Relationships Between Students' Conceptions of Proof and Classroom Factors

By

Zekiye Ozgur

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The dissertation is approved by the following committee members of the Final Oral Committee:

Amy B. Ellis, Associate Professor, Curriculum and Instruction
Eric Knuth, Professor, Curriculum and Instruction
Hala Ghousseini, Assistant Professor, Curriculum and Instruction
Erica Halverson, Associate Professor, Curriculum and Instruction
Mitchell Nathan, Professor, Educational Psychology

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PART I:

AN INVESTIGATION OF PROOF CONCEPTIONS IN A HIGH SCHOOL MATHEMATICS CLASSROOM

Abstract

Maintaining that students' conceptions of proof can be better understood if studied as they evolve in a classroom community over an extended time, I conducted an in-depth case study of one high school mathematics class and examined students' conceptions of proof in relation to their teacher's conceptions of proof and their classroom experiences. In this paper, I focus on the interviews with the students and their teacher and examine to what extent and in what ways the students' conceptions of proof resemble or differ from their teacher's conceptions of proof. Specifically, I present the results of the analyses of the participants' conceptions of proof as understood through (a) their descriptions of proof and proving (i.e., *proof description*), (b) their evaluations of hypothetical student proofs (i.e., *proof evaluation*), and (c) the arguments they produce to prove a given statement and their evaluations of their own arguments as a proof (i.e., *proof production*), respectively. I then discuss how the triangulation of the results from each measure collectively afforded a better understanding of the participants' conceptions of proof, by providing a more complete and nuanced understanding of their conceptions. The study also shows that the analyses of the teacher interviews provide context and background to the students' conceptions of proof, and thus strengthens our understanding of the students' ways of thinking and understandings about proof. In conclusion, the results indicate a close alignment between the students' and their teacher's conceptions of proof.

1. Introduction

The importance of proof for doing and learning mathematics is well recognized, which is evident by the extensive body of literature on proof, as well as the increased emphasis placed on proof and proving in the policy documents (e.g., The Principles and Standards for School Mathematics [NCTM, 2000]; Common Core State Standards in Mathematics [National Governors Association/Council of Chief State School Officers, 2010]). While a vast majority of this scholarship focuses on students' conceptions of proof (e.g., Balacheff, 1988; Bell, 1976; Chazan, 1993; Harel & Sowder, 1998; Healy & Hoyles, 2000; Knuth, Choppin, & Bieda, 2009), some research focuses on teachers' conceptions of proof (e.g., Dickerson & Doerr, 2014; Knuth 2002a, 2002b; Martin & Harel, 1989; Simon & Blume, 1996), with both lines of research often documenting that students and teachers hold narrow conceptions (or even misconceptions) of proof.

However, the existing literature usually documents students' or teachers' conceptions of proof based on a survey or interview administered at one point in time. Moreover, students' and teachers' conceptions of proof are often studied separately and thus documented independent of each other. While these studies inform how proof is viewed and understood by each group, more research is needed to better understand how students' conceptions of proof develop through their schooling experiences, in general, and how students' conceptions of proof are related to their teachers' conceptions of proof, in particular. Hence, I conducted an in-depth case study of one high school mathematics class to investigate students' conceptions of proof in relation to their teacher's conceptions of proof and their classroom experiences, assuming there is greater potential to more thoroughly understand students' conceptions of proof if studied as they evolve in a classroom community over an extended period.

More specifically, this study assumes that a teacher's conception of proof is a helpful lens for better understanding students' experiences with proof in school mathematics and seeks to understand to what extent the students' conceptions of proof resemble or differ from their teacher's conceptions– with the ultimate goal of identifying classroom experiences that may support the observed proof conceptions in class, though the latter focus is outside the scope of this paper (see Paper #2, *Classroom Factors Supporting Students' Conceptions of Proof: Classroom Norms, Instructional Practices, and Curriculum*).

Furthermore, the existing research on proof conceptions often asks individuals to prove a given mathematical statement to determine their conceptions of proof; however, Stylianides and Stylianides (2009) point to the distinction between an individual's ability to produce a proof and their understanding of what a valid proof is. Taking into account this distinction, Stylianides and Stylianides asked pre-service teachers to construct their own arguments to prove a statement and then to evaluate their own argument in terms of whether they thought it counted as proof. By doing so, they sought to differentiate between individuals who produce an empirical argument to prove a given statement and believe they proved it and others who produce empirical arguments, but know that it does not count as proof. Stylianides and Stylianides found that half of the participants who produced empirical arguments were aware that they did not prove the statement, and highlighted the importance of the ways in which individuals' conceptions of proof are studied, showing that individuals' conceptions may be interpreted differently depending on the research methods used. Hence, Stylianides and Stylianides called for further research to examine teachers' conceptions of proof through both proof production and proof evaluation activities. Following their recommendation, I explored both the students' and the teacher's conceptions of proof through multiple measures to gain a better sense of their proof conceptions. Specifically, I examined the participants' proof conceptions through three activities– proof production, proof evaluation, and proof description.

In this paper, I present the results of the analyses of the students' and the teacher's conceptions of proof as understood through (a) their descriptions of proof and proving (i.e., *proof description*), (b) their evaluations of hypothetical student proofs (i.e., *proof evaluation*), and (c) the arguments they produce to prove a given statement and their evaluations of their own arguments as a proof (i.e., *proof production*), respectively. I then discuss how the triangulation of the results from each measure collectively enabled a better understanding of the participants' conceptions of proof, by providing a more complete and nuanced understanding of their conceptions. The study also shows that the analyses of the teacher interviews provide context and background to the students' conceptions of proof, and thus strengthens our understanding of the students' ways of thinking and understandings about proof. Finally, the study offers evidence that students can develop more robust and desired conceptions of proof if the learning environment is conducive to sharing and justifying mathematical ideas where the teacher values proof as an important aspect of doing and learning mathematics.

In the following sections I provide a brief theoretical background of the study, by outlining some key constructs and findings in the literature that informed this study.

2. Theoretical Background and Relevant Literature

2.1. Defining Proof and Proving

While the importance of poof is agreed on, there is not an agreed-upon definition of proof within the mathematics education community. However, three characteristics of proof stand out. That is, proving is a mathematical, cognitive, and social activity. Proving includes looking for relationships, making conjectures, generalizing, and justifying (*mathematical aspect*); proof may take different forms and have different levels of sophistication based on students' ages and cognitive development (*cognitive aspect*); and proof is socially constructed and thus its validity is determined according to the norms of the community (*social aspect*).

Traditional definitions of proof tend to foreground the mathematical aspect of it, emphasizing certainty, form and rigor. For instance, the *Curriculum and Evaluation Standards* defines proof as "a careful sequence of steps with each step following logically from an assumed or previously proved statement and from previous steps" (NCTM, 1989, p. 144). Such a definition gives prominence to deductive arguments and judges the validity of an argument according to the logical inferences made. While this is a mathematically valid definition, it falls short of accounting for other important aspects of proof. As many scholars argue (e.g., Balacheff, 1988; Hanna, 1990), proving is a social process and the validity of proof is determined according to the norms of the community.

Accordingly, Stylianides (2007) offers a proof definition that blends the social aspects of proof with the mathematical aspects. He defines proof as an argument that (a) "uses statements accepted by the classroom community that are true and available without further justification– "*set of accepted statements"*; (b) employs forms of reasoning that are valid and known to, or within the conceptual reach of, the classroom community– "*modes of argumentation";* and (c) is communicated with forms of expression that are appropriate and known to the classroom community, or within the conceptual reach of, the classroom community – "*modes of argument representation"* (p. 291). Thus, this definition is more helpful to study and understand proofs that occur in classrooms.

Harel and Sowder (1998) offer another useful proof definition, one that highlights both the cognitive and social aspects of proof. They define proving as the process employed by an individual (or a community) to remove doubts about the truth of an assertion, which includes two sub-processes: one is a cognitive process (*ascertaining*), and the other is a social process (*persuading*). Ascertaining is the process by which one removes his or her doubts about the truth of an assertion, and persuading is the process of removing others' doubts. So, the process of persuading accounts for the social aspect of proof, whereas the processes of ascertaining accounts for the cognitive aspect of proof.

The first definition, which highlights the mathematical aspect of proof, may be the one most reminiscent of how people typically view proof, since the traditional treatment of proof in school mathematics is focused on mathematical formalism. However, the second and third definitions are more useful for studying proof and proving in school mathematics as these definitions allow one to view proving as a collective classroom practice. Therefore, I take a view of proof and proving that blends the definitions provided by Harel and Sowder (1998) and Stylianides (2007), maintaining that proving is a mathematical, cognitive and a social activity.

Accordingly, I view *proving* as a process (a) of establishing conviction about the validity of mathematical statements and constructing mathematical meanings, which has individual and social aspects as Harel and Sowder (1998) describe (i.e., *ascertaining* and *persuading,* respectively); and (b) of communicating those meanings via arguments as outlined by Stylianides (2007) (i.e., through *sets of accepted statements*, *modes of argumentation*, and *modes of argument representation*). I view a *proof* as the product of the proving process that is accepted by the classroom community.

2.2. Conception of Proof

Pointing to the importance of studying knowledge and beliefs together, Thompson (1992) defines *conception* as "a general notion or mental structure encompassing beliefs, meanings, concepts, propositions, rules, mental images, and preferences" (p. 130). Following this definition, I use *conception of proof* to refer to what meanings individuals hold for proof, what they believe the role and function of proof is, what constitutes proof for them, and what they understand about proof, as well as the kinds of arguments they produce and consider to be a proof.

2.2.1. Students' conceptions of proof

2.2.1.a. *Meanings of proof: What is the role of proof?*

Many studies report that students do not see the value of proving for learning mathematics, but rather see it as a requirement– a formal, meaningless ritual in which they are asked to either *verify* something that they are told to be true (by their teacher or textbook) or *confirm* something that is obvious (Schoenfeld, 1985). Similar findings occur even among undergraduate students. Coe and Ruthven (1994) found that undergraduate freshmen's notion of proof was mainly limited to verification and confirmation. Some of their students viewed proof as "a general formula" that helps one rigorously check the veracity of statements. Yet, there are various functions that a proof can serve in learning mathematics that can contribute to students' understanding. In particular, researchers (e.g., Bell, 1976; deVillers, 2012; Healy & Hoyles, 2000; Porteous, 1990) emphasize three main purposes of proof as follows: (a) *verification* (i.e., a proof verifies the truth of a proposition), (b) *explanation* (i.e., illumination and communication; a proof should also give insight into why the proposition is true), and (c) *discovery* (i.e., discovery and systematization; the arguments put forth for the proof– axioms, concepts, and the derived results– should be organized into a deductive system). Indeed, Hanna (2000) highlights the verification and explanation as the fundamental functions of proof, arguing that the main purpose of proof in school should be to explain. Hanna asserts that rigor is secondary to understanding (especially in school mathematics), arguing that the focus should be on the educational value of proof rather than its formal correctness. However, studies document that few students recognize the explanation role of proof, and even fewer students (if any at all) recognize the discovery role of proof (Healy & Hoyles, 2000; Porteous, 1990), underscoring that students (even advanced students such as senior high school students and undergraduate students) have a very limited idea regarding the purpose and role of proof.

2.2.1.b. ***Students' proof evaluations: What counts as a proof?***

It has been frequently shown that students' accounts of what counts as a proof seem to depend more on the form of the arguments rather than on the correctness and completeness of the arguments, with many students thinking that a proof must be presented in a particular form, written in a clear and logical format (Almeida, 2000; Healy & Hoyles, 2000; Smith, 2006; Vinner, 1983). For instance, Healy and Hoyles (2000) explored the characteristics of arguments that students recognized as proofs and the reasons behind their judgments, by surveying nearly 2500 high school students. The researchers provided students with several arguments including narrative, empirical, visual, and deductive arguments– both valid and invalid deductive arguments– for one familiar and one unfamiliar algebra problem and asked them to choose the argument closest to what they would produce themselves and the argument that they thought their teacher would give the best mark. The study revealed that students had different conceptions of what constitutes proof for themselves and for their teachers– a similar finding to Chazan's (1993) study. The researchers reported that the students considered algebraic arguments as appropriate proofs for their teachers, while they themselves viewed other forms of arguments as proof as well.

More specifically, when choosing an argument that they would produce themselves to prove the familiar conjecture, 46% of the students chose a narrative argument, 24% chose the empirical argument, 16% chose the visual argument, and 14% chose an algebraic argument, although 2% chose an invalid deductive argument. However, 64% of them picked an algebraic argument as the one that would receive the best mark, with 42% choosing the invalid algebraic argument. Tendency to choose an algebraic argument increased for the unfamiliar conjecture: 20% chose an algebraic argument as the one that they would produce, and 79% thought an algebraic argument would receive the best mark. These findings indicate that students choose arguments that they can understand and find to be convincing and explanatory for themselves, while they focus on the form of the arguments when it comes to choosing an argument that would count as a proof in the eyes of their teachers. However, the students' low tendency to choose algebraic arguments as their personal choices may be related to the distinction that Stylianides and Stylianides (2009) pointed to; that is, the distinction between producing and recognizing a valid proof. Indeed, the students were not precisely asked to choose the argument that they would personally consider as a proof, but rather they were asked to choose the one that they would 'produce' themselves. Hence, these findings may instead be pointing to the students' difficulties in constructing algebraic proofs, rather than understanding what a valid proof is.

2.2.1.c. ***Students' proof schemes: How students go about proving?***

Students' proof competencies are often examined by classifying their proof productions as either empirical or deductive arguments (Bell, 1976; Coe & Ruthven, 1994; Porteous, 1990). Several researchers have also examined students' proof competencies from the learner's perspective, specifically by trying to identify how students go about proving and what students consider as proof (e.g., Balacheff, 1988; Harel & Sowder, 1998; van Dormolen, 1977). For instance, Balacheff (1988) identified two main categories for student proofs as ‘pragmatic’ and ‘conceptual’, which roughly correlates to empirical and deductive proofs. Within the pragmatic class, he identified three levels: naïve empiricism, the crucial experiment, and the generic example. Both naïve empiricism and the crucial experiment involve empirical arguments; however, *crucial experiment* refers to deliberately testing the conjecture with non-special or extreme cases to gain conviction that the conjecture will hold in general, whereas *naïve empiricism* refers to being convinced by only a few confirming examples. Balacheff defined *generic example* as students proving the conjecture with an example that is seen as the representative of a class. Lastly, he described student arguments that demonstrate an understanding of the key idea of a proof as a *thought experiment*, the only member of the conceptual proof category.

Harel and Sowder (1998) further elaborated on what students consider as proof and proposed a framework for students’ proof schemes, including three main categories: external conviction, empirical, and analytical proof schemes. By proof scheme, they mean “what constitutes ascertaining and persuading for that person” (p. 244). Hence, their categorization of proof schemes is based on individuals’ doubts, truths, and convictions in a social context. More recently, Harel (2006) offered a revision of the framework by incorporating a historical and philosophical analysis on proof, resulting in some revisions on the third class. Hence, the revised framework includes the external conviction, empirical, and deductive proof scheme classes.

External conviction proof schemes include student proofs in which students’ conviction comes from external factors such as a teacher or a textbook (i.e., the authoritative proof scheme), appearance of a proof (i.e., the ritual proof scheme), or the symbol manipulation without a coherent system of referents (i.e., the non-referential symbolic proof scheme) (Harel & Sowder, 1998).

Students holding an *empirical proof scheme* validate conjectures by appealing to examples (i.e., the inductive proof schemes) or sensory experiences (i.e., the perceptual proof scheme). It is a very common proof scheme observed among students across grade bands, including high school students (e.g., Balacheff, 1988; Chazan, 1993; Edwards, 1999; Porteous, 1990). The *inductive proof scheme* is particularly common among students, where students are convinced by testing the conjecture with one or more specific cases. The literature is abounded with studies documenting that many students prefer numerical calculations as justification and find a few confirming examples as sufficient to prove (e.g., Balacheff, 1988, Chazan, 1993; Coe & Ruthven, 1994; Edwards, 1999; Harel & Sowder, 1998; Knuth et al, 2002; Knuth, Choppin, & Bieda, 2009; Porteous, 1990). It is also documented that some students are convinced by examples only if the conjecture also holds true for non-special examples such as an extreme or random case– that is, in Balacheff's terms, if the conjecture pass the "crucial experiment" (e.g., Balacheff, 1988; Chazan, 1993; Knuth et al., 2002). The *perceptual proof scheme*, on the other hand, involves coming to a conviction based on observation or the appearance of a visual figure. For instance, a student may conclude that two triangles are similar because they look similar (Harel & Sowder, 1998).

The *deductive proof scheme* class basically entails validating conjectures by means of logical deductions. It consists of the *transformational proof scheme*, which is characterized by generality, operational thought, and logical inference, and the *axiomatic proof scheme*, which requires further understanding that any proving process, in principle, must start from accepted axioms (Harel, 2006). More precisely, the deductive proof scheme requires that an individual: (a) understands that a proof should account for all cases it is given for (*generality*), (b) sets goals and sub-goals and attempts to anticipate the outcomes of his or her actions during the proving process (*operational thought),* and (c) understands that mathematical justification should be based on the rules of logical inference (*logical inference*). Although it is desired that students develop deductive proof schemes as they go through schooling, research has often portrayed a disappointing picture. Studies repeatedly document that only a small percentage of students exhibit deductive proof schemes (e.g., Bell, 1976; Edwards, 1999; Healy & Hoyles, 2000; Ususkin, 1987).

2.2.2. Teachers' conceptions of proof

Research on teachers' conceptions of proof is relatively limited compared to students' conceptions of proof. Moreover, most such studies have investigated pre-service teachers' conceptions of proof, thus, even less is known about in-service teachers' conceptions of proof. The existing research, however, documents that pre-service and in-service teachers' conceptions of proof are somewhat limited. In particular, teachers seem to have narrow views about the nature of proof in school mathematics, and often think of two-column, formal proofs that are typical in geometry when they think of proof (Knuth, 2002b). Studies found that teachers tend to view proof as a separate topic of study reserved for upper mathematics classes, and thus appropriate only for the most able students who are likely to pursue mathematics-related majors in college (Furinghetti & Morselli, 2011; Knuth, 2002b; Varghese, 2009). Further, Knuth (2002b) found that most teachers in his study did not think that proof (formal or less formal proofs) was appropriate for all students, thus they did not view proof as a central idea in school mathematics.

Knuth investigated secondary mathematics teachers' views about the role of proof in the context of mathematics as a discipline (2002a) and in the context of school mathematics (2002b), and found that the teachers had different views about the roles and purposes of proof in each context. Within the context of mathematics as a discipline, Knuth (2002a) found that many teachers expressed various roles of proof such as verification, communication, and systemization, but they did not recognize promoting understanding as a role of proof. Knuth reported that only a slim percent of the teachers considered explaining why something was true as a role of proof, while no teacher stated promoting understanding as a role of proof.

In the context of school mathematics, on the other hand, teachers described the role of proof as developing logical thinking skills, communicating mathematics, displaying students' thinking processes, explaining why, and creating mathematical knowledge (Knuth, 2002b). However, Knuth found that teachers had two meanings for *explain why*; for some teachers, he noted, 'explain why' simply meant to show how the statement came to be true, rather than focusing on the underlying mathematical reasons (e.g., relationships, concepts) that makes the statement true. Likewise, Dickerson and Doerr (2014) found that the teachers in their study considered the two pedagogical purposes of proof in secondary school mathematics as "to enhance students' mathematical understandings" and "to develop generalizable thinking skills that were transferable to other fields of endeavor" (p. 711). Thus, it seems that teachers have more positive views about the role of proof in school mathematics compared to their views of the role of proof in disciplinary mathematics. But, this discrepancy may be due to teachers having different meanings for proof in each context, which may have important implications for their treatment of proof and what they accept as a proof in class.

Several studies indicate that pre-service and in-service teachers accept both empirical and deductive arguments as proof (e.g., Knuth, 2002a; Martin & Harel, 1989; Morris, 2002). For instance, Martin and Harel (1989) found that 46% of pre-service teachers concurrently rated a general deductive proof high (as a valid proof) while also rating at least one inductive argument high. Similarly, Morris (2002) found that pre-service (elementary and middle school) teachers considered inductive arguments, as well as deductive arguments, to constitute proof, believing that they both assured the certainty of the statements. Furthermore, when evaluating arguments, some teachers also accepted false deductive arguments as proofs (e.g., Knuth, 2002a, 2002b; Martin & Harel, 1989), suggesting that teachers, too, may focus on the form of the argument rather than the plausibility of the argument when determining validity.

For example, in-service teachers in Knuth's study (2002b) were overall successful in identifying the arguments that were proofs; however, one third of them also accepted non-proof arguments as a proof. Interestingly, Knuth (2002a) also noticed that what teachers found most convincing were often arguments that were not proofs; the teachers were more convinced by arguments that included specific examples or a visual to accompany the argument, which would fall under the empirical proof scheme class. Moreover, Dickerson and Doerr (2014) found that less experienced teachers emphasized the form and rigor of proof, specifically expressing that proof in high school should conform to the standard form of proof (i.e., two-column proof), while more experienced teachers asserted that visual and concrete proofs are well suited for high school students. Hence, further research is needed to uncover teachers' reasons for accepting non-proof arguments as proofs. Given that teachers favored specific examples and visuals to supplement the arguments, their choices may be related to their views of the nature and pedagogic purposes of proof in school mathematics, again pointing to a possible discrepancy between teachers' meanings for proof in the context of school mathematics and mathematics discipline.

In sum, together these two lines of research portray how students and teachers view of proof and what they understand about proof, but several questions emerge needing further attention: What lies beneath students' and teachers' conceptions of proof? How students' conceptions of proof develop through their schooling experiences, and particularly, how they are related to their teacher's conceptions of proof? Hence, there is a pressing need to go beyond identifying students' conceptions of proof and to focus on understanding why students conceive of proof the way they do. Maintaining that studying students' conceptions of proof in the context of their mathematics class (together with a focus on their teacher's conceptions) has a great potential to gain a better understanding of their conceptions, I sought to examine both the students' and their teacher's conceptions of proof through multiple measures to shed some light on these questions.

3. Methods

To explore students' conceptions of proof from multiple perspectives, I conducted an in-depth case study (Yin, 2003) of one high school mathematics class. As I situate the teacher's conceptions of proof (as well as the classroom norms and practices) as helpful lenses to better understand students' ways of thinking and understandings about proof, I also conducted teacher interviews and classroom observations, in addition to interviewing students. In this paper, I focus on the interview data and report findings about the students' conceptions of proof, supplementing them with analyses of their teacher's conception of proof. (See Paper #2, *Classroom Factors Supporting Students' Conceptions of Proof: Classroom Norms, Instructional Practices, and*

Curriculum, for findings on the relationship between the students' proof conceptions and their classroom experiences).

3.1. Context of the Study

The study is situated in an honors, integrated Algebra-II, Geometry, and Pre-Calculus course in a public school district in the Midwest. After observing various mathematics classes in different schools, I chose this class as the study site due to its emphasis on mathematical discussions and justifying ideas. Specifically, I needed a classroom in which proof and justification would be a regular part of the classroom discourse to obtain rich data about not only students' proof conceptions, but also about the ways in which students' classroom experiences may have shaped their understanding of proof.

Designed by a team of mathematics teachers in the school (including the participant teacher, Ms. V), this hybrid class was a recently created course. The team designed the course in part to allow students to take more advanced mathematics courses before they graduate from high school. As such, the course was part of a sequence that enables students to take Algebra-II, Geometry, and Pre-calculus in two years. Students who have already passed an Algebra-I course are eligible to take this course if they choose an accelerated math path. Thus, unlike the traditional honors courses, multiple sessions of the course were available due to high student interest. In this study, I examine one session of this two-year hybrid course during the first semester of Year-1of the course sequence.

3.2. Data Collection and Participants

The teacher, Ms. V, was part of a team of three teachers who designed and implemented this hybrid course for the first time. I had conducted informal classroom observations of this course during its initial implementation the year before the actual study was carried out. Thus, the study took place during the second implementation of this course. Ms. V had four years of teaching experience and had previously taught Algebra-I, Algebra-II, and Geometry classes. At the time of the study, Ms. V was teaching both Year-1 and Year-2 of the course sequence. The class included 31 students, who were a mix of 9th and 10th grade students. The students typically worked on problems in teams of three or four students, and thus small group discussions constituted an essential characteristic of the class.

Data sources of the study include interviews with students and the teacher, videotapes of lessons, audiotapes of small group discussions of a subset of students, field notes, reflections, and artifacts (e.g., lesson plans, tasks, student work, etc.). Figure 1 below summarizes the entire data collection process.

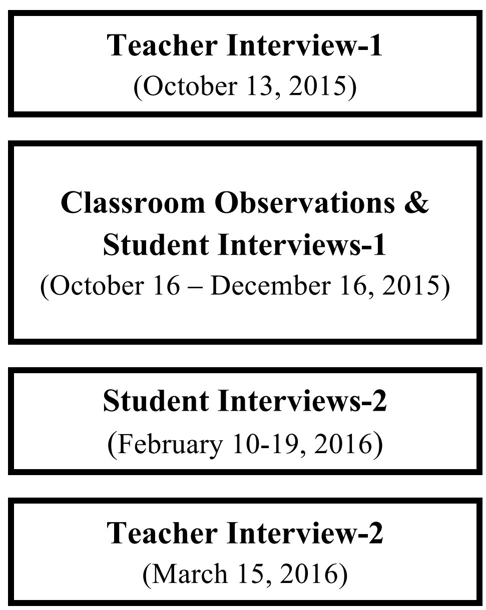

Figure 1. Data collection process

3.2.1. Teacher interview-1

Data collection began in mid-October with an hour-long, semi-structured interview with the teacher to understand her conceptions of proof, with a focus on her views about the notion of proof, her views about the learning and teaching of proof, her goals for her students with regards to proof and proving, as well as background information about the class (see Appendix A for the interview protocol).

3.2.2. Classroom observations

To examine students' proof conceptions in situ, I observed mathematics lessons for two months, consisting of 18 lessons covering two units in geometry (i.e., Unit 3: 2D Figures & Unit 4: Similarity and Congruence) and one unit in pre-calculus (i.e., Unit 5: Intro to Trigonometry). I had started classroom visits in September and informally observed the class several times prior to beginning the data collection. This allowed me both to familiarize myself with the class, and to become a somewhat regular member of the class by the time the study officially started, after which I visited each lesson for two months. I videotaped the lessons, collected classroom artifacts (such as classwork, homework assignments, and sample student work), and took field notes to document classroom norms and practices related to proof and proving, as well as any remarks revealing individuals' conceptions of proof. I also wrote reflections after each classroom observation. Table 1 below summarizes the mathematical topics that the class studied throughout the observations.

Table 1. Mathematical topics of lessons

Day #	**Unit# Day#**	**Topics**
Unit 3: *2D Figures*		
Day 1	U3D3	Pythagoras
Day 2	U3D4	Shapes, Definitions, Properties
Day 3	U3D5	Circles, Triangles, Composites, Parallelograms
Day 4	U3D6	Constructions
Unit 4: *Similarity and Congruence*		
Day 5	U4D1	Similarity
Day 6	U4D2	Proofs
Day 7	U4D3	Congruence
Day 8	U4D4	Ratios and Similarity
Day 9	U4D5	Quadrilateral Proofs
Day 10	U4D6	Coordinate Proofs
Day 11	U4D7	Team Test

Unit 5: *Intro to Trigonometry*		
Day 12	U5D1	Special Rights and Angles of Polygons
Day 13	U5D2	SOHCAHTOA
Day 14	U5D3	Inverse Trigonometric Functions
Day 15	U5D4	Law of Sines and Cosines
Day 16	U5D5	Choosing A Trig Tool
Day 17	U5D6	Finding Area of Regular Polygons
Day 18	U5D7	Team Test

3.2.3. Student interviews-1

Alongside the classroom observations, I conducted two student interviews, Interview 1 and Interview 2. Interview 1 was an hour-long, semi-structured individual interview with 18 students. I interviewed every student who agreed to be interviewed. The interview protocol consisted of three parts (i.e., proof evaluation, proof description, and proof production) in order to examine students' conceptions of proof from multiple perspectives (see Appendix B for the entire interview protocol). Specifically, I asked the students (a) to evaluate six hypothetical student proofs given for an algebra task by judging whether each argument constitutes a proof or not and explaining why they think so– *proof evaluation* (see Figure 2), (b) to describe what proof and proving means to them– *proof description*, and (c) to prove a mathematical statement and to evaluate their own proof productions in terms of whether they consider it to be a proof– *proof production* (see Figure 3).

<table>
<tr><td colspan="2">Arthur, Bonnie, Ceri, Duncan, Eric, and Yvonne were trying to prove whether the following statement is true or false:

When you add any two even numbers, your answer is always even.</td></tr>
<tr><td>Arthur's answer

a is any whole number
b is any whole number
$2a$ and $2b$ are any two even numbers
$2a + 2b = 2(a + b)$</td><td>Bonnie's answer

2 + 2 = 4   4 + 2 = 6

2 + 4 = 6   4 + 4 = 8

2 + 6 = 8   4 + 6 = 10</td></tr>
</table>

So, Arthur says it's true.	*So, Bonnie says it's true.*
Ceri's answer Even numbers are numbers that can be divided by 2. When you add numbers with a common factor, 2 in this case, the answer will have the same common factor. *So, Ceri says it's true.*	*Duncan's answer* Even numbers end in 0, 2, 4, 6, or 8. When you add any two of these, the answer will still end in 0, 2, 4, 6, or 8. *So, Duncan says it's true.*
Eric's answer Let x = any whole number y = any whole number $x + y = z$ $z - x = y$ $z - y = x$ $z + z - (x + y) = x + y = 2z$ *So, Eric says it's true.*	*Yvonne's answer* ■■■■■ ■■■ ■■■■■■■■ ■■■■■ ■■■ ■■■■■■■■ *So, Yvonne says it's true.*

Figure 2. Student interview-1, Proof evaluation task, *adopted from Healy & Hoyles (2000)*

How would you prove the following statement?

If p and q are any two odd numbers, (p + q) x (p - q) is always a multiple of 4.

- Do you think your argument counts as proof?
- How confident are you in terms of the validity of your proof?
- How do you know your proof is sufficient?
- Do you think your teacher would agree that your proof is valid?

Figure 3. Student interview-1, Proof production task, *adapted from Healy & Hoyles (2000)*

3.2.4. Student interviews-2

Of the 18 students from Interview 1, 7 students (who also consented to audiotaping their small-group discussions and were grouped together) were chosen as focus-group students, and were interviewed a second time two months after the classroom observations were completed to further probe their conceptions of proof. Thus, when reporting the results about students' conceptions of proof, I restrict my analyses to the focus-group students for whom I have a more thorough account of their conceptions of proof. Furthermore, the focus-group students also happened to be quite representative of the class in terms of their mathematical background and abilities, as confirmed both by their teacher and by their first interviews.

Like the class in general, the focus-group students were a mix of 9^{th} and 10^{th} grade students, with Julie, Mark, and Molly being the 10^{th} graders. They all had taken algebra (either in 8^{th} grade or 9^{th} grade), which was the pre-requisite to take this hybrid course. Only two students (Julie and Mark) had taken an honors mathematics course before, while one student (Mark) had taken a geometry course. Except one student (Molly), all focus-group students stated that they liked mathematics, expressing mathematics as one of their favorite subjects. In addition, all of them considered themselves good at mathematics. Overall, they had a positive disposition to mathematics; many explained that they wanted to take this hybrid course because they liked taking accelerated mathematics to challenge themselves, and so that they could take more advanced courses (like calculus) in high school. Furthermore, they all studied the same mathematics curriculum (Core Connections series by CPM Educational Program) in middle school– the curriculum they continue to use in high school.

The initial student interviews started with the focus-group students, completing their interviews during the first month of the classroom observations. The focus-group students were interviewed a second time about three months after their initial interviews, with an aim to further probe their conceptions of proof. Hence, the preliminary analyses of the first interviews (with 18 students) were used to inform the design of the second interview protocol. This time using a geometry task, I first asked the students to prove the given task, and then provided four hypothetical student proofs for them to evaluate. Additionally, I compiled a list of proof statements based on the preliminary analysis of the students' proof descriptions and asked the focus-group students to mark whether they agree, disagree, or somewhat agree with those statements and then to pick three statements that best describe what proof and proving means to them (see Appendix C for the full interview protocol). With this task, my goal was to see to what extent the students share similar understandings and views about proof and proving. Lastly, I revisited the proof evaluation task from the first interview by providing the students with two additional hypothetical student proofs, which was designed based on the preliminary analysis of the first interviews (see Figure 4).

Do you think Sam and Abby proved the statement? Why or why not?

Sam:

2 + 4 = 6
60 + 26 = 86
406 + 262 = 668

I tested it with different examples, both small and large numbers. It worked each time. So, it works for any two even numbers.

Abby:

Say x and y are any two even numbers. By definition, x and y can be represented as follows:

$x = 2 \bullet a$ (a is any whole number)
$y = 2 \bullet b$ (b is any whole number)

Then, $x + y = 2 \bullet a + 2 \bullet b = 2 \bullet (a + b)$

Because the sum of x and y has a factor of 2, it is divisible by 2. Therefore, the sum of any two even numbers is always an even number.

Figure 4. Two additional hypothetical student proofs for the algebra task given in Interview-1

I developed these additional student proofs to better understand the nuances of the students' conceptions of proof; namely, whether they would accept an empirical argument as a proof when it is followed with a narrative explanation of reasoning (Sam's argument), and whether the students would be more likely to accept a deductive argument as a proof when the warrants are made explicit (Abby's argument). Below, Table 2 summarizes the structure of the student interview protocols.

Table 2. Structure of the student interview protocols

Student Interview-1	Student Interview-2

Proof Evaluation (Algebra task)
• Evaluating 6 hypothetical student proofs
• Favorite argument
  ○ Personal favorite
  ○ Teacher's favorite
	Proof Production (Geometry task)
	• Making a conjecture
	• Proving their conjecture

Proof Description
• Defining proof and proving
• Views on proof and proving in class
• Importance of proof & purposes of proof
	Proof Evaluation (Geometry task)
	• Evaluating 4 hypothetical student proofs
	• Favorite argument
	  ○ Personal favorite
	  ○ Teacher's favorite

Proof Production (Algebra task) • What do they produce as proof? • How do they evaluate their own proof? • What difficulties do students have in proving?	**Proof Description** • Proof statements table (Agree/Disagree/Somewhat agree) • Top 3 proof statements

	Proof Evaluation (Revisiting the algebra task from Interview-1)
	• Evaluating 2 additional hypothetical student proofs
	• Favorite argument
	  ○ Personal favorite
	  ○ Teacher's favorite

3.2.5. Teacher interview-2

One way to draw parallels between the students' and the teacher's conceptions of proof is to provide the teacher with the student interview protocols and to ask her how she would respond to them, as well as her expectations regarding how her students would have responded to the same items. Therefore, I completed the data collection with a second teacher interview, during which I also asked clarifying questions that emerged from the initial analysis of the teacher's first interview and the classroom observations (for more details, see Appendix-D). Hence, this interview enabled me to examine how the teacher engaged with those tasks and to probe her expectations about the students' ways of thinking and their abilities to prove.

Table 3. Structure of the teacher interview protocols

Teacher Interview-1	Teacher Interview-2
Background Information	**Proof Production** (Algebra task) • Proving the statement • Explaining how she would prove in class

Course information	**Proof Evaluation** (Algebra task)
	• Evaluating 8 hypothetical student proofs
	• Favorite argument
	  ○ Personal favorite
	  ○ Students' favorite

Curriculum – Is the curriculum supportive of reasoning and proof?	**Proof Production** (Geometry task) • Making a conjecture • Proving her conjecture • Explaining how she would prove in class

Proof Description
• Defining proof and proving
• Views on proof and proving in class
• Importance of proof and purposes of proof
	Proof Evaluation (Geometry task)
	• Evaluating 4 hypothetical student proofs
	• Favorite argument
	  ○ Personal favorite
	  ○ Students' favorite

Students' proof competencies	**Proof Description** • Proof statements table (Agree/Disagree/Somewhat agree) • Top 3 proof statements

Teaching proof – Strategies for teaching proof	*Clarification questions* – How does she view "verify", "justify", "explain reasoning", and "prove? Same? Different? – Formal proof vs. informal proof?

In sum, together these data sources enabled me to investigate the students' and their teacher's conceptions of proof in relation to each other and from multiple aspects. Specifically, I examined the participants' conceptions of proof through (a) their descriptions of proof and proving *(proof description*), their evaluations of hypothetical student proofs (*proof evaluation*), and (c) the arguments they produce to prove given statements (*proof productions*). By triangulating various data sources and measures of individuals' proof conceptions, I could develop a more complete and nuanced understanding of the participants' conceptions of proof.

3.3. Data Analysis

All interviews were videotaped and transcribed verbatim for analysis. Transcripts of each interview were first parsed into chunks per each distinct task, by using the three main categories (i.e., proof description, proof evaluation, and proof production*)*, as well as other related sub-categories. For instance, the proof evaluation task was further divided by each student argument (e.g., evaluating Arthur's argument, etc.), favorite argument, and teacher's choice.

The analysis began with open coding (Charmaz, 2006), starting with the first teacher interview, continuing with the first set of interviews with the focus-group students (Interview-1), followed up by the second set of student interviews with the focus-group students (Interview-2), and finished with the second teacher interview. The first round of the analysis was more explorative in nature. Although the three main categories loosely framed the analysis, and the literature (particularly as reviewed in the theoretical background section) broadly informed the analysis, in this initial phase the analysis was essentially open to emergent codes, aiming to identify aspects of the participants' conceptions of proof as freely and thoroughly as possible. During analysis, I continually compared evidence of aspects of the participants' proof conceptions with previously coded instances. As new insights about the participants' conceptions were gained, I wrote analytical memos to keep track of the ideas and emerging themes. Thus, codes were revised accordingly throughout the analysis. By the end of the first round of analysis, I created an emergent coding scheme by organizing the codes around themes through constant comparison (Glaser & Strauss, 1967) and reflections on the analytical memos, and by eliminating idiosyncratic codes.

I then used this coding scheme for focused coding (Charmaz, 2006) to analyze the data set by using the most recent codes. Thus, I re-analyzed all the interviews, continuing to write analytical memos to further distinguish nuances to the codes, and to maintain consistency and coherence in coding. After the second round of analysis, I aggregated the frequencies of codes within each category (i.e., proof description, proof evaluation, proof production) for each interview per participants; codes with low frequencies were further eliminated. The most salient aspects of the participants' conceptions of proof are presented in the results section.

4. Results

I will present and discuss the results in three parts, following the three main measures used for investigating the participants' proof conceptions. First, I will present the participants' proof descriptions, focusing on their views about proof and proving. Specifically, I will present the students' meanings of proof revealed in their first interviews, and then juxtapose them with their teacher's description of proof. I will then present further details about the students' meanings of proof that are understood through the second interviews. Second, I will report the results regarding the participants' proof evaluations, highlighting the types of arguments they consider to be proofs, their understandings about proof, and their criteria for accepting (or rejecting) an argument as a proof. Third, I will elaborate on the students' proof productions and argue that what students produce as proof may not truly reflect their ways of thinking and understandings about proof. In reporting the results, particularly about the proof evaluation and proof production parts, the students' conceptions will be foregrounded, with the teacher's conceptions providing context for and supplementing the students' conceptions.

4.1. Proof Description: Participants' Meanings of Proof

One of the measures I used to determine how the students conceived of proof was asking them to describe what proof and proving means to them, their views about proof and proving in mathematics, and their experiences with proof in class. The students' responses revealed that, in contrast to the existing literature, all the focus-group students deemed proof important in mathematics. For example, Julie expressed that proving advances mathematical understanding: "When you prove something, you gain a better knowledge of that and if you know how to apply that topic well because of the knowledge that you have, then it allows you to do better in other things that get gradually more complicated". As for the roles and purposes of proof, all the students stressed the verification role of proof, while three students (Brett, Julie, and Neil) also mentioned the explanation role of proof, and only one student (Brett) referred to the systematization role of proof. Hence, the students' descriptions of the roles of proof during the first interviews were consistent with the findings reported in the literature (e.g., Healy & Hoyles, 2000; Porteous, 1990).

4.1.1. Students' views of proof and proving: The first interview

During the first interviews the students' descriptions of proof emerged around three themes as follows: (a) *proof is backing up* statements/conclusion, (b) *proof is evidence* that shows the statement is true, and (c) *proof is explaining/showing how* you know that your work/answer is true. Table 4 below shows each focus-group student's description of proof, presenting students in the chronological order of the interviews. The first three interviews (Tyson, Hera, and Neil) occurred before the proof-related lessons had started in Unit 4 (in which the lessons emphasized similarity and congruence), while the remaining interviews (Molly, Julie, Mark, and Brett) occurred after the proof-related lessons had started.

Table 4. Students' descriptions of what proof and proving means to them

Students' Descriptions of Proof	Tyson	Hera	Neil	Molly	Julie	Mark	Brett
Proof is *backing up* statements/conclusions.	✓	✓					
Proof is *evidence* that shows the statement is true.		✓				✓	✓
Proof is *showing how* you know that your work/answer is true.	✓	✓	✓	✓	✓	✓	✓

4.1.1.a. ***Proof is backing up statements/conclusions***

The notion of proof as *backing up* statements or conclusions was uttered by only the first two students, Tyson and Hera. However, this notion of proof seemed to be closely related to the students' view of proof as showing how you know that your work or answer is true– another theme of the students' proof descriptions. For instance, Tyson explained:

> Proof is like backing up your statements, and like you can't just say like I like purple or something; you have to explain like why you like purple. Or like in the case of math, this equals 13, how does it equal 13? Well you show them your work, and that's proof.

Judging from the example of a mathematical proof that Tyson gave, one may argue that he was referring to a view of proof as explaining how one got his or her answer, rather than showing why the answer was true, which is certainly less sophisticated than proof. However, as will be elaborated in the following sections, the students seemed to hold a broad view of what it means to show something is true, which at times included both *how* and *why* something was true. It was also clear that the students were curious to understand why a conjecture was true during the interviews, and they demonstrated a profound emphasis on *explaining why* as a critical aspect of proof when they evaluated hypothetical proofs.

4.1.1.b. *Proof is evidence that shows the statement is true*

Another meaning of proof that emerged from the students' descriptions was proof as *evidence*; Hera, Mark, and Brett articulated proof as evidence. For example, Brett described proof as "Something like a piece of evidence that you can use, and then the proving is like the explanation of the evidence, the argument that I'm making for why it might be true". On the other hand, Hera's description of proof as evidence resembled an inductive way of thinking about proof, distinguishing her from Mark and Brett, who also viewed proof as evidence. Hera described proof as similar to a scientific experiment, emphasizing the need for testing many times and the accumulation of evidence leading to proof: "To me to prove something is to show that nothing else works, like, it's just that I guess. So, there's like no doubt almost in what proof is. So, if I, it's kinda like a science experiment". Hera went on to comment that, "I kind of interchange proof and a rule because if you have enough proof, you have a rule, like something that is always true". Although Hera's description hints that she understood that a proof needs to show that a statement is always true, her phrasing of having "enough proof" suggests that Hera was considering inductive testing as a viable approach to "prove". Indeed, when evaluating the hypothetical student proofs Hera considered that a statement can be shown to be always true through testing various examples, as well as through deductive reasoning. More specifically, her use of "always true" may possibly refer to something holding true for *each e*xample tested against a claim, rather than a logical implication necessitating the general truth. Hence, even though both Brett and Hera described proof as providing evidence, what students considered to be a legitimate evidence for proof seemed to differ among students, thus pointing to an important nuance to further probe during their proof evaluations and proof productions.

4.1.1.c. ***Proof is explaining/showing how you know that your work/answer is true***

All focus-group students had a shared notion of proof as *explaining* or *showing* how you know that your answer or work is true. In fact, this view seemed to be the overarching idea that the students had of proving, which the other two descriptions of proof – *proof as backing up* and *proof as evidence*– could be deemed as part of. However, being such a broad and encompassing description (as seen in Tyson's account above), the students' descriptions fell short in clarifying what is entailed in showing how one knows that his or her answer is true. For example, Julie described, "I feel in math it's proving that your answer is correct through the work that you show." Julie's description does not elucidate whether she referred to simply showing how one got the answer, or whether it also included justifying why something must be true– an important distinction similar to the distinction that Knuth (2002b) observed among teachers' accounts of "explaining why".

Other students echoed similarly inexplicit views. For example, the students described proving as a common practice in class in the form of explaining one's answer, emphasizing that it was never called as proving, though. For instance, Mark expressed that, "I suppose it [proving] was commonly used, but it wasn't ever referred to as proving or anything like that. Probably more like explain your answer, it should be a form of proving." Molly confirmed a similar view when she stated, "But a lot of teachers would always be like, show proof of how you got it, which is just like explain." Tyson also noted that they were always engaged in proving "in the form of show work", noting that they were always expected to show their work in their math class. Thus, the students' descriptions appear to reflect an informal meaning of proof that the students seemed to appropriate from their classroom experiences.

Therefore, a critical question arises regarding the all three meanings of proof that the students uttered; that is, what is the nature of legitimate evidence, backing up, and showing the truth of something? Would the students consider an example a valid evidence or back up? Or, would they accept explaining the steps of one's solution as a proof? The students' descriptions of proof in the first interview did not explicitly disclose what it meant for them to explain how one knows that something is true, leaving it uncertain whether they would consider showing *how* one got their answer as a case of proving, which is different than showing something must be true through deduction. On the other hand, the analysis of the first set of interviews in their entirety hinted that the students possibly had a notion of proof that conflated showing why something must be true with showing the steps of one's answer. Thus, the second interviews aimed to elucidate some subtleties about the students' meanings of proof, which will be discussed next, after presenting the teacher's description and views of proof and proving to provide some context to the students' descriptions of proof.

4.1.2. Ms. V's description of proof

During the first interview, Ms. V expressed that her perspective of proof has changed as a teacher; while she used to think of only formal proofs (which she associates with her high school geometry and college mathematics courses) as proof, Ms. V stated that as a teacher she now adopts a broader view of proof as *justifying ones' reasoning or claims*:

> I think what I learned to be as a proof in high school is definitely different from how I perceive proofs today. So, as a teacher, I guess, when I think of proofs- just in general, students saying an answer to something and then explaining how they know that that answer is correct.

Ms. V's description reveals that she had two meanings of proof; one that she associated with formal proofs in the context of mathematics discipline (considering from the perspective of a *learner* of mathematics), and one that is related to the context of school mathematics, which appears to be a broader, informal meaning of proof (considering from the perspective of a *teacher* of mathematics). Note that this broad meaning of proof is reminiscent to the students' notion of proof as *explaining* how you know that your answer is true. Although explaining one's conclusions and claims is a practice that students are encouraged to engage with (NGA/CCSSO, 2010), it does not necessarily constitute proof; more specifications are needed for proving.

In response to the question of whether proving is expected of students in the class, Ms. V affirmed that students were expected to prove in her class, noting that the answer to this question depends on how one thinks about proof. Thus, Ms. V elaborated that she considers proof as "being able to always say the answer is this *because*, or I know this *because*- being able to justify your reasoning and explain it in some way, I think that's always something that we're expecting students to do." Here Ms. V emphasized proof as justifying reasoning with warrants, suggesting that she views proving as backing up one's claims– which will be more evident in her second interview.

On the other hand, Ms. V also explained that she does not use the word “proof” in class, but rather asks students questions such as “How do you know that?”. She articulated, “I don't say, ‘Prove it’. I don't say, ‘Write me a proof that explains why this is true’, I just say: ‘How do you know?’, ‘Can you explain that?’, ‘How do you know that's true?’, ‘How do you know that's the answer’?” Indeed, Ms. V frequently asked such questions when monitoring small group discussions or leading whole class discussions. However, while such questions were often intended to press students to justify and back up their claims deductively, there were also times that they were simply meant to have students explain their work or strategy. Hence, this dual emphasis on both justifying claims deductively and explaining one’s work seem to account for why the students had developed an informal meaning of proof as explaining or showing how you know that your answer is true.

Furthermore, despite to her emphasis on “proving” in class, Ms. V anticipated that the students would not recognize that they prove regularly in their math class, assuming the students did not have the perspective that she had of proving. Ms. V expected that the students would have a narrow view of proof that is limited to geometry units. Hence, she expressed that her goal was to broaden students’ view of proof. More specifically, she asserted:

> I guess that is one thing I would like to change as a teacher. Because from my experience, once I learned about proofs in those specific units, that's all I associated with proof… So, I guess one thing that I would like to do with my students is make them see that proving is not just those two units that we do in geometry; it's something that you're constantly doing.

Contrary to the teacher’s expectation, however, the student interviews revealed that the students did share the same perspective of proving with their teacher; that is, proving is not limited to geometry and that they are always expected to justify their ideas in class. Given that research (e.g., Furinghetti & Morselli, 2011; Knuth, 2002b; Varghese, 2009) documents that teachers often have impoverished conceptions of proof (and so do students), viewing proof as limited to geometry, appropriate only for advanced students, and thus not a central idea in school mathematics, this is an interesting finding as it shows that when a teacher has a more encompassing conception of proof, this may be powerful in influencing students' conceptions of proof.

In sum, while the students' descriptions of proof disclosed broad accounts of how they viewed proof, their articulation of what proof and proving meant to them often left out important details, showing that individuals' descriptions of proof alone is not enough to draw conclusions on their conceptions of proof. Articulating what proof means is not an easy task, especially for students; thus, the students' definitions of proof may not adequately portray their views of proof. Therefore, I sought to further probe the students' meanings of proof in the second interview through a list of proof statements that was developed based on the preliminary analyses of the first interviews. In what follows I present some additional findings regarding the students' meanings of proof that emerged from the second interviews, and discuss to what extent the students shared similar meanings of proof among each other and with their teacher.

4.1.3. Further probing into the students' views of proof and proving: The second interview

4.1.3.a. *Proof is backing up statements and providing evidence*

Students' descriptions of proof in the first interview pointed to two views of proof that may have been interchangeably used by the students; that is, proof is *backing up statements* and proof is *providing evidence*. It is possible that, for both views of proof, the students may have intended to express that proof means providing warrants to the claims or statements that explain why a statement or a claim is true. Moreover, these views of proof may still be part of what students believed a proof is, even though they had not expressed it during the first interview. Hence, seeking to identify some nuances of the students' views of proof, I had developed a list of 20 proof statements based on the remarks that the students made during the first interviews, considering all 18 student interviews. I asked both the students and their teacher to mark whether they agree, disagree, or somewhat agree with each statement (see Appendix C for the full list). Two of those statements were related to the students' views of *proof as backing up* and *proof as evidence*, and Table 5 below presents how the students evaluated each statement. ✓ indicates agreement with the statement, ✗ indicates disagreement, and ~ indicates that the student somewhat agrees with the statement. Grey shading signifies that the statement was also one of the top three statements that was picked as best describing what proof means to the interviewee.

Table 5. Proof statements regarding students' meanings of proof and proving

Proof Statements	Tyson	Hera	Neil	Molly	Julie	Mark	Brett
Proving is backing up your statements or claims.	✓	✓	✓	~	✓	✓	✓
Proving is providing evidence, such as an example, that the statement is true.	✓	~	✓	~	✓	✓	✓

Perhaps not surprisingly, the number of students who agreed with each statement exceeded the number of students who articulated the corresponding views of proof in the first interview. While only Tyson and Hera described *proof as backing up* claims in the first interview, almost all students, except Molly, agreed with the statement that proof is backing up statements or conclusion. Similarly, while Hera, Mark, and Brett described *proof as evidence* in the first interview, five students agreed with that statement in the second interview, and two students somewhat agreed. As seen in the table, all students, but Hera, marked both statements the same, hinting that the students might have considered these two views to be very similar. In fact, one student, Mark, commented that the second statement was the same as the first statement. Note that the second statement, proof as evidence, also indicates examples as a means to prove; the aim was to see how students would react to it. Given that most students agreed with the statement, the inclusion of examples as an evidence, and therefore as a means to prove, was not problematic for the students. However, this was not necessarily because the students accepted examples as proof, as will be seen in the next sections, but rather because the students considered examples helpful in the process of proving, and thus an important part of the process of proving.

Although Hera's description of proof in the first interview involved both views of proof, *proof as backing up* and *proof as providing evidence*, interestingly in the second interview she somewhat agreed with the statement that proof is providing evidence. Hera thought that it would be "really hard to give evidence that applies to all scenarios", indicating that she understood that proof must account for all cases and that one example is insufficient to prove. On the other hand, Hera strongly agreed with the statement that proving is backing up statements or claims, which was one of her favorite statements. In fact, that statement was one of the favorite statements of the teacher, as well. Ms. V. remarked that, "That's awesome. I'm glad that the first thing wasn't that they had to make a flowchart", and commented that, "This would be my most basic explanation of what a proof is. And I hope that's what my students think as well; proving is just being able to justify what you're saying is true." Thus, the analysis of the interviews overall showed that the students did share the same perspective of proving with their teacher; that is, proving is justifying one's ideas with reasons. But what did they consider to be legitimate reasons? A definite answer to this question was not possible based on their descriptions of proof, but will become evident in their evaluations of hypothetical proofs, which will be discussed later.

Only one student, Molly, did not fully agree with either statement, pointing to the limitations of the statements. She somewhat agreed with the statement that *proving is backing up*, expressing that proof is more than backing up your statements or claims, “Because there are so many ways that you could back it up.” She also somewhat agreed with the statement about *proof as providing evidence*, highlighting that evidence shows the truth for specific cases instead of showing that the statement is always true. Explicitly, Molly expressed that, “You're not really supposed to use evidence, like when they used evidence in the ones like Clara and Ben and all them [referring to the proof evaluation task], it only made it seem true for specific numbers and not for every single number.” Ms. V also somewhat agreed with this statement due to the same limitation that Molly expressed. As Molly pointed out, the given statements were still inherently somewhat vague as they were based on the student responses, but they allowed the students to comment on them and enabled one to see to what extent they were shared understandings among the students. In sum, both *proof as backing up* and *proof as providing evidence* were commonly held views of proof among the students and their teacher, as no students disagreed with them.

4.1.3.b. *Proof is explaining/showing how you know that your work/answer is true*

During the first interview, all focus-group students described proof as explaining or showing how you know that your work or answer is true, but what it exactly entailed was not clear. To understand more nuances as to the students’ meanings of ‘showing work’ or ‘explaining’, three statements, based on the students’ remarks, were added to the list of proof statements. Specifically, the statements aimed at understanding whether the students regarded (a) showing how one got an answer, (b) checking one’s solution, and (c) explaining one’s thought process as forms of proof. As seen in Table 6, the number of students who agreed that proof is explaining one's answer dropped to five in the second interview; the other two students asserted that the statement was somewhat true. Molly, one of the two students, articulated that "I mean it's not necessarily showing how you got it. It's just showing why it's true". Thus, although most students still perceived showing how one got an answer as a kind of proof, providing students with proof statements enabled me to uncover some nuanced (and more sophisticated) meanings associated with students' descriptions of proof.

Table 6. Proof statements regarding the meaning of *proof as showing work and explaining*

Proof Statements	Tyson	Hera	Neil	Molly	Julie	Mark	Brett
Proof is like showing how you got your answer; explaining your answer.	✓	✓	✓	~	✓	~	✓
Proving is like checking your work to make sure that it is correct.	✓	✓	✓	✓	~	✓	✗
Proof is explaining your thought process.	✓	✗	~	✗	~	✗	~

As suspected, for some students *showing how you know* that your work is true seems to involve *checking one's work* to make sure that it is correct, which was also agreed by five students. So, most students regarded checking one's work as a form of proof, as they believed that proof ensures the accuracy of the work. For example, Molly explained that it was an accurate statement because, she asserted, "When we have to do proofs, it helps you make sure that your stuff is correct because you have to follow all the rules of math when you make one". Her remarks suggest that she deemed a role of proof was to verify the steps of one's solution, on the basis that proof is built on the mathematical rules. Hence, it is also possible that students might have perceived verifying each step with a warrant– a reason that made an argument or an operation valid. In addition, Ms. V was pleased that the students had said that proving was like checking one's work, asserting that verifying the solution of an equation algebraically could be one way of proving, whereby 'algebraically verifying' means plugging a solution into an equation. Ms. V's comments show that her notion of proof was, indeed, quite broad as she stated, accepting the verification of the solution of an equation as a kind of proving. In fact, several students mentioned checking the solution of an equation as an example of proof, suggesting that this broad notion of proof was a shared view across the students and the teacher.

Yet, the students' notion of proof was not too broad to accept explanation of thought process as a proof; only one student agreed that proof is *explaining one's thought process*, which suggests that the students understood that proof must have certain characteristics. For instance, Brett argued:

> Proof is like showing that there's no holes or like missing pieces in why you think that, but your thought process … could be missing a lot of things. I need to have examples and evidence and like proof that it works, not just like why I think it.

Brett's remarks indicate that he understands that a proof requires reasons that validate each step or claim, albeit also suggesting examples as a possible means of warrant. Nevertheless, the students overall understood that proving entails providing warrants to claims; although what they accepted as legitimate warrants is equally, if not more, important, as will be discussed in the following sections. The teacher somewhat agreed with this statement, asserting that although explaining thought process does not count as proof, it is still important during proving. Thus, Ms. V's comment reflects her view of proving *as a process* that includes exploring a conjecture with examples, figuring out reasons that make the conjecture true, and communicating the reasoning.

4.1.3.c. ***Proving is showing something is true based on known facts, rules, and definitions***

Even though during the first interviews no students defined *proof as showing that a mathematical statement is true based on definitions, known facts and properties*, their engagement with proof evaluation and proof production tasks hinted that they might have held this meaning of proof as well. Aiming to see to what extent the students would agree with this view, the list of proof statements that was given to the students included the following statement: "Proving is showing that something is right based on the known facts, rules, definitions, and properties." All students agreed that it was a valid statement. Furthermore, all students, except Hera, also picked this statement as one of the top three statements that best described what proof meant to them, asserting that they use rules and definition when they do proofs in class. Tyson, for example, remarked, "Yeah, when we were trying to prove that triangles were similar, we would have to do these exact things, like we could prove that they're similar because of definitions". Those classroom experiences seem to be ingrained in Tyson so much that he picked this statement as one of his favorite proof statements, saying that he sees that in class every day. Additionally, Hera expressed her agreement by making reference to her teacher: **"**Yeah, at least that's what my teacher says." Evidently, the students' classroom experiences with proving and their teacher's emphasis on using known facts, definitions, and rules to prove mathematical statements seem to have made an influence on the students' views of proof and proving. Thus, the second interview not only provided a context for uncovering some nuances about the students' meanings of proof but also exposed some of the students' evolving notions of proof that were influenced by the classroom factors. By the time the second interviews were conducted the students had been in Ms. V's class for a substantially long time, and thus they had been subject to certain emphasis on what it means to prove a mathematical statement. Through such influences, the students' views of proof appear to have aligned more closely with their teacher's view of proof.

In conclusion, the students' proof descriptions revealed that they hold various meanings of proof, including proof as *evidence*, *backing up claims*, and *showing that something is true*. In addition, the students knew that proof is constructed based on known facts, rules, and definitions– a clear influence of their classroom experiences. However, proof descriptions alone were not sufficient to distinguish students in terms of their conceptions of proof, but they provided a helpful foundation for the other two measures (proof evaluation and proof production). Specifically, asking the participants to evaluate hypothetical student proofs turned out to be a fruitful context for uncovering more details regarding their ways of thinking and understandings about proof.

4.2. Proof Evaluation: What Do Students Understand About Proof?

Asking students to evaluate hypothetical student proofs (ranging from empirical arguments to algebraic deductive arguments) in terms of whether they constitute a proof or not was informative about (a) what kinds of arguments the students accepted as a proof, (b) what understandings about proof informed their evaluations, and (c) what criteria they considered for accepting an argument as a proof (or for rejecting when certain criteria were not met). First, I present what kinds of arguments the students accepted as a proof for the algebra conjecture given in the first interview, along with a discussion of the sources of difficulties that seemed to affect the students' evaluations. I then present the students' proof evaluations for two additional hypothetical proofs given for the same task during the second interview, which is then followed up by the students' favorite proof selections. Next, I focus on the students' proof understandings and the criteria they used for accepting an argument as a proof to provide a more detailed account of the students' conceptions of proof, by also considering their evaluations of their own proof productions in addition to their evaluations of the hypothetical student proofs.

4.2.1. What kinds of arguments did the students accept as a proof?

In the first interview, the students were asked to evaluate six hypothetical student proofs given for the conjecture that the sum of any two even numbers is an even number (Figure 2), expecting that the conjecture would be a familiar mathematical statement for high school students and thus its proof would be accessible to them in terms of the mathematical content knowledge it requires. To recap, the given hypothetical student proofs included a deductive algebraic argument (Arthur), an empirical argument (Bonnie), a deductive narrative argument (Ceri), a narrative argument of proof by exhaustion (Duncan), an incorrect algebraic argument (Eric), and a visual argument that could be viewed as a generic example (Yvonne).

The proper evaluation of Arthur's argument requires students to recognize how the definition of even numbers is used in algebraically representing even numbers and building the mathematical statement. Similarly, Ceri's argument also requires students to recognize how the definition of even numbers is used in constructing the statement, but in a narrative form. In both arguments, students need to be able to follow the logical chains between the premises, the definition of even numbers, and the claim. Duncan's argument draws on the fact that even numbers can be characterized by the unit digits of numbers and presents a proof by exhaustion by listing all possible unit digits of even numbers; thus, students need to see that connection and be aware of proof by exhaustion as an acceptable method of proving. The proper evaluation of Eric's argument requires students to focus on the meaning of algebra, rather than on the form of the argument, by decomposing the algebraic expressions and evaluating the logical chains between premises and the claim. Lastly, the evaluation of Yvonne's argument would depend on whether students view it as a generic example or just a particular example. For students to view it as a generic example they need to recognize that the definition of even numbers is represented visually and that the visual representation shows that the sum of *any* two even numbers is always an even number.

As shown in the Table 7, the students tended to choose deductive arguments, albeit more frequently in narrative form than algebraic form, as a proof, but not the empirical argument. More specifically, almost all the students considered Ceri's deductive narrative argument as a proof, while more than half of the students also considered Arthur's deductive algebraic argument and Yvonne's visual argument as a proof. Overall, the students were aware of the limitations of empirical arguments, with only two students accepting Bonnie's empirical argument as a proof.

Table 7. Types of arguments accepted as proof

Hypothetical student proofs	**Number of students who accepted it as a proof**
Deductive/narrative argument (Ceri)	6
Deductive/algebraic argument (Arthur)	4
Visual argument/generic example (Yvonne)	4
Narrative argument/proof by exhaustion (Duncan)	2
Empirical argument (Bonnie)	2
Invalid algebraic argument (Eric)	1

Furthermore, although the students found the use of variables and equations to be sophisticated and important in proving, they were not, however, influenced by the form of the argument alone. This is more evidently seen in their evaluation of the incorrect algebraic argument given by Eric; only one student (Julie) accepted it as a proof. These findings are encouraging as they demonstrate more sophisticated proof conceptions than reported in the literature (e.g., Healy & Hoyles, 2000). On the other hand, almost half of the students did not consider the correct algebraic argument as a proof either. Thus, while these findings, taken together, suggest that the students did not base their decisions solely on the form of the argument, yet a question remains: Why did some of the students not accept the valid algebraic argument as a proof?

4.2.1.a. *Student difficulties in evaluating hypothetical student proofs*

A close examination of the students' evaluation of the hypothetical proofs revealed several student difficulties, which affected their ability to accurately evaluate the arguments, in general, and led some students to reject Arthur's algebraic deductive argument as a proof, in particular. The two main difficulties that affected the students the most were (a) understanding how the definitions and previously established results were used in constructing arguments and (b) understanding the mathematical properties of numbers. Clearly, the students' ability to correctly evaluate the arguments as a proof was closely linked to their knowledge of mathematics involved in the argument to be evaluated, in addition to their conception of what a proof is.

Arthur's argument had been particularly challenging for the students to evaluate; all students had difficulty in understanding how Arthur had algebraically represented any two even numbers by using the definition of even numbers, and how he then used it to construct the mathematical statement and showed that the sum of any two even numbers is an even number. The warrant for each step of Arthur's argument was present, but implicitly, which (for some students) obscured how and why the argument proved the conjecture. For instance, Neil was confused why Arthur began his proof by defining *a* and *b* as any two whole numbers instead of defining them as any two even numbers: “I don't get why it says– wait $2a$ and $2b$ are any two even numbers, but usually a is just a single variable, so shouldn't it be a plus b equals an even number? I don't know if I'm not understanding that right.” I followed up his comment by asking him to describe what he was specifically confused about, and suggested that he use an example to explain his confusion. Neil described:

> Well, when it says any whole number I think it can mean 1 or 3 because it's whole. So, you multiply it by 2 and it goes to $2a$, so that would be 2, then plus 6, and then. But its saying– Oh, it's just an extra process to say any number! If they're even, that means they always add up together because if you multiply an even number by 2, it becomes, it's still even. If you multiply an odd number by 2, it becomes even. So, now I understand where he's going; he's saying when you add them together it's still going to be even when you multiply it by 2.

Neil initially had difficulty in understanding why Arthur began his proof with any whole numbers, and therefore did not understand how Arthur proved the statement. But, through explaining his confusion Neil recognized that by doing so Arthur constructed two even numbers, which can then be used in the next step where the sum of any two even numbers were shown to be an even number. Thus, after overcoming his initial difficulty, Neil accepted Arthur’s argument as a proof, asserting that it showed that the statement was always true and also explained why it was true. But, Neil also added that had Arthur provided an example to illustrate his proof, it would be helpful, indicating his initial struggle to understand Arthur’s argument.

As shown in the case of Neil, some students were able to overcome their difficulties in understanding how the definitions and previously established results were used in constructing arguments and subsequently accepted Arthur’s argument as a proof. But, three students, Mark, Hera, and Tyson, were not able to fully overcome this difficulty and thus did not consider it as a proof. Hera and Tyson claimed that it showed that the statement was *always true*– because it used variables– but it did not *explain why* it was true, therefore, they rejected it as a proof. This is reminiscent to Harel's (2006) discussion of the need for causality of a proof, which had been a contested debate between mathematicians, with some arguing that proofs that do not show causality, such as proof by contradiction, are not acceptable as a proof. Thus, in his revision of the proof schemes framework, Harel (2006) introduced the causal proof scheme as a sub-category of the transformational proof scheme, describing *causal proof* as "an enlightening proof that gives not just mere evidence for the truth of the theorem but the cause of the theorem's assertion" (p. 71). Hence, Hera and Tyson's emphasis on *explaining why* as a decisive criterion for their judgment of Arthur's argument seems to be a manifestation of the causal proof scheme.

On the other hand, Mark thought that it neither showed that the statement was *always true* nor *explained why* it was true, thus, he did not accept it as a proof. Mark remarked: "I think that all sounds very sound, but I don't quite see how it proves that it's true… it only– it tells the obvious, but it doesn't tell why two even numbers always add up to even numbers." Apparently, Mark failed to see how the argument was connected to the statement being proved, as he claimed that the argument was not connected to *all numbers*, that is "All even numbers add up to an even number when they're added together." Mark had difficulty in understanding how the variables were set to establish the given statement, and thus showed the statement was true in general, difficulties arising from not seeing the logical implications inherent in Arthur's argument. In addition, by pointing to the need for the argument to explain *why* the statement was *always* true, Mark also signals his inability to recognize that Arthur established any two even numbers by using the definition of even numbers.

Similar to Neil and Mark, Hera was also confused that Arthur began his proof by defining *a* and *b* as any whole numbers, which she considered to be redundant: “I'm confused why he did that I guess to prove his answer. Oh, but then, add two even numbers. I guess he didn't really need to do that step.” Her puzzlement was also coupled with a temporary confusion of considering *whole number* to imply being an *even number*. Hera exclaimed: “We already know that those are even, because when he says it’s a whole number, well it's also an even number, right?” Clearly, Hera did not see the logical necessity of establishing any two even numbers by multiplying any whole number by 2, and thus she considered the first step of Arthur’s proof to be redundant. So, Hera initially thought that Arthur did not prove: “He didn’t really prove, like show why that’s, I feel like that just has no place in there almost. They already said that two times *a* plus *b*, that's just rewriting it, it's not really proving it.” But, as she further explored the argument, Hera suddenly realized that Arthur’s “purpose was to represent even numbers” in writing 2*a* and 2*b,* and that Arthur showed that the sum of two even numbers will be an even number. With this insight, Hera was able to make connections between the definition of even numbers and the algebraic argument, which was evident in her comments when she elaborated on how Arthur showed that the sum of two even numbers was an even number, the second step of Arthur’s proof: “Even numbers are numbers that can be divided by two. So, we know that if that's a number that has already been multiplied by two, then that's an even statement.”

Consequently, Hera thought that Arthur *explained why* the statement was true and accepted it as a proof. However, shortly after that, Hera changed her mind again and said that, “I'm still kind of like iffy about it though for some reason, just because it's like if *a* plus *b* is equal to *a* plus *b,* like if I divide it by two that's– I don't think it proved it.” In conclusion, Hera claimed that although Arthur’s argument showed that the statement was *always* true (since it used variables), it was not a proof because it did not *explain why* the statement was true. Like Neil, Hera also indicated that using examples would have been helpful to explain Arthur's argument. Hence, Hera's case displays shifting changes in her understanding of the deductive algebraic argument; although she was able to grasp the logic in Arthur's proof for a brief period of time, this insight was not a solid understanding yet. Specifically, Hera struggled to make sense of the chain of deductions because she struggled to understand the necessity of the type of algebraic representations Arthur used, reflecting her difficulty in understanding the mathematics of this particular argument– not necessarily reflecting a deficit in her conception of what a valid proof is.

In conclusion, these examples of student difficulties show that what students accept as a proof depend not only on their conception of what a valid proof is but also on their comprehension of the mathematics involved in an argument. In other words, it is evident that students' evaluation of arguments as a proof occurs at the intersection of their understanding of mathematics and their notion of what a proof is more broadly. Recall that two students (Mark and Tyson) accepted Ceri's narrative deductive argument as a proof, but not Arthur's algebraic argument, because they did not see how Arthur's argument explained why the statement was true due to their difficulty in understanding how the stated definitions and mathematical properties were used to construct the statement. Apparently, Arthur's argument was harder for students to understand, possibly due to lack of experience in algebraically representing even numbers and constructing mathematical arguments based on definitions and established results. In sum, these findings do not reveal impoverished understandings of proof, but rather show that the students struggled to make sense of the algebraic connections and that some students had higher expectations regarding what counts as proof, including the need for causality.

4.2.1.b. *Favorite proof choices*

To see what types of arguments students considered to be the best proof I asked the students which argument was their favorite proof among the six hypothetical proofs given in the first interview. The students' favorite proof choices, as shown in Table 8, reflect what they considered to be the most *explanatory* argument for the given conjecture. Again, Ceri's deductive narrative argument was chosen as the favorite proof by most students, either by itself or in combination with Yvonne's visual proof or with Arthur's algebraic argument. It is noteworthy to mention that the students often considered an imagined audience when they evaluated arguments, so the students' evaluations reflect not only what was personally explanatory to them but also what, they believed, would be explanatory to others. For example, some students found Yvonne's visual proof very explanatory but did not accept it as a proof because they were not sure whether it would be clear to other students how that visual representation could be applied to all even numbers, and thus, showed the truth for all cases. In other words, some students viewed Yvonne's argument as a *generic example* (Mason & Pimm, 1984), recognizing that it showed the truth in general, but they were cautious that others may not view it as a generic example. Hence, consideration of explanatory power of an argument was essential in the students' favorite proof choices, as well as consideration of an imagined audience, highlighting that the students viewed proof as a social process. The students' low preference of the algebraic argument as a favorite proof, on the other hand, was likely due to their difficulties in understanding the mathematics involved in that particular argument (Arthur's argument), and thus should not be taken as an evidence that students in general do not consider algebraic arguments as favorite proof choices. In fact, the students did pick an algebraic argument as their favorite proof when they were able to comprehend the argument during the second interview, as will be shown in the following section.

Table 8. Students' favorite argument choices

Favorite argument as a proof in Interview-1	**# of students**
Deductive/narrative (Ceri)	2
Narrative argument/proof by exhaustion (Duncan)	2
Visual (Yvonne)	1
Deductive/narrative (Ceri) + Visual (Yvonne)	1
Deductive/algebraic (Arthur) + Deductive/narrative (Ceri) + Visual (Yvonne)	1

In addition, I asked the students which argument they thought would be their teacher's favorite proof, aiming to see if the students' personal choices differed from what they expected that their teacher would consider as the best proof. In contrast to the existing findings (e.g., Healy & Hoyles, 2000), it turned out that the students' expectations for their teachers' favorite proof aligned to the large extent with their personal favorite proofs, suggesting that the students had appropriated their teacher's proof values. Six students in total thought that their teacher would choose Ceri's narrative argument as the best proof, while two students believed that Arthur's argument would be their teacher's favorite. One of the students who thought Ceri's argument would be the teacher's favorite suggested that her teacher would also choose Duncan's argument, which was her personal favorite proof. All in all, the students' guesses for their teacher's favorite proof points to alignment between their personal views of proof and the view of proof that they perceived that their teacher had. The students' predictions of their teacher's favorite proof, a deductive narrative argument, suggests that the students had developed an impression that a proof needs to be an explanatory general argument, which does not need to be in algebraic form.

4.2.1.c. *Proof evaluation task revisited: The second interview*

During the first interviews the students appeared to believe that testing a wide range of examples would be a more sophisticated way to show the truth of a statement compared to just testing a few small numbers. Also, in their descriptions of proof and their evaluation of arguments they praised the importance of explaining reasoning in proof. Therefore, I wanted to test whether the students would accept an empirical argument consisting of a diverse set of examples that also included a narrative explanation of the thought process as a proof. Furthermore, given that the students commonly had difficulty in understanding the implicit warrants in Arthur's algebraic argument, I also wanted to see if the students would be more likely to accept an algebraic deductive argument as a proof when the warrants in the argument were made more explicit. Therefore, I created two additional hypothetical student arguments for the same mathematical statement given in the first interview: one was an empirical argument– what would be classified as *crucial experiment* in Balacheff's (1988) terms– that was presented as Sam's answer, and the other one was a deductive algebraic argument that was presented as Abby's answer (see Figure 4 re-presented below). During the second interview, I reminded the students of the proof evaluation task from the first interview and asked them to similarly evaluate these two additional hypothetical student proofs.

Do you think Sam and Abby proved the statement? Why or why not?

Sam:

2 + 4 = 6
60 + 26 = 86
406 + 262 = 668

I tested it with different examples, both small and large numbers. It worked each time. So, it works for any two even numbers.

Abby:

Say *x* and *y* are any two even numbers. By definition, *x* and *y* can be represented as follows:

$x = 2 \bullet a$ (*a* is any whole number)
$y = 2 \bullet b$ (*b* is any whole number)

Then, $x + y = 2 \bullet a + 2 \bullet b = 2 \bullet (a + b)$

Because the sum of *x* and *y* has a factor of 2, it is divisible by 2. Therefore, the sum of any two even numbers is always an even number.

Figure 4. Two additional hypothetical student proofs for the algebra task given in Interview-1 (*re-presented*)

The students (except Hera) did not accept Sam's empirical argument as a proof, although it included a diverse set of examples and an explanation. The students explicitly stated that Sam's argument showed the truth only for a particular set of examples, not for all cases, and that his explanation did not show *why* the statement was true. For instance, Tyson argued that, "It's just, it shows, like he's showing work, but I don't think he's really explaining it just because … he needs to provide more, like he needs to explain it… It does not show that it is always true." Similarly, Molly contended that, "I mean there's an explanation, but there's not a good reasoning behind the explanation, so there's not anything." On the other hand, all the students accepted

Abby's algebraic argument as a proof, with no hesitation. Tyson's reaction exemplifies the students' evaluation of Abby's argument in general: "See I like this new one! Because it shows like the equation, the thinking behind it; because even numbers is [*sic*] always going to be divisible by two… So, the statement is always true? I agree. It shows to me for all." Furthermore, all, but Hera, thought that their teacher would not accept Sam's empirical argument as a proof, while all the students asserted that their teacher would accept Abby's argument as a proof, providing further evidence for the close alignment between the students' personal proof evaluations and what they perceived to be their teacher's proof values are. Moreover, revisiting the proof evaluation task showed that the students had developed a sharpened understanding of the distinction between empirical arguments and deductive arguments, revealing that the students did not consider simply any narrative explanation as sufficient for proof; they deemed the explanation devoid of any warrants illuminating why the statement was true as inadequate. Overall, the students had a combined need for an argument to *explain why* something was true and to show that it is true for *all cases* to consider it as a proof.

In addition, during the second interview I asked the students which argument was their favorite proof out of the eight student arguments. Students' solidified understanding of what counts as a proof was also manifested in their favorite proof choices, as this time they chose deductive algebraic proofs (either Abby or Arthur). Three students picked Abby's and two students picked Arthur's argument as their favorite proof (data are not available for Brett and Tyson's favorite proof choices in the second interview). It is notable that all students for which we have available data selected an algebraic argument in the second interview, considering that most students had preferred a narrative argument in the first interview.

I posed the same question to the teacher and inquired her personal favorite proof, as well as what she thought the students would have picked as their favorite proof. Ms. V selected Abby's proof as her favorite proof because "It included a lot of justifications that verified each step". Yet, Ms. V predicted that her students would have picked either Abby's or maybe Eric's argument as their favorite proofs. Her prediction that the students may have picked Eric's as their favorite proof reflects her awareness that students are often swayed by the form of the argument when it comes to proof, so Ms. V expected that her students might have considered it to be more sophisticated and thus to be the best proof.

In conclusion, considering the results from the first and second interviews, the students overall had sophisticated conceptions of proof in terms of the kinds of arguments that may count as a proof, by mostly accepting deductive arguments– both narrative and algebraic arguments– (and rejecting empirical arguments) as a proof. Notably, the students' proof evaluations were not reliant on the form of the argument– as commonly observed in the literature, but rather depended on the perceived explanatory power of the argument, which was largely influenced by the students' understanding of mathematics involved in each argument. Essentially, the findings suggest that students' proof evaluations exist at the intersection of their understanding of mathematics and their notion of what a valid proof is, underscoring that the types of arguments students accept as a proof should not be taken as what students universally consider to be a proof. As we have seen in the second interview, the students unanimously accepted an algebraic argument when the warrants for each step of the argument was made explicit, and thus the students had no difficulty in understanding the content of the argument. What was more important for the students is whether the argument explained why the statement was true in general, regardless of its form. But, the students were also aware that an algebraic proof is what is expected of them, and thus they considered it highly. Ms. V, on the other hand, was generally aware of common student difficulties with proof. Finally, the students' proof evaluations and favorite proof choices were closely aligned with what they thought their teacher would accept as a proof and consider as the best proof, indicating that the students had appropriated their teacher's proof values. This is also a promising finding as it shows that students can develop desired conceptions of proof when their teachers appreciate the value of proof and have robust conceptions of proof. (The classroom norms and practices that appeared to have supported the students' evolving conceptions of proof will be unpacked in paper #2, *Classroom Factors Supporting Students' Conceptions of Proof: Classroom Norms, Instructional Practices, and Curriculum*). Next, I present results regarding the students' understandings about proof that informed their proof evaluations.

4.2.2. Students' understandings about proof and proving

In addition to the types of arguments that the students accepted as a proof, the students' comments in interviews, especially during their evaluation of the hypothetical student proofs, unveiled some key proof understandings that the students possessed, which are presented in Table 9 in a somewhat gradually increasing order in sophistication. The number of references column displays the total frequency of instances that the students made an explicit remark about a proof understanding during the interviews, including the first and the second interviews collectively.

Table 9. Students' proof understandings

Students' Proof Understandings	# of students	# of references
Appeal to form or appearance is not accepted in mathematics.	2	2
Examples are insufficient for proof.	6	19

One counterexample is sufficient to disprove.	5	7
Testing a diverse set of examples is more convincing than testing a small set of similar examples.	7	18
Proof shows the truth for all cases.	5	9
Proof explains why a conjecture/statement is true.	5	10
Examples are helpful/needed to:		
• communicate/illustrate one's proof	5	10
• verify one's proof	4	11

4.2.2.a. ***Appeal to form or appearance is not accepted***

Although during the first interviews only one student (Mark) explicitly remarked that making assumptions about the truth of a conjecture based on its appearance is not accepted when proving, I suspected that other students might have shared this understanding as well because Mark's comment referred to an example of proof they did in class, thus pointed to a possible classroom influence. He specifically noted that, "When we have shapes that we have to prove they are congruent– you can really see… You can assume they're congruent, but you have to prove it. And to prove it, you need to have facts that are undeniable." Hence, I included the following statement to the list of proof statements that was given to the students in the second interview: "*In math you cannot build your work on assumptions. That's why we prove things in math*". In the second interview, one more student explicitly remarked that making assumptions based on appearance is not acceptable, yet all the students agreed with the statement presented to them. In short, despite that not many students had articulated this understanding, presenting the corresponding statement to the students revealed that it was, in fact, a shared understanding among the students, underscoring the importance of triangulation of multiple data sources when determining students' conceptions of proof.

4.2.2.b. ***Examples are insufficient for proving***

All the students, but Hera, made explicit remarks indicating their understanding of examples as insufficient for proving. Mark, for example, provided an elaborated expression of this understanding when evaluating hypothetical proofs as follows:

> When you're just using values to prove something, then that is not really a proof. Yeah, because when you're just using integers or actual values and not variables, then you're not proving it. You're just showing that for that case that this works, but not for every case like a proof should.

On the other hand, Hera (the only student who did not exhibit a clear understanding of examples being insufficient for proof), nevertheless recognized the limitations of examples when the argument involved a single example or limited range of examples.

Moreover, while attempting to prove a conjecture, all of the students indicated that they found examples helpful for gaining conviction about the truth of a statement, but four students (Brett, Mark, Molly, and Tyson) subsequently also declared that examples were insufficient for proving. Once the students believed the conjecture was true, such comments were often followed by a remark indicating that the students were aware that they needed to algebraically prove the statement. For instance, during the proof production task in the first interview, Molly explored whether the conjecture was true by testing two examples and found that it held true for both examples, leading her to assume that the conjecture was true. But, Molly knew that those examples did not prove that the conjecture was true; instead, they simply suggested that the conjecture was probably true and now she needed to find a way to algebraically prove it: "I'm just going to assume that that works with all other odd numbers… I'm just going to say that it's true. I don't know how I would prove it algebraically, but I sense that it's true." This is a particularly pleasant finding that the students recognized examples as helpful entry points in exploring mathematical conjectures, yet they knew that simply testing and confirming the conjecture with examples do not constitute proof– counter to what is frequently found in the literature (e.g., Balacheff, 1988, Harel & Sowder, 1998; Knuth, Choppin, & Bieda, 2009).

In addition, the students generally had a clear understanding of the role of counterexamples for disproving; five students stated that only one counterexample was sufficient to disprove, while only one student (Hera) viewed counterexamples as exceptions. For instance, Brett articulated that to disprove a conjecture "All you need is like one in certain situations. If it uses like an absolute verb like always or something like that, then you need to– it's just one thing to prove it wrong, and that would– it would be false" (Interview-2). Whereas, Hera contended that if a counterexample is found "You're not necessarily disproving it, you're just finding exceptions" (Interview-2)– a common student misconception (Harel & Sowder, 1998).

4.2.2.c. ***A diverse set of examples is more convincing than a few small examples***

The analysis of the interviews revealed a distinction between students' views of diverse set of examples *as more convincing* for the truth of a statement and viewing it *as a proof* for the truth of the statement. While all the students considered a diverse range of examples more convincing for gaining conviction about the veracity of a statement, which was evidenced in their critique of Bonnie's empirical argument and was also reflected in their deliberate test of large and diverse numbers during their proof attempt in the first interview, it was not clear whether they would consider it a proof as well. Specifically, Julie and Hera's comments in the first interview led me to explore if the students would accept a diverse set of examples as a legitimate proof. Hence, as described earlier in Section 4.2.1.c., in the second interview I asked the students to evaluate an additional hypothetical student proof, Sam's empirical argument.

While in the first interview Julie commented that Bonnie's argument would have been more accurate if she had shown other numbers beyond the limited scope of the numbers that she had tested, commenting that Bonnie showed "A pretty limited window of numbers because she keeps everything under ten", Julie did not, however, accept Sam's argument as a proof in the second interview, arguing that Sam showed that the argument was true only for some even numbers even though he tested "a variety of examples… in different ranges". The only exception was Hera, as she thought that Sam proved the conjecture by showing that it was true for a range of numbers.

Hence, the findings indicate that the students overall found testing a diverse set of examples (or *crucial experiment* in Balacheff's (1988) terms) more sophisticated and informative than testing a small limited range of examples, yet most of the students understood that it did not count as a proof. This points to a nuance to the way 'crucial experiment' is previously discussed in the literature that presented it as if what students accepted as a proof. While this may be the case for some students, it also seems possible that some students simply find it more convincing and sophisticated. Indeed, in this study both the students and the teacher considered a 'crucial experiment' as a more sophisticated and valued approach to proving compared to just testing a few examples, but they were aware that it is still insufficient for proof.

4.2.2.d. ***Proof is a general argument that explains why the conjecture is true***

A particularly encouraging finding of the study was that many students articulated proof as a general explanatory argument as they evaluated or described what proof meant to them. As seen in Table 9, five students (excluding Julie and Neil) explicitly remarked that proof shows the truth for all cases, while five students (except Hera and Neil) stated that proof explains why a statement is true or false. For instance, Brett's remark, "If I prove something, then it's true in all cases”, indicates that he understood that proof accounts for all cases. Molly, on the other hand, emphasized the need to provide warrants when proving, as she described: “You need to say because of this and this, this will be this, kind of like in geometry like a proof like that.” Important to note here, however, is that these frequencies reflect the explicit remarks students made, and thus, they do not necessarily mean that those students who did not explicitly remark such comments do not share those understandings. As will be seen in the next section, all the students considered *generality* and *explaining why* as critical aspects of proof when they evaluated arguments.

Furthermore, while only five students explicitly articulated that proof needs to account for all cases, all the students repeatedly remarked that the use of variables or equation enables one to show the truth for all cases. For example, Mark tried to come up with a general argument to prove the conjecture about odd numbers during the first interview. When I inquired of his goal, Mark responded that he was looking for a general explanation, “One with variables that could account for all odd numbers”. Although all the students considered using variables or equations as a means to show the truth in general, most of them also expressed that a proof can be in any form, and that it did not need to be an algebraic argument. Recall that all of the students accepted Ceri’s narrative argument as a proof in the first interview, for example. Molly’s remark nicely sums up the students’ shared understanding: “A proof isn't like, no one says a proof is this, like a proof has to be an equation. A proof can kind of be however you can prove it. And you could do that through an equation, you could do that through whatever.” Hence, it is pleasing that the students’ conceptions of proof were not limited by the form of the arguments.

Additionally, the students had a shared understanding of proof as an explanatory argument, as evidenced by their unanimous approval of the statement, “Proof shows why something is true or false by showing the reasons behind it.” Hence, even though two students did not make an explicit remark about proof as explaining why, when considering multiple data sources, it was evident that the students collectively understood proof as an argument illuminating the reasons that makes an argument true.

4.2.2.e. *The need for examples in proving: To illustrate the proof and to verify the proof*

The analysis of the student interviews revealed interesting nuances about the students’ need to use examples after accepting an argument as a proof (or even after producing a general deductive proof). In the proof literature students’ tendency to use an example after accepting an argument as a proof is often discussed as a deficiency in students’ understanding of proof (e.g., Vinner, 1983), since there is no need to further check with an example given that proof guarantees the truth in general. However, I found the students’ intentions to use an example to be different than to further test the veracity of the statement to be proved. Instead, the students’ need to use an example appeared to be either to *illustrate* a *proof* or to *verify the accuracy of a proof*.

The students’ need to use examples to *illustrate a proof* was related to their view of examples as helpful tools to convey meaning of deductive arguments, similar to generic examples that reveal the structure of an argument (Mason & Pimm, 1984). In addition, the students often suggested using an example to illustrate a proof when they considered the proof being communicated to an audience. More specifically, five students expressed that an example would be helpful to include in a proof so that others could better understand it. This was more frequently uttered when evaluating an algebraic deductive argument where it was not easy for the students to immediately comprehend it. Similarly, when responding to the statement, “*A proof* *should include why a statement is true, the reasoning, and an example*", several students alluded to the need for examples to illustrate one's proof. Molly, for instance, discussed that, "I think a solid proof would have all of them, just because it would help the whoever is looking at it better understand what you're trying to get across." Likewise, Julie agreed with the statement, asserting that, "Because that's kind of the way we are taught to do them". Hence, the students' need to use examples turned out to be more nuanced and sophisticated, also reflecting an influence of classroom experiences. Ms. V's comments indicated that she considers and draws on examples as a pedagogical tool to better illustrate to students how a proof makes sense. Given that the students are accustomed to seeing examples being used to illustrate a proof, they also found examples helpful to include in their proofs to better communicate their arguments. Hence, the students valued illustrative examples highly to communicate their argument.

Additionally, four students expressed that examples were helpful to *verify the accuracy* of one's proof. This is reminiscent to how Porteous (1990) argued the use of an example after accepting an argument as a proof on logical grounds; he described this phenomenon as a type of "checking", stressing that students' aim is to check the validity of the proof itself, not to check a particular instance of the conjecture as an individual case. Thus, contrary to its typical treatment by researchers, Porteous advocated that this is a sophisticated behavior. Moreover, this view also seemed to reflect a classroom influence, resulting from the emphasis placed on checking answers in students' (previous and current) mathematics classes. For instance, this emphasis can be seen in Tyson's remarks, "You should always double check everything that you do". Similarly, Hera was unsure whether her teacher would accept an algebraic argument as a proof "Because they didn't really test any numbers", indicating a similar need to check the accuracy of one's work. Hera explained her hesitation by referring to how she could discover an error in her proof attempt by means of an example: “Because I know when I tested mine, it didn't work and then I was like oh, I made an error. This was… we call it like checking”. Hence, some students regarded the use of an example as a way to show that the proof they produced was actually correct, which they considered a validation of their proof. In fact, students were frequently asked to check their answers (particularly, their solutions to equation systems because some of the solutions could be an extraneous solution) through their classwork and homework assignments as well as by their teacher. Recall that Ms. V had a broad notion of proof that included verification of the algebraic solution(s) of an equation system as a type of proof, a view that was also shared by the students. It seems that the teacher and the students have transferred the verification role of proof to verifying the accuracy of everything they do, including verifying the accuracy of one’s proof as well as verifying the solutions of an equation system.

4.2.2.f. *The teacher’s evaluation of the hypothetical student proofs*

These five key proof understandings also reflect how the teacher viewed proving in class. More specifically, when presented the same proof evaluation tasks to the teacher, Ms. V accepted all the hypothetical student proofs as students’ versions of proving, stressing that they varied from incomplete to complete proofs. In her evaluation, Ms. V pointed to what was missing in each argument and explained how she would provide feedback to the students to improve their proofs. Thus, her evaluation of the student arguments corresponded to the progression of proof understandings as outlined in Table 9. To recap, like the students, Ms. V viewed examples as first steps into proving. Ms. V described examples as incomplete proofs, but also stressed that examples are helpful for understanding what a conjecture says and for gaining conviction about the truth of the conjecture. Ms. V also considered testing a diverse set of examples more sophisticated and more strategic than simply testing a few similar examples, but also emphasized that it was still an incomplete proof. In addition, she accepted general deductive arguments, either in algebraic form or in a narrative form, as a complete proof, highlighting the need for a proof to account for all cases. Furthermore, Ms. V also considered the inclusion of an example as a helpful pedagogical tool to illustrate a proof to students. All in all, there was a close alignment between the students' proof understandings and their teacher's views about proof and proving in class.

Indeed, Ms. V's orientation that some of the student proofs were incomplete, rather than incorrect, and her approach of providing feedback to the students about how to enhance their proofs seem to be productive in supporting students' developing proof conceptions. As she reported during the interviews, Ms. V's approach to providing students feedback to improve their proofs was a recurrent emphasis in class discussions as well as in her written feedbacks to the students' homework. Hence, the teacher's consistent emphasis on such feedback seems to have contributed to the students' ways of making sense of proof, given that the students' understandings of proof resemble many similarities with their teacher's view of proof in class.

4.2.3. Students' criteria for accepting an argument as a proof

While the students' proof understandings reported above largely explicate how the students conceived of proof, they are restricted to the students' explicit remarks, and thus, may not adequately portray how each student conceived of proof. In fact, the reasons that the students had for accepting an argument as a proof as they evaluated the hypothetical student proofs or their own proof attempts provided another window into how they conceived of proof and proving. In some cases, the students also discussed why they rejected an argument as a proof, stressing the aspects of a valid proof that was missing in the argument being judged to not be a proof. For instance, when explaining why he thought that Bonnie's empirical argument was not a proof, Mark expressed that, “In order to prove it, he [*sic*] needs to have some kind of theorem that uses all possible integer numbers that are even to explain his answer.” Thus, Mark emphasized generality as a missing aspect of Bonnie’ s argument, as well as the need to build on theorems in proving.

Hence, the criteria that the students used for accepting or rejecting an argument as a proof, which were outlined in Table 10, shed light on what the students considered to be the aspects of a valid proof and thus contributed to more thoroughly uncovering the students’ conceptions of proof. As seen in the table, the students’ criteria for proof were grouped into four categories as they related to (a) the generality of the argument, (b) explanation of the reasons for why a statement is true, and the connections between premises and the claims, (c) forms of the argument, and (d) some other aspects of proof that the students deemed important. In what follows, each group of criteria will be elaborated respectively.

Table 10. Students’ criteria for accepting (or rejecting) an argument as a proof

Students’ Criteria for Accepting an Argument as a Proof	# of students	# of references
Generality	7	33
Explains why	7	40
Provide warrants	7	47
• Definitions	7	18
• Known facts/properties	3	9
• Logical inferences	3	9
• Theorems	2	2
• Empirical evidence	2	2
Assertions lead to the conclusion	4	9
Forms of arguments:		
• Variables/equation	7	16

• Narrative explanation	5	13
• Visual representation	5	10
• Deductive/algebraic argument	4	5
• Examples	3	10
• Examples (*crucial experiment*) + Deductive/algebraic	2	6
• Examples (*crucial experiment*) + Narrative explanation	2	4
Other aspects of arguments:		
• Easy to understand	7	16
• Thorough/sophisticated argument	3	5
• Familiar argument	2	2

4.2.3.a. *Generality*

Although only five students expressed that a proof must account for all cases, all the students, however, considered generality as an aspect of a valid proof. In fact, generality was one of the main criteria that the students typically considered when evaluating arguments, which was also evident by the high frequency of its occurrences. For example, in explaining why one of the hypothetical student proofs presented in the second interview was not a proof, Hera emphasized the generality as a missing aspect of the argument, simultaneously also pointing out that it was not built on theorems. Specifically, Hera argued:

> Because you really didn't say like– shows us applying to all cases because … when we're doing like similarity or congruency theorems for triangles, we have like specific theorems that work for every triangle that we base our conjectures off of, but when you don't base yours off of anything, then you can't really say that it's true all the time.

Hera's emphasis on showing the truth for all cases as a missing aspect of the argument is particularly important, given that she was the only person who accepted an empirical argument that included a diverse set of examples (Sam's argument) as a proof during the same interview. Her remarks suggest that Hera was aware that a proof needs to account for all cases, but she was not always consistent in considering this aspect when evaluating arguments. Nevertheless, Hera often acknowledged generality as a criterion for accepting an argument as a proof, or for rejecting an argument when it was not general, as seen above. This is also important as it points to the complexity of studying individual's conceptions of proof, and thus underscores the necessity of using multiple measures to uncover individuals' conceptions as thoroughly as possible.

4.2.3.b. ***Explaining why something is true with warrants***

In addition to the generality of arguments**,** all of the students also frequently considered whether an argument explained why a statement was true as a critical feature of proof, such that this criterion largely affecting their acceptance of an argument as a proof. For instance, Neil asserted, "If you have something that's being proved, you have to be able to say why it works" (Interview-1). Clearly, *explaining why* constituted an important criterion for the students throughout the interviews. Simultaneously, the students commonly emphasized *providing warrants*, which often fulfilled a supportive role to explain why a statement was true. As seen in Table 10, the students recognized various types of warrants including definitions, known mathematical facts, rules and properties, logical inferences, theorems, and even empirical evidence. Among these various types of warrants that the students considered as a criterion for evaluating arguments as proof, backing up an argument with definitions was the most common type of warrant, as all of the students unanimously emphasized it. To continue with Neil, he specifically emphasized the use of definitions, as well as explaining why, as reasons that made Ceri's argument a proof. Neil described that, "Well, it defines first … what an even number is. And then it shows like the factor that can be divided by when added, so that means it shows that it's an even number they're adding together." When evaluating Yvonne's argument, on the other hand, Neil rejected it as a proof on the basis that the argument did not include the definition of even numbers: "He [*sic*] doesn't really state anything about what the even number is. What makes the even number and why in the end the answer is even." Similarly, when Julie was evaluating Eric's incorrect algebraic argument, she pointed to the fact that Eric did not "actually specifically mention the even numbers at all". Moreover, several students picked Abby's algebraic deductive argument as their favorite proof for the reasons that included generality, explaining why, and providing warrants such as the definition of even numbers, and known facts and properties. For example, Hera justified her favorite proof choice of Abby as follows:

> Because they [Abby] didn't test anything, but the way that, they very logically– they used like all three facts. *a* is always going to be a whole number because that's the definition of an even number, and then the way like she plugged it in using the information. It's all like, there would be no exceptions to that rule.

Related to the students' criteria of explaining why a statement was true and providing warrants was another criterion that four students considered in their evaluation of proofs; that is, whether the assertions made in an argument followed logically from premises to conclusion. In Brett's words, whether the assertions "leads you to a conclusion". Specifically, Brett explained:

> Ceri's [argument] is like a walk through and like that's kind of easy because it kind of like, it kind of explains what an even number is sort of and that kind of

> leads you to a conclusion, while a lot of them kind of just lead to it by– like
> Bonnie's definitely not, and I still don't know what's up with Eric's answer.

As seen in Brett's excerpt, this criterion, assertions leading to a conclusion, indeed embraces the other two criteria, explaining why and providing warrants, since the process of arriving at a conclusion from premises naturally involves providing warrants, and as a result, expose viewers what reasons make the statement true, provided that the warrants are valid and complete.

Hence, explanation of the reasons for why a statement is true was one of the most frequently referenced proof criteria that the students used when evaluating arguments, which seems to be related to the students' and the teacher's notion of proof *as backing up claims*. Indeed, explaining why something is true was a recurrent emphasis in class– both in small group discussions and whole class discussions.

4.2.3.c. ***Forms of arguments***

When it comes to the forms of arguments, there appears to be a diversity of forms that the students valued in accepting as a proof. Foremost, all the students indicated the use of variables (or equations) as favorably affecting their evaluation of an argument. But, it is important to note that the form of an argument as a criterion was never sufficient by itself to accept an argument as a proof. In other words, even though all the students considered highly the use of variables in proving, the use of variables alone was not enough for the students to accept an argument as a proof– recall that only one student accepted Eric's incorrect algebraic argument. Moreover, many students also praised the inclusion of a narrative explanation or a visual representation as contributing to the acceptability of an argument as a proof, which seemed to be an influence of classroom norms set by their course syllabus as well as their classroom experiences. In addition, four students (Brett, Mark, Molly, and Tyson) specifically emphasized deductive algebraic arguments as a valid and sophisticated form of proof, highlighting it as a reason for their evaluation.

On the other hand, three students (Hera, Julie, and Neil) also considered the inclusion of examples as contributing to the validity of arguments as a proof. For those students, examples often served the role of illustrating the argument by exposing the structure of the argument (such as a generic example) or verifying that the argument was correct. Additionally, Hera and Neil also suggested that a diverse set of examples together with an algebraic argument would be an acceptable proof, while Hera and Julie suggested a diverse set of examples combined with a narrative explanation would be acceptable, too. In short, although for the most part the students understood that examples alone were not a legitimate way to prove, the students, especially those who had more difficulty in understanding the content of a given argument, nevertheless found examples quite helpful (and even necessary) in proving. Given that the students also indicated complementing a diverse set of examples either with a narrative or algebraic argument, it is fair to argue that they viewed examples as a tool for proving, rather than as an end in proving. This may also be a result of the students' experiences with proving, in which students often explore a situation with a variety of examples, try to find a pattern, and then express that pattern either algebraically or more often with a narrative explanation. Thus, for those students who suggested a diverse set of examples coupled with an explanation as a proof, the examples were likely a medium for explicitly showing that the statement was true for various cases, while the explanation provided a backing up for why the statement was true in general.

4.2.3.d. *Other aspects of proof*

The students also considered some additional aspects of arguments, which in combination with the other criteria outlined above, contributed to the students' evaluation of arguments as a proof. These aspects included (a) *accessibility of arguments*– whether an argument was easy to understand, (b) *sophistication of arguments*, and (c) *familiarity of arguments*– whether the argument being evaluated resembled to the students' own approaches. Of those, perhaps the most important aspect was the accessibility of arguments, as it was considered by all the students quite frequently. However, it is important to note that the students' sense of an easy-to-understand argument was not restricted to a particular form, as it was uttered for all forms of arguments (including algebraic deductive arguments), but instead it was mediated by the students' ability to understand the content of an argument. That said, however, it was generally easier for students to unpack the meaning of narrative arguments compared to algebraic arguments– this finding is consistent with the literature (e.g., Healy & Hoyles, 2000).

About half of the students also considered an argument's thoroughness and sophistication as contributing to its acceptability as a proof. Whereas, two students stated that the argument they accepted as a proof was similar to their own proof approaches, indicating it as an additional reason for why they considered that the argument was a proof. Hence, accessibility, sophistication, and familiarity of arguments functioned as supplementary criteria supporting the students' proof evaluations.

4.2.4. Conclusion: Students' proof evaluations

All in all, when the students' proof evaluations are considered together with the kinds of arguments they accepted as a proof, the proof understandings they expressed, and the criteria they used for accepting an argument as a proof, two loosely defined groups emerged in which the students can be grouped together: (a) students with a robust understanding of proof as a deductive argument, and (b) students who understood proof as a deductive argument, but also occasionally valued empirical arguments. Brett, Mark, Molly, and Tyson demonstrated a robust understanding of proof as a deductive argument, which was evidenced by their explicit articulation of the need for deductive argument for proving, their understanding of generality and explaining why a statement is true as key characteristics of proof, and consistent emphasis on examples as being insufficient for proof. On the other hand, while Hera, Julie, and Neil also understood proof as a deductive argument that explains why a statement is true in general, they did not consistently (or clearly) consider examples as insufficient for proof, which was evidenced by their occasionally accepting an empirical argument as a proof or viewing examples as an aspect of a valid proof.

Even though all of the students considered whether an argument accounted for all cases and whether it explained why the statement was true, the students in the first group more frequently and explicitly emphasized those aspects. For Brett, Mark, Molly, and Tyson, *generality* and *explaining why* were the two main criteria that they consistently used to evaluate hypothetical student proofs, as well as their own proofs. Moreover, these students explicitly highlighted that showing something is true (i.e., merely verifying the truth of a statement) is different than showing *why* something is true (i.e., explaining the reasons that make a statement true). But, despite some variations across the students in terms of their thinking and understandings about proof, there were remarkable shared understandings about proof (among the students as well as with their teacher) too, which can be attributed to classroom norms and practices as briefly described above (for a detailed discussion of the interplay between the students' proof conceptions and the classroom norms and practices, see Paper #2).

In the following section, I will focus on students' proof productions to complement the analysis of their proof conceptions and argue that what students produce as proof may not truly reflect what they understand about proof. I will also show how students' ability to prove is largely dependent on the conjecture to be proved and what mathematical knowledge it entails.

4.3. Proof Production: What Does Students' "Proofs" Tell Us?

As a third measure for studying students' proof conceptions, I asked all of the participants to prove a conjecture in each interview, an algebra task in the first interview and a geometry task in the second interview (as seen in Table 11). If time permitted, I provided students with an additional proof production task during the first interview to see students' proving approaches and competencies through multiple tasks. Thus, four students (Hera, Julie, Molly, and Tyson) attempted to prove a second conjecture (i.e., the bonus task). For each task, students' proof attempts were followed up with questions probing whether they think that their argument counts as a proof, how confident they are in terms of the validity of their proof, how they know that their proof is sufficient, and whether they think that their teacher would accept it as a proof. While the algebra task required students to prove a mathematical statement about odd numbers, the geometry task required students first to explore a given problem and develop a conjecture, and then to attempt to prove their own conjecture. For the bonus task, students first examined a hypothetical student argument and then attempted to prove or disprove the given conjecture about the sum of consecutive numbers. Hence, the proof production tasks varied by their domain (algebra vs. geometry), the activities they require students to engage with (e.g., exploring, conjecturing, proving), and the mathematics content knowledge it requires.

Table 11. Proof production tasks

Algebra Task	How would you prove the following statement? *If p and q are any two odd numbers, (p + q) x (p - q) is always a multiple of 4.*
Bonus Task	Can you comment on what Georgia now knows as a result of what she noticed?

	"Georgia is asked to prove or disprove that **the sum of any *n* consecutive integers is divisible by *n*.** In order to test whether or not the statement is true, she tries a few examples. She notices that 1 + 2 + 3 is divisible by 3, but 7 + 8 + 9 + 10 is not divisible by 4." How would you prove or disprove the conjecture? Let's try to prove or disprove it.
Geometry Task	Say you have a square and you add a certain amount to its length and take away that same amount from its width. What happens to the area?

Students' proof productions for each task are presented in Table 12. To see if there is any pattern between students' thinking and understandings about proof and their ability to prove, the results about students' proof productions are presented per the two main groups that emerged from the analysis of the other two measures– students' proof descriptions and proof evaluations (as described in section 4.2.4.); that is, (a) students with strong understanding of proof as deductive argument (Brett, Mark, Molly, and Tyson), and (b) students with relatively moderate understanding of proof as deductive argument who sometimes also regarded empirical arguments as a proof (Hera, Julie, and Neil). All in all, there does not appear to be a clear link between students' proof understandings and their proof productions. Specifically, contrary to popular belief, the students with strong understanding of proof as a deductive argument did not necessarily perform better (by producing deductive arguments) than the students who demonstrated a relatively mixed understanding of proof as deductive and empirical arguments.

Table 12. Students' proof productions per task

	Students with Strong Understanding of Proof as Deductive Argument *(Brett, Mark, Molly, and Tyson)*	Students with Relatively Moderate Understanding of Proof as Deductive Argument *(Hera, Julie, and Neil)*
Algebra Task	– Partial narrative argument supported with empirical evidence (*Molly, Tyson*) – Empirical evidence (*Brett, Mark*)	– Partial narrative argument supported with empirical evidence (*Hera, Julie, Neil*)

Bonus Task	– Deductive (narrative) proof (*Molly, Tyson)* – N/A (*Brett, Mark*)	– Deductive (algebraic) proof (*Hera*) – Partial narrative argument (*Julie)* – N/A (*Neil*)
Geometry Task	– Developed correct conjecture (*Brett, Mark, Tyson)* – Partial deductive (algebraic) argument (*Tyson*)	– Developed correct conjecture (*Hera*) – Deductive (algebraic) proof (*Hera*)

In the algebra task, although no students could fully prove the conjecture, half of the students with a strong understanding of proof and all the students with relatively moderate understanding of proof developed a partial narrative proof, which was supported with empirical evidence, to explain why the conjecture is true. Their arguments were considered a partial proof because the arguments were on the right track to be a proof, but they had logical holes or missing parts, and thus, were incomplete. On the other hand, the two students who demonstrated the most robust understanding of proof (Brett and Mark), who consistently and strongly emphasized generality and explaining why as two critical aspects of proof, were not able to produce a proof. These students explored the conjecture with examples and once convinced that the conjecture was true they attempted to prove it algebraically, yet they could not construct an algebraic argument. Hence, Brett and Mark could produce only empirical evidence; but, importantly, they were aware that they had not proved the conjecture. In fact, all of the students with strong proof understanding maintained that they could not prove the conjecture, while two students with relatively moderate proof understanding (Hera and Julie) asserted that they partially proved it, and one student (Neil) claimed that he proved it. Therefore, there was not much difference between the two groups in terms of the arguments they produced, and the students were largely aware that they had not proved the conjecture, with one distinction: the students with a strong proof understanding more firmly acknowledged that they had not proven it, reflecting a higher set of expectations for proof.

Interestingly, in the same interview all four students who were given the extra proving task could produce a deductive proof, albeit one of them (Julie) was a partial proof. While Molly and Tyson (students with strong proof understanding) could develop a narrative deductive proof, Hera, who was classified as one of the students with relatively moderate proof understandings, could construct an algebraic proof. This is an important finding for couple of reasons. First, it confirms that students' ability to produce proofs (i.e., types and sophistication of their arguments) vary depending on the context (i.e., the nature of the proving task). Second, it also shows that students' views and understandings about proof do not necessarily determine what they are able to produce as a proof, as Hera was one of the three students who had accepted an empirical argument as a proof, yet she was also the one who could produce the most sophisticated proofs across the two interviews.

Lastly, the geometry task was of a different nature; in addition to being a geometry problem (a different domain), the students also needed to form a conjecture first. Four students could make a correct conjecture; three of them (Brett, Mark, and Tyson) were the students who had strong proof understandings; but, only one of them (Tyson) could –partially– prove it. On the other hand, Hera, the only student from the second group who could develop a conjecture, could also prove her conjecture. Hence, although the students with a strong understanding of proof as a deductive argument were more successful at developing a conjecture, that was not the case for proving their conjectures. It was Hera again who produced the most sophisticated proof, rebutting an implicit (but common) assumption that students' views about proof and their proof productions are analogous, one determining the other, and therefore could be used interchangeably.

Hence, the findings underscore that what students can produce as a proof may not truly reflect their ways of thinking and understandings about proof. For instance, Brett and Mark produced an empirical argument, yet they knew that it does not constitute a proof. Focusing only on students' "proofs" would have resulted in underrating what Brett and Mark knew about proof. By asking students to evaluate their "proofs", one can see that what is regarded as "students' proofs" may not always be what students themselves consider to be "proof"; instead, they are simply what students are able to produce at a given time. Using multiple measures, on the other hand, allowed me to see the complexity and richness of each individual's conceptions, and thus enabled me to more accurately represent the students' conceptions of proof.

Moreover, students' proof understandings and their proof productions did not appear to be linked in a way that favored the students with strong proof understandings to produce deductive proofs. In fact, while the students who had a strong deductive understanding of proof were not able to produce a deductive proof, a student whose understanding of proof included both deductive and inductive ways of thinking (Hera) was able to produce an algebraic proof when presented with an additional proving task. Thus, taken together, the results show that students' "proofs" can be different from what they find convincing as a proof and their beliefs about proof, and therefore what students produce as a proof should not be taken as the only evidence for determining their proof schemes. Students' beliefs and views about proof may inform their approach to proving, but nevertheless students' ability to prove a given conjecture depends on various factors, the conjecture to be proved and the mathematics content knowledge necessary for its proof being one of the major factors.

Indeed, the students' increased success in proving the bonus task during the first interview speaks to that issue. Although the conjecture given as the algebra task was expected to be accessible to the students, it required representing odd numbers, which turned out to be a major student difficulty and a critical obstacle to algebraically proving the statement. Unable to represent odd numbers generically, the students were left to explore the conjecture with examples, aiming to identify a pattern that can justify the claim. But, another common student difficulty emerged as the students largely focused on the results (outcome) of the operations rather than focusing on understanding how the operands might be related to the claim. More specifically, especially during their initial exploration, the students generally tested the conjecture with examples to see whether the statement was true; thus, they focused on the result of the multiplication of the sum of two odd numbers and the difference of two odd numbers, checking if it was divisible by 4. By focusing on the calculation and not on the factors in that multiplication operation, many students (at least initially) missed an important opportunity to notice that a factor of 4 is produced from the multiplication operation, a key insight that could lead to proof of the conjecture. On the other hand, the conjecture about the sum of consecutive integers happened to be easier for students to express symbolically, making it possible for students to generalize the insights they gained from their example exploration. Hence, as seen in these examples, each proving task presents unique mathematical challenges and opportunities for students that inevitably affect their ability to prove depending on their mathematical abilities as well as their readiness, willingness, and perseverance to deal with them; some tasks, for instance, may require algebraically representing mathematical propositions– which may or may not be within the mathematical resources students possess, while in some cases the mathematical structure that is core to the proof of a given conjecture might be easier to detect, which in turn may influence students' approach to prove the conjecture.

4.3.1. Shifting nature of student proofs by task: An illustrative case

To further illustrate how students' proof productions were not necessarily dependent on students' views of what a valid proof is, but rather varied depending on the task and the mathematics involved in its proof, I present an overview of Hera's proving process for both proof production tasks given in the first interview– the algebra task and the bonus task. Hera's case, namely her approach to prove (together with the difficulties she had) during these two tasks, portray quite different proving processes. The difficulties Hera had during the first task did not appear to be an issue during the second task, showing that the difficulties Hera experienced were related to the specific conjecture. Through this illustrative case, I show that students (who even sometimes value empirical evidence as proof) can produce algebraic deductive proofs when the conjecture and the mathematics involved in proving it is within their reach. In presenting Hera's proving processes, I will also exemplify some of the common student difficulties that hindered students' ability to prove.

4.3.1.a. ***Hera's proving process during the algebra task***

Hera's exploration of the first proving task, the algebra task, began with testing the veracity of the statement with an example. Noticing that the difference and the sum of two odd numbers (premises of the conjecture) yield even numbers, Hera then shifted to thinking about the conjecture generically. Using the known facts and properties as warrant, Hera justified that the product of the sum and the difference of two odd numbers would be an even number, but was still unaware why the product might be divisible by 4:

> If you take an even plus an even, or an odd plus an odd, you would always get an even. And if you take an odd minus an odd, you would always get an even. So, if you multiply two evens, I don't know if you would always get a number that is divisible by four; I know it would always be divisible by two, though.

A turning point in Hera's approach was when she started focusing more on the result of the calculations, aiming to verify the statement, as she continued her example exploration by assuming random numbers to be an example *of* (p + q) and (p – q), neglecting the premises of the conjecture: "Let's pick like two even numbers, like 12 and 4 assuming that p plus q is 12 and then p minus q is 4. I don't know if that would actually work out. Let's just say it does to make two evens." Thus, by making such unwarranted assumptions, Hera unwittingly continued her exploration with irrelevant examples (e.g., p = 8 and q = 4, violating the premise of the conjecture), which, of course, was unhelpful to progress towards proof. Hence, during this phase a fixed focus on verification together with a disregard of the premises and the reasons that make the conjecture true became sources of difficulties for Hera.

In response to the question of whether she proved the conjecture, Hera avoided to make a claim about proving; instead, she replied that she "could definitely show evidence". This prompt led her to focus on the premises again and to explore the conjecture with relevant examples, but she did not go beyond providing empirical evidence. Hence, I followed up by asking whether she thinks that the conjecture is *always* true, which shifted her approach to algebra. Hera tried to justify why the statement is true through algebraic expressions (as seen in Figure 5), but was again unable to fully explain why the conjecture must be divisible by 4.

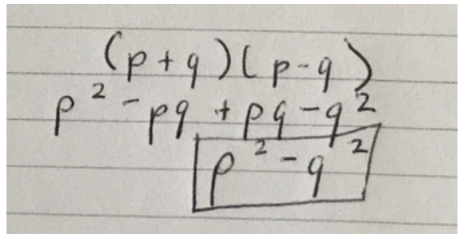

Figure 5. Shifting to algebra to prove

In brief, throughout her attempt to prove the conjecture about odd numbers, Hera made use of examples, developed a narrative argument that justifies that the product is an even number, and later attempted to construct an algebraic argument, but was unable to fully explain why the conjecture must be true. She had several difficulties along the way, including representing odd numbers, focusing on the results of the examples used (and verification of the conjecture), and neglecting the given premises that led to a disconnect between the premises and the claim of the conjecture. These are difficulties that occurred in the context of this task; they may or may not appear in other tasks.

4.3.1.b. ***Hera's proving process during the bonus task***

The second proving task involved a hypothetical student argument exploring the conjecture that the sum of any n consecutive integers is divisible by n for the cases of 3 and 4 consecutive integers by testing each case with an example (1 + 2 + 3 and 7 + 8 + 9 + 10, respectively) and showing that the conjecture held true for the case of 3, but not for the case of 4. In response to what conclusions could be made about the conjecture, Hera stated that it means that the conjecture is not true, but she was also hesitant that it might not be true only for the particular example that was tested. Therefore, Hera went on to further explore the conjecture with her own examples, testing 7 + 8 + 9 and 1 + 2 + 3 + 4. Unlike her tendency in the previous task, Hera was curious to understand why the conjecture worked for the case of 3 but not for the case of 4. Her eagerness to understand *why* paid off, as she noticed a structural element in her example. Hera exclaimed:

> I think I know why. Because oh, the difference here, yeah, so the difference here is one. So, you have two extras so if you're already at seven plus seven plus seven plus two. (pause) Is that? And then two, oh so it'd be a 3. So, then that's why you know it's true because you have these. So, like three times seven, you know that's going to be divisible by 3, and then you have three which you know is going to be divisible by 3.

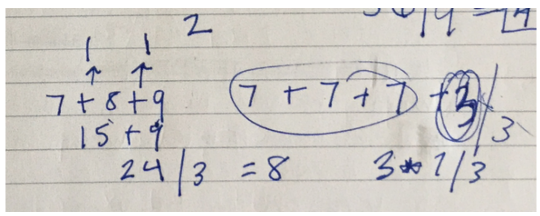

Figure 6. Generic example for the case of 3

Hence, through one example Hera gained a key insight accounting for why the conjecture is true for the case of 3 consecutive numbers. It was clear that Hera viewed the example as a generic example (Mason & Pimm, 1984), as she elaborated that the conjecture is true for *any* three consecutive numbers as follows:

> Because we know that if we try to make every one of these numbers the same so that we know that, you know, that you have three of the same number added together that's going to be divisible by 3, and you want to get a number that is divisible by 3 as the extra numbers… You want to make that divisible by 3 too, so that, you know, when you add those together, you're gonna get divisible by 3.

Eager to construct “a rule type thing”, Hera went on to develop an algebraic proof for the case of 3 by using variables (a, b, c) and representing consecutive numbers in relation to each other (i.e., b = a + 1, c = a + 2):

> So, if we did like a plus b plus c, just trying to come up with one that would be true… So, then a plus 1 plus a plus 2 is 3a. One two three and then plus three. So, we know that whatever number you start out with that is always going to be true.

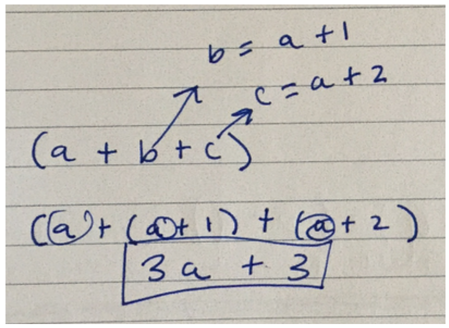

Figure 7. Algebraic proof for the case of 3

Therefore, for the sum of the consecutive numbers conjecture, Hera could develop an algebraic proof; unlike representing odd numbers, Hera was familiar with representing consecutive numbers. Generalizing her reasoning and the insight she gained, Hera developed similar proofs for why the conjecture was not true for the case of 4 and 6 consecutive numbers, but true for the cases of 5 and 7 consecutive numbers. Thus, by examining each case individually, Hera developed general arguments for each sub-conjecture, and correctly concluded that the conjecture was true for odd number of string of consecutive numbers: "When you add consecutive numbers and the number of consecutive numbers that you're adding is seven, or an odd number I mean, then it will be divisible by *n,* and then if it's an even number it won't work". Furthermore, in contrast to the previous task, this time Hera confidently claimed that she proved the conjecture. She argued:

> I tested many numbers and I also came up with a rule type thing… I didn't give a definitive number that will, like definitive consecutive numbers. I didn't say one two three like Georgia did. I said like a plus a plus one plus a plus 2 plus a plus 3 and then **proved why** that was right. So, then I think that's why I proved my

statement because I showed that for **any** number**,** that is *a,* it would work, for the specific ones that I made up like for three, five, and seven [consecutive numbers].

In conclusion, each proving task presents students with different challenges and opportunities for proving based on their mathematical background and the resources available to them. This illustrative case provides evidence that students' ability to prove is essentially dependent on the conjecture to be proved and the mathematical resources its proof requires. That is, students may produce different types of arguments (with varying degree of sophistication) depending on the nature of the task and their familiarity with its mathematics content. As seen, students can produce algebraic proofs if the mathematics involved in the task is accessible to them, irrespective of their conception of what a valid proof is.

5. Discussion and Implications

By investigating conceptions of proof in a mathematics class, this study takes a novel approach and examines both the students' and their teacher's conceptions of proof. Analysis of the participants' conceptions of proof as situated in their mathematics class allowed me to identify in what ways the students' conceptions were related to their teacher's conceptions of proof. Further, studying the class over a long time enabled me to examine the participants' proof conceptions through multiple measures and at different time points, which contributed to a better characterization of the participant's ways of thinking and understandings about proof. Both the teacher and the students turned out to have more sophisticated and advanced conceptions of proof than commonly reported in the literature, portraying a positive case to further probe what accounts for these positive findings, which will be the focus of the next paper. In what follows, I discuss some of the key findings that differ from the literature, point to some nuances detected that offer alternative interpretations to the existing findings, and highlight the methodological contribution of the study; that is, the triangulation of the data.

Although the existing research indicates that students typically do not recognize the value of proving for learning mathematics and view proof as restricted to geometry or certain forms, the students in this study regarded proof important in learning and doing mathematics. They deemed proof as a central practice that they commonly engaged in their mathematics class because the students' notion of proof was not limited to a particular form such as an algebraic proof or a two-column proof, even though writing flowchart and two-column proofs was a strong focus of the geometry unit that they learned during this study. Instead, the students viewed proof more broadly as justifying one' claims by supporting their reasoning with warrants and showing that their claim is true, which is precisely the perspective that their teacher wanted her students to have. Furthermore, unlike the existing findings (e.g., Almeida, 2000; Healy & Hoyles, 2000; Smith, 2006; Vinner, 1983), the students' evaluations of an argument as a proof was not based on the form of the argument; they accepted narrative arguments, algebraic arguments, and visual arguments as a proof if they believed that the argument explained why the given statement must be true. In addition, the students (except one) rejected an invalid algebraic argument as a proof, further indicating that the form of an argument was not decisive for the students' proof evaluations.

Contrary to the common finding that students across grade levels (including undergraduate students) are often convinced that a conjecture will hold true in general after testing a few examples (Balacheff, 1988, Chazan, 1993; Coe & Ruthven, 1994; Edwards, 1999; Harel & Sowder, 1998; Knuth et al., 2002), I found that the students in this study overall understood that examples are insufficient to prove. They did not accept empirical arguments as a proof, and in their attempts to prove a given conjecture they tried to go beyond empirical evidence and sought to develop a general argument. Instead, the students viewed examples helpful to investigate whether a conjecture is true, and valued examples to illustrate their proof to others or to verify that their proof is accurate. This is a nuance to the interpretation of students' recourse to an example after accepting or producing an argument as a proof.

While this behavior has been recognized as an indication of students' poor understanding of the generality of proof (e.g., Fischbein & Kedem, 1982; Vinner, 1983), the findings of this study indicate that students' inclination to use an example does not arise from the need to further check the veracity of a conjecture. But rather, the students' intention was either to better communicate their proof to someone else or to make sure that their proof was correct, both of which were (directly and indirectly) encouraged through their classroom experiences. Because team work and group discussions were essential features of the class, the students were accustomed to explaining and justifying their ideas to their teammates, leading to view proving as a social process such that they considered an imagined audience even when evaluating hypothetical student arguments. As Porteous (1990) argues, students' resort to examples does not necessarily indicate a weakness in their conceptions of proof. Indeed, students' tendency to use an example to illustrate a proof resonates with teachers' proclivity to favor arguments that are supplemented with specific examples or visuals (Knuth, 2002a). Hence, these findings point to the nuances in how students can and do think with examples in a way that does not necessarily indicate that they have impoverish views of proof.

Perhaps the most prominent aspect of the students' conceptions of proof was their strong emphasis on explaining why something must be true as a defining criterion for an argument to count as a proof. This is a novel finding, diverging starkly from the literature, as research often reports that many students do not consider seeking explanations for why their observations are true as important or relevant to what they are expected to do (e.g., Coe & Ruthven, 1994; Porteous, 1990; Vinner, 1983). In contrast, the students in this study were curious to understand why the given conjectures were true (or false) and deliberately sought to figure out the underlying reasons that make a conjecture true (or false). In Harel's (2006) terms, the students in general attempted to develop a *causal proof,* one that not only shows that a conjecture is true but also illuminates the reasons that make the conjecture true.

There may be various factors accounting for this positive finding, but one principal factor seems to be related to the students' meanings of proof and their views of the roles of proof, which are shaped by their classroom experiences. A shared notion of proof among the students and their teacher was that proof is justifying one's reasoning by backing up their claims with reasons that shows that their claim is valid. Accordingly, explaining why something is true was a recurrent emphasis in their mathematics class– both in small group discussions and whole class discussions, which will be elaborated in the subsequent paper. Furthermore, if students believe that the sole purpose of proof is to verify the truth of mathematical statements, then it is no surprise that a few confirming examples will suffice as a proof for them; even though multiple confirming examples (no matter how diverse they are) does not verify that the statement holds true in general, yet students may consider it to be "enough evidence" and be satisfied with a few confirming examples. Thus, cultivating a need for understanding why something is true should be an important instructional goal for teachers to support their students' conceptions of proof and their abilities to prove, in particular, and learning of mathematics, in general.

Considering all these promising findings, a natural question follows: What enabled the students to have such conceptions of proof? While this question will be more thoroughly addressed in the ensuing paper, I argue that the triangulation of data employed in this study also contributed to uncovering a more comprehensive and nuanced understanding about the participants' conceptions of proof. More specifically, in this study I sought to go beyond identifying students' conceptions of proof, aiming to understand why students hold such conceptions by exploring what may account for their conceptions so that we can be better off at finding ways to support students in advancing their understandings of proof. In accomplishing this goal, triangulation of data has been crucial and is achieved through various ways; specifically, by means of (a) different measures used for studying individuals' conceptions of proof (e.g., proof description, proof evaluation, and proof production), (b) different data sources/subjects (e.g., students and their teacher), and (c) multiple interviews.

Each measure used for exploring the participants' conceptions of proof revealed different – though not mutually exclusive– aspects of their conceptions and thus complemented and informed each other, enabling me to more thoroughly and accurately represent the students' conceptions of proof. In other words, any single measure was inadequate to truly expose the students' conceptions of proof. For instance, the students' proof descriptions were often too vague and brief, leaving out many important details to confidently draw conclusions about their proof conceptions or to distinguish students in terms of their meanings of proof. Students' proof evaluations, on the other hand, provided an expanded set of information about their ways of thinking and understandings about proof, by considering the types of arguments the students accepted as proof, the proof understandings that informed their decisions, as well as the criteria they used for accepting (or rejecting) an argument as a proof. Furthermore, considering students' proof productions alone was misleading as to what they think and understand about proof, given that the students' proof productions did not appear to be linked to their proof understandings in a way that favored the students with strong proof understandings to produce deductive proofs, but rather the students' proof productions were found to be largely affected by their understanding of the mathematics involved in a proving task. Thus, the results underscored that that what students can produce as a proof may not truly reflect their ways of thinking and understandings about proof. By comparing findings obtained from each measure, I looked at the overall patterns in students' ways of thinking and understandings about proof, distinguishing two groups of students; that is, (a) students with strong understanding of proof as a deductive argument and (b) students with relatively moderate understanding of proof as deductive argument, who also occasionally valued empirical arguments. But, the combination of these two measures (proof description and proof evaluation) does not account for students' abilities to develop proofs. Thus, the third measure (proof production) complemented the analysis of students' proof conceptions, showing that students' ability to develop proofs (i.e., types and sophistication of their argument) depends on the task and the mathematics involved in its proof.

The three measures used for examining students' conceptions of proof do not reveal disconnected aspects of their conceptions, but rather they present interrelated characteristics of students' ways of thinking and understandings about proof. For instance, students who described proof as *backing up* also emphasized *explaining why* as a key criterion when accepting (or rejecting) an argument as a proof by pointing to the use of (or lack of) warrants such as definitions or known properties, which was also reflected in their own proof productions where they tried to come up with an explanation for why the conjecture must be true.

On the other hand, triangulation of the different measures showed that students' proof productions, as well as their evaluations of proofs, largely occur at the intersection of their understandings about proof with their understanding of mathematics involved in a task. Attention to students' difficulties when evaluating and producing proofs revealed various difficulties that hampered students' judgments or their progress on a proof. Important to note is that many common difficulties were related to students' mathematical facilities, such as expressing mathematical ideas symbolically, familiarity with mathematical properties of numbers, and understanding how definitions and previously established results are used in constructing deductive arguments, to name a few. Hence, just like proof and proving need to be continually emphasized across grade levels, topics, and domains, these are areas that need to be always attended to by creating opportunities for students to engage with them and thus to develop their mathematical facilities. Apparently, development of students' proof abilities depends on a simultaneous emphasis on and support for their understandings about proof as well as their mathematical facilities.

A novelty of the study was its use of multiple data sources to develop a comprehensive understanding of students' conceptions of proof, by drawing on supplementary data sources in addition to the student interviews. To be precise, students' conceptions were examined through the interviews with students, the interviews with their teacher, as well as the classroom observations, although the latter is minimally used in this paper– only to point to the classroom influences on students' conceptions. Specifically, the interviews with the teacher helped to situate the identified conceptions of proof that the students had, revealing a close alignment between the students' ways of thinking and understandings about proof and their teachers' views on proof and proving. This close alignment was a rather surprising finding given that research often documented a mismatch between students' personal notions of proof and their perceptions regarding their teachers' notion of proof. Because both the students' and their teacher's conceptions of proof were found to be more sophisticated than reported in the literature, this study presents an encouraging case, suggesting that teachers can be influential in helping students to develop robust conceptions of proof.

Furthermore, an essential part of the data triangulation was achieved via multiple interviews. Both the focus-group students and the teacher were interviewed twice (one during the beginning phase of the classroom observations and one at the end). Interviewing the participants twice was helpful to more thoroughly and adequately study their conceptions of proof. Specifically, it allowed me to study their conceptions in multiple domains and contexts, but also allowed me to ask probing questions through clarifying questions or new tasks designed based on the preliminary results of the first set of interviews as well as the classroom observations. The second interviews also enabled me to check agreement among the participants (both among students and between the students and their teacher), discovering numerous shared understandings across the students and the teacher. Although the two interviews were not intentionally designed to show growth in the students' conceptions of proof (thus, it is not possible to make claims about quantifiable changes in students' conceptions of proof between two interviews), the two interviews nevertheless were far apart in time that some qualitative changes were evident across two interviews. Investigating the same aspect of students' conceptions of proof (e.g., students' meanings of proof) through multiple measures and multiple interviews, therefore, enhanced our understanding of students' conceptions by uncovering nuances and increasing credibility of the interpretations of the data.

In sum, consideration of the findings obtained from each measure together informed a more detailed and nuanced understanding of the students' conceptions of proof. Different data sources and measures revealed different aspects of the individuals' proof conceptions, helping to corroborate findings as well as pointing to mismatches, which was further strengthened with two interviews per participant at different time points. All together, they attest to the intricacy and complexity of individuals' conceptions; any single method alone or data source is likely to yield incomplete (or even misleading) conclusions.

The findings reported in this paper regarding the students' conceptions of proof by no means represent freshmen and junior high school students in general; instead, these seven focus-group students offer a fair representation of a select mathematics classroom. Thus, the findings of the study are significant not because it presents generalizable results about students' conceptions of proof, but rather because it offers an illustrative case, examined in-depth, that shows that students' conceptions of proof are related to their teacher's conception of proof, as well as to their classroom experiences. While this paper mainly focused on the links between the students' conceptions and their teacher's conceptions of proof, the following paper specifically focuses on how the classroom norms, instructional practices, and curriculum mediated the identified alignment between the students' and their teacher's conceptions of proof.

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PART II:

CLASSROOM FACTORS SUPPORTING STUDENTS' CONCEPTIONS OF PROOF: CLASSROOM NORMS, INSTRUCTIONAL PRACTICES, AND CURRICULUM

Abstract

Given that the students' and their teacher's conceptions of proof were examined in Part #1 and were found to be both closely aligned to each other and also more sophisticated than what is typically found in the proof literature, this paper specifically focuses on the ways in which the classroom factors might have supported the students' conceptions of proof, and thus the close alignment between the students' and their teacher's conceptions. In addressing this question, I first turned to the curriculum used in the class and examined what opportunities for proving were present in the mathematical activities and tasks given to the students, and then to the teacher's instructional emphases and practices and examined to what extent and in what ways the teacher emphasized proving and supported students' conceptions of proof. I share the results about the classroom factors related to the students' proof conceptions by considering both of these analyses. Specifically, I discuss how the curricular materials and the teacher (through her instructional emphases and practices) supported the students' notions of proof, including the students' views about the roles of proof. I also consider the classroom factors with respect to their potential support for the development of the students' proof schemes. Specifically, I discuss how the classroom factors may have fostered the development of the deductive proof scheme, while discouraging the development of the authoritative and empirical proof schemes. In conclusion, the study offers evidence that students can develop robust conceptions of proof if the learning environment is conducive to sharing and justifying mathematical ideas where the teacher values proof as an important aspect of doing and learning mathematics.

1. Introduction

Although there is a wealth of research on students' conceptions of proof, less is known about how students' conceptions of proof relate to their experiences in classrooms. Through task selection, instructional emphases, and classroom practices, teachers are influential on what meanings and understandings of proof and proving students construct, but very few studies (e.g., Harel & Rabin, 2010; Martin, McCrone, Bower, & Dindyal, 2005) have specifically investigated the relationship between instructional practices and students' understanding of proof. Moreover, the Common Core State Standards for Mathematics (CCSSM) (NGA/CCSSO, 2010) charges teachers to incorporate proof into all of their mathematics classes, yet proof is still not adequately emphasized in many mathematics classrooms (e.g., Bieda, 2010). Hence, it is sorely needed to find ways to help teachers make proof a consistent emphasis in their teaching, and thus, support their students in developing productive conceptions of proof.

To achieve this, as a field we need studies that unpack the relationship between instructional practices and students' evolving ideas and understandings about proof. Specifically, we need to better understand how to positively influence students' conceptions of proof through instruction. Thus, as Harel and Fuller (2009) urge, more research is needed to investigate classroom activities and interactions that foster desirable conceptions of proof. This study addresses this call by examining the links between students' conceptions of proof and their classroom experiences, by focusing on the classroom norms and instructional practices related to proof, as well as their teacher's conceptions of proof.

As Pajares (1992) argues, conceptions can act as filters through which individuals make decisions; thus, it is important to examine how teachers' conceptions (or misconceptions) of proof are reflected in their instructional practices. Indeed, it is widely acknowledged that teachers' conceptions shape their instructional goals and practices (Ambrose, 2004; Ball, Lubienski & Mewborn, 2001; Cross, 2009; Even, 1993; Hammerness et al., 2005; Hill, Rowan, & Ball, 2005; Sowder, 2007; Thompson, 1984, 1992). Accordingly, it follows that what teachers believe about the role of proof and what counts as proof is likely to influence their instructional treatment of proof. More precisely, teachers' pedagogical decisions regarding what to include, exclude, or encourage regarding proving related activities may be filtered through their conceptions of proof. Put differently, how one defines and conceives of proof, including its perceived role in school mathematics, is likely to influence one's treatment of proof in one's mathematics class. However, empirical findings are needed to better understand in what ways those constructs are related.

Conner (2007) provided preliminary findings regarding the link between teachers' conception of proof and their instructional supports for proof. Specifically, Conner worked with three secondary pre-service mathematics teachers and examined how they facilitated classroom argumentation, a practice closely related to proving. Conner found that the student teachers' conceptions of proof, especially their view of the role and function of proof, were closely aligned with the way each student teacher supported classroom argumentation. Thus, Conner's work provides evidence that teachers' conceptions of the role of proof can influence their instructional practices in important ways, calling for more research to unpack the nature of those connections and the ways in which teachers' practices may influence student understanding.

Hence, this study investigates how students' views and understandings about proof and proving are related to the classroom factors. By observing a high school mathematics class for over two months and interviewing the students and their teacher, I examined the possible links between the students' developing conceptions of proof and the proof-related classroom norms and practices that occurred in their mathematics class. Given that the students had more sophisticated proof conceptions than what is typically found in the proof literature, this paper is specifically concerned with the question of what classroom factors might have supported those proof conceptions. In addressing this question, I first turned to the curriculum used in the class and examined what opportunities for proving were present in the mathematical activities and tasks given to the students, and then to the teacher's instructional emphases and practices and examined to what extent and in what ways the teacher emphasized proving and supported students' conceptions of proof. I will present the findings about the classroom factors related to the students' proof conceptions by considering both of these analyses, and discuss them in relation to what proof schemes they are likely to promote.

2. Theoretical Background and Relevant Literature

2.1. The Emergent Perspective on Learning

With its emphasis on coordinating psychological analyses of students' individual activity with interactionist analyses of classroom discourse in which an individual's activity is embedded, *the emergent perspective* on learning mathematics (Cobb & Bauersfeld, 1995; Cobb & Yackel, 1996) frames this study. Drawing on the tenets of constructivism and social interactionism, *the emergent perspective* views students as active creators of the mathematical meanings they construct, but also considers those meanings to be shaped by the social interactions in a classroom culture (Cobb & Bauersfeld, 1995). Thus, learning occurs through negotiation of meanings in social interactions, which also includes indirect learning that occurs when the negotiation of meanings is implicit. As Cobb and Bauersfeld argue, when participating in mathematical practices in a classroom the teacher and students often implicitly negotiate meanings without being aware of it. Hence, the emergent perspective maintains that the individual's mathematical activity and the classroom culture are reflexively related to each other. In other words, individual students' active participation contributes to the construction of classroom mathematical practices and the classroom culture, and in turn, those practices shape students' subsequent mathematical activities by allowing or restricting their participation in certain ways. Thus, due to this reflexive relationship, Cobb and Bauersfeld argue that one cannot adequately account for either the social or psychological aspects of learning alone, unless they are considered in relation to each other.

Because *the emergent perspective* attends to both the social aspects (i.e., classroom social norms, sociomathematical norms, classroom mathematical practices) and the psychological aspects of individuals (i.e., beliefs about one's own role, others' role and nature of the mathematical tasks, mathematical beliefs, values and conceptions), it is a useful perspective for examining individuals' mathematical activity situated in a classroom community. Hence, I have adapted Cobb and Yackel's (1996) interpretive framework for studying individuals' proof conceptions as situated in their mathematics class as shown in Table 1.

Table 1. An interpretive framework for analyzing individual and collective proof-related activities at the classroom level, *adapted from Cobb & Yackel, 1996*

Social Perspective	**Psychological Perspective**
Classroom social norms	Beliefs about own role, others' roles, and nature of mathematical activity
Sociomathematical norms *(related to proving)*	Mathematical beliefs and values *(related to proving)*
Classroom mathematical practices *(related to proving)*	Mathematical conceptions and activity *(related to proving)*

Classroom social norms refer to the regularities observed in collective classroom activities that are established together by the teacher and students. These norms are not specific to mathematics classrooms, but can be observed in other classes as well. For example, "explaining and justifying solution methods, attempting to make sense of other students' solution methods, and asking clarifying questions whenever a conflict in interpretations arose" (Stephan & Cobb, 2003, p. 38) are norms that can be observed in other courses as well. In contrast, sociomathematical norms refer to the normative aspects of the students' and teacher's activity that are specific to mathematics, such as what counts as "a different mathematical solution, a sophisticated mathematical solution, an efficient mathematical solution, and an acceptable mathematical explanation" (Cobb, Stephan, McClain, & Gravemeijer, 2001, p. 124). Each row of the table presents the related psychological and social aspects; for example, mathematical beliefs and values that pertain to proving are assumed to be the psychological correlate of the sociomathematical norms related to proving. When a teacher initiates and facilitates a discussion on whether a given student justification is a valid and sufficient proof, this may simultaneously lead students to reconsider their corresponding beliefs about what proof means and entails. Conversely, students' beliefs about what counts as proof influence the collective establishment of the sociomathematical norms about proving in that classroom.

Lastly, classroom mathematical practices are described as taken-as-shared ways of reasoning and arguing mathematically about particular mathematical ideas, which evolve as the teacher and students discuss situations, problems and solution methods and often include aspects of symbolizing and notating (Cobb et al., 2001). Thus, classroom mathematical practices are specific to particular mathematical ideas under discussion, and refer to the collective understanding of the mathematical content as a class, which is the correlate of individuals' mathematical conceptions. For instance, one proof-related classroom mathematical practice in a high school geometry class could be creating a proof for showing that two triangles are congruent by using the taken-as-shared understandings (such as the known properties, relationships, theorems, and definitions) at a given time in that classroom community.

In accordance with the emergent perspective, I focused both on the *psychological aspects*– the individual students' and their teacher's conceptions of proof– and the *social aspects*– the classroom practices situated in the classroom interactions and governed by the collectively set classroom norms. While the first paper focuses on the psychological aspects, this paper mainly focuses on the social aspects, with an aim to coordinate the analyses of the students' and the teacher's conceptions of proof with the analyses of the classroom factors.

2.2. Relationship Between Teachers' Proof Conceptions and Their Instructional Practices

Teachers' knowledge of and beliefs about mathematics, constructs that are hard to separate and thus referred as *conceptions* in this study, have been identified as important factors that shape teachers' instructional practices (Ambrose, 2004; Ball, Lubienski & Mewborn, 2001; Cross, 2009; Even, 1993; Fennema & Franke, 1992; Hammerness et al., 2005; Hill, Rowan, & Ball, 2005; Sowder, 2007; Thompson, 1992). While there is limited existing research that empirically establishes this relationship specifically for the case of proof, it naturally follows that teachers' support for student learning of proof is likely to be determined by teachers' conceptions of what proving entails and the roles proof plays in mathematics. In other words, teachers' conceptions of what proving entails and the roles proof plays in learning mathematics are likely to influence what kind of a notion of proof they cultivate in their classes; that is, teachers can incorporate proof into their instruction to the extent commensurate with their understanding of proof. Thus, to what extent, and in what ways, a teacher requests students to justify their claims; what he or she accepts as a sufficient proof; and in what ways he or she supports students to develop proofs in class is likely to align with his or her conceptions of proof, which may be mediated together by other personal, classroom, or institutional factors.

As mentioned before, in her study with three secondary mathematics pre-service teachers Conner (2007) examined each pre-service teacher's support for classroom argumentation and provided empirical support to the alluded link between teachers' conceptions of proof and their instructional practices. Specifically, Connor examined the nature and the frequency of the components of the arguments (i.e., data, claim, warrant, and backings) constructed in teachers' classrooms, and then compared those to their conceptions of proof. She found that the pre-service teachers' conceptions of proof, especially their view of the role and function of proof, were closely aligned with the way each student teacher supported classroom argumentation. For example, a pre-service teacher who viewed proof as explaining *why* something works used more warrants and the warrants were usually theorems and definitions. Whereas another pre-service teacher who viewed proof as explaining *how* things work had facilitated arguments with less warrants and of those warrants she used were usually either a rule or procedure.

Likewise, Knipping (2008) reported that teachers' proof-related classroom practices were linked to their rationales for how individuals learn to prove. In comparing six French and German secondary mathematics teachers' collective classroom argumentations on the same content (i.e., the Pythagorean theorem), Knipping examined the warrants (and backings) used for the arguments developed in each class and classified the provided warrants according to their nature (i.e., empirical-visual, conceptual-visual, conceptual-deductive). Her analyses resulted in identification of two distinct types of global argumentation structures (i.e., source-structure and reservoir-structure) within these six classes, suggesting that learning to prove had different meanings in each class. Hence, by establishing links between different types of argumentation structures and teachers' purposes of proving, Knipping's work offers further support to the claim that teachers' (conscious or unconscious) conceptions of proof and goals for proving are manifested in their instruction through emphasizing and fostering different notions of proof.

2.3. Curricular Opportunities for Proof and Their Classroom Implementation

A critical factor in students' experiences with, and thus opportunities to learn to prove, is the curriculum they study (Stylianides, 2009). Curriculum also has a key influence on teachers' instructional emphases and practices, as it essentially frames the mathematical tasks and activities that students are to engage with (Cai, Ni, & Lester, 2011), and thus sets the expectations for conceptual demand, in general, and proving, in particular. Hence, to what extent a curriculum offers opportunities for reasoning and proof is crucial to achieve the goal of making reasoning and proof central to students' mathematical experiences as the policy documents recommend (e.g., NGA/CCSSO, 2010; NCTM, 2000). This critical role of curriculum has led to copious research on the designed opportunities for proof related activities in textbooks across grade levels, ranging from elementary grades (e.g., Bieda, Ji, Drwencke, & Picard, 2014) to middle school (e.g., Stylianides, 2009), to high school (e.g., Davis, Smith, Roy, & Bilgic, 2014; Johnson, Thompson, and Senk, 2010), and to teacher education programs at the undergraduate level (e.g., McCrory & Stylianides, 2014). The findings across this body of research show that secondary textbooks typically include significantly more opportunities for reasoning-and-proving compared to elementary textbooks and that those opportunities for reasoning-and-proving vary across different content areas, with geometry being one of the content areas that most frequently include reasoning-and-proving tasks.

Given that the study of proof is typically regarded as a subject of high school geometry course in the US (Herbst, 2002), it is not surprising that more opportunities for proof is found in high school textbooks, and particularly in geometry textbooks. Because of the explicit emphasis on proof in geometry courses, Otten, Males and Gilbertson (2014) examined six high school geometry textbooks that are commonly used in the US, and analyzed the introduction to proof chapter of each textbook in terms of the nature and frequency of "reasoning-and-proving opportunities" (Stylianides, 2009) included. Their analysis showed that the reasoning-and-proving opportunities in the introduction to proof chapters included investigating conjectures, making conjectures, and developing rationales for the conjectures, but the requests for constructing a proof was not common. Further, the frequency of reasoning-and-proving opportunities varied across the six textbooks, ranging from 27% to 65% of the total number of tasks analyzed in each textbook. Note that these frequencies reflect only the introduction to proof chapters (in which proof is the explicit focus), hence one would expect to find more opportunities for reasoning and proof.

Johnson, Thompson and Senk (2010) examined the proof-related reasoning opportunities present in non-geometry high school textbooks and analyzed various algebra and pre-calculus textbooks (including both conventional and reform-oriented textbooks). In their analysis, the researchers considered the proof-related reasoning opportunities to include: investigating a conjecture, making a conjecture, developing an argument, evaluating an argument, correcting a mistake in an argument, finding a counterexample, and reading arguments and proofs. Their findings indicate that the opportunities for proof-related reasoning increased from Algebra 1 to Algebra 2 to pre-calculus, but nevertheless the percentage of proof-related reasoning opportunities was quite low (3.4 %, 5.4 %, and 7.7 %, respectively). This is a concerning finding, if students are to develop appropriate notions of proof and skills to prove, and appreciate the value of proof in doing and learning mathematics.

On the other hand, Stylianides (2009) examined a reform-oriented curriculum, Connected Mathematics Project (CMP), that claims to be aligned to the recommendations set by the policy documents (i.e., NCTM, 2000), so one may expect to find more opportunities for reasoning and proof. Stylianides analyzed 12 units of the CMP curriculum across three content areas (algebra, number theory, and geometry) and across three grade levels (grades 6 to 8). He found that about 40% of the tasks included at least one opportunity for students to engage with a proof-related activity (such as identifying a pattern, making a conjecture, developing a proof (generic example or demonstration), and developing non-proof arguments, including empirical arguments and rationale). Of those opportunities, 62% required students to give a rationale, 24% required to identify patterns (definite or plausible patterns), and only 12% required a demonstration (i.e., a proof). However, the distribution of the reasoning-and-proving tasks significantly varied across the units and grade levels.

Although Stylianides's (2009) analysis of the CMP curriculum reveals promising results with respect to the opportunities for learning to prove when compared with the considerably fewer opportunities that were typically found in other textbooks (even at the high school level), the implementation of the curriculum matters the most for the actual opportunities students have for reasoning and proving. For instance, Stylianides's curriculum analysis pointed out that very little percentage of the tasks in CMP required students to develop empirical arguments (3%); yet, Knuth, Choppin, Slaughter, and Sutherland (2002) studied the proof understandings of students from the CMP classrooms and found that most students across all three grade levels relied on empirical arguments to justify the truth of mathematical claims. This mismatch between a low emphasis on empirical arguments in the curriculum and the high tendency of students to use empirical arguments to justify underscores a potential issue in the implementation of the curriculum; that is, the implemented curriculum might have diverged significantly from the intended (written) curriculum. Indeed, Bieda (2010) investigated how experienced CMP teachers implemented proof-related tasks in their classrooms. Studying seven middle school mathematics classrooms (grades 6 to 8), Bieda found that the designed opportunities for proof in the written curriculum were not fully realized during the implementation of the tasks, pointing out that the classroom implementations fell short of developing a robust understanding of proof, and thus, were insufficient to support students' abilities to prove.

In sum, Stylianides' work on curriculum analysis and Bieda's study of the classroom implementation of the same curriculum nicely complement each other, contributing to our understanding of the relationships between the intended curriculum and its classroom implementation. These studies together suggest that the classroom implementation of a curriculum may further reduce the opportunities for reasoning and proving designed by the curriculum, and thus point to the critical role teachers have in maintaining or improving the opportunities available in a curriculum so that students can develop a robust understanding of proof. But, more research is needed to identify how a curriculum, its classroom implementation, and student learning is related to each other (Thompson, 2014); specifically, research "to conduct a detailed analysis connecting the intended opportunities to learn reasoning-and-proving with the practices that are actually enacted in classrooms, and further, to connect students' development of reasoning-and-proving skills with various curricula" (Cai & Cirillo, 2014, p. 138) is needed. This study is in line with this call, as it examines the interplay between students' conceptions of proof and the related classroom factors, including the curriculum used.

2.4. Relationship Between Instructional Practices and Students' Conceptions of Proof

Research has documented that teachers' instructional practices significantly influence students' learning opportunities (e.g., Blanton & Kaput, 2005; Mason, 2000; Mewborn, 2003). In a recent review of research on teachers' roles in mathematical classroom discourse, Walshaw and Anthony (2008) provided an extensive review of the studies that presented an explicit link between teacher practices and student learning and concluded that teachers' pedagogical decisions related to classroom discourse significantly influenced student learning. Furthermore, the authors contend that studies that investigate the link between teacher practices and student leaning help us understand not only what discourse practices work, but also how and why they work. However, they underscore that "We do not know as much about quality classroom discourse at the high (secondary) school level as we do about the elementary (primary) level" (Walshaw & Anthony, 2008, p. 541).

Accordingly, researchers (e.g., Herbst, 2002; Hoyles, 1997; Martin et al., 2005; Selden & Selden, 2008) acknowledge the role of teachers in shaping students' proof schemes through their instructional emphasis on proof and justification, but relatively few studies have specifically investigated the relationship between instructional practices related to proving and students' understanding of proof. Furthermore, most of them were conducted either at the elementary level (e.g., Stylianides, Ball, 2008; Reid & Zack, 2009) or at the undergraduate level (e.g., Blanton, Stylianou, & David, 2009; Smith, 2006; Stylianides & Stylianides, 2009), and thus less is known about how instructional practices in secondary grade mathematics classes relate to students' proof conceptions.

At the undergraduate level, Blanton, Stylianou and David (2009) conducted a year-long teaching experiment in a discrete mathematics course, with an explicit focus on learning to prove. The researchers examined the teacher-researcher's discourse and the students' development of proof competencies with an aim to relate two analyses. They specifically examined the instructional scaffolding in classroom discourse during proof development, building on the premise that the nature of instructional scaffolding is critical in understanding how students learn to prove. Hence, they developed a framework of instructional support for proof development, consisting of four types of instructional scaffolding: (a) transactive utterances, (b) facilitative utterances, (c) didactic utterances, and (d) directive utterances. Transactive utterances consist of "requests for critique, explanations, justifications, clarifications, elaborations, and strategies, where the teacher's intent is to prompt students' transactive reasoning" (p. 294), thus, they are the building blocks for proof development. Facilitative utterances are used to structure classroom discussions and consist of re-voicing or confirming student contributions. Through facilitative moves such as summarizing a discussion or leading students to focus their arguments on certain aspects, teachers structure classroom discussions. Didactive utterances refer to teacher explanations related to the nature of mathematical knowledge (e.g., nature of mathematical proof, proof methods, etc.). Lastly, directive utterances are more leading and are aimed at correcting student errors or explicitly leading students towards the solution path. Blanton and colleagues investigated the effects of these four types of utterances on the development of students' understanding of proof. As a result, they hypothesized that teacher prompts that requested students to share new ideas and to clarify, elaborate and justify their ideas were pivotal in students' development in learning to prove. Furthermore, they stressed that transactive and facilitative prompts, which were key to making student ideas public and negotiating mathematical meanings, enabled students to engage in conjecturing.

At the secondary level, Harel and Rabin (2010) examined two high school algebra classrooms, with an aim to identify teaching practices that contributed to the development of the authoritative proof scheme. The researchers observed 8 class sessions for Teacher A and 9 sessions for Teacher B, approximately once in every three weeks over the course of an academic year. Harel and Rabin identified three categories of teaching practices that can promote the authoritative proof scheme: (a) answering students' questions, (b) responding to students' ideas, and (c) lecturing. Specifically, the researchers identified several instructional practices (such as telling students how to proceed with the solution, evaluating student ideas rather than asking the class to evaluate) that conveyed the message that the teacher is the sole authority in the classroom who decides what counts as mathematically correct, and thus they argued that these practices are likely to promote the development of the authoritative proof scheme in students. Harel and Rabin also noted that the teachers' lectures often began with a general rule, followed by examples and then continued with asking students to solve similar tasks, which, they argued, further contributed to an image of teacher as the authority figure in class. In addition, they reported that teachers' justifications were usually authoritative and sometimes empirical in nature, which is likely to reinforce students' authoritative and empirical proof schemes. They also identified some productive teacher moves (such as probing student reasoning, prompting error correction, encouraging argumentation among peers, and offering deductive justifications) that can support more advanced conceptions of proof.

In another exemplar study conducted at the secondary school level, Martin, McCrone, Bower, and Dindyal (2005) investigated the classroom factors related to the development of students' proof understandings. Specifically, the researchers linked a high school geometry teacher's instructional practices to students' developing axiomatic proof schemes. In their analysis of the classroom observations, the researchers identified the actions the teacher and students did and visualized those actions in a chronological order. They discussed that the teacher supported students' reasoning and proof abilities by (a) asking open-ended tasks, (b) encouraging student argumentations in which students were responsible for reasoning, (c) analyzing student arguments, and (d) coaching students during their proof constructions. The researchers argued that the teacher moves such as evaluating, re-voicing, exposing students' flaws, and pressing students to provide justifications based on the axiomatic system had supported the students' movement towards the axiomatic proof scheme. Additionally, they maintained that giving students responsibility to assess each other's arguments helped students move away from the authoritative proof scheme.

This study builds on the body of literature described here, with an aim to contribute to our collective understanding of the relationships between classroom factors (such as sociomathematical norms, the teacher's emphasis on proof and justifying, and instructional practices) and students' developing conceptions of proof. To achieve this, I complement classroom observations with interviews with students and their teacher, as well as an analysis of the curriculum they study. As Steele and Rogers (2012) argue, each method (interview and classroom observation) enables different aspects of individuals' conceptions of proof to surface, allowing one to better understand what the participants understand about proof and how that understanding is manifested and supported in the classroom.

3. Methods

As described in the previous chapter, this study examines a high school mathematics class, an honors integrated algebra-II, geometry, and pre-calculus course, situated in a public school district in the Midwest (for more details about the context and the participants of the study, see Paper #1, *An Investigation of Proof Conceptions in a High School Mathematics Classroom*). Being an honors course, this class may be considered a non-typical class; however, since proof and proving often receive little attention, and thus, assume limited roles in most classrooms, for the purposes of this study it was essential to examine a non-typical classroom in which proof and proving were emphasized. As Stylianides (2007) argues, "To study how proof and proving can be cultivated we need to look at what might be considered as successful rather than typical teaching practices" (p. 301).

As part of the overall study, this paper focuses on the classroom factors related to proof and proving (as highlighted in Figure 1), with an aim to relate the analyses of classroom factors with the analyses of the students' and their teacher's conceptions of proof. Therefore, the videotapes of mathematics lessons and the associated mathematical tasks (both classwork and homework assignments) constituted the main data sources for this paper, while the audiotapes of small group discussions of two focus-groups, field notes, and reflections were used to supplement the data analysis.

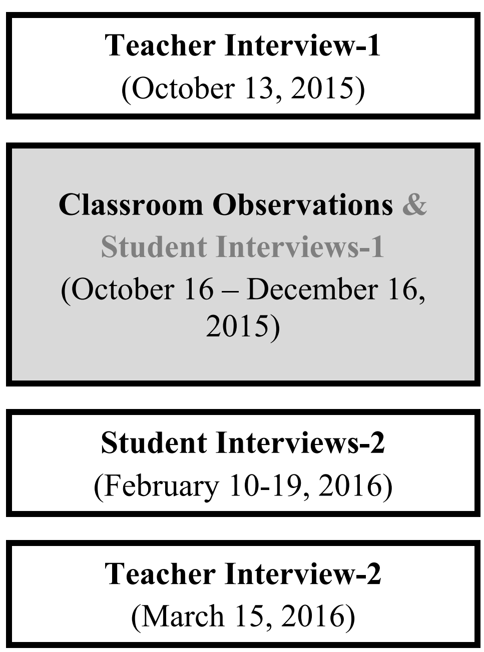


Figure 1. Data collection process

3.1. Classroom Observations

I observed 18 consecutive lessons over two months, covering two units in geometry (Unit 3: 2D Figures & Unit 4: Similarity and Congruence) and one unit in pre-calculus (Unit 5: Intro to Trigonometry). In this paper, I report the analyses of the two geometry units, consisting of 10 lessons, in order to focus on the students' introduction to and experiences with proof within one mathematical domain. Table 2 summarizes the mathematical topics of each lesson. The class met every other day for a period of 87 minutes.

Table 2. Mathematical topics of lessons analyzed

Lesson #	Unit # Day #	Topics
Unit 3: *2D Figures*		
Lesson 1	U3D3	Pythagoras
Lesson 2	U3D4	Shapes, Definitions, Properties
Lesson 3	U3D5	Circles, Triangles, Composites, Parallelograms
Lesson 4	U3D6	Constructions
Unit 4: *Similarity and Congruence*		
Lesson 5	U4D1	Similarity
Lesson 6	U4D2	Proofs
Lesson 7	U4D3	Congruence
Lesson 8	U4D4	Ratios and Similarity
Lesson 9	U4D5	Quadrilateral Proofs
Lesson 10	U4D6	Coordinate Proofs

I videotaped each lesson with a camera placed at the back of the room, by focusing on the teacher and following her as she shifted from monitoring small group discussions to leading whole-class discussions. Additionally, a clip mic was used to better record the teacher's discourse. Because the teacher frequently moved between different groups of students, it was not possible to record individual students' discussions in its entirety as the camera followed the teacher. Thus, I audiotaped two focus-groups' team discussions (who consented to audiotaping) so that I could also have access to a subset of students' discourse at length. There were 7 focus-group students in total, who happened to be fairly representative of the class in general (for more information on the focus-group students, see Paper #1, *An Investigation of Proof Conceptions in a High School Mathematics Classroom*). The focus-group students remained stable, except that they were grouped differently across the two units in order to conform to the classroom norm (that is, students worked in different teams in each unit), yet still maintaining that the focus-group students were teamed together. During the observations, I also took field notes to document the classroom norms and practices related to proof and proving, and wrote reflections after the observations. Further, I collected classroom artifacts such as mathematical tasks used in the class and a sample of students' written work to draw on multiple sources of data and triangulate the findings. The mathematical tasks (both classwork and homework assignments) were mainly drawn from the College Preparatory Mathematics (CPM) curriculum, which emphasizes team work and discovery of core mathematical ideas through problem based lessons, and identifies itself as well-aligned to the CCSSM. While the three books of *Core Connections* series of the CPM curriculum (i.e., Algebra-2, Geometry, and PreCalculus with Trigonometry) were alternately used throughout the semester, during the focal units of this paper the class relied on the *Core Connections Geometry* book.

3.2. Data Analysis

3.2.1. Analysis of curricular materials

Prior to analyzing the lessons, I first examined the curricular materials available for the class in order to better understand the nature of the class and the opportunities available in the tasks in which the students engaged. Specifically, I analyzed the mathematical tasks and activities that the students engaged in (both as classwork and homework) for the entire semester, covering units 1 through 5, as well as the syllabus of the course. Examining the syllabus and the curricular materials of the previous units (i.e., Units 1 and 2) enabled me to better understand the classroom norms and expectations as well as the students' prior mathematical experiences pertinent to this class that will be foundational for the rest of the semester.

For each unit, I populated the tasks in classwork and homework per each lesson and then coded each task for any proof-related emphases (such as not making assumptions based on appearance) or requests for proof-related activities (such as identifying a pattern, making a generalization, providing a justification, etc.), with the unit of analysis being the entire task. For instance, if a task had several parts, I considered all parts together constituting one distinct task. Table 3 shows the total number of tasks presented as classwork and homework in units 3 and 4– the units that are the focus of this paper. Because the tasks usually included multiple aspects of proof-related emphases or practices, multiple codes were permitted to capture them all. I coded each task to identify the mathematical activities it required and the mathematical emphases it included. In coding, I mainly followed the language of the tasks, although I was also sensitive to the proof-related practices identified in the literature described in section 2.3. In other words, I did not use an existing analytical framework for analyzing the tasks, but rather the codes were emergent. By following the language of the tasks, I wanted to see how the language used in the tasks might have been taken up by the students. Specifically, I was interested in seeing to what extent and in what ways the tasks asked students to *justify* or *prove* something. I used the findings of the analysis of the curricular materials to supplement the findings of the analysis of the lessons.

Table 3. The number of tasks per unit

	# of Classwork Task	**# of Homework Tasks**
Unit 3 (= 4 lessons)	34	30

Unit 4 (= 6 lessons)	61	46
Total (= 10 lessons)	**95**	**76**

3.2.2. Analysis of lessons

I used the findings about the students' and the teacher's meanings of proof, their views about the roles of proof, and their understandings about proof– the findings reported in Paper #1– as a guide for the analysis of the mathematics lessons. Specifically, based on the coding scheme that emerged from the analysis of the interview data, I developed a list of possible classroom influences related to those findings (see Appendix-E for the full list). For instance, given that the students commonly described the role of proof as verification and explanation, in the classroom data I looked for a potential instance in which a justification is given to *verify* a claim or a justification is given to *explain* a claim.

Although this list provided me with an initial focus for the analysis, I also aimed to attend to various aspects of the lessons that might have supported the students' conceptions of proof; therefore, I conducted an open coding (Charmaz, 2006) of the videotaped lessons, by being open to emergent codes. My unit of analysis was a full exchange between the teacher and students either in a small-group discussion or in a whole-class discussion setting, which I call an *episode*. The students worked on mathematical tasks in teams of three or four students, with a total of 8 groups (31 students), and the teacher frequently monitored and facilitated small-group discussions. Thus, an episode often lasted a few minutes, enabling me to parse the data into small-enough pieces that were manageable to code, yet captured a complete exchange between students and the teacher, allowing me to see how the teacher elicited students' reasoning, pushed them to justify their ideas, and facilitated their proving abilities. I identified and analyzed 132 episodes in unit 3, and 167 episodes in unit 4, summing to 299 episodes in total.

I carefully watched each videotaped lesson and identified the episodes in which a proof-related emphasis or practice occurred, by keeping track of the time it occurred and transcribing the parts of the episode that were particularly revealing. Following the identification and transcription of an episode, I coded the episode based on the existing codes as well as emergent codes at a given time during the analysis. I also noted the classroom norms that might have supported the students' notions of proof (e.g., encouraging team collaboration/discussion, giving students responsibility), as well as other teacher moves for supporting student reasoning (Ozgur, Reiten, & Ellis, 2015), such as re-voicing and prompting error correction, that may have had a supplementary role to explain the learning environment and the teacher's instructional support. Other general aspects of instruction were excluded from the analysis.

After the analysis of each lesson, I updated the current coding scheme with new codes and organized the codes around themes. At the end of the analysis of all 10 lessons, the codes were organized around four main categories as follows: (a) the meanings of proof promoted in the class, (b) the roles of proof emphasized in the class, (c) other proof-related emphases (such as emphasizing not making assumptions based on appearance or using accurate mathematical notations), and (d) the proof-related practices, which is further organized around instructional practices in which *the teacher requests* students to engage in a mathematical activity such as identifying a pattern, making a generalization, developing a justification, and thus placing the emphasis on the student; and the instructional practices in which *the teacher facilitates* and supports students' engagement in those proof-related activities by, for example, scaffolding how to write a proof or by providing a justification. While I allowed multiple codes for any given episode in order to capture all proof-related emphases and practices, I applied each unique code only once within the same episode even if it occurred multiple times. For instance, even though the teacher might have asked three questions in succession to press a student to give a justification, the "request for justification" code was applied only once. I aggregated the frequencies of codes per unit, and will present some of the main findings in the ensuing section.

4. Results

I will report the findings in two parts. First, I will present the connections between the classroom factors and select findings about the students' conceptions of proof reported in the first paper. More specifically, I will discuss how the curricular materials and the teacher (through her instructional emphases and practices) supported the students' notions of proof, including the students' views about the roles of proof. Next, I will consider the classroom factors with respect to their potential support for the development of the students' proof schemes. Specifically, I will discuss how the classroom factors may have fostered the development of the deductive proof scheme, while discouraging the development of the authoritative and empirical proof schemes. First, I begin with a brief description of the nature of the class.

4.1. Overview of the Course: Classroom Norms and Expectations

The course syllabus describes the course as having a "strong emphasis on group-based discovery and teamwork", further elaborating that, in this course "Students will be making connections, discovering relationships, figuring out what strategies can be used to solve problems, and explaining their thinking". My observations of the class concur with these descriptions. Investigating problems in teams and coming to a conclusion through small-group discussions was an essential feature of the class; the teacher regularly encouraged team collaboration and discussion, and the students were both expected to justify their ideas and also to ask their teammates to explain their ideas and back up their claims. Another essential feature of the class is that the students were given responsibility for figuring out mathematical ideas by themselves. This was supported by both the curriculum and the teacher. For instance, instead of responding to the questions right away, the teacher redirected student questions back to the team or class. Furthermore, new mathematical ideas and concepts were introduced as a foundation for more advanced concepts, but were also connected back to prior knowledge. Previous concepts were also continually revisited through homework questions. Thus, mathematical ideas were introduced and discussed in relation to other ideas and concepts, maintaining a view of mathematics as a growing body of knowledge, rather than as a collection of disconnected concepts and procedures.

Typically, each lesson started with a brief review of homework questions, with the teacher helping students individually and also leading a whole class discussion on the common student questions. Homework check was followed by a warm-up task, by which students began to think about the mathematical ideas to be studied in that lesson, and thus getting ready for the ensuing classwork. After a quick discussion of what the students had come up with in the warm-up task, the teacher summarized the discussion and introduced the goal of the lesson, launching the classwork. In a typical lesson, there were usually two or three parts of classwork, with each part consisting of multiple mathematical tasks and activities. During the classwork, students worked in their teams and the teacher frequently monitored the teams' work, elicited their reasoning, pressed for explanation and justification of their ideas, and provided guidance, if needed. Each classwork concluded with a quick whole-class discussion in which students shared their ideas and the teacher summarized the main ideas.

4.2. Classroom Factors Related to the Students' Notions of Proof

The interview data revealed that both the teacher and the students used proof and justification interchangeably, indicating that they viewed proving synonymous to justifying. As shown in Table 4, the students described proof in various ways, including *proof as backing up*, *proof as evidence*, and *proof as explaining* how one knows that his or her answer is true. The second interview showed that the students also viewed proof as showing that something is true based on known facts, rules, and definitions, though it was not articulated in the first interview. (for more information on the students' meanings of proof, see Paper #1, *An Investigation of Proof Conceptions in a High School Mathematics Classroom*).

Table 4. The students' meanings of proof

Students' Meanings of Proof	**# of students** (Interview-1)	**# of students** (Interview-2)
Proof is backing up statements/conclusions.	2	6
Proof is evidence that shows that something is true.	3	5
Proof is explaining/showing how you know that your work/answer is true.	7	5
Proof is showing that something is right based on known facts, rules, definitions and properties.	-	7

Given that all of the students described proof as *explaining how one knows that his or her answer is true* during the first interview, I will begin with presenting the possible classroom influences on the students' notion of proof as *explaining*. I will then focus on the treatment of proof and justification in the curricular materials and by the teacher, respectively, before I further discuss the possible classroom factors in shaping the other meanings of proof that the students had.

4.2.1. Classroom factors related to the notion of proof as *explaining*

Interestingly, all of the students in the first interview described proof as "explaining (or *showing*) how you know that your answer (or your work) is true", leaving it uncertain whether the students conflated proof with simply explaining one's solution steps. The analysis of the curricular materials and the lessons illuminated why the students unanimously described proof that way.

As part of the syllabus, homework expectations set for students required them to show all of their work; the same document also noted that, “Problems will often ask you to **describe**, **explain**, or **justify** your work. Given these directions, it is important to show any mathematical steps *and* use full sentences to support your reasoning” (*emphasis in the original*). Because such directions (i.e., to describe, explain, or justify your work) were commonly included in the tasks, the students seem to have associated proof with explaining how one knows that his or her work or answer is true. The teacher also followed the language of the textbook when requesting students to justify their ideas. In fact, during the first interview, Ms. V told me that she does not use the word “proof” in class, but rather asks students questions such as, “How do you know that?” She further explained: “I don't say, ‘Prove it’. I don't say, ‘Write me a proof that explains why this is true’, I just say ‘How do you know?’, ‘Can you explain that?’, ‘How do you know that's true?’, ‘How do you know that's the answer?’” Note that the questions that the teacher listed to exemplify how she typically asks for a justification in class also includes a request to explain, similar to the directions found in the tasks, which may have reinforced the students’ notion of proof as explaining one’s work or answer. However, what is important to highlight is that a request to explain can serve two functions: (a) to describe one’s reasoning or solution, when a teacher’s intention is to figure out students’ reasoning, or (b) to provide a justification in support of a claim, when the teacher’s intention is to push students to provide a more thorough argument. In my analysis, I found that both the task directions and the teacher’s requests for an explanation included both types of purposes; while the task directions to “explain an answer” were often in service of requesting a justification, the teacher frequently prompted students to explain their work or solution both to figure out their reasoning and also to press them to justify their ideas. Although both purposes of asking students to explain are pedagogically valuable and desired, students may not necessarily recognize the distinction between those two functions, and thus may view explaining one's reasoning same as justifying one's claims. Out of seven students, however, only one of them agreed that proof is explaining one's thought process, while three students somewhat agreed. Three students' partial agreement indicates an influence of the prevalence of requests for explaining their thinking in the class, yet the students were largely aware that explaining one's reasoning is not sufficient to prove. For instance, Brett disagreed with the statement that proof is explaining one's though process by arguing that, "Proof is like showing that there's no holes or like missing pieces in why you think that, but your thought process … could be missing a lot of things." As Ms. V argued, although explaining one's reasoning does not constitute a proof, nevertheless, it is helpful in proving and also pedagogically valuable as it allows teachers to understand how their students are reasoning about a given task or concept.

Next, I will elaborate to what extent and in what ways the tasks and the teacher requested for a proof and a justification, and the possible influence of those on the students' meanings of proof, but first how the textbook distinguishes between a proof and a justification is due.

4.2.2. Classroom support for proof and justification

4.2.2.a. Curricular support for proof and justification

In its glossary section, the textbook (Core Connections Geometry) defines proof as "A convincing logical argument that uses definitions and previously proven conjectures in an organized sequence to show that a conjecture is true"; whereas it defines *justification* as "To give a logical reason supporting a statement or step in a proof. More generally to use facts, definitions, rules, and/or previously proven conjectures in an organized sequence to convincingly demonstrate that your claim (or your answer) is valid (true)”. The first sentence in the definition of justification indicates that the difference between a proof and a justification is that justification refers to one piece of an argument that is part of a sequence of logically organized arguments; that is, a proof consists of multiple justifications corresponding to each constituent statement (or step) of the proof. But, the second sentence in the definition of justification blurs the distinction between a justification and a proof, making it possible to use them interchangeably.

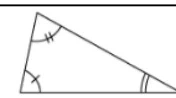
I examined the tasks to see to what extent and in what ways the tasks requested a justification or a proof, and found that in general requests for justifications were more common than requests for a proof, with the requests for a proof being almost always reserved for requesting a full argument that shows that something is true or explain why something must be always true. Table 5 below shows the number of times a justification or a proof was requested within the 171 tasks given (in classwork and homework) in units 3 and 4. As seen, there were a total of 67 requests for a justification, and 35 total requests for a proof within the 171 tasks analyzed. Note that the unit 3 was about 2D figures and unit 4 was about similarity and congruence, in which the students were introduced to writing flowchart and two-column proofs. Thus, the unit 3 was a foundational unit in which students learned about the definitions and properties of geometric shapes, which they would need to use in constructing proofs to show that two given shapes are (or, are not) similar or congruent. Thus, the specific focus of the unit 4 on proof explains the sharp increase in the number of requests for justifications and proof.

Table 5. Frequency of requests for a justification and a proof in tasks

	Request for a Justification	Request for a Proof
Unit 3 (= 4 lessons)	11	1
Unit 4 (= 6 lessons)	56	34
Total (= 10 lessons)	**67**	**35**

The requests for a proof in the (classwork and homework) tasks occurred either by explicitly asking students to prove that something is true or to write a convincing argument, or in most cases (29 out of 35 instances) by requesting a specific form of proof. Of those 29 instances, 22 of them were a request to create a flowchart proof, while there were only two requests for an algebraic proof, one of which occurred in unit 3. The remaining five instances asked students either to create a two-column proof or to construct a proof in a form that they choose among a flowchart, two-column, or paragraph form. Figure 2 shows a sample task in which students were asked to create a flowchart proof– the most common type of a request for a proof found in the tasks.

7. (CCG 7-49) Carefully trace the triangle at right onto tracing paper. Be sure to copy the angle markings as well. Then rotate the triangle about a *midpoint of any side* of the shape to make a new shape that looks like a parallelogram. Trace the new shape on your paper.



a. Is the shape truly a parallelogram? Think about the *definition* of a parallelogram and use the angles in your diagram to write a convincing argument.

b. What can the two congruent triangles tell you about a parallelogram? Look for any relationships you can find between the *angles* and *sides* of a parallelogram and list them on your paper. (You may already KNOW these attributes to be true, but now you can use triangles to show WHY these attributes are true)

c. Does the diagonal of a parallelogram always split the shape into two congruent triangles? Draw the parallelogram at left on your paper. Knowing only that the opposite sides of a parallelogram are parallel, create a **flowchart** to show that the triangles are congruent. Be sure to include a fact inside of each bubble and a reason outside of each bubble in your flowchart proof.

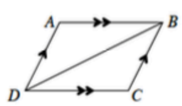


Figure 2. A sample task that requests students to create a flowchart proof

As part of a homework assignment in unit 4, the task asks students to create a flowchart proof. But, note that the task includes several parts, and thus, through the earlier parts of the task, it also scaffolds students' proof constructions by giving directions about how to proceed in proving, suggesting what to consider and posing guiding questions. Note that the task also encourages students to use definitions in proving and to look for relationships, and emphasizes showing why something is true.

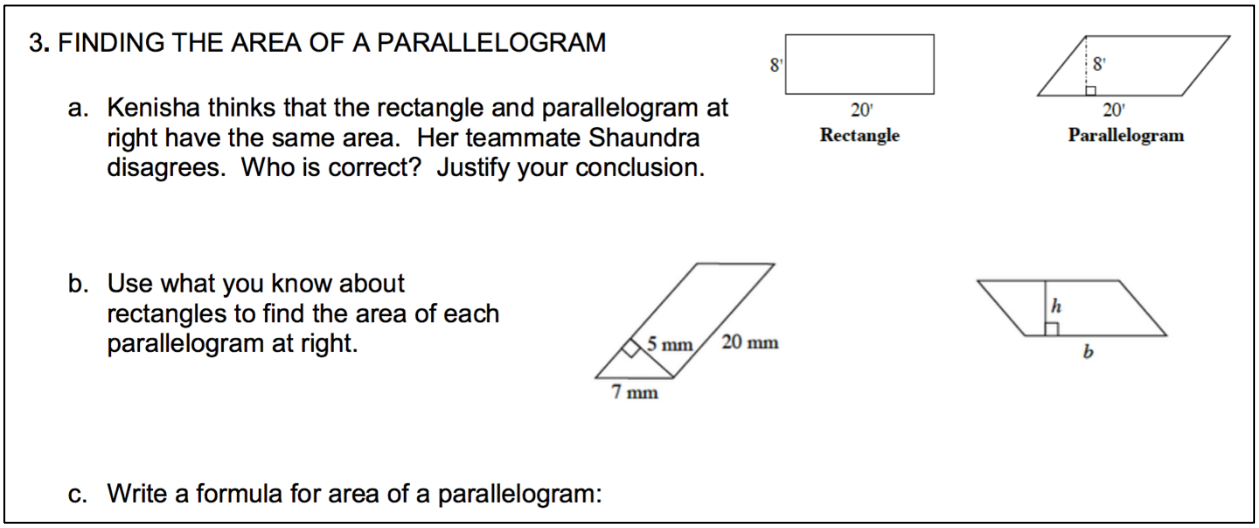
On the other hand, as seen in Table 5, the number of requests for a justification almost doubled the number of requests for a proof. Furthermore, the requests for a justification appeared in a variety of ways in the task directions, which could be grouped into three categories as shown in Table 6.

Table 6. Frequency of the types of requests for a justification

Types of Request for a Justification	Frequency	%
Request a warrant (Explain how you know that something is true)	32	48%
Justify your conclusion/answer/reasoning	23	34%
Explain why an argument or claim is true	12	18%

About half of the requests for a justification (48%) occurred in the form of asking student to explain (or state) how they know that something is true, where the requested justification was concerned only with a particular part of a problem, rather than requiring a logically connected sequence of arguments. This is closely related to the distinction made in the textbook between a proof and a justification, which emphasizes that a justification is given to support one constituent statement (or step) of a proof. In other words, these were rather requests for a warrant for one particular claim or statement. For example, tasks requiring students to create a flowchart often included directions to provide a warrant as well, by reminding students "to justify each statement with a reason". Another common way to request a justification in the tasks was by asking students to "explain why an argument or claim is true"; 12 instances (18%) of the requests for a justification were in this form.

Lastly, about one third of the total number of requests for a justification (23 instances out of 67) appeared in tasks that explicitly ask students to *justify*, through phrases such as "justify your conclusion", "justify your answer (response or solution)", and "justify your reasoning". While only 2 requests to justify were asked for one's reasoning, the rest of the requests to justify were almost evenly split between for a conclusion (11) and an answer (10). Such phrasings of requests to justify may appear as possibly problematic since there is an uncertainty as to what is specifically asked for; for instance, whether an empirical evidence would count as sufficient for justifying an answer. But, such requests were usually meant to push students to provide a logical reason and support their claims or work. To give an example, a sample task from unit 3 is provided below in Figure 3.


3. FINDING THE AREA OF A PARALLELOGRAM

a. Kenisha thinks that the rectangle and parallelogram at right have the same area. Her teammate Shaundra disagrees. Who is correct? Justify your conclusion.

b. Use what you know about rectangles to find the area of each parallelogram at right.

c. Write a formula for area of a parallelogram:

Figure 3. A sample task that requests students to justify their conclusion

The task aims to have students investigate the relationship between the area of a rectangle and the area of a parallelogram, and make a generalization based on the identified relationship. By setting the problem as a disagreement between two hypothetical students, the task requires students to defend their conclusion. What is important here is that students could “justify” their conclusion by simply calculating the area of the given rectangle and parallelogram and show that they are the same, and thus contend that they “justified” their conclusion. Hence, this possibility places an important responsibility on the teacher to make sure that appropriate standards are set for justifying. Ms. V set the standards for providing a justification by emphasizing the need to know why their conclusion is true. For instance, when monitoring the students’ work on the task, Ms. V pressed students to think about why the area of the parallelogram was the same as the area of the rectangle: “So, here it says Kenisha thinks that the rectangle and parallelogram at right have the same area. But I want to know why.” To another team, Ms. V asked students if they could “visually show that the area of that parallelogram is the same as if it were a rectangle?”, helping them to move away from relying on one specific example.

In sum, in the curricular materials the requests for a proof and requests for a justification often occurred together, and by means of various task directions such as, “Justify your conclusion”, “Explain why your answer is true”, or “Write a convincing argument”. Moreover, the task narratives also tacitly promoted particular notions of proof such as (a) proof is defending one’s answer, (b) proof confirms that something is true, (c) proof explains why something is true, (d) proof involves logical conclusions that are based on the known facts, and (e) proof is constructed by using known facts, properties, definitions, relationships, and theorems. For instance, the task presented in Figure 3 can be taken as indicating that proof is defending one’s answer, while the task in Figure 2 can be taken as indicating that proof is constructed by using definitions and that proof explains why something is true. Hence, the discourse used in the tasks seems to have contributed to the meanings of proof that the students had developed, given that students’ descriptions of proof included corresponding notions of proof as follows: (a) proof is backing up statements/conclusions, (b) proof is evidence that shows that something is true, (c) proof is explaining/showing how you know that your work is true, and (d) proof is showing that something is true based on known facts, rules, definitions, and properties.

4.2.2.b. The teacher’s support for proof and justification

Consistent with the curricular materials, proof and justification were also interchangeably used in the class, where justification meant to back up claims by providing a reason (a warrant) to explain why a claim is true, which was often requested by asking how one knows that a claim made is true. Yet, the distinction made in the textbook between a proof and a justification (that is, a justification is given for one particular statement (or step) in a proof (which is usually a set of logically sequenced arguments)) was evident in the teacher's discourse as well. For instance, when leading a whole-class discussion, Ms. V told her students that, "In order to prove something, we have to be able to justify everything that we're saying is true" (U4D5). Table 7 below shows the distribution of Ms. V's requests for a justification and a proof across the two units. Note that the frequency of the requests for a proof reflects the teacher's explicit use of the words *proof* or *prove* when asking students to construct a proof.

Table 7. Frequency of the teacher's requests for a justification and a proof in class

	Request for a Justification	Request for a Proof
Unit 3 (= 4 lessons)	35	0
Unit 4 (= 6 lessons)	70	8
Total (= 10 lessons)	**105**	**8**

As seen in Table 7, while Ms. V quiet frequently requested a justification (105 times in total), she rarely requested a proof (only 8 times). The low occurrence of the teacher's explicit request to prove can be explained by a few factors. First, as mentioned earlier, during the interview Ms. V explained that she typically does not use the word "proof" in class, but instead asks students to explain why something is true or how they know that it is true, which concurs with what was observed. Underlying her practice of not explicitly asking students to prove lie two inter-related reasons: (a) she thinks that students are often intimidated by the word "proof", and (b) she views proving as synonymous to justifying; hence, Ms. V used proof and justification interchangeably. Second, Ms. V's requests for a proof often occurred when launching the classwork, but because a classwork included multiple tasks, the frequency of the teacher's requests for a proof was fewer than the actual number of tasks that required students to create a proof. Lastly, and more importantly, given that the students worked on the mathematical tasks in teams and the teacher monitored and facilitated their group discussions during the majority of the class time, the teacher predominantly requested a justification, rather than requesting to prove. Namely, when facilitating group discussions, Ms. V naturally focused on a particular claim or a step in a proof. For instance, even though the teacher may have been monitoring the students' construction of a flowchart proof, she would assist students by asking them to justify a specific claim or statement that was part of the proof.

Indeed, Ms. V commonly requested justifications, 105 times in total across the two units. As seen in Table 8, Ms. V's requests for a justification were grouped into three sub-categories as follows: (a) request for a warrant, (b) request to explain why, and (c) request to verify. There were five additional requests for a justification that were not coded as one of those sub-categories.

Table 8. Frequency of the types of teacher requests for a justification

Types of Teacher Requests for a Justification	**Frequency**	**%**
Request a warrant for a claim	72	69%
Request to explain why a claim is true or false	18	17%
Request to verify that a claim is true	10	10%

Ms. V most frequently requested a justification by asking students how they know that something is true, which was actually requesting a warrant for a claim. Specifically, Ms. V pressed students to support their claims with known facts, properties, relationships, definitions, or theorems. 69% of the requests for a justification (72 of the 105 instances) were in this form, where 55 of them occurred in unit 4. For example, while leading a whole-class discussion in the first lesson of the unit 4 (U4D1), Ms. V pressed students to support their conclusion that the given figures were not similar. She said, “The more important question is how do we know that two figures aren't similar. So, what certain characteristics have to happen in order for us to be able to say 'Yes, this figure is similar to this figure.' Ok, Rachael, what did your team talk about?” In this excerpt, Ms. V was specifically asking students to support their claim that the figures were not similar by using the relevant definitions and properties of the geometric shapes and the relationships they had learned in class. Note that this type of requests for a justification described here promotes the notion of proof as backing up claims with known facts, properties, and definitions, which was a shared view of proof among the students and their teacher.

In fact, Ms. V described proof as backing up one’s claims with warrants in the first interview, saying that, “[To me proof is] being able to always say the answer is this *because*, or I know this *because*- being able to justify your reasoning and explain it in some way, I think that's always something that we're expecting students to do.” In line with her description of proof, in her introduction to writing proofs in unit 4, Ms. V emphasized proof as a common mathematical activity that the class has been regularly engaging in, and stressed that they will just learn a new way to organize reasoning and information:

> So, today's focus is going to be those two things that I just mentioned; talking about similarity and ratios of similarity between two similar triangles and then being able to write proofs to justify how do we know that they are either similar or not similar. And a lot of the time when people see that word 'proof', they get really intimidated by it, but I think what you'll see today is that a proof is basically something that we are already doing all year. It's stating a fact and then saying how you know it is true. Not just saying, ‘Oh, the triangles are similar’, but saying they are similar because I know this, this, and this. So, that's really what we're going to

be doing today. But we're going to kind of learn a way to organize our thoughts and organize information. (U4D2)

Note that Ms. V's introduction of proof is very similar to the way she described proof in the first interview, and it is also consistent with her goal to broaden students' views of proof, as she expressed in the interview. More precisely, Ms. V thought that students might have a narrow view of proof that is limited to flowchart or two-column proofs, and asserted that, "One thing that I would like to do with my students is make them see that proving is not just those two units that we do in geometry; it's something that you're constantly doing." Thus, her emphasis on proving as a common mathematical activity that the class has regularly engaged is a manifestation of her explicit goal that she set for supporting students' notion of proof. Ms. V seems to be successful at her goal to have students recognize that they engage in proving outside of geometry units as well, given that during the interviews all of the students asserted that they were always expected to justify their ideas in class, noting that it may never be referred to as proof, though. Figure 4 summarizes how Ms. V's instructional emphases and practices were related to her notion of proof and the goal she had for supporting students' notions of proof.

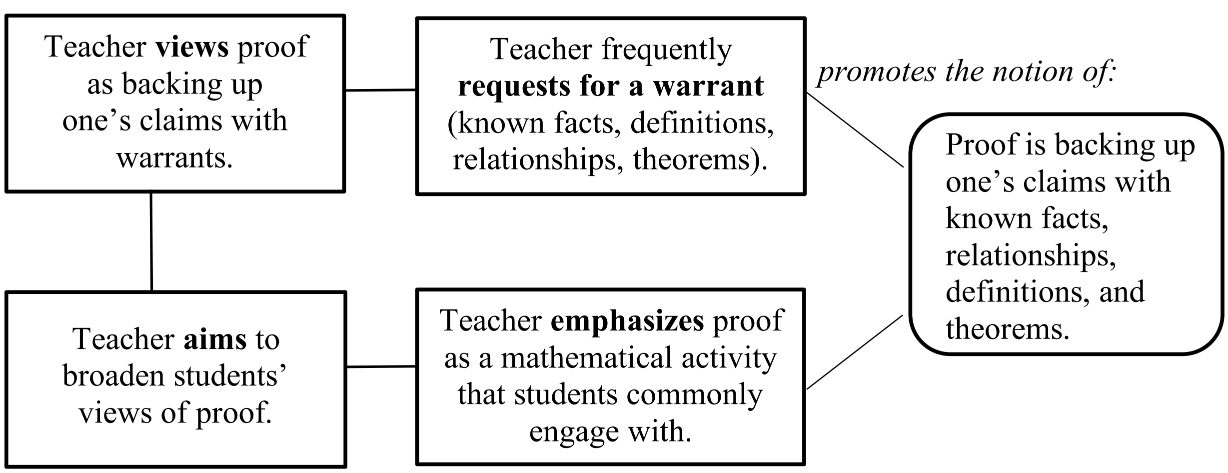


Figure 4. Links between Ms. V's notion of proof and her instructional goals, emphases, and practices

As mentioned earlier, during the second interview all of the students agreed with the statement that, “Proof is showing that something is right based on known facts, rules, definitions and properties”, while this notion of proof was not evident in the first interview. Moreover, all the students, but one, picked this statement as one of the top three statements that best describes what proof meant to them. For example, one student remarked that, “Yeah, when we were trying to prove that triangles were similar, we would have to do these exact things, like we could prove that they're similar because of definitions”. Hence, Ms. V’s instructional emphases and her frequent requests for a warrant, coupled with the tasks requiring students to support their claims with a warrant, seem to have jointly supported the students’ conceptions of proof to evolve to encompass a new notion of proof– one that aligns with their teacher’s notion of proof.

Another common way that Ms. V requested a justification was by pressing students to explain why something was true or false, which accounted for about 17% of her requests for a justification. For instance, when observing a team’s work on a task and finding out that the students in the team had different answers, Ms. V asked the team members to discuss their answers not only to figure out the correct answer but also to figure out why it was true: “You just said two different things. Talk it out, figure out which because it is one of those, but not both; figure out which one it is and why” (U4D4). Lastly, about 10% of Ms. V’s requests for a justification were phrased in a way that asked students to verify that something is true. For example, Ms. V asked students: “How could we verify that the opposite sides in Julie's diagram are actually parallel?” (U3D6). Hence, while the teacher’s emphasis on asking students to explain why a claim or answer is true may have supported students’ meaning of proof as backing up, her emphasis on verifying that a claim or answer is true may have contributed to the development of the notion of proof as evidence, as depicted in Figure 5. As shown in Table 4, during the second interview, the majority of the students agreed that proof is backing up statements or conclusions and that proof is evidence that shows that something is true.

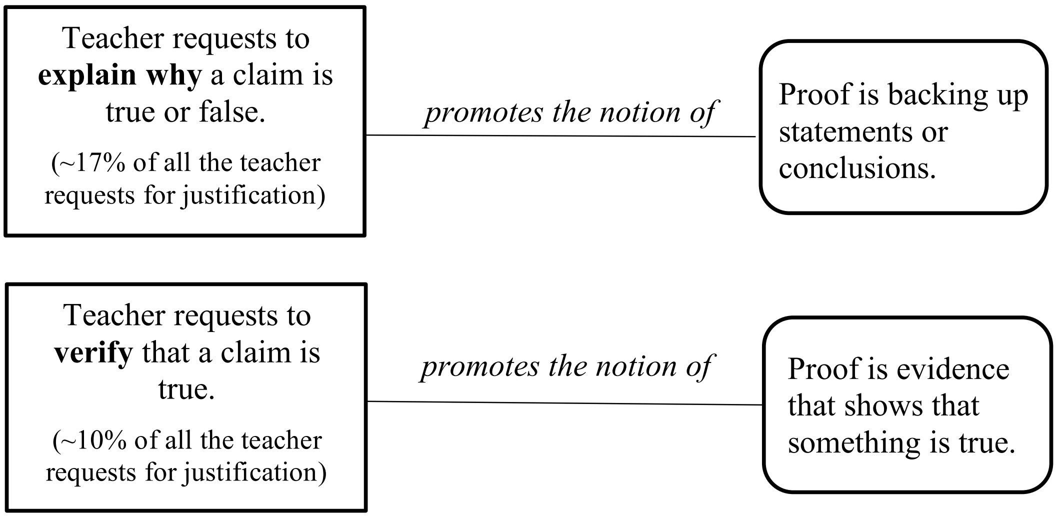


Figure 5. Links between the types of teacher requests for a justification and the students' meanings of proof

4.2.3. Instructional emphases promoting the verification and explanation roles of proof

The emphases embedded in the teacher's requests for a justification, whether asking to (a) explain why something is true or (b) verify that something is true, may also convey messages about the purposes of proof and the roles a proof plays in mathematics (such as *explanation* and *verification*, respectively). Teachers may also implicitly convey such messages about the roles of proof through other instructional emphases; for instance, by highlighting (a) the need for and the importance of understanding why something is true, or (b) verifying each mathematical claim. Teachers can induce such instructional emphases when eliciting students' answers and thinking, evaluating their work, or providing guidance. For example, Ms. V often stressed that students should always check and confirm that their claim or work is true. In facilitating a team discussion, for instance, she advised students to "Constantly think about 'How could we check this?', okay?", as she moved to another team. Such emphases on verifying one's answer or work was also common in more procedural questions such as solving an equation system. Thus, the students seem to have associated such emphasis on checking and verifying one's work with proving, as all of the students highlighted the verification role of proof during the first interview.

Ms. V also promoted the explanation role of proof by highlighting the need and importance of explaining why something is true when monitoring and facilitating students' teamwork and summarizing discussions. For example, the following excerpt shows how Ms. V concluded a class activity in which the teams explored why the Pythagorean Theorem is true, by pointing to the importance of understanding why something is true. Ms. V said:

> I know that this theorem may be not new to you, but I think that the quick investigation that you just did is a really valuable way for you to reinforce why the theorem works. Maybe you knew the theorem, but you hadn't actually seen why it worked before, so hopefully that helped you reinforced that a little bit. (U3D3)

Hence, by emphasizing the value of understanding why something is true rather than merely showing that a claim is true, the teacher promoted the explanation role of proof through such comments and feedback as well as the explicit requests for a justification that explains why something is true. While in the first interview only three students stated explanation as a role of proof, all of the students considered explaining why as an essential aspect of proof.

4.3. Classroom Factors Related to the Students' Proof Schemes

Now that we have drawn some connections between the students' notions of proof and the classroom factors, I move on to discuss the classroom factors in relation to their potential influence on the development of students' proof schemes, by using Harel and Sowder's (1998) proof scheme taxonomy to structure the findings. Specifically, I will discuss how the curricular materials and the teacher's instructional emphases and practices discouraged the development of the authoritative and empirical proof schemes, but fostered the development of the deductive proof scheme (Harel, 2006).

4.3.1. Classroom factors discouraging the authoritative proof scheme

In the authoritative proof scheme, individuals' conviction of the truth of a statement or claim is rooted in external factors such as a teacher or a textbook. For instance, students may assume that a conjecture must be true if it appears in a textbook or is presented by someone who is viewed as an authority, such as their teacher. Although this is not an uncommon proof scheme among students, I have not found evidence that any of the students in this study had an authoritative proof scheme. What classroom factors, if at all, might have supported students not to develop this proof scheme?

Harel and Sowder (1998) argue that viewing mathematics as a collection of truth (whose validity need not be questioned as long as it comes from an authority) seems to be the underlying characteristic of this proof scheme. Further, focusing on the truth of statements rather than exploring why the statements are true might also contribute to the development of this proof scheme. In Ms. V's classroom, there was evidence that the teacher discouraged this view. For example, while checking students' homework questions, Neil asked Ms. V if he could use a formula that he had seen elsewhere to solve a homework question. In response, Ms. V discouraged Neil from using the mathematical formula unless he fully understood why it worked and could justify it:

Neil: So, I was wondering if we could use this equation, c minus b squared over 4a, to get the maximum minimum point of a quadratic equation?

Ms. V: Do you know why that works?

Neil: Not necessarily.

Ms. V: So, here's my rule of thumb. If you don't know why it works, don't use it. Because you want to be able to always justify what you're doing, and so if you're, if you found a formula that somebody says is true, which is great, but if you don't understand fully why it works or if for sure even that works, then I wouldn't use it. Okay? So, if you can reason through that and figure it out-

Neil: - figure out how it works?

Ms. V: -yeah, then by all means use it. Yeah, but you want to be able to justify it.

Neil: So, if I try to, if I figure out how this works, can I use it?

Ms. V: Yes, of course, yep.

The teacher's emphasis on the importance of understanding why a claim (or a formula, as it was the case in this excerpt) is true seemed to be influential on the students' understandings and views about proof. As seen in the interview data, all of the students considered *explaining why* as a critical feature of proof, which affected the students' acceptance of an argument as a proof. For example, Neil maintained that, "If you have something that's being proved, you have to be able to say why it works". Thus, the students' eagerness to understand why a mathematical statement or claim is true, which was clearly evidenced both in their evaluation of hypothetical proofs and in their attempts to prove a given statement as reported in the first paper, indicates that the students did not possess the authoritative proof scheme.

Furthermore, Harel and Rabin (2010) hypothesized that answering students' questions, responding to their ideas, evaluating student ideas rather than asking the class to evaluate, lecturing, and telling students how to proceed with the solution are the instructional practices that can promote the development of authoritative proof scheme. Instead of evaluating students' ideas directly, Ms. V often asked students to discuss in their teams, or she had them explain their argument and help them notice their error, if a student had a flaw or error in his or her argument. Instead of answering students' questions right away, Ms. V usually redirected a student question to the student's teammates or to the whole class to discuss first, thereby giving students responsibility for their own reasoning. For instance, while Ms. V was monitoring teams' work in U4D3, a student asked to Ms. V if the AAS and ASA were the same similarity condition. She started responding by saying that, "They are not the same because–", and then directed the question to the team to explain why there are not the same: "Well, can somebody in the team explain why the AAS and ASA are not the same?".

Additionally, it is a very common practice in mathematics classes that students are typically asked to prove true mathematical statements, which may lead students to assume that a statement must be true if it is asked to be proven, and thus student may inadvertently develop an appeal to the textbook. Although most of the requests for proof involved true mathematical statements in this curriculum as well, there were also tasks that asked students to prove an incorrect mathematical statement, challenging the idea that if a statement is asked to be proven, then it must be a true statement.

4.3.2. Classroom factors discouraging the empirical proof schemes

The empirical proof schemes class includes two sub-categories: (a) the inductive proof scheme, in which an individual's conviction comes from appeal to examples, and (b) the perceptual proof scheme, in which the conviction is based on sensory experiences (Harel & Sowder, 1998). In the inductive proof scheme, students' convictions come from examples through testing a conjecture with one or more specific cases. It is a very common proof scheme observed among students across grade bands, including high school students (e.g., Balacheff, 1988; Chazan, 1993; Edwards, 1999; Porteous, 1990). For instance, Porteous (1990) found that 75% of the students believed a number theory conjecture was true based on empirical evidence alone. Of those, only 10% offered additional logical support on their own for the claim that the conjecture was true. However, as discussed in detail in Paper #1, *An Investigation of Proof Conceptions in a High School Mathematics Classroom,* the students in this study were mostly aware that examples are insufficient for proof; but rather, they believed that a proof must be a general explanatory argument. In other words, the students had evidence of possessing the deductive proof scheme rather than the inductive proof scheme. For instance, one student remarked that:

> When you're just using values to prove something, then that is not really a proof. Yeah, because when you're just using integers or actual values and not variables, then you're not proving it; you're just showing that for that case that this works, but not for every case like a proof should.

Hence, to keep it concise, I will reserve the discussion of how the classroom factors supported students to move away from the inductive proof scheme to the deductive proof scheme to the next section (sec. 4.3.3), in which I will focus on the classroom factors that may have promoted deductive proof scheme. Instead, here I would like to focus on the classroom factors that helped students move away from the perceptual proof scheme– another sub-category of the empirical proof schemes class.

The perceptual proof scheme involves coming to a conviction based on an observation or the appearance of a visual figure. For instance, a student may conclude that two triangles are similar because they look similar. Indeed, Harel and Sowder (1998) noted that the perceptual proof scheme is children's first source of internal conviction. In what follows I present how the tasks and the teacher discouraged the development of the perceptual proof scheme by emphasizing the importance of not making assumptions based on appearance.

First of all, the task directions included an explicit emphasis on not making assumptions; more specifically, I identified eight tasks (within 171 tasks that the students engaged with), in which making assumptions was explicitly discouraged. For instance, to deter students from making assumptions based on appearance, the tasks deliberately included figures that were not drawn to scale, with an accompanying warning to students. There were also tasks that clearly set the expectation that the claims need to be supported with known properties, relationships, and definitions. For example, in one task, students were given four equations and asked to draw the lines corresponding to the given equations, and then to identify what geometric shape the intersection of these lines would form and to prove their conclusion (see Figure 6). The task direction explicitly stated that, "It is not enough to say that a quadrilateral *looks* like it is of certain type or *looks* like it has a certain property." (*emphasis on the original*) (U4D6). Hence, the task made it clear that making claims based on the appearance of diagrams are not acceptable in mathematics.

7. THE SHAPE FACTORY
You just got a job in the Quadrilaterals Division of your uncle's Shape Factory. In the old days, customers called up your uncle and described the quadrilaterals they wanted over the phone: "*I'd like a parallelogram with…*".

"*But nowadays,*" your uncle says, "*customers using computers have been emailing orders in lots of different ways.*" Your uncle needs your team to help analyze his most recent orders listed below to identify the quadrilaterals and help the shape-makers know what to produce.

Your Task: For each of the quadrilateral orders listed below,

(i) Create a diagram of the quadrilateral.
(ii) Calculate all side lengths and all slopes. Show all work and clearly identify all side lengths and slopes.
(iii) Decide if the quadrilateral ordered has a special name. To help the shape-makers, your name must be as specific as possible. (For example, do not just call a shape a rectangle when it is also a square!)
(iv) Customers will want to be sure they get the type of quadrilateral they ordered! Make a *proof* that the quadrilateral ordered must be the kind you say it is. This can be in the form of a flowchart or two-column proof OR it can be written out in sentences. It is not enough to say that a quadrilateral *looks* like it is of a certain type or *looks* like it has a certain property. Include your work in your proof.

The orders:
a. A quadrilateral formed by the intersection of these lines:

$$y = -\frac{3}{2}x + 3$$
$$y = \frac{3}{2}x - 3$$
$$y = -\frac{3}{2}x + 9$$
$$y = \frac{3}{2}x + 3$$

Name:
Proof (include a flowchart or 2-column proof or full sentences defending your answer.)

Figure 6. A sample task discouraging making assumptions based on appearance

The teacher further reinforced such emphases found in the tasks by regularly emphasizing the importance of not making assumptions. Specifically, I identified 13 instances in which Ms. V explicitly asked students not to make assumptions, which occurred in six lessons spreading over the two units. For instance, during the first lesson that I observed, students had just investigated why the Pythagorean theorem is true and then moved to a new task, which required the use of the theorem. As she was introducing the task to the students, Ms. V advised them, "Really make sure that you're never making assumptions; use what you know about the Pythagorean theorem to determine where the hypotenuse has to be" (U3D3). In another lesson, as she was monitoring students' work on a task, Ms. V pressed a group of students to base their argument (that the given triangles were similar) based on the known facts, properties, and relationships, instead of the appearance of the figures:

> Ms. V: So, what if these aren't drawn to scale? … I only asked because a lot of time these figures aren't necessarily drawn to scale.
>
> *(Harry described how he knew that the triangles were similar, but it is inaudible.)*
>
> Ms. V: Okay (*gesturing at Harry to validate his answer*). So, instead of going off of based on the look, like which angles look smallest or which side is the smallest, we have to really go off of what we know. So, (*gesturing at Harry*) using that scale factor, so recognizing that the scale factor is 2 and then showing which angles are congruent to which angles is the way to go. (U4D1).

Therefore, both the tasks and the teacher helped students refrain from making assumptions based on appearance. The task directions and Ms. V's feedback highlighted that making unwarranted assumptions is not acceptable in mathematics, and stressed that mathematical claims need to be supported with known facts, properties, relationships, and definitions. Hence, these classroom influences seem to have collectively discouraged the development of the perceptual proof scheme, given that the students unanimously agreed with the statement that building one's work on assumptions is not acceptable in mathematics during the second interview. Moreover, when describing what proof and proving meant to him (during the first interview), one student stated that, "When we have shapes that we have to prove they are congruent– you can really see… you can assume they're congruent, but you have to prove it. And to prove it, you need to have facts that are undeniable", indicating that the emphasis on not making assumptions based on appearance has been taken up by the students.

4.3.3. Classroom factors supporting the deductive proof scheme

The deductive proof scheme entails validating conjectures by means of logical deductions. More precisely, the deductive proof scheme requires that an individual: (a) understands that a proof should account for all cases it is given for (*generality*); (b) sets goals and sub-goals and attempts to anticipate the outcomes of his or her actions during the proving process (*operational thought);* and (c) understands that mathematical justification should be based on the rules of logical inference (*logical inference*) (Harel, 2006). Unlike the common finding that students, including high-attaining high school students, typically do not exhibit deductive proof schemes (e.g., Bell, 1976; Edwards, 1999; Healy & Hoyles, 2000; Ususkin, 1987), the students in this study overall understood that proof is a general argument that explains why a mathematical statement or claim is true (for more details, see Paper #1, *An Investigation of Proof Conceptions in a High School Mathematics Classroom).* In addition to the classroom factors that discouraged the authoritative and empirical proof schemes, what other classroom factors might have supported or reinforced the development of the deductive proof scheme?

First of all, in order to cultivate a deductive proof scheme, students must be given ample opportunities to engage in proof-related mathematical activities, such as investigating conjectures, making conjectures, generalizing, justifying claims, and proving (one's own or given) conjectures, and also appreciate the need for proof; that is, the need for deductive arguments rather than empirical evidence that is based on a specific case. The mathematical tasks with which students engaged included several opportunities for searching for a pattern, identifying a general relationship, making conjectures, and generalizing, although not as many as the requests for a justification and proof. The tasks, however, had a strong emphasis on requests for a justification and proof; out of 171 tasks in total, there were 67 requests for a justification and 35 requests for a proof, summing to a total of 102 opportunities for developing a justification and proof. As described earlier (in section 4.2.2.a.), these opportunities were further supported with the teacher's requests for a justification (105 instances) or a proof (8 instances), as she monitored and facilitated small-group discussions or led a whole class discussion.

However, equally important to the presence of the opportunities for justification and proof is the expectations set for what an acceptable justification and proof is. Recall that about one third of the requests for a justification found in the tasks were phrased as "Justify your conclusion" or "Justify your answer", leaving it ambiguous as to what is a satisfactory justification. This ambiguity in the task language was cleared with the teacher's emphasis on pushing students to move beyond empirical verification to providing a logical argument that explains *why* their conclusion or answer is true. In fact, the teacher's recurrent emphasis on explaining why a claim or statement is true appear to be a key factor in supporting (or reinforcing) the development of the deductive proof scheme, as all of the students considered explaining why as a critical criterion for proof, creating an intellectual need (Harel, 2007) to go beyond empirical evidence and merely showing that a claim or statement is true. Indeed, researchers maintain that intellectual need for proof is crucial in order for students to appreciate the value of proof and develop more advanced proof conceptions (e.g., Harel, 2007; Stylianides, 2011; Zaslavsky, Nickerson, Stylianides, Kidron, & Winicki-Landman, 2012). Additionally, some tasks set the expectation to not assume that a mathematical property is true unless it is proven. For instance, students were told that, "Although we know it to be true in isosceles triangles, we CANNOT ASSUME THAT $\angle A \cong \angle C$ until we *prove* that it is true!" (*emphasis on the original*) (U4D5). Hence, through such task directions and the complementing teacher emphases, the need for proof was fostered in class.

Having pointed out to the importance of the opportunities for proof-related activities and creating a need for proof, to being with, let us now consider the three pillars of the deductive proof scheme (i.e., generality, operational thought, and logical inference) to discuss the classroom norms and teacher practices that may have supported the development of the deductive proof scheme. First, as clearly evidenced in the interview data, the students had a strong understanding that a proof needs to account for all cases it is given for (*generality*). For instance, one student remarked that, “If I prove something, then it's true in all cases.” Moreover, another student, Hera, made a connection to her classroom experiences with proof, as she rejected a hypothetical student proof on the basis that it did fail to account for all cases, also noting that it was not built on theorems. Specifically, Hera expressed that the argument did not “show us applying to all cases”, and argued that, “Because when we're doing like similarity or congruency theorems for triangles, we have like specific theorems that work for every triangle that we base our conjectures off of, but when you don't base yours off of anything, then you can't really say that it’s true all the time”.

In addition to creating opportunities for students to develop proofs based on the known definitions, properties, and theorems in class, Ms. V also supported the students’ understanding of the generality aspect of proof through her careful attention to setting the norms for an acceptable justification. As described earlier, Ms. V did not accept empirical verification as a sufficient justification, instead she asked students to develop a general argument by encouraging them to think about why a claim must *always* be true or suggesting they use a generic representation to think about all cases rather than relying on one specific case (as explained in section 4.2.2.a.). For example, while facilitating one team’s investigation of the relationship between the area of a rectangle and a parallelogram, instead of calculating the area by using the given dimensions, Ms. V asked students to develop a visual argument, which could lead to noticing the general relationship: “How could you visually show that the area of that parallelogram is the same as if it were a rectangle?” In addition, there were also several tasks that specifically asked students to show that something is “always” true, further supporting the need for generality by pressing students to develop arguments that cannot be restricted to particular cases.

The second pillar of the deductive proof scheme requires students to engage in *operational thought* by setting goals and sub-goals and attempting to anticipate the outcomes of their actions during the proving process. Ms. V supported this by highlighting the goal of a proving task when introducing tasks, and by clarifying the goal or focusing students’ attention on the goal of the proving task when monitoring their work, if students seem to be struggling. For instance, in one task the students were asked to investigate whether a line segment that proportionally subdivided the two sides of a triangle was parallel to the base of the triangle. As part of the task, the students were asked to algebraically show that $\frac{a+b}{a} = \frac{c+d}{c}$, given that $\frac{b}{a} = \frac{d}{c}$. Not knowing what to do, one team asked for help from Ms. V:

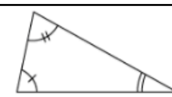
Hera: Ms. V, can you please help me? It says, start with b over a equals to d over c, what does that mean?

Ms. V: We're trying to show this is true, but we have to start with what we know, and what we know is true is that these are equal because they [the task] told us that was given. So, we're starting with what we know and our goal is to make it look like this. That's our goal, okay?

Hera: Okay.

Furthermore, there was a substantial scaffolding built in the tasks that required students to develop a proof by breaking the task into sub-tasks through which the ultimate goal of the task (e.g., creating a flowchart proof) was divided into sub-goals (e.g., justifying constituent arguments of the proof). For an example, consider the task presented in Figure 2 (re-presented below).

7. (CCG 7-49) Carefully trace the triangle at right onto tracing paper. Be sure to copy the angle markings as well. Then rotate the triangle about a *midpoint of any side* of the shape to make a new shape that looks like a parallelogram. Trace the new shape on your paper.



a. Is the shape truly a parallelogram? Think about the *definition* of a parallelogram and use the angles in your diagram to write a convincing argument.

b. What can the two congruent triangles tell you about a parallelogram? Look for any relationships you can find between the *angles* and *sides* of a parallelogram and list them on your paper. (You may already KNOW these attributes to be true, but now you can use triangles to show WHY these attributes are true)

c. Does the diagonal of a parallelogram always split the shape into two congruent triangles? Draw the parallelogram at left on your paper. Knowing only that the opposite sides of a parallelogram are parallel, create a **flowchart** to show that the triangles are congruent. Be sure to include a fact inside of each bubble and a reason outside of each bubble in your flowchart proof.

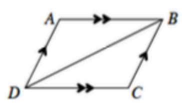


Figure 2. A sample task that requests students to create a flowchart proof (*re-presented*)

The task leads students through the steps of the proof with the accompanying questions that cue students' attention, asks students to anticipate the outcomes of their actions, and to provide a warrant for each mathematical statement used in the proof. Thus, by partitioning a task into sub-parts and through guiding questions, the curriculum helped students develop a proof. However, while this may help students recognize the importance of setting goals and sub-goals in proving, it may also make students too reliant on such scaffolding when proving, given that identifying the sub-goals in a proving task is an important intellectual work, yet it was often carried out by the tasks, instead of the students.

The teacher further scaffolded the students' proof development by frequently asking students what information they could use to prove a given mathematical statement and asking for a warrant to support each statement. Indeed, about 69% of Ms. V's requests for a justification were a request for a warrant that justifies a mathematical claim or statement. For example, in a construction task, the students were asked to construct a quadrilateral by following the directions that were given in the task, and then identify what type of a quadrilateral it was and justify how they know it (see Figure 7). While monitoring a team's work, the students stated that the quadrilateral was a rhombus. In response, Ms. V asked the students how they knew that it was a rhombus:

Ms. V: How do we know… because what else has to happen if it is a rhombus? Not only do they have to have the same side lengths, but–

Molly: –the opposites sides have to be parallel.

Ms. V: Yes. So how could we verify that characteristics?

Problem	Construction
9-75 a. In the space at right, construct two congruent intersecting circles so that each passes through the other's center. Label the centers *A* and *B*. b. Locate the two points where the circles intersect each other. Label these points *C* and *D*. Then construct quadrilateral *ACBD*. What type of quadrilateral is *ACBD*? Explain how you know. c. Use what you know about the diagonals of *ACBD* to describe the relationship of $\overline{AB}$ and $\overline{CD}$. Make as many statements as you can in the space at right. d. What else can this diagram help you construct when given a line segment such as $\overline{AB}$?	

Figure 7. The construction task (U3D6)

The third fundamental aspect of the deductive proof scheme is that students must understand that mathematical justification should be based on the rules of logical inference. The students' understanding of this aspect of the deductive proof scheme was fostered through the continual emphasis on using definitions, properties, relationships, and theorems as a warrant in developing a proof. As discussed earlier, the teacher regularly requested a warrant to support claims made, but also provided a warrant as she validated students' responses, re-voiced their arguments, or summarized a discussion in order to make sure that the given justifications were made available to other students as well as to enhance a given justification to make it a more sophisticated and mathematically accurate argument, if needed. More specifically, there were 60 episodes in which Ms. V used mathematical definitions, properties, relationships, and theorems as a warrant through re-voicing students' arguments or summarizing discussions, as compared with 72 episodes in which Ms. V requested a warrant. Hence, this frequent co-occurrence of requesting a warrant from students and providing a warrant by the teacher appears to have contributed to the development of a shared understanding of proof as based on the known facts, properties, relationships, definitions, and theorems.

Below, I provide a vignette from a whole-class discussion on a warm-up task, which meant to review the similarity conditions and then transition to the congruency conditions, in order to exemplify how Ms. V asked students to justify their claims and then provided warrants in response to the students' justifications.

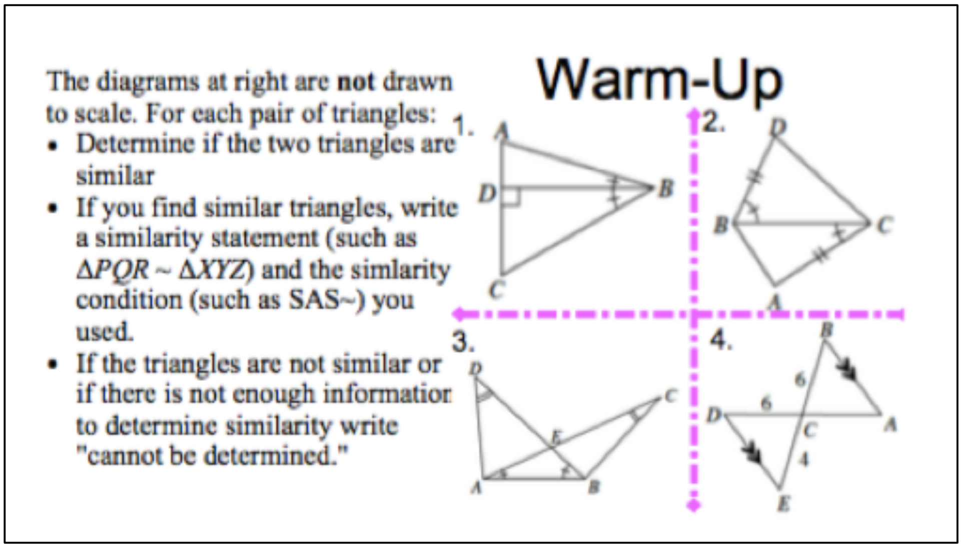


Figure 8. Warm-up task, U4D3

Ms. V: Okay, last but not least, we've got number four. Any team who wants to share what they came up with number four? Different team. Already had Team 8, Team 5, and Team 7. Can we get a different team to talk about number four? Even- (*a student volunteers*) yeah, thank you!

Mark: We did Side-Angle-Side.

Ms. V: Okay. Can you show– can you elaborate a little bit because I only see like one set of sides that is equal in this diagram and I don't see any angles marked as equal, so can you share a little bit about what your team talked about?

Mark: For angle C, there is like, the two angles are equal.

Ms. V pushed students to be precise in naming angles, stressing that there are four different angles that meet at point C. As a result of the class discussion on how to appropriately specify the angles, Mark clarified that he was referring to angles CED and ACB.

Ms. V: So, you said, Mark, that we had vertical angles here (*marks the angle on the diagram*), right? Okay, so we know those are equal. And then what other angles did you have?

Mark: That's the only angle.

Ms. V: Oh, that's the only angle you had, okay. So, then you had the side. But, then what about the other side? I see this is labeled (*circles EC on the diagram*), but then I don't see this labeled (*circles CA on the diagram*).

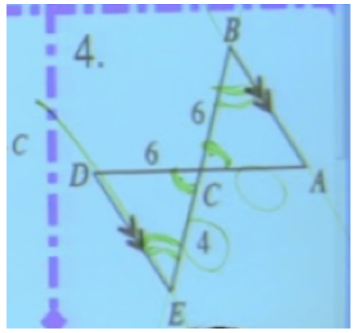


Figure 9. Warm-up task #4

Mark: That one has to equal to 4.

Ms. V: Why?

Mark: Because the other side is parallel to the other one.

Ms. V: Okay, so we're using the fact that these lines are parallel. We need to be really careful because when we learned about parallel lines, we learned conditional statements that said if these lines are parallel, what do we know is true? This isn't just for Mark; this is for the whole class. If we have two lines cut by a transversal and we know that those lines are parallel, what do we know it has to be true?

Several students responded that alternate angles need to be congruent.

Ms. V: Yeah, alternate interior angles have to be congruent. So, we can't jump from the parallel lines to side lengths being equal; we can only use theorems that we have learned in this class. So, we have learned that if these are parallel, then angle CED is congruent to angle ABC because those are alternate interior angles. So, just be really careful about what information you're using. So, which condition would we be using then, if we use that information that highlighted up there?

Ray: AA.

Ms. V: Yeah, yeah, you can just use AA because you’ve got two angles here. So, yeah AA would suffice. Awesome.

As seen in the vignette, Mark claimed that the triangles were similar due to the Side-Angle-Side similarity condition. Although this was not a valid condition in this task based on what was known to the classroom community at that time, Ms. V asked Mark to elaborate on his reasoning, instead of evaluating his answer as incorrect right away. She then pushed him to justify how he knew that CA was congruent to CE. Mark used the fact that DE was parallel to AB as a warrant for his claim that the length of AC was equal to 4, same as the length of CE. While, indeed, AC is congruent to CE, this could be deduced from the fact that the triangle CED is congruent to the triangle CAB, which the class had not proven yet. Hence, Ms. V asked the class what could be reasonably concluded from the given premise. Once the students responded that the alternate interior angles must be congruent, Ms. V then emphasized the connection between the premises and the conclusions made, as she summarized the discussion. Hence, Ms. V also supported students' understanding that a proof must be based on logical inferences, by emphasizing not to draw conclusions more than what could be reasonably deduced from the available information.

Additionally, there was a profound emphasis on using definitions as a warrant for justifications in the class, beginning in unit 3. For instance, Ms. V stressed the importance of definitions by informing students that, "Definitions will come into play a lot because we need to use definitions when we are explaining, justifying, and trying to back up our solutions or answers" (U3D4). In fact, by focusing on the definitions and properties of 2D shapes, the Unit 3 could be considered as a foundational unit for the ensuing unit in which proof was the main focus. This special emphasis given on using definitions as a warrant explains why the students unanimously praised the hypothetical student arguments in which definitions were used to back up a claim. When evaluating the hypothetical student proofs, the students commonly praised the use of a warrant as a valid aspect of a proof, recognizing various types of warrants such as definitions, known mathematical facts, rules and properties, logical inferences, theorems, and even empirical evidence.

5. Discussion and Implications

By studying a high school mathematics classroom over an extended period of time, I investigated students' evolving conceptions of proof in the context of their mathematics class, with the goal to better understand how the students' conceptions of proof were related to the classroom factors (which are organized by their teacher). Specifically, I explored the links between (a) the students' and their teacher's conceptions of proof, (b) the teacher's conceptions of proof and her instructional practices, and (c) the classroom factors and the students' conceptions of proof, where by *classroom factors* I refer to the classroom norms, the teacher's instructional practices, and the curriculum used in the class. Because curriculum has an essential role in establishing students' potential learning experiences with respect to proof through the opportunities for proof-related activities it includes (or excludes), I considered curriculum as part of the classroom factors as well. While in the first paper I primarily focused on the connections between the students' and their teacher's conceptions of proof, in this paper I focused on the connections between the classroom factors and the students' conceptions of proof, by also attending to how the teacher's conceptions of proof were manifested in her instructional emphases and practices.

The first paper not only illustrated the students' and their teacher's conceptions in detail, but also showed that the students' proof conceptions were in large part similar to their teacher's conceptions, indicating that the students may have appropriated their teacher's notions of proof. Moreover, both the teacher and the students had more sophisticated conceptions of proof than typically documented in the respective proof literature, leading naturally to the examination of classroom environment to uncover how the teacher's conceptions might have manifested in her instruction and also to identify the possible classroom factors that might have supported the students' developing proof conceptions to resemble their teacher's proof conceptions. Hence, based on the analyses of the curricular materials and the lessons, in this paper I offered evidence of connections between the teacher's conception of proof and her instructional emphases and practices, as well as the connections between those instructional emphases and practices (together with the curricular opportunities) and the students' conceptions of proof.

For example, in the first interview, all of the students described proof as explaining or showing that one's answer or work is true, which is a quite vague description of proof, leaving one to wonder whether the students had a mathematically rigorous understanding of proof. Yet, in the second interview, the students also unanimously agreed with the description of proof that, "Proof is showing that something is true based on known facts, rules, definitions and properties". Unlike the first description, the latter description specifies the means to establish the mathematical truth. The analysis of the curricular materials and the lessons pointed to several classroom influences on the students' notions of proof. First, about one third of the requests for a justification found in the tasks specifically asked students to justify their conclusion or answer, supporting an informal and vague notion of what it means to justify, as reflected in the students' description of proof during the first interview. On the other hand, about half of the requests for a justification (48% of 67 requests) found in the tasks asked students to provide a warrant for a particular claim or statement.

The teacher also frequently asked students to justify their conclusions or the claims made. In addition, Ms. V emphasized the need to understand why a claim is true, and pushed students to go beyond providing an empirical evidence and encouraged them to think about why a claim must be true in general, therefore setting the expectation for an acceptable justification. Moreover, similar to the tasks, most of Ms. V's requests for a justification (69% of 105 requests) was in the form of asking students to provide a warrant for a claim; namely, she asked students to support their claims with known facts, properties, relationships, definitions, or theorems. Ms. V's instructional emphases on providing warrants to mathematical claims and her frequent requests for a warrant are compatible with her notion of proof as *backing up claims* by providing a reason (i.e., known facts, definitions, properties, relationships, or theorems) to explain why a claim is true. Indeed, Ms. V's introduction of proof to the class precisely reflected her description of proof in the interview, which highlighted proof as backing up one's claims with warrants. Hence, Ms. V introduced the notion of proof and supported the students' conceptions of proof in accordance with her view of proof, which was found to be appropriated by the students as evidenced in the second interview. In short, Ms. V's instructional emphases and her frequent requests for a warrant, coupled with the tasks requiring students to support their claims with a warrant, seem to have jointly supported the students' conceptions of proof to evolve to encompass a new notion of proof– one that aligns with their teacher's notion of proof.

These findings are noteworthy for two reasons. First, it shows that a teacher and curriculum collectively create opportunities for students to prove and justify, and thus convey messages about what an acceptable justification or proof is. Further, it also underscores that teachers have an important role not only to maintain the opportunities for proof-related activities designed in the curriculum but also to improve them, if they are open to different interpretations (than the intended goal of the task) that could promote less sophisticated or undesirable notions of proof. Second, the findings also provide further empirical support that students' conceptions of proof are related to the classroom factors, offering some specific instantiations of this relationship.

In sum, the findings confirm the assumption made in the study that a mathematics teacher has an important role in shaping students' conceptions of proof through her instructional emphases and practices, which are influenced by her conceptions of proof. In particular, Ms. V's notion of proof as *backing up claims with warrants* to explain why a claim is true was influential on the way she introduced proof to the class, in what ways she requested a justification and what she accepted as a sufficient justification, as well as how she supported students to develop a justification, concurring with the findings of Conner's study (2007). In addition, the analyses of the interviews revealed that the students also have developed the same notion of proof as their teacher; that is, proof is backing up one's claims with known facts, definitions, and theorems.

Therefore, the results regarding the connections between the teacher's conceptions of proof and her instructional practices concur with the studies that identify teachers' knowledge and beliefs (which are referred to as *conceptions* in this study) about mathematics as important determinants shaping their instructional practices in general (e.g., Ball, Lubienski & Mewborn, 2001; Hill, Rowan, & Ball, 2005; Sowder, 2007), and lend credence to the studies that establish such links precisely for proof (e.g., Conner, 2007; Knipping, 2008). Moreover, the close alignment found between the teacher's and the students' conceptions of proof is in agreement with the body of work that shows that teachers' pedagogical practices significantly influence students' learning (e.g., Blanton & Kaput, 2005; Mason, 2000; Mewborn, 2003), expanding this work in the area of proof. More precisely, this study provides further empirical support to the link between teacher's instructional practices and students' conceptions of proof (Harel & Rabin, 2010; Martin et al., 2005) and exemplifies specific instantiations of such links, which are much needed for sharing with pre-service and in-service mathematics teachers.

In addition to establishing links between particular aspects of the students' notions of proof and the classroom factors, I also discussed the classroom factors in relation to their potential support for the development of particular proof schemes. Specifically, I identified and described the classroom factors that seemed to have discouraged the authoritative and empirical proof schemes (such as, emphasizing not making assumptions based on appearance, and rejecting empirical verification as a valid justification), as well as the possible classroom factors that supported the development of the deductive proof scheme, though they are not mutually exclusive factors. Indeed, research has identified various teacher practices that can support development of the deductive proof scheme, such as requesting for a justification; encouraging argumentation among peers; eliciting students' ideas; responding to their ideas through probing their reasoning, re-voicing, prompting error correction, or correcting students' flaws in a justification; and facilitating their reasoning through cueing their attention on specific aspects, summarizing a discussion, or offering a justification (Blanton et al., 2009; Harel & Rabin, 2010; Martin et al., 2005). While I agree that these are productive teacher moves for advancing students' proof conceptions, and I found that Ms. V commonly engaged in these practices, I was, however, interested in identifying classroom factors that might have specifically supported the three characteristics of the deductive proof scheme; that is, generality, operational thought, and logical inference. Thus, I discussed the classroom support for the development of deductive proof scheme by focusing on those three characteristics and presented corresponding classroom evidence for each, offering some novel findings and further empirical support for the links between instructional practices and students' proof schemes.

6. Conclusion, Limitations, and Future Directions

The findings reported in these two papers jointly show that students can develop robust conceptions of proof if the learning environment is conducive to sharing and justifying mathematical ideas where the teacher values proof as an important aspect of doing and learning mathematics. In particular, the study offers evidence that students can develop desired conceptions of proof when they are provided with appropriate tasks and instructional supports, suggesting that students form their notions of proof through implicit or explicit messages conveyed by the nature of mathematical tasks with which they engage, the norms established in class, and teachers' instructional emphasis on proof. In addition, the findings underscore the crucial role of teachers in setting clear norms about what a valid justification or proof is, because unless there are rigorous standards set in class, opportunities for proof present in a curriculum may be missed and students may develop improper conceptions of proof. Thus, the empirical findings reported in these papers may be valuable to teacher educators in informing and exemplifying pre-service and in-service mathematics teachers about in what ways teachers' own conceptions of proof may be reflected in their instructional support for proof, and in turn, how they may shape students' developing conceptions of proof.

More specifically, if students are to appreciate the value of proof in doing and learning mathematics and advance their notions of proof and skills to prove, they must be provided with ample opportunities for engaging in proof-related mathematical activities, such as investigating conjectures, making conjectures, generalizing, and justifying claims, and must be given responsibility for their own reasoning. Teachers can support their students' conceptions of proof by emphasizing the importance of understanding why a claim is true, rather than merely showing that it is true, and setting clear expectations for what an acceptable justification and proof is.

While examples can be useful in exploring a problem and in developing a justification, teachers should set explicit norms that empirical arguments alone are insufficient for proof. Accordingly, teachers should push students to move beyond empirical verification to developing a general deductive argument, by encouraging them to think why a claim must always be true and stressing that mathematical claims need to be supported with known facts, properties, definitions, and theorems. Additionally, teachers can emphasize the importance of not making assumptions based on appearance and not to draw conclusions more than what could be reasonably deduced from the available information.

That said, however, it is important to acknowledge a caveat that, being a case study of one mathematics classroom, the empirical findings reported in this study regarding the interplay between students' proof conceptions and the classroom factors cannot be generalized to make conclusive claims about those alluded relationships in general. Instead, this case study characterizes the nature of those relationships in one particular context. Specifically, this study offers an example of how a teacher who did not hold a narrow view of proof (that is restricted to certain mathematics classes or forms) and values proving in class may support students' developing proof conceptions in the domain of geometry. In addition, the students in this study were not typical high school students, but rather– considering that they were in an honors level, accelerated mathematics course– they were advanced mathematics students who had positive dispositions to mathematics and who viewed themselves good at mathematics. Hence, further research is needed to examine and compare those relationships in different classrooms, including different domains of mathematics and teachers with different conceptions of proof, as well as more typical mathematics classes that are not an advanced level mathematics course.

Moreover, the identified classroom factors cannot be conclusively claimed to be responsible for the students' conceptions of proof. Individuals' conceptions are formed over long period of time through various sources. Conceptions of proof are no exception; students come to mathematics classes with preconceived ideas about what it means to prove or what is a valid proof (though, they may be tacit), influenced not only by their previous mathematical experiences but also by their everyday experiences, such as proving one's argument in a debate. Hence, it is not possible to attribute the students' conceptions of proof solely to the identified classroom factors, but instead I argue that those classroom factors are possible classroom influences that might have supported or reinforced the development of students' conceptions of proof. Although we cannot make conclusive claims about causality, these hypothesized relationships, nevertheless, provide a helpful foundation on which to build future research to study their effects.

In conclusion, this study contributes to our collective understanding of the ways in which students' views and understandings about proof are related to the classroom factors, such as curriculum and teachers' instructional emphases and practices, and thus, to the teacher's conceptions of proof. Hence, this study was explorative in nature, aiming to unpack those relationships in the context of one high school mathematics course. By building on the results of this explorative study and the related body of knowledge in the field, a much-needed next step is sharing those results with mathematics teachers and collaboratively designing intervention studies in order to help students develop robust conceptions of proof and to support their proving competencies.

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APPENDICES

Appendix-A. Teacher Interview-1 Protocol

1. Tell me about **yourself**. How many years have you been teaching? What courses are you teaching this year? What other courses and grade levels have you taught previously?

2. Tell me a little bit about the **course**. How was the course designed? Were you part of the design team? What are the goals of this course?

3. Can you tell me a little bit about **students** who take this course? What math courses do they typically take prior to this course?

4. Do you consider this course to be an advanced level math course among the math courses offered at the school?

5. Tell me about the **curriculum** you use. Do you follow the textbooks strictly or do you also design your own tasks? If so, how do you come up with the tasks?

6. Because this is an integrated course, I understand that you try to carefully select which tasks to use in class, which ones to assign as a homework, etc. Can you tell me how you choose the tasks?

7. Another aspect of the class that I would like to talk about is the team assessments. What is the goal of these assessments? Is this something common to all math classes or is it unique to this class?

8. I noticed that students work on math projects as well. Can you tell me what these projects are about and what are their goals?

9. Do you think the **curriculum** is supportive of engaging students in reasoning and proof? If yes, can you give a specific example?

10. Can you describe what **the notion of proof** means to you? What purpose do you think proof serve in mathematics? What does it mean to prove something?

    - What makes something a proof?
        - Is there a requirement for a particular format?
        - Are there different types of proofs?
        - Are some types more valid than others?

11. Many mathematicians think proof is a big idea in mathematics. What do you think about this view? Why do you think that?

12. How would you describe the notion of proof in the context of secondary school mathematics?
    - How do you (or would you) describe the notion of proof to your students?

13. What purpose do you think proof serves in secondary school mathematics?

    - Why teach it?
    - Do you think some courses or content better suited to including proof? Why?
    - Does proof look different in different classes?

14. What role should proof play in secondary school mathematics curricula? Why?
    - Do you think proof is a separate topic in a course or an integral part of a course?

15. What do you think is important for students to **learn about proof**? Why?
    - When should students be introduced to proof?

16. How important reasoning and proof for this class?

17. What goals do you have for your students regarding justification and proof in your class?
    - Does your expectation vary depending on the course you teach?

18. We know from research that students find learning proof very challenging, why do you think this is the case?

19. Considering your experience **teaching** high school mathematics courses, what kinds of activities do you think work well to get students to explain their reasoning about problems or mathematical ideas?

20. In your experience, have you found that high school students generate explanations that could be considered proof?

21. What are some strategies that you think are useful in helping students justify and prove their reasoning?

Appendix-B. Student Interview-1 Protocol

Introduction

Thank you for agreeing to interview. I am interested in how you are thinking about mathematics, so I am going to ask you a few mathematics problems. You do not need to worry about being right or wrong, I am just interested in hearing your ideas and how you are reasoning. So, the more you can talk aloud how you are thinking the better.

Background information

- What mathematics courses have you taken so far?
- How would you describe your experiences with mathematics?
- Do you like/dislike mathematics?
- Do you consider yourself good at mathematics?

Evaluation of hypothetical student proofs

<table>
<tr><td colspan="2">Arthur, Bonnie, Ceri, Duncan, Eric, and Yvonne were trying to prove whether the following statement is true or false:

When you add any two even numbers, your answer is always even.</td></tr>
<tr><td>Arthur's answer

a is any whole number
b is any whole number
$2a$ and $2b$ are any two even numbers
$2a + 2b = 2(a + b)$

So, Arthur says it's true.</td><td>Bonnie's answer

2 + 2 = 4   4 + 2 = 6

2 + 4 = 6   4 + 4 = 8

2 + 6 = 8   4 + 6 = 10

So, Bonnie says it's true.</td></tr>
<tr><td>Ceri's answer

Even numbers are numbers that can be divided by 2. When you add numbers with a common factor, 2 in this case, the answer will have the same common factor.

So, Ceri says it's true.</td><td>Duncan's answer

Even numbers end in 0, 2, 4, 6, or 8. When you add any two of these, the answer will still end in 0, 2, 4, 6, or 8.

So, Duncan says it's true.</td></tr>
<tr><td>Eric's answer

Let x = any whole number
y = any whole number

$x + y = z$
$z - x = y$
$z - y = x$
$z + z - (x + y) = x + y = 2z$

So, Eric says it's true.</td><td>Yvonne's answer

So, Yvonne says it's true.</td></tr>
</table>

Arthur's answer

	Agree	Disagree	Don't know
Has a **mistake** in it			
Shows that the statement is **always true**			
Only shows that the statement is true for some even numbers			
Shows you **why** the statement is true			
Is an easy way to **explain** to someone in your class who is unsure			

Bonnie's answer

	Agree	Disagree	Don't know
Has a **mistake** in it			
Shows that the statement is **always true**			
Only shows that the statement is true for some even numbers			
Shows you **why** the statement is true			
Is an easy way to **explain** to someone in your class who is unsure			

Ceri's answer

	Agree	Disagree	Don't know
Has a **mistake** in it			
Shows that the statement is **always true**			
Only shows that the statement is true for some even numbers			
Shows you **why** the statement is true			
Is an easy way to **explain** to someone in your class who is unsure			

Duncan's answer

	Agree	Disagree	Don't know
Has a **mistake** in it			
Shows that the statement is **always true**			
Only shows that the statement is true for some even numbers			
Shows you **why** the statement is true			
Is an easy way to **explain** to someone in your class who is unsure			

Eric's answer

	Agree	Disagree	Don't know
Has a **mistake** in it			
Shows that the statement is **always true**			
Only shows that the statement is true for some even numbers			
Shows you **why** the statement is true			
Is an easy way to **explain** to someone in your class who is unsure			

Yvonne's answer

	Agree	Disagree	Don't know
Has a **mistake** in it			
Shows that the statement is **always true**			
Only shows that the statement is true for some even numbers			
Shows you **why** the statement is true			
Is an easy way to **explain** to someone in your class who is unsure			

Meanings and view of proof and proving

- Can you describe what proof means to you? What does it mean to prove something in mathematics? (*Suggest explaining what proof means in the context of the previous task, if student has difficulty articulating his or her views.)*

- How important do you think proving is in mathematics?

- Has the term "proof" come up in any of your math classes?
    - *If, yes*: Which classes? Do you recall in what ways or in what context the term proof has been used?

- Are you asked to prove or justify mathematical statements or your ideas in mathematics classes?

- Why do you think you are asked to prove in mathematics classes?

- Why do you think mathematicians prove mathematical statements?

Proof production task

How would you prove the following statement?

If p and q are any two odd numbers, (p + q) x (p - q) is always a multiple of 4.

- Do you think your argument counts as proof?
- How confident are you in terms of the validity of your proof?
- How do you know your proof is sufficient?
- Do you think your teacher would agree that your proof is valid?

Extra proof production task

(*Note that not all students received this task due to time constraints*.)

What do you think about the following task?

> "Georgia is asked to prove or disprove that **the sum of any *n* consecutive integers is divisible by *n*.** In order to test whether or not the statement is true, she tries a few examples. She notices that 1 + 2 + 3 is divisible by 3, but 7 + 8 + 9 + 10 is not divisible by 4."

Can you comment on what Georgia now knows as a result of what she noticed?

Ask the following questions based on the student's response.

- Should she try other numbers to prove or disprove the statement?
  - If so, which numbers?
  - If not, why not?
- How would you prove or disprove it?
  - Let's try to prove or disprove it.
- Do you think you proved the statement?
- How confident are you in terms of the validity of your proof?
- How do you know your proof is sufficient?
- Do you think your teacher would agree that your proof is valid?

Appendix-C. Student Interview-2 Protocol

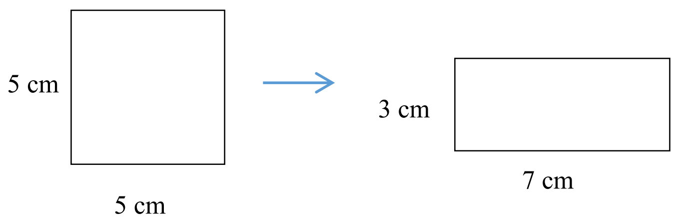
Proof production task

Say you have a square and you add a certain amount to its length and take away that same amount from its width. What happens to the area?

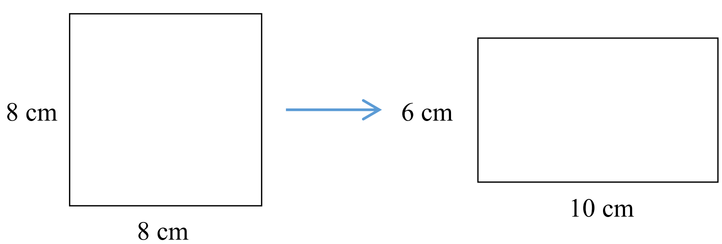
Evaluation of hypothetical student proofs

Ali, Ben, Clara, and Dylan were working on the same task. Here are each student's arguments:

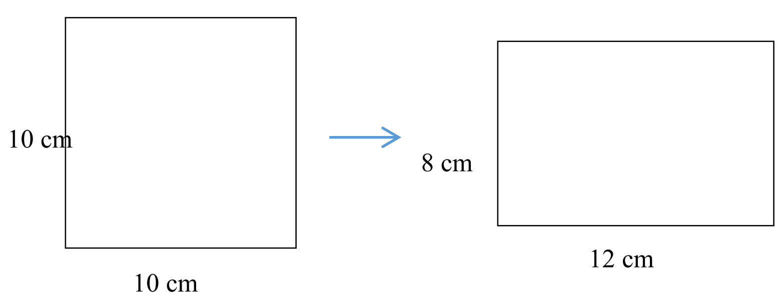
Ali



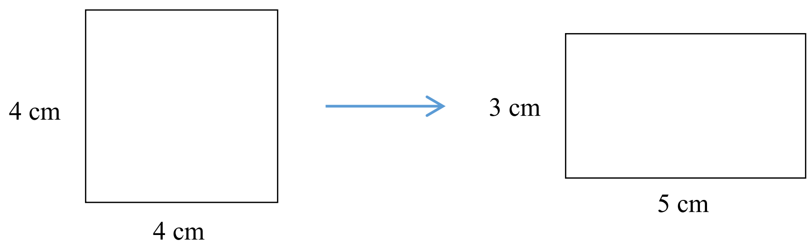
$A_1 = 5 \text{ x } 5 = 25 \text{ cm}^2$ $A_2 = 3 \text{ x } 7 = 21 \text{ cm}^2$ $A_1 - A_2 = 25 – 21 = \mathbf{4 \text{ cm}^2}$



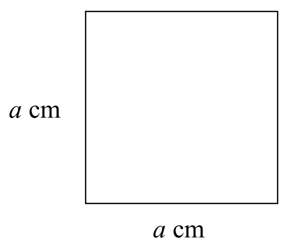
$A_1 = 8 \text{ x } 8 = 64 \text{ cm}^2$ $A_2 = 6 \text{ x } 10 = 60 \text{ cm}^2$ $A_1 - A_2 = 64 – 60 = \mathbf{4 \text{ cm}^2}$



$A_1 = 10 \text{ x } 10 = 100 \text{ cm}^2$ $A_2 = 8 \text{ x } 12 = 96 \text{ cm}^2$ $A_1 – A_2 = 100 - 96 = \mathbf{4 \text{ cm}^2}$

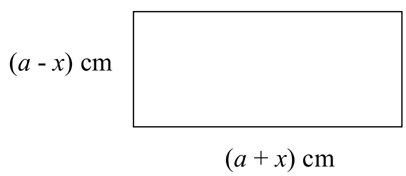
	So, the area decreases by **4 cm²**.
Ben	Say the original square is 4 cm by 4 cm, and you take away 1 cm from the length and add 1 cm to the width. Then, you get a rectangle that is 3 cm by 5 cm.  Original area = 4 x 4 = 16 cm^2, New Area = 3 x 5 = 15 cm^2 → Difference between areas = 16 - 15 = **1 cm^2** Say the original square is 5 cm by 5 cm, and you take away 2 cm from the length and add 2 cm to the width. Then, you get a rectangle that is 3 cm by 7 cm. Original area = 5 x 5 = 25 cm^2, New Area = 3 x 7 = 21 cm^2 → Difference between areas = 25 - 21 = **4 cm^2** Say the original square is 6 cm by 6 cm, and you take away 3 cm from the length and add 3 cm to the width. Then, you get a rectangle that is 3 cm by 9 cm. Original area = 6 x 6 = 36 cm^2, New Area = 3 x 9 = 27 cm^2 → Difference between areas = 36 - 27 = **9 cm^2** Say the original square is 7 cm by 7 cm, and you take away 4 cm from the length and add 4 cm to the width. Then, you get a rectangle that is 3 cm by 11 cm. Original area = 7 x 7 = 49 cm^2, New Area = 3 x 11 = 33 cm^2 → Difference between areas = 49 - 33 = **16 cm^2** Say the original square is 8 cm by 8 cm and you take away 5 cm from the length and add 5 cm to the width. Then, you get a rectangle that is 3 cm by 13 cm. Original area = 8 x 8 = 64 cm^2, New Area = 3 x 13 = 39 cm^2 → Difference between areas = 64 - 39 = **25 cm^2** I noticed a pattern between the amount you take/add and the difference between the areas. When you take/add 1 cm, the area decreases by 1 cm^2; when you take/add 2 cm, the area decreases by 4 cm^2; when you take/add 3 cm, the area decreases by 9 cm^2; when you take/add 4 cm, the area decreases by 16 cm^2, and when you

<table>
<tr><td></td><td>take/add 5 cm, the area decreases by 25 cm^2. So, the area decreases by the square of the amount you add/take away.</td></tr>
<tr><td>Clara</td><td>Say the original square was a cm by a cm. The area of the original square would be a^2 cm^2.



$A_1 = a * a = a^2$ cm^2

If you take x cm from the length and add x cm to the width of the original square, then the new dimensions will be $(a - x)$ cm and $(a + x)$ cm.

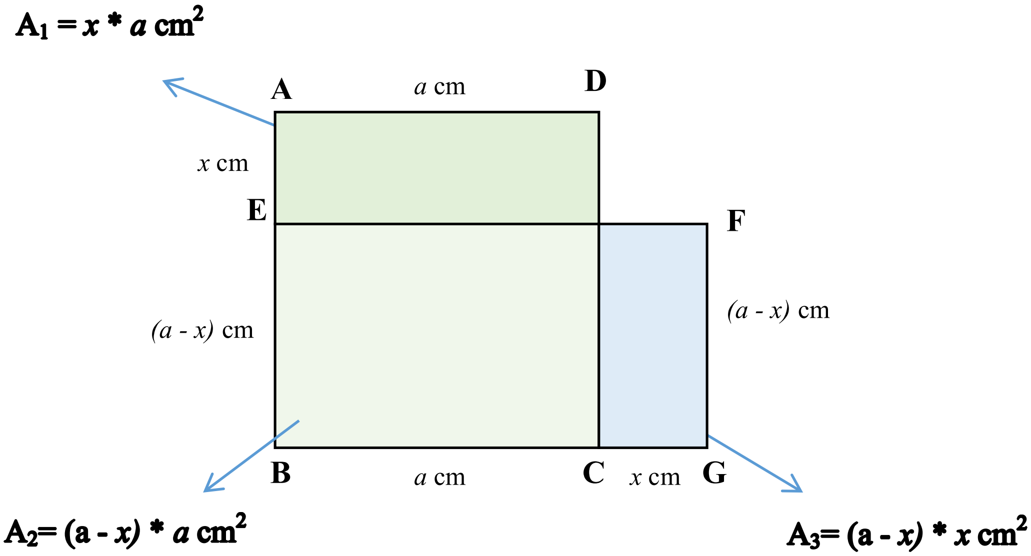


$A_2 = (a - x) * (a + x)$
$A_2 = a^2 - a * x + a * x - x^2$
$A_2 = (a^2 - x^2)$ cm^2

$A_1 - A_2 = a^2 - (a^2 - x^2)$
$= a^2 - a^2 + x^2$
$= x^2$ cm^2

So, the difference between the original square and the new rectangle will always be the square of the amount added/taken. Therefore, the area will always decrease by the square of the amount added/taken away.</td></tr>
<tr><td>Dylan</td><td>Say the original square, ABCD, is a cm by a cm. When you take away x cm from its length and add x cm to its width, you get a rectangle, EBGF, which is $(a - x)$ cm by $(a + x)$ cm, as shown in the figure below.

The original area (the area of ABCD) is the sum of A_1 and A_2, and the new area (the area of EBGF) is the sum of A_2 and A_3.</td></tr>
</table>



Because A_2 is common in both the original and the new area, the difference between A_1 and A_3 determines how the area changes.

$$A_1 - A_3 = x * a - (a - x) * x$$
$$= x * a - a * x + x^2$$
$$= x^2 \text{ cm}^2$$

Therefore, the original area is x^2 cm^2 bigger than the new area. Thus, the area will always decrease by the square of the amount added/taken away.

Ali's answer

	Agree	Disagree	Don't know
Has a **mistake** in it			
Shows that her conjecture is **always true**			
Only shows that her conjecture is true for some cases			
Shows you **why** her conjecture is true			
Is an easy way to **explain** to someone in your class who is unsure			
My math teacher would accept this as a proof			

Ben's answer

	Agree	Disagree	Don't know
Has a **mistake** in it			
Shows that his conjecture is **always true**			
Only shows that his conjecture is true for some cases			
Shows you **why** his conjecture is true			
Is an easy way to **explain** to someone in your class who is unsure			
My math teacher would accept this as a proof			

Clara's answer

	Agree	Disagree	Don't know
Has a **mistake** in it			
Shows that her conjecture is **always true**			
Only shows that her conjecture is true for some cases			
Shows you **why** her conjecture is true			
Is an easy way to **explain** to someone in your class who is unsure			
My math teacher would accept this as a proof			

Dylan's answer

	Agree	Disagree	Don't know
Has a **mistake** in it			
Shows that his conjecture is **always true**			
Only shows that his conjecture is true for some cases			
Shows you **why** his conjecture is true			
Is an easy way to **explain** to someone in your class who is unsure			
My math teacher would accept this as a proof			

Meanings and views of proof and proving

Below is a collection of student statements about proof and proving. Please mark each statement whether you *agree*, *somewhat agree*, or *disagree* with it.

#	Statements about Proof and Proving	Agree	Somewhat agree	Disagree
1	Proving is backing up your statements or claims.			
2	Proving is providing evidence, such as an example, that the statement is true.			
3	When you prove something, you gain a better knowledge of that.			
4	Proof is explaining your thought process.			
5	Proving in math is like a science experiment; you need to test a statement many times to make a claim.			
6	Proof is like a rule, something that is always true.			
7	To disprove something, you need to find at least three counterexamples.			
8	Proving is like checking your work to make sure that it is correct.			
9	Proof shows why something is true or false by showing the reasons behind it.			
10	If proof is accurate, you cannot find an example that would disprove it.			
11	In math, you cannot build your work on assumptions. That's why we prove things in math.			
12	Proof is like showing how you got your answer; explaining your answer.			
13	When proving a mathematical statement, you need to show that it works in all cases.			
14	A proof should include why a statement is true, your reasoning, and an example.			
15	The purpose of proof is to make sure that you made no errors.			
16	Proof is important when you're learning a new concept, but it's not important when you know the concept.			
17	Every equation is a proof, but proof is not limited to equations.			
18	Proving is showing that something is right based on the known facts, rules, definitions, and properties.			
19	You need to try several examples to know that your proof is valid.			
20	We are asked to prove in class so that teachers can see how we got our answers.			

Pick **3 statements** that you think best describes what proof and proving means to you or that are key aspects of what proof and proving means. If you think there are important aspects that are missing in this collection, please write it down.

Evaluation of hypothetical student proofs: Revisiting the algebra task in interview-1

Remember the student arguments that were trying to prove the mathematical statement that, “When you add 2 even numbers, your answer is always even”, from the 1st interview? Here are two additional student arguments. Do you think Sam and/or Abby proved the statement? Why or why not?

Sam’s answer

2 + 4 = 6

60 + 26 = 86

406 + 262 = 668

I tested it with different numbers, both small and large numbers. It worked each time. So, it works for any two even numbers.

So, Sam says it’s true.

	Agree	Disagree	Don’t know
Has a **mistake** in it			
Shows that the statement is **always true**			
Only shows that the statement is true for some even numbers			
Shows you **why** the statement is true			
Is an easy way to **explain** to someone in your class who is unsure			
My math teacher would accept this as a proof.			

Abby's answer

Say x and y are any two even numbers. By definition, x and y can be represented as follows:

$x = 2 \bullet a$ (a is any whole number)
$y = 2 \bullet b$ (b is any whole number)

Then, $x + y = 2 \bullet a + 2 \bullet b = 2 \bullet (\mathrm{a} + b)$

Because the sum of x and y has a factor of 2, it is divisible by 2. Therefore, the sum of any two even numbers is always an even number.

So, Abby says it's true.

	Agree	Disagree	Don't know
Has a **mistake** in it			
Shows that the statement is **always true**			
Only shows that the statement is true for some even numbers			
Shows you **why** the statement is true			
Is an easy way to **explain** to someone in your class who is unsure			
My math teacher would accept this as a proof.			

Appendix-D. Teacher Interview-2 Protocol

Introduction

In our first interview, we talked about the course as well as what you think about proof and proving. Today, I would like to continue our conversation around proof and proving, and this time I would like to share with you some of the tasks that I gave to the students, and I am eager to know what you think about them, especially in relation to your students.

Proof production task: Algebra task

First, I will give you a conjecture and ask you to prove it the way you would do in class, and tell me why you would prove it that way.

> "When you add any two even numbers, your answer is always even."

Evaluation of hypothetical student proofs

I provided the students with several hypothetical student arguments for the proof of this conjecture and asked them to evaluate those arguments. I would like you to evaluate those student arguments and for each argument tell me whether you think it proves the conjecture or not, and why.

<table>
<tr><td colspan="2">Arthur, Bonnie, Ceri, Duncan, Eric, and Yvonne were trying to prove whether the following statement is true or false:

When you add any two even numbers, your answer is always even.</td></tr>
<tr><td>Arthur's answer

a is any whole number
b is any whole number
$2a$ and $2b$ are any two even numbers
$2a + 2b = 2(a + b)$

So, Arthur says it's true.</td><td>Bonnie's answer

2 + 2 = 4   4 + 2 = 6

2 + 4 = 6   4 + 4 = 8

2 + 6 = 8   4 + 6 = 10

So, Bonnie says it's true.</td></tr>
<tr><td>Ceri's answer

Even numbers are numbers that can be divided by 2. When you add numbers with a common factor, 2 in this case, the answer will have the same common factor.

So, Ceri says it's true.</td><td>Duncan's answer

Even numbers end in 0, 2, 4, 6, or 8. When you add any two of these, the answer will still end in 0, 2, 4, 6, or 8.

So, Duncan says it's true.</td></tr>
</table>

Eric's answer Let x = any whole number y = any whole number $x + y = z$ $z - x = y$ $z - y = x$ $z + z - (x + y) = x + y = 2z$ *So, Eric says it's true.*	*Yvonne's answer*  *So, Yvonne says it's true.*
Sam's answer 2 + 4 = 6 60 + 26 = 86 406 + 262 = 668 I tested it with different numbers, both small and large numbers. It worked each time. So, it works for any two even numbers. *So, Sam says it's true.*	*Abby's answer* Say x and y are any two even numbers. By definition, x and y can be represented as follows: $x = 2 \bullet a$ (a is any whole number) $y = 2 \bullet b$ (b is any whole number) Then, $x + y = 2 \bullet a + 2 \bullet b = 2 \bullet (a + b)$ Because the sum of x and y has a factor of 2, it is divisible by 2. Therefore, the sum of any two even numbers is always an even number. *So, Abby says it's true.*

- How would you evaluate these arguments in terms of whether the argument shows that the conjecture is always true, whether it only shows that the conjecture is true for some cases, whether it shows why the conjecture is true, and whether it is an easy way to explain?
- Which one is your favorite proof? Why?
- What arguments do you think your students might have picked as proof? Why?

Proof production task: Geometry task

I would like you to explore the following problem and then prove your conjecture the way you would do in class, and tell me why you would prove it that way.

> "Say you have a square and you add a certain amount to its length and take away that same amount from its width. What happens to the area?"

Evaluation of hypothetical student proofs

Here are a few hypothetical student proofs that I provided to the students. Again, I would like you to evaluate these student arguments and tell me whether you think they are valid proofs or not, and why. *(The teacher is provided with the hypothetical student proofs—Ali, Ben, Clara, and Dylan—that were given to the students.)*

- How would you evaluate these arguments in terms of whether the argument shows that the conjecture is always true, whether it only shows that the conjecture is true for some cases, whether it shows why the conjecture is true, and whether it is an easy way to explain?
- Which one is your favorite proof? Why?
- What arguments do you think your students might have picked as proof? Why?

Views about proof and proving

Below is a collection of student statements about proof and proving. Please mark each statement whether you *agree*, *somewhat agree*, or *disagree* with it. (*The teacher is provided with the same list of proof statements that was given to the students*.)

Other clarification questions

- When I was in your class, I often heard you asking students to "verify", "justify", "explain their reasoning", and "prove". I'm curious to know what these terms mean to you. Can you tell me what you expect students to do when you ask them to "verify", "justify", "explain their reasoning", and "prove"? Are they basically the same thing or do you see them differently?
    - Can you think of a case where explaining reasoning would not be considered as proof?

- I have also heard you saying "formally proving" and "formal proofs". What do you refer to when you say formal proofs? And how different are they from proofs in general?

- Do you have any criteria for student explanations to be satisfactory or sufficient for proving?

Appendix-E. List of Potential Classroom Influences

Roles and purposes of proof:

- A justification is given to *verify* a claim.
- A justification is given to *explain* a claim.

Meanings of proof:

- Proving is showing something is true based on known facts, rules, and definitions.
- Proof is evidence.
- Proof is backing up.
    - Proof requires warrants. (What kinds of warrants are acceptable?)
- Proof is justifying one's reasoning or claims.
- Proof is checking your work to make sure that it is correct.
- Proof is like showing how you got your answer; explaining your answer.
- Proof is explaining your thought process.

Forms of arguments:

- Narrative arguments are valued as valid justification/proof.
- Visual representations are valued and students are encouraged to use them.
- Deductive arguments are valued and the students are expected of constructing deductive arguments.
- Empirical arguments are accepted as a proof.

Related to proof understandings:

- Assumptions without warrants are not acceptable.
- Making claims based on appearance is not acceptable.
- Examples are insufficient for proof.
- One counterexample is sufficient for disproving.
- Crucial experiment is more convincing.
- Proof shows the truth for all cases.
- Proof explains why a conjecture/statement is true.
- An example is used to illustrate/communicate one's argument.
- An example is used to verify the accuracy of one's argument.
- An example is used to check/verify one's work/solution/answer

Proof-related activities:

- Students are asked to make a conjecture.
- Students are asked to verify their conjectures.
- Students are asked to make a generalization.
- Students are asked to algebraically represent mathematical statements.
- Students are asked to explain why something is true.

List of Potential Manifestations of the Teacher's Conceptions of Proof in her Instruction

- Teacher values proof and proving.

- Teacher requests a warrant/backing. (List the questions and classify the expected type of warrant.)
    - How do you know that?
    - How do you know that is true?
    - Can you explain that?
    - How do you know that is the answer?

- Teacher provides a warrant.
    - Definitions
    - Known facts/properties
    - Logical inferences
    - Theorems
    - Empirical evidence

- Teacher requests for a justification. (Is the expectation a deductive answer justifying the claim or just describing one's steps to solve the problem?)

- Teacher provides a justification.