ESTIMATES FOR THE SZEGŐ KERNEL ON UNBOUNDED CONVEX DOMAINS

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Abstract

We study the size of the Szegő kernel on the boundary of unbounded domains defined by convex polynomials. Given a convex polynomial $b : \mathbb{R}^n \to \mathbb{R}$ such that its mixed terms are dominated by its pure terms, we consider the domain $\Omega_b = \{\boldsymbol{z} \in \mathbb{C}^{n+1} : \mathrm{Im}[z_{n+1}] > b(\mathrm{Re}[z_1], \ldots, \mathrm{Re}[z_n])\}$.

Given two points $(\boldsymbol{x}, \boldsymbol{y}, t)$ and $(\boldsymbol{x'}, \boldsymbol{y'}, t')$ in $\partial\Omega_b$, define $\tilde{b}(\boldsymbol{v}) = b\left(\boldsymbol{v} + \frac{\boldsymbol{x}+\boldsymbol{x'}}{2}\right) - \nabla b\left(\frac{\boldsymbol{x}+\boldsymbol{x'}}{2}\right) \cdot \boldsymbol{v} - b\left(\frac{\boldsymbol{x}+\boldsymbol{x'}}{2}\right)$; $\delta(\boldsymbol{x}, \boldsymbol{x'}) = b(\boldsymbol{x}) + b(\boldsymbol{x'}) - 2b\left(\frac{\boldsymbol{x}+\boldsymbol{x'}}{2}\right)$; and $w = (t'-t) + \nabla b\left(\frac{\boldsymbol{x}+\boldsymbol{x'}}{2}\right) \cdot (\boldsymbol{y'} - \boldsymbol{y})$.

We obtain the following estimate for the Szegő kernel associated to the domain Ω_b :

$$|S\left((\boldsymbol{x}, \boldsymbol{y}, t); (\boldsymbol{x'}, \boldsymbol{y'}, t')\right)| \lesssim \frac{1}{\sqrt{\delta^2 + \tilde{b}(\boldsymbol{y} - \boldsymbol{y'})^2 + w^2}\left|\left\{\boldsymbol{v} \,:\, \tilde{b}(\boldsymbol{v}) < \sqrt{\delta^2 + \tilde{b}(\boldsymbol{y} - \boldsymbol{y'})^2 + w^2}\right\}\right|^2},$$

where the constant depends on the degrees of the highest order pure terms of b and the dimension of the space, but is independent of the two given points.

This is a generalization of the one-dimensional result by Nagel [23].

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Contents

Chapter 1

Introduction

1.1 The Szegő kernel

1.1.1 Definitions

The Szegő projection was first introduced by Gabor Szegő in 1921 in the paper *Über orthogonale Polynome, die zu einer gegebenen Kurve der komplexen Ebene gehören* [30]. Almost concurrently, Stefan Bergman introduced the Bergman projection as part of his doctoral dissertation. Since then, these projections have been extensively studied.

The Bergman projection associated to a domain $\Omega \subset \mathbb{C}^n$ is the orthogonal projection of $L^2(\Omega)$ onto the space $A_2(\Omega) = \{f \in L^2(\Omega) \mid f \text{ is holomorphic in } \Omega\}$. If T is the Bergman projection and $f \in L^2(\Omega)$, then there exists a function $B(\cdot\,,\cdot) \in C^\infty(\Omega \times \Omega)$ such that

$$Tf(z) = \int_\Omega B(z,w)f(w)d\mu,$$

where $d\mu$ is the volume measure and $B(z,w)$ is holomorphic in z and antiholomorphic in w. This function $B(\cdot\,,\cdot) \in C^\infty(\Omega \times \Omega)$ is called the Bergman kernel.

There are several (equivalent) ways of defining the Szegő kernel for bounded domains

$\Omega \subset\subset \mathbb{C}^n$ (see, e.g., [18]). The purpose of this thesis, however, is to obtain a bound for the Szegő kernel on a class of *unbounded* domains $\Omega \subset \mathbb{C}^{n+1}$, $n \geq 1$, defined by convex polynomials. Given a convex polynomial $b : \mathbb{R}^n \to \mathbb{R}$, consider the domain

$$\Omega_b = \{(z_1, \ldots, z_{n+1}) \in \mathbb{C}^{n+1} \,:\, \mathrm{Im}[z_{n+1}] > b(\mathrm{Re}[z_1], \ldots, \mathrm{Re}[z_n])\}.$$

For these domains, it is convenient to define the Szegő projection as in [13]. We can identify the boundary $\partial\Omega_b$ with $\mathbb{C}^n \times \mathbb{R}$ so that a point $(\boldsymbol{z}, t) \in \mathbb{C}^n \times \mathbb{R}$ corresponds to $(\boldsymbol{z}, t + ib(\mathrm{Re}[z_1], \ldots, \mathrm{Re}[z_n])) \in \partial\Omega_b$.

Let $\mathcal{O}(\Omega_b)$ be the set of holomorphic functions in Ω_b. Given $F \in \mathcal{O}(\Omega_b)$ and $\epsilon > 0$, set

$$F_\epsilon(\boldsymbol{z}, t) = F(\boldsymbol{z}, t + ib(\mathrm{Re}[z_1], \ldots, \mathrm{Re}[z_n]) + i\epsilon).$$

The Hardy space $\mathcal{H}^2(\Omega_b)$ is defined as

$$\mathcal{H}^2(\Omega_b) = \left\{ F \in \mathcal{O}(\Omega_b) \,:\, \sup_{\epsilon > 0} \int_{\mathbb{C}^n \times \mathbb{R}} |F_\epsilon(\boldsymbol{z}, t)|^2 \, d\boldsymbol{z} \, dt \equiv ||F||^2_{\mathcal{H}^2} < \infty \right\}.$$

Now let ρ be a defining function for the domain, i.e., $\Omega_b = \{\boldsymbol{x} \in \mathbb{C}^{n+1} \,:\, \rho(\boldsymbol{x}) < 0\}$ where $\rho \in C^\infty(\mathbb{C}^{n+1})$ is such that $\nabla\rho \neq 0$ when $\rho = 0$. A Cauchy-Riemann operator is an operator of the form

$$L = \sum_{j=1}^{n+1} a_j \frac{\partial}{\partial \overline{z_j}}.$$

We say that L is tangential if in addition $L(\rho) = 0$ (notice that writing $L = (a_1, \ldots, a_{n+1})$ the condition $L(\rho) = 0$ can actually be written as $< L, \partial\rho >= 0$, where $< \cdot, \cdot >$ is the Hermitian inner product).

For a class of convex polynomials $b : \mathbb{R}^n \to \mathbb{R}$ we will define the Szegő projection $S : L^2(\partial\Omega_b) \to \mathcal{H}^2(\Omega_b)$ to be the orthogonal projection from $L^2(\partial\Omega_b)$ to the closed subspace of functions $f \in L^2(\partial\Omega_b)$ that are annihilated in the sense of distributions by all tangential Cauchy-Riemann operators on $\partial\Omega_b$. To show that such a map exists, some work is required. We will present a brief outline of the basic facts leading up to this definition.

For this derivation to hold, it does not suffice to simply require that the polynomials $b : \mathbb{R}^n \to \mathbb{R}$ that define the domains be convex. In particular, the result does not follow for convex polynomials that are flat along some directions (for example, consider the convex polynomial $b(x_1, x_2) = x_1^2$ in $\mathbb{R}^2$). We introduce a growth condition via the following definition:

Definition 1.1. *Let $m_1, \ldots, m_n$ be positive integers. We will say that a polynomial $p : \mathbb{R}^n \to \mathbb{R}$ is of "combined degree" $(m_1, \ldots, m_n)$ if it is of the form*

$$p(\boldsymbol{x}) = \sum_{\alpha} c_\alpha \boldsymbol{x}^\alpha,$$

where each index $\alpha = (\alpha_1, \ldots, \alpha_n)$ satisfies

1. $\frac{\alpha_1}{2m_1} + \ldots + \frac{\alpha_n}{2m_n} \leq 1$;
2. $\frac{\alpha_1}{2m_1} + \ldots + \frac{\alpha_n}{2m_n} = 1$ *if and only if there exists some j such that $\alpha_j = 2m_j$;*

and the exponents of its pure terms of highest order are $2m_1, \ldots, 2m_n$ respectively.

Example 1.2. *The polynomial $p(x_1, x_2) = x_1^2 + x_1x_2 + x_1^2x_2^2 + x_1^4 + x_2^6$ is of "combined degree" $(2, 3)$. However, the polynomial $\tilde{p}(x_1, x_2) = x_1^2 + x_1x_2 + x_1^2x_2^3 + x_1^4 + x_2^6$ is not.*

Throughout the rest of this work we will assume that

$$\Omega_b = \{(z_1, \ldots, z_{n+1}) \in \mathbb{C}^{n+1} \, : \, \mathrm{Im}[z_{n+1}] > b(\mathrm{Re}[z_1], \ldots, \mathrm{Re}[z_n])\},$$

where $b : \mathbb{R}^n \to \mathbb{R}$ is a convex polynomial of "combined degree" $(m_1, \ldots, m_n)$.

There are two key steps needed to justify that the Szegő kernel as given above is well-defined: that the space $\mathcal{H}^2(\Omega_b)$ is a closed subspace of $L^2(\partial\Omega_b)$, and that $\mathcal{H}^2(\Omega_b)$ is equivalent to the space of functions on $L^2(\partial\Omega_b)$ that are annihilated in the sense of distributions by the tangential Cauchy-Riemann operators. The first follows as in Lemma A.5. in [13].

Lemma. *Let $F \in \mathcal{H}^2(\Omega_b)$. Then there exists $F^b \in L^2(\partial\Omega_b)$ with the following properties.*

a) *For almost every $(\boldsymbol{z}, t) \in \mathbb{C}^n \times \mathbb{R}$, $\lim_{\epsilon\to 0^+} F(\boldsymbol{z}, t + ib(\mathrm{Re}[z_1], \ldots, \mathrm{Re}[z_n]) + i\epsilon) = F^b(\boldsymbol{z}, t + ib(\mathrm{Re}[z_1], \ldots, \mathrm{Re}[z_n]))$;*

b) $||F^b||_{L^2(\partial\Omega_b)} = ||F||_{\mathcal{H}^2(\Omega_b)}$;

c) $\lim_{\epsilon\to 0^+} ||F_\epsilon - F^b||_{L^2(\partial\Omega_b)} = 0$;

d) *There is a constant C_0 independent of F with $||\mathcal{N}_0[F]||_{L^2(\partial\Omega_b)} \le C_0||F||_{\mathcal{H}^2(\Omega_b)}$, where $\mathcal{N}_0[F](\boldsymbol{z}, t) = \sup_{\epsilon>0} |F(\boldsymbol{z}, t + ib(\mathrm{Re}[z_1], \ldots, \mathrm{Re}[z_n]) + i\epsilon)|$;*

e) *The boundary function F^b is annihilated (in the sense of distributions) by all tangential Cauchy-Riemann operators on $\partial\Omega_b$.*

f) *For any compact subset $K \subset \Omega_b$ there is a positive constant $C(K)$ independent of F such that*

$$\sup_{z\in K}|F(z)| \leq C(K)||F||_{\mathcal{H}^2(\Omega_b)}.$$

This result does not require the growth condition imposed on the polynomials b. However, this condition is used in the proposition that follows.

Define the partial Fourier transform $\mathcal{F} : L^2(\mathbb{R}^{2n+1}) \to L^2(\mathbb{R}^{2n+1})$ by the integral

$$\mathcal{F}[f](\boldsymbol{x}, \boldsymbol{y}, t) = \hat{f}(\boldsymbol{x}, \boldsymbol{\eta}, \tau) = \int\limits_{\mathbb{R}^{n+1}} e^{-2\pi i(\boldsymbol{y}\cdot\boldsymbol{\eta}+t\tau)} f(\boldsymbol{x}, \boldsymbol{y}, t)\, d\boldsymbol{y}\, dt.$$

Then, as in Proposition 2.5 in [13], one can show the following:

Proposition. *Let $f \in L^2(\partial\Omega_b)$. Then*

a) the function f is annihilated by the tangential Cauchy-Riemann operators on $\mathbb{C}^n \times \mathbb{R}$ in the distributional sense if and only if for all $1 \leq j \leq n$ the partial Fourier transform $\mathcal{F}[f] = \hat{f}$ satisfies

$$\frac{\partial}{\partial x_j}\left(e^{-2\pi[\boldsymbol{\eta}\cdot\boldsymbol{x}-b(\boldsymbol{x})\tau]}\hat{f}(\boldsymbol{x}, \boldsymbol{\eta}, \tau)\right) = 0$$

on $\mathbb{R}^{2n+1}$ in the sense of distributions;

b) if f is annihilated by the tangential Cauchy-Riemann operators in the distributional sense, then $\hat{f}(\boldsymbol{x}, \boldsymbol{\eta}, \tau) = 0$ almost everywhere when $\tau < 0$. In particular, if we set $h_s(\boldsymbol{x}, \boldsymbol{\eta}, \tau) = e^{-2\pi\tau s}\hat{f}(\boldsymbol{x}, \boldsymbol{\eta}, \tau)$, then $h_s \in L^2(\mathbb{R}^{2n+1})$ for $s \geq 0$;

c) if f is annihilated by the tangential Cauchy-Riemann operators in the distributional sense and if

$$F(\boldsymbol{z}, z_{n+1}) = F(\boldsymbol{z}, t + ib(\mathrm{Re}[z_1], \ldots, \mathrm{Re}[z_n]) + is) = \mathcal{F}^{-1}[h_s](\boldsymbol{x}, \boldsymbol{y}, t),$$

then $F \in \mathcal{H}^2(\Omega_b)$ and $F^b = f$.

The proof of this proposition is analogous to that in [13]. We have included an appendix where a detailed explanation can be found. It follows from these propositions that the set of functions $f \in L^2(\partial\Omega)$ such that there exists $F \in \mathcal{H}^2(\Omega)$ with $F^b = f$ is the set of functions that are annihilated in the sense of distributions by the tangential Cauchy-Riemann operators.

Furthermore, starting from the projection onto the null space of the operators $\left\{\frac{\partial}{\partial x_j}\right\}$ on the weighted space $L^2(\mathbb{R}^{2n+1}, e^{4\pi[\boldsymbol{\eta}\cdot\boldsymbol{x} - b(\boldsymbol{x})\tau]} d\boldsymbol{x}\, d\boldsymbol{\eta}\, d\tau\,)$ we are able to retrieve the Szegő projection. We show in the Appendix that for $f \in L^2(\partial\Omega)$, the Szegő projection is given by

$$\Pi[f](\boldsymbol{x}, \boldsymbol{y}, t) = \int_{\mathbb{R}^{2n+1}} f(\boldsymbol{x'}, \boldsymbol{y'}, t') S\left((\boldsymbol{x}, \boldsymbol{y}, t); (\boldsymbol{x'}, \boldsymbol{y'}, t')\right) d\boldsymbol{x'}\, d\boldsymbol{y'}\, dt',$$

where

$$S((\boldsymbol{x}, \boldsymbol{y}, t); (\boldsymbol{x'}, \boldsymbol{y'}, t')) = \int_0^\infty e^{-2\pi\tau[b(\boldsymbol{x'}) + b(\boldsymbol{x}) + i(t'-t)]} \left(\int_{\mathbb{R}^n} \frac{e^{2\pi\boldsymbol{\eta}\cdot[\boldsymbol{x} + \boldsymbol{x'} - i(\boldsymbol{y'} - \boldsymbol{y})]}}{\int\limits_{\mathbb{R}^n} e^{4\pi[\boldsymbol{\eta}\cdot\boldsymbol{v} - b(\boldsymbol{v})\tau]}\, d\boldsymbol{v}}\, d\boldsymbol{\eta} \right) d\tau$$

is the Szegő kernel.

1.1.2 A survey of the literature

For very simple domains, such as the unit ball, the Bergman and Szegő kernels can be computed explicitly. In fact, if $D \subset \mathbb{C}^n$ is the unit ball, then the Bergman kernel is given by

$$B(z,\xi) = \frac{n!}{\pi^n}\frac{1}{(1 - z\cdot\overline{\xi})^{n+1}}$$

and the Szegő kernel is given by

$$S(z,\xi) = \frac{(n-1)!}{2\pi^n}\frac{1}{(1 - z\cdot\overline{\xi})^{n}}.$$

A derivation of these formulas can be found, e.g., in [29].

Even for some more complex domains closed formulas have been obtained. Greiner and Stein [12] compute an explicit formula for the Szegő kernel in domains of the type $\Omega_k = \{(z, z_1) \in \mathbb{C}^2 : \mathrm{Im}[z_1] > |z|^{2k}\}$ for any positive integer k. They show that for $\xi = (z, t + i(|z|^{2k} + \mu))$ and $\omega = (w, s + i(|w|^{2k} + \nu))$, with $\mu, \nu > 0$,

$$\begin{aligned} S(\xi,\omega) = \frac{1}{4\pi^2}&\left[\left(\frac{i}{2}[s-t] + \frac{|z|^{2k}+|w|^{2k}}{2} + \frac{\mu+\nu}{2}\right) - z\overline{w}\right]^{-2} \\ &\times \frac{1}{4\pi^2}\left[\frac{i}{2}[s-t] + \frac{|z|^{2k}+|w|^{2k}}{2} + \frac{\mu+\nu}{2}\right]^{\frac{1-k}{k}}. \end{aligned}$$

Díaz [8] shows that for these domains, the Szegő Projection is bounded in L^p, for $1 < p < \infty$.

Francsics and Hanges [10] generalized this result to domains in $\mathbb{C}^n$. They compute an explicit formula for the Szegő kernel in domains of the type $\Omega = \{(z, \xi, w) \in \mathbb{C}^{n+m+1} : \mathrm{Im}[w] > ||z||^2 + ||\xi||^{2p}\}$. They show that

$$S(z,\xi,t;z',\xi',t') = \sum_{k=1}^{n+1} c_k \frac{(A - z\cdot\overline{z}')^{\frac{k}{p}-n-1}}{[(A - z\cdot\overline{z}')^{\frac{1}{p}} - \xi\cdot\overline{\xi}']^{m+k}}, \tag{1.1}$$

where $A = \frac{1}{2}[||z||^2 + ||z'||^2 + ||\xi||^{2p} + ||\xi'||^{2p} - i(t-t')]$.

More recently, Park obtains closed formulas for the Bergman kernel in domains of the form $\{(z_1,z_2) \in \mathbb{C}^2 : |z_1|^{\frac{4}{q_1}} + |z_2|^{\frac{4}{q_2}} < 1\}$ for any positive integers q_1 and q_2 in [26] and for domains of the form $\{(z_1,z_2,z_3) \in \mathbb{C}^3 : |z_1|^4 + |z_2|^4 + |z_3|^4 < 1\}$ and $\{(z_1,z_2,z_3) \in \mathbb{C}^3 : |z_1|^4 + |z_2|^4 + |z_3|^2 < 1\}$ in [27]. Furthermore, he shows [26] that among the domains $\{(z_1,z_2) \in \mathbb{C}^2 : |z_1|^{2p_1} + |z_2|^{2p_2} < 1\}$ with $p = (p_1,p_2) \in \mathbb{N}^2$, the Bergman kernel is represented in terms of closed forms if and only if $p = (p_1,1),(1,p_2)$ or $(2,2)$.

Since for most domains it is not feasible to obtain closed formulas for the Szegő kernel, one of the main questions in the field is whether one can obtain an estimate for these kernels in terms of the geometry of the domains defining them. Much progress has been done in the case of bounded domains. In fact, if Ω is a bounded strongly pseudoconvex domain, complete asymptotic expressions for the Bergman and Szegő projections are known from the work of Fefferman [9] (see also [3]). Even though no such results are known if the domain is only weakly pseudoconvex, there has been much work showing that in special cases one can obtain estimates for these kernels in terms of the geometry of the domains.

Machedon [19] shows that the magnitude of the Szegő kernel for smooth bounded pseudoconvex domains of finite type in $\mathbb{C}^n$ with one degenerate eigenvalue is bounded by the reciprocal of the volume of the non-isotropic ball defined by the domain (a discussion on balls defined by non-isotropic distances given in terms of vector fields can be found in [25]).

McNeal [20] obtains sharp upper bounds for the Bergman kernel for smoothly bounded convex domains Ω of finite type in $\mathbb{C}^n$ in terms of the volumes of polydiscs fitting inside $\partial\Omega$. More precisely, if Ω is convex in some neighborhood U of a point $p \in \partial\Omega$ of finite type M then there exists a defining function r for $U \cap \Omega$ such that the sets $\{r : r(z) < \eta\}$ are convex on η for some range $-\eta_0 < \eta < \eta_0$, $\eta_0 > 0$. For $q \in \Omega$ near p, he defines a set of coordinates $(z_1, \ldots, z_n)$ and distances $\tau_1(q,\epsilon), \ldots, \tau_n(q,\epsilon)$ by extremizing the distance from q to the level set $\{z \in U \;:\; r(z) = r(q) + \epsilon\}$ and defines the polydisc $P_\epsilon(q) = \{z \in U \;:\; |z_1| < \tau_1(q,\epsilon), \ldots, |z_n| < \tau_n(q,\epsilon)\}$. McNeal shows that there exists a constant C so that for all $q_1, q_2 \in U \cap \Omega$

$$|K(q_1, q_2)| \leq \frac{C}{|P_\delta(q_1)|},$$

where $K(q_1, q_2)$ is the Bergman kernel and $\delta = |r(q_1)| + |r(q_2)| + \inf\{\epsilon > 0 \;:\; q_2 \in P_\epsilon(q_1)\}$.

By integrating this result along "normal directions", McNeal and Stein [21] obtain a bound for the Szegő kernel $S(z, w)$ for these domains in terms of the smallest "tent" in $\partial\Omega$ containing z and w. That is, they show that for smoothly bounded convex domains of finite type in $\mathbb{C}^n$, there exists a constant C so that for all $z, w \in \overline{\Omega} \times \overline{\Omega} \setminus \{\text{diagonal in } \partial\Omega\}$,

$$|S(z, w)| \leq \frac{C}{|T(z, \gamma)|}.$$

Here $T(z, \gamma) = P_\gamma(\pi(z)) \cap \overline{\Omega}$; the projection $\pi : U \to \partial\Omega$ is a smooth map such that if $b \in \partial\Omega$, $\pi(b) = b$ and $\pi^{-1}(b)$ is a smooth curve, transversally intersecting $\partial\Omega$ at b; and $\gamma = |r(z)| + |r(w)| + \inf\{\epsilon > 0 \;:\; w \in T(z, \epsilon)\}$. Similar estimates are obtained in both papers for the derivatives of these kernels.

Although no general results have been obtained for the kernels defined over unbounded convex domains, some particular classes of domains have been studied. In the last section of [23], Nagel studies the Szegő kernel on the boundary of domains of the kind $\Omega = \{\boldsymbol{z} \in \mathbb{C}^2 : \text{Im}[z_2] > \phi(\text{Re}[z_1])\}$, where ϕ is a subharmonic, non-harmonic polynomial with the property that $\Delta\phi(z) = \Delta\phi(x + iy)$ is independent of y. He shows that in this case the Szegő kernel is bounded by $|B|^{-1}$, where $|B|$ is the volume of the non-isotropic ball. That is, for $((x, y, t), (r, s, u)) \in \partial\Omega \times \partial\Omega$,

$$|S((x, y, t); (r, s, u))| \leq C|B((x, y, t); \delta)|^{-1},$$

where δ is the non-isotropic distance between (x, y, t) and (r, s, u). Nagel, Rosay, Stein and Wainger [24] generalize this result to domains of the form $\Omega = \{(z_1, z_2) \in \mathbb{C}^2 : \text{Im}[z_2] > P(z_1)\}$, where P is a subharmonic, non-harmonic polynomial in $\mathbb{C}$.

Halfpap, Nagel and Wainger [13] study the singular behavior of the Bergman and Szegő kernel in domains of this kind, i.e., domains of the form $\Omega = \{(z_1, z_2) \in \mathbb{C}^2 \,|\, \text{Im}[z_2] > b(\text{Re}[z_1])\}$, but where $b \in C^\infty$ belongs to a particular class of convex functions (a model example of which is $b(r) = \exp(-|r|^{-a})$ for $|r|$ small, and $b(r) = r^{2m}$ for $|r|$ large, with $a,\ m > 0$). They show that if $\Delta \subset \partial\Omega \times \partial\Omega$ is the diagonal of the boundary, then if $0 < a < 1$ the Bergman and Szegő kernels extend smoothly to $\overline{\Omega} \times \overline{\Omega} \setminus \Delta$, while if $a \geq 1$, the kernels are singular at points on $\overline{\Omega} \times \overline{\Omega} \setminus \Delta$.

The Szegő kernel has also been studied in some non-pseudoconvex domains of this type, i.e., domains of the kind $\Omega = \{\boldsymbol{z} \in \mathbb{C}^2 : \text{Im}[z_2] > b(\text{Re}[z_1])\}$, but where the function

b is not convex. In [5], Carracino studies a domain given by a particular choice of a non-convex b, and shows that there are singularities not only on the diagonal (which is true in the pseudoconvex case), but also off the diagonal. In [11], Gilliam and Halfpap consider domains where b is a non-convex quartic polynomial with positive leading coefficient, and show that there are points off the diagonal of $\partial\Omega \times \partial\Omega$ at which the Szegő kernel is infinite, as well as points on the diagonal at which it is finite.

Haslinger studies the Szegő kernel in domains of the kind $\Omega = \{\boldsymbol{z} \in \mathbb{C}^2 : \text{Im}[z_2] > p(z_1)\}$, for functions $p : \mathbb{C} \to \mathbb{R}_+$. In [14] he finds an integral expression for the Szegő kernel in terms of the Bergman kernel and in [15] he computes an asymptotic expansion for the Szegő kernel when $p(z_1) = |\text{Re}[z_1]|^\alpha$, where $\alpha > \frac{4}{3}$. He then uses this result to study the singularities on the diagonal of $\partial\Omega \times \partial\Omega$.

By definition the Szegő projection is bounded in L^2. It is a relevant question in the field, however, to study the L^p boundedness of this operator for $p \neq 2$. Estimates for the Szegő kernel in terms of the volumes of non-isotropic balls can be used to obtain such bounds. Bonami and Lohoué [2] show that the Szegő projection defined over domains of the kind $D_\alpha = \{z \in \mathbb{C}^n : |z_1|^{\frac{2}{\alpha_1}} + \ldots + |z_n|^{\frac{2}{\alpha_n}} < 1\}$ where $0 < \alpha_j < 1$ for all j is weak type $(1,1)$ and strong type (p,p) for $1 < p < \infty$. Phong and Stein [28] obtain Sobolev bounds for the Bergman and Szegő projections for smoothly bounded strongly pseudoconvex domains in $\mathbb{C}^n$, $n \geq 2$. Christ [6] shows that if M is a compact pseudoconvex real 3-dimensional C^∞ Cauchy-Riemann manifold of finite type and the associated first-order differential operator $\overline{\partial}_b$ has closed range, then the Szegő projection extends to an operator bounded in $L^p(M)$ for $1 < p < \infty$. Mitchell [22] proves L^p estimates for the Szegő projection for a class of bounded symmetric domains; and McNeal and Stein [21] generalize this result to smoothly bounded convex domains of finite type, showing that the Szegő projection maps $L^p_s(\partial\Omega)$ to $L^p_s(\partial\Omega)$ for $1 < p < \infty$ and $s \in \mathbb{N}$.

1.2 Convexity

There is much that can be said about convexity, but we will limit our discussion to a few classical results that we use repeatedly throughout our work. The proof of our main theorem relies heavily on the fact that given a compact convex set, one can find a maximal ellipsoid (called a John ellipsoid) that can be inscribed in said body. We will begin this section by giving a brief summary of the history of this result. We will then go on to describe two lemmas by Bruna, Nagel and Wainger on their paper *Convex Hypersurfaces and Fourier Transforms* [4] which describe the size of convex polynomials and their derivatives relative to the absolute value of their coefficients. In the course of this thesis we will repeatedly use these two results together, since applying John's results to these two lemmas yields absolute bounds for the coefficients of compact convex polynomials of the form we study. On section 2 we generalize the result by Bruna, Nagel and Wainger to several variables. We will finish the discussion of convexity by deriving an estimate for sets defined by convex functions that will be relevant to our study.

1.2.1 Loewner-John Ellipsoids

Since antiquity mathematicians have estimated the size of regions in terms of simple geometric shapes of similar size. In particular, on the mid 20^{th} century, the ellipsoids (i.e., the images under invertible linear transformations of the unit ball) that can be inscribed and circumscribed in compact convex bodies were extensively studied. We will restrict our discussion to the question of uniqueness of the inscribed ellipsoid of maximal volume (called a John ellipsoid) and the circumscribed ellipsoid of minimal volume (called a Loewner ellipsoid), as well as the ratio between their volumes. For a more complete exposition of the development and applications of this problem see, e.g., [16].

In his paper *Über die kleinste umbeschriebene und die größte einbesehriebene Ellipse eines konvexen Bereichs* [1], Behrend shows that given a convex body in $\mathbb{R}^2$ there exists a unique maximal inscribed ellipse and a unique minimal circumscribed ellipse. This result was generalized to n dimensions by Danzer, Laugwitz and Lenz [7] as well as by Zaguskin [31].

In 1948, in his paper *Extremum problems with inequalities as subsidiary conditions* [17], John used an application of Lagrange multipliers (or rather, an extension of Lagrange multipliers where the constraints are given by inequalities) to study the volume of these ellipsoids. In particular, he showed that the ratio between these ellipsoids is independent of the given convex body, and depends only on the dimension of the space.

More precisely, let K be a compact convex body in $\mathbb{R}^n$, B_n be the unit ball and $\mathfrak{E}$ be the minimal circumscribed ellipsoid to K. Notice that there exists a $t \in \mathbb{R}^n$ and an invertible linear transformation $T \in \mathbb{R}^{n\times n}$ such that $\mathfrak{E} = t + TB_n$. By John, it follows that

$$t + \frac{1}{n}TB_n \subseteq K \subseteq t + TB_n.$$

Furthermore, if K has a center of symmetry (i.e., there exists a $c \in \mathbb{R}^n$ such that

$K = c - K = \{c - y \,:\, y \in K\}$), then the ratio can be improved to $\frac{1}{\sqrt{n}}$. That is to say, for a symmetric compact convex body K, the containment can be sharpened to

$$t + \frac{1}{\sqrt{n}} T B_n \subseteq K \subseteq t + T B_n.$$

1.2.2 Coefficients of convex polynomials of one variable

Bruna, Nagel and Wainger obtained estimates on [4] for convex polynomials of one variable in terms of the absolute values of their coefficients. We generalize these results to polynomials of several variables in Section 2.

The one-dimensional results (which can be found on Section 2, page 338, of the aforementioned paper) are as follows:

Let $C(m, T)$ denote the space of polynomials

$$P(t) = \sum_{j=0}^{m} a_j t^j$$

which satisfy:

1. The degree of P is no bigger than m;
2. $P(0) = a_0 = 0$; $P'(0) = a_1 = 0$;
3. P is convex for $0 \leq t \leq T$.

Lemma 2.1. *There is a constant C_m, independent of T, so that if $P \in C(m, T)$, $P(t) = \sum_{j=2}^{m} a_j t^j$, then*

$$P(t) \geq C_m \sum_{j=2}^{m} |a_j| t^j \tag{1.2}$$

for $0 \leq t \leq T$*. In particular, since* $a_0 = a_1 = 0$,

$$P(t) \geq C_m t^m \sum_{j=2}^{m} |a_j| \qquad \textit{if} \qquad 0 \leq t \leq 1, \qquad \textit{and}$$
$$P(t) \geq C_m t^2 \sum_{j=2}^{m} |a_j| \qquad \textit{if} \qquad 1 \leq t \leq T.$$

Remark 1.3. *Notice that since* $C_m \sum_{j=2}^{m} |a_j| t^j \leq P(t) \leq \sum_{j=2}^{m} |a_j| t^j$ *it follows that* $C_m \leq 1$.

Lemma 2.2. *There is a constant* C_m*, independent of* T*, so that if* $P \in C(m, T)$, $P(t) = \sum_{j=2}^{m} a_j t^j$*, then*

$$P'(t) \geq C_m \sum_{j=2}^{m} |a_j| t^{j-1}$$

for $0 \leq t \leq T$*. In particular,*

$$P'(t) \geq C_m t^{m-1} \sum_{j=2}^{m} |a_j| \qquad \textit{if} \qquad 0 \leq t \leq 1, \qquad \textit{and}$$
$$P'(t) \geq C_m t \sum_{j=2}^{m} |a_j| \qquad \textit{if} \qquad 1 \leq t \leq T.$$

1.2.3 An estimate for sets defined by convex functions

The third and last topic we would like to cover in this overview of convexity is the following estimate, which plays a crucial role in the proof of one of our lemmas.

Proposition 1.4. *If $f : \mathbb{R}^n \to \mathbb{R}$ is a convex function such that $f(\mathbf{0}) = 0$ and $\nabla f(\mathbf{0}) = 0$ then*

$$I = \int_{\mathbb{R}^n} e^{-f(\boldsymbol{w})}\, d\boldsymbol{w} \approx |\{\boldsymbol{w} \,:\, f(\boldsymbol{w}) \leq 1\}|.$$

In the proof of the above estimate, we will use the following inequality:

Claim 1.5. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function such that $f(\mathbf{0}) = 0$. Given $x > 0$, let*

$$A_x = \{\boldsymbol{w} \in \mathbb{R}^n \,:\, f(\boldsymbol{w}) \leq x\}.$$

Then for any constant $0 \leq \lambda \leq 1$, $vol(A_{\lambda x}) \geq \lambda^n vol(A_x)$.

Proof. By convexity of f, for any vectors $\boldsymbol{w}$ and $\boldsymbol{u}$ in $\mathbb{R}^n$, and any constant $0 \leq \lambda \leq 1$,

$$f(\lambda\boldsymbol{w} + (1-\lambda)\boldsymbol{u}) \leq \lambda f(\boldsymbol{w}) + (1-\lambda) f(\boldsymbol{u}).$$

In particular, taking $\boldsymbol{u} = \mathbf{0}$, and since by hypothesis $f(\mathbf{0}) = 0$, it follows that $f(\lambda\boldsymbol{w}) \leq \lambda f(\boldsymbol{w})$.

Thus, if $\boldsymbol{w} \in A_x$, then $f(\lambda\boldsymbol{w}) \leq \lambda f(\boldsymbol{w}) \leq \lambda x$. That is, $\lambda \cdot A_x \subseteq A_{\lambda x}$. It follows that

$$vol(A_{\lambda x}) \geq vol(\lambda \cdot A_x) = \lambda^n vol(A_x).$$

□

The proof of Proposition 1.4 is as follows:

Proof. Without loss of generality we can assume $f \not\equiv 0$.

Notice that under these hypothesis $f(\boldsymbol{v}) \geq 0\ \forall \boldsymbol{v} \in \mathbb{R}^n$. In fact, by the Fundamental Theorem of Calculus we can write

$$f(\boldsymbol{u}+\boldsymbol{v}) - f(\boldsymbol{u}) = -\int_0^1 \nabla f(\boldsymbol{u}+\boldsymbol{v}t) \cdot \boldsymbol{v} \left(\frac{d}{dt}(1-t)\right) dt.$$

And integrating by parts, we have that

$$\int_0^1 \nabla f(\boldsymbol{u}+t\boldsymbol{v}) \cdot \boldsymbol{v}\, dt = -\nabla f(\boldsymbol{u}+t\boldsymbol{v}) \cdot \boldsymbol{v}(1-t)\Big|_0^1 + \int_0^1 \frac{d}{dt}\left[\nabla f(\boldsymbol{u}+t\boldsymbol{v}) \cdot \boldsymbol{v}\right](1-t)\, dt.$$

But

$$\frac{d}{dt}\left[\nabla f(\boldsymbol{u}+t\boldsymbol{v}) \cdot \boldsymbol{v}\right] = \sum_{i,j=1}^{n} f_{ij}(\boldsymbol{u}+t\boldsymbol{v}) v_i v_j.$$

It follows that

$$f(\boldsymbol{u}+\boldsymbol{v}) - f(\boldsymbol{u}) = \nabla f(\boldsymbol{u}) \cdot \boldsymbol{v} + \int_0^1 \sum_{i,j=1}^{n} f_{ij} v_i v_j (1-t)\, dt. \tag{1.3}$$

Letting $\boldsymbol{u} = \boldsymbol{0}$ we have by convexity that

$$f(\boldsymbol{v}) = \int_0^1 \sum_{i,j=1}^{n} f_{ij} v_i v_j (1-t)\, dt \geq 0. \tag{1.4}$$

A lower bound for I can be easily obtained, since

$$\int_{\mathbb{R}^n} e^{-f(\boldsymbol{w})}\, d\boldsymbol{w} \geq \int_{\{\boldsymbol{w}\,:\, f(\boldsymbol{w}) \leq 1\}} e^{-f(\boldsymbol{w})}\, d\boldsymbol{w} \geq \frac{1}{e}\left|\{\boldsymbol{w}\ :\ f(\boldsymbol{w}) \leq 1\}\right|.$$

To obtain an upper bound we can write

$$\int_{\mathbb{R}^n} e^{-f(\boldsymbol{w})}\, d\boldsymbol{w} = \int_{\{\boldsymbol{w}\,:\, f(\boldsymbol{w})\le 1\}} e^{-f(\boldsymbol{w})}\, d\boldsymbol{w} \,+ \sum_{j=1}^{\infty} \int_{\{\boldsymbol{w}\,:\, j\le f(\boldsymbol{w})\le j+1\}} e^{-f(\boldsymbol{w})}\, d\boldsymbol{w}. \tag{1.5}$$

But

$$\int_{\{\boldsymbol{w}\,:\, j\le f(\boldsymbol{w})\le j+1\}} e^{-f(\boldsymbol{w})}\, d\boldsymbol{w} \le e^{-j}\, |\{\boldsymbol{w} \,:\, f(\boldsymbol{w}) \le j+1\}|.$$

But by Claim 1.5, and taking $\lambda = \frac{1}{j+1}$ and $x = j+1$, it follows that

$$|\{\boldsymbol{w} \,:\, f(\boldsymbol{w}) \le j+1\}| \le (j+1)^n |\{\boldsymbol{w} \,:\, f(\boldsymbol{w}) \le 1\}|.$$

Hence, by equation (1.5), we have that

$$I \le |\{\boldsymbol{w} \,:\, f(\boldsymbol{w}) \le 1\}| \left(1 + \sum_{j=1}^{\infty} e^{-j}(j+1)^n\right).$$

Since the sum converges we get the desired upper bound.

□

1.3 Main results

The purpose of this thesis is to study the size of the Szegő kernel on the boundary of certain convex domains in $\mathbb{C}^{n+1}$. We consider domains of the kind

$$\Omega_b = \{\boldsymbol{z} \in \mathbb{C}^{n+1} : \mathrm{Im}[z_{n+1}] > b(\mathrm{Re}[z_1], \ldots, \mathrm{Re}[z_n])\}$$

for convex polynomial functions $b : \mathbb{R}^n \to \mathbb{R}$ of "combined degree" (refer to definition on page 3). We generalize Nagel's one-dimensional size estimate for the Szegő kernel established in [23] to several variables. Throughout this paper we will let $z_j = x_j + iy_j$, and we will use boldfonts to denote vectors $\boldsymbol{x} = (x_1, \ldots, x_n)$ in $\mathbb{R}^n$.

It can be shown, as in [23], that the Szegő kernel for the domains under consideration can be written as an integral formula.

Proposition A.1. *The Szegő kernel on* Ω_b *is given by*

$$S((\boldsymbol{x},\boldsymbol{y},t);(\boldsymbol{x'},\boldsymbol{y'},t')) = \int_0^\infty e^{-2\pi\tau[b(\boldsymbol{x'})+b(\boldsymbol{x})+i(t'-t)]} \left(\int_{\mathbb{R}^n} \frac{e^{2\pi\boldsymbol{\eta}\cdot[\boldsymbol{x}+\boldsymbol{x'}-i(\boldsymbol{y'}-\boldsymbol{y})]}}{\int\limits_{\mathbb{R}^n} e^{4\pi[\boldsymbol{\eta}\cdot\boldsymbol{v}-b(\boldsymbol{v})\tau]}\, d\boldsymbol{v}}\, d\boldsymbol{\eta} \right) d\tau, \tag{1.6}$$

where $(\boldsymbol{x},\boldsymbol{y},t)$ *and* $(\boldsymbol{x'},\boldsymbol{y'},t')$ *are points in* $\partial\Omega_b$.

We have included an Appendix with the proof of this proposition. Our estimates will all follow from a study of this integral expression for the Szegő kernel.

Perhaps one of the most striking differences between the one-dimensional case studied in [23] and the n-dimensional case I study is the fact that in the former it is enough to assume that b is a convex polynomial, whereas in the latter that assumption is not enough. Convexity alone will not ensure that $\int_{\mathbb{R}^n} e^{\boldsymbol{\eta}\cdot\boldsymbol{v}-\tau b(\boldsymbol{v})}\, dv$ converges in $\mathbb{R}^n$ for $n \geq 2$ (in fact, for $n = 2$ consider the polynomial $b(x_1, x_2) = x_1^2$). The “combined degree” condition ensures that the class of convex polynomials under consideration satisfy that the set $R = \{\boldsymbol{v} \in \mathbb{R}^n : b(\boldsymbol{v}) \leq 1\}$ is compact. This guarantees the convergence of the denominator integral.

We devote the second section to a study of the coefficients of convex polynomials in several variables. In the one-variable case it was shown by Bruna, Nagel and Wainger (as we explained in detail in the previous subsection) that the absolute value of the coefficients of a convex polynomial with no constant or linear terms can be bounded by the value of the polynomial at 1. It is not possible to obtain such a bound in more variables, since the polynomial might be growing in some directions but not along others. However, we show that the absolute value of the coefficients can be bounded by the average of the polynomial over a circle of arbitrary positive radius. We obtain the following result:

Proposition 2.1. *Let* $\Gamma(M) = \{(\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n \,:\, 2 \leq |\alpha| \leq M\}$. *Let* $S(M)$ *be the set of convex polynomials* $g(\boldsymbol{v}) = \sum_{\alpha \in \Gamma(M)} c_\alpha \boldsymbol{v}^\alpha$. *Then for any fixed* $a > 0$, *there exists a positive constant* $C(M, a)$ *that depends only on* M *and the constant* a *such that if* $g \in S(M)$,

$$\sum_{\alpha \in \Gamma(M)} |c_\alpha| \leq C(M, a) \int_{|\sigma| = a} g(\sigma)\, d\sigma.$$

Remark 1.6. *Notice that with* $\Gamma(M)$ *defined as above, polynomials of the form* $g(\boldsymbol{v}) = \sum_{\alpha \in \Gamma(M)} c_\alpha \boldsymbol{v}^\alpha$ *are such that the degree of* g *is no bigger than* M; $g(\mathbf{0}) = 0$; *and* $\nabla g(\mathbf{0}) = \mathbf{0}$.

In Sections 3 and 4 we present the proof of our main result. That is, we derive an estimate for the size of the Szegő kernel in terms of the geometry of the domain that defines it.

Our goal is to obtain a geometric bound for the Szegő kernel from a study of the integral expression in equation (1.6). One of the main technical difficulties comes from handling the denominator integral. After some manipulation, this denominator integral can be expressed as a function that we will denote as $\theta(\boldsymbol{\eta})$. In the following lemma we show that $\theta(\boldsymbol{\eta})$ is a Schwartz function.

Lemma 3.1. *Let $g : \mathbb{R}^n \to \mathbb{R}$ be a convex polynomial such that*

i) $g(\mathbf{0}) = 0$;

ii) $\nabla g(\mathbf{0}) = \mathbf{0}$;

iii) there exists a constant $0 < A < 1$ *such that* $\{\boldsymbol{v} : |\boldsymbol{v}| \leq A\} \subseteq \{\boldsymbol{v} : g(\boldsymbol{v}) \leq 1\} \subseteq \{\boldsymbol{v} : |\boldsymbol{v}| \leq 1\}$; *and*

iv) there exist positive integers $m_1, \ldots, m_n$ *such that the "combined degree" of* g *is* $(m_1, \ldots, m_n)$.

Then

$$\theta(\boldsymbol{\eta}) = \left[\int_{\mathbb{R}^n} e^{\boldsymbol{\eta}\cdot\boldsymbol{v} - g(\boldsymbol{v})}\, d\boldsymbol{v} \right]^{-1}$$

is a Schwartz function. Moreover, its decay depends only on the constant A and the exponents $\{m_1, \ldots, m_n\}$.

We devote an entire section to the study of the decay of $\theta(\boldsymbol{\eta})$, rather than incorporating it in the proof of the main theorem, which is discussed in Section 4. The estimates for convex polynomials of several variables obtained in Section 2 will play a crucial role in the proof of this lemma. The bound obtained for the coefficients of the polynomials combined with the compactness requirement (given by hypothesis (iii)) ensure that the decay of $\theta(\boldsymbol{\eta})$ is independent of the coefficients of g.

Our main result is the following:

Main Theorem. *Let $b : \mathbb{R}^n \to \mathbb{R}$ be a convex polynomial of "combined degree" $(m_1, \ldots, m_n)$. Define* $\Omega_b = \{\boldsymbol{z} \in \mathbb{C}^{n+1} : \mathrm{Im}[z_{n+1}] > b(\mathrm{Re}[z_1], \ldots, \mathrm{Re}[z_n])\}$ *and let* $(\boldsymbol{x}, \boldsymbol{y}, t)$ *and* $(\boldsymbol{x}', \boldsymbol{y}', t')$ *be any two points in* $\partial\Omega_b$. *Define*

$$\tilde{b}(\boldsymbol{v}) = b\left(\boldsymbol{v} + \frac{\boldsymbol{x}+\boldsymbol{x'}}{2}\right) - \nabla b\left(\frac{\boldsymbol{x}+\boldsymbol{x'}}{2}\right)\cdot \boldsymbol{v} - b\left(\frac{\boldsymbol{x}+\boldsymbol{x'}}{2}\right);$$

$$\delta(\boldsymbol{x},\boldsymbol{x'}) = b(\boldsymbol{x}) + b(\boldsymbol{x'}) - 2b\left(\frac{\boldsymbol{x}+\boldsymbol{x'}}{2}\right);$$

and

$$w = (t'-t) + \nabla b\left(\frac{\boldsymbol{x}+\boldsymbol{x'}}{2}\right)\cdot(\boldsymbol{y'}-\boldsymbol{y}).$$

We obtain the following estimate for the Szegő kernel associated to the domain Ω_b *:*

$$|S\left((\boldsymbol{x},\boldsymbol{y},t);(\boldsymbol{x'},\boldsymbol{y'},t')\right)| \lesssim \frac{1}{\sqrt{\delta^2+\widetilde{b}(\boldsymbol{y}-\boldsymbol{y'})^2+w^2}\left|\left\{\boldsymbol{v}\,:\,\tilde{b}(\boldsymbol{v}) < \sqrt{\delta^2+\widetilde{b}(\boldsymbol{y}-\boldsymbol{y'})^2+w^2}\right\}\right|^2}.$$

Here the constant depends on the exponents $\{m_1,\ldots,m_n\}$ *and the dimension of the space, but is independent of the two given points.*

Remark 1.7. *Notice that since* b *is convex,* $\delta(\boldsymbol{x},\boldsymbol{x'}) \geq 0$*. It is convenient to think of* δ *as a measure of the curvature between* $\boldsymbol{x}$ *and* $\boldsymbol{x'}$*. The more curved the domain is, the larger the value of* δ*.*

The proof of the main theorem is discussed in Section 4. The proof is, at its core, an application of John's ellipsoids. We introduce a change of variables in the integral expression for the Szegő kernel comprised of factors defined by the length of the axes of the unique maximal inscribed ellipsoid associated to a symmetrization of the convex body $R = \left\{\boldsymbol{v}\,:\,\tilde{b}(\boldsymbol{v}) < \frac{1}{\tau}\right\}$.

Chapter 2

Coefficients of convex polynomials of several variables

In this section we obtain bounds for the absolute value of the coefficients of convex polynomials of several variables. We begin by showing that for polynomials whose constant and linear terms are zero, the sum of the absolute value of the coefficients can be bounded (up to a universal constant) by the average of the polynomial on a sphere of arbitrary positive radius. We then consider convex polynomials $g : \mathbb{R}^n \to \mathbb{R}$ with no constant or linear terms such that there exist some positive constants A, B so that the set $\{\boldsymbol{v} \,:\, g(\boldsymbol{v}) \leq 1\}$ contains the ball of radius A and is contained in the ball of radius B. For these, we show that the sum of the absolute value of the coefficients is bounded by a universal constant. Moreover, we show that the absolute value of the coefficients is bounded (up to a universal constant) by the value of g at any point on the boundary of the ball of radius A.

These results are needed for the proof of the geometric bound for the Szegő kernel on the domains we study. In particular, we will use these results in Section 3 to show that the decay of the function $\theta(\boldsymbol{\eta})$ (which arises naturally from a study of the denominator integral of the expression for the Szegő kernel obtained in Proposition A.1) does not depend on the coefficients of the polynomial which defines it.

Proposition 2.1. *Let* $\Gamma(M) = \{(\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n : 2 \le |\alpha| \le M\}$*. Let* $S(M)$ *be the set of convex polynomials* $g(\boldsymbol{v}) = \sum_{\alpha \in \Gamma(M)} c_\alpha \boldsymbol{v}^\alpha$*. Then for any fixed* $a > 0$*, there exists a positive constant* $C(M, a)$ *that depends only on* M *and the constant* a *such that if* $g \in S(M)$*,*

$$\sum_{\alpha \in \Gamma(M)} |c_\alpha| \le C(M, a) \int_{|\sigma| = a} g(\sigma)\, d\sigma. \tag{2.1}$$

Remark 2.2. *This is a generalization of the result in one variable by Bruna, Nagel and Wainger in [4] (Lemma 2.1).*

Proof. Let $a > 0$ be a fixed positive constant and let $|\Gamma(M)|$ denote the cardinality of the set of indices $\Gamma(M)$. We identify the space $S(M)$ with $\mathbb{R}^{|\Gamma(M)|}$ via the identification

$$g(\boldsymbol{v}) = \sum_{\alpha \in \Gamma(M)} c_\alpha \boldsymbol{v}^\alpha \in S(M) \leftrightarrow (c_1, \ldots, c_{|\Gamma(M)|}) \in \mathbb{R}^{|\Gamma(M)|},$$

where we have ordered all the $\alpha \in \Gamma(M)$ so that c_j corresponds to the coefficient c_α for the j^{th} element α with this ordering.

Let

$$\Sigma_M = \{g(\boldsymbol{v}) = \sum_{\alpha \in \Gamma(M)} c_\alpha \boldsymbol{v}^\alpha \in S(M) \,:\, \sum_{\alpha \in \Gamma(M)} |c_\alpha| = 1\}. \tag{2.2}$$

We claim that Σ_M is a compact subset of $S(M)$. In fact, let $\{\boldsymbol{c_n}\}_{n \in \mathbb{N}}$ in $\mathbb{R}^{|\Gamma(M)|}$ be a sequence of tuples associated to a sequence of polynomials $\{q_n\}_{n \in \mathbb{N}}$ in Σ_M. Since $\{\boldsymbol{c_n}\}_{n \in \mathbb{N}}$ is a sequence contained in the compact set $B_M = \{(c_1, \ldots, c_{|\Gamma(M)|}) \in \mathbb{R}^{|\Gamma(m)|} : \sum_{1 \le j \le |\Gamma(M)|} |c_j| = 1\}$, it has a convergent subsequence $\{\boldsymbol{c_{n_i}}\}_{n_i \in \mathbb{N}}$. Let $\boldsymbol{c}$ be the limit of this subsequence, and let q be the polynomial associated to this tuple. We claim that q is an element of Σ_M. In fact, the identification preserves the degree of the polynomial and the fact that there are no constant or linear terms. Also, since $\boldsymbol{c}$ is an element of B_M, it satisfies that $\sum_{1\leq j\leq|\Gamma(M)|}|c_j| = 1$. Thus, is suffices to show that q is convex. This follows easily, since given any polynomial q_{n_i} associated to an element of the convergent subsequence $\{\boldsymbol{c}_{\boldsymbol{n_i}}\}_{n_i\in\mathbb{N}}$, we have that $q_{n_i}(\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y}) \leq \alpha q_{n_i}(\boldsymbol{x}) + (1-\alpha)q_{n_i}(\boldsymbol{y})$ for all $0 \leq \alpha \leq 1$ and for all points $\boldsymbol{x}, \boldsymbol{y}$ in $\mathbb{R}^n$. Thus, and since $q_{n_i}(\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y}) \to q(\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y})$; $\alpha q_{n_1}(\boldsymbol{x}) \to \alpha q(\boldsymbol{x})$; and $(1-\alpha)q_{n_i}(\boldsymbol{y}) \to (1-\alpha)q(\boldsymbol{y})$, the convexity of q follows immediately.

Let

$$\Phi_I(g) = \frac{1}{\omega_n(a)}\int_{|\sigma|=a} g(\sigma)\,d\sigma,$$

where $\omega_n(a)$ is the surface area of the sphere of radius a in $\mathbb{R}^n$ and

$$\Phi_{II}(g) = \sum_{\alpha\in\Gamma(M)} |c_\alpha|.$$

Notice that these functions are continuous on $S(M)$, and that $\Phi_{II}(g) = 1$ on Σ_M.

We claim that $\Phi_I(g)$ is strictly positive on Σ_M. In fact, since g is convex, $g(\mathbf{0}) = 0$ and $\nabla g(\mathbf{0}) = \mathbf{0}$ it follows that g is nonnegative (see, e.g., the proof of Proposition 1.4). Moreover, on Σ_M at least one of the coefficients of g must be different from zero, so g can not be the zero polynomial. Thus g must be positive almost everywhere. In particular, the average over the circle of radius a must be strictly positive.

Therefore, and since $\Phi_I(g)$ is continuous as a function of g, it attains a minimum in Σ_M, and this minimum is strictly positive. Thus, and since $\Phi_{II}(g) = 1$ on Σ_M, there exists a constant $C > 0$ such that for any $g \in \Sigma_M$,

$$\Phi_I(g) \geq C = C\Phi_{II}(g).$$

That is,

$$\frac{1}{\omega_n(a)} \int_{|\sigma|=a} g(\sigma)\, d\sigma \geq C\Phi_{II}(g) = C \sum_{\alpha\in\Gamma(M)} |c_\alpha|,$$

as desired.

Consider now a polynomial $g(\boldsymbol{v}) = \sum_{\alpha\in\Gamma(M)} c_\alpha \boldsymbol{v}^\alpha \in S(M)$, but which is not necessarily in Σ_M. Then let $h(\boldsymbol{v}) = \sum_{\alpha\in\Gamma(M)} b_\alpha \boldsymbol{v}^\alpha$ where

$$b_\alpha = \frac{c_\alpha}{\sum_{\beta\in\Gamma(M)} |c_\beta|}$$

so that $\sum_{\alpha\in\Gamma(M)} |b_\alpha| = 1$ and $h(\boldsymbol{v}) \in \Sigma_M$. It follows from the previous case that

$$\frac{1}{\omega_n(a)} \int_{|\sigma|=a} h(\sigma)\, d\sigma \geq C.$$

That is,

$$\frac{1}{\omega_n(a)} \int_{|\sigma|=a} \frac{g(\sigma)}{\sum_{\beta\in\Gamma} |c_\beta|}\, d\sigma \geq C.$$

This gives the desired inequality.

□

Corollary 2.3. *Let* $g(\boldsymbol{v}) = \sum_\alpha c_\alpha \boldsymbol{v}^\alpha$ *be a convex polynomial such that* $g(\mathbf{0}) = 0$ *and* $\nabla g(\mathbf{0}) = \mathbf{0}$. *Suppose there exist two positive constants* A *and* B *such that*

$$\{\boldsymbol{v} : |\boldsymbol{v}| \leq A\} \subseteq \{\boldsymbol{v} : g(\boldsymbol{v}) \leq 1\} \subseteq \{\boldsymbol{v} : |\boldsymbol{v}| \leq B\}. \tag{2.3}$$

Then there exists a constant that depends only on A *and the degree of* g *such that*

$$\sum_{\alpha} |c_\alpha| \le C. \tag{2.4}$$

Moreover, for any point $\boldsymbol{x} = (x_1, \dots, x_n)$ *on the boundary of the circle of radius* A*, there exist constants* $C_1 > 0$*,* $C_2 > 0$ *that depend only on* A*,* B *and the degree of* g *such that*

$$g(\boldsymbol{x}) \ge C_1 \ge C_2 \sum_{\alpha} |c_\alpha|. \tag{2.5}$$

Remark 2.4. *The bound given by equation (2.4) can be obtained using just the left containment, i.e, the existence of a constant* $A > 0$ *such that* $\{\boldsymbol{v} : |\boldsymbol{v}| \le A\} \subseteq \{\boldsymbol{v} : g(\boldsymbol{v}) \le 1\}$*. The second bound, however, requires the existence of both an inner and an outer ball.*

Proof. The first result follows immediately from the previous claim. In fact, we showed that

$$\sum_{\alpha} |c_\alpha| \le C \int_{|\sigma|=A} g(\sigma)\, d\sigma.$$

But by (2.3) we have that $g(\sigma) \le 1$ for all σ such that $|\sigma| = A$. The result follows.

Observe that the bound $g(\boldsymbol{x}) \ge C_2 \sum_\alpha |c_\alpha|$ will be an immediate consequence of the above bound on the coefficients once we show that $g(\boldsymbol{x}) \ge C_1$.

The proof of equation (2.5) requires the use of Lemma 2.1 on the paper *Convex hypersurfaces and Fourier Transforms* by Bruna, Nagel and Wainger [4]. The lemma states that given a convex polynomial of one variable of degree M of the form

$$p(t) = \sum_{j=2}^{M} a_j t^j$$

there exists a constant $C_M > 0$ that depends only on M such that

$$C_M \sum_{j=2}^{M} |a_j| t^j \leq p(t) \leq \sum_{j=2}^{M} |a_j| t^j \qquad \forall t \geq 0. \tag{2.6}$$

In particular, this result implies that for any $\lambda > 1$ and $t \geq 0$,

$$p(\lambda t) \leq \sum_{j=2}^{M} |a_j| \lambda^j t^j \leq \lambda^M \sum_{j=2}^{M} |a_j| t^j \leq \frac{\lambda^M}{C_M} p(t). \tag{2.7}$$

Given a point $\boldsymbol{x}$ on the boundary of the circle centered at the origin of radius A, we will let

$$p(t) = g(t\boldsymbol{x}).$$

Notice that this defines a convex polynomial of one variable for which the bounds in equation (2.6) apply.

Taking $t = 1$ and $\lambda = \frac{B}{A}$ (where A and B are the radius of the inner and outer ball respectively) in equation (2.7), we have that

$$p(1) \geq \frac{C_M}{\lambda^M} \, p\left(\frac{B}{A}\right).$$

That is,

$$g(\boldsymbol{x}) \geq C_M \frac{A^M}{B^M} \, g\left(\frac{B\boldsymbol{x}}{A}\right). \tag{2.8}$$

Since $\boldsymbol{x}$ is a point on the boundary of the inner circle, the point $\frac{B\boldsymbol{x}}{A}$ is on the border of the outer circle (the circle of radius B). Thus, by hypothesis, it follows that

$$g\left(\frac{B\boldsymbol{x}}{A}\right) \geq 1.$$

This together with equation (2.8) implies the desired result.

□

Remark 2.5. *Notice that the convexity of g implies that*

$$g(\boldsymbol{x}) \geq C_1 \geq C_2 \sum_{\alpha} |c_\alpha|.$$

for any point $\boldsymbol{x}$ such that $|\boldsymbol{x}| \geq A$.

Chapter 3

Decay of $\theta(\boldsymbol{\eta})$

In this section we study the decay of the function $\theta(\boldsymbol{\eta}) = \left[\int\limits_{\mathbb{R}^n} e^{\boldsymbol{\eta}\cdot\boldsymbol{v}-g(\boldsymbol{v})}\,d\boldsymbol{v}\right]^{-1}$, where g is a convex polynomial. This function arises naturally from the study of the denominator integral of the integral expression obtained for the Szegő kernel in Appendix A. We show that $\theta(\boldsymbol{\eta})$ is a Schwartz function. This implies, in particular, that $\int_{\mathbb{R}^n} \theta(\boldsymbol{\eta})\,d\boldsymbol{\eta}$ converges. We use this fact in the proof of the geometric bound for the Szegő kernel in Section 4.

Lemma 3.1. *Let*

$$g(\boldsymbol{v}) = \sum_{\alpha\in\Gamma} c_\alpha \boldsymbol{v}^\alpha \tag{3.1}$$

be a convex polynomial in $\mathbb{R}^n$ *such that*

i) $g(\mathbf{0}) = 0$;

ii) $\nabla g(\mathbf{0}) = 0$;

iii) there exists a constant $0 < A < 1$ *such that* $\{\boldsymbol{v} : |\boldsymbol{v}| \leq A\} \subseteq \{\boldsymbol{v} : g(\boldsymbol{v}) \leq 1\} \subseteq \{\boldsymbol{v} : |\boldsymbol{v}| \leq 1\}$; *and*

iv) there exist positive integers $m_1, \ldots, m_n$ *such that the "combined degree" of* g *is* $(m_1, \ldots, m_n)$ *(refer to definition on page 3).*

Then

$$\theta(\boldsymbol{\eta}) = \left[\int\limits_{\mathbb{R}^n} e^{\boldsymbol{\eta}\cdot\boldsymbol{v} - g(\boldsymbol{v})}\, d\boldsymbol{v} \right]^{-1}$$

is a Schwartz function. Moreover, its decay depends only on the constant A and the exponents $\{m_1, \ldots, m_n\}$.

Remark 3.2. *Notice that under these assumptions, the coefficients of the polynomial $g(\boldsymbol{v}) = \sum_{\alpha\in\Gamma} c_\alpha \boldsymbol{v}^\alpha$ satisfy $\sum_{\alpha\in\Gamma} |c_\alpha| \leq C$, where C depends only on the constant A, on the degree of the polynomial and on the dimension of the space. This was shown in Corollary 2.3 on page 26.*

Let $I = \int_{\mathbb{R}^n} e^{\boldsymbol{\eta}\cdot\boldsymbol{v} - g(\boldsymbol{v})}\, d\boldsymbol{v}$. We will show that I grows at an exponential rate. We can write

$$I = e^{h(\boldsymbol{v_0})} \int_{\mathbb{R}^n} e^{h(\boldsymbol{v}) - h(\boldsymbol{v_0})}\, d\boldsymbol{v}, \tag{3.2}$$

where $h(\boldsymbol{v}) = \boldsymbol{\eta}\cdot\boldsymbol{v} - g(\boldsymbol{v})$ and $\boldsymbol{v_0}$ is the point where $h(\boldsymbol{v})$ attains its maximum (notice that $\boldsymbol{\eta} = \nabla g(\boldsymbol{v_0})$).

We will show that the dominant term, $e^{h(\boldsymbol{v_0})}$, grows at an exponential rate in $\boldsymbol{\eta}$. This term will provide the desired decay for I^{-1}. We will then show that $\int e^{h(\boldsymbol{v}) - h(\boldsymbol{v_0})}\, d\boldsymbol{v}$ does not decrease too fast, that is, that it does not annul the growth of the dominant term. Notice that $h(\boldsymbol{v_0}) = L(\boldsymbol{\eta}) = \sup_{\boldsymbol{v}} \{\boldsymbol{\eta}\cdot\boldsymbol{v} - g(\boldsymbol{v})\}$ is the Legendre Transform of g.

3.1 The dominant term

We begin by studying the growth of the term $e^{h(\boldsymbol{v_0})}$, where $h(\boldsymbol{v}) = \boldsymbol{\eta} \cdot \boldsymbol{v} - g(\boldsymbol{v})$ and $\boldsymbol{v_0}$ is the point where $h(\boldsymbol{v})$ attains its maximum. We show that $e^{h(\boldsymbol{v_0})}$ grows exponentially as a function of $\boldsymbol{\eta}$. Moreover, we claim that the growth is independent of the choice of g, but rather depends only on the constant A, on the "combined degree" of g and on the dimension of the space. More precisely, we show that there exist positive constants $C, \widetilde{C}$ which depend only on the "combined degree" of g, the dimension of the space and the constant A, such that

$$e^{h(\boldsymbol{v_0})} \geq \exp\left[\widetilde{C}\left(|\eta_1|^{\frac{2m_1}{2m_1-1}} + \ldots + |\eta_n|^{\frac{2m_n}{2m_n-1}}\right) - C\right]. \tag{3.3}$$

We begin by showing that the polynomial g is dominated, independently of its coefficients, by its pure terms of highest order.

Claim 3.3. *If $g(\boldsymbol{v})$ is as in the statement of Lemma 3.1, then there exists a constant $C > 0$ that depends only on the constant A, on the "combined degree" of the polynomial and on the dimension of the space such that*

$$g(\boldsymbol{v}) \leq C(1 + v_1^{2m_1} + \ldots + v_n^{2m_n}).$$

Proof. Let

$$r(\boldsymbol{v}) = v_1^{2m_1} + \ldots + v_n^{2m_n}.$$

Notice that for any $\boldsymbol{v} \in \mathbb{R}^n$,

$$v_1^{\alpha_1} \cdots v_n^{\alpha_n} \leq r(\boldsymbol{v})^{\frac{\alpha_1}{2m_1}+\cdots+\frac{\alpha_n}{2m_n}}. \tag{3.4}$$

In fact, if any of the components of $\boldsymbol{v}$ are zero, the statement is trivial. Otherwise, dividing equation (3.4) by $v_1^{\alpha_1} \cdots v_n^{\alpha_n}$, it suffices to show that

$$1 \leq \frac{r(\boldsymbol{v})^{\frac{\alpha_1}{2m_1}}}{v_1^{\alpha_1}} \cdots \frac{r(\boldsymbol{v})^{\frac{\alpha_n}{2m_n}}}{v_n^{\alpha_n}} = \left(\frac{r(\boldsymbol{v})}{v_1^{2m_1}}\right)^{\frac{\alpha_1}{2m_1}} \cdots \left(\frac{r(\boldsymbol{v})}{v_n^{2m_n}}\right)^{\frac{\alpha_n}{2m_n}}.$$

That is, it suffices to show that

$$1 \leq \left(1 + \left(\frac{v_2^{2m_1}}{v_1^{2m_1}}\right) + \cdots + \left(\frac{v_n^{2m_n}}{v_1^{2m_1}}\right)\right)^{\frac{\alpha_1}{2m_1}} \cdots \left(\left(\frac{v_1^{2m_1}}{v_n^{2m_n}}\right) + \cdots + \left(\frac{v_{n-1}^{2m_{n-1}}}{v_n^{2m_n}}\right) + 1\right)^{\frac{\alpha_n}{2m_n}}.$$

This last inequality is trivial, since each factor in the right hand size is larger than 1.

Furthermore, since g is a convex function such that $g(\mathbf{0}) = \nabla g(\mathbf{0}) = 0$, we have that $g \geq 0$. This was discussed on page 17, equation (1.4). Thus,

$$\begin{aligned} g(\boldsymbol{v}) = |g(\boldsymbol{v})| &= \left|\sum_{\alpha\in\Gamma} c_\alpha v_1^{\alpha_1} \cdots v_n^{\alpha_n}\right| \\ &\leq \sum_{\alpha\in\Gamma} |c_\alpha| \left|v_1^{\alpha_1} \cdots v_n^{\alpha_n}\right| \\ &\leq \sum_{\alpha\in\Gamma} |c_\alpha| \left| r(\boldsymbol{v})^{\frac{\alpha_1}{2m_1}+\cdots+\frac{\alpha_n}{2m_n}}\right|. \end{aligned} \tag{3.5}$$

Moreover, recall that since g is of "combined degree" $(m_1, \ldots, m_n)$, any index $\alpha \in \Gamma$ satisfies that

$$\frac{\alpha_1}{2m_1} + \ldots + \frac{\alpha_n}{2m_n} \leq 1.$$

Hence,

$$r(\boldsymbol{v})^{\frac{\alpha_1}{2m_1}+\cdots+\frac{\alpha_n}{2m_n}} \leq 1 + r(\boldsymbol{v}). \tag{3.6}$$

Thus, and since $\sum_{\alpha\in\Gamma} |c_\alpha| \leq C$, it follows from equations (3.5) and (3.6) that

$$g(\boldsymbol{v}) \leq \sum_{\alpha\in\Gamma} |c_\alpha|(1 + r(\boldsymbol{v})) \leq C(1 + r(\boldsymbol{v})).$$

This finishes the proof of Claim 3.3

□

Since this estimate does not depend on the coefficients of g, it is now easy to obtain a lower bound for $h(\boldsymbol{v_0})$ in terms of $\boldsymbol{\eta}$ which does not depend on the choice of g.

Claim 3.4. *The Legendre Transform of $g(\boldsymbol{v})$ where $\boldsymbol{v} \in \mathbb{R}^n$ is large for large values of $|\boldsymbol{\eta}|$. More precisely,*

$$L(\boldsymbol{\eta}) \geq \widetilde{C}\left(|\eta_1|^{\frac{2m_1}{2m_1-1}} + \ldots + |\eta_n|^{\frac{2m_n}{2m_n-1}} \right) - C,$$

where C and $\widetilde{C}$ are positive constants that depend only on the constant A, on the "combined degree" of g and on the dimension of the space.

Proof. It follows from the previous claim that

$$\begin{aligned} L(\boldsymbol{\eta}) &= \sup_{\boldsymbol{v}} \{\boldsymbol{\eta}\cdot\boldsymbol{v} - g(\boldsymbol{v})\} \\ &\geq \sup_{\boldsymbol{v}} \{\boldsymbol{\eta}\cdot\boldsymbol{v} - C - C|v_1|^{2m_1} - \ldots - C|v_n|^{2m_n}\} \\ &= -C + \sup_{v_1} \{\eta_1 v_1 - C|v_1|^{2m_1}\} + \ldots + \sup_{v_n} \{\eta_n v_n - C|v_n|^{2m_n}\} \end{aligned}$$

But given $w \in \mathbb{R}$, the Legendre Transform of $\frac{B}{2k}|w|^{2k}$ is $\widetilde{B}|\eta|^{\frac{2k}{2k-1}}$, where

$$\widetilde{B} = B^{\frac{-1}{2k-1}} \left(\frac{2k-1}{2k} \right).$$

Thus,

$$L(\boldsymbol{\eta}) \geq \widetilde{C} \left(|\eta_1|^{\frac{2m_1}{2m_1-1}} + \ldots + |\eta_n|^{\frac{2m_n}{2m_n-1}} \right) - C,$$

where $\widetilde{C} = \min \left\{ B_1^{\frac{-1}{2m_1-1}} \left(\frac{2m_1-1}{2m_1} \right), \ldots, B_n^{\frac{-1}{2m_n-1}} \left(\frac{2m_n-1}{2m_n} \right) \right\}$, and $B_j = C2m_j$.

□

This finishes the proof that the dominant term, $e^{h(\boldsymbol{v_0})}$, grows at an exponential rate in $\boldsymbol{\eta}$, independently of the coefficients of g. More precisely, we have shown that

$$e^{h(\boldsymbol{v_0})} \geq \exp \left[\widetilde{C} \left(|\eta_1|^{\frac{2m_1}{2m_1-1}} + \ldots + |\eta_n|^{\frac{2m_n}{2m_n-1}} \right) - C \right]. \tag{3.7}$$

3.2 A polynomial bound for the remaining terms

It suffices now to show that $J = \int_{\mathbb{R}^n} e^{h(\boldsymbol{v}) - h(\boldsymbol{v_0})} \, d\boldsymbol{v}$ is not too small to obtain the desired decay for I^{-1}. Recall that

$$J = \int_{\mathbb{R}^n} e^{\boldsymbol{\eta} \cdot (\boldsymbol{v} - \boldsymbol{v_0}) + g(\boldsymbol{v_0}) - g(\boldsymbol{v})} \, d\boldsymbol{v}.$$

In order to estimate this integral, we will approximate it by an area by using Claim 1.5. Recall that if $f : \mathbb{R}^n \to \mathbb{R}$ is a convex function such that $f(\mathbf{0}) = 0$ and $\nabla f(\mathbf{0}) = 0$ then

$$\int_{\mathbb{R}^n} e^{-f(\boldsymbol{w})}\, d\boldsymbol{w} \approx |\{\boldsymbol{w} \,:\, f(\boldsymbol{w}) \leq 1\}|.$$

This was discussed in the introduction (page 16).

We must start by rewriting J as stated in the above result. Since $\boldsymbol{\eta} = \nabla g(\boldsymbol{v_0})$, and making the change of variables $\boldsymbol{w} = \boldsymbol{v} - \boldsymbol{v_0}$, we can write

$$J = \int_{\mathbb{R}^n} e^{-f(\boldsymbol{w})}\, d\boldsymbol{w},$$

where

$$f(\boldsymbol{w}) = -\nabla g(\boldsymbol{v_0}) \cdot \boldsymbol{w} - g(\boldsymbol{v_0}) + g(\boldsymbol{v_0} + \boldsymbol{w}). \tag{3.8}$$

Clearly $f(\mathbf{0}) = 0$ and $\nabla f(\mathbf{0}) = 0$. Also, since g is convex, so is f. Thus, it follows by Claim 1.5 that

$$J \approx |\{\boldsymbol{w} : f(\boldsymbol{w}) \leq 1\}|. \tag{3.9}$$

That is,

$$I = \int_{\mathbb{R}^n} e^{\boldsymbol{\eta}\cdot\boldsymbol{v} - g(\boldsymbol{v})}\, d\boldsymbol{v} \approx e^{h(\boldsymbol{v_0})}|\{\boldsymbol{w} : f(\boldsymbol{w}) \leq 1\}|. \tag{3.10}$$

Our goal is to show that as $|\boldsymbol{\eta}|$ grows, this area decreases slower than the rate of growth we obtained for $e^{h(\boldsymbol{v_0})}$. We begin by obtaining an upper bound for f that is independent of the choice of g, but rather that depends only on its “combined degree”, on the dimension of the space and on the constant A. To do so we will write f as an integral in terms of the quadratic form associated to the Hessian of g. In Claim 3.5

we obtain an upper bound for this quadratic form in terms of a polynomial that is independent of the coefficients of g. In Claim 3.6 we use this estimate to obtain the desired bound for f.

Claim 3.5. *The quadratic form associated to the Hessian of g is bounded in terms of a polynomial that depends on the "combined degree" of g, but which is otherwise independent of the choice of g. More precisely,*

$$\sum_{i,j=1}^{n} g_{ij}(\boldsymbol{v})w_i w_j \lesssim (1 + r(\boldsymbol{v}))\,|\boldsymbol{w}|^2,$$

where $r(\boldsymbol{v}) = v_1^{2m_1} + \ldots + v_n^{2m_n}$ *and* $g_{ij}(\boldsymbol{v}) = \frac{\partial^2 g}{\partial v_i \partial v_j}(\boldsymbol{v})$.

Proof. Let L be the Hessian matrix of g. That is,

$$L = \begin{pmatrix} g_{11} & g_{12} & \cdots & g_{1n} \\ g_{21} & g_{22} & \cdots & g_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ g_{n1} & g_{n2} & \cdots & g_{nn} \end{pmatrix}$$

so that $\boldsymbol{w}^{\mathbf{T}}\, L\, \boldsymbol{w} = \sum_{i,j=1}^{n} g_{ij} w_i w_j$. Since L is symmetric, it has n linearly independent eigenvectors. Let $\boldsymbol{u_i}$, $i = 1, \ldots, n$ be the eigenvectors of L, and λ_i, $i = 1, \ldots, n$ be the corresponding eigenvalues. Since g is convex, the matrix L is positive definite, so its eigenvalues are positive.

Let $P = Tr(L)I$, where $Tr(L) = \lambda_1 + \ldots + \lambda_n$ is the trace of the matrix L and I is the identity matrix.

Let $Q = P - L$. We claim that Q is positive definite, and hence that $L \leq P$ as quadratic forms. In fact, notice that for $i = 1, \dots, n$

$$Q\boldsymbol{u_i} = P\boldsymbol{u_i} - L\boldsymbol{u_i} = Tr(L)\boldsymbol{u_i} - \lambda_i \boldsymbol{u_i} = (Tr(L) - \lambda_i)\,\boldsymbol{u_i} = \sum_{\substack{1 \leq j \leq n \\ j \neq i}} \lambda_j \boldsymbol{u_i}.$$

Thus, for $i = 1, \dots, n$, $\boldsymbol{u_i}$ is an eigenvector of Q, with eigenvalue

$$\mu_i = \sum_{\substack{1 \leq j \leq n \\ j \neq i}} \lambda_j.$$

Notice that for $i = 1, \dots, n$, μ_i is positive. Also, since P is a diagonal matrix and L is symmetric, Q is symmetric. Thus, since Q is a symmetric matrix whose eigenvalues are all positive, Q is positive definite.

Hence, since

$$P = Tr(L)I = \begin{pmatrix} g_{11} + \ldots + g_{nn} & 0 & \cdots & 0 \\ 0 & g_{11} + \ldots + g_{nn} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & g_{11} + \ldots + g_{nn} \end{pmatrix}$$

and $\boldsymbol{w}^{\mathbf{T}}\, L\, \boldsymbol{w} \;\leq\; \boldsymbol{w}^{\mathbf{T}}\, P\, \boldsymbol{w}$, it follows that

$$\begin{aligned} 0 &\leq \sum_{i,j=1}^{n} g_{ij}(\boldsymbol{v}) w_i w_j \\ &\leq (g_{11}(\boldsymbol{v}) + \ldots + g_{nn}(\boldsymbol{v}))|w|^2 \\ &\leq (|g_{11}(\boldsymbol{v})| + \ldots + |g_{nn}(\boldsymbol{v})|)\, |w|^2. \end{aligned}$$

Notice that each $g_{jj}(\boldsymbol{v})$ is a polynomial of the form $\sum_\beta \widetilde{c_\beta} \boldsymbol{v}^\beta$ where the indexes β satisfy

$$\frac{\beta_1}{2m_1} + \cdots + \frac{\beta_n}{2m_n} \leq 1 - \frac{1}{m_j} < 1.$$

In particular, this implies that

$$r(\boldsymbol{v})^{\frac{\beta_1}{2m_1} + \cdots + \frac{\beta_n}{2m_n}} \leq 1 + r(\boldsymbol{v}).$$

Moreover, recall that, as was shown on equation (3.4) on page 33,

$$v_1^{\beta_1} \cdots v_n^{\beta_n} \leq r(\boldsymbol{v})^{\frac{\beta_1}{2m_1} + \cdots + \frac{\beta_n}{2m_n}}.$$

It follows that

$$\begin{aligned} |g_{jj}(\boldsymbol{v})| &\leq \sum_\beta |\widetilde{c_\beta}| \left| v_1^{\beta_1} \cdots v_n^{\beta_n} \right| \\ &\leq \sum_\beta |\widetilde{c_\beta}| \, r(\boldsymbol{v})^{\frac{\beta_1}{2m_1} + \cdots + \frac{\beta_n}{2m_n}} \\ &\leq \sum_\beta |\widetilde{c_\beta}| \left(1 + r(\boldsymbol{v})\right). \end{aligned}$$

Furthermore, the coefficients $\widetilde{c_\beta}$ are constant multiples of the c_α, where the factors are bounded by a factorial of the degree of the polynomial. Since $\sum_{\alpha \in \Gamma} |c_\alpha| \leq C$, the coefficients $\widetilde{c_\beta}$ are bounded by a constant that does not depend on the choice of g, but rather on the "combined degree" of g. Then, and by the previous inequality, we have that

$$|g_{11}(\boldsymbol{v})| + \ldots + |g_{nn}(\boldsymbol{v})| \leq C(1 + r(\boldsymbol{v})).$$

for a universal constant C.

That is,

$$\sum_{i,j=1}^{n} g_{ij}(\boldsymbol{v})w_i w_j \lesssim (1+r(\boldsymbol{v}))|\boldsymbol{w}|^2.$$

□

Using this result it is now possible to obtain an upper bound for f which is independent of the choice of g. We do so in the following claim.

Claim 3.6. *If*

$$r(\boldsymbol{v}) = v_1^{2m_1} + \ldots + v_n^{2m_n}$$

and

$$f(\boldsymbol{w}) = -\nabla g(\boldsymbol{v_0}) \cdot \boldsymbol{w} - g(\boldsymbol{v_0}) + g(\boldsymbol{v_0} + \boldsymbol{w})$$

then,

$$f(\boldsymbol{w}) \lesssim |\boldsymbol{w}|^2(1 + r(\boldsymbol{v_0}) + r(\boldsymbol{w})),$$

where the constant depends only on the constant A, the "combined degree" of g and the dimension of the space.

Proof. We begin by rewriting f as an integral in terms of the quadratic form associated to the Hessian, so that we can apply our previous estimate.

We can write

$$g(\boldsymbol{v_0}+\boldsymbol{w})-g(\boldsymbol{v_0})=\nabla g(\boldsymbol{v_0})\cdot\boldsymbol{w}+\int_0^1\sum_{i,j=1}^n g_{ij}(\boldsymbol{v_0}+t\boldsymbol{w})w_iw_j(1-t)\,dt.$$

This was shown in detail on page 17 (equation (1.3)). It follows that

$$f(\boldsymbol{w})=\int_0^1\sum_{i,j=1}^n g_{ij}(\boldsymbol{v_0}+t\boldsymbol{w})w_iw_j(1-t)dt. \tag{3.11}$$

In particular, by convexity of g we have that $f\geq 0$.

We can now use the bound for the Hessian obtained in Claim 3.5. It follows that

$$f(\boldsymbol{w})\lesssim\int_0^1(1+r(\boldsymbol{v_0}+t\boldsymbol{w}))|\boldsymbol{w}|^2(1-t)\,dt. \tag{3.12}$$

We would like to get rid of the dependence on t of the term $r(\boldsymbol{v_0}+t\boldsymbol{w})$. To do so, we use a triangle inequality. We claim that given vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, the function r satisfies

$$r(\boldsymbol{u}+\boldsymbol{v})\leq C[r(\boldsymbol{u})+r(\boldsymbol{v})],$$

where the constant C depends on the exponents $\{m_1,\ldots,m_n\}$. In fact, by convexity it follows that for $j=1,\ldots,n$

$$(u_j+v_j)^{2m_j}=2^{2m_j}\left(\frac{u_j+v_j}{2}\right)^{2m_j}\leq 2^{2m_j}\left(\frac{u_j^{2m_j}}{2}+\frac{v_j^{2m_j}}{2}\right)=2^{2m_j-1}\left(u_j^{2m_j}+v_j^{2m_j}\right).$$

Thus,

$$\begin{aligned}r(\boldsymbol{u}+\boldsymbol{v})&=(u_1+v_1)^{2m_1}+\ldots+(u_n+v_n)^{2m_n}\\&\leq\max\{2^{2m_1-1},\ldots,2^{2m_n-1}\}\left[r(\boldsymbol{u})+r(\boldsymbol{v})\right].\end{aligned}$$

Applying this inequality to $r(\boldsymbol{v_0}+t\boldsymbol{w})$, and since $0 \le t \le 1$, we have that

$$r(\boldsymbol{v_0}+t\boldsymbol{w}) \lesssim r(\boldsymbol{v_0}) + r(t\boldsymbol{w}) \le r(\boldsymbol{v_0}) + r(\boldsymbol{w}).$$

Hence, it follows from equation (3.12) that

$$f(\boldsymbol{w}) \lesssim |\boldsymbol{w}|^2(1 + r(\boldsymbol{v_0}) + r(\boldsymbol{w})).$$

□

Recall that we are studying the decay of I^{-1}, where $I \approx e^{h(\boldsymbol{v_0})}|\{\boldsymbol{w} : f(\boldsymbol{w}) \le 1\}|$. Thus far we have shown that the dominant term, $e^{h(\boldsymbol{v_0})}$, grows at an exponential rate in $\boldsymbol{\eta}$ (where by definition $\boldsymbol{\eta} = \nabla g(\boldsymbol{v_0})$). Our goal is to show that $|\{\boldsymbol{w} : f(\boldsymbol{w}) \le 1\}|$ is not too small. More precisely, that it does not decay exponentially, so that it does not annul the growth of the dominant term. In the next three claims we show that there is a polynomial P and a constant C depending only on the degrees $\{m_1, \ldots, m_n\}$ so that

$$|\{\boldsymbol{w} : f(\boldsymbol{w}) \le 1\}|^{-1} \le C(1 + P(|\boldsymbol{\eta}|))^{\frac{n}{2}}.$$

In Claim 3.7 we show that $|\{\boldsymbol{w} : f(\boldsymbol{w}) \le 1\}|^{-1}$ is bounded by a polynomial in terms of $r(\boldsymbol{v_0})$, and in Claim 3.8 we compare the sizes of $|\boldsymbol{v_0}|$ and $|\boldsymbol{\eta}|$. In Claim 3.9 we conclude that $r(\boldsymbol{v_0})$ grows at most at a polynomial rate in $|\boldsymbol{\eta}|$.

Claim 3.7. *If* $f(\boldsymbol{w}) \lesssim |\boldsymbol{w}|^2(1 + r(\boldsymbol{v_0}) + r(\boldsymbol{w}))$, *then*

$$|\{\boldsymbol{w} : f(\boldsymbol{w}) \le 1\}| \gtrsim (1 + r(\boldsymbol{v_0}))^{-\frac{n}{2}}.$$

Proof. Let C be such that $f(\boldsymbol{w}) \leq C|\boldsymbol{w}|^2(1+r(\boldsymbol{v_0})+r(\boldsymbol{w}))$ and let

$$T(\boldsymbol{w}) = C|\boldsymbol{w}|^2(1+r(\boldsymbol{v_0})+r(\boldsymbol{w})).$$

Then,

$$|\{\boldsymbol{w} : f(\boldsymbol{w}) \leq 1\}| \geq |\{\boldsymbol{w} : T(\boldsymbol{w}) \leq 1\}|.$$

Let $\Sigma = \{\boldsymbol{u} : T(\boldsymbol{u}) = 1\}$ and let $m = \min\{ |\boldsymbol{u}| : \boldsymbol{u} \in \Sigma\}$. Choose $\boldsymbol{w_T}$ such that $T(\boldsymbol{w_T}) = 1$ and $|\boldsymbol{w_T}| = m$. Then the set $|\{\boldsymbol{w} : T(\boldsymbol{w}) \leq 1\}|$ is bounded from below by the volume of the ball of radius $|\boldsymbol{w_T}|$. That is,

$$|\{\boldsymbol{w} : T(\boldsymbol{w}) \leq 1\}| \geq C_n|\boldsymbol{w_T}|^n,$$

where $C_n = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)}$.

Let $a = 1 + r(\boldsymbol{v_0})$. Our goal is to show that $|\boldsymbol{w_T}|^n \gtrsim a^{-\frac{n}{2}}$. If $|\boldsymbol{w_T}|^2 \geq \frac{1}{2Ca}$, then $|\boldsymbol{w_T}|^n \gtrsim a^{-\frac{n}{2}}$, as desired. Otherwise, we have that $|\boldsymbol{w_T}|^2 < \frac{1}{2Ca}$. Since $1 = T(\boldsymbol{w_T}) = C|\boldsymbol{w_T}|^2(1 + r(\boldsymbol{v_0}) + r(\boldsymbol{w_T}))$ it follows that

$$a|\boldsymbol{w_T}|^2 + |\boldsymbol{w_T}|^2 r(\boldsymbol{w_T}) = \frac{1}{C}.$$

But since $|\boldsymbol{w_T}|^2 < \frac{1}{2Ca}$, it follows that

$$\frac{1}{2C} < |\boldsymbol{w_T}|^2 r(\boldsymbol{w_T}). \tag{3.13}$$

Also, since $a \geq 1$, we have that $|\boldsymbol{w_T}|^2 < \frac{1}{2C}$. Thus, $w_{Tj}^2 < \frac{1}{2C}$ for every $1 \leq j \leq n$. Hence,

$$r(\boldsymbol{w_T}) < \left(\frac{1}{2C}\right)^{m_1} + \cdots + \left(\frac{1}{2C}\right)^{m_n}.$$

Using this in equation (3.13) we have that

$$\frac{1}{2C} < |\boldsymbol{w_T}|^2 \left(\left(\frac{1}{2C}\right)^{m_1} + \cdots + \left(\frac{1}{2C}\right)^{m_n}\right).$$

Thus, and since $a \geq 1$, it follows that

$$|\boldsymbol{w_T}|^2 > \Lambda \geq \frac{\Lambda}{a},$$

where $\Lambda = \frac{1}{2C}\left(\left(\frac{1}{2C}\right)^{m_1} + \cdots + \left(\frac{1}{2C}\right)^{m_n}\right)^{-1}$ is a strictly positive constant.

Therefore,

$$|\{\boldsymbol{w} : f(\boldsymbol{w}) \leq 1\}| \gtrsim |\boldsymbol{w_T}|^n \gtrsim a^{-\frac{n}{2}}.$$

This finishes the proof of Claim 3.7.

□

It follows from the estimate for the dominant term as well as from this bound for $|\{\boldsymbol{w} \,:\, f(\boldsymbol{w}) \leq 1\}|$ that

$$\theta(\boldsymbol{\eta}) = I^{-1} \lesssim \exp\left[-C\left(|\eta_1|^{\frac{2m_1}{2m_1-1}} + \ldots + |\eta_n|^{\frac{2m_n}{2m_n-1}}\right)\right](1 + r(\boldsymbol{v_0}))^{\frac{n}{2}},$$

where the constants only depend on A, the "combined degree" of g and the dimension of the space. We must show that $r(\boldsymbol{v_0})$ is not too large as a function of $\boldsymbol{\eta}$ in order to obtain the desired decay.

Claim 3.8. *There exist positive constants* β_1 *and* β_2 *such that*

$$|\boldsymbol{v_0}| \leq \beta_1 |\boldsymbol{\eta}| + \beta_2.$$

The constants depend only on $m_1, \ldots, m_n$ *and the dimension of the space.*

Proof. Recall that by definition $\nabla g(\boldsymbol{v_0}) = \boldsymbol{\eta}$, and that by hypothesis

$$\{\boldsymbol{v} \,:\, g(\boldsymbol{v}) \leq 1\} \subseteq \{\boldsymbol{v} \,:\, |\boldsymbol{v}| \leq 1\}.$$

The statement is trivial if $|\boldsymbol{v_0}| \leq 1$, so we will assume that $|\boldsymbol{v_0}| > 1$.

Let $G(t) = g\left(\frac{t\boldsymbol{v_0}}{|\boldsymbol{v_0}|}\right)$. Then

$$G'(t) = \nabla g\left(\frac{t\boldsymbol{v_0}}{|\boldsymbol{v_0}|}\right) \cdot \left(\frac{\boldsymbol{v_0}}{|\boldsymbol{v_0}|}\right).$$

Thus, since $\nabla g(\mathbf{0}) = 0$ by hypothesis, $G'(0) = \nabla g(\mathbf{0}) \cdot \left(\frac{\boldsymbol{v_0}}{|\boldsymbol{v_0}|}\right) = 0$. Also, notice that since g is convex, so is G. Thus, $G'(t) > 0$ if $t > 0$.

By Cauchy-Schwarz,

$$|G'(t)| \leq \left|\nabla g\left(\frac{t\boldsymbol{v_0}}{|\boldsymbol{v_0}|}\right)\right|.$$

Evaluating at $t = |\boldsymbol{v_0}|$ we have that

$$|G'(|\boldsymbol{v_0}|)| \leq |\nabla g(\boldsymbol{v_0})| = |\boldsymbol{\eta}|.$$

But since $|\boldsymbol{v_0}| > 0$, it follows that

$$G'(|\boldsymbol{v_0}|) = |G'(|\boldsymbol{v_0}|)| \leq |\boldsymbol{\eta}|. \tag{3.14}$$

It suffices now to obtain a polynomial lower bound for $G'(|\boldsymbol{v_0}|)$ in terms of $|\boldsymbol{v_0}|$.

Nagel, Bruna and Wainger proved in [4] (Lemma 2.2) that given a convex polynomial of one variable of the form

$$P(t) = \sum_{j=2}^{m} a_j t^j$$

there exists a constant C_m such that for $t \geq 0$,

$$P'(t) \geq C_m \sum_{j=2}^{m} |a_j| t^{j-1}.$$

In particular, if $t \geq 1$,

$$P'(t) \geq C_m t \sum_{j=2}^{m} |a_j|. \tag{3.15}$$

Notice that $G(t)$ is a convex polynomial of one variable such that $G(0) = G'(0) = 0$, so we can use the aforementioned result. Write

$$G(t) = \sum_{j=2}^{m} a_j t^j.$$

Since we are considering $|\boldsymbol{v_0}| > 1$, it follows from equations (3.14) and (3.15) that

$$|\boldsymbol{v_0}| \sum_{j=2}^{m} |a_j| \lesssim G'(|\boldsymbol{v_0}|) \leq |\boldsymbol{\eta}|. \tag{3.16}$$

It suffices now to obtain a lower bound for $\sum_{j=2}^{m} |a_j|$, which must be independent of the choice of g. To do so, we use the fact that

$$\{\boldsymbol{v} \,:\, g(\boldsymbol{v}) \leq 1\} \subseteq \{\boldsymbol{v} \,:\, |\boldsymbol{v}| \leq 1\}.$$

In particular, if $|\boldsymbol{v}| = 1$, it must follow that $g(\boldsymbol{v}) \geq 1$. Thus, evaluating at $t = 1$, it follows that

$$G(1) = g\left(\frac{\boldsymbol{v_0}}{|\boldsymbol{v_0}|}\right) \geq 1.$$

But $G(1) = \sum_{j=2}^{m} a_j \leq \sum_{j=2}^{m} |a_j|$. This implies that

$$1 \leq \sum_{j=2}^{m} |a_j|.$$

Using this last bound on equation (3.16) yields $|\boldsymbol{v_0}| \lesssim |\boldsymbol{\eta}|$. This finishes the proof of Claim 3.8.

□

Claim 3.9. $(1 + r(\boldsymbol{v_0})^p)^{\frac{n}{2}}$ *is at most of polynomial growth in* $|\boldsymbol{\eta}|$.

Proof. The proof is trivial. In fact, since $|\boldsymbol{v_0}| \leq \beta_1|\boldsymbol{\eta}| + \beta_2$, it is clear that $r(\boldsymbol{v_0}) = v_{01}^{2m_1} + \ldots + v_{0n}^{2m_n} \leq |\boldsymbol{v_0}|^{2m_1} + \ldots + |\boldsymbol{v_0}|^{2m_n}$ is bounded from above by a polynomial in $|\boldsymbol{\eta}|$. We will call this polynomial $P(|\boldsymbol{\eta}|)$.

□

Recall that we had shown that

$$I^{-1} \lesssim \exp\left[-C\left(|\eta_1|^{\frac{2m_1}{2m_1-1}} + \ldots + |\eta_n|^{\frac{2m_n}{2m_n-1}}\right)\right](1 + r(\boldsymbol{v_0}))^{\frac{n}{2}}.$$

It follows from this last claim that there exists a polynomial $P(|\boldsymbol{\eta}|)$, which does not depend on the choice of g, such that

$$\theta(\boldsymbol{\eta}) = I^{-1} \lesssim \exp\left[-C\left(|\eta_1|^{\frac{2m_1}{2m_1-1}} + \ldots + |\eta_n|^{\frac{2m_n}{2m_n-1}}\right)\right](1+P(|\boldsymbol{\eta}|))^{\frac{n}{2}}.$$

This finishes the proof that $\theta(\boldsymbol{\eta})$ decays at an exponential rate. Moreover, this decay is independent of the coefficients of the polynomial g that defines it.

We must now show that the same is true of all the derivatives of $\theta(\boldsymbol{\eta})$.

3.3 Decay of the derivatives

Claim 3.10. *The derivatives of $\theta(\boldsymbol{\eta})$ consist of sums of terms of the form*

$$\frac{C\left[\int_{\mathbb{R}^n} e^{\boldsymbol{\eta}\boldsymbol{v}-g(\boldsymbol{v})} v_1^{i_{1,1}} \cdots v_n^{i_{n,1}}\, d\boldsymbol{v}\right]^{a_1} \cdots \left[\int_{\mathbb{R}^n} e^{\boldsymbol{\eta}\boldsymbol{v}-g(\boldsymbol{v})} v_1^{i_{1,r}} \cdots v_n^{i_{n,r}}\, d\boldsymbol{v}\right]^{a_r}}{\left[\int_{\mathbb{R}^n} e^{\boldsymbol{\eta}\boldsymbol{v}-g(\boldsymbol{v})}\, d\boldsymbol{v}\right]^{d}}, \tag{3.17}$$

where $i_{1,1}, \ldots, i_{n,r}, a_1, \ldots, a_r, d \in \mathbb{N}$ and $a_1 + \ldots + a_r + 1 = d$.

Remark 3.11. *The fact that $a_1 + \ldots + a_r - d < 0$ is crucial. As before (equation (3.2)), we can factor out a term $e^{h(\boldsymbol{v_0})}$ for each of these integrals. That is, we will factor out $\left(e^{h(\boldsymbol{v_0})}\right)^{a_1+\ldots+a_r-d} = e^{-h(\boldsymbol{v_0})}$. This term will provide the desired decay.*

Proof. We will prove it by induction. We have that for any $j = 1, \ldots, n$,

$$\frac{\partial \theta}{\partial \eta_j}(\boldsymbol{\eta}) = \frac{-\int_{\mathbb{R}^n} e^{\boldsymbol{\eta}\boldsymbol{v}-g(\boldsymbol{v})} v_j\, d\boldsymbol{v}}{\left[\int_{\mathbb{R}^n} e^{\boldsymbol{\eta}\boldsymbol{v}-g(\boldsymbol{v})}\, d\boldsymbol{v}\right]^2}$$

is of this form.

Suppose $\frac{\partial^k \theta}{\partial \eta_1^{c_1} \cdots \partial \eta_n^{c_n}}(\boldsymbol{\eta})$ is of this form. Then, for any $j = 1, \ldots, n$, $\frac{\partial^{k+1} \theta}{\partial \eta_j \partial \eta_1^{c_1} \cdots \partial \eta_n^{c_n}}(\boldsymbol{\eta})$ consists of sums of terms of the form

$$\frac{\partial}{\partial \eta_j}\left(\frac{C\left[\int_{\mathbb{R}^n} e^{\boldsymbol{\eta v}-g(\boldsymbol{v})} v_1^{i_{1,1}} \cdots v_n^{i_{n,1}}\, d\boldsymbol{v}\right]^{a_1} \cdots \left[\int_{\mathbb{R}^n} e^{\boldsymbol{\eta v}-g(\boldsymbol{v})} v_1^{i_{1,r}} \cdots v_n^{i_{n,r}}\, d\boldsymbol{v}\right]^{a_r}}{\left[\int_{\mathbb{R}^n} e^{\boldsymbol{\eta v}-g(\boldsymbol{v})}\, d\boldsymbol{v}\right]^d}\right).$$

Let $f_s(\boldsymbol{\eta}) = \left[\int_{\mathbb{R}^n} e^{\boldsymbol{\eta v}-g(\boldsymbol{v})} v_1^{i_{1,s}} \cdots v_n^{i_{n,s}}\, d\boldsymbol{v}\right]^{a_s}$ and $\gamma(\boldsymbol{\eta}) = \left[\int_{\mathbb{R}^n} e^{\boldsymbol{\eta v}-g(\boldsymbol{v})}\, d\boldsymbol{v}\right]^d$. By the quotient rule we have that

$$\frac{\partial}{\partial \eta_j}\left(\frac{C\left[\int_{\mathbb{R}^n} e^{\boldsymbol{\eta v}-g(\boldsymbol{v})} v_1^{i_{1,1}} \cdots v_n^{i_{n,1}}\, d\boldsymbol{v}\right]^{a_1} \cdots \left[\int_{\mathbb{R}^n} e^{\boldsymbol{\eta v}-g(\boldsymbol{v})} v_1^{i_{1,r}} \cdots v_n^{i_{n,r}}\, d\boldsymbol{v}\right]^{a_r}}{\left[\int_{\mathbb{R}^n} e^{\boldsymbol{\eta v}-g(\boldsymbol{v})}\, d\boldsymbol{v}\right]^d}\right)$$

consists of sums of terms of the form

$$\frac{C f_1 \cdots f_{s-1} \frac{\partial f_s}{\partial \eta_j} f_{s+1} \cdots f_r}{\gamma} \tag{3.18}$$

and

$$\frac{C(f_1 \cdots f_r)(\frac{\partial \gamma}{\partial \eta_j})}{\gamma^2}. \tag{3.19}$$

But

$$\frac{\partial f_s}{\partial \eta_j}(\boldsymbol{\eta}) = a_s \left[\int_{\mathbb{R}^n} e^{\boldsymbol{\eta v}-g(\boldsymbol{v})} v_1^{i_{1,s}} \cdots v_n^{i_{n,s}}\, d\boldsymbol{v}\right]^{a_s - 1} \left[\int_{\mathbb{R}^n} e^{\boldsymbol{\eta v}-g(\boldsymbol{v})} v_1^{i_{1,s}} \cdots v_n^{i_{n,s}} v_j\, d\boldsymbol{v}\right],$$

and

$$\frac{\partial \gamma}{\partial \eta_j}(\boldsymbol{\eta}) = d\left[\int_{\mathbb{R}^n} e^{\boldsymbol{\eta v}-g(\boldsymbol{v})}\, d\boldsymbol{v}\right]^{d-1} \left[\int_{\mathbb{R}^n} e^{\boldsymbol{\eta v}-g(\boldsymbol{v})} v_j\, d\boldsymbol{v}\right].$$

Thus, a generic term of the form given in equation (3.18) is given by

$$\frac{\left[\int\limits_{\mathbb{R}^n} e^{\boldsymbol{\eta}\boldsymbol{v}-g(\boldsymbol{v})} v_1^{i_{1,1}}\cdots v_n^{i_{n,1}}\,d\boldsymbol{v}\right]^{a_1}\cdots\left[\int\limits_{\mathbb{R}^n} e^{\boldsymbol{\eta}\boldsymbol{v}-g(\boldsymbol{v})} v_1^{i_{1,s}}\cdots v_n^{i_{n,s}}\,d\boldsymbol{v}\right]^{a_s-1}\left[\int\limits_{\mathbb{R}^n} e^{\boldsymbol{\eta}\boldsymbol{v}-g(\boldsymbol{v})} v_1^{i_{1,1}}\cdots v_n^{i_{n,1}} v_j\,d\boldsymbol{v}\right]}{\left[\int\limits_{\mathbb{R}^n} e^{\boldsymbol{\eta}\boldsymbol{v}-g(\boldsymbol{v})}\,dv\right]^{d}}$$

$$\times\left[\int\limits_{\mathbb{R}^n} e^{\boldsymbol{\eta}\boldsymbol{v}-g(\boldsymbol{v})} v_1^{i_{1,1}}\cdots v_n^{i_{n,1}}\,d\boldsymbol{v}\right]^{a_{s+1}}\cdots\left[\int\limits_{\mathbb{R}^n} e^{\boldsymbol{\eta}\boldsymbol{v}-g(\boldsymbol{v})} v_1^{i_{1,1}}\cdots v_n^{i_{n,1}}\,d\boldsymbol{v}\right]^{a_r}.$$

The sum of the exponents of the numerator is $a_1 + a_2 + \ldots + a_{s-1} + (a_s - 1) + 1 + a_{s+1} + \ldots + a_r = a_1 + \ldots + a_r = d - 1$. Thus this term has the desired form.

In the same way, a generic term of the form given in equation (3.19) is given by

$$\frac{\left(\left[\int\limits_{\mathbb{R}^n} e^{\boldsymbol{\eta}\boldsymbol{v}-g(\boldsymbol{v})} v_1^{i_{1,1}}\cdots v_n^{i_{n,1}}\,d\boldsymbol{v}\right]^{a_1}\cdots\left[\int\limits_{\mathbb{R}^n} e^{\boldsymbol{\eta}\boldsymbol{v}-g(\boldsymbol{v})} v_1^{i_{1,r}}\cdots v_n^{i_{n,r}}\,d\boldsymbol{v}\right]^{a_r}\right)}{\left(\left[\int\limits_{\mathbb{R}^n} e^{\boldsymbol{\eta}\boldsymbol{v}-g(\boldsymbol{v})}\,d\boldsymbol{v}\right]^{d}\right)^2}$$

$$\times\left(\left[\int\limits_{\mathbb{R}^n} e^{\boldsymbol{\eta}\boldsymbol{v}-g(\boldsymbol{v})}\,d\boldsymbol{v}\right]^{d-1}\left[\int\limits_{\mathbb{R}^n} e^{\boldsymbol{\eta}\boldsymbol{v}-g(\boldsymbol{v})} v_j\,d\boldsymbol{v}\right]\right).$$

The sum of the exponents of the numerator is $a_1 + a_2 + \ldots + a_r + (d-1) + 1 = 2d - 1$. Since the exponent of the denominator is $2d$, this term also has the desired form. This completes the proof of Claim 3.10. □

By the previous claim, in order to understand the decay of the derivatives of $\theta(\boldsymbol{\eta})$ we need to study integrals of the form $\int_{\mathbb{R}^n} e^{\boldsymbol{\eta}\boldsymbol{v}-g(\boldsymbol{v})} v_1^{i_1}\cdots v_n^{i_n}\,d\boldsymbol{v}$.

Claim 3.12.

$$\int_{\mathbb{R}^n} e^{\boldsymbol{\eta v}-g(\boldsymbol{v})} v_1^{i_1} \cdots v_n^{i_n}\, d\boldsymbol{v} \lesssim e^{h(\boldsymbol{v_0})} H_f[|\boldsymbol{v_0}|],$$

where

$$H_f\left[|\boldsymbol{v_0}|\right] = \sum_{s_1=0}^{i_1} \cdots \sum_{s_n=0}^{i_n} \binom{i_1}{s_1} \cdots \binom{i_n}{s_n} |\boldsymbol{v_0}|^{i_1+\ldots+i_n-s}$$
$$\times \Big(|\{\boldsymbol{w} \,:\, f(\boldsymbol{w}) \le 1\}| \left(1 + |\boldsymbol{v_0}|^{sB}\right) + \Theta \Big);$$

Θ is a constant that depends only on the "combined degree" of g and the dimension of the space; $s = s_1 + \cdots + s_n$; and $B = 4\max\{m_1, \ldots, m_n\}$.

Proof. As before (equation (3.2)), we can write

$$\widetilde{I} = \int_{\mathbb{R}^n} e^{\boldsymbol{\eta v}-g(\boldsymbol{v})} v_1^{i_1} \cdots v_n^{i_n}\, d\boldsymbol{v} = e^{h(\boldsymbol{v_0})} \int_{\mathbb{R}^n} e^{h(\boldsymbol{v})-h(\boldsymbol{v_0})} v_1^{i_1} \cdots v_n^{i_n}\, d\boldsymbol{v}, \tag{3.20}$$

where $h(\boldsymbol{v}) = \boldsymbol{\eta} \cdot \boldsymbol{v} - g(\boldsymbol{v})$ and $\boldsymbol{v_0}$ is the point where $h(\boldsymbol{v})$ attains its maximum; and

$$\widetilde{J} = \int_{\mathbb{R}^n} e^{h(\boldsymbol{v})-h(\boldsymbol{v_0})} v_1^{i_1} \cdots v_n^{i_n}\, d\boldsymbol{v} = \int_{\mathbb{R}^n} (w_1 + v_{01})^{i_1} \cdots (w_n + v_{0n})^{i_n} e^{-f(\boldsymbol{w})}\, d\boldsymbol{w},$$

where $f(\boldsymbol{w})$ is as in equation (3.8). That is,

$$f(\boldsymbol{w}) = g(\boldsymbol{v_0} + \boldsymbol{w}) - g(\boldsymbol{v_0}) - \nabla g(\boldsymbol{v_0}) \cdot \boldsymbol{w}.$$

But $(w_j + v_{0j})^l = \sum_{s=0}^{l} \binom{l}{s} v_{0j}^{l-s} w_j^s$. Thus, we can write

$$\widetilde{J} = \sum_{s_1=0}^{i_1} \cdots \sum_{s_n=0}^{i_n} \binom{i_1}{s_1} \cdots \binom{i_n}{s_n} v_{01}^{i_1-s_1} \cdots v_{0n}^{i_n-s_n} \int_{\mathbb{R}^n} w_1^{s_1} \cdots w_n^{s_n} e^{-f(\boldsymbol{w})}\, d\boldsymbol{w}.$$

It follows that

$$|\widetilde{J}| \leq \sum_{s_1=0}^{i_1} \cdots \sum_{s_n=0}^{i_n} \binom{i_1}{s_1} \cdots \binom{i_n}{s_n} |\boldsymbol{v_0}|^{i_1+\ldots+i_n-(s_1+\ldots+s_n)} \int_{\mathbb{R}^n} |\boldsymbol{w}|^{s_1+\ldots+s_n} e^{-f(\boldsymbol{w})}\, d\boldsymbol{w}.$$

Let $s = s_1 + \cdots + s_n$ and $J_s = \int_{\mathbb{R}^n} |\boldsymbol{w}|^s e^{-f(\boldsymbol{w})}\, d\boldsymbol{w}$. Write

$$J_s = \int_{\{\boldsymbol{w}\in\mathbb{R}^n \,:\, |\boldsymbol{w}|\leq 1\}} |\boldsymbol{w}|^s e^{-f(\boldsymbol{w})}\, d\boldsymbol{w} + \int_{\{\boldsymbol{w}\in\mathbb{R}^n \,:\, |\boldsymbol{w}|>1\}} |\boldsymbol{w}|^s e^{-f(\boldsymbol{w})}\, d\boldsymbol{w} = J_{s_1} + J_{s_2}. \tag{3.21}$$

Then

$$J_{s_1} \leq \int_{\mathbb{R}^n} e^{-f(\boldsymbol{w})}\, d\boldsymbol{w} \approx |\{\boldsymbol{w} \,:\, f(\boldsymbol{w}) \leq 1\}|. \tag{3.22}$$

Given $\boldsymbol{v_0}$, we can estimate the size of J_{s_2} by splitting the integral into the two following regions:

$$J_{s_2} = \int\limits_{\substack{|\boldsymbol{w}|>1 \\ |\boldsymbol{w}|\leq\lambda|\boldsymbol{v_0}|^B}} |\boldsymbol{w}|^s e^{-f(\boldsymbol{w})}\, d\boldsymbol{w} + \int\limits_{\substack{|\boldsymbol{w}|>1 \\ |\boldsymbol{w}|>\lambda|\boldsymbol{v_0}|^B}} |\boldsymbol{w}|^s e^{-f(\boldsymbol{w})}\, d\boldsymbol{w}, \tag{3.23}$$

for some large constant λ yet to be determined and $B = 4\max\{m_1, \ldots, m_n\}$. Then

$$\int\limits_{\substack{|\boldsymbol{w}|>1 \\ |\boldsymbol{w}|\leq\lambda|\boldsymbol{v_0}|^B}} |\boldsymbol{w}|^s e^{-f(\boldsymbol{w})}\, d\boldsymbol{w} \leq\ \lambda^s |\boldsymbol{v_0}|^{sB} \int_{\mathbb{R}^n} e^{-f(\boldsymbol{w})}\, d\boldsymbol{w} \approx \lambda^s |\boldsymbol{v_0}|^{sB} |\{\boldsymbol{w} \,:\, f(\boldsymbol{w}) \leq 1\}|. \tag{3.24}$$

In order to estimate $\int_{\{\boldsymbol{w}\,:\,|\boldsymbol{w}|>1, |\boldsymbol{w}|>\lambda|\boldsymbol{v_0}|^B\}} |\boldsymbol{w}|^s e^{-f(\boldsymbol{w})}\, d\boldsymbol{w}$, we will find a lower bound in this region for $f(\boldsymbol{w})$ in terms of $|\boldsymbol{w}|^2$ and we will then bound the integral by a constant.

Recall that

$$f(\boldsymbol{w}) = g(\boldsymbol{v_0} + \boldsymbol{w}) - g(\boldsymbol{v_0}) - \nabla g(\boldsymbol{v_0}) \cdot \boldsymbol{w}.$$

Thus (and since $f \geq 0$),

$$f(\boldsymbol{w}) \geq |g(\boldsymbol{v_0} + \boldsymbol{w})| - |g(\boldsymbol{v_0})| - |\nabla g(\boldsymbol{v_0}) \cdot \boldsymbol{w}|. \tag{3.25}$$

We will show that $g(\boldsymbol{v_0} + \boldsymbol{w})$ is bounded from below by a constant multiple of $|\boldsymbol{w}|^2$. It will then suffice to show that the remaining terms in the above expression can be dominated by this bound.

Let

$$F(t) = g\left(\frac{t(\boldsymbol{v_0} + \boldsymbol{w})}{|\boldsymbol{v_0} + \boldsymbol{w}|}\right),$$

where $t \in \mathbb{R}$. Then $F(t)$ is a convex polynomial in one variable, such that $F(0) = F'(0) = 0$. We will write

$$F(t) = \sum_{j=2}^{M} a_j t^j.$$

By Lemma 2.1 of the paper by Bruna Nagel and Wainger [4], we know that there exists a constant $C_M > 0$ that depends only on the degree of F such that for all $t \geq 1$,

$$F(t) \geq C_M t^2 \sum_{j=2}^{M} |a_j|. \tag{3.26}$$

Thus,

$$F(|\boldsymbol{v_0} + \boldsymbol{w}|) \geq C_M |\boldsymbol{v_0} + \boldsymbol{w}|^2 \sum_{j=2}^{M} |a_j|.$$

Furthermore, we claim that $\sum_{j=2}^{M} |a_j| \geq 1$ so that $F(|\boldsymbol{v_0} + \boldsymbol{w}|) \geq C_M |\boldsymbol{v_0} + \boldsymbol{w}|^2$. In fact, by hypothesis

$$\{\boldsymbol{v} \,:\, g(\boldsymbol{v}) \leq 1\} \subseteq \{\boldsymbol{v} \,:\, |\boldsymbol{v}| \leq 1\}.$$

It follows that

$$F(1) = g\left(\frac{\boldsymbol{v_0} + \boldsymbol{w}}{|\boldsymbol{v_0} + \boldsymbol{w}|}\right).$$

Thus, since the vector $\frac{\boldsymbol{v_0}+\boldsymbol{w}}{|\boldsymbol{v_0}+\boldsymbol{w}|}$ is on the unit sphere and g is convex,

$$F(1) = g\left(\frac{\boldsymbol{v_0} + \boldsymbol{w}}{|\boldsymbol{v_0} + \boldsymbol{w}|}\right) \geq 1.$$

Therefore,

$$\sum_{j=2}^{M} |a_j| \geq \sum_{j=2}^{M} a_j = F(1) \geq 1.$$

Notice that

$$g(\boldsymbol{v_0} + \boldsymbol{w}) = F(|\boldsymbol{v_0} + \boldsymbol{w}|) \geq C_M |\boldsymbol{v_0} + \boldsymbol{w}|^2. \tag{3.27}$$

Also, since in the region we are considering $|\boldsymbol{w}| > \lambda |\boldsymbol{v_0}|^B$ (i.e., $-|\boldsymbol{v_0}| > -\frac{|\boldsymbol{w}|^{\frac{1}{B}}}{\lambda^{\frac{1}{B}}}$) and $|\boldsymbol{w}| > 1$, and since $B = 4\max\{m_1, \ldots, m_n\} > 1$, it follows that

$$|\boldsymbol{v_0} + \boldsymbol{w}| \geq |\boldsymbol{w}| - |\boldsymbol{v_0}| > |\boldsymbol{w}| - \frac{|\boldsymbol{w}|^{\frac{1}{B}}}{\lambda^{\frac{1}{B}}} \geq |\boldsymbol{w}| - \frac{|\boldsymbol{w}|}{\lambda^{\frac{1}{B}}} \geq |\boldsymbol{w}| \left(1 - \frac{1}{\lambda^{\frac{1}{B}}}\right). \tag{3.28}$$

Thus, by equations (3.27) and (3.28) it follows that

$$g(\boldsymbol{v_0} + \boldsymbol{w}) \geq C_M |\boldsymbol{v_0} + \boldsymbol{w}|^2 \geq C_M |\boldsymbol{w}|^2 \left(1 - \frac{1}{\lambda^{\frac{1}{B}}}\right)^2.$$

Choosing $\lambda > \left(\frac{\sqrt{2}}{\sqrt{2}-1}\right)^B$, it follows that

$$g(\boldsymbol{v_0}+\boldsymbol{w}) \geq \frac{C_M|\boldsymbol{w}|^2}{2}. \tag{3.29}$$

We would now like to obtain an upper bound for $|g(\boldsymbol{v_0})|$. Recall that by Claim 3.3, for any $\boldsymbol{v} \in \mathbb{R}^n$ we have that $g(\boldsymbol{v}) \leq C(1+r(\boldsymbol{v}))$. Thus, and since $\max\{2m_1, \ldots, 2m_n\} = \frac{B}{2} < B$,

$$\begin{aligned}
|g(\boldsymbol{v_0})| &\leq C(1+r(\boldsymbol{v_0})) = C(1+v_{01}^{2m_1}+\ldots+v_{0n}^{2m_n}) \\
&\leq C(1+|\boldsymbol{v_0}|^{2m_1}+\ldots+|\boldsymbol{v_0}|^{2m_n}) \\
&\leq C(1+(1+|\boldsymbol{v_0}|^B)+\ldots+(1+|\boldsymbol{v_0}|^B)) \\
&= C(1+n+n|\boldsymbol{v_0}|^B) \\
&< C\left(1+n+\frac{n|\boldsymbol{w}|}{\lambda}\right).
\end{aligned}$$

But since $|\boldsymbol{w}| \leq |\boldsymbol{w}|^2+1$, it follows that

$$|g(\boldsymbol{v_0})| \leq C\left(1+n+\frac{n}{\lambda}+\frac{n|\boldsymbol{w}|^2}{\lambda}\right).$$

For $\lambda > \frac{8Cn}{C_M}$, it follows that

$$|g(\boldsymbol{v_0})| \leq C\left(1+n+\frac{n}{\lambda}\right)+\frac{Cn|\boldsymbol{w}|^2}{\lambda} \leq C\left(1+n+\frac{n}{\lambda}\right)+\frac{C_M|\boldsymbol{w}|^2}{8}. \tag{3.30}$$

It now suffices to obtain an upper bound for $|\nabla g(\boldsymbol{v_0})\cdot\boldsymbol{w}|$. Notice that for each $1 \leq j \leq n$, the j^{th} entry of ∇g is a polynomial whose exponents satisfy

$$\frac{2\alpha_1}{B} + \cdots + \frac{2\alpha_n}{B} \leq \frac{\alpha_1}{2m_1} + \cdots + \frac{\alpha_n}{2m_n} \leq 1 - \frac{1}{2m_j} < 1.$$

That is, $\alpha_1 + \ldots + \alpha_n < \frac{B}{2}$. Thus, we can bound each entry of $|\nabla g(\boldsymbol{v_0})|$ by a constant multiple of $1 + |\boldsymbol{v_0}|^{\frac{B}{2}}$. The coefficients of each of these entries are multiples of the coefficients of g, where the factors depend only on the degree of g. Thus, since $\sum_{\alpha \in \Gamma} |c_\alpha| \leq C$, there exists a constant C_1 that depends only on C and the degree of g such that

$$|\nabla g(\boldsymbol{v_0})| \leq C_1(1 + |\boldsymbol{v_0}|^{\frac{B}{2}}).$$

Hence, in the region under consideration we have

$$|\nabla g(\boldsymbol{v_0})| \leq C_1 \left(1 + \frac{|\boldsymbol{w}|^{\frac{1}{2}}}{\lambda^{\frac{1}{2}}}\right).$$

It follows that

$$|\nabla g(\boldsymbol{v_0}) \cdot \boldsymbol{w}| \leq |\nabla g(\boldsymbol{v_0})||\boldsymbol{w}| \leq C_1 \left(1 + \frac{|\boldsymbol{w}|^{\frac{1}{2}}}{\lambda^{\frac{1}{2}}}\right) |\boldsymbol{w}| = C_1 \left(|\boldsymbol{w}| + \frac{|\boldsymbol{w}|^{\frac{3}{2}}}{\lambda^{\frac{1}{2}}}\right).$$

But given $x > 0$, $0 \leq j < d$, and any constant $A > 0$, it follows that

$$x^j \leq \frac{1}{A} x^d + A^d. \tag{3.31}$$

Hence, $|\boldsymbol{w}| \leq \frac{1}{A}|\boldsymbol{w}|^2 + A^2$ for an arbitrary constant $A > 0$. Also, $|\boldsymbol{w}|^{\frac{3}{2}} \leq |\boldsymbol{w}|^2 + 1$. Thus,

$$|\nabla g(\boldsymbol{v_0}) \cdot \boldsymbol{w}| \leq C_1 \left(\frac{1}{A}|\boldsymbol{w}|^2 + A^2 + \frac{|\boldsymbol{w}|^2}{\lambda^{\frac{1}{2}}} + \frac{1}{\lambda^{\frac{1}{2}}}\right).$$

Then for $A > \frac{16C_1}{C_M}$ and $\lambda > \left(\frac{16C_1}{C_M}\right)^2$ it follows that

$$|\nabla g(\boldsymbol{v_0}) \cdot \boldsymbol{w}| \leq C_1 \left(A^2 + \frac{1}{\lambda^{\frac{1}{2}}} \right) + \frac{C_M |\boldsymbol{w}|^2}{8}. \tag{3.32}$$

Therefore, by equations (3.29),(3.30) and (3.32), and taking

$$\lambda > \max \left\{ \left(\frac{\sqrt{2}}{\sqrt{2}-1} \right)^B, \frac{8Cn}{C_M}, \left(\frac{16C_1}{C_M} \right)^2 \right\}$$

it follows that

$$f(\boldsymbol{w}) \geq |\boldsymbol{w}|^2 \left(\frac{C_M}{2} - \frac{C_M}{8} - \frac{C_M}{8} \right) - E = \frac{C_M |\boldsymbol{w}|^2}{4} - E, \tag{3.33}$$

where E is a constant that depends on the "combined degree" of g and the dimension of the space, but is otherwise independent.

Recall that our goal is to obtain an upper bound for

$$I_s = \int_{\substack{|\boldsymbol{w}|>1 \\ |\boldsymbol{w}|>\lambda|\boldsymbol{v_0}|^B}} |\boldsymbol{w}|^s e^{-f(\boldsymbol{w})} \, d\boldsymbol{w}.$$

Using the lower bound for $f(\boldsymbol{w})$ obtained in equation(3.33) we have that

$$I_s \lesssim \int_{\substack{|\boldsymbol{w}|>1 \\ |\boldsymbol{w}|>\lambda|\boldsymbol{v_0}|^B}} |\boldsymbol{w}|^s e^{\frac{-C_M|\boldsymbol{w}|^2}{4}} \, d\boldsymbol{w} \lesssim \int_0^\infty r^s e^{\frac{-C_M r^2}{4}} \, dr.$$

Since C_M is a strictly positive constant, the above integral converges. This finishes the proof of Claim 3.12.

□

It follows from Claim 3.10, equation (3.10) and Claim 3.12 that the derivatives of $\theta(\boldsymbol{\eta})$ are bounded from above by a sum of terms of the form

$$\frac{H_f[|\boldsymbol{v_0}|]^{d-1}}{e^{h(\boldsymbol{v_0})}|\{\boldsymbol{w} \,:\, f(\boldsymbol{w}) \leq 1\}|^d}.$$

Moreover, by Claim 3.12 these terms can be bounded by terms of the form

$$\frac{q(|\boldsymbol{v_0}|)}{e^{h(\boldsymbol{v_0})}|\{\boldsymbol{w} \,:\, f(\boldsymbol{w}) \leq 1\}|^k},$$

where $q : \mathbb{R} \to \mathbb{R}$ is a polynomial, and $k \in [1, d] \times \mathbb{Z}$. By Claim 3.8 it follows that $q(|\boldsymbol{v_0}|)$ is bounded by a polynomial in $|\boldsymbol{\eta}|$. Furthermore, by Claim 3.7, $|\{\boldsymbol{w} : f(\boldsymbol{w}) \leq 1\}|^{-k} \lesssim (1+r(\boldsymbol{v_0}))^{\frac{kn}{2}}$. By Claim 3.9 this latter bound is at most of polynomial growth in $|\boldsymbol{\eta}|$. Thus, the derivatives of $\theta(\boldsymbol{\eta})$ are bounded by sums of terms of the form $e^{-h(\boldsymbol{v_0})}\tilde{q}(|\boldsymbol{\eta}|)$, where $\tilde{q}$ grows at a polynomial rate. Finally, by equation (3.7) $e^{-h(\boldsymbol{v_0})}$ decays at an exponential rate in $|\boldsymbol{\eta}|$. This finishes the proof that the derivatives of $\theta(\boldsymbol{\eta})$ decay exponentially. Thus, θ is a Schwartz function. Moreover, it follows from the previous computations that its decay is independent of the coefficients of g.

□

Chapter 4

A Geometric Bound for the Szegő kernel

In this section we present the proof of our main result. We consider convex unbounded domains of the kind $\Omega_b = \{\boldsymbol{z} \in \mathbb{C}^{n+1} : \mathrm{Im}[z_{n+1}] > b(\mathrm{Re}[z_1], \ldots, \mathrm{Re}[z_n])\}$, where $b : \mathbb{R}^n \to \mathbb{R}$ are convex polynomials of "combined degree" $(m_1, \ldots, m_n)$ (refer to definition on page 3).

Let Ω_b be one such domain, and let $(\boldsymbol{x}, \boldsymbol{y}, t)$ and $(\boldsymbol{x'}, \boldsymbol{y'}, t')$ be any two points in $\partial\Omega_b$. Define

$$\tilde{b}(\boldsymbol{v}) = b\left(\boldsymbol{v} + \frac{\boldsymbol{x}+\boldsymbol{x'}}{2}\right) - \nabla b\left(\frac{\boldsymbol{x}+\boldsymbol{x'}}{2}\right) \cdot \boldsymbol{v} - b\left(\frac{\boldsymbol{x}+\boldsymbol{x'}}{2}\right);$$

$$\delta(\boldsymbol{x}, \boldsymbol{x'}) = b(\boldsymbol{x}) + b(\boldsymbol{x'}) - 2b\left(\frac{\boldsymbol{x}+\boldsymbol{x'}}{2}\right);$$

and

$$w = (t' - t) + \nabla b\left(\frac{\boldsymbol{x}+\boldsymbol{x'}}{2}\right) \cdot (\boldsymbol{y'} - \boldsymbol{y}).$$

We obtain the following estimate for the Szegő kernel associated to the domain Ω_b :

$$|S\left((\boldsymbol{x},\boldsymbol{y},t);(\boldsymbol{x'},\boldsymbol{y'},t')\right)| \lesssim \frac{1}{\sqrt{\delta^2+\widetilde{b}(\boldsymbol{y}-\boldsymbol{y'})^2+w^2}\left|\left\{\boldsymbol{v}\,:\,\tilde{b}(\boldsymbol{v})<\sqrt{\delta^2+\widetilde{b}(\boldsymbol{y}-\boldsymbol{y'})^2+w^2}\right\}\right|^2}.$$

Here the constant depends on the exponents $\{m_1,\ldots,m_n\}$ and the dimension of the space, but is independent of the two given points.

Notice that since

$$\tilde{b}(\boldsymbol{v}) = b\left(\boldsymbol{v}+\frac{\boldsymbol{x}+\boldsymbol{x'}}{2}\right) - \nabla b\left(\frac{\boldsymbol{x}+\boldsymbol{x'}}{2}\right)\cdot\boldsymbol{v} - b\left(\frac{\boldsymbol{x}+\boldsymbol{x'}}{2}\right)$$

we can write

$$\tilde{b}(\boldsymbol{v}) = f(\boldsymbol{v}) - L(\boldsymbol{v}),$$

where $f(\boldsymbol{v}) = b\left(\boldsymbol{v}+\frac{\boldsymbol{x}+\boldsymbol{x'}}{2}\right)$ and L is the tangent hyperplane to f at $\boldsymbol{v}=0$. Thus, given any $M>0$, $|\{\boldsymbol{v}\,:\,\tilde{b}(\boldsymbol{v})\le M\}|$ is depicted in the following figure:

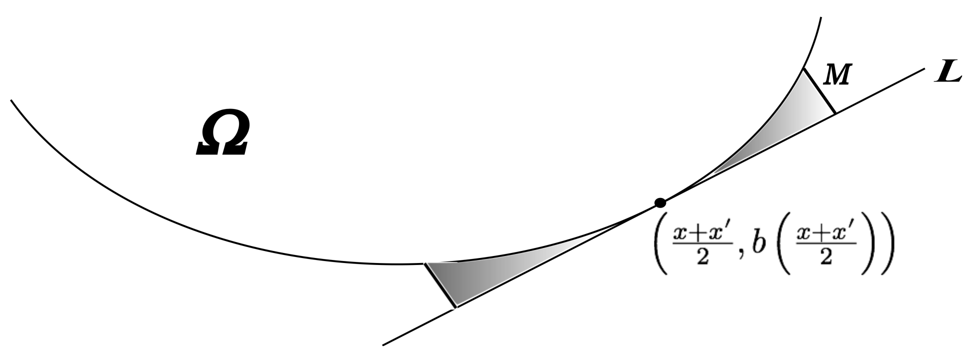

Figure 4.1.

We obtain these bounds by estimating the integral expression for the Szegő kernel obtained in Appendix A. That is, we study

$$S((\boldsymbol{x},\boldsymbol{y},t);(\boldsymbol{x'},\boldsymbol{y'},t')) = \int_0^\infty e^{-2\pi\tau[b(\boldsymbol{x'})+b(\boldsymbol{x})+i(t'-t)]}\left(\int_{\mathbb{R}^n}\frac{e^{2\pi\boldsymbol{\eta}\cdot[\boldsymbol{x}+\boldsymbol{x'}-i(\boldsymbol{y'}-\boldsymbol{y})]}}{\int\limits_{\mathbb{R}^n} e^{4\pi[\boldsymbol{\eta}\cdot\boldsymbol{v}-b(\boldsymbol{v})\tau]}\,d\boldsymbol{v}}\,d\boldsymbol{\eta}\right)d\tau.$$

The proof is in essence an application of John ellipsoids. Recall that by John [17], given a symmetric convex compact region, there exists a maximal inscribed ellipsoid $\mathfrak{E}$ in that region (centered at the center of symmetry) such that $\sqrt{n}\,\mathfrak{E}$ contains the region, where n is the dimension of the space. The key step of our proof consists in introducing factors $\mu_1(\boldsymbol{x},\boldsymbol{x'},\tau)$, ..., $\mu_n(\boldsymbol{x},\boldsymbol{x'},\tau)$ via a change of variable so that

$$\mu_1\cdots\mu_n \approx \left|\left\{\boldsymbol{v}\,:\,\tilde{b}(\boldsymbol{v}) \le \frac{1}{\tau}\right\}\right|.$$

These factors are chosen to be the length of the axis of the John ellipsoid associated to a symmetrization of the convex region $\left\{\boldsymbol{v}\,:\,\tilde{b}(\boldsymbol{v}) \le \frac{1}{\tau}\right\}$.

4.1 Construction of the factors $\mu_1\ldots,\mu_n$

We will begin by discussing how to construct these factors $\mu_1\ldots,\mu_n$. Let

$$R = \left\{\boldsymbol{v}\,:\,\tilde{b}(\boldsymbol{v}) \le \frac{1}{\tau}\right\}.$$

Notice that since b is convex, so is $\tilde{b}$, and the region R is convex. In order to be able to use John's bounds, we need to show that the set R is also compact. We do so in the following claim.

Claim 4.2. *Let* $b : \mathbb{R}^n \to \mathbb{R}$ *be a convex polynomial of "combined degree"* $(m_1, \ldots, m_n)$. *Let*

$$\tilde{b}(\boldsymbol{v}) = b\left(\boldsymbol{v} + \frac{\boldsymbol{x}+\boldsymbol{x'}}{2}\right) - \nabla b\left(\frac{\boldsymbol{x}+\boldsymbol{x'}}{2}\right) \cdot \boldsymbol{v} - b\left(\frac{\boldsymbol{x}+\boldsymbol{x'}}{2}\right).$$

Then for any $M > 0$, *the set* $\{\boldsymbol{v} : \tilde{b}(\boldsymbol{v}) \leq M\}$ *is compact.*

Proof. Notice that if b is a convex polynomial of "combined degree" $(m_1, \ldots, m_n)$, so is $\tilde{b}$. Also, since $\tilde{b}(\mathbf{0}) = \mathbf{0}$ and $\nabla b(\mathbf{0}) = 0$, it follows that $\tilde{b} \geq 0$ (this was already shown in equation (1.4) on page 17). Observe that $\tilde{b}$ is not the zero polynomial, since by definition of "combined degree", $\tilde{b}$ is of strictly positive degree.

Fix $M > 0$. Since $\tilde{b}$ is continuous and $\{\boldsymbol{v} : \tilde{b}(\boldsymbol{v}) \leq M\} \subset \mathbb{R}^n$, it suffices to show that this set is bounded. Suppose that this is not so.

We claim that if $\{\boldsymbol{v} : \tilde{b}(\boldsymbol{v}) \leq M\}$ is unbounded, then there exists a $\boldsymbol{x} \in \mathbb{R}^n$ such that $\forall c > 0$, $\tilde{b}(c\boldsymbol{x}) \leq M$. In particular, since $\tilde{b}(\mathbf{0}) = 0$ and $\tilde{b}$ is convex, $\forall c > 0$, $\tilde{b}(c\boldsymbol{x}) \equiv 0$.

In fact, if the set is unbounded, then there exists a sequence $\{\boldsymbol{v}_i\}_{i\in\mathbb{N}}$ in $\mathbb{R}^n$ such that $|\boldsymbol{v}_i| > i$ and $\tilde{b}(\boldsymbol{v_i}) \leq M$. Define $\boldsymbol{y}_i = \frac{\boldsymbol{v}_i}{|\boldsymbol{v}_i|}$. Let $\boldsymbol{x}$ be any limit point of $\{\boldsymbol{y}_i\}_{i\in\mathbb{N}}$. Notice that since $\{\boldsymbol{y}_i\}_{i\in\mathbb{N}}$ is a sequence in the unit ball, which is compact, it has a convergent subsequence, and so there exists at least one suitable $\boldsymbol{x}$.

Let c be given and let $N \in \mathbb{N}$ so that $N > c$. Then $\tilde{b}(c\boldsymbol{y}_i) \leq M$ for all $i \geq N$. In fact, $c\boldsymbol{y}_i = \frac{c}{|\boldsymbol{v}_i|} \cdot \boldsymbol{v}_i$, but the sequence $\{\boldsymbol{v}_i\}_{i\in\mathbb{N}}$ was chosen so that $\frac{i}{|\boldsymbol{v}_i|} < 1$, so $\frac{c}{|\boldsymbol{v}_i|} \leq \frac{i}{|\boldsymbol{v}_i|} < 1$. Thus, it follows from the convexity of $\tilde{b}$ that $\tilde{b}(c\boldsymbol{y}_i) \leq \tilde{b}(\boldsymbol{v}_i) \leq M$.

Furthermore, if $\{\boldsymbol{y}_{i_j}\}$ is a subsequence of $\{\boldsymbol{y}_i\}$ which converges to $\boldsymbol{x}$, then, and since $\tilde{b}$ is continuous, $\tilde{b}(c\boldsymbol{y}_{i_j})$ converges to $\tilde{b}(c\boldsymbol{x})$. But since $\tilde{b}(c\boldsymbol{y}_{i_j}) \leq M$ for all $i_j \geq N$, it follows that $\tilde{b}(c\boldsymbol{x}) \leq M$ for all $c > 0$.

Moreover, from the above and the convexity of $\tilde{b}$, it follows that $\tilde{b}(c\boldsymbol{x}) \equiv 0$. In fact, suppose there exists $0 < r \leq M$ such that $\tilde{b}(c\boldsymbol{x}) = r$. Then by convexity, $\tilde{b}(\lambda c\boldsymbol{x}) \geq \lambda\tilde{b}(c\boldsymbol{x}) = \lambda r$ for all $\lambda > 1$. In particular, taking $\lambda = \frac{M+1}{r}$ it follows that $\tilde{b}\left(\frac{M+1}{r}c\boldsymbol{x}\right) \geq \frac{M+1}{r}r = M + 1$. This contradicts the fact that $\tilde{b}(c\boldsymbol{x}) \leq M$ for all $c > 0$.

Now let $\boldsymbol{x} = (x_1, \ldots, x_n)$ be as above, and define $\boldsymbol{w}(t) = t \cdot (x_1, \ldots, x_n)$. Then since $\tilde{b}(c\boldsymbol{x}) \equiv 0$ for all $c > 0$, it follows that $\tilde{b}(\boldsymbol{w}(t)) \equiv 0$, and so the polynomial of one variable $\tilde{b} \circ \boldsymbol{w}$ is the zero polynomial.

On the other hand, if $c_\alpha(tx_1)^{\alpha_1} \cdots (tx_n)^{\alpha_n}$ is a term of $\tilde{b}(\boldsymbol{w}(t))$, then the corresponding term in $\tilde{b} \circ \boldsymbol{w}$ is of degree $|\alpha| = \alpha_1 + \cdots + \alpha_n$, and its coefficient is $c_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. Thus, the maximal degree terms in $\tilde{b}$ correspond to the maximal degree terms in $\tilde{b} \circ \boldsymbol{w}$. But since $\tilde{b}$ is of "combined degree," its highest order term (or terms) corresponds to one of the pure terms. That is, the highest degree term of $\tilde{b}(\boldsymbol{w}(t))$ is of the form $c_\alpha(tx_i)^{2m_i}$ (or a sum of terms of this form). It is easy to check that the coefficients of the highest degree pure terms of $\tilde{b}$ are positive. This follows from the fact that $\tilde{b}(0, \ldots, 0, tx_i, 0, \ldots, 0) : \mathbb{R}^n \to \mathbb{R}^+$. Since for these terms the α_i are even, it follows that the coefficient of the highest degree term of $\tilde{b} \circ \boldsymbol{w}$ is positive. Thus, $\tilde{b} \circ \boldsymbol{w}$ is not the zero polynomial.

This yields the desired contradiction. It follows that $\{\boldsymbol{v} : \tilde{b}(\boldsymbol{v}) \leq M\}$ is bounded, and therefore compact.

□

Recall that by construction $\tilde{b}(\mathbf{0}) = 0$, so the region R contains the origin. We would now like to show that there exists an ellipsoid $\mathfrak{E}$ centered at the origin such that

$$\mathfrak{E} \subseteq R \subseteq C\mathfrak{E},$$

for some independent positive constant C. If this were the case, we could choose μ_1 to be the length of the largest semi–axis of $\mathfrak{E}$, μ_2 to be the length of the second largest semi–axis of $\mathfrak{E}$, etc. It would follow that

$$\mu_1 \cdots \mu_n \approx Vol(\mathfrak{E}) \approx Vol(R).$$

That is, we would have found factors $\mu_1, \ldots, \mu_n$ such that

$$\mu_1 \cdots \mu_n \approx \left| \left\{ \boldsymbol{v} \,:\, \tilde{b}(\boldsymbol{v}) \leq \frac{1}{\tau} \right\} \right|.$$

The existence of such an ellipsoid would follow immediately by John if the region were symmetric (with the origin as center of symmetry). However, we have made no symmetry assumptions on our domain. Nevertheless, we can show the following:

Claim 4.3. *Let L be any line through the origin (recall the the origin is contained in the set R). This line L will intersect R in two points. Let d_1 be the shortest distance along L from the origin to the boundary of R, and let d_2 be the largest distance. Then there exist constants m, M depending only on the degree of the polynomial b such that*

$$0 < m \leq \frac{d_2}{d_1} \leq M < +\infty.$$

Proof. Let $h(\boldsymbol{v}) = \tau\tilde{b}(\boldsymbol{v})$. Along the line L, the polynomial $h(\boldsymbol{v})$ is a polynomial of one variable which we will call $h_L(t)$. This polynomial satisfies $h_L(0) = h_L'(0) = 0$. Write

$$h_L(t) = \sum_{j=2}^{N} c_j t^j.$$

Then there exists some $2 \leq k \leq N$ such that

$$h_L(d_1) \leq \sum_{j=2}^{N} |c_j| d_1^j \leq (N-1)|c_k| d_1^k.$$

On the other hand, it follows from Lemma 2.1 on [4] (refer to page 15) that there exists a constant $0 < C_N \leq 1$ such that

$$h_L(d_2) \geq C_N \sum_{j=2}^{N} |c_j| d_2^j.$$

For k as above, it follows that

$$C_N |c_k| d_2^k \leq C_N \sum_{j=2}^{N} |c_j| d_2^j \leq h_L(d_2).$$

Moreover, $h_L(d_1) = h_L(d_2) = 1$, since d_1 and d_2 where chosen as the distances where L intersects the boundary of the region $R = \{\boldsymbol{v} : h(\boldsymbol{v}) \leq 1\}$. Thus,

$$C_N |c_k| d_2^k \leq h_L(d_2) = 1 = h_L(d_1) \leq (N-1)|c_k| d_1^k.$$

Therefore,

$$\frac{d_2}{d_1} \leq \left(\frac{N-1}{C_N}\right)^{\frac{1}{k}}.$$

On the other hand, since d_1 is the shortest distance along L from R to the origin and d_2 is the largest, it follows that $d_1 \leq d_2$.

Choosing $M = \max_{2\leq k\leq N}\left\{\left(\frac{N-1}{C_N}\right)^{\frac{1}{k}}\right\}$ and $m = 1$ it follows that

$$m \leq \frac{d_2}{d_1} \leq M.$$

This finishes the proof of Claim 4.3.

□

We have shown that even though the region R is not symmetric, the ratio between rays passing through the origin is bounded by universal constants that only depend on the degree of b. In the following lemma we will show that this is enough to guarantee the existence of an ellipsoid centered at the origin contained in R and such that a dilation by a universal constant contains R.

Lemma 4.4. *Let* $R = \left\{\boldsymbol{v} : \tilde{b}(\boldsymbol{v}) \leq \frac{1}{\tau}\right\}$ *and* $\widetilde{R} = \{\boldsymbol{x} : -\boldsymbol{x} \in R\}$. *Let* $\mathfrak{E}$ *be the maximal inscribed ellipsoid in the region* $R \cap \widetilde{R}$. *Then*

$$\mathfrak{E} \subseteq R \subseteq M\sqrt{n}\mathfrak{E}$$

where M *is as in Claim 4.3, and* n *is the dimension of the space.*

Proof. By definition, the set $R \cap \widetilde{R}$ is symmetric. Moreover, since R and $\widetilde{R}$ are compact and convex, their intersection is also compact and convex. It follows from John that there exists an ellipsoid $\mathfrak{E}$ centered at the origin such that

$$\mathfrak{E} \subseteq R \cap \widetilde{R} \subseteq \sqrt{n}\mathfrak{E}.$$

It is clear that $\mathfrak{E} \subseteq R$. We would like to show that there is a dilation of $\mathfrak{E}$ which contains R. Let $\boldsymbol{x}$ be any point in R. Then, if $-\boldsymbol{x} \in R$, it follows by definition that $x \in \sqrt{n}\mathfrak{E}$. Now suppose that $-\boldsymbol{x} \notin R$. Let L be the line that goes through the origin and $\boldsymbol{x}$. Using the notation of the previous claim, we have that $|\boldsymbol{x}| \leq d_2$. We would like to find a constant $\rho > 0$ such that $-\rho\boldsymbol{x} \in R$.

Given M as in Claim 4.3, let $0 < \rho \leq \frac{1}{M}$. Then $|-\rho\boldsymbol{x}| \leq \rho d_2 \leq \rho M d_1 \leq d_1$. But since d_1 is the minimum distance from the boundary of R to the origin along line L, it follows that $-\rho\boldsymbol{x} \in R$.

It follows that given any point $\boldsymbol{x} \in R$ the point $-\frac{1}{M}\boldsymbol{x}$ is also contained in R. That is, $\frac{1}{M}\boldsymbol{x} \in R \cap \widetilde{R} \subseteq \sqrt{n}\mathfrak{E}$. Thus,

$$\mathfrak{E} \subseteq R \subseteq M\sqrt{n}\mathfrak{E}.$$

This finishes the proof of Lemma 4.4.

□

It follows from the previous lemma that

$$Vol(\mathfrak{E}) \approx \left|\left\{\boldsymbol{v} \,:\, \tilde{b}(\boldsymbol{v}) \leq \frac{1}{\tau}\right\}\right|.$$

Thus, choosing μ_1 to be the length of the largest semi–axis of $\sqrt{n}M\mathfrak{E}$, μ_2 to be the length of the second largest semi–axis of $\sqrt{n}M\mathfrak{E}$, etc., we have that

$$\mu_1 \cdots \mu_n \approx \left| \left\{ \boldsymbol{v} \,:\, \tilde{b}(\boldsymbol{v}) \leq \frac{1}{\tau} \right\} \right|. \tag{4.1}$$

Remark 4.5. *The reason we have chosen the components of $\boldsymbol{\mu}$ to be the lengths of the axes of $\sqrt{n}M\mathfrak{E}$ rather than those of $\mathfrak{E}$ will become apparent in the proof of the Main Theorem. This normalization will be used to construct a function satisfying hypothesis (iii) in Lemma 3.1.*

4.2 Proof of the Main Theorem

We are now ready to present the proof of the Main Theorem. For the reader's convenience, we have divided the proof into three subsections, corresponding to a bound in terms of δ, a bound in terms of $\widetilde{b}(\boldsymbol{y}-\boldsymbol{y'})$ and a bound in terms of w. We finish by combining all three bounds to obtain the estimate stated in the Main Theorem. It will be convenient in the course of the proof of all three bounds to rearrange the terms of the integral expression for the Szegő kernel obtained in Theorem A.1 as follows:

Making the change of variables $\boldsymbol{v} \to \boldsymbol{v} + \frac{\boldsymbol{x}}{2} + \frac{\boldsymbol{x'}}{2}$ to get rid of the term $e^{2\pi\boldsymbol{\eta}\cdot(\boldsymbol{x}+\boldsymbol{x'})}$ in the original expression obtained for the Szegő kernel given by

$$S((\boldsymbol{x},\boldsymbol{y},t);(\boldsymbol{x'},\boldsymbol{y'},t')) = \int_0^\infty e^{-2\pi\tau[b(\boldsymbol{x'})+b(\boldsymbol{x})+i(t'-t)]} \left(\int_{\mathbb{R}^n} \frac{e^{2\pi\boldsymbol{\eta}\cdot[\boldsymbol{x}+\boldsymbol{x'}-i(\boldsymbol{y'}-\boldsymbol{y})]}}{\int_{\mathbb{R}^n} e^{4\pi[\boldsymbol{\eta}\cdot\boldsymbol{v}-b(\boldsymbol{v})\tau]}\,d\boldsymbol{v}}\,d\boldsymbol{\eta} \right) d\tau$$

it follows that

$$S = \int_0^\infty e^{-2\pi\tau[b(\boldsymbol{x'})+b(\boldsymbol{x})+i(t'-t)]} \int_{\mathbb{R}^n} \frac{e^{2\pi i \boldsymbol{\eta}\cdot(\boldsymbol{y}-\boldsymbol{y'})}}{\int_{\mathbb{R}^n} e^{4\pi\left[\boldsymbol{\eta}\cdot\boldsymbol{v}-\tau b\left(\boldsymbol{v}+\frac{\boldsymbol{x}+\boldsymbol{x'}}{2}\right)\right]} d\boldsymbol{v}} \, d\boldsymbol{\eta} \, d\tau.$$

We will modify the denominator integral so as to change it into an integral of the form $[\theta(\boldsymbol{\eta})]^{-1}$, where θ is the function studied in Section 3.

In particular, the exponent of our integral must be of the form $\boldsymbol{\eta}\cdot\boldsymbol{v} - g(\boldsymbol{v})$, where $g(\mathbf{0}) = 0$ and $\nabla g(\mathbf{0}) = 0$. Since

$$\nabla \left[\tau b\left(\boldsymbol{v} + \frac{\boldsymbol{x}+\boldsymbol{x'}}{2}\right)\right]\Bigg|_{\{\boldsymbol{v}=\mathbf{0}\}} = \nabla b\left(\frac{\boldsymbol{x}+\boldsymbol{x'}}{2}\right)\tau$$

we can make the change of variables $\boldsymbol{\eta} \to \boldsymbol{\eta} + \nabla b\left(\frac{\boldsymbol{x}+\boldsymbol{x'}}{2}\right)\tau$. Then,

$$S = \int_0^\infty e^{-2\pi\tau[b(\boldsymbol{x'})+b(\boldsymbol{x})+i(t'-t)]} e^{4\pi\tau\, b\left(\frac{\boldsymbol{x}+\boldsymbol{x'}}{2}\right)} e^{2\pi i\tau\nabla b\left(\frac{\boldsymbol{x}+\boldsymbol{x'}}{2}\right)\cdot(\boldsymbol{y}-\boldsymbol{y'})}$$
$$\times \int_{\mathbb{R}^n} \frac{e^{2\pi i \boldsymbol{\eta}\cdot(\boldsymbol{y}-\boldsymbol{y'})}}{\int_{\mathbb{R}^n} e^{4\pi[\boldsymbol{\eta}\cdot\boldsymbol{v}-\tau\tilde{b}(\boldsymbol{v})]} d\boldsymbol{v}} \, d\boldsymbol{\eta} \, d\tau,$$

where

$$\tilde{b}(\boldsymbol{v}) = b\left(\boldsymbol{v} + \frac{\boldsymbol{x}+\boldsymbol{x'}}{2}\right) - \nabla b\left(\frac{\boldsymbol{x}+\boldsymbol{x'}}{2}\right)\cdot\boldsymbol{v} - b\left(\frac{\boldsymbol{x}+\boldsymbol{x'}}{2}\right)$$

and the term $b\left(\frac{\boldsymbol{x}+\boldsymbol{x'}}{2}\right)$ has been added so that $\tilde{b}(\mathbf{0}) = 0$.

Letting

$$\delta(\boldsymbol{x}, \boldsymbol{x'}) = b(\boldsymbol{x}) + b(\boldsymbol{x'}) - 2b\left(\frac{\boldsymbol{x}+\boldsymbol{x'}}{2}\right)$$

and

$$w = (t' - t) + \nabla b\left(\frac{\boldsymbol{x} + \boldsymbol{x'}}{2}\right) \cdot (\boldsymbol{y'} - \boldsymbol{y})$$

it follows that

$$S = \int_0^\infty e^{-2\pi\tau\delta} e^{-2\pi\tau i w} \int_{\mathbb{R}^n} \frac{e^{2\pi i \boldsymbol{\eta}\cdot(\boldsymbol{y}-\boldsymbol{y'})}}{\int_{\mathbb{R}^n} e^{4\pi[\boldsymbol{\eta}\cdot\boldsymbol{v} - \tau\tilde{b}(\boldsymbol{v})]}\, d\boldsymbol{v}}\, d\boldsymbol{\eta}\, d\tau. \tag{4.2}$$

Notice that since b is convex, $\tau\tilde{b}$ is also convex. Moreover, if b is of "combined degree" $(m_1, \dots, m_n)$, so is $\tau\tilde{b}$. Thus, $\tau\tilde{b}$ is a convex polynomial satisfying conditions (i), (ii) and (iv) of Lemma 3.1. It remains to renormalize $\tau\tilde{b}$ so that it also satisfies condition (iii).

With $\mu_1, \dots, \mu_n$ chosen as in equation (4.1), let

$$g(\boldsymbol{v}) = \tau\tilde{b}(\boldsymbol{\mu}\boldsymbol{v}). \tag{4.3}$$

Notice that since

$$\mathfrak{E} \subseteq \left\{\boldsymbol{v} \,:\, \tau\tilde{b}(\boldsymbol{v}) \le 1\right\} \subseteq \sqrt{n}M\,\mathfrak{E}$$

and letting $A = (\sqrt{n}M)^{-1}$ condition (iii) of Lemma 3.1 is satisfied. That is,

$$\{\boldsymbol{v} \,:\, |\boldsymbol{v}| \le A\} \subseteq \{\boldsymbol{v} \,:\, g(\boldsymbol{v}) \le 1\} \subseteq \{\boldsymbol{v} \,:\, |\boldsymbol{v}| \le 1\}. \tag{4.4}$$

By making the change of variables $\boldsymbol{v} \to \boldsymbol{\mu}\boldsymbol{v}$ so as to introduce the factors $\mu_1, \dots, \mu_n$ in the denominator integral of equation (4.2), as well as the change of variables $\boldsymbol{\eta} \to \frac{\boldsymbol{\eta}}{\boldsymbol{\mu}}$ we have that

$$S = \int_0^\infty \frac{e^{-2\pi\tau\delta} e^{-2\pi\tau i w}}{\mu_1^2 \cdots \mu_n^2} \int_{\mathbb{R}^n} \frac{e^{2\pi i \boldsymbol{\eta} \cdot \left(\frac{\boldsymbol{y}-\boldsymbol{y'}}{\boldsymbol{\mu}}\right)}}{\int_{\mathbb{R}^n} e^{4\pi[\boldsymbol{\eta}\cdot\boldsymbol{v} - \tau\tilde{b}(\boldsymbol{\mu}\boldsymbol{v})]}\, d\boldsymbol{v}}\, d\boldsymbol{\eta}\, d\tau. \tag{4.5}$$

The term $e^{-2\pi\tau\delta}$ will provide the necessary decay to obtain the bound in terms of δ. However, to obtain the bound in terms of $\widetilde{b}(\boldsymbol{y}-\boldsymbol{y'})$ and the bound in terms of w we will need to use the oscillation of the terms $e^{2\pi i \boldsymbol{\eta}\cdot\left(\frac{\boldsymbol{y}-\boldsymbol{y'}}{\boldsymbol{\mu}}\right)}$ and $e^{-2\pi\tau i w}$ respectively.

4.2.1 The bound in terms of δ

Proposition 4.6. *Let $b : \mathbb{R}^n \to \mathbb{R}$ be a convex polynomial of "combined degree" $(m_1, \ldots, m_n)$. Let $(\boldsymbol{x}, \boldsymbol{y}, t)$ and $(\boldsymbol{x'}, \boldsymbol{y'}, t')$ be any two points in $\partial\Omega_b$. Define*

$$\tilde{b}(\boldsymbol{v}) = b\left(\boldsymbol{v} + \frac{\boldsymbol{x}+\boldsymbol{x'}}{2}\right) - \nabla b\left(\frac{\boldsymbol{x}+\boldsymbol{x'}}{2}\right)\cdot\boldsymbol{v} - b\left(\frac{\boldsymbol{x}+\boldsymbol{x'}}{2}\right);$$

and

$$\delta(\boldsymbol{x}, \boldsymbol{x'}) = b(\boldsymbol{x}) + b(\boldsymbol{x'}) - 2b\left(\frac{\boldsymbol{x}+\boldsymbol{x'}}{2}\right). \tag{4.6}$$

Then,

$$|S\left((\boldsymbol{x}, \boldsymbol{y}, t); (\boldsymbol{x'}, \boldsymbol{y'}, t')\right)| \lesssim \frac{1}{\delta |\{\boldsymbol{v} : \tilde{b}(\boldsymbol{v}) < \delta\}|^2},$$

where the constant may depend on the "combined degree" of b and the dimension of the space, but is independent of the two given points.

Remark 4.7. *Notice that since b is convex, $\delta(\boldsymbol{x}, \boldsymbol{x'}) \geq 0$.*

Proof. It follows from equation (4.5) that

$$|S| \leq \int_0^\infty \frac{e^{-2\pi\tau\delta}}{\mu_1^2 \cdots \mu_n^2} \int_{\mathbb{R}^n} \frac{1}{\int_{\mathbb{R}^n} e^{4\pi[\boldsymbol{\eta}\cdot\boldsymbol{v} - \tau\tilde{b}(\mu\boldsymbol{v})]}\, d\boldsymbol{v}}\, d\boldsymbol{\eta}\, d\tau.$$

But

$$\mu_1 \cdots \mu_n \approx \left| \left\{ \boldsymbol{v} \,:\, \tilde{b}(\boldsymbol{v}) \leq \frac{1}{\tau} \right\} \right|.$$

Thus, it follows that

$$|S| \lesssim \int_0^\infty \frac{e^{-2\pi\tau\delta}}{\left|\left\{ \boldsymbol{v} \,:\, \tilde{b}(\boldsymbol{v}) \leq \frac{1}{\tau} \right\}\right|^2} \int_{\mathbb{R}^n} \frac{1}{\int\limits_{\mathbb{R}^n} e^{4\pi[\boldsymbol{\eta}\cdot\boldsymbol{v} - g(\boldsymbol{v})]}\, d\boldsymbol{v}}\, d\boldsymbol{\eta}\, d\tau. \tag{4.7}$$

We showed in Lemma 3.1 that

$$\theta(\boldsymbol{\eta}) = \left[\int\limits_{\mathbb{R}^n} e^{4\pi[\boldsymbol{\eta}\cdot\boldsymbol{v} - g(\boldsymbol{v})]}\, d\boldsymbol{v} \right]^{-1}$$

decays at an exponential rate, where the decay is independent of the coefficients of g. In particular, the decay does not depend on τ. Thus, $\int_{\mathbb{R}^n} \theta(\boldsymbol{\eta})\, d\boldsymbol{\eta}$ converges. Hence,

$$|S| \lesssim \int_0^\infty \frac{e^{-2\pi\tau\delta}}{\left|\left\{ \boldsymbol{v} \,:\, \tilde{b}(\boldsymbol{v}) \leq \frac{1}{\tau} \right\}\right|^2}\, d\tau.$$

We can write

$$\int_0^\infty \frac{e^{-2\pi\tau\delta}}{\left|\left\{ \boldsymbol{v} \,:\, \tilde{b}(\boldsymbol{v}) \leq \frac{1}{\tau} \right\}\right|^2}\, d\tau = \sum_{j=-\infty}^{\infty} \int_{\tau\delta=2^j}^{\tau\delta=2^{j+1}} \frac{e^{-2\pi\tau\delta}}{\left|\left\{ \boldsymbol{v} \,:\, \tilde{b}(\boldsymbol{v}) \leq \frac{1}{\tau} \right\}\right|^2}\, d\tau.$$

Thus,

$$|S| \lesssim \sum_{j=-\infty}^{\infty} \frac{e^{-2\pi 2^j}}{\left|\left\{\boldsymbol{v} \,:\, \tilde{b}(\boldsymbol{v}) \leq \frac{\delta}{2^{j+1}}\right\}\right|^2} \int_{\tau=2^j\delta^{-1}}^{\tau=2^{j+1}\delta^{-1}} d\tau = \sum_{j=-\infty}^{\infty} \frac{2^j e^{-2\pi 2^j}}{\delta \left|\left\{\boldsymbol{v} \,:\, \tilde{b}(\boldsymbol{v}) \leq \frac{\delta}{2^{j+1}}\right\}\right|^2}.$$

In order to get rid of the dependence on j of $\left|\left\{\boldsymbol{v} \,:\, \tilde{b}(\boldsymbol{v}) \leq \frac{\delta}{2^{j+1}}\right\}\right|$ we can write

$$|S| \lesssim \sum_{-\infty \leq j < 0} \frac{e^{-2\pi 2^j} 2^j}{\delta |\{\boldsymbol{v} \,:\, \tilde{b}(\boldsymbol{v}) \leq \delta\}|^2} + \sum_{0 \leq j \leq \infty} \frac{e^{-2\pi 2^j} 2^j}{\delta \left|\left\{\boldsymbol{v} \,:\, \tilde{b}(\boldsymbol{v}) \leq \frac{\delta}{2^{j+1}}\right\}\right|^2}. \tag{4.8}$$

But by Claim 1.5 on page 16, for $j \geq 0$ we have that

$$|\{\boldsymbol{v} \,:\, \tilde{b}(\boldsymbol{v}) \leq \delta\}| \leq 2^{n(j+1)} \left|\left\{\boldsymbol{v} \,:\, \tilde{b}(\boldsymbol{v}) \leq \frac{\delta}{2^{j+1}}\right\}\right|.$$

Thus, it follows that

$$|S| \lesssim \frac{1}{\delta |\{\boldsymbol{v} \,:\, \tilde{b}(\boldsymbol{v}) \leq \delta\}|^2} \left(\sum_{-\infty \leq j < 0} e^{-2^{j+1}\pi} 2^j + \sum_{0 \leq j \leq \infty} e^{-2^{j+1}\pi} 2^{2n(j+1)+j} \right).$$

Since both sums converge, we obtain the desired estimate. That is,

$$|S| \lesssim \frac{1}{\delta |\{\boldsymbol{v} \,:\, \tilde{b}(\boldsymbol{v}) \leq \delta\}|^2}.$$

This finishes the proof Proposition 4.6.

□

4.2.2 The bound in terms of $\tilde{b}(\boldsymbol{y} - \boldsymbol{y'})$

Proposition 4.8. *Let $b : \mathbb{R}^n \to \mathbb{R}$ be a convex polynomial of "combined degree" $(m_1, \ldots, m_n)$. Let $(\boldsymbol{x}, \boldsymbol{y}, t)$ and $(\boldsymbol{x'}, \boldsymbol{y'}, t')$ be any two points in $\partial\Omega_b$. Define*

$$\tilde{b}(\boldsymbol{v}) = b\left(\boldsymbol{v} + \frac{\boldsymbol{x}+\boldsymbol{x'}}{2}\right) - \nabla b\left(\frac{\boldsymbol{x}+\boldsymbol{x'}}{2}\right)\cdot\boldsymbol{v} - b\left(\frac{\boldsymbol{x}+\boldsymbol{x'}}{2}\right).$$

Then

$$|S\left((\boldsymbol{x},\boldsymbol{y},t);(\boldsymbol{x'},\boldsymbol{y'},t')\right)| \lesssim \frac{1}{\tilde{b}(\boldsymbol{y}-\boldsymbol{y'})|\{\boldsymbol{v} : \tilde{b}(\boldsymbol{v}) < \tilde{b}(\boldsymbol{y}-\boldsymbol{y'})\}|^2},$$

where the constant may depend on the "combined degree" of b *and the dimension of the space, but is independent of the two given points.*

Proof. We had shown in equation (4.5) on page 71 that

$$S = \int_0^\infty \frac{e^{-2\pi\tau\delta}e^{-2\pi\tau iw}}{\mu_1^2\cdots\mu_n^2}\int_{\mathbb{R}^n}\frac{e^{2\pi i\boldsymbol{\eta}\cdot\left(\frac{\boldsymbol{y}-\boldsymbol{y'}}{\boldsymbol{\mu}}\right)}}{\int\limits_{\mathbb{R}^n} e^{4\pi[\boldsymbol{\eta}\cdot\boldsymbol{v}-\tau\tilde{b}(\boldsymbol{\mu}\boldsymbol{v})]}\,d\boldsymbol{v}}\,d\boldsymbol{\eta}\,d\tau.$$

Thus, and since $\delta \geq 0$,

$$|S| \leq \int_0^\infty \frac{1}{\mu_1^2\cdots\mu_n^2}\left|\int_{\mathbb{R}^n}\frac{e^{2\pi i\boldsymbol{\eta}\cdot\left(\frac{\boldsymbol{y}-\boldsymbol{y'}}{\boldsymbol{\mu}}\right)}}{\int_{\mathbb{R}^n} e^{4\pi[\boldsymbol{\eta}\cdot\boldsymbol{v}-\tau\tilde{b}(\boldsymbol{\mu}\boldsymbol{v})]}\,d\boldsymbol{v}}\,d\boldsymbol{\eta}\right|\,d\tau.$$

But by Lemma 3.1,

$$\theta(\boldsymbol{\eta}) = \left[\int_{\mathbb{R}^n} e^{4\pi[\boldsymbol{\eta}\cdot\boldsymbol{v}-\tau\tilde{b}(\boldsymbol{\mu}\boldsymbol{v})]}\,d\boldsymbol{v}\right]^{-1}$$

is Schwartz. Moreover, its decay is independent of τ and the coefficients of b. The same is true of its Fourier transform, $\hat{\theta}$. We can write

$$|S| \leq \int_0^\infty \frac{1}{\mu_1^2\cdots\mu_n^2}\left|\hat{\theta}\left(\frac{\boldsymbol{y}-\boldsymbol{y'}}{\boldsymbol{\mu}}\right)\right|\,d\tau.$$

Recall that the factors μ_j were chosen so that

$$\mu_1 \cdots \mu_n \approx \left| \left\{ \boldsymbol{v} \,:\, \widetilde{b}(\boldsymbol{v}) \leq \frac{1}{\tau} \right\} \right|.$$

Hence,

$$|S| \lesssim \int_0^\infty \frac{1}{\left| \left\{ \boldsymbol{v} \,:\, \widetilde{b}(\boldsymbol{v}) \leq \frac{1}{\tau} \right\} \right|^2} \left| \hat{\theta} \left(\frac{\boldsymbol{y} - \boldsymbol{y'}}{\boldsymbol{\mu}} \right) \right| d\tau.$$

Let $\boldsymbol{u} = \boldsymbol{y} - \boldsymbol{y'}$. We can split the integral into integrals defined over smaller intervals in the following way:

$$\int_0^\infty \frac{1}{\left| \left\{ \boldsymbol{v} \,:\, \widetilde{b}(\boldsymbol{v}) \leq \frac{1}{\tau} \right\} \right|^2} \left| \hat{\theta} \left(\frac{\boldsymbol{y} - \boldsymbol{y'}}{\boldsymbol{\mu}} \right) \right| d\tau = \sum_{j=-\infty}^{\infty} \int_{\frac{2^j}{\widetilde{b}(\boldsymbol{u})}}^{\frac{2^{j+1}}{\widetilde{b}(\boldsymbol{u})}} \frac{1}{\left| \left\{ \boldsymbol{v} \,:\, \widetilde{b}(\boldsymbol{v}) \leq \frac{1}{\tau} \right\} \right|^2} \left| \hat{\theta} \left(\frac{\boldsymbol{u}}{\boldsymbol{\mu}} \right) \right| d\tau.$$

But if

$$\frac{\widetilde{b}(\boldsymbol{u})}{2^{j+1}} \leq \frac{1}{\tau} \leq \frac{\widetilde{b}(\boldsymbol{u})}{2^j},$$

then

$$\left| \left\{ \boldsymbol{v} \,:\, \widetilde{b}(\boldsymbol{v}) \leq \frac{\widetilde{b}(\boldsymbol{u})}{2^{j+1}} \right\} \right| \leq \left| \left\{ \boldsymbol{v} \,:\, \widetilde{b}(\boldsymbol{v}) \leq \frac{1}{\tau} \right\} \right|.$$

It follows that

$$|S| \lesssim \sum_{j=-\infty}^{\infty} \int_{\frac{2^j}{\widetilde{b}(\boldsymbol{u})}}^{\frac{2^{j+1}}{\widetilde{b}(\boldsymbol{u})}} \frac{1}{\left| \left\{ \boldsymbol{v} \,:\, \widetilde{b}(\boldsymbol{v}) \leq \frac{\widetilde{b}(\boldsymbol{u})}{2^{j+1}} \right\} \right|^2} \left| \hat{\theta} \left(\frac{\boldsymbol{u}}{\boldsymbol{\mu}} \right) \right| d\tau.$$

Splitting the sum for positive and negative values of j, we can write

$$|S| \lesssim \sum_{-\infty \leq j < 0} \int_{\frac{2^j}{\widetilde{b}(\boldsymbol{u})}}^{\frac{2^{j+1}}{\widetilde{b}(\boldsymbol{u})}} \frac{1}{\left|\left\{\boldsymbol{v} \,:\, \widetilde{b}(\boldsymbol{v}) \leq \frac{\widetilde{b}(\boldsymbol{u})}{2^{j+1}}\right\}\right|^2} \left|\hat{\theta}\left(\frac{\boldsymbol{u}}{\boldsymbol{\mu}}\right)\right| d\tau$$

$$+ \sum_{0 \leq j \leq \infty} \int_{\frac{2^j}{\widetilde{b}(\boldsymbol{u})}}^{\frac{2^{j+1}}{\widetilde{b}(\boldsymbol{u})}} \frac{1}{\left|\left\{\boldsymbol{v} \,:\, \widetilde{b}(\boldsymbol{v}) \leq \frac{\widetilde{b}(\boldsymbol{u})}{2^{j+1}}\right\}\right|^2} \left|\hat{\theta}\left(\frac{\boldsymbol{u}}{\boldsymbol{\mu}}\right)\right| d\tau.$$

But for $j < 0$ we have that $\widetilde{b}(\boldsymbol{u}) \leq \frac{\widetilde{b}(\boldsymbol{u})}{2^{j+1}}$, so

$$\left|\left\{\boldsymbol{v} \,:\, \widetilde{b}(\boldsymbol{v}) \leq \frac{\widetilde{b}(\boldsymbol{u})}{2^{j+1}}\right\}\right|^{-1} \leq \left|\left\{\boldsymbol{v} \,:\, \widetilde{b}(\boldsymbol{v}) \leq \widetilde{b}(\boldsymbol{u})\right\}\right|^{-1}.$$

On the other hand, for $j \geq 0$, we can use Claim 1.5 on page 16 to show that

$$\left|\left\{\boldsymbol{v} \,:\, \widetilde{b}(\boldsymbol{v}) \leq \frac{\widetilde{b}(\boldsymbol{u})}{2^{j+1}}\right\}\right|^{-1} \leq 2^{n(j+1)} \left|\left\{\boldsymbol{v} \,:\, \widetilde{b}(\boldsymbol{v}) \leq \widetilde{b}(\boldsymbol{u})\right\}\right|^{-1}.$$

Hence,

$$|S| \lesssim \sum_{-\infty \leq j < 0} \frac{1}{\left|\left\{\boldsymbol{v} \,:\, \widetilde{b}(\boldsymbol{v}) \leq \widetilde{b}(\boldsymbol{u})\right\}\right|^2} \int_{\frac{2^j}{\widetilde{b}(\boldsymbol{u})}}^{\frac{2^{j+1}}{\widetilde{b}(\boldsymbol{u})}} \left|\hat{\theta}\left(\frac{\boldsymbol{u}}{\boldsymbol{\mu}}\right)\right| d\tau$$

$$+ \sum_{0 \leq j \leq \infty} \frac{2^{2n(j+1)}}{\left|\left\{\boldsymbol{v} \,:\, \widetilde{b}(\boldsymbol{v}) \leq \widetilde{b}(\boldsymbol{u})\right\}\right|^2} \int_{\frac{2^j}{\widetilde{b}(\boldsymbol{u})}}^{\frac{2^{j+1}}{\widetilde{b}(\boldsymbol{u})}} \left|\hat{\theta}\left(\frac{\boldsymbol{u}}{\boldsymbol{\mu}}\right)\right| d\tau.$$

The term 2^j that will com from the bounds of integration will be enough to obtain the convergence of the first sum. Thus, for the first sum it suffices to bound $|\hat{\theta}|$ by a universal constant. It follows that

$$\sum_{-\infty\le j<0} \frac{1}{\left|\left\{\boldsymbol{v}\,:\,\widetilde{b}(\boldsymbol{v})\le \widetilde{b}(\boldsymbol{u})\right\}\right|^2} \int_{\frac{2^j}{\widetilde{b}(\boldsymbol{u})}}^{\frac{2^{j+1}}{\widetilde{b}(\boldsymbol{u})}} \left|\hat{\theta}\left(\frac{\boldsymbol{u}}{\boldsymbol{\mu}}\right)\right| d\tau \lesssim \sum_{-\infty\le j<0} \frac{1}{\left|\left\{\boldsymbol{v}\,:\,\widetilde{b}(\boldsymbol{v})\le \widetilde{b}(\boldsymbol{u})\right\}\right|^2} \int_{\frac{2^j}{\widetilde{b}(\boldsymbol{u})}}^{\frac{2^{j+1}}{\widetilde{b}(\boldsymbol{u})}} d\tau$$
$$= \sum_{-\infty\le j<0} \frac{2^j}{\widetilde{b}(\boldsymbol{u})\left|\left\{\boldsymbol{v}\,:\,\widetilde{b}(\boldsymbol{v})\le \widetilde{b}(\boldsymbol{u})\right\}\right|^2} \approx \frac{1}{\widetilde{b}(\boldsymbol{u})\left|\left\{\boldsymbol{v}\,:\,\widetilde{b}(\boldsymbol{v})\le \widetilde{b}(\boldsymbol{u})\right\}\right|^2}.$$

We would like to obtain a similar bound for the second sum. The main obstacle is obtaining decay in j to counteract the growth of the term $2^{2n(j+1)}$, thus ensuring the convergence of the series. Our goal is to show that

$$\int_{\frac{2^j}{\widetilde{b}(\boldsymbol{u})}}^{\frac{2^{j+1}}{\widetilde{b}(\boldsymbol{u})}} \left|\hat{\theta}\left(\frac{\boldsymbol{u}}{\boldsymbol{\mu}}\right)\right| d\tau \lesssim \frac{1}{2^{Kj}\widetilde{b}(\boldsymbol{u})},$$

for some arbitrarily large constant K. Notice that in this interval,

$$\frac{1}{\tau\widetilde{b}(\boldsymbol{u})} \le \frac{1}{2^j}.$$

Also,

$$\left|\hat{\theta}\left(\frac{\boldsymbol{u}}{\boldsymbol{\mu}}\right)\right| \lesssim \frac{1}{\left|1+\left|p\left(\frac{\boldsymbol{u}}{\boldsymbol{\mu}}\right)\right|\right|^N}$$

for any polynomial p. Thus, it suffices to show that there exists some polynomial p such that $\tau\widetilde{b}(\boldsymbol{u}) \le 1+\left|p\left(\frac{\boldsymbol{u}}{\boldsymbol{\mu}}\right)\right|$. In fact, it would follow that

$$\left|\hat{\theta}\left(\frac{\boldsymbol{u}}{\boldsymbol{\mu}}\right)\right| \lesssim \frac{1}{\left|\tau\widetilde{b}(\boldsymbol{u})\right|^N} \le \frac{1}{2^{Nj}},$$

so that

$$\sum_{j=0}^{\infty} \frac{2^{2n(j+1)}}{\left|\left\{\boldsymbol{v} : \widetilde{b}(\boldsymbol{v}) \leq \widetilde{b}(\boldsymbol{u})\right\}\right|^2} \int_{\frac{2^j}{\widetilde{b}(\boldsymbol{u})}}^{\frac{2^{j+1}}{\widetilde{b}(\boldsymbol{u})}} \left|\hat{\theta}\left(\frac{\boldsymbol{u}}{\boldsymbol{\mu}}\right)\right| d\tau$$
$$\leq \sum_{j=0}^{\infty} \frac{2^j 2^{2n(j+1)}}{2^{Nj}\widetilde{b}(\boldsymbol{u}) \left|\left\{\boldsymbol{v} : \widetilde{b}(\boldsymbol{v}) \leq \widetilde{b}(\boldsymbol{u})\right\}\right|^2}.$$

For sufficiently large N, the series converges, and we would obtain the desired estimate.

We will now find a polynomial p such that $\tau\widetilde{b}(\boldsymbol{u}) \leq 1 + \left|p\left(\frac{\boldsymbol{u}}{\boldsymbol{\mu}}\right)\right|$. Letting $\boldsymbol{s} = \frac{\boldsymbol{u}}{\boldsymbol{\mu}}$, we can write the above requirement as $\tau\widetilde{b}(\boldsymbol{\mu s}) \leq 1 + |p(\boldsymbol{s})|$. Recall that by equation (4.4), there exists a universal constant $A < 1$ such that

$$\{\boldsymbol{v} : |\boldsymbol{v}| \leq A\} \subseteq \{\boldsymbol{v} : \tau\widetilde{b}(\boldsymbol{\mu v}) \leq 1\} \subseteq \{\boldsymbol{v} : |\boldsymbol{v}| \leq 1\}. \tag{4.9}$$

Then the polynomial $g(\boldsymbol{s}) = \tau\widetilde{b}(\boldsymbol{\mu s})$ satisfies all the hypothesis of Lemma 3.1. In particular, Claim 3.3 on page 32 holds. Thus, there exists a universal constant such that

$$\tau\widetilde{b}(\boldsymbol{\mu s}) \leq C(1 + s_1^{2m_1} + \ldots + s_n^{2m_n}).$$

This finishes the proof of Proposition 4.8.

□

4.2.3 The bound in terms of w

Proposition 4.9. *Let $b : \mathbb{R}^n \to \mathbb{R}$ be a convex polynomial of "combined degree"* $(m_1, \ldots, m_n)$. *Let $(\boldsymbol{x}, \boldsymbol{y}, t)$ and $(\boldsymbol{x'}, \boldsymbol{y'}, t')$ be any two points in $\partial\Omega_b$. Define*

$$\tilde{b}(\boldsymbol{v}) = b\left(\boldsymbol{v} + \frac{\boldsymbol{x} + \boldsymbol{x'}}{2}\right) - \nabla b\left(\frac{\boldsymbol{x} + \boldsymbol{x'}}{2}\right) \cdot \boldsymbol{v} - b\left(\frac{\boldsymbol{x} + \boldsymbol{x'}}{2}\right)$$

and

$$w = (t' - t) + \nabla b\left(\frac{\boldsymbol{x} + \boldsymbol{x'}}{2}\right) \cdot (\boldsymbol{y'} - \boldsymbol{y}).$$

Then,

$$|S\left((\boldsymbol{x}, \boldsymbol{y}, t); (\boldsymbol{x'}, \boldsymbol{y'}, t')\right)| \lesssim \frac{1}{|w|\ |\{\boldsymbol{v} : \widetilde{b}(\boldsymbol{v}) < |w|\}|^2},$$

where the constant may depend on the "combined degree" of b *and the dimension of the space, but is independent of the two given points.*

The derivation of this last bound is rather long and technical. Before giving all the technical details, however, we shall begin by briefly outlining the main ideas behind the proof. It follows from equation (4.2) on page 70 that

$$S = \int_0^{\frac{\pi}{|u|}} e^{-iu\tau} F(\tau)\, d\tau + \int_{\frac{\pi}{|u|}}^{\infty} e^{-iu\tau} F(\tau)\, d\tau. \tag{4.10}$$

where $u = 2\pi\left[(t' - t) + \nabla b\left(\frac{\boldsymbol{x}+\boldsymbol{x'}}{2}\right) \cdot (\boldsymbol{y'} - \boldsymbol{y})\right]$ and

$$F(\tau) = e^{-2\pi\tau\delta} \int\limits_{\mathbb{R}^n} \frac{e^{2\pi i \boldsymbol{\eta}\cdot(\boldsymbol{y}-\boldsymbol{y'})}}{\int\limits_{\mathbb{R}^n} e^{4\pi[\boldsymbol{\eta}\cdot\boldsymbol{v} - \tau\tilde{b}(\boldsymbol{v})]}\, d\boldsymbol{v}}\, d\boldsymbol{\eta}. \tag{4.11}$$

The integral for $0 \leq \tau \leq \frac{\pi}{|u|}$ in equation (4.10) yields the desired estimate by using similar techniques as those detailed in the proof of the previous two bounds. Thus, the main difficulty lies in estimating the integral for $\frac{\pi}{|u|} \leq \tau \leq \infty$. In particular, we must show that the integral converges. To do so, we will take advantage of the oscillation of the term $e^{-iu\tau}$. Integrating the latter by parts N times, for an arbitrary positive integer N, we obtain formally an equation of the form

$$\int_{\frac{\pi}{|u|}}^{\infty} \frac{e^{-iu\tau}}{|u|^N} F^{(N)}(\tau)\, d\tau.$$

We then show that after introducing the factors $\boldsymbol{\mu}$ as in the two previous bounds, every derivative of $F(\tau)$ yields a factor of $\frac{1}{\tau}$ times a bounded function, so that

$$\begin{aligned}\left|\int_{\frac{\pi}{|u|}}^{\infty} \frac{e^{-iu\tau}}{|u|^N} F^{(N)}(\tau)\, d\tau\right| &\approx \frac{1}{|u|^N}\int_{\frac{\pi}{|u|}}^{\infty} \frac{1}{\mu_1^2 \cdots \mu_n^2} \cdot \frac{1}{\tau^N}\, d\tau \\ &\approx \frac{1}{|u|^N}\int_{\frac{\pi}{|u|}}^{\infty} \frac{1}{\left|\left\{\boldsymbol{v} \,:\, \widetilde{b}(\boldsymbol{v}) \le \frac{1}{\tau}\right\}\right|^2} \cdot \frac{1}{\tau^N}\, d\tau.\end{aligned}$$

Finally, using Claim 1.5, we show that this last integral is bounded by an expression of the form

$$\frac{1}{|u|^N \left|\left\{\boldsymbol{v} \,:\, \widetilde{b}(\boldsymbol{v}) \le |u|\right\}\right|^2} \int_{\frac{\pi}{|u|}}^{\infty} \frac{|u|^{2n}\tau^{2n}}{\tau^N}\, d\tau,$$

yielding the desired estimate for large enough values of N.

Before presenting a rigorous proof of Proposition 4.9, we discuss three technical results that will be used in the course of the proof. In Claim 4.10 we obtain an upper bound for $\int_0^\infty e^{-iu\tau} F(\tau)\, d\tau$ in terms of the $N+1^{th}$ derivative of F. The method we use is analogous to integration by parts, but does not yield boundary terms, making the computation slightly simpler. In Claim 4.11 we compute the N^{th} derivative of F. In Claim 4.12 we show that, after introducing the factors $\boldsymbol{\mu}$, the N^{th} derivative of F is dominated by $\frac{1}{\tau^N}$ times a bounded function.

Claim 4.10. *Let* $t \in \mathbb{R}$ *and*

$$I(t) = \int_0^\infty e^{-it\tau} F(\tau)\, d\tau,$$

where $F \in C^\infty(\mathbb{R})$. *Then given* $N > 0$ *there exist positive coefficients* $c_1, \ldots, c_{N+1}$ *such that*

$$|I(t)| \le \sum_{j=0}^{N+1} c_j \left| \int_0^{\frac{\pi}{|t|}} e^{-it\tau} F\left(\tau + \frac{j\pi}{|t|}\right) d\tau \right| + c_{N+1} \left| \int_{\frac{\pi}{|t|}}^\infty e^{-it\tau} \int_0^{\frac{\pi}{|t|}} \cdots \int_0^{\frac{\pi}{|t|}} F^{(N+1)}(\tau + s_1 + \ldots + s_{N+1})\, ds_1 \cdots ds_{N+1}\, d\tau \right|. \tag{4.12}$$

Proof. We can write

$$I(t) = \int_0^{\frac{\pi}{|t|}} e^{-it\tau} F(\tau)\, d\tau + \int_{\frac{\pi}{|t|}}^\infty e^{-it\tau} F(\tau)\, d\tau = S + L. \tag{4.13}$$

Introducing a factor of $e^{i\,\mathrm{sgn}(t)\pi}$, we can split L as follows:

$$\begin{aligned} L &= \frac{1}{2}\left[\int_{\frac{\pi}{|t|}}^\infty e^{-it\tau} F(\tau)\, d\tau - \int_{\frac{\pi}{|t|}}^\infty e^{i\,\mathrm{sgn}(t)\pi} e^{-it\tau} F(\tau)\, d\tau\right] \\ &= \frac{1}{2}\left[\int_{\frac{\pi}{|t|}}^\infty e^{-it\tau} F(\tau)\, d\tau - \int_{\frac{\pi}{|t|}}^\infty e^{-it\left(\tau - \frac{\pi}{|t|}\right)} F(\tau)\, d\tau\right] \\ &= \frac{1}{2}\left[\int_{\frac{\pi}{|t|}}^\infty e^{-it\tau} F(\tau)\, d\tau - \int_0^\infty e^{-it\tau} F\left(\tau + \frac{\pi}{|t|}\right) d\tau\right]. \end{aligned}$$

Writing $F\left(\tau + \frac{\pi}{|t|}\right) = \left[F\left(\tau + \frac{\pi}{|t|}\right) - F(\tau)\right] + F(\tau)$, we have that

$$\begin{aligned} L &= \frac{1}{2}\left(\int_{\frac{\pi}{|t|}}^\infty e^{-it\tau} F(\tau)\, d\tau - \int_0^\infty e^{-it\tau} \left[F\left(\tau + \frac{\pi}{|t|}\right) - F(\tau)\right] d\tau - \int_0^\infty e^{-it\tau} F(\tau)\, d\tau\right) \\ &= \frac{1}{2}\left(-\int_0^{\frac{\pi}{|t|}} e^{-it\tau} F(\tau)\, d\tau - \int_0^\infty e^{-it\tau} \left[F\left(\tau + \frac{\pi}{|t|}\right) - F(\tau)\right] d\tau\right). \end{aligned}$$

Using this last expression in equation (4.13), it follows that

$$I(t) = \frac{1}{2}\left(\int_0^{\frac{\pi}{|t|}} e^{-it\tau} F(\tau)\, d\tau - \int_0^{\infty} e^{-it\tau}\left[F\left(\tau + \frac{\pi}{|t|}\right) - F(\tau)\right] d\tau\right).$$

Now let $F_1(\tau) = F\left(\tau + \frac{\pi}{|t|}\right) - F(\tau)$ and let

$$I_1(t) = \int_0^{\infty} e^{-it\tau}\left[F\left(\tau + \frac{\pi}{|t|}\right) - F(\tau)\right] d\tau = \int_0^{\infty} e^{-it\tau} F_1(\tau)\, d\tau.$$

Then, by the same argument, it follows that

$$I_1(t) = \frac{1}{2}\left(\int_0^{\frac{\pi}{|t|}} e^{-it\tau} F_1(\tau)\, d\tau - \int_0^{\infty} e^{-it\tau}\left[F_1\left(\tau + \frac{\pi}{|t|}\right) - F_1(\tau)\right] d\tau\right).$$

After N times of repeating this process, we would have that

$$\begin{aligned} I_N(t) &= \int_0^{\infty} e^{-it\tau} F_N(\tau)\, d\tau \\ &= \frac{1}{2}\left(\int_0^{\frac{\pi}{|t|}} e^{-it\tau} F_N(\tau)\, d\tau - \int_0^{\infty} e^{-it\tau}\left[F_N\left(\tau + \frac{\pi}{|t|}\right) - F_N(\tau)\right] d\tau\right), \end{aligned}$$

where $F_N(\tau) = F_{N-1}\left(\tau + \frac{\pi}{|t|}\right) - F_{N-1}(\tau)$.

Letting

$$S_j(t) = \int_0^{\frac{\pi}{|t|}} e^{-it\tau} F_j(\tau)\, d\tau$$

for $1 \leq j \leq N-1$, it follows that

$$\begin{aligned} I(t) &= \frac{1}{2}\left[S(t) - I_1(t)\right] \\ &= \frac{1}{2}\left[S(t) - \frac{1}{2}\left[S_1(t) - I_2(t)\right]\right] \\ &\quad\vdots \\ &= \frac{1}{2}\left[S(t) - \frac{1}{2}\left[S_1(t) - \frac{1}{2}\left[S_2(t) \cdots - \frac{1}{2}\left[S_{N-1}(t) - I_N(t)\right]\right]\right]\right]. \end{aligned}$$

That is,

$$\begin{aligned} I(t) &= \frac{1}{2}S(t) - \frac{1}{2^2}S_1(t) + \frac{1}{2^3}S_2(t) + \ldots + \frac{(-1)^{N-1}}{2^N}S_{N-1}(t) + \frac{(-1)^N}{2^N}I_N(t) \\ &= \frac{1}{2}S(t) + \sum_{k=1}^{N}\frac{(-1)^k}{2^{k+1}}\int_0^{\frac{\pi}{|t|}} e^{-it\tau}F_k(\tau)\,d\tau + \frac{(-1)^{N+1}}{2^{N+1}}\int_0^{\infty} e^{-it\tau}F_{N+1}(\tau)\,d\tau. \end{aligned}$$

Notice that after expanding and rearranging terms, we can write for $1 \leq k \leq N$

$$F_k(\tau) = \sum_{j=0}^{k-1}(-1)^{k+j+1}\binom{k-1}{j}\left[F\left(\tau + \frac{(j+1)\pi}{|t|}\right) - F\left(\tau + \frac{j\pi}{|t|}\right)\right].$$

It follows that

$$\begin{aligned} I(t) = \frac{1}{2}S(t) &+ \sum_{k=1}^{N}\sum_{j=0}^{k-1}\binom{k-1}{j}\frac{(-1)^{j+1}}{2^{k+1}}\int_0^{\frac{\pi}{|t|}} e^{-it\tau}\left[F\left(\tau + \frac{(j+1)\pi}{|t|}\right) - F\left(\tau + \frac{j\pi}{|t|}\right)\right]d\tau \\ &+ \sum_{j=0}^{N}\frac{(-1)^{j+1}}{2^{N+1}}\binom{N}{j}\int_0^{\frac{\pi}{|t|}} e^{-it\tau}\left[F\left(\tau + \frac{(j+1)\pi}{|t|}\right) - F\left(\tau + \frac{j\pi}{|t|}\right)\right]d\tau \\ &+ \frac{(-1)^{N+1}}{2^{N+1}}\int_{\frac{\pi}{|t|}}^{\infty} e^{-it\tau}F_{N+1}(\tau)\,d\tau. \end{aligned}$$

Changing the order of summation, it follows that

$$\begin{aligned}
I(t) = \frac{1}{2}S(t) &+ \sum_{j=0}^{N-1}\sum_{k=j+1}^{N}\binom{k-1}{j}\frac{(-1)^{j+1}}{2^{k+1}}\int_0^{\frac{\pi}{|t|}} e^{-it\tau}F\left(\tau+\frac{(j+1)\pi}{|t|}\right)\,d\tau \\
&+ \sum_{j=0}^{N-1}\sum_{k=j+1}^{N}\binom{k-1}{j}\frac{(-1)^{j}}{2^{k+1}}\int_0^{\frac{\pi}{|t|}} e^{-it\tau}F\left(\tau+\frac{j\pi}{|t|}\right)\,d\tau \\
&+ \sum_{j=0}^{N}\frac{(-1)^{j+1}}{2^{N+1}}\binom{N}{j}\int_0^{\frac{\pi}{|t|}} e^{-it\tau}F\left(\tau+\frac{(j+1)\pi}{|t|}\right)\,d\tau \\
&+ \sum_{j=0}^{N}\frac{(-1)^{j}}{2^{N+1}}\binom{N}{j}\int_0^{\frac{\pi}{|t|}} e^{-it\tau}F\left(\tau+\frac{j\pi}{|t|}\right)\,d\tau + \frac{(-1)^{N+1}}{2^{N+1}}\int_{\frac{\pi}{|t|}}^{\infty} e^{-it\tau}F_{N+1}(\tau)\,d\tau.
\end{aligned}$$

That is,

$$\begin{aligned}
I(t) = \frac{1}{2}S(t) &+ \sum_{j=1}^{N}\sum_{k=j}^{N}\binom{k-1}{j-1}\frac{(-1)^{j}}{2^{k+1}}\int_0^{\frac{\pi}{|t|}} e^{-it\tau}F\left(\tau+\frac{j\pi}{|t|}\right)\,d\tau \\
&+ \sum_{j=0}^{N-1}\sum_{k=j+1}^{N}\binom{k-1}{j}\frac{(-1)^{j}}{2^{k+1}}\int_0^{\frac{\pi}{|t|}} e^{-it\tau}F\left(\tau+\frac{j\pi}{t}\right)\,d\tau \\
&+ \sum_{j=1}^{N+1}\frac{(-1)^{j}}{2^{N+1}}\binom{N}{j-1}\int_0^{\frac{\pi}{|t|}} e^{-it\tau}F\left(\tau+\frac{j\pi}{|t|}\right)\,d\tau \\
&+ \sum_{j=0}^{N}\frac{(-1)^{j}}{2^{N+1}}\binom{N}{j}\int_0^{\frac{\pi}{|t|}} e^{-it\tau}F\left(\tau+\frac{j\pi}{|t|}\right)\,d\tau + \frac{(-1)^{N+1}}{2^{N+1}}\int_{\frac{\pi}{|t|}}^{\infty} e^{-it\tau}F_{N+1}(\tau)\,d\tau.
\end{aligned}$$

Let

$c_0 = \frac{1}{2} + \sum_{k=1}^{N}\frac{1}{2^{k+1}} + \frac{1}{2^{N+1}}$;

$c_j = \sum_{k=j}^{N}\binom{k-1}{j-1}\frac{1}{2^{k+1}} + \sum_{k=j+1}^{N}\binom{k-1}{j}\frac{1}{2^{k+1}} + \binom{N}{j-1}\frac{1}{2^{N+1}} + \binom{N}{j}\frac{1}{2^{N+1}}$ for $1 \le j \le N-1$;

$c_N = \frac{N+2}{2^{N+1}}$; and

$c_{N+1} = \frac{1}{2^{N+1}}$.

Then it follows that

$$|I(t)| \leq \sum_{j=0}^{N+1} c_j \left| \int_0^{\frac{\pi}{|t|}} e^{-it\tau} F\left(\tau + \frac{j\pi}{|t|}\right) d\tau \right| + \frac{1}{2^{N+1}} \left| \int_0^{\infty} e^{-it\tau} F_{N+1}(\tau)\, d\tau \right|. \tag{4.14}$$

It suffices now to show that

$$F_{N+1}(\tau) = \int_0^{\frac{\pi}{|t|}} \cdots \int_0^{\frac{\pi}{|t|}} F^{(N+1)}(\tau + s_1 + \ldots + s_{N+1})\, ds_1 \cdots ds_{N+1},$$

where

$$\begin{aligned} F_{N+1}(\tau) &= \sum_{j=0}^{N} (-1)^{N+j} \binom{N}{j} \left[F\left(\tau + \frac{(j+1)\pi}{|t|}\right) - F\left(\tau + \frac{j\pi}{|t|}\right) \right] \\ &= \sum_{j=0}^{N} (-1)^{N+j} \binom{N}{j} \int_0^{\frac{\pi}{|t|}} F'\left(\tau + s + \frac{j\pi}{|t|}\right) ds. \end{aligned} \tag{4.15}$$

Using the identity

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1},$$

it follows that for any integer $M > 0$ and for any function h,

$$\begin{aligned} &\sum_{j=0}^{M} (-1)^{M+j} \binom{M}{j} h(j) \\ &= h(M) + (-1)^M h(0) + \sum_{j=1}^{M-1} (-1)^{M+j} \left(\binom{M-1}{j} + \binom{M-1}{j-1} \right) h(j) \\ &= h(M) - h(M-1) + \sum_{j=1}^{M-2} (-1)^{M+j} \binom{M-1}{j} h(j) \\ &+ \sum_{j=1}^{M-2} (-1)^{M+j+1} \binom{M-1}{j} h(j+1) + (-1)^{M+1} h(1) + (-1)^M h(0). \end{aligned}$$

That is,

$$\sum_{j=0}^{M}(-1)^{M+j}\binom{M}{j}h(j) = \sum_{j=0}^{M-1}(-1)^{M+j+1}\binom{M-1}{j}\left[h(j+1)-h(j)\right]. \tag{4.16}$$

Let $h(j) = \int_0^{\frac{\pi}{|t|}} F'\left(\tau + s + \frac{j\pi}{|t|}\right) ds$ and $M = N$. Notice that we can write

$$\begin{aligned} h(j+1)-h(j) &= \int_0^{\frac{\pi}{|t|}} F'\left(\tau + s + \frac{j\pi}{|t|} + \frac{\pi}{|t|}\right) - F'\left(\tau + s + \frac{j\pi}{|t|}\right) ds \\ &= \int_0^{\frac{\pi}{|t|}}\int_0^{\frac{\pi}{|t|}} F''\left(\tau + s_1 + s_2 + \frac{j\pi}{|t|}\right) ds_1\, ds_2. \end{aligned} \tag{4.17}$$

It follows from equations (3.8), (4.16) and (4.17) that

$$F_{N+1}(\tau) = \sum_{j=0}^{N-1}(-1)^{N+j+1}\binom{N-1}{j}\int_0^{\frac{\pi}{|t|}}\int_0^{\frac{\pi}{|t|}} F''\left(\tau + s_1 + s_2 + \frac{j\pi}{|t|}\right) ds_1\, ds_2.$$

Now let $h(j) = \int_0^{\frac{\pi}{|t|}}\int_0^{\frac{\pi}{|t|}} F''\left(\tau + s_1 + s_2 + \frac{j\pi}{|t|}\right) ds_1\, ds_2$ and $M = N-1$, and use equation (4.16) once more. Since

$$h(j+1)-h(j) = \int_0^{\frac{\pi}{|t|}}\int_0^{\frac{\pi}{|t|}}\int_0^{\frac{\pi}{|t|}} F'''\left(\tau + s_1 + s_2 + s_3 + \frac{j\pi}{|t|}\right) ds_1\, ds_2\, ds_3$$

it follows that

$$F_{N+1}(\tau) = -\sum_{j=0}^{N-2}(-1)^{N+j}\binom{N-2}{j}\int_0^{\frac{\pi}{|t|}}\int_0^{\frac{\pi}{|t|}}\int_0^{\frac{\pi}{|t|}} F'''\left(\tau + s_1 + s_2 + s_3 + \frac{j\pi}{|t|}\right) ds_1\, ds_2\, ds_3.$$

Repeating this process $N-2$ more times, we obtain

$$F_{N+1}(\tau) = \int_0^{\frac{\pi}{|t|}} \cdots \int_0^{\frac{\pi}{|t|}} F^{(N+1)}\left(\tau + s_1 + \ldots + s_{N+1}\right) ds_1 \cdots ds_{N+1}.$$

Thus, it follows from equation (4.14) that

$$|I(t)| \leq \sum_{j=0}^{N+1} c_j \left| \int_0^{\frac{\pi}{|t|}} e^{-it\tau} F\left(\tau + \frac{j\pi}{|t|}\right) d\tau \right|$$
$$+ \frac{1}{2^{N+1}} \left| \int_0^{\infty} e^{-it\tau} \int_0^{\frac{\pi}{|t|}} \cdots \int_0^{\frac{\pi}{|t|}} F^{(N+1)} \left(\tau + s_1 + \ldots + s_{N+1}\right) ds_1 \cdots ds_{N+1}(\tau)\, d\tau \right|.$$

This finishes the proof of Claim 4.10.

□

Claim 4.11. *The N^{th} derivative of*

$$F(\tau) = e^{-2\pi\tau\delta} \int_{\mathbb{R}^n} \frac{e^{2\pi i \boldsymbol{\eta}\cdot(\boldsymbol{y}-\boldsymbol{y'})}}{\int_{\mathbb{R}^n} e^{4\pi[\boldsymbol{\eta}\cdot\boldsymbol{v} - \tau\tilde{b}(\boldsymbol{v})]}\, d\boldsymbol{v}}\, d\boldsymbol{\eta} \tag{4.18}$$

consists of sums of terms of the form

$$\frac{C(\tau\delta)^{N-k} e^{-2\pi\tau\delta}}{\tau^N} \int_{\mathbb{R}^n} \frac{e^{2\pi i \boldsymbol{\eta}\cdot(\boldsymbol{y}-\boldsymbol{y'})} f_1(\tau) \cdots f_k(\tau)}{\gamma(\tau)}\, d\boldsymbol{\eta},$$

where

$$f_s(\tau) = \left[\int_{\mathbb{R}^n} (\tau\tilde{b}(\boldsymbol{v}))^s e^{4\pi[\boldsymbol{\eta}\cdot\boldsymbol{v} - \tau\tilde{b}(\boldsymbol{v})]}\, d\boldsymbol{v} \right]^{a_s};$$

$$\gamma(\tau) = \left[\int_{\mathbb{R}^n} e^{4\pi[\boldsymbol{\eta}\cdot\boldsymbol{v} - \tau\tilde{b}(\boldsymbol{v})]}\, d\boldsymbol{v} \right]^{d};$$

$a_1, \ldots, a_k, k, d \in \mathbb{N}$; $0 \leq k \leq N$; $a_1 + \ldots + a_k = d - 1$; *and* $a_1 + 2a_2 + \ldots + ka_k = k$.

Proof. We will begin by showing by induction that the k^{th} derivative of

$$J(\tau) = \int_{\mathbb{R}^n} \frac{e^{2\pi i \boldsymbol{\eta}\cdot(\boldsymbol{y}-\boldsymbol{y'})}}{\int\limits_{\mathbb{R}^n} e^{4\pi[\boldsymbol{\eta}\cdot\boldsymbol{v}-\tau\tilde{b}(\boldsymbol{v})]}\, d\boldsymbol{v}}\, d\boldsymbol{\eta}$$

consists of sums of terms of the form

$$C\int_{\mathbb{R}^n} \frac{e^{2\pi i \boldsymbol{\eta}\cdot(\boldsymbol{y}-\boldsymbol{y'})}\left[\int_{\mathbb{R}^n} \widetilde{b}(\boldsymbol{v}) e^{4\pi[\boldsymbol{\eta}\cdot\boldsymbol{v}-\tau\tilde{b}(\boldsymbol{v})]}\, d\boldsymbol{v}\right]^{a_1} \cdots \left[\int_{\mathbb{R}^n} \widetilde{b}(\boldsymbol{v})^k e^{4\pi[\boldsymbol{\eta}\cdot\boldsymbol{v}-\tau\tilde{b}(\boldsymbol{v})]}\, d\boldsymbol{v}\right]^{a_k}}{\left[\int_{\mathbb{R}^n} e^{4\pi[\boldsymbol{\eta}\cdot\boldsymbol{v}-\tau\tilde{b}(\boldsymbol{v})]}\, d\boldsymbol{v}\right]^{d}}\, d\boldsymbol{\eta},$$

where $a_1 + \ldots + a_k = d - 1$; and $a_1 + 2a_2 + \ldots + ka_k = k$.

Notice that

$$J'(t) = 4\pi \int_{\mathbb{R}^n} \frac{e^{2\pi i \boldsymbol{\eta}\cdot(\boldsymbol{y}-\boldsymbol{y'})}\left[\int_{\mathbb{R}^n} \widetilde{b}(\boldsymbol{v}) e^{4\pi[\boldsymbol{\eta}\cdot\boldsymbol{v}-\tau\tilde{b}(\boldsymbol{v})]}\, d\boldsymbol{v}\right]}{\left[\int_{\mathbb{R}^n} e^{4\pi[\boldsymbol{\eta}\cdot\boldsymbol{v}-\tau\tilde{b}(\boldsymbol{v})]}\, d\boldsymbol{v}\right]^2}\, d\boldsymbol{\eta}$$

is of this form.

Suppose $J^{(k)}(\tau)$ is of this form. We will show that $J^{(k+1)}(\tau)$ is of this form. Let

$$g_s(\tau) = \left[\int\limits_{\mathbb{R}^n} \widetilde{b}(\boldsymbol{v})^s e^{4\pi[\boldsymbol{\eta}\cdot\boldsymbol{v}-\tau\tilde{b}(\boldsymbol{v})]}\, d\boldsymbol{v}\right]^{a_s};$$

and

$$\gamma(\tau) = \left[\int\limits_{\mathbb{R}^n} e^{4\pi[\boldsymbol{\eta}\cdot\boldsymbol{v}-\tau\tilde{b}(\boldsymbol{v})]}\, d\boldsymbol{v}\right]^{d}.$$

Then, by the quotient rule, $\frac{d}{d\tau}\left[J^{(k)}(\tau)\right]$ consists of sums of terms of the form

$$C\int_{\mathbb{R}^n} \frac{e^{2\pi i \boldsymbol{\eta}\cdot(\boldsymbol{y}-\boldsymbol{y'})} g_1 \cdots g_{s-1} \frac{d}{d\tau}(g_s) g_{s+1} \cdots g_k}{\gamma}\, d\boldsymbol{\eta} \tag{4.19}$$

or

$$C\int_{\mathbb{R}^n} \frac{e^{2\pi i\boldsymbol{\eta}\cdot(\boldsymbol{y}-\boldsymbol{y}')} g_1\cdots g_k \frac{d}{d\tau}(\gamma)}{\gamma^2}\,d\boldsymbol{\eta}. \tag{4.20}$$

But

$$\frac{d}{d\tau}(g_s) = -4\pi a_s\left[\int_{\mathbb{R}^n}\widetilde{b}(\boldsymbol{v})^s e^{4\pi[\boldsymbol{\eta}\cdot\boldsymbol{v}-\tau\tilde{b}(\boldsymbol{v})]}\,d\boldsymbol{v}\right]^{a_s-1}\left[\int_{\mathbb{R}^n}\widetilde{b}(\boldsymbol{v})^{s+1} e^{4\pi[\boldsymbol{\eta}\cdot\boldsymbol{v}-\tau\tilde{b}(\boldsymbol{v})]}\,d\boldsymbol{v}\right],$$

and

$$\begin{aligned}\frac{d}{d\tau}(\gamma) &= -4\pi d\left[\int_{\mathbb{R}^n} e^{4\pi[\boldsymbol{\eta}\cdot\boldsymbol{v}-\tau\tilde{b}(\boldsymbol{v})]}\,d\boldsymbol{v}\right]^{d-1}\int_{\mathbb{R}^n}\widetilde{b}(\boldsymbol{v})e^{4\pi[\boldsymbol{\eta}\cdot\boldsymbol{v}-\tau\tilde{b}(\boldsymbol{v})]}\,d\boldsymbol{v}\\ &= -4\pi d\gamma^{\frac{d-1}{d}}\int_{\mathbb{R}^n}\widetilde{b}(\boldsymbol{v})e^{4\pi[\boldsymbol{\eta}\cdot\boldsymbol{v}-\tau\tilde{b}(\boldsymbol{v})]}\,d\boldsymbol{v}.\end{aligned}$$

Thus, a generic term of the form given in equation (4.19) is given by

$$C\int_{\mathbb{R}^n}\frac{e^{2\pi i\boldsymbol{\eta}\cdot(\boldsymbol{y}-\boldsymbol{y}')}g_1\cdots g_k}{\gamma}\left[\int_{\mathbb{R}^n}\widetilde{b}(\boldsymbol{v})^s e^{4\pi[\boldsymbol{\eta}\cdot\boldsymbol{v}-\tau\tilde{b}(\boldsymbol{v})]}\,d\boldsymbol{v}\right]^{-1}\left[\int_{\mathbb{R}^n}\widetilde{b}(\boldsymbol{v})^{s+1}e^{4\pi[\boldsymbol{\eta}\cdot\boldsymbol{v}-\tau\tilde{b}(\boldsymbol{v})]}\,d\boldsymbol{v}\right]d\boldsymbol{\eta}. \tag{4.21}$$

Let $h_j = g_j$ for $j \neq s, s+1$; $h_s = \left[\int_{\mathbb{R}^n}\widetilde{b}(\boldsymbol{v})^s e^{4\pi[\boldsymbol{\eta}\cdot\boldsymbol{v}-\tau\tilde{b}(\boldsymbol{v})]}\,d\boldsymbol{v}\right]^{\widetilde{a}_s}$, where $\widetilde{a}_s = a_s - 1$; and $h_{s+1} = \left[\int_{\mathbb{R}^n}\widetilde{b}(\boldsymbol{v})^{s+1}e^{4\pi[\boldsymbol{\eta}\cdot\boldsymbol{v}-\tau\tilde{b}(\boldsymbol{v})]}\,d\boldsymbol{v}\right]^{\widetilde{a}_{s+1}}$, where $\widetilde{a}_{s+1} = a_{s+1}+1$. Then equation (4.21) can be written as

$$C\int_{\mathbb{R}^n}\frac{e^{2\pi i\boldsymbol{\eta}\cdot(\boldsymbol{y}-\boldsymbol{y}')}h_1\cdots h_k}{\gamma}\,d\boldsymbol{\eta}.$$

For this term to have the desired form, the exponents must satisfy $a_1+\ldots+a_{s-1}+\widetilde{a}_s+\widetilde{a}_{s+1}+a_{s+2}+\ldots+a_k = d-1$, and $a_1+\ldots+(s-1)a_{s-1}+s\widetilde{a}_s+(s+1)\widetilde{a}_{s+1}+(s+2)a_{s+2}+\ldots+$

$ka_k = k+1$. The former holds, since by inductive hypothesis $a_1+\ldots+a_{s-1}+\widetilde{a}_s+\widetilde{a}_{s+1}+a_{s+2}+\ldots+a_k = a_1+\ldots+a_s-1+a_s+1+\ldots+a_k = a_1+\ldots+a_k = d-1$. The latter also holds, since by inductive hypothesis, $a_1+\ldots+(s-1)a_{s-1}+s\widetilde{a}_s+(s+1)\widetilde{a}_{s+1}+(s+2)a_{s+2}+\ldots+ka_k = a_1+\ldots+(s-1)a_{s-1}+sa_s-s+(s+1)a_{s+1}+s+1+(s+2)a_{s+2}+\ldots+ka_k = a_1+\ldots+ka_k+1 = k+1$.

In the same way, a generic term of the form given in equation (4.20) is given by

$$C\int_{\mathbb{R}^n} \frac{e^{2\pi i\boldsymbol{\eta}\cdot(\boldsymbol{y}-\boldsymbol{y'})} g_1\cdots g_k \int_{\mathbb{R}^n} \widetilde{b}(\boldsymbol{v}) e^{4\pi[\boldsymbol{\eta}\cdot\boldsymbol{v}-\tau\tilde{b}(\boldsymbol{v})]}\, d\boldsymbol{v}}{\left[\int_{\mathbb{R}^n} e^{4\pi[\boldsymbol{\eta}\cdot\boldsymbol{v}-\tau\tilde{b}(\boldsymbol{v})]}\, d\boldsymbol{v}\right]^{d+1}}\, d\boldsymbol{\eta}. \tag{4.22}$$

Let $h_j = g_j$ for $j \neq 1$, and $h_1 = \left[\int_{\mathbb{R}^n} \widetilde{b}(\boldsymbol{v}) e^{4\pi[\boldsymbol{\eta}\cdot\boldsymbol{v}-\tau\tilde{b}(\boldsymbol{v})]}\, d\boldsymbol{v}\right]^{\widetilde{a}_1}$, where $\widetilde{a}_1 = a_1+1$ so that equation (4.22) can be written as

$$C\int_{\mathbb{R}^n} \frac{e^{2\pi i\boldsymbol{\eta}\cdot(\boldsymbol{y}-\boldsymbol{y'})} h_1\cdots h_k}{\left[\int_{\mathbb{R}^n} e^{4\pi[\boldsymbol{\eta}\cdot\boldsymbol{v}-\tau\tilde{b}(\boldsymbol{v})]}\, d\boldsymbol{v}\right]^{d+1}}\, d\boldsymbol{\eta}.$$

For this term to have the desired form, the exponents must satisfy $\widetilde{a}_1+a_2+\ldots+a_k = d$, and $\widetilde{a}_1+2a_2+\ldots+ka_k = k+1$. The former holds, since by inductive hypothesis, $\widetilde{a}_1+a_2+\ldots+a_k = a_1+\ldots+a_k+1 = (d-1)+1 = d$. The latter also holds, since by inductive hypothesis $\widetilde{a}_1+2a_2+\ldots+ka_k = a_1+2a_2+\ldots+ka_k+1 = k+1$.

It follows that for any $k \in \mathbb{N}$, the k^{th} derivative of $J(\tau)$ consists of sums of terms of the form

$$C\int_{\mathbb{R}^n} \frac{e^{2\pi i\boldsymbol{\eta}\cdot(\boldsymbol{y}-\boldsymbol{y'})}\left[\int_{\mathbb{R}^n} \widetilde{b}(\boldsymbol{v}) e^{4\pi[\boldsymbol{\eta}\cdot\boldsymbol{v}-\tau\tilde{b}(\boldsymbol{v})]}\, d\boldsymbol{v}\right]^{a_1}\cdots\left[\int_{\mathbb{R}^n} \widetilde{b}(\boldsymbol{v})^k e^{4\pi[\boldsymbol{\eta}\cdot\boldsymbol{v}-\tau\tilde{b}(\boldsymbol{v})]}\, d\boldsymbol{v}\right]^{a_k}}{\left[\int_{\mathbb{R}^n} e^{4\pi[\boldsymbol{\eta}\cdot\boldsymbol{v}-\tau\tilde{b}(\boldsymbol{v})]}\, d\boldsymbol{v}\right]^{d}}\, d\boldsymbol{\eta},$$

where $a_1 + \ldots + a_k = d - 1$; and $a_1 + 2a_2 + \ldots + ka_k = k$.

Since $F(\tau) = e^{-2\pi\tau\delta} J(\tau)$, the N^{th} derivative of F is given by

$$F^{(N)}(\tau) = \sum_{k=0}^{N} \binom{N}{k} \left(e^{-2\pi\tau\delta}\right)^{(N-k)} J^{(k)}(\tau).$$

But $\left(e^{-2\pi\tau\delta}\right)^{(N-k)} = C\delta^{N-k} e^{-2\pi\tau\delta}$. Thus, the N^{th} derivative of F consists of sums of multiples of terms of the form

$$\delta^{N-k} e^{-2\pi\tau\delta} \int_{\mathbb{R}^n} \frac{e^{2\pi i \boldsymbol{\eta}\cdot(\boldsymbol{y}-\boldsymbol{y'})} \left[\int_{\mathbb{R}^n} \widetilde{b}(\boldsymbol{v}) e^{4\pi[\boldsymbol{\eta}\cdot\boldsymbol{v} - \tau\tilde{b}(\boldsymbol{v})]}\, d\boldsymbol{v} \right]^{a_1} \cdots \left[\int_{\mathbb{R}^n} \widetilde{b}(\boldsymbol{v})^k e^{4\pi[\boldsymbol{\eta}\cdot\boldsymbol{v} - \tau\tilde{b}(\boldsymbol{v})]}\, d\boldsymbol{v} \right]^{a_k}}{\left[\int_{\mathbb{R}^n} e^{4\pi[\boldsymbol{\eta}\cdot\boldsymbol{v} - \tau\tilde{b}(\boldsymbol{v})]}\, d\boldsymbol{v} \right]^{d}}\, d\boldsymbol{\eta},$$

where $a_1 + \ldots + a_k = d - 1$ and $a_1 + 2a_2 + \ldots + ka_k = k$.

Finally, writing for $1 \le s \le k$,

$$\left[\int_{\mathbb{R}^n} \widetilde{b}(\boldsymbol{v})^s e^{4\pi[\boldsymbol{\eta}\cdot\boldsymbol{v} - \tau\tilde{b}(\boldsymbol{v})]}\, d\boldsymbol{v} \right]^{a_s} = \frac{1}{\tau^{s a_s}} \left[\int_{\mathbb{R}^n} (\tau\widetilde{b}(\boldsymbol{v}))^s e^{4\pi[\boldsymbol{\eta}\cdot\boldsymbol{v} - \tau\tilde{b}(\boldsymbol{v})]}\, d\boldsymbol{v} \right]^{a_s}$$

yields the desired expression. This finishes the proof of Claim 4.11.

□

Claim 4.12. *Let*

$$\Delta^{\boldsymbol{\mu}}_{N+1,k}(\tau) = (\tau\delta)^{N+1-k} e^{-2\pi\tau\delta} \int_{\mathbb{R}^n} \frac{e^{2\pi i \frac{\boldsymbol{\eta}}{\boldsymbol{\mu}}\cdot(\boldsymbol{y}-\boldsymbol{y'})} f_1^{\mu}(\tau) \cdots f_k^{\mu}(\tau)}{\gamma^{\mu}(\tau)}\, d\boldsymbol{\eta}, \tag{4.23}$$

where

$$f_s^{\mu}(\tau) = \left[\int_{\mathbb{R}^n} (\tau\widetilde{b}(\boldsymbol{\mu v}))^s e^{4\pi[\boldsymbol{\eta}\cdot\boldsymbol{v} - \tau\tilde{b}(\boldsymbol{\mu v})]}\, d\boldsymbol{v} \right]^{a_s};$$

$$\gamma^{\mu}(\tau) = \left[\int\limits_{\mathbb{R}^n} e^{4\pi[\boldsymbol{\eta}\cdot\boldsymbol{v} - \tau\tilde{b}(\boldsymbol{\mu}\boldsymbol{v})]} \, d\boldsymbol{v} \right]^d ;$$

$a_1, \ldots, a_k, k, d \in \mathbb{N}$; $0 \leq k \leq N+1$; $a_1 + \ldots + a_k = d-1$; and $a_1 + 2a_2 + \ldots + ka_k = k$.

Then there exists a constant C that depends only on N, k, the "combined degree" of b and the dimension of the space such that

$$|\Delta^{\boldsymbol{\mu}}_{N+1,k}(\tau)| \leq C.$$

Proof. It is easy to check that $(\tau\delta)^{N+1-k} e^{-2\pi\tau\delta}$ is bounded. In fact, for $x > 0$ and $A > 0$, define $h(x) = x^A e^{-x}$. Then $h'(x) = x^{A-1} e^{-x}(A - x)$. It follows that $h(x)$ attains its maximum at $x = A$. That is, $h(x) \leq A^A e^{-A}$. It follows that

$$(\tau\delta)^{N+1-k} e^{-2\pi\tau\delta} \leq \frac{(N+1-k)^{N+1-k} e^{-(N+1-k)}}{(2\pi)^{N+1-k}}.$$

Thus, it suffices to show that

$$\int\limits_{\mathbb{R}^n} \frac{f_1^{\mu}(\tau) \cdots f_k^{\mu}(\tau)}{\gamma^{\mu}(\tau)} \, d\boldsymbol{\eta}$$

is bounded.

We will begin by studying integrals of the form

$$f_s^{\mu}(\tau) = \left[\int\limits_{\mathbb{R}^n} (\tau\widetilde{b}(\boldsymbol{\mu}\boldsymbol{v}))^s e^{4\pi[\boldsymbol{\eta}\cdot\boldsymbol{v} - \tau\tilde{b}(\boldsymbol{\mu}\boldsymbol{v})]} \, d\boldsymbol{v} \right]^{a_s}.$$

By Claim 3.3 on page 32 there exists a universal constant such that

$$\tau\widetilde{b}(\boldsymbol{\mu}\boldsymbol{v}) \leq C(1 + v_1^{2m_1} + \ldots + v_n^{2m_n}).$$

Thus, we must study the behavior of integrals of the form

$$L_s = \int_{\mathbb{R}^n} p_s(\boldsymbol{v}) e^{\boldsymbol{\eta v} - g(\boldsymbol{v})}\, d\boldsymbol{v},$$

for polynomials $p_s : \mathbb{R}^n \to \mathbb{R}$ with non-negative coefficients and $g(\boldsymbol{v}) = \tau\tilde{b}(\boldsymbol{\mu v})$.

As usual (see, e.g., equation (3.2) on page 31), we can write

$$L_s = e^{h(\boldsymbol{v_0})} \int_{\mathbb{R}^n} e^{h(\boldsymbol{v}) - h(\boldsymbol{v_0})} p_s(\boldsymbol{v})\, d\boldsymbol{v},$$

where $h(\boldsymbol{v}) = \boldsymbol{\eta} \cdot \boldsymbol{v} - g(\boldsymbol{v})$ and $\boldsymbol{v_0}$ is the point where $h(\boldsymbol{v})$ attains its maximum. Making the change of variables $\boldsymbol{v} = \boldsymbol{w} + \boldsymbol{v_0}$, it follows that

$$L_s = e^{h(\boldsymbol{v_0})} \int_{\mathbb{R}^n} e^{-f(\boldsymbol{w})} p_s(\boldsymbol{w} + \boldsymbol{v_0})\, d\boldsymbol{w},$$

where $f(\boldsymbol{w})$ is as in equation (3.8). That is,

$$f(\boldsymbol{w}) = g(\boldsymbol{v_0} + \boldsymbol{w}) - g(\boldsymbol{v_0}) - \nabla g(\boldsymbol{v_0}) \cdot \boldsymbol{w}.$$

Since the coefficients of p are non-negative, it follows that

$$p_s(w_1 + v_{01}, \ldots, w_n + v_{0n}) \leq p_s(|\boldsymbol{w}| + |\boldsymbol{v_0}|, \ldots, |\boldsymbol{w}| + |\boldsymbol{v_0}|).$$

This last polynomial is now a polynomial of just one variable, and after expanding and regrouping all the terms, it consists of sums of terms of the form $|\boldsymbol{w}|^{j_s} |\boldsymbol{v_0}|^{i_s}$ for indices i_s and j_s. Hence, there exist positive coefficients c_{i_s, j_s} such that

$$L_s \leq e^{h(\boldsymbol{v_0})} \sum_{i_s, j_s} c_{i_s, j_s} |\boldsymbol{v_0}|^{i_s} \int_{\mathbb{R}^n} |\boldsymbol{w}|^{j_s} e^{-f(\boldsymbol{w})}\, d\boldsymbol{w}.$$

Let

$$J_{j_s} = \int_{\mathbb{R}^n} |\boldsymbol{w}|^{j_s} e^{-f(\boldsymbol{w})}\, d\boldsymbol{w}.$$

This integral is identical to the one studied in equation (3.21) in Claim 3.12 on page 52. Thus,

$$J_{j_s} \lesssim |\{\boldsymbol{w} \,:\, f(\boldsymbol{w}) \leq 1\}|\,(1 + |\boldsymbol{v_0}|^{j_s B}) + \Theta,$$

where Θ is a constant that depends only on $m_1, \ldots, m_n$ and the dimension of the space; and $B = 4\max\{m_1, \ldots, m_n\}$. It follows that

$$L_s \lesssim e^{h(\boldsymbol{v_0})} \sum_{i_s, j_s} c_{i_s, j_s} |\boldsymbol{v_0}|^{i_s} \left[|\{\boldsymbol{w} \,:\, f(\boldsymbol{w}) \leq 1\}|\,(1 + |\boldsymbol{v_0}|^{j_s B}) + \Theta \right].$$

That is,

$$L_s \lesssim e^{h(\boldsymbol{v_0})} \left[\, |\{\boldsymbol{w} \,:\, f(\boldsymbol{w}) \leq 1\}|\, \phi_s(|\boldsymbol{v_0}|) + \psi_s(|\boldsymbol{v_0}|) \,\right],$$

for some polynomials $\phi_s : \mathbb{R} \to \mathbb{R}^+$, $\psi_s : \mathbb{R} \to \mathbb{R}^+$.

On the other hand, by equation (3.10)

$$\int_{\mathbb{R}^n} e^{4\pi[\boldsymbol{\eta}\cdot\boldsymbol{v} - \tau\tilde{b}(\boldsymbol{\mu v})]}\, d\boldsymbol{v} \approx e^{h(\boldsymbol{v_0})} |\{\boldsymbol{w} \,:\, f(\boldsymbol{w}) \leq 1\}|.$$

Therefore, there exists some polynomial $q_j : \mathbb{R} \to \mathbb{R}^+$ such that

$$\int_{\mathbb{R}^n} \frac{f_1^{\mu}(\tau) \cdots f_k^{\mu}(\tau)}{\gamma^{\mu}(\tau)}\, d\boldsymbol{\eta} \lesssim \frac{[e^{h(\boldsymbol{v_0})}]^{a_1 + \ldots + a_k} \left(\sum_{j=0}^{a_1 + \ldots + a_k} |\{\boldsymbol{w} \,:\, f(\boldsymbol{w}) \leq 1\}|^j q_j(|\boldsymbol{v_0}|) \right)}{[e^{h(\boldsymbol{v_0})}]^d |\{\boldsymbol{w} \,:\, f(\boldsymbol{w}) \leq 1\}|^d}.$$

But $a_1 + \ldots + a_k = d - 1$, so

$$\int_{\mathbb{R}^n} \frac{f_1^\mu(\tau)\cdots f_k^\mu(\tau)}{\gamma^\mu(\tau)}\, d\boldsymbol{\eta} \lesssim e^{-h(\boldsymbol{v_0})} \left(\sum_{j=0}^{d-1} |\{\boldsymbol{w} \,:\, f(\boldsymbol{w}) \leq 1\}|^{j-d} q_j(|\boldsymbol{v_0}|) \right).$$

Moreover, by Claim 3.7,

$$|\{\boldsymbol{w} : f(\boldsymbol{w}) \leq 1\}| \gtrsim (1 + r(\boldsymbol{v_0}))^{-\frac{n}{2}}.$$

where $r(\boldsymbol{v}) = v_1^{2m_1} + \ldots + v_n^{2m_n}$. Thus, and since $j - d < 0$,

$$\int_{\mathbb{R}^n} \frac{f_1^\mu(\tau)\cdots f_k^\mu(\tau)}{\gamma^\mu(\tau)}\, d\boldsymbol{\eta} \lesssim e^{-h(\boldsymbol{v_0})} \left(\sum_{j=0}^{d-1} (1 + r(\boldsymbol{v_0}))^{\frac{n(d-j)}{2}} q_j(|\boldsymbol{v_0}|) \right).$$

By Claims 3.9 and 3.8, $\sum_{j=0}^{d-1}(1+r(\boldsymbol{v_0}))^{\frac{n(d-j)}{2}} q_j(|\boldsymbol{v_0}|)$ is at most of polynomial growth in $|\boldsymbol{\eta}|$. On the other hand, by equation (3.7), $e^{-h(\boldsymbol{v_0})}$ decays exponentially in $|\boldsymbol{\eta}|$. Hence,

$$\int_{\mathbb{R}^n} \frac{f_1^\mu(\tau)\cdots f_k^\mu(\tau)}{\gamma^\mu(\tau)}\, d\boldsymbol{\eta}$$

is bounded. This finishes the proof of Claim 4.12.

□

We are now ready to present the proof of Proposition 4.9.

Proof. We had shown in equation (4.2) on page 70 that

$$S = \int_0^\infty e^{-2\pi\tau\delta} e^{-2\pi i(t'-t)\tau} e^{2\pi i\tau\nabla b\left(\frac{\boldsymbol{x}+\boldsymbol{x'}}{2}\right)\cdot(\boldsymbol{y}-\boldsymbol{y'})} \int_{\mathbb{R}^n} \frac{e^{2\pi i\boldsymbol{\eta}\cdot(\boldsymbol{y}-\boldsymbol{y'})}}{\int_{\mathbb{R}^n} e^{4\pi[\boldsymbol{\eta}\cdot\boldsymbol{v}-\tau\tilde{b}(\boldsymbol{v})]}\, d\boldsymbol{v}}\, d\boldsymbol{\eta}\, d\tau.$$

Letting $u = 2\pi\left[(t'-t) + \nabla b\left(\frac{\boldsymbol{x}+\boldsymbol{x'}}{2}\right)\cdot(\boldsymbol{y'}-\boldsymbol{y})\right]$ we have that

$$S = \int_0^\infty e^{-iu\tau} e^{-2\pi\tau\delta} \int_{\mathbb{R}^n} \frac{e^{2\pi i \boldsymbol{\eta}\cdot(\boldsymbol{y}-\boldsymbol{y'})}}{\int_{\mathbb{R}^n} e^{4\pi[\boldsymbol{\eta}\cdot\boldsymbol{v}-\tau\tilde{b}(\boldsymbol{v})]}\, d\boldsymbol{v}}\, d\boldsymbol{\eta}\, d\tau.$$

Let

$$F(\tau) = e^{-2\pi\tau\delta} \int_{\mathbb{R}^n} \frac{e^{2\pi i \boldsymbol{\eta}\cdot(\boldsymbol{y}-\boldsymbol{y'})}}{\int_{\mathbb{R}^n} e^{4\pi[\boldsymbol{\eta}\cdot\boldsymbol{v}-\tau\tilde{b}(\boldsymbol{v})]}\, d\boldsymbol{v}}\, d\boldsymbol{\eta} \tag{4.24}$$

so that

$$S = \int_0^\infty e^{-iu\tau} F(\tau)\, d\tau.$$

Then by Claim 4.10 it follows that

$$\begin{aligned} |S| \le& \sum_{j=0}^{N+1} c_j \left| \int_0^{\frac{\pi}{|u|}} e^{-iu\tau} F\left(\tau + \frac{j\pi}{|u|}\right)\, d\tau \right| \\ &+ \frac{1}{2^{N+1}} \left| \int_{\frac{\pi}{|u|}}^\infty e^{-iu\tau} \int_0^{\frac{\pi}{|u|}} \cdots \int_0^{\frac{\pi}{|u|}} F^{(N+1)}\left(\tau + s_1 + \ldots + s_{N+1}\right)\, ds_1 \, \cdots \, ds_{N+1}\, d\tau \right|. \end{aligned}$$

We will begin by obtaining the desired bound for the terms of the form

$$I_j(u) = \int_0^{\frac{\pi}{|u|}} e^{-iu\tau} F\left(\tau + \frac{j\pi}{|u|}\right)\, d\tau.$$

We can write

$$|I_j| \le \int_0^{\frac{\pi}{|u|}} \left| F\left(\tau + \frac{j\pi}{|u|}\right) \right| d\tau = \int_0^{\frac{\pi}{|u|}} \left| e^{-2\pi\delta\left(\tau+\frac{j\pi}{|u|}\right)} \int_{\mathbb{R}^n} \frac{e^{2\pi i \boldsymbol{\eta}\cdot(\boldsymbol{y}-\boldsymbol{y'})}}{\int\limits_{\mathbb{R}^n} e^{4\pi[\boldsymbol{\eta}\cdot\boldsymbol{v}-\left(\tau+\frac{j\pi}{|u|}\right)\tilde{b}(\boldsymbol{v})]}\, d\boldsymbol{v}}\, d\boldsymbol{\eta} \right| d\tau.$$

Making the change of variables $s = \tau + \frac{j\pi}{|u|}$ as well as $\boldsymbol{\eta} \to \frac{\boldsymbol{\eta}}{\boldsymbol{\mu}}$ and $\boldsymbol{v} \to \boldsymbol{\mu v}$, where $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)$ are chosen as in equation (4.1) on page 68, it follows that

$$|I_j(u)| \le \int_{\frac{j\pi}{|u|}}^{\frac{(j+1)\pi}{|u|}} \frac{e^{-2\pi\delta s}}{\mu_1^2 \cdots \mu_n^2} \int_{\mathbb{R}^n} \frac{1}{\int_{\mathbb{R}^n} e^{4\pi[\boldsymbol{\eta}\cdot\boldsymbol{v} - s\tilde{b}(\boldsymbol{\mu v})]}\, d\boldsymbol{v}}\, d\boldsymbol{\eta}\, ds.$$

By Lemma 3.1,

$$\int_{\mathbb{R}^n} \frac{1}{\int_{\mathbb{R}^n} e^{4\pi[\boldsymbol{\eta}\cdot\boldsymbol{v} - s\tilde{b}(\boldsymbol{\mu v})]}\, d\boldsymbol{v}}\, d\boldsymbol{\eta}$$

converges. Also, by convexity of b, $\delta \ge 0$. Thus,

$$|I_j(u)| \lesssim \int_{\frac{j\pi}{|u|}}^{\frac{(j+1)\pi}{|u|}} \frac{1}{\mu_1^2 \cdots \mu_n^2}\, ds.$$

Since the factors μ_j were chosen so that

$$\mu_1(\tau) \cdots \mu_n(\tau) \approx \left| \left\{ \boldsymbol{v} \,:\, \tilde{b}(\boldsymbol{v}) \le \frac{1}{\tau} \right\} \right|$$

it follows that

$$|I_j(u)| \lesssim \int_{\frac{j\pi}{|u|}}^{\frac{(j+1)\pi}{|u|}} \frac{1}{\left| \left\{ \boldsymbol{v} \,:\, \tilde{b}(\boldsymbol{v}) \le \frac{1}{\tau} \right\} \right|^2}\, d\tau.$$

But since we are considering $\tau \le \frac{(j+1)\pi}{|u|}$, then

$$\left\{ \boldsymbol{v} \,:\, \tilde{b}(\boldsymbol{v}) \le \frac{|u|}{\pi(j+1)} \right\} \subseteq \left\{ \boldsymbol{v} \,:\, \tilde{b}(\boldsymbol{v}) \le \frac{1}{\tau} \right\}.$$

Thus,

$$|I_j(u)| \lesssim \int_{\frac{j\pi}{|u|}}^{\frac{(j+1)\pi}{|u|}} \frac{1}{\left|\left\{\boldsymbol{v} \,:\, \widetilde{b}(\boldsymbol{v}) \leq \frac{|u|}{\pi(j+1)}\right\}\right|^2} \, d\tau = \frac{\pi}{|u| \left|\left\{\boldsymbol{v} \,:\, \widetilde{b}(\boldsymbol{v}) \leq \frac{|u|}{\pi(j+1)}\right\}\right|^2}.$$

Let $w = \frac{u}{2\pi} = \left[(t' - t) + \nabla b\left(\frac{\boldsymbol{x}+\boldsymbol{x'}}{2}\right) \cdot (\boldsymbol{y'} - \boldsymbol{y})\right]$. Then,

$$|I_j(w)| \lesssim \frac{1}{|w| \left|\left\{\boldsymbol{v} \,:\, \widetilde{b}(\boldsymbol{v}) \leq \frac{2|w|}{(j+1)}\right\}\right|^2}.$$

Since $\left\{\boldsymbol{v} \,:\, \widetilde{b}(\boldsymbol{v}) \leq |w|\right\} \subseteq \left\{\boldsymbol{v} \,:\, \widetilde{b}(\boldsymbol{v}) \leq 2|w|\right\}$ it follows that

$$|I_0(w)| \lesssim \frac{1}{|w| \left|\left\{\boldsymbol{v} \,:\, \widetilde{b}(\boldsymbol{v}) \leq |w|\right\}\right|^2}.$$

The same bound is obtained trivially for $j = 1$. For $j > 1$, we can use Claim 1.5 on page 16 to show that

$$\left|\left\{\boldsymbol{v} \,:\, \widetilde{b}(\boldsymbol{v}) \leq |w|\right\}\right| \leq \frac{(j+1)^n}{2^n} \left|\left\{\boldsymbol{v} \,:\, \widetilde{b}(\boldsymbol{v}) \leq \frac{2|w|}{(j+1)}\right\}\right|.$$

Hence,

$$|I_j(w)| \lesssim \frac{(j+1)^{2n}}{|w| \left|\left\{\boldsymbol{v} \,:\, \widetilde{b}(\boldsymbol{v}) \leq |w|\right\}\right|^2}.$$

Since the sum over j is finite, it follows that

$$\begin{aligned} |S| \lesssim & \frac{1}{|w| \left|\left\{\boldsymbol{v} \,:\, \widetilde{b}(\boldsymbol{v}) \leq |w|\right\}\right|^2} \\ & + \frac{1}{2^{N+1}} \left| \int_{\frac{\pi}{|u|}}^{\infty} e^{-iu\tau} \int_0^{\frac{\pi}{|u|}} \cdots \int_0^{\frac{\pi}{|u|}} F^{(N+1)}\left(\tau + s_1 + \ldots + s_{N+1}\right) ds_1 \cdots ds_{N+1} \, d\tau \right|. \end{aligned} \tag{4.25}$$

We must now bound the term

$$\left| \int_{\frac{\pi}{|u|}}^{\infty} e^{-iu\tau} \int_0^{\frac{\pi}{|u|}} \cdots \int_0^{\frac{\pi}{|u|}} F^{(N+1)} \left(\tau + s_1 + \ldots + s_{N+1} \right) ds_1 \, \cdots \, ds_{N+1} \, d\tau \right|$$

We showed in Claim 4.11 that the $(N+1)^{th}$ derivative of

$$F(\tau) = e^{-2\pi\tau\delta} \int_{\mathbb{R}^n} \frac{e^{2\pi i \boldsymbol{\eta} \cdot (\boldsymbol{y}-\boldsymbol{y'})}}{\int_{\mathbb{R}^n} e^{4\pi[\boldsymbol{\eta}\cdot\boldsymbol{v} - \tau\tilde{b}(\boldsymbol{v})]} \, d\boldsymbol{v}} \, d\boldsymbol{\eta}.$$

consists of sums of terms of the form

$$\frac{C(\tau\delta)^{N+1-k} e^{-2\pi\tau\delta}}{\tau^{N+1}} \int_{\mathbb{R}^n} \frac{e^{2\pi i \boldsymbol{\eta} \cdot (\boldsymbol{y}-\boldsymbol{y'})} f_1(\tau) \cdots f_k(\tau)}{\gamma(\tau)} \, d\boldsymbol{\eta},$$

where

$$f_s(\tau) = \left[\int\limits_{\mathbb{R}^n} (\tau\widetilde{b}(\boldsymbol{v}))^s e^{4\pi[\boldsymbol{\eta}\cdot\boldsymbol{v} - \tau\tilde{b}(\boldsymbol{v})]} \, d\boldsymbol{v} \right]^{a_s};$$

$$\gamma(\tau) = \left[\int\limits_{\mathbb{R}^n} e^{4\pi[\boldsymbol{\eta}\cdot\boldsymbol{v} - \tau\tilde{b}(\boldsymbol{v})]} \, d\boldsymbol{v} \right]^{d};$$

$a_1, \ldots, a_k, k, d \in \mathbb{N}$; $0 \leq k \leq N+1$; $a_1 + \ldots + a_k = d - 1$; and $a_1 + 2a_2 + \ldots + ka_k = k$.

We can write these terms as $\frac{C}{\tau^{N+1}} \Delta_{N+1,k}(\tau)$, where

$$\Delta_{N+1,k}(\tau) = (\tau\delta)^{N+1-k} e^{-2\pi\tau\delta} \int_{\mathbb{R}^n} \frac{e^{2\pi i \boldsymbol{\eta} \cdot (\boldsymbol{y}-\boldsymbol{y'})} f_1(\tau) \cdots f_k(\tau)}{\gamma(\tau)} \, d\boldsymbol{\eta}.$$

Thus, we must study integrals of the form

$$J = \int_{\frac{\pi}{|u|}}^{\infty} \frac{e^{-iu\tau}}{\tau^{N+1}} \int_0^{\frac{\pi}{|u|}} \cdots \int_0^{\frac{\pi}{|u|}} \Delta_{N+1,k} \left(\tau + s_1 + \ldots + s_{N+1} \right) ds_1 \, \cdots \, ds_{N+1} \, d\tau.$$

With $\mu_1, \ldots, \mu_n$ chosen as in equation (4.1) on page 68 we can make the changes of variable $\boldsymbol{\eta} \to \frac{\boldsymbol{\eta}}{\boldsymbol{\mu}}$ and $\boldsymbol{v} \to \boldsymbol{\mu}\boldsymbol{v}$. We obtain an integral of the form

$$J = \int_{\frac{\pi}{|u|}}^{\infty} \frac{(\mu_1 \cdots \mu_n)^{a_1+\ldots+a_k}}{\tau^{N+1}(\mu_1 \cdots \mu_n)^{d+1}} \int_0^{\frac{\pi}{|u|}} \cdots \int_0^{\frac{\pi}{|u|}} \Delta^{\boldsymbol{\mu}}_{N+1,k}\left(\tau + s_1 + \ldots + s_{N+1}\right) ds_1 \cdots ds_{N+1}\, d\tau,$$

where now

$$\Delta^{\boldsymbol{\mu}}_{N+1,k}(\tau) = (\tau\delta)^{N+1-k} e^{-2\pi\tau\delta} \int_{\mathbb{R}^n} \frac{e^{2\pi i \frac{\boldsymbol{\eta}}{\boldsymbol{\mu}}\cdot(\boldsymbol{y}-\boldsymbol{y'})} f_1^{\mu}(\tau) \cdots f_k^{\mu}(\tau)}{\gamma^{\mu}(\tau)}\, d\boldsymbol{\eta}; \tag{4.26}$$

$$f_s^{\mu}(\tau) = \left[\int_{\mathbb{R}^n} (\tau \widetilde{b}(\boldsymbol{\mu v}))^s e^{4\pi[\boldsymbol{\eta}\cdot\boldsymbol{v} - \tau\tilde{b}(\boldsymbol{\mu v})]}\, d\boldsymbol{v}\right]^{a_s};$$

and

$$\gamma^{\mu}(\tau) = \left[\int_{\mathbb{R}^n} e^{4\pi[\boldsymbol{\eta}\cdot\boldsymbol{v} - \tau\tilde{b}(\boldsymbol{\mu v})]}\, d\boldsymbol{v}\right]^{d}.$$

But by Claim 4.12, $|\Delta^{\boldsymbol{\mu}}_{N+1,k}(\tau)| \leq C$. It follows that

$$|J| \lesssim \int_{\frac{\pi}{|u|}}^{\infty} \frac{(\mu_1 \cdots \mu_n)^{a_1+\ldots+a_k}}{|u|^{N+1}\tau^{N+1}(\mu_1 \cdots \mu_n)^{d+1}}\, d\tau.$$

Also, $a_1 + \ldots + a_k = d - 1$. Thus,

$$|J| \lesssim \frac{1}{|u|^{N+1}} \int_{\frac{\pi}{|u|}}^{\infty} \frac{1}{\tau^{N+1}(\mu_1 \cdots \mu_n)^2}\, d\tau.$$

Since $\boldsymbol{\mu}$ was chosen so that

$$\mu_1(\tau) \cdots \mu_n(\tau) \approx \left|\left\{\boldsymbol{v} : \widetilde{b}(\boldsymbol{v}) \leq \frac{1}{\tau}\right\}\right|,$$

and taking $w = \frac{u}{2\pi}$ as before, it follows that

$$|J| \lesssim \frac{1}{|w|^{N+1}} \int_{\frac{1}{2|w|}}^{\infty} \frac{1}{\tau^{N+1} \left|\left\{\boldsymbol{v} : \widetilde{b}(\boldsymbol{v}) \leq \frac{1}{\tau}\right\}\right|^2} \, d\tau.$$

Notice that on the interval under consideration, $(2|w|\tau)^{-1} \leq 1$. Using Claim 1.5, it follows that

$$\left|\left\{\boldsymbol{v} : \widetilde{b}(\boldsymbol{v}) \leq \frac{1}{\tau}\right\}\right| = \left|\left\{\boldsymbol{v} : \widetilde{b}(\boldsymbol{v}) \leq \frac{2|w|}{2|w|\tau}\right\}\right| \geq \frac{1}{(2|w|\tau)^n} \left|\left\{\boldsymbol{v} : \widetilde{b}(\boldsymbol{v}) \leq 2|w|\right\}\right|.$$

Also, as mentioned earlier,

$$\left|\left\{\boldsymbol{v} : \widetilde{b}(\boldsymbol{v}) \leq 2|w|\right\}\right| \geq \left|\left\{\boldsymbol{v} : \widetilde{b}(\boldsymbol{v}) \leq |w|\right\}\right|$$

It follows that

$$\left|\left\{\boldsymbol{v} : \widetilde{b}(\boldsymbol{v}) \leq \frac{1}{\tau}\right\}\right| \geq \frac{1}{(2|w|\tau)^n} \left|\left\{\boldsymbol{v} : \widetilde{b}(\boldsymbol{v}) \leq |w|\right\}\right|.$$

Thus, taking $N \geq 2n+1$,

$$\begin{aligned} |J| &\lesssim \frac{|w|^{2n}}{|w|^{N+1}} \int_{\frac{1}{2|w|}}^{\infty} \frac{\tau^{2n}}{\tau^{N+1} \left|\left\{\boldsymbol{v} : \widetilde{b}(\boldsymbol{v}) \leq |w|\right\}\right|^2} \, d\tau \\ &\approx \frac{1}{|w|^{N+1-2n} \left|\left\{\boldsymbol{v} : \widetilde{b}(\boldsymbol{v}) \leq |w|\right\}\right|^2} \cdot \tau^{2n-N} \Big|_{\frac{1}{2|w|}}^{\infty}. \end{aligned}$$

That is,

$$|J| \lesssim \frac{1}{|w| \left|\left\{\boldsymbol{v} : \tilde{b}(\boldsymbol{v}) \leq |w|\right\}\right|^2}.$$

It follows from equation (4.25) that

$$|S| \lesssim \frac{1}{|w| \left|\left\{\boldsymbol{v} \,:\, \tilde{b}(\boldsymbol{v}) \leq |w|\right\}\right|^2}.$$

This finishes the proof of our third and last bound.

□

We have shown that

$$|S\left((\boldsymbol{x}, \boldsymbol{y}, t); (\boldsymbol{x'}, \boldsymbol{y'}, t')\right)| \lesssim \min\left\{A, B, C\right\},$$

where

$$A = \frac{1}{\delta|\{\boldsymbol{v} \,:\, \tilde{b}(\boldsymbol{v}) < \delta\}|^2};$$

$$B = \frac{1}{\widetilde{b}(\boldsymbol{y} - \boldsymbol{y'})|\{\boldsymbol{v} \,:\, \tilde{b}(\boldsymbol{v}) < \widetilde{b}(\boldsymbol{y} - \boldsymbol{y'})\}|^2};$$

and

$$C = \frac{1}{|w| \; |\{\boldsymbol{v} \,:\, \widetilde{b}(\boldsymbol{v}) < |w|\}|^2}.$$

Thus, to conclude the proof of the Main Theorem, it suffices to show that

$$\min\{A, B, C\} \lesssim \frac{1}{\sqrt{\delta^2 + \widetilde{b}(\boldsymbol{y} - \boldsymbol{y'})^2 + w^2}\left|\left\{\boldsymbol{v} \,:\, \tilde{b}(\boldsymbol{v}) < \sqrt{\delta^2 + \widetilde{b}(\boldsymbol{y} - \boldsymbol{y'})^2 + w^2}\right\}\right|^2}.$$

Without loss of generality, suppose that $\delta \leq \widetilde{b}(\boldsymbol{y} - \boldsymbol{y'}) \leq |w|$. Then, and since $\tilde{b}$ is non-negative,

$$\{\boldsymbol{v} : \tilde{b}(\boldsymbol{v}) < \delta\} \subseteq \{\boldsymbol{v} : \tilde{b}(\boldsymbol{v}) < \tilde{b}(\boldsymbol{y} - \boldsymbol{y'})\} \subseteq \{\boldsymbol{v} : \tilde{b}(\boldsymbol{v}) < |w|\}.$$

It follows that

$$\frac{1}{|w|\ |\{\boldsymbol{v} : \tilde{b}(\boldsymbol{v}) < |w|\}|^2} \leq \frac{1}{\tilde{b}(\boldsymbol{y} - \boldsymbol{y'})|\{\boldsymbol{v} : \tilde{b}(\boldsymbol{v}) < \tilde{b}(\boldsymbol{y} - \boldsymbol{y'})\}|^2} \leq \frac{1}{\delta|\{\boldsymbol{v} : \tilde{b}(\boldsymbol{v}) < \delta\}|^2},$$

so

$$|S| \lesssim \frac{1}{|w|\ |\{\boldsymbol{v} : \tilde{b}(\boldsymbol{v}) < |w|\}|^2}.$$

Moreover, since $\delta \leq \tilde{b}(\boldsymbol{y} - \boldsymbol{y'}) \leq |w|$, then $\sqrt{\delta^2 + \tilde{b}(\boldsymbol{y} - \boldsymbol{y'})^2 + w^2} \leq \sqrt{3w^2}$. That is,

$$\frac{1}{|w|} \leq \frac{\sqrt{3}}{\sqrt{\delta^2 + \tilde{b}(\boldsymbol{y} - \boldsymbol{y'})^2 + w^2}}.$$

and

$$\left\{\boldsymbol{v} : \tilde{b}(\boldsymbol{v}) < \sqrt{\delta^2 + \tilde{b}(\boldsymbol{y} - \boldsymbol{y'})^2 + w^2}\right\} \subseteq \left\{\boldsymbol{v} : \tilde{b}(\boldsymbol{v}) < \sqrt{3}|w|\right\}.$$

But by Claim 1.5,

$$\left|\left\{\boldsymbol{v} : \tilde{b}(\boldsymbol{v}) < \sqrt{3}|w|\right\}\right|^2 \leq (\sqrt{3})^{2n} \left|\left\{\boldsymbol{v} : \tilde{b}(\boldsymbol{v}) < |w|\right\}\right|^2.$$

Therefore,

$$\begin{aligned} |S| &\lesssim \frac{1}{|w|\ |\{\boldsymbol{v} : \tilde{b}(\boldsymbol{v}) < |w|\}|^2} \\ &\leq \frac{(\sqrt{3})^{2n+1}}{\sqrt{\delta^2 + \tilde{b}(\boldsymbol{y} - \boldsymbol{y'})^2 + w^2}\left|\left\{\boldsymbol{v} : \tilde{b}(\boldsymbol{v}) < \sqrt{\delta^2 + \tilde{b}(\boldsymbol{y} - \boldsymbol{y'})^2 + w^2}\right\}\right|^2}. \end{aligned}$$

This finishes the proof of the Main Theorem.

Appendix A

An integral expression for the Szegő kernel

In this appendix we derive an integral expression for the Szegő kernel for a class of unbounded domains defined by convex polynomials.

Proposition A.1. *The Szegő kernel on the boundary of domains of the kind* $\Omega = \{\boldsymbol{z} \in \mathbb{C}^{n+1} : \mathrm{Im}[z_{n+1}] > b(\mathrm{Re}[z_1], \ldots, \mathrm{Re}[z_n])\}$ *where* $b : \mathbb{R}^n \to \mathbb{R}$ *is a convex function of "combined degree" is given by*

$$S((\boldsymbol{x}, \boldsymbol{y}, t); (\boldsymbol{x'}, \boldsymbol{y'}, t')) = \int_0^\infty e^{-2\pi\tau[b(\boldsymbol{x'})+b(\boldsymbol{x})+i(t'-t)]} \left(\int_{\mathbb{R}^n} \frac{e^{2\pi\boldsymbol{\eta}\cdot[\boldsymbol{x}+\boldsymbol{x'}-i(\boldsymbol{y'}-\boldsymbol{y})]}}{\int\limits_{\mathbb{R}^n} e^{4\pi[\boldsymbol{\eta}\cdot\boldsymbol{v}-b(\boldsymbol{v})\tau]}\, d\boldsymbol{v}}\, d\boldsymbol{\eta} \right) d\tau, \tag{A.1}$$

where $(\boldsymbol{x}, \boldsymbol{y}, t)$ *and* $(\boldsymbol{x'}, \boldsymbol{y'}, t')$ *are any two points on* $\partial\Omega$.

Let $\Omega = \{\boldsymbol{z} \in \mathbb{C}^{n+1} : \mathrm{Im}[z_{n+1}] > b(\mathrm{Re}[z_1], \ldots, \mathrm{Re}[z_n])\}$ and

$$\begin{aligned} \rho(z_1, \ldots, z_{n+1}) &= b(\mathrm{Re}[z_1], \ldots, \mathrm{Re}[z_n]) - \mathrm{Im}[z_{n+1}] \\ &= b\left(\frac{z_1 + \overline{z_1}}{2}, \ldots, \frac{z_n + \overline{z_n}}{2}\right) - \frac{z_{n+1} - \overline{z_{n+1}}}{2i} \end{aligned} \tag{A.2}$$

be a defining function for our domain. Recall that the Szegő Projection is the orthogonal projection $\Pi_b : L^2(\partial\Omega) \to H^2(\partial\Omega)$, where $H^2(\partial\Omega) = \{f \in L^2(\partial\Omega) : L(f) = 0$ as a distribution, for all tangential Cauchy-Riemann operators $L\}$. We begin by finding a base for the tangential Cauchy-Riemann operators. We can let

$$\overline{Z_j} = 2\left(\frac{\partial}{\partial \overline{z_j}} + A_j(z_1, \ldots, z_n)\frac{\partial}{\partial \overline{z_{n+1}}}\right) \qquad j = 1, \ldots, n.$$

For these operators to be tangential they must satisfy $\overline{Z_j}(\rho) = 0$. Thus,

$$\overline{Z_j} = 2\left(\frac{\partial}{\partial \overline{z_j}} - i\frac{\partial b}{\partial x_j}(\boldsymbol{x})\frac{\partial}{\partial \overline{z_{n+1}}}\right) \qquad j = 1, \ldots, n$$

are a basis for the space of tangential Cauchy-Riemann operators for our domain in $\mathbb{C}^{n+1}$.

We can identify $\partial\Omega$ with $\mathbb{C}^n \times \mathbb{R}$ via the map

$$(z_1, \ldots, z_n, t) \in \mathbb{C}^n \times \mathbb{R} \leftrightarrow (z_1, \ldots, z_n, t + ib(\mathrm{Re}[z_1], \ldots, \mathrm{Re}[z_n])) \in \partial\Omega.$$

Our operators $\overline{Z_j}$ are operators in $\mathbb{C}^{n+1}$. The restriction of these operators to $\mathbb{C}^n \times \mathbb{R}$ is

$$\overline{Z_j} = \frac{\partial}{\partial x_j} + i\left(\frac{\partial}{\partial y_j} - \frac{\partial b}{\partial x_j}(\boldsymbol{x})\frac{\partial}{\partial t}\right). \tag{A.3}$$

Lemma A.2. *Let*

$$\mathcal{M}[g](\boldsymbol{x}, \boldsymbol{\eta}, \tau) = e^{-2\pi[\boldsymbol{\eta}\cdot\boldsymbol{x} - b(\boldsymbol{x})\tau]} g(\boldsymbol{x}, \boldsymbol{\eta}, \tau),$$

and define the partial Fourier transform

$$\mathcal{F}[f](\boldsymbol{x}, \boldsymbol{y}, t) = \hat{f}(\boldsymbol{x}, \boldsymbol{\eta}, \tau) = \int\limits_{\mathbb{R}^{n+1}} e^{-2\pi i(\boldsymbol{y}\cdot\boldsymbol{\eta} + t\tau)} f(\boldsymbol{x}, \boldsymbol{y}, t)\, d\boldsymbol{y}\, dt.$$

Then

$$\mathcal{M} : L^2(\mathbb{R}^{2n+1}, d\boldsymbol{x} d\boldsymbol{\eta}\, d\tau\,) \to L^2(\mathbb{R}^{2n+1}, e^{4\pi[\boldsymbol{\eta}\cdot\boldsymbol{x}-b(\boldsymbol{x})\tau]} d\boldsymbol{x}\, d\boldsymbol{\eta}\, d\tau\,)$$

is an isometry, and

$$\overline{Z_j}[f] = \mathcal{F}^{-1}\mathcal{M}^{-1}\frac{\partial}{\partial x_j}\mathcal{M}\mathcal{F}[f] \qquad j = 1, \ldots n.$$

Proof. It is easy to check that $\mathcal{M}$ is an isometry in this weighted L^2 space, in fact

$$\begin{aligned} ||\mathcal{M}[g]||^2_{L^2(e^{4\pi[\boldsymbol{\eta}\cdot\boldsymbol{x}-b(\boldsymbol{x})\tau]})} &= \int\limits_{\mathbb{R}^{2n+1}} \left| g(\boldsymbol{x}, \boldsymbol{\eta}, \tau) e^{-2\pi[\boldsymbol{\eta}\cdot\boldsymbol{x}-b(\boldsymbol{x})\tau]} \right|^2 e^{4\pi[\boldsymbol{\eta}\cdot\boldsymbol{x}-b(\boldsymbol{x})\tau]} d\boldsymbol{x}\, d\boldsymbol{\eta}\, d\tau \\ &= \int\limits_{\mathbb{R}^{2n+1}} |g(\boldsymbol{x}, \boldsymbol{\eta}, \tau)|^2 \, d\boldsymbol{x}\, d\boldsymbol{\eta}\, d\tau\,. \end{aligned}$$

Also,

$$\begin{aligned} \overline{Z_j}[f] =& \overline{Z_j}\mathcal{F}^{-1}(\hat{f}) \\ =& \int\limits_{\mathbb{R}^{n+1}} e^{2\pi i(\boldsymbol{y}\cdot\boldsymbol{\eta}+t\tau)} \left(\frac{\partial \hat{f}(\boldsymbol{x}, \boldsymbol{\eta}, \tau)}{\partial x_j} - 2\pi\eta_j \hat{f}(\boldsymbol{x}, \boldsymbol{\eta}, \tau) + \frac{\partial b}{\partial x_j}(\boldsymbol{x}) 2\pi\tau \hat{f}(\boldsymbol{x}, \boldsymbol{\eta}, \tau) \right) d\boldsymbol{\eta}\, d\tau \\ =& \int\limits_{\mathbb{R}^{n+1}} e^{2\pi i(\boldsymbol{y}\cdot\boldsymbol{\eta}+t\tau)} e^{2\pi[\boldsymbol{\eta}\boldsymbol{x}-b(\boldsymbol{x})\tau]} \frac{\partial}{\partial x_j} \left(e^{-2\pi[\boldsymbol{\eta}\cdot\boldsymbol{x}-b(\boldsymbol{x})\tau]} \hat{f}(\boldsymbol{x}, \boldsymbol{\eta}, \tau) \right) d\boldsymbol{\eta}\, d\tau \\ =& \mathcal{F}^{-1}\mathcal{M}^{-1}\frac{\partial}{\partial x_j}\mathcal{M}\mathcal{F}[f]. \end{aligned}$$

□

Since $\mathcal{F}$ and $\mathcal{M}$ are isometries, instead of projecting onto the null space of the tangential Cauchy-Riemann operators we can project onto the null space of the operators $\left\{ \frac{\partial}{\partial x_j} \right\}$. That is, we project onto functions $\hat{f}(\boldsymbol{x}, \boldsymbol{\eta}, \tau) \in L^2(\mathbb{R}^{2n+1}, e^{4\pi[\boldsymbol{\eta}\cdot\boldsymbol{x}-b(\boldsymbol{x})\tau]} d\boldsymbol{x}\, d\boldsymbol{\eta}\, d\tau\,)$ such that the $\{x_j\}$ are constants. Let $\widehat{\Pi}$ be this projection. Then the Szegő projection is given by $\Pi[f] = \mathcal{F}^{-1}\mathcal{M}^{-1}\widehat{\Pi}\mathcal{M}\mathcal{F}[f]$.

Remark A.3. *i) Notice that for these functions $\hat{f}$ to be in $L^2(\mathbb{R}^{2n+1}, e^{4\pi[\boldsymbol{\eta}\cdot\boldsymbol{x}-b(\boldsymbol{x})\tau]}d\boldsymbol{x}\,d\boldsymbol{\eta}\,d\tau)$ one must have that*

$$\int\limits_{\mathbb{R}^{n+1}} |\hat{f}|^2 \left(\int\limits_{\mathbb{R}^n} e^{4\pi[\boldsymbol{\eta}\cdot\boldsymbol{x}-b(\boldsymbol{x})\tau]}\, d\boldsymbol{x} \right) d\boldsymbol{\eta}\, d\tau \; < \infty.$$

The inner integral $\int\limits_{\mathbb{R}^n} e^{4\pi[\boldsymbol{\eta}\cdot\boldsymbol{x}-b(\boldsymbol{x})\tau]}\, d\boldsymbol{x}$ diverges if $\tau \leq 0$ because of the growth hypothesis on b. Thus we set $\widehat{\Pi}\hat{f}(\boldsymbol{x},\boldsymbol{\eta},\tau) \equiv 0$ if $\tau \leq 0$.

ii) Notice that under these hypothesis the constant functions belong to $L^2(\mathbb{R}^n, e^{4\pi[\boldsymbol{\eta}\cdot\boldsymbol{x}-b(\boldsymbol{x})\tau]}d\boldsymbol{x}\,)$.

Since the basis for the null space of the operators $\left\{\frac{\partial}{\partial x_j}\right\}$ is just the constant function, the projection $\widehat{\Pi}$ for $\tau > 0$ is given by

$$\begin{aligned}
\widehat{\Pi}[g] \;&=\; \frac{<g,1>1}{<1,1>} \;=\; \frac{\int_{\mathbb{R}^n} g(\boldsymbol{x'},\boldsymbol{\eta},\tau)e^{4\pi[\boldsymbol{\eta}\cdot\boldsymbol{x'}-b(\boldsymbol{x'})\tau]}\, d\boldsymbol{x'}}{\int_{\mathbb{R}^n} e^{4\pi[\boldsymbol{\eta}\cdot\boldsymbol{v}-b(\boldsymbol{v})\tau]}\, d\boldsymbol{v}} \\
&= \int_{\mathbb{R}^n} g(\boldsymbol{x'},\boldsymbol{\eta},\tau) \left(\frac{e^{4\pi[\boldsymbol{\eta}\cdot\boldsymbol{x'}-b(\boldsymbol{x'})\tau]}}{\int_{\mathbb{R}^n} e^{4\pi[\boldsymbol{\eta}\cdot\boldsymbol{v}-b(\boldsymbol{v})\tau]}\, d\boldsymbol{v}} \right) d\boldsymbol{x'}\,.
\end{aligned}$$

The Szegő Projection, then, is given by

$$\begin{aligned}
&\Pi[f](\boldsymbol{x},\boldsymbol{y},t) = \mathcal{F}^{-1}\mathcal{M}^{-1}\widehat{\Pi}\mathcal{M}\mathcal{F}[f](\boldsymbol{x'},\boldsymbol{y'},t') \\
&= \mathcal{F}^{-1}\mathcal{M}^{-1}\widehat{\Pi}\mathcal{M}\int_{\mathbb{R}^{n+1}} e^{-2\pi i(\boldsymbol{y'}\cdot\boldsymbol{\eta}+t'\tau)} f(\boldsymbol{x'},\boldsymbol{y'},t')\, d\boldsymbol{y'}\, dt' \\
&= \mathcal{F}^{-1}\mathcal{M}^{-1}\widehat{\Pi}e^{-2\pi[\boldsymbol{\eta}\cdot\boldsymbol{x'}-b(\boldsymbol{x'})\tau]}\int_{\mathbb{R}^{n+1}} e^{-2\pi i(\boldsymbol{y'}\cdot\boldsymbol{\eta}+t'\tau)} f(\boldsymbol{x'},\boldsymbol{y'},t')\, d\boldsymbol{y'}\, dt' \\
&= \mathcal{F}^{-1}\mathcal{M}^{-1}\int_{\mathbb{R}^n} e^{-2\pi[\boldsymbol{\eta}\cdot\boldsymbol{x'}-b(\boldsymbol{x'})\tau]}\int_{\mathbb{R}^{n+1}} \frac{e^{-2\pi i(\boldsymbol{y'}\cdot\boldsymbol{\eta}+t'\tau)} f(\boldsymbol{x'},\boldsymbol{y'},t')e^{4\pi[\boldsymbol{\eta}\cdot\boldsymbol{x'}-b(\boldsymbol{x'})\tau]}}{\int\limits_{\mathbb{R}^n} e^{4\pi[\boldsymbol{\eta}\cdot\boldsymbol{v}-b(\boldsymbol{v})\tau]}\, d\boldsymbol{v}} d\boldsymbol{y'}\, dt'\, d\boldsymbol{x'}
\end{aligned}$$

$$= \mathcal{F}^{-1}\ e^{2\pi[\boldsymbol{\eta}\cdot\boldsymbol{x}-b(\boldsymbol{x})\tau]}\int_{\mathbb{R}^n} e^{-2\pi[\boldsymbol{\eta}\cdot\boldsymbol{x'}-b(\boldsymbol{x'})\tau]}$$

$$\int_{\mathbb{R}^{n+1}} \frac{e^{-2\pi i(\boldsymbol{y'}\cdot\boldsymbol{\eta}+t'\tau)} f(\boldsymbol{x'},\boldsymbol{y'},t') e^{4\pi[\boldsymbol{\eta}\cdot\boldsymbol{x'}-b(\boldsymbol{x'})\tau]}}{\int_{\mathbb{R}^n} e^{4\pi[\boldsymbol{\eta}\cdot\boldsymbol{v}-b(\boldsymbol{v})\tau]}\, d\boldsymbol{v}} d\boldsymbol{y'}\, dt'\, d\boldsymbol{x'}$$

$$= \int_0^\infty \int_{\mathbb{R}^n} e^{2\pi i(\boldsymbol{y}\cdot\boldsymbol{\eta}+t\tau)} e^{2\pi[\boldsymbol{\eta}\cdot\boldsymbol{x}-b(\boldsymbol{x})\tau]} \int_{\mathbb{R}^n} e^{-2\pi[\boldsymbol{\eta}\cdot\boldsymbol{x'}-b(\boldsymbol{x'})\tau]}$$

$$\int_{\mathbb{R}^{n+1}} \frac{e^{-2\pi i(\boldsymbol{y'}\cdot\boldsymbol{\eta}+t'\tau)} f(\boldsymbol{x'},\boldsymbol{y'},t') e^{4\pi[\boldsymbol{\eta}\cdot\boldsymbol{x'}-b(\boldsymbol{x'})\tau]}}{\int_{\mathbb{R}^n} e^{4\pi[\boldsymbol{\eta}\cdot\boldsymbol{v}-b(\boldsymbol{v})\tau]}\, d\boldsymbol{v}} d\boldsymbol{y'}\, dt'\, d\boldsymbol{x'}\, d\boldsymbol{\eta}\, d\tau$$

$$= \int_{\mathbb{R}^{2n+1}} f(\boldsymbol{x'},\boldsymbol{y'},t') \int_0^\infty e^{-2\pi\tau[b(\boldsymbol{x'})+b(\boldsymbol{x})+i(t'-t)]}$$

$$\left(\int_{\mathbb{R}^n} \frac{e^{2\pi\boldsymbol{\eta}\cdot[\boldsymbol{x}+\boldsymbol{x'}-i(\boldsymbol{y'}-\boldsymbol{y})]}}{\int_{\mathbb{R}^n} e^{4\pi[\boldsymbol{\eta}\cdot\boldsymbol{v}-b(\boldsymbol{v})\tau]}\, d\boldsymbol{v}}\, d\boldsymbol{\eta} \right) d\tau d\boldsymbol{x'}\, d\boldsymbol{y'}\, dt'.$$

Therefore, the Szegő kernel is given by

$$\int_0^\infty e^{-2\pi\tau[b(\boldsymbol{x'})+b(\boldsymbol{x})+i(t'-t)]} \left(\int_{\mathbb{R}^n} \frac{e^{2\pi\boldsymbol{\eta}\cdot[\boldsymbol{x}+\boldsymbol{x'}-i(y'-y)]}}{\int_{\mathbb{R}^n} e^{4\pi[\boldsymbol{\eta}\cdot\boldsymbol{v}-b(\boldsymbol{v})\tau]}\, d\boldsymbol{v}}\, d\boldsymbol{\eta} \right) d\tau.$$

This finishes the proof of Proposition A.1. □

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