that a lattice ring is necessarily contained in $k[[t]]$, otherwise it would not be contained in any standard lattice. Furthermore, because a lattice ring contains 1, the standard lower bound of any lattice ring is 0. The **delta invariant** of a lattice ring R, denoted $\delta(R)$, is $w(R) - s(R)$.

Note that if L_1, L_2 are lattices, then $L_1 \cap L_2$ and $L_1 + L_2$, the smallest vector subspace of $k((t))$ containing L_1 and L_2 are both also lattices.

We begin by establishing some lemmas about relative dimension of lattices.

Lemma 2.1.1. *For lattices L_1, L_2, $\dim(L_1 : L_2) = \dim(L_1/M) - \dim(L_2/M)$ for any lattice M contained in both L_1 and L_2.*

Proof. With L_1, L_2, M as above, we have $L_i/(L_1 \cap L_2) \cong (L_i/M)/((L_1 \cap L_2)/M)$ as vector spaces for $i = 1, 2$. Thus

$$\begin{aligned}
\dim(L_1 : L_2) &= \dim(L_1/(L_1 \cap L_2)) - \dim(L_2/(L_1 \cap L_2)) \\
&= \dim(L_1/M)/((L_1 \cap L_2)/M)) - \dim((L_2/M)/((L_1 \cap L_2)/M)) \\
&= \dim(L_1/M) - \dim((L_1 \cap L_2)/M) - (\dim(L_2/M) - \dim((L_1 \cap L_2)/M)) \\
&= \dim(L_1/M) - \dim(L_2/M)
\end{aligned}$$

□

Lemma 2.1.2. *Given lattices L_1, L_2, L_3, we have $\dim(L_1 : L_2) = \dim(L_1 : L_3) - \dim(L_2 : L_3)$.*

Proof.

$$\begin{aligned}
\dim(L_1 : L_2) =& \dim(L_1/(L_1 \cap L_2 \cap L_3)) - \dim(L_2/(L_1 \cap L_2 \cap L_3) \\
=& (\dim(L_1/(L_1 \cap L_2 \cap L_3)) - \dim(L_3/(L_1 \cap L_2 \cap L_3)) \\
& - (\dim(L_2/(L_1 \cap L_2 \cap L_3)) - \dim(L_3/(L_1 \cap L_2 \cap L_3)) \\
=& \dim(L_1 : L_3) - \dim(L_2 : L_3)
\end{aligned}$$

□

Corollary 2.1.1. *Suppose* $\dim(L_1 : L_1') = 0$ *and* $\dim(L_2 : L_2') = 0$*. Then* $\dim(L_1 : L_2) = \dim(L_1' : L_2')$

Proof.

$$\begin{aligned}\dim(L_1 : L_2) &= \dim(L_1 : L_1') - \dim(L_2 : L_1') \\ &= \dim(L_1 : L_1') - (\dim(L_2 : L_2') - \dim(L_1' : L_2')) \\ &= 0 - (0 - \dim(L_1' : L_2')) = \dim(L_1' : L_2')\end{aligned}$$

□

In addition to the relative dimension of two lattices, we will want to compare a given lattice to standard lattices.

Definition 2.1.2. The **index** of a lattice L, denoted $\operatorname{ind} L$, is $\dim(k[\![t]\!] : L)$.

Note that $\operatorname{ind} L$ is the unique $n \in \mathbb{Z}$ such that $\dim(L : t^n k[\![t]\!]) = 0$, and in particular $\operatorname{ind}(t^n k[\![t]\!]) = n$. If $L_1 \subset L_2$, then $\operatorname{ind}(L_1) \geq \operatorname{ind}(L_2)$.

2.2 The Geometry of Lattices

We will now construct a variety whose points are in bijection with the set of lattices of a given index and whose width is bounded by a given integer.

Suppose that L_1, L_2 are lattices and $d \in \mathbb{Z}$. Consider the set of lattices L with $L_1 \subset L \subset L_2$ and $\operatorname{ind} L = d$ (which is nonempty iff $L_1 \subset L_2$ and $\operatorname{ind}(L_1) \geq d \geq \operatorname{ind}(L_2)$). This can be identified with the set of subspaces of L_2/L_1 of dimension $\operatorname{ind} L - d$, i.e. the Grassmannian $\operatorname{Gr}(L_2/L_1, \operatorname{ind} L - d)$. We refer to this set as $\mathcal{G}_d^{L_1 \subset L_2}$. When $L_1 = t^d k[\![t]\!]$ and $L_2 = t^m k[\![t]\!]$, we will also call it $\mathcal{G}_d^{n>m}$.

Note that for lattices $L_1 \subset L_1' \subset L_2' \subset L_2$ and $d \in \mathbb{Z}$ with $\operatorname{ind} L_1 \geq d \geq \operatorname{ind} L_2$, we have $\mathcal{G}_d^{L_1'\subset L_2'} \subset \mathcal{G}_d^{L_1\subset L_2}$. We identify this inclusion with the inclusion

$$\mathrm{Gr}(L_2'/L_1', \mathrm{ind}(L_1') - d) \hookrightarrow \mathrm{Gr}(L_2/L_1, \mathrm{ind}(L_1) - d)$$

which comes from the standard closed embedding of Grassmannians

$$\mathrm{Gr}(W_1/W_2, \mathrm{ind}(W_2) - d) \hookrightarrow \mathrm{Gr}(V, d)$$

for $W_2 \subset W_1 \subset V$. Note also that for $d \in \mathbb{Z}$, and lattices L_1, L_2, L_1', L_2' with $\mathrm{ind}(L_1) \geq d \geq \mathrm{ind}(L_2)$ and $\mathrm{ind}(L_1') \geq d \geq \mathrm{ind}(L_2')$, we have that as sets, $\mathcal{G}_d^{L_1\subset L_2} \cap \mathcal{G}_d^{L_1'\subset L_2'} = \mathcal{G}_d^{L_1+L_1'\subset L_2\cap L_2'}$. In particular,

$$\mathcal{G}_d^{n>m} \cap \mathcal{G}_d^{n'>m'} = \mathcal{G}_d^{\min\{n,n'\}>\max\{m,m'\}}\,. \tag{2.1}$$

Fix an integer $w > 0$. We will construct a space whose points are in bijection with the set of lattices L with $w(L) < w$. Every such lattice is contained in some $\mathcal{G}_d^{n+w>n}$, so as a set, our space will be $\bigcup_{d\in\mathbb{Z},n\in\mathbb{Z}} \mathcal{G}_d^{n+w>n}$. In order to geometrize this set, we view it as a nondisjoint union of the Grassmannians identified with each $\mathcal{G}_d^{n+w>n}$. By Equation 2.1, all intersections between varying $\mathcal{G}_d^{n+w>n}$'s will be of the form $\mathcal{G}_d^{m>n}$ with $m - n < w$. Therefore we can consider the diagram $\mathcal{D}_w$ in the category of k-schemes, indexed by the poset

$$\{(m, n, d) \in \mathbb{Z}^3 : 0 \leq m - n \leq w\}$$

where $(m, n, d) \leq (m', n', d')$ iff $d = d'$, $m \leq m'$, and $n \leq n'$, such that the morphism indexed by $(m, n, d) \leq (m', n', d)$ is the inclusion

$$\mathcal{G}_d^{m>n} \hookrightarrow \mathcal{G}_d^{m'>n'}\,.$$

Definition 2.2.1. The *lattice space of width* w, denoted $\mathcal{L}_w$ is $\operatorname{colim} \mathcal{D}_w$, where $\mathcal{D}_w$ is considered as a diagram in the category of varieties by identifying each $\mathcal{G}_d^{n+a>n}$ with a

Grassmannian as above.

In $\mathcal{D}_w$, there are no maps between $\mathcal{G}_d^{n+a>n}$'s with different indices d. Therefore we can consider $\mathcal{D}_w$ as a disjoint union of diagrams $\mathcal{D}_{w,d}$ for $d \in \mathbb{Z}$, and thus also $\mathcal{L}_w$ is a disjoint union of the finite colimits $\mathcal{L}_{w,d} := \operatorname{colim} \mathcal{D}_{w,d}$. The action of $\mathbb{Z}$ on the set of lattices induces an action of $\mathbb{Z}$ on $\mathcal{L}_w$, and the action of $n \in \mathbb{Z}$ maps $\mathcal{L}_{w,d}$ isomorphically onto $\mathcal{L}_{w,d+n}$. Therefore $\mathcal{L}_w$ can be identified with $\mathcal{L}_{w,d} \times D$ for any fixed $d \in \mathbb{Z}$.

Fix $d \in \mathbb{Z}$ and $w > 0$. Then the nonempty Grassmannians whose union is $\mathcal{L}_{w,d}$ are $\mathcal{G}_d^{d+i>d-w+i}$ for $i = 0, \ldots, w$. Each such $\mathcal{G}_d^{d+i>d-w+i}$ embeds in $\mathcal{G}_d^{d+w>d-w}$. We will show that $\mathcal{L}_{w,d}$ can be realized as the union of the $\mathcal{G}_d^{d+i>d-w+i}$ inside $\mathcal{G}_d^{d+w>d-w}$. In particular:

Proposition 2.2.1. *The morphism $\mathcal{L}_{w,d} \to \mathcal{G}_d^{d+w>d-w}$ induced by the canonical morphisms $\mathcal{G}_d^{d+i>d-w+i} \to \mathcal{G}_d^{d+w>d-w}$ is an isomorphism onto its image.*

We will use the following general statement:

Lemma 2.2.1. *Let X be a scheme and let $Y_1, \ldots, Y_n \subset X$ be closed subschemes. For $S \subset \{1, \ldots, n\}$, let Y_S denote the scheme-theoretic intersection of all the Y_i, $i \in S$. Let $P(n)$ be the poset of nonempty subsets of $\{1, \ldots, n\}$ and inclusions, and let $\mathcal{D}_{Y_1,\ldots,Y_n}$ be the diagram indexed by $P(n)$, consisting of all inclusions $Y_S \hookrightarrow Y_{S'}$ for $\emptyset \neq S' \subset S \subset \{1, \ldots, n\}$. Let Y denote the union of all the Y_i inside X. Then the natural morphism* $\operatorname{colim} \mathcal{D}_{Y_1,\ldots,Y_n} \to Y$ *is an isomorphism.*

Proof. We proceed by induction on n. For $n = 1$, there is nothing to prove. Let $n = 2$, so we wish to compute the pushout of Y_1 and Y_2 along their scheme-theoretic intersection $Y_1 \cap Y_2$ in X. First, we assume X is affine and compute the pushout in the category of affine schemes. Let $X = \operatorname{Spec} A$, and $Y_i = \operatorname{Spec} A/I_i$ for $i = 1, 2$. Then $Y_1 \cap Y_2 = A/(I_1 + I_2)$. To compute the pushout, we can compute the fiber product of A/I_1 and A/I_2 over the maps $A/I_i \to A/(I_1 + I_2)$. This fiber product is

$$\{(a + I_1, b + I_2) \in (A/I_1) \times (A/I_2) : a + I_1 + I_2 = b + I_1 + I_2\}$$

which, by the Chinese Remainder Theorem, is isomorphic to $A/(I_1 \cap I_2)$. Now $\operatorname{Spec}(A/(I_1 \cap I_2)) = Y_1 \cup Y_2$, so the result holds in the category of affine schemes.

Next, consider X in general. Let Z be another scheme, and let $f_i : Y_i \to Z$ be morphisms for $i = 1, 2$ such that $f_1 \circ j_1 = f_2 \circ j_2$, where $j_i : Y_1 \cap Y_2 \hookrightarrow Y_i$ is the inclusion. Let $p \in Y_1$ be a point, and let U be an affine open in Z containing $f_1(p)$. Let V_1 be an open subset of X such that $V_1 \cap Y_1 = f_1^{-1}(U)$. Let V_2' be an open subset of X such that $V_2' \cap Y_2 = f_2^{-1}(U)$, and let $V_2 = V_2' \cup X - Y_2$, so that $V_2 \cap Y_2 = f_2^{-1}(U)$ and $p \in V_2$. Now let V be an affine open subset of $V_1 \cap V_2$ that contains p. Then $V \cap Y_i$ is an affine open subset of Y_i for $i = 1, 2$, and $p \in V \cap Y_1$, and $f_i(V \cap Y_i) \subset U$, so $f_i' := f_i|_{V \cap Y_i} : V \cap Y_i \to U$ is a morphism of affine schemes for $i = 1, 2$. Thus f_1' and f_2' factor uniquely through $(V \cap Y_1) \cup (V \cap Y_2) = V \cap (Y_1 \cup Y_2)$. Letting p vary accross all points in Y_1 and Y_2, we see that the f_1 and f_2 locally factor uniquely through $Y_1 \cup Y_2$, but this means that in fact f_1 and f_2 factor uniquely through $Y_1 \cup Y_2$, so the desired pushout is $Y_1 \cup Y_2$.

Now suppose that $n \geq 3$, and assume that we know the result for $n-1$. Let $\mathcal{D}'$ be the subdiagram of $\mathcal{D}_{Y_1,\dots,Y_n}$ indexed by $P(n-1)$, and let $\mathcal{D}''$ be the subdiagram of $\mathcal{D}_{Y_1,\dots,Y_n}$ indexed by the elements of $P(n)$ containing n. Then $\mathcal{D}'$ can be identified with $\mathcal{D}_{Y_1,\dots,Y_{n-1}}$, and $\mathcal{D}''$ can be identified with $\mathcal{D}_{Y_{\{1,n\}},\dots,Y_{\{n-1,n\}}}$. By the inductive hypothesis, $\operatorname{colim}\mathcal{D}'$ is identified with the union of $Y_1, \dots, Y_{n-1}$ inside X, and $\operatorname{colim} D''$ is identified with the union of $Y_{\{1,n\}}, \dots, Y_{\{n-1,n\}}$ inside X, which coincides with the scheme-theoretic intersection of Y_n with the union of $Y_1, \dots, Y_{n-1}$. Furthermore, $\operatorname{colim}\mathcal{D}_{Y_1,\dots,Y_n}$ can be computed as the colimit of

$$\begin{array}{ccc} & & \operatorname{colim}\mathcal{D}' \\ & \nearrow & \\ \operatorname{colim}\mathcal{D}'' & & \\ & \searrow & \\ & & Y_n \end{array}$$

where the morphism $\operatorname{colim}\mathcal{D}'' \to \operatorname{colim}\mathcal{D}'$ is induced by the morphisms $Y_S \to Y_{S-\{n\}}$ for $S \ni n$. Now by the case $n = 2$, this is the union of Y_n with the union of $Y_1, \dots, Y_{n-1}$, which is the union of $Y_1, \dots, Y_n$. $\square$

To apply Lemma 2.2.1, first notice that set-theoretically, for $i < j$,

$$\mathcal{G}_d^{n+i>n-w+i} \cap \mathcal{G}_d^{n+j>n-w+j} = \mathcal{G}_d^{n+i>n-w+j}.$$

We must show that this holds scheme-theoretically as well, so that the diagram $\mathcal{D}_{w,d}$ coincides with the diagram $\mathcal{D}$ in Lemma 2.2.1. In general:

Lemma 2.2.2. *Let V be a vector space and $W_1 \subset W_2 \subset V$, $W_1' \subset W_2' \subset V$ subspaces. Let $G \subset \mathrm{Gr}(V, d)$ denote the scheme-theoretic intersection of $\mathrm{Gr}(W_2/W_1, d - \dim W_1)$ and $\mathrm{Gr}(W_2'/W_1', d - \dim W_1')$ in $\mathrm{Gr}(V, d)$. Then the canonical morphism*

$$G \to \mathrm{Gr}((W_2 \cap W_2')/(W_1 + W_1'), d - \dim(W_1 + W_1'))$$

is an isomorphism.

Note that the canonical morphism exists because

$$\mathrm{Gr}((W_2 \cap W_2')/(W_1 + W_1'), d - \mathrm{codim}(W_2 \cap W_2'))$$

is the set-theoretic intersection with the canonical reduced closed subscheme structure.

Proof. Let $G_1 := \mathrm{Gr}(W_2/W_1, d - \dim W_1)$ and $G_2 := \mathrm{Gr}(W_2'/W_1', d - \dim W_1')$. For a scheme X, let h_X denote its functor of points. For an arbitrary test scheme T, we have $h_{\mathrm{Gr}(V,d)}(T) =$

$$\{\text{isomorphism classes of vector bundle surjections } \mathcal{O}_T \otimes_k V \twoheadrightarrow \mathcal{E}, \mathrm{rank}(\mathcal{E}) = \dim V - d\}$$

and we can identify h_{G_1} with the subfunctor consisting of isomorphism classes of vector bundle surjections ϕ such that $\mathcal{O}_T \otimes_k W_1 \subset \ker \phi \subset \mathcal{O}_T \otimes_k W_2$, and make a similar identification for h_{G_2}. The scheme-theoretic intersection of G_1 and G_2 is the fiber product $G_1 \times_{\mathrm{Gr}(V,d)} G_2$, which can be computed on the level of the functor of points, i.e. $G_1 \times_{\mathrm{Gr}(V,d)} G_2(T) = G_1(T) \times_{\mathrm{Gr}(V,d)(T)} G_2(T)$. This is $G_1(T) \cap G_2(T)$ as subsets of $\mathrm{Gr}(V, d)(T)$, so

$G_1 \times_{\mathrm{Gr}(V,d)} G_2(T)$ consists of isomorphism classes of vector bundle surjections ϕ to bundles of rank d such that $\mathcal{O}_T \otimes_k W_1 \subset \ker\phi \subset \mathcal{O}_T \otimes_k W_2$ and $\mathcal{O}_T \otimes_k W_1' \subset \ker\phi \subset \mathcal{O}_T \otimes_k W_2'$, which is equivalent to the condition that $\mathcal{O}_T \otimes_k (W_1 + W_1') \subset \ker\phi \subset \mathcal{O}_T \otimes_k (W_2 \cap W_2')$. This is exactly the functor of points of $\mathrm{Gr}((W_2 \cap W_2')/(W_1 + W_1'), d - \dim(W_1 \cap W_1'))$, as a subfunctor of $h_{\mathrm{Gr}(V,d)}$. □

We now can prove Proposition 2.2.1:

Proof. Lemma 2.2.2 shows that the objects in the diagram defining $\mathcal{L}_{w,d}$ are exactly the $\mathcal{G}_d^{n+i>n-w+i}$ and their the scheme-theoretic intersections, so we can apply Lemma 2.2.1 to the closed embeddings $\mathcal{G}_d^{n+i>n-w+i} \to \mathcal{G}_d^{n+w>n-w}$ to obtain the result. □

Thus, $\mathcal{L}_{w,d}$ can be realized both as an abstract colimit, or gluing, of Grassmannians along closed subschemes, and as a closed subvariety of a Grassmannian. Both perspectives will be useful later on.

2.3 Algebraic Properties of Lattices

In the previous section, we viewed lattices simply as a type of vector space. We will now explore some of the properties of lattices as they relate to the multiplicative structure inherited from $k((t))$.

Lemma 2.3.1. *Let L, L' be lattices and let $f \in k((t))$.*

a) $\dim(L : fL') = \dim(f^{-1}L : L')$

b) $\dim(L : fL) = \mathrm{val}(f)$

Proof. 1. Multiplication by f is a linear automorphism of $k((t))$, and takes the pair $(f^{-1}L, L')$ to the pair (L, fL').

2. First suppose L is a standard lattice $t^n k[\![t]\!]$. Then $fL = t^{\mathrm{val}(f)}L$, and so $\dim(L :$

$fL) = \dim(t^n k[\![t]\!] : t^{n+\mathrm{val}(f)} k[\![t]\!]) = \mathrm{val}(f)$. For general L, we have

$$\begin{aligned}\dim(L : fL) &= \dim(L : k[\![t]\!]) - \dim(fL : k[\![t]\!])\\ &= \dim(L : k[\![t]\!]) - \dim(L : f^{-1}k[\![t]\!])\\ &= \dim(L : k[\![t]\!]) - \dim(L : t^{-\mathrm{val}(f)}k[\![t]\!])\\ &= \dim(L : k[\![t]\!]) - (\dim(L : k[\![t]\!]) - \mathrm{val}(f))\\ &= \mathrm{val}(f).\end{aligned}$$

□

Let R be a lattice ring. Then for every $r \in R$, the $t^{w(R)-1}$ coefficient of r is 0.

The next important ingredient is duality: for a lattice, we wish to define a suitable notion of dual lattice. Consider the pairing $\langle \cdot, \cdot \rangle : k((t)) \times k((t)) \to k$ given by $\langle f, g \rangle = \mathrm{res}(fg)$ where $\mathrm{res}(h)$ is the t^{-1} coefficient of h.

Definition 2.3.1. Given a lattice L, the **dual lattice** is $L^\vee := \{f \in k((t)) | \langle f, g \rangle = 0 \text{ for all } g \in L\}$.

Observe that if $L_1 \subset L_2$ are lattices, then $L_2^\vee \subset L_1^\vee$. Furthermore, $\langle \cdot, \cdot \rangle$ induces a pairing $(L_2/L_1) \times (L_1^\vee/L_2^\vee) \to k$. Indeed, suppose $f \in L_2$, $f' \in L_1$, $g \in L_1^\vee$, $g' \in L_2^\vee$. Then $\langle f + f', g + g' \rangle = \langle f, g \rangle + \langle f, g' \rangle + \langle f', g \rangle + \langle f', g' \rangle$. The latter three terms vanish because they are pairings between elements of a lattice and its dual–for the final term, use the fact that $L_2^\vee \subset L_1^\vee$. Thus $\langle f + f', g + g' \rangle = \langle f, g \rangle$, so the restriction of $\langle \cdot, \cdot \rangle$ to $L_1 \times L_2^\vee$ descends to a bilinear pairing $(L_2/L_1) \times (L_1^\vee/L_2^\vee) \to k$ as claimed.

Lemma 2.3.2. *a)* $(t^n k[\![t]\!])^\vee = t^{-n} k[\![t]\!]$

b) for $n > m$, $\langle \cdot, \cdot \rangle : (t^m k[\![t]\!]/t^n k[\![t]\!]) \times (t^{-n} k[\![t]\!]/t^{-m} k[\![t]\!]) \to k$ *is a perfect pairing.*

Proof. a) For any $f \in t^n k[\![t]\!]$ and $g \in t^{-n} k[\![t]\!]$, $fg \in k[\![t]\!]$ so $\mathrm{res}(fg) = 0$. So $t^{-n} k[\![t]\!] \subset (t^n k[\![t]\!])^\vee$. Conversely, if $g \notin t^{-n} k[\![t]\!]$, then $\mathrm{val}(g) < -n$, and so $t^{-\mathrm{val}(g)+1} \in t^n k[\![t]\!]$, but $\mathrm{res}(t^{-\mathrm{val}(g)+1} g) \neq 0$, so $g \notin (t^n k[\![t]\!])$.

b) Suppose $v \in t^m k[\![t]\!]/t^n k[\![t]\!]$ so that for every $w \in t^{-n}k[\![t]\!]/t^{-m}k[\![t]\!]$, $\langle v, w\rangle = 0$. Lift v to $f \in t^m k[\![t]\!]$. Then we have that $\langle f, g\rangle = 0$ for every $g \in t^{-n}k[\![t]\!]$. Thus $f \in (t^{-n}k[\![t]\!])^\vee = t^n k[\![t]\!]$, so $v = 0$ in $t^m k[\![t]\!]/t^n k[\![t]\!]$. Thus $\langle\cdot,\cdot\rangle$ is nondegenerate on the left. By symmetry, it is also nondegenerate on the right and thus a perfect pairing.

□

Lemma 2.3.3. *Let L, L_0, $L_1 \subset L_2$ be lattices. Then*

a) $(L^\vee)^\vee = L$

b) $\langle\cdot,\cdot\rangle : (L_2/L_1) \times (L_1^\vee/L_2^\vee) \to k$ *is a perfect pairing*

c) $w(L^\vee) = w(L)$

d) $s(L^\vee) = w(L) - s(L)$.

e) $\dim(L_0 : L) = \dim(L^\vee : L_0^\vee)$

f) $\operatorname{ind}(L^\vee) = -\operatorname{ind}(L)$

Proof. a) Suppose that $t^n k[\![t]\!] \subset L \subset t^m k[\![t]\!]$. Then $t^{-m}k[\![t]\!] \subset L^\vee \subset t^{-n}k[\![t]\!]$. Observe that $L^\vee/t^{-m}k[\![t]\!] = (L/t^n k[\![t]\!])^\perp$ as subspaces of $t^{-n}k[\![t]\!]/t^{-m}k[\![t]\!]$, where the orthogonal complement is taken with respect to the pairing in b. Applying this to $L^\vee$, we have that

$$((L^\vee)^\vee/t^n k[\![t]\!]) = (L^\vee/t^{-m}k[\![t]\!])^\perp = ((L/t^n k[\![t]\!])^\perp)^\perp = L/t^n k[\![t]\!]$$

from which it follows that $(L^\vee)^\vee = L$.

b) Suppose that $v \in L_1^\vee/L_2^\vee$ such that for all $w \in L_2/L_1$, $\langle w, v\rangle = 0$. Lift v to $f \in L_1^\vee$ such that for all $g \in L_2$, $\langle g, f\rangle = 0$. By definition, $f \in L_2^\vee$, so we have $v = 0$ as an element of $L_1^\vee/L_2^\vee$.

Suppose that $v \in L_2/L_1$ such that for all $w \in L_1^\vee/L_2^\vee$, $\langle v, w\rangle = 0$. Lift v to $f \in L_2$ such that for all $g \in L_1^\vee$, $\langle f, g\rangle = 0$. Then $f \in (L_1^\vee)^\vee = L_1$, so $v = 0$ in L_2/L_1.

c) If $t^n k[\![t]\!] \subset L \subset t^m k[\![t]\!]$, then $t^{-m} k[\![t]\!] \subset L \subset t^{-n} k[\![t]\!]$. Because $-m - (-n) = n - m$, this shows that $w(L^\vee) \leq w(L)$. Therefore $w(L) = w((L^\vee)^\vee) \leq w(L^\vee)$ also, so $w(L) = w(L^\vee)$.

d) Let (m, n) be the standard bounds of L, so $t^n k[\![t]\!] \subset L \subset t^m k[\![t]\!]$. Then $t^{-m} k[\![t]\!] \subset L^\vee \subset t^{-n} k[\![t]\!]$, and because $w(L^\vee) = w(L) = -m - (-n)$, we see that $(-n, -m)$ are the standard bounds of $L^\vee$. Thus

$$\begin{aligned} s(L^\vee) = \dim(L^\vee / t^m k[\![t]\!]) &= \dim((L/t^n k[\![t]\!])^\perp) \\ &= \dim(t^m k[\![t]\!]/t^n k[\![t]\!]) - \dim(L/t^n k[\![t]\!]) = w(L) - s(L). \end{aligned}$$

e) If $L \subset L_0$, this follows from b. In general, let $M \subset L, L_0$. Then

$$\begin{aligned} \dim(L_0 : L) = \dim(L_0 : M) - \dim(L : M) &= \dim(M^\vee : L_0^\vee) - \dim(M^\vee : L^\vee) \\ &= \dim(L^\vee : M^\vee) - \dim(L_0^\vee : M^\vee) = \dim(L^\vee : L_0^\vee) \end{aligned}$$

f) We have $\operatorname{ind}(L^\vee) = \dim(k[\![t]\!] : L^\vee) = \dim(L : k[\![t]\!]) = -\dim(k[\![t]\!] : L) = -\operatorname{ind}(L)$.

□

We can view dualization as a morphism between lattice spaces. In particular, dualization takes $\mathcal{G}_d^{L_1 \subset L_2}$ to $\mathcal{G}_{-d}^{L_2^\vee \subset L_1^\vee}$.

Lemma 2.3.4. *a)* $-^\vee : \mathcal{G}_d^{L_1 \subset L_2} \to \mathcal{G}_{-d}^{L_2^\vee \subset L_1^\vee}$ *is an isomorphism.*

b) $-^\vee$ *gives an isomorphism* $\mathcal{L}_{w,d} \to \mathcal{L}_{w,-d}$*, which assemble into an involution of* $\mathcal{L}_w$*.*

Proof. a) This is an application of the more general fact that given vector spaces V, W with $\dim V = \dim W = n$ and a perfect pairing $\langle \cdot, \cdot \rangle : V \times W \to k$, taking the orthogonal complement with respect to $\langle \cdot, \cdot \rangle$ gives an isomorphism $\operatorname{Gr}(V, d) \to \operatorname{Gr}(W, n - d)$. This is simply a canonical form of the standard isomorphism $\operatorname{Gr}(k, n) \to \operatorname{Gr}(n - k, n)$.

b) For each $n \in \mathbb{Z}$ with $n + w \geq d \geq n$, we have $\mathcal{G}_d^{n+w > n} \xrightarrow{\vee} \mathcal{G}_{-d}^{-n > -n-w} \hookrightarrow \mathcal{L}_{w,-d}$, and it is clear that these maps form a cocone over $\mathcal{D}_{w,d}$, so they give a map $\mathcal{L}_{w,d} \to$

$\mathcal{L}_{w,-d}$. Composing with the map $\mathcal{L}_{w,-d} \to \mathcal{L}_{w,d}$ gives the identity, so these maps are isomorphisms. It is clear that the induced morphism $\mathcal{L}_w \to \mathcal{L}_w$ is an involution.

□

Next, we explore how duality interacts with lattice rings and lattices with actions of lattice rings.

Lemma 2.3.5. *Let R be a lattice ring, and let L, L' be R-lattices. Suppose $\phi : L \to L'$ is an homomorphism of R-modules. Then ϕ is multiplication by an element of $k((t))$.*

Proof. Note that the embedding $R \hookrightarrow k((t))$ is the inclusion of R into its fraction field. Let $\ell \in L$. We can write $\phi(\ell)$ as $f\ell$ for some $f \in k((t))$. Pick $\ell' \in L$; we wish to show that $\phi(\ell') = f\ell'$. Write $\ell' = g\ell$ with $g \in k((t))$, and write $g = p/q$ with $p, q \in R$. Then $q\phi(\ell') = \phi(q\ell') = \phi(p\ell) = p\phi(\ell) = pf\ell$, so $\phi(\ell') = gf\ell = f\ell'$. □

This means that we can identify $\mathrm{Hom}_R(L, L')$ with the subset

$$H := \{f \in k((t)) | f\ell \in L' \ \forall \ell \in L\}.$$

In fact this subset is itself a lattice, for any two lattices L, L'. Indeed, suppose $t^n k[\![t]\!] \subset L \subset t^m k[\![t]\!]$ and $t^{n'} k[\![t]\!] \subset L' \subset t^{m'} k[\![t]\!]$. Then $t^{n'-m} k[\![t]\!] \subset H \subset t^{m'-n} k[\![t]\!]$, and it is clear that H is closed under addition. We take this as the definition of $\mathrm{Hom}(L, L')$ as a lattice. The above shows that if L, L' are R-lattices, $\mathrm{Hom}_R(L, L') = \mathrm{Hom}(L, L')$, so the Hom-set between two R-lattices only depends on the lattices themselves, not the lattice ring R.

Lemma 2.3.6. *We have the following equalities of lattices:*

a) $\mathrm{Hom}(L_1, L_2) = \mathrm{Hom}(L_2^\vee, L_1^\vee)$ *for arbitrary lattices* L_1, L_2

b) $\mathrm{Hom}(R, L) = L$ *for* R *a lattice ring and* L *an* R*-lattice*

c) $\mathrm{Hom}(L, R^\vee) = L^\vee$ *for* R *a lattice ring and* L *an* R*-lattice.*

Proof. a) Let $f \in \operatorname{Hom}(L_1, L_2)$, i.e. $f \in k((t))$ such that $f\ell \in L_2$ for all $\ell \in L_1$. Let $m \in L_2^\vee$. Then for any $\ell \in L_1$, $\operatorname{res}(mf\ell) = 0$, so $fm \in L_1^\vee$. Therefore $\operatorname{Hom}(L_1, L_2) \subset \operatorname{Hom}(L_2^\vee, L_1^\vee)$. The opposite inclusion holds by duality.

b) We have $L \xrightarrow{\sim} \operatorname{Hom}_R(R, L)$ as R-modules, and this map agrees with the inclusion of lattices $L \hookrightarrow \operatorname{Hom}(R, L)$, so this map must be an equality.

c) $\operatorname{Hom}(L, R^\vee) = \operatorname{Hom}(R, L^\vee) = L^\vee$.

□

Recall that a local Cohen-Macaulay k-algebra R admits a unique (up to isomorphism) *canonical module* Ω_R, with the defining property that

$$\operatorname{Ext}^n_R(k, \Omega_R) = \begin{cases} k & n = \dim R \\ 0 & n \neq \dim R \end{cases}$$

and R is Gorenstein if and only if $\Omega_R \cong R$. Given an R-module M, the R-module $\operatorname{Hom}_R(M, \Omega_R)$ is called the *dual module* of M. Since a lattice ring is a reduced one-dimensional Noetherian ring, it is Cohen-Macaulay.

Lemma 2.3.7. *Let R be a lattice ring and let L be an R-lattice. Then $L^\vee$ is an R-lattice isomorphic to the dual module of L.*

Proof. For $n > 0$ and $i \in \mathbb{Z}$, let $k((t))_i$ be the vector subspace of $k((t))$ consisting of formal series whose terms only include exponents congruent to i mod n. If $f \in k((t))_i$ and $g \in k((t))_j$, then $fg \in k((t))_{i+j}$, so $k((t))$ has a $\mathbb{Z}/n\mathbb{Z}$-grading $k((t)) = \bigoplus_{i=0}^{n-1} k((t))_i$. Given a lattice L, we write $L_i := L \cap k((t))_i$, and given $f \in k((t))$, we write f_i for the projection of f to $k((t))_i$. Note that $(fg)_i = \sum_{j=0}^{n-1} f_j g_{i-j}$. In particular, $(t^m f)_i = t^m f_{i-m}$.

Choose $n > 0$ so that $t^{n-1}k[[t]] \subset R$. Then $k[[t^n]]$ is a subring of R, so we can consider L as a module over $k[[t^n]]$. Since $k[[t^n]] \cong k[[z]]$ is Gorenstein, $\operatorname{Hom}_{k[[t^n]]}(L, k[[t^n]])$ is the dual of L as a $k[[t^n]]$-module. The inclusion $k[[t^n]] \hookrightarrow R$ corresponds to the n-fold cover $\operatorname{Spec} R \to \operatorname{Spec} k[[t^n]]$, which is a finite morphism. The pushforward by a finite morphism is exact, and thus commutes with dualization. Therefore $\mathrm{Hom}_{k[\![t^n]\!]}(L, k[\![t^n]\!])$ is isomorphic as an R-module to the dual module of L. We will show that $L^\vee$ is isomorphic as an R-module to $\mathrm{Hom}_{k[\![t^n]\!]}(L, k[\![t^n]\!])$.

For $\ell \in L$ and $f \in L^\vee$, we know $f\ell \in R^\vee$ and its t^{-1} coefficient is 0, and because $w(R) \leq n$, all coefficients of t^d with $d < -n$ are 0 as well. Thus $(f\ell)_{-1} \in t^{n-1}k[\![t^n]\!]$. Define $\phi : L^\vee \to \mathrm{Hom}_{k[\![t^n]\!]}(L, k[\![t^n]\!])$ by $\phi(f)(\ell) = (f\ell)_{-1}t^{1-n}$. This is a homomorphism of R-modules.

Next, note that

$$\begin{aligned}\mathrm{Hom}_{k[\![t^n]\!]}(L, k[\![t^n]\!]) &= \mathrm{Hom}_{k[\![t^n]\!]}\left(\bigoplus_{i=0}^{n-1} L_i, k[\![t^n]\!]\right)\\ &= \bigoplus_{i=0}^{n-1} \mathrm{Hom}_{k[\![t^n]\!]}(L_i, k[\![t^n]\!]).\end{aligned}$$

Furthermore, $t^{-i}L_i$ is a lattice in $k[\![t^n]\!]$, so we can make the identification

$$\mathrm{Hom}_{k[\![t^n]\!]}(L_i, k[\![t^n]\!]) = \{f \in t^{-i}k[\![t^n]\!] | f\ell \in k[\![t^n]\!] \text{ for all } f \in L_i\}$$

where a homomorphism on the left is given by multiplication by an element on the right. Now given an element $\alpha \in \mathrm{Hom}_{k[\![t^n]\!]}(L, k[\![t^n]\!])$, let α_i denote the element of $t^{-i}k[\![t^n]\!]$ corresponding to the projection of α to $\mathrm{Hom}_{k[\![t^n]\!]}(L_i, k[\![t^n]\!])$, so that $\alpha(\ell) = \sum_{i=0}^{n-1} \ell_i\alpha_{n-i}$ (here $\alpha_n = \alpha_0$). Given $\alpha \in \mathrm{Hom}_{k[\![t^n]\!]}(L, k[\![t^n]\!])$ and $\ell \in L$, note that $(\left(\sum_{i=0}^{n-1} \alpha_i\right) t^{n-1}\ell)_{-1} = t^{n-1}?(\left(\sum_{i=0}^{n-1} \alpha_i\right) \ell)_0 = t^{n-1}\sum_{i=0}^{n-1} \alpha_i\ell_{n-i} = t^{n-1}\alpha(\ell)$. Because $\alpha(\ell) \in [\![t^n]\!]$, $\mathrm{res}(t^{n-1}\alpha(\ell)) = 0$. Thus $\left(\sum_{i=0}^{n-1} \alpha_i\right) t^{n-1} \in L^\vee$. Let $\psi : \mathrm{Hom}_{k[\![t^n]\!]}(L, k[\![t^n]\!]) \to L^\vee$ be given by $\psi(\alpha) = \left(\sum_{i=0}^{n-1} \alpha_i\right) t^{n-1}$.

Then ϕ and ψ are mutually inverse. Indeed, let $f \in L^\vee$. Then $\phi(f)(\ell) = (f\ell)_{-1}t^{1-n} = \sum_{i=0}^{n-1} \ell_i f_{-1-i}t^{1-n}$, so $(\phi(f))_i = f_{-1-i}t^{1-n}$. So

$$\psi(\phi(f)) = \left(\sum_{i=0}^{n-1} (\phi(f))_i\right) t^{n-1} = \left(\sum_{i=0}^{n-1} f_{-1-i}t^{1-n}\right) t^{n-1} = \sum_{i=0}^{n-1} f_{-1-i} = f.$$

Next, let $\alpha \in \operatorname{Hom}_{k[\![t^n]\!]}(L, k[\![t^n]\!])$. Then for $\ell \in L$,

$$\begin{aligned}\phi(\psi(\alpha))(\ell) &= \phi\left(t^{n-1}\sum_{i=0}^{n-1}\alpha_i\right)(\ell) = \left(t^{n-1}\ell\sum_{i=0}^{n-1}\alpha_i\right)_{-1} t^{1-n} = t^{n-1}\left(\ell\sum_{i=0}^{n-1}\alpha_i\right)_0 t^{1-n} \\ &= \left(\ell\sum_{i=0}^{n-1}\alpha_i\right)_0 = \sum_{i=0}^{n-1}\ell_{n-i}\alpha_i = \alpha(\ell)\end{aligned}$$

so $\phi(\psi(\alpha)) = \alpha$. Therefore ϕ is a bijective homomorphism, and thus is an isomorphism. □

Next, we explore some properties demonstrating the rigidity of the structure of R-lattices.

Lemma 2.3.8. *Let R be a lattice ring, and let L, L' be R-lattices.*

a) If $L \cong L'$, then $w(L) = w(L')$ and $s(L) = s(L')$

b) $w(L) \leq w(R)$.

Proof. a) Let $\phi : L \to L'$ be an isomorphism, so it is multiplication by some $f \in k((t))$. Suppose $t^n k[\![t]\!] \subset L$. Then $t^{n+\text{val}(f)}k[\![t]\!] = ft^n k[\![t]\!] \subset fL = L'$. Similarly, if $L \subset t^m k[\![t]\!]$, then $L' \subset t^{m+\text{val}(f)}k[\![t]\!]$. Therefore, $w(L') \leq w(L)$. By symmetry, we conclude that $w(L) = w(L')$. Furthermore, if $t^n k[\![t]\!] \subset L$ with n minimal, the above shows that ϕ takes $t^n k[\![t]\!]$ to the maximal standard lattice contained in L'. Therefore $s(L) = \dim(L/t^n k[\![t]\!]) = \dim(\phi(L)/\phi(t^n k[\![t]\!])) = s(L')$.

b) Let $f \in L$ such that $\text{val}(f) = \min\{\text{val}(g) : g \in L\} = m$, the standard lower bound of L. Then fR is a sublattice of L, and m is the standard lower bound of both fR and L, and since $fR \subset L$, the standard upper bound of L is at most that of fR. It follows that $w(L) \leq w(fR) = w(R)$.

□

Lemma 2.3.9. *Let R be a lattice ring and let L be an R-lattice. Suppose that $w(L) = w(R)$ and $s(L) = s(R)$. Then $L \cong R$ as R-modules.*

Proof. Suppose $L \subset t^m k[\![t]\!]$ with m maximal. Let $\ell \in L$ with $\mathrm{val}(\ell) = m$. Then $\ell R \subset L$. But $w(\ell R) = w(R) = w(L)$ and $s(\ell R) = s(R) = w(R)$. From this it follows that the inclusion $\ell R \subset L$ is an equality. □

Remark 2.3.1. Lattice rings are intimately linked to *numerical semigroups*, i.e. subsets of $\mathbb{N}$ containing 0, with finite complement in $\mathbb{N}$, which are closed under addition. Indeed, given a lattice ring R, the set $\mathrm{val}(R)$ of valuations of elements of R is a numerical semigroup, and given a numerical semigroup S, $k[\![t^s]\!]_{s \in S}$ is a lattice ring. In [8], Kunz shows that R is Gorenstein iff $\mathrm{val}(R)$ is a *symmetric* numerical semigroup, i.e. there is some $n \in \mathbb{N}$ such that $a \mapsto n - a$ interchanges S with its complement in $\mathbb{Z}$. Thus we can view much of the study of lattice rings and R-modules as an extension of the study of numerical semigroups.

Corollary 2.3.1. *If R is a Gorenstein lattice ring, then $w(R) = w(R^\vee)$ and $s(R) = s(R^\vee)$.*

Proof. Because R is Gorenstein, R is isomorphic to its own dual R-module. Then by Lemma 2.3.7, $R \cong R^\vee$ as R-modules. The result follows by Lemma 2.3.8. □

The following can be viewed as the lattice version of the Lemma in [Kunz].

Corollary 2.3.2. *Let R be a lattice ring. Then $s(R) \leq w(R)/2$, with equality iff R is Gorenstein.*

Proof. First, by Lemma 3.3, we have $w(R^\vee) = w(R)$ and $s(R) + s(R^\vee) = w(R)$. For $f \in R^\vee$ an element of minimal valuation, fR is a lattice with $w(fR) = w(R) = w(R^\vee)$, and $fR \subset R^\vee$. Thus $s(R^\vee) \geq s(fR) = s(R)$. Then $2s(R) \leq s(R) + s(R^\vee) = w(R)$, so $s(R) \leq w(R)/2$.

Suppose R is Gorenstein. Then $R \cong R^\vee$ as R-modules, so $s(R) = s(R^\vee) = w(R) - s(R)$, so $s(R) = w(R)/2$.

Next, suppose $s(R) = w(R)/2$. Then $s(R^\vee) = w(R) - s(R) = s(R)$. So we have $w(R) = w(R^\vee)$ and $s(R) = s(R^\vee)$. Thus by Lemma 2.3.9, $R \cong R^\vee$ as R-modules. So R is Gorenstein. □

This gives a better description of $R^\vee$ for Gorenstein R:

Lemma 2.3.10. *Let R be a Gorenstein Lattice ring. Then $R^\vee = t^{\dim(R^\vee : R)}R = t^{-w(R)}R$.*

Proof. First, $\dim(R^\vee : R) = \dim(R^\vee : k[\![t]\!]) - \dim(R : k[\![t]\!]) = -\dim(k[\![t]\!] : R) - \dim(R : k[\![t]\!]) = -s(R) - s(R) = w(R)$. Next, $\dim(t^{\dim(R^\vee : R)}R : R^\vee) = 0$, so it suffices to show that $t^{\dim(R^\vee : R)}R \subset R^\vee$, i.e. $t^{-w(R)} \in R^\vee$. This is true because $\operatorname{res}(t^{-w(R)}r)$ is the $t^{w(R)-1)}$ coefficient of r, which is 0 for all $r \in R$. □

Finally, we characterize the units of a lattice ring.

Lemma 2.3.11. *Let R be a lattice ring. Then $R^\times = R \cap (k[\![t]\!]^\times)$ as subsets of $k[\![t]\!]$.*

Proof. The inclusion $R^\times \subset R \cap (k[\![t]\!]^\times)$ is obvious. Suppose $f \in R \cap (k[\![t]\!]^\times)$. Then $\operatorname{val}(f) = 0$, so $\dim(R : fR) = 0$. But $fR \subset R$, so we have $fR = R$. Thus $f \in R^\times$. □

Chapter 3

The Local Compactified Jacobian

3.1 Construction and Analysis of the Local Compactified Jacobian

Let R be a lattice ring. We saw above that any R-lattice has width at most $w(R)$, and thus $\mathcal{L}_{w(R)}$ contains all R-lattices. Thus we may consider the subset of $\mathcal{L}_{w(R)}$ consisting of all R-lattices.

Definition 3.1.1. The **local compactified Picard scheme** of R, denoted $\overline{\mathcal{P}}_R$, is the subset of $\mathcal{L}_{w(R)}$ consisting of the R-lattices, i.e. those lattices that are closed under the multiplication action of R. The **local compactified Jacobian**, denoted $\overline{\mathcal{P}}_R^0$, is $\overline{\mathcal{P}}_R \cap \mathcal{L}_{w(R),\mathrm{ind}(R)}$, i.e. the intersection of $\overline{\mathcal{P}}_R$ with the component of $\mathcal{L}_{w(R)}$ containing R.

Proposition 3.1.1. *$\overline{\mathcal{P}}_R$ is a closed subset of $\mathcal{L}_{w(R)}$.*

Proof. It suffices to check that $\mathcal{P}_R \cap \mathcal{G}_d^{n<n+w(R)}$ is closed in $\mathcal{G}_d^{n<n+w(R)}$, because the irreducible components of $\mathcal{L}_{w(R)}$ are the $\mathcal{G}_d^{n<n+w(R)}$. The ring R is generated as a complete ring by finitely many elements, so a lattice L is an R-lattice if and only if L is closed under multiplication by the generators of R. Each generator acts as a linear operator on the finite-dimensional vector space $t^{n+w(R)}k[\![t]\!]/t^n k[\![t]\!]$, and we can identify $\mathcal{P}_R \cap \mathcal{G}_d^{n<n+w(R)}$ with the subset of the Grassmannian consisting of subspaces closed under these linear op-

erators. Given a linear operator T on a vector space V, the subset of $\mathrm{Gr}(V, d)$ consisting of subspaces closed under T is closed, so the proposition follows. $\square$

Since we realize $\overline{\mathcal{P}}_R$ and $\overline{\mathcal{P}}_R^0$ as closed subsets of varieties, we give them the natural induced variety structure.

Recall that $\mathcal{L}_{w(R),d}$ can be embedded in the Grassmannian $\mathcal{G}_d^{n+w(R)>n-w(R)}$. The component $\overline{\mathcal{P}}_R \cap \mathcal{L}_{w(R),d}$ can be alternately described as the locus of points of $\mathcal{G}_d^{n+w(R)>n-w(R)}$ stable under the action of R when viewed as lattices: indeed, any R-lattice L has $w(L) \leq w(R)$, and so any lattice in $\mathcal{G}_d^{n+w(R)>n-w(R)}$ must lie on $\mathcal{L}_{w(R),d}$. This is in some sense a simpler description, and appeared in [2]. However, we claim that having access to the components of $\mathcal{L}_{w(R),d}$ allows for an analysis of $\overline{\mathcal{P}}_R^0$ unavailable in the other perspective.

Lemma 3.1.1. *The subset $\mathcal{P}_R^0 \subset \overline{\mathcal{P}}_R^0$ corresponding to the principal R-lattices is open, and is a connected algebraic group of dimension $\delta(R)$, where the group operation is given by elementwise multiplication.*

Proof. First, the principal R-lattices are exactly the R-lattices isomorphic to R. By Lemma 2.3.9, these are exactly the R-lattices L with $w(L) = w(R)$ and $s(L) = s(R)$. Thus the principal R-lattices in $\overline{\mathcal{P}}_R^0$ are exactly the R-lattices with the same bounds $(0, w(R))$ and relative dimension as R. Because $\mathcal{G}_{\dim(R)}^{n<n+w(R)}$ consists of lattices whose bounds are bounded by $(n, n+w(R))$, an element of $\overline{\mathcal{P}}_R^0$ is a principal R-lattice if and only if it lies on $\mathcal{G}_{\dim(R)}^{0<w(R)}$ and no other $\mathcal{G}_{\dim(R)}^{n<n+w(R)}$. Therefore $\mathcal{P}_R^0 = \overline{\mathcal{P}}_R^0 \cap \left(\mathcal{L}_{w(R),\dim(R)} - \bigcup_{n>0} \mathcal{G}_{\dim(R)}^{n<n+w(R)}\right)$, taken inside $\mathcal{L}_{w(R),\dim(R)}$. Each $\mathcal{G}_{\dim(R)}^{n<n+w(R)}$ is closed in $\mathcal{L}_{w(R),\dim(R)}$, so this tells us that $\mathcal{L}_{w(R),\dim(R)} - \bigcup_{n>0} \mathcal{G}_{\dim(R)}^{n<n+w(R)}$ is open in $\mathcal{L}_{w(R),\dim(R)}$, so $\mathcal{P}_R^0$ is open in $\overline{\mathcal{P}}_R^0$.

Next, first note that the group operation on $\mathcal{P}_R^0$ gives $fR \cdot gR = fgR$ for $f, g \in k[\![t]\!]^\times$. Consider the algebraic group $G = (k[t]/(t^n))^\times$, which acts algebraically on the affine space $\mathbb{A}(k[t]/(t^n))$. This induces an algebraic action of G on $\mathrm{Gr}(k[t]/(t^n), d)$. Taking $n = w(R)$, we get an action of G on $\mathrm{Gr}(k[t]/(t^{w(R)}), s(R)) \cong \mathcal{G}_{\dim(R)}^{0<w(R)}$. Given $f \in k[\![t]\!]$, the lattice fR is determined by $fR/t^{w(R)}k[\![t]\!]$, which is determined by the image of f in $k[\![t]\!]/t^{w(R)}k[\![t]\!] = k[t]/(t^{w(R)})$. Therefore the orbit of $R \in \mathcal{G}_{\dim(R)}^{0<w(R)}$ under the action of G is exactly the set of all lattices of the form fR, i.e. $\mathcal{P}_R^0$. Therefore we can identify $\mathcal{P}_R^0$ with

the quotient of G by the stabilizer G_R of R, which is the image of $R^\times$ in $k[\![t]\!]/t^{w(R)}k[\![t]\!]$. Because G is an abelian group, this gives $\mathcal{P}_R^0$ the structure of an algebraic group. Because $R^\times = k[\![t]\!]^\times \cap R$, $G_R = G \cap (R/t^{w(R)}k[\![t]\!])$. Now $\dim G = w(R)$ and $\dim G_R = s(R)$, so $\dim \mathcal{P}_R^0 = w(R) - s(R) = \delta(R)$. Because G is connected surjects onto $\mathcal{P}_R^0$, $\mathcal{P}_R^0$ is also connected. □

Recall that the map $L \mapsto L^\vee$ induces an involution on the lattice space $\mathcal{L}_w$. The involution restricts to an involution local compactified Picard scheme, because L is an R-lattice if and only if $L^\vee$ is. Furthermore, dualization gives isomorphisms between parts of $\overline{\mathcal{P}}_R$ lying on dual components of $\mathcal{L}_{w(R)}$. In fact:

Lemma 3.1.2. *The map $\overline{\mathcal{P}}_R^0 \to \overline{\mathcal{P}}_R^0$ given by $L \mapsto t^{2\,\mathrm{ind}(R)}L^\vee$ is an involution.*

Proof. Both the dualization map and the map $L \mapsto tL$ are automorphisms of $\mathcal{L}_{w(R)}$ that restrict to $\overline{\mathcal{P}}_R$. It is clear that the given map is well-defined and squares to the identity. □

The following lemma is used in the main theorem for this section.

Lemma 3.1.3. *Let A be a local k-algebra with maximal ideal m, and let I be an ideal such that $\dim_k(m/I) = 1$. Then $m^2 \subset I$.*

Proof. Note that A/I is local with maximal ideal m/I, and $\dim_k(m/I) = 1$, so m/I is a principal ideal of A/I. Let $m/I = (x)$. Then we need to show that $x^2 = 0$. Suppose $x^2 \neq 0$. Then $x^2 \in (x)$, so $x^2 = ax$ for $a \in k$. But then $x(x-a) = 0$, but $x - a$ is a unit in A/I. Thus $x = 0$, a contradiction. □

The next theorem is analogous to Proposition (4) of [7]. The precise relationship between the two results will be explored in the following section.

Theorem 3.1.1. *Let R be a lattice ring whose embedding dimension is at least 3. Then the local compactified Jacobian $\overline{\mathcal{P}}_R^0$ is reducible.*

Proof. First, suppose that R is non-Gorenstein. Then we have $w(R^\vee) = w(R)$. Let $D = t^{\dim(R:R^\vee)}R^\vee$, so $\operatorname{ind}(D) = \operatorname{ind}(R)$. Now

$$\dim(R : R^\vee) = \dim(R/t^{-w(R)}k[\![t]\!]) - \dim(R^\vee/t^{-w(R)}k[\![t]\!]) =$$
$$\dim(R/k[\![t]\!]) + w(R) - \dim(R^\vee/t^{-w(R)}k[\![t]\!]) =$$
$$(w(R) - s(R)) + w(R) - (w(R) - s(R^\vee)) = w(R) + s(R^\vee) - s(R).$$

But because R is non-Gorenstein, $s(R^\vee) > s(R)$, so $\dim(R : R^\vee) > w(R)$. Now because $w(R) = w(D)$, R and D as points of $\overline{\mathcal{P}}^0_R$ each lie on a single component of $\mathcal{L}_{w(R),\operatorname{ind}(R)}$. Furthermore, the standard bounds of D are $(\dim(R : R^\vee) - w(R), \dim(R : R^\vee))$, while the standard bounds of R are $(0, w(R))$, and these are not equal as observed above. So $R \in \mathcal{G}^{w(R)>0}_{\operatorname{ind}(R)}$ and $D \in \mathcal{G}^{\dim(R:R^\vee)>\dim(R:R^\vee)-w(R)}_{\operatorname{ind}(R)}$, which are two different components of $\mathcal{L}_{w(R),\operatorname{ind}(R)}$. Thus the component of $\overline{\mathcal{P}}^0_R$ containing R is contained in $\mathcal{G}^{w(R)>0}_{\operatorname{ind}(R)}$ and the component containing D is contained in $\mathcal{G}^{\dim(R:R^\vee)>\dim(R:R^\vee)-w(R)}_{\operatorname{ind}(R)}$, and R and D are not in the intersections of these Grassmannians, so R and D cannot be on the same component of $\overline{\mathcal{P}}^0_R$. Thus $\overline{\mathcal{P}}^0_R$ is reducible.

Next, suppose that R is Gorenstein. Because the set $\mathcal{P}^0_R$ of principal R-lattices is an open subset of $\overline{\mathcal{P}}^0_R$ of dimension $\delta(R)$, it suffices to find another subset of $\overline{\mathcal{P}}^0_R$ of dimension at least $\delta(R)$, disjoint from $\mathcal{P}^0_R$. Let m denote the maximal ideal of R. We will show that the set X of lattices L such that $\dim(L : R) = 0$ and there is $f \in k[\![t]\!]$ so that $m \supset fL \supset m^2$ and $\dim_k m/fL = 1$, is such a subset. First, we show that X is disjoint from $\mathcal{P}^0_R$, i.e. any lattice in X is nonprincipal. If some principal lattice is in X, then there is some $f \in k[\![t]\!]$ such that $fR \subset R$, with $m/fR = 1$. But then m is generated by f and one other element, which means the embedding dimension of R is at most 2, contrary to our assumption.

Now we need to show that $\dim(X) \geq \delta(R)$. Note that for a lattice L, $m \supset L \supset m^2$ with $\dim_k m/L = 1$ iff $L \subset R$ with $\dim_k R/L = 2$. The left-to-right implication is clear because $\dim_k R/m = 1$. For the other direction, suppose $L \subset R$ with $\dim_k R/L = 2$. Since

R is local, it is clear that $L \subset m$, and $\dim_k m/L = \dim_k R/L - \dim_k R/m = 2 - 1 = 1$, and $m^2 \subset L$ by Lemma 4.3.

Now, consider the surjection $\mathcal{P}_R^0 \times \mathrm{Gr}(m/m^2, 1) \twoheadrightarrow X$, given by $(fR, V) \mapsto t^{-2}fV$, viewing $V \in \mathrm{Gr}(m/m^2, 1)$ as a lattice with $m \supset V \supset m^2$. The left side has dimension $\delta(R) + e - 1$ (where $e \geq 3$ is the embedding dimension of R), so we must bound the dimension of a fiber. That is, given a lattice $L \in X$, we want to bound

$$\begin{aligned} &\dim\{(fR \in \mathcal{P}_R^0, V \in \mathrm{Gr}(m/m^2, 1) | t^{-2}fV = L\} \\ =&\dim\{fR \in \mathcal{P}_R^0 \,| t^2 f^{-1} L \in \mathrm{Gr}(m/m^2, 1)\} \\ =&\dim\{fR \in \mathcal{P}_R^0 \,| t^2 f^{-1} L \subset R,\ \dim_k R/t^2 f^{-1} L = 2\} \\ =&\dim\{fR \in \mathcal{P}_R^0 \,| t^2 f L \subset R,\ \dim_k R/t^2 f L = 2\} \end{aligned}$$

Note that $t^2 fL \subset R \iff f \in \mathrm{Hom}(t^2 L, R) = \mathrm{Hom}(t^2 L, t^{w(R)} R^\vee) = \mathrm{Hom}(t^{2-w(R)} L, R^\vee) = (t^{2-w(R)} L)^\vee = t^{w(R)-2} L^\vee$. Thus we are trying to find the dimension of the set of elements of $\mathcal{P}_R^0$ that are generated as principal lattices by elements of $t^{w(R)-2} L^\vee$. Without loss of generality, assume $t^2 L \subset R$; this can be done by replacing L with fL for suitable f, which doesn't change the dimension count. Then $t^{-w(R)} R = R^\vee \subset (t^2 L)^\vee = t^{-2} L^\vee$, so $R \subset t^{w(R)-2} L^\vee$. Now we have $\dim_k(t^{w(R)-2} L^\vee / R) = \dim(t^{w(R)-2} L^\vee : R) = \dim(R : t^2 L) = 2$. Finally, just as we can think of $\mathcal{P}_R^0$ as $k[\![t]\!]^\times / R^\times$, we can think of $\{fR \in \mathcal{P}_R^0 \,| f \in t^{w(R)-2} L^\vee\}$ as $(k[\![t]\!]^\times \cap t^{w(R)-2})/R^\times$. As in the proof of Lemma 4.1, the dimension of this is equal to $\dim_k(k[\![t]\!] \cap t^{w(R)-2} L^\vee / R) \leq \dim_k(t^{w(R)-2} L^\vee / R) = 2$. Therefore $\dim(X) \geq \delta(R) + e - 3 \geq \delta(R)$, as desired. □

In the previous proof, the strategies to handle the non-Gorenstein and Gorenstein cases differed dramatically. One may ask whether the simpler proof method used in the non-Gorenstein case could possibly work for the Gorenstein case as well, i.e. whether $\overline{\mathcal{P}}_R^0$ intersects different components of $\mathcal{L}_{w(R),\mathrm{ind}(R)}$ nontrivially for R Gorenstein. We show that this is not the case.

Proposition 3.1.2. *Let R be a Gorenstein lattice ring. Then the inclusion $\overline{\mathcal{P}}_R^0 \hookrightarrow$*

$\mathcal{L}_{w(R),\mathrm{ind}(R)}$ *factors through* $\mathcal{G}^{w(R)>0}_{\mathrm{ind}(R)} \hookrightarrow \mathcal{L}_{w(R),\mathrm{ind}(R)}$.

Proof. We will prove the following equivalent statement: let L be an R-lattice with $\dim(L : R) = 0$; then $t^{w(R)}k[\![t]\!] \subset L \subset k[\![t]\!]$. Let $f \in L$. Then $fR \subset L$, so $\dim(L : fR) \geq 0$. Now $\dim(L : fR) = \dim(L : R) - \dim(fR : R) = 0 - (-\operatorname{val}(f)) = \operatorname{val}(f)$. Thus $\operatorname{val}(f) \geq 0$ for all $f \in L$, so $L \subset k[\![t]\!]$. On the other hand, $0 = \dim(L^\vee : R^\vee) = \dim(L^\vee : t^{-w(R)}R) = \dim(t^{w(R)}L^\vee : R)$. Applying the above to $t^{w(R)}L^\vee$, we see that $t^{w(R)}L^\vee \subset k[\![t]\!]$, so $L^\vee \subset t^{-w(R)}k[\![t]\!]$, so dually, $t^{w(R)}k[\![t]\!] \subset L$. □

This means that for R Gorenstein, we can identify $\mathcal{P}^0_R$ with the subset of $\mathcal{G}^{w(R)>0}_{\mathrm{ind}(R)}$ consisting of lattices closed under the action of R.

3.2 Comparison to the Compactified Jacobian

In this section, we will relate the compactified Jacobian of a curve with unibranch singularities to the local compactified Jacobians at the singularities. First, we will see that lattice rings correspond to completed local rings of unibranch curve singularities, and points of the local compactified Jacobian correspond to descent data for rank-one torsion-free sheaves at curve singularities. Then, we will construct a family of rank-one torsion free sheaves parameterized by the local compactified Jacobian, and analyze the corresponding morphism to the compactified Jacobian.

Lemma 3.2.1. *Let X be a curve and $x \in X$ be a unibranch singular point. Let R_x be the image of the inclusion of $\widehat{\mathcal{O}}_{X,x}$ into its fraction field, identified with $k((t))$. Then R_x is a lattice ring.*

Proof. The integral closure of $\widehat{\mathcal{O}}_{X,x}$ is $k[\![t]\!]$, so we have $R_x \subset k[\![t]\!]$. Next, because $\operatorname{Frac}\widehat{\mathcal{O}}_{X,x} = k((t))$, we have $f, g \in R_x$ with $\frac{f}{g} = t$, and in particular $n := \operatorname{val}(f) = \operatorname{val}(g) + 1$. Then by the Frobenius coin problem, for all $m > n^2 - n - 1$, there is $a, b > 0$ such that $\operatorname{val}(f^a g^b) = m$. Thus $h \in k((t))$ with $\operatorname{val}(h) > n^2 - n - 1$ can be expressed as a formal sum of elements of the form $cf^a g^b$. Thus $t^{n^2-n}k[\![t]\!] \subset R_x$. Thus R_x is a lattice ring. □

Conversely, given a lattice ring R and a curve X with a smooth point x, we can "insert R at x", i.e. form a new curve X' isomorphic to X away from x but with the completed local ring at x replaced with R, by taking the pushout of the morphisms $\operatorname{Spec} k((t)) \cong \operatorname{Spec} \operatorname{Frac} \widehat{\mathcal{O}}_{C,p} \to C - \{p\}$ and $\operatorname{Spec} k((t)) \to \operatorname{Spec} R$.

In the above, the identifications $k((t)) \cong \operatorname{Frac} \widehat{\mathcal{O}}_{X,x}$ amount to choices of uniformizing parameters. In other words, a lattice ring is equivalent to a completed local ring of a unibranch point on a curve, along with a choice of uniformizing parameter at the lift of the point to the normalization.

Let X be a singular curve, and let $x_1, \ldots, x_n$ be the singular points of X. Then there is an fpqc cover of X given by $X^{\mathrm{sm}} = X - \{x_1, \ldots, x_n\}$ together with $\operatorname{Spec} \widehat{\mathcal{O}}_{X,x_i}$ for all i. The intersection of X^{sm} and $\operatorname{Spec} \widehat{\mathcal{O}}_{X,x_i}$ is the punctured formal neighborhood of x_i, $\operatorname{Spec} \operatorname{Frac}(\widehat{\mathcal{O}}_{X,x_i})$; this is isomorphic to $k((t))^n$, where n is the number of branches of the singularity at x_i and t is a local parameter of each branch of the normalization. No other pairs of elements of the cover intersect, and so no three intersect. If all the singularities of X are unibranch, then all of the intersections are isomorphic to $\operatorname{Spec} k((t))$. In this case, a descent datum for a sheaf defined on this cover consists of a sheaf $\mathcal{F}$ on X^{sm}, a module M_i over each $\widehat{\mathcal{O}}_{X,x_i}$, along with isomorphisms over the intersections. If the sheaf is coherent and torsion-free of generic rank 1, then its restriction to X^{sm} is a line bundle, so in this case the restriction of $\mathcal{F}$ to the punctured formal neighborhood of each x_i is isomorphic to $k((t))$. Thus, for each i we must have an isomorphism $M_i \otimes k((t)) \to k((t))$.

Given modules M_1, M_2 over a lattice ring R and isomorphisms $\phi_i : M_i \otimes k((t)) \xrightarrow{\sim} k((t))$ for $i = 1, 2$, say $(M_1, \phi_1) \sim (M_2, \phi_2)$ if there is an isomorphism $\psi : M_1 \to M_2$ such that

$$\begin{array}{ccc} M_1 \otimes k((t)) & \xrightarrow{\psi \otimes k((t))} & M_2 \otimes k((t))) \\ & \searrow^{\phi_1} \quad \swarrow_{\phi_2} & \\ & k((t)) & \end{array}$$

commutes.

Note that given a lattice ring $R \subset k[\![t]\!]$, the R-submodules of $k((t))$ are exactly 0, the R-lattices, and $k((t))$. Indeed, suppose $t^n k[\![t]\!] \subset R$. If $M \subset k((t))$ is a nonzero R-submodule and $m \in M$, then $m \cdot t^n k[\![t]\!] = t^{n+\mathrm{val}(m)} k[\![t]\!] \subset M$. If M is not a lattice, then for all $a \in \mathbb{Z}$ we can find $m \in M$ with $\mathrm{val}(m) < a$, and thus we have $t^{n+a} k[\![t]\!] \subset M$ for all $a \in \mathbb{Z}$. So $M \supset \bigcup_{a \in \mathbb{Z}} t^{n+a} k[\![t]\!] = k((t))$, so $M = k((t))$.

Lemma 3.2.2. *Let $R \subset k((t))$ be a lattice ring. There there is a bijection between the sets*

$$A_R = \{(M, \phi) : M \text{ a f.g. rank-1 torsion-free } R\text{-module}, \phi : M \otimes k((t)) \xrightarrow{\sim} k((t))\}/\sim$$

and

$$B_R = \{R\text{-lattices in } k((t))\}$$

Proof. Define $f : A_R \to B_R$ by sending (M, ϕ) to the R-lattice given by the composition $M \hookrightarrow M \otimes k((t)) \xrightarrow{\phi} k((t))$. Clearly this respects $\sim$ and thus is well-defined. The map $M \to M \otimes k((t))$ given by $m \mapsto m \otimes 1$ is an injection because M is torsion-free. The image of the composite map is a finitely generated R-submodule of $k((t))$, which must be an R-lattice.

Define $g : B_R \to A_R$ by sending an R-lattice $L \subset k((t))$ to the pair $(L, L \otimes k((t)) \to k((t)))$, where the map $\psi : L \otimes k((t)) \to k((t))$ is given by $\ell \otimes f \mapsto \ell f$. We must check that this map is an isomorphism. For any $f \in k((t))$, we can choose an arbirtary $\ell \in L$, then $\psi(\ell \otimes f\ell^{-1}) = f$, so ψ is surjective. Next, let $\ell \in L$ and $f \in k((t))$. Because $\operatorname{Frac} R_x = k((t))$, we can write $f = \frac{f_1}{f_2}$ where $f_1, f_2 \in R_x$. Next, because R_x is a lattice ring, we can find n such that $t^n f_2^{-1} \in R_x$. Thus $\ell \otimes f = \ell \otimes f_1 f_2^{-1} t^n t^{-n} = \ell f_1 f_2^{-1} t^n \otimes t^{-n} = \ell f t^n \otimes t^{-n}$. Furthermore, in the argument we can replace n with any $m > n$. It follows that any simple tensor can be written in the form $\ell \otimes t^n$, and in fact any linear combination of simple tensors can be written so that the powers of t on the right are all the same. Thus every element of $L \otimes k((t))$ is of the form $\ell \otimes t^n$. Now suppose $\psi(\ell \otimes t^n) = \psi(\ell' \otimes t^m)$, so $\ell t^n = \ell' t^m$. As above, we can rewrite $\ell \otimes t^n = j \otimes t^p$ and $\ell' \otimes t^m = j' \otimes t^p$ for some p. Then $jt^p = j't^p$, so $j = j'$, and it follows that ψ is injective. Thus ψ is an isomorphism.

Finally, we check that f and g are mutually inverse. If $(M, \phi) \in A_R$, then the inclusion $f(M) \subset k((t))$ is given by $m \mapsto \phi(m \otimes 1)$. Then $g(f(M)) = (M, \phi')$ where $\phi' : M \otimes k((t)) \to k((t))$ is given by $m \otimes f \mapsto \phi(m \otimes 1)f = \phi(m \otimes f)$. So $\phi' = \phi$, so $g \circ f = \mathrm{id}_{A_R}$. On the other hand, if $L \subset k((t))$ is an R-lattice, then the inclusion $f(g(L)) \hookrightarrow k((t))$ is given by $L \hookrightarrow L \otimes k((t)) \to k((t))$, sending $\ell \mapsto \ell \otimes 1 \mapsto \ell \cdot 1 = \ell$. Thus $f(g(L)) = L$. □

We will use A_x and B_x to refer to A_{R_x} and B_{R_x}, respectively.

Now suppose X is a curve with only unibranch singularities at $x_1, \ldots, x_n \in X$. A choice of $(M_i, \phi_i) \in A_{x_i}$ for each i can be extended to a descent datum of a sheaf over X by adding the trivial line bundle $\mathcal{O}_{X^{\mathrm{sm}}}$ along with the natural restriction isomorphisms $\mathcal{O}_{X^{\mathrm{sm}}} \otimes \operatorname{Frac} \widehat{\mathcal{O}}_{X,x_i} \xrightarrow{\sim} \operatorname{Frac} \widehat{\mathcal{O}}_{X,x_i} \cong k((t))$ over the punctured neighborhoods of the x_i. This is exactly a descent datum for a rank-1 torsion-free sheaf which is trivial over X^{sm}. Furthermore, the equivalence relation defining A_R is exactly the condition of isomorphism of descent data. Furthermore, because we are working with an fpqc cover, all descent data is effective. Thus we can conclude the following:

Corollary 3.2.1. *Let X be a curve with unibranch singularities at $x_1, \ldots, x_n \in X$ and no other singularities. Then $\prod_{i=1}^n B_{x_i}$ is in bijection with the set of isomorphism classes of rank-1 torsion free sheaves on X which are trivial over X^{sm}.*

If, in addition, X is rational, then every line bundle on X^{sm} is trivial. Thus we get the following:

Corollary 3.2.2. *Let X be a rational curve with unibranch singularities at $x_1, \ldots, x_n \in X$ and no other singularities. Then $\prod_{i=1}^n B_{x_i}$ is in bijection with the set of isomorphism classes of rank-1 torsion free sheaves on X.*

Given $(L_1, \ldots, L_n) \in \prod_{i=1}^n B_{x_i}$, we will refer to the corresponding sheaf as $\mathcal{F}_{(L_1,\ldots,L_n)}$. By the construction above, the completed stalk of $\mathcal{F}_{(L_1,\ldots,L_n)}$ at x_i can be identified with L_i.

Lemma 3.2.3. *For each i, let $L_i, L'_i \in B_{x_i}$. Then*

$$\chi(\mathcal{F}_{(L_1,\ldots,L_n)}) - \chi(\mathcal{F}_{(L'_1,\ldots,L'_n)}) = \sum_{i=1}^{n} \dim(L_i : L'_i)$$

where χ is the Euler characteristic.

Proof. First suppose $L'_i \subset L_i$ for each i. Then $\mathcal{F}_{(L_1,\ldots,L_n)} \,/\, \mathcal{F}_{(L'_1,\ldots,L'_n)}$ is the direct sum of the skyscraper sheaves supported at x_i with stalk L_i/L'_i for each i. Therefore

$$\begin{aligned}\chi(\mathcal{F}_{(L_1,\ldots,L_n)}) - \chi(\mathcal{F}_{(L'_1,\ldots,L'_n)}) = \chi(\mathcal{F}_{(L_1,\ldots,L_n)} \,/\, \mathcal{F}_{(L'_1,\ldots,L'_n)}) &= \sum_{i=1}^{n} \dim(L_i/L'_i) \\ &= \sum_{i=1}^{n} \dim(L_i : L'_i).\end{aligned}$$

In general, let $L''_i = L_i \cap L'_i$. Then $L''_i \subset L_i, L'_i$, so

$$\begin{aligned}\chi(\mathcal{F}_{(L_1,\ldots,L_n)}) - \chi(\mathcal{F}_{(L'_1,\ldots,L'_n)}) &= \\ \chi(\mathcal{F}_{(L_1,\ldots,L_n)}) - \chi(\mathcal{F}_{(L''_1,\ldots,L''_n)}) - (\chi(\mathcal{F}_{(L'_1,\ldots,L'_n)}) - \chi(\mathcal{F}_{(L''_1,\ldots,L''_n)})) &= \\ \sum_{i=1}^{n} \dim(L_i : L''_i) - \dim(L'_i : L''_i) &= \sum_{i=1}^{n} \dim(L_i : L'_i).\end{aligned}$$

□

Note that if each $L_i \in \overline{\mathcal{P}}^0_{R_{x_i}}$, i.e. $\dim(L_i : R_{x_i}) = 0$, then $\chi(\mathcal{F}_{(L_1,\ldots,L_n)}) - \chi(\mathcal{O}_x) = \chi(\mathcal{F}_{(L_1,\ldots,L_n)}) - \chi(\mathcal{F}_{(R_{x_1},\ldots,R_{x_n})}) = \sum_{i=1}^{n} \dim(L_i : R_{x_i}) = 0$, so $\mathcal{F}_{(L_1,\ldots,L_n)}$ is a point of the compactified Jacobian $\overline{J}_X$.

Lemma 3.2.4. *Let X be a curve with unibranch singularities at $x_1, \ldots, x_n \in X$. The map $\prod_{i=1}^{n} B^{d_i}_{x_i} \to \overline{J}^{d_1+\cdots+d_n}_X(k)$ given by $(L_1, \ldots, L_n) \mapsto \mathcal{F}_{(L_1,\ldots,L_n)}$ is injective.*

Proof. Given $(L_1, \ldots, L_n), (L'_1, \ldots, L'_n) \in \prod_{i=1}^{n} B^{d_i}_{x_i}$, a morphism of the corresponding descent data consists of a $\widehat{\mathcal{O}}_{X,x_i}$-module homomorphism $L_i \to L'_i$ for each i, and an endomorphism of $\mathcal{O}_{X^{\mathrm{sm}}}$, with compatibility conditions. Equivalently, this is $\ell_i \in k((t))$ for each i and $f \in k[X^{\mathrm{sm}}]$ such that the restriction of f to $\operatorname{Spec} \widehat{\mathcal{O}}_{X,x_i}$ is ℓ_i. If $\mathcal{F}_{(L_1,\ldots,L_n)} \cong \mathcal{F}_{(L'_1,\ldots,L'_n)}$, then there is an isomorphism of descent data. If $(\ell_1, \dots, \ell_n, f)$ is an isomorphism of descent data, then we have $\mathrm{val}(\ell_i) = 0$ for each i, because $\dim(L_i) = \dim(L_i')$. In particular, each $\ell_i \in k[\![t]\!]$. Because f restricts to ℓ_i, this means that f extends over the punctures of X^{sm}, and thus is a globally defined function on the normalization X^ν. But X^ν is projective, so f is constant and each ℓ_i is constant, thus $L_i' = L_i$. □

Let X be a rational curve with unibranch singularities at $x_1, \dots, x_n \in X$. As B_{x_i} can be identified with the k-points of $\overline{\mathcal{P}}_{R_{x_i}}$, we have established that the map $(L_1, \dots, L_n) \mapsto \mathcal{F}_{(L_1,\dots,L_n)}$ gives a bijection between the k-points of $\prod_{i=1}^n \overline{\mathcal{P}}_{R_{x_i}}^{d_i}$ and $\overline{J}_X^{d_1+\cdots+d_n}$. We now show that this extends to a morphism of schemes $\prod_{i=1}^n \overline{\mathcal{P}}_{R_{x_i}}^{d_i} \to \overline{J}_X^{d_1+\cdots+d_n}$. To do so, we construct a sheaf $\mathcal{F}$ on $\prod_{i=1}^n \overline{\mathcal{P}}_{R_{x_i}}^{d_i} \times X$ such that for each $(L_1, \dots, L_n) \in \prod_{i=1}^n \overline{\mathcal{P}}_{R_{x_i}}^{d_i}(k)$, the restriction of $\mathcal{F}$ to $\{(L_1, \dots, L_n)\} \times X \cong X$ is $\mathcal{F}_{(L_1,\dots,L_n)}$.

To construct $\mathcal{F}$, recall that for each i, there are $a_i < b_i$ so that $\overline{\mathcal{P}}_{R_{x_i}}^{d_i}$ is identified with the subset of $\mathrm{Gr}(t^{a_i}k[\![t]\!]/t^{b_i}k[\![t]\!], d_i)$ consisting of lattices closed under the action of R_{x_i}. In particular, for each $L_i \in \overline{\mathcal{P}}_{R_{x_i}}^{d_i}$, we have $t^{b_i}k[\![t]\!] \subset L_i \subset t^{a_i}k[\![t]\!]$. Let $S = \prod_{i=1}^n \overline{\mathcal{P}}_{R_{x_i}}^{d_i}$, and let $\mathcal{F}_1 = \mathcal{O}_S \boxtimes \mathcal{F}_{(t^{b_1}k[\![t]\!],\dots,t^{b_n}k[\![t]\!])}$ and $\mathcal{F}_2 = \mathcal{O}_S \boxtimes \mathcal{F}_{(t^{a_1}k[\![t]\!],\dots,t^{a_n}k[\![t]\!])}$. The restriction of $\mathcal{F}_1$ to $\{(L_1, \dots, L_n)\} \times X$ is $\mathcal{F}_{(t^{b_1}k[\![t]\!],\dots,t^{b_n}k[\![t]\!])}$, and similarly for $\mathcal{F}_2$. Thus we want to construct $\mathcal{F}$ as a subsheaf of $\mathcal{F}_2$ containing $\mathcal{F}_1$, or equivalently as a subsheaf of $\mathcal{F}_2 / \mathcal{F}_1$. Because $\mathcal{F}_{(t^{a_1}k[\![t]\!],\dots,t^{a_n}k[\![t]\!])}$ and $\mathcal{F}_{(t^{b_1}k[\![t]\!],\dots,t^{b_n}k[\![t]\!])}$ are both trivial over the smooth locus of X, we can see that $\mathcal{F}_2 / \mathcal{F}_1$ is 0 over $S \times X^{\mathrm{sm}}$, and over $S \times \{x_i\}$, $\mathcal{F}_2 / \mathcal{F}_1$ restricts to $\mathcal{O}_S \otimes (t^{a_i}k[\![t]\!]/t^{b_i}k[\![t]\!])$. Now we have $S \times \{x_i\} \to S \to \overline{\mathcal{P}}_{R_{x_i}}^{d_i} \to \mathrm{Gr}(t^{a_i}k[\![t]\!]/t^{b_i}k[\![t]\!], d_i)$, so we can pull back the universal family of the Grassmannian and get a subsheaf of $\mathcal{O}_S \otimes (t^{a_i}k[\![t]\!]/t^{b_i}k[\![t]\!])$. Doing this over all $\{x_i\}$ gives the desired subsheaf $\mathcal{F}'$ of $\mathcal{F}_2 / \mathcal{F}_1$. Note that by construction, for each $(s, x) \in S \times X$, $\mathcal{F}'_{(s,x)}$ is closed under the action of $\mathcal{O}_{X,x}$, so $\mathcal{F}'$ is a $\mathcal{O}_{S\times X}$-submodule of $\mathcal{F}_2 / \mathcal{F}_1$. Now $\mathcal{F} = \pi^{-1}(\mathcal{F}')$ is the desired sheaf, where $\pi : \mathcal{F}_2 \to \mathcal{F}_2 / \mathcal{F}_1$ is the quotient map.

Let $\phi : \prod_{i=1}^n \overline{\mathcal{P}}_{R_{x_i}}^{d_i} \to \overline{J}_X^{d_1+\cdots+d_n}$ be the morphism of schemes determined by $\mathcal{F}$.

Theorem 3.2.1. *Write $P := \prod_{i=1}^n \overline{\mathcal{P}}_{R_{x_i}}^{d_i}$ and $J := \overline{J}_X^{d_1+\cdots+d_n}$. Then $\phi : P \to J$ is a universal homeomorphism, i.e. for any morphism of k-schemes $S \to J$, the pullback* *morphism* $\pi_1 : P \underset{J}{\times} S \to S$ *is a homeomorphism.*

Proof. A closed, continuous bijection between topological spaces is a homeomorphism. Because P is projective, it is proper, so any morphism $P \to Y$ is proper, so in particular ϕ is proper. Thus ϕ is universally closed. We must show that ϕ is a universal bijection, i.e. for any morphism of k-schemes $S \to J$, $\pi_1 : S \underset{J}{\times} P \to S$ is a bijection on the sets of schematic points. Note that above, we showed that ϕ is a induces a bijection $P(k) \to J(k)$. In fact, the argument does not use anything about the field k, so the same argument shows that ϕ induces a bijection $P(K) \to J(K)$ for any algebraically closed extension $K \supset k$.

Let $f : S \to J$ be an arbitrary morphism of k-schemes. Let p be a point of S. Then there is an algebraically closed field K and $\alpha \in S(K)$ such that $\alpha(\operatorname{Spec} K) = p$. Now $f \circ \alpha \in J(K)$, so there is $\beta \in S(K)$ such that $\phi \circ \beta = f \circ \alpha$. This means that there is $\gamma \in S \underset{J}{\times} P(K)$ that maps to α under the projection, and $\gamma(K)$ is a point of $S \underset{J}{\times} P$ such that $\pi_1(\gamma(K)) = p$. Thus π_1 is surjective.

Finally, let p, q be points of $S \underset{J}{\times} P$ and suppose that $\pi_1(p) = \pi_1(q)$. Let L be the field of definition of $\pi_1(p) = \pi_1(q)$, so there is a unique $\gamma \in S(L)$ such that $\gamma(\operatorname{Spec} L) = \pi_1(p) = \pi_1(q)$, and let K be an algebraically closed extension of L over which p and q are defined, so there are $\alpha, \beta \in S \underset{J}{\times} P(K)$ such that $\alpha(\operatorname{Spec} K) = p$ and $\beta(\operatorname{Spec} K) = q$. Then $\pi_1 \circ \alpha$ and $\pi_1 \circ \beta$ both factor as $\gamma \circ i$, where $i : \operatorname{Spec} K \to \operatorname{Spec} L$ is dual to the inclusion $L \hookrightarrow K$. So $\pi_1 \circ \alpha = \pi_1 \circ \beta$, and thus because $f \circ \pi_1 = \phi \circ \pi_2$, we have $\phi \circ \pi_2 \circ \alpha = \phi \circ \pi_2 \circ \beta$. But we know that ϕ is injective on K-points, so $\pi_2 \circ \beta = \pi_2 \circ \alpha$. But $\pi_1 \circ \alpha = \pi_1 \circ \beta$ and $\pi_2 \circ \beta = \pi_2 \circ \alpha$ imply that $\alpha = \beta$ by the unversal property of the fiber product. Thus $p = q$, so π_1 is injective. □

In particular, ϕ is a homeomorphism. Above, we proved that $\overline{\mathcal{P}}^0_X$ has some of the same topological properties as a compactified Jacobian, and so this homeomorphism shows that that behavior is necessary. However, we have demonstrated that the local perspective gives an alternate method for proving facts about the topology of compactified Jacobians, distinct to those in the literature.

Chapter 4

Examples

4.1 A class of trivial examples

Let $R_2 = k[\![t^2,t^3]\!] = k + t^2k[\![t]\!] \subset k[\![t]\!]$. We have $t^2k[\![t]\!] \subset R_2 \subset k[\![t]\!]$ tightly, so $w(R_2) = 2$. Let $t^2k[\![t]\!] \subset V \subset k[\![t]\!]$ be any vector subspace, $v \in V$, and $r \in R_2$. Then $r = a + t^2f$ with $a \in k$ and $f \in k[\![t]\!]$, so $rv = (av + t^2fv) \in V$ because $av \in V$ and $t^2fv \in t^2k[\![t]\!] \subset V$. Thus we have that $\overline{\mathcal{P}}^0_{R_2} = \mathcal{L}_{2,1}$, because $1 = \operatorname{ind} R_2 = \operatorname{ind}(tk[\![t]\!])$. Now $\mathcal{L}_{2,d}$ is the union of the Grassmannians $\mathcal{G}_1^{1>-1}$, $\mathcal{G}_1^{2>0}$, and $\mathcal{G}_1^{3>1}$. Explicitly, these are $\operatorname{Gr}(t^{-1}k[\![t]\!]/tk[\![t]\!],0)$, $\operatorname{Gr}(k[\![t]\!]/t^2k[\![t]\!],1)$, and $\operatorname{Gr}(tk[\![t]\!]/t^3k[\![t]\!],2)$, respectively. The first and last are points identified with $\{tk[\![t]\!]\}$, while the middle is identified with $\mathbb{P}^1$. Thus we see that $\overline{\mathcal{P}}^0_R \cong \mathbb{P}^1$.

Furthermore, a rational curve with a unibranch singularity whose completed local ring is isomorphic to R is isomorphic to the cuspidal cubic X. It is known (see e.g. [6]) that the compactified Jacobian of the cuspidal cubic is again the cuspidal cubic, with the singular point corresponding to the unique torsion-free sheaf of generic rank 1. Thus the universal homeomorphism $\overline{\mathcal{P}}^0_R \to \overline{J}_X$ can be identified with the normalization morphism $\mathbb{P}^1 \to X$.

Next, let $R_3 = k[\![t^3,t^4,t^5]\!] = k + t^3k[\![t]\!] \subset k[\![t]\!]$. We have $w(R_3) = 3$ with $t^3k[\![t]\!] \subset R \subset k[\![t]\!]$. For exactly the same reason as above, we have $\overline{\mathcal{P}}^0_{R_3} = \mathcal{L}_{3,2}$, as $2 = \dim R = \dim(t^2k[\![t]\!])$. As above, $\mathcal{L}_{3,2}$ is the union of $\mathcal{G}_2^{2>-1}$, $\mathcal{G}_2^{3>0}$, $\mathcal{G}_2^{4>1}$, and $\mathcal{G}_2^{5>2}$, which are equal to $\operatorname{Gr}(t^{-1}k[\![t]\!]/t^2k[\![t]\!],0)$, $\operatorname{Gr}(k[\![t]\!]/t^3k[\![t]\!],1)$, $\operatorname{Gr}(tk[\![t]\!]/t^4k[\![t]\!],2)$, and $\operatorname{Gr}(t^2k[\![t]\!]/t^5k[\![t]\!],3)$.

These are isomorphic to a point, $\mathbb{P}^2$, $\mathbb{P}^2$, and a point, respectively. The two points are both identified with $t^2k[\![t]\!]$. The intersection of the two middle terms is $\mathcal{G}_2^{3>1} = \mathrm{Gr}(tk[\![t]\!]/t^3k[\![t]\!], 1) \cong \mathbb{P}^1$. The embedding $\mathrm{Gr}(tk[\![t]\!]/t^3k[\![t]\!], 1) \hookrightarrow \mathrm{Gr}(k[\![t]\!]/t^3k[\![t]\!], 1)$ can be identified with the embedding $\mathbb{P}^1 \hookrightarrow \mathbb{P}^2$ which just includes a line in $tk[\![t]\!]/t^3k[\![t]\!]$ into $k[\![t]\!]/t^3k[\![t]\!]$ via the inclusion $tk[\![t]\!]/t^3k[\![t]\!] \hookrightarrow k[\![t]\!]/t^3k[\![t]\!]$, which is given in coordinates by $[x : y] \mapsto [0 : x : y]$, where we take $\{1, t, t^2\}$ as a basis for $k[\![t]\!]/t^3k[\![t]\!]$. The embedding $\mathrm{Gr}(tk[\![t]\!]/t^3k[\![t]\!], 1) \hookrightarrow \mathrm{Gr}(tk[\![t]\!]/t^4k[\![t]\!], 2)$ can be identified with the map $\mathbb{P}^1 \hookrightarrow \mathbb{P}^2$ which takes a line in $tk[\![t]\!]/t^3k[\![t]\!]$ and maps it to its preimage under the quotient map $tk[\![t]\!]/t^4k[\![t]\!] \twoheadrightarrow tk[\![t]\!]/t^3k[\![t]\!]$, then outputs the line orthogonal to the preimage; in coordinates this is $[x : y] \mapsto [-y : x : 0]$, using the basis $\{t, t^2, t^3\}$ for $tk[\![t]\!]/t^4k[\![t]\!]$. Therefore $\overline{\mathcal{P}}^0_{R_3}$ has two components, each isomorphic to $\mathbb{P}^2$, which are glued along the two embeddings $\mathbb{P}^1 \hookrightarrow \mathbb{P}^2$ given above.

In general, let $R_n = k[\![t^n, t^{n+1}, \ldots, t^{2n-1}]\!] = k + t^nk[\![t]\!] \subset k[\![t]\!]$. Following the same analysis as above, we see that $\overline{\mathcal{P}}^0_{R_n} \cong \mathcal{L}_{n,n-1}$, so the components of $\overline{\mathcal{P}}^0_R$ are $\mathcal{G}_{n-1}^{n+i>i}$ for $i = 0, \ldots, n-2$, which are isomorphic to $\mathrm{Gr}(i, n)$ for $i = 1, \ldots, n-1$. As discussed in section 2.2, the components $\mathcal{G}_{n-1}^{n+i>i}$ and $\mathcal{G}_{n-1}^{n+j>j}$, with $i < j$, intersect at $\mathcal{G}_{n-1}^{n+i>j} \cong \mathrm{Gr}(n-1-j, n+i-j)$.

In summary, we have the following:

Proposition 4.1.1. *For R_n defined as above, $\overline{\mathcal{P}}^0_{R_n}$ has exactly $n-2$ components, which we call $G_1, \ldots, G_{n-1}$. The component G_i is isomorphic to* $\mathrm{Gr}(i, n)$*, and the intersection $G_i \cap G_j$, with $i < j$, is isomorphic to* $\mathrm{Gr}(n-1-j, n+i-j)$*.*

Combining this with Theorem 5.1, we have:

Corollary 4.1.1. *Let X be a rational curve with a single singularity whose completed local ring is isomorphic to R_n. Then the compactified Jacobian of C has exactly $n-2$ components $H_1, \ldots, H_{n-1}$. The component H_i is homeomorphic to* $\mathrm{Gr}(i, n)$*, and the intersection $H_i \cap H_j$, with $i < j$, is homeomorphic to* $\mathrm{Gr}(n-1-j, n+i-j)$*.*

4.2 A nontrivial reducible example

Let $R = k[\![t^3, t^5, t^7]\!] = k + k\langle t^3\rangle + t^5k[\![t]\!] = k + k \cdot t^3 + t^5k[\![t]\!]$. Then $w(R) = 5$ and $\mathrm{ind}(R) = 3$. Thus $\overline{\mathcal{P}}^0_R \subset \mathcal{L}_{5,3}$, whose components are $\mathcal{G}_3^{4>-1}$, $\mathcal{G}_3^{5>0}$, $\mathcal{G}_3^{6>1}$, and $\mathcal{G}_3^{7>2}$.

Proposition 4.2.1. *$\overline{\mathcal{P}}^0_R$ has exactly two components, given by $\overline{\mathcal{P}}^0_R \cap \mathcal{G}_3^{5>0}$ and $\overline{\mathcal{P}}^0_R \cap \mathcal{G}_3^{6>1}$.*

Proof. We will show that First, we show that $\overline{\mathcal{P}}^0_R \cap \mathcal{G}_3^{4>-1} \subset \mathcal{G}_3^{5>0}$. Let $L \in \overline{\mathcal{P}}^0_R \cap \mathcal{G}_3^{4>-1}$. Then $\dim_k(L/t^4k[\![t]\!]) = \dim(L : t^4k[\![t]\!]) = \dim(t^3k[\![t]\!] : t^4k[\![t]\!]) = 1$, so $L/t^4k[\![t]\!]$ is the subspace of $t^{-1}k[\![t]\!]/t^4k[\![t]\!]$ spanned by a single vector $v \in t^{-1}k[\![t]\!]/t^4k[\![t]\!]$. But $t^3v \in L/t^4k[\![t]\!]$ because $L \in \overline{\mathcal{P}}^0_R$, and t^3v cannot be a nonzero scalar multiple of v, so $t^3v = 0$. Thus $v \in tk[\![t]\!] \subset k[\![t]\!]$, so in fact $L \in \mathcal{G}_3^{4>1} \subset \mathcal{G}_3^{5>0}$.

Next, we show that $\overline{\mathcal{P}}^0_R \cap \mathcal{G}_3^{5>0}$ is irreducible. Because $\mathcal{P}^0_R \subset \mathcal{G}_3^{5>0}$, this is equivalent to the claim that every lattice in $\overline{\mathcal{P}}^0_R \cap \mathcal{G}_3^{5>0}$ is in the closure of $\mathcal{P}^0_R$. A lattice L is an R-lattice if and only if L is closed under multiplication by t^3 and t^5, and any lattice L with $w(L) \le 5$ is automatically closed under multiplication by t^5, so we can identify $\overline{\mathcal{P}}^0_R \cap \mathcal{G}_3^{5>0}$ with the set of linear subspaces of $k[\![t]\!]/t^5k[\![t]\!]$ of dimension 2 which are closed under multiplication by t^3. Let V be such a subspace. Suppose $v \in V$ with $\mathrm{val}(v) = 0$ or 1. Then $t^3v \neq 0$ and t^3v is not a scalar multiple of v. It follows that $V = \mathrm{span}\{v, t^3v\}$. Thus the points of $\overline{\mathcal{P}}^0_R \cap \mathcal{G}_3^{5>0}$ are exactly those of this form, or of the form $\mathrm{span}\{v, w\}$ for $v, w \in V$ linearly independent with $\mathrm{val}(v), \mathrm{val}(w) \ge 2$, so that $t^3v = t^3w = 0$. If $V = \mathrm{span}\{v, t^3v\}$ with $\mathrm{val}(v) = 0$, then V corresponds to the principal R-lattice Rf where $f \in k[\![t]\!]^\times$ such that $f/t^5k[\![t]\!] = v$. Suppose $V = \mathrm{span}\{v, t^3v\}$ with $\mathrm{val}(v) = 1$. Let $w_s = 1 + sv$, where s is a parameter taking values in k, which we identify with $\mathbb{P}^1$ punctured at ∞. Then as s approaches ∞, the subspace $\mathrm{span}\{w_s, t^3w_s\}$, which lies in $\mathcal{P}^0_R$, approaches $\mathrm{span}\{v, t^3v\}$. Thus V is in the closure of $\mathcal{P}^0_R$. Next, suppose $V = \mathrm{span}\{v, w\}$ with $\mathrm{val}(v), \mathrm{val}(w) \ge 2$, so $v, w \in \mathrm{span}\{t^2, t^3, t^4\}$. Without loss of generality, suppose $w \in \mathrm{span}\{t^3, t^4\}$. Now let $x_s = t^{-3}w + sv$. Then $t^3x_s = w$, so as s approaches ∞, $\mathrm{span}\{x_s, t^3x_s\}$ approaches $\mathrm{span}\{v, w\}$, so again V is in the closure of $\mathcal{P}^0_R$. This verifies the claim.

Next, by Lemma 3.1.2, the dual claims that $\overline{\mathcal{P}}^0_R \cap \mathcal{G}_3^{7>2} \subset \mathcal{G}_3^{6>1}$ and that $\overline{\mathcal{P}}^0_R \cap \mathcal{G}_3^{6>1}$ is irreducible follow immediately. Finally, by observing that R and $t^6R^\vee$ are in different components as in Theorem 3.1.1, we can see that $\overline{\mathcal{P}}^0_R \cap \mathcal{G}_3^{5>0}$ and $\overline{\mathcal{P}}^0_R \cap \mathcal{G}_3^{6>1}$ are different components, so they are exactly the two components of $\overline{\mathcal{P}}^0_R$. □

In [5], Kass shows that the compactified Jacobian of a curve with a single singularity whose completed local ring is isomorphic to R (among other examples) has two components. This example shows the utility of the local perspective in proving similar results.

4.3 Irreducible examples

The previous examples were applied to lattice rings R with $e(R) > 2$, and obtained precise component counts of the corresponding local compactified Jacobians, and thus yield precise component counts for certain compactified Jacobians. Because compactified Jacobians of curves with planar singularities are irreducible, the analysis in section 5 tells us that if $e(R) = 2$, then $\overline{\mathcal{P}}^0_R$ is irreducible. It would be desirable to have a direct proof of this fact from the local point of view in addition to Theorem 3.1.1. We do not present such a proof, but we can still use similar techniques to the previous example to obtain the result for specific R.

Example 4.3.1. Let $n \geq 2$ and let $R = k[\![t^2, t^{2n+1}]\!]$. Then $t^{2n}k[\![t]\!] \subset R \subset k[\![t]\!]$ tightly, so $w(R) = 2n$ and $\mathrm{ind}(R) = n$. Because R is Gorenstein, we have $\overline{\mathcal{P}}^0_R \subset \mathcal{G}_n^{2n>0}$. Then we can identify $\overline{\mathcal{P}}^0_R$ with the set of linear subspaces of $k[\![t]\!]/t^{2n}k[\![t]\!]$ of dimension n, closed under multiplication by t^2. First, we claim that any linear subspace of $k[\![t]\!]/t^{2n}k[\![t]\!]$ closed under multiplication by t^2, i.e. an R-submodule of $k[\![t]\!]/t^{2n}k[\![t]\!]$, is generated by at most two elements. Indeed, suppose V is generated by two elements v, w. Without loss of generality, $\mathrm{val}(v)$ and $\mathrm{val}(w)$ have opposite parity. Assume $\mathrm{val}(v) < \mathrm{val}(w)$, with $\mathrm{val}(w) = \mathrm{val}(v) + 2a + 1$. Then we can see that $V = \mathrm{span}\{t^{2i}v, t^{2j}w\} = \mathrm{span}\{v, t^2v, \ldots, t^{2a-2}v, t^{\mathrm{val}(w)-1}, t^{\mathrm{val}(w)}, \ldots, t^{2n-1}\}$. Now let $v, w, x \in V$ and consider the submodule W generated by v, w, x, with $\mathrm{val}(v) < \mathrm{val}(w) < \mathrm{val}(x)$. If v and w have opposite parity, then the above immediately shows that x is in the submodule generated by v and w. Otherwise, we can replace w with $w' = w - t^{2i}v$ with

$\operatorname{val}(w') = \operatorname{val}(w) + 1$ for an appropriate i. Then again we see that x is in the submodule generated by v and w'. This proves the claim.

Now let V be a subspace of $k[\![t]\!]/t^{2n}k[\![t]\!]$ of dimension n, closed under multiplication by t^2. If V is principal, then $V = \operatorname{span}\{v, t^2v, \ldots, t^{2n-2}v\}$ for some $v \in k[\![t]\!]^\times$. Otherwise, V is generated by v and w, and we can see that $\operatorname{val}(v) + \operatorname{val}(w) = 2n + 1$. Suppose $\operatorname{val}(v)$ is even. Then we can take $z_s = t^{-\operatorname{val}(v)}v + sw$.

Example 4.3.2. Let $R = k[\![t^3, t^4]\!] = k + k \cdot t^3 + k \cdot t^4 + t^6k[\![t]\!]$. Then $w(R) = 6$ and $\operatorname{ind}(R) = 3$, so as above we can identify $\overline{\mathcal{P}}^0_R$ with the set of linear subspaces of $k[\![t]\!]/t^6k[\![t]\!]$ closed under multiplication by t^3 and t^4. Here there are 3 cases for such a subspace V:

1. $V = \operatorname{span}\{v, t^3v, t^4v\}$ for $v \in V$ with $\operatorname{val}(v) = 0$ or 1.

2. $V = \operatorname{span}\{v, t^3v, w\}$ for $v \in V$ with $\operatorname{val}(v) = 2$ and $w \in V$ with $\operatorname{val}(w) = 3, 4$

3. $V = \operatorname{span}\{v, w, x\}$ for $v, w, x \in V$ with $\operatorname{val}(v), \operatorname{val}(w), \operatorname{val}(x) \geq 3$

Again, in each case we want to construct a family z_s of vectors with $\operatorname{val}(z_s) = 0$, such that $\operatorname{span}\{z_s, t^3z_s, t^4z_s\}$ approaches V as s approaches ∞. We can do that as follows:

1. If $\operatorname{val}(v) = 0$, then $z_s = v$. If $\operatorname{val}(v) = 1$, then $z_s = 1 + sv$.

2. Without loss of generality, w has no t^5 term. Then $z_s = t^{-3}w + sv$.

3. Without loss of generality, $\operatorname{val}(v) = 3$, $\operatorname{val}(w) = 4$, $\operatorname{val}(x) = 5$. We can cancel off terms and get $v = t^3$, $w = t^4$, and $x = t^5$. Then let $z_s = 1 + st^5$.

Note that given a lattice ring R and an R-lattice L, $\operatorname{val}(L)$ is a $\operatorname{val}(R)$-*semimodule*, i.e. a subset of Z bounded below and closed under the addition action of the numerical semigroup $\operatorname{val}(R)$. In the previous examples, what we effectively do is construct a stratification of $\overline{\mathcal{P}}^0_R$, where each stratum is indexed by a particular $\operatorname{val}(R)$-semimodule M, and consists of those R-lattices L such that $\operatorname{val}(L) = M$. Further analysis of this stratification could lead to a local proof of irreducibility of $\overline{\mathcal{P}}^0_R$ for $e(R) = 2$.

Chapter 5

Conclusion

We conclude by presenting some further questions raised by the analysis above. In particular, the connection between lattice rings and numerical semigroups. A broad guiding question is:

Question 1. *To what extent is the structure of* $\overline{\mathcal{P}}^0_R$ *determined by the numerical semigroup* $\mathrm{val}(R)$*?*

For example, the following conjecture seems natural:

Conjecture 1. *If* R *and* R' *are lattice rings with* $\mathrm{val}(R) = \mathrm{val}(R')$*, then* $\overline{\mathcal{P}}^0_R$ *and* $\overline{\mathcal{P}}^0_{R'}$ *have the same number of irreducible components.*

If this is true, then given a numerical semigroup S, we can define $n(S)$ as the number of irreducible components of $\overline{\mathcal{P}}^0_R$ for R with $\mathrm{val}(R) = S$. (Note that even if Conjecture 1 is false, we can still define $n(S)$ as the number of components of $\overline{\mathcal{P}}^0_R$ where $R = k[\![t^a]\!]_{a \in S}$.) This is a numerical invariant of S. There are many well-studied numerical invariants of numerical semigroups, including the embedding dimension, multiplicity, Frobenius number, type (see [9] for an introduction to these invariants and the theory of numerical semigroups in general).

Question 2. *Is there a formula for* $n(S)$ *in terms of other known numerical invariants of* S*?*

This appears to be a difficult question, as a solution would give counts of components for a wide class of compactified Jacobians.

Next, we return to the discussion at the end of Chapter 4. We observed that in some examples, there appeared to be a stratification of $\overline{\mathcal{P}}^0_R$ where the strata are indexed by the isomorphism classes semimodules over $\mathrm{val}(R)$. In particular, the stratum corresponding to a semimodule S are those R-lattices L such that $\mathrm{val}(L) \cong S$. Denote this subset of $\overline{\mathcal{P}}^0_R$ by P_S. For example, $P_{\mathrm{val}(R)} = \mathcal{P}^0_R$, an open subset of $\overline{\mathcal{P}}^0_R$.

In our examples, we saw that $\overline{\mathcal{P}}^0_R$ was irreducible for some examples of R by showing that every P_S is in the closure of $P_{\mathrm{val}(R)}$.

Conjecture 2. *The P_S form a stratification of $\overline{\mathcal{P}}^0_R$. In particular, each P_S is locally closed, and there is a partial order $\preceq$ on the set $\mathcal{S}$ of isomorphism classes of* $\mathrm{val}(R)$*-semimodules such that*

$$\overline{P}_S \subset \bigcup_{S' \preceq S} P_{S'}$$

See [10] for a discussion of stratifications.

We noted above that $P_{\mathrm{val}(R)} = \mathcal{P}^0_R$, a connected, irreducible subset of $\overline{\mathcal{P}}^0_R$ of dimension $\delta(R)$.

Question 3. *What can be said about each P_S? For example, are the P_S connected? irreducible? What is their dimension?*

On the other hand, the poset $(\mathcal{S}, \preceq)$ also gives some information about $\overline{\mathcal{P}}^0_R$. For example, the fact that $\mathcal{P}^0_R$ is irreducible if $e(R) = 2$ tells us that the class of R is the unique top element of the poset. In general, the number of top elements gives a lower bound on the number of components of $\overline{\mathcal{P}}^0_R$.

Question 4. *Can the number of top elements of $(\mathcal{S}, \preceq)$ be computed in terms of* $\mathrm{val}(R)$*?*

Answering these questions would give insight into the topology of $\overline{\mathcal{P}}^0_S$, and thus of compactified Jacobians as well.

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