TRIDIAGONAL PAIRS, DOUBLE LOWERING OPERATORS, AND $U_q(\mathfrak{sl}_2)$

By

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Abstract

This thesis is about a linear algebraic object called a tridiagonal pair. The concept of a tridiagonal pair was introduced in [10] and is defined as follows. Let $\mathbb{K}$ denote a field and let V denote a vector space over $\mathbb{K}$ with finite positive dimension. By a *tridiagonal pair* on V, we mean an ordered pair of linear transformations $A : V \to V$ and $A^* : V \to V$ that satisfy the following four conditions: (i) Each of A, A^* is diagonalizable; (ii) there exists an ordering $\{V_i\}_{i=0}^d$ of the eigenspaces of A such that $A^* V_i \subseteq V_{i-1} + V_i + V_{i+1}$ for $0 \leq i \leq d$, where $V_{-1} = 0$ and $V_{d+1} = 0$; (iii) there exists an ordering $\{V_i^*\}_{i=0}^\delta$ of the eigenspaces of A^* such that $AV_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^*$ for $0 \leq i \leq \delta$, where $V_{-1}^* = 0$ and $V_{\delta+1}^* = 0$; (iv) there does not exist a subspace W of V such that $AW \subseteq W$, $A^*W \subseteq W$, $W \neq 0$, $W \neq V$.

Let A, A^* denote a tridiagonal pair on V as above. It is known that the integers d and δ from (ii) and (iii) are equal; we call this common value the *diameter* of A, A^*. To avoid trivialities, assume that the diameter is at least one.

This thesis is divided into two parts. The first part of this thesis is about two commuting linear transformations associated with A, A^*. The second part of this thesis explores a connection between A, A^* and the quantum enveloping algebra $U_q(\mathfrak{sl}_2)$. We now summarize our results for each part.

In the first part of this thesis, we show that there exists a unique linear transformation $\Delta : V \to V$ such that $(\Delta - I)V_i^* \subseteq V_0^* + V_1^* + \cdots + V_{i-1}^*$ and $\Delta(V_i + V_{i+1} + \cdots + V_d) \subseteq V_0 + V_1 + \cdots + V_{d-i}$ for $0 \leq i \leq d$. We consider two well-known decompositions of V called the first and second split decomposition. They are denoted $\{U_i\}_{i=0}^d$ and

$\{U_i^{\Downarrow}\}_{i=0}^d$ respectively. We show that $\Delta U_i = U_i^{\Downarrow}$, $(\Delta - I)U_i \subseteq U_0 + U_1 + \cdots + U_{i-1}$, and $(\Delta - I)U_i^{\Downarrow} \subseteq U_0^{\Downarrow} + U_1^{\Downarrow} + \cdots + U_{i-1}^{\Downarrow}$ for $0 \leq i \leq d$. We introduce a second linear transformation $\Psi : V \to V$; one feature of Ψ is that $\Psi U_i \subseteq U_{i-1}$ and $\Psi U_i^{\Downarrow} \subseteq U_{i-1}^{\Downarrow}$ for $1 \leq i \leq d$ and both $\Psi U_0 = 0$, $\Psi U_0^{\Downarrow} = 0$. We describe Δ, Ψ from several points of view and show how they are related to each other. Along this line we have two main results. Our first main result is that Δ, Ψ commute. In the literature on tridiagonal pairs, there is a scalar β used to describe the eigenvalues. Our second main result is that each of $\Delta^{\pm 1}$ is a polynomial of degree d in Ψ, under a minor assumption on β.

In the second part of this thesis, we assume that A, A^* has q-Racah type. This is the most general type of tridiagonal pair. For simplicity, we also assume that $\mathbb{K}$ is algebraically closed. We define linear transformations $K : V \to V$ and $B : V \to V$ such that $(K - q^{d-2i}I)U_i = 0$ and $(B - q^{d-2i}I)U_i^{\Downarrow} = 0$ for $0 \leq i \leq d$. Our results are summarized as follows. Recall the quantum enveloping algebra $U_q(\mathfrak{sl}_2)$ with Chevalley generators e, f, k, k^{-1}. We obtain two $U_q(\mathfrak{sl}_2)$-module structures on V. For the first $U_q(\mathfrak{sl}_2)$-module structure, the generator k acts as K and the generator e acts as a nonzero scalar multiple of Ψ. For the second, the generator k acts as B and the generator e acts as the same scalar multiple of Ψ. In each case, we express the action of f in terms of A. For each of the $U_q(\mathfrak{sl}_2)$-module structures, we compute the action of the Casimir element on V. We show that these two actions agree. Using this fact, we express Ψ as a rational function of $K^{\pm 1}, B^{\pm 1}$ in several ways. Eliminating Ψ from these equations we find that K and B are related by a quadratic equation.

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Contents

Chapter 1

Introduction

This thesis is about a linear algebraic object called a tridiagonal pair. Before summarizing our results, we recall the definition of a tridiagonal pair.

Throughout this thesis, let $\mathbb{K}$ denote a field and let V denote a vector space over $\mathbb{K}$ with finite positive dimension. For a linear transformation $A : V \to V$ and a subspace $W \subseteq V$, we say that W is an *eigenspace* of A whenever $W \neq 0$ and there exists $\theta \in \mathbb{K}$ such that $W = \{v \in V | Av = \theta v\}$. In this case, θ is called the *eigenvalue* of A associated with W. We say that A is *diagonalizable* whenever V is spanned by the eigenspaces of A.

We now recall the notion of a tridiagonal pair.

Definition 1.0.1. [10, Definition 1.1]. By a *tridiagonal pair* (or *TD pair*) on V we mean an ordered pair of linear transformations $A : V \to V$ and $A^* : V \to V$ that satisfy the following four conditions.

(i) Each of A, A^* is diagonalizable.

(ii) There exists an ordering $\{V_i\}_{i=0}^{d}$ of the eigenspaces of A such that

$$A^* V_i \subseteq V_{i-1} + V_i + V_{i+1} \qquad (0 \leq i \leq d), \tag{1.1}$$

where $V_{-1} = 0$ and $V_{d+1} = 0$.

(iii) There exists an ordering $\{V_i^*\}_{i=0}^{\delta}$ of the eigenspaces of A^* such that

$$AV_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^* \qquad (0 \leq i \leq \delta), \tag{1.2}$$

where $V_{-1}^* = 0$ and $V_{\delta+1}^* = 0$.

(iv) There does not exist a subspace W of V such that $AW \subseteq W$, $A^*W \subseteq W$, $W \neq 0$, $W \neq V$.

We say the TD pair A, A^* is *over* $\mathbb{K}$.

Note 1.0.2. According to a common notational convention A^* denotes the conjugate-transpose of A. We are not using this convention. In a TD pair A, A^* the linear transformations A and A^* are arbitrary subject to (i)–(iv) above.

Referring to the TD pair A, A^* in Definition 1.0.1, by [10, Lemma 4.5] the integers d and δ are equal. We call this common value the *diameter* of A, A^*. To avoid trivialities, throughout this thesis we assume that the diameter is at least one.

In Definition 1.0.1 we do not assume that the spaces $\{V_i\}_{i=0}^{d}, \{V_i^*\}_{i=0}^{d}$ all have dimension one. A TD pair for which each of these spaces has dimension one is called a *Leonard pair* [33]. A TD pair for which each of V_0, V_0^*, V_d, V_d^* has dimension one is said to be *sharp* [9]. Note that all Leonard pairs are sharp.

We comment on the history of the TD pair concept. This concept originated in the theory of Q-polynomial distance-regular graphs [32] and was formally introduced in [10]. Ever since, researchers have been exploring various ramifications and connections. For example, in [10, 37], the eigenvalues of a TD pair were investigated. The paper [11] concerns the shape of a TD pair. In [10, 25, 38], the split decompositions were discussed. The Drinfel'd polynomial of a TD pair was examined in [16]. The papers [33, 35, 37, 38, 39] are about Leonard pairs. These papers led to several characterizations of Leonard pairs involving orthogonal polynomials [36, 37], the Lie algebra $\mathfrak{sl}_2$ [8], parameter arrays [33], upper/lower bidiagonal matrices [37, 39], and tridiagonal/diagonal matrices [39]. The papers [9, 14, 15, 18, 26, 28, 29, 42] are about sharp TD pairs. These papers ultimately led to the classification of sharp TD pairs up to isomorphism [9, Theorem 3.1]. As a corollary, the TD pairs over an algebraically closed field were classified up to isomorphism [9, Corollary 18.1]. In [19] it is shown how to "sharpen" a TD pair.

We mention some notable connections between TD pairs and other areas of mathematics and physics. Connections have been found between TD pairs and representation theory [1, 7, 12, 13, 17, 22, 23, 41], orthogonal polynomials [37, 40], partially ordered sets [35], statistical mechanical models [2, 6, 31], and other areas of physics [24, 30]. Among the above papers on representation theory, there are several works that connect TD pairs to quantum groups [1, 7, 13, 17]. These papers consider certain special classes of TD pairs. In [1], Curtin and Al-Najjar considered the class of mild TD pairs of q-Serre type. They showed that these TD pairs induce an action of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$ on the underlying vector space. In [7], Funk-Neubauer extended this construction to TD pairs of q-Hahn type. In [13], Ito and Terwilliger extended the construction to the entire q-Serre class. In [17], Ito and Terwilliger extended the construction to the q-Racah class.

We now summarize the contents of this thesis. This thesis is divided into two parts. Below we give a brief overview of what each part is about. Later in the Introduction, we give a more detailed summary.

The first part of this thesis is about two commuting linear transformations associated with a TD pair. Given a TD pair A, A^* on V, we introduce two linear transformations

$\Delta : V \to V$ and $\Psi : V \to V$ that we find attractive. We discuss how Δ, Ψ act on the first and second split decomposition of V. We describe Δ, Ψ from several points of view and show how they are related to each other. Along this line we have two main results. Our first main result is that Δ, Ψ commute. In the literature on tridiagonal pairs, there is a scalar β used to describe the eigenvalues. Our second main result is that each of $\Delta^{\pm 1}$ is a polynomial of degree d in Ψ, under a minor assumption on β.

The second part of this thesis explores a connection between TD pairs and the quantum enveloping algebra $U_q(\mathfrak{sl}_2)$. In this part, we focus on TD pairs of q-Racah type. For simplicity, we also assume that $\mathbb{K}$ is algebraically closed. We define two linear transformations $K : V \to V$ and $B : V \to V$ which act on the split decompositions in an attractive way. Using Ψ, K, B we obtain two $U_q(\mathfrak{sl}_2)$-module structures on V. For each of the $U_q(\mathfrak{sl}_2)$-module structures, we compute the action of the Casimir element on V. We show that these two actions agree. Using this fact, we express Ψ as a rational function of $K^{\pm 1}, B^{\pm 1}$ in several ways. Eliminating Ψ from these equations we find that K and B are related by a quadratic equation.

We now describe our main results in more detail. To set the stage, we give some additional background on TD pairs. For the rest of the Introduction, let A, A^* denote a TD pair on V. We fix an ordering $\{V_i\}_{i=0}^d$ (resp. $\{V_i^*\}_{i=0}^d$) of the eigenspaces of A (resp. A^*) which satisfies (1.1) (resp. (1.2)). For $0 \leq i \leq d$ let θ_i (resp. θ_i^*) denote the eigenvalue of A (resp. A^*) corresponding to V_i (resp. V_i^*). By [10, Theorem 11.1] the ratios

$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}, \qquad \frac{\theta^*_{i-2} - \theta^*_{i+1}}{\theta^*_{i-1} - \theta^*_i} \tag{1.3}$$

are equal and independent of i for $2 \leq i \leq d-1$. This gives two recurrence relations whose solutions can be written in closed form. There are several cases [10, Theorem

11.2]. The most general case is called the q-Racah case [17, Section 1]. For convenience, let $\beta+1$ denote the common value of (1.3). In the q-Racah case, there exists a nonzero scalar q in the algebraic closure $\overline{\mathbb{K}}$ such that $q^4 \neq 1$ and $\beta = q^2 + q^{-2}$.

We now recall the split decompositions of V [10]. For $0 \leq i \leq d$ define

$$U_i = (V_0^* + V_1^* + \cdots + V_i^*) \cap (V_i + V_{i+1} + \cdots + V_d),$$
$$U_i^{\Downarrow} = (V_0^* + V_1^* + \cdots + V_i^*) \cap (V_0 + V_1 + \cdots + V_{d-i}).$$

By [10, Theorem 4.6], both the sums $V = \sum_{i=0}^d U_i$ and $V = \sum_{i=0}^d U_i^{\Downarrow}$ are direct. We call $\{U_i\}_{i=0}^d$ (resp. $\{U_i^{\Downarrow}\}_{i=0}^d$) the first split decomposition (resp. second split decomposition) of V. The split decompositions are of interest to us because A, A^* act on them in the following attractive manner. By [10, Theorem 4.6], A, A^* act on the first split decomposition in the following way:

$$(A - \theta_i I)U_i \subseteq U_{i+1} \qquad (0 \leq i \leq d-1), \qquad (A - \theta_d I)U_d = 0,$$
$$(A^* - \theta_i^* I)U_i \subseteq U_{i-1} \qquad (1 \leq i \leq d), \qquad (A^* - \theta_0^* I)U_0 = 0.$$

By [10, Theorem 4.6], A, A^* act on the second split decomposition in the following way:

$$(A - \theta_{d-i} I)U_i^{\Downarrow} \subseteq U_{i+1}^{\Downarrow} \qquad (0 \leq i \leq d-1), \qquad (A - \theta_0 I)U_d^{\Downarrow} = 0,$$
$$(A^* - \theta_i^* I)U_i^{\Downarrow} \subseteq U_{i-1}^{\Downarrow} \qquad (1 \leq i \leq d), \qquad (A^* - \theta_0^* I)U_0^{\Downarrow} = 0.$$

We now recall the raising maps $R, R^{\Downarrow}$ [10, Definition 6.1]. Let $R : V \to V$ denote the linear transformation that acts on U_i as $A - \theta_i I$ for $0 \leq i \leq d$. Let $R^{\Downarrow} : V \to V$ denote the linear transformation that acts on $U_i^{\Downarrow}$ as $A - \theta_{d-i} I$ for $0 \leq i \leq d$. By construction,

$$RU_i \subseteq U_{i+1}, \qquad R^{\Downarrow}U_i^{\Downarrow} \subseteq U_{i+1}^{\Downarrow}$$

for $0 \leq i \leq d-1$ and both $RU_d = 0$ and $R^{\Downarrow}U^{\Downarrow}_d = 0$. We refer to R (resp. $R^{\Downarrow}$) as the *first* (resp. *second*) *raising map*.

We now discuss the main results from the first part of this thesis. We show that there exists a unique linear transformation $\Delta : V \to V$ such that both

$$(\Delta - I)V^*_i \subseteq V^*_0 + V^*_1 + \cdots + V^*_{i-1}, \tag{1.4}$$

$$\Delta(V_i + V_{i+1} + \cdots + V_d) \subseteq V_0 + V_1 + \cdots + V_{d-i} \tag{1.5}$$

for $0 \leq i \leq d$. We also show that there exists a unique linear transformation $\Psi : V \to V$ such that both

$$\Psi V_i \subseteq V_{i-1} + V_i + V_{i+1}, \tag{1.6}$$

$$\left(\Psi - \frac{\Delta - I}{\theta_0 - \theta_d}\right) V^*_i \subseteq V^*_0 + V^*_1 + \cdots + V^*_{i-2} \tag{1.7}$$

for $0 \leq i \leq d$. By construction,

$$\Psi V^*_i \subseteq V^*_0 + V^*_1 + \cdots + V^*_{i-1} \qquad (0 \leq i \leq d). \tag{1.8}$$

We investigate the actions of Δ, Ψ on the split decompositions of V. We show that

$$\Delta U_i = U^{\Downarrow}_i, \tag{1.9}$$

$$(\Delta - I)U_i \subseteq U_0 + U_1 + \cdots + U_{i-1}, \tag{1.10}$$

$$(\Delta - I)U^{\Downarrow}_i \subseteq U^{\Downarrow}_0 + U^{\Downarrow}_1 + \cdots + U^{\Downarrow}_{i-1} \tag{1.11}$$

for $0 \leq i \leq d$. We also show that

$$\Psi U_i \subseteq U_{i-1}, \qquad \Psi U^{\Downarrow}_i \subseteq U^{\Downarrow}_{i-1} \tag{1.12}$$

for $1 \leq i \leq d$ and both $\Psi U_0 = 0$, $\Psi U^{\Downarrow}_0 = 0$. In light of this, we refer to Ψ as the *double lowering map*.

Let $0 \leq i \leq d$. We show that on U_i,

$$\Psi R - R\Psi = \frac{\theta_i - \theta_{d-i}}{\theta_0 - \theta_d} I. \tag{1.13}$$

We show that on $U_i^{\Downarrow}$,

$$\Psi R^{\Downarrow} - R^{\Downarrow}\Psi = \frac{\theta_i - \theta_{d-i}}{\theta_0 - \theta_d} I. \tag{1.14}$$

We now discuss how Δ, Ψ are related to each other. Along this line we have two main results. Our first main result is that Δ, Ψ commute. In order to state the second result, we define

$$\vartheta_i = \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} \qquad\qquad (1 \leq i \leq d).$$

Our second main result is that both

$$\Delta = I + \frac{\eta_1(\theta_0)}{\vartheta_1}\Psi + \frac{\eta_2(\theta_0)}{\vartheta_1\vartheta_2}\Psi^2 + \cdots + \frac{\eta_d(\theta_0)}{\vartheta_1\vartheta_2\cdots\vartheta_d}\Psi^d, \tag{1.15}$$

$$\Delta^{-1} = I + \frac{\tau_1(\theta_d)}{\vartheta_1}\Psi + \frac{\tau_2(\theta_d)}{\vartheta_1\vartheta_2}\Psi^2 + \cdots + \frac{\tau_d(\theta_d)}{\vartheta_1\vartheta_2\cdots\vartheta_d}\Psi^d \tag{1.16}$$

provided that each of $\vartheta_1, \vartheta_2, \ldots, \vartheta_d$ is nonzero. Here τ_i, η_i are the polynomials

$$\tau_i = (x - \theta_0)(x - \theta_1)\cdots(x - \theta_{i-1}),$$

$$\eta_i = (x - \theta_d)(x - \theta_{d-1})\cdots(x - \theta_{d-i+1})$$

for $0 \leq i \leq d$. We show that each of $\vartheta_1, \vartheta_2, \ldots, \vartheta_d$ is nonzero if and only if neither of the following holds: (i) $\beta = -2$, d is odd, and $\text{Char}(\mathbb{K}) \neq 2$; (ii) $\beta = 0$, $d = 3$, and $\text{Char}(\mathbb{K}) = 2$. In particular, (1.15), (1.16) hold whenever A, A^* has q-Racah type.

We now discuss the main results of the second part of this thesis. We focus on TD pairs of q-Racah type. As we will see in Section 3.1, for these TD pairs, there exist nonzero scalars $a, b \in \overline{\mathbb{K}}$ such that

$$\theta_i = aq^{d-2i} + a^{-1}q^{2i-d}, \qquad\qquad \theta^*_i = bq^{d-2i} + b^{-1}q^{2i-d}$$

for $0 \leq i \leq d$. For the rest of the Introduction, assume that A, A^* has q-Racah type. For simplicity, we also assume that $\mathbb{K}$ is algebraically closed.

Motivated by [18, Section 1.1], we define some maps K, B as follows. Let $K : V \to V$ and $B : V \to V$ denote the linear transformations such that $(K - q^{d-2i}I)U_i = 0$ and $(B - q^{d-2i}I)U_i^{\Downarrow} = 0$ for $0 \leq i \leq d$. In [18, Section 1.1] it is shown how each of K, B is related to A and A^*, but the relationship between K and B is not discussed. One of our results describes how K, B are related. We show that

$$aK^2 - \frac{a^{-1}q - aq^{-1}}{q - q^{-1}}\, KB - \frac{aq - a^{-1}q^{-1}}{q - q^{-1}}\, BK + a^{-1}B^2 = 0. \qquad (1.17)$$

In order to state the remaining results concisely, we work with the normalization $\psi = (q - q^{-1})(q^d - q^{-d})\Psi$. Drawing on the results in the first part of this thesis, we obtain some equations that link ψ to the maps $K, B, R, R^{\Downarrow}$. From these equations we obtain two $U_q(\mathfrak{sl}_2)$-module structures on V. For the first $U_q(\mathfrak{sl}_2)$-module structure, the Chevalley generators e, f, k, k^{-1} act as follows:

element of $U_q(\mathfrak{sl}_2)$	e	f	k	k^{-1}
action on V	$(q - q^{-1})^{-1}\psi$	$(q - q^{-1})^{-1}R$	K	K^{-1}

For the second $U_q(\mathfrak{sl}_2)$-module structure, the Chevalley generators act as follows:

element of $U_q(\mathfrak{sl}_2)$	e	f	k	k^{-1}
action on V	$(q - q^{-1})^{-1}\psi$	$(q - q^{-1})^{-1}R^{\Downarrow}$	B	B^{-1}

For each of the above $U_q(\mathfrak{sl}_2)$-module structures we obtain a direct sum decomposition of V into irreducible $U_q(\mathfrak{sl}_2)$-submodules. In each case, we compute the action of the Casimir element on V. We show that these two actions agree. Using this information we show that ψ is equal to each of the following:

$$\frac{I - BK^{-1}}{q(aI - a^{-1}BK^{-1})}, \qquad \frac{I - KB^{-1}}{q(a^{-1}I - aKB^{-1})}, \tag{1.18}$$

$$\frac{q(I - K^{-1}B)}{aI - a^{-1}K^{-1}B}, \qquad \frac{q(I - B^{-1}K)}{a^{-1}I - aB^{-1}K}. \tag{1.19}$$

Line (1.17) is a consequence of the fact that the four expressions in (1.18), (1.19) are equal.

This thesis is based on [3, 4].

Chapter 2

The operators Δ and Ψ

This part of the thesis is about two commuting linear transformations associated with a TD pair. Given a TD pair A, A^* on V, we introduce two linear transformations $\Delta : V \to V$ and $\Psi : V \to V$ that we find attractive. We discuss how Δ, Ψ act on the first and second split decomposition of V. We describe Δ, Ψ from several points of view and show how they are related to each other. Along this line we have two main results. Our first main result is that Δ, Ψ commute. In the literature on tridiagonal pairs, there is a scalar β used to describe the eigenvalues. Our second main result is that each of $\Delta^{\pm 1}$ is a polynomial of degree d in Ψ, under a minor assumption on β.

2.1 Preliminaries

When working with a tridiagonal pair, it is useful to consider a closely related object called a tridiagonal system. In order to define this, we first recall some facts from elementary linear algebra.

Let V denote a vector space over $\mathbb{K}$ with finite positive dimension. Let $\mathrm{End}(V)$ denote the $\mathbb{K}$-algebra consisting of all linear transformations from V to V. Let A denote a diagonalizable element in $\mathrm{End}(V)$. Let $\{V_i\}_{i=0}^d$ denote an ordering of the eigenspaces of A. For $0 \leq i \leq d$ let θ_i be the eigenvalue of A corresponding to V_i. Define $E_i \in \mathrm{End}(V)$

by

$$(E_i - I)V_i = 0, \tag{2.1}$$

$$E_iV_j = 0 \quad \text{if} \quad j \neq i, \qquad (0 \leq j \leq d). \tag{2.2}$$

In other words, E_i is the projection map from V onto V_i. We refer to E_i as the *primitive idempotent* of A associated with θ_i. By elementary linear algebra, we have

$$AE_i = E_iA = \theta_iE_i \qquad (0 \leq i \leq d), \tag{2.3}$$

$$E_iE_j = \delta_{ij}E_i \qquad (0 \leq i, j \leq d), \tag{2.4}$$

$$V_i = E_iV \qquad (0 \leq i \leq d), \tag{2.5}$$

$$I = \sum_{i=0}^{d} E_i. \tag{2.6}$$

One readily checks that

$$E_i = \prod_{\substack{0 \leq j \leq d \\ j \neq i}} \frac{A - \theta_j I}{\theta_i - \theta_j} \qquad (0 \leq i \leq d).$$

Let M denote the $\mathbb{K}$-subalgebra of $\text{End}(V)$ generated by A. We note that each of $\{A^i\}_{i=0}^d$, $\{E_i\}_{i=0}^d$ is a basis for the $\mathbb{K}$-vector space M.

Given a TD pair A, A^* on V, an ordering of the eigenspaces of A (resp. A^*) is said to be *standard* whenever (1.1) (resp. (1.2)) holds. Let $\{V_i\}_{i=0}^d$ denote a standard ordering of the eigenspaces of A. By [10, Lemma 2.4], the ordering $\{V_{d-i}\}_{i=0}^d$ is standard and no further ordering is standard. A similar result holds for the eigenspaces of A^*. An ordering of the primitive idempotents of A (resp. A^*) is said to be *standard* whenever the corresponding ordering of the eigenspaces of A (resp. A^*) is standard.

Definition 2.1.1. [28, Definition 2.1] Let V denote a vector space over $\mathbb{K}$ with finite positive dimension. By a *tridiagonal system* (or *TD system*) on V, we mean a sequence

$$\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$$

that satisfies (i)–(iii) below.

(i) A, A^* is a tridiagonal pair on V.

(ii) $\{E_i\}_{i=0}^d$ is a standard ordering of the primitive idempotents of A.

(iii) $\{E_i^*\}_{i=0}^d$ is a standard ordering of the primitive idempotents of A^*.

We call d the *diameter* of Φ, and say Φ is *over* $\mathbb{K}$. For notational convenience, set $E_{-1} = 0$, $E_{d+1} = 0$, $E_{-1}^* = 0$, $E_{d+1}^* = 0$.

For the rest of the present paper, we fix a TD system Φ as in Definition 2.1.1.

Definition 2.1.2. For $0 \le i \le d$ let θ_i (resp. θ_i^*) denote the eigenvalue of A (resp. A^*) associated with E_i (resp. E_i^*). We refer to $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^*\}_{i=0}^d$) as the *eigenvalue sequence* (resp. *dual eigenvalue sequence*) of Φ.

A given TD system can be modified in a number of ways to get a new TD system. For example, given the TD system Φ in Definition 2.1.1, the sequence

$$\Phi^{\Downarrow} = (A; \{E_{d-i}\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$$

is a TD system on V. Following [10, Section 3], we call $\Phi^{\Downarrow}$ the *second inversion* of Φ. When discussing $\Phi^{\Downarrow}$, we use the following notational convention. For any object f associated with Φ we let $f^{\Downarrow}$ denote the corresponding object for $\Phi^{\Downarrow}$.

For later use, we associate with Φ two families of polynomials as follows. Let x be an indeterminate. Let $\mathbb{K}[x]$ denote the $\mathbb{K}$-algebra consisting of the polynomials in x that have all coefficients in $\mathbb{K}$. For $0 \leq i \leq j \leq d+1$, we define the polynomials $\tau_{ij} = \tau_{ij}(\Phi)$, $\eta_{ij} = \eta_{ij}(\Phi)$ in $\mathbb{K}[x]$ by

$$\tau_{ij} = (x - \theta_i)(x - \theta_{i+1}) \cdots (x - \theta_{j-1}), \tag{2.7}$$

$$\eta_{ij} = (x - \theta_{d-i})(x - \theta_{d-i-1}) \cdots (x - \theta_{d-j+1}). \tag{2.8}$$

We interpret $\tau_{i,i-1} = 0$ and $\eta_{i,i-1} = 0$. Note that each of τ_{ij}, η_{ij} is monic with degree $j - i$. In particular, $\tau_{ii} = 1$ and $\eta_{ii} = 1$. We remark that $\tau_{ij}^{\Downarrow} = \eta_{ij}$ and $\eta_{ij}^{\Downarrow} = \tau_{ij}$.

Observe that for $0 \leq i \leq j \leq k \leq d+1$,

$$\tau_{ij}\tau_{jk} = \tau_{ik}, \qquad \eta_{ij}\eta_{jk} = \eta_{ik}. \tag{2.9}$$

As we proceed through the paper, we will focus on τ_{ij}. We will develop a number of results concerning τ_{ij}. Similar results hold for η_{ij}, although we will not state them explicitly.

Lemma 2.1.3. *For $0 \leq i \leq j \leq d+1$, the kernel of $\tau_{ij}(A)$ is*

$$E_iV + E_{i+1}V + \cdots + E_{j-1}V.$$

Proof. For $0 \leq h \leq d$, E_hV is the eigenspace of A corresponding to θ_h. The result follows from this and (2.7). ∎

For $0 \leq j \leq d+1$, we abbreviate

$$\tau_j = \tau_{0j}, \qquad \eta_j = \eta_{0j}.$$

Thus

$$\tau_j = (x - \theta_0)(x - \theta_1) \cdots (x - \theta_{j-1}), \tag{2.10}$$

$$\eta_j = (x - \theta_d)(x - \theta_{d-1}) \cdots (x - \theta_{d-j+1}). \tag{2.11}$$

In our discussion of Ψ, the following scalars will be useful.

Definition 2.1.4. [33, Section 10] For $0 \leq i \leq d+1$, define

$$\vartheta_i = \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d}.$$

We observe that

$$\vartheta_{i+1} - \vartheta_i = \frac{\theta_i - \theta_{d-i}}{\theta_0 - \theta_d} \qquad (0 \leq i \leq d). \tag{2.12}$$

These scalars will be discussed further in Section 2.12.

2.2 The first split decomposition of V

We continue to discuss the TD system Φ from Definition 2.1.1.

We use the following concept. By a *decomposition* of V, we mean a sequence of subspaces whose direct sum is V. For example, $\{E_iV\}_{i=0}^d$ and $\{E_i^*V\}_{i=0}^d$ are decompositions of V. There are two more decompositions of V of interest called the first and second split decomposition. In this section, we discuss the first split decomposition of V. In Section 2.3, we will discuss the second split decomposition of V.

Definition 2.2.1. For $0 \leq i \leq d$ define

$$U_i = (E_0^*V + E_1^*V + \cdots + E_i^*V) \cap (E_iV + E_{i+1}V + \cdots + E_dV).$$

For notational convenience, define $U_{-1} = 0$ and $U_{d+1} = 0$.

Theorem 2.2.2. [10, Theorem 4.6] *The sequence* $\{U_i\}_{i=0}^d$ *is a decomposition of* V*. Moreover, the following* (i)–(iii) *hold.*

(i) $(A - \theta_i I)U_i \subseteq U_{i+1} \qquad (0 \leq i \leq d-1), \qquad (A - \theta_d I)U_d = 0.$

(ii) $(A^* - \theta^*_i I)U_i \subseteq U_{i-1} \qquad (1 \leq i \leq d), \qquad (A^* - \theta^*_0 I)U_0 = 0.$

(iii) *For* $0 \leq i \leq d$ *both*

$$U_i + U_{i+1} + \cdots + U_d = E_i V + E_{i+1} V + \cdots + E_d V,$$

$$U_0 + U_1 + \cdots + U_i = E^*_0 V + E^*_1 V + \cdots + E^*_i V.$$

Definition 2.2.3. With reference to Definition 2.2.1, we refer to the sequence $\{U_i\}_{i=0}^d$ as the *first split decomposition* of V.

Lemma 2.2.4. [10, Corollary 5.7] *For* $0 \leq i \leq d$ *the dimensions of* $E_i V$, $E^*_i V$, U_i *coincide. Denoting this common dimension by* ρ_i, *we have* $\rho_i = \rho_{d-i}$.

Definition 2.2.5. [11, Section 1] With reference to Lemma 2.2.4, we refer to the sequence $\{\rho_i\}_{i=0}^d$ as the *shape* of Φ. Note that Φ and $\Phi^{\Downarrow}$ have the same shape.

Lemma 2.2.6. *Both*

$$AU_i \subseteq U_i + U_{i+1} \qquad (0 \leq i \leq d-1), \qquad AU_d \subseteq U_d, \tag{2.13}$$

$$A^* U_i \subseteq U_i + U_{i-1} \qquad (1 \leq i \leq d), \qquad A^* U_0 \subseteq U_0.$$

Proof. Use Theorem 2.2.2(i),(ii). ■

Corollary 2.2.7. *For* $0 \leq i \leq d$ *both*

$$A^k U_i \subseteq U_i + U_{i+1} + \cdots + U_{i+k} \qquad (0 \leq k \leq d-i), \tag{2.14}$$

$$(A^*)^k U_i \subseteq U_i + U_{i-1} + \cdots + U_{i-k} \qquad (0 \leq k \leq i). \tag{2.15}$$

Proof. Use Lemma 2.2.6. ■

Definition 2.2.8. [10, Definition 5.2] For $0 \leq i \leq d$ define $F_i \in \mathrm{End}(V)$ by

$$(F_i - I)U_i = 0, \tag{2.16}$$

$$F_i U_j = 0 \quad \text{if} \quad j \neq i, \qquad (0 \leq j \leq d). \tag{2.17}$$

In other words, F_i is the projection map from V onto U_i. For notational convenience, define $F_{-1} = 0$ and $F_{d+1} = 0$.

Lemma 2.2.9. [10, Lemma 5.3] *With reference to Definition 2.2.8, both*

$$F_i F_j = \delta_{ij} F_i \qquad (0 \leq i, j \leq d), \tag{2.18}$$
$$I = \sum_{i=0}^{d} F_i.$$

Definition 2.2.10. [10, Definition 6.1] Define

$$R = A - \sum_{h=0}^{d} \theta_h F_h, \qquad L = A^* - \sum_{h=0}^{d} \theta_h^* F_h.$$

We refer to R (resp. L) as the *first raising map* (resp. *first lowering map*) for Φ.

Lemma 2.2.11. [10, Lemma 6.2] *For $0 \leq i \leq d$ the following hold on U_i.*

$$R = A - \theta_i I, \qquad L = A^* - \theta_i^* I. \tag{2.19}$$

Combining Theorem 2.2.2(i),(ii) with Lemma 2.2.11 we obtain the following result.

Lemma 2.2.12. *Both*

$$RU_i \subseteq U_{i+1} \qquad (0 \leq i \leq d-1), \qquad RU_d = 0, \tag{2.20}$$

$$LU_i \subseteq U_{i-1} \qquad (1 \leq i \leq d), \qquad LU_0 = 0. \tag{2.21}$$

Corollary 2.2.13. *The expression*

$$R^{j-i} - \tau_{ij}(A)$$

vanishes on U_i *for* $0 \leq i \leq j \leq d+1$.

Proof. Use (2.7), (2.20), and Lemma 2.2.11. ■

Lemma 2.2.14. *For* $0 \leq i \leq j \leq d+1$,

$$\tau_{ij}(A)U_i \subseteq U_j.$$

Proof. Use Lemma 2.2.12 and Corollary 2.2.13. ■

The following result is a reformulation of [10, Lemma 6.5].

Lemma 2.2.15. [10, Lemma 6.5] *For* $0 \leq i \leq j \leq d$ *the linear transformation*

$$\begin{aligned} U_i &\to U_j \\ v &\mapsto \tau_{ij}(A)v \end{aligned}$$

is an injection if $i+j \leq d$, *a bijection if* $i+j=d$, *and a surjection if* $i+j \geq d$.

Proof. By [10, Lemma 6.5] the linear transformation $U_i \to U_j$, $v \mapsto R^{j-i}v$ is an injection if $i+j \leq d$, a bijection if $i+j=d$, and a surjection if $i+j \geq d$. The result follows from this and Corollary 2.2.13. ■

Corollary 2.2.16. *The restriction of* $A - \theta_i I$ *to* U_i *is injective for* $0 \leq i < d/2$.

2.3 The second split decomposition of V

We continue to discuss the TD system Φ from Definition 2.1.1. Since $\Phi^{\Downarrow}$ is a TD system on V, all the results from Section 2.2 apply to it. For later use, we now emphasize a few of these results. By definition,

$$U_i^{\Downarrow} = (E_0^*V + E_1^*V + \cdots + E_i^*V) \cap (E_0V + E_1V + \cdots + E_{d-i}V) \qquad (2.22)$$

for $0 \le i \le d$. Applying Theorem 2.2.2 to $\Phi^{\Downarrow}$ we obtain the following facts. The subspaces $\{U_i^{\Downarrow}\}_{i=0}^d$ form a decomposition of V which we call the *second split decomposition* of V. We also have that

$$(A - \theta_{d-i}I)U_i^{\Downarrow} \subseteq U_{i+1}^{\Downarrow} \qquad (0 \le i \le d-1), \qquad (A - \theta_0 I)U_d^{\Downarrow} = 0,$$

$$(A^* - \theta_i^* I)U_i^{\Downarrow} \subseteq U_{i-1}^{\Downarrow} \qquad (1 \le i \le d), \qquad (A^* - \theta_0^* I)U_0^{\Downarrow} = 0.$$

In addition, for $0 \le i \le d$ both

$$U_i^{\Downarrow} + U_{i+1}^{\Downarrow} + \cdots + U_d^{\Downarrow} = E_0V + E_1V + \cdots + E_{d-i}V,$$

$$U_0^{\Downarrow} + U_1^{\Downarrow} + \cdots + U_i^{\Downarrow} = E_0^*V + E_1^*V + \cdots + E_i^*V.$$

Lemma 2.3.1. *For $0 \le i \le d$,*

$$U_0 + U_1 + \cdots + U_i = U_0^{\Downarrow} + U_1^{\Downarrow} + \cdots + U_i^{\Downarrow}.$$

Proof. Both sides equal $E_0^*V + E_1^*V + \cdots + E_i^*V$ by Theorem 2.2.2(iii). ■

We now make some comments concerning $\{F_i^{\Downarrow}\}_{i=0}^d$ and the second raising map $R^{\Downarrow}$. For $0 \le i \le d$, $F_i^{\Downarrow}$ is the projection of V onto $U_i^{\Downarrow}$. Observe that

$$R^{\Downarrow} = A - \sum_{h=0}^{d} \theta_{d-h} F_h^{\Downarrow}. \qquad (2.23)$$

For $0 \leq i \leq j \leq d+1$, the action of $\left(R^{\Downarrow}\right)^{j-i}$ on $U_i^{\Downarrow}$ agrees with the action of $\eta_{ij}(A)$ on $U_i^{\Downarrow}$. In addition,

$$R^{\Downarrow} U_i^{\Downarrow} \subseteq U_{i+1}^{\Downarrow} \qquad (0 \leq i \leq d-1), \qquad R^{\Downarrow} U_d^{\Downarrow} = 0.$$

2.4 The projections F_i, $F_i^{\Downarrow}$

We continue to discuss the TD system Φ from Definition 2.1.1. In this section, we consider how the maps $\{F_i\}_{i=0}^d$ and $\{F_i^{\Downarrow}\}_{i=0}^d$ interact. In [10, Section 5], there are a number of results concerning how the maps $\{E_i\}_{i=0}^d$ and $\{F_i\}_{i=0}^d$ interact. The results given in this section are reformulations of those results.

Lemma 2.4.1. *For $0 \leq i < j \leq d$ both*

$$F_j F_i^{\Downarrow} = 0, \qquad F_j^{\Downarrow} F_i = 0. \tag{2.24}$$

Proof. We first verify the equation on the left in (2.24). By Definition 2.2.8 and Lemma 2.3.1,

$$\begin{aligned} F_j F_i^{\Downarrow} V &= F_j U_i^{\Downarrow} \\ &\subseteq F_j (U_0^{\Downarrow} + U_1^{\Downarrow} + \cdots + U_i^{\Downarrow}) \\ &= F_j \left(U_0 + U_1 + \cdots + U_i\right). \end{aligned} \tag{2.25}$$

Since $i < j$, it follows from (2.17) that (2.25) equals 0. So $F_j F_i^{\Downarrow}$ vanishes on V.

The proof for the equation on the right in (2.24) is similar. ■

Lemma 2.4.2. *For $0 \leq i \leq d$ both*

$$F_i F_i^{\Downarrow} F_i = F_i, \tag{2.26}$$

$$F_i^{\Downarrow} F_i F_i^{\Downarrow} = F_i^{\Downarrow}. \tag{2.27}$$

Proof. We first show (2.26). By Lemma 2.2.9 and Lemma 2.4.1,

$$F_i = F_i F_i = F_i \left(\sum_{h=0}^{d} F_h^{\Downarrow} \right) F_i = F_i F_i^{\Downarrow} F_i.$$

The proof of (2.27) is similar. ∎

Lemma 2.4.3. *For $0 \le i \le d$ the restrictions*

$$F_i^{\Downarrow}|_{U_i} : U_i \to U_i^{\Downarrow}, \qquad\qquad F_i|_{U_i^{\Downarrow}} : U_i^{\Downarrow} \to U_i$$

are bijections. Moreover, these bijections are inverses.

Proof. We first show that the map $F_i F_i^{\Downarrow}$ acts as the identity on U_i. Let $v \in U_i$. By (2.16) and (2.26),

$$F_i F_i^{\Downarrow} v = F_i F_i^{\Downarrow} F_i v = F_i v = v.$$

We have shown $F_i F_i^{\Downarrow}$ acts as the identity on U_i. One can show similarly that $F_i^{\Downarrow} F_i$ acts as the identity on $U_i^{\Downarrow}$. The result follows. ∎

Lemma 2.4.4. [10, Lemma 6.4] *We have*

(i) $RF_i = F_{i+1}R \qquad (0 \le i \le d-1), \qquad RF_d = 0, \qquad F_0 R = 0,$

(ii) $LF_i = F_{i-1}L \qquad (1 \le i \le d), \qquad LF_0 = 0, \qquad F_d L = 0.$

Lemma 2.4.5. *For $0 \le i \le d-1$,*

$$R^{\Downarrow} F_i^{\Downarrow} F_i = F_{i+1}^{\Downarrow} F_{i+1} R.$$

Proof. We show $R^{\Downarrow} F_i^{\Downarrow} F_i - F_{i+1}^{\Downarrow} F_{i+1} R = 0$. By Lemma 2.4.4(i) (applied to both Φ and $\Phi^{\Downarrow}$),

$$\begin{aligned} R^{\Downarrow} F_i^{\Downarrow} F_i - F_{i+1}^{\Downarrow} F_{i+1} R &= F_{i+1}^{\Downarrow} R^{\Downarrow} F_i - F_{i+1}^{\Downarrow} R F_i \\ &= F_{i+1}^{\Downarrow} \left(R^{\Downarrow} - R \right) F_i. \end{aligned} \tag{2.28}$$

By Definition 2.2.10,

$$R^{\Downarrow} - R = \sum_{h=0}^{d} \theta_h F_h - \sum_{h=0}^{d} \theta_{d-h} F_h^{\Downarrow}. \tag{2.29}$$

Eliminate $R^{\Downarrow} - R$ in (2.28) using (2.29). Simplify the resulting expression using (2.18) (applied to both Φ and $\Phi^{\Downarrow}$) and Lemma 2.4.1 to get 0. ∎

2.5 The subspaces K_i

We continue to discuss the TD system Φ from Definition 2.1.1. Shortly we will define the linear transformation Ψ. In our discussion of Ψ, it will be useful to consider a certain refinement of the first and second split decomposition of V. This refinement was introduced in [25]. In order to describe this refinement, we introduce a sequence of subspaces $\{K_i\}_{i=0}^{r}$, where $r = \lfloor d/2 \rfloor$.

Definition 2.5.1. For $0 \le i \le d/2$, define the subspace $K_i \subseteq V$ by

$$K_i = (E_0^* V + E_1^* V + \cdots + E_i^* V) \cap (E_i V + E_{i+1} V + \cdots + E_{d-i} V).$$

Observe that $K_0 = E_0^* V = U_0$.

Lemma 2.5.2. *We have*

$$K_i = U_i \cap U_i^{\Downarrow} \qquad (0 \le i \le d/2).$$

Proof. Use (2.22), Definition 2.2.1, and Definition 2.5.1. ∎

Lemma 2.5.3. [25, Lemma 4.1(iii)] *For $0 \le i \le d/2$, the restriction of $\tau_{i,d-i+1}(A)$ to U_i has kernel K_i.*

Proof. Use Lemma 2.1.3 and Definition 2.2.1. ∎

We now consider the spaces

$$\tau_{ij}(A)K_i$$

where $0 \leq i \leq d/2$ and $i \leq j \leq d-i$. We start with an observation.

Lemma 2.5.4. [25, Lemma 4.1(vi)] *For $0 \leq i \leq d/2$ and $i \leq j \leq d-i$, the linear transformation*

$$K_i \to \tau_{ij}(A)K_i$$
$$v \mapsto \tau_{ij}(A)v$$

is a bijection.

Proof. By construction the map is surjective. By Lemma 2.2.15 the restriction of $\tau_{ij}(A)$ to K_i is injective. The result follows. ∎

From Lemma 2.5.4, we draw two corollaries.

Corollary 2.5.5. *For $0 \leq i \leq d/2$ and $i \leq j \leq k \leq d-i$, the linear transformation*

$$\tau_{ij}(A)K_i \to \tau_{ik}(A)K_i$$
$$v \mapsto \tau_{jk}(A)v$$

is a bijection.

Proof. Use Lemma 2.5.4 and the equation on the left in (2.9). ∎

Corollary 2.5.6. *For $0 \leq i \leq d/2$ and $i \leq j \leq d-i$, the dimension of $\tau_{ij}(A)K_i$ coincides with the dimension of K_i.*

2.6 Concerning the decomposition $\{U_i\}_{i=0}^d$

We continue to discuss the TD system Φ from Definition 2.1.1. Recall the first split decomposition $\{U_i\}_{i=0}^d$ of V from Definition 2.2.1. We know that $K_0 = U_0$ and $K_i \subseteq U_i$ for $1 \leq i \leq d$. We will use this fact along with information about the first raising map R to give a decomposition of each U_i.

The following result is essentially due to K. Nomura [25, Theorem 4.2]. We give an alternate proof.

Lemma 2.6.1. [25, Theorem 4.2] *For $1 \leq i \leq d/2$, each of the following sums is direct.*

(i) $U_i = K_i + RU_{i-1}$,

(ii) $U_i = K_i + (A - \theta_{i-1}I)U_{i-1}$.

Proof. (i) We first show that $U_i = K_i + RU_{i-1}$. By Lemma 2.2.12 and Lemma 2.5.2, $U_i \supseteq K_i + RU_{i-1}$. We now show $U_i \subseteq K_i + RU_{i-1}$. Let $v \in U_i$. By Lemma 2.2.12 we get $R^{d-2i+1}v \in U_{d-i+1}$. By Corollary 2.2.13 and Lemma 2.2.15 there exists $w \in U_{i-1}$ such that $R^{d-2i+2}w = R^{d-2i+1}v$. Rearranging terms we obtain $R^{d-2i+1}(Rw - v) = 0$. So $Rw - v$ is in the kernel of R^{d-2i+1}. By Lemma 2.2.12, $Rw - v \in U_i$. By Corollary 2.2.13 and Lemma 2.5.3, K_i is the intersection of U_i and the kernel of R^{d-2i+1}. By these comments $Rw - v \in K_i$. Therefore

$$\begin{aligned} v &= -(Rw - v) + Rw \\ &\in K_i + RU_{i-1}. \end{aligned}$$

Hence $U_i \subseteq K_i + RU_{i-1}$. We have shown $U_i = K_i + RU_{i-1}$. We now show that this sum is direct. Let $v \in K_i \cap RU_{i-1}$. Since $v \in RU_{i-1}$, there exists $w \in U_{i-1}$ such that

$v = Rw$. Recall $v \in K_i$ so $R^{d-2i+1}v = 0$. Therefore $R^{d-2i+2}w = 0$. By Lemma 2.2.15, the restriction of R^{d-2i+2} to U_{i-1} is injective. So $w = 0$ and thus $v = 0$. We have shown that the sum $U_i = K_i + RU_{i-1}$ is direct.

(ii) Use (i) and Lemma 2.2.11. ■

From Lemma 2.6.1 we obtain the following two corollaries.

Corollary 2.6.2. [10, Corollary 6.6] *With reference to Lemma 2.2.4,*

(i) $\rho_i \leq \rho_{i+1}$ *for* $0 \leq i < d/2$,

(ii) $\rho_i \geq \rho_{i+1}$ *for* $d/2 \leq i \leq d-1$.

Proof. (i) Use Lemma 2.6.1(i) and Lemma 2.2.15.
(ii) Use Corollary 2.6.2(i) and Lemma 2.2.4. ■

Corollary 2.6.3. [25, Lemma 4.3] *For* $1 \leq i \leq d/2$, *the dimension of* K_i *equals* $\rho_i - \rho_{i-1}$ *(this dimension could be zero). Moreover, the dimension of* K_0 *equals* ρ_0.

Lemma 2.6.4. [25, Theorem 4.7]

(i) *For* $0 \leq i \leq d/2$, *the following sum is direct.*

$$U_i = K_i + \tau_{i-1,i}(A)K_{i-1} + \tau_{i-2,i}(A)K_{i-2} + \cdots + \tau_{0,i}(A)K_0. \qquad (2.30)$$

(ii) *For* $d/2 \leq i \leq d$, *the following sum is direct.*

$$U_i = \tau_{d-i,i}(A)K_{d-i} + \tau_{d-i-1,i}(A)K_{d-i-1} + \cdots + \tau_{0,i}(A)K_0.$$

Proof. (i) Recall $U_0 = K_0$. By Lemma 2.6.1(ii), the sum $U_j = K_j + (A - \theta_{j-1}I)U_{j-1}$ is direct for $1 \leq j \leq i$. Combining these equations and simplifying the result using (2.7), we get (2.30). The directness of the sum (2.30) follows in view of Corollary 2.2.16.

(ii) Observe that $0 \leq d-i \leq d/2$. So (2.30) gives a decomposition of U_{d-i}. By Lemma 2.2.15, the restriction of $\tau_{d-i,i}(A)$ to U_{d-i} gives a bijection $U_{d-i} \to U_i$. Apply this bijection to each term in the above mentioned decomposition for U_{d-i} and simplify the result using the equation on the left in (2.9). ■

Combining parts (i) and (ii) of Lemma 2.6.4 we have

$$U_j = \sum_{i=0}^{\min\{j,d-j\}} \tau_{ij}(A)K_i \qquad \text{(direct sum)} \tag{2.31}$$

for $0 \leq j \leq d$.

Corollary 2.6.5. [25, Theorem 4.8] *The following sum is direct.*

$$V = \sum_{i=0}^{r} \sum_{j=i}^{d-i} \tau_{ij}(A)K_i, \tag{2.32}$$

where $r = \lfloor d/2 \rfloor$.

Proof. In the decomposition of V from Theorem 2.2.2, evaluate each summand using (2.31). In the resulting double summation, invert the order of summation. ■

The refinement of the first split decomposition given above yields the following description of the kernel of the map R from Section 2.2.

Lemma 2.6.6. *For* $0 \leq i < d/2$, *the restriction of* R *to* U_i *is injective. For* $d/2 \leq i \leq d$, *the restriction of* R *to* U_i *is surjective with kernel* $\tau_{d-i,i}(A)K_{d-i}$ *and image* U_{i+1}. *Moreover the kernel of* R *on* V *is* $\sum_{i=0}^{\lfloor d/2 \rfloor} \tau_{i,d-i}(A)K_i$.

Proof. The claims concerning injectivity and surjectivity follow from [10, Lemma 6.5]. Let $d/2 \leq i \leq d$. We now show that the kernel of R on U_i is $\tau_{d-i,i}(A)K_{d-i}$. Recall that R acts on U_i as $A - \theta_i I$ and $RU_i \subseteq U_{i+1}$. The result follows from this along with (2.7) and Lemma 2.6.4. ■

2.7 The subalgebra M

We continue to discuss the TD system Φ from Definition 2.1.1. Recall from Section 2.1 the subalgebra M of $\mathrm{End}(V)$ generated by A. In our discussion of M, we mentioned that each of $\{E_i\}_{i=0}^d$, $\{A^i\}_{i=0}^d$ is a basis for M. In this section, we give a third basis for M and use it to realize V as a direct sum of M-modules.

Lemma 2.7.1. *For $0 \leq i \leq d/2$, the vector space M has basis*

$$\{E_0, E_1, \ldots, E_{i-1}\} \cup \{E_{d-i+1}, E_{d-i+2}, \ldots, E_d\} \cup \{\tau_{ij}(A) | i \leq j \leq d-i\}. \tag{2.33}$$

Proof. By [28, Lemma 5.1],

$$\{E_0, E_1, \ldots, E_{i-1}\} \cup \{E_{d-i+1}, E_{d-i+2}, \ldots, E_d\} \cup \{A^{j-i} | i \leq j \leq d-i\}$$

is a basis for M. By the comments following (2.8),

$$\mathrm{Span}\{A^{j-i} | i \leq j \leq d-i\} = \mathrm{Span}\{\tau_{ij}(A) | i \leq j \leq d-i\}.$$

The result follows. ■

For the rest of this section, we view V as an M-module. For $0 \leq i \leq d/2$, let MK_i denote the M-submodule of V generated by K_i. Our goal in this section is to show that the sum $V = \sum_{i=0}^r MK_i$ is direct, where $r = \lfloor d/2 \rfloor$. We start by giving a detailed description of the MK_i.

Lemma 2.7.2. *For $0 \leq i \leq d/2$ such that $K_i \neq 0$, the sum*

$$MK_i = K_i + \tau_{i,i+1}(A)K_i + \tau_{i,i+2}(A)K_i + \cdots + \tau_{i,d-i}(A)K_i \tag{2.34}$$

is direct. Moreover $\tau_{i,d-i+1}$ is the minimal polynomial for the action of A on MK_i.

Proof. For the basis of M given in (2.33), apply each element to K_i. By Definition 2.5.1, each primitive idempotent in (2.33) vanishes on K_i. This gives equation (2.34). We now show that the sum on the right in (2.34) is direct. By Lemma 2.2.14, we have $\tau_{ij}(A)K_i \subseteq U_j$ for $i \leq j \leq d-i$. The sum (2.34) is direct by this and Theorem 2.2.2.

It remains to show that $\tau_{i,d-i+1}$ is the minimal polynomial for the action of A on MK_i. Let P denote the minimal polynomial for the action of A on MK_i and let k denote the degree of P. By Lemma 2.1.3 and Definition 2.5.1, $\tau_{i,d-i+1}(A)K_i = 0$. Since $A \in M$ and M is commutative, it follows that $\tau_{i,d-i+1}(A)MK_i = 0$. So P divides $\tau_{i,d-i+1}$ and hence $k \leq d-2i+1$.

Suppose now that $k < d-2i+1$ to get a contradiction. Since the degree of P is k,

$$MK_i = K_i + AK_i + \cdots + A^{k-1}K_i. \tag{2.35}$$

By (2.14), the right-hand side of (2.35) is contained in $U_i + U_{i+1} + \cdots + U_{i+k-1}$. By Lemma 2.5.4, the restriction of $\tau_{i,d-i}(A)$ to K_i is an injection. It follows from this and $K_i \neq 0$ that $\tau_{i,d-i}(A)K_i \neq 0$. Recall that $\tau_{i,d-i}(A)K_i \subseteq U_{d-i}$. By (2.34) and the above comments we find that $\tau_{i,d-i}(A)K_i$ is contained in the intersection of $U_i + U_{i+1} + \cdots + U_{i+k-1}$ and U_{d-i}. Since $k < d-2i+1$, this intersection is zero by Theorem 2.2.2. Therefore $\tau_{i,d-i}(A)K_i = 0$ for a contradiction. Thus $k = d-2i+1$ and therefore $P = \tau_{i,d-i+1}$ since $\tau_{i,d-i+1}$ is monic. ∎

Corollary 2.7.3. *For $0 \leq i \leq d/2$ and $0 \neq v \in K_i$, the vector space Mv has basis*

$$v, \quad \tau_{i,i+1}(A)v, \quad \tau_{i,i+2}(A)v, \quad \ldots, \quad \tau_{i,d-i}(A)v.$$

Lemma 2.7.4. *The following is a direct sum of M-modules.*

$$V = \sum_{i=0}^{r} MK_i, \tag{2.36}$$

where $r = \lfloor d/2 \rfloor$.

Proof. Equation (2.36) follows from Corollary 2.6.5 and Lemma 2.7.2. The directness of the sum follows from the directness of the sum in Corollary 2.6.5. ■

2.8 The linear transformation Δ

We continue to discuss the TD system Φ from Definition 2.1.1. In this section we will construct a linear transformation $\Delta \in \mathrm{End}(V)$ that has certain properties which we find attractive. It will turn out that Δ is the unique element of $\mathrm{End}(V)$ such that both

$$(\Delta - I)E_i^*V \subseteq E_0^*V + E_1^*V + \cdots + E_{i-1}^*V,$$
$$\Delta(E_iV + E_{i+1}V + \cdots + E_dV) \subseteq E_0V + E_1V + \cdots + E_{d-i}V$$

for $0 \leq i \leq d$.

Definition 2.8.1. Define $\Delta \in \mathrm{End}(V)$ by

$$\Delta = \sum_{h=0}^{d} F_h^{\Downarrow} F_h,$$

where F_h, $F_h^{\Downarrow}$ are from Definition 2.2.8.

Lemma 2.8.2. *With reference to Definition 2.8.1,*

$$F_i^{\Downarrow}\Delta = \Delta F_i \qquad (0 \leq i \leq d).$$

Proof. Use (2.18) and Definition 2.8.1. ■

Lemma 2.8.3. *With reference to Definition 2.8.1,* Δ^{-1} *exists and*

$$\Delta^{-1} = \Delta^{\Downarrow}.$$

Proof. Observe that $\Delta^{\Downarrow} = \sum_{h=0}^{d} F_h F_h^{\Downarrow}$. Consider the product $\Delta\Delta^{\Downarrow}$. Simplify this product using Lemma 2.2.9 and Lemma 2.4.2 to obtain $\Delta\Delta^{\Downarrow} = I$. ■

Lemma 2.8.4. *With reference to Definition 2.8.1,*

$$\Delta U_i = U_i^{\Downarrow} \qquad (0 \leq i \leq d), \tag{2.37}$$

$$(\Delta - I)U_i \subseteq U_0 + U_1 + \cdots + U_{i-1} \qquad (0 \leq i \leq d). \tag{2.38}$$

Proof. Line (2.37) follows from Definition 2.2.8, Lemma 2.4.3 and Definition 2.8.1.

We now verify (2.38). By Definition 2.2.8, it suffices to show that $F_j\,(\Delta - I)\,U_i = 0$ for $i \leq j \leq d$. For $i = j$, this follows from Definition 2.2.8, Definition 2.8.1, and (2.26). For $i + 1 \leq j \leq d$, this follows from Definition 2.2.8, Definition 2.8.1, and (2.24). ■

We now show that (2.37), (2.38) characterize Δ.

Lemma 2.8.5. *Given* $\Delta' \in \mathrm{End}(V)$ *such that*

$$\Delta' U_i \subseteq U_i^{\Downarrow} \qquad (0 \leq i \leq d), \tag{2.39}$$

$$(\Delta' - I)U_i \subseteq U_0 + U_1 + \cdots + U_{i-1} \qquad (0 \leq i \leq d). \tag{2.40}$$

Then $\Delta' = \Delta$.

Proof. In view of Theorem 2.2.2, it suffices to show that Δ, Δ' agree on U_i for $0 \leq i \leq d$. Let i be given. By (2.37) and (2.39),

$$(\Delta - \Delta')\,U_i \subseteq U_i^{\Downarrow}. \tag{2.41}$$

By (2.38), (2.40), and Lemma 2.3.1,

$$\begin{aligned} (\Delta - \Delta')\,U_i &\subseteq U_0 + U_1 + \cdots + U_{i-1} \\ &= U_0^{\Downarrow} + U_1^{\Downarrow} + \cdots + U_{i-1}^{\Downarrow}. \end{aligned} \tag{2.42}$$

Combining (2.41) and (2.42) we find that $(\Delta - \Delta')\, U_i$ is contained in the intersection of $U_i^{\Downarrow}$ and $U_0^{\Downarrow} + U_1^{\Downarrow} + \cdots + U_{i-1}^{\Downarrow}$. This intersection is zero by Theorem 2.2.2 (applied to $\Phi^{\Downarrow}$). Therefore $(\Delta - \Delta')U_i = 0$. So Δ, Δ' agree on U_i. ■

Lemma 2.8.6. *With reference to Definition 2.8.1,*

$$(\Delta^{-1} - I)U_i \subseteq U_0 + U_1 + \cdots + U_{i-1} \qquad (0 \le i \le d).$$

Proof. Apply Δ^{-1} to both sides in (2.38). In the resulting containment, simplify the right-hand side using Lemma 2.3.1 and (2.37). ■

Lemma 2.8.7. *With reference to Definition 2.8.1,*

$$(\Delta - I)U_i^{\Downarrow} \subseteq U_0^{\Downarrow} + U_1^{\Downarrow} + \cdots + U_{i-1}^{\Downarrow} \qquad (0 \le i \le d).$$

Proof. Apply Lemma 2.8.6 to $\Phi^{\Downarrow}$. Use Lemma 2.8.3 to simplify the result. ■

We now obtain the characterization of Δ given in the Introduction.

Lemma 2.8.8. *With reference to Definition 2.8.1,*

$$(\Delta - I)E_i^*V \subseteq E_0^*V + E_1^*V + \cdots + E_{i-1}^*V \qquad (0 \le i \le d), \tag{2.43}$$

$$\Delta(E_iV + E_{i+1}V + \cdots + E_dV) = E_0V + E_1V + \cdots + E_{d-i}V \qquad (0 \le i \le d). \tag{2.44}$$

Proof. We first show (2.43). By Theorem 2.2.2(iii) and (2.38),

$$\begin{aligned}
(\Delta - I)E_i^*V &\subseteq (\Delta - I)(E_0^*V + E_1^*V + \cdots + E_i^*V) \\
&= (\Delta - I)(U_0 + U_1 + \cdots + U_i) \\
&\subseteq U_0 + U_1 + \cdots + U_{i-1} \\
&= E_0^*V + E_1^*V + \cdots + E_{i-1}^*V.
\end{aligned}$$

We now show (2.44). Applying Theorem 2.2.2(iii) to both Φ and $\Phi^{\Downarrow}$, and also using (2.37), we obtain

$$\begin{aligned}\Delta(E_iV + E_{i+1}V + \cdots + E_dV) &= \Delta(U_i + U_{i+1} + \cdots + U_d)\\ &= U_i^{\Downarrow} + U_{i+1}^{\Downarrow} + \cdots + U_d^{\Downarrow}\\ &= E_0V + E_1V + \cdots + E_{d-i}V.\end{aligned}$$

■

We now show that (2.43), (2.44) characterize Δ.

Lemma 2.8.9. *Given* $\Delta' \in \mathrm{End}(V)$ *such that*

$$(\Delta' - I)E_i^*V \subseteq E_0^*V + E_1^*V + \cdots + E_{i-1}^*V \qquad (0 \le i \le d), \qquad (2.45)$$

$$\Delta'(E_iV + E_{i+1}V + \cdots + E_dV) \subseteq E_0V + E_1V + \cdots + E_{d-i}V \qquad (0 \le i \le d). \qquad (2.46)$$

Then $\Delta' = \Delta$.

Proof. By Lemma 2.8.5, it suffices to show that Δ' satisfies (2.39) and (2.40). These lines are routinely verified using Theorem 2.2.2(iii) (applied to both Φ and $\Phi^{\Downarrow}$) and Lemma 2.3.1. ■

We now derive some relations involving Δ that will be of use later.

Lemma 2.8.10. *With reference to Definition 2.8.1,*

$$R^{\Downarrow}\Delta = \Delta R.$$

Proof. In the expression $R^{\Downarrow}\Delta - \Delta R$, eliminate Δ using Definition 2.8.1. Simplify the result using Lemma 2.4.4(i) and Lemma 2.4.5 to obtain $R^{\Downarrow}\Delta - \Delta R = 0$. ■

Lemma 2.8.11. *With reference to Definition 2.8.1,*

$$\Delta A - A\Delta = \sum_{h=0}^{d} (\theta_h - \theta_{d-h}) F_h^{\Downarrow} F_h. \tag{2.47}$$

Proof. By Lemma 2.8.10,

$$\Delta R - R^{\Downarrow}\Delta = 0. \tag{2.48}$$

In (2.48), eliminate R and $R^{\Downarrow}$ using Definition 2.2.10 and (2.23) to get

$$\Delta A - A\Delta = \sum_{h=0}^{d} \theta_h \Delta F_h - \sum_{h=0}^{d} \theta_{d-h} F_h^{\Downarrow} \Delta. \tag{2.49}$$

Simplify the right-hand side of (2.49) using Definition 2.8.1 and (2.18) to get the result. ■

We now express Lemma 2.8.11 from a slightly different perspective.

Corollary 2.8.12. *With reference to Definition 2.8.1,*

$$A - \Delta^{-1} A \Delta = \sum_{h=0}^{d} (\theta_h - \theta_{d-h}) F_h.$$

Proof. Apply Δ^{-1} to both sides of (2.47). Simplify the resulting right-hand side using Lemma 2.4.2, Definition 2.8.1, and (2.18). ■

Lemma 2.8.13. *With reference to Definition 2.8.1,*

$$L^{\Downarrow}\Delta - \Delta L = A^*\Delta - \Delta A^*. \tag{2.50}$$

Proof. In the left-hand side of (2.50), eliminate L and $L^{\Downarrow}$ using Definition 2.2.10. Evaluate the result using Lemma 2.8.2. ■

Lemma 2.8.14. *With reference to Definition 2.8.1,*

$$(\Delta^{-1} A^* \Delta - A^*) U_i \subseteq U_{i-1} \qquad (1 \le i \le d), \qquad (\Delta^{-1} A^* \Delta - A^*) U_0 = 0.$$

Proof. By Lemma 2.8.13,

$$\Delta^{-1}A^*\Delta - A^* = \Delta^{-1}L^{\Downarrow}\Delta - L.$$

Let $1 \leq i \leq d$. By (2.37) and (2.21) (applied to $\Phi^{\Downarrow}$), $\Delta^{-1}L^{\Downarrow}\Delta U_i \subseteq U_{i-1}$. By (2.21), $LU_i \subseteq U_{i-1}$. Thus $(\Delta^{-1}L^{\Downarrow}\Delta - L)U_i \subseteq U_{i-1}$. By these comments, $(\Delta^{-1}A^*\Delta - A^*)U_i \subseteq U_{i-1}$.

To obtain $(\Delta^{-1}A^*\Delta - A^*)U_0 = 0$, use (2.21) (applied to both Φ and $\Phi^{\Downarrow}$) and (2.37).

■

2.9 More on Δ

We continue to discuss the TD system Φ from Definition 2.1.1. Recall the decomposition of V given in Corollary 2.6.5. In this section, we consider the action of Δ on each of the summands of this decomposition.

Lemma 2.9.1. *Let $0 \leq i \leq d/2$. For $v \in K_i$ and $i \leq j \leq d-i$, both*

$$F_j^{\Downarrow}\tau_{ij}(A)v = \eta_{ij}(A)v, \tag{2.51}$$

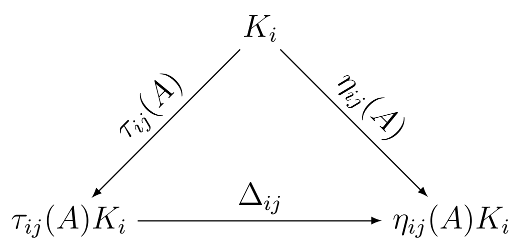
$$F_j\eta_{ij}(A)v = \tau_{ij}(A)v. \tag{2.52}$$

Proof. We first show (2.51). First suppose $i = j$. Use (2.16), Lemma 2.5.2, and the fact that both τ_{ii} and η_{ii} equal 1. Now suppose $i < j$. By the comments following (2.8), $\tau_{ij} - \eta_{ij}$ has degree at most $j-i-1$ and is therefore in $\mathrm{Span}\{\eta_{ih}\}_{h=i}^{j-1}$. From this and Lemma 2.2.14 (applied to $\Phi^{\Downarrow}$) we find that

$$(\tau_{ij}(A) - \eta_{ij}(A))\, v \in U_i^{\Downarrow} + U_{i+1}^{\Downarrow} + \cdots + U_{j-1}^{\Downarrow}. \tag{2.53}$$

Apply $F_j^{\Downarrow}$ to each side of (2.53). By Definition 2.2.8 (applied to $\Phi^{\Downarrow}$), $F_j^{\Downarrow}$ applied to the right-hand side of (2.53) is zero. By (2.16) and Lemma 2.2.14 (applied to $\Phi^{\Downarrow}$), $F_j^{\Downarrow}\eta_{ij}(A)v = \eta_{ij}(A)v$. Line (2.51) follows from the above comments.
Line (2.52) is similarly obtained. ■

Lemma 2.9.2. *For $0 \le i \le d/2$ and $i \le j \le d-i$, let Δ_{ij} denote the restriction of Δ to the subspace $\tau_{ij}(A)K_i$. Then the following diagram commutes.*



Proof. Let $v \in K_i$. We push v around the diagram. Observe that $\Delta_{ij}\tau_{ij}(A)v = \Delta\tau_{ij}(A)v$. Consider $\Delta\tau_{ij}(A)v$. In this expression, eliminate Δ using Definition 2.8.1. Then simplify the result using Definition 2.2.8, Lemma 2.2.14 (applied to both Φ and $\Phi^{\Downarrow}$), and Lemma 2.9.1. By these comments we find $\Delta\tau_{ij}(A)v = \eta_{ij}(A)v$. ■

We emphasize a point for later use. By Lemma 2.9.2, we see that for $0 \le i \le d/2$ and $i \le j \le d-i$,

$$\Delta\tau_{ij}(A)v = \eta_{ij}(A)v \qquad\qquad (v \in K_i). \tag{2.54}$$

Setting $j = i$ in the above argument, we see that

$$(\Delta - I)K_i = 0. \tag{2.55}$$

2.10 The linear transformation Ψ

We continue to discuss the TD system Φ from Definition 2.1.1. We now introduce a certain linear transformation $\Psi \in \mathrm{End}(V)$ which has properties that we find attractive.

To define Ψ we give its action on each summand in the decomposition of V from Corollary 2.6.5. It will turn out that Ψ is the unique linear transformation such that both

$$\Psi E_i V \subseteq E_{i-1}V + E_iV + E_{i+1}V,$$
$$\left(\Psi - \frac{\Delta - I}{\theta_0 - \theta_d}\right) E_i^* V \subseteq E_0^* V + E_1^* V + \cdots + E_{i-2}^* V$$

for $0 \leq i \leq d$. This characterization of Ψ will be discussed in Section 2.15.

Lemma 2.10.1. *There exists a unique linear transformation $\Psi \in \mathrm{End}(V)$ such that*

$$\Psi \tau_{ij}(A) - (\vartheta_j - \vartheta_i)\, \tau_{i,j-1}(A) \tag{2.56}$$

vanishes on K_i for $0 \leq i \leq d/2$ and $i \leq j \leq d - i$. Recall that $\tau_{i,i-1} = 0$.

Proof. By Corollary 2.6.5 the sum in (2.32) is a decomposition of V. In the statement of the lemma, we specified the action of Ψ on each summand and therefore Ψ exists. The uniqueness assertion is clear. ■

We clarify the meaning of Ψ. Fix an integer i $(0 \leq i \leq d/2)$. Lemma 2.10.1 implies that $\Psi K_i = 0$. More generally, for $i \leq j \leq d - i$ and $v \in K_i$,

$$\Psi \tau_{ij}(A) v = (\vartheta_j - \vartheta_i) \tau_{i,j-1}(A) v. \tag{2.57}$$

We look at Ψ from several perspectives.

Lemma 2.10.2. *With reference to Lemma 2.10.1,*

$$\Psi U_j \subseteq U_{j-1} \qquad (1 \leq j \leq d), \qquad \Psi U_0 = 0.$$

Proof. We first show $\Psi U_j \subseteq U_{j-1}$ for $1 \leq j \leq d$. Let j be given. Recall from (2.31) the direct sum $U_j = \sum_{i=0}^{\min\{j,d-j\}} \tau_{ij}(A) K_i$. Referring to this sum, we will show Ψ sends each summand into U_{j-1}. Consider the i^{th} summand $\tau_{ij}(A)K_i$. First suppose $i = j$. Then Ψ sends this summand to zero because $\Psi K_i = 0$. Next suppose $i < j$. Using Lemma 2.2.14 and (2.57), we obtain

$$\Psi \tau_{ij}(A)K_i \subseteq \tau_{i,j-1}(A)K_i \subseteq U_{j-1}.$$

We now show $\Psi U_0 = 0$. Recall that $\Psi K_0 = 0$. The result follows since $K_0 = U_0$.

■

Lemma 2.10.3. *With reference to Lemma 2.10.1,*

$$F_i \Psi = \Psi F_{i+1} \qquad (0 \leq i \leq d-1), \qquad \Psi F_0 = 0, \qquad F_d \Psi = 0. \tag{2.58}$$

Proof. We first show that $F_i \Psi = \Psi F_{i+1}$ for $0 \leq i \leq d-1$. Let i be given. Recall the decomposition $\{U_j\}_{j=0}^d$ of V from Theorem 2.2.2. We will show that $F_i \Psi - \Psi F_{i+1}$ vanishes on each U_j. Observe that

$$F_i \Psi - \Psi F_{i+1} = (F_i - I)\Psi - \Psi(F_{i+1} - I). \tag{2.59}$$

The right-hand side of (2.59) vanishes on U_j by Definition 2.2.8 and Lemma 2.10.2. Thus $F_i \Psi - \Psi F_{i+1}$ vanishes U_j and hence on V. The equation on the left in (2.58) follows from the above comments.

The assertions $\Psi F_0 = 0$, $F_d \Psi = 0$ follow from Lemma 2.10.2. ■

Lemma 2.10.4. *With reference to Definition 2.8.1 and Lemma 2.10.1, for $0 \leq j \leq d$ apply either of*

$$\Delta - I - (\theta_0 - \theta_d)\Psi, \qquad \Delta^{-1} - I + (\theta_0 - \theta_d)\Psi \tag{2.60}$$

to U_j and consider the image. This image is contained in $U_0 + U_1 + \cdots + U_{j-2}$ if $j \geq 2$ and equals 0 if $j < 2$.

Proof. We first consider the expression on the left in (2.60). Recall the direct sum $U_j = \sum_{i=0}^{\min\{j,d-j\}} \tau_{ij}(A)K_i$ from (2.31). Consider a summand $\tau_{ij}(A)K_i$. We show that the image of $\tau_{ij}(A)K_i$ under the expression on the left in (2.60) is contained in $U_0+U_1+\cdots+U_{j-2}$ if $j \geq 2$ and equals 0 if $j < 2$. By (2.54) and Lemma 2.10.1, the actions of the expression on the left in (2.60) times $\tau_{ij}(A)$ and

$$\eta_{ij}(A) - \tau_{ij}(A) - (\theta_0 - \theta_d)(\vartheta_j - \vartheta_i)\tau_{i,j-1}(A) \tag{2.61}$$

agree on K_i. By (2.7), (2.8), and Definition 2.1.4, (2.61) is a polynomial in A of degree at most $j-i-2$ if $j \geq i+2$ and equals 0 if $j < i+2$. The result follows from the above comments and (2.14).

We now consider the expression on the right in (2.60). We will use the fact that the result holds for the expression on the left in (2.60). Observe that

$$\Delta^{-1} - I + (\theta_0 - \theta_d)\Psi = \Delta^{-1}(\Delta - I)^2 - \Delta + I + (\theta_0 - \theta_d)\Psi.$$

The result follows from the above comments, (2.38) and Lemma 2.8.6. ■

Lemma 2.10.5. *With reference to Lemma 2.10.1, Ψ satisfies*

$$\Psi R - R\Psi = \sum_{h=0}^{d} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} F_h. \tag{2.62}$$

Proof. Referring to the decomposition of V given in Corollary 2.6.5, consider any summand $\tau_{ij}(A)K_i$. We apply each side of (2.62) to this summand. We claim that on this summand, each side of (2.62) acts as $(\theta_j - \theta_{d-j})(\theta_0 - \theta_d)^{-1}I$.

The claim holds for the right-hand side of (2.62) by Definition 2.2.8 and the fact that $\tau_{ij}(A)K_i \subseteq U_j$. Concerning the left-hand side of (2.62), we routinely carry out this application using (2.7), (2.12), Lemma 2.2.11, and Lemma 2.10.1. ■

Corollary 2.10.6. *With reference to Definition 2.8.1 and Lemma 2.10.1,*

$$\frac{A - \Delta^{-1} A \Delta}{\theta_0 - \theta_d} = \Psi R - R\Psi.$$

Proof. Use Corollary 2.8.12 and Lemma 2.10.5. ■

We now give a characterization of Ψ.

Lemma 2.10.7. *Given $\Psi' \in \mathrm{End}(V)$ such that*

$$\Psi' R - R\Psi' = \sum_{h=0}^{d} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} F_h \tag{2.63}$$

and $\Psi' K_i = 0$ for $0 \leq i \leq d/2$. Then $\Psi' = \Psi$.

Proof. Recall from Corollary 2.6.5 the decomposition $V = \sum_{i=0}^{r} \sum_{j=i}^{d-i} \tau_{ij}(A) K_i$, where $r = \lfloor d/2 \rfloor$. We show that $\Psi - \Psi'$ vanishes on each summand by fixing i and inducting on j. Let i be given. Recall that $\Psi K_i = 0$. Thus $\Psi - \Psi'$ vanishes on $\tau_{ii}(A) K_i = K_i$. Now suppose $\Psi - \Psi'$ vanishes on $\tau_{ij}(A) K_i$. We show that $\Psi - \Psi'$ vanishes on $\tau_{i,j+1}(A) K_i$. By (2.62) and (2.63), we see that

$$(\Psi - \Psi')R = R(\Psi - \Psi').$$

By the above comments, $\Psi - \Psi'$ vanishes on $R\tau_{ij}(A) K_i$. By (2.7) and Lemma 2.2.11, $R\tau_{ij}(A) K_i = \tau_{i,j+1}(A) K_i$. Thus $\Psi - \Psi'$ vanishes on $\tau_{i,j+1}(A) K_i$. So $\Psi - \Psi'$ vanishes on V. ■

Shortly we will give a second characterization of Ψ. That characterization will be based on the following result.

Lemma 2.10.8. *Given $X \in \mathrm{End}(V)$ such that $XR = RX$ and $XU_i \subseteq U_{i-1}$ for $0 \leq i \leq d$. Then $X = 0$.*

Proof. By (2.7), Lemma 2.2.11, and Corollary 2.6.5, it suffices to show that $XR^hK_i = 0$ for $0 \leq i \leq d/2$ and $0 \leq h \leq d-2i$. Since $XR = RX$, it suffices to show that $XK_i = 0$ for $0 \leq i \leq d/2$. Let i be given. First assume that $i = 0$. Then $XK_0 = 0$, since $K_0 = U_0$ and $XU_0 = 0$. Next assume that $i \geq 1$. By Lemma 2.6.6 and $R^{d-2i}K_i = \tau_{i,d-i}(A)K_i$, we obtain $R^{d-2i+1}K_i = 0$. From this and since $XR = RX$, it follows that $R^{d-2i+1}XK_i = 0$. By Lemma 2.5.2, $K_i \subseteq U_i$ and hence $XK_i \subseteq U_{i-1}$. By Lemma 2.6.6 the action of R^{d-2i+1} on U_{i-1} is injective. By these comments, $XK_i = 0$. We have now shown that $X = 0$. ■

Lemma 2.10.9. *Given $\Psi' \in \mathrm{End}(V)$ such that* (2.63) *is satisfied and $\Psi' U_i \subseteq U_{i-1}$ for $0 \leq i \leq d$. Then $\Psi' = \Psi$.*

Proof. By Lemma 2.10.2 and (2.62), Ψ satisfies these conditions. We now show the uniqueness assertion. Assume $\Psi' \in \mathrm{End}(V)$ satisfies the conditions in the statement of the lemma. Observe that $(\Psi - \Psi')R = R(\Psi - \Psi')$ and $(\Psi - \Psi')U_i \subseteq U_{i-1}$ for $0 \leq i \leq d$. The result follows from these comments along with Lemma 2.10.8. ■

Lemma 2.10.10. *With reference to Definition 2.8.1 and Lemma 2.10.1, $\Delta^{-1}A^*\Delta - A^*$ acts on U_i as*

$$(\theta^*_{i-1} - \theta^*_i)(\theta_0 - \theta_d)\Psi$$

for $1 \leq i \leq d$ and as 0 for $i = 0$.

Proof. First assume $1 \leq i \leq d$. For notational convenience, we abbreviate $\Omega = (\theta_0 - \theta_d)\Psi$. We will show that

$$\Delta^{-1}A^*\Delta - A^* - (\theta^*_{i-1} - \theta^*_i)\Omega \tag{2.64}$$

vanishes on U_i. To accomplish this, we show that the image of U_i under (2.64) is contained in both U_{i-1} and $\sum_{h=0}^{i-2} U_h$.

We first show that the image of U_i under (2.64) is contained in U_{i-1}. This follows from Lemma 2.8.14 and Lemma 2.10.2.

We now show that the image of U_i under (2.64) is contained in $\sum_{h=0}^{i-2} U_h$. Observe that (2.64) is equal to

$$\begin{aligned} \theta^*_{i-1}(\Delta^{-1} - I)\Omega \;+\; \Delta^{-1}(A^* - \theta^*_{i-1}I)\Omega \;+\; (\Delta^{-1} - I)(A^* - \theta^*_i I) \\ +\; \Delta^{-1}A^*(\Delta - I - \Omega) \;+\; \theta^*_i(\Delta^{-1} - I + \Omega). \end{aligned} \tag{2.65}$$

We will argue that each of the five terms in this sum sends U_i into $\sum_{h=0}^{i-2} U_h$. We begin by recalling some facts. For $0 \leq j \leq d$ each of

$$A^* - \theta^*_j I, \qquad \Delta - I, \qquad \Delta^{-1} - I, \qquad \Omega$$

sends U_j into $\sum_{h=0}^{j-1} U_h$. This is a consequence of Theorem 2.2.2(ii), (2.38), Lemma 2.8.6, and Lemma 2.10.2 respectively. It follows from these comments that for $0 \leq j \leq d$, each of A^*, Δ, Δ^{-1}, Ω sends U_j into $\sum_{h=0}^{j} U_h$. Using the above facts, we find that each of

$$(\Delta^{-1} - I)\Omega, \qquad \Delta^{-1}(A^* - \theta^*_{i-1}I)\Omega, \qquad (\Delta^{-1} - I)(A^* - \theta^*_i I)$$

sends U_i into $\sum_{h=0}^{i-2} U_h$. Thus each of the first three terms in the sum (2.65) sends U_i into $\sum_{h=0}^{i-2} U_h$. By Lemma 2.10.4, each of

$$\Delta - I - \Omega, \qquad \Delta^{-1} - I + \Omega$$

sends U_i into $\sum_{h=0}^{i-2} U_h$. By the above facts, each of the last two terms in the sum (2.65) sends U_i into $\sum_{h=0}^{i-2} U_h$. We have now shown that each of the five terms in the sum (2.65) sends U_i into $\sum_{h=0}^{i-2} U_h$. Therefore, the image of U_i under (2.65) is contained in $\sum_{h=0}^{i-2} U_h$.

By the above comments and Theorem 2.2.2, the expression (2.64) vanishes on U_i. The proof is complete for $1 \leq i \leq d$.

The case when $i = 0$ follows from Lemma 2.8.14. ■

Combining Lemma 2.10.10 with Lemma 2.8.13, we obtain the following corollary.

Corollary 2.10.11. *With reference to Definition 2.8.1 and Lemma 2.10.1, $\Delta^{-1}L^{\Downarrow}\Delta - L$ acts on U_i as*

$$(\theta^*_{i-1} - \theta^*_i)(\theta_0 - \theta_d)\Psi$$

for $1 \leq i \leq d$ and as 0 for $i = 0$.

2.11 The eigenvalue and dual eigenvalue sequences

We continue to discuss the TD system Φ from Definition 2.1.1. In Sections 2.14, 2.15, and 2.16, we will obtain some detailed results about Δ and Ψ. In order to do so, we must first recall some facts concerning the eigenvalues and dual eigenvalues of Φ.

Theorem 2.11.1. [10, Theorem 11.1] *The expressions*

$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}, \qquad \frac{\theta^*_{i-2} - \theta^*_{i+1}}{\theta^*_{i-1} - \theta^*_i} \tag{2.66}$$

are equal and independent of i for $2 \leq i \leq d-1$.

Definition 2.11.2. We associate a scalar β with Φ as follows. If $d \geq 3$ let $\beta + 1$ denote the common value of (3.1). If $d \leq 2$ let β denote any nonzero scalar in $\mathbb{K}$. We call β the *base* of Φ.

Theorem 2.11.3. [10, Theorem 11.2] *With reference to Definition 2.11.2, the following* (i)–(iv) *hold.*

(i) *Suppose* $\beta \neq \pm 2$, *and pick* $q \in \overline{\mathbb{K}}$ *such that* $q^2 + q^{-2} = \beta$. *Then there exist scalars* $\alpha_1, \alpha_2, \alpha_3, \alpha_1^*, \alpha_2^*, \alpha_3^*$ *in* $\overline{\mathbb{K}}$ *such that*

$$\begin{aligned} \theta_i &= \alpha_1 + \alpha_2 q^{2i} + \alpha_3 q^{-2i}, \\ \theta_i^* &= \alpha_1^* + \alpha_2^* q^{2i} + \alpha_3^* q^{-2i}, \end{aligned}$$

for $0 \leq i \leq d$. *Moreover* $q^{2i} \neq 1$ *for* $1 \leq i \leq d$.

(ii) *Suppose* $\beta = 2$ *and* $\mathrm{Char}(\mathbb{K}) \neq 2$. *Then there exist scalars* $\alpha_1, \alpha_2, \alpha_3, \alpha_1^*, \alpha_2^*, \alpha_3^*$ *in* $\mathbb{K}$ *such that*

$$\begin{aligned} \theta_i &= \alpha_1 + \alpha_2 i + \alpha_3 i^2, \\ \theta_i^* &= \alpha_1^* + \alpha_2^* i + \alpha_3^* i^2, \end{aligned}$$

for $0 \leq i \leq d$. *Moreover* $\mathrm{Char}(\mathbb{K}) = 0$ *or* $\mathrm{Char}(\mathbb{K}) > d$.

(iii) *Suppose* $\beta = -2$ *and* $\mathrm{Char}(\mathbb{K}) \neq 2$. *Then there exist scalars* $\alpha_1, \alpha_2, \alpha_3, \alpha_1^*, \alpha_2^*, \alpha_3^*$ *in* $\mathbb{K}$ *such that*

$$\begin{aligned} \theta_i &= \alpha_1 + \alpha_2 (-1)^i + \alpha_3 i (-1)^i, \\ \theta_i^* &= \alpha_1^* + \alpha_2^* (-1)^i + \alpha_3^* i (-1)^i, \end{aligned}$$

for $0 \leq i \leq d$. *Moreover* $\mathrm{Char}(\mathbb{K}) = 0$ *or* $\mathrm{Char}(\mathbb{K}) > d/2$.

(iv) *Suppose* $\beta = 0$ *and* $\mathrm{Char}(\mathbb{K}) = 2$. *Then* $d = 3$.

Lemma 2.11.4. [33, Lemma 9.4] *With reference to Definition 2.11.2, pick integers* i, j, r, s $(0 \leq i, j, r, s \leq d)$ *and assume* $i + j = r + s$, $i \neq j$. *Then the following* (i)–(iv) *hold.*

(i) *Suppose* $\beta \neq \pm 2$. *Then*

$$\frac{\theta_r - \theta_s}{\theta_i - \theta_j} = \frac{q^{r-s} - q^{s-r}}{q^{i-j} - q^{j-i}},$$

where $q^2 + q^{-2} = \beta$.

(ii) *Suppose* $\beta = 2$ *and* $\mathrm{Char}(\mathbb{K}) \neq 2$. *Then*

$$\frac{\theta_r - \theta_s}{\theta_i - \theta_j} = \frac{r - s}{i - j}.$$

(iii) *Suppose* $\beta = -2$ *and* $\mathrm{Char}(\mathbb{K}) \neq 2$. *Then*

$$\frac{\theta_r - \theta_s}{\theta_i - \theta_j} = \begin{cases} (-1)^{r+i}\frac{r-s}{i-j} & \text{if } i+j \text{ is even}, \\ (-1)^{r+i} & \text{if } i+j \text{ is odd}. \end{cases}$$

(iv) *Suppose* $\beta = 0$ *and* $\mathrm{Char}(\mathbb{K}) = 2$. *Then*

$$\frac{\theta_r - \theta_s}{\theta_i - \theta_j} = \begin{cases} 0 & \text{if } r = s, \\ 1 & \text{if } r \neq s. \end{cases}$$

Proof. Use Theorem 2.11.3. ■

2.12 Some scalars

We continue to discuss the TD system Φ from Definition 2.1.1. In Section 2.1, we used Φ to define the scalars $\{\vartheta_i\}_{i=0}^{d+1}$. In this section we discuss some properties of these scalars which will be of use later.

Recall from Definition 2.1.4 that

$$\vartheta_i = \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} \qquad (0 \leq i \leq d+1).$$

We remark that

$$\vartheta_0 = 0, \qquad \vartheta_1 = 1, \qquad \vartheta_d = 1, \qquad \vartheta_{d+1} = 0.$$

Moreover,

$$\vartheta_i = \vartheta_{d-i+1} \qquad\qquad (0 \le i \le d+1). \tag{2.67}$$

Lemma 2.12.1. *For $0 \le i \le d$,*

$$\vartheta_{d-i} - \vartheta_i = \frac{\theta_i - \theta_{d-i}}{\theta_0 - \theta_d}.$$

Proof. Use (2.12) and (2.67). ∎

We now express the ϑ_i in closed form.

Lemma 2.12.2. [33, Lemma 10.2] *With reference to Definition 2.11.2, the following holds for $0 \le i \le d+1$.*

(i) *Suppose $\beta \neq \pm 2$. Then*

$$\vartheta_i = \frac{(q^i - q^{-i})(q^{d-i+1} - q^{i-d-1})}{(q - q^{-1})(q^d - q^{-d})},$$

where $q^2 + q^{-2} = \beta$.

(ii) *Suppose $\beta = 2$ and $\mathrm{Char}(\mathbb{K}) \neq 2$. Then*

$$\vartheta_i = \frac{i(d-i+1)}{d}.$$

(iii) *Suppose $\beta = -2$, $\mathrm{Char}(\mathbb{K}) \neq 2$, and d is odd. Then*

$$\vartheta_i = \begin{cases} 0 & \text{if } i \text{ is even,} \\ 1 & \text{if } i \text{ is odd.} \end{cases}$$

(iv) *Suppose* $\beta = -2$, $\mathrm{Char}(\mathbb{K}) \neq 2$, *and* d *is even. Then*

$$\vartheta_i = \begin{cases} i/d & \text{if } i \text{ is even,} \\ (d-i+1)/d & \text{if } i \text{ is odd.} \end{cases}$$

(v) *Suppose* $\beta = 0$, $\mathrm{Char}(\mathbb{K}) = 2$, *and* $d = 3$. *Then*

$$\vartheta_i = \begin{cases} 0 & \text{if } i \text{ is even,} \\ 1 & \text{if } i \text{ is odd.} \end{cases}$$

Proof. The above sums can be computed directly using Lemma 2.11.4. ∎

Corollary 2.12.3. *With reference to Lemma 2.12.2, assume we are in the situation of* (i), (ii) *or* (iv). *Then* $\vartheta_i \neq 0$ *for* $1 \leq i \leq d$.

When we were working with the eigenvalues of Φ, a key feature was that they are mutually distinct. So it is natural to ask if there are any duplications in the sequence $\{\vartheta_i\}_{i=0}^{d+1}$. In (2.67) we already saw that $\vartheta_i = \vartheta_{d-i+1}$ for $0 \leq i \leq d+1$. So we would like to know if the $\{\vartheta_i\}_{i=0}^{r}$ are mutually distinct, where $r = \lfloor \frac{d+1}{2} \rfloor$. It turns out that this is false in general, but something can be said in certain cases. We now explain the details.

Corollary 2.12.4. *With reference to Definition 2.11.2, the following holds for* $0 \leq i, j \leq d+1$.

(i) *Suppose* $\beta \neq \pm 2$. *Then*

$$\vartheta_i - \vartheta_j = \frac{\left(q^{i-j} - q^{j-i}\right)\left(q^{d-i-j+1} - q^{i+j-d-1}\right)}{\left(q - q^{-1}\right)\left(q^d - q^{-d}\right)}.$$

(ii) *Suppose* $\beta = 2$ *and* $\mathrm{Char}(\mathbb{K}) \neq 2$. *Then*

$$\vartheta_i - \vartheta_j = \frac{(i-j)(d-i-j+1)}{d}.$$

(iii) *Suppose* $\beta = -2$, $\mathrm{Char}(\mathbb{K}) \neq 2$, *and* d *odd. Then*

$$\vartheta_i - \vartheta_j = \begin{cases} 0 & \text{if } i+j \text{ is even,} \\ (-1)^j & \text{if } i+j \text{ is odd.} \end{cases}$$

(iv) *Suppose* $\beta = -2$, $\mathrm{Char}(\mathbb{K}) \neq 2$, *and* d *even. Then*

$$\vartheta_i - \vartheta_j = \begin{cases} (-1)^j \frac{i-j}{d} & \text{if } i+j \text{ is even,} \\ (-1)^j \frac{d-i-j+1}{d} & \text{if } i+j \text{ is odd.} \end{cases}$$

(v) *Suppose* $\beta = 0$, $\mathrm{Char}(\mathbb{K}) = 2$, *and* $d = 3$. *Then*

$$\vartheta_i - \vartheta_j = \begin{cases} 0 & \text{if } i+j \text{ is even,} \\ 1 & \text{if } i+j \text{ is odd.} \end{cases}$$

Proof. Use Lemma 2.12.2. ■

Lemma 2.12.5. *With reference to Lemma 2.12.2, assume we are in the situation of* (i), (ii) *or* (iv). *Then the following are equivalent for* $0 \leq i, j \leq d+1$.

(i) $\vartheta_i = \vartheta_j$.

(ii) $i = j$ *or* $i + j = d + 1$.

Proof. Use Theorem 2.11.3 and Corollary 2.12.4. ■

We finish this section with a comment.

Lemma 2.12.6. *For* $0 \leq i, j, r, s \leq d$ *we have*

$$(\theta_r - \theta_s)(\vartheta_i - \vartheta_j) = (\theta_i - \theta_j)(\vartheta_r - \vartheta_s),$$

provided that $i + j = r + s$.

Proof. Use Lemma 2.11.4 and Corollary 2.12.4. ■

2.13 The scalars $[r, s, t]$

We continue to discuss the TD system Φ from Definition 2.1.1. To motivate our results in this section, for the moment fix an integer i $(0 \leq i \leq d/2)$. As we proceed, it will be convenient to express each of $\{\tau_{ij}\}_{j=i}^{d-i}$ as a linear combination of $\{\eta_{ij}\}_{j=i}^{d-i}$. In order to describe the coefficients, we will use the following notation.

For all $a, q \in \overline{\mathbb{K}}$ define

$$(a; q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}), \qquad n = 0, 1, 2, \ldots \tag{2.68}$$

and interpret $(a; q)_0 = 1$.

In [34] Terwilliger defined some scalars $[r, s, t]_q \in \mathbb{K}$ for nonnegative integers r, s, t such that $r + s + t \leq d$. By [34, Lemma 13.2] these scalars are rational functions of the base β. In this paper we are going to drop the subscript q and just write $[r, s, t]$. For further discussion of these scalars see [16] and [34].

Definition 2.13.1. [34, Lemma 13.2] With reference to Definition 2.11.2, let r, s, t denote nonnegative integers such that $r + s + t \leq d$. We define $[r, s, t]$ as follows.

(i) Suppose $\beta \neq \pm 2$. Then

$$[r, s, t] = \frac{(q^2; q^2)_{r+s}(q^2; q^2)_{r+t}(q^2; q^2)_{s+t}}{(q^2; q^2)_r(q^2; q^2)_s(q^2; q^2)_t(q^2; q^2)_{r+s+t}},$$

where $q^2 + q^{-2} = \beta$.

(ii) Suppose $\beta = 2$ and $\mathrm{Char}(\mathbb{K}) \neq 2$. Then

$$[r, s, t] = \frac{(r+s)!\,(r+t)!\,(s+t)!}{r!\,s!\,t!\,(r+s+t)!}.$$

(iii) Suppose $\beta = -2$ and $\mathrm{Char}(\mathbb{K}) \neq 2$. If each of r, s, t is odd, then $[r, s, t] = 0$. If at least one of r, s, t is even, then

$$[r, s, t] = \frac{\lfloor \frac{r+s}{2} \rfloor ! \lfloor \frac{r+t}{2} \rfloor ! \lfloor \frac{s+t}{2} \rfloor !}{\lfloor \frac{r}{2} \rfloor ! \lfloor \frac{s}{2} \rfloor ! \lfloor \frac{t}{2} \rfloor ! \lfloor \frac{r+s+t}{2} \rfloor !}.$$

The expression $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

(iv) Suppose $\beta = 0$, $\mathrm{Char}(\mathbb{K}) = 2$, and $d = 3$. If each of r, s, t equals 1, then $[r, s, t] = 0$. If at least one of r, s, t equals 0, then $[r, s, t] = 1$.

We make a few observations. The expression $[r, s, t]$ is symmetric in r, s, t. Also, $[r, s, t] = 1$ if at least one of r, s, t equals zero.

Lemma 2.13.2. [16, Lemma 5.3] *Let r, s, t, u denote nonnegative integers such that $r + s + t + u \leq d$. Then*

$$[r, s, t + u][t, u, r + s] = [s, u, r + t][r, t, s + u].$$

The following result is a modification of [27, Lemma 12.4].

Lemma 2.13.3. *Let $0 \leq i \leq d/2$ and $i \leq j \leq d - i$. Both*

$$\tau_{ij} = \sum_{h=0}^{j-i} [h, j - i - h, d - i - j] \tau_{i,i+h}(\theta_{d-i}) \eta_{i,j-h}, \tag{2.69}$$

$$\eta_{ij} = \sum_{h=0}^{j-i} [h, j - i - h, d - i - j] \eta_{i,i+h}(\theta_i) \tau_{i,j-h}. \tag{2.70}$$

Proof. Apply [27, Lemma 12.4] to the sequence $\{\theta_k\}_{k=i}^{d-i}$. ■

Later in the paper, we will be doing some computations involving the coefficients in (2.69) and (2.70). The following results will aid in these computations.

Corollary 2.13.4. *For $0 \leq i \leq d/2$ and $i+1 \leq j \leq d-i$,*

$$(\theta_0 - \theta_d)(\vartheta_j - \vartheta_i) = (\theta_i - \theta_{d-i})\,[1, j-i-1, d-i-j].$$

Proof. Let C denote the coefficient of x^{j-i-1} on either side of (2.69). From the left-hand side of (2.69), we see

$$C = -\sum_{h=i}^{j-1} \theta_h. \tag{2.71}$$

From the right-hand side of (2.69), we see

$$C = (\theta_{d-i} - \theta_i)\,[j-i-1, 1, d-i-j] - \sum_{h=i}^{j-1} \theta_{d-h}. \tag{2.72}$$

Subtract (2.71) from (2.72) and invoke the symmetry of $[r, s, t]$ as well as Definition 2.1.4 to get the result. ■

Lemma 2.13.5. *For $0 \leq i \leq d/2$ and $i+1 \leq j \leq d-i$ and $0 \leq h \leq j-i-1$,*

$$\begin{aligned}(\vartheta_j - \vartheta_i)&[h, j-i-h-1, d-i-j+1] \\ &= (\vartheta_{j-h} - \vartheta_i)\,[h, j-i-h, d-i-j]\end{aligned} \tag{2.73}$$

and

$$\begin{aligned}(\vartheta_j - \vartheta_i)&[h, j-i-h-1, d-i-j+1] \\ &= (\vartheta_{i+h+1} - \vartheta_i)\,[h+1, j-i-h-1, d-i-j].\end{aligned} \tag{2.74}$$

Proof. For (2.73), use Lemma 2.13.2 with $r=1$, $s=j-i-h-1$, $t=d-i-j$, $u=h$. Simplify the result using Corollary 2.13.4 and the fact that $[r, s, t]$ is symmetric in r, s, t.

Line (2.74) is similarly obtained. ■

Corollary 2.13.6. *With reference to Lemma 2.12.2, assume we are in the situation of* (i), (ii) *or* (iv). *For $0 \leq i \leq d/2$ and $i \leq j \leq d-i$ and $0 \leq h \leq j-i$,*

$$[h, j-i-h, d-i-j] = \prod_{k=0}^{h-1} \frac{\vartheta_{j-k} - \vartheta_i}{\vartheta_{d-i-k} - \vartheta_i}. \tag{2.75}$$

In (2.75) the denominators are nonzero by Lemma 2.12.5.

Proof. Assume $h \geq 1$; otherwise both sides of (2.75) equal 1. From (2.74) we obtain

$$[h, j-i-h, d-i-j] = \frac{\vartheta_j - \vartheta_i}{\vartheta_{i+h} - \vartheta_i}[h-1, j-i-h, d-i-j+1].$$

Iterating this we get

$$[h, j-i-h, d-i-j] = \prod_{k=0}^{h-1} \frac{\vartheta_{j-k} - \vartheta_i}{\vartheta_{i+k+1} - \vartheta_i}.$$

Evaluating the denominator using Lemma 2.12.5 we obtain the result. ■

2.14 The maps Δ, Ψ commute

We continue to discuss the TD system Φ from Definition 2.1.1. In Section 2.8, we introduced the linear transformation Δ and discussed some of its properties. In Section 2.10, we introduced the linear transformation Ψ and discussed some of its properties. We now discuss how Δ, Ψ relate to each other. Along this line we have two main results. They are Theorem 2.14.1 and Theorem 2.16.1. We prove Theorem 2.14.1 in this section. Before proving Theorem 2.16.1, it will be convenient to give the characterization of Ψ discussed in the Introduction. This will be done in Section 2.15.

Theorem 2.14.1. *With reference to Definition 2.8.1 and Lemma 2.10.1, the operators* Δ, Ψ *commute.*

Proof. Recall the decomposition of V given in Corollary 2.6.5. We will show that $\Psi\Delta$, $\Delta\Psi$ agree on each summand $\tau_{ij}(A)K_i$.

First assume that $i = j$. Recall that τ_{ii} and η_{ii} both equal 1. Using (2.55) and the fact that $\Psi K_i = 0$, we routinely find that each of $\Psi\Delta$, $\Delta\Psi$ vanishes on $\tau_{ii}(A)K_i$.

Next assume that $i < j$. In order to show that $\Psi\Delta$, $\Delta\Psi$ agree on $\tau_{ij}(A)K_i$, it suffices to show that $\Psi\Delta\tau_{ij}(A)$ and $\Delta\Psi\tau_{ij}(A)$ agree on K_i. By (2.70), Lemma 2.9.2, and Lemma 2.10.1, the operators $\Psi\Delta\tau_{ij}(A)$ and

$$\sum_{h=0}^{j-i-1} (\vartheta_{j-h} - \vartheta_i)\,[h, j-i-h, d-i-j]\eta_{i,i+h}(\theta_i)\tau_{i,j-h-1}(A) \qquad (2.76)$$

agree on K_i. By (2.70), Lemma 2.9.2, and Lemma 2.10.1, the operators $\Delta\Psi\tau_{ij}(A)$ and

$$(\vartheta_j - \vartheta_i) \sum_{h=0}^{j-i-1} [h, j-i-h-1, d-i-j+1]\eta_{i,i+h}(\theta_i)\tau_{i,j-h-1}(A) \qquad (2.77)$$

agree on K_i. In order to show (2.76), (2.77) agree on K_i, we will need the fact that

$$(\vartheta_{j-h} - \vartheta_i)\,[h, j-i-h, d-i-j]$$

and

$$(\vartheta_j - \vartheta_i)\,[h, j-i-h-1, d-i-j+1]$$

are equal for $0 \le h \le j-i-1$. This equality is (2.73). Therefore (2.76) and (2.77) agree on K_i. Thus $\Psi\Delta\tau_{ij}(A)$ and $\Delta\Psi\tau_{ij}(A)$ agree on K_i. Hence $\Psi\Delta$, $\Delta\Psi$ agree on $\tau_{ij}(A)K_i$. By Corollary 2.6.5, $\Psi\Delta$, $\Delta\Psi$ agree on V. ■

From Theorem 2.14.1, we derive a number of corollaries.

Corollary 2.14.2. *With reference to Lemma 2.10.1, $\Psi^{\Downarrow} = \Psi$.*

Proof. We first show that $\Psi^{\Downarrow}\Delta = \Delta\Psi$. Recall the decomposition of V given in Corollary 2.6.5. We will show that $\Psi^{\Downarrow}\Delta$, $\Delta\Psi$ agree on each summand $\tau_{ij}(A)K_i$. By (2.54) and (2.57) (applied to both Φ and $\Phi^{\Downarrow}$), $\Psi^{\Downarrow}\Delta\tau_{ij}(A)$ and $\Delta\Psi\tau_{ij}(A)$ agree on K_i. Hence $\Psi^{\Downarrow}\Delta$, $\Delta\Psi$ agree on $\tau_{ij}(A)K_i$. By Corollary 2.6.5, $\Psi^{\Downarrow}\Delta$, $\Delta\Psi$ agree on V. Thus $\Psi^{\Downarrow}\Delta = \Delta\Psi$. Combine this fact with Theorem 2.14.1 and the fact that Δ is invertible to get the result. ■

Corollary 2.14.3. *With reference to Lemma 2.10.1, we have*

$$\Psi U_i^{\Downarrow} \subseteq U_{i-1}^{\Downarrow} \qquad (1 \leq i \leq d), \qquad \Psi U_0^{\Downarrow} = 0.$$

Proof. Combine Corollary 2.14.2 with Lemma 2.10.2. ∎

Corollary 2.14.4. *With reference to Lemma 2.10.1, we have*

$$\Psi E_i V \subseteq E_{i-1}V + E_iV + E_{i+1}V \qquad (0 \leq i \leq d).$$

Proof. Let i be given. On the one hand, by Theorem 2.2.2(iii) and Lemma 2.10.2, we have

$$\begin{aligned} \Psi E_i V &\subseteq \Psi(E_iV + E_{i+1}V + \cdots + E_dV) \\ &= \Psi(U_i + U_{i+1} + \cdots + U_d) \\ &\subseteq U_{i-1} + U_i + \cdots + U_d \\ &= E_{i-1}V + E_{i+1}V + \cdots + E_dV. \end{aligned} \tag{2.78}$$

On the other hand, by Theorem 2.2.2(iii) applied to $\Phi^{\Downarrow}$ and Corollary 2.14.3, we have

$$\begin{aligned} \Psi E_i V &\subseteq \Psi(E_0V + E_1V + \cdots + E_iV) \\ &= \Psi(U_{d-i}^{\Downarrow} + U_{d-i+1}^{\Downarrow} + \cdots + U_d^{\Downarrow}) \\ &\subseteq U_{d-i-1}^{\Downarrow} + U_{d-i}^{\Downarrow} + \cdots + U_d^{\Downarrow} \\ &= E_0V + E_1V + \cdots + E_{i+1}V. \end{aligned} \tag{2.79}$$

Observe that $\Psi E_i V$ is in the intersection of (2.78) and (2.79). This intersection equals $E_{i-1}V + E_iV + E_{i+1}V$, and the result follows. ∎

2.15 A characterization of Ψ

We continue to discuss the TD system Φ from Definition 2.1.1. Our goal in this section is to obtain the characterization of Ψ given in the Introduction.

Lemma 2.15.1. *With reference to Lemma 2.10.1, we have*

$$\Psi E_i^*V \subseteq E_0^*V + E_1^*V + \cdots + E_{i-1}^*V \qquad (0 \leq i \leq d).$$

Proof. Using Theorem 2.2.2(iii) and Lemma 2.10.2, we obtain

$$\begin{aligned}\Psi E_i^*V &\subseteq \Psi\left(E_0^*V + E_1^*V + \cdots + E_i^*V\right)\\ &= \Psi\left(U_0 + U_1 + \cdots + U_i\right)\\ &\subseteq U_0 + U_1 + \cdots + U_{i-1}\\ &= E_0^*V + E_1^*V + \cdots + E_{i-1}^*V.\end{aligned}$$

■

Lemma 2.15.2. *With reference to Definition 2.8.1 and Lemma 2.10.1, for* $0 \leq j \leq d$ *apply either of*

$$\Delta - I - (\theta_0 - \theta_d)\Psi, \qquad \Delta^{-1} - I + (\theta_0 - \theta_d)\Psi$$

to E_j^*V *and consider the image. This image is contained in* $E_0^*V + E_1^*V + \cdots + E_{j-2}^*V$ *if* $j \geq 2$ *and equals* 0 *if* $j < 2$.

Proof. Use Theorem 2.2.2(iii) and Lemma 2.10.4. ■

By Corollary 2.14.4 and Lemma 2.15.2, both

$$\Psi E_iV \subseteq E_{i-1}V + E_iV + E_{i+1}V,$$
$$\left(\Psi - \frac{\Delta - I}{\theta_0 - \theta_d}\right) E_i^*V \subseteq E_0^*V + E_1^*V + \cdots + E_{i-2}^*V$$

for $0 \leq i \leq d$. We show that these two properties characterize Ψ.

Lemma 2.15.3. *Given $\Psi' \in \mathrm{End}(V)$ such that both*

$$\Psi' E_i V \subseteq E_{i-1}V + E_i V + E_{i+1}V,$$
$$\left(\Psi' - \frac{\Delta - I}{\theta_0 - \theta_d}\right) E_i^* V \subseteq E_0^* V + E_1^* V + \cdots + E_{i-2}^* V$$

for $0 \leq i \leq d$. Then $\Psi' = \Psi$.

Proof. Recall from Theorem 2.2.2 that $\{U_i\}_{i=0}^d$ is a decomposition of V. It suffices to show that Ψ, Ψ' agree on U_i for $0 \leq i \leq d$. Let i be given. Observe that

$$\Psi - \Psi' = \Psi - \frac{\Delta - I}{\theta_0 - \theta_d} - \Psi' + \frac{\Delta - I}{\theta_0 - \theta_d}. \tag{2.80}$$

Using (2.80) along with Theorem 2.2.2(iii) and Lemma 2.15.2, we obtain

$$\begin{aligned}
(\Psi - \Psi')U_i &\subseteq (\Psi - \Psi')(U_0 + U_1 + \cdots + U_i) \\
&= (\Psi - \Psi')(E_0^* V + E_1^* V + \cdots + E_i^* V) \\
&\subseteq E_0^* V + E_1^* V + \cdots + E_{i-2}^* V \\
&= U_0 + U_1 + \cdots + U_{i-2}.
\end{aligned}$$

By Theorem 2.2.2(iii) and Corollary 2.14.4,

$$\begin{aligned}
(\Psi - \Psi')U_i &\subseteq (\Psi - \Psi')(U_i + U_{i+1} + \cdots + U_d) \\
&= (\Psi - \Psi')(E_i V + E_{i+1} V + \cdots + E_d V) \\
&\subseteq E_{i-1} V + E_i V + \cdots + E_d V \\
&= U_{i-1} + U_i + \cdots + U_d.
\end{aligned}$$

Thus $(\Psi - \Psi')U_i$ is contained in the intersection of $U_0 + U_1 + \cdots + U_{i-2}$ and $U_{i-1} + U_i + \cdots + U_d$. This intersection is zero since $\{U_i\}_{i=0}^d$ is a decomposition of V. So $\Psi - \Psi'$ vanishes on U_i. Therefore Ψ, Ψ' agree on U_i. ∎

2.16 In general, $\Delta^{\pm 1}$ are polynomials in Ψ

We continue to discuss the TD system Φ from Definition 2.1.1. Recall the map Δ from Definition 2.8.1 and the map Ψ from Lemma 2.10.1. In Section 2.14, we saw that Δ, Ψ commute. In this section, we show that $\Delta^{\pm 1}$ are polynomials in Ψ provided that each of $\vartheta_1, \vartheta_2, \ldots, \vartheta_d$ is nonzero.

Theorem 2.16.1. *Let $\Delta \in \mathrm{End}(V)$ be as in Definition 2.8.1 and let $\Psi \in \mathrm{End}(V)$ be as in Lemma 2.10.1. With reference to Lemma 2.12.2, assume we are in the situation of* (i), (ii), *or* (iv) *so that the scalars $\{\vartheta_i\}_{i=1}^d$ from Definition 2.1.4 are nonzero. Then both*

$$\Delta = I + \frac{\eta_1(\theta_0)}{\vartheta_1}\Psi + \frac{\eta_2(\theta_0)}{\vartheta_1\vartheta_2}\Psi^2 + \cdots + \frac{\eta_d(\theta_0)}{\vartheta_1\vartheta_2\cdots\vartheta_d}\Psi^d, \tag{2.81}$$

$$\Delta^{-1} = I + \frac{\tau_1(\theta_d)}{\vartheta_1}\Psi + \frac{\tau_2(\theta_d)}{\vartheta_1\vartheta_2}\Psi^2 + \cdots + \frac{\tau_d(\theta_d)}{\vartheta_1\vartheta_2\cdots\vartheta_d}\Psi^d. \tag{2.82}$$

Proof. We first show (2.81). Recall the decomposition of V from Corollary 2.6.5. We show that each side of (2.81) agrees on each summand $\tau_{ij}(A)K_i$. Let $v \in K_i$. We apply each side of (2.81) to the vector $\tau_{ij}(A)v$ and show that the results agree.

We first apply the left-hand side of (2.81) to $\tau_{ij}(A)v$. By Lemma 2.9.2 and (2.70), $\Delta\tau_{ij}(A)v$ is a linear combination of $\{\tau_{i,j-h}(A)v\}_{h=0}^{j-i}$ such that the coefficient of $\tau_{i,j-h}(A)v$ is

$$[h, j-i-h, d-i-j]\eta_{i,i+h}(\theta_i) \tag{2.83}$$

for $0 \leq h \leq j-i$. We now apply the right-hand side of (2.81) to $\tau_{ij}(A)v$. For the sum on the right-hand side of (2.81), the action of each term on $\tau_{ij}(A)v$ is computed using (2.57). From this computation, one finds that the right-hand side of (2.81) applied to $\tau_{ij}(A)v$ is a linear combination of $\{\tau_{i,j-h}(A)v\}_{h=0}^{j-i}$ such that the coefficient of $\tau_{i,j-h}(A)v$

is

$$\frac{\eta_h(\theta_0)}{\vartheta_1 \vartheta_2 \cdots \vartheta_h} \prod_{k=0}^{h-1} (\vartheta_{j-k} - \vartheta_i) \tag{2.84}$$

for $0 \leq h \leq j-i$. It remains to show that (2.83) is equal to (2.84) for $0 \leq h \leq j-i$. Let h be given. By (2.8) and Corollary 2.13.6, the scalar (2.83) is equal to

$$\prod_{k=0}^{h-1} \frac{(\theta_i - \theta_{d-i-k})\,(\vartheta_{j-k} - \vartheta_i)}{\vartheta_{d-i-k} - \vartheta_i}. \tag{2.85}$$

By (2.11) and since $\vartheta_\ell = \vartheta_{d-\ell+1}$ for $1 \leq \ell \leq h$, the scalar (2.84) is equal to

$$\prod_{k=0}^{h-1} \frac{(\theta_0 - \theta_{d-k})\,(\vartheta_{j-k} - \vartheta_i)}{\vartheta_{d-k}}. \tag{2.86}$$

By Lemma 2.12.6 and since $\vartheta_0 = 0$,

$$\frac{\theta_i - \theta_{d-i-k}}{\vartheta_{d-i-k} - \vartheta_i} = \frac{\theta_0 - \theta_{d-k}}{\vartheta_{d-k}} \qquad (0 \leq k \leq h-1).$$

Using this we find that (2.85) is equal to (2.86). Therefore (2.83) is equal to (2.84) for $0 \leq h \leq j-i$ as desired. We have shown (2.81).

To get (2.82), apply (2.81) to $\Phi^{\Downarrow}$ and use Corollary 2.14.2 along with the fact that $\vartheta_k^{\Downarrow} = \vartheta_k$ for $1 \leq k \leq d$. ■

Chapter 3

Tridiagonal systems and $U_q(\mathfrak{sl}_2)$

This part of the thesis explores a connection between TD pairs and the quantum enveloping algebra $U_q(\mathfrak{sl}_2)$. In this part, we focus on TD pairs of q-Racah type. For simplicity, we also assume that $\mathbb{K}$ is algebraically closed. We define two linear transformations $K : V \to V$ and $B : V \to V$ which act on the split decompositions in an attractive way. Using Ψ, K, B we obtain two $U_q(\mathfrak{sl}_2)$-module structures on V. For each of the $U_q(\mathfrak{sl}_2)$-module structures, we compute the action of the Casimir element on V. We show that these two actions agree. Using this fact, we express Ψ as a rational function of $K^{\pm 1}, B^{\pm 1}$ in several ways. Eliminating Ψ from these equations we find that K and B are related by a quadratic equation.

3.1 The q-Racah case

We now focus our attention on a special class of TD systems said to have q-Racah type. Recall from Section 2.11 that the expressions

$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}, \qquad \frac{\theta^*_{i-2} - \theta^*_{i+1}}{\theta^*_{i-1} - \theta^*_i} \tag{3.1}$$

are equal and independent of i for $2 \leq i \leq d-1$. This gives two recurrence relations whose solutions can be written in closed form. There are several cases [10, Theorem 11.2]. The most general case is known as the q-Racah case [17, Section 1] and is described as follows.

We say that the TD system Φ has *q-Racah type* whenever there exist nonzero scalars $q, a, b \in \overline{\mathbb{K}}$ such that $q^4 \neq 1$ and

$$\theta_i = aq^{d-2i} + a^{-1}q^{2i-d}, \qquad \theta^*_i = bq^{d-2i} + b^{-1}q^{2i-d}$$

for $0 \leq i \leq d$.

Throughout the rest of this thesis, we make the following assumption.

Assumption 3.1.1. Assume that $\mathbb{K}$ is algebraically closed and that the TD system Φ has q-Racah type. Thus, there exist nonzero scalars $q, a, b \in \mathbb{K}$ such that $q^4 \neq 1$ and

$$\theta_i = aq^{d-2i} + a^{-1}q^{2i-d}, \qquad \theta^*_i = bq^{d-2i} + b^{-1}q^{2i-d} \tag{3.2}$$

for $0 \leq i \leq d$. To avoid trivialities, we also assume that the diameter d is at least three.

Lemma 3.1.2. *With reference to Assumption 3.1.1, the following hold.*

(i) *Neither of a^2, b^2 is among $q^{2d-2}, q^{2d-4}, \ldots, q^{2-2d}$.*

(ii) *$q^{2i} \neq 1$ for $1 \leq i \leq d$.*

Proof. By Definition 2.1.2 the $\{\theta_i\}_{i=0}^d$ are mutually distinct and the $\{\theta^*_i\}_{i=0}^d$ are mutually distinct. ∎

3.2 The linear transformations K, B

We continue to discuss the situation of Assumption 3.1.1. In this section, we introduce two linear transformations $K : V \to V$ and $B : V \to V$ and discuss their actions on the split decompositions.

Definition 3.2.1. Define $K \in \mathrm{End}(V)$ such that for $0 \leq i \leq d$, U_i is the eigenspace of K with eigenvalue q^{d-2i}. In other words,

$$(K - q^{d-2i} I)U_i = 0 \qquad (0 \leq i \leq d). \tag{3.3}$$

Definition 3.2.2. Define $B \in \mathrm{End}(V)$ such that for $0 \leq i \leq d$, $U_i^{\Downarrow}$ is the eigenspace of B with eigenvalue q^{d-2i}. In other words,

$$(B - q^{d-2i} I)U_i^{\Downarrow} = 0 \qquad (0 \leq i \leq d). \tag{3.4}$$

Observe that $B = K^{\Downarrow}$.

By construction each of K, B is invertible and diagonalizable on V.

Lemma 3.2.3. *For* $0 \leq i \leq d$,

$$(B - q^{d-2i} I)U_i \subseteq U_0 + U_1 + \cdots + U_{i-1}, \tag{3.5}$$

$$(K - q^{d-2i} I)U_i^{\Downarrow} \subseteq U_0^{\Downarrow} + U_1^{\Downarrow} + \cdots + U_{i-1}^{\Downarrow}. \tag{3.6}$$

Proof. We first show (3.5). By Lemma 2.3.1, $U_i \subseteq U_0^{\Downarrow} + U_1^{\Downarrow} + \cdots + U_i^{\Downarrow}$. Use this fact along with (3.4).

The proof of (3.6) is similar. ■

Recall the raising maps $R, R^{\Downarrow}$ from Sections 2.2 and 2.3. We now express $R, R^{\Downarrow}$ in terms of A, K, B.

Lemma 3.2.4. *We have*

$$R = A - aK - a^{-1}K^{-1}, \tag{3.7}$$

$$R^{\Downarrow} = A - a^{-1}B - aB^{-1}. \tag{3.8}$$

Proof. To obtain (3.7), observe that $K = \sum_{i=0}^{d} q^{d-2i} F_i$ and $K^{-1} = \sum_{i=0}^{d} q^{2i-d} F_i$. The result follows from this and Definition 2.2.10.

The proof of (3.8) is similar. ■

We now recall some results concerning K and B.

Lemma 3.2.5. [18, Section 1.1]. *Both*

$$KRK^{-1} = q^{-2}R, \qquad BR^{\Downarrow}B^{-1} = q^{-2}R^{\Downarrow}. \tag{3.9}$$

Proof. We first show the equation on the left in (3.9). Recall that for $0 \leq i \leq d$, U_i is an eigenspace for K with eigenvalue q^{d-2i}. Use this fact along with (2.20).

The proof is similar for the equation on the right in (3.9). ■

Lemma 3.2.6. [18, Section 1.1]. *Both*

$$\frac{qKA - q^{-1}AK}{q - q^{-1}} = aK^2 + a^{-1}I, \qquad \frac{qBA - q^{-1}AB}{q - q^{-1}} = a^{-1}B^2 + aI. \tag{3.10}$$

Proof. First we show the equation on the left in (3.10). By Lemma 3.2.5, $qKR - q^{-1}RK = 0$. In this equation, eliminate R using (3.7).

The proof is similar for the equation on the right in (3.10). ■

We conclude this section by giving a result which relates R and $R^{\Downarrow}$. Combining (3.7), (3.8) we obtain

$$R^{\Downarrow} - R = aK + a^{-1}K^{-1} - a^{-1}B - aB^{-1}. \tag{3.11}$$

3.3 The linear transformation ψ

We continue to discuss the situation of Assumption 3.1.1. In Section 2.10 we introduced an element $\Psi \in \mathrm{End}(V)$. For our present purpose it is convenient to use the normalization $\psi = (q - q^{-1})(q^d - q^{-d})\Psi$. We note that Lemma 2.10.5 can now be reformulated as follows.

Lemma 3.3.1. *The map ψ is the unique element of $\mathrm{End}(V)$ such that both*

$$\psi R - R\psi = (q - q^{-1})(K - K^{-1}) \tag{3.12}$$

and $\psi K_i = 0$ for $0 \leq i \leq d/2$.

Recall the decomposition of V given in Corollary 2.6.5 and consider the summand $\tau_{ij}(A)K_i$. We describe the action of ψ on this summand. By (2.57), for $v \in K_i$,

$$\psi \tau_{ij}(A)v = (q^{j-i} - q^{i-j})(q^{d-i-j+1} - q^{i+j-d-1})\tau_{i,j-1}(A)v. \tag{3.13}$$

We also note that the following hold on $\tau_{ij}(A)K_i$:

$$R\psi = (q^{j-i} - q^{i-j})(q^{d-i-j+1} - q^{i+j-d-1})I,$$

$$\psi R = (q^{j-i+1} - q^{i-j-1})(q^{d-i-j} - q^{i+j-d})I.$$

Lemma 3.3.2. *With reference to Lemma 3.3.1,*

$$K\psi K^{-1} = q^2\psi, \qquad B\psi B^{-1} = q^2\psi. \tag{3.14}$$

Proof. Use Lemma 2.10.2 together with the definitions of K and B. ∎

Lemma 3.3.3. *For $0 \leq i \leq d/2$, K_i is the kernel of ψ acting on U_i.*

Proof. Use Lemma 2.6.4 along with (3.13) and the fact that $q^{2j} \neq 1$ for $1 \leq j \leq d$. ∎

3.4 The algebra $U_q(\mathfrak{sl}_2)$

In this section we recall the quantum universal enveloping algebra $U_q(\mathfrak{sl}_2)$. See [20], [21] for background information.

Definition 3.4.1. Let $U_q(\mathfrak{sl}_2)$ denote the $\mathbb{K}$-algebra with generators e, f, k, k^{-1} and relations

$$kk^{-1} = k^{-1}k = 1,$$

$$kek^{-1} = q^2 e, \qquad kfk^{-1} = q^{-2} f, \tag{3.15}$$

$$ef - fe = \frac{k - k^{-1}}{q - q^{-1}}. \tag{3.16}$$

We refer to $e, f, k^{\pm 1}$ as the *Chevalley generators* for $U_q(\mathfrak{sl}_2)$.

Following [20, p. 21], we define the normalized Casimir element Λ for $U_q(\mathfrak{sl}_2)$ by

$$\Lambda = (q - q^{-1})^2 ef + q^{-1}k + qk^{-1}, \tag{3.17}$$

$$= (q - q^{-1})^2 fe + qk + q^{-1}k^{-1}. \tag{3.18}$$

By [20, Lemma 2.7], Λ is central in $U_q(\mathfrak{sl}_2)$.

Lemma 3.4.2. *With reference to Definition 3.4.1, both*

$$f^2 e - (q^2 + q^{-2}) fef + ef^2 = -\Lambda f, \tag{3.19}$$

$$e^2 f - (q^2 + q^{-2}) efe + fe^2 = -\Lambda e. \tag{3.20}$$

Proof. We first prove (3.19). The left-hand side of (3.19) is equal to

$$-f(ef - fe) - (q - q^{-1})^2 fef + (ef - fe)f. \tag{3.21}$$

By (3.16), the element (3.21) is equal to

$$-f\frac{k-k^{-1}}{q-q^{-1}} - (q-q^{-1})^2 fef + \frac{k-k^{-1}}{q-q^{-1}}f.$$

Line (3.19) follows from this along with (3.15) and (3.18).

The proof is similar for (3.20). ∎

We now discuss the finite-dimensional modules for $U_q(\mathfrak{sl}_2)$. Recall the natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$ and the integers $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$. For $n \in \mathbb{Z}$ define

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}.$$

For $n \in \mathbb{N}$ define

$$[n]_q^! = [n]_q [n-1]_q \cdots [1]_q,$$

where we interpret $[0]_q^! = 1$.

Lemma 3.4.3. [20, Theorem 2.6]. *For $n \in \mathbb{N}$ and $\varepsilon \in \{1, -1\}$, there exists a $U_q(\mathfrak{sl}_2)$-module $L(n, \varepsilon)$ with the following properties. $L(n, \varepsilon)$ has a basis $\{v_i\}_{i=0}^n$ such that*

$$ev_i = \varepsilon [n+1-i]_q v_{i-1} \qquad (1 \le i \le n), \qquad ev_0 = 0, \tag{3.22}$$

$$fv_i = [i+1]_q v_{i+1} \qquad (0 \le i \le n-1), \qquad fv_n = 0, \tag{3.23}$$

$$kv_i = \varepsilon q^{n-2i} v_i \qquad (0 \le i \le n). \tag{3.24}$$

The $U_q(\mathfrak{sl}_2)$-module $L(n, \varepsilon)$ is irreducible provided that $q^{2i} \neq 1$ for $1 \le i \le n$.

With reference to Lemma 3.4.3 we refer to ε as the *type* of $L(n, \varepsilon)$.

Lemma 3.4.4. [20, Lemma 2.7]. *With reference to Lemma 3.4.3, for $n \in \mathbb{N}$ and $\varepsilon \in \{1, -1\}$, Λ acts on $L(n, \varepsilon)$ as $\varepsilon(q^{n+1} + q^{-n-1})$ times the identity.*

If q is not a root of unity, then the $L(n,\varepsilon)$ ($n \in \mathbb{N}, \varepsilon \in \{1,-1\}$) are the only finite-dimensional irreducible $U_q(\mathfrak{sl}_2)$-modules. If q is a root of unity, there are other types of finite-dimensional irreducible $U_q(\mathfrak{sl}_2)$-modules. See [20, Chapter 2] for a complete classification. In our application, we will only be concerned with the finite-dimensional irreducible $U_q(\mathfrak{sl}_2)$-modules of type $L(n,\varepsilon)$.

We now consider finite-dimensional $U_q(\mathfrak{sl}_2)$-modules which are not necessarily irreducible.

Definition 3.4.5. Let V denote a finite-dimensional $U_q(\mathfrak{sl}_2)$-module. We say that V is *semisimple* whenever it is a direct sum of irreducible $U_q(\mathfrak{sl}_2)$-modules.

Definition 3.4.6. [20, Section 2.2]. Let V denote a finite-dimensional $U_q(\mathfrak{sl}_2)$-module. For $\lambda \in \mathbb{K}$, let $V_\lambda = \{v \in V | kv = \lambda v\}$. We call λ a *weight* of V whenever $V_\lambda \neq 0$. In this case we call V_λ the *weight space of* V *associated with* λ.

Referring to Lemma 3.4.3, assume $q^{2i} \neq 1$ for $1 \leq i \leq n$. Observe that the weights of $L(n,\varepsilon)$ are $\varepsilon q^n, \varepsilon q^{n-2}, \ldots, \varepsilon q^{-n}$. We note that for $0 \leq i \leq n$, v_i is a basis for the weight space of $L(n,\varepsilon)$ associated with the weight εq^{n-2i}.

Definition 3.4.7. Let V denote a finite-dimensional $U_q(\mathfrak{sl}_2)$-module. Let λ denote a weight of V. By the *highest weight space of* V *associated with* λ, we mean the kernel of the action of e on V_λ. We refer to λ as a *highest weight* of V whenever the corresponding highest weight space is nonzero.

Referring to Lemma 3.4.3, assume $q^{2i} \neq 1$ for $1 \leq i \leq n$. We note that εq^n is the unique highest weight of $L(n,\varepsilon)$. For $L(n,\varepsilon)$, the highest weight space associated with the weight εq^n is equal to the weight space associated with εq^n.

Definition 3.4.8. Let V denote a finite-dimensional $U_q(\mathfrak{sl}_2)$-module. For $n \in \mathbb{N}$ and $\varepsilon \in \{1, -1\}$, consider the subspace of V spanned by the $U_q(\mathfrak{sl}_2)$-submodules of V which are isomorphic to $L(n, \varepsilon)$. We call this subspace the *homogeneous component of* V *associated with* $L(n, \varepsilon)$.

3.5 A $U_q(\mathfrak{sl}_2)$-module structure on V associated with Φ

We now return to the situation of Assumption 3.1.1. Recall from Lemma 3.3.1 the equation

$$\psi R - R\psi = (q - q^{-1})(K - K^{-1}). \tag{3.25}$$

Recall from Lemma 3.2.5 and Lemma 3.3.2 that

$$KRK^{-1} = q^{-2}R, \qquad K\psi K^{-1} = q^2\psi. \tag{3.26}$$

These relations are reminiscent of the defining relations for $U_q(\mathfrak{sl}_2)$. In this section we use the above relations to obtain a $U_q(\mathfrak{sl}_2)$-module structure on V. Then we will discuss this $U_q(\mathfrak{sl}_2)$-module structure from various points of view.

Lemma 3.5.1. *With reference to Definition 3.4.1, there exists a $U_q(\mathfrak{sl}_2)$-module structure on V for which the Chevalley generators act as follows:*

element of $U_q(\mathfrak{sl}_2)$	e	f	k	k^{-1}
action on V	$(q - q^{-1})^{-1}\psi$	$(q - q^{-1})^{-1}R$	K	K^{-1}

Proof. Use (3.25), (3.26), and Definition 3.4.1. ∎

For the rest of this section, we will discuss the $U_q(\mathfrak{sl}_2)$-module V from Lemma 3.5.1. Recall the Casimir element Λ of $U_q(\mathfrak{sl}_2)$ from (3.17), (3.18).

Lemma 3.5.2. *The action of Λ on V is equal to both*

$$\psi R + q^{-1}K + qK^{-1}, \tag{3.27}$$

$$R\psi + qK + q^{-1}K^{-1}. \tag{3.28}$$

Proof. Use (3.17), (3.18), and Lemma 3.5.1. ∎

Lemma 3.5.3. *The action of Λ on V commutes with each of*

$$\psi, \qquad R, \qquad K, \qquad A.$$

Proof. Since Λ is central in $U_q(\mathfrak{sl}_2)$, Λ commutes with each of e, f, k. So the action of Λ on V commutes with each of ψ, R, K in view of Lemma 3.5.1. The action of Λ on V commutes with A by (3.7). ∎

Lemma 3.5.4. *The following equations hold on V:*

$$R^2\psi - (q^2 + q^{-2})R\psi R + \psi R^2 = -(q - q^{-1})^2 \Lambda R, \tag{3.29}$$

$$\psi^2 R - (q^2 + q^{-2})\psi R\psi + R\psi^2 = -(q - q^{-1})^2 \Lambda \psi. \tag{3.30}$$

Proof. Use Lemma 3.4.2 and Lemma 3.5.1. ∎

Lemma 3.5.5. *For $0 \leq i \leq d$, U_i is the weight space of the $U_q(\mathfrak{sl}_2)$-module V associated with the weight q^{d-2i}.*

Proof. Recall from Definition 3.2.1 that U_i is an eigenspace of K with corresponding eigenvalue q^{d-2i}. The result follows. ∎

Corollary 3.5.6. *The weights of the $U_q(\mathfrak{sl}_2)$-module V are $q^d, q^{d-2}, \ldots, q^{-d}$.*

Lemma 3.5.7. *For $0 \leq i \leq d/2$, K_i is the highest weight space of the $U_q(\mathfrak{sl}_2)$-module V associated with the weight q^{d-2i}.*

Proof. Use Lemma 3.3.3 and Lemma 3.5.5. ■

Lemma 3.5.8. *Let $0 \leq i \leq d/2$ and $0 \neq v \in K_i$. Then Mv is an irreducible $U_q(\mathfrak{sl}_2)$-submodule of V. The $U_q(\mathfrak{sl}_2)$-module Mv is isomorphic to $L(d-2i, 1)$.*

Proof. For $0 \leq j \leq d-2i$, let $v_j = \gamma_j^{-1}\tau_{i,i+j}(A)v$, where $\gamma_j = (q-q^{-1})^j [j]_q^!$. By Corollary 2.7.3, $\{v_j\}_{j=0}^{d-2i}$ is a basis for Mv. By Lemma 2.2.11 and Lemma 2.6.4, $Rv_j = (q-q^{-1})[j+1]_q v_{j+1}$ for $0 \leq j \leq d-2i-1$ and $Rv_{d-2i} = 0$. By (3.13), $\psi v_0 = 0$ and $\psi v_j = (q-q^{-1})[d-2i+1-j]_q v_{j-1}$ for $1 \leq j \leq d-2i$. By Lemma 2.6.4, $Kv_j = q^{d-2i-2j}v_j$ for $0 \leq j \leq d-2i$. The result follows from the above comments along with Lemma 3.4.3 and Lemma 3.5.1. ■

Lemma 3.5.9. *For $0 \leq i \leq d/2$, MK_i is a $U_q(\mathfrak{sl}_2)$-submodule of V. Moreover MK_i is the homogeneous component of V associated with $L(d-2i, 1)$.*

Proof. Use Lemma 2.7.4 and Lemma 3.5.8. ■

Lemma 3.5.10. *For $0 \leq i \leq d/2$, MK_i is an eigenspace for Λ with corresponding eigenvalue $q^{d-2i+1} + q^{2i-d-1}$.*

Proof. By Lemma 3.1.2, the scalars $\{q^{d-2j+1} + q^{2j-d-1}\}_{j=0}^{\lfloor d/2 \rfloor}$ are mutually distinct. The result follows from this along with Lemma 2.7.4, Lemma 3.4.4, and Lemma 3.5.9. ■

Lemma 3.5.11. *The $U_q(\mathfrak{sl}_2)$-module V is semisimple. Let W denote an irreducible $U_q(\mathfrak{sl}_2)$-submodule of V. Then there exists an integer i $(0 \leq i \leq d/2)$ such that W is isomorphic to $L(d-2i, 1)$.*

Proof. Use Lemma 2.7.4, Lemma 3.5.8, and Lemma 3.5.9. ■

3.6 A $U_q(\mathfrak{sl}_2)$-module structure on V associated with $\Phi^{\Downarrow}$

We continue to discuss the situation of Assumption 3.1.1. In Section 3.5 we used Φ to obtain a $U_q(\mathfrak{sl}_2)$-module structure on V. In the present section, we consider the corresponding $U_q(\mathfrak{sl}_2)$-module structure on V associated with $\Phi^{\Downarrow}$.

Recall from Corollary 2.14.2 that $\psi^{\Downarrow} = \psi$. Applying Lemma 3.3.1 to $\Phi^{\Downarrow}$ we obtain

$$\psi R^{\Downarrow} - R^{\Downarrow}\psi = (q - q^{-1})(B - B^{-1}). \tag{3.31}$$

Recall from Lemma 3.2.5 and Lemma 3.3.2 that

$$B\psi B^{-1} = q^2\psi, \qquad BR^{\Downarrow}B^{-1} = q^{-2}R^{\Downarrow}. \tag{3.32}$$

Lemma 3.6.1. *With reference to Definition 3.4.1, there exists a $U_q(\mathfrak{sl}_2)$-module structure on V for which the Chevalley generators act as follows:*

element of $U_q(\mathfrak{sl}_2)$	e	f	k	k^{-1}
action on V	$(q-q^{-1})^{-1}\psi$	$(q-q^{-1})^{-1}R^{\Downarrow}$	B	B^{-1}

Proof. Use (3.31), (3.32), and Definition 3.4.1. ■

For the rest of this section, we will discuss the $U_q(\mathfrak{sl}_2)$-module V from Lemma 3.6.1. Recall the Casimir element Λ.

Lemma 3.6.2. *The action of Λ on V is equal to both*

$$\psi R^{\Downarrow} + q^{-1}B + qB^{-1}, \tag{3.33}$$

$$R^{\Downarrow}\psi + qB + q^{-1}B^{-1}. \tag{3.34}$$

Lemma 3.6.3. *The action of* Λ *on* V *commutes with each of*

$$\psi, \qquad R^{\Downarrow}, \qquad B, \qquad A.$$

Lemma 3.6.4. *The following equations hold on* V*:*

$$(R^{\Downarrow})^2\psi - (q^2 + q^{-2})R^{\Downarrow}\psi R^{\Downarrow} + \psi(R^{\Downarrow})^2 = -(q - q^{-1})^2\Lambda R^{\Downarrow}, \tag{3.35}$$

$$\psi^2 R^{\Downarrow} - (q^2 + q^{-2})\psi R^{\Downarrow}\psi + R^{\Downarrow}\psi^2 = -(q - q^{-1})^2\Lambda\psi. \tag{3.36}$$

Lemma 3.6.5. *For* $0 \leq i \leq d$, $U_i^{\Downarrow}$ *is the weight space of the* $U_q(\mathfrak{sl}_2)$*-module* V *associated with the weight* q^{d-2i}.

Corollary 3.6.6. *The weights of the* $U_q(\mathfrak{sl}_2)$*-module* V *are* $q^d, q^{d-2}, \ldots, q^{-d}$.

Lemma 3.6.7. *For* $0 \leq i \leq d/2$, K_i *is the highest weight space of the* $U_q(\mathfrak{sl}_2)$*-module* V *associated with the weight* q^{d-2i}.

Lemma 3.6.8. *Let* $0 \leq i \leq d/2$ *and* $0 \neq v \in K_i$. *Then* Mv *is an irreducible* $U_q(\mathfrak{sl}_2)$*-submodule of* V. *The* $U_q(\mathfrak{sl}_2)$*-module* Mv *is isomorphic to* $L(d-2i, 1)$.

Lemma 3.6.9. *For* $0 \leq i \leq d/2$, MK_i *is a* $U_q(\mathfrak{sl}_2)$*-submodule of* V. *Moreover* MK_i *is the homogeneous component of* V *associated with* $L(d-2i, 1)$.

Lemma 3.6.10. *For* $0 \leq i \leq d/2$, MK_i *is an eigenspace for* Λ *with corresponding eigenvalue* $q^{d-2i+1} + q^{2i-d-1}$.

Lemma 3.6.11. *The* $U_q(\mathfrak{sl}_2)$*-module* V *is semisimple. Let* W *denote an irreducible* $U_q(\mathfrak{sl}_2)$*-submodule of* V. *Then there exists an integer* i $(0 \leq i \leq d/2)$ *such that* W *is isomorphic to* $L(d-2i, 1)$.

Note 3.6.12. The paper [17] describes an action of $U_q(\widehat{\mathfrak{sl}}_2)$ on V. Roughly speaking, $U_q(\widehat{\mathfrak{sl}}_2)$ is generated by two copies of $U_q(\mathfrak{sl}_2)$ that are glued together in a certain way [5, p. 262]. Thus the action of $U_q(\widehat{\mathfrak{sl}}_2)$ on V induces two actions of $U_q(\mathfrak{sl}_2)$ on V. For these actions the Chevalley generator e does not act as a scalar multiple of ψ [17, Lines (28) and (30)]. Therefore the two $U_q(\mathfrak{sl}_2)$-actions from [17] are not the same as the two $U_q(\mathfrak{sl}_2)$-actions from Lemma 3.5.1 and Lemma 3.6.1. As far as we know, the two $U_q(\mathfrak{sl}_2)$-actions from [17] are not directly related to the $U_q(\mathfrak{sl}_2)$-actions from Lemma 3.5.1 and Lemma 3.6.1.

3.7 How $\psi, K^{\pm 1}, B^{\pm 1}$ are related

We continue to discuss the situation of Assumption 3.1.1. In Sections 3.5 and 3.6 we introduced two $U_q(\mathfrak{sl}_2)$-module structures on V. In this section we compare these module structures. From this comparison, we obtain several equations relating $\psi, K^{\pm 1}, B^{\pm 1}$.

Recall the Casimir element Λ of $U_q(\mathfrak{sl}_2)$ from Section 3.4.

Lemma 3.7.1. *The following coincide:*

(i) *the action of Λ on V for the $U_q(\mathfrak{sl}_2)$-module structure from Lemma 3.5.1,*

(ii) *the action of Λ on V for the $U_q(\mathfrak{sl}_2)$-module structure from Lemma 3.6.1.*

Proof. Use Lemma 2.7.4, Lemma 3.5.10, and Lemma 3.6.10. ■

Proposition 3.7.2. The following coincide:

$$(I - aq\psi)K, \qquad (I - a^{-1}q\psi)B, \qquad K(I - aq^{-1}\psi), \qquad B(1 - a^{-1}q^{-1}\psi).$$

Moreover the following coincide:

$$(I - a^{-1}q^{-1}\psi)K^{-1}, \qquad (I - aq^{-1}\psi)B^{-1}, \qquad K^{-1}(I - a^{-1}q\psi), \qquad B^{-1}(1 - aq\psi).$$

Proof. Consider the expression q^{-1} times (3.27) minus q times (3.28) minus q^{-1} times (3.33) plus q times (3.34). We evaluate this expression in two ways. First, by Lemma 3.7.1 this expression is equal to zero. Second, eliminate R and $R^{\Downarrow}$ using (3.7) and (3.8) and simplify the result using Lemma 3.3.2. By these comments,

$$(1 - aq\psi)K = (1 - qa^{-1}\psi)B. \tag{3.37}$$

The remaining assertions follow from (3.37) and Lemma 3.3.2. ■

Shortly we will write KB^{-1}, $K^{-1}B$, and their inverses in terms of ψ. In order to do this, we will need that certain elements of $\mathrm{End}(V)$ are invertible.

Lemma 3.7.3. *Each of the following is invertible:*

$$I - aq\psi, \qquad I - a^{-1}q\psi, \qquad I - aq^{-1}\psi, \qquad I - a^{-1}q^{-1}\psi. \tag{3.38}$$

Their inverses are as follows:

$$(I - aq\psi)^{-1} = \sum_{i=0}^{d} a^i q^i \psi^i, \qquad (I - a^{-1}q\psi)^{-1} = \sum_{i=0}^{d} a^{-i} q^i \psi^i, \tag{3.39}$$

$$(I - aq^{-1}\psi)^{-1} = \sum_{i=0}^{d} a^i q^{-i} \psi^i, \qquad (I - a^{-1}q^{-1}\psi)^{-1} = \sum_{i=0}^{d} a^{-i} q^{-i} \psi^i. \tag{3.40}$$

Proof. Recall from Lemma 2.10.2 that $\psi^{d+1} = 0$. ■

Theorem 3.7.4. *The following hold:*

$$BK^{-1} = \frac{I - aq\psi}{I - a^{-1}q\psi}, \qquad KB^{-1} = \frac{I - a^{-1}q\psi}{I - aq\psi}, \tag{3.41}$$

$$K^{-1}B = \frac{I - aq^{-1}\psi}{I - a^{-1}q^{-1}\psi}, \qquad B^{-1}K = \frac{I - a^{-1}q^{-1}\psi}{I - aq^{-1}\psi}. \tag{3.42}$$

In (3.41), (3.42) *the denominators are invertible by Lemma 3.7.3.*

Proof. Use Proposition 3.7.2 and Lemma 3.7.3. ■

Lemma 3.7.5. *The following mutually commute:*

$$\psi, \qquad BK^{-1}, \qquad KB^{-1}, \qquad K^{-1}B, \qquad B^{-1}K.$$

Proof. By Theorem 3.7.4 each of the four expressions on the right is a polynomial in ψ. ■

Shortly we will give four ways to write ψ in terms of K, B. In order to do this, we will need that certain elements of $\mathrm{End}(V)$ are invertible.

Lemma 3.7.6. *Each of*

$$I - BK^{-1}, \qquad I - KB^{-1}, \qquad I - K^{-1}B, \qquad I - B^{-1}K \tag{3.43}$$

sends U_i *into* $U_0 + U_1 + \cdots + U_{i-1}$ *for* $0 \leq i \leq d$. *Moreover each of* (3.43) *is nilpotent.*

Proof. We first consider $I - BK^{-1}$. On U_i,

$$I - BK^{-1} = I - q^{2i-d}B.$$

By this and Lemma 3.2.3, $I - BK^{-1}$ sends U_i into $U_0 + U_1 + \cdots + U_{i-1}$.

Since KB^{-1} and BK^{-1} are inverses, we see that $I - KB^{-1}$ sends U_i into $U_0 + U_1 + \cdots + U_{i-1}$.

The proof is similar for $I - K^{-1}B$ and $I - B^{-1}K$. ■

Lemma 3.7.7. *Each of the following is invertible:*

$$aI - a^{-1}BK^{-1}, \qquad a^{-1}I - aKB^{-1},$$

$$aI - a^{-1}K^{-1}B, \qquad a^{-1}I - aB^{-1}K.$$

Proof. We show that $aI - a^{-1}BK^{-1}$ is invertible. Observe that

$$aI - a^{-1}BK^{-1} = (a - a^{-1})I + a^{-1}(I - BK^{-1}).$$

Now $aI - a^{-1}BK^{-1}$ is invertible by Lemma 3.7.6 and the fact that $a^2 \neq 1$.

The remaining assertions are similarly proved. ■

Theorem 3.7.8. *The map ψ is equal to each of the following:*

$$\frac{I - BK^{-1}}{q(aI - a^{-1}BK^{-1})}, \qquad \frac{I - KB^{-1}}{q(a^{-1}I - aKB^{-1})}, \tag{3.44}$$

$$\frac{q(I - K^{-1}B)}{aI - a^{-1}K^{-1}B}, \qquad \frac{q(I - B^{-1}K)}{a^{-1}I - aB^{-1}K}. \tag{3.45}$$

In (3.44), (3.45) *the denominators are invertible by Lemma 3.7.7.*

Proof. In each equation of Theorem 3.7.4, solve for ψ. ■

Theorem 3.7.9. *We have*

$$aK^2 - \frac{a^{-1}q - aq^{-1}}{q - q^{-1}} KB - \frac{aq - a^{-1}q^{-1}}{q - q^{-1}} BK + a^{-1}B^2 = 0. \tag{3.46}$$

Proof. Equate the expression on the left in (3.44) and the expression on the right in (3.45). For every term in the resulting equation, multiply on the left by $B(a^{-1}I - aB^{-1}K)$ and on the right by $(aI - a^{-1}BK^{-1})K$. ■

We mention a reformulation of Theorem 3.7.9.

Theorem 3.7.10. *We have*

$$aB^{-2} - \frac{a^{-1}q - aq^{-1}}{q - q^{-1}} K^{-1}B^{-1} - \frac{aq - a^{-1}q^{-1}}{q - q^{-1}} B^{-1}K^{-1} + a^{-1}K^{-2} = 0. \tag{3.47}$$

Proof. For every term in (3.46), multiply on the left by $B^{-1}K^{-1}$ and on the right by $K^{-1}B^{-1}$. Simplify the result using Lemma 3.7.5. ■

Equations (3.46) and (3.47) can be put in the following attractive forms.

Lemma 3.7.11. *The following equations hold:*

$$q(K-B)(aK-a^{-1}B) = q^{-1}(aK-a^{-1}B)(K-B), \tag{3.48}$$

$$q(a^{-1}K^{-1}-aB^{-1})(K^{-1}-B^{-1}) = q^{-1}(K^{-1}-B^{-1})(a^{-1}K^{-1}-aB^{-1}), \tag{3.49}$$

$$q(I-K^{-1}B)(aI-a^{-1}BK^{-1}) = q^{-1}(aI-a^{-1}K^{-1}B)(I-BK^{-1}), \tag{3.50}$$

$$q(a^{-1}I-aKB^{-1})(I-B^{-1}K) = q^{-1}(I-KB^{-1})(a^{-1}I-aB^{-1}K). \tag{3.51}$$

Proof. To verify (3.48), multiply out each side and compare the result with (3.46). Equation (3.49) is similarly verified using (3.47). To verify (3.50), multiply each term in (3.48) on the left by K^{-1} and on the right by K^{-1}. To verify (3.51), multiply each term in (3.49) on the left by K and on the right by K. ■

3.8 How $R, K^{\pm 1}$ and $R^{\Downarrow}, B^{\pm 1}$ are related

We continue to discuss the situation of Assumption 3.1.1. In Sections 3.5 and 3.6 we displayed two $U_q(\mathfrak{sl}_2)$-actions on V. A natural question is, can we write each of the operators for one action in terms of the operators for the other action. In this section we demonstrate that this can be done.

Recall from Corollary 2.14.2 that

$$\psi = \psi^{\Downarrow}.$$

We now give $R^{\Downarrow}, B^{\pm 1}$ in terms of $\psi, R, K^{\pm 1}$.

Lemma 3.8.1. *The following equations hold:*

$$B = a^2 K + (1 - a^2) K \sum_{i=0}^{d} a^{-i} q^{-i} \psi^i, \tag{3.52}$$

$$B^{-1} = a^{-2} K^{-1} + (1 - a^{-2}) K^{-1} \sum_{i=0}^{d} a^{i} q^{i} \psi^i, \tag{3.53}$$

$$R^{\Downarrow} = R + (a - a^{-1}) \sum_{i=0}^{d} (a^{-i} q^{-i} K - a^{i} q^{i} K^{-1}) \psi^i. \tag{3.54}$$

Proof. To obtain (3.52) and (3.53), use Theorem 3.7.4 along with (3.39), (3.40), and the fact that $\psi^{d+1} = 0$. To obtain (3.54), use (3.11) along with (3.52) and (3.53). ■

We now give $R, K^{\pm 1}$ in terms of $\psi, R^{\Downarrow}, B^{\pm 1}$.

Lemma 3.8.2. *The following equations hold:*

$$K = a^{-2} B + (1 - a^{-2}) B \sum_{i=0}^{d} a^{i} q^{-i} \psi^i, \tag{3.55}$$

$$K^{-1} = a^{2} B^{-1} + (1 - a^{2}) B^{-1} \sum_{i=0}^{d} a^{-i} q^{i} \psi^i, \tag{3.56}$$

$$R = R^{\Downarrow} + (a - a^{-1}) \sum_{i=0}^{d} (a^{-i} q^{i} B^{-1} - a^{i} q^{-i} B) \psi^i. \tag{3.57}$$

Proof. To obtain (3.55) and (3.56), use (3.52) and (3.53) along with Lemma 3.3.2. To obtain (3.57), use (3.11) along with (3.55) and (3.56). ■

3.9 How A, ψ are related

We continue to discuss the situation of Assumption 3.1.1. In this section we show how A and ψ are related. In what follows we refer to the Λ-action from Lemma 3.7.1.

Lemma 3.9.1. *On V, we have*

$$A^2\psi - (q^2+q^{-2})A\psi A + \psi A^2 + (q^2-q^{-2})^2\psi \\ = -(q-q^{-1})^2\Lambda A + (a+a^{-1})(q-q^{-1})^2(q+q^{-1})I \tag{3.58}$$

and also

$$\psi^2 A - (q^2+q^{-2})\psi A\psi + A\psi^2 = -(q-q^{-1})^2\Lambda\psi. \tag{3.59}$$

Proof. We first prove (3.58). Let L denote the expression on the left-hand side in (3.58). In L, eliminate A using $A = R + aK + a^{-1}K^{-1}$. Simplify the result using (3.26) and (3.29). This shows that L is equal to $-(q-q^{-1})^2\Lambda A$ plus $\Lambda - \psi R$ times

$$a(q^2-1)K + a^{-1}(q^{-2}-1)K^{-1}$$

plus $\Lambda - R\psi$ times

$$a(q^{-2}-1)K + a^{-1}(q^2-1)K^{-1}.$$

In this expression, eliminate $\Lambda - \psi R$ and $\Lambda - R\psi$ using Lemma 3.5.2. The resulting expression for L is the right-hand side of (3.58).

The proof of (3.59) is similar. ∎

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