Modular Forms, Borcherds Lifts and Gross–Zagier Type CM Value Formulas

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Dedicated to the memory of my grandparents.

Abstract

In this thesis, we employ the theory of modular forms and the theory of Borcherds lifts to study three topics related to number theory, namely, representations of integers by quadratic forms, Gross–Zagier type CM value formulas, and explicit constructions of Borcherds products on a Picard modular surface.

In Chapter 1, we give an overview of this thesis and briefly describe the background, motivation and main results.

In Chapter 2, we establish Ramanujan–Mordell type formulas for for the number of representations of integers by the quadratic forms

$$x_1^2 + \cdots + x_k^2 + m\left(x_{k+1}^2 + \cdots + x_{2k}^2\right)$$

for $m \in \{2, 3, 4, 7, 11, 23\}$.

In Chapter 3, we show that the difference of a Hauptmodul for a genus zero group $\Gamma_0(N)$ as a Hilbert modular function on $Y_0(N) \times Y_0(N)$ is a Borcherds lift of type $(2, 2)$. As applications, we derive Monster denominator formula like product expansions for these Hilbert modular functions and certain Gross–Zagier type CM value formulas.

In Chapter 4, we construct canonical bases for the spaces of weakly holomorphic modular forms with poles supported at the cusp ∞ for $\Gamma_0(4)$ of integral weight k for $k \leq -1$, and we make use of the basis elements for the case $k = -1$ to construct explicit Borcherds products on unitary group $\mathrm{U}(2, 1)$.

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Contents

Chapter 1

Introduction

1.1 Modular Forms and Number Theory

During the course of the past decades, the theory of modular forms has played an increasingly central role in number theory. For example, Wile's proof [91] of the Fermat's Last Theorem relies on a deep connection between elliptic curves and modular forms, the so-called modularity theorem which also serves as a key ingredient in the famous Gross–Zagier–Zhang formula [101]. In addition, the theory of modular forms is very useful and powerful in revealing arithmetic information of certain number theoretic objects. For example, Zagier's traces of singular moduli,

$$\sum_{Q \in \mathcal{Q}_D^{prim,+}/\mathrm{SL}_2(\mathbb{Z})} \frac{j(\tau_Q) - 744}{|\mathrm{PSL}_2(\mathbb{Z})_Q|},$$

can be viewed [103] as the Fourier coefficients of a weight $3/2$ modular form on $\Gamma_0(4)$, and thus it can be interpreted by the number of representations of an integer by a sum of three squares since the generating functions of both of these two number theoretic sequences lie in the same one-dimensional space of modular forms. Such a simple interpretation indeed follows from those very special properties owned by a modular form, which have somehow endowed it with some remarkable connections with number theory via its Fourier coefficients that may encode some significant arithmetic consequence or may be interpreted number theoretically. Similar to how we interpret Zagier's traces of singular moduli above, for a sequence of numerics of particular interest in number theory, if one can realize its generating function as a modular form, one may expect some arithmetic information to be revealed or an interesting formulation of the sequence via investigating the associated modular form. For example, if one lets $\mathcal{R}_4(n)$ denotes the number of representations of an integer n by a sum of four squares, i.e.,

$$\mathcal{R}_4(n) = \#\left\{(x_i) \in \mathbb{Z}^4 \middle|\, x_1^2 + x_2^2 + x_3^2 + x_4^2 = n\right\},$$

then one can check that the generating function $\sum_{n=0}^{\infty} \mathcal{R}_4(n) q^n$ is a weight 2 modular forms on $\Gamma_0(4)$. The theory of modular forms tells us that the space of such modular forms is spanned by $4E_2(4\tau) - E_2(\tau)$ and $2E_2(2\tau) - E(\tau)$ where

$$E_2(\tau) = 1 - 24 \sum_{n=1}^{\infty} \sigma(n) q^n$$

is the normalized weight 2 Eisenstein series, and $\sigma(n) = \sum_{d|n} d$ is the divisor function of order 1, and thus comparing coefficients, one can deduce that

$$\sum_{n=0}^{\infty} \mathcal{R}_4(n) q^n = \frac{1}{3}\left(4E_2(4\tau) - E_2(\tau)\right),$$

which implies that

$$\mathcal{R}_4(n) = 8\sigma(n) - 32\sigma(n/4) > 0, \tag{1.1.1}$$

and simply recovers the famous Lagrange's four-squares theorem stating that every integer is representable by a sum of four squares. Similarly, Fermat's two-squares theorem can be interpreted in terms of modular forms. In the most general consideration, for the number of representations $\mathcal{R}_m(n)$ of an integer n by a sum of m squares, one of the most traditional and classic numeric sequences in number theory, it is known that the associated generating function $\theta_m(z) := \sum_{n=0}^{\infty} \mathcal{R}_m(n) q^n$ is a weight $\frac{m}{2}$ modular form on $\Gamma_0(4)$ with some character. Such an observation has motivated number theorists to study the modular form $\theta_m(z)$ and its Fourier coefficients in order to obtain some information about $\mathcal{R}_m(n)$. For instance, when $m = 2k$ is even, through investigating the function $\theta_{2k}(z)$, one of the greatest Indian mathematicians, Ramanujan [78, 80], discovered an interesting and incredible closed formula called the Ramanujan–Mordell formula (see Theorem 2.1.1) for $\mathcal{R}_{2k}(n)$, which explicitly expresses $\mathcal{R}_{2k}(n)$ in terms of divisor functions like (1.1.1) and certain lower order terms.

Now in the view of modular forms and algebraic geometry, we can tell that the essence of the existence of such a formulation simply follows from the structure formula [39] of the space of modular forms, which tells that the space of modular forms of integral weight can be written as a direct sum of the space of Eisenstein series and the space of cusp forms, together with the fact that the modular curve for $\Gamma_0(4)$, the matrix group generated by $\Gamma_0(4)$ and its Fricke involution $\begin{pmatrix} 0 & -1 \\ 4 & 0 \end{pmatrix}$, is a genus zero smooth projective curve over $\mathbb{Q}$. Following this idea, in Chapter 2 (see Theorem 2.2.1), we establish Ramanujan–Mordell type formulas for the number of solutions in integers of the equations

$$x_1^2 + \cdots + x_k^2 + m\left(x_{k+1}^2 + \cdots + x_{2k}^2\right) = n$$

for $m \in \{2, 3, 4, 7, 11, 23\}$.

1.2 Borcherds Lifts

As we have known so far, classical modular forms are originally defined on $\mathbb{H}$ for congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$. We should also note, in the view of algebraic groups, that $\mathrm{SL}_2 \cong \mathrm{Spin}(1,2)$, where the spin group of type $(1,2)$ satisfies the exact sequence

$$1 \to \{\pm 1\} \to \mathrm{Spin}(1,2) \to \mathrm{SO}(1,2) \to k^*/(k^*)^2.$$

So it is very natural for us to ask if we can define and construct modular forms on locally symmetric spaces for orthogonal groups or unitary groups. The study of this subject originates from the seminal work [20, 21] of Borcherds, in which he develops a remarkable method to construct meromorphic modular forms on a locally symmetric space for $\mathrm{O}(n,2)$, whose associated variety is the well known Shimura variety of orthogonal type $(n,2)$, from some weakly holomorphic modular form f for the Weil representation of $\mathrm{SL}_2(\mathbb{Z})$ via regularizing integrals called theta-lifts of f against the Siegel theta function. We now call the resulting regularized integrals Borcherds lifts (see Subsection 3.2.1 for details) denoted by $\Phi(z,f)$ of type $(n,2)$ and the associated meromorphic modular forms Borcherds forms $\Psi(z,f)$ of type $(n,2)$. Moreover, Borcherds shows that the Borcherds form $\Psi(z,f)$ has a beautiful product representation called Borcherds product for its Fourier expansion near a cusp of the orthogonal Shimura variety.

In the study of the so-called Monster Lie Algebra, a famous formula called monster denominator formula [18] is related to one of the most famous modular forms, the Klein's modular j-invariant, via

$$j(z_1) - j(z_2) = (q_1^{-1} - q_2^{-1}) \prod_{m,n>0} (1 - q_1^m q_2^n)^{c(mn)}, \tag{1.2.1}$$

where $j(\tau) = q^{-1} + \sum_{n=0}^{\infty} c(n)q^n$. It has been proved by some mathematicians using various method, but using the theory of Borcherds lifts, we can now see a much simpler explanation. As we can show that $j(z_1) - j(z_2)$ as a Hilbert modular function for $\mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z})$ is indeed a Borcherds form $\Psi((z_1, z_2), f)$ of type $(2, 2)$, whose input f can be realized as $j - 744$. Then the right hand side of (1.2.1) follows from the Borcherds product representation of $\Psi((z_1, z_2), j - 744)$ at the cusp of the associated Shimura variety that is identified with the cusp $(i\infty, i\infty)$ of $Y(1) \times Y(1)$.

The essence of the fact that $j(z_1) - j(z_2)$ is a Borcherds form is partially due to the fact that the projective curve $X(1) = \mathrm{SL}_2(\mathbb{Z}) \backslash (\mathbb{H} \cup \mathbb{P}^1(\mathbb{Q}))$ is of genus zero and $j(z)$ is a uniformizer from $X(1)$ to $\mathbb{P}^1(\mathbb{C})$, where we call a congruence subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ a genus zero group if the associated modular curve of Γ is of genus zero. Similarly, there exists a uniformizer also called Hauptmodul mapping from the modular curve $X(\Gamma)$ to $\mathbb{P}^1(\mathbb{C})$. A natural question is to ask if the difference of a Hauptmodul for a genus zero group is a Borcherds form. Motivated by this interesting question, in Chapter 3, we extend the above result to a family of genus zero subgroups called Hecke subgroups $\Gamma_0(N)$ of $\mathrm{SL}_2(\mathbb{Z})$ by a uniform approach.

Theorem 1.2.1 (see Theorem 3.1.5)**.** *Let $\pi_N(\tau)$ be a Hauptmodul for a genus zero group $\Gamma_0(N)$ for $N \geq 2$. Then $\pi_N(z_1) - \pi_N(z_2)$ is a Borcherds form $\Psi((z_1, z_2), F_N)$ of type $(2, 2)$ for some weakly holomorphic modular function F_N for the Weil representation of $\mathrm{SL}_2(\mathbb{Z})$.*

This result will play a crucial role in extending the celebrated Gross–Zagier CM value formula [46] to $X_0(N)$, which describes a very interesting number theoretic phenomenon, the prime factorization of the rational norm of the difference of singular moduli. We will say more details about this in the following section. As a quick consequence of

Theorem 1.2.1, one will also see in Chapter 3 that we derive monster denominator identity like product expansion for $\pi_N(z_1) - \pi_N(z_2)$, which is related [19] to the twisted denominator formula for certain generalized Kac–Moody superalgebra.

Corollary 1.2.2 (see Corollary 3.1.6). *Let $\pi_N(\tau)$ be a Hauptmodul for a genus zero group $\Gamma_0(N)$ and define for $d|N$,*

$$\sum_{\ell=-1}^{\infty} A(\ell, d) q^{\ell}$$

$$= \frac{2}{\lambda_{2,N}|\Gamma_0(N):\Gamma_1(N)|} \left\{ \sum_{\substack{s\in\mathcal{C}(\Gamma_1(N)) \\ s \text{ regular} \\ m_s=d}} [(\pi_N|\, M_s)_0 - A_s(0)] + \sum_{\substack{s\in\mathcal{C}(\Gamma_1(N)) \\ s \text{ irregular} \\ m_s=d}} \frac{1}{h_s} [(\pi_N|\, M_s)_0 - A_s(0)] \right\},$$

where $\lambda_{2,N} = 2$ or 1 depending on whether $N = 2$ or not. Then we have

$$\pi_N(z_1) - \pi_N(z_2) = (q_1^{-1} - q_2^{-1}) \prod_{m,n>0} \prod_{d|N} \left(1 - (q_1^m q_2^n)^{\frac{N}{d}}\right)^{A(mn,d)} \tag{1.2.2}$$

where $q_j = \exp(2\pi i z_j)$.

1.3 CM Values of Modular Functions

In the study of number theory, it is traditionally of particular interest in investigating the values of modular functions evaluated at imaginary quadratic points, which are called CM values of modular functions. By the theory of complex multiplication (see, e.g., [85] or [34]), we know that CM values of modular functions are somehow related to abelian extensions of number fields. For example, let $-d$ be a negative fundamental discriminant, and let τ_Q be the unique imaginary quadratic point in $\mathbb{H}$ arising from a quadratic form $Q(X,Y) = aX^2 + bXY + cY^2$ of discriminant $-d$, then the value $j(\tau_Q)$ generates the

Hilbert class field (the maximal unramified abelian extension) of $\mathbb{Q}(\tau_Q) = \mathbb{Q}(\sqrt{-d})$. For instance, if $d = -3$ which is of class number 1, then the Hilbert class field of $\mathbb{Q}(\sqrt{-3})$ is itself. So it is expected that the CM value $j\left(\frac{1+\sqrt{-3}}{2}\right)$ belongs to $\mathbb{Q}(\sqrt{-3})$, which is actually 0. More interestingly, the Galois conjugates of $j(\tau_Q)$ are simply $j(\tau_{Q'})$, where Q' runs over the equivalence classes of the set $\mathcal{Q}_d^{prim,+}$ of primitive and positive-definite quadratic forms of discriminant $-d$ modulo the action of $\mathrm{SL}_2(\mathbb{Z})$. Such CM values are algebraic integers of degree $h(-d)$, the class number of $\mathbb{Q}(\sqrt{-d})$, and are classically called singular moduli (see, e.g., [46]). It was discovered by many mathematicians that some interesting number theoretic phenomena happens to the rational norm of the difference of singular moduli. For example, by a glance at the following data

$$\begin{aligned} j(i) - j\left(\frac{1+\sqrt{-3}}{2}\right) &= 1728 = 2^6 3^3, \\ j\left(\frac{1+\sqrt{-163}}{2}\right) - j\left(\frac{1+\sqrt{-3}}{2}\right) &= -262537412640768000 = -2^{18}3^3 5^3 23^3 29^3, \\ j\left(\frac{1+\sqrt{-163}}{2}\right) - j(i) &= -262537412640769728 = -2^6 3^6 7^2 11^2 19^2 127^2 163, \end{aligned}$$

one may note that all of these big numbers have only very small prime factors. In fact, the integrality of these three numbers follow from the fact that -3, -4 and -163 are all fundamental discriminants of class number 1. For higher class number, for example,

$$j\left(\frac{1+\sqrt{-267}}{2}\right) - j\left(\frac{1+\sqrt{-3}}{2}\right) \in \mathbb{Q}(\sqrt{-267}, \sqrt{-3}) \backslash \mathbb{Q},$$

where -267 is of class number 2, considering its rational norm, we have

$$\begin{aligned} &\mathrm{Norm}\left(j\left(\frac{1+\sqrt{-267}}{2}\right) - j\left(\frac{1+\sqrt{-3}}{2}\right)\right) \\ &= -400141877656135887171571050086 4 = -2^{36}3^6 7^6 11^3 17^3 47^3, \end{aligned}$$

which again can be factorized into very small prime factors. All of these interesting phenomena have motivated number theorists to look for an explanation and even a description of the general pattern. In their celebrated work [46], Gross and Zagier first solved this mystery, and they discover and proved a remarkable formula that (up to sign) gives the factorization of the rational norm of the difference of singular moduli, namely,

$$\left|\mathrm{Norm}\left(j\left(\frac{D+\sqrt{-D}}{2}\right)-j\left(\frac{d+\sqrt{-d}}{2}\right)\right)\right| = \prod_{\substack{x,n,n'\in\mathbb{Z}\\ n,n'>0\\ x^2+4nn'=dD}} n^{\epsilon(n')}, \tag{1.3.1}$$

where $-D$ and $-d$ are negative fundamental discriminants with $(D,d)=1$, $\epsilon(n')$ is multiplicative, and $\epsilon(p)$ is defined via the local Hilbert symbol at a prime p, and here we have assumed $-d,-D<-4$ for simplicity. We now call such a formula the Gross–Zagier CM value formula. It was first proved by Gross and Zagier by two different methods, algebraic and analytic, and since then, it has been reproved by many mathematicians by various different methods (see, e.g., [40, 58, 84, 94]), of which an interesting one is from the point of view of Borcherds lifts.

In their recent work [28], Bruinier, Kudla and Yang compute and express the value of a Borcherds lift of type O(n, 2) summing over a so-called big CM-cycle, which is a 0-cycle associated to a CM field by some means, in terms of the Fourier coefficients of an incoherent Eisenstein series of weight one. We now call their formula big CM value formula (see Subsection 3.3.1 for details). This can be viewed as a generalization of Gross–Zagier CM value formula in the view of that

$$\log|j(z_1)-j(z_2)|^4 = -\Phi((z_1,z_2);j-744) \tag{1.3.2}$$

is a Borcherds lift of type $(2,2)$, which is a variant of what we have seen in the previous section, and the rational norm (1.3.1) is equivalent to

$$\sum_{(Q_D,Q_d)\in\mathcal{Q}_D^{prim,+}/\mathrm{SL}_2(\mathbb{Z})\times\mathcal{Q}_d^{prim,+}/\mathrm{SL}_2(\mathbb{Z})} \log|j(\tau_{Q_D})-j(\tau_{Q_d})|^4 = -\sum_{(z_1,z_2)\in Z(U)_K} \Phi((z_1,z_2);j-744)$$

where the sum on the right is over a CM-cycle $Z(U)_K$ of the underlying Shimura variety for $\mathrm{O}(2,2)$ that can be identified with $\mathcal{Q}_D^{prim,+}/\mathrm{SL}_2(\mathbb{Z}) \times \mathcal{Q}_d^{prim,+}/\mathrm{SL}_2(\mathbb{Z})$ on the left by some means. Then applying Bruinier, Kudla and Yang's big CM value formula to the sum on the right and computing relevant Fourier coefficients of an incoherent Eisenstein series of weight one, one can obtain an equivalent version of Gross–Zagier CM value formula.

Inspired by the idea mentioned above and Theorem 1.2.1, we look into the $\Gamma_0(N)$ cases and compute several examples one of which tells that

$$\prod_{(Q_D,Q_d)\in\mathcal{Q}_D(3)/\Gamma_0(3)\times\mathcal{Q}_d(3)/\Gamma_0(3)} |\pi_3(\tau_{Q_D})-\pi_3(\tau_{Q_d})| = 885699026404416 = 2^6 3^{24} 7^2,$$

where $\mathcal{Q}_d(3)$ denotes the set of primitive and positive definite binary quadratic forms $aX^2+bXY+cY^2$ of discriminant $-d$ with $(a,3)=1$, as a big number, also has very small prime factors. This observation indicates that the interesting prime factorization phenomenon may not merely happen to $j(z)$, but also to $\pi_N(z)$. Such an observation has greatly motivated us to look for this pattern and describe it mathematically, which results in the second main result of Chapter 3.

Theorem 1.3.1 (see Theorem 3.1.10)**.** *Let $E_i=\mathbb{Q}(\sqrt{d_i})$ be two imaginary quadratic fields of fundamental discriminants d_i with $(d_1,d_2)=1$. Let $F=\mathbb{Q}(\sqrt{D})$ with $D=d_1d_2$ and $E=E_1E_2=\mathbb{Q}(\sqrt{d_1},\sqrt{d_2})$. Let $\pi_p(\tau)$ be a Hauptmodul of $\Gamma_0(p)$. Then*

$$\sum_{([\mathfrak{a}_1],[\mathfrak{a}_2])\in S(p,d_1,d_2)} \log|\pi_p(\tau_{\varphi(\mathfrak{a}_1)})-\pi_p(\tau_{\varphi(\mathfrak{a}_2)})|$$

$$= -\frac{|S(p,d_1,d_2)|w_1 w_2}{32h(E_1)h(E_2)} \left(\sum_{\substack{t=\frac{2m+D+\sqrt{D}}{2} \\ |2m+D|<\sqrt{D} \\ m\in\mathbb{Z}}} a\left(\frac{t}{\sqrt{D}}, \phi_{0,0}\right) + \frac{24}{p-1}\sum_{k=1}^{p-1} a_0(\phi_{0,k}) \right) \tag{1.3.3}$$

where

$$S(p,d_1,d_2) := \{([\mathfrak{a}_1],[\mathfrak{a}_2]) \in \mathrm{Cl}(E_{1,p}) \times \mathrm{Cl}(E_{2,p}) : \exists \textit{ fractional ideals } \mathfrak{a}_i \textit{ such that } \mathrm{N}(\mathfrak{a}_1) = \mathrm{N}(\mathfrak{a}_2)\},$$

$\mathrm{Cl}(E_{i,p})$ *denote the ring class group of conductor* p *of* E_i*,* w_i *is the number of units of* E_i*, the map* φ *is defined by* $\varphi([a, p\frac{b+\sqrt{d}}{2}]) = [a, \frac{b+\sqrt{d}}{2}]$ *for an integral ideal representative* $\mathfrak{a} = [a, p\frac{b+\sqrt{d}}{2}]$ *in* $\mathrm{Cl}(E_{i,p})$*, and* $a\left(\frac{t}{\sqrt{D}}, \phi_{0,0}\right)$ *and* $a_0(\phi_{0,k})$ *are computed and expressed explicitly in Subsection* 3.3.3 *of Chapter* 3*.*

It is worthwhile remarking that the way we use to obtain Theorem 1.3.1 would work for all $\Gamma_0(N)$. The primality is merely assumed for simplicity.

1.4 Unitary Borcherds Products

Finally, we conclude this chapter with a brief description of the motivation of Chapter 4.

With the study of Borcherds lifts of orthogonal type getting deeper and deeper, lots of explicit examples besides (1.2.1) have been computed, and one finds that the theory of Borcherds lifts is somehow very helpful in the study of the arithmetic aspects of varieties associated to orthogonal groups. For example, the theory of Borcherds lifts can imply the modularity of a generating function of a sequence of certain special divisors [30, Thm. 3.51]. Interestingly, the theory has been recently generalized to unitary case by Hofmann in [50] (see, also, [27]), in which he established the theory of Borcherds lifts on unitary groups $\mathrm{U}(m, 1)$ by constructing an embedding from a Shimura variety of unitary type $(m, 1)$ to a Shimura variety of orthogonal type $(2m, 2)$. Such a generalization gives ones a broader view of the relationship between the theory of Borcherds lifts and arithmetic geometry, and is somehow helpful in studying the arithmetic aspects of varieties associated to unitary groups. However, unlike the theory of orthogonal Borcherds lifts, such a generalization is still at a beginning stage, whose concrete examples are still underrepresented in the literature. In general, there are two key ingredients in computing Borcherds products; one is the lift input, a weakly holomorphic modular forms of weight $1 - m$ for the Weil representation, which is usually easy to construct directly, and the other is the so-called Weyl chamber, a connected component of a $(2m - 1)$-dimensional real manifold, which is in general not easy to compute and describe explicitly. As a prototype, in Chapter 4 (see Theorem 4.3.5), we compute and construct the first family of explicit and concrete examples of unitary Borcherds products on the so-called Picard modular surface which is a 2-dimensional unitary Shimura variety for some finite index subgroup of $\mathrm{U}(2, 1)$. As one will see in Chapter 4, to achieve our goal, we do a bit more as we also construct canonical bases for the spaces of weakly holomorphic modular forms of weight $-k \leq -1$ on $\Gamma_0(4)$ with poles supported only at $i\infty$ (see Theorem 4.2.1), and explicitly compute the Weyl chamber (see Lemma 4.3.2) containing the cusp at which we compute the associated Borcherds products.

This thesis is based on the work [37], [93], [96] and [100].

Chapter 2

Representations of Integers by Certain $2k$-ary Quadratic Forms

2.1 Introduction

In the long history of number theory, one of the classical problems is to give an explicit formula for the number of ways that one can express a positive integer n as a sum of $2k$ squares, that is, the number of integral solutions of

$$x_1^2 + x_2^2 + \cdots + x_{2k-1}^2 + x_{2k}^2 = n,$$

which we denote by $\mathcal{R}_k(n)$. It is known from the theory of modular forms that, in general,

$$\mathcal{R}_k(n) = \Delta_k(n) + \mathcal{E}_k(n)$$

where $\Delta_k(n)$ is a generalized divisor function and $\mathcal{E}_k(n)$ is a function of order substantially lower than that of $\Delta_k(n)$. Formulas for $\mathcal{R}_k(n)$ in this fashion have been found and studied by many mathematicians. For $k = 1$, 2, 3 and 4, i.e., sums of 2, 4, 6 and 8 squares, (reformulated) formulas for $\mathcal{R}_k(n)$ were originally due to Jacobi [53],

$$\mathcal{R}_1(n) = 4\sum_{d|n}\left(\frac{-4}{d}\right), \tag{2.1.1}$$

$$\mathcal{R}_2(n) = 8\sum_{d|n} d - 32\sum_{d|\frac{n}{4}} d, \tag{2.1.2}$$

$$\mathcal{R}_3(n) = -4\sum_{d|n}\left(\frac{-4}{d}\right)d^2 + 16\sum_{d|n}\left(\frac{-4}{n/d}\right)d^2, \tag{2.1.3}$$

$$\mathcal{R}_4(n) = 16\sum_{d|n} d^3 - 32\sum_{d|\frac{n}{2}} d^3 + 256\sum_{d|\frac{n}{4}} d^3 \tag{2.1.4}$$

where, here and throughout this chapter, $\left(\frac{\cdot}{\cdot}\right)$ denotes the Jacobi symbol. The result for $k = 5$, i.e., sum of 10 squares, was due (without proof) in part to Eisenstein [42], and fully described (without proof) by Liouville [64]. The results for $1 \leq k \leq 9$ were all proved by Glaisher [44]. In around 1916, this classical problem was "completely" solved by Ramanujan [78], [80, Eqs. (145)–(147)]. To state Ramanujan's result, we need the well-known Dedekind eta function

$$\eta(\tau) := q^{1/24}\prod_{j=1}^{\infty}(1 - q^j)$$

where, here and throughout this chapter, τ denotes a complex number with positive imaginary part and $q = e^{2\pi i\tau}$. For brevity, in the remainder of this chapter, we write η_m for $\eta(m\tau)$ for any positive integer m. In addition, for brevity, we define $\sigma_k(n)$, $\sigma^{\infty}_{k,\chi_{-4}}(n)$ and $\sigma^{0}_{k,\chi_{-4}}(n)$ to be the divisor functions,

$$\sigma_k(n) = \sum_{d|n} d^k,$$

$$\sigma^{\infty}_{k,\chi_{-4}}(n) = \sum_{d|n}\left(\frac{-4}{d}\right)d^k,$$

$$\sigma^{0}_{k,\chi_{-4}}(n) = \sum_{d|n}\left(\frac{-4}{n/d}\right)d^k,$$

with the convention that they are defined to be 0 if n is not a positive integer. In order to be consistent with previous notation, we now reformulate and summarize Ramanujan's result in Theorem 2.1.1 below.

Theorem 2.1.1 (Ramanujan–Mordell). *Suppose k is a positive integer. Then there are unique rational numbers $c_{j,k}$ depending on j and k such that*

$$\mathcal{R}_k(n) \tag{2.1.5}$$
$$= \begin{cases} -\dfrac{2k}{B_{k,4}}\left(\dfrac{\sigma^{\infty}_{k-1,\chi_{-4}}(n)+(-1)^{(k-1)/2}2^{k-1}\sigma^{0}_{k-1,\chi_{-4}}(n)}{1+\delta_{k,1}}\right) & \text{if } k \text{ is odd,}\\ -\dfrac{2k}{B_k}\left(\dfrac{(-1)^{k/2}\sigma_{k-1}(n)-(1+(-1)^{k/2})\sigma_{k-1}(n/2)+2^k\sigma_{k-1}(n/4)}{-1+2^k}\right) & \text{if } k \text{ is even} \end{cases}$$
$$+ \sum_{1\le j\le (k-1)/4} c_{j,k}a_{j,k}(n)$$

where, here and throughout this chapter, $\delta_{\cdot,\cdot}$ denotes the Kronecker delta by

$$\delta_{m,n} = \begin{cases} 1 & \text{if } m=n,\\ 0 & \text{otherwise,} \end{cases}$$

B_k is the kth ordinary Bernoulli number, $B_{k,4}$ is the kth generalized Bernoulli number of order 4 defined by

$$\frac{t}{e^{4t}-1}\sum_{j=1}^{4}\chi_{-4}(j)e^{jt} = \sum_{n=0}^{\infty} B_{n,4}\frac{t^n}{n!}, \tag{2.1.6}$$

and the numbers $a_{j,k}(n)$ are defined by

$$\sum_{n=0}^{\infty} a_{j,k}(n)q^n = \frac{\eta_2^{10k}}{(\eta_1\eta_4)^{4k}} \times \frac{(\eta_1\eta_4)^{24j}}{\eta_2^{48j}}.$$

However, Ramanujan had never given a proof to Theorem 2.1.1. It was in fact proved first by Mordell [68] utilizing the theory of modular forms. Thus, it has become a common practice to give credits to him, and we now call it the Ramanujan–Mordell Theorem. An elementary proof of the Ramanujan–Mordell Theorem was given by Cooper in [35] by making skillful use of Ramanujan's ${}_1\psi_1$ formula. If we consider the generating functions associated to both sides of (2.1.5), the Ramanujan–Mordell Theorem can be interpreted in terms of well-known Eisenstein series and is equivalent to the following, which is Ramanujan's original formulation [78], [80, Eqs. (145)–(147)].

Theorem 2.1.2 (Ramanujan–Mordell)**.** *Suppose k is a positive integer. Let $\theta(\tau)$ be Ramanujan's theta function defined by*

$$\theta(\tau) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{\eta_2^5}{\eta_1^2\eta_4^2} \tag{2.1.7}$$

where the η-quotient representation after the second equality is due to Jacobi [53]. Let $E_k(\tau)$ be the normalized Eisenstein series of weight k on $\mathrm{SL}_2(\mathbb{Z})$ defined by

$$E_k(\tau) = 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty} \sigma_{k-1}(n)q^n = \textit{normalization of} \sum_{(c,d)\neq(0,0)} \frac{1}{(c\tau+d)^k}. \tag{2.1.8}$$

Furthermore, let $F_k(\tau)$ be defined by

$$F_k(\tau) = \begin{cases} \dfrac{E^{\infty}_{k,\chi_{-4}}(\tau) + (-1)^{(k-1)/2}2^{k-1}E^{0}_{k,\chi_{-4}}(\tau)}{1+\delta_{k,1}} & \textit{if } k \textit{ is odd,} \\ \dfrac{(-1)^{k/2}E_k(\tau) - (1+(-1)^{k/2})E_k(2\tau) + 2^kE_k(4\tau)}{2^k-1} & \textit{if } k \textit{ even} \end{cases}$$

where $E^{\infty}_{k,\chi_{-4}}(\tau)$ and $E^{0}_{k,\chi_{-4}}(\tau)$ are the normalized Eisenstein series of weight k on $\Gamma_0(4)$ with character χ_{-4} defined by

$$E^{\infty}_{k,\chi_{-4}}(\tau) = 1 - \frac{2k}{B_{k,4}}\sum_{n=1}^{\infty} \sigma^{\infty}_{k-1,\chi_{-4}}(n)q^n = \textit{normalization of} \sum_{(c,d)\neq(0,0)} \frac{\left(\frac{-4}{c}\right)}{(c\tau+d)^k}, \tag{2.1.9}$$

and

$$E^{0}_{k,\chi_{-4}}(\tau) = \delta_{k,1} - \frac{2k}{B_{k,4}}\sum_{n=1}^{\infty} \sigma^{0}_{k-1,\chi_{-4}}(n)q^n = \textit{normalization of} \sum_{(c,d)\neq(0,0)} \frac{\left(\frac{-4}{d}\right)}{(4c\tau+d)^k}, \tag{2.1.10}$$

and let $x = x(\tau)$ be defined by

$$x = x(\tau) := \frac{\eta_1^{24}\eta_4^{24}}{\eta_2^{48}}.$$

Then there are unique rational numbers $c_{j,k}$ depending on j and k such that

$$\theta(\tau)^{2k} = F_k(\tau) + \theta(\tau)^{2k} \sum_{j=1}^{\frac{(k-1)}{4}} c_{j,k} x^j. \tag{2.1.11}$$

Motivated by the work of Ramanujan, in this chapter, we aim to extend the Ramanujan–Mordell Theorem to the cases

$$x_1^2 + \cdots + x_k^2 + m\left(x_{k+1}^2 + \cdots + x_{2k}^2\right) = n$$

for $m \in \{2, 3, 4, 7, 11, 23\}$, and establish analogous formulas for $(\theta(\tau)\theta(m\tau))^k$.

This chapter is organized as follows. In Section 2.2, we state our main results, and as illustrations, we also present some examples that follow from the general case we obtain. Proof of the main result will be given in Section 2.3. In the last section, we conclude this chapter with some remarks, which explain the essence of the existence of these Ramanujan–Mordell type formulas.

2.2 Statement of Results

In this section, we state the main result of this chapter and give various examples as special cases of our general formula. Before that, throughout the remainder of this chapter, we write $\chi_D(\cdot) = \left(\frac{D}{\cdot}\right)$ and $\psi_m = \left(\frac{\cdot}{m}\right)$, and in addition, we define $\sigma_{k,\varepsilon}^{\infty}(n)$ and $\sigma_{k,\varepsilon}^{0}(n)$ to be the divisor functions by

$$\sigma_{k,\varepsilon}^{\infty}(n) = \sum_{d|n} \varepsilon(d) d^k \quad \text{and} \quad \sigma_{k,\varepsilon}^{0}(n) = \sum_{d|n} \varepsilon(n/d) d^k$$

for a given character ε. Now we are ready to state the main result of this chapter in the following theorem.

Theorem 2.2.1. *Suppose k is a positive integer. Let $E_k(\tau)$ be the normalized Eisenstein series of weight k on $\mathrm{SL}_2(\mathbb{Z})$ defined as in (2.1.8).*

1. *For $m \in \{3, 7, 11, 23\}$, let ℓ_m be defined by*

$$\ell_m = \begin{cases} \frac{m+1}{8} - \frac{1}{2} & \text{if } m = 3 \text{ or } 11, \text{ and } k \text{ is odd,} \\ \frac{m+1}{8} - 1 & \text{otherwise.} \end{cases}$$

Let $F_{k,m}(\tau)$ be defined by

$$F_{k,m}(\tau) = \begin{cases} \dfrac{G_{k,m}(\tau) + 2^k G_{k,m}(4\tau)}{(2^k+1)(1+\delta_{k,1})} & \text{if } m = 3 \text{ or } 11 \text{ and } k \text{ is odd,} \\ \dfrac{G_{k,m}(\tau) - 2G_k(2\tau) + 2^k G_k(4\tau)}{(2^k-1)(1+\delta_{k,1})} & \text{if } m = 7 \text{ or } 23 \text{ and } k \text{ is odd,} \\ \dfrac{G_{k,m}(\tau) - 2G_k(2\tau) + 2^k G_{k,m}(4\tau)}{(2^k-1)(1+(-m)^{k/2})} & \text{if } k \text{ is even,} \end{cases}$$

where

$$G_{k,m} := \begin{cases} E^{\infty}_{k,\psi_m}(\tau) + (-m)^{(k-1)/2} E^{0}_{k,\psi_m}(\tau) & \text{if } k \text{ is odd,} \\ E_k(\tau) + (-m)^{k/2} E_k(m\tau) & \text{if } k \text{ is even,} \end{cases}$$

$E^{\infty}_{k,\psi_m}(\tau)$ and $E^{0}_{k,\psi_m}(\tau)$ are the normalized Eisenstein series of weight 2 on $\Gamma_0(m)$ with character ψ_m defined by

$$E^{\infty}_{k,\psi_m}(\tau) = 1 - \frac{2k}{B_{k,m}} \sum_{n=1}^{\infty} \sigma^{\infty}_{k-1,\psi_m}(n) q^n$$

and

$$E^{0}_{k,\psi_m}(\tau) = \delta_{k,1} - \frac{2k}{B_{k,m}} \sum_{n=1}^{\infty} \sigma^{0}_{k-1,\psi_m}(n) q^n,$$

and $B_{k,m}$ is the kth generalized Bernoulli number of order m defined by

$$\frac{t}{e^{mt}-1} \sum_{j=1}^{m-1} \psi_m(j) e^{jt} = \sum_{n=0}^{\infty} B_{n,m} \frac{t^n}{n!}, \tag{2.2.1}$$

and finally, let $x_m = x_m(\tau)$ be defined by

$$x_m = x_m(\tau) := \frac{(\eta_1\eta_4\eta_m\eta_{4m})^{24/(m+1)}}{(\eta_2\eta_{2m})^{48/(m+1)}}.$$

Then there are unique rational numbers $c_{j,k,m}$ depending on j, k and m such that

$$(\theta(\tau)\theta(m\tau))^k = F_{k,m}(\tau) + (\theta(\tau)\theta(m\tau))^k \sum_{j=1}^{\ell_m} c_{j,k,m} x_m^j. \tag{2.2.2}$$

2. *For $m = 2$, let ℓ_2 be defined by*

$$\ell_2 = \begin{cases} \frac{k-1}{2} & \text{if } k \text{ is odd,} \\ \frac{k-2}{2} & \text{if } k \text{ is even.} \end{cases}$$

Let $F_{k,2}(\tau)$ be defined by

$$F_{k,2}(\tau) = \begin{cases} \dfrac{E^{\infty}_{k,\chi_{-2}}(\tau) + (-8)^{(k-1)/2} E^{0}_{k,\chi_{-2}}(\tau)}{1+\delta_{k,1}} & \text{if } k \text{ is odd,} \\ \dfrac{(-1)^{\frac{k}{2}} E_k(\tau) - (-1)^{\frac{k}{2}} E_k(2\tau) - 2^{\frac{k}{2}} E_k(4\tau) + 8^{\frac{k}{2}} E_k(8\tau)}{2^{\frac{k}{2}}(2^k - 1)} & \text{if } k \text{ is even,} \end{cases}$$

where $E^{\infty}_{k,\chi_{-2}}(\tau)$ and $E^{0}_{k,\chi_{-2}}(\tau)$ are the normalized Eisenstein series of weight k on $\Gamma_0(8)$ with character χ_{-2} defined by

$$E^{\infty}_{k,\chi_{-2}}(\tau) = 1 - \frac{2k}{B_{k,8}} \sum_{n=1}^{\infty} \sigma^{\infty}_{k-1,\chi_{-2}}(n) q^n$$

and

$$E^{0}_{k,\chi_{-2}}(\tau) = \delta_{k,1} - \frac{2k}{B_{k,8}} \sum_{n=1}^{\infty} \sigma^{0}_{k-1,\chi_{-2}}(n) q^n,$$

and $B_{k,8}$ is the kth generalized Bernoulli number of order 8 defined by

$$\frac{t}{e^{8t}-1} \sum_{j=1}^{7} \chi_{-2}(j) e^{jt} = \sum_{n=0}^{\infty} B_{n,8} \frac{t^n}{n!}, \tag{2.2.3}$$

and let $x_2 = x_2(\tau)$ *be defined by*

$$x_2(\tau) = \left(\frac{\eta_1\eta_8}{\eta_2\eta_4}\right)^8.$$

Then there are unique rational numbers $c_{j,k,2}$ *depending on* j *and* k *such that*

$$(\theta(\tau)\theta(2\tau))^k = F_{k,2}(\tau) + (\theta(\tau)\theta(2\tau))^k \sum_{j=1}^{\ell_2} c_{j,k,2} x_2^j. \tag{2.2.4}$$

3. *For* $m = 4$, *let* ℓ_4 *be defined by*

$$\ell_4 = \begin{cases} k-2 & \text{if } k \text{ is odd,} \\ k-1 & \text{if } k \text{ is even.} \end{cases}$$

Let $F_{k,4}(\tau)$ *be defined by*

$$F_{k,4}(\tau)$$
$$= \begin{cases} \frac{1}{2}\left(E^{\infty}_{1,\chi_2}(\tau) - E^{\infty}_{1,\chi_2}(2\tau) + 2E^{\infty}_{1,\chi_{-2}}(4\tau)\right) & \text{if } k = 1, \\ \frac{1}{2}\begin{pmatrix} (-1)^{\frac{k+1}{2}} E^{\infty}_{k,\chi_{-2}}(\tau) - (-1)^{\frac{k+1}{2}} E^{\infty}_{k,\chi_{-2}}(2\tau) + 2E^{\infty}_{k,\chi_{-2}}(4\tau) \\ -(-1)^{\frac{k+1}{2}} E^{0}_{k,\chi_{-2}}(\tau) + 2^{k-1}E^{0}_{k,\chi_{-2}}(2\tau) - 2^{2k-1}E^{0}_{k,\chi_{-2}}(4\tau) \end{pmatrix} & \text{if } k \geq 3 \text{ and is odd,} \\ \dfrac{(-1)^{\frac{k}{2}} E_k(\tau) - (-1)^{\frac{k}{2}} E_k(2\tau) - 2^k E_k(8\tau) + 4^k E_k(16\tau)}{2^k(2^k-1)} & \text{if } k \text{ is even,} \end{cases}$$

where $E^{\infty}_{k,\chi_{-4}}(\tau)$ *and* $E^{0}_{k,\chi_{-4}}(\tau)$ *are the normalized Eisenstein series of weight* k *on* $\Gamma_0(4)$ *with character* χ_{-4} *defined as in* (2.1.9)–(2.1.10), *and let* $x_4 = x_4(\tau)$ *be defined by*

$$x_4(\tau) = \frac{(\eta_1\eta_4\eta_{16})^4}{(\eta_2\eta_8)^6}.$$

Then there are unique rational numbers $c_{j,k,4}$ *depending on* j *and* k *such that*

$$(\theta(\tau)\theta(4\tau))^k = F_{k,4}(\tau) + (\theta(\tau)\theta(4\tau))^k \sum_{j=1}^{\ell_4} c_{j,k,4} x_4^j. \tag{2.2.5}$$

In the remainder of this section we describe some well-known results that can obtained as special cases from Theorem 2.2.1. To make the examples be in analogy with (2.1.1)–(2.1.4), let us first denote by $r(1^k m^k; n)$ the number of integral solutions of the equation

$$x_1^2 + \cdots + x_k^2 + m\left(x_{k+1}^2 + \cdots + x_{2k}^2\right) = n.$$

Then from Theorem 2.2.1, we can easily deduce the following examples.

Example 2.2.2. *For $k = 1$, the cases $m = 3$, 7, 11 or 23 of Theorem 2.2.1 give*

$$r(1^1 3^1; n) = 2\sum_{d|n}\left(\frac{d}{3}\right) + 4\sum_{d|\frac{n}{4}}\left(\frac{d}{3}\right), \tag{2.2.6}$$

$$r(1^1 7^1; n) = 2\sum_{d|n}\left(\frac{d}{7}\right) - 4\sum_{d|\frac{n}{2}}\left(\frac{d}{7}\right) + 4\sum_{d|\frac{n}{4}}\left(\frac{d}{7}\right), \tag{2.2.7}$$

$$r(1^1 11^1; n) = \frac{2}{3}\sum_{d|n}\left(\frac{d}{11}\right) + \frac{4}{3}\sum_{d|\frac{n}{4}}\left(\frac{d}{11}\right) + \frac{4}{3}a(n), \tag{2.2.8}$$

$$\textit{where} \sum_{n=1}^{\infty} a(n)q^n = \eta_2\eta_{22},$$

$$\begin{aligned} r(1^1 23^1; n) = {} & \frac{2}{3}\sum_{d|n}\left(\frac{d}{23}\right) - \frac{4}{3}\sum_{d|\frac{n}{2}}\left(\frac{d}{23}\right) + \frac{4}{3}\sum_{d|\frac{n}{4}}\left(\frac{d}{23}\right) \\ & + \frac{4}{3}a_1(n) - \frac{4}{3}a_2(n), \end{aligned} \tag{2.2.9}$$

$$\textit{where} \sum_{n=1}^{\infty} a_1(n)q^n = \frac{\eta_2^3\eta_{46}^3}{\eta_1\eta_4\eta_{23}\eta_{92}} \ \textit{and} \sum_{n=1}^{\infty} a_2(n)q^n = \eta_2\eta_{46}.$$

The identity (2.2.6) was first stated in an equivalent form by Lorenz [65, p. 420] and was also given by Ramanujan independently in his second notebook [79, Ch. 19, Entry 3]. See Berndt [14, pp. 223–224], Fine [43, p. 73, (31.16), (31.22)] and Hirschhorn [49] for proofs and further information.

The identities (2.2.7) also appears in Ramanujan's second notebook [79, Ch. 19, Entry 17] and proofs have been given by Berndt [14, pp. 302–304]. The identity (2.2.7) has also been proved by Pall [72].

The identities (2.2.8) and (2.2.9) have been recently proved by the author of this thesis in [99].

Example 2.2.3. *For $m = 3$, the cases $k = 2$, 3, 4 and 6 of Theorem 2.2.1 give*

$$r(1^2 3^2; n) = 4\left(\sum_{d|n} d - 2\sum_{d|\frac{n}{2}} d - 3\sum_{d|\frac{n}{3}} d + 4\sum_{d|\frac{n}{4}} d + 6\sum_{d|\frac{n}{6}} d - 12\sum_{d|\frac{n}{12}} d\right), \tag{2.2.10}$$

$$r(1^3 3^3; n) = 3\sum_{d|n}\left(\frac{n/d}{3}\right) d^2 + 24\sum_{d|\frac{n}{4}}\left(\frac{n/4d}{3}\right) - \sum_{d|n}\left(\frac{d}{3}\right) d^2 \tag{2.2.11}$$

$$- 8\sum_{d|\frac{n}{4}}\left(\frac{d}{3}\right) d^2 + 4a(n), \quad \textit{where} \sum_{n=1}^{\infty} a(n)q^n = \eta_2^3\eta_6^3,$$

$$r(1^4 3^4; n) = \frac{8}{5}\left(\sum_{d|n} d^3 - 2\sum_{d|\frac{n}{2}} d^3 + 9\sum_{d|\frac{n}{3}} d^3 + 16\sum_{d|\frac{n}{4}} d^3 - 18\sum_{d|\frac{n}{6}} d^3\right. \tag{2.2.12}$$

$$\left. + 144\sum_{d|\frac{n}{12}} d^3\right) + \frac{32}{5}a(n), \quad \textit{where} \sum_{n=1}^{\infty} a(n)q^n = \frac{\eta_2^8\eta_6^8}{\eta_1^2\eta_3^2\eta_4^2\eta_{12}^2},$$

$$r(1^6 3^6; n) = 1 + \frac{4}{13}\left(\sum_{d|n} d^5 - 2\sum_{d|\frac{n}{2}} d^5 - 27\sum_{d|\frac{n}{3}} d^5 + 64\sum_{d|\frac{n}{4}} d^5\right. \tag{2.2.13}$$

$$\left. + 54\sum_{d|\frac{n}{6}} d^5 - 1728\sum_{d|\frac{n}{12}} d^5\right) + \frac{152}{13}a_1(n) - \frac{256}{13}a_2(n),$$

$$\textit{where} \sum_{n=1}^{\infty} a_1(n)q^n = \frac{\eta_2^{18}\eta_6^{18}}{\eta_1^6\eta_3^6\eta_4^6\eta_{12}^6} \ \textit{and} \sum_{n=1}^{\infty} a_2(n)q^n = \eta_2^6\eta_6^6.$$

The identity (2.2.10) was first stated without proof in an equivalent form by Liouville [61, 63]. See Pepin [74], Bachmann [11], Kloosterman [54] and Alaca et al. [2] for proofs. The identity (2.2.10) also appears in Ramanujan's second notebook [79, Ch. 19, Entry 3]. Proofs have been given by Fine [43, (31.4)–(31.43) and (33.2)] and Berndt [14, pp. 223–226].

A formula equivalent to (2.2.11) was proved by Alaca et al. in [5], where it was attributed to Berkovich and Yeśilyurt.

The identity (2.2.12) was proved by Alaca and Williams [8]. A formula similar to (2.2.12), in which the coefficients in $\eta_2^2\eta_2^2\eta_3^2\eta_6^2$ are given as a quadruple sum, has been given by Beridze [12].

A formula equivalent to (2.2.13) was given by Alaca [1]; that formula involves three cusp forms on the right hand side, while ours involves only two[1].

Example 2.2.4. *For $m = 7$, the cases $k = 2$ and 3 of Theorem 2.2.1 give*

$$r(1^2 7^2; n) = \frac{4}{3}\left(\sum_{d|n} d - 2\sum_{d|\frac{n}{2}} d + 4\sum_{d|\frac{n}{4}} d - 7\sum_{d|\frac{n}{7}} d + 14\sum_{d|\frac{n}{14}} d - 28\sum_{d|\frac{n}{28}} d\right) + \frac{8}{3}a(n), \tag{2.2.14}$$

$$\text{where } \sum_{n=1}^{\infty} a(n)q^n = \frac{\eta_2^4\eta_{14}^4}{\eta_1\eta_4\eta_7\eta_{28}},$$

$$\begin{aligned} r(1^3 7^3; n) = {} & \frac{7}{8}\sum_{d|n}\left(\frac{n/d}{7}\right)d^2 - \frac{7}{4}\sum_{d|\frac{n}{2}}\left(\frac{n/2d}{7}\right)d^2 + 7\sum_{d|\frac{n}{4}}\left(\frac{n/4d}{7}\right)d^2 \\ & - \frac{1}{8}\left(\sum_{d|n}\left(\frac{d}{7}\right)d^2 - 2\sum_{d|\frac{n}{2}}\left(\frac{d}{7}\right)d^2 + 8\sum_{d|\frac{n}{4}}\left(\frac{d}{7}\right)d^2\right) \\ & + \frac{21}{4}a_1(n) - 6a_2(n), \end{aligned} \tag{2.2.15}$$

$$\text{where } \sum_{n=1}^{\infty} a_1(n)q^n = \frac{\eta_2^9\eta_{14}^9}{\eta_1^3\eta_4^3\eta_7^3\eta_{28}^3} \text{ and } \sum_{n=1}^{\infty} a_2(n)q^n = \eta_2^3\eta_{14}^3.$$

Identity equivalent to (2.2.14) has been proved in [38]. The identity (2.2.15) arises in the theory of 7-cores and were proved by Berkovich and Yesilyurt [13].

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[1] This is because, if $b(q) = \eta_1^6\eta_3^6$, then $b(q) + 12b(q^2) + 64b(q^4) + b(-q) = 0$.

Example 2.2.5. *For $m = 2$ or 4, the cases $k = 1$, 2, 3 or 4 of Theorem 2.2.1 give*

$$r(1^1 2^1; n) = 2 \sum_{d|n} \left(\frac{-2}{d}\right), \tag{2.2.16}$$

$$r(1^1 4^1; n) = 2 \sum_{d|n} \left(\frac{-4}{d}\right) - 2 \sum_{d|\frac{n}{2}} \left(\frac{-4}{d}\right) + 4 \sum_{d|\frac{n}{4}} \left(\frac{-4}{d}\right), \tag{2.2.17}$$

$$r(1^2 2^2; n) = 4 \sum_{d|n} d - 4 \sum_{d|\frac{n}{2}} d + 8 \sum_{d|\frac{n}{4}} d - 32 \sum_{d|\frac{n}{8}} d, \tag{2.2.18}$$

$$r(1^2 4^2; n) = 2 \sum_{d|n} d - 2 \sum_{d|\frac{n}{2}} d + 8 \sum_{d|\frac{n}{8}} d - 32 \sum_{d|\frac{n}{16}} d + 2a(n), \tag{2.2.19}$$

$$\text{where } \sum_{n=1}^{\infty} a(n) q^n = \frac{\eta_2^4 \eta_8^4}{\eta_4^4},$$

$$r(1^3 2^3; n) = -\frac{2}{3} \sum_{d|n} \left(\frac{-2}{d}\right) d^2 + \frac{16}{3} \sum_{d|n} \left(\frac{-2}{n/d}\right) d^2 + \frac{4}{3} a(n), \tag{2.2.20}$$

$$\text{where } \sum_{n=1}^{\infty} a(n) q^n = \eta_1^2 \eta_2 \eta_4 \eta_8^2,$$

$$\begin{aligned} r(1^3 4^3; n) = {} & -2 \sum_{d|n} \left(\frac{-4}{d}\right) d^2 + 2 \sum_{d|\frac{n}{2}} \left(\frac{-4}{d}\right) d^2 - 4 \sum_{d|\frac{n}{4}} \left(\frac{-4}{d}\right) d^2 \\ & + 2 \sum_{d|n} \left(\frac{-4}{n/d}\right) d^2 - 8 \sum_{d|\frac{n}{2}} \left(\frac{-4}{n/2d}\right) d^2 + 64 \sum_{d|\frac{n}{4}} \left(\frac{-4}{n/4d}\right) d^2 + 6a(n), \end{aligned} \tag{2.2.21}$$

$$\text{where } \sum_{n=1}^{\infty} a(n) q^n = \frac{\eta_2^9 \eta_8^9}{\eta_1^2 \eta_4^8 \eta_{16}^2},$$

$$r(1^4 2^4; n) = 4 \sum_{d|n} d^3 - 4 \sum_{d|\frac{n}{2}} d^3 - 16 \sum_{d|\frac{n}{4}} d^3 + 64 \sum_{d|\frac{n}{8}} d^3 + 4a(n), \tag{2.2.22}$$

$$\text{where } \sum_{n=1}^{\infty} a(n) q^n = \eta_2^4 \eta_4^4,$$

$$r(1^4 4^4; n) = \sum_{d|n} d^3 - \sum_{d|\frac{n}{2}} d^3 - 16 \sum_{d|\frac{n}{8}} d^3 + 256 \sum_{d|\frac{n}{16}} d^3 + 7a_1(n) - 12a_2(n), \tag{2.2.23}$$

$$\text{where } \sum_{n=1}^{\infty} a_1(n) q^n = \frac{\eta_2^{14} \eta_8^{14}}{\eta_1^4 \eta_4^{12} \eta_{16}^4} \text{ and } \sum_{n=1}^{\infty} a_2(n) q^n = \frac{\eta_2^8 \eta_8^8}{\eta_4^8}.$$

The formula (2.2.16) was proved by Shen [87]. The formula (2.2.17) was due in part to Ramanujan [14, Entry 25(i), (iii), p. 40], [15, Entry 18, p. 152]. The formulas (2.2.18) and (2.2.19) were stated without proof by Liouville [60, 62] and proved by Pepin [73, 75], Bachmann [11], and Alaca *et al.* [2] . The formulas (2.2.20), and (2.2.21)–(2.2.23) are due to Alaca *et al.* [3] and Alaca *et al.* [6, 7, 4], respectively.

2.3 Proof of Theorem 2.2.1

This section is devoted to proving Theorem 2.2.1. The proof hinges on the following preliminary results.

Lemma 2.3.1. *For $m \in \{3, 7, 11, 23\}$ and any odd integer $k \geq 0$, and for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(m)$, we have*

$$E^0_{k,\psi_m}\left(\frac{a\tau+b}{c\tau+d}\right) = \psi_m(d)(c\tau+d)^k E^0_{k,\psi_m}(\tau), \tag{2.3.1}$$

$$E^\infty_{k,\psi_m}\left(\frac{a\tau+b}{c\tau+d}\right) = \psi_m(d)(c\tau+d)^k E^\infty_{k,\psi_m}(\tau), \tag{2.3.2}$$

$$E^\infty_{k,\psi_m}\left(-\frac{1}{m\tau}\right) = \frac{1}{i\sqrt{m}}(m\tau)^k E^0_{k,\psi_m}(\tau), \tag{2.3.3}$$

$$E^0_{k,\psi_m}\left(-\frac{1}{m\tau}\right) = \frac{\sqrt{m}}{i}\tau^k E^\infty_{k,\psi_m}(\tau). \tag{2.3.4}$$

For $m = 2$, and any odd integer $k \geq 0$, and for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(8)$, we have

$$E^\infty_{k,\chi_{-2}}\left(\frac{a\tau+b}{c\tau+d}\right) = \chi_{-2}(d)(c\tau+d)^k E^\infty_{k,\chi_{-2}}(\tau), \tag{2.3.5}$$

$$E^0_{k,\chi_{-2}}\left(\frac{a\tau+b}{c\tau+d}\right) = \chi_{-2}(d)(c\tau+d)^k E^0_{k,\chi_{-2}}(\tau), \tag{2.3.6}$$

$$E^\infty_{k,\chi_{-2}}\left(-\frac{1}{8\tau}\right) = \left(\frac{\tau}{i}\right)^k E^0_{k,\chi_{-2}}(\tau), \tag{2.3.7}$$

$$E^0_{k,\chi_{-2}}\left(-\frac{1}{8\tau}\right) = (-i8\tau)^k E^\infty_{k,\chi_{-2}}(\tau). \tag{2.3.8}$$

For $m = 4$, and any odd integer $k \geq 0$, and for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$, *we have*

$$E^\infty_{k,\chi_{-4}}\left(\frac{a\tau+b}{c\tau+d}\right) = \chi_{-4}(d)(c\tau+d)^k E^\infty_{k,\chi_{-4}}(\tau), \tag{2.3.9}$$

$$E^0_{k,\chi_{-4}}\left(\frac{a\tau+b}{c\tau+d}\right) = \chi_{-4}(d)(c\tau+d)^k E^0_{k,\chi_{-4}}(\tau), \tag{2.3.10}$$

$$E^\infty_{k,\chi_{-4}}\left(-\frac{1}{4\tau}\right) = \left(\frac{\tau}{i}\right)^k E^0_{k,\chi_{-4}}(\tau), \tag{2.3.11}$$

$$E^0_{k,\chi_{-4}}\left(-\frac{1}{4\tau}\right) = (-i4\tau)^k E^\infty_{k,\chi_{-4}}(\tau). \tag{2.3.12}$$

Proof. These are well-known results, see, e.g., Cooper [36], Kolberg [55], Miyake [67, Ch. 7], Serre [86, pp. 79–83] or Stein [89, Sec. 5]. □

Lemma 2.3.2. *For any even positive integer k, we have*

$$E_k\left(\tau+\frac{1}{2}\right) = -E_k(\tau) + (2^k+2)E_k(2\tau) - 2^k E_k(4\tau). \tag{2.3.13}$$

For $m = 3$ or 11 and any odd positive integer k, we have

$$E^0_{k,\psi_m}\left(\tau+\frac{1}{2}\right) = -E^0_{k,\psi_m}(\tau) + (2^k-2)E^0_{k,\psi_m}(2\tau) + 2^k E^0_{k,\psi_m}(4\tau), \tag{2.3.14}$$

$$E^\infty_{k,\psi_m}\left(\tau+\frac{1}{2}\right) = -E^\infty_{k,\psi_m}(\tau) + (2-2^k)E^\infty_{k,\psi_m}(2\tau) + 2^k E^\infty_{k,\psi_m}(4\tau). \tag{2.3.15}$$

For $m = 7$ or 23 and any odd positive integer k, we have

$$E^0_{k,\psi_m}\left(\tau+\frac{1}{2}\right) = -E^0_{k,\psi_m}(\tau) + (2^k+2)E^0_{k,\psi_m}(2\tau) - 2^k E^0_{k,\psi_m}(4\tau), \tag{2.3.16}$$

$$E^\infty_{k,\psi_m}\left(\tau+\frac{1}{2}\right) = -E^\infty_{k,\psi_m}(\tau) + (2^k+2)E^\infty_{k,\psi_m}(2\tau) - 2^k E^\infty_{k,\psi_m}(4\tau). \tag{2.3.17}$$

For any odd positive integer k, we have

$$E^{\infty}_{k,\chi_{-2}}\left(\tau+\frac{1}{2}\right)=-E^{\infty}_{k,\chi_{-2}}(\tau)+2E^{\infty}_{k,\chi_{-2}}(2\tau), \tag{2.3.18}$$

$$E^{0}_{k,\chi_{-2}}\left(\tau+\frac{1}{2}\right)=-E^{0}_{k,\chi_{-2}}(\tau)+2^{k}E^{0}_{k,\chi_{-2}}(2\tau), \tag{2.3.19}$$

$$E^{\infty}_{k,\chi_{-4}}\left(\tau+\frac{1}{2}\right)=-E^{\infty}_{k,\chi_{-4}}(\tau)+2E^{\infty}_{k,\chi_{-4}}(2\tau), \tag{2.3.20}$$

$$E^{0}_{k,\chi_{-4}}\left(\tau+\frac{1}{2}\right)=-E^{0}_{k,\chi_{-4}}(\tau)+2^{k}E^{0}_{k,\chi_{-4}}(2\tau). \tag{2.3.21}$$

Proof. For (2.3.13), we first observe that

$$\sigma_k(2n)=\left(1+2^k\right)\sigma_k(n)-2^k\sigma_k\left(n/2\right),$$

where $\sigma_k(n/2)$ is defined to be zero if $n/2$ is not a positive integer. Then for k even,

$$\begin{aligned}
&E_k\left(\tau+\frac{1}{2}\right)+E_k(\tau)\\
&=2\left(1-\frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(2n)q^{2n}\right)\\
&=2\left(1-\frac{2k}{B_k}\sum_{n=1}^{\infty}\left((1+2^{k-1})\sigma_{k-1}(n)-2^{k-1}\sigma_{k-1}\left(n/2\right)\right)q^{2n}\right)\\
&=2\left(\left(1+2^{k-1}\right)\left(1-\frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^{2n}\right)-2^{k-1}\left(1-\frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^{4n}\right)\right)\\
&=\left(2+2^k\right)E_k(2\tau)-2^kE_k(4\tau).
\end{aligned}$$

Similarly, we find that

$$\sigma^{0}_{k,\psi_m}(2n)=\begin{cases}(2^{k-1}-1)\sigma^{0}_{k,\psi_m}(n)+2^{k-1}\sigma^{0}_{k,\psi_m}(n/2), & \text{for } m=3 \text{ or } 11,\\[2ex] (1+2^{k-1})\sigma^{0}_{k,\psi_m}(n)-2^{k-1}\sigma^{0}_{k,\psi_m}(n/2), & \text{for } m=7 \text{ or } 23,\end{cases}$$

$$\sigma^{\infty}_{k,\psi_m}(2n)=\begin{cases}(1-2^{k-1})\sigma^{\infty}_{k,\psi_m}(n)+2^{k-1}\sigma^{\infty}_{k,\psi_m}(n/2), & \text{for } m=3 \text{ or } 11,\\[2ex] (1+2^{k-1})\sigma^{\infty}_{k,\psi_m}(n)-2^{k-1}\sigma^{\infty}_{k,\psi_m}(n/2), & \text{for } m=7 \text{ or } 23,\end{cases}$$

and for $m = 2$ or 4,

$$\sigma^{\infty}_{k,\chi_{-m}}(2n) = \sigma^{\infty}_{k,\chi_{-m}}(n) \quad \text{and} \quad \sigma^{0}_{k,\chi_{-m}}(2n) = 2^k \sigma^{0}_{k,\chi_{-m}}(n).$$

By the above observations, identities (2.3.14)–(2.3.21) can be proved in the same fashion, so we omit the details. $\square$

Lemma 2.3.3. *Let* $\mathrm{ord}_z(f)$ *denote the order of vanishing of* $f(\tau)$ *at* $\tau = z$*. Let* $F_{k,m}(\tau)$ *be defined as in Theorem* 2.2.1 *for* $m \in \{2, 3, 4, 7, 11, 23\}$*. We view* $F_{k,m}(\tau)$ *as a modular form on the associated discrete subgroups of the underlying Eisenstein series. Then we have*

$$\mathrm{ord}_{1/2}(F_{k,m}) = \begin{cases} \frac{1}{2} & \textit{if } m = 2 \textit{ and } k \textit{ is odd,} \\ 1 & \textit{if } m = 2 \textit{ and } k \textit{ is even,} \\ 2 & \textit{if } m = 4 \textit{ and } k \textit{ is odd,} \\ 1 & \textit{if } m = 4 \textit{ and } k \textit{ is even,} \\ 1 & \textit{if } m = 3 \textit{ or } 11 \textit{ and } k \textit{ is odd,} \\ 2 & \textit{otherwise.} \end{cases} \tag{2.3.22}$$

Proof. By (2.3.18) and (2.3.19) of Lemma 2.3.2 together with Lemma 2.3.1, we can deduce that for k odd,

$$\begin{aligned} &\left[E^{\infty}_{k,\chi_{-2}}(\tau) + (-8)^{(k-1)/2} E^{0}_{k,\chi_{-2}}(\tau)\right]\Big|_k \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \\ &= (2\tau+1)^{-k} \left[E^{\infty}_{k,\chi_{-2}}\left(\frac{1}{2} + \frac{-1}{4\tau+2}\right) + (-8)^{(k-1)/2} E^{0}_{k,\chi_{-2}}\left(\frac{1}{2} + \frac{-1}{4\tau+2}\right)\right] \\ &= (2\tau+1)^{-k} \Bigg\{ - E^{\infty}_{k,\chi_{-2}}\left(\frac{-1}{4\tau+2}\right) + 2E^{\infty}_{k,\chi_{-2}}\left(\frac{-1}{2\tau+1}\right) \end{aligned}$$

$$+ (-8)^{(k-1)/2}\left[-E^0_{k,\chi_{-2}}\left(\frac{-1}{4\tau+2}\right) + 2^k E^0_{k,\chi_{-2}}\left(\frac{-1}{2\tau+1}\right)\right]\Bigg\}$$

$$= \frac{i2^k}{8^{1/2}}E^0_{k,\chi_{-2}}\left(\frac{\tau}{2}+\frac{1}{4}\right) - \frac{2i}{8^{1/2}}E^0_{k,\chi_{-2}}\left(\frac{\tau}{4}+\frac{1}{8}\right)$$
$$+ \frac{i8^{1/2}(-8)^{(k-1)/2}}{4^k}\left[E^\infty_{k,\chi_{-2}}\left(\frac{\tau}{2}+\frac{1}{4}\right) - E^\infty_{k,\chi_{-2}}\left(\frac{\tau}{4}+\frac{1}{8}\right)\right]$$
$$= Cq^{1/4} + O(q^{1/2})$$

for some nonzero constant C as $\tau \to i\infty$. For $k = 2$,

$$[-E_2(\tau) + E_2(2\tau) - 2E_2(4\tau) + 8E_2(8\tau)]\Big|_2 \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$
$$= (2\tau+1)^{-2}\Bigg[-E_2\left(\frac{1}{2}+\frac{-1}{4\tau+2}\right) + E_2\left(1+\frac{-1}{2\tau+1}\right)$$
$$- 2E_2\left(2+\frac{-1}{(2\tau+1)/2}\right) + 8E_2\left(4+\frac{-1}{(2\tau+1)/4}\right)\Bigg]$$
$$= (2\tau+1)^{-2}\Bigg[E_2\left(\frac{-1}{4\tau+2}\right) - 5E_2\left(\frac{-1}{2\tau+1}\right)$$
$$+ 2E_2\left(\frac{-1}{(2\tau+1)/2}\right) + 8E_2\left(\frac{-1}{(2\tau+1)/4}\right)\Bigg]$$
$$= 4E_2\left(4\tau+2\right) + \frac{12}{\pi i(2\tau+1)} - 5E_2\left(2\tau+1\right) - \frac{30}{\pi i(2\tau+1)}$$
$$+ \frac{1}{2}E_2\left(\tau+\frac{1}{2}\right) + \frac{6}{\pi i(2\tau+1)} + \frac{1}{2}E_2\left(\frac{\tau}{2}+\frac{1}{4}\right) + \frac{12}{\pi i(2\tau+1)}$$
$$= 4E_2\left(4\tau+2\right) - 5E_2\left(2\tau+1\right) + \frac{1}{2}E_2\left(\tau+\frac{1}{2}\right) + \frac{1}{2}E_2\left(\frac{\tau}{2}+\frac{1}{4}\right)$$
$$= Cq^{1/2} + O(q)$$

for some nonzero constant C as $\tau \to i\infty$, and for $k \geq 4$ and even,

$$\left[(-1)^{\frac{k}{2}}E_k(\tau) - (-1)^{\frac{k}{2}}E_k(2\tau) - 2^{\frac{k}{2}}E_k(4\tau) + 8^{\frac{k}{2}}E_k(8\tau)\right]\Big|_k \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

$$
\begin{aligned}
&= (2\tau+1)^{-k}\Bigg[(-1)^{\frac{k}{2}}E_k\left(\frac{1}{2}+\frac{-1}{4\tau+2}\right)-(-1)^{\frac{k}{2}}E_k\left(1+\frac{-1}{2\tau+1}\right)\\
&\qquad\qquad -2^{\frac{k}{2}}E_k\left(2+\frac{-1}{(2\tau+1)/2}\right)+8^{\frac{k}{2}}E_k\left(4+\frac{-1}{(2\tau+1)/4}\right)\Bigg]\\
&= (2\tau+1)^{-k}\Bigg[-(-1)^{\frac{k}{2}}E_k\left(\frac{-1}{4\tau+2}\right)+(-1)^{\frac{k}{2}}(2^k+2)E_k\left(\frac{-1}{2\tau+1}\right)\\
&\qquad\qquad -(-1)^{\frac{k}{2}}2^kE_k\left(\frac{-1}{(2\tau+1)/2}\right)-(-1)^{\frac{k}{2}}E_k\left(\frac{-1}{2\tau+1}\right)\\
&\qquad\qquad -2^{\frac{k}{2}}E_k\left(\frac{-1}{(2\tau+1)/2}\right)+8^{\frac{k}{2}}E_k\left(\frac{-1}{(2\tau+1)/4}\right)\Bigg]\\
&= -(-1)^{\frac{k}{2}}2^kE_k(4\tau+2)+(-1)^{\frac{k}{2}}(2^k+2)E_k(2\tau+1)-(-1)^{\frac{k}{2}}E_k\left(\tau+\frac{1}{2}\right)\\
&\qquad -(-1)^{\frac{k}{2}}E_k(2\tau+1)-2^{-\frac{k}{2}}E_k\left(\tau+\frac{1}{2}\right)+2^{-\frac{k}{2}}E_k\left(\frac{\tau}{2}+\frac{1}{4}\right)\\
&= Cq^{1/2}+O(q)
\end{aligned}
$$

for some nonzero constant C as $\tau \to i\infty$. Together with the fact that the width of $\frac{1}{2}$ of $\Gamma_0(8)$ is 2, the above observations conclude the first two results of (2.3.22).

Making use of corresponding transformation formulas given in Lemma 2.3.1 of $E^{\infty}_{k,\psi_m}(\tau)$, $E^{0}_{k,\psi_m}(\tau)$ for $m \in \{3,7,11,23\}$, $E^{\infty}_{k,\chi_{-4}}(\tau)$ and $E^{0}_{k,\chi_{-4}}(\tau)$ and Lemma 2.3.2 together with the fact that the width of $\frac{1}{2}$ of $\Gamma_0(4m)$ is m, we can deduce the other results of (2.3.22) in a similar fashion, so we omit the details. $\square$

Lemma 2.3.4. *If $f(\tau)=\prod_{d|N}\eta_d^{r_d}$ for some positive integer N with $k=\frac{1}{2}\sum_{d|N}r_d \in \mathbb{Z}$, with the additional properties that*

$$\sum_{d|N} d r_d \equiv 0 \pmod{24}$$

and

$$\sum_{d|N} \frac{N}{d} r_d \equiv 0 \pmod{24},$$

then $f(\tau)$ *satisfies*

$$f\left(\frac{a\tau+b}{c\tau+d}\right)=\chi(d)(c\tau+d)^k f(\tau)$$

for every $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$. *Here the character* χ *is defined by Jacobi symbol* $\chi(d) = \left(\frac{(-1)^k s}{d}\right)$ *where* $s = \prod_{d|N} d^{r_d}$.

Proof. See Gordon and Hughes [45], or Newman [69, 70]. □

Lemma 2.3.5. *Let* a, c *and* N *be positive integers with* $c|N$ *and* $\gcd(a,c)=1$. *If* $f(\tau) = \prod_{d|N} \eta_d^{r_d}$ *satisfies the conditions of Lemma 2.3.4 for* N, *then the order of vanishing* $\mathrm{ord}_{a/c}(f)$ *of* $f(\tau)$ *at the cusp* a/c *is*

$$\frac{N}{24}\sum_{d|N}\frac{\gcd(c,d)^2 r_d}{\gcd(c,N/c)cd}.$$

Proof. See Biagioli [17], Ligozat [59] or Martin [66]. □

Proof of Theorem 2.2.1. Let ℓ_m be defined as in Theorem 2.2.1. Consider the functions

$$f_m(\tau)=\frac{F_{k,m}(\tau)}{\left(\theta(\tau)\theta(m\tau)\right)^k x_m(\tau)^{\ell_m}} \quad \text{and} \quad g_m(\tau)=\frac{1}{x_m(\tau)}.$$

Both $f_m(\tau)$ and $g_m(\tau)$ are analytic on the upper half plane $\mathbb{H}$. By Lemma 2.3.1 and Lemma 2.3.4 we may verify that both $f_m(\tau)$ and $g_m(\tau)$ are invariant under $\Gamma_0(4m)$ and

$$W_e = \left\{ \begin{pmatrix} ae & b \\ 4mc & de \end{pmatrix} \middle| \; a,\, b,\, c,\, d \in \mathbb{Z}, \text{ the determinant is } e \right\}$$

for $e \parallel 4m$, i.e., $e|4m$ and $\gcd(e, 4m/e) = 1$. Therefore, both $f_m(\tau)$ and $g_m(\tau)$ are invariant under $\Gamma_0(4m)^+$, the group obtained from $\Gamma_0(4m)$ by adjoining all of its Atkin-Lehner involutions W_e. Let us analyze their behavior at $\tau = i\infty$. By observing the

Fourier expansions at $\tau = i\infty$, we find that $f_m(\tau)$ has rational coefficients, and

$$f_m(\tau) = \frac{1 + O(q)}{(1 + O(q))^k q^{\ell_m} (1 + O(q))^{\ell_m}} = q^{-\ell_m} + O(q^{-\ell_m+1}).$$

Therefore $f_m(\tau)$ has a pole of order ℓ_m at $i\infty$. Similarly, we note that $g_m(\tau)$ has a simple pole at $\tau = i\infty$. It implies that there exist rational constants $a_{1,k,m}, \ldots, a_{\ell_m,k,m}$ such that the function

$$h_m(\tau) := f_m(\tau) - \sum_{j=1}^{\ell_m} a_{j,k,m} g_m(\tau)^j$$

has no pole at $\tau = i\infty$, that is,

$$h_m(\tau) = a_{0,k,m} + O(q) \quad \text{as } \tau \to i\infty$$

for some constant $a_{0,k,m}$. Let us consider the behavior of $h_m(\tau)$ at $\tau = \frac{1}{2}$. In Lemma 2.3.3, we have shown that

$$\mathrm{ord}_{1/2}(F_{k,m}) = \begin{cases} \frac{1}{2} & \text{if } m = 2 \text{ and } k \text{ is odd,} \\ 1 & \text{if } m = 2 \text{ and } k \text{ is even,} \\ 2 & \text{if } m = 4 \text{ and } k \text{ is odd,} \\ 1 & \text{if } m = 4 \text{ and } k \text{ is even,} \\ 1 & \text{if } m = 3 \text{ or } 11 \text{ and } k \text{ is odd,} \\ 2 & \text{otherwise.} \end{cases}$$

Moreover, by Lemma 2.3.5 together with the η-quotient representation (2.1.7) of $\theta(\tau)$, i.e.,

$$\theta(\tau) = \frac{\eta_2^5}{\eta_1^2 \eta_4^2},$$

we can compute and show that

$$\mathrm{ord}_{\mathfrak{c}_m}(\theta(\tau)\theta(m\tau)) = \begin{cases} \frac{1}{2} & \text{if } m = 2 \text{ and } \mathfrak{c}_m = \frac{1}{2}, \\ 1 & \text{if } m = 4 \text{ and } \mathfrak{c}_m = \frac{1}{2}, \\ 0 & \text{if } m = 4 \text{ and } \mathfrak{c}_m = \frac{1}{4}, \\ \frac{m+1}{4} & \text{if } m \in \{3, 7, 11, 23\} \text{ and } \mathfrak{c}_m = \frac{1}{2}, \end{cases}$$

and

$$\mathrm{ord}_{\mathfrak{c}_m}(x_m) = \begin{cases} -1 & \text{if } m = 2 \text{ or } 4 \text{ and } \mathfrak{c}_m = \frac{1}{2}, \\ 0 & \text{if } m = 4 \text{ and } \mathfrak{c}_m = \frac{1}{4}, \\ -2 & \text{if } m \in \{3, 7, 11, 23\} \text{ and } \mathfrak{c}_m = \frac{1}{2}. \end{cases}$$

Here $\mathfrak{c}_m$ denotes a cusp of $\Gamma_0(4m)$. It is clear that $\mathrm{ord}_{\mathfrak{c}_m}(F_{k,m}) \geq 0$ for $m = 4$ and $\mathfrak{c}_m = \frac{1}{4}$. Thus we can summarize that

$$\mathrm{ord}_{\mathfrak{c}_m}(h_m) = 0 \quad \text{if } m \in \{2, 3, 4, 7, 11, 23\} \text{ and } \mathfrak{c}_m = \frac{1}{2},$$

and

$$\mathrm{ord}_{\mathfrak{c}_m}(h_m) \geq 0 \quad \text{if } m = 4 \text{ and } \mathfrak{c}_m = \frac{1}{4}.$$

Since the set of inequvalent cusps of $\Gamma_0(4m)^+$ is $\{\infty, \frac{1}{2}\}$ for $m \in \{2, 3, 7, 11, 23\}$, or is $\{\infty, \frac{1}{2}, \frac{1}{4}\}$ for $m = 4$, it follows that $h_m(\tau)$ is holomorphic on $X(\Gamma_0(4m)^+) = \Gamma_0(4m)^+ \backslash (\mathbb{H} \cup \mathbb{P}^1(\mathbb{Q}))$, and thus $h_m(\tau)$ is a constant, that is, $h_m(\tau) \equiv a_{0,k,m}$. Moreover, since $\mathrm{ord}_{1/2}(h_m) = 0$, $h_m(\tau)$ does not vanish at $\frac{1}{2}$ and thus $a_{0,k,m} \neq 0$. Therefore, we have

$$f_m(\tau) = \sum_{j=0}^{\ell_m} a_{j,k,m} g_m(\tau)^j,$$

which is equivalent to

$$F_{k,m}(\tau) = (\theta(\tau)\theta(m\tau))^k \sum_{j=0}^{\ell_m} a_{j,k,m} x_m^{\ell - j} = (\theta(\tau)\theta(m\tau))^k \sum_{j=0}^{\ell_m} b_{j,k,m} x_m^j,$$

where $b_{j,k,m} = a_{\ell_m - j,k,m}$. Equating the constant term shows that $b_{0,k,m} = 1$. Now take $b_{j,k,m} = -c_{j,k,m}$ to complete the proof. $\square$

2.4 Concluding Remarks

In this section, we conclude this chapter with some remarks on the essence of existence of the Ramanujan–Mordell type formulas (2.2.2), (2.2.4) and (2.2.5), and an explanation of why the upper indices of the sums on the right hand sides of (2.2.2), (2.2.4) and (2.2.5) cannot be improved further based on the functions $x_m(\tau)$ we use.

1. The essence of existence of (2.2.2), (2.2.4) and (2.2.5) is that for $m \in \{2, 3, 4, 7, 11, 23\}$, $X(\Gamma_0(4m)^+)$ is of genus zero, and the function $\frac{1}{x_m(\tau)}$ is a Hauptmodul of $X(\Gamma_0(4m)^+)$, i.e., a generator of the function field $\mathbb{C}(X(\Gamma_0(4m)^+))$. Since the function $\frac{F_{k,m}(\tau)}{(\theta(\tau)\theta(m\tau))^k x_m^{\ell_m}}$ is in $\mathbb{C}(X(\Gamma_0(4m)^+))$, and it has poles only at $\tau = i\infty$, then $\frac{F_{k,m}(\tau)}{(\theta(\tau)\theta(m\tau))^k x_m^{\ell_m}}$ is a polynomial in $\frac{1}{x_m(\tau)}$.

   In general, for a positive integer m, we may obtain identities for $(\theta(\tau)\theta(m\tau))^k$ similar to (2.2.4) and (2.2.5) if we could construct a function $F_k(\tau)$ by a linear combination of Eisenstein series of weight k such that $\frac{F_k(\tau)}{(\theta(\tau)\theta(m\tau))^k}$ is invariant under some genus zero discrete subgroup Γ of $\mathrm{SL}_2(\mathbb{R})$, and could construct a generator $\pi(\tau)$ for the function field $\mathbb{C}(X(\Gamma))$ such that the locations of zeros of $\pi(\tau)$ are the same as that of $(\theta(\tau)\theta(m\tau))^k$.

2. We now explain that with the Hauptmodul $\frac{1}{x_m(\tau)}$ we use in this chapter, the upper indices ℓ_m cannot be improved any further, i.e., cannot be smaller. From the proof

of Theorem 2.2.1, we can first note that for k fixed, the size of ℓ_m is determined by the order of vanishing of the function $F_{k,m}(\tau)$ at $\frac{1}{2}$; the higher the order is, the smaller ℓ_m will be. Then according to the proof of Lemma 2.3.3, $\mathrm{ord}_{1/2}(F_{k,m})$ is directly related to the definition of $F_{k,m}(\tau)$ as a linear combination of Eisenstein series. Thus it is natural ask whether one could redefine $F_{k,m}(\tau)$ to have higher order at $\frac{1}{2}$. In our cases, this is impossible. For example, for $m = 2$, and $k \geq 4$ and even, first we know that $(\theta(\tau)\theta(2\tau))^k$ and $x_2(\tau)$ are modular forms of weight k on $\Gamma_0(8)$ with trivial character by Lemma 2.3.4. Then we must have $F_{k,2}(\tau)$ be a linear combination of Eisenstein series of weight k on $\Gamma_0(8)$ with trivial character, thus we must have

$$F_{k,2}(\tau) = C_1 E_k(\tau) + C_2 E_k(2\tau) + C_3 E_k(4\tau) + C_4 E_k(8\tau)$$

for some constants $C_1, \ldots, C_4$ since $E_k(m\tau)$ for $m \in \{1, 2, 4, 8\}$ are linearly independent Eisenstein series of weight k on $\Gamma_0(8)$ with trivial character and the dimension of the space spanned by such Eisenstein series is 4. According to the proof of Lemma 2.3.3, in order to have $\mathrm{ord}_{1/2}(F_{k,2}) \geq 1$, we must have $C_4 = 0$. In addition, since $x_2(\tau)$ is invariant under $\Gamma_0(8)^+$, then $\frac{F_{k,2}(\tau)}{(\theta(\tau)\theta(2\tau))^k}$ must also be invariant under $\Gamma_0(8)^+$. Following such modularity, we can deduce that $C_1 = 0$ and $C_2 = 2^{-k/2} C_3$, and thus we have

$$F_{k,2}(\tau) = C_3 \left(2^{-k/2} E_k(2\tau) + E_k(4\tau)\right).$$

However, similar to the proof of Lemma 2.3.3, we can show that the order of vanishing of $2^{-k/2} E_k(2\tau) + E_k(4\tau)$ at $\frac{1}{2}$ is 0. This demonstrates our claim for the case $m = 2$, and $k \geq 4$ and even. For the cases ($m = 2$ and $k = 2$), ($m = 2$ or 4 and $k = 1$), and ($m = 3$ or 7 and $k = 1$), the formulas we obtained do not involve any lower order term in x_m. For the other cases, similar arguments can be applied by the facts [89, Chapter 5] that

(i) the space of Eisenstein series of weight 2 on $\Gamma_0(16)$ is spanned by $\ell E_2(\ell\tau) - E_2(\tau)$ for $\ell|16$ and $\ell \neq 1$, and $\sum_{n=1}^{\infty} \left(\frac{-4}{n}\right) \sigma_1(n)q^n$;

(ii) the space of Eisenstein series of even weight $k \geq 4$ on $\Gamma_0(16)$ is spanned by $E_k(\ell\tau)$ for $\ell|16$ and $\sum_{n=1}^{\infty} \chi_{-4}(n)\sigma_{k-1}(n)q^n$;

(iii) the space of Eisenstein series of odd weight $k \geq 3$ on $\Gamma_0(8)$ with character χ_{-2} is spanned by $E^{\infty}_{k,\chi_{-2}}(\tau)$ and $E^{0}_{k,\chi_{-2}}(\tau)$;

(iv) the space of Eisenstein series of odd weight $k \geq 3$ on $\Gamma_0(16)$ with character χ_{-4} is spanned by $E^{\infty}_{k,\chi_{-4}}(\ell\tau)$ and $E^{0}_{k,\chi_{-4}}(\ell\tau)$ for $\ell|4$;

(v) the space of Eisenstein series of weight 2 on $\Gamma_0(4m)$ for $m \in \{3, 7, 11, 23\}$ is spanned by $\ell E_2(\ell\tau) - E_2(\tau)$ for $\ell|4m$ and $\ell \neq 0$;

(vi) the space of Eisenstein series of even weight $k \geq 4$ on $\Gamma_0(4m)$ for $m \in \{3, 7, 11, 23\}$ is spanned by $E_k(\ell\tau)$ for $\ell|4m$;

(vii) the space of Eisenstein series of odd weight 1 on $\Gamma_0(4m)$ for $m \in \{3, 7, 11, 23\}$ is spanned by $E^{\infty}_{k,\psi_m}(\ell\tau)$ for $\ell|4$;

(viii) the space of Eisenstein series of odd weight $k \geq 3$ on $\Gamma_0(4m)$ for $m \in \{3, 7, 11, 23\}$ is spanned by $E^{\infty}_{k,\psi_m}(\ell\tau)$ and $E^{0}_{k,\psi_m}(\ell\tau)$ for $\ell|4$.

Chapter 3

Difference of a Hauptmodul for $\Gamma_0(N)$ and Gross–Zagier Type CM Value Formulas

3.1 Introduction

In his seminal work [20, 21], Borcherds develops a remarkable method to construct meromorphic modular forms $\Psi(z, f)$ on an orthogonal Shimura variety associated to some rational quadratic space of signature $(n, 2)$ from some weakly holomorphic modular form f for the Weil representation of $\mathrm{SL}_2(\mathbb{Z})$ via regularizing some integrals called theta-lifts against the Siegel theta function. We now call such regularized integrals Borcherds lifts of type $(n, 2)$ denoted by $\Phi(z, f)$ and the associated meromorphic modular forms Borcherds forms of type $(n, 2)$. Moreover, Borcherds shows that $\Psi(z, f)$ has a beautiful product representation called Borcherds product for its Fourier expansion near a cusp of the orthogonal Shimura variety. One of the most famous Borcherds forms is the difference $j(\tau_1)-j(\tau_2)$ of the well-known Klein's modular j-invariant as a Hilbert modular function for $\mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z})$, which results from the Borcherds lift of type $(2, 2)$ with $j - 744$ as its theta-lift input, namely, $j(z_1) - j(z_2) = \Psi(z, j - 744)$. If we write $j(\tau) = q^{-1} +$

$\sum_{n\geq 0} c(n)q^n$ with $q := \exp(2\pi i\tau)$, then computing the Borcherds product of $\Psi(z, j-744)$ near the cusp of the underlying Shimura variety of type $(2,2)$, which is identified with the cusp $(i\infty, i\infty)$ of $Y(1)\times Y(1)$, we can recover the famous Monster denominator formula [18], namely,

$$j(z_1)-j(z_2)=(q_1^{-1}-q_2^{-1})\prod_{m,n>0}(1-q_1^m q_2^n)^{c(mn)}\,.$$

Furthermore, another interesting application of the fact that $j(z_1)-j(z_2)$ is a Borcherds form $\Psi(z, j-744)$ is to serve as a key ingredient in the use of the so called big CM value formula [28] in reproving the interesting and famous Gross–Zagier CM value formula [46], namely,

$$\left|\mathrm{Norm}\left(j\left(\frac{d_1+\sqrt{d_1}}{2}\right)-j\left(\frac{d_2+\sqrt{d_2}}{2}\right)\right)\right|^{\frac{8}{w_1 w_2}}=\prod_{\substack{x,n,n'\in\mathbb{Z}\\ n,n'>0\\ x^2+4nn'=d_1d_2}} n^{\epsilon(n')}, \qquad (3.1.1)$$

where d_1 and d_2 are coprime negative fundamental discriminants, $\epsilon(n')$ is multiplicative, and $\epsilon(p)$ is defined via the local Hilbert symbol at a prime p. Such a formula was first discovered and proved by Gross and Zagier [46], and it presents a beautiful and remarkable prime factorization formula for the rational norm of the algebraic integer $j\left(\frac{d_1+\sqrt{d_1}}{2}\right)-j\left(\frac{d_2+\sqrt{d_2}}{2}\right)$ and reveals the arithmetic information encoded in the exponents of the prime factors via Hilbert symbols. The reproof of the Gross–Zagier CM value formula mentioned above has been recently worked out by Yang and Yin [94], and it yields an equivalent form of (3.1.1) as follows.

Theorem 3.1.1 (Gross and Zagier)**.** *Let* $E_i=\mathbb{Q}(\sqrt{d_i})$ *be two imaginary quadratic fields of fundamental discriminants* d_i *with* $(d_1,d_2)=1$*. Let* $F=\mathbb{Q}(\sqrt{D})$ *with* $D=d_1d_2$ *and*

$E = E_1E_2 = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$. *Then*

$$\sum_{([\mathfrak{a}_1],[\mathfrak{a}_2])\in \mathrm{Cl}(E_1)\times \mathrm{Cl}(E_2)} \log |j(\tau_{\mathfrak{a}_1}) - j(\tau_{\mathfrak{a}_2})|^{\frac{8}{w_1w_2}}$$
$$= \sum_{\substack{t=\frac{2m+D+\sqrt{D}}{2} \\ |2m+D|<\sqrt{D} \\ m\in\mathbb{Z}}} \sum_{\mathfrak{p} \text{ inert in } E/F} \frac{1+\mathrm{ord}_{\mathfrak{p}}(t\mathcal{O}_F)}{2}\rho(t\mathfrak{p}^{-1})\log(N(\mathfrak{p})), \tag{3.1.2}$$

where $\mathrm{Cl}(E_i)$ *is the ideal class group of* E_i, $\tau_{\mathfrak{a}} = \frac{b+\sqrt{d}}{2a}$ *is the unique CM point in the upper half plane* $\mathbb{H}$ *given by the integral ideal* $\mathfrak{a} = [a, \frac{b+\sqrt{d}}{2}]$, w_i *is the number of roots of unity in* E_i, *and for an integral ideal* $\mathfrak{a}$ *of* F

$$\rho(\mathfrak{a}) := |\{\mathfrak{B} \subset \mathcal{O}_E : \mathrm{N}_{E/F}(\mathfrak{B}) = \mathfrak{a}\}|$$

which can be computed via calculating its local factors $\rho_{\mathfrak{q}}(\mathfrak{a})$ *(see* (3.3.14)*).*

Remark 3.1.2. *To see how* (3.1.2) *is equivalent to* (3.1.1)*, we refer the reader to [46, Eq. (7.1)] and [94, Remark 4.1].*

All of these interesting phenomenon and relation mentioned above that are related to $j(\tau)$ have greatly motivated us to look into the genus zero Hecke subgroups $\Gamma_0(N)$ cases. Now recall that for a genus zero congruence subgroup Γ of $\mathrm{SL}_2(\mathbb{R})$ commensurable with $\mathrm{SL}_2(\mathbb{Z})$, the function field on $X(\Gamma)$ can be generated by a single modular function, and such function is called a Hauptmodul for Γ if it has a unique simple pole of residue 1 at the cusp $i\infty$, i.e., it has Fourier expansion of the form $q^{-1/h} + c(0) + c(1)q^{1/h} + \cdots$ with $q = \exp(2\pi i\tau)$ at the cusp $i\infty$ where h is the width of the cusp $i\infty$. in this chapter, we first aim to extend the fact that $j(z_1) - j(z_2)$ is a Borcherds form of type $(2, 2)$ to any genus zero groups $\Gamma_0(N)$ and show that the difference of a Hauptmodul $\pi_N(\tau)$ for a genus zero group $\Gamma_0(N)$ as a Hilbert modular function for $\Gamma_0(N) \times \Gamma_0(N)$ is a Borcherds form of type $(2,2)$ by a uniform approach. These extend Scheithauer's results [81], in which he works only on N square free using the twisted denominator identity of the Monster Lie algebra. The method we use is different from Scheithauer's and is more natural from the analytic point of view. Later, employing the big CM value formula [28] together with the fact that $\pi_N(z_1) - \pi_N(z_2)$ is a Borcherds form shown in this chapter, we derive Gross–Zagier type CM value formulas for $\pi_p(\tau)$ for $p \in \{3, 5, 7, 13\}$, the only odd primes such that $\Gamma_0(p)$ are of genus zero.

Remark 3.1.3. *The way we compute the Gross–Zagier type CM value formulas in this chapter was first initiated in [94] in which Yang and Yin prove a Gross–Zagier type CM value formula for $\pi_2(\tau)$ first conjectured by Yui and Zagier [102]. In general, one can similarly obtain Gross–Zagier type CM value formulas for all $\pi_N(\tau)$ by carefully computing the constant terms of the corresponding theta-lift input (see Lemma* 3.2.8*) and relevant local Whittaker functions (see Subsection* 3.3.1*). We refer the reader to [94, 95] for comprehensive descriptions of these computations, and leave the details to the reader.*

Remark 3.1.4. *Similar formulas for $\Gamma_0(N)+$ with N square free have first been worked out in [19] by Borcherds. In recent work [32], Carnahan obtains similar formulas for completely replicable modular functions.*

Now we state the first main result of this chapter.

Theorem 3.1.5. *Let $\pi_N(\tau)$ be a Hauptmodul for a genus zero group $\Gamma_0(N)$ for $N \geq 2$. Then $\pi_N(z_1) - \pi_N(z_2)$ is a Borcherds form $\Psi(z, F_N)$ of type $(2,2)$ for some weakly holomorphic modular function F_N for the Weil representation of $\mathrm{SL}_2(\mathbb{Z})$.*

Define, here and throughout the remainder of this chapter, $\mathcal{C}(\Gamma_1(N))$ to be the set of inequivalent cusps $s = a_s/c_s \in \mathbb{Q}$ of $\Gamma_1(N)$ with $(a_s, c_s) = 1$, $M_s = \begin{pmatrix} a_s & b_s \\ c_s & d_s \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, $m_s = (c_s, N)$, and write $h_s = N/m_s$. Note that when $N \neq 4$, $\Gamma_1(N)$ has no irregular cusps, and then $h_s = N/m_s$ is the width of cusp $s \in \mathcal{C}(\Gamma_1(N))$. If s is regular write for $0 \le t < h_s$

$$(f|\, M_s)_t = \sum_{n \gg -\infty} A_s(nh_s + t) q^{(nh_s+t)/h_s}$$

where

$$f|\, M_s = \sum_{n \gg -\infty} A_s(n) q^{n/h_s} \quad \text{and} \quad f|\, M_s := f\left(\frac{a_s\tau + b_s}{c_s\tau + d_s}\right).$$

Then we can derive a Monster denominator formula like product expansion for $\pi_N(z_1) - \pi_N(z_2)$ as follows.

Corollary 3.1.6. *Let $\pi_N(\tau)$ be a Hauptmodul for a genus zero group $\Gamma_0(N)$ and define for $d|N$,*

$$\sum_{\ell=-1}^{\infty} A(\ell, d) q^{\ell} = \frac{2}{\lambda_{2,N} |\Gamma_0(N) : \Gamma_1(N)|} \times \left\{ \sum_{\substack{s \in \mathcal{C}(\Gamma_1(N)) \\ s \text{ regular} \\ m_s = d}} [(\pi_N|\, M_s)_0 - A_s(0)] + \sum_{\substack{s \in \mathcal{C}(\Gamma_1(N)) \\ s \text{ irregular} \\ m_s = d}} \frac{1}{h_s} [(\pi_N|\, M_s)_0 - A_s(0)] \right\},$$

where $\lambda_{2,N} = 2$ or 1 depending on whether $N = 2$ or not. Then we have

$$\pi_N(z_1) - \pi_N(z_2) = (q_1^{-1} - q_2^{-1}) \prod_{m,n>0} \prod_{d|N} \left(1 - (q_1^m q_2^n)^{\frac{N}{d}}\right)^{A(mn,d)} \tag{3.1.3}$$

where $q_j = \exp(2\pi i z_j)$.

Here are two concrete examples following from Corollary 3.1.6.

Example 3.1.7. *Let $\pi_5(\tau)$ be a Hauptmodul for $\Gamma_0(5)$ given by*

$$\pi_5(\tau) = \left(\frac{\eta(\tau)}{\eta(5\tau)}\right)^6,$$

where $\eta(\tau)$ is the Dedekind eta function. Then

$$\left(\frac{\eta(z_1)}{\eta(5z_1)}\right)^6 - \left(\frac{\eta(z_2)}{\eta(5z_2)}\right)^6 = (q_1^{-1} - q_2^{-1}) \prod_{m,n>0} (1 - q_1^m q_2^n)^{A(mn,5)} (1 - q_1^{5m} q_2^{5n})^{A(mn,1)}.$$

where

$$\sum_{\ell=-1}^{\infty} A(\ell,5) q^\ell = \left(\frac{\eta(\tau)}{\eta(5\tau)}\right)^6 + 6,$$

$$A(\ell,1) = a(5\ell) \quad \textit{and} \quad \sum_{\ell=0}^{\infty} a(\ell) q^\ell = 125 \left(\frac{\eta(5\tau)}{\eta(\tau)}\right)^6.$$

Example 3.1.8. *Let $\pi_8(\tau)$ be a Hauptmodul for $\Gamma_0(8)$ given by*

$$\pi_8(\tau) = \frac{\eta(\tau)^4 \eta(4\tau)^2}{\eta(2\tau)^2 \eta(8\tau)^4}.$$

Then

$$\frac{\eta(z_1)^4 \eta(4z_1)^2}{\eta(2z_1)^2 \eta(8z_1)^4} - \frac{\eta(z_2)^4 \eta(4z_2)^2}{\eta(2z_2)^2 \eta(8z_2)^4} = \left(q_1^{-1} - q_2^{-1}\right) \prod_{m,n>0} \prod_{d|8} \left(1 - (q_1^m q_2^n)^{\frac{8}{d}}\right)^{A(mn,d)}.$$

where

$$\sum_{\ell=-1}^{\infty} A(\ell,8) q^\ell = \frac{\eta(\tau)^4 \eta(4\tau)^2}{\eta(2\tau)^2 \eta(8\tau)^4} + 4,$$

$$\sum_{\ell=1}^{\infty} A(\ell,4) q^\ell = 4 - 4\frac{\eta(\tau)^4 \eta(8\tau)^4 \eta(2\tau)^2}{\eta(4\tau)^{10}},$$

$$A(\ell,2) = a(2\ell) \quad \textit{and} \quad \sum_{\ell=1}^{\infty} a(\ell) q^\ell = 8 - 8e\left(1/12\right) \frac{\eta(2\tau)^2 \eta(4\tau)^4}{\eta(8\tau)^2 \eta\left(\tau + \frac{1}{4}\right)^4} = 8 - 8 \prod_{n=1}^{\infty} \frac{(1-q^{2n})^2 (1-q^{4n})^4}{(1-q^{8n})^2 (1-(iq)^n)^4},$$

$$A(\ell,1) = b(8\ell) \quad \textit{and} \quad \sum_{\ell=1}^{\infty} b(\ell) q^\ell = 32 \frac{\eta(8\tau)^4 \eta(2\tau)^2}{\eta(4\tau)^2 \eta(\tau)^4}.$$

Following from Corollary 3.1.6, we can relate the canonical basis elements in z_2 of the space of weakly holomorphic modular functions for $\Gamma_0(N)$ with pole supported only at $i\infty$ to the logarithmic derivative of $\pi_N(z_1) - \pi_N(z_2)$ with respect to z_1 as follows.

Corollary 3.1.9. *Let $\pi_N(\tau)$ be a Hauptmodul for a genus zero $\Gamma_0(N)$. Then*

$$-\frac{1}{2\pi i}\frac{\pi_N'(z_1)}{\pi_N(z_1)-\pi_N(z_2)} = \sum_{n=0}^{\infty} P_{N,n}(\pi_N(z_2))q_1^n$$

for some polynomial $P_{N,n}$ of degree n. Then $P_{N,n}(\pi_N(z_2)) = q_2^{-n} + O(q_2)$.

The second main results of this chapter are the Gross–Zagier type CM value formulas in the following.

Theorem 3.1.10. *Let $E_i = \mathbb{Q}(\sqrt{d_i})$ be two imaginary quadratic fields of fundamental discriminants d_i with $(d_1, d_2) = 1$. Let $F = \mathbb{Q}(\sqrt{D})$ with $D = d_1 d_2$ and $E = E_1 E_2 = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$. Let $\pi_p(\tau)$ be a Hauptmodul of $\Gamma_0(p)$. Then*

$$\begin{aligned}\sum_{([\mathfrak{a}_1],[\mathfrak{a}_2])\in S(p,d_1,d_2)} &\log|\pi_p(\tau_{\varphi(\mathfrak{a}_1)}) - \pi_p(\tau_{\varphi(\mathfrak{a}_2)})| \\ &= -\frac{|S(p,d_1,d_2)|w_1w_2}{32h(E_1)h(E_2)}\left(\sum_{\substack{t=\frac{2m+D+\sqrt{D}}{2}\\ |2m+D|<\sqrt{D}\\ m\in\mathbb{Z}}} a\left(\frac{t}{\sqrt{D}},\phi_{0,0}\right) + \frac{24}{p-1}\sum_{k=1}^{p-1} a_0(\phi_{0,k})\right) \qquad (3.1.4)\end{aligned}$$

where

$$S(p,d_1,d_2) := \{([\mathfrak{a}_1],[\mathfrak{a}_2]) \in \mathrm{Cl}(E_{1,p}) \times \mathrm{Cl}(E_{2,p}) : \exists \textit{ fractional ideals } \mathfrak{a}_i \textit{ such that } \mathrm{N}(\mathfrak{a}_1) = \mathrm{N}(\mathfrak{a}_2)\},$$

$\mathrm{Cl}(E_{i,p})$ *denote the ring class group of conductor p of E_i, w_i is the number of units of E_i, the map φ is defined by $\varphi([a, p\frac{b+\sqrt{d}}{2}]) = [a, \frac{b+\sqrt{d}}{2}]$ for an integral ideal representative*

$\mathfrak{a} = [a, p\frac{b+\sqrt{d}}{2}]$ *in* $\mathrm{Cl}(E_{i,p})$*, and* $a\left(\frac{t}{\sqrt{D}}, \phi_{0,0}\right)$ *and* $a_0(\phi_{0,k})$ *are computed and expressed explicitly in Section* 6*.*

In particular, the left hand side of (3.1.4) *can be reformulated in the language of quadratic forms*

$$\sum_{([\mathfrak{a}_1],[\mathfrak{a}_2])\in S(p,d_1,d_2)} \log|\pi_p(\tau_{\varphi(\mathfrak{a}_1)}) - \pi_p(\tau_{\varphi(\mathfrak{a}_2)})| = \sum_{(Q_1,Q_2)\in S_{\mathcal{Q}}(p,d_1,d_2)} \log|\pi_p(\tau_{Q_1}) - \pi_p(\tau_{Q_2})|,$$

where

$$S_{\mathcal{Q}}(p,d_1,d_2) := \{(Q_1,Q_2) \in \mathcal{Q}_{d_1}(p)/\Gamma_0(p) \times \mathcal{Q}_{d_2}(p)/\Gamma_0(p)|\ Q_1(1,0) = Q_2(1,0)\},$$

$\mathcal{Q}_d(p)$ *denotes the set of primitive and positive definite binary quadratic forms* $aX^2 + bXY + cY^2$ *of discriminant* d *with* $(a,p) = 1$*, and* τ_Q *is the unique CM point in* $\mathbb{H}$ *given by* $Q(\tau,1) = 0$ *for a quadratic form* $Q(X,Y)$*.*

Example 3.1.11. *Taking* $p = 3$*,* $d_1 = -8$ *and* $d_2 = -11$*, we can first check that* $S(3,-8,-11) = \mathrm{Cl}(E_{1,3}) \times \mathrm{Cl}(E_{2,3})$*, and then by computing the corresponding* $a\left(\frac{t}{\sqrt{D}}, \phi_{0,0}\right)$ *and* $a_0(\phi_{0,k})$ *via Section 6, we obtain the following prime factorization for the product of CM values*

$$\prod_{([\mathfrak{a}_1],[\mathfrak{a}_2])\in \mathrm{Cl}(E_{1,3})\times\mathrm{Cl}(E_{2,3})} |\pi_3(\tau_{\mathfrak{a}_1}) - \pi_3(\tau_{\mathfrak{a}_2})| = 2^6 3^{24} 7^2.$$

Remark 3.1.12. *Although we call* (3.1.4) *a Gross–Zagier type CM value formula, one might note that in general, it only tells a formula for the value*

$$\frac{1}{|S(p,d_1,d_2)|} \sum_{([\mathfrak{a}_1],[\mathfrak{a}_2])\in S(p,d_1,d_2)} \log|\pi_p(\tau_{\varphi(\mathfrak{a}_1)}) - \pi_p(\tau_{\varphi(\mathfrak{a}_2)})|,$$

where the set $S(p,d_1,d_2)$ *is not explicit in general. The appearance of* $S(p,d_1,d_2)$ *follows from an injection of a big CM cycle of a Shimura variety into* $\mathrm{Cl}(E_{1,p}) \times \mathrm{Cl}(E_{2,p})$ *(see*

Lemma 3.3.3*). If one had* $S(p, d_1, d_2) = \mathrm{Cl}(E_{1,p}) \times \mathrm{Cl}(E_{2,p})$*, then* (3.1.4) *does give a formula for the product of CM values*

$$\prod_{([\mathfrak{a}_1],[\mathfrak{a}_2]) \in Cl(E_{1,p}) \times Cl(E_{2,p})} |\pi_p(\tau_{\mathfrak{a}_1}) - \pi_p(\tau_{\mathfrak{a}_2})|$$

as we have seen in Example 3.1.11*, and it is more "Gross–Zagier-type". According to a number of computational calculations, it seems to us that such an equality may be true for any* $p \in \{3, 5, 7, 13\}$ *and any coprime negative fundamental discriminants* d_1 *and* d_2*. However, we were unable to prove or disprove this claim in the present work. Moreover, a natural question may be raised for general integer* N *instead of merely prime* p*. We leave these interesting observations as a conjecture and an open question below, and these can be the topics for further investigation.*

Conjecture 3.1.13. *For* $p \in \{3, 5, 7, 13\}$ *and any coprime negative fundamental discriminants* d_1 *and* d_2*, we have* $S(p, d_1, d_2) = \mathrm{Cl}(E_{1,p}) \times \mathrm{Cl}(E_{2,p})$*. In the language of quadratic form, it is equivalent to say that for every pair of primitive and positive definite quadratic forms* $a_iX^2 + b_iXY + c_iY^2$ *of* $\mathcal{Q}_{d_i}(p)$*, there always exists a positive integer* n *such that* $a_iX^2 + b_iXY + c_iY^2 = n$ *both have integral solutions* (X_i, Y_i) *with* $\gcd(X_i, Y_i) = 1$*.*

Question 3.1.14. *It is known that* $S(1, d_1, d_2) = \mathrm{Cl}(E_{1,1}) \times \mathrm{Cl}(E_{2,1}) = \mathrm{Cl}(E_1) \times \mathrm{Cl}(E_2)$*. In general, when does* $S(N, d_1, d_2) = \mathrm{Cl}(E_{1,N}) \times \mathrm{Cl}(E_{2,N})$ *hold?*

Remark 3.1.15. *Gross–Zagier type CM value formulas for the genus zero Fricke subgroups* $\Gamma_0^*(p)$ *have been recently derived in another work of the author [98] using the so called small CM value formula [85]. We refer the reader to [28] for a brief explanation on the distinction between the concepts related to "big CM" and "small CM" and to [98] for a brief explanation on why the small CM value formula may not work in our case.*

This chapter is divided into two parts and is organized as follows. In the first part, we briefly review the theory of Borcherds lifts, realize a family of Shimura varieties as Hilbert modular surfaces, set up several preliminary results and construct the desired weakly holomorhic modular form F_N for the Weil representation. Proofs of Theorem 3.1.5, Corollaries 3.1.6 and 3.1.9 are given in Subsection 3.2.3. In the second part of this chapter, we will briefly review the concepts of big CM cycles and big CM value formula and show how to employ such formula together with Theorem 3.1.5 to obtain Theorem 3.1.10 in Subsection 3.3.2. Computations of $a(\frac{t}{\sqrt{D}}, \phi_{0,0})$ and $a_0(\phi_{0,k})$ mentioned in Theorem 3.1.10 are carried out in Subsection 3.3.3.

3.2 Part I: Difference of a Hauptmodul for $\Gamma_0(N)$

3.2.1 Review of Borcherds Lifts

In this section, we briefly review the theory of Borcherds lifts in the adelic setting and relevant concepts such as Shimura variety and special divisor, and we also realize a family of Shimura varieties as Hilbert modular surfaces on which the difference of Hauptmoduls are defined. We rely heavily on [56] (also see [94]).

Shimura Variety of Type $(n, 2)$

Let V be a rational quadratic space with a quadratic form $Q(\cdot)$ of signature $(n, 2)$ and the associated bilinear form $(\cdot, \cdot)$. Let $H = \mathrm{GSpin}(V)$ be the general spin group of V, then there is an exact sequence

$$1 \to \mathbb{G}_m \to H \to \mathrm{SO}(V) \to 1.$$

For a $\mathbb{Q}$-algebra F, we write V_F for $V \otimes_{\mathbb{Q}} F$. A Hermitian symmetric domain for $H(\mathbb{R})$ is the oriented Grassmannian of negative 2-planes of $V_{\mathbb{R}}$, denoted by $\mathbb{D}$. Denote by $\mathbb{A}_{\mathcal{K}}$ the adele ring over a number field $\mathcal{K}$ and by $\mathbb{A}_{\mathcal{K},f}$ the associated finite adele ring. For a compact open subgroup $K \subset H(\mathbb{A}_{\mathbb{Q},f})$, there is an associated Shimura variety X_K over $\mathbb{Q}$ such that

$$X_K(\mathbb{C}) = H(\mathbb{Q})\backslash (\mathbb{D} \times H(\mathbb{A}_{\mathbb{Q},f})/K).$$

Let

$$\mathcal{L} = \{Z \in V_{\mathbb{C}} |\, (Z, Z) = 0 \text{ and } (Z, \bar{Z}) < 0\}.$$

Then we can see that $\mathbb{D}$ possesses a complex structure via the isomorphism $pr : \mathcal{L}/\mathbb{C}^* \to \mathbb{D}$ sending $Z = X + iY$ to $\mathbb{R}X + \mathbb{R}(-Y)$. There is another useful realization for $\mathbb{D}$ as follows. Take two isotropic (zero-norm) elements ℓ and ℓ' of V with $(\ell, \ell') = 1$, and let $V_0 = (\mathbb{Q}\ell + \mathbb{Q}\ell')^{\perp}$ be the orthogonal complement of the plane spanned by ℓ and ℓ'. Then we define a so-called tube domain associated to ℓ and ℓ' by

$$\mathcal{H} = \{Z = X + iY \in V_{0,\mathbb{C}} |\, X, Y \in V_{0,\mathbb{R}} \text{ and } Q(Y) < 0\}$$

which is isomorphic to $\mathcal{L}/\mathbb{C}^*$ via $w(Z) = \ell' - Q(Z)\ell + Z$. Then the map w induces an action of $\Gamma = K \cap H(\mathbb{Q})^+$ on $\mathcal{H}$, where $H(\mathbb{Q})^+ = H(\mathbb{Q}) \cap H(\mathbb{R})^+$ and $H(\mathbb{R})^+$ is the identity component of $H(\mathbb{R})$, and induces an automorphy factor $j(g, Z)$ characterized by the following identity

$$g \cdot w(Z) = \nu(g) j(g, Z) w(g \cdot Z),$$

where $\nu(g)$ is the spinor norm of g. Note that this action preserves the two connected components $\mathcal{H}^{\pm}$ of $\mathcal{H}$. Fix one of these two connected components, say, $\mathcal{H}^+$. Assuming for simplicity that $H(\mathbb{A}_{\mathbb{Q}}) = H(\mathbb{Q})H(\mathbb{R})^+K$, which is guaranteed by an appropriate choice of K and the strong approximation theorem (see, e.g., [76, Ch. 7]), we have the identification $X_K \cong \Gamma \backslash \mathcal{H}^+$.

Definition 3.2.1. *A meromorphic modular form on $\mathcal{H}^+$ of weight k for Γ is a meromorphic function $f : \mathcal{H}^+ \to \mathbb{C}$ such that $f(\gamma \cdot Z) = j(\gamma, Z)^k f(Z)$ for all $\gamma \in \Gamma$. Moreover, under the identification given above, such a function f can be equivalently viewed as a meromorphic modular form defined on $\mathbb{D} \times H(\mathbb{A}_{\mathbb{Q},f})$ satisfying*

(1) $f(Z, hk) = f(Z, h)$ for all $k \in K$,

(2) $f(\gamma \cdot Z, \gamma h) = j(\gamma, z)^k f(Z, h)$ for all $\gamma \in H(\mathbb{Q})$.

Special Divisor

For a vector $X \in V$ with $Q(X) > 0$ and $h \in H(\mathbb{A}_{\mathbb{Q},f})$, let

$H_X = \{g \in H |\, g{\cdot}X = X\}$, $\quad \mathbb{D}_X = \{Z \in \mathbb{D} |\, (Z, X) = 0\}$, $\quad$ and $\quad K_{X,h} = H_X(\mathbb{A}_{\mathbb{Q},f}) \cap hKh^{-1}$.

Then the map

$$H_X(\mathbb{Q}) \backslash \left(\mathbb{D}_X \times H_X(\mathbb{A}_{\mathbb{Q},f})/K_{X,h}\right) \to X_K(\mathbb{C}), \quad [Z, g] \to [Z, gh]$$

gives a divisor $Z(X, h)$ in X_K which is defined over $\mathbb{Q}$. For a positive rational number m and $\varphi \in \mathcal{S}(V_{\mathbb{A}_{\mathbb{Q},f}})$, the Schwartz-Bruhat space of $V_{\mathbb{A}_{\mathbb{Q},f}}$, if there is a $X \in V$ with $Q(X) = m$, the special divisor of index (m, φ) in X_K is defined by

$$Z(m, \varphi) = \sum_{h \in H_X(\mathbb{Q}) \backslash H(\mathbb{A}_{\mathbb{Q},f})/K} \varphi(h^{-1} \cdot X) Z(X, h).$$

When there is no such X, we simply set $Z(m, \varphi) = 0$.

Theta-lift and Borcherds Theorem

For $Z \in \mathbb{D}$, let $\mathrm{pr}_Z : V_{\mathbb{R}} \to Z$ be the projection map, and for $X \in V_{\mathbb{R}}$, let $R(X,Z) = -(\mathrm{pr}_Z(X), \mathrm{pr}_Z(X))$. Then we define

$$(X,X)_Z = (X,X) + 2R(X,Z),$$

and our Gaussian for V is the function

$$\varphi_\infty(X,Z) = e^{-\pi(X,X)_Z}.$$

For $\tau \in \mathbb{H}$ with $\tau = u + iv$, let

$$g_\tau = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v^{\frac{1}{2}} & 0 \\ 0 & v^{-\frac{1}{2}} \end{pmatrix},$$

and $g'_\tau = (g_\tau, 1) \in \mathrm{Mp}_2(\mathbb{R})$, the metaplectic group. Let $l = \frac{n}{2} - 1$, $G = \mathrm{SL}_2$ and ρ be the Weil representation of the metaplectic group $G'(\mathbb{A}_{\mathbb{Q}})$ on $\mathcal{S}(V_{\mathbb{A}_{\mathbb{Q},f}})$. Then for the linear action of $H(\mathbb{A}_{\mathbb{Q},f})$ we write $\rho(h)\varphi(X) = \rho(h^{-1} \cdot X)$ for $\varphi \in \mathcal{S}(V_{\mathbb{A}_{\mathbb{Q},f}})$. For $Z \in \mathbb{D}$ and $h \in H(\mathbb{A}_{\mathbb{Q},f})$, we have the linear functional on $\mathcal{S}(V_{\mathbb{A}_{\mathbb{Q},f}})$ given by

$$\varphi \to \theta(\tau, Z, h; \varphi) := v^{-\frac{1}{2}} \sum_{X \in V_{\mathbb{Q}}} \rho(g'_\tau) \left(\varphi_\infty(\cdot, Z) \otimes \rho(h)\varphi\right)(X).$$

Let L be a lattice of V, and let L' be the dual lattice of L defined by

$$L' = \{X \in V \,|\, (X,L) \subset \mathbb{Z}\}.$$

Let $\mathcal{S}_L$ be the subspace of $\mathcal{S}(V_{\mathbb{A}_{\mathbb{Q},f}})$ consisting of functions with support in $\hat{L}'$ and constant on cosets of $\hat{L}$, where $\hat{L} = L \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$. Then

$$\mathcal{S}_L = \bigoplus_{\eta \in L'/L} \mathbb{C}\phi_\eta, \quad \phi_\eta = \mathrm{Char}(\eta + \hat{L}),$$

where Char$(\cdot)$ denotes the characteristic function. One can check that $\mathcal{S}_L$ is $\mathrm{SL}_2(\mathbb{Z})$-invariant under the Weil representation ρ. Associated to the lattice L, we denote by ρ_L, and one has

$$\rho_L(T)\phi_\mu = e(-Q(\mu))\phi_\mu,$$
$$\rho_L(S)\phi_\mu = \frac{e((n-2)/8)}{\sqrt{|L'/L|}} \sum_{\gamma \in L'/L} e((\gamma,\mu))\phi_\gamma$$

where $e(\tau) := \exp(2\pi i \tau)$, $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Let $\Gamma' = \mathrm{Mp}_2(\mathbb{Z})$ be the full inverse image of $\mathrm{SL}_2(\mathbb{Z}) \subset G(\mathbb{R})$ in $\mathrm{Mp}_2(\mathbb{R})$.

Definition 3.2.2. *A holomorphic function $F : \mathbb{H} \to \mathcal{S}_L$ is a weakly holomorphic modular form of weight k for the Weil representation ρ_L if*

(i) $F(\gamma'\tau) = (c\tau + d)^k \rho_L(\gamma')F(\tau)$ *for all* $\gamma' \in \Gamma'$,

(ii) $F(\tau)$ *has a Fourier expansion*

$$F(\tau) = \sum_{\eta \in L'/L} \sum_{\substack{m \in -Q(\eta)+\mathbb{Z} \\ m \gg -\infty}} c(m,\eta) q^m \phi_\eta$$

where the condition $m \equiv -Q(\eta) \pmod{\mathbb{Z}}$ *follows from the transformation law for* $T' \to \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

Furthermore, denote by $M^!_{k,\rho_L}$ the space of weakly holormophic modular forms of weight k for the Weil representation ρ_L.

For the theta function called Siegel theta function

$$\theta(\tau, Z, h) = \sum_{\mu \in L'/L} \theta(\tau, Z, h; \phi_\mu),$$

we can pair it with $F(\tau)$ by the following $\mathbb{C}$-bilinear pairing

$$\langle F(\tau), \theta(\tau, Z, h)\rangle = \sum_{\mu\in L'/L}\sum_{m\in Q(\mu)+\mathbb{Z}} c(m,\mu)q^m\theta(\tau, Z, h; \phi_\mu).$$

Using this pairing, we define a regularized integral as in [21], called Borcherds lift,

$$\Phi(Z, h; F) := \underset{s=0}{\mathrm{CT}}\left\{\lim_{t\to\infty}\int_{\mathcal{F}_t}\langle F(\tau), \theta(\tau, Z, h)\rangle v^{-2-s}dudv\right\}$$

where $\underset{s=0}{\mathrm{CT}}$ denotes the constant term in the Laurent expansion at $s = 0$ of

$$\lim_{t\to\infty}\int_{\mathcal{F}_t}\langle F(\tau), \theta(\tau, Z, h)\rangle v^{-2-s}dudv,$$

$\mathcal{F}_t$ is the truncated fundamental domain defined by

$$\mathcal{F}_t := \{\tau \in \mathcal{F} |\, \mathrm{Im}(\tau) \leq t\}$$

and $\mathcal{F}$ is the usual fundamental domain for the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$.

Now we are ready to state the celebrated Borcherds Theorem [21, 25, 56]. Under the assumption $H(\mathbb{A}_\mathbb{Q}) = H(\mathbb{Q})H(\mathbb{R})^+K$, we can simply set $h = 1$ and omit it for the simplicity of notation.

Theorem 3.2.3 (Borcherds). *For a given $f \in M^!_{1-\frac{n}{2},\rho_L}$ with*

$$f(\tau) = \sum_{\mu\in L'/L}\sum_{\substack{n\in -Q(\mu)+\mathbb{Z}\\ n\gg-\infty}} c(n,\mu)q^n\phi_\mu,$$

and $c(n,\mu) \in \mathbb{Z}$ for $n < 0$, and assuming $\Gamma \subset \mathrm{Aut}(L, f)$ where $\mathrm{Aut}(L, f)$ is the subgroup of automorphism group of L that stablizes f, there is a meromorphic modular form $\Psi(Z, f)$ called Borcherds form on $\mathbb{D} \times H(\mathbb{A}_{\mathbb{Q},f})$ (or say on $\mathcal{H}^+$) of weight $c(0, 0)/2$ for Γ such that

1. the divisor of $\Psi(Z,f)$ *on* X_K *is given by*

$$\operatorname{div}(\Psi(Z,f)^2) = \sum_{\mu \in L'/L} \sum_{\substack{n \in Q(\mu)+\mathbb{Z} \\ 0<n}} c(-n,\mu)\delta(\mu)Z(n,\mu),$$

where $Z(n,\mu)$ *is the special divisor of index* (n,μ) *and* $\delta(\mu) = 2$ *or* 1 *depending on whether* $2\mu \in L$ *or not, and can be written as*

$$Z(n,\mu) = \{Z \in \mathbb{D}^+ | \, (Z,X) = 0 \textit{ for some } X \in \mu + L \textit{ with } Q(X) = n\},$$

2. the following relation

$$\Phi(Z,f) = -4\log|\Psi(Z,f)| - c(0,0)\left(2\log|Y| + \Gamma'(1) + \log(2\pi)\right)$$

holds,

3. near each cusp $\mathbb{Q}\ell$ *of* X_K*, the meromorphic function* $\Psi(Z,f)$ *has a product expansion called Borcherds product of the form*

$$\Psi(Z,f) = Ce\left((Z,\rho(W_{f,\ell_M},f)\right) \prod_{\substack{\lambda \in M'_\ell \\ (\lambda, W_{f,\ell_M})>0}} \prod_{\substack{\mu \in L'_\ell/L \\ p(\mu)=\lambda+M_\ell}} \left[1 - e\left((\lambda,Z) + (\mu,\ell')\right)\right]^{c(-Q(\mu),\mu)}$$

where C *is a constant with absolute value*

$$\left| \prod_{\substack{\delta \in \mathbb{Z}/N \\ \delta \neq 0}} (1 - e(\delta/N))^{\frac{c(0,\frac{\delta}{N}\ell)}{2}} \right|.$$

For the definitions of the notation, we refer the reader to [94, Subsection 2.1].

A Shimura Variety as a Hilbert Modular Surface

In this subsection, we will see how to realize a family of Shimura varieties X_{K_N} of type $(2,2)$ for some open compact subgroup K_N as Hilbert modular surfaces $Y_0(N) \times Y_0(N)$.

Let $V = M_2(\mathbb{Q})$ be a rational quadratic space with the quadratic form $Q(\cdot) := \det(\cdot)$ of signature $(2,2)$. Then the general spin group $H = \mathrm{GSpin}(V)$ of V is

$$H = \{(g_1, g_2) \in \mathrm{GL}_2 \times \mathrm{GL}_2 | \det g_1 = \det g_2\}$$

and acts on V via $(g_1, g_2) \cdot X = g_1 X g_2^{-1}$. Taking $\ell = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$ and $\ell' = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, one can check that

$$\mathcal{H} = \left\{ \begin{pmatrix} z_1 & 0 \\ 0 & -z_2 \end{pmatrix} \middle| \, \Im(z_1)\Im(z_2) > 0 \right\}.$$

The following proposition is useful and well-known (see, e.g., [94, Proposition 3.1]).

Proposition 3.2.4. *Define*

$$\tilde{w} : \mathbb{H}^2 \cup (\mathbb{H}^-)^2 \to \mathcal{L}$$

by $\tilde{w}((z_1, z_2)) = \begin{pmatrix} z_1 & -z_1 z_2 \\ 1 & -z_2 \end{pmatrix}$. *Then the composition* $pr \circ \tilde{w}$ *gives an isomorphism between* $\mathbb{H}^2 \cup (\mathbb{H}^-)^2$ *and* $\mathbb{D}$. *One can check that such an isomorphism induces an action of* $H(\mathbb{R})$ *on* $\mathbb{H}^2 \cup (\mathbb{H}^-)^2$ *via the usual fractional linear transformation, i.e.,*

$$(g_1, g_2) \cdot (z_1, z_2) = (g_1 \cdot z_1,\, g_2 \cdot z_2),$$

and an automorphy factor $j(g_1, g_2; z_1, z_2) = (c_1 z_1 + d_1)(c_2 z_2 + d_2)$ *for* $g_j = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix}$.

Now let

$$K_0(N) = \left\{ g \in \mathrm{GL}_2(\hat{\mathbb{Z}}) \middle| \, g \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod N \right\}$$

and denote $K = (K_0(N) \times K_0(N)) \cap H(\mathbb{A}_f)$ by K_N. Then $\Gamma = H(\mathbb{Q})^+ \cap K_N = \Gamma_0(N) \times \Gamma_0(N)$ denoted by Γ_N. The strong approximation theorem tells that $H(\mathbb{A}_{\mathbb{Q}}) =$

$H(\mathbb{Q})H(\mathbb{R})^+K_N$, and together with Proposition 3.2.4 it implies the realization of $X_{K_N} \cong Y_0(N) \times Y_0(N)$ as a Hilbert modular surface. Moreover, set $L = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & \mathbb{Z} \end{pmatrix}$ and denote it by L_N. Then one can check that $L_N' = \begin{pmatrix} \mathbb{Z} & \mathbb{Z}/N \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix}$ and the special divisor of index (n, μ) can be expressed as a divisor on $Y_0(N) \times Y_0(N)$

$$Z(n,\mu) = (\Gamma_0(N) \times \Gamma_0(N)) \setminus \left\{ z = (z_1, z_2) \in \mathbb{H}^2 | \, \tilde{w}(z) \perp X \text{ for some } X \in \mu + L_N \text{ with } Q(\mu) = n \right\}.$$

Finally, one can check that $\Gamma_N \subset \mathrm{Aut}(L_N'/L_N) \cap \mathrm{Aut}(L_N)$, where $\mathrm{Aut}(L_N'/L_N)$ and $\mathrm{Aut}(L_N)$ denote the automorphism groups of L_N'/L_N and L_N, respectively.

3.2.2 Preliminary Results

In this section, we aim to construct an appropriate Borcherds lift input F_N for $\pi_N(z_1) - \pi_N(z_2)$. First, with $L_N = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & \mathbb{Z} \end{pmatrix}$ and $L_N' = \begin{pmatrix} \mathbb{Z} & \mathbb{Z}/N \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix}$, we note that $\{\mu\}_{\mu \in L_N'/L_N} = \left\{ \begin{pmatrix} 0 & j/N \\ k & 0 \end{pmatrix} \right\}_{0 \le j,k \le N-1}$. Then to simplify our notation, we write $\mu_{j,k}$ for $\begin{pmatrix} 0 & j/N \\ k & 0 \end{pmatrix}$, and write $L_N'/L_N = \{\mu_{j,k}\}_{0 \le j,k \le N-1}$. We also write $\phi_{j,k}$ for $\phi_{\mu_{j,k}} \in \mathcal{S}_{L_N}$.

Lemma 3.2.5. *Let $Z(1, \mu_{0,0})$ be the special divisor of index $(1, \mu_{0,0})$. Then*

$$Z(1, \mu_{0,0}) = \{ (\tau, \tau) | \, \tau \in Y_0(N) \}.$$

Proof. By the identification given in Subsection 3.2.1, we have

$$Z(1, \mu_{0,0})$$
$$= (\Gamma_0(N) \times \Gamma_0(N)) \setminus \left\{ (z_1, z_2) \in \mathbb{H}^2 | \, \tilde{w}(z_1, z_2) \perp X \text{ for some } X \in \mu_{0,0} + L_N \text{ with } Q(X) = 1 \right\}$$

$$= (\Gamma_0(N) \times \Gamma_0(N)) \backslash \left\{ (z_1, z_2) \in \mathbb{H}^2 |\, z_2 = \frac{az_1 + b}{cz_1 + d} \text{ for some } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \right\}$$

$$= \{ (\tau, \tau) |\, \tau \in Y_0(N) \} .$$

$\square$

Note that the divisor of $\pi_N(z_1) - \pi_N(z_2)$ is $\{ (\tau, \tau) |\, \tau \in Y_0(N) \}$. Then Lemma 3.2.5 tells us that $\mathrm{div}(\Psi(z, F_N))$ must be $Z(1, \mu_{0,0})$, and thus the $\phi_{0,0}$-component function of F_N must have a simple pole at $i\infty$ of residue 1, and the other component functions are all holomorphic at $i\infty$.

Lemma 3.2.6. *For $N \neq 4$, the following set*

$$\bigcup_{d|N} \left\{ \sum_{k=0}^{d-1} \sum_{j=0}^{\frac{N}{d}-1} \phi_{jd,\, k\frac{N}{d}} \right\}$$

is a basis for the subspace of $M^!_{0,\rho_{L_N}}$ consisting of elements with constant component functions.

For $N = 4$, the following set

$$\left\{ \begin{matrix} \phi_{0,0} + \phi_{1,0} + \phi_{2,0} + \phi_{3,0}, \\ \phi_{0,0} + \phi_{0,1} + \phi_{0,2} + \phi_{0,3}, \\ \phi_{2,0} - \phi_{0,1} + \phi_{2,2} - \phi_{0,3} \end{matrix} \right\}$$

is a basis for the subspace of $M^!_{0,\rho_{L_4}}$ consisting of elements with constant component functions.

Proof. Setting up the equations

$$\rho_{L_N}(T) \left(\sum_{k=0}^{N-1} \sum_{j=0}^{N-1} a_{j,k} \phi_{j,k} \right) = \sum_{k=0}^{N-1} \sum_{j=0}^{N-1} a_{j,k} \phi_{j,k},$$

$$\rho_{L_N}(S)\left(\sum_{k=0}^{N-1}\sum_{j=0}^{N-1} a_{j,k}\phi_{j,k}\right) = \sum_{k=0}^{N-1}\sum_{j=0}^{N-1} a_{j,k}\phi_{j,k},$$

expanding the left hand side by the Weil representation, and equating the coefficients by the linear independence of $\phi_{j,k}$, we can obtain the desired results after some routine calculations. □

Lemma 3.2.7. *Let $c_{s,N}^{-1}$ be the inverse of c_s in $(\mathbb{Z}/N\mathbb{Z})^\times$ when $m_s = 1$, and $c_{s,N}^{-1} = m_s$ otherwise. For $s \in \mathcal{C}(\Gamma_1(N))$, and integers j and k, define $t_{s,j,k}$ by $t_{s,j,k}/h_s \equiv jk/N \pmod 1$. Let $\pi_N = \pi_N(\tau)$ be a Hauptmodul for a genus zero group $\Gamma_0(N)$ and $f_N = f_N(\tau)$ be the $\Gamma_0(N)$-induction of $\pi_N(\tau)$ against $\phi_{0,0}$ defined by*

$$f_N = \sum_{M\in\Gamma_0(N)\backslash SL_2(\mathbb{Z})} \pi_N|\, M \cdot \rho_{L_N}(M^{-1})\phi_{0,0}. \tag{3.2.1}$$

Then f_N is in $M^!_{0,\rho_{L_N}}$, and

$$f_N = \frac{2}{|\Gamma_0(N):\Gamma_1(N)|} \sum_{M\in\Gamma_1(N)\backslash SL_2(\mathbb{Z})} \frac{1}{2}\,\pi_N|\, M \cdot \rho(M^{-1})\phi_{0,0}$$

where

$$\begin{aligned} f_N = \frac{2}{\lambda_{2,N}|\Gamma_0(N):\Gamma_1(N)|}\Bigg[&\sum_{\substack{s\in\mathcal{C}(\Gamma_1(N))\\ s \text{ regular}}} (\pi_N|\, M_s)_0\, \phi_{0,0} + \sum_{\substack{s\in\mathcal{C}(\Gamma_1(N))\\ s \text{ regular}\\ m_s\neq N}} (\pi_N|\, M_s)_0 \sum_{j=1}^{h_s-1} \phi_{jm_s,0} \\ &+ \sum_{\substack{s\in\mathcal{C}(\Gamma_1(N))\\ s \text{ regular}\\ m_s\neq N}} (\pi_N|\, M_s)_0 \sum_{k=1}^{h_s-1} \phi_{0,km_s} \\ &+ \sum_{\substack{s\in\mathcal{C}(\Gamma_1(N))\\ s \text{ regular}\\ m_s\neq N}} \sum_{k=1}^{h_s-1}\sum_{j=1}^{h_s-1} e\left(-\frac{d_s c_{s,N}^{-1} m_s^2 jk}{N}\right) (\pi_N|\, M_s)_{t_{s,jm_s,km_s}}\, \phi_{jm_s,km_s} \end{aligned} \tag{3.2.2}$$

$$+\sum_{\substack{s\in\mathcal{C}(\Gamma_1(N))\\ s\ irregular}}\frac{1}{h_s}\left(\pi_N|\,M_s\right)\sum_{k=0}^{h_s-1}\sum_{j=0}^{h_s-1}e\left(-\frac{d_s m_s jk}{N}\right)\phi_{jm_s,km_s}\Bigg],$$

and $\lambda_{2,N}=2$ *or* 1 *depending on whether* $N=2$ *or not.*

Proof. These follow from [23, Theorem 5.4] and [83, Theorem 3.7] by setting $D=L_N'/L_N$ with quadratic form $Q(\cdot)=\det(\cdot)$ and $e^{\gamma}=\phi_{0,0}$, and realizing that

$$D_{c_s}=\{\phi_{jh_s,\,kh_s}\}_{0\le j,\,k\le m_s-1}\quad\text{and}\quad D^{c_s*}=\{\phi_{jm_s,\,km_s}\}_{0\le j,\,k\le h_s-1}\,.$$

□

Lemma 3.2.8. *Let* $\pi_N=\pi_N(\tau)$ *be a Hauptmodul for genus zero group* $\Gamma_0(N)$*, and let*

$$\pi_N|\,M_s=\sum_{n=-1}^{\infty}A_s(n)q^{n/\tilde{h}_s}$$

where $\tilde{h}_s=1$ *if* $N=4$ *and* $s=1/2$*, otherwise,* $\tilde{h}_s=h_s$ *as defined in Section 3.1. Let* $f_N=f_N(\tau)$ *be the* $\Gamma_0(N)$*-induction as defined in Lemma* 3.2.6*. For* $N\neq 4$*, let* $F_N=F_N(\tau)$ *be defined by*

$$F_N=f_N-\frac{2}{\lambda_{2,N}|\Gamma_0(N):\Gamma_1(N)|}\times\left[\sum_{s\in\mathcal{C}(\Gamma_1(N))}A_s(0)\sum_{k=0}^{N-1}\phi_{0,k}+\sum_{\substack{s\in\mathcal{C}(\Gamma_1(N))\\ m_s\neq N}}A_s(0)\left(\sum_{k=0}^{m_s-1}\sum_{j=0}^{h_s-1}\phi_{jm_s,\,kh_s}-\sum_{k=0}^{N-1}\phi_{0,\,k}\right)\right].\tag{3.2.3}$$

For $N=4$*, let* $F_4=F_4(\tau)$ *be defined by*

$$F_4=f_4-\left(A_{1/4}(0)+A_{0/1}(0)+\frac{1}{2}A_{1/2}(0)\right)\sum_{k=0}^{3}\phi_{0,k}-A_{0/1}(0)\sum_{k=1}^{3}(\phi_{k,0}-\phi_{0,k})-\frac{1}{2}A_{1/2}(0)\left(\phi_{2,0}-\phi_{0,1}+\phi_{2,2}-\phi_{0,3}\right).\tag{3.2.4}$$

Then we have

1. *$F_N(\tau)$ is in $M^!_{0,\rho_{L_N}}$, and is invariant under* $\mathrm{Aut}(L'_N/L_N)$,

2. *$c(0,\mu_{0,0}) = 0$ and the $\phi_{0,0}$–component of F_N has Fourier expansion*

$$q^{-1} + \frac{2}{\lambda_{2,N}|\Gamma_0(N):\Gamma_1(N)|}\left[\sum_{\substack{s\in\mathcal{C}(\Gamma_1(N))\\ s \text{ regular}}}\sum_{n=1}^{\infty} A_s(nh_s)q^n + \sum_{\substack{s\in\mathcal{C}(\Gamma_1(N))\\ s \text{ irregular}}}\frac{1}{h_s}\sum_{n=1}^{\infty} A_s(n)q^n\right],$$

3. *$c(0,\mu_{j,0}) = 0$ for $0 \le j \le N-1$,*

4. *for $d|N$ and $d \ne N$,*

$$\sum_{k=0}^{\frac{N}{d}-1}\sum_{j=0}^{d-1} c(0,\mu_{j\frac{N}{d},kd}) = 24.$$

Proof. Assertion (1) follows from Lemmas 3.2.6 and 3.2.7.

By collecting the terms attached to $\phi_{0,0}$ in (3.2.2), we obtain that the $\phi_{0,0}$–component of f_N is

$$\begin{aligned}
&\frac{2}{\lambda_{2,N}|\Gamma_0(N):\Gamma_1(N)|}\left[\sum_{\substack{s\in\mathcal{C}(\Gamma_1(N))\\ s \text{ regular}}}(\pi_N|\,M_s)_0 + \sum_{\substack{s\in\mathcal{C}(\Gamma_1(N))\\ s \text{ irregular}}}\frac{1}{h_s}(\pi_N|\,M_s)\right]\\
&= q^{-1} + \frac{2}{\lambda_{2,N}|\Gamma_0(N):\Gamma_1(N)|}\left[\sum_{\substack{s\in\mathcal{C}(\Gamma_1(N))\\ s \text{ regular}}}\sum_{n=0}^{\infty} A_s(nh_s)q^n + \sum_{\substack{s\in\mathcal{C}(\Gamma_1(N))\\ s \text{ irregular}}}\frac{1}{h_s}\sum_{n=0}^{\infty} A_s(n)q^n\right]\\
&= q^{-1} + \frac{2}{\lambda_{2,N}|\Gamma_0(N):\Gamma_1(N)|}\left[\sum_{\substack{s\in\mathcal{C}(\Gamma_1(N))\\ s \text{ regular}}}A_s(0) + \sum_{\substack{s\in\mathcal{C}(\Gamma_1(N))\\ s \text{ irregular}}}\frac{1}{h_s}A_s(0)\right. \qquad (3.2.5)\\
&\qquad\left. + \sum_{\substack{s\in\mathcal{C}(\Gamma_1(N))\\ s \text{ regular}}}\sum_{n=1}^{\infty} A_s(nh_s)q^n + \sum_{\substack{s\in\mathcal{C}(\Gamma_1(N))\\ s \text{ irregular}}}\frac{1}{h_s}\sum_{n=1}^{\infty} A_s(n)q^n\right].
\end{aligned}$$

Then Assertion (2) follows from (3.2.5) and the definition of F_N.

By extracting the constant terms attached to $\phi_{j,0}$ for $1 \le j \le N-1$ in (3.2.2) of F_N, we obtain

$$\frac{2}{\lambda_{2,N}|\Gamma_0(N):\Gamma_1(N)|}\left[\sum_{\substack{s\in\mathcal{C}(\Gamma_1(N))\\ s \text{ regular}\\ m_s\neq N}} A_s(0)\sum_{j=1}^{h_s-1}\phi_{jm_s,0} + \sum_{\substack{s\in\mathcal{C}(\Gamma_1(N))\\ s \text{ irregular}}} \frac{1}{h_s}A_s(0)\sum_{j=1}^{h_s-1}\phi_{jm_s,0}\right]. \tag{3.2.6}$$

Then Assertion (3) follows from (3.2.6) and the definition of F_N.

For Assertion (4), by (3.2.2), (3.2.3) and (3.2.4), it can be verified case by case that

$$\sum_{k=0}^{\frac{N}{d}-1}\sum_{j=0}^{d-1} c(0,\phi_{j\frac{N}{d},kd}) = -\frac{2}{\lambda_{2,N}|\Gamma_0(N):\Gamma_1(N)|}\sum_{s\in\mathcal{C}(\Gamma_1(N))} A_s(0)\left(\frac{d(m_s,N/d)^2-N}{m_s}\right). \tag{3.2.7}$$

Let

$$H_d(\tau) = \frac{N}{d}E_2\left(\frac{N}{d}\tau\right) - E_2(\tau)$$

where $E_2(\tau)$ is the normalized weight 2 Eisenstein series. It is well-known [39, Section 1.2] that $H_d(\tau)$ is a weight 2 modular form for $\Gamma_0(N/d)$, so it is a weight 2 modular form for $\Gamma_1(N)$. Let $B_s(0)$ be the constant term of the Fourier expansion of $H_d(\tau)$ at the cusp $s = a_s/c_s \in \mathcal{C}(\Gamma_1(N))$. It is easy to show that

$$B_s(0) = \frac{(m_s, N/d)^2}{N/d} - 1$$

and

$$H_d(\tau) = \frac{N}{d} - 1 + 24q + O(q^2).$$

Since $\pi_N(\tau)$ is a weight 0 weakly holomorphic modular form for $\Gamma_0(N)$, then it is also on $\Gamma_1(N)$, and by Serre duality [22, Theorem 3.1], we have

$$0 = 24 \times \frac{\lambda_{2,N}|\Gamma_0(N):\Gamma_1(N)|}{2} + \sum_{s\in\mathcal{C}(\Gamma_1(N))} h_sB_s(0)A_s(0)$$

$$= 24 \times \frac{\lambda_{2,N}|\Gamma_0(N):\Gamma_1(N)|}{2} + \sum_{s\in\mathcal{C}(\Gamma_1(N))} h_s \left(\frac{(m_s,N/d)^2}{N/d} - 1\right) A_s(0)$$
$$= 24 \times \frac{\lambda_{2,N}|\Gamma_0(N):\Gamma_1(N)|}{2} + \sum_{s\in\mathcal{C}(\Gamma_1(N))} A_s(0) \left(\frac{d(m_s,N/d)^2 - N}{m_s}\right). \qquad (3.2.8)$$

Finally, Assertion (4) follows from (3.2.7) and (3.2.8). □

3.2.3 Proofs of Theorem 3.1.5 and Corollaries 3.1.6 and 3.1.9

Proof of Theorem 3.1.5

This subsection is devoted to the proof of our first main result, Theorem 3.1.5. We rely heavily on Borcherds Theorem, especially the third part of it. To assist in understanding the proof, we first review the definitions of the notation used in Theorem 3.2.3(3) for our case. Let $M_\ell = L \cap (\mathbb{Q}\ell + \mathbb{Q}\ell')^\perp$ be the Lorentzian lattice of L associated to ℓ and ℓ'. Assume that $(\ell, L) = N_\ell\mathbb{Z}$. Choose a $\xi \in L$ such that $(\xi, \ell) = N_\ell$. Let L'_ℓ be a sublattice of L' defined by

$$L'_\ell = \{x \in L' \,|\, (\ell, x) \equiv 0 \pmod{N_\ell}\}.$$

Then there is a projection

$$p : L'_\ell \to M'_\ell, \qquad p(x) = x_M + \frac{(x,\ell)}{N_\ell}\xi_M,$$

where x_M and ξ_M are the orthogonal projections of $x, \xi \in V$ to $M_{\ell,\mathbb{Q}}$. So it induces a projection, which is also denoted by p, from L'_ℓ/L to M'_ℓ/M_ℓ. Next, we define the Weyl chamber W_{ℓ_M} for

$$f = \sum_{\mu\in L'/L} f_\mu \phi_\mu \in M^!_{0,\rho_L}.$$

Let $\mathrm{Gr}(M_\ell)$ be the Grassmannian of negative lines of $M_{\ell,\mathbb{R}}$, which is a real manifold of dimension 1. For $\lambda \in M'_\ell/M_\ell$ and $n \in \mathbb{Q}$, let

$$Z_M(n,\lambda) = \{z \in \mathrm{Gr}(M_\ell) |\, z \perp x \text{ for some } x \in \lambda + M_\ell \text{ with } Q(x) = n\},$$

which is either empty or a real divisor of $\mathrm{Gr}(M_\ell)$. The Weyl chambers W_f associated with f are the connected components of

$$\mathrm{Gr}(M_\ell) - \bigcup_{\mu \in L'_\ell/L} \bigcup_{\substack{n>0 \\ c(-n,\mu)\neq 0}} Z_M(n, p(\mu)).$$

Let $\ell_M \in M_\ell$ and $\ell'_M \in M'_\ell$ be isotropic elements with $(\ell_M, \ell'_M) = 1$, whose existence is guaranteed in our case. For general case, we refer the reader to [21, Section 9] and [29, Section 5]. Choose a Weyl chamber such that ℓ_M is contained in its closure, and denote such Weyl chamber by W_{f,ℓ_M}. We now define its Weyl vector $\rho(W_{f,\ell_M}, f)$. Define

$$f_M = \sum_{\lambda \in M'_\ell/M_\ell} f_{M,\lambda}\phi_{M,\lambda}$$

where $\phi_{M,\lambda} \in \mathbb{C}[M'_\ell/M_\ell]$ and

$$f_{M,\lambda} = \sum_{\substack{\mu \in L'_\ell/L \\ p(\mu)=\lambda+M}} f_\mu.$$

Let P_{ℓ_M} be the Lorentzian sublattice of M_ℓ associated with ℓ_M. Clearly, $P_{\ell_M} = \{0\}$ and P'_{ℓ_M}/P_{ℓ_M} is trivial. Construct f_P from f_M in the same way as f_M constructed from f, and obtain

$$f_P = \sum_{\lambda \in M'_{\ell,\ell_M}/M_\ell} \sum_{\substack{\mu \in L'_\ell/L \\ p(\mu)=\lambda+M_\ell}} f_\mu.$$

Then the Weyl vector $\rho(W_{f,\ell_M}, f)$ is defined by

$$\rho(W_{f,\ell_M}, f) = \rho_{\ell_M}\ell_M + \rho_{\ell'_M}\ell'_M$$

where

$$\rho_{\ell'_M} = -1 + \frac{c_P(0)}{24} = -1 + \frac{1}{24} \sum_{\lambda \in M'_{\ell,\ell_M}/M_\ell} \sum_{\substack{\mu \in L'_\ell/L \\ p(\mu)=\lambda+M_\ell}} c(0,\mu),$$

$$\rho_{\ell_M} = -\frac{1}{4} \sum_{\lambda \in M'_{\ell,\ell_M}/M_\ell} c_M(0,\lambda) B_2((\lambda, \ell'_M))$$

and $B_2(x) = x^2 - x + \frac{1}{6}$ is the second Bernoulli polynomial.

Proof of Theorem 3.1.5. Let $\ell_s = \begin{pmatrix} 0 & a_s \\ 0 & c_s \end{pmatrix} \in L_N$ and $\ell'_s = \begin{pmatrix} -b_s & 0 \\ -d_s & 0 \end{pmatrix} \in L'_N$. Then we have

$$L'_{N,\ell_s} = \begin{pmatrix} \mathbb{Z} & \mathbb{Z}/N \\ m_s\mathbb{Z} & \mathbb{Z} \end{pmatrix}, \qquad L'_{N,\ell_s}/L_N = \left\{ \mu_{j,m_s k} = \begin{pmatrix} 0 & j/N \\ m_s k & 0 \end{pmatrix} \right\}_{\substack{0 \le j \le N-1 \\ 0 \le k \le h_s - 1}}$$

$$M_s := M_{\ell_s} = \left\{ \begin{pmatrix} a_s h_s x & b_s y \\ c_s h_s x & d_s y \end{pmatrix} \middle| \, x, y \in \mathbb{Z} \right\} \quad \text{and} \quad M'_s := M'_{\ell_s} = \left\{ \begin{pmatrix} a_s x & \frac{b_s}{h_s} y \\ c_s x & \frac{d_s}{h_s} y \end{pmatrix} \middle| \, x, y \in \mathbb{Z} \right\},$$

and

$$z_{\ell_s} = \begin{pmatrix} -a_s z_1 & b_s z_2 \\ -c_s z_1 & d_s z_2 \end{pmatrix} \quad \text{and} \quad w(z_{\ell_s}) = \begin{pmatrix} -a_s z_1 - b_s & b_s z_2 + a_s z_1 z_2 \\ -c_s z_1 - d_s & d_s z_2 + c_s z_1 z_2 \end{pmatrix}.$$

Also, we deduce that

$$\mathrm{Gr}(M_s) = \left\{ \mathbb{R} \begin{pmatrix} -a_s x & b_s \\ -c_s x & d_s \end{pmatrix} \middle| \, x > 0 \right\}$$

and

$$Z_{M_s}(1, \mu_{0,0}) = \{ Z \in \mathrm{Gr}(M_s) |\, Z \perp X \text{ for some } X \in \mu_{0,0} + M_s \text{ with } Q(X) = 1 \}$$

$$= \begin{cases} \varnothing & \text{if } m_s \neq N, \\ \left\{ \mathbb{R} \begin{pmatrix} -a_s & b_s \\ -c_s & d_s \end{pmatrix} \right\} & \text{if } m_s = N. \end{cases}$$

And thus

$$\mathrm{Gr}(M_s) - Z_{M_s}(1, \mu_{0,0}) = \begin{cases} \mathrm{Gr}(M_s) & \text{if } m_s \neq N, \\ \left\{ \mathbb{R} \begin{pmatrix} -a_s x & b_s \\ -c_s x & d_s \end{pmatrix} \middle| \ 0 < x < 1 \right\} \cup \left\{ \mathbb{R} \begin{pmatrix} -a_s x & b_s \\ -c_s x & d_s \end{pmatrix} \middle| \ x > 1 \right\} & \text{if } m_s = N. \end{cases}$$

Let $\ell_{M_s} = \begin{pmatrix} a_s h_s & 0 \\ c_s h_s & 0 \end{pmatrix}$ and $\ell'_{M_s} = \begin{pmatrix} 0 & \frac{b_s}{h_s} \\ 0 & \frac{d_s}{h_s} \end{pmatrix}$. Then the Weyl chamber whose closure contains $\mathbb{Q}\ell_{M_s}$ is

$$W_{\ell_{M_s}} = \begin{cases} \left\{ \mathbb{R} \begin{pmatrix} -a_s x & b_s \\ -c_s x & d_s \end{pmatrix} \middle| \ x > 0 \right\} & \text{if } m_s \neq N, \\ \\ \left\{ \mathbb{R} \begin{pmatrix} -a_s x & b_s \\ -c_s x & d_s \end{pmatrix} \middle| \ x > 1 \right\} & \text{if } m_s = N, \end{cases}$$

and thus for $X = \begin{pmatrix} a_s n & -\frac{b_s}{h_s} m \\ c_s n & -\frac{d_s}{h_s} m \end{pmatrix} \in M'_s$, we have that

$$(X, W_{\ell_{M_s}}) > 0 \quad \text{if and only if} \quad \begin{cases} m, n \geq 0 \text{ and } m^2 + n^2 > 0 & \text{if } m_s \neq N, \\ m \geq 0,\ m + n \geq 0 \text{ and } m^2 + n^2 > 0 & \text{if } m_s = N, \end{cases} \tag{3.2.9}$$

and we can check that $(X, z_{\ell_s}) = \frac{m_s}{N} z_1 m + z_2 n = \frac{1}{h_s} z_1 m + z_2 n$. We also have

$$M'_{s,\ell_{M_s}} = \left\{ \begin{pmatrix} a_s x & b_s y \\ c_s x & d_s y \end{pmatrix} \middle| \, x, y \in \mathbb{Z} \right\} \quad \text{and} \quad M'_{s,\ell_{M_s}}/M_s = \left\{ \begin{pmatrix} a_s k & 0 \\ c_s k & 0 \end{pmatrix} \right\}_{0 \le k \le h_s - 1}.$$

Let $\tilde{x}, \tilde{y} \in \mathbb{Z}$ such that $c_s \tilde{x} - aN\tilde{y} = m_s$. Then $p : L'_{N,\ell_s}/L_N \to M'_s/M_s$ is

$$p\left(\begin{pmatrix} 0 & j/N \\ m_s k & 0 \end{pmatrix} \right) = \begin{pmatrix} a_s k \tilde{x} & -\frac{b_s c_s}{N} j \\ c_s k \tilde{x} & -\frac{d_s c_s}{N} j \end{pmatrix},$$

and thus

$$F_{N,P} = \sum_{\lambda \in M'_{s,\ell_{M_s}}/M} \left(\sum_{\substack{\mu \in L'_{N,\ell_s}/L_N \\ p(\mu) = \lambda}} F_{N,\phi_\mu} \right) = \sum_{k=0}^{h_s - 1} \sum_{j=0}^{m_s - 1} F_{N,\phi_{jh_s,km_s}}.$$

Thus,

$$c_P(0, \phi_{0,0}) = \sum_{k=0}^{h_s-1} \sum_{j=0}^{m_s-1} c(0, \phi_{jh_s,km_s}) = \begin{cases} 24 & \text{if } m_s \neq N, \\ 0 & \text{if } m_s = N, \end{cases}$$

and

$$\rho_{\ell'_{M_s}} = \begin{cases} 0 & \text{if } m_s \neq N, \\ -1 & \text{if } m_s = N. \end{cases}$$

Therefore, the Weyl vector of $W_{\ell_{M_s}}$ associated to F_N is

$$\rho(W_{\ell_{M_s}}, F_N) = \begin{cases} \rho_{\ell_{M_s}} \ell_{M_s} & \text{if } m_s \neq N, \\ -\ell'_{M_s} & \text{if } m_s = N, \end{cases}$$

and thus

$$(z_{\ell_s}, \rho(W_{\ell_{M_s}}, F_N)) = \begin{cases} -h_s \rho_{\ell_{M_s}} z_2 & \text{if } m_s \neq N, \\ -z_1 & \text{if } m_s = N. \end{cases} \tag{3.2.10}$$

We can also show that $h_s\rho_{\ell_{M_s}} = 1$, but we do not need this fact in our proof, so we leave the details of computations to the reader. Similarly, if we let $\ell_s = \begin{pmatrix} c_s & -a_s \\ 0 & 0 \end{pmatrix} \in L_N$ and $\ell'_s = \begin{pmatrix} 0 & 0 \\ d_s & -b_s \end{pmatrix} \in L'_N$, then we have

$$M_s = \left\{ \begin{pmatrix} -d_s y & b_s y \\ c_s h_s x & -a_s h_s x \end{pmatrix} \middle| \, x, y \in \mathbb{Z} \right\} \quad \text{and} \quad M'_s = \left\{ \begin{pmatrix} -\frac{d_s}{h_s} y & \frac{b_s}{h_s} y \\ c_s x & -a_s x \end{pmatrix} \middle| \, x, y \in \mathbb{Z} \right\},$$

$$\ell_{M_s} = \begin{pmatrix} 0 & 0 \\ c_s h_s & -a_s h_s \end{pmatrix} \quad \text{and} \quad \ell'_{M_s} = \begin{pmatrix} -\frac{d_s}{h_s} & \frac{b_s}{h_s} \\ 0 & 0 \end{pmatrix},$$

$$z_{\ell_s} = \begin{pmatrix} d_s z_1 & -b_s z_1 \\ c_s z_2 & -a_s z_2 \end{pmatrix} \quad \text{and} \quad w(z_{\ell_s}) = \begin{pmatrix} d_s z_1 + c_s z_1 z_2 & -b_s z_1 - a_s z_1 z_2 \\ c_s z_2 + d_s & -a_s z_2 - b_s \end{pmatrix},$$

$$W_{\ell_{M_s}} = \begin{cases} \left\{ \mathbb{R} \begin{pmatrix} d_s & -b_s \\ c_s x & -a_s x \end{pmatrix} \middle| \, x > 0 \right\} & \text{if } m_s \neq N, \\ \left\{ \mathbb{R} \begin{pmatrix} d_s & -b_s \\ c_s x & -a_s x \end{pmatrix} \middle| \, x > 1 \right\} & \text{if } m_s = N, \end{cases}$$

$$(X, W_{\ell_{M_s}}) > 0 \quad \text{if and only if} \quad \begin{cases} m,\, n \geq 0 \text{ and } m^2 + n^2 > 0 & \text{if } m_s \neq N, \\ n \geq 0,\, m + n \geq 0 \text{ and } m^2 + n^2 > 0 & \text{if } m_s = N, \end{cases} \tag{3.2.11}$$

for $X = \begin{pmatrix} -\frac{d_s}{h_s}n & \frac{b_s}{h_s}n \\ -c_s m & a_s m \end{pmatrix} \in M'_s$, and

$$\rho(W_{\ell_{M_s}}, F_N) = \begin{cases} \rho_{\ell_{M_s}} \ell_{M_s} & \text{if } m_s \neq N, \\ -\ell'_{M_s} & \text{if } m_s = N. \end{cases}$$

Also, we can check $(X, z_{\ell_s}) = z_1 m + \frac{1}{h_s} z_2 n$, and

$$(z_{\ell_s}, \rho(W_{\ell_{M_s}}, F_N)) = \begin{cases} -h_s \rho_{\ell_{M_s}} z_1 & \text{if } m_s \neq N, \\ -z_2 & \text{if } m_s = N. \end{cases} \tag{3.2.12}$$

Now we are ready for the proof of Theorem 3.1.5. We aim to show that

$$\pi_N(z_1) - \pi_N(z_2) = \Psi(z, F_N).$$

We first note by Borcherds Theorem, Proposition 3.2.4 and Lemma 3.2.8(ii) that $\Psi(z, F_N)$ can be viewed as either a meromorphic function on the Shimura variety X_{K_N} or a meromorphic function on $Y_0(N) \times Y_0(N)$, and $h_N(z_1, z_2) := \pi_N(z_1) - \pi_N(z_2)$ is a meromorphic function on $Y_0(N) \times Y_0(N)$. Let

$$g(z_1, z_2) = \frac{\Psi(z, F_N)}{h_N(z_1, z_2)}.$$

Then $g(z_1, z_2)$ is a meromorphic function on $Y_0(N) \times Y_0(N)$ with no zeros or poles by Lemma 3.2.5. Let us fix $z_2 \in \mathbb{H}$. Then $g_1(z_1) := g(z_1, z_2)$ is a meromorphic function in z_1 on $Y_0(N)$. Let us investigate the behavior of $g_1(z_1)$ at the cusps of $Y_0(N)$. By Proposition 3.2.4, we can know that the Fourier expansion of $\Psi(z, F_N)$ at the cusp $(s = a_s/c_s, i\infty)$ is equal to the Borcherds product expansion of $\Psi(z, F_N)$ at the cusp $\mathbb{Q}\ell_s$ where $\ell_s = \begin{pmatrix} 0 & a_s \\ 0 & c_s \end{pmatrix}$. Then by Theorem 3.2.3(3) together with (3.2.9) and (3.2.10), we have

$$\Psi(z, F_N)$$

$$= \begin{cases} C_s q_2^{-h_s \rho_{\ell_{M_s}}} \prod\limits_{\substack{m,n\geq 0 \\ m^2+n^2>0}} \prod\limits_{\substack{\mu \in L'_{N,\ell_s}/L_N \\ p(\mu)=\begin{pmatrix} -a_s n & \frac{b_s}{h_s} m \\ -c_s n & \frac{d_s}{h_s} m \end{pmatrix}}} \left(1 - q_1^{m/h_s} q_2^n e((\mu, \ell'_s))\right)^{c(mn,\phi_\mu)} & \text{if } m_s \neq N, \\ C_s(q_1^{-1} - q_2^{-1}) \prod\limits_{\substack{m\geq 0 \\ m+n\geq 0 \\ m^2+n^2>0}} \prod\limits_{\substack{\mu \in L'_{N,\ell_s}/L_N \\ p(\mu)=\begin{pmatrix} -a_s n & \frac{b_s}{h_s} m \\ -c_s n & \frac{d_s}{h_s} m \end{pmatrix}}} \left(1 - q_1^{m/h_s} q_2^n e((\mu, \ell'_s))\right)^{c(mn,\phi_\mu)} & \text{if } m_s = N, \end{cases}$$

near the cusp $\mathbb{Q}\ell_s$ for some nonzero constant C_s depending on the choice of cusp s, which can be identified by the condition given in Borhcerds' Theorem. Then when $z_2 \in \mathbb{H}$ is fixed, as a meromorphic function on $Y_0(N)$, the order of vanishing of $\Psi(z, F_N)$ at the cusp $s = a_s/c_s$ can be computed

$$\operatorname{ord}_s(\Psi(z, F_N)) = \begin{cases} 0 & \text{if } m_s \neq N, \\ -1 & \text{if } m_s = N. \end{cases}$$

Also, we can easily compute the Fourier expansion of $h_N(z_1, z_2)$ at the same cusp, and obtain the order of vanishing of $h_N(z_1, z_2)$ at the cusp $s = a_s/c_s$ when $z_2 \in \mathbb{H}$ is fixed, which is

$$\operatorname{ord}_s(h_N) = \begin{cases} 0 & \text{if } m_s \neq N, \\ -1 & \text{if } m_s = N. \end{cases}$$

Therefore, as a meromorphic function on $Y_0(N)$, the modular function $g_1(z_1)$ is holomorphic at all of the cusps, and thus $g(z_1, z_2)$ is constant on $Y_0(N) \times \{z_2\}$. Similarly, if we fix $z_1 \in \mathbb{H}$, by Theorem 3.2.3(3), (3.2.11) and (3.2.12), we can show that $g(z_1, z_2)$ is constant on $\{z_1\} \times Y_0(N)$. Hence, $g(z_1, z_2)$ is constant on $Y_0(N) \times Y_0(N)$, which is 1 by comparing the Fourier expansions of $\Psi(z, F_N)$ and $h_N(z_1, z_2)$ at $(i\infty, i\infty) = (1/N, 1/N)$, and this completes the proof.

□

Proofs of Corollaries 3.1.6 and 3.1.9

We end this section with the proofs to Corollaries 3.1.6 and 3.1.9.

Proof of Corollary 3.1.6. We first note (3.2.2)–(3.2.4) that the $\phi_{j,0}$-component for $0 \leq j \leq N-1$ of F_N is

$$\sum_{\ell=-1}^{\infty} c(\ell, \mu_{j,0}) q^\ell \tag{3.2.13}$$

$$= \frac{2}{\lambda_{2,N}|\Gamma_0(N):\Gamma_1(N)|} \left\{ \sum_{\substack{s\in\mathcal{C}(\Gamma_1(N)) \\ s \text{ regular} \\ m_s|j}} [(\pi_N|\, M_s)_0 - A_s(0)] + \sum_{\substack{s\in\mathcal{C}(\Gamma_1(N)) \\ s \text{ irregular} \\ m_s|j}} \frac{1}{h_s} [(\pi_N|\, M_s) - A_s(0)] \right\}$$

$$= \sum_{\substack{d|N \\ d|j}} \frac{2}{\lambda_{2,N}|\Gamma_0(N):\Gamma_1(N)|} \left\{ \sum_{\substack{s\in\mathcal{C}(\Gamma_1(N)) \\ s \text{ regular} \\ m_s=d}} [(\pi_N|\, M_s)_0 - A_s(0)] + \sum_{\substack{s\in\mathcal{C}(\Gamma_1(N)) \\ s \text{ irregular} \\ m_s=d}} \frac{1}{h_s} [(\pi_N|\, M_s) - A_s(0)] \right\}$$

$$= \sum_{\substack{d|N \\ d|j}} \sum_{\ell=-1}^{\infty} A(\ell, d) q^\ell$$

$$= \sum_{\ell=-1}^{\infty} \left(\sum_{\substack{d|N \\ d|j}} A(\ell, d) \right) q^\ell. \tag{3.2.14}$$

Now from the proof of Theorem 3.1.5 and (3.2.13) together with Borcherds Theorem, we deduce that

$$
\begin{aligned}
&(\pi_N(z_1)-\pi_N(z_2)) \prod_{\substack{\gamma\in\Gamma_1(N)\backslash\bar{\Gamma}_0(N)\\ \gamma\neq I}} \left(1-\pi_N(z_1)^{-1}\pi_N(\gamma z_2)\right)\\
&=(q_1^{-1}-q_2^{-1})\prod_{m,n>0}\prod_{j=0}^{N-1}\left(1-q_1^m q_2^n e\left(-j/N\right)\right)^{c(mn,\phi_{j,0})}\\
&=(q_1^{-1}-q_2^{-1})\prod_{m,n>0}\prod_{j=0}^{N-1}\prod_{\substack{d|N\\ d|j}}\left(1-q_1^m q_2^n e\left(-j/N\right)\right)^{A(mn,d)}\\
&=(q_1^{-1}-q_2^{-1})\prod_{m,n>0}\prod_{d|N}\prod_{j'=0}^{\frac{N}{d}-1}\left(1-q_1^m q_2^n e\left(-j'd/N\right)\right)^{A(mn,d)}\\
&=(q_1^{-1}-q_2^{-1})\prod_{m,n>0}\prod_{d|N}\left(1-\left(q_1^m q_2^n\right)^{\frac{N}{d}}\right)^{A(mn,d)}.
\end{aligned}
$$

□

Proof of Corollary 3.1.9. Taking the logarithmic derivative of both sides of (3.1.3), we have that

$$
\begin{aligned}
&-\frac{1}{2\pi i}\frac{\pi_N'(z_1)}{\pi_N(z_1)-\pi_N(z_2)}\\
&=\frac{1}{1-q_2^{-1}q_1}+\sum_{m,n>0}\sum_{d|N}\frac{A(mn,d)\left(q_1^m q_2^n\right)^{\frac{N}{d}}}{1-\left(q_1^m q_2^n\right)^{\frac{N}{d}}}\\
&=\sum_{n=0}^{\infty}(q_2^{-1}q_1)^n+\sum_{m,n>0}\sum_{d|N}\sum_{\ell=1}^{\infty}A(mn,d)\left(q_1^m q_2^n\right)^{\frac{N}{d}\ell},
\end{aligned}
$$

and this proves the corollary. □

Remark 3.2.9. *Corollary* 3.1.9 *has recently been generalized to arbitrary Fuchsian subgroups of first kind of* $\mathrm{SL}_2(\mathbb{R})$ *and proved by the author [97] using a pure analytic method.*

3.3 Part II: Gross–Zagier Type CM Value Formulas

3.3.1 Big CM Cycles and Big CM Value Formula

In this section, we briefly review the concepts of big CM cycles and big CM value formula (see [28, Sec. 2–4] for details), based on which we realize a big CM cycle in the Hilbert modular surface $Y_0(N) \times Y_0(N)$ and prove Theorem 3.1.10 at the end of Subsection 3.3.2.

Big CM Cycles in X_K

Let F be a totally real number field of degree $d+1$ and W be an F-quadratic space with an F-quadratic form $Q_F(\cdot)$ of signature $((2,0),\dots,(2,0),(0,2))$ with respect to the $d+1$ embeddings $\{\sigma_i\}_{i=1}^{d+1}$ of F such that $\mathrm{Res}_{F/\mathbb{Q}}W$ is a rational quadratic space of signature $(2d,2)$ with the quadratic form $Q(\cdot) = \mathrm{tr}_{F/\mathbb{Q}} \circ Q_F(\cdot)$ induced from $Q_F(\cdot)$. Then we have an orthogonal direct sum decomposition

$$\mathrm{Res}_{F/\mathbb{Q}}W = \bigoplus_{i=1}^{d+1} W_{\sigma_i}$$

where $W_{\sigma_i} = W \otimes_{F,\sigma_i} \mathbb{R}$. The negative 2-plane $W_{\sigma_{d+1}}$ gives rise to a two-point (big CM points) subset $\{z^{\pm}_{\sigma_{d+1}}\}$ of $\mathbb{D}$. Let T be the preimage of $\mathrm{Res}_{F/\mathbb{Q}}\mathrm{SO}(W) \subset \mathrm{SO}(\mathrm{Res}_{F/\mathbb{Q}}W)$ in H, the general spin group of $\mathrm{Res}_{F/\mathbb{Q}}W$. Then we have the following commutative diagram.

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{G}_m & \longrightarrow & T & \longrightarrow & \mathrm{Res}_{F/\mathbb{Q}}\mathrm{SO}(W) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathbb{G}_m & \longrightarrow & H & \longrightarrow & \mathrm{SO}(V) & \longrightarrow & 1
\end{array}$$

and this implies that T is a maximal torus associated to the CM number field $E = F(\sqrt{-\det W})$. And we obtain a so called big CM cycle in X_K, the Shimura variety associated to the compact open subgroup K,

$$Z(W, z^{\pm}_{\sigma_{d+1}}) = T(\mathbb{Q})\backslash \left(\{z^{\pm}_{\sigma_{d+1}}\} \times T(\mathbb{A}_{\mathbb{Q},f})/K_T \right),$$

where $K_T = K \cap T(\mathbb{A}_{\mathbb{Q},f})$. The CM cycle $Z(W, z^{\pm}_{\sigma_{d+1}})$ is defined over F, and the formal sum $Z(W)$ of all of its Galois conjugates is a 0-cycle in X_K defined over $\mathbb{Q}$.

Big CM Value Formula

Associated to the F-quadratic space W and the additive adelic character $\psi_F = \psi \circ \mathrm{tr}_{F/\mathbb{Q}}$ is a Weil representation $\omega = \omega_{\psi_F}$ of $\mathrm{SL}_2(\mathbb{A}_{\mathbb{Q},f})$ (and thus $T(\mathbb{A}_{\mathbb{Q}})$) on $\mathcal{S}(W_{\mathbb{A}_F}) = \mathcal{S}(V_{\mathbb{A}_{\mathbb{Q}}})$. Let $\chi = \chi_{E/F}$ be the quadratic Hecke character of F associated to E/F. Then $\chi = \chi_W$ is also the quadratic Hecke character of F associated to W, and there is an $\mathrm{SL}_2(\mathbb{A}_F)$-equivalent map

$$\lambda = \prod_{\nu} \lambda_\nu : \mathcal{S}(W_{\mathbb{A}_F}) \to I(0, \chi)$$

via $\lambda(\phi)(g) = \omega(g)\phi(0)$. where $I(s,\chi) = \mathrm{Ind}_{B_{\mathbb{A}_F}}^{\mathrm{SL}_2(\mathbb{A}_F)} \chi|\cdot|^s$ is the principal series whose elements are smooth functions Φ on $\mathrm{SL}_2(\mathbb{A}_F)$ satisfying

$$\Phi(n(b)m(a)g, s) = \chi(a)|a|^{s+1}\Phi(g,s)$$

for $b \in \mathbb{A}_F$ and $a \in \mathbb{A}_F^\times$ with $n(b)m(a) \in B_F$, where

$$n(b) := \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad m(a) := \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$$

and $B_{\mathbb{A}_F}$ the standard Borel subgroup of $\mathrm{SL}_2(\mathbb{A}_F)$. Such an element is called factorizable if $\Phi = \otimes \Phi_\nu$ with $\Phi_\nu \in I(s,\chi)$. It is called standard if $\Phi_{\mathrm{SL}_2(\hat{\mathcal{O}}_F)\mathrm{SO}_2(\mathbb{R})^2}$ is independent of s. For a standard element Φ, its associated Eisenstein series is defined for $\Re(s) \gg 0$ by

$$E(g,s,\Phi) = \sum_{\gamma \in B_{\mathbb{A}_F}\backslash \mathrm{SL}_2(\mathbb{A}_F)} \Phi(\gamma g, s).$$

For $\phi \in \mathcal{S}(V_{\mathbb{A}_{\mathbb{Q},f}}) = \mathcal{S}(W_{\mathbb{A}_{F,f}})$, let Φ_f be the standard element associated to $\lambda_f(\phi) \in I(0,\chi)$. For each real embedding σ_i of F, let $\Phi_{\sigma_i} \in I(s, \chi_{\mathbb{C}/\mathbb{R}}) = I(s, \chi_{E_{\sigma_i}/F_{\sigma_i}})$ be the unique 'weight one' eigenvector of $\mathrm{SL}_2(\mathbb{R})$ given by

$$\Phi_{\sigma_i}(n(b)m(a)k_\theta) = \chi_{\mathbb{C}/\mathbb{R}}(a)|a|^{s+1}e^{i\theta}$$

for $b \in \mathbb{R}$, $a \in \mathbb{R}^\times$ and $k_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \in \mathrm{SO}_2(\mathbb{R})$. We define for $\vec{\tau} = (\tau_1, \dots, \tau_{d+1}) \in \mathbb{H}^{d+1}$,

$$E(\vec{\tau}, s, \phi) = \left(\prod_{i=1}^{d+1} v_i\right)^{-\frac{1}{2}} E(g_{\vec{\tau}}, s, \Phi_f \otimes \Phi_{\sigma_1} \otimes \Phi_{\sigma_2}),$$

where $\tau_i = u_i + iv_i$ and $g_{\vec{\tau}} = (n(u_i)m(\sqrt{v_i}))_{1\le i\le d+1}$. It is a non-holomorphic Hilbert modular form of scalar weight 1 for some congruence subgroup of $\mathrm{SL}_2(\mathcal{O}_F)$. We normalize $E(\vec{\tau}, s, \phi)$ by

$$E^*(\vec{\tau}, s, \phi) = \Lambda(s+1, \chi)E(\vec{\tau}, s, \phi)$$

where

$$\Lambda(s, \chi) = \left(\mathrm{N}_{F/\mathbb{Q}}(\partial_F d_{E/F})\right)^{\frac{s}{2}} \left(\pi^{-\frac{s+1}{2}}\Gamma\left(\frac{s+1}{2}\right)\right)^{d+1} L(s, \chi).$$

We write the Fourier expansion of $E^*(\vec{\tau}, s, \phi)$ as

$$E^*(\vec{\tau}, s, \phi) = E_0^*(\vec{\tau}, s, \phi) + \sum_{t\in F^\times} E_t^*(\vec{\tau}, s, \phi). \tag{3.3.1}$$

If one assumes that ϕ is factorizable, then

$$E_t^*(\vec{\tau}, s, \phi) = \prod_{\mathfrak{p}\nmid\infty} W_{t,\mathfrak{p}}^{*,\psi_F}(s, \phi) \prod_{i=1}^{d+1} W_{t,\sigma_i}^*(\tau_i, s, \Phi_{\sigma_i}) \tag{3.3.2}$$

and

$$E_0^*(\vec{\tau}, s, \phi) = \phi(0)\Lambda(s+1, \chi)\left(\prod_{i=1}^{d+1} v_i\right)^{\frac{s}{2}} \tag{3.3.3}$$

$$+\left(\prod_{i=1}^{d+1} v_i\right)^{-\frac{s}{2}} \Lambda(s,\chi) \prod_{\mathfrak{p}<\infty} W_{0,\mathfrak{p}}^{*,\psi_F}(s,\phi) \prod_{i=1}^{d+1} \gamma(W_{\sigma_i}), \tag{3.3.4}$$

where

$$W_{t,\mathfrak{p}}^{*,\psi_F}(s,\phi) = |\mathrm{N}_{F/\mathbb{Q}}(\partial_F d_{E/F})|_{\mathfrak{p}}^{-\frac{s+1}{2}} L_{\mathfrak{p}}(1+s,\chi) W_{t,\mathfrak{p}}^{\psi_F}(s,\phi) \tag{3.3.5}$$

and

$$W_{t,\sigma_i}^{*}(\tau_i, s, \Phi_{\sigma_i}) = v_i^{-1/2} \pi^{-\frac{s+2}{2}} \Gamma\left(\frac{s+2}{2}\right) W_{t,\sigma_i}^{\psi_F}(\tau_i, s, \Phi_{\sigma_i}). \tag{3.3.6}$$

are the normalized local Whittaker functions, and $\gamma(W_{\sigma_i})$ are Weil indices (see, e.g., [57, 92]). Moreover, for $m > 0$, we write

$$a_m(\phi) = \sum_{\substack{t \in F_+^\times \\ \mathrm{tr}_{F/\mathbb{Q}} t = m}} a(t,\phi),$$

where $F_+^\times$ consists of all totally positive elements in F, the term $a(t,\phi)$ is the t-th Fourier coefficient of $E^{*,\prime}(\tau^\Delta, 0, \phi)$ and $\tau^\Delta = (\tau, \ldots, \tau)$ the diagonal element, and write the constant term of $E^{*,\prime}(\vec{\tau}, s, \phi)$ as

$$\phi(0)\Lambda(0,\chi) \log\left(\prod_{i=1}^{d+1} v_i\right) + a_0(\phi)$$

for a constant $a_0(\phi)$ depending on ϕ. One can check that $a(t,\phi) = 0$ unless $t - Q_F(\mu) \in \partial_F^{-1}$ where μ is the element in F associated with ϕ (see., e.g., [57, 92]).

Now we are ready to state the big CM value formula due to Bruinier, Kudla and Yang [28, Thm. 5.2], which expresses the sum of the values of a theta-lift on a Shimura variety X_K over a big CM cycle in terms of the coefficients of an incoherent Eisenstein series of weight 1.

Theorem 3.3.1 (Bruinier, Kudla, and Yang)**.** *For a given $f \in M^!_{1-d,\rho_L}$ with*

$$f(\tau) = \sum_{\mu \in L'/L} \sum_{\substack{n \in -Q(\mu)+\mathbb{Z} \\ n \gg -\infty}} c(n,\mu) q^n \phi_\mu,$$

let $\Phi(Z,f)$ be the Borcherds lift for f defined as in Subsection 3.2.1*. Then we have*

$$\sum_{Z \in Z(W)} \Phi(Z,f) = \frac{\deg(Z(W, z^{\pm}_{\sigma_{d+1}}))}{\Lambda(0,\chi)} \sum_{\substack{\mu \in L'/L \\ m \in Q(\mu)+\mathbb{Z} \\ m \geq 0}} c(-m,\mu) a_m(\phi_\mu).$$

3.3.2 Proof of Theorem 3.1.10

This subsection is devoted to the proof of our second main result, Theorem 3.1.10. We first realize and interpret big CM cycles in the Hilbert modular surfaces for $\Gamma_0(N)$ under the identification $Y_0(N) \times Y_0(N) \cong X_{K_N}$ as we have seen in Subsection 3.2.1. Then we conclude this section with the proof.

Let $E_i = \mathbb{Q}(\sqrt{d_i})$ for $i = 1, 2$ with $(d_1, d_2) = 1$ be two imaginary quadratic fields of fundamental discriminants d_i, and let $E = E_1 \otimes E_2 = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$. Let $F = \mathbb{Q}(\sqrt{D})$ with $D = d_1 d_2$ be the maximal totally real subfield of E. We can view E as a F-quadratic space W with F-quadratic form $Q_F(z) = \frac{z\bar{z}}{\sqrt{D}}$. Then we can also view it as a rational quadratic space $\mathrm{Res}_{F/\mathbb{Q}} W$ with quadratic form $Q(z) = \mathrm{tr}_{F/\mathbb{Q}} \circ Q_F(z)$. Let σ_i for $i = 1, 2$ be the two real embeddings of F with $\sigma_i(\sqrt{D}) = (-1)^{i-1}\sqrt{D}$. Then we can see that W_{σ_1} has signature $(2, 0)$ at σ_1 and W_{σ_2} has signature $(0, 2)$ at σ_2. We choose a $\mathbb{Z}$-basis for $\mathcal{O}_E$ as follows

$$e_1 = 1 \otimes 1, \quad e_2 = \frac{-d_1 + \sqrt{d_1}}{2} \otimes 1, \quad e_3 = 1 \otimes \frac{d_2 + \sqrt{d_2}}{2}, \quad e_4 = e_2 e_3,$$

and throughout the remainder of this section, we will drop $\otimes$ when there is no ambiguity. Then we can identify $(\mathrm{Res}_{F/\mathbb{Q}} W, Q(\cdot))$ with $(V = M_2(\mathbb{Q}), \det)$ considered in Subsection

3.2.1 by

$$\sum_{i=1}^{4} x_i e_i \to \begin{pmatrix} x_3 & x_1 \\ x_4 & x_2 \end{pmatrix}.$$

In particular, under this identification, we have

$$L_N \cong \mathbb{Z} + \mathbb{Z}\frac{-d_1 + \sqrt{d_1}}{2} + \mathbb{Z}\frac{d_2 + \sqrt{d_2}}{2} + \mathbb{Z}\frac{N(-d_1 + \sqrt{d_1})(d_2 + \sqrt{d_2})}{4} \tag{3.3.7}$$

which is of index N in $\mathcal{O}_E$.

In such a case, the maximal torus T over $\mathbb{Q}$ is given by (see [52] or [28, Section 6])

$$T(R) = \{(t_1, t_2) \in (E_1 \otimes_{\mathbb{Q}} R)^\times \times (E_2 \otimes_{\mathbb{Q}} R)^\times | \, t_1 \bar{t}_1 = t_2 \bar{t}_2\}$$

for any $\mathbb{Q}$-algebra R. Then the map from T to E is given by $(t_1, t_2) \to t_1/\bar{t}_2$. By the theory of complex multiplication [88], there is an embedding

$$\imath_i : E_i \to M_2(\mathbb{Q})$$

such that

$$\imath_i(t) \begin{pmatrix} e_{i+1} \\ e_1 \end{pmatrix} = \begin{pmatrix} te_{i+1} \\ te_1 \end{pmatrix}.$$

Then $\imath = (\imath_1, \imath_2)$ gives the embedding from T to H, and one has

$$K_{N,T} := K_N \cap T(\mathbb{Q}) = \{(t_1, t_2) \in T(\mathbb{A}_f) | \, t_i \in \imath_i^{-1}(K_0(N))\}.$$

In the following, we will interpret the big CM cycle

$$Z(W, z_{\sigma_2}^{\pm}) = T(\mathbb{Q}) \backslash \left(\{z_{\sigma_2}^{\pm}\} \times T(\mathbb{A}_{\mathbb{Q},f})/K_{N,T}\right)$$

in X_{K_N} as a 0-cycle in $Y_0(N) \times Y_0(N)$.

Lemma 3.3.2. *Under the identification between* $\mathbb{D}$ *and* $\mathbb{H}^2 \cup (\mathbb{H}^-)^2$ *(see Proposition 3.2.4), the big CM points* z_{σ_2} *and* $z_{\sigma_2}^-$ *are identified with* $(\tau_1, \tau_2) \in \mathbb{H}^2$ *and* $(-\bar{\tau}_1, -\bar{\tau}_2) \in (\mathbb{H}^-)^2$*, respectively, where*

$$\tau_i = \frac{d_i + \sqrt{d_i}}{2}.$$

Proof. See [94, Lemma 3.4]. □

Lemma 3.3.3. *Let* $\mathrm{Cl}(E_{i,N})$ *be the ring class group of conductor* N *of* E_i *for* $i = 1, 2$. *Then there is an injection*

$$\jmath : T(\mathbb{Q})\backslash T(\mathbb{A}_f)/K_{N,T} \hookrightarrow \mathrm{Cl}(E_{1,N}) \times \mathrm{Cl}(E_{2,N})$$

with image

$$\begin{aligned} S(N, d_1, d_2) &:= \{([\mathfrak{a}_1], [\mathfrak{a}_2]) \in \mathrm{Cl}(E_{1,N}) \times \mathrm{Cl}(E_{2,N}) : \exists \textit{ fractional ideals } \mathfrak{a}_i \textit{ such that } \mathrm{N}(\mathfrak{a}_1) = \mathrm{N}(\mathfrak{a}_2)\} \\ &\cong \{(Q_1, Q_2) \in \mathcal{Q}_{d_1}(N)/\Gamma_0(N) \times \mathcal{Q}_{d_2}(N)/\Gamma_0(N) |\ a_1 = a_2,\ \textit{i.e.},\ Q_1(1,0) = Q_2(1,0)\} \end{aligned}$$

where $\mathcal{Q}_d(N)$ *denotes the set of primitive and positive definite binary quadratic forms* $aX^2 + bXY + cY^2$ *of discriminant* d *with* $(a, N) = 1$. *The isomorphism is given by*

$$[aX^2 + bXY + cY^2] \to \left[a, N\frac{-b + \sqrt{d}}{2}\right].$$

Proof. It is not hard to check that

$$\imath_i\left(x + y\frac{d_i + \sqrt{d_i}}{2}\right) = \begin{pmatrix} x + yd_i & y\frac{d_i - d_i^2}{4} \\ y & x \end{pmatrix}$$

for $x, y \in \mathbb{Q}$. Thus, one has $\imath_i^{-1}(K_0(N)) = \hat{\mathcal{O}}_{i,N}^\times$ where $\mathcal{O}_{i,N}$ is the order of conductor N of E_i and $E_i^\times\backslash\mathbb{A}_{E_i,f}^\times/\imath_i^{-1}(K_0(N)) \cong \mathrm{Cl}(E_{i,N})$ by [24, Section 4.4]. This fact together with [94, Lemma 3.5] implies the first assertion of the lemma. The isomorphism is due to Chen and Yui [33, Thm. 4.4]. □

Proposition 3.3.4. *Let G_i be the associated Galois group of the ring class field of conductor N of E_i. Then the points*

$$[z_{\sigma_2}^{\pm}, (t_1^{-1}, t_2^{-1})] \in T(\mathbb{Q})\backslash \left(\{z_{\sigma_2}^{\pm}\} \times T(\mathbb{A}_{\mathbb{Q},f})/K_{N,T}\right)$$

are identified with

$$[\tau_1^{\sigma_{\mathfrak{a}_1}}, \tau_2^{\sigma_{\mathfrak{a}_2}}],\ [(-\bar{\tau}_1)^{\sigma_{\mathfrak{a}_1}}, (-\bar{\tau}_2)^{\sigma_{\mathfrak{a}_2}}] \in Y_0(N) \times Y_0(N),$$

respectively, where $\sigma_{\mathfrak{a}_i} \in G_i$ is the Galois element associated to $[\mathfrak{a}_i] \in \mathrm{Cl}(E_{i,N})$ via Artin map, and $[\mathfrak{a}_i] \in \mathrm{Cl}(E_{i,N})$ is the ideal class associated to the idele class $[t_i] \in T(\mathbb{Q})\backslash T(\mathbb{A}_{\mathbb{Q},f})/K_{N,T}$.

In particular, by the Shimura reciprocity law, one has $\tau_i^{\sigma_{\mathfrak{a}_i}} = \tau_{\varphi(\mathfrak{a}_i)}$ where $\tau_{\mathfrak{a}}$ is the CM point associated to the ideal $\mathfrak{a}$, and $\varphi([a, N\frac{b+\sqrt{d}}{2}]) = [a, \frac{b+\sqrt{d}}{2}]$.

Proof. The former results follow from Lemma 3.3.3 and [94, Proposition 3.6]. The latter result follows from [88, Chapter 6] (or see [90] for a nice summary of Shimura reciprocity law) and [33, Theorem III]. □

Now we are ready for

Proof of Theorem 3.1.10. By Theorem 3.2.3(2) and Theorem 3.1.5, we deduce that

$$\Phi(z, F_N) = -4 \log |\Psi(z, F_N)| = -4 \log |\pi_N(z_1) - \pi_N(z_2)| \tag{3.3.8}$$

since $c(0,0) = 0$ by Lemma 3.2.8. Then together with Proposition 3.3.4 and Theorem 3.3.1, equation (3.3.8) implies that

$$4 \sum_{([\mathfrak{a}_1],[\mathfrak{a}_2]) \in S(p,d_1,d_2)} \Bigg(\log |\pi_N(\tau_1^{\sigma_{\mathfrak{a}_1}}) - \pi_N(\tau_2^{\sigma_{\mathfrak{a}_2}})| + \log |\pi_N((-\bar{\tau}_1)^{\sigma_{\mathfrak{a}_1}}) - \pi_N(\tau_2^{\sigma_{\mathfrak{a}_2}})|$$

$$+\log|\pi_N(\tau_1^{\sigma_{\mathfrak{a}_1}})-\pi_N((-\bar{\tau}_2)^{\sigma_{\mathfrak{a}_2}})|+\log|\pi_N((-\bar{\tau}_1)^{\sigma_{\mathfrak{a}_1}})-\pi_N((-\bar{\tau}_2)^{\sigma_{\mathfrak{a}_2}})|\Bigg)$$

$$=-\frac{2|S(N,d_1,d_2)|}{\Lambda(0,\chi)}\left(a_1(\phi_{0,0})+\sum_{\mu\in L'/L}c(0,\mu)a_0(\phi_\mu)\right)$$

$$=-\frac{|S(N,d_1,d_2)|w_1w_2}{2h(d_1)h(d_2)}\left(a_1(\phi_{0,0})+\sum_{\mu\in L'/L}c(0,\mu)a_0(\phi_\mu)\right), \tag{3.3.9}$$

where the simplification in the second equality follows from Lemma 3.2.8 which tells that $c(-m,\mu)=0$ except $c(-1,\mu_{0,0})=1$ for $m>0$, and the last equality follows from the fact that $\Lambda(0,\chi)=\Lambda(0,\chi_{E_1/\mathbb{Q}})\Lambda(0,\chi_{E_2/\mathbb{Q}})$. Since $\pi_N(\tau+1)=\pi_N(\tau)$ and $-\bar{\tau}_i=-d_i+\tau_i$, then the left hand side of (3.3.9) is simply

$$16\sum_{([\mathfrak{a}_1],[\mathfrak{a}_2])\in S(p,d_1,d_2)}\log|\pi_N(\tau_1^{\sigma_{\mathfrak{a}_1}})-\pi_N(\tau_2^{\sigma_{\mathfrak{a}_2}})|,$$

and thus we have

$$\sum_{([\mathfrak{a}_1],[\mathfrak{a}_2])\in S(p,d_1,d_2)}\log|\pi_N(\tau_1^{\sigma_{\mathfrak{a}_1}})-\pi_N(\tau_2^{\sigma_{\mathfrak{a}_2}})|$$

$$=-\frac{|S(N,d_1,d_2)|w_1w_2}{32h(d_1)h(d_2)}\left(a_1(\phi_{0,0})+\sum_{\mu\in L'/L}c(0,\mu)a_0(\phi_\mu)\right)$$

$$=-\frac{|S(N,d_1,d_2)|w_1w_2}{32h(d_1)h(d_2)}\left(\sum_{\substack{t\in F_+^\times\\ \mathrm{tr}_{F/\mathbb{Q}}t=m}}a(t,\phi_{0,0})+\sum_{\mu\in L'/L}c(0,\mu)a_0(\phi_\mu)\right)$$

$$=-\frac{|S(N,d_1,d_2)|w_1w_2}{32h(d_1)h(d_2)}\left(\sum_{\substack{t=\frac{2m+D+\sqrt{D}}{2}\\ |2m+D|<\sqrt{D}\\ m\in\mathbb{Z}}}a\left(\frac{t}{\sqrt{D}},\phi_{0,0}\right)+\sum_{\mu\in L'/L}c(0,\mu)a_0(\phi_\mu)\right), \tag{3.3.10}$$

where (3.3.10) follows from the rescaling $t\to\frac{t}{\sqrt{D}}$ and the fact that $a\left(\frac{t}{\sqrt{D}},\phi_{0,0}\right)=0$ unless $\frac{t}{\sqrt{D}}\in\partial_F^{-1}=\frac{1}{\sqrt{D}}\mathcal{O}_F$. Moreover, by Lemma 3.2.8, we can easily show that for

$p \in \{3,5,7,13\}$, the constant terms $c(0,\mu_{0,k}) = \frac{24}{p-1}$ for $1 \le k \le p-1$ and $c(0,\mu) = 0$ for $\mu \neq \mu_{0,k}$, and these simplify the right hand side of (3.3.10) and yield (3.1.4) in Theorem 3.1.10. Finally, the quadratic form interpretation of the left hand side of (3.1.4) follows from the isomorphism given in Lemma 3.3.3. $\square$

3.3.3 Computations of $a_1\left(\frac{t}{\sqrt{D}}, \phi_{0,0}\right)$ and $a_0(\phi_{0,k})$

In this section, we compute $a_1\left(\frac{t}{\sqrt{D}}, \phi_{0,0}\right)$ and $a_0(\phi_{0,k})$ explicitly. To compute $a_1\left(\frac{t}{\sqrt{D}}, \phi_{0,0}\right)$, one needs to calculate the local Whittaker functions $W^{\psi_F}_{\frac{t}{\sqrt{D}},\mathfrak{p}}(s,\phi)$. We note that if one lets $\psi'_F(t) = \psi_F(\frac{t}{\sqrt{D}})$ and $W' = W$ with F-quadratic form $Q'_F(z) = z\bar{z}$, then the Weil representations associated to (W, Q_F, ψ_F) and (W', Q'_F, ψ'_F) are the same, and one has by [52, Lemma 4.2.2],

$$W^{\psi_F}_{\frac{t}{\sqrt{D}},\mathfrak{p}}(s,\phi) = |D|_{\mathfrak{p}}^{\frac{1}{2}} W^{\psi'_F}_{t,\mathfrak{p}}(s,\phi). \tag{3.3.11}$$

Thus

$$\frac{W^{*}_{\frac{t}{\sqrt{D}},\mathfrak{p}}(s,\phi)}{\gamma(W_{\mathfrak{p}})} = |D|_{\mathfrak{p}}^{-\frac{s}{2}} L_{\mathfrak{p}}(s+1,\chi) \frac{W^{\psi'_F}_{t,\mathfrak{p}}(s,\phi)}{\gamma(W'_{\mathfrak{p}})}, \tag{3.3.12}$$

and we will compute $a_1\left(\frac{t}{\sqrt{D}}, \phi_{0,0}\right)$ via $W^{\psi'_F}_{t,\mathfrak{p}}(s,\phi)$ whose calculations are tidier due to the normalization (see [95, Sec. 6]). In addition, for $a_0(\phi_{0,k})$, by the definition (3.3.3), one has

$$a_0(\phi_{0,k}) = -\tilde{W}'_{0,f}(0,\phi_{0,k})$$

with

$$\tilde{W}_{0,f}(s,\phi_\mu) = \prod_{\mathfrak{p}\nmid\infty} \frac{|D|_{\mathfrak{p}}^{-\frac{1}{2}} L_{\mathfrak{p}}(s+1,\chi)}{L_{\mathfrak{p}}(s,\chi)} \frac{W^{\psi_F}_{0,\mathfrak{p}}(s,\phi_\mu)}{\gamma(W_{\mathfrak{p}})}$$

provided ϕ_μ is factorizable.

We will compute $a_1\left(\frac{t}{\sqrt{D}}, \phi_{0,0}\right)$ and $a_0(\phi_{0,k})$ case by case according to the ramification of p in F and E. Throughout the remainder of this section, for $0 \le k \le p-1$, we write

$\phi_{0,k}$ for $\mathrm{Char}(\mu_{0,k} + L_p \otimes \hat{\mathbb{Z}})$. Clearly, by (3.3.7), one has

$$\phi_{0,k} = \phi_{0,k,p} \prod_{\mathfrak{p} \nmid p} \phi_{0,0,\mathfrak{p}}$$

where $\phi_{0,k,p} = \mathrm{Char}(\mu_{0,k} + \mathcal{L}_p)$, $\mathcal{L}_p = L_p \otimes \mathbb{Z}_p$ and $\phi_{0,0,\mathfrak{p}} = \mathrm{Char}(\mathcal{O}_{E_\mathfrak{p}})$ for $\mathfrak{p} \nmid p$. One key step to computing these coefficients via local Whittaker functions is the factorizability of ϕ_μ, so our main strategy is to write $\phi_{0,k,p}$ as a sum of products factorizable over $\mathfrak{p}|p$.

Case 1. When $\left(\frac{d_1}{p}\right) = \left(\frac{d_2}{p}\right) = 1$, then p is completely split in F and in E, that is, $p\mathcal{O}_F = \mathfrak{p}_1\mathfrak{p}_2$ and $p\mathcal{O}_E = \mathfrak{B}_1\bar{\mathfrak{B}}_1\mathfrak{B}_2\bar{\mathfrak{B}}_2$. Similar to [94, Section 5], it is not hard to check that

$$\mathcal{L}_p = \coprod_{i=0}^{p-1} (M_i \times M_i)$$

where $M_i = \left\{ (x_1, x_2) \in \mathbb{Z}_p^2 \middle| \; x_1 + x_2 \equiv i \pmod{p} \right\}$, and thus for $0 \le k \le p-1$,

$$\phi_{0,k} = \sum_{i=0}^{p-1} \phi_{0,k}^{(i)}$$

where

$$\phi_{0,k}^{(i)} = \phi_{\mathfrak{p}_1}^{(i)} \phi_{\mathfrak{p}_2}^{(i+k)} \prod_{\mathfrak{p} \nmid p} \phi_{0,0,\mathfrak{p}}$$

and $\phi^{(i)} = \mathrm{Char}(M_i)$. In this case, we aim to compute

$$a_1(\phi_{0,0}) = \sum_{i=0}^{p-1} a_1(\phi_{0,0}^{(i)})$$

where

$$a_1(\phi_{0,0}^{(i)}) = \sum_{\substack{t = \frac{2m+D+\sqrt{D}}{2} \\ |2m+D| < \sqrt{D} \\ m \in \mathbb{Z}}} a\left(\frac{t}{\sqrt{D}}, \phi_{0,0}^{(i)}\right),$$

and for $1 \le k \le p-1$,

$$a_0(\phi_{0,k}) = \sum_{i=0}^{p-1} a_0(\phi_{0,k}^{(i)})$$

where

$$a_0(\phi_{0,k}^{(i)}) = -\tilde{W}'_{0,f}(0, \phi_{0,k}^{(i)}).$$

Now we briefly explain how to compute $a\left(\frac{t}{\sqrt{D}}, \phi_{0,0}^{(i)}\right)$. First denote by $\mathrm{Diff}(W, t/\sqrt{D})$ the set of prime ideals $\mathfrak{p}$ of F such that $W_{\mathfrak{p}}$ does not represent $t/\sqrt{D}$, i.e., $\mathfrak{p} \in \mathrm{Diff}(W, t/\sqrt{D})$ if and only if $t \neq z\bar{z}$ for any $z \in E_{\mathfrak{p}}^{\times}$ if and only if $\mathfrak{p}$ is inert in E/F and $\mathrm{ord}_{\mathfrak{p}}(t)$ is odd. By [94, Prop. 2.7(1)] and (3.3.2), we can see that $a\left(\frac{t}{\sqrt{D}}, \phi_{0,0}^{(i)}\right) = 0$ when $|\mathrm{Diff}(W, t/\sqrt{D})| > 1$ since $W^*_{\frac{t}{\sqrt{D}},\mathfrak{p}}(0, \phi_{0,0}^{(i)}) = 0$ for $\mathfrak{p} \in \mathrm{Diff}(W, t/\sqrt{D})$. Then when $\mathrm{Diff}(W, t/\sqrt{D}) = \{\mathfrak{p}\}$, by [94, Prop. 2.7(2)] or direct calculations from (3.3.2), one has

$$a\left(\frac{t}{\sqrt{D}}, \phi_{0,0}^{(i)}\right) = -4W^{*,\prime}_{\frac{t}{\sqrt{D}},\mathfrak{p}}(0, \phi_{0,0}^{(i)}) \prod_{\mathfrak{q}\nmid\mathfrak{p}\infty} W^*_{\frac{t}{\sqrt{D}},\mathfrak{q}}(0, \phi_{0,0}^{(i)}). \tag{3.3.13}$$

Clearly, such a prime ideal $\mathfrak{p}$ must be neither $\mathfrak{p}_1$ nor $\mathfrak{p}_2$. In addition, by [94, Prop. 2.7(3), (4)], we have

$$\frac{W^{*,\prime}_{\frac{t}{\sqrt{D}},\mathfrak{p}}(0, \phi_{0,0}^{(i)})}{\gamma(W_{\mathfrak{p}})} = \frac{1 + \mathrm{ord}_{\mathfrak{p}}(t)}{2} \log N(\mathfrak{p})$$

and

$$\frac{W^*_{\frac{t}{\sqrt{D}},\mathfrak{q}}(0, \phi_{0,0}^{(i)})}{\gamma(W_{\mathfrak{q}})} = \rho_{\mathfrak{q}}(t\mathfrak{p}^{-1})$$

for $\mathfrak{q} \neq \mathfrak{p}$, where

$$\rho_{\mathfrak{q}}(\mathfrak{a}) = \begin{cases} 1 & \text{if } \mathfrak{q} \text{ is ramified in } E, \\ \frac{1+(-1)^{ord_{\mathfrak{q}}\mathfrak{a}}}{2} & \text{if } \mathfrak{q} \text{ is inert in } E, \\ 1 + ord_{\mathfrak{q}}\mathfrak{a} & \text{if } \mathfrak{q} \text{ is split in } E. \end{cases} \tag{3.3.14}$$

Then together with the facts that $\prod_{\mathfrak{q}\neq\mathfrak{p}} \rho_{\mathfrak{p}}(t\mathfrak{p}^{-1}) = 1$, we can rewrite (3.3.13) as

$$a\left(\frac{t}{\sqrt{D}}, \phi_{0,0}^{(i)}\right)$$

$$= -4\frac{1+\mathrm{ord}_{\mathfrak{p}}(t)}{2}\log N(\mathfrak{p})\prod_{\mathfrak{q}\nmid\mathfrak{p}p}\frac{W^*_{\frac{t}{\sqrt{D}},\mathfrak{q}}(0,\phi^{(i)}_{0,0})}{\gamma(W_{\mathfrak{q}})}\times\prod_{v}\gamma(W_v)\times\prod_{j=1}^{2}\frac{W^*_{\frac{t}{\sqrt{D}},\mathfrak{p}_j}(0,\phi^{(i)}_{\mathfrak{p}_j})}{\gamma(W_{\mathfrak{p}_j})}$$

$$= -4\frac{1+\mathrm{ord}_{\mathfrak{p}}(t)}{2}\log N(\mathfrak{p})\prod_{\mathfrak{q}\nmid p}\rho_{\mathfrak{q}}(t\mathfrak{p}^{-1})\prod_{j=1}^{2}\frac{W^*_{\frac{t}{\sqrt{D}},\mathfrak{p}_j}(0,\phi^{(i)}_{\mathfrak{p}_j})}{\gamma(W_{\mathfrak{p}_j})}$$

$$= -4\frac{1+\mathrm{ord}_{\mathfrak{p}}(t)}{2}\log(N(\mathfrak{p}))\prod_{\mathfrak{q}\nmid p}\rho_{\mathfrak{q}}(t\mathfrak{p}^{-1})\prod_{j=1}^{2}L(1,\chi_{\mathfrak{p}_j})\frac{W^{\psi'_F}_{t,\mathfrak{p}_j}(0,\phi^{(i)}_{\mathfrak{p}_j})}{\gamma(W'_{\mathfrak{p}_j})}$$

$$= -4\frac{p^2}{(p-1)^2}\frac{1+\mathrm{ord}_{\mathfrak{p}}(t)}{2}\log(N(\mathfrak{p}))\prod_{\mathfrak{q}\nmid p}\rho_{\mathfrak{q}}(t\mathfrak{p}^{-1})\prod_{j=1}^{2}\frac{W^{\psi'_F}_{t,\mathfrak{p}_j}(0,\phi^{(i)}_{\mathfrak{p}_j})}{\gamma(W'_{\mathfrak{p}_j})}$$

where the third equality follows from (3.3.12). Finally, since this happens for exactly one prime ideal $\mathfrak{p}$ such that $\mathrm{Diff}(W,t/\sqrt{D})=\{\mathfrak{p}\}$, we can write it in a unified form as

$$a\left(\frac{t}{\sqrt{D}},\phi^{(i)}_{0,0}\right) = -4\frac{p^2}{(p-1)^2}\sum_{\mathfrak{p}\text{ inert in }E/F}\frac{1+\mathrm{ord}_{\mathfrak{p}}(t)}{2}\log(N(\mathfrak{p}))\prod_{\mathfrak{q}\nmid p}\rho_{\mathfrak{q}}(t\mathfrak{p}^{-1})\prod_{j=1}^{2}\frac{W^{\psi'_F}_{t,\mathfrak{p}_j}(0,\phi^{(i)}_{\mathfrak{p}_j})}{\gamma(W'_{\mathfrak{p}_j})}.$$

The local Whittaker functions $W^{\psi'_F}_{t,\mathfrak{p}_j}(0,\phi^{(i)}_{\mathfrak{p}_j})/\gamma(W'_{\mathfrak{p}_j})$ are computed explicitly in Corollary 3.3.6.

For $a_0(\phi^{(i)}_{0,k})$, by definition, we first know that

$$a_0(\phi^{(i)}_{0,k}) = -\left(\frac{L(1+s,\chi_{\mathfrak{p}_1})W^{\psi_F}_{0,\mathfrak{p}_1}(s,\phi^{(i)}_{\mathfrak{p}_1})}{L(s,\chi_{\mathfrak{p}_1})\gamma(W_{\mathfrak{p}_1})}\frac{L(1+s,\chi_{\mathfrak{p}_2})W^{\psi_F}_{0,\mathfrak{p}_2}(s,\phi^{(i+k)}_{\mathfrak{p}_2})}{L(s,\chi_{\mathfrak{p}_2})\gamma(W_{\mathfrak{p}_2})}\times\prod_{\mathfrak{p}\nmid p}\frac{L(1+s,\chi_{\mathfrak{p}})W^{\psi_F}_{0,\mathfrak{p}}(s,\phi_{0,0,\mathfrak{p}})}{L(s,\chi_{\mathfrak{p}})\gamma(W_{\mathfrak{p}})}\right)'\Bigg|_{s=0}.$$

By [95, Prop. 5.7], one can check that

$$\left(\frac{L(1+s,\chi_{\mathfrak{p}})W^{\psi_F}_{0,\mathfrak{p}}(s,\phi_{0,0,\mathfrak{p}})}{L(s,\chi_{\mathfrak{p}})\gamma(W_{\mathfrak{p}})}\right)'\Bigg|_{s=0} = 0$$

and

$$\left.\frac{L(1+s,\chi_{\mathfrak{p}})W_{0,\mathfrak{p}}^{\psi_F}(s,\phi_{0,0,\mathfrak{p}})}{L(s,\chi_{\mathfrak{p}})\gamma(W_{\mathfrak{p}})}\right|_{s=0} = 1.$$

Thus, we can deduce that

$$a_0(\phi_{0,k}^{(i)}) = -\left.\left(\frac{L(1+s,\chi_{\mathfrak{p}_1})W_{0,\mathfrak{p}_1}^{\psi_F}(s,\phi_{\mathfrak{p}_1}^{(i)})}{L(s,\chi_{\mathfrak{p}_1})\gamma(W_{\mathfrak{p}_1})}\frac{L(1+s,\chi_{\mathfrak{p}_2})W_{0,\mathfrak{p}}^{\psi_F}(s,\phi_{\mathfrak{p}_2}^{(i+k)})}{L(s,\chi_{\mathfrak{p}})\gamma(W_{\mathfrak{p}})}\right)'\right|_{s=0},$$

and by Lemma 3.3.5, we know that for $i \neq 0$

$$\left.\left(\frac{L(1+s,\chi_{\mathfrak{p}_j})W_{0,\mathfrak{p}_j}^{\psi_F}(s,\phi_{\mathfrak{p}_j}^{(i)})}{L(s,\chi_{\mathfrak{p}_j})\gamma(W_{\mathfrak{p}_j})}\right)\right|_{s=0} = 0$$

and

$$\left.\left(\frac{L(1+s,\chi_{\mathfrak{p}_j})W_{0,\mathfrak{p}_j}^{\psi_F}(s,\phi_{\mathfrak{p}_j}^{(i)})}{L(s,\chi_{\mathfrak{p}_j})\gamma(W_{\mathfrak{p}_j})}\right)'\right|_{s=0} = \frac{2\log p}{p-1}.$$

Also, by Lemma 3.3.5, one can check that

$$\left.\left(\frac{L(1+s,\chi_{\mathfrak{p}_j})W_{0,\mathfrak{p}_j}^{\psi_F}(s,\phi_{\mathfrak{p}_j}^{(0)})}{L(s,\chi_{\mathfrak{p}_j})\gamma(W_{\mathfrak{p}_j})}\right)\right|_{s=0} = 1.$$

Therefore, we have

$$a_0(\phi_{0,k}) = -\frac{4\log p}{p-1}.$$

Case 2. When $\left(\frac{d_1}{p}\right) = \left(\frac{d_2}{p}\right) = -1$, then $p\mathcal{O}_F = \mathfrak{p}_1\mathfrak{p}_2$ and $\mathfrak{p}_i$ are inert in E/F. We have

$$E_p \cong E_{\mathfrak{p}_1} \times E_{\mathfrak{p}_2}$$

and the following identification (see, e.g., [77, Prop. 4.31])

$$\begin{aligned}(\lambda_{\mathfrak{p}_1},\lambda_{\mathfrak{p}_2}) : E_p &\to E_{\mathfrak{p}_1} \times E_{\mathfrak{p}_2}\\ \sqrt{d_1} &\to (\sqrt{d_1},\sqrt{d_1}),\\ \sqrt{d_2} &\to (\sqrt{d_2},-\sqrt{d_2}).\end{aligned}$$

Since $p\mathcal{O}_E \subset L_p$, then it is easy to see that

$$\mathcal{L}_p = \coprod_{h,i,j\in\mathbb{Z}/p\mathbb{Z}} \left\{(\lambda_{\mathfrak{p}_1}(\gamma_{hij}), \lambda_{\mathfrak{p}_2}(\gamma_{hij})) + p\mathcal{O}_{E_{\mathfrak{p}_1}} \times p\mathcal{O}_{E_{\mathfrak{p}_2}}\right\}$$

where

$$\gamma_{hij} = h + i\frac{-d_1 + \sqrt{d_1}}{2} + j\frac{d_2 + \sqrt{d_2}}{2}.$$

Thus, one has

$$\phi_{0,k,p} = \sum_{h,i,j\in\mathbb{Z}/p\mathbb{Z}} \phi^{(k)}_{hij,\mathfrak{p}_1}\phi^{(k)}_{hij,\mathfrak{p}_2}$$

where for $l = 1, 2$,

$$\phi^{(k)}_{hij,\mathfrak{p}_l} = \mathrm{Char}(\lambda_{\mathfrak{p}_l}(\gamma^{(k)}_{hij}) + p\mathcal{O}_{E_{\mathfrak{p}_l}})$$

and

$$\gamma^{(k)}_{hij} = h + i\frac{-d_1 + \sqrt{d_1}}{2} + j\frac{d_2 + \sqrt{d_2}}{2} + k\frac{(-d_1 + \sqrt{d_1})(d_2 + \sqrt{d_2})}{4}.$$

Then we can write

$$\phi_{0,k} = \sum_{h,i,j\in\mathbb{Z}/p\mathbb{Z}} \Phi^{(k)}_{hij}$$

where

$$\Phi^{(k)}_{hij} = \phi^{(k)}_{hij,\mathfrak{p}_1}\phi^{(k)}_{hij,\mathfrak{p}_2}\prod_{\mathfrak{p}\nmid p}\phi_{0,0,\mathfrak{p}}.$$

and we need to compute

$$a_1(\phi_{0,0}) = \sum_{h,i,j\in\mathbb{Z}/p\mathbb{Z}} a_1(\Phi^{(0)}_{hij})$$

where

$$a_1(\Phi^{(0)}_{hij}) = \sum_{\substack{t=\frac{2m+D+\sqrt{D}}{2} \\ |2m+D|<\sqrt{D} \\ m\in\mathbb{Z}}} a\left(\frac{t}{\sqrt{D}}, \Phi^{(0)}_{hij}\right),$$

and for $1 \leq k \leq p-1$,

$$a_0(\phi_{0,k}) = \sum_{h,i,j \in \mathbb{Z}/p\mathbb{Z}} a_0(\Phi_{hij}^{(k)})$$

where

$$a_0(\Phi_{hij}^{(k)}) = -\tilde{W}'_{0,f}(0, \Phi_{hij}^{(k)}).$$

Similar to Case 1, it is not hard to deduce that

$$a\left(\frac{t}{\sqrt{D}}, \Phi_{hij}^{(0)}\right)$$
$$= -4 \begin{cases} \dfrac{p^2}{(p-1)^2} \dfrac{1+\mathrm{ord}_{\mathfrak{p}}(t)}{2} \log(N(\mathfrak{p})) \prod_{\mathfrak{q} \nmid p} \rho_{\mathfrak{q}}(t\mathfrak{p}^{-1}) \prod_{l=1}^{2} \dfrac{W_{t,\mathfrak{p}_l}^{\psi'_F}(0, \phi_{hij,\mathfrak{p}_l}^{(0)})}{\gamma(W'_{\mathfrak{p}_l})} & \text{if } \mathfrak{p} \nmid p, \\ \\ \dfrac{p}{p-1} \left(\dfrac{p^{s+1}}{p^{s+1}-1} \dfrac{W_{t,\mathfrak{p}_t}^{\psi'_F}\left(s, \phi_{hij,\mathfrak{p}_t}^{(0)}\right)}{\gamma(W'_{\mathfrak{p}_t})} \right)' \Bigg|_{s=0} \prod_{\mathfrak{q} \nmid p} \rho_{\mathfrak{q}}(t) \dfrac{W_{t,\tilde{\mathfrak{p}}_t}^{\psi'_F}(0, \phi_{hij,\tilde{\mathfrak{p}}_t}^{(0)})}{\gamma(W'_{\tilde{\mathfrak{p}}_t})} & \text{if } \mathfrak{p}_t | p \text{ and above } t, \end{cases}$$

and for $1 \leq k \leq p-1$,

$$a_0(\Phi_{hij}^{(k)})$$
$$= -\,p^2 \left(\prod_{l=1}^{2} \frac{(p^s-1)}{p^{s+1}-1} \frac{W_{0,\mathfrak{p}_l}^{\psi_F}(s, \phi_{hij,\mathfrak{p}_l}^{(k)})}{\gamma(W_{\mathfrak{p}_l})} \right)' \Bigg|_{s=0} = -\,p^2 \left(\prod_{l=1}^{2} \frac{(p^s-1)}{p^{s+1}-1} \frac{W_{0,\mathfrak{p}_l}^{\psi'_F}(s, \phi_{hij,\mathfrak{p}_l}^{(k)})}{\gamma(W'_{\mathfrak{p}_l})} \right)' \Bigg|_{s=0},$$

where, by [95, Cor. 5.3],

$$\frac{W_{t,\mathfrak{p}_l}^{\psi'_F}\left(s, \phi_{hij,\mathfrak{p}_l}^{(k)}\right)}{\gamma(W'_{\mathfrak{p}_l})}$$

$$= \begin{cases} 0 & \text{if } o_{\mathfrak{p}_l}(p^2\eta^{(k)}_{hij,\mathfrak{p}_l}\bar{\eta}^{(k)}_{hij,\mathfrak{p}_l} - t) < 0, \\ (1-\frac{1}{p^s}) \displaystyle\sum_{n=0}^{o_{\mathfrak{p}_l}(p^2\eta^{(k)}_{hij,\mathfrak{p}_l}\bar{\eta}^{(k)}_{hij,\mathfrak{p}_l}-t)} p^{n(1-s)} & \text{if } 0 \le o_{\mathfrak{p}_l}(p^2\eta^{(k)}_{hij,\mathfrak{p}_l}\bar{\eta}^{(k)}_{hij,\mathfrak{p}_l} - t) < o_{E_{\mathfrak{p}_l}}(\eta^{(k)}_{hij,\mathfrak{p}_l})+2, \\ (1-\frac{1}{p^s}) \sum_{n=0}^{o_{\mathfrak{p}_l}(p^2\eta^{(k)}_{hij,\mathfrak{p}_l}\bar{\eta}^{(k)}_{hij,\mathfrak{p}_l}-t)} p^{n(1-s)-2} \\ \quad + p^{(o_{E_{\mathfrak{p}_l}}(\eta^{(k)}_{hij,\mathfrak{p}_l})+2)(1-s)-2} & \text{if } o_{\mathfrak{p}_l}(p^2\eta^{(k)}_{hij,\mathfrak{p}_l}\bar{\eta}^{(k)}_{hij,\mathfrak{p}_l} - t) \ge o_{E_{\mathfrak{p}_l}}(\eta^{(k)}_{hij,\mathfrak{p}_l}) + 2, \end{cases}$$

and $\eta^{(k)}_{hij,\mathfrak{p}_l} = \frac{1}{p}\lambda_{\mathfrak{p}_l}(\gamma^{(k)}_{hij})$, $o_{\mathfrak{p}_l}(x) = \mathrm{ord}_{\pi_{F_{\mathfrak{p}_l}}}(x)$, $\pi_{F_{\mathfrak{p}_l}}$ is the uniformizer of $F_{\mathfrak{p}_l}$, $o_{E_{\mathfrak{p}_l}}(x) = \mathrm{ord}_{\pi_{E_{\mathfrak{p}_l}}}(x)$, $\pi_{E_{\mathfrak{p}_l}}$ is the uniformizer of $E_{\mathfrak{p}_l}$. One can easily see that in this case, $a_0(\phi_{0,k}) = 0$.

Case 3. When $\left(\frac{d_1}{p}\right) = 1$ and $\left(\frac{d_2}{p}\right) = -1$, then p is inert in F and is split in E, that is, $p\mathcal{O}_F = p$ and $p\mathcal{O}_E = \mathcal{B}\bar{\mathcal{B}}$. Then

$$E_p \cong F_p \times F_p$$

and we have the following identification

$$\begin{aligned} \lambda : E_p &\to F_p \times F_p \\ \sqrt{d_1} &\to (\sqrt{d_1}, -\sqrt{d_1}), \\ \sqrt{d_2} &\to (\sqrt{d_2}, -\sqrt{d_2}). \end{aligned}$$

Clearly, the characteristic function $\phi_k = \mathrm{Char}(\mu_{0,k} + L_p \otimes \hat{\mathbb{Z}})$ is factorizable over the spectrum of F, namely,

$$\phi_k = \phi_{0,k,p} \prod_{\mathfrak{p} \nmid p} \phi_{0,0,\mathfrak{p}}$$

where $\phi_{0,k,p} = \mathrm{Char}(\mu_{0,k} + \mathcal{L}_p)$ and $\phi_{0,0,\mathfrak{p}} = \mathrm{Char}(\mathcal{O}_{E,\mathfrak{p}})$. Similar to Case 2, we have

$$\phi_{0,k,p} = \sum_{h,i,j \in \mathbb{Z}/p\mathbb{Z}} \phi_{hij}^{(k)}$$

where

$$\phi_{hij}^{(k)} = \mathrm{Char}(\lambda(\gamma_{hij}^{(k)}) + p\mathcal{O}_{E_p})$$

and

$$\gamma_{hij}^{(k)} = h + i\frac{-d_1 + \sqrt{d_1}}{2} + j\frac{d_2 + \sqrt{d_2}}{2} + k\frac{(-d_1 + \sqrt{d_1})(d_2 + \sqrt{d_2})}{4}.$$

And similarly, we write

$$\phi_{0,k} = \sum_{h,i,j \in \mathbb{Z}/p\mathbb{Z}} \Phi_{hij}^{(k)}$$

where

$$\Phi_{hij}^{(k)} = \phi_{hij}^{(k)} \prod_{\mathfrak{p} \nmid p} \phi_{0,0,\mathfrak{p}}.$$

and we aim to compute

$$a_1(\phi_{0,0}) = \sum_{h,i,j \in \mathbb{Z}/p\mathbb{Z}} a_1(\Phi_{hij}^{(0)})$$

where

$$a_1(\Phi_{hij}^{(0)}) = \sum_{\substack{t = \frac{2m+D+\sqrt{D}}{2} \\ |2m+D| < \sqrt{D} \\ m \in \mathbb{Z}}} a\left(\frac{t}{\sqrt{D}}, \Phi_{hij}^{(0)}\right),$$

and for $1 \leq k \leq p-1$,

$$a_0(\phi_{0,k}) = \sum_{h,i,j \in \mathbb{Z}/p\mathbb{Z}} a_0(\Phi_{hij}^{(k)})$$

where

$$a_0(\Phi_{hij}^{(k)}) = -\tilde{W}'_{0,f}(0, \Phi_{hij}^{(k)}).$$

Similar to Case 1, we can easily deduce that

$$a\left(\frac{t}{\sqrt{D}},\Phi^{(0)}_{hij}\right)=-4\frac{p^2}{p^2-1}\sum_{\mathfrak{p}\text{ inert in }E/F}\frac{1+\mathrm{ord}_{\mathfrak{p}}(t)}{2}\log(N(\mathfrak{p}))\prod_{\mathfrak{q}\nmid p}\rho_{\mathfrak{q}}(t\mathfrak{p}^{-1})\frac{W^{\psi'_F}_{t,p}\left(0,\phi^{(0)}_{hij}\right)}{\gamma(W'_p)}.$$

Also, for $a_0(\Phi^{(k)}_{hij})$, we aim to compute

$$a_0(\Phi^{(k)}_{hij})=-\left.\left(\frac{L(1+s,\chi_p)}{L(s,\chi_p)}\frac{W^{\psi_F}_{0,p}\left(s,\phi^{(k)}_{hij}\right)}{\gamma(W_p)}\prod_{\mathfrak{p}\nmid p}\frac{L(1+s,\chi_{\mathfrak{p}})W^{\psi_F}_{0,\mathfrak{p}}(s,\phi_{0,0,\mathfrak{p}})}{L(s,\chi_{\mathfrak{p}})\gamma(W_{\mathfrak{p}})}\right)'\right|_{s=0}.$$

Again, similar to Case 1, we can easily deduce that

$$a_0(\Phi^{(k)}_{hij})=-\frac{2p^2\log p}{p^2-1}\frac{W^{\psi_F}_{0,p}\left(0,\phi^{(k)}_{hij}\right)}{\gamma(W_p)}=-\frac{2p^2\log p}{p^2-1}\frac{W^{\psi'_F}_{0,p}\left(0,\phi^{(k)}_{hij}\right)}{\gamma(W'_p)}$$

where the second equality follows from (3.3.11). Finally, by [95, Cor. 5.3], we have

$$\frac{W^{\psi'_F}_{t,p}\left(0,\phi^{(k)}_{hij}\right)}{\gamma(W'_p)}=\begin{cases}0 & \text{if } o(p^2\eta^{(k)}_{hij}\bar{\eta}^{(k)}_{hij}-t)<o_{E_p}(\eta^{(k)}_{hij})+2,\\ \\ p^{2o_{E_p}(\eta^{(k)}_{hij})+2} & \text{if } o(p^2\eta^{(k)}_{hij}\bar{\eta}^{(k)}_{hij}-t)\geq o_{E_p}(\eta^{(k)}_{hij})+2,\end{cases}$$

where $\eta^{(k)}_{hij}=\frac{1}{p}\gamma^{(k)}_{hij}$, $o(x)=\mathrm{ord}_{\pi_{F_p}}(x)$, π_{F_p} is the uniformizer of F_p, $o_{E_p}(x)=\min\{o(x_1),o(x_2)\}$ for $x=(x_1,x_2)$ under the given identification $E_p=F_p\times F_p$.

Case 4. When $p\nmid d_1$ and $\left(\frac{d_2}{p}\right)=1$, then $p\mathcal{O}_F=\tilde{\mathfrak{p}}^2$ and $\tilde{\mathfrak{p}}$ is split in E/F. Then we have $F_p\cong F_{\tilde{\mathfrak{p}}}$ and

$$E_p\cong F_p\times F_p$$

with the following identification

$$\begin{aligned}\lambda: E_p&\to F_p\times F_p\\ \sqrt{d_1}&\to(\sqrt{d_1},-\sqrt{d_1}),\end{aligned}$$

$$\sqrt{d_2} \to (\sqrt{d_2}, -\sqrt{d_2}).$$

The treatment if essentially the same as Case 3, and in this case, we have

$$a\left(\frac{t}{\sqrt{D}}, \Phi_{hij}^{(0)}\right) = -4\frac{p}{p-1}\sum_{\mathfrak{p}\text{ inert in }E/F}\frac{1+\mathrm{ord}_{\mathfrak{p}}(t)}{2}\log(N(\mathfrak{p}))\prod_{\mathfrak{q}\nmid p}\rho_{\mathfrak{q}}(t\mathfrak{p}^{-1})\frac{W_{t,\tilde{\mathfrak{p}}}^{\psi'_F}\left(0,\phi_{hij}^{(0)}\right)}{\gamma(W'_{\tilde{\mathfrak{p}}})}$$

and

$$a_0(\Phi_{hij}^{(k)}) = -\frac{p\log p}{p-1}\frac{W_{0,\tilde{\mathfrak{p}}}^{\psi'_F}\left(0,\phi_{hij}^{(k)}\right)}{\gamma(W'_{\tilde{\mathfrak{p}}})},$$

where

$$\frac{W_{t,\tilde{\mathfrak{p}}}^{\psi'_F}\left(0,\tilde{\phi}_{hij}^{(k)}\right)}{\gamma(W'_{\tilde{\mathfrak{p}}})} = \begin{cases} 0 & \text{if } o(p^2\eta_{hij}^{(k)}\bar{\eta}_{hij}^{(k)}-t) < o_{E_p}(\eta_{hij}^{(k)})+4, \\ \\ p^{o_{E_p}(\eta_{hij}^{(k)})+2} & \text{if } o(p^2\eta_{hij}^{(k)}\bar{\eta}_{hij}^{(k)}-t) \geq o_{E_p}(\eta_{hij}^{(k)})+4, \end{cases}$$

$$\eta_{hij}^{(k)} = \frac{1}{p}\left(h + i\frac{-d_1+\sqrt{d_1}}{2} + j\frac{d_2+\sqrt{d_2}}{2} + k\frac{(-d_1+\sqrt{d_1})(d_2+\sqrt{d_2})}{4}\right),$$

and π_{F_p} is the uniformizer of F_p, $o_{E_p}(x) = \min\{o(x_1), o(x_2)\}$ for $x = (x_1, x_2)$ under the given identification $E_p = F_p \times F_p$.

Case 5. When $p \nmid d_1$ and $\left(\frac{d_2}{p}\right) = -1$, then $p\mathcal{O}_F = \tilde{\mathfrak{p}}^2$ and $\tilde{\mathfrak{p}}$ is inert in E/F. Similarly, the treatment of this case is essentially the same as the previous two, but one needs to consider an extra subcase, say $\mathrm{Diff}(W, t/\sqrt{D}) = \{\tilde{\mathfrak{p}}\}$. Thus we have

$$a\left(\frac{t}{\sqrt{D}}, \Phi_{hij}^{(0)}\right)$$

$$= -4 \begin{cases} \frac{p}{p-1}\frac{1+\mathrm{ord}_{\mathfrak{p}}(t)}{2}\log(N(\mathfrak{p})) \\ \times \prod_{\mathfrak{q}\nmid p}\rho_{\mathfrak{q}}(t\mathfrak{p}^{-1})\frac{W^{\psi'_F}_{t,\tilde{\mathfrak{p}}}\left(0,\phi^{(0)}_{hij}\right)}{\gamma(W'_{\tilde{\mathfrak{p}}})} & \text{if } \mathrm{Diff}(W,t/\sqrt{D}) = \{\mathfrak{p}\} \text{ and } \mathfrak{p} \neq \tilde{\mathfrak{p}}, \\ \left(\frac{p^{2s+1}}{p^{s+1}-1}\frac{W^{\psi'_F}_{t,\tilde{\mathfrak{p}}}\left(s,\phi^{(0)}_{hij}\right)}{\gamma(W'_{\tilde{\mathfrak{p}}})}\right)'\Bigg|_{s=0}\prod_{\mathfrak{q}\nmid p}\rho_{\mathfrak{q}}(t) & \text{if } \mathrm{Diff}(W,t/\sqrt{D}) = \{\tilde{\mathfrak{p}}\}, \end{cases}$$

and

$$a_0(\Phi^{(k)}_{hij}) = -\frac{p\log p}{p-1}\frac{W^{\psi'_F}_{0,\tilde{\mathfrak{p}}}\left(0,\phi^{(k)}_{hij}\right)}{\gamma(W'_{\tilde{\mathfrak{p}}})},$$

where

$$\frac{W^{\psi'_F}_{t,\tilde{\mathfrak{p}}}\left(s,\phi^{(k)}_{hij}\right)}{\gamma(W'_{\tilde{\mathfrak{p}}})}$$

$$= \begin{cases} 0 & \text{if } o(p^2\eta^{(k)}_{hij}\bar{\eta}^{(k)}_{hij} - t) < 0, \\ (1-\frac{1}{p^s})\sum_{n=0}^{o(p^2\eta^{(k)}_{hij}\bar{\eta}^{(k)}_{hij}-t)} p^{n(1-s)} & \text{if } 0 \le o(p^2\eta^{(k)}_{hij}\bar{\eta}^{(k)}_{hij} - t) < o_{E_p}(\eta^{(k)}_{hij})+4, \\ (1-\frac{1}{p^s})\sum_{n=0}^{o(p^2\eta^{(k)}_{hij}\bar{\eta}^{(k)}_{hij}-t)} p^{n(1-s)-4} + p^{(o_{E_p}(\eta^{(k)}_{hij})+4)(1-s)-4} & \text{if } o(p^2\eta^{(k)}_{hij}\bar{\eta}^{(k)}_{hij} - t) \ge o_{E_p}(\eta^{(k)}_{hij}) + 4, \end{cases}$$

$$\eta^{(k)}_{hij} = \frac{1}{p}\left(h + i\frac{-d_1+\sqrt{d_1}}{2} + j\frac{d_2+\sqrt{d_2}}{2} + k\frac{(-d_1+\sqrt{d_1})(d_2+\sqrt{d_2})}{4}\right),$$

and π_{F_p} is the uniformizer of F_p, $o_{E_p}(x) = \mathrm{ord}_{\pi_{E_{\tilde{\mathfrak{p}}}}}(x)$, $\pi_{E_{\tilde{\mathfrak{p}}}}$ is the uniformizer of $E_{\tilde{\mathfrak{p}}}$.

Finally, it remains to compute the local Whittaker functions $W^{\psi'_F}_{t,\mathfrak{p}_j}(0,\phi^{(i)}_{\mathfrak{p}_j})/\gamma(W'_{\mathfrak{p}_j})$ mentioned in Case 1. The following lemma essentially gives what we need.

Lemma 3.3.5. *Let $W = \mathbb{Q}_p^2$ with the quadratic form $Q(x) = x_1x_2$ and $\phi^{(i)}$ be defined as above. Let $N(k,m) = |\{i \in (\mathbb{Z}/p\mathbb{Z})^\times \,|\, i(k-i) \equiv m \pmod{p}\}|$. Then the local Whittaker function $W_t(s,\phi^{(i)})$ for $t \in \mathbb{Z}_p$ is given by*

$$\frac{W_t(s,\phi^{(0)})}{\gamma(W)} = \begin{cases} \frac{1}{p} - \frac{1}{p^{1+s}} & \text{if } t \in k + p\mathbb{Z}_p \text{ and } \left(\frac{k}{p}\right) = 1, \\ \frac{1}{p} + \frac{1}{p^{1+s}} & \text{if } t \in k + p\mathbb{Z}_p \text{ and } \left(\frac{k}{p}\right) = -1, \\ \frac{1}{p} + \frac{p-1}{p}\sum_{n=2}^{\mathrm{ord}_p(t)} \frac{1}{p^{ns}} - \frac{1}{p^{1+(1+\mathrm{ord}_p(t))s}} & \text{if } t \in p\mathbb{Z}_p, \end{cases}$$

and for $1 \le i \le p-1$,

$$\frac{W_t(s,\phi^{(i)})}{\gamma(W)} = \begin{cases} \frac{1}{p} + \frac{1}{p^{1+s}} & \text{if } t \in p\mathbb{Z}_p, \\ \frac{1}{p} + \frac{1}{p^{1+s}}(N(i,m)-1) & \text{if } t \in m + p\mathbb{Z}_p \subset \mathbb{Z}_p^\times. \end{cases}$$

Especially, when $s = 0$, we have

$$\frac{W_t(0,\phi^{(0)})}{\gamma(W)} = \begin{cases} 0 & \text{if } t \in j + p\mathbb{Z}_p \text{ and } \left(\frac{-j}{p}\right) = -1, \\ \frac{2}{p} & \text{if } t \in j + p\mathbb{Z}_p \text{ and } \left(\frac{-j}{p}\right) = 1, \\ \frac{p-1}{p}(\mathrm{ord}_p(t) - 1) & \text{if } t \in p\mathbb{Z}_p, \end{cases}$$

and for $1 \le i \le p-1$,

$$\frac{W_t(0,\phi^{(i)})}{\gamma(W)} = \begin{cases} \frac{N(i,m)}{p} & \text{if } t \in m + p\mathbb{Z}_p \subset \mathbb{Z}_p^\times, \\ \frac{2}{p} & \text{if } t \in p\mathbb{Z}_p. \end{cases}$$

Proof. By the definition of $W_t(0,\phi)/\gamma(W)$ and unfolding, we have

$$\begin{aligned} \frac{W_t(0,\phi^{(i)})}{\gamma(W)} &= \int_{\mathbb{Q}_p} J_i(b)\psi(-tb)|a(wn(b))|^2 db \\ &= \int_{\mathbb{Z}_p} J_i(b)\psi(-tb)db + \sum_{n=1}^{\infty} p^n \int_{\mathbb{Z}_p^\times} J_i(p^{-n}b)\psi(-p^{-n}tb)|a(wn(p^{-n}b))|^s db \end{aligned}$$

where

$$J_i(b) = \int_{M_i} \psi(bx_1x_2)dx_1dx_2.$$

We first compute the integral $J_i(b)$. For $i = 0$, we can deduce that

$$\begin{aligned}
J_0(b) &= \int_{x_1+x_2\equiv 0 \pmod p} \psi(bx_1x_2)dx_1dx_2 \\
&= \frac{1}{p}\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p} \psi(b(py - x_1)x_1)dydx_1 \\
&= \frac{1}{p}\int_{\mathbb{Z}_p} \psi(-bx_1^2)\left(\int_{\mathbb{Z}_p} \psi(bpyx_1)dy\right)dx_1 \\
&= \frac{1}{p}\int_{\mathbb{Z}_p} \psi(-bx_1^2)\mathrm{Char}(\mathbb{Z}_p)(bpx_1)dx_1.
\end{aligned}$$

Now we compute the last integral case by case.

1. For $b \in \mathbb{Z}_p$,

$$\begin{aligned}
J_0(b) &= \frac{1}{p}\int_{\mathbb{Z}_p} \psi(-bx_1^2)\mathrm{Char}(\mathbb{Z}_p)(bpx_1)dx_1 \\
&= \frac{1}{p}\int_{\mathbb{Z}_p} \psi(-bx_1^2)dx_1 \\
&= \frac{1}{p},
\end{aligned}$$

2. for $b \in \frac{k}{p} + \mathbb{Z}_p$ with $1 \le k \le p-1$,

$$\begin{aligned}
J_0(b) &= \frac{1}{p}\int_{\mathbb{Z}_p} \psi(-bx_1^2)\mathrm{Char}(\mathbb{Z}_p)(bpx_1)dx_1 \\
&= \frac{1}{p}\int_{\mathbb{Z}_p} \psi(-bx_1^2)dx_1 \\
&= \frac{1}{p}\left(\int_{p\mathbb{Z}_p} 1dx_1 + \int_{\mathbb{Z}_p^\times} \psi(-bx_1^2)dx_1\right)
\end{aligned}$$

$$= \frac{1}{p}\left(\frac{1}{p} + \sum_{j=1}^{p-1}\int_{j+p\mathbb{Z}_p}\psi(-bx_1^2)dx_1\right)$$

$$= \frac{1}{p}\left(\frac{1}{p} + \frac{1}{p}\sum_{j=1}^{p-1} e^{-\frac{2\pi i j^2 k}{p}}\right),$$

3. for $b \notin \frac{1}{p}\mathbb{Z}_p$,

$$\begin{aligned} J_0(b) &= \frac{1}{p}\int_{\mathbb{Z}_p}\psi(-bx_1^2)\mathrm{Char}(\mathbb{Z}_p)(bpx_1)dx_1 \\ &= \frac{1}{p}\int_{\frac{1}{bp}\mathbb{Z}_p}\psi(-bx_1^2)dx_1 \\ &= \frac{1}{p}\int_{\frac{1}{bp}\mathbb{Z}_p} 1dx_1 \\ &= |b|_p^{-1}. \end{aligned}$$

In summary, one has

$$J_0(b) = \begin{cases} \frac{1}{p} & \text{if } b \in \mathbb{Z}_p, \\ \frac{1}{p^2}\left(1 + \sum_{j=1}^{p-1} e^{-\frac{2\pi i j^2 k}{p}}\right) & \text{if } b \in \frac{k}{p} + p\mathbb{Z}_p \subset \frac{1}{p}\mathbb{Z}_p^\times, \\ |b|_p^{-1} & \text{if } b \notin \frac{1}{p}\mathbb{Z}_p. \end{cases}$$

Now we are ready to compute

$$\begin{aligned} &\frac{W_t(s,\phi^{(0)})}{\gamma(W)} \\ &= \int_{\mathbb{Z}_p} J_0(b)\psi(-tb)db + \sum_{n=1}^{\infty} p^{n-ns}\int_{\mathbb{Z}_p^\times} J_0(p^{-n}b)\psi(-p^{-n}tb)db \\ &= \frac{1}{p} + p^{1-s}\sum_{j=1}^{p-1}\int_{j+p\mathbb{Z}_p} J_0(p^{-1}b)\psi(-p^{-1}tb)db + \sum_{n=2}^{\infty} p^{n-ns}\sum_{j=1}^{p-1}\int_{j+p\mathbb{Z}_p} J_0(p^{-n}b)\psi(-p^{-n}tb)db, \end{aligned}$$

where

1. for $t \in -k + p\mathbb{Z}_p \subset \mathbb{Z}_p^\times$,

$$\frac{1}{p} + p^{1-s} \sum_{j=1}^{p-1} \int_{j+p\mathbb{Z}_p} J_0(p^{-1}b)\psi(-p^{-1}tb)db + \sum_{n=2}^{\infty} p^{n-ns} \sum_{j=1}^{p-1} \int_{j+p\mathbb{Z}_p} J_0(p^{-n}b)\psi(-p^{-n}tb)db$$
$$= \frac{1}{p} + \frac{1}{p^{2+s}} \sum_{j=1}^{p-1} \left(1 + \sum_{m=1}^{p-1} e^{-\frac{2\pi i m^2 j}{p}}\right) e^{\frac{2\pi i j k}{p}}$$
$$= \frac{1}{p} + \frac{1}{p^{2+s}} \left(-1 + \sum_{\substack{1 \le m \le p-1 \\ m^2 \equiv k \pmod p}} (p-1) - \sum_{\substack{1 \le m \le p-1 \\ m^2 \not\equiv k \pmod p}} 1\right)$$
$$= \begin{cases} \frac{1}{p} - \frac{1}{p^{1+s}} & \text{if } \left(\frac{k}{p}\right) = -1, \\ \frac{1}{p} + \frac{1}{p^{1+s}} & \text{if } \left(\frac{k}{p}\right) = 1, \end{cases}$$

2. for $t \in p\mathbb{Z}_p$,

$$\frac{1}{p} + p^{1-s} \sum_{j=1}^{p-1} \int_{j+p\mathbb{Z}_p} J_0(p^{-1}b)\psi(-p^{-1}tb)db + \sum_{n=2}^{\infty} p^{n-ns} \sum_{j=1}^{p-1} \int_{j+p\mathbb{Z}_p} J_0(p^{-n}b)\psi(-p^{-n}tb)db$$
$$= \frac{1}{p} + \sum_{n=2}^{\mathrm{ord}_p(t)+1} \frac{1}{p^{ns}} \int_{\mathbb{Z}_p^\times} \psi(p^{-n+\mathrm{ord}_p(t)}b)db$$
$$= \frac{1}{p} + \frac{p-1}{p} \sum_{n=2}^{\mathrm{ord}_p(t)} \frac{1}{p^{ns}} - \frac{1}{p^{1+(1+\mathrm{ord}_p(t))s}}.$$

Therefore, one has

$$\frac{W_t(s,\phi^{(0)})}{\gamma(W)} = \begin{cases} \frac{1}{p} - \frac{1}{p^{1+s}} & \text{if } t \in k + p\mathbb{Z}_p \text{ and } \left(\frac{k}{p}\right) = 1, \\ \frac{1}{p} + \frac{1}{p^{1+s}} & \text{if } t \in k + p\mathbb{Z}_p \text{ and } \left(\frac{k}{p}\right) = -1, \\ \frac{1}{p} + \frac{p-1}{p} \sum_{n=2}^{\mathrm{ord}_p(t)} \frac{1}{p^{ns}} - \frac{1}{p^{1+(1+\mathrm{ord}_p(t))s}} & \text{if } t \in p\mathbb{Z}_p. \end{cases}$$

Now for $1 \le i \le p-1$, similarly, we have

$$J_i(b) = \int_{x_1+x_2 \equiv i \pmod p} \psi(bx_1x_2)dx_1dx_2$$

$$
\begin{aligned}
&= \frac{1}{p}\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\psi(bx_1(i+py-x_1)dydx_1 \\
&= \frac{1}{p}\int_{\mathbb{Z}_p}\psi(bx_1(i-x_1))\mathrm{Char}(\mathbb{Z}_p)(bpx_1)dx_1.
\end{aligned}
$$

Then

1. for $b \in \mathbb{Z}_p$,

$$
\begin{aligned}
J_i(b) &= \frac{1}{p}\int_{\mathbb{Z}_p}\psi(bx_1(i-x_1))\mathrm{Char}(\mathbb{Z}_p)(bpx_1)dx_1 \\
&= \frac{1}{p}\int_{\mathbb{Z}_p} 1dx_1 \\
&= \frac{1}{p},
\end{aligned}
$$

2. for $b \in \frac{k}{p} + \mathbb{Z}_p \subset \frac{1}{p}\mathbb{Z}_p^{\times}$,

$$
\begin{aligned}
J_i(b) &= \frac{1}{p}\int_{\mathbb{Z}_p}\psi(bx_1(i-x_1))\mathrm{Char}(\mathbb{Z}_p)(bpx_1)dx_1 \\
&= \frac{1}{p}\int_{\mathbb{Z}_p}\psi(bx_1(i-x_1))dx_1 \\
&= \frac{1}{p}\left(\int_{p\mathbb{Z}_p} 1dx_1 + \sum_{j=1}^{p-1}\int_{j+p\mathbb{Z}_p}\psi(bx_1(i-x_1))dx_1\right) \\
&= \frac{1}{p}\left(\frac{1}{p} + \frac{1}{p}\sum_{j=1}^{p-1} e^{\frac{2\pi i}{p}(kj(i-j))}\right),
\end{aligned}
$$

3. for $b \in \frac{1}{p}\mathbb{Z}_p$,

$$
\begin{aligned}
J_i(b) &= \frac{1}{p}\int_{\mathbb{Z}_p}\psi(bx_1(i-x_1))\mathrm{Char}(\mathbb{Z}_p)(bpx_1)dx_1 \\
&= \frac{1}{p}\int_{\frac{1}{bp}\mathbb{Z}_p}\psi(bx_1(i-x_1))dx_1
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{p}\left(\int_{\frac{1}{b}\mathbb{Z}_p} \psi(bx_1(i-x_1))dx_1 + \int_{\frac{1}{bp}\mathbb{Z}_p^\times} \psi(bx_1(i-x_1))dx_1\right) \\
&= \frac{1}{p}\left(|b|_p^{-1} + \sum_{j=1}^{p-1} |b|_p^{-1} e^{\frac{2\pi i}{p}(ji)}\right) \\
&= 0.
\end{aligned}
$$

These can be summarized as follows.

$$
J_i(b) = \begin{cases} \frac{1}{p} & \text{if } b \in \mathbb{Z}_p, \\ \frac{1}{p^2}\left(1 + \sum_{j=1}^{p-1} e^{\frac{2\pi i}{p}(kj(i-j))}\right) & \text{if } b \in \frac{k}{p} + \mathbb{Z}_p \subset \frac{1}{p}\mathbb{Z}_p^\times, \\ 0 & \text{if } b \notin \frac{1}{p}\mathbb{Z}_p. \end{cases}
$$

Then for $1 \le i \le p-1$, $W_t(s, \phi^{(i)})/\gamma(W)$ can be computed in a similar way as $i = 0$.

$$
\begin{aligned}
&\frac{W_t(s,\phi^{(i)})}{\gamma(W)} \\
&= \int_{\mathbb{Z}_p} J_i(b)\psi(-tb)db + \sum_{n=1}^{\infty} p^{n-ns} \int_{\mathbb{Z}_p^\times} J_i(p^{-n}b)\psi(-p^{-n}tb)db \\
&= \frac{1}{p} + \frac{1}{p^{1+s}} \sum_{k=1}^{p-1}\left(1 + \sum_{j=1}^{p-1} e^{\frac{2\pi i}{p}(kj(i-j))}\right)\int_{-j+p\mathbb{Z}_p} \psi(p^{-1}tb)db,
\end{aligned}
$$

where

1. for $t \in p\mathbb{Z}_p$,

$$
\begin{aligned}
&\frac{1}{p} + \frac{1}{p^{1+s}} \sum_{k=1}^{p-1}\left(1 + \sum_{j=1}^{p-1} e^{\frac{2\pi i}{p}(kj(i-j))}\right)\int_{-j+p\mathbb{Z}_p} \psi(p^{-1}tb)db \\
&= \frac{1}{p} + \frac{1}{p^{2+s}} \sum_{k=1}^{p-1}\left(1 + \sum_{j=1}^{p-1} e^{\frac{2\pi i}{p}(kj(i-j))}\right) \\
&= \frac{1}{p} + \frac{1}{p^{2+s}}\left(p - 1 - \sum_{\substack{1\le j\le p-1 \\ j\ne i}} 1 + p - 1\right)
\end{aligned}
$$

$$= \frac{1}{p} + \frac{1}{p^{1+s}},$$

2. for $t \in m + p\mathbb{Z}_p \subset \mathbb{Z}_p^\times$,

$$\begin{aligned}
&\frac{1}{p} + \frac{1}{p^{1+s}} \sum_{k=1}^{p-1} \left(1 + \sum_{j=1}^{p-1} e^{\frac{2\pi i}{p}(kj(i-j))}\right) \int_{-j+p\mathbb{Z}_p} \psi(p^{-1}tb)db \\
&= \frac{1}{p} + \frac{1}{p^{2+s}} \sum_{k=1}^{p-1} \left(1 + \sum_{j=1}^{p-1} e^{\frac{2\pi i}{p}(kj(i-j))}\right) e^{-\frac{2\pi imk}{p}} \\
&= \frac{1}{p} + \frac{1}{p^{2+s}} \sum_{k=1}^{p-1} \left(e^{-\frac{2\pi imk}{p}} + \sum_{j=1}^{p-1} e^{\frac{2\pi i(j(i-j)-m)k}{p}}\right) \\
&= \frac{1}{p} + \frac{1}{p^{2+s}} \left(-1 + \sum_{\substack{1\le j\le p-1 \\ j(i-j)\equiv m \pmod p}} (p-1) - \sum_{\substack{1\le j\le p-1 \\ j(i-j)\not\equiv m \pmod p}} 1\right) \\
&= \frac{1}{p} + \frac{1}{p^{2+s}} \left(-1 + (p-1)N(i,m) - (p-1) + N(i,m)\right) \\
&= \frac{1}{p} + \frac{1}{p^{1+s}}(N(i,m) - 1).
\end{aligned}$$

Therefore, we have

$$\frac{W_t(s,\phi^{(i)})}{\gamma(W)} = \begin{cases} \frac{1}{p} + \frac{1}{p^{1+s}} & \text{if } t \in p\mathbb{Z}_p, \\ \frac{1}{p} + \frac{1}{p^{1+s}}(N(i,m)-1) & \text{if } t \in m + p\mathbb{Z}_p \subset \mathbb{Z}_p^\times. \end{cases}$$

Hence, when $s = 0$,

$$\frac{W_t(0,\phi^{(0)})}{\gamma(W)} = \begin{cases} 0 & \text{if } t \in k + p\mathbb{Z}_p \text{ and } \left(\frac{k}{p}\right) = 1, \\ \frac{2}{p} & \text{if } t \in k + p\mathbb{Z}_p \text{ and } \left(\frac{k}{p}\right) = -1, \\ \frac{p-1}{p}(\mathrm{ord}_p(t) - 1) & \text{if } t \in p\mathbb{Z}_p \end{cases}$$

and for $1 \le i \le p-1$,

$$\frac{W_t(0,\phi^{(i)})}{\gamma(W)} = \begin{cases} \frac{2}{p} & \text{if } t \in p\mathbb{Z}_p, \\ \frac{N(i,m)}{p} & \text{if } t \in m + p\mathbb{Z}_p \subset \mathbb{Z}_p^\times. \end{cases}$$

$\square$

Following from Lemma 3.3.5, the local Whittaker functions $W^{\psi'_F}_{t,\mathfrak{p}_j}(0,\phi^{(i)}_{\mathfrak{p}_j})/\gamma(W'_{\mathfrak{p}_j})$ can be computed explicitly as follows.

Corollary 3.3.6. *Fix $d \in (\mathbb{Z}/p\mathbb{Z})^\times$ such that $\sqrt{D} \equiv d \pmod{p\mathbb{Z}_p}$. Write $t = \frac{2m+D+\sqrt{D}}{2}$ and $\tilde{d} = \frac{d^2+d}{2}$. Then for $1 \le i \le p-1$,*

1. *when $m \equiv -\tilde{d} \pmod{p}$,*

$$\frac{W^{\psi'_F}_{t,\mathfrak{p}_1}(0,\phi^{(i)}_{\mathfrak{p}_1})}{\gamma(W'_{\mathfrak{p}_1})} = \frac{2}{p} \quad \textit{and} \quad \frac{W^{\psi'_F}_{t,\mathfrak{p}_2}(0,\phi^{(i)}_{\mathfrak{p}_2})}{\gamma(W'_{\mathfrak{p}_2})} = \frac{N(i,p-d)}{p};$$

2. *when $m \equiv -\tilde{d} + d \pmod{p}$,*

$$\frac{W^{\psi'_F}_{t,\mathfrak{p}_1}(0,\phi^{(i)}_{\mathfrak{p}_1})}{\gamma(W'_{\mathfrak{p}_1})} = \frac{N(i,d)}{p} \quad \textit{and} \quad \frac{W^{\psi'_F}_{t,\mathfrak{p}_2}(0,\phi^{(i)}_{\mathfrak{p}_2})}{\gamma(W'_{\mathfrak{p}_2})} = \frac{2}{p};$$

3. *when $m \equiv -\tilde{d} + k \pmod{p}$ with $k \not\equiv 0, d \pmod{p}$,*

$$\frac{W^{\psi'_F}_{t,\mathfrak{p}_1}(0,\phi^{(i)}_{\mathfrak{p}_1})}{\gamma(W'_{\mathfrak{p}_1})} = \frac{N(i,k)}{p} \quad \textit{and} \quad \frac{W^{\psi'_F}_{t,\mathfrak{p}_2}(0,\phi^{(i)}_{\mathfrak{p}_2})}{\gamma(W'_{\mathfrak{p}_2})} = \frac{N(i,k-d)}{p},$$

and for $i = 0$,

1. *when* $m \equiv -\tilde{d} \pmod{p}$ *and* $\left(\frac{d}{p}\right) = -1$,

$$\frac{W^{\psi'_F}_{t,\mathfrak{p}_1}(0, \phi^{(0)}_{\mathfrak{p}_1})}{\gamma(W'_{\mathfrak{p}_1})} = \frac{p-1}{p}(\mathrm{ord}_{\mathfrak{p}_1}(\sigma_1(t)) - 1) \quad \text{and} \quad \frac{W^{\psi'_F}_{t,\mathfrak{p}_2}(0, \phi^{(0)}_{\mathfrak{p}_2})}{\gamma(W'_{\mathfrak{p}_2})} = 0;$$

2. *when* $m \equiv -\tilde{d} \pmod{p}$ *and* $\left(\frac{d}{p}\right) = 1$,

$$\frac{W^{\psi'_F}_{t,\mathfrak{p}_1}(0, \phi^{(0)}_{\mathfrak{p}_1})}{\gamma(W'_{\mathfrak{p}_1})} = \frac{p-1}{p}(\mathrm{ord}_{\mathfrak{p}_1}(\sigma_1(t)) - 1) \quad \text{and} \quad \frac{W^{\psi'_F}_{t,\mathfrak{p}_2}(0, \phi^{(0)}_{\mathfrak{p}_2})}{\gamma(W'_{\mathfrak{p}_2})} = \frac{2}{p};$$

3. *when* $m \equiv -\tilde{d} + d \pmod{p}$ *and* $\left(\frac{-d}{p}\right) = -1$,

$$\frac{W^{\psi'_F}_{t,\mathfrak{p}_1}(0, \phi^{(0)}_{\mathfrak{p}_1})}{\gamma(W'_{\mathfrak{p}_1})} = 0 \quad \text{and} \quad \frac{W^{\psi'_F}_{t,\mathfrak{p}_2}(0, \phi^{(0)}_{\mathfrak{p}_2})}{\gamma(W'_{\mathfrak{p}_2})} = \frac{p-1}{p}(\mathrm{ord}_{\mathfrak{p}_2}(\sigma_2(t)) - 1);$$

4. *when* $m \equiv -\tilde{d} + d \pmod{p}$ *and* $\left(\frac{-d}{p}\right) = 1$,

$$\frac{W^{\psi'_F}_{t,\mathfrak{p}_1}(0, \phi^{(0)}_{\mathfrak{p}_1})}{\gamma(W'_{\mathfrak{p}_1})} = \frac{2}{p} \quad \text{and} \quad \frac{W^{\psi'_F}_{t,\mathfrak{p}_2}(0, \phi^{(0)}_{\mathfrak{p}_2})}{\gamma(W'_{\mathfrak{p}_2})} = \frac{p-1}{p}(\mathrm{ord}_{\mathfrak{p}_2}(\sigma_2(t)) - 1);$$

5. *when* $m \equiv -\tilde{d} + k$ *with* $k \not\equiv 0, d \pmod{p}$, $\left(\frac{-k}{p}\right) = -1$ *and* $\left(\frac{d-k}{p}\right) = 1$,

$$\frac{W^{\psi'_F}_{t,\mathfrak{p}_1}(0, \phi^{(0)}_{\mathfrak{p}_1})}{\gamma(W'_{\mathfrak{p}_1})} = 0 \quad \text{and} \quad \frac{W^{\psi'_F}_{t,\mathfrak{p}_2}(0, \phi^{(0)}_{\mathfrak{p}_2})}{\gamma(W'_{\mathfrak{p}_2})} = \frac{2}{p};$$

6. *when* $m \equiv -\tilde{d} + k$ *with* $k \not\equiv 0, d \pmod{p}$, $\left(\frac{-k}{p}\right) = -1$ *and* $\left(\frac{d-k}{p}\right) = -1$,

$$\frac{W^{\psi'_F}_{t,\mathfrak{p}_1}(0, \phi^{(0)}_{\mathfrak{p}_1})}{\gamma(W'_{\mathfrak{p}_1})} = 0 \quad \text{and} \quad \frac{W^{\psi'_F}_{t,\mathfrak{p}_2}(0, \phi^{(0)}_{\mathfrak{p}_2})}{\gamma(W'_{\mathfrak{p}_2})} = 0;$$

7. *when* $m \equiv -\tilde{d} + k$ *with* $k \not\equiv 0, d \pmod{p}$, $\left(\frac{-k}{p}\right) = 1$ *and* $\left(\frac{d-k}{p}\right) = 1$,

$$\frac{W_{t,\mathfrak{p}_1}^{\psi'_F}(0, \phi_{\mathfrak{p}_1}^{(0)})}{\gamma(W'_{\mathfrak{p}_1})} = \frac{2}{p} \quad \text{and} \quad \frac{W_{t,\mathfrak{p}_2}^{\psi'_F}(0, \phi_{\mathfrak{p}_2}^{(0)})}{\gamma(W'_{\mathfrak{p}_2})} = \frac{2}{p};$$

8. *when* $m \equiv -\tilde{d} + k$ *with* $k \not\equiv 0, d \pmod{p}$, $\left(\frac{-k}{p}\right) = 1$ *and* $\left(\frac{d-k}{p}\right) = -1$,

$$\frac{W_{t,\mathfrak{p}_1}^{\psi'_F}(0, \phi_{\mathfrak{p}_1}^{(0)})}{\gamma(W'_{\mathfrak{p}_1})} = \frac{2}{p} \quad \text{and} \quad \frac{W_{t,\mathfrak{p}_2}^{\psi'_F}(0, \phi_{\mathfrak{p}_2}^{(0)})}{\gamma(W'_{\mathfrak{p}_2})} = 0.$$

Proof. These follow directly from Lemma 3.3.6. □

Chapter 4

Weakly Holomorphic Modular Forms on $\Gamma_0(4)$ and Borcherds Products on Unitary Group $\mathrm{U}(2,1)$

4.1 Introduction

As we have seen in Chapter 3, the meromorphic modular forms constructed from Borcherds lifts have two distinct properties. The first one is the so-called Borcherds product expansion at a cusp of the Shimura variety–his original motivation to prove the Moonshine conjecture. The second is that the divisor of these modular forms are known to be a linear combination of special divisors dictated by the principal part of the input weakly holomorphic forms. The second feature has been extended to produce so-called automorphic green functions for special divisors using harmonic Maass forms via regularized theta lifting by Bruinier ([25]) and Bruinier–Funke ([26]), which turned out to be very useful to generalization of the well-known Grosss–Zagier formula ([47]) and the beautiful Grosss–Zagier factorization formula of singular moduli ([46]) to Shimura varieties of orthogonal type $(n, 2)$ and unitary type $(n, 1)$ (see for example [31], [28], [10], [9], [85], [94]). On the other hand, the Borcherds product expansion and in particular its integral structure is essential to prove modularity of some generating functions of arithmetic divisors on these Shimura varieties ([27], [51]). Borcherds products are also closely related to Mock theta functions (see for example [71] and references therein).

We should mention that the analogue of the Borcherds product to unitary Shimura varieties of type $(n, 1)$ has been recently worked out by Hofmann ([50]) (see also [27]). The Borcherds product expansion in the unitary case is a little more complicated as it is a Fourier–Jacobi expansion rather than Fourier expansion; the coefficients are theta functions rather than numbers. The purpose of this chapter is to give some explicit examples of these Borcherds product expansion in concrete term. For this reason, we focus on the Picard modular surface $X_{\Gamma_L} = \Gamma_L \backslash \mathcal{H}$ associated to the Hermitian lattice $L = \mathbb{Z}[i] \oplus \mathbb{Z}[i] \oplus \frac{1}{2}\mathbb{Z}[i]$ with Hermitian form

$$\langle x, y \rangle = x_1 \bar{y}_3 + x_3 \bar{y}_1 + x_2 \bar{y}_2.$$

Here

$$\mathcal{H} = \{(\tau, \sigma) \in \mathbb{H} \times \mathbb{C} |\, 4\mathrm{Im}(\tau) > |\sigma|^2\},$$

and Γ_L is a subgroup of $\mathrm{U}(L)$ defined by (4.3.4). Our inputs are weakly holomorphic modular forms for $\Gamma_0(4)$ of weight -1, character $\chi_{-4} := \left(\frac{-4}{\cdot}\right)$ which have poles only at the cusp $i\infty$, which we denote by $M^{!,\infty}_{-k}(\Gamma_0(4), \chi^k_{-4})$ with $k = 1$. Our first result (Theorem 4.2.1) is to give a canonical basis $F_{k,m}$ $(m \geq 1)$ for the infinitely dimensional vector space for every $k \geq 1$. The even k case was given by Haddock and Jenkins in [48] in a slightly different fashion. Similar method can be applied to yield a canonical basis for the space of weakly holomorphic forms of $\Gamma_0(4)$ with weight $-k$, character χ^k_{-4}, and having poles only at the cusp 0 (resp. $\frac{1}{2}$).

Next, we use a standard induction procedure to produce vector-valued weakly holomorphic modular forms for $\mathrm{SL}_2(\mathbb{Z})$ using our lattice L which will be used to construct Picard modular forms on $\mathrm{U}(2,1)$ (described above). Although the resulting vector-valued modular forms for $\mathrm{SL}_2(\mathbb{Z})$ from the three different scalar valued spaces $M^{!,P}_{-k}(\Gamma_0(4), \chi^k_{-4})$, $P = i\infty, 0, \frac{1}{2}$ are linearly independent, they don't generate the whole space. This concludes Part I of our note, which should be of independent interest.

In Part II, we focus on the unitary group $\mathrm{U}(2,1)$ associated to the above Hermitian form and give explicit Borcherds product expansion of the Picard modular forms constructed from $F_m = F_{1,m}$. The delicate part is to choose a proper Weyl chamber, which is a dimensional 3 real manifold and described it explicitly and carefully. Our main formula is Theorem 4.3.5. We remark that the same method also applies to high dimensional unitary Shimura varieties of unitary type $(n,1)$ using forms in $M^{!,P}_{1-n}(\Gamma_0(4), \chi^k_{-4})$ where P is a cusp for $\Gamma_0(4)$. We restrict to $\mathrm{U}(2,1)$ for being as explicit as possible.

4.2 Part I: Vector-valued Modular Forms

In this part, we derive a canonical basis for the space $M^{!,\infty}_{-k}(\Gamma_0(4), \chi^k_{-4})$ for any integer $k \geq 0$, and investigate the properties of the vector-valued modular forms arising from $M^{!,\infty}_{-k}(\Gamma_0(4), \chi^k_{-4})$. For completeness, we will also give canonical bases for $M^{!,0}_{-k}(\Gamma_0(4), \chi^k_{-4})$ and $M^{!,\frac{1}{2}}_{-k}(\Gamma_0(4), \chi^k_{-4})$.

4.2.1 A Canonical Basis for $M^{!,\infty}_{-k}(\Gamma_0(4), \chi^k_{-4})$

Let $\chi_{-4}(\cdot) := \left(\frac{-4}{\cdot}\right)$ be the Kronecker symbol modulo 4. Recall that $X_0(4)$ has 3 cusps, represented by $i\infty$, 0, and $\frac{1}{2}$. For each cusp P, let $M^{!,P}_{-k}(\Gamma_0(4),\chi^k_{-4})$ denote the space of weakly holomorphic modular forms, which are holomorphic everywhere except at the cusp P, of weight $-k$ on $\Gamma_0(4)$ with character χ^k_{-4}. We will focus mainly on the cusp $i\infty$ and will remark on other cusps (very similar) in the end. We will also denote $M^!_{-k}(\Gamma_0(4),\chi^k_{-4})$ for the space of weakly holomorphic modular forms for $\Gamma_0(4)$ of weight $-k$ and character χ^k_{-4}.

Let τ be a complex number with positive imaginary part, and set $q = e(\tau) = e^{2\pi i\tau}$, and $q_r = e^{2\pi i\tau/r}$. The Dedekind eta function is defined by

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n).$$

Throughout this chapter, we write η_m for $\eta(m\tau)$. The well known Jacobi theta functions are defined by

$$\vartheta_{00}(\tau) = \sum_{n=-\infty}^{\infty} q^{n^2}, \quad \vartheta_{01}(\tau) = \sum_{n=-\infty}^{\infty} (-q)^{n^2}, \quad \vartheta_{10}(\tau) = \sum_{n=-\infty}^{\infty} q^{\left(n+\frac{1}{2}\right)^2}.$$

Now we define three functions as follows.

$$\theta_1 = \theta_1(\tau) := \frac{1}{16}\vartheta_{10}^4(\tau) = \frac{\eta_4^8}{\eta_2^4} = q + O(q^2), \tag{4.2.1}$$

$$\theta_2 = \theta_2(\tau) := \vartheta_{00}^2(\tau) = \frac{\eta_2^{10}}{\eta_1^4\eta_4^4} = 1 + O(q), \tag{4.2.2}$$

$$\varphi_\infty = \varphi_\infty(\tau) := \left(\frac{\eta_1}{\eta_4}\right)^8 = q^{-1} + O(1). \tag{4.2.3}$$

Here are some basic facts [48] about the functions θ_1, θ_2 and φ_∞.

1. $\theta_1(\tau)$ is a holomorphic modular form of weight 2 on $\Gamma_0(4)$ with trivial character, has a simple zero at the cusp $i\infty$, and vanishes nowhere else.

2. $\theta_2(\tau)$ is a holomorphic modular form of weight 1 on $\Gamma_0(4)$ with character χ_{-4}, has a zero of order $\frac{1}{2}$ at the irregular cusp $\frac{1}{2}$, and vanishes nowhere else.

3. $\varphi_\infty(\tau)$ is a modular form of weight 0 on $\Gamma_0(4)$ with trivial character, has exactly one simple pole at the cusp $i\infty$ and a simple zero at the cusp 0.

The following is a variant of [48] where the case even k has been treated by Haddock and Jenkins. We should mention that similar results for the space of weakly holomorphic modular forms for $\mathrm{SL}_2(\mathbb{Z})$ were first obtained in [41] by Duke and Jenkins.

Theorem 4.2.1. *1. For $k \geq 1$ odd, there is a (canonical) basis $F_{k,m}$ $(m \geq 1)$ of $M^{!,\infty}_{-k}(\Gamma_0(4), \chi_{-4})$ whose Fourier expansion has the following form:*

$$F_{k,m} = q^{-\frac{k+1}{2}-m+1} + \sum_{n \geq -\frac{k-1}{2}} c(n)q^n.$$

2. For $k > 1$ even, there is a (canonical) basis $F_{k,m}$ $(m \geq 1)$ of $M^{!,\infty}_{-k}(\Gamma_0(4))$ whose Fourier expansion has the following form:

$$F_{k,m} = q^{-\frac{k}{2}-m+1} + \sum_{n \geq -\frac{k}{2}+1} c(n)q^n,$$

Proof of Theorem 4.2.1. The proof is similar to those given in [41] and [48], and we include it for completeness. We prove (1) first. Notice that $X_0(4)$ has no elliptic points [39, Section 3.9]. For $F \in M^{!,\infty}_{-k}(\Gamma_0(4), \chi_{-4})$, the valence formula for $\Gamma_0(4)$ asserts that

$$\sum_{z \in \Gamma_0(4)\backslash \mathbb{H}} \mathrm{ord}_z(F) + \mathrm{ord}_\infty(F) + \mathrm{ord}_0(F) + \mathrm{ord}_{1/2}(F) = -\frac{k}{2}.$$

This implies $\mathrm{ord}_{1/2} F \geq \frac{1}{2}$ (1/2 is the unique irregular cusp), $\mathrm{ord}_\infty(F) \leq -\frac{k+1}{2}$. This implies the uniqueness of the basis $\{F_{k,m}\}$ if it exists. We prove the existence by inductively constructing a sequence of monic polynomials $P_{k,m}(x)$ of degree m $(m \geq 0)$ such that $F_{k,m+1} = \theta_2\theta_1^{-\frac{k+1}{2}} P_{k,m}(\varphi_\infty)$ give the basis we seek, i.e., with the following property

$$F_{k,m+1} = \theta_2\theta_1^{-\frac{k+1}{2}} P_{k,m}(\varphi_\infty) = q^{-\frac{k+1}{2}-m} + \sum_{n \geq -\frac{k-1}{2}} c(n)q^n. \tag{4.2.4}$$

1. Notice that $\theta_2\theta_1^{-\frac{k+1}{2}} \in M^{!,\infty}_{-k}(\Gamma_0(4),\chi_{-4})$ with
$$\theta_2\theta_1^{-\frac{k+1}{2}} = q^{-\frac{k+1}{2}} + \sum_{n\geq -\frac{k-1}{2}} c(n)q^n.$$
So we can and will first define $P_{k,0} = 1$.

2. For $m \geq 1$, assume that $P_{k,m-1}(x) \in \mathbb{C}[x]$ is constructed with degree $m-1$, leading coefficient 1, and the property
$$F_{k,m} = \theta_2\theta_1^{-\frac{k+1}{2}} P_{k,m-1}(\varphi_\infty) = q^{-\frac{k+1}{2}-m+1} + \sum_{n\geq -\frac{k-1}{2}} c(n)q^n.$$
Then it is easy to see
$$F_{k,m}\varphi_\infty = q^{-\frac{k+1}{2}-m} + \sum_{n> -\frac{k+1}{2}-m} d(n)q^n.$$
Let
$$P_{k,m} = xP_{k,m-1} - \sum_{n=-\frac{k+1}{2}-m+1}^{-\frac{k+1}{2}} d(n)P_{k,-n},$$
and
$$F_{k,m+1} = \theta_2\theta_1^{-\frac{k+1}{2}} P_{k,m}(\varphi_\infty).$$
Then $F_{k,m+1}$ satisfies (4.2.4). By induction, we prove the existence of the basis $\{F_{k,m}\}$, and (1).

The proof of (2) is similar and is left to the reader. In this case, the basis $\{F_{k,m+1}\}$, $m \geq 0$, has the form
$$F_{k,m+1} = \theta_1^{-\frac{k}{2}} Q_{k,m}(\varphi_\infty) = q^{-\frac{k}{2}-m} + \sum_{n=-\frac{k}{2}+1}^{\infty} c(n)q^n \tag{4.2.5}$$
for a unique monic polynomial $Q_{k,m}$ of degree m.

$\square$

The following corollary follows directly from the proof of Theorem 4.2.1(1).

Corollary 4.2.2. *Every weakly holomorphic modular form* $f(\tau) \in M^{!,\infty}_{-k}(\Gamma_0(4), \chi^k_{-4})$ *with* k *odd, vanishes at the cusp* $1/2$.

4.2.2 Vector-valued Modular Form Arising from $M^{!,\infty}_{-k}(\Gamma_0(4), \chi^k_{-4})$

Let L be an even lattice over $\mathbb{Z}$ with symmetric non-degenerate bilinear form $(\cdot,\cdot)$ and associated quadratic form $Q(x) = \frac{1}{2}(x,x)$. Let L' be the dual lattice of L. Assume that L has rank $2m+2$ and signature $(2m,2)$. Then the Weil representation of the metaplectic group $\mathrm{Mp}_2(\mathbb{Z})$ on the group algebra $\mathbb{C}[L'/L]$ factors through $\mathrm{SL}_2(\mathbb{Z})$. Thus we have a unitary representation ρ_L of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{C}[L'/L]$, defined by

$$\rho_L(T)\phi_\mu = e(-Q(\mu))\phi_\mu, \tag{4.2.6}$$

$$\rho_L(S)\phi_\mu = \frac{\sqrt{i}^{2m-2}}{\sqrt{|L'/L|}} \sum_{\beta \in L'/L} e((\mu,\beta))\phi_\beta \tag{4.2.7}$$

where $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, ϕ_μ for $\mu \in L'/L$ are the standard basis elements of $\mathbb{C}[L'/L]$ and $e(z) = e^{2\pi i z}$. We remark that the Weil representation ρ_L depends only on the finite quadratic module $(L'/L, Q)$ (called the discriminant group of L), where $Q(x+L) = Q(x) \pmod 1 \in \mathbb{Q}/\mathbb{Z}$.

Let k be an integer and $\vec{F}$ be a $\mathbb{C}[L'/L]$ valued function on $\mathbb{H}$ and let $\rho = \rho_L$ be a representation of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{C}[L'/L]$. For $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ we define the slash operator by

$$\left(\vec{F}\Big|_{k,\rho} \gamma \right)(\tau) = (c\tau+d)^{-k}\rho(\gamma)^{-1}\vec{F}(\gamma\tau),$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ acts on $\mathbb{H}$ via $\gamma\tau = \frac{a\tau+b}{c\tau+d}$.

Definition 4.2.3. *Let k be an integer. A function $\vec{F} : \mathbb{H} \to \mathbb{C}[L'/L]$ is called a weakly holomorphic vector-valued modular form of weight k with respect to $\rho = \rho_L$ if it satisfies*

1. $\vec{F}\Big|_{k,\rho} \gamma = F$ *for all* $\gamma \in SL_2(\mathbb{Z})$,
2. $\vec{F}$ *is holomorphic on* $\mathbb{H}$,
3. $\vec{F}$ *is meromorphic at the cusp* $i\infty$.

The space of such forms is denoted by $M^!_{k,\rho}$.

The invariance of T-action implies that $\vec{F} \in M^!_{k,\rho}$ has a Fourier expansion of the form

$$\vec{F} = \sum_{\mu \in L'/L} \sum_{\substack{n \in \mathbb{Q} \\ n \gg -\infty}} c(n, \phi_\mu) q^n \phi_\mu.$$

Note that $c(n, \phi_\mu) = 0$ unless $n \equiv -Q(\mu) \pmod 1$.

From now on, we focus on the special case with the discriminant group $L'/L \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ with quadratic form $Q(x, y) = \frac{1}{4}(x^2 + y^2) \pmod 1$. For our purpose (in Section 4.3), it is convenient to identify $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}[i]/2\mathbb{Z}[i]$, where $Q(z) = \frac{1}{4} z\bar{z} \in \mathbb{Q}/\mathbb{Z}$. We write ϕ_0, ϕ_1, ϕ_i and ϕ_{1+i} for the basis elements of $\mathbb{C}[L'/L]$ corresponding to $(0, 0)$, $(1, 0)$, $(0, 1)$ and $(1, 1)$ respectively.

Let $F = F(\tau) \in M^{!,\infty}_{-k}(\Gamma_0(4), \chi_{-4})$ with k odd and positive. Then using $\Gamma_0(4)$-lifting, we can construct a vector-valued modular form $\vec{F} = \vec{F}(\tau)$ arising from $F(\tau)$ as follows:

$$\vec{F}(\tau) = \sum_{\gamma \in \Gamma_0(4) \backslash \mathrm{SL}_2(\mathbb{Z})} (F|_{-k}\,\gamma) \rho_L(\gamma)^{-1} \phi_0 = \frac{1}{2} \sum_{\gamma \in \Gamma_1(4) \backslash \mathrm{SL}_2(\mathbb{Z})} (F|_{-k}\,\gamma) \rho_L(\gamma)^{-1} \phi_0. \tag{4.2.8}$$

Define modular forms F_0, F_2 and F_3 as follows. Recall that $q_r = e^{2\pi i\tau/r}$. Let

$$F|_{-k}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} = \sum_{n=0}^{\infty} a(n)q_4^n.$$

Then for $j \in \{0, 2, 3\}$, we write

$$F_j = \sum_{n=0}^{\infty} a(4n+j)q_4^{4n+j}. \tag{4.2.9}$$

We also define $F_{1/2}$ to be

$$F_{1/2} = F|_{-k}\begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix} = \sum_{n=0}^{\infty} b(n)q_2^n. \tag{4.2.10}$$

In addition, taking the coset representatives $\{I, S, ST^{-1}, ST, ST^2, ST^2S^{-1}\}$ for $\Gamma_0(4)\backslash\mathrm{SL}_2(\mathbb{Z})$, it is easy to check by (4.2.6)–(4.2.7) that

$$\begin{aligned}
\rho_L(S)^{-1}\phi_0 &= -\frac{i}{2}\left(\phi_0 + \phi_1 + \phi_i + \phi_{1+i}\right),\\
\rho_L(ST^{-1})^{-1}\phi_0 &= -\frac{i}{2}\left(\phi_0 - i\phi_1 - i\phi_i - \phi_{1+i}\right),\\
\rho_L(ST)^{-1}\phi_0 &= -\frac{i}{2}\left(\phi_0 + i\phi_1 + i\phi_i - \phi_{1+i}\right),\\
\rho_L(ST^2)^{-1}\phi_0 &= -\frac{i}{2}\left(\phi_0 - \phi_1 - \phi_i + \phi_{1+i}\right),\\
\rho_L(ST^2S^{-1})^{-1}\phi_0 &= \phi_{1+i}.
\end{aligned}$$

Finally, direct calculations yield

$$\vec{F}(\tau) = \left(-2iF_0 + F\right)\phi_0 - 2iF_3\phi_1 - 2iF_3\phi_i + \left(-2iF_2 - F_{1/2}\right)\phi_{1+i}. \tag{4.2.11}$$

The following theorem gives some basic facts about F_0, F_2, F_3 and $F_{1/2}$.

Theorem 4.2.4. *With the above definitions, we have*

$$F_0 \in M^!_{-k}(\Gamma_0(4), \chi_{-4}), \tag{4.2.12}$$

$$F_3 \in M^!_{-k}(\Gamma_0(4), \chi_1) \tag{4.2.13}$$

where $\chi_1(\gamma) = \chi_{-4}(d)e(-ab/4)$ *for* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$,

$$(2iF_2 + F_{1/2}) \in M^!_{-k}(\Gamma_0(4), \chi_2) \tag{4.2.14}$$

where $\chi_2(\gamma) = \chi_{-4}(d)e(-ab/2)$ *for* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$,

and

$$F_{1/2} \in M^!_{-k}(\delta^{-1}\Gamma_0(4)\delta, \chi_{-4}) \tag{4.2.15}$$

where $\delta = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$.

Proof. By (4.2.11), and [83, Section 3, p. 6] or [82, Proposition 4.5], we can show that for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$,

$$(-2iF_0 + F)|_{-k}\,\gamma = \chi_{-4}(d)(-2iF_0 + F), \tag{4.2.16}$$

$$F_3|_{-k}\,\gamma = \chi_{-4}(d)e(-ab/4)F_3, \tag{4.2.17}$$

$$(-2iF_2 - F_{1/2})\big|_{-k}\,\gamma = \chi_{-4}(d)e(-ab/2)(-2iF_2 - F_{1/2}). \tag{4.2.18}$$

Since $F \in M^!_{-k}(\Gamma_0(4), \chi_{-4})$, then (4.2.16) implies (4.2.12). Relations (4.2.13) and (4.2.14) follow directly from (4.2.17) and (4.2.18), respectively. The last relation (4.2.15) follows from the definition of $F_{1/2}$,

$$F_{1/2} = F|_{-k} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

$\square$

Theorem 4.2.5. *Let k be odd. Let $F = F(\tau) \in M^{!,\infty}_{-k}(\Gamma_0(4), \chi_{-4})$ with*

$$F(\tau) = \sum_{n=-m}^{\infty} c(n) q^n.$$

Write

$$F|_{-k} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \sum_{n=0}^{\infty} a(n) q_4^n \quad \text{and} \quad F|_{-k} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = \sum_{n=0}^{\infty} b(n) q_2^n.$$

And let the $\Gamma_0(4)$-lifting of F be

$$\vec{F}(\tau) = \sum_{\mu \in L'/L} \sum_{\substack{n \in \mathbb{Q} \\ n \gg -\infty}} c(n, \phi_\mu) q^n \phi_\mu.$$

Then we have

(i)

$$c(n, \phi_0) = -2ia(4n) + c(n),$$

$$c(n, \phi_1) = c(n, \phi_i) = -2ia(4n),$$

$$c(n, \phi_{1+i}) = -2ia(4n) - b(2n),$$

(ii) the principal part of the vector-valued modular form $\vec{F}(\tau)$ is

$$\left(c(-m) q^{-m} + \cdots + c(-1) q^{-1} \right) \phi_0,$$

(iii) the constant term of the ϕ_0-component of $\vec{F}(\tau)$ is

$$c(0,\phi_0) = -(8i)^{k+1} \sum_{n=\frac{k+1}{2}}^{m} c(-n) P_{k,n-\frac{k+1}{2}}(0) + c(0),$$

where $P_{k,n}(x)$ are the polynomials defined as in the proof of Theorem 4.2.1.

In particular, when $k=1$, the constant term of the ϕ_0-component of $\vec{F}(\tau)$ is

$$c(0,\phi_0) = \sum_{n=1}^{m} c(-n) \left(\sum_{d|n} \left(64\chi_{-4}(n/d) + 4\chi_{-4}(d)\right) d^2 \right). \tag{4.2.19}$$

Proof. Assertion (i) follows directly from (4.2.11). For the assertion (ii), since F is holomorphic at 0 and $\frac{1}{2}$, then F_j for $j \in \{0,2,3\}$ and $F_{1/2}$ will not contribute anything to the principal part of $\vec{F}$. So the principal part of $\vec{F}$ is given by

$$\left(c(-m)q^{-m} + \cdots + c(-1)q^{-1} \right) \phi_0.$$

For the assertion (iii), we first note by (i) that

$$c(0,\phi_0) = -2ia(0) + c(0).$$

By Theorem 4.2.1(1), we have

$$F = c(-m)\theta_2\theta_1^{-\frac{k+1}{2}} P_{k,m-\frac{k+1}{2}}(\varphi_\infty) + \cdots + c\left(-\frac{k+1}{2}\right) \theta_2\theta_1^{-\frac{k+1}{2}} P_{k,0}(\varphi_\infty) \tag{4.2.20}$$

Since θ_1 and θ_2 do not vanish at the cusp 0, and φ_∞ has a simple zero at 0 of width 4, then we have

$$\theta_2\theta_1^{-\frac{k+1}{2}} \varphi_\infty^l \Big|_{-k} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = O(q^{\frac{l}{4}}),$$

and thus $\theta_2\theta_1^{-\frac{k+1}{2}} \varphi_\infty^l \Big|_{-1} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ will not contribute anything to the constant term of F_0 when $l \geq 1$. Moreover, simple calculation using the transformation formula for the

Dedekind eta function shows that the constant term of the Fourier expansion at the cusp 0 of $\theta_2\theta_1^{-\frac{k+1}{2}}$ is $-(8i)^{k+1}$. Therefore,

$$\begin{aligned} a(0) &= \left(\sum_{n=\frac{k+1}{2}}^{m} c(-n) P_{k,n-\frac{k+1}{2}}(0)\theta_2\theta_1^{-\frac{k+1}{2}} \Big|_{-k} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right)_0 \\ &= -(8i)^{k+1} \sum_{n=\frac{k+1}{2}}^{m} c(-n) P_{k,n-\frac{k+1}{2}}(0) \end{aligned}$$

where $(f)_0$ denotes the constant term of the q-expansion of f. Hence, we have

$$c(0,\phi_0) = -(8i)^{k+1} \sum_{n=\frac{k+1}{2}}^{m} c(-n) P_{k,n-\frac{k+1}{2}}(0) + c(0).$$

For (4.2.19), according to (iii), we need to show that

$$P_{1,m}(0) = \sum_{d|(m+1)} \chi_{-4}((m+1)/d)d^2 \quad \text{and} \quad c(0) = \sum_{n=1}^{m} c(-n) \left(4 \sum_{d|n} \chi_{-4}(d) d^2 \right).$$

For the first formula, we first observe that

$$\theta_2\theta_1^{-1}\varphi_\infty^\ell = q^{-\ell-1} + \sum_{j=1}^{\ell} c_\ell(-j) q^{-j} + O(1)$$

for $0 \le \ell \le m$. Thus there are $b_1, \ldots, b_{m-1}$ such that

$$h(\tau) := \theta_2\theta_1^{-1}\varphi_\infty^m + b_{m-1}\theta_2\theta_1^{-1}\varphi_\infty^{m-1} + \cdots + b_1\theta_2\theta_1^{-1}\varphi_\infty = q^{-m-1} + a(-1)q^{-1} + O(1)$$

for some constant $a(-1)$. Let $g(\tau)$ be defined by

$$g(\tau) = \sum_{n=1}^{\infty} \left(\sum_{d|n} \chi_{-4}(n/d) d^2 \right) q^n = \sum_{n=1}^{\infty} d_n q^n.$$

It is known [55] that $g(\tau)$ is a weight 3 modular form on $\Gamma_0(4)$ with character χ_{-4}. We note by the basic facts about θ_1, θ_2 and φ_∞ that $h(\tau)$ vanishes at the cusps 1/2 and 0. Then by [22, Theorem 3.1], we have

$$d_{m+1} + a(-1) = 0, \ \textit{i.e.}, \ d_{m+1} = -a(-1).$$

Therefore

$$P_{1,m}(0) = d_{m+1} = \sum_{d|(m+1)} \chi_{-4}((m+1)/d)d^2.$$

This proves the first formula. For the second one, the proof is similar by noting that

$$h_1(\tau) := \theta_2\theta_1^{-1} P_{1,m}(\varphi_\infty) = q^{-m-1} + C + O(q)$$

and

$$g_1(\tau) = 1 + 4\sum_{n=1}^{\infty}\left(\sum_{d|n}\chi_{-4}(d)d^2\right)q^n$$

is [55] a weight 3 modular form on $\Gamma_0(4)$ with character χ_{-4}. Then again [22, Theorem 3.1] shows that

$$C = 4\sum_{d|(m+1)} \chi_{-4}(d)d^2.$$

This together with (4.2.20) proves the second formula. $\square$

Example 4.2.6. *Let $k = 1$ and $F(\tau) = \theta_2\theta_1^{-1} = \frac{\eta_2^{14}}{\eta_1^4\eta_4^{12}} \in M_{-1}^{!,\infty}(\Gamma_0(4), \chi_{-4})$. Then we have*

$$\vec{F}(\tau) = (-2iF_0 + F)\,\phi_0 - 2iF_3\phi_1 - 2iF_3\phi_i + \left(-2iF_2 - F_{1/2}\right)\phi_{1+i} \tag{4.2.21}$$

where F_0, F_2, F_3 and $F_{1/2}$ are defined as in (4.2.9) *and* (4.2.10)*. We have*

$$\begin{aligned}
F|_{-1}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} &= 32i\frac{\eta(\tau/2)^{14}}{\eta(\tau/4)^4\eta(\tau)^{12}} \\
&= 32i\big(1 + 12q^{1/4} + 76q^{2/4} + 352q^{3/4} + 1356q + 4600q^{5/4} \\
&\qquad + 14176q^{6/4} + 40512q^{7/4} + \cdots\big) \\
&= 32i\left(1 + 1356q + O(q^2)\right) \\
&\quad + 32i\left(12q^{1/4} + 4600q^{5/4} + O(q^{9/4})\right)
\end{aligned}$$

$$+ 32i\left(76q^{2/4} + 14176q^{6/4} + O(q^{10/4})\right)$$
$$+ 32i\left(352q^{3/4} + 40512q^{7/4} + O(q^{11/4})\right),$$

then

$$F_0 = 32i\left(1 + 1356q + O(q^2)\right),$$
$$F_2 = 32i\left(76q^{2/4} + 14176q^{6/4} + O(q^{10/4})\right)$$
$$F_3 = 32i\left(352q^{3/4} + 40512q^{7/4} + O(q^{11/4})\right).$$

And

$$F_{1/2} = F|_{-1}\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = 64\left(q^{1/2} - 8q^{3/2} + 42q^{5/2} + O(q^{7/2})\right).$$

From (4.2.21), *we note that the principal part of* F *is* $e(-\tau)\phi_0$ *and the constant term of the* ϕ_0*-component is* $c(0, \phi_0) = 68$.

4.2.3 Canonical Bases for $M^{!,0}_{-k}(\Gamma_0(4), \chi^k_{-4})$ and $M^{!,\frac{1}{2}}_{-k}(\Gamma_0(4), \chi^k_{-4})$

We complete this section by giving canonical bases for the other two companions of $M^{!,\infty}_{-k}(\Gamma_0(4), \chi^k_{-4})$.

Let $\theta_3(\tau)$, $\varphi_0(\tau)$ and $\varphi_{1/2}(\tau)$ be defined by

$$\theta_3 = \theta_3(\tau) := \vartheta_{01}^4(\tau) = \frac{\eta_1^8}{\eta_2^4} = 1 + O(q), \tag{4.2.22}$$

$$\varphi_0 = \varphi_0(\tau) := \left(\frac{\eta_4}{\eta_1}\right)^8 = q + O(q^2), \tag{4.2.23}$$

$$\varphi_{1/2} = \varphi_{1/2}(\tau) := \frac{\eta_1^8\eta_4^{16}}{\eta_2^{24}} = q + O(q^2). \tag{4.2.24}$$

Here are some basic facts about θ_3, φ_0 and $\varphi_{1/2}$:

1. $\theta_3(\tau)$ is a weight 2 modular form on $\Gamma_0(4)$ with trivial character, has a simple zero at the cusp 0, and vanishes nowhere else;

2. $\varphi_0(\tau)$ is a weight 0 modular form on $\Gamma_0(4)$ with trivial character, has a simple pole at the cusp 0 and a simple zero at the cusp $i\infty$, and vanishes nowhere else;

3. $\varphi_{1/2}(\tau)$ is a weight 0 modular form on $\Gamma_0(4)$ with trivial character, has a simple pole at the cusp $\frac{1}{2}$ and a simple zero at the cusp $i\infty$, and vanishes nowhere else.

Theorem 4.2.7. *Let θ_2, θ_3 and φ_0 be as defined in* (4.2.2), (4.2.22) *and* (4.2.23), *respectively.*

1. *For k odd, the set $\{\theta_2\theta_3^{-\frac{k+1}{2}}P_{k,m}(\varphi_0)\}_{m=0}^{\infty}$ where $P_{k,m}$ is a monic polynomial of degree m such that*

$$\theta_2\theta_3^{-\frac{k+1}{2}}P_{k,m}(\varphi_0)\Big|_{-k}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} = q_4^{-\frac{k+1}{2}-m} + \sum_{n=-\frac{k-1}{2}}^{\infty} c(n)q_4^n,$$

is a canonical basis for $M_{-k}^{!,0}(\Gamma_0(4),\chi_{-4})$.

2. *For k even, the set $\{\theta_3^{-\frac{k}{2}}P_{k,m}(\varphi_0)\}_{m=0}^{\infty}$ where $P_{k,m}$ is a monic polynomial of degree m such that*

$$\theta_3^{-\frac{k}{2}}P_{k,m}(\varphi_0)\Big|_{-k}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} = q_4^{-\frac{k}{2}-m} + \sum_{n=-\frac{k}{2}+1}^{\infty} c(n)q_4^n,$$

is a canonical basis for $M_{-k}^{!,0}(\Gamma_0(4))$.

Theorem 4.2.8. *Let θ_2 and $\varphi_{1/2}$ be as defined in* (4.2.2) *and* (4.2.24), *respectively. Then the set $\{\theta_2^{-k}P_{k,m}(\varphi_{1/2})\}_{m=0}^{\infty}$ where $P_{k,m}$ is a monic polynomial of degree m such* *that*

$$\theta_2^{-k} P_{k,m}(\varphi_{1/2})\big|_{-k}\begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix} = q^{-\frac{k}{2}-m} + \sum_{n=-\frac{k}{2}+1}^{\infty} c(n)q^n,$$

is a canonical basis for $M^{!,\frac{1}{2}}_{-k}(\Gamma_0(4), \chi_{-4}^k)$.

Proofs of Theorems 4.2.7 and 4.2.8 are similar to that of Theorem 4.2.1, so we omit the details.

Remark 4.2.9. *For a cusp* P*, denote by* $M^{!,P}_{-k,\rho_L}$ *the space of vector-valued modular forms induced from* $M^{!,P}_{-k}(\Gamma_0(4), \chi_{-4}^k)$ *via* $\Gamma_0(4)$*-lifting. We have, by* (4.2.11)*,*

$$M^{!,\infty}_{-k,\rho_L} + M^{!,0}_{-k,\rho_L} + M^{!,\frac{1}{2}}_{-k,\rho_L} = M^{!,\infty}_{-k,\rho_L} \oplus M^{!,0}_{-k,\rho_L} \oplus M^{!,\frac{1}{2}}_{-k,\rho_L}.$$

Clearly, $M^{!,\infty}_{-k,\rho_L} + M^{!,0}_{-k,\rho_L} + M^{!,\frac{1}{2}}_{-k,\rho_L}$ *is a subspace of* $M^{!}_{-k,\rho_L}$*. In general, the former space may not be equal to the latter one. We first note that every vector-valued modular form in* $M^{!,\infty}_{-k,\rho_L} + M^{!,0}_{-k,\rho_L} + M^{!,\frac{1}{2}}_{-k,\rho_L}$ *must have the same component functions at* ϕ_1 *and* ϕ_i*. We now give an example of functions in* $M^{!}_{-1,\rho_L}$ *that does not have this property. Let* $F(\tau) = \theta_2\theta_1^{-1} \in M^{!,\infty}_{-1}(\Gamma_0(4), \chi_{-4})$*. Then as above we write the* $\Gamma_0(4)$*-lifting of* $F(\tau)$ *as*

$$\vec{F}(\tau) = (-2iF_0 + F)\,\phi_0 - 2iF_3\phi_1 - 2iF_3\phi_i + \left(-2iF_2 - F_{1/2}\right)\phi_{1+i}$$

where

$$F_j = \sum_{n=0}^{\infty} a(4n+j)q_4^{4n+j},$$

$$F\big|_{-k}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} = \sum_{n=0}^{\infty} a(n)q_4^n$$

and

$$F_{1/2} = F\big|_{-1}\begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix}.$$

By (4.2.13), we know that $F_3(\tau) \in M^!_{-1}(\Gamma_1(4), \chi)$ *where* $\chi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = e(-b/4)$. *Now we do* $\Gamma_1(4)$*-lifting on* $F_3(\tau)$ *against* ϕ_1*, namely,*

$$\vec{F}_3(\tau) = \sum_{\gamma \in \Gamma_1(4) \backslash \mathrm{SL}_2(\mathbb{Z})} \left(F_3|_{-1}\, \gamma\right) \rho_L(\gamma)^{-1} \phi_1,$$

and get

$$\vec{F}_3(\tau) = -4if_0\phi_0 + (2F_3 + 4if_3)\phi_1 + (-4if_3 - 2f_{1/2})\phi_i + 4if_2\phi_{1+i}$$

where

$$f_j = \sum_{\substack{n \in \mathbb{Z} \\ n \gg -\infty}} \tilde{a}(4n+j) q_4^{4n+j},$$

$$F_3|_{-1} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \sum_{\substack{n \in \mathbb{Z} \\ n \gg -\infty}} \tilde{a}(n) q_4^n$$

and

$$f_{1/2} = F_3|_{-1} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

Now the component functions at ϕ_1 *and* ϕ_i *are* $2F_3 + 4if_3$ *and* $-4if_3 - 2f_{1/2}$*, respectively. We can compute and verify that they are not the same. Therefore,* $\vec{F}_3(\tau)$ *is not in the space* $M^{!,\infty}_{-k,\rho_L} + M^{!,0}_{-k,\rho_L} + M^{!,\frac{1}{2}}_{-k,\rho_L}$.

4.3 Part II: Borcherds Products on $\mathrm{U}(2,1)$

It is well-known that the vector-valued weakly modular forms construction in Part I can be used to construct memomophic modular forms on Shimura varieties of orthogonal type $(n, 2)$ and unitary type $(n, 1)$ with Borcherds product formulas and known divisors.

In this part, we focus on one special case to make it very explicitly—the Picard modular surfaces over $\boldsymbol{k} = \mathbb{Q}(i)$. In particular, we describe a Weyl chamber explicitly and write down the Borcherds product expression concretely.

This part is devoted to deriving Borcherds products lifted from a vector-valued modular form arising from $M^{!,\infty}_{-1}(\Gamma_0(4), \chi_{-4})$.

4.3.1 Picard modular surfaces over $\boldsymbol{k} = \mathbb{Q}(i)$

Let $(V, \langle\,,\rangle)$ be a Hermitian vector space over $\boldsymbol{k}$ of signature $(2, 1)$ and let $H = \mathrm{U}(V)$, where $\mathrm{U}(V)$ denotes the unitary group associated to V. Let $V_{\mathbb{C}} = V \otimes_{\boldsymbol{k}} \mathbb{C}$, and

$$\mathcal{L} = \{w \in V_{\mathbb{C}} |\, \langle w, w\rangle < 0\}.$$

Then $\mathcal{K} = \mathcal{L}/\mathbb{C}^{\times}$ is the Hermitian domain for $H(\mathbb{R})$, and $\mathcal{L}$ is the tautological line bundle over $\mathcal{K}$. For a congruence subgroup Γ of $H(\mathbb{Q})$, the associated Picard modular surface $X_{\Gamma} = \Gamma\backslash\mathcal{K}$ is defined over some number field.

Given an isotropic line $\boldsymbol{k}\ell$ (i.e., a cusp), choose another isotropic element ℓ' with $\langle \ell, \ell'\rangle \neq 0$. Let $V_0 = (\boldsymbol{k}\ell + \boldsymbol{k}\ell')^{\perp}$, and let

$$\mathcal{H} = \mathcal{H}_{\ell,\ell'} = \left\{(\tau, \sigma) \in \mathbb{H} \times V_{0,\mathbb{C}} \,\middle|\, \Im\tau > \frac{\langle \sigma, \sigma\rangle}{4|\langle \ell', \ell\rangle|^2}\right\}.$$

Then the map

$$\mathcal{H} \to \mathcal{L},\ (\tau, \sigma) \mapsto z(\tau, \sigma) = 2i\langle \ell', \ell\rangle\tau\ell + \sigma + \ell' \tag{4.3.1}$$

gives rise to an isomorphism $\mathcal{H} \cong \mathcal{K}$. It is also a nowhere vanishing section of the line bundle $\mathcal{L}$. Using this map, we can define action of $H(\mathbb{R})$ on $\mathcal{H}$ and automorphy factor $j(\gamma, \tau, \sigma)$ via the equation

$$\gamma z(\tau, \sigma) = j(\gamma, \tau, \sigma)z(\gamma(\tau, \sigma)). \tag{4.3.2}$$

Indeed, both $\gamma z(\tau, \sigma)$ and $z(\gamma(\tau, \sigma))$ are in $\mathcal{L}$ and they become the same in $\mathcal{K}$, so they are different by a multiplication constant, namely, the automorphy factor $j(\tau, \sigma)$.

Definition 4.3.1. *Let Γ be a unitary modular group. A holomorphic automorphic form of weight k and with character χ for Γ is a function $g : \mathcal{H} \to \mathbb{C}$, with the following properties:*

1. *g is holomorphic on $\mathcal{H}$,*

2. *$g(\gamma(\tau, \sigma)) = j(\gamma; \tau, \sigma)^k \chi(\gamma) g(\tau, \sigma)$ for all $\gamma \in \Gamma$.*

We remark that a holomorphic modular form g for Γ is automatically holomorphic at the cusps.

Now we make everything concrete and explicit. First choose a basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ of V with Gram matrix

$$J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

so $V = \bigoplus_{i=1}^3 \boldsymbol{k}\mathbf{e}_i \cong \boldsymbol{k}^3$ with Hermitian form

$$\langle x, y \rangle = x_1 \bar{y}_3 + x_2 \bar{y}_2 + x_3 \bar{y}_1 = {}^t x J \bar{y}, \tag{4.3.3}$$

and

$$H = H(\mathbb{Q}) = \{h \in \mathrm{GL}_3(\boldsymbol{k}) |\, hJ\, {}^t\bar{h} = J\}.$$

We take the lattice

$$L = \mathbb{Z}[i] \oplus \mathbb{Z}[i] \oplus \frac{1}{2}\mathbb{Z}[i]$$

(instead of the typical $\mathbb{Z}[i]^3$). Its $\mathbb{Z}$-dual lattice is

$$L' = \{v \in V |\, \mathrm{Tr}_{\boldsymbol{k}/\mathbb{Q}} \langle v, L \rangle \subset \mathbb{Z}\} = \mathbb{Z}[i] \oplus \frac{1}{2}\mathbb{Z}[i] \oplus \frac{1}{2}\mathbb{Z}[i]$$

So $L'/L \cong \frac{1}{2}\mathbb{Z}[i]/\mathbb{Z}[i]$ with quadratic form $Q(x) = x\bar{x} \in \frac{1}{4}\mathbb{Z}/\mathbb{Z}$, which is the same finite quadratic module considered in Part I. Let

$$\begin{aligned}\mathrm{U}(L) &= \{g \in H |\, gL = L\} \\ &= H \cap \left\{ \begin{pmatrix} \mathbb{Z}[i] & \mathbb{Z}[i] & 2\mathbb{Z}[i] \\ \mathbb{Z}[i] & \mathbb{Z}[i] & 2\mathbb{Z}[i] \\ \frac{1}{2}\mathbb{Z}[i] & \frac{1}{2}\mathbb{Z}[i] & \mathbb{Z}[i] \end{pmatrix} \right\}.\end{aligned}$$

be the stabilizer of L in H, and Γ_L be the subgroup of $\mathrm{U}(L)$ which acts on the discriminant group L'/L trivially:

$$\Gamma_L = \mathrm{U}(L) \cap \left\{ \begin{pmatrix} \mathbb{Z}[i] & 2\mathbb{Z}[i] & 2\mathbb{Z}[i] \\ \mathbb{Z}[i] & 1+2\mathbb{Z}[i] & 2\mathbb{Z}[i] \\ \mathbb{Z}[i] & 2\mathbb{Z}[i] & \mathbb{Z}[i] \end{pmatrix} \right\}. \tag{4.3.4}$$

Take the cusp $\ell = \mathbf{e}_1$ and $\ell' = \mathbf{e}_3$. Then $V_0 \cong \boldsymbol{k}$ with Hermitian form $\langle x, y\rangle = x\bar{y}$, and

$$\mathcal{H} = \{(\tau, \sigma) \in \mathbb{H} \times \mathbb{C} |\, 4\mathrm{Im}(\tau) > |\sigma|^2\}.$$

Moreover, one has for $\gamma = (a_{ij}) \in H$

$$\gamma(\tau, \sigma) = \left(\frac{a_{11}\tau + (2i)^{-1}a_{12}\sigma + (2i)^{-1}a_{13}}{2ia_{31}\tau + a_{32}\sigma + a_{33}}, \frac{2ia_{21}\tau + a_{22}\sigma + a_{23}}{2ia_{31}\tau + a_{32}\sigma + a_{33}} \right).$$

and

$$j(\gamma, \tau, \sigma) = \frac{\langle \gamma z, \ell\rangle}{\langle \ell', \ell\rangle} = 2i\tau a_{31} + a_{32}\sigma + a_{33}.$$

Our Picard modular surface is the quotient space $X_{\Gamma_L} = \Gamma_L \backslash \mathcal{H}$ of $\mathcal{H}$ modulo the action of Γ_L.

Let P_ℓ be the stabilizer of the cusp $\boldsymbol{k}\ell$ in H. Then $P_\ell = N_\ell M_\ell$ with

$$M_\ell = \{m(a, b) = \mathrm{Diag}(a, b, \bar{a}^{-1}) |\, a \in \boldsymbol{k}^\times, b \in \boldsymbol{k}^1\},$$

$$N_\ell = \left\{ n(b,c) = \begin{pmatrix} 1 & -2\bar{b} & -2b\bar{b}+2ic \\ 0 & 1 & 2b \\ 0 & 0 & 1 \end{pmatrix} \middle| \, b \in \boldsymbol{k},\ c \in \mathbb{Q} \right\},$$

where $\boldsymbol{k}^1 = \{a \in \boldsymbol{k} | a\bar{a} = 1\}$ is the norm one group. Notice that N_ℓ is a Heisenberg group action on $\mathcal{H}_{\ell,\ell'}$ via

$$n(b,c)(\tau,\sigma) = (\tau + c + i\bar{b}(\sigma + b), \sigma + b).$$

In particular

$$n(0,c)(\tau,\sigma) = (\tau + c, \sigma).$$

Let

$$\Gamma_{L,\ell} = \Gamma_L \cap N_\ell = \{n(b,c) | \, b \in \mathbb{Z}[i], c \in \mathbb{Z}\}.$$

Then for a holomorphic modular form $f(\tau,\sigma)$ for Γ_L, we have a Fourier–Jacobi expansion at the cusp $\boldsymbol{k}\ell$:

$$f(\tau,\sigma) = \sum_{n\geq 0} f_n(\sigma) q^n. \tag{4.3.5}$$

4.3.2 The Hermitian Space V as a Quadratic Space

As mentioned in the previous subsection, the hermitian space V can be viewed as a quadratic space $V_{\mathbb{Q}}$ of signature (4,2) associated with bilinear form induced from the hermitian form:

$$(x,y) = \mathrm{Tr}_{\boldsymbol{k}/\mathbb{Q}} \langle x, y \rangle.$$

Then the lattice L can be considered as a quadratic $\mathbb{Z}$-lattice in $V_{\mathbb{Q}}$. Denote by

$$\mathrm{SO}(V_{\mathbb{Q}}) = \{g \in \mathrm{SL}(V_{\mathbb{Q}}) | \, (gx, gy) = (x,y) \text{ for all } x, y \in V_{\mathbb{Q}}\}$$

the special orthogonal group of $V_{\mathbb{Q}}$ and its set of real points as $\mathrm{SO}(V_{\mathbb{Q}})(\mathbb{R}) \cong \mathrm{SO}(4,2)$. A model for the symmetric domain of $\mathrm{SO}(V_{\mathbb{Q}})(\mathbb{R})$ is the Grassmannian of two-dimensional negative definite subspaces of $V_{\mathbb{Q}}$, denoted by Gr_O. It can be realized as a tube domain $\mathcal{H}_O$ as follows. Denote by $V_{\mathbb{Q}}(\mathbb{C})$ the complex quadratic space $V_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C}$ with $(\cdot,\cdot)$ extended to a $\mathbb{C}$-valued bilinear form.

Now we view L as a $\mathbb{Z}$-lattice. Let $e_1 \in L$ be a primitive isotropic lattice vector and choose an isotropic dual vector $e_2 \in L'$ with $(e_1, e_2) = 1$. Denote by K the Lorentzian $\mathbb{Z}$-sublattice $K = L \cap e_1^{\perp} \cap e_2^{\perp}$ with respect to $(\cdot,\cdot)$. The tube domain model $\mathcal{H}_O$ is one of the the two connected components of the following subset of $K \otimes_{\mathbb{Z}} \mathbb{C}$

$$\{Z = X + iY \,|\, X, Y \in K \otimes_{\mathbb{Z}} \mathbb{R},\, Q(Y) < 0\}.$$

Recall that $\ell = \mathbf{e}_1$ and $\ell' = \mathbf{e}_3$. We define

$$e_1 = \ell,\, e_2 = \frac{\hat{1}}{2}\ell',\, e_3 = -\hat{i}\ell,\, e_4 = -\frac{\hat{i}}{2}\ell'$$

where we denote by $\hat{\mu}$ the endomorphism of $V_{\mathbb{Q}}(\mathbb{R})$ induced from the scalar multiplication with μ. Then we can check that $\{e_1, e_2, e_3, e_4\}$ is a basis for $(\mathbb{Z}[i]\ell + \mathbb{Z}[i]\ell') \otimes_{\mathbb{Z}} \mathbb{Q}$ and we can see that $K \otimes_{\mathbb{Z}} \mathbb{R} = ((\mathbb{Q}e_3 + \mathbb{Q}e_4) \otimes_{\mathbb{Z}} \mathbb{R}) \oplus (V_0 \otimes_{\mathbb{Z}} \mathbb{R})$. Thus we can identify Y with $y_1 e_3 + y_2 e_4 + \sigma \in K \otimes_{\mathbb{Z}} \mathbb{R}$. Now denote by $\mathcal{C}$ the set of $Y = y_1 e_3 + y_2 e_4 + \sigma$ with $y_1 y_2 + Q(\sigma) < 0$, $y_1 < 0$ and $y_2 > 0$. We can fix $\mathcal{H}_O$ as the component for which $Y \in \mathcal{C}$. Therefore, $\mathcal{H}_O = K \otimes_{\mathbb{Z}} \mathbb{R} + i\mathcal{C}$.

In addition, the tube domain $\mathcal{H}_O$ can be mapped biholomorphically to any one of the two connected components of a negative cone of $\mathbb{P}^1(V_{\mathbb{Q}})(\mathbb{C})$ given by

$$\{[Z_L] \in \mathbb{P}^1(V_{\mathbb{Q}})(\mathbb{C}) \,|\, (Z_L, Z_L) = 0,\, (Z_L, \bar{Z}_L) < 0\}.$$

We fix this component and denote it by $\mathcal{K}_O$. For each $[Z_L]$, we can uniquely represent it as

$$Z_L = e_2 - q(Z)e_1 + Z$$

with $Z \in \mathcal{H}_O$.

4.3.3 Embedding of $\mathcal{H}$ into $\mathcal{H}_O$

As in [50, Section 4], we can embed $\mathcal{H}$ into $\mathcal{H}_O$ via

$$(\tau, \sigma) \to \iota(\tau, \sigma) = -\tau e_3 + ie_4 + \mathfrak{z}(\sigma) \tag{4.3.6}$$

where

$$\mathfrak{z}(\sigma) = \frac{\hat{1}}{2}\sigma + i\left(-\frac{\hat{i}}{2}\right)\sigma. \tag{4.3.7}$$

Similarly, $\mathcal{K}_U$ can be embedded into $\mathcal{K}_O$ through the identifications between $\mathcal{K}_U$ and $\mathcal{H}$, and between $\mathcal{K}_O$ and $\mathcal{H}_O$. Namely,

$$z = \ell' + 2i\tau\ell + \sigma \to Z_L = -i\tau e_1 + e_2 - \tau e_3 + ie_4 + \mathfrak{z}(\sigma). \tag{4.3.8}$$

4.3.4 Weyl Chambers of $K \otimes_{\mathbb{Z}} \mathbb{R}$

In Theorem 4.2.1 (1), we have shown that $F_{1,m} = q^{-m} + O(1)$ for $m \geq 1$, form a canonical basis for $M^{!,\infty}_{-1}(\Gamma_0(4), \chi_{-4})$. Therefore, to study the Borcherds product lifted from $M^{!,\infty}_{-1,\rho_L}$, it suffices to start with $F_{1,m}$. Since we only deal with weight -1 in the rest of this chapter, we will simply write $F_m = F_{1,m}$, and $\vec{F}_m = \vec{F}_{1,m}$

For general definitions of the following, we refer the reader to [25, Chapter 3.1]. For $\kappa \in K$ with $q(\kappa) > 0$, denote by $\kappa^\perp$ the orthogonal complement of κ in $K \otimes_{\mathbb{Z}} \mathbb{R}$. Denote by $\mathcal{D}_K$ the Grassmannian of negative 1-lines of $K \otimes_{\mathbb{Z}} \mathbb{R}$, which can be realized as

$$\begin{aligned}\mathcal{D}_K &= \{\mathbb{R}w \subset K_{\mathbb{R}} |\, q(w) < 0\} \\ &\cong \{w = y_1 e_3 + e_4 + (y_3 + iy_4) |\, y_i \in \mathbb{R}, q(w) < 0\}.\end{aligned}$$

Then by considering the Grassmannian of negative 1-lines of $\kappa^{\perp}$, it corresponds to a codimension 1 sub-manifold of the Grassmannian $\mathcal{D}_K$ of $K \otimes_{\mathbb{Z}} \mathbb{R}$.

In our case, a Heegner divisor of index $(m, 0)$, $H_K(m, 0)$, is a locally finite union of codimension 1 sub-manifolds of $\mathcal{D}_K$, namely,

$$H_K(m, 0) = \{z \in \mathcal{D}_K |\, \exists \kappa \in K \text{ with } q(\kappa) = m \text{ and } (z, \kappa) = 0\}$$

Let $\vec{F}_m(\tau)$ be the vector-valued modular form arising from F_m. It is known by Theorem 4.2.5 that the principal part of $\vec{F}_m(\tau)$ is $q^{-m}\phi_0$. The Weyl chambers attached to $\vec{F}_m(\tau)$ are the connected components W_m of

$$\mathcal{D}_K - H_K(m, 0).$$

Fix a Weyl chamber W_m of $\mathcal{D}_K$, we can also define the corresponding Weyl chambers of $K \otimes_{\mathbb{Z}} \mathbb{R}$ and $\mathcal{H}$ by

$$\begin{aligned}W_{m,K} &= \{w \in K \otimes_{\mathbb{Z}} \mathbb{R} |\, \mathbb{R}w \in W_m\}, \\ W_{m,U} &= \left\{(\tau, \sigma) \in \mathcal{H} \,\middle|\, \Im(\iota(\tau, \sigma)) = -\Im\tau e_3 + e_4 - \frac{\hat{i}}{2}\sigma \in W_{m,K}\right\},\end{aligned}$$

respectively. In the following lemma, we give an explicit description of the Weyl chamber that we use to construct Borcherds product in Theorem 4.3.5.

Lemma 4.3.2. *(1) Let*

$$W_m = \left\{ y_1 e_3 + e_4 + (y_3 + iy_4) \in \mathcal{D}_K \,\middle|\, \begin{array}{c} y_1 < r^2 + s^2 - m + 2ry_3 + 2sy_4 \ \forall\, r, s \in \mathbb{Z}, \\ 1 + 2ty_3 + 2hy_4 > 0 \ \forall\, t, h \in \mathbb{Z}, t^2 + h^2 = m, \\ ty_3 + hy_4 > 0, \ \forall\, t, h \in \mathbb{Z}, t^2 + h^2 = m, t > 0, \\ y_4 > 0 \textit{ if } m \textit{ is a square.} \end{array} \right\} \tag{4.3.9}$$

$$\subset \left\{ y_1e_3 + e_4 + (y_3 + iy_4) \in \mathcal{D}_K \,\middle|\, \begin{array}{c} k_2y_1 < -k_1 + 2k_3y_3 + 2k_4y_4 \ \forall\, k_i \in \mathbb{Z},\, k_2 > 0,\, k_1k_2 + k_3^2 + k_4^2 = m, \\ k + 2ty_3 + 2hy_4 > 0 \ \forall\, k, t, h \in \mathbb{Z},\, k > 0,\, t^2 + h^2 = m, \\ ty_3 + hy_4 > 0,\ \forall\, t, h \in \mathbb{Z},\, t^2 + h^2 = m,\, t > 0, \\ y_4 > 0 \text{ if } m \text{ is a square} \end{array} \right\}.$$

Then W_m is a Weyl chamber containing e_3.

(2) Let

$$K_m = \left\{ \lambda = \lambda_1e_3 - \lambda_2e_4 + \frac{1}{2}(\lambda_3 + i\lambda_4) \in K' \,\middle|\, \begin{array}{c} (\lambda, W_m) > 0, \\ (Q(\lambda) = m \text{ with } \lambda_3, \lambda_4 \in 2\mathbb{Z}) \\ \text{or } (Q(\lambda) \le 0) \end{array} \right\}$$

where $(\lambda, W_m) > 0$ means that $(\lambda, w) > 0$ for all $w \in W_m$. Then

$$K_m = \left\{ \lambda = \lambda_1e_3 - \lambda_2e_4 + \frac{1}{2}(\lambda_3 + i\lambda_4) \,\middle|\, \begin{array}{c} \lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{Z}, \\ \lambda_2 > 0, \\ \text{or } (\lambda_2 = 0 \text{ and } \lambda_1 > 0), \\ \text{or } (\lambda_2 = \lambda_1 = 0 \text{ and } \lambda_3 > 0), \\ \text{or } (\lambda_2 = \lambda_1 = \lambda_3 = 0 \text{ and } \lambda_4 > 0) \end{array} \right\}.$$

Proof. For Assertion (1), it is clear that W_m contains e_3 since the set of (y_3, y_4, y_1) determined by the inequalities in W_m contains $y_1 = -\infty$. We only need to show W_m is actually a Weyl chamber.

Write $\kappa = k_1e_3 + k_2e_4 + k_3 + ik_4 \in K$ with $k_i \in \mathbb{Z}$. Since $(-\kappa)^\perp = \kappa^\perp$, we can assume $k_2 \ge 0$. By the definition of Weyl chamber W_m, we can see that a Weyl chamber W_m can be viewed as a connected component of $\mathbb{R}^3$ cut out by the planes

$$k_2y_1 + k_1 + 2k_3y_3 + 2k_4y_4 = 0$$

for all $k_1, \ldots, k_4 \in \mathbb{Z}$ with $k_2 \ge 0$ and $k_1k_2 + k_3^2 + k_4^2 = m$.

When $k_2 = 0$ and m is representable as a sum of two squares, then we have planes

$$k_1 + 2k_3y_3 + 2k_4y_4 = 0$$

perpendicularly passing through the (y_3, y_4)-plane. In this case, the connected components are determined by the connected components of the $y_3 - y_4$ plane cut out by the lines

$$k_1 + 2k_3y_3 + 2k_4y_4 = 0,$$

and it is easy to find that one of the connected components $\mathcal{C}_1$ can be identified as

$$\left\{ (y_3, y_4) \in \mathbb{R}^2 \,\middle|\, \begin{smallmatrix} 1+2ty_3+2hy_4>0 \ \forall\, t,h\in\mathbb{Z},\, t^2+h^2=m, \\ ty_3+hy_4>0,\ \forall\, t,h\in\mathbb{Z},\, t^2+h^2=m,\, t>0, \\ y_4>0 \text{ if } m \text{ is a square} \end{smallmatrix} \right\}$$

which is a subset of

$$\left\{ (y_3, y_4) \in \mathbb{R}^2 \,\middle|\, \begin{smallmatrix} k+2ty_3+2hy_4>0 \ \forall\, k,t,h\in\mathbb{Z},\, k>0,\, t^2+h^2=m, \\ ty_3+hy_4>0,\ \forall\, t,h\in\mathbb{Z},\, t^2+h^2=m,\, t>0, \\ y_4>0 \text{ if } m \text{ is a square} \end{smallmatrix} \right\}.$$

When $k_2 > 0$, with the aid of MAPLE, we can check that there is a connected component $\mathcal{C}_2$ of $\mathbb{R}^3$ covered by

$$y_1 = r^2 + s^2 - m + 2ry_3 + 2sy_4$$

for r, $s \in \mathbb{Z}$. Such a connected component contains $y_1 < -m$, and all the other planes

$$k_2y_1 = -k_1 + 2k_3y_3 + 2k_4y_4$$

for $k_1, \ldots, k_4 \in \mathbb{Z}$ with $k_2 > 0$ and $k_1k_2 + k_3^2 + k_4^2 = m$. In conclusion, $W_m = \mathcal{C}_1 \cap \mathcal{C}_2$ is a connected component of $\mathbb{R}^3$ cut out by the planes

$$k_2y_1 + k_1 + 2k_3y_3 + 2k_4y_4 = 0$$

for all $k_1, \ldots, k_4 \in \mathbb{Z}$ with $k_2 \geq 0$ and $k_1k_2 + k_3^2 + k_4^2 = m$, and thus W_m is a Weyl chamber. For the case $m = 1$, we can visualize it by a 3D-plot. See Figure 4.3.4.

Now let us prove Assertion (2).

(i) Suppose that $Q(\lambda) = m$ and λ_3, $\lambda_4 \in 2\mathbb{Z}$ which imply that $\lambda \in K$. By (4.3.9), we note that $y_1e_3 + e_4 + (y_3 + iy_4) \in W_m$ implies that

$$k_2y_1 < -k_1 + 2k_3y_3 + 2k_4y_4$$

for all $k_i \in \mathbb{Z}$ with $k_2 > 0$ and $k_1k_2 + k_3^2 + k_4^2 = m$, which is equivalent to

$$k_2y_1 + k_1 + 2k_3y_3 + 2k_4y_4 > 0$$

for all $k_i \in \mathbb{Z}$ with $k_2 < 0$ and $k_1k_2 + k_3^2 + k_4^2 = m$. Therefore, when $\lambda_2 \neq 0$ and $Q(\lambda) = m$, that is, $\lambda_1(-\lambda_2) + \frac{1}{4}(\lambda_3^2 + \lambda_4^2) = m$, $(\lambda, W_m) > 0$ if and only if $-\lambda_2 < 0$, that is, $\lambda_2 > 0$. Similarly, by the other conditions given in (4.3.9), we can conclude that when $Q(\lambda) = m$, $(\lambda, W_m) > 0$ if and only if $\lambda_2 < 0$, or ($\lambda_2 = 0$ and $\lambda_1 > 0$), or ($\lambda_2 = \lambda_1 = 0$ and $\lambda_3 > 0$), or ($\lambda_2 = \lambda_1 = \lambda_3 = 0$ and $\lambda_4 > 0$).

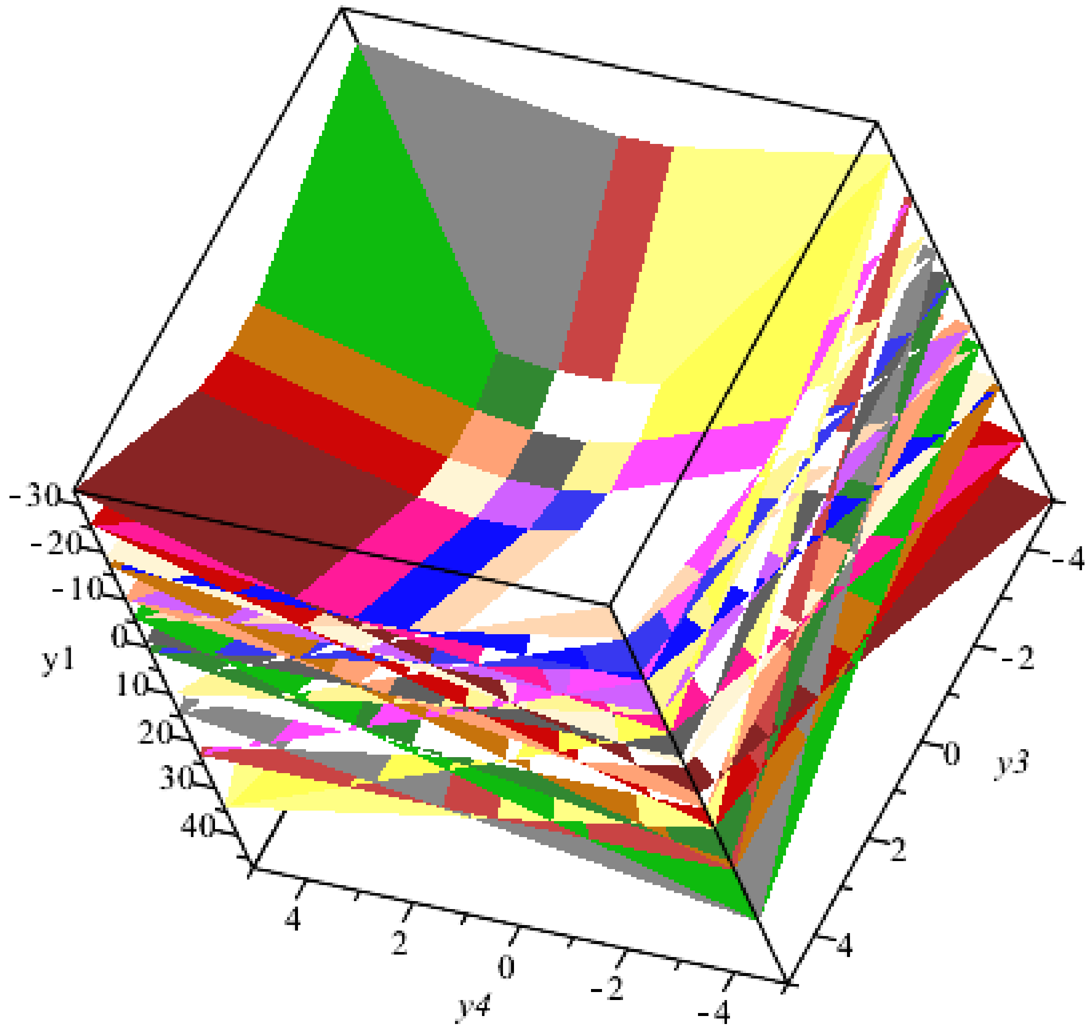
(ii) Now suppose that $Q(\lambda) \leq 0$, that is, $\lambda_1\lambda_2 + \frac{1}{4}(\lambda_3^2 + \lambda_4^2) \leq 0$. By (4.3.9), we know that

$$y_1 < r^2 + s^2 - m + 2ry_3 + 2sy_4$$

for all r, $s \in \mathbb{Z}$. By [25, Lemma 3.2], it is known that if $(\lambda, w_0) > 0$ for a $w_0 \in W_m$, then $(\lambda, W_m) > 0$. Thus $(\lambda, W_m) > 0$ if and only if $\lambda_2 > 0$. When $\lambda_2 = 0$, since $Q(\lambda) \leq 0$, then $\lambda_3 = \lambda_4 = 0$, and thus $(\lambda, w) = \lambda_1$ for $w \in W_m$. This implies that $(\lambda, W_m) > 0$ if and only if $\lambda_1 > 0$ when $\lambda_2 = 0$.

□

A 3D-plot for the Weyl chamber W_1



4.3.5 The Weyl Vector for $\vec{F}_m$

In this subsection, we aim to compute the Weyl vector $\rho(W_m, \vec{F}_m)$. We first recall a nice summary of the explicit computations of Weyl vector given in [94, Subsection 2.1] (also see [21, Thm. 10.4] for original definitions).

Let L be a $\mathbb{Z}$-lattice with quadratic form $Q(\cdot)$ of a quadratic space V of type $(n, 2)$ and L' be its dual lattice. Take $\ell_L \in L$ and $\ell'_L \in L'$ to be such that $Q(\ell_L) = Q(\ell'_L) = 0$

and $(\ell_L, \ell'_L) = 1$. Assume that $(\ell_L, L) = N_L\mathbb{Z}$ for some unique positive integer and choose $\xi \in L$ with $(\ell_L, \xi) = N_L$. Let $K = L \cap (\mathbb{Q}\ell_L + \mathbb{Q}\ell'_L)^\perp$ and let

$$L'_0 = \{x \in L' | \, (\ell_L, x) \equiv 0 \pmod{N_L}\} \subset L'.$$

Then there is a projection

$$p : L'_0 \to K', \quad p(x) = x_K + \frac{(x, \ell_L)}{N_L}\xi_K,$$

where x_K and ξ_K are the orthogonal projections of $x, \xi \in V$ to $K_\mathbb{Q} = K \otimes_\mathbb{Z} \mathbb{Q}$. So it induces a projection from L'_0/L to K'/K. Next, for

$$\vec{f} = \sum f_\mu \phi_\mu = \sum c(m, \phi_\mu) q^m \phi_\mu \in M^!_{1-\frac{n}{2}, \rho_L},$$

define

$$\vec{f}_K = \sum_{\lambda \in K'/K} f_\lambda \phi_{\lambda,K} = \sum c_K(m, \lambda) q^m \phi_{\lambda,K},$$

where $\phi_{\lambda,K}$ is the basis element associated to λ of $\mathbb{C}[K'/K]$, and

$$f_\lambda = \sum_{\substack{\mu \in L'_0/L \\ p(\mu)=\lambda}} f_\mu.$$

For a Weyl chamber W, take $\ell_K \in K \cap \overline{W}$, where $\overline{W}$ denotes the closure of the Weyl chamber W, and $\ell'_K \in K'$ with $Q(\ell_K) = Q(\ell'_K) = 0$ and $(\ell_K, \ell'_K) = 1$, and let $P = K \cap (\mathbb{Q}\ell_K + \mathbb{Q}\ell'_K)^\perp$, which is positive definite of rank $n-2$. Similar to the projection p from L'_0/L to K'/K, one also has a projection, also denoted by p, from K'_0/K to P'/P defined in the same way. Similarly, we have a weakly holomorphic modular form $\vec{f}_P$ induced by $\vec{f}_K$. Then we can compute and express the Weyl vector $\rho(W, \vec{f})$ associated to W and $\vec{f}$ as

$$\rho(W, \vec{f}) = \rho_{\ell_K}\ell_K + \rho_{\ell'_K}\ell'_K + \rho_P,$$

where

$$\rho_{\ell_K} = -\frac{1}{4}\sum_{\substack{\lambda\in K_0'/K\\ p(\lambda)=0+P}} c_K(0,\lambda)B_2((\lambda,\ell_K')) \\ -\frac{1}{2}\sum_{\substack{\gamma\in P'\\ (\gamma,W)>0}}\sum_{\substack{\lambda\in K_0'/K\\ p(\lambda)=\gamma+P}} c_K(-Q(\gamma),\lambda)B_2((\lambda,\ell_K')), \tag{4.3.10}$$

$B_2(x) := \{x\}^2 - \{x\} + \frac{1}{6}$ is the second Bernoulli polynomial, $\{x\}$ is the fractional part of x,

$$\rho_{\ell_K'} = \text{constant term of } \langle \vec{\theta}_P, \vec{f}_P\rangle E_2/24, \tag{4.3.11}$$

$$\vec{\theta}_P := \sum_{\gamma\in P'/P}\sum_{\lambda\in\gamma+P} e(Q(\lambda)\tau)\phi_\gamma,$$

$E_2 := 1 - 24\sum_{n=1}^{\infty}\sigma_1(n)q^n$ is the holomorphic Eisenstein series of weight 2,

$$\rho_P = -\frac{1}{2}\sum_{\substack{\gamma\in P'\cap K'\\ (\gamma,W)>0}} c_K(-Q(\gamma),\gamma)\gamma. \tag{4.3.12}$$

Now for our case, we set $L = \mathbb{Z}[i]\oplus\mathbb{Z}[i]\oplus\frac{1}{2}\mathbb{Z}[i]$, $\ell_L = e_1$, $\ell_L' = e_2$, $\vec{f} = \vec{F}_m$, $W = W_m$, $\ell_K = e_3$ and $\ell_K' = e_4$, where e_1, e_2, e_3 and e_4 are defined as in Subsection 4.3.2. It is easy to check that $K = \mathbb{Z}e_3\oplus\mathbb{Z}e_4\oplus P$ and $P = \mathbb{Z}[i]\mathbf{e}_2$ where $\mathbf{e}_2$ is defined as in Subsection 4.3.1. Direct calculations show that $L_0' = L'$, $K_0' = K'$ and $L'/L \cong K'/K \cong P'/P$. Write

$$\vec{F}_m = \sum_{\mu\in L'/L} F_{m,\mu}\phi_\mu = \sum_{\mu\in L'/L}\sum_{n\in\mathbb{Q}} c(n,\mu)q^n\phi_\mu.$$

Then under $L'/L \cong K'/K \cong P'/P$, direct calculations show that

$$\vec{F}_{m,K} = \vec{F}_{m,P} = \vec{F}_m.$$

Note that by Theorem 4.2.5, for $n > 0$, $c_K(-n,\lambda) \neq 0$ if and only if $c_K(-n,\lambda) = c_K(-m,0)$, which equals 1. Also, we can compute and express $\vec{\theta}_P$ as

$$\vec{\theta}_P = \left(\sum_{r,s\in\mathbb{Z}} e\left(\left(r^2+s^2\right)\tau\right)\right)\phi_0 + \left(\sum_{r,s\in\mathbb{Z}} e\left(\left(\frac{1}{4}+r+r^2+s^2\right)\tau\right)\right)\phi_1 \qquad (4.3.13)$$
$$+\left(\sum_{r,s\in\mathbb{Z}} e\left(\left(r^2+s^2+s+\frac{1}{4}\right)\tau\right)\right)\phi_i + \left(\sum_{r,s\in\mathbb{Z}} e\left(\left(\frac{1}{2}+r+s+r^2+s^2\right)\tau\right)\right)\phi_{1+i}.$$

Now let us first compute ρ_{e_3}. Since $K_0'/K = K'/K \cong P'/P$, then $\lambda \in K_0'/K$ such that $p(\lambda) = 0 + P$ if and only if $\lambda = 0 + K$. In addition, as we point out above that for $n > 0$, $c_K(-n,\lambda) \neq 0$ if and only if $c_K(-n,\lambda) = c_K(-m,0) = 1$, then we can see that $c_K(-Q(\gamma),\lambda) = 1$ if and only if $\gamma \in P$ with $Q(\gamma) = m$, i.e., $\gamma = (r+si)\mathbf{e}_2$ with $r,s \in \mathbb{Z}$ and $r^2+s^2 = m$. In addition, by Lemma 4.3.9, to have $(\gamma, W_m) > 0$, we must have ($r > 0$) or ($r = 0$ and $s > 0$). Now by the definition of ρ_{e_3} and the above analysis, we have

$$\begin{aligned}\rho_{e_3} &= -\frac{1}{24}c_K(0,0) - \frac{1}{2}\sum_{\substack{r^2+s^2=m\\ r>0\\ \text{or } (r=0 \text{ and } s>0)}} \frac{1}{6}\\ &= -\frac{1}{6}\sum_{d|m}\left(16\chi_{-4}(m/d)+\chi_{-4}(d)\right)d^2 - \frac{1}{6}\sigma_{\chi_{-4}}(m),\end{aligned}$$

where $\sigma_{\chi_{-4}}(m) = \sum_{d|m}\chi_{-4}(d)$ follows from the well known fact (see, e.g., [16, Thm. 3.2.1]) that the number of integral solutions of $r^2+s^2 = m$ is given by $4\sigma_{\chi_{-4}}(m)$.

For the e_4-component ρ_{e_4}, we first note that the non-ϕ_0-component functions of θ_P have no constant terms, and the non-ϕ_0-component functions of $\vec{F}_P$ have no negative power terms. In addition, the ϕ_0-component function of θ_P is

$$\sum_{r,s\in\mathbb{Z}} e((r^2+s^2)\tau) = 1 + 4\sum_{n=1}^{\infty}\sigma_{\chi_{-4}}(n)q^n,$$

and the ϕ_0-component function of $\vec{F}_P$ is $q^{-m} + c(0,0) + O(q)$. Therefore, the constant term of $\langle \vec{\theta}_P, \vec{f}_P \rangle E_2$ is the constant term of

$$\left(1 + 4\sum_{n=1}^{\infty} \sigma_{\chi_{-4}}(n) q^n\right) \left(q^{-m} + c(0,0) + O(q)\right) \left(1 - 24\sum_{n=1}^{\infty} \sigma_1(n) q^n\right),$$

which is

$$4\sigma_{\chi_{-4}}(m) - 24\sigma_1(m) - 96\left(\sum_{\substack{k+l=m\\ k,l\geq 1}} \sigma_{\chi_{-4}}(k)\sigma_1(l)\right) + c(0,0).$$

Thus by Theorem 4.2.5, we have

$$\rho_{e_4} = \frac{1}{6}\left[\sigma_{\chi_{-4}}(m) - 6\sigma_1(m) - 24\left(\sum_{\substack{k+l=m\\ k,l\geq 1}} \sigma_{\chi_{-4}}(k)\sigma_1(l)\right)\right.$$
$$\left. + \sum_{d|m} \left(16\chi_{-4}(m/d) + \chi_{-4}(d)\right) d^2\right].$$

For ρ, notice that for $n > 0$, $c_K(-n, \lambda) \neq 0$ if and only if $\gamma \in P$ with $Q(\gamma) = m$, i.e., $\gamma = (r + si)\mathbf{e}_2$ with $r, s \in \mathbb{Z}$ and $r^2 + s^2 = m$. Notice also that $(\gamma, W_m) > 0$ implies $(r > 0)$ or $(r = 0$ and $s > 0)$. So by similar calculations, we have

$$\rho_P = -\frac{1}{2} \sum_{\substack{r^2+s^2=m\\ r>0\\ \text{or } (r=0 \text{ and } s>0)}} (r + si)\mathbf{e}_2.$$

Summing up, we conclude with the following proposition.

Proposition 4.3.3. *Let $\vec{F}_m$ be defined as in Subsection* 4.3.4*, and let W_m be the Weyl chamber given by* (4.3.9)*. Then the Weyl vector associated to the Weyl chamber W_m and the weakly holomorphic modular form $\vec{F}_m$ is*

$$\rho(W_m, \vec{F}_m) = \rho_{e_3} e_3 + \rho_{e_4} e_4 + \rho_P,$$

where

$$\rho_{e_3} = -\frac{1}{6}\sum_{d|m}\left(16\chi_{-4}(m/d)+\chi_{-4}(d)\right)d^2 - \frac{1}{24}\sigma_{\chi_{-4}}(m),$$

$$\rho_{e_4} = \frac{1}{6}\Bigg[\sigma_{\chi_{-4}}(m) - 6\sigma_1(m) - 24\left(\sum_{\substack{k+l=m\\k,l\geq 1}}\sigma_{\chi_{-4}}(k)\sigma_1(l)\right) + \sum_{d|m}\left(16\chi_{-4}(m/d)+\chi_{-4}(d)\right)d^2\Bigg],$$

$$\rho_P = -\frac{1}{2}\sum_{\substack{r^2+s^2=m\\ r>0\\ or\ (r=0\ and\ s>0)}}(r+si)\mathbf{e}_2,$$

and $\sigma_{\chi_{-4}}(m) = \sum_{d|m}\chi_{-4}(d)$.

4.3.6 Heegner Divisors for Γ_L

Let $\lambda \in L'$ be a lattice vector with positive norm, i.e., $\langle\lambda,\lambda\rangle > 0$. The orthogonal complement of λ in $\mathcal{K}_U$ is a closed analytic subset of comdimension 1, which we denote as follows.

$$\mathbf{H}(\lambda) = \{[z]\in\mathcal{K}_U|\,\langle z,\lambda\rangle = 0\}.$$

By identification between $\mathcal{K}_U$ and $\mathcal{H}$, $\mathbf{H}(\lambda)$ can also be considered as a closed analytic subset of $\mathcal{H}$, and we call such set a prime Heegner divisor on $\mathcal{H}$. Given $\beta \in L'/L$ and $m \in \mathbb{Z}_{>0}$, a Heegner divisor of index (m,β) in $\mathcal{H}$ is defined as the locally finite sum

$$\mathbf{H}(m,\beta) = \sum_{\substack{\lambda\in\beta+L\\Q(\lambda)=m}}\mathbf{H}(\lambda).$$

The associated Heegner divisor in $X_{\Gamma_L} = \Gamma_L\backslash\mathcal{H}$ is $\mathbf{Z}(m,\beta) = \Gamma_L\backslash\mathbf{H}(m,\beta)$.

4.3.7 Borcherds Products

In this section, we give a family of new Borcherds products explicitly by using the results of Hofmann [50, Thm. 4, Thm. 5 and Cor. 1]. We first summarize Hofmann's results as follows.

Theorem 4.3.4 (Hofmann). *Let $\mathbb{F}$ be an imaginary quadratic field. Let L be an even hermitian lattice of signature $(m, 1)$ with $m \geq 1$, and $\ell \in L$ a primitive isotropic vector. Let $\ell' \in L'$ an isotropic vector with $\langle \ell, \ell' \rangle \neq 0$. Further assume that L is the direct sum of a hyperbolic plane $H \cong \mathcal{O}_{\mathbb{F}} \oplus \partial_{\mathbb{F}}^{-1}$ and a definite part D with $\langle D, H \rangle = 0$.*

Given a weakly holomorphic modular form $f \in M^!_{1-m,\rho_L}$ with Fourier coefficients $c(n, \beta)$ satisfying $c(n, \beta) \in \mathbb{Z}$ for $n < 0$, there is a meromorphic function $\Psi(\tau, \sigma; f)$ on $\mathcal{H}$ with the following properties:

1. *$\Psi(\tau, \sigma; f)$ is an automorphic form of weight $c(0, \phi_0)/2$ for Γ_L with some multiplier system χ of finite order.*

2. *The zeros and poles of $\Psi(\tau, \sigma; f)$ lie on Heegner divisors. The divisor of $\Psi(\tau, \sigma; f)$ on $X_{\Gamma_L} = \Gamma_L \backslash \mathcal{H}$ is given by*

$$div(\Psi(\tau, \sigma; f)) = \frac{1}{2} \sum_{\beta \in L'/L} \sum_{\substack{n \in \mathbb{Z} - Q(\beta) \\ n > 0}} c(-n, \phi_\beta) \mathbf{H}(n, \beta).$$

   The multiplicities of $\mathbf{H}(n, \beta)$ are 2 if $2\beta = 0$ in L'/L, and 1 otherwise.

3. *For a Weyl chamber W whose closure contains the cusp $\mathbb{Q}e_3$, $\Psi(\tau, \sigma; f)$ has an infinite product expansion of the form*

$$\Psi(\tau, \sigma; f) = Ce\left(\frac{\langle z, \rho(W, f) \rangle}{\langle \ell, \ell' \rangle}\right) \prod_{\substack{\lambda \in K' \\ (\lambda, W) > 0}} \left[1 - e\left(\frac{\langle z, \lambda \rangle}{\langle \ell, \ell' \rangle}\right)\right]^{c(-Q(\lambda), \lambda)},$$

where $z = z(\tau,\sigma) = \ell' + \delta\langle \ell, \ell'\rangle \tau\ell + \sigma$, δ *is the square root of the discriminant of* $\mathbb{F}$, *the constant* C *has absolute value* 1 *and* $\rho(W,f)$ *is the Weyl vector attached to* W *and* f.

4. *The lifting is multiplicative:* $\Psi(\tau,\sigma; f+g) = \Psi(\tau,\sigma;f)\Psi(\tau,\sigma;g)$.

5. *Let* W *be a Weyl chamber such that the cusp corresponding to* ℓ *is contained in the closure of* W. *If this cusp is neither a pole nor a zero of* $\Psi(\tau,\sigma;f)$, *then we have*

$$\lim_{\tau\to\infty} \Psi(\tau,\sigma;f) = Ce\left(\overline{\rho(W,f)_\ell}\right) \prod_{\substack{\lambda\in K' \\ \lambda=\frac{1}{2}\kappa\delta\ell \\ \kappa\in\mathbb{Q}_{>0}}} \left(1 - e\left(-\frac{1}{2}\kappa\bar{\delta}\right)\right)^{c(0,\lambda)}$$

where $\overline{\rho(W,f)_\ell}$ *denotes the complex conjugate of the* ℓ*-component of the Weyl vector* $\rho(W,f)$.

By specializing Theorem 4.3.4 in our case, we obtain the main result of this chapter.

Theorem 4.3.5. *Let* $L = \mathbb{Z}[i] \oplus \mathbb{Z}[i] \oplus \frac{1}{2}\mathbb{Z}[i]$ *with respect to the standard basis over* $\mathbb{Z}[i]$ *with hermitian form defined in* (4.3.3). *We set* $\ell = (1,0,0)$ *and* $\ell' = (0,0,1)$. *Let* $\vec{F}_m$ *be the vector-valued modular form arising from* $F_m = \theta_2\theta_1^{-1}P_{1,m-1}(\varphi_\infty)$ *and denote by* $c(n,\phi_\mu)$ *the Fourier coefficient of index* (n,ϕ_μ) *of* $\vec{F}_m$. *Then there is a meromorphic function* $\Psi(\tau,\sigma;F_m) = \Psi(\tau,\sigma;\vec{F}_m)$ *on* $\mathcal{H}$ *with the following properties:*

1. $\Psi(\tau,\sigma;\vec{F}_m)$ *is an automorphic form of weight*

$$32\sum_{d|m}\chi_{-4}(n/d)d^2 + 2\sum_{d|m}\chi_{-4}(d)d^2$$

for Γ_L, *with some multiplier system* χ *of finite order.*

2. *The zeros and poles of* $\Psi(\tau,\sigma;\vec{F}_m)$ *lie on Heegner divisors. The divisor of* $\Psi(\tau,\sigma;\vec{F}_m)$ *on* $X_{\Gamma_L}=\Gamma_L\backslash\mathcal{H}$ *is given by*

$$div(\Psi(\tau,\sigma;\vec{F}_m))=\mathbf{Z}(m,0)=\Gamma_L\backslash\mathbf{H}(m,0),$$

where

$$\mathbf{H}(m,0)=\sum_{\substack{(r_1,s_1,r_2,s_2,r_3,s_3)\in\mathbb{Z}^6\\ r_1r_3+s_1s_3+r_2^2+s_2^2=m}}\left\{(\tau,\sigma)\in\mathcal{H}\,\middle|\,\begin{smallmatrix} r_1+2r_2\Re\sigma+2s_2\Im\sigma+s_3\Re\tau-r_3\Im\tau=0,\\ s_1+2r_2\Im\sigma-2s_2\Re\sigma+s_3\Im\tau+r_3\Re\tau=0\end{smallmatrix}\right\}.$$

3. *For the Weyl chamber* W_m *described in* (4.3.9), $\Psi(\tau,\sigma;\vec{F}_m)$ *has an infinite product expansion near the cusp* $\mathbb{Q}e_3$ *(precisely, when* $(\tau,\sigma)\in W_{m,U}$ *with* $\Im\tau$ *sufficiently large):*

$$\Psi(\tau,\sigma;F_m)=A_1(\tau,\sigma)A_2(\sigma)A_3(\sigma)A_4(\sigma)A_5(\tau,\sigma), \tag{4.3.14}$$

where

(i)

$$A_1(\tau,\sigma)=e(i\rho_{e_3}-\rho_{e_4}\tau+\bar{\rho}\sigma)$$

where ρ_{e_3}, ρ_{e_4} *and* ρ *are defined as in Proposition 4.3.3,*

(ii)

$$A_2(\sigma)=\begin{cases}\left[1-e\left(-i\sigma\sqrt{m}\right)\right] & \text{if } m \text{ is a square,}\\ 1 & \text{otherwise,}\end{cases}$$

(iii)

$$A_3(\sigma)=\prod_{\substack{(k_3,k_4)\in\mathbb{Z}^2_{>0}\\ k_3^2+k_4^2=m}}\left[1-e\left(\sigma\left(k_3+ik_4\right)\right)\right]\left[1-e\left(\sigma\left(k_3-ik_4\right)\right)\right],$$

(iv)

$$A_4(\sigma) = \prod_{\substack{n_3,n_4\in\mathbb{Z}\\ n_3^2+n_4^2=m}} \prod_{n_2\in\mathbb{Z}_{>0}} \left[1-e(in_2)e\left(\sigma\left(n_3-in_4\right)\right)\right]$$
$$\times \prod_{n_2\in\mathbb{Z}_{>0}} \left(1-e(in_2)\right)^{c(0,0)}$$

with

$$c(0,0) = c(0,\phi_0) = \sum_{d|m} \left(64\chi_{-4}(m/d) + 4\chi_{-4}(d)\right) d^2,$$

(v)

$$A_5(\tau,\sigma) = \prod_{\substack{(n_1,n_2,n_3,n_4)\in\mathbb{Z}^4\\ n_1>0}} \left[1-e\left(n_1\tau+\sigma\left(\frac{n_3}{2}-i\frac{n_4}{2}\right)+in_2\right)\right]^{c(n_1n_2-\frac{1}{4}(n_3^2+n_4^2),\phi_{\vec{n}})}$$

with $\vec{n} = n_2e_3 - n_1e_4 + \frac{1}{2}(n_3+in_4)$.

4. *If the cusp corresponding to* ℓ *is neither a pole nor a zero of* $\Psi(\tau,\sigma;\vec{F}_m)$*, then we have*

$$\lim_{\tau\to i\infty} \Psi(\tau,\sigma;\vec{F}_m) = e(i\rho_{e_3}) \prod_{k=1}^{\infty} \left(1-e(ki)\right)^{c(0,\phi_0)}$$

where

$$\rho_{e_3} = -\frac{1}{6}\sum_{d|m} \left(16\chi_{-4}(m/d) + \chi_{-4}(d)\right) d^2 - \frac{1}{24}\sigma_{\chi_4}(m)$$

is defined as in Proposition 4.3.3, and

$$c(0,\phi_0) = \sum_{d|m} \left(64\chi_{-4}(m/d) + 4\chi_{-4}(d)\right) d^2$$

is defined as in Theorem 4.2.5.

Proof. Assertion (1) follows from Theorem 4.2.5 and Theorem 4.3.4 (1). Assertion (2) follows directly from Theorem 4.3.4 (2).

Then by Theorem 4.3.4 (3) together with Lemma 4.3.2 and Proposition 4.3.3, we have that $\Psi(\tau, \sigma; \vec{F}_m)$ has the following infinite product expansion near the cusp $\mathbb{Q}e_3$

$$\begin{aligned}
&\Psi(\tau, \sigma; \vec{F}_m) \\
&= e(i\rho_{e_3} - \rho_{e_4}\tau + \bar{\rho}\sigma) \\
&\times \prod_{\substack{(\lambda_1,\lambda_2,\lambda_3,\lambda_4)\in\mathbb{Z}^4 \\ \lambda_2>0, \\ or\ \lambda_2=0\ and\ \lambda_1>0, \\ or\ \lambda_2=\lambda_1=0\ and\ \lambda_3>0, \\ or\ \lambda_2=\lambda_1=\lambda_3=0\ and\ \lambda_4>0.}} \left[1 - e\left(\lambda_2\tau + \sigma\left(\frac{\lambda_3}{2} - i\frac{\lambda_4}{2}\right) + i\lambda_1\right)\right]^{c\left(\lambda_1\lambda_2 - \frac{1}{4}(\lambda_3^2+\lambda_4^2),\, \phi_\lambda\right)}
\end{aligned}$$

where $\lambda = \lambda_1 e_3 - \lambda_2 e_4 + \frac{1}{2}(\lambda_3 + i\lambda_4)$, and ρ_{e_3}, ρ_{e_4} and ρ are given as in Proposition 4.3.3. We first set $A_1(\tau, \sigma) = e(i\rho_{e_3} - \rho_{e_4}\tau + \bar{\rho}\sigma)$. Then by decomposing the infinite product according to the four cases in its product index set, we can easily rewrite it as (4.3.14).

Finally, for Assertion (4), we first note that in our case, $K' = \mathbb{Z}i \oplus \mathbb{Z}[i] \oplus \frac{1}{2}\mathbb{Z}i$ and $\delta = 2i$, then $\lambda \in K'$ and $\lambda = \frac{1}{2}\kappa\delta\ell = \kappa i\ell$ with $\kappa \in \mathbb{Q}_{>0}$ imply that $\kappa \in \mathbb{Z}_{>0}$ and $c(0, \lambda) = c(0, \phi_0)$. Together with the Weyl vector attached to W_m and $\vec{F}_m$ shown in Subsection 4.3.5, Theorem 4.3.4 (5) proves Assertion (4).

□

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