SOME RESULTS ON LATTICE DIRECTED POLYMER MODELS

By

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Abstract

In this work we study 1+1 dimensional lattice directed polymer models. After introducing the general model in Chapter 1, in Chapter 2 (joint work with Christian Noack) we develop a Mellin transform framework which allows us to simultaneously analyze the four known exactly solvable 1+1 dimensional lattice polymer models: the log-gamma, strict-weak, beta, and inverse-beta models. Using this framework we prove the conjectured fluctuation exponents of the free energy and the polymer path for the stationary point-to-point versions of these four models. In Chapter 3 (joint work with Christian Noack) we define an integrability property shared by each of the log-gamma, strict-weak, beta, and inverse-beta models. This integrability property encapsulates a preservation in distribution of ratios of partition functions which in turn implies the so called Burke property. We show that under some regularity assumptions, up to trivial modifications, there exist no other models possessing this property. In Chapter 4 we further study the log-gamma directed polymer and consider a model with multiple paths. We formulate an environment in which the ratios of multi-path partition functions satisfy a Burke-type stationarity property. This stationarity is then used to derive a formula for the variance of the multi-path partition function.

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List of Figures

Notation and Symbols

$\mathbb{N}$	The natural numbers $\{1, 2, \ldots\}$.
$\mathbb{Z}$	The integers.
$\mathbb{Z}_+$	The nonnegative integers $\{0, 1, 2, \ldots\}$.
$\mathbb{R}$	The real numbers.
$\lfloor x \rfloor$	The greatest integer less than or equal to x.
$a \vee b$	The maximum of a and b.
$a \wedge b$	The minimum of a and b.
$\mathrm{supp}(f)$	For a real valued function f, $\mathrm{supp}(f) = \{x : f(x) \neq 0\}$.
$\otimes$	Denotes (independent) product distribution.
$-A$	For $A \subset \mathbb{R}$, $-A = \{-a : a \in A\}$.
A^{-1}	For $A \subset \mathbb{R}$ such that $0 \notin A$, $A^{-1} = \{\frac{1}{a} : a \in A\}$.
$\overline{X}$	For a random variable X with finite expectation, $\overline{X} = X - \mathbb{E}[X]$.

Contents

Chapter 1

Introduction

1.1 Background

A polymer is a large molecule consisting of a long chain of repeated subunits held together by chemical bonds. Nylon, polyethylene, and polystyrene are some well-known examples of synthetic polymers. Proteins, nucleic acids, and polysaccharides are examples of natural polymers. Due to the ubiquity of polymer structures, polymers are a subject of intense study in chemistry, biology, physics, and mathematics.

In statistical physics and mathematics, a polymer is typically modeled as a random nearest neighbor path in a lattice. A typical formulation of a polymer model is to specify a *configuration space* Ω_n, a finite collection of allowable n-step polymer paths, and a *Hamiltonian function* H_n, which associates to each path $x \in \Omega_n$ an energy $H_n(x)$. The *Gibbs measure* on Ω_n associated to H_n is the probability measure defined by

$$P_n(x) = \frac{1}{Z_n} \exp(-H_n(x)),$$

where $Z_n := \sum_{x \in \Omega_n} e^{-H_n(x)}$ is the normalizing *partition function*.

The flexibility in the choice of the configuration space and Hamiltonian allows for the modeling of a wide range of physical systems. The lecture notes of den Hollander [21] provide an excellent introduction to the study of a variety of random polymer models and the techniques used in their analysis.

In this work we will consider *directed polymers models*, for which the allowable paths are directed in time. The paths considered are trajectories of nearest neighbor walks in $\mathbb{Z}^d$, with time-space representation $\{(t, x_t)\}_{t=0}^n$, where $(x_t)_{t=0}^n$ is a nearest-neighbor walk in $\mathbb{Z}^d$ starting at the origin at time $t = 0$. The Hamiltonian is defined in terms of a random environment. A path is weighted according to the random weights that it traverses. Our setup will be made precise in the following section. The model of the directed polymer in a random environment can exhibit many interesting behaviors including phase transitions when a temperature parameter is used to tune the strength of the random environment. See the lecture notes of Comets [15] for a survey of the mathematical work on directed polymers.

The Kardar-Parisi-Zhang [30] or KPZ equation is a stochastic partial differential equation which, in the 1+1 dimensional setting, is commonly used as a model equation in physics for surface growth and random interfaces. A broad class of 1+1 dimensional random growth and interface models are expected to lie in the *KPZ universality class*. For such models, certain observables of interest are expected to have random fluctuations that grow as a power of the system size or time, N. As opposed to the Gaussian universality class, where the order of the fluctuations is $N^{1/2}$ and the scaled fluctuations have a Gaussian limiting distribution, models in the KPZ universality class exhibit fluctuations of order $N^{1/3}$ and non-Gaussian limiting distributions arising from random matrix theory. For an introduction to the KPZ equation and its universality class, see the articles by Corwin [16],[17] and survey lecture notes by Quastel [43].

There is a vast body of physics literature on the KPZ equation and universality class, including heuristic derivations of fluctuation exponents and even physical experimental evidence for KPZ universality. See the survey article of Halpin-Healy and Takeuchi [26] and the many references within for an entry point into the literature. Despite the strong evidence for KPZ universality, there are relatively few models for which KPZ behavior has been rigorously proved. In the setting of lattice polymers there are four such models. These are described in Section 1.2.1.

1.1.1 Summary of results

In Chapter 2 we formulate a framework which allows us to prove the KPZ fluctuation exponents for the four models in a unified manner. The techniques used in the analysis depend upon a stationarity property that each of the four models exhibits. In Chapter 3 we show that, up to trivial modifications, these four models are the only 1+1 dimensional lattice polymers that have this stationarity property. In Chapter 4 we further consider the log-gamma polymer, which is special in its exact solvability through the geometric RSK correspondence. We formulate a stationary setup for a multi-path polymer, where the configuration space now consists of k-tuples of non-intersecting walks, and use this to prove a variance formula in the two-path case.

1.2 1+1 Dimensional Lattice Polymer Models

We now specify the model which is the subject of this work. We consider a 1+1 dimensional lattice directed polymer. By making the change of coordinates $(t, z) \mapsto (\frac{t-z}{2}, \frac{t+z}{2})$ (illustrated in Figure 1), we can consider paths in the nonnegative quadrant of $\mathbb{Z}^2$. We study the point to point polymer, meaning that the endpoint is fixed in space-time.

On each edge e of the $\mathbb{Z}_+^2$ lattice we place a positive random weight. The superscripts 1 and 2 will be used to denote horizontal and vertical edge weights, respectively. For $z \in$

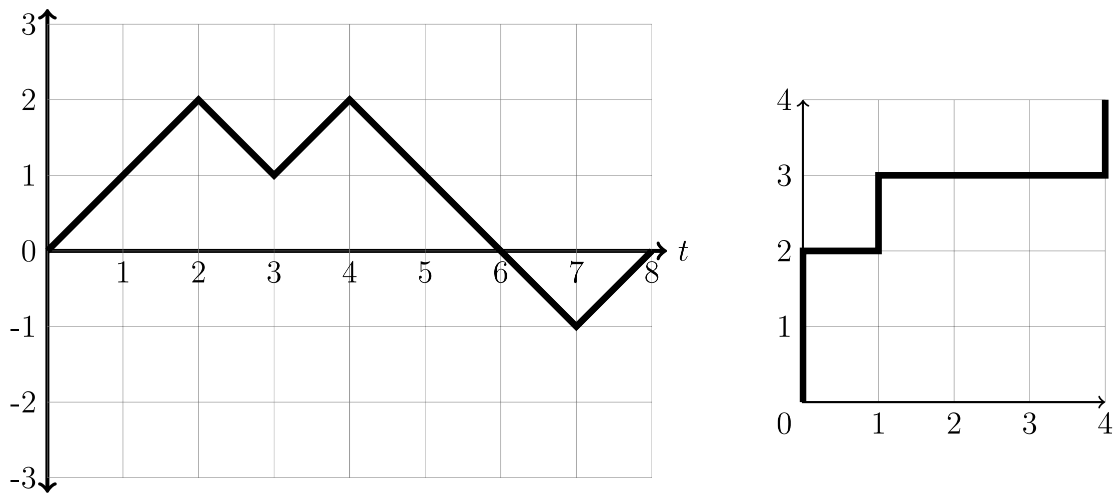

Figure 1: An example polymer trajectory in (t, z) coordinates and the corresponding path in $\mathbb{Z}^2_+$ obtained by rotating 45 degrees counter-clockwise then rescaling.

$\mathbb{N}^2$, let Y^1_z and Y^2_z denote the horizontal and vertical incoming edge weights. We assume that the collection of pairs $\{(Y^1_z, Y^2_z)\}_{z\in\mathbb{N}^2}$ is independent and identically distributed with common distribution (Y^1, Y^2), but do not insist that Y^1_z is independent of Y^2_z. Call this collection the *bulk weights*. For $x \in \mathbb{N} \times \{0\}$, let R^1_x denote the horizontal incoming edge weight, and for $y \in \{0\} \times \mathbb{N}$, let R^2_y denote the vertical incoming edge weight. We assume the collections $\{R^1_x\}_{x\in\mathbb{N}\times\{0\}}$ and $\{R^2_y\}_{y\in\{0\}\times\mathbb{N}}$ are independent and identically distributed with common distributions R^1 and R^2, and refer to them as the *horizontal* and *vertical boundary weights*, respectively. We further assume that the horizontal boundary weights, the vertical boundary weights, and the bulk weights are independent of each other. This assignment of edge weights is illustrated in Figure 2. We call

$$\omega = \{R^1_x, R^2_y, (Y^1_z, Y^2_z) : x \in \mathbb{N} \times \{0\}, y \in \{0\} \times \mathbb{N}, z \in \mathbb{N}^2\} \tag{1.2.1}$$

the *polymer environment*. We use $\mathbb{P}$ and $\mathbb{E}$ to denote the probability measure and corresponding expectation of the polymer environment.

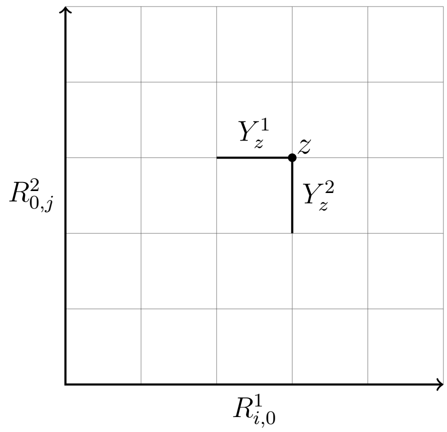

Figure 2: Assignment of edge weights.

A path is weighted according to the product of the weights along its edges. For $(m,n) \in \mathbb{Z}^2_+ \backslash \{(0,0)\}$ we define a probability measure on all up-right paths from $(0,0)$ to (m,n). See Figure 3 for an example of an up-right path. Let $\Pi_{m,n}$ denote the collection of all such paths. We identify paths $x_\bullet = (x_0, x_1, \ldots, x_{m+n})$ by their sequence of vertices, but also associate to paths their sequence of edges $(e_1, \ldots, e_{m+n})$, where $e_i = \{x_{i-1}, x_i\}$. Define the quenched polymer measure on $\Pi_{m,n}$,

$$Q_{m,n}(x_\bullet) := \frac{1}{Z_{m,n}} \prod_{i=1}^{m+n} \omega_{e_i},$$

where ω_e is the weight associated to the edge e and

$$Z_{m,n} := \sum_{x_\bullet \in \Pi_{m,n}} \prod_{i=1}^{m+n} \omega_{e_i}$$

is the associated partition function. At the origin, define $Z_{0,0} := 1$. Taking the expectation $\mathbb{E}$ of the quenched measure with respect to the edge weights gives the annealed measure on $\Pi_{m,n}$,

$$P_{m,n}(x_\bullet) := \mathbb{E}[Q_{m,n}(x_\bullet)].$$

The annealed expectation will be denoted by $E_{m,n}$.

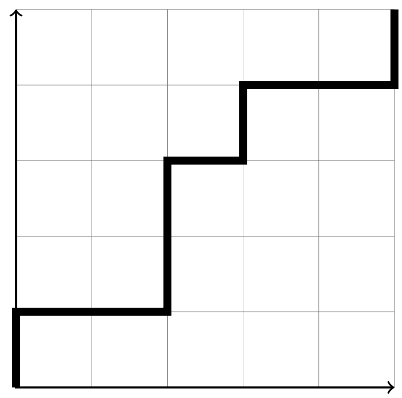

Figure 3: An up-right path from $(0,0)$ to $(5,5)$.

Note that this formulation of the polymer measure is equivalent to using the Hamiltonian $H(x_\bullet) = -\sum_{i=1}^{m+n} \log \omega_{e_i}$.

The logarithm of the partition function is called the *free energy* of the polymer model and is an important object of study. Many of the thermodynamic quantities of interest of an equilibrium statistical mechanical model can be expressed in terms of the free energy.

1.2.1 Four basic beta-gamma models

In the setting of lattice directed polymers, there are four models for which results about the scaling exponents or limit distributions are known. The log-gamma directed polymer was introduced by Seppäläinen in [45], where the fluctuation exponents were proved. A limit distribution result for the scaled free energy was proved by Borodin, Corwin, and Remenik [11]. The strict-weak polymer model was simultaneously introduced by Corwin, Seppäläinen, Shen [20] and O'Connell, Ortmann [38] and its limit distribution was proved through different methods in these two papers. The beta directed polymer was introduced by Barraquand and Corwin in [7], where its limit distribution was also calculated. The fourth model is the inverse-beta model, introduced by Thiery and Le Doussal in [50], in which they conjecture a formula for the Laplace transform of the polymer partition function and, contingent on this conjecture, show Tracy-Widom limit distribution for the rescaled free energy.

We specify the edge weight distributions of the random variables (R^1, R^2, Y^1, Y^2) to define the four stationary polymer models. The notation $X \sim \mathrm{Ga}(\alpha, \beta)$ is used to denote that a random variable is gamma(α, β) distributed, i.e. has density $\Gamma(\alpha)^{-1}\beta^\alpha x^{\alpha-1}e^{-\beta x}$ supported on $(0, \infty)$, where $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}dx$ is the gamma function. $X \sim \mathrm{Be}(\alpha, \beta)$ is used to say that X is beta(α, β) distributed, i.e. has density $\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$ supported on $(0, 1)$. We then use $X \sim \mathrm{Ga}^{-1}(\alpha, \beta)$ and $X \sim \mathrm{Be}^{-1}(\alpha, \beta)$ to denote that $X^{-1} \sim \mathrm{Ga}(\alpha, \beta)$ and $X^{-1} \sim \mathrm{Be}(\alpha, \beta)$, respectively. We also use $X \sim \big(\mathrm{Be}^{-1}(\alpha, \beta) - 1\big)$ to denote that $X + 1 \sim \mathrm{Be}^{-1}(\alpha, \beta)$.

- **Inverse-gamma (IG)**: This is also known as the log-gamma model. Assume $\mu > \theta > 0$, $\beta > 0$ and

$$\begin{aligned} R^1 &\sim \mathrm{Ga}^{-1}(\mu-\theta, \beta) \qquad R^2 \sim \mathrm{Ga}^{-1}(\theta, \beta) \\ (Y^1, Y^2) &= (X, X) \qquad \text{where} \qquad X \sim \mathrm{Ga}^{-1}(\mu, \beta). \end{aligned} \tag{1.2.2}$$

- **Gamma (G)**: This is also known as the strict-weak model. Assume θ, μ, $\beta > 0$ and

$$\begin{aligned} R^1 &\sim \mathrm{Ga}(\mu+\theta, \beta) \qquad R^2 \sim \mathrm{Be}^{-1}(\theta, \mu) \\ (Y^1, Y^2) &= (X, 1) \qquad \text{where} \qquad X \sim \mathrm{Ga}(\mu, \beta). \end{aligned} \tag{1.2.3}$$

- **Beta (B)**: Assume θ, μ, $\beta > 0$ and

$$\begin{aligned} R^1 &\sim \mathrm{Be}(\mu+\theta, \beta) \qquad R^2 \sim \mathrm{Be}^{-1}(\theta, \mu) \\ (Y^1, Y^2) &= (X, 1-X) \qquad \text{where} \qquad X \sim \mathrm{Be}(\mu, \beta). \end{aligned} \tag{1.2.4}$$

- **Inverse-beta (IB)**: Assume $\mu > \theta > 0$, $\beta > 0$ and

$$\begin{aligned} R^1 &\sim \mathrm{Be}^{-1}(\mu-\theta,\beta) \qquad R^2 \sim \big(\mathrm{Be}^{-1}(\theta,\beta+\mu-\theta)-1\big) \\ (Y^1,Y^2) &= (X,X-1) \qquad \text{where} \qquad X \sim \mathrm{Be}^{-1}(\mu,\beta). \end{aligned} \tag{1.2.5}$$

The name of each model refers to the distribution of the bulk weights. We call these models the **four basic beta-gamma models**.

The method used to obtain the fluctuation exponent results depends up the stationarity property of the four basic beta-gamma models, which we formulate as a down-right property, stated below. Section 2.2 of Chapter 2 elaborates upon this property.

Write $\alpha_1 = (1,0)$, $\alpha_2 = (0,1)$. For $k = 1,2$ define ratios of partition functions

$$R^k_x := \frac{Z_x}{Z_{x-\alpha_k}} \qquad \text{for all } x \text{ such that } x - \alpha_k \in \mathbb{Z}^2_+.$$

Note that these extend the definitions of $R^1_{i,0}$ and $R^2_{0,j}$, since for example $Z_{i,0} = \prod_{k=1}^i R^1_{k,0}$. We say that $\pi = \{\pi_k\}_{k\in\mathbb{Z}}$ is a down-right path in $\mathbb{Z}^2_+$ if $\pi_k \in \mathbb{Z}^2_+$ and $\pi_{k+1} - \pi_k \in \{\alpha_1, -\alpha_2\}$ for each $k \in \mathbb{Z}$. To each edge along a down-right path we associate the random variable

$$\Lambda_{\{\pi_{k-1},\pi_k\}} := \begin{cases} R^1_{\pi_k} & \text{if } \{\pi_{k-1},\pi_k\} \text{ is horizontal,} \\ R^2_{\pi_{k-1}} & \text{if } \{\pi_{k-1},\pi_k\} \text{ is vertical.} \end{cases}$$

An example down-right path and the associated random variables are given in Figure 4.

The following definition is a weaker form of the Burke property (see Theorem 3.3 of [45]).

Definition 1.1. *Say the polymer model has the* down-right property *if for all down-right paths $\pi = \{\pi_k\}_{k\in\mathbb{Z}}$, the random variables*

$$\Lambda(\pi) := \{\Lambda_{\{\pi_{k-1},\pi_k\}} : k \in \mathbb{Z}\}$$

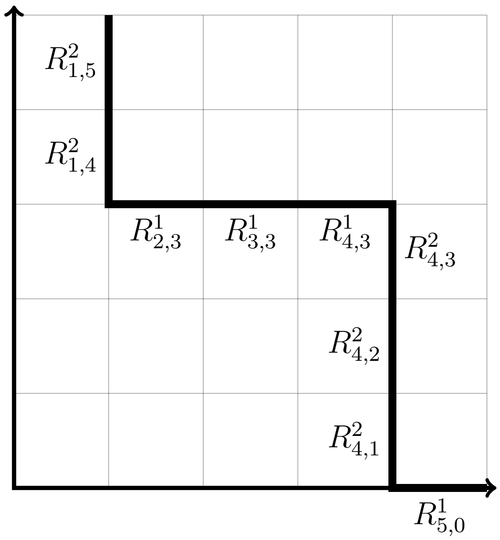

Figure 4: A section of a down-right path in $\mathbb{Z}_+^2$ and the associated random variables.

are independent and each $R^1_{\pi_k}$ *and* $R^2_{\pi_k}$ *appearing in the collection are respectively distributed as* R^1 *and* R^2*.*

To illustrate the usefulness of this down-right property, we use it to compute the expectation of the free energy. By writing the partition function as a product of ratios and taking a logarithm, the free energy can be expressed as

$$\log Z_{m,n} = \sum_{i=1}^{m} \log R^1_{i,0} + \sum_{j=1}^{n} \log R^2_{m,j}.$$

By assumption the terms $R^1_{i,0} \overset{d}{=} R^1$. Applying the down-right property to any down-right path passing through the edges from (m,n) to $(m,0)$ shows that $R^2_{m,j} \overset{d}{=} R^2$. Thus we have

$$\mathbb{E}[\log Z_{m,n}] = m\mathbb{E}[\log R^1] + n\mathbb{E}[\log R^2].$$

1.3 Results

1.3.1 Fluctuation exponent results

In the setting of $1+1$ dimensional directed polymers, KPZ universality predicts that the polymer path and free energy fluctuation exponents are 2/3 and 1/3, respectively.

The first main theorem of Chapter 2 shows that the free energy of the four basic beta-gamma models has the conjectured order of fluctuations. The theorem is restated below.

Theorem 1.2. *Assume that the polymer environment has edge weight distributions* $R^1, R^2, (Y^1, Y^2)$ *as in one of* (1.2.2) *through* (1.2.5)*, and let* $(m,n) = (m_N, n_N)_{N=1}^{\infty}$ *be a sequence such that*

$$|m_N - N\mathbb{V}ar[\log R^2]| \leqslant \gamma N^{2/3} \qquad \text{and} \qquad |n_N - N\mathbb{V}ar[\log R^1]| \leqslant \gamma N^{2/3} \tag{1.3.1}$$

for some fixed $\gamma > 0$*. Then there exist positive constants* c, C, *and* N_0 *depending only on* $\mu, \theta, \beta, \gamma$ *such that for all* $N \geqslant N_0$,

$$cN^{2/3} \leqslant \mathbb{V}ar[\log Z_{m,n}] \leqslant CN^{2/3}.$$

The same constants c, C, N_0 *can be taken for all* $\mu, \theta, \beta, \gamma$ *varying in a compact set.*

The assumption (1.3.1) comes from the fact that we are considering the stationary version of each model, and the boundary weights are distributed differently from the bulk weights. The polymer needs to be scaled in a characteristic direction in order to see the appropriate behavior. Corollary 2.4 of Chapter 2 shows that we see Gaussian fluctuations and limit distribution when we scale in a non-characteristic direction.

The second main theorem of Chapter 2 shows that the polymer paths have order $N^{2/3}$ fluctuations. The theorem is restated below.

Given a path $x_\bullet \in \Pi_{m,n}$, for $0 \leqslant k \leqslant m$ and $0 \leqslant l \leqslant n$ define

$$v_0(l) := \min\{i : (i,l) \in x_\bullet\} \qquad v_1(l) := \max\{i : (i,l) \in x_\bullet\}$$

$$w_0(k) := \min\{j : (k,j) \in x_\bullet\} \qquad w_1(k) := \max\{j : (k,j) \in x_\bullet\}.$$

This is illustrated in figure 5.

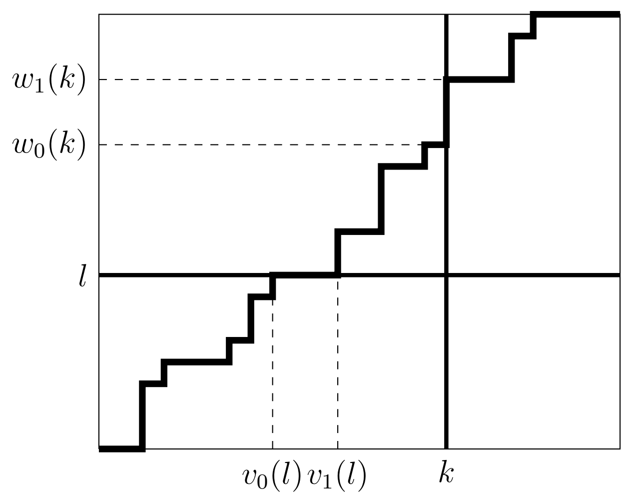

Figure 5: Example path with v_0, v_1, w_0, w_1 illustrated.

Theorem 1.3. *Assume that the polymer environment has edge weight distributions* $R^1, R^2, (Y^1, Y^2)$ *as in one of* (1.2.2) *through* (1.2.5), *and let* $(m,n) = (m_N, n_N)_{N=1}^{\infty}$ *be a sequence satisfying* (1.3.1) *for some fixed* $\gamma > 0$. *Let* $0 \leqslant \tau < 1$. *Then there exist positive constants* b_0, C, c_0, c_1, N_0 *depending only on* μ, θ, β, γ, τ *such that for* $b \geqslant b_0$ *and* $N \in \mathbb{N}$,

$$\begin{aligned} P_{m,n}(v_0(\lfloor \tau n \rfloor) \leqslant \tau m - bN^{2/3} \text{ or } v_1(\lfloor \tau n \rfloor) \geqslant \tau m + bN^{2/3}) &\leqslant \frac{C}{b^3}, \\ P_{m,n}(w_0(\lfloor \tau m \rfloor) \leqslant \tau n - bN^{2/3} \text{ or } w_1(\lfloor \tau m \rfloor) \geqslant \tau n + bN^{2/3}) &\leqslant \frac{C}{b^3}, \end{aligned} \tag{1.3.2}$$

and for all $N \geqslant N_0$*,*

$$c_0 \leqslant P_{m,n}(v_1(\lfloor \tau n \rfloor) \geqslant \tau m + c_1 N^{2/3} \text{ or } w_1(\lfloor \tau m \rfloor) \geqslant \tau n + c_1 N^{2/3}). \tag{1.3.3}$$

The same constants can be taken for all μ*,* θ*,* β*,* γ*,* τ *varying in a compact set.*

The content of the upper bounds (1.3.2) can be thought of as a form of tightness for the annealed polymer measure. Given a point $(\tau m, \tau n)$ on the diagonal, one can choose a horizontal or vertical window of order $N^{2/3}$ about $(\tau m, \tau n)$ (illustrated by the blue line segments in Figure 6) such that the annealed polymer measure assigns most of its mass to paths crossing through that window.

The lower bound (1.3.3) then shows that the correct order of the path fluctuations is no smaller than $N^{2/3}$. Under the annealed polymer measure, with probability of at least c_0 the polymer paths avoid a small window of order $N^{2/3}$ about $(\tau m, \tau n)$ (illustrated by the red shaded box in Figure 6).

1.3.2 Characterization results

Since each up-right path ending at $x \in \mathbb{N}^2$ uses the edge $\{x - \alpha_1, x\}$ or $\{x - \alpha_2, x\}$ in the last step, the partition functions satisfy the recurrence relation

$$Z_x = Y_x^1 Z_{x-\alpha_1} + Y_x^2 Z_{x-\alpha_2} \qquad \text{for } x \in \mathbb{N}^2. \tag{1.3.4}$$

This recurrence relation then implies the recursions

$$\begin{aligned} R_x^1 &= Y_x^1 + Y_x^2 \frac{R_{x-\alpha_2}^1}{R_{x-\alpha_1}^2} \\ R_x^2 &= Y_x^1 \frac{R_{x-\alpha_1}^2}{R_{x-\alpha_2}^1} + Y_x^2 \end{aligned} \qquad \text{for } x \in \mathbb{N}^2. \tag{1.3.5}$$

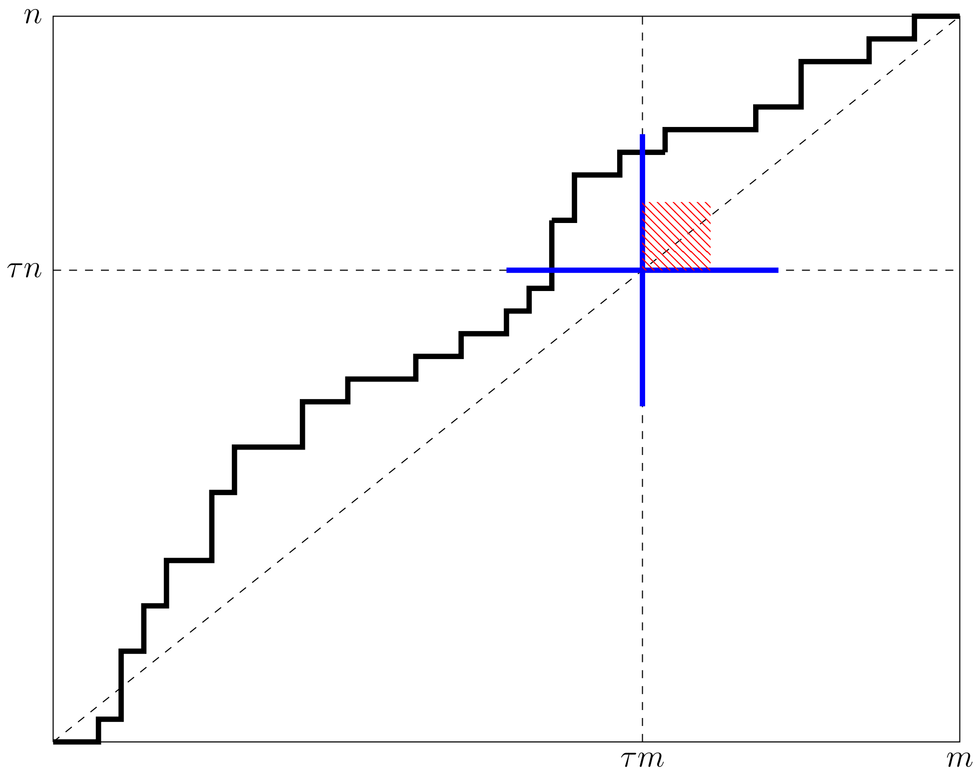

Figure 6: The blue line segments indicate windows of size $bN^{2/3}$ from upper bound (1.3.2). The red shaded box is the window of size $c_1 N^{2/3}$ which the polymer path avoids in lower bound (1.3.3).

Using the recursions (1.3.5) we can reduce the down-right property to the following distributional identity:

$$(R^1, R^2) \stackrel{d}{=} (Y^1 + Y^2 R^1/R^2,\ Y^1 R^2/R^1 + Y^2). \tag{1.3.6}$$

The content of Chapter 3 is to give a characterization of stationary polymer models (i.e. those with the down-right property) by characterizing random variables $R^1, R^2, (Y^1, Y^2)$ which satisfy (1.3.6).

We further assume that Y^1 and Y^2 have a functional dependence of the form $(Y^1, Y^2) = \big(Y, h(Y)\big)$ for some positive random variable Y and positive function h. Notice that each of the four basic beta-gamma models has this form.

When Y is a random variable taking values in the domain of h and (R^1, R^2) is a random vector taking values in $(0,\infty)^2$, define the random vector

$$T^{h,Y}(R^1, R^2) := \left(Y + h(Y)\frac{R^1}{R^2}, Y\frac{R^2}{R^1} + h(Y)\right). \tag{1.3.7}$$

Note that with $(Y^1, Y^2) = \big(Y_x, h(Y_x)\big)$, the recursive equations 1.3.5 imply

$$\big(R^1_x, R^2_x\big) = T^{h,Y_x}\big(R^1_{x-\alpha_2}, R^2_{x-\alpha_1}\big) \qquad \text{for all } x \in \mathbb{N}^2. \tag{1.3.8}$$

Definition 1.4. *Let $O_3 \subset (0,\infty)$, $h : O_3 \to (0,\infty)$, and assume the random variable Y takes values in O_3. Let (R^1, R^2) be a random vector taking values in $(0,\infty)^2$ that is independent of Y. We say that (R^1, R^2) is $T^{h,Y}$-*invariant *if $T^{h,Y}(R^1, R^2) \overset{d}{=} (R^1, R^2)$.*

The first main result of Chapter 3, Theorem 1.5, consists of showing that, under some regularity assumptions, $T^{h,Y}$-invariance can only occur if h is of the form $h(y) = a + by$ for real numbers a, b satisfying $a \vee b > 0$. This theorem is restated below.

Define the non-random analogue of (1.3.7),

$$T^{h,y}(r_1, r_2) := \big(y + h(y)\tfrac{r_1}{r_2}, y\tfrac{r_2}{r_1} + h(y)\big). \tag{1.3.9}$$

Theorem 1.5. *Let R^1, R^2, Y be positive, independent random variables with respective densities f_1, f_2, f_3. Assume that the support of f_j is $O_j \subset (0,\infty)$ for $j = 1, 2, 3$, where each O_j is open and O_3 is connected. Assume f_1, f_2 are twice differentiable on O_1 and O_2 respectively and that f_3 is three times differentiable on O_3. Suppose $h : O_3 \to (0,\infty)$ is four times differentiable, the mapping $O_1 \times O_2 \times O_3 \ni (r_1, r_2, y) \mapsto T^{h,y}(r_1, r_2)$ surjects onto $O_1 \times O_2$, and $\frac{r_2}{r_1} + h'(y) \neq 0$ for all $(r_1, r_2, y) \in O_1 \times O_2 \times O_3$. If (R^1, R^2) is $T^{h,Y}$-invariant, then h must be of the form $h(y) = a + by$, where a, b are real numbers satisfying $a \vee b > 0$.*

The second main result of the chapter, Theorem 3.4, consists of showing that if h has this form, then $T^{h,Y}$-invariance only arises as a modification of the four known invariant models (described in (1.2.2) through (1.2.5)). Details of the modifications are given in Section 3.4, but can loosely be described as either interchanging the x and y axes or scaling the weights by constants. This theorem is restated below.

Theorem 1.6. *Let $O_j \subset (0,\infty)$ for $j = 1,2,3$ and assume $h : O_3 \to (0,\infty)$ has the form $h(y) = a + by$, where a, b are real numbers satisfying $a \vee b > 0$. Assume the mapping $O_1 \times O_2 \times O_3 \ni (r_1, r_2, y) \mapsto T^{h,y}(r_1, r_2)$ surjects onto $O_1 \times O_2$, and R^1, R^2, Y are non-degenerate, independent random variables taking values in O_1, O_2, O_3 respectively.*

(a) If $a = 0$ and $b > 0$, then (R^1, R^2) is $T^{h,Y}$-invariant if and only if $\left(R^1, \frac{1}{b}R^2, Y, Y\right)$ is distributed as in (1.2.2)*.*

(b) If $a > 0$ and $b = 0$, then (R^1, R^2) is $T^{h,Y}$-invariant if and only if $\left(R^1, \frac{1}{a}R^2, Y, 1\right)$ is distributed as in (1.2.3)*.*

(c) If $a > 0$, $b < 0$, and $-b \notin \{\frac{y}{x} : (x,y) \in O_1 \times O_2\}$, then (R^1, R^2) is $T^{h,Y}$-invariant if and only if either $\left(-\frac{b}{a}R^1, \frac{1}{a}R^2, -\frac{b}{a}Y, 1 + \frac{b}{a}Y\right)$ or $\left(\frac{1}{a}R^2, -\frac{b}{a}R^1, 1 + \frac{b}{a}Y, -\frac{b}{a}Y\right)$ is distributed as in (1.2.4)*.*

(d) If $a < 0$ and $b > 0$, then (R^1, R^2) is $T^{h,Y}$-invariant if and only if $\left(-\frac{b}{a}R^1, -\frac{1}{a}R^2, -\frac{b}{a}Y, -\frac{b}{a}Y - 1\right)$ is distributed as in (1.2.5)*.*

(e) If $a, b > 0$, then (R^1, R^2) is $T^{h,Y}$-invariant if and only if $\left(\frac{1}{a}R^2, \frac{b}{a}R^1, 1 + \frac{b}{a}Y, \frac{b}{a}Y\right)$ is distributed as in (1.2.5)*.*

Figure 7 illustrates which one of the four basic beta-gamma models corresponds to each choice of parameters a, b.

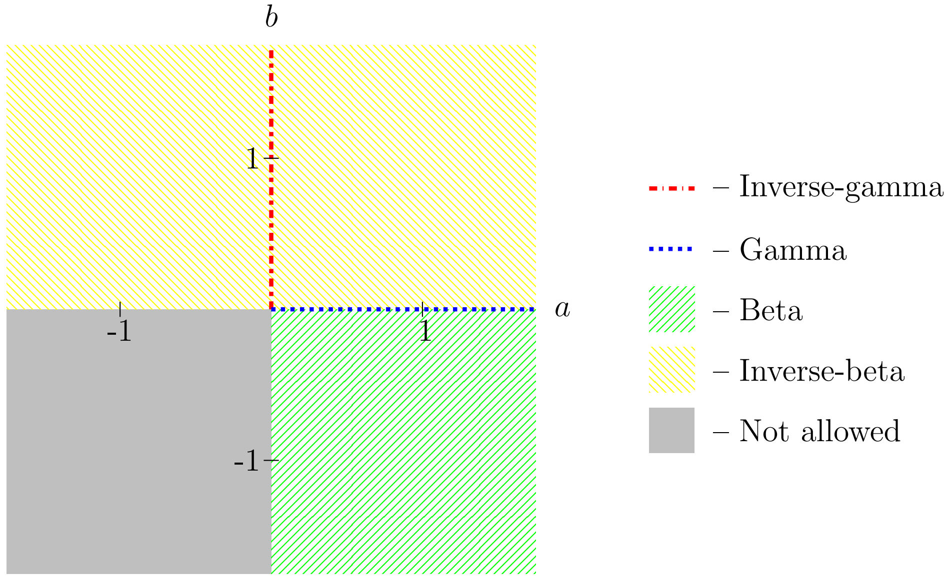

Figure 7: Modifications of the four beta-gamma models.

1.3.3 Multi-path polymer results

In Chapter 4 we further study the inverse-gamma (log-gamma) directed polymer, which is exactly solvable using the geometric Robinson-Schensted-Knuth (gRSK) correspondence. We consider multi-path polymers, which are represented as tuples of non-intersecting up-right lattice paths.

The classical RSK correspondence is a combinatorial mapping between matrices with entries in $\mathbb{Z}_+$ and pairs of semi-standard Young tableaux of the same shape. See the texts by Fulton [24] and Stanley [48] for constructions of the correspondence and some of its applications. The RSK correspondence is the combinatorial structure that lies behind the solvability of the problem of the length of the longest increasing subsequence of a random permutation [3] and the solvability of directed last passage percolation with geometric or exponential weights [29].

The geometric RSK correspondence (gRSK) is a mapping from matrices with positive real entries to pairs of triangular arrays. The mapping was first introduced by Kirillov [32]. In [19], Corwin, O'Connell, Seppäläinen, and Zygouras use the matrix product formulation for gRSK developed by Noumi and Yamada [37] to apply the gRSK correspondence to the study of directed polymers. If the entries of the input matrix to gRSK are used as weights for a polymer environment, the output triangular arrays can be expressed in terms of multi-path polymer partition functions. In the case where the input matrix has inverse-gamma distributed entries with certain parameters, the pushforward probability measure on triangular arrays has an explicit form.

The first result of Chapter 4 utilizes a theorem from [19] about the invariant distributions of a certain Markov process on triangular arrays to specify an inverse-gamma polymer environment for which ratios of multi-path polymer partition functions have a form of stationarity similar to the down-right property of Definition 1.1.

The other main result of Chapter 4 applies the stationary setup and the coupling method used in Section 2.3 of Chapter 2 to derive a formula for the variance of the two-path partition function.

Chapter 2

Fluctuation exponents

The content of this chapter is joint work with Christian Noack and is a modified form of an article which has been published in the *Latin American Journal of Probability and Mathematical Statistics* [14].

2.1 Introduction

In the $1 + 1$ dimensional case, a large class of polymer models are expected to lie in the KPZ universality class. For this class, the polymer path and free energy fluctuation exponents are conjectured to be 2/3 and 1/3, respectively, and the probability distribution of the rescaled free energy is conjectured to converge to the Tracy-Widom GUE distribution.

There are a few $1+1$ dimensional polymer models for which these results have been proved. Balázs, Quastel, and Seppäläinen [5] prove the fluctuation exponents for a Hopf-Cole solution to the KPZ equation with Brownian initial condition. This solution can be interpreted as the free energy of a stationary continuum directed polymer. Amir, Corwin, and Quastel [2] study the Hopf-Cole solution to the KPZ equation with narrow-wedge initial condition and prove Tracy-Widom limit distribution for large time. For the O'Connell-Yor semi-discrete Brownian directed polymer [41], the fluctuation exponents are proved by Seppäläinen and Valkó [46], and the limit distribution is proved in Borodin, Corwin [9] and Borodin, Corwin, Ferrari [10].

In the setting of lattice directed polymers, there are four models for which results about the scaling exponents or limit distributions are known. The log-gamma directed polymer was introduced by Seppäläinen in [45], where the fluctuation exponents were proved. The limit distribution result was proved by Borodin, Corwin, and Remenik [11]. The strict-weak polymer model was simultaneously introduced by Corwin, Seppäläinen, Shen [20] and O'Connell, Ortmann [38] and its limit distribution was proved through different methods in these two papers. The beta directed polymer was introduced by Barraquand and Corwin in [7], where its limit distribution was also calculated. The fourth model is the inverse-beta model, introduced by Thiery and Le Doussal in [50], in which they conjecture a formula for the Laplace transform of the polymer partition function and, contingent on this conjecture, show Tracy-Widom limit distribution for the rescaled free energy.

In this work we provide a Mellin transform framework with which we are able to treat these four lattice polymer models simultaneously and prove the fluctuation exponents of the free energy and the polymer path for their stationary versions. While for the log-gamma model these results were previously shown by [45], for the strict-weak, beta, and inverse-beta models, the path fluctuation results are new. An independent and concurrent work by Balázs, Rassoul-Agha, and Seppäläinen [6] also gives the path fluctuation result for the beta model.

Our methods rely upon a Burke-type stationarity property that each of these models possesses. This stationarity, along with a coupling argument, are used to prove a variance formula which is then amenable to analysis. This method was first used by

Cator and Groeneboom [12] to prove the order of the variance of the length of the longest weakly North-East path in Hammersley's process with sources and sinks. In [4], Balázs, Cator, and Seppäläinen adapt this method to prove the order of the fluctuations of the passage time and the fluctuations of the maximal path for last passage percolation with exponential weights. Seppäläinen [45] used this method to prove the order of the fluctuation of the free energy and the polymer path fluctuations for the point-to-point log-gamma model with stationary boundary conditions, and upper bounds on the fluctuation exponents for the non-stationary point-to-point and point-to-line models. Seppäläinen and Valkó [46] prove the scaling exponents for the O'Connell-Yor polymer, and Flores, Seppäläinen, and Valkó [34] extend the result to the intermediate disorder regime. Our work closely follows the methods in [45]; the Mellin transform framework provides a unified way to apply these methods to the four models.

2.1.1 Results

Recall the model and notation defined in Section 1.2 of Chapter 1.

If X is a positive random variable with density ρ, define

$$L_X(x) := -\frac{1}{x\rho(x)}\mathbb{C}\mathrm{ov}(\log X, \mathbb{1}_{\{X\leqslant x\}}) \tag{2.1.1}$$

for all x such that $\rho(x) > 0$. Given a path $x_\bullet \in \Pi_{m,n}$, define the *exit points* of the path from the horizontal and vertical axes by

$$t_1 := \max\{i : (i,0) \in x_\bullet\} \qquad \text{and} \qquad t_2 := \max\{j : (0,j) \in x_\bullet\}. \tag{2.1.2}$$

The following proposition gives exact formulas for the expectation and variance of the free energy, which is a starting point for analysis of these four models.

Proposition 2.1. *Assume that the polymer environment has edge weight distributions $R^1, R^2, (Y^1, Y^2)$ as in one of* (1.2.2) *through* (1.2.5)*. Then for all $(m,n) \in \mathbb{Z}_+^2$,*

$$\begin{aligned}
\mathbb{E}[\log Z_{m,n}] &= m\mathbb{E}[\log R^1] + n\mathbb{E}[\log R^2], \\
\mathbb{V}ar[\log Z_{m,n}] &= -m\mathbb{V}ar[\log R^1] + n\mathbb{V}ar[\log R^2] + 2E_{m,n}\left[\sum_{i=1}^{t_1} L_{R^1}(R^1_{i,0})\right], \qquad (2.1.3) \\
\mathbb{V}ar[\log Z_{m,n}] &= m\mathbb{V}ar[\log R^1] - n\mathbb{V}ar[\log R^2] + 2E_{m,n}\left[\sum_{j=1}^{t_2} L_{R^2}(R^2_{0,j})\right]. \qquad (2.1.4)
\end{aligned}$$

Using these exact formulas, we can obtain the following bounds on the variance of the free energy when (m,n) grow in a characteristic direction.

Theorem 2.2. *Assume that the polymer environment has edge weight distributions $R^1, R^2, (Y^1, Y^2)$ as in one of* (1.2.2) *through* (1.2.5)*, and let $(m,n) = (m_N, n_N)_{N=1}^{\infty}$ be a sequence such that*

$$|m_N - N\mathbb{V}ar[\log R^2]| \leqslant \gamma N^{2/3} \qquad \textit{and} \qquad |n_N - N\mathbb{V}ar[\log R^1]| \leqslant \gamma N^{2/3} \qquad (2.1.5)$$

for some fixed $\gamma > 0$. Then there exist positive constants c, C, and N_0 depending only on $\mu, \theta, \beta, \gamma$ such that for all $N \geqslant N_0$,

$$cN^{2/3} \leqslant \mathbb{V}ar[\log Z_{m,n}] \leqslant CN^{2/3}.$$

The same constants c, C, N_0 can be taken for all $\mu, \theta, \beta, \gamma$ varying in a compact set.

Theorem 2.2 and a Borel-Cantelli argument give the following law of large numbers.

Corollary 2.3. *With assumptions as in Theorem 2.2 the following limit holds $\mathbb{P}$-almost surely*

$$\lim_{N\to\infty} \frac{\log Z_{m,n}}{N} = \mathbb{E}[\log R^1]\mathbb{V}ar[\log R^2] + \mathbb{E}[\log R^2]\mathbb{V}ar[\log R^1]. \qquad (2.1.6)$$

For the four basic beta-gamma models, the right-hand side of (2.1.6) is given by

$$A(\mu,\theta,\beta)\Big(\frac{\partial}{\partial\theta}B(\mu,\theta)\Big) - \Big(\frac{\partial}{\partial\theta}A(\mu,\theta,\beta)\Big)B(\mu,\theta) + C(\mu,\theta),$$

where the functions A, B, and C are given in Table 8 and $\Psi_n(x) := \frac{\partial^{n+1}}{\partial x^{n+1}}\log\Gamma(x)$ denotes the polygamma function of order n.

Model	$A(\mu,\theta,\beta)$	$B(\mu,\theta)$	$C(\mu,\theta)$
IG	$\log\beta$	$\Psi_0(\theta)-\Psi_0(\mu-\theta)$	$-\Psi_0(\mu-\theta)\Psi_1(\theta)-\Psi_0(\theta)\Psi_1(\mu-\theta)$
G	$\log\beta$	$\Psi_0(\mu+\theta)-\Psi_0(\theta)$	$\Psi_0(\mu+\theta)\Psi_1(\theta)-\Psi_0(\theta)\Psi_1(\mu+\theta)$
B	$\Psi_0(\mu+\theta+\beta)$	$\Psi_0(\mu+\theta)-\Psi_0(\theta)$	$\Psi_0(\mu+\theta)\Psi_1(\theta)-\Psi_0(\theta)\Psi_1(\mu+\theta)$
IB	$-\Psi_0(\mu-\theta+\beta)$	$\Psi_0(\theta)-\Psi_0(\mu-\theta)$	$-\Psi_0(\mu-\theta)\Psi_1(\theta)-\Psi_0(\theta)\Psi_1(\mu-\theta)$

Figure 8: Functions for the limiting rescaled free energy of the four basic beta-gamma models.

The following is a result for when the sequence (m_N, n_N) does not satisfy condition (2.1.5). The statement is given for when the horizontal direction is too large, but an analogous result holds for the vertical direction.

Corollary 2.4. *Assume that the polymer environment has edge weight distributions $R^1, R^2, (Y^1, Y^2)$ as in one of* (1.2.2) *through* (1.2.5) *and that $m, n \to \infty$. Define N by $n = N\mathbb{V}ar[\log R^1]$ and assume*

$$N^{-\alpha}(m - N\mathbb{V}ar[\log R^2]) \to c_1 > 0$$

for some $\alpha > 2/3$. Then as $N \to \infty$

$$N^{-\alpha/2}\left(\log Z_{m,n} - \mathbb{E}[\log Z_{m,n}]\right)$$

converges in distribution to a centered normal with variance $c_1\mathbb{V}ar[\log R^1]$.

The variance formulas in Proposition 2.1 connect the variance of the free energy to the exit points of the path from the boundaries (2.1.2). This allows us to obtain bounds on the polymer path fluctuations under the annealed measure.

Given a path $x_\bullet \in \Pi_{m,n}$, for $0 \leqslant k \leqslant m$ and $0 \leqslant l \leqslant n$ define

$$\begin{aligned} v_0(l) &:= \min\{i : (i,l) \in x_\bullet\} \qquad & v_1(l) &:= \max\{i : (i,l) \in x_\bullet\} \\ w_0(k) &:= \min\{j : (k,j) \in x_\bullet\} \qquad & w_1(k) &:= \max\{j : (k,j) \in x_\bullet\}. \end{aligned} \tag{2.1.7}$$

This is illustrated in Figure 9.

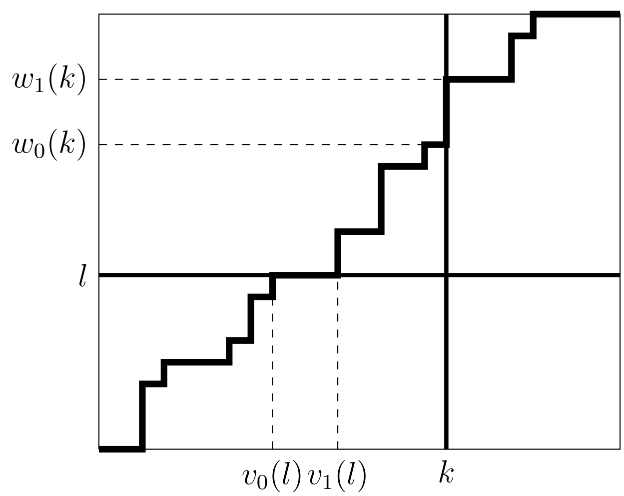

Figure 9: Example path with v_0, v_1, w_0, w_1 illustrated.

Theorem 2.5. *Assume that the polymer environment has edge weight distributions* $R^1, R^2, (Y^1, Y^2)$ *as in one of* (1.2.2) *through* (1.2.5)*, and let* $(m,n) = (m_N, n_N)_{N=1}^{\infty}$ *be a sequence satisfying* (2.1.5) *for some fixed* $\gamma > 0$*. Let* $0 \leqslant \tau < 1$*. Then there exist positive constants* b_0, C, c_0, c_1, N_0 *depending only on* μ, θ, β, γ, τ *such that for* $b \geqslant b_0$ *and* $N \in \mathbb{N}$*,*

$$P_{m,n}(v_0(\lfloor \tau n \rfloor) \leqslant \tau m - bN^{2/3} \textit{ or } v_1(\lfloor \tau n \rfloor) \geqslant \tau m + bN^{2/3}) \leqslant \frac{C}{b^3}, \tag{2.1.8}$$

$$P_{m,n}(w_0(\lfloor \tau m \rfloor) \leqslant \tau n - bN^{2/3} \textit{ or } w_1(\lfloor \tau m \rfloor) \geqslant \tau n + bN^{2/3}) \leqslant \frac{C}{b^3}, \tag{2.1.9}$$

and for all $N \geqslant N_0$,

$$c_0 \leqslant P_{m,n}(v_1(\lfloor \tau n \rfloor) \geqslant \tau m + c_1 N^{2/3} \text{ or } w_1(\lfloor \tau m \rfloor) \geqslant \tau n + c_1 N^{2/3}). \tag{2.1.10}$$

The same constants can be taken for all μ, θ, β, γ, τ *varying in a compact set.*

Structure of the chapter: In Section 2.2 we define the down-right property then state and prove consequences of this property. In Section 2.3 we introduce the Mellin transform framework, which allows us to treat the four basic beta-gamma models simultaneously, and prove Proposition 2.1. In Section 2.4 we prove the upper bound of Theorem 2.2. In Section 2.5 we prove bounds (2.1.8) and (2.1.9) of Theorem 2.5. In Section 2.6 we prove the lower bound of Theorem 2.2 and bound (2.1.10) of Theorem 2.5. In Appendix A we verify that each of the four basic beta-gamma models satisfies the conditions of Hypothesis 2.15. Appendix B collects technical lemmas used in Sections 2.3 and 2.4. Appendix C collects facts used in the proof of Proposition 2.26.

2.2 The down-right property

Write $\alpha_1 = (1,0)$, $\alpha_2 = (0,1)$. For $k = 1,2$ define ratios of partition functions

$$R^k_x := \frac{Z_x}{Z_{x-\alpha_k}} \qquad \text{for all } x \text{ such that } x - \alpha_k \in \mathbb{Z}^2_+.$$

Note that these extend the definitions of $R^1_{i,0}$ and $R^2_{0,j}$, since for example $Z_{i,0} = \prod_{k=1}^i R^1_{k,0}$. We say that $\pi = \{\pi_k\}_{k\in\mathbb{Z}}$ is a down-right path in $\mathbb{Z}^2_+$ if $\pi_k \in \mathbb{Z}^2_+$ and $\pi_{k+1} - \pi_k \in \{\alpha_1, -\alpha_2\}$ for each $k \in \mathbb{Z}$. To each edge along a down-right path we associate the random variable

$$\Lambda_{\{\pi_{k-1},\pi_k\}} := \begin{cases} R^1_{\pi_k} & \text{if } \{\pi_{k-1},\pi_k\} \text{ is horizontal,} \\ R^2_{\pi_{k-1}} & \text{if } \{\pi_{k-1},\pi_k\} \text{ is vertical.} \end{cases}$$

The following definition is a weaker form of the Burke property (see Theorem 3.3 of [45]).

Definition 2.6. *Say the polymer model has the* down-right property *if for all down-right paths* $\pi = \{\pi_k\}_{k\in\mathbb{Z}}$*, the random variables*

$$\Lambda(\pi) := \{\Lambda_{\{\pi_{k-1},\pi_k\}} : k \in \mathbb{Z}\}$$

are independent and each $R^1_{\pi_k}$ *and* $R^2_{\pi_k}$ *appearing in the collection are respectively distributed as* R^1 *and* R^2*.*

The partition functions satisfy the recurrence relation

$$Z_x = Y^1_x Z_{x-\alpha_1} + Y^2_x Z_{x-\alpha_2} \qquad \text{for } x \in \mathbb{N}^2. \tag{2.2.1}$$

This recurrence relation then implies the recursions

$$\begin{aligned} R^1_x &= Y^1_x + Y^2_x \frac{R^1_{x-\alpha_2}}{R^2_{x-\alpha_1}} \\ R^2_x &= Y^1_x \frac{R^2_{x-\alpha_1}}{R^1_{x-\alpha_2}} + Y^2_x \end{aligned} \qquad \text{for } x \in \mathbb{N}^2. \tag{2.2.2}$$

Using the recursions (2.2.2) we can reduce the down-right property to a simple preservation in distribution.

Lemma 2.7. *Let* $R^1, R^2, (Y^1, Y^2)$ *be positive random variables such that* R^1, R^2 *and the pair* (Y^1, Y^2) *are independent. Put*

$$\left(\widetilde{R}^1, \widetilde{R}^2\right) := (Y^1 + Y^2 R^1/R^2,\ Y^1 R^2/R^1 + Y^2).$$

Then the polymer model with edge weights $R^1, R^2, (Y^1, Y^2)$ *has the down-right property if and only if* $\left(\widetilde{R}^1, \widetilde{R}^2\right) \overset{d}{=} \left(R^1, R^2\right)$*.*

Proof of Lemma 2.7. Given a down-right path π, define its lower-left interior

$$\mathrm{Int}(\pi) := \{x \in \mathbb{Z}_+^2 \text{ such that } x + (m,n) \in \{\pi\} \text{ for some } m,n \in \mathbb{N}\}.$$

If the polymer model with edge weights $R^1, R^2, (Y^1, Y^2)$ has the down-right property, taking π to be the unique down-right path with interior $\{(0,0)\}$ implies that $(R^1_{1,1}, R^2_{1,1}) \stackrel{d}{=} (R^1, R^2)$. Then (2.2.2) and the fact that $\big(R^1_{1,0}, R^2_{0,1}, (Y^1_{1,1}, Y^2_{1,1})\big) \stackrel{d}{=} \big(R^1, R^2, (Y^1, Y^2)\big)$ imply that $(\widetilde{R}^1, \widetilde{R}^2) \stackrel{d}{=} (R^1, R^2)$.

For the converse direction, we first prove the statement for π with finite interior. The case when the interior is empty is true by assumption. Assume that the down-right property holds for all paths π with $|\mathrm{Int}(\pi)| = n$. Given a path π with $|\mathrm{Int}(\pi)| = n + 1$ there exists x such that π traverses the right-down corner $\{x - \alpha_1, x, x - \alpha_2\}$. Let $\widetilde{\pi}$ be the path which traverses the same points as π with the exception of instead passing through the down-right corner $\{x - \alpha_1, x - \alpha_1 - \alpha_2, x - \alpha_2\}$. Then $|\mathrm{Int}(\widetilde{\pi})| = n$ and so $\big(R^1_{x-\alpha_2}, R^2_{x-\alpha_1}\big) \stackrel{d}{=} \big(R^1, R^2\big)$. Using (2.2.2), the assumption that $(\widetilde{R}^1, \widetilde{R}^2) \stackrel{d}{=} (R^1, R^2)$ and the independence of (Y^1_x, Y^2_x) from the collection $\Lambda(\widetilde{\pi})$ gives us that the collection $\Lambda(\pi)$ has the desired property.

To prove the statement for arbitrary π, pick a finite sub-collection F of $\Lambda(\pi)$. Then there exists $\widetilde{\pi}$ such that $\mathrm{Int}(\widetilde{\pi})$ is finite and $F \subset \Lambda(\widetilde{\pi})$. Since the statement holds for down-right paths with finite interior, we are done. □

Proposition 2.8. *Each of the four basic beta-gamma models,* (1.2.2) *through* (1.2.5)*, possesses the down-right property.*

Proof. The $(\widetilde{R}^1, \widetilde{R}^2) \stackrel{d}{=} (R^1, R^2)$ condition in Lemma 2.7 has been checked for the inverse-gamma, gamma, beta, and inverse-beta models by Lemma 3.2 of [45], Lemma 6.3 of [20], Lemma 3.1 of [6], and Proposition 3.1 of [49] respectively. □

The following lemma is an immediate consequence of the down-right property and the starting point for the proof of Proposition 2.1.

Lemma 2.9. *If the polymer model with edge weights R^1, R^2, (Y^1, Y^2) possesses the down-right property and $\log R^1$, $\log R^2$ both have finite variance, then for all $(m,n) \in \mathbb{Z}_+^2$,*

(a) $\mathbb{E}[\log Z_{m,n}] = m\mathbb{E}[\log R^1] + n\mathbb{E}[\log R^2]$,

(b) $\mathbb{V}ar[\log Z_{m,n}] = -m\mathbb{V}ar[\log R^1] + n\mathbb{V}ar[\log R^2] + 2\mathbb{C}ov(S_N, S_S)$,

(c) $\mathbb{V}ar[\log Z_{m,n}] = m\mathbb{V}ar[\log R^1] - n\mathbb{V}ar[\log R^2] + 2\mathbb{C}ov(S_E, S_W)$,

where

$$\begin{aligned} S_N &:= \log Z_{m,n} - \log Z_{0,n} = \sum_{i=1}^{m} \log R^1_{i,n}, \qquad S_S := \log Z_{m,0} = \sum_{i=1}^{m} \log R^1_{i,0}, \\ S_E &:= \log Z_{m,n} - \log Z_{m,0} = \sum_{j=1}^{n} \log R^2_{m,j}, \quad S_W := \log Z_{0,n} = \sum_{j=1}^{n} \log R^2_{0,j}. \end{aligned} \tag{2.2.3}$$

Proof. By the down-right property S_S is independent of S_W, S_N is independent of S_E, and

$$\mathbb{V}\text{ar}[S_N] = \mathbb{V}\text{ar}[S_S] = m\mathbb{V}\text{ar}[\log R^1], \qquad \mathbb{V}\text{ar}[S_E] = \mathbb{V}\text{ar}[S_W] = n\mathbb{V}\text{ar}[\log R^2].$$

These facts along with the equalities $\log Z_{m,n} = S_N + S_W = S_E + S_S$ gives (a) and

$$\begin{aligned} \mathbb{V}\text{ar}[\log Z_{m,n}] &= \mathbb{V}\text{ar}[S_N] + \mathbb{V}\text{ar}[S_W] + 2\mathbb{C}\text{ov}(S_N, S_W) \\ &= \mathbb{V}\text{ar}[S_N] + \mathbb{V}\text{ar}[S_W] + 2\mathbb{C}\text{ov}(S_N, S_E + S_S - S_N) \\ &= -\mathbb{V}\text{ar}[S_N] + \mathbb{V}\text{ar}[S_W] + 2\mathbb{C}\text{ov}(S_N, S_S) \\ &= -m\mathbb{V}\text{ar}[\log R^1] + n\mathbb{V}\text{ar}[\log R^2] + 2\mathbb{C}\text{ov}(S_N, S_S). \end{aligned}$$

Similarly,

$$\mathbb{V}\mathrm{ar}[\log Z_{m,n}] = -n\mathbb{V}\mathrm{ar}[\log R^2] + m\mathbb{V}\mathrm{ar}[\log R^1] + 2\mathbb{C}\mathrm{ov}(S_E, S_W).$$

$\square$

2.3 The Mellin transform framework

In this section we develop a framework which allows us to treat the four basic beta-gamma models simultaneously.

Given a function $f : (0,\infty) \to [0,\infty)$, write M_f for its Mellin transform

$$M_f(a) := \int_0^\infty x^{a-1} f(x) dx$$

for any $a \in \mathbb{R}$ such that the integral converges. Define

$$D(M_f) := \mathrm{interior}(\{a \in \mathbb{R} : 0 < M_f(a) < \infty\}).$$

Definition 2.10. *Given a function $f : (0,\infty) \to [0,\infty)$ such that $D(M_f)$ is non-empty, we define a family of densities on $(0,\infty)$ parametrized by $a \in D(M_f)$:*

$$\rho_{f,a}(x) := M_f(a)^{-1} x^{a-1} f(x). \tag{2.3.1}$$

We write $X \sim m_f(a)$ to denote that the random variable X has density $\rho_{f,a}$.

Remark 2.11. *If $f : (0,\infty) \to [0,\infty)$ is such that $D(M_f)$ is non-empty, then M_f is C^∞ throughout $D(M_f)$. Furthermore, if $X \sim m_f(a)$, then*

(a) $\log X$ has finite exponential moments. That is, there exists some $\epsilon > 0$ such that

$$\mathbb{E}[e^{\epsilon|\log X|}] \leqslant \mathbb{E}[X^\epsilon] + \mathbb{E}[X^{-\epsilon}] = \frac{M_f(a+\epsilon) + M_f(a-\epsilon)}{M_f(a)} < \infty.$$

(b) *For all* $k \in \mathbb{N}$,

$$\frac{\partial^k}{\partial a^k} M_f(a) = M_f(a)\mathbb{E}[(\log X)^k].$$

(c) $\mathbb{E}[\log X] = \psi_0^f(a)$ *and* $\mathbb{V}ar[\log X] = \psi_1^f(a)$, *where*

$$\psi_n^f(a) := \frac{\partial^{n+1}}{\partial a^{n+1}} \log M_f(a) \text{ for } n \in \mathbb{Z}_+.$$

The following remark says that random variables with densities of the form (2.3.1) are closed under inversion.

Remark 2.12. *If* $f : (0, \infty) \to [0, \infty)$ *is such that* $D(M_f)$ *is non-empty and* $g(x) := f(\frac{1}{x})$ *for* $x \in (0, \infty)$, *then for all* $a \in D(M_f)$,

(a) $X \sim m_f(a) \Leftrightarrow X^{-1} \sim m_g(-a)$,

(b) $M_f(a) = M_g(-a)$ *and therefore* $D(M_g) = -D(M_f)$,

(c) $\psi_n^f(a) = (-1)^{n+1}\psi_n^g(-a)$ *for all* $n \in \mathbb{N}$.

Definition 2.13. *Let* $f^j : (0, \infty) \to [0, \infty)$ *be such that* $D(M_{f^j})$ *is non-empty for* $j = 1, 2$. *We say that the polymer environment is* Mellin-type with respect to (f^1, f^2) *if* $(R^1, R^2) \sim m_{f^1}(a_1) \otimes m_{f^2}(a_2)$ *for some* $a_j \in D(M_{f^j})$.

When the polymer environment is Mellin-type with parameters (a_1, a_2), we use $\mathbb{P}^{(a_1,a_2)}$, $\mathbb{E}^{(a_1,a_2)}$, $\mathbb{V}\mathrm{ar}^{(a_1,a_2)}$, $\mathbb{C}\mathrm{ov}^{(a_1,a_2)}$ in place of $\mathbb{P}$, $\mathbb{E}$, $\mathbb{V}\mathrm{ar}$, $\mathbb{C}\mathrm{ov}$ respectively.

2.3.1 The four basic beta-gamma models are Mellin-type

We first specify functions f to obtain each of the random variables appearing in the four basic beta-gamma models. Note that the fourth column in Table 10 specifies the distribution of the random variable corresponding to f. We let $\mathrm{B}(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ denote the beta function and recall that $\Psi_n(x) = \frac{\partial^{n+1}}{\partial x^{n+1}} \log \Gamma(x)$. For the Table 10 we assume $b > 0$ and $a \in D(M_f)$.

$f(x)$	$D(M_f)$	$M_f(a)$	$m_f(a)$
e^{-bx}	$(0,\infty)$	$\Gamma(a)/b^a$	$\mathrm{Ga}(a,b)$
$e^{-b/x}$	$(-\infty,0)$	$\Gamma(-a)b^a$	$\mathrm{Ga}^{-1}(-a,b)$
$(1-x)^{b-1}\mathbb{1}_{\{0<x<1\}}$	$(0,\infty)$	$\mathrm{B}(a,b)$	$\mathrm{Be}(a,b)$
$(1-\frac{1}{x})^{b-1}\mathbb{1}_{\{x>1\}}$	$(-\infty,0)$	$\mathrm{B}(-a,b)$	$\mathrm{Be}^{-1}(-a,b)$
$(\frac{x}{x+1})^b$	$(-b,0)$	$\mathrm{B}(-a,b+a)$	$\mathrm{Be}^{-1}(-a,b+a)-1$

$f(x)$	$\psi_n^f(a)$
e^{-bx}	$\Psi_n(a) - \delta_{n,0}\log b$
$e^{-b/x}$	$(-1)^{n+1}(\Psi_n(-a) - \delta_{n,0}\log b)$
$(1-x)^{b-1}\mathbb{1}_{\{0<x<1\}}$	$\Psi_n(a) - \Psi_n(a+b)$
$(1-\frac{1}{x})^{b-1}\mathbb{1}_{\{x>1\}}$	$(-1)^{n+1}(\Psi_n(-a) - \Psi_n(-a+b))$
$(\frac{x}{x+1})^b$	$\Psi_n(a+b) + (-1)^{n+1}\Psi_n(-a)$

Figure 10: Mellin framework data for the distributions appearing in the four basic beta-gamma models.

To express the distribution of the polymer environment in each of the four basic beta-gamma models given in (1.2.2) through (1.2.5) within this Mellin framework, we let

$$(R^1, R^2, X) \sim m_{f^1}(a_1) \otimes m_{f^2}(a_2) \otimes m_{f^1}(a_3), \tag{2.3.2}$$

where the functions f^1, f^2 and parameters a_j, $j = 1,2,3$ are given in Table 11. Recall that in each of the models, (Y^1, Y^2) are given in terms of X. For Table 11 we assume $\mu, \beta > 0$.

Remark 2.14. *For each fixed value of the bulk parameter a_3, we obtain a family of models with boundary parameters a_1 and a_2 satisfying $a_1 + a_2 = a_3$. For any such a_1 and a_2, by Proposition 2.8 these models will have the down-right property.*

Model	$f^1(x)$	$f^2(x)$	(a_1, a_2, a_3)	
IG	$e^{-\beta/x}$	$e^{-\beta/x}$	$(\theta-\mu, -\theta, -\mu)$	$\theta \in (0, \mu)$
G	$e^{-\beta x}$	$(1-\frac{1}{x})^{\mu-1}\mathbb{1}_{\{x>1\}}$	$(\mu+\theta, -\theta, \mu)$	$\theta \in (0, \infty)$
B	$(1-x)^{\beta-1}\mathbb{1}_{\{0<x<1\}}$	$(1-\frac{1}{x})^{\mu-1}\mathbb{1}_{\{x>1\}}$	$(\mu+\theta, -\theta, \mu)$	$\theta \in (0, \infty)$
IB	$(1-\frac{1}{x})^{\beta-1}\mathbb{1}_{\{x>1\}}$	$(\frac{x}{x+1})^{(\beta+\mu)}$	$(\theta-\mu, -\theta, -\mu)$	$\theta \in (0, \mu)$

Figure 11: Functions and parameters to fit the four basic beta-gamma models into the Mellin framework.

2.3.2 Coupling of polymer environments

In order to compare polymer environments with different parameters, we use a coupling to express the boundary weights as functions of i.i.d. uniform$(0,1)$ random variables.

If $f : (0,\infty) \to [0,\infty)$ is such that $D(M_f)$ is non-empty, write F^f for the CDF of the random variable $X \sim m_f(a)$. Specifically, $F^f : D(M_f) \times [0,\infty) \to [0,1]$ is given by

$$F^f(a,x) = \frac{1}{M_f(a)} \int_0^x y^{a-1} f(y) dy.$$

Define the quantile function

$$H^f(a,p) := \inf\{x : p \leqslant F^f(a,x)\}. \tag{2.3.3}$$

If the random variable η is uniformly distributed on the interval $(0,1)$, then $H^f(a,\eta)$ $\sim m_f(a)$.

Suppose that a polymer environment ω is Mellin-type with respect to (f^1, f^2) with parameters (b_1, b_2). Let $\{\eta_i^1, \eta_j^2 : i,j \in \mathbb{N}\}$ be i.i.d. uniform$(0,1)$ random variables that are independent of the bulk weights $\{(Y_z^1, Y_z^2) : z \in \mathbb{N}^2\}$. Write $\widehat{\mathbb{P}}$, $\widehat{\mathbb{E}}$, and $\widehat{\mathbb{V}\mathrm{ar}}$ for the probability measure and the corresponding expectation and variance of these uniform random variables and the bulk weights. Define the coupled environment

$$\omega^{(b_1,b_2)} := \{H^{f^1}(b_1, \eta_i^1), H^{f^2}(b_2, \eta_j^2), (Y_z^1, Y_z^2) : i \in \mathbb{N}, j \in \mathbb{N}, z \in \mathbb{N}^2\}. \tag{2.3.4}$$

Note that this environment is equal in distribution to the original environment ω.

To specifically denote weights accumulated by a path, the partition function, the quenched measure, and the annealed expectation, associated to the coupled environment $\omega^{(b_1,b_2)}$, define

$$\begin{aligned} W(b_1,b_2)(x_\bullet) &:= \prod_{k=1}^{m+n} \omega^{(b_1,b_2)}_{(x_{k-1},x_k)} \qquad \text{for } x_\bullet \in \Pi_{m,n} \\ Z_{m,n}(b_1,b_2) &:= \sum_{x_\bullet \in \Pi_{m,n}} W(b_1,b_2)(x_\bullet) \\ Q^{(b_1,b_2)}_{m,n}(A) &:= \frac{1}{Z_{m,n}(b_1,b_2)} \sum_{x_\bullet \in A} W(b_1,b_2)(x_\bullet) \qquad \text{for } A \subset \Pi_{m,n} \\ E^{(b_1,b_2)}_{m,n}[\bullet] &:= \widehat{\mathbb{E}}\left[E^{Q^{(b_1,b_2)}_{m,n}}[\bullet]\right]. \end{aligned} \tag{2.3.5}$$

Recall the definition of the exit points t_j (2.1.2). We can decompose the weight accumulated along a path to isolate the dependence on boundary weights

$$W(b_1,b_2)(x_\bullet) = \prod_{i=1}^{t_1} H^{f^1}(b_1,\eta^1_i) \prod_{j=1}^{t_2} H^{f^2}(b_2,\eta^2_j) \prod_{k=(t_1\vee t_2)+1}^{m+n} \omega^{(b_1,b_2)}_{(x_{k-1},x_k)}. \tag{2.3.6}$$

Notice that one of the first two products will be empty and the third product involves only the bulk weights.

If we assume that $f:(0,\infty)\to[0,\infty)$ has open support, is continuous on its support, and $D(M_f)$ is non-empty, then F^f is continuously differentiable on the set $D(M_f)\times \mathrm{supp}(f)$. By the implicit function theorem, H^f is continuously differentiable and for all $(a,p)\in D(M_f)\times(0,1)$, we have

$$\frac{\partial H^f}{\partial a}(a,p) = \frac{-\frac{\partial F^f}{\partial a}(a,H^f(a,p))}{\frac{\partial F^f}{\partial x}(a,H^f(a,p))} = H^f(a,p)L^f(a,H^f(a,p)) \tag{2.3.7}$$

where L^f is given by

$$\begin{aligned} L^f(a,x) &:= \frac{x^{-a}}{f(x)} \int_0^x (\psi^f_0(a) - \log y) y^{a-1} f(y)dy \\ &= -\frac{x^{-a}}{f(x)} \int_x^\infty (\psi^f_0(a) - \log y) y^{a-1} f(y)dy. \end{aligned} \tag{2.3.8}$$

The second equality follows from the definition of $\psi_0^f(a)$. Notice that

$$L^f(a,x) = -\frac{1}{x\rho_{f,a}(x)}\mathbb{C}\mathrm{ov}(\log X, \mathbb{1}_{\{X\leqslant x\}}) = L_X(x) \text{ (as defined in (2.1.1))}$$

when $X \sim m_f(a)$, and therefore $L^f(a,x) \geqslant 0$.

The following hypothesis collects technical conditions for the function f used in the sequel.

Hypothesis 2.15. *Suppose that $f : (0,\infty) \to [0,\infty)$ is such that $D(M_f)$ is non-empty, f has open support, is differentiable on its support, and for all compact $K \subset D(M_f)$ there exists a constant C depending only on K such that the following hold for all $a \in K$:*

$$L^f(a,x) \leqslant C(1 + |\log x|) \qquad \text{for all } x \in supp(f), \tag{2.3.9}$$

$$\int_0^1 \left|\frac{\partial}{\partial a}L^f(a, H^f(a,p))\right| dp \leqslant C. \tag{2.3.10}$$

Remark 2.16. *If $X \sim m_f(a)$ where f satisfies Hypothesis 2.15, then by* (2.3.9) *and Remark 2.11, $L_X(X)$ has finite exponential moments. By Lemma A.2 in the appendix, each of the functions f corresponding to the random variables appearing in the four basic beta-gamma models (see Table 10) satisfies Hypothesis 2.15.*

Lemma 2.17. *Assume that the polymer environment is Mellin-type with respect to (f^1, f^2), where f^1 and f^2 satisfy Hypothesis 2.15. Further assume that $\log Y^1$ and $\log Y^2$ have finite variance. Recall the notation* (2.2.3)*. Then for all $(m,n) \in \mathbb{Z}_+^2$,*

$$\mathbb{C}ov(S_N, S_S) = E_{m,n}[\sum_{i=1}^{t_1} L_{R^1}(R^1_{i,0})], \tag{2.3.11}$$

$$\mathbb{C}ov(S_E, S_W) = E_{m,n}[\sum_{j=1}^{t_2} L_{R^2}(R^2_{0,j})]. \tag{2.3.12}$$

Proof. By assumption, there exists $(a_1, a_2) \in D(M_{f^1}) \times D(M_{f^2})$ such that $(R^1, R^2) \sim m_{f^1}(a_1) \otimes m_{f^2}(a_2)$. There exist open neighborhoods U_j about a_j contained in $D(M_{f^j})$ for $j = 1, 2$. We then show that

$$\frac{\partial}{\partial b_1}\mathbb{E}^{(b_1,a_2)}[S_N] = \mathbb{C}\mathrm{ov}^{(b_1,a_2)}(S_N, S_S) \text{ for all } b_1 \in U_1, \tag{2.3.13}$$

$$\frac{\partial}{\partial b_2}\mathbb{E}^{(a_1,b_2)}[S_E] = \mathbb{C}\mathrm{ov}^{(a_1,b_2)}(S_E, S_W) \text{ for all } b_2 \in U_2, \tag{2.3.14}$$

and that the mappings $b_1 \mapsto \mathbb{C}\mathrm{ov}^{(b_1,a_2)}(S_N, S_S)$ and $b_2 \mapsto \mathbb{C}\mathrm{ov}^{(a_1,b_2)}(S_E, S_W)$ are continuous. We begin with (2.3.13). We will vary the parameter b_1 of the weights $R^1_{i,0}$ while keeping the parameter a_2 of the weights $R^2_{0,j}$ fixed. Let $\widetilde{\mathbb{E}}$ be the expectation over $\{R^2_{0,j}, (Y^1_x, Y^2_x)\}_{j\in\mathbb{N}, x\in\mathbb{N}\times\mathbb{N}}$. By Remark 2.11 and Lemma B.1, $\mathbb{E}^{(b_1,a_2)}[S_N^2] < \infty$ for all $b_1 \in U_1$. Then $\mathbb{E}^{(b_1,a_2)}[S_N] = \mathbb{E}^{b_1}[\widetilde{\mathbb{E}}[S_N]]$ where $\mathbb{E}^{b_1}$ denotes the expectation over $\{R^1_{i,0}\}_{i=1}^m$ when $R^1 \sim m_{f^1}(b_1)$. We now invoke Lemma B.2. Specifically, we use $r = m$, $X_k = R^1_{k,0}$, $f_k = f^1$ for all $k = 1, \ldots, m$ and $A(R^1_{1,0}, \ldots, R^1_{m,0}) = \widetilde{\mathbb{E}}[S_N]$ to get, for all $b_1 \in U_1$,

$$\begin{aligned}\frac{\partial}{\partial b_1}\mathbb{E}^{(b_1,a_2)}[S_N] = \frac{\partial}{\partial b_1}\mathbb{E}^{b_1}[A(X_1, \ldots, X_m)] &= \mathbb{C}\mathrm{ov}^{b_1}(A(X_1, \ldots, X_m), S_S) \\ &= \mathbb{C}\mathrm{ov}^{(b_1,a_2)}(S_N, S_S)\end{aligned}$$

and $U_1 \ni b_1 \mapsto \mathbb{C}\mathrm{ov}^{(b_1,a_2)}(S_N, S_S)$ is continuous. The third equality follows from the fact that the collection $\{R^2_{0,j}, (Y^1_x, Y^2_x)\}_{j\in\mathbb{N}, x\in\mathbb{N}\times\mathbb{N}}$ is independent of S_S. The second moment condition of Lemma B.2 is satisfied since for all $b_1 \in U_1$,

$$\mathbb{E}^{b_1}[A(X_1, \ldots, X_r)^2] = \mathbb{E}^{b_1}[(\widetilde{\mathbb{E}}[S_N])^2] \leqslant \mathbb{E}^{b_1}[\widetilde{\mathbb{E}}[S_N^2]] = \mathbb{E}^{(b_1,a_2)}[S_N^2] < \infty.$$

A similar argument yields (2.3.14).

Using the coupling (2.3.4)

$$E_{m,n}[\sum_{i=1}^{t_1} L_{R^1}(R^1_{i,0})] = E^{(a_1,a_2)}_{m,n}[\sum_{i=1}^{t_1} L^{f^1}(a_1, H^{f^1}(a_1,\eta^1_i))]. \tag{2.3.15}$$

Taking the derivative of (2.3.6) and using (2.3.7), for $j = 1, 2$

$$\frac{\partial}{\partial b_j}\log(W(b_1,b_2)(x_\bullet)) = \sum_{k=1}^{t_j}\frac{\partial}{\partial b_j}\log H^{f^j}(b_j,\eta^j_k) = \sum_{k=1}^{t_j} L^{f^j}(b_j, H^{f^j}(b_j\eta^j_k)). \tag{2.3.16}$$

Therefore

$$\frac{\partial}{\partial b_j}W(b_1,b_2)(x_\bullet) = W(b_1,b_2)(x_\bullet)\sum_{k=1}^{t_j} L^{f^j}(b_j, H^{f^j}(b_j\eta^j_k)) \tag{2.3.17}$$

which implies that

$$\frac{\partial}{\partial b_j}\log Z_{m,n}(b_1,b_2) = E^{Q^{(b_1,b_2)}_{m,n}}[\sum_{k=1}^{t_j} L^{f^j}(b_j, H^{f^j}(b_j,\eta^j_k))]. \tag{2.3.18}$$

We now prove (2.3.11). Similar to (2.3.5), in the coupled environment we use $S_\bullet(b_1,b_2)$ to make explicit the dependence of $S_\bullet$ on the parameters b_1 and b_2. Recall that $\widehat{\mathbb{E}}$ is the expectation of the coupled environment. For $\epsilon > 0$ small enough such that $[a_1-\epsilon, a_1+\epsilon] \subset U_1$,

$$\begin{aligned}\int_{a_1-\epsilon}^{a_1+\epsilon}\mathbb{C}\mathrm{ov}^{(b_1,a_2)}(S_N,S_S)db_1 &= \mathbb{E}^{(a_1+\epsilon,a_2)}[S_N] - \mathbb{E}^{(a_1-\epsilon,a_2)}[S_N]\\ &= \widehat{\mathbb{E}}[S_N(a_1+\epsilon,a_2) - S_N(a_1-\epsilon,a_2)]\\ &= \widehat{\mathbb{E}}[\int_{a_1-\epsilon}^{a_1+\epsilon}\frac{\partial}{\partial b_1}\log Z_{m,n}(b_1,a_2)db_1]\\ &= \int_{a_1-\epsilon}^{a_1+\epsilon}\widehat{\mathbb{E}}[\frac{\partial}{\partial b_1}\log Z_{m,n}(b_1,a_2)]db_1\end{aligned} \tag{2.3.19}$$

where the first equality follows from (2.3.13), the third equality follows because S_W does not depend on b_1 and $S_N(b_1,a_2) = \log Z_{m,n}(b_1,a_2) - S_W(a_2)$. The last equality follows from (2.3.18) and Tonelli's theorem (by the non-negativity of L^{f^1}).

Recall that $b_1 \mapsto \mathbb{C}\mathrm{ov}^{(b_1,a_2)}(S_N, S_S)$ is continuous. Once we show that the mapping

$$b_1 \mapsto \widehat{\mathbb{E}}[\frac{\partial}{\partial b_1} \log Z_{m,n}(b_1, a_2)] = E^{(b_1,a_2)}_{m,n}[\sum_{i=1}^{t_1} L^{f^1}(b_1, H^{f^1}(b_1, \eta^1_i))] \tag{2.3.20}$$

is continuous, using (2.3.19) and (2.3.15) we will have (2.3.11). The continuity of (2.3.20) follows from the continuity of $b_1 \mapsto E^{Q^{(b_1,a_2)}_{m,n}}[\sum_{k=1}^{t_1} L^{f^1}(b_1, H^{f^1}(b_1, \eta^1_k))]$, the dominated convergence theorem, and the bound

$$\begin{aligned}
&\widehat{\mathbb{E}}[\sup_{|b_1-a_1|\leqslant\epsilon} E^{Q^{(b_1,a_2)}_{m,n}}[\sum_{k=1}^{t_1} L^{f^1}(b_1, H^{f^1}(b_1, \eta^1_k))]] \\
&\leqslant \widehat{\mathbb{E}}[\sup_{|b_1-a_1|\leqslant\epsilon} \sum_{k=1}^{m} L^{f^1}(b_1, H^{f^1}(b_1, \eta^1_k))] \\
&\leqslant C\widehat{\mathbb{E}}[\sum_{k=1}^{m} 1 + |\log H^{f^1}(a_1 - \epsilon, \eta^1_k)| + |\log H^{f^1}(a_1 + \epsilon, \eta^1_k)|] < \infty
\end{aligned}$$

where we use the non-negativity of L^{f^1} to replace t_1 by its upper bound m, then use assumption (2.3.9) of Hypothesis 2.15 (with the fact that $H^{f^1}(b, x)$ is non-decreasing in b) and part (a) of Remark 2.11.

A similar argument shows that

$$\mathbb{C}\mathrm{ov}^{(a_1,a_2)}(S_E, S_W) = E^{(a_1,a_2)}_{m,n}[\sum_{j=1}^{t_2} L^{f^2}(a_2, R^2_{0,j})].$$

This completes the proof. □

We can now give the proof of Proposition 2.1.

Proof of Proposition 2.1. By assumption, the polymer environment is distributed as in (2.3.2), where f^1 and f^2 satisfy Hypothesis 2.15 by Remark 2.16. By Remark 2.11, for each of the four models $\log u$ and $\log v$ have finite variance. Thus the conditions of Lemma 2.17 are satisfied. Combining Proposition 2.8 with Lemma 2.9, and Lemma 2.17 yields the result. □

2.4 Proof of the variance upper bound

The first lemma of this section allows us to compare the variance of the free energy at different parameter values.

Lemma 2.18. *Assume that the polymer environment is distributed as in* (2.3.2)*. Let* ϵ *be small enough such that for all* $|\lambda| \leqslant \epsilon$*,* $a_1 + \lambda \in D(M_{f^1})$ *and* $a_2 - \lambda \in D(M_{f^2})$*. Then there exists a positive constant* C *depending only on* (a_1, a_2)*,* β*, and* ϵ *such that for all* $(m, n) \in \mathbb{Z}^2_+$*, the following holds for all* $|\lambda| \leqslant \epsilon$*,*

$$\left|\mathbb{V}ar^{(a_1,a_2)}[\log Z_{m,n}] - \mathbb{V}ar^{(a_1+\lambda,a_2-\lambda)}[\log Z_{m,n}]\right| \leqslant C(m+n)|\lambda|$$

Proof. Let $\tilde{a}_1 = a_1 + \lambda$ and $\tilde{a}_2 = a_2 - \lambda$. Applying Proposition 2.1 (recalling that $\psi_1^f(a) = \mathbb{V}\mathrm{ar}[\log X]$ when $X \sim m_f(a)$) then using the coupling (2.3.5) yields, for $j = 1, 2$:

$$\frac{1}{2}\left(\mathbb{V}\mathrm{ar}^{(\tilde{a}_1,\tilde{a}_2)}[\log Z_{m,n}] - \mathbb{V}\mathrm{ar}^{(a_1,a_2)}[\log Z_{m,n}]\right) \tag{2.4.1}$$
$$= \frac{(-1)^j}{2}\left[m\left(\psi_1^{f^1}(\tilde{a}_1) - \psi_1^{f^1}(a_1)\right) - n\left(\psi_1^{f^2}(\tilde{a}_2) - \psi_1^{f^2}(a_2)\right)\right] \tag{2.4.2}$$
$$+ E_{m,n}^{(\tilde{a}_1,\tilde{a}_2)}\left[\sum_{k=1}^{t_j} L^{f^j}(\tilde{a}_j, H^{f^j}(\tilde{a}_j, \eta_k^j))\right] - E_{m,n}^{(a_1,a_2)}\left[\sum_{k=1}^{t_j} L^{f^j}(a_j, H^{f^j}(a_j, \eta_k^j))\right]. \tag{2.4.3}$$

Since $\psi_1^{f^1}$ and $\psi_1^{f^2}$ are continuously differentiable, there is a constant C_1 such that line (2.4.2) is bounded by $C_1(m+n)|\lambda|$. Suppressing the m, n dependence, we then split line (2.4.3) as

$$= \widehat{\mathbb{E}}E^{Q^{(\tilde{a}_1,\tilde{a}_2)}}\left[\sum_{k=1}^{t_j} L^{f^j}(\tilde{a}_j, H^{f^j}(\tilde{a}_j, \eta_k^j))\right] - \widehat{\mathbb{E}}E^{Q^{(\tilde{a}_1,\tilde{a}_2)}}\left[\sum_{k=1}^{t_j} L^{f^j}(a_j, H^{f^j}(a_j, \eta_k^j))\right] \tag{2.4.4}$$
$$+ \widehat{\mathbb{E}}E^{Q^{(\tilde{a}_1,\tilde{a}_2)}}\left[\sum_{k=1}^{t_j} L^{f^j}(a_j, H^{f^j}(a_j, \eta_k^j))\right] - \widehat{\mathbb{E}}E^{Q^{(a_1,a_2)}}\left[\sum_{k=1}^{t_j} L^{f^j}(a_j, H^{f^j}(a_j, \eta_k^j))\right] \tag{2.4.5}$$

For line (2.4.4), since t_j is all that is random under $E^{Q^{(\tilde{a}_1,\tilde{a}_2)}}$, we can replace t_j by $m \vee n$. Thus

$$\begin{aligned}
|\,\text{line (2.4.4)}| &\leqslant \widehat{\mathbb{E}} \sum_{k=1}^{m\vee n} \left| L^{f^j}(\tilde{a}_j, H^{f^j}(\tilde{a}_j, \eta_k^j)) - L^{f^j}(a_j, H^{f^j}(a_j, \eta_k^j)) \right| \\
&= (m \vee n) \int_0^1 \left| L^{f^j}(\tilde{a}_j, H^{f^j}(\tilde{a}_j, \eta)) - L^{f^j}(a_j, H^{f^j}(a_j, \eta)) \right| d\eta \\
&= (m \vee n) \int_0^1 \left| \int_{a_j}^{\tilde{a}_j} \frac{\partial}{\partial a} L^{f^j}(a, H^{f^j}(a, \eta)) da \right| d\eta \\
&\leqslant (m \vee n) \left| \int_{a_j}^{\tilde{a}_j} \int_0^1 \left| \frac{\partial}{\partial a} L^{f^j}(a, H^{f^j}(a, \eta)) \right| d\eta da \right| \\
&\leqslant (m \vee n) C_2 |\lambda|. \qquad (2.4.6)
\end{aligned}$$

In the last step we used the fact that f^j satisfy assumption (2.3.10) in Hypothesis 2.15 by Remark 2.16.

We can write line (2.4.5) as

$$\widehat{\mathbb{E}}\Big[\sum_{k=1}^{\ell_j} L^{f^j}(a_j, H^{f^j}(a_j, \eta_k^j)) \big(Q^{(\tilde{a}_1,\tilde{a}_2)}(t_j \geqslant k) - Q^{(a_1,a_2)}(t_j \geqslant k) \big) \Big],$$

where $\ell_1 = m$ and $\ell_2 = n$. By Lemma B.3, $Q^{(a_1+\lambda, a_2-\lambda)}(t_1 \geqslant k)$ is stochastically non-decreasing in λ, and $Q^{(a_1+\lambda, a_2-\lambda)}(t_2 \geqslant k)$ is stochastically non-increasing in λ. Using the bound on (2.4.2), the bound (2.4.6), and the non-negativity of L^{f^j}, line (2.4.5) is non-negative if $j = 1$ and $\lambda > 0$ or $j = 2$ and $\lambda < 0$. This implies

$$(2.4.1) \geqslant -C(m+n)|\lambda|.$$

If $j = 2$ and $\lambda > 0$ or $j = 1$ and $\lambda < 0$, then (2.4.5) is non-positive, so

$$(2.4.1) \leqslant C(m+n)|\lambda|.$$

This completes the proof. $\square$

Lemma 2.19. *Assume that the polymer environment is distributed as in* (2.3.2)*. Then there exists a positive constant C depending only on (a_1, a_2) and β such that for all $(m,n) \in \mathbb{Z}_+^2$ the following two inequalities hold:*

$$E_{m,n}[\sum_{i=1}^{t_1} L_{R^1}(R^1_{i,0})] \leqslant C(E_{m,n}[t_1]+1), \qquad E_{m,n}[\sum_{j=1}^{t_2} L_{R^2}(R^2_{0,j})] \leqslant C(E_{m,n}[t_2]+1).$$

Proof. Let $L_i = L_{R^1}(R^1_{i,0})$, $\overline{L}_i = L_i - \mathbb{E}[L_i]$, and $S_k = \sum_{i=1}^k \overline{L}_i$. Note that $L_i \sim L_{R^1}(R^1)$ has finite exponential moments by Remark 2.16. Using Cauchy-Schwarz, Markov's inequality, and the bound $\mathbb{E}[S_k^8] \leqslant Ck^4$, we estimate

$$\mathbb{E}\left[\mathbb{1}_{\{S_k>k\}} S_k\right] \leqslant \left(\mathbb{P}\{S_k > k\}\right)^{1/2} \left(k \mathbb{V}\mathrm{ar} L_1\right)^{1/2} \leqslant \left(\frac{\mathbb{E}[S_k^8]}{k^8}\right)^{1/2} (kC)^{1/2} \leqslant Ck^{-3/2}.$$

Thus

$$\sum_{k=1}^{\infty} \mathbb{E}\left[\mathbb{1}_{\{S_k>k\}} S_k\right] \leqslant C.$$

Using this, we then get

$$\begin{aligned}
E_{m,n}\left[\sum_{i=1}^{t_1} L_{R^1}(R^1_{i,0})\right] &= E_{m,n}\left[\sum_{i=1}^{t_1} \overline{L}_i + \mathbb{E} L_i\right] \\
&= E_{m,n}[t_1]\mathbb{E}[L_1] + E_{m,n}\left[\sum_{i=1}^{t_1} \overline{L}_i\right] \\
&= E_{m,n}[t_1]\mathbb{E}[L_1] + \sum_{k=1}^{m} \mathbb{E}\left[Q_{m,n}(t_1 = k) S_k\right] \\
&\leqslant E_{m,n}[t_1]\mathbb{E}[L_1] + \sum_{k=1}^{m} \left(k\mathbb{E}\left[Q_{m,n}(t_1 = k)\right] + \mathbb{E}\left[\mathbb{1}_{\{S_k>k\}} S_k\right]\right) \\
&\leqslant E_{m,n}[t_1]\mathbb{E}[L_1] + E_{m,n}[t_1] + C \\
&\leqslant C\left(E_{m,n}[t_1] + 1\right).
\end{aligned}$$

The proof for t_2 is analogous. ∎

Proposition 2.20. *Assume that the polymer environment is distributed as in* (2.3.2). *Assume that the sequence* $(m,n) = (m_N, n_N)_{N=1}^{\infty}$ *satisfies*

$$|m - N\psi_1^{f^2}(a_2)| \vee |n - N\psi_1^{f^1}(a_1)| \leqslant \kappa_N$$

where $\kappa_N \leqslant \gamma N^{2/3}$ *and* γ *is some positive constant.*

Then there exist positive constants $C_1, C_2, C_3, \delta, \delta_1$ *depending only on* (a_1, a_2), β, *and* γ *such that for* $N \in \mathbb{N}$ *and* $1 \vee C_1\kappa_N \leqslant u \leqslant \delta N$,

$$\mathbb{P}\big\{Q_{m,n}(t_j \geqslant u) \geqslant e^{\frac{-\delta u^2}{N}}\big\} \leqslant C_2\Big(\frac{N^2}{u^4}E_{m,n}[t_j] + \frac{N^2}{u^3}\Big) \qquad \textit{for } j = 1, 2,$$

while for $N \in \mathbb{N}$ *and* $u \geqslant 1 \vee C_1\kappa_N \vee \delta N$,

$$\mathbb{P}\big\{Q_{m,n}(t_j \geqslant u) \geqslant e^{-\delta_1 u}\big\} \leqslant 2e^{-C_3 u} \qquad \textit{for } j = 1, 2.$$

Proof. Let $\epsilon > 0$ be small enough such that for all $|\lambda| \leqslant \epsilon$, $a_1(\lambda) := a_1 + \lambda \in D(M_{f^1})$ and $a_2(\lambda) := a_2 - \lambda \in D(M_{f^2})$. For the moment fix $\lambda_1 \in [0, \epsilon]$, $\lambda_2 \in [-\epsilon, 0]$, and $u \geqslant 1$. The λ_j will give the perturbation $(a_1(\lambda_j), a_2(\lambda_j))$ of the parameters (a_1, a_2) which will be used when dealing with the exit time t_j. Using the coupling in (2.3.5), (2.3.6) gives: for both $j = 1, 2$ and any path $x_{\bullet}$ such that $t_j(x_{\bullet}) \geqslant u$,

$$\frac{W(a_1, a_2)(x_{\bullet})}{W(a_1(\lambda_j), a_2(\lambda_j))(x_{\bullet})} = \prod_{k=1}^{t_j} \frac{H^{f^j}(a_j, \eta_k^j)}{H^{f^j}(a_j(\lambda_j), \eta_k^j)} \leqslant \prod_{k=1}^{\lfloor u \rfloor} \frac{H^{f^j}(a_j, \eta_k^j)}{H^{f^j}(a_j(\lambda_j), \eta_k^j)},$$

since $H^f(a, x)$ is non-decreasing in a. Therefore

$$\begin{aligned} Q_{m,n}^{(a_1,a_2)}(t_j \geqslant u) =& \frac{1}{Z_{m,n}(a_1, a_2)} \sum_{x_{\bullet} \in \Pi_{m,n}} \mathbb{1}_{\{x_{\bullet} \geqslant u\}} W(a_1, a_2)(x_{\bullet}) \\ \leqslant& \frac{Z_{m,n}(a_1(\lambda_j), a_2(\lambda_j))}{Z_{m,n}(a_1, a_2)} \prod_{k=1}^{\lfloor u \rfloor} \frac{H^{f^j}(a_j, \eta_k^j)}{H^{f^j}(a_j(\lambda_j), \eta_k^j)}. \end{aligned}$$

Then for all real numbers z, r

$$\mathbb{P}\Big\{Q_{m,n}(t_j \geqslant u) \geqslant e^{-z}\Big\} \leqslant \widehat{\mathbb{P}}\Big\{\prod_{k=1}^{\lfloor u \rfloor} \frac{H^{f^j}(a_j, \eta_k^j)}{H^{f^j}(a_j(\lambda_j), \eta_k^j)} \geqslant e^{-r}\Big\} \tag{2.4.7}$$

$$+ \widehat{\mathbb{P}}\Big\{\frac{Z_{m,n}(a_1(\lambda_j), a_2(\lambda_j))}{Z_{m,n}(a_1, a_2)} \geqslant e^{r-z}\Big\}. \tag{2.4.8}$$

We now split the proof into two cases.

Case 1: $1 \vee C_1\kappa_N \leqslant u \leqslant \delta N$. Let $b, \delta > 0$ be small enough such that $b\delta \leqslant \epsilon$. These constants will be determined through the course of the proof. Put $\lambda_1 = \frac{bu}{N}$ and $\lambda_2 = -\frac{bu}{N}$. The condition $u \leqslant \delta N$ guarantees that $-\epsilon \leqslant \lambda_2 < 0 < \lambda_1 \leqslant \epsilon$. Now plug in $r = \lfloor u \rfloor\Big(\psi_0^{f^j}(a_j(\lambda_j)) - \psi_0^{f^j}(a_j)\Big) - \frac{\delta u^2}{N}$ and $z = \frac{\delta u^2}{N}$ to obtain

$$\text{RHS of (2.4.7)} = \widehat{\mathbb{P}}\Big\{\sum_{k=1}^{\lfloor u \rfloor} \overline{\log H^{f^j}(a_j, \eta_k^j) - \log H^{f^j}(a_j(\lambda_j), \eta_k^j)} \geqslant \frac{\delta u^2}{N}\Big\} \leqslant C\frac{N^2}{u^3} \tag{2.4.9}$$

by Chebyshev's inequality and the fact that $H^f(a, \eta) \sim m_f(a)$. The constant C here depends only on (a_1, a_2), ϵ, and δ. We will now show how to tune b and δ as functions of (a_1, a_2) and ϵ to get a meaningful bound on

$$\begin{aligned}(2.4.8) =& \widehat{\mathbb{P}}\Big\{\overline{\log Z_{m,n}(a_1(\lambda_j), a_2(\lambda_j))} - \overline{\log Z_{m,n}(a_1, a_2)} \geqslant \\ & \widehat{\mathbb{E}}\big[\log Z_{m,n}(a_1, a_2) - \log Z_{m,n}(a_1(\lambda_j), a_2(\lambda_j))\big] + r - z\Big\}.\end{aligned} \tag{2.4.10}$$

Since the parameters satisfy $a_1(\lambda_j) + a_2(\lambda_j) = a_3$, by Remark 2.14, the down-right property is still satisfied for the perturbed model with parameters $\big(a_1(\lambda_j), a_2(\lambda_j)\big)$. Using

Proposition 2.1 we can evaluate the right-hand side inside the above probability

$$
\begin{aligned}
=& m\Big(\psi_0^{f^1}(a_1) - \psi_0^{f^1}(a_1(\lambda_j))\Big) + n\Big(\psi_0^{f^2}(a_2) - \psi_0^{f^2}(a_2(\lambda_j))\Big) \\
&+ \lfloor u \rfloor \Big(\psi_0^{f^j}(a_j(\lambda_j)) - \psi_0^{f^j}(a_j)\Big) - 2\delta\frac{u^2}{N} \\
=&(m - N\psi_1^{f^2}(a_2))\Big(\psi_0^{f^1}(a_1) - \psi_0^{f^1}(a_1(\lambda_j))\Big) \\
&+ (n - N\psi_1^{f^1}(a_1))\Big(\psi_0^{f^2}(a_2) - \psi_0^{f^2}(a_2(\lambda_j))\Big) \\
&+ N\Big[\psi_1^{f^2}(a_2)\Big(\psi_0^{f^1}(a_1) - \psi_0^{f^1}(a_1(\lambda_j))\Big) + \psi_1^{f^1}(a_1)\Big(\psi_0^{f^2}(a_2) - \psi_0^{f^2}(a_2(\lambda_j))\Big)\Big] \\
&+ \lfloor u \rfloor \Big(\psi_0^{f^j}(a_j(\lambda_j)) - \psi_0^{f^j}(a_j)\Big) - 2\delta\frac{u^2}{N} \\
\geqslant& -\kappa_N\frac{bu}{N}C' - N(\frac{bu}{N})^2C' + u(\frac{bu}{N})C'' - 2\delta\frac{u^2}{N} \\
=&\frac{u}{N}\Big[C''bu - C'b^2u - 2\delta u - C'b\kappa_N\Big] \qquad (2.4.11)
\end{aligned}
$$

for some positive constants C' and C''. This can be obtained by taking a 2nd-order Taylor expansion of the functions $\psi_0^{f^j}$, keeping in mind that $\psi_1^{f^j} > 0$. In the last inequality we also used $u \geqslant 1$.

Now fixing b small enough followed by then fixing δ small enough we can ensure that the entire quantity (2.4.11) is $\geqslant C'''\frac{u^2}{N}$ for some positive constant C''' as long as $u \geqslant C_1\kappa_N$ for some positive C_1. With these restrictions,

$$
\begin{aligned}
(2.4.8) \leqslant& \widehat{\mathbb{P}}\Big\{\overline{\log Z_{m,n}(a_1(\lambda_j), a_2(\lambda_j)) - \log Z_{m,n}(a_1, a_2)} \geqslant C'''\frac{u^2}{N}\Big\} \\
\leqslant& \frac{N^2}{(C''')^2u^4}\widehat{\mathbb{V}\mathrm{ar}}\big[\log Z_{m,n}(a_1(\lambda_j), a_2(\lambda_j)) - \log Z_{m,n}(a_1, a_2)\big] \\
\leqslant& C\frac{N^2}{u^4}\Big(\widehat{\mathbb{V}\mathrm{ar}}\big[\log Z_{m,n}(a_1, a_2)\big] + (m+n)\frac{bu}{N}\Big) \\
\leqslant& C\Big(\frac{N^2}{u^4}E_{m,n}[t_j] + \frac{N^2}{u^3}\Big).
\end{aligned}
$$

The second to last and last inequalities are applications of Lemma 2.18, Proposition 2.1, and Lemma 2.19. Combining this result with (2.4.9) finishes the first case.

Case 2: $1 \vee C_1\kappa_N \vee \delta N \leqslant u$. Take δ, ϵ fixed from the first case, let $\delta_1 \in (0,\delta]$, and $\epsilon_1 \in (0,\epsilon]$. The constants δ_1 and ϵ_1 will be determined throughout the course of the proof. This time, put $\lambda_1 = \epsilon_1$, $\lambda_2 = -\epsilon_1$, $r = \lfloor u \rfloor \big(\psi_0^{f^j}(a_j(\lambda_j)) - \psi_0^{f^j}(a_j)\big) - \delta_1 u$, and $z = \delta_1 u$. Then

$$(2.4.7) = \widehat{\mathbb{P}}\Big\{ \sum_{k=1}^{\lfloor u \rfloor} \overline{\log H^{f^j}(a_j, \eta_k^j) - \log H^{f^j}(a_j(\lambda_j), \eta_k^j))} \geqslant \delta_1 u \Big\}. \tag{2.4.12}$$

By Remark 2.11 the random variables in the summation have finite exponential moments. A large deviation estimate gives us the existence of a positive constant C_3 such that (2.4.12)$\leqslant e^{-C_3 u}$.

We now consider (2.4.10). A similar analysis to that in Case 1 tells us that the right-hand side inside of the above probability

$$\begin{aligned} &\geqslant -C'\epsilon_0\kappa_N - C'\epsilon_0^2 N + C''\epsilon_0 u - 2\delta_1 u \\ &\geqslant u\Big(C''\epsilon_0 - \frac{C'\epsilon_0^2}{\delta} - 2\delta_1\Big) - C'\epsilon_0\kappa_N \end{aligned} \tag{2.4.13}$$

for some positive constants C' and C'' (the second inequality follows from $u \geqslant \delta N$). Now fixing ϵ_0 small enough followed by then fixing δ_1 small enough we can ensure that (2.4.13) $\geqslant Cu$ for some positive constant C as long as $u \geqslant C_1\kappa_N$ for some positive C_1 (here we increase the previous constant C_1 found in Case 1 if necessary). With these constraints,

$$(2.4.8) \leqslant \widehat{\mathbb{P}}\Big\{ \overline{\log Z_{m,n}(a_1(\lambda_j), a_2(\lambda_j)) - \log Z_{m,n}(a_1, a_2)} \geqslant Cu \Big\}.$$

Since the perturbed parameters are such that the polymer environment still has the down-right property, the random variable inside the above probability can be expressed as two sums of i.i.d. random variables, each of which has entries with finite exponential moments. Therefore a large deviation estimate gives the existence of a positive constant C_3 such that (2.4.8) $\leqslant e^{-uC_3}$. Combining this with (2.4.12) completes the proof. □

Remark 2.21. *If $\epsilon > 0$ is small enough such that for all $|\lambda| \leqslant \epsilon$, $a_1 + \lambda \in D(M_{f^1})$ and $a_2 - \lambda \in D(M_{f^2})$, then the constants in Proposition 2.20 can be chosen such that the conclusion also holds for the polymer environment with parameters $(a_1 + \lambda, a_2 - \lambda, a_3)$ for any $|\lambda| \leqslant \epsilon$.*

Using the previous proposition, we can now bound the annealed expectation of the exit points of the polymer path from the axes.

Corollary 2.22. *Suppose all of the assumptions of Proposition 2.20 hold. Then there exists a positive constant C depending only on (a_1, a_2), β, and γ such that for both $j = 1, 2$,*

$$E_{m,n}[t_j] \leqslant CN^{2/3} \qquad \text{for all } N \in \mathbb{N}.$$

Proof of Corollary 2.22. Since all of the constants C_1, C_2, C_3, δ, δ_1 determined by Proposition 2.20 depend only on (a_1, a_2), β, and γ, it is sufficient to show that the constant C to be determined in this proof depends only on these five constants and γ. Let $r \geqslant 1 \vee C_1\gamma$. Then $rN^{2/3} \geqslant 1 \vee C_1\kappa_N$. Suppressing the m, n dependence,

$$\begin{aligned} E[t_j] &= \int_0^\infty P(t_j \geqslant u)du \\ &\leqslant rN^{2/3} + \int_{rN^{2/3}}^{rN^{2/3} \vee \delta N} P(t_j \geqslant u)du + \int_{rN^{2/3} \vee \delta N}^{\infty} P(t_j \geqslant u)du. \end{aligned} \tag{2.4.14}$$

We now bound the integrals in line (2.4.14) individually.

$$\begin{aligned}
\int_{rN^{2/3}\vee\delta N}^{\infty} P(t_j \geqslant u)du &= \int_{rN^{2/3}\vee\delta N}^{\infty}\int_0^{e^{-\delta_1 u}} \mathbb{P}\{Q(t_j \geqslant u) \geqslant x\}dxdu \\
&\quad + \int_{rN^{2/3}\vee\delta N}^{\infty}\int_{e^{-\delta_1 u}}^{1} \mathbb{P}\{Q(t_j \geqslant u) \geqslant x\}dxdu. \\
&\leqslant \int_{\delta N}^{\infty} e^{-\delta_1 u}du \\
&\quad + \int_{rN^{2/3}\vee\delta N}^{\infty}\int_0^{\delta_1} \mathbb{P}\{Q(t_j \geqslant u) \geqslant e^{-su}\}e^{-su}u\, dsdu \\
&\leqslant \frac{1}{\delta_1}e^{-\delta_1\delta N} + \int_{rN^{2/3}\vee\delta N}^{\infty}\int_0^{\delta_1} 2e^{-(C_3+s)u}u\, dsdu \leqslant C, \qquad (2.4.15)
\end{aligned}$$

where in the first inequality we bounded the first integrand by one and made the substitution $x = e^{-su}$ for the second. For the second inequality, we apply Proposition 2.20 to get that $\mathbb{P}\{Q(t_j \geqslant u) \geqslant e^{-su}\} \leqslant \mathbb{P}\{Q(t_j \geqslant u) \geqslant e^{-\delta_1 u}\} \leqslant 2e^{-C_3 u}$ for all $u \geqslant rN^{2/3} \vee \delta N$ and all $0 < s \leqslant \delta_1$.

We now bound the first integral of (2.4.14). Without loss of generality, assume that $rN^{2/3} < \delta N$. Then

$$\begin{aligned}
\int_{rN^{2/3}}^{rN^{2/3}\vee\delta N} P(t_j \geqslant u)du &= \int_{rN^{2/3}}^{\delta N}\int_0^{e^{-\delta\frac{u^2}{N}}} \mathbb{P}\{Q(t_j \geqslant u) \geqslant x\}dxdu \\
&\quad + \int_{rN^{2/3}}^{\delta N}\int_{e^{-\delta\frac{u^2}{N}}}^{1} \mathbb{P}\{Q(t_j \geqslant u) \geqslant x\}dxdu \\
&\leqslant \int_{rN^{2/3}}^{\delta N} e^{-\delta\frac{u^2}{N}}du \\
&\quad + \int_{rN^{2/3}}^{\delta N}\int_0^{\delta} \mathbb{P}\{Q(t_j \geqslant u) \geqslant e^{-s\frac{u^2}{N}}\}e^{-s\frac{u^2}{N}}\frac{u^2}{N}dsdu \\
&\leqslant \delta N e^{-\delta r^2 N^{1/3}} + \int_{rN^{2/3}}^{\infty}\int_{e^{-s\frac{u^2}{N}}}^{1} C_2(\frac{N^2}{u^4}E[t_j] + \frac{N^2}{u^3})dsdu \\
&\leqslant C + C_2\Big(\frac{E[t_j]}{3r^3} + \frac{N^{2/3}}{2r^2}\Big), \qquad (2.4.16)
\end{aligned}$$

where for the first inequality we bound the first integrand by one and make the substitution $x = e^{-s\frac{u^2}{N}}$ for the second. For the second inequality we apply Proposition 2.20

to get that $\mathbb{P}\{Q(t_j \geqslant u) \geqslant e^{-s\frac{u^2}{N}}\} \leqslant \mathbb{P}\{Q(t_j \geqslant u) \geqslant e^{-\delta\frac{u^2}{N}}\} \leqslant C_2\Big(\frac{N^2}{u^4}E[t_j] + \frac{N^2}{u^3}\Big)$ for all $rN^{2/3} \leqslant u \leqslant \delta N$ and all $0 < s \leqslant \delta$.

Combining the bounds on (2.4.15), (2.4.16) and (2.4.14), we have: for all $r \geqslant 1 \vee C_1\gamma$,

$$E[t_j] \leqslant rN^{2/3} + C + C_2\big(\frac{E[t_j]}{3r^3} + \frac{N^{2/3}}{2r^2}\big).$$

We can now fix r large enough with respect to C and C_2 then rearrange to get the desired result. $\square$

We can now give the proof of the upper bound of the variance of the free energy.

Proof of upper bound of Theorem 2.2. Averaging (2.1.3) and (2.1.4) of Proposition 2.1 then applying Lemma 2.19 followed by Corollary 2.22 (recalling that $\psi_1^{fj}(a_j) = \mathbb{V}\mathrm{ar}[\log R^j]$) gives

$$\begin{aligned}\mathbb{V}\mathrm{ar}[\log Z_{m,n}] &= E_{m,n}[\sum_{i=1}^{t_1} L_{R^1}(R^1_{i,0})] + E_{m,n}[\sum_{j=1}^{t_2} L_{R^2}(R^2_{0,j})] \\ &\leqslant C(E_{m,n}[t_1] + E_{m,n}[t_2] + 2) \\ &\leqslant CN^{2/3},\end{aligned}$$

which concludes the proof. $\square$

The following corollary is obtained by combining Proposition 2.20 and Corollary 2.22.

Corollary 2.23. *Assume that the polymer environment is distributed as in* (2.3.2) *and the sequence* $(m,n) = (m_N, n_N)_{N=1}^{\infty}$ *satisfies* (2.1.5) *for some positive constant* γ. *Then there exists positive constants* b_0, C_2, C_3, δ, *and* δ_1 *depending only on* (a_1, a_2), β, *and* γ *such that for all* $N \in \mathbb{N}$ *and* $b_0 \leqslant b \leqslant \delta N^{1/3}$,

$$\mathbb{P}\big\{Q_{m,n}(t_j \geqslant bN^{2/3}) \geqslant e^{-\delta b^2 N^{1/3}}\big\} \leqslant \frac{2C_2}{b^3} \quad \textit{for } j = 1, 2, \tag{2.4.17}$$

while for all $N \in \mathbb{N}$ *and* $b \geqslant b_0 \vee \delta N^{1/3}$,

$$\mathbb{P}\big\{Q_{m,n}(t_j \geqslant bN^{2/3}) \geqslant e^{-\delta_1 bN^{2/3}}\big\} \leqslant 2e^{-C_3 bN^{2/3}} \quad \text{for } j = 1, 2. \tag{2.4.18}$$

Lemma 2.24. *Assume that the polymer environment is distributed as in* (2.3.2) *and the sequence* $(m,n) = (m_N, n_N)_{N=1}^{\infty}$ *satisfies* (2.1.5) *for some positive constant* γ. *Then there exist constants* $b_0 \geqslant 1$ *and* $C > 0$ *depending only on* (a_1, a_2), β, *and* γ *such that for all* $b \geqslant b_0$ *and* $N \in \mathbb{N}$,

$$P_{m,n}(t_j \geqslant bN^{2/3}) \leqslant \frac{C}{b^3} \quad \text{for } j = 1, 2.$$

Therefore, for all $0 < p < 3$ *there exists a positive constant* C' *depending on* (a_1, a_2), β, γ, *and* p *such that for all* $N \in \mathbb{N}$,

$$E_{m,n}\Big[\Big(\frac{t_j}{N^{2/3}}\Big)^p\Big] \leqslant C' \quad \text{for } j = 1, 2.$$

Proof of Lemma 2.24. By Corollary (2.23) there exist positive constants b_0, C_2, C_3, δ, δ_1 with $b_0 \geqslant 1$ such that (2.4.17) holds for $b_0 \leqslant b \leqslant \delta N^{1/3}$ while (2.4.18) holds for $b \geqslant \delta N^{1/3} \vee b_0$.

We first estimate for $b \leqslant \delta N^{1/3}$,

$$\begin{aligned} P_{m,n}(t_j \geqslant bN^{2/3}) &= \int_0^1 \mathbb{P}\big\{Q_{m,n}(t_j \geqslant bN^{2/3}) \geqslant x\big\}dx \\ &= \int_0^\delta \mathbb{P}\big\{Q_{m,n}(t_j \geqslant bN^{2/3}) \geqslant e^{-sb^2N^{1/3}}\big\}b^2N^{1/3}e^{-sb^2N^{1/3}}ds \qquad (2.4.19) \\ &\quad + \int_\delta^\infty \mathbb{P}\big\{Q_{m,n}(t_j \geqslant bN^{2/3}) \geqslant e^{-sb^2N^{1/3}}\big\}b^2N^{1/3}e^{-sb^2N^{1/3}}ds \qquad (2.4.20) \\ &\leqslant \frac{2C_2}{b^3} + e^{-\delta b^2 N^{1/3}} \leqslant \frac{C}{b^3} \end{aligned}$$

for some positive constant C, where we made the substitution $x = e^{-sb^2N^{1/3}}$, used (2.4.17) to bound the probability inside the integral of (2.4.19), and bounded the probability inside the integral of (2.4.20) by 1. For $b \geqslant \delta N^{1/3}$, we make the substitution $x = e^{-sbN^{2/3}}$ to get

$$
\begin{aligned}
P_{m,n}(t_j \geqslant bN^{2/3}) &= \int_0^1 \mathbb{P}\{Q_{m,n}(t_j \geqslant bN^{2/3}) \geqslant x\}dx \\
&= \int_0^{\delta_1} \mathbb{P}\{Q_{m,n}(t_j \geqslant bN^{2/3}) \geqslant e^{-sbN^{2/3}}\}bN^{2/3}e^{-sbN^{2/3}}ds \qquad (2.4.21)\\
&\quad + \int_{\delta_1}^{\infty} \mathbb{P}\{Q_{m,n}(t_j \geqslant bN^{2/3}) \geqslant e^{-sbN^{2/3}}\}bN^{2/3}e^{-sbN^{2/3}}ds \qquad (2.4.22)\\
&\leqslant 2e^{-C_3 bN^{2/3}} + e^{-\delta_1 bN^{2/3}} \leqslant \frac{C}{b^3}
\end{aligned}
$$

increasing the constant C if necessary, where we used (2.4.18) to bound the probability inside the integral of (2.4.21) and bounded the probability inside the integral of (2.4.22) by 1. □

Proof of Corollary 2.4. Let $m_1 = \lfloor N\mathbb{V}\mathrm{ar}[\log R^2]\rfloor$. Since $Z_{m,n} = Z_{m_1,n} \prod_{i=m_1+1}^{m} R^1_{i,n}$,

$$N^{-\alpha/2}\overline{\log Z_{m,n}} = N^{-\alpha/2}\overline{\log Z_{m_1,n}} + N^{-\alpha/2} \sum_{i=m_1+1}^{m} \overline{\log R^1_{i,n}}.$$

The sequence (m_1, n) satisfies (2.1.5). Using Chebyshev's inequality and the upper bound of Theorem 2.2 shows that the term $N^{-\alpha/2}\overline{\log Z_{m_1,n}}$ converges to zero in probability. By the down-right property, the summands in the second term are i.i.d. with mean zero and variance $\mathbb{V}\mathrm{ar}[\log R^1]$. By the central limit theorem, $N^{-\alpha/2}\sum_{i=m_1+1}^{m} \overline{\log R^1_{i,n}}$ converges in distribution to a centered normal with variance $c_1\mathbb{V}\mathrm{ar}[\log R^1]$. □

2.5 Proof of the path fluctuation upper bound

Given $0 \leqslant k < m$ and $0 \leqslant l < n$, we define a partition function $Z^{(k,l)}_{m,n}$ and quenched polymer measure $Q^{(k,l)}_{m,n}$ on up-right paths from (k, l) to (m, n) by using the collections

$\{R^1_{i,l} : k+1 \leqslant i \leqslant m\}$ and $\{R^2_{k,j} : l+1 \leqslant j \leqslant n\}$ as weights along the edges of the south and west boundaries of the rectangle $[k,m] \times [l,n]$ respectively, and the weights $\{(Y^1_z, Y^2_z) : z \in \{k+1,\dots,m\} \times \{l+1,\dots,n\}\}$ for the remaining edges. When the original polymer environment (1.2.1) has the down-right property, it follows that $Z^{(k,l)}_{m,n}$ has the same distribution as $Z_{m-k,n-l}$.

For an up-right path $x_\bullet$ from (k,l) to (m,n), define

$$t^{(k,l)}_1(x_\bullet) := \max\{i : (k+i,l) \in x_\bullet\}, \qquad t^{(k,l)}_2(x_\bullet) := \max\{j : (k,l+j) \in x_\bullet\}.$$

Recall the definition (2.1.7).

Lemma 2.25. *Assume that the polymer environment satisfies the down-right property. Then for all* $0 \leqslant k < m$, $0 \leqslant l < n$, *and* $u \geqslant 0$,

$$Q_{m,n}(v_1(l) \geqslant k+u) = Q^{(k,l)}_{m,n}(t^{(k,l)}_1 \geqslant u) \overset{d}{=} Q_{m-k,n-l}(t_1 \geqslant u), \tag{2.5.1}$$

$$Q_{m,n}(w_1(k) \geqslant l+u) = Q^{(k,l)}_{m,n}(t^{(k,l)}_2 \geqslant u) \overset{d}{=} Q_{m-k,n-l}(t_2 \geqslant u). \tag{2.5.2}$$

Proof. For $0 \leqslant i < m$ and $0 \leqslant j < n$, we let

$$Z_{(i,j),(m,n)} := \sum_{x_\bullet} \prod_{k=1}^{(m-i)+(n-j)} \omega_{(x_{k-1},x_k)}$$

denote the partition function for up-right paths from (i,j) to (m,n), where the sum is taken over all such paths. A decomposition shows that

$$\begin{aligned} Z^{(k,l)}_{m,n} &= \sum_{i=k+1}^{m} \left(\prod_{a=k+1}^{i} R^1_{a,l} \right) Y^2_{i,l+1} Z_{(i,l+1),(m,n)} \\ &\quad + \sum_{j=l+1}^{n} \left(\prod_{b=l+1}^{j} R^2_{k,b} \right) Y^1_{k+1,j} Z_{(k+1,j),(m,n)} \\ &= \sum_{i=k+1}^{m} \frac{Z_{i,l}}{Z_{k,l}} Y^2_{i,l+1} Z_{(i,l+1),(m,n)} + \sum_{j=l+1}^{n} \frac{Z_{k,j}}{Z_{k,l}} Y^1_{k+1,j} Z_{(k+1,j),(m,n)} = \frac{Z_{m,n}}{Z_{k,l}}. \end{aligned}$$

We then have that for $r \in \{0, \ldots, m-k\}$,

$$\begin{aligned} Q_{m,n}^{(k,l)}(t_1^{(k,l)} = r) &= \frac{1}{Z_{m,n}^{(k,l)}} \left(\prod_{i=1}^{r} R_{k+i,l}^1 \right) Y_{k+r,l+1}^2 Z_{(k+r,l+1),(m,n)} \\ &= \frac{1}{Z_{m,n}^{(k,l)}} \frac{Z_{k+r,l}}{Z_{k,l}} Y_{k+r,l+1}^2 Z_{(k+r,l+1),(m,n)} \\ &= \frac{Z_{k+r,l} Y_{k+r,l+1}^2 Z_{(k+r,l+1),(m,n)}}{Z_{m,n}} \\ &= Q_{m,n}(v_1(l) = k+r). \end{aligned}$$

Summing over $r \geqslant u$ gives the first equality in (2.5.1). The equality in distribution follows from the down-right property. An analogous argument gives (2.5.2). □

We can now prove the upper bound on the polymer path fluctuations under the annealed measure.

Proof of Theorem 2.5. By assumption, the polymer environment is distributed as in (2.3.2). If $\tau = 0$ this reduces to Lemma 2.24. If $\tau \in (0,1)$ put $(k,l) = (\lfloor \tau m \rfloor, \lfloor \tau n \rfloor)$. By part (c) of Remark 2.11, $\mathbb{V}\mathrm{ar}[R^i] = \psi_1^{f^i}(a_i)$ for $i = 1,2$. Multiplying (2.1.5) by $(1-\tau)$, up to integer corrections the sequence $(m-k, n-l)$ satisfies

$$|m - k - M\psi_1^{f^2}(a_2)| \vee |n - l - M\psi_1^{f^1}(a_1)| \leqslant \gamma_0 M^{2/3}, \tag{2.5.3}$$

where $M = (1-\tau)N$ and $\gamma_0 = \gamma(1-\tau)^{1/3}$. We then apply Lemma 2.25 to get

$$\begin{aligned} Q_{m,n}(v_1(\lfloor \tau n \rfloor) \geqslant \tau m + bN^{2/3}) &\leqslant Q_{m,n}(v_1(\lfloor \tau n \rfloor) \geqslant \lfloor \tau m \rfloor + bN^{2/3}) \\ &\overset{d}{=} Q_{m-k,n-l}(t_1 \geqslant bN^{2/3}). \end{aligned}$$

Applying Lemma 2.24, we get

$$P_{m,n}(v_1(\lfloor \tau n \rfloor) \geqslant \tau m + bN^{2/3}) \leqslant \frac{C}{b^3}. \tag{2.5.4}$$

The same argument in the vertical direction gives us

$$P_{m,n}(w_1(\lfloor \tau m \rfloor) \geqslant \tau n + bN^{2/3}) \leqslant \frac{C}{b^3}. \tag{2.5.5}$$

To prove the corresponding bounds for v_0 and w_0 we now let $k = \lfloor \tau m - bN^{2/3} \rfloor$ and $l = \lfloor \tau n - bN^{2/3} \frac{n}{m} \rfloor$. Again $(m-k, n-l)$ will satisfy (2.5.3) for a different constant γ_0. Since $w_1(k) \geqslant \lfloor \tau n \rfloor$ implies that $v_0(\lfloor \tau n \rfloor) \leqslant k$, it follows that

$$\begin{aligned} Q_{m,n}(v_0(\lfloor \tau n \rfloor) \leqslant \tau m - bN^{2/3}) &\leqslant Q_{m,n}(w_1(k) \geqslant \lfloor \tau n \rfloor) \\ &= Q_{m,n}^{(k,l)}(t_2^{(k,l)} \geqslant \lfloor \tau n \rfloor - l) \\ &\leqslant Q_{m,n}^{(k,l)}(t_2^{(k,l)} \geqslant CbN^{2/3}) \\ &\overset{d}{=} Q_{m-k,n-l}(t_2 \geqslant CbN^{2/3}), \end{aligned}$$

for some constant C depending on (a_1, a_2), β, and γ. Applying Lemma 2.24 gives

$$P_{m,n}(v_0(\lfloor \tau n \rfloor) \leqslant \tau m - bN^{2/3}) \leqslant \frac{C}{b^3}. \tag{2.5.6}$$

An analogous argument shows that

$$P_{m,n}(w_0(\lfloor \tau m \rfloor) \leqslant \tau n - bN^{2/3}) \leqslant \frac{C}{b^3}. \tag{2.5.7}$$

Combining bounds (2.5.4) and (2.5.6) gives (2.1.8), and (2.5.5) with (2.5.7) gives (2.1.9), completing the proof. □

2.6 Proof of the variance and path fluctuation lower bounds

Proposition 2.26. *Assume that the polymer environment is distributed as in* (2.3.2) *and the sequence* $(m, n) = (m_N, n_N)_{N=1}^{\infty}$ *satisfies* (2.1.5) *for some positive constant* γ.

Then there exist positive constants c_0, ϵ_0, N_0 depending only on (a_1, a_2), β and γ such that for all $N \geqslant N_0$,

$$\mathbb{P}\big(\overline{\log Z_{m,n}} \geqslant c_0 N^{1/3}\big) \geqslant \epsilon_0.$$

From this proposition we can obtain the lower bound of Theorem 2.2:

$$\begin{aligned}\mathbb{V}\mathrm{ar}[\log Z_{m,n}] &\geqslant \mathbb{E}\big[(\overline{\log Z_{m,n}})^2 : \overline{\log Z_{m,n}} \geqslant c_0 N^{1/3}\big] \\ &\geqslant \mathbb{P}\big(\overline{\log Z_{m,n}} \geqslant c_0 N^{1/3}\big)(c_0 N^{1/3})^2 \\ &\geqslant \epsilon_0 c_0^2 N^{2/3}.\end{aligned}$$

Proof of Proposition 2.26. Let $\epsilon > 0$ be small enough such that for all $|\lambda| \leqslant \epsilon$, $a_1 + \lambda \in D(M_{f^1})$ and $a_2 - \lambda \in D(M_{f^2})$. Define

$$\widetilde{m} = \lfloor m \frac{\psi_1^{f^2}(a_2 - \lambda)}{\psi_1^{f^2}(a_2)} \rfloor, \qquad \widetilde{n} = \lfloor n \frac{\psi_1^{f^1}(a_1 + \lambda)}{\psi_1^{f^1}(a_1)} \rfloor.$$

Taking Taylor expansions gives

$$\begin{aligned} m - \widetilde{m} &= \lambda \frac{\psi_2^{f^2}(a_2)}{\psi_1^{f^2}(a_2)} m + o(\lambda) m \\ \widetilde{n} - n &= \lambda \frac{\psi_2^{f^1}(a_1)}{\psi_1^{f^1}(a_1)} n + o(\lambda) n. \end{aligned} \tag{2.6.1}$$

Let b be a large fixed constant which will be determined through the course of the proof. Then there exists $N_0 \in \mathbb{N}$ such that for all $N \geqslant N_0$, $bN^{-1/3} \leqslant \epsilon$. Then with $\lambda = bN^{-1/3}$, the sequence $(\widetilde{m}, \widetilde{n})$ satisfies

$$|\widetilde{m} - N\psi_1^{f^2}(a_2 - \lambda)| \vee |\widetilde{n} - N\psi_1^{f^1}(a_1 + \lambda)| \leqslant \gamma_0 N^{2/3}$$

for some positive constant γ_0. By Table 22 and (C.0.1) in the Appendix, in each of the four basic beta-gamma models, either $\psi_2^{f^1}(a_1)$ and $\psi_2^{f^2}(a_2)$ are both positive (inverse-beta model for certain choices of parameters and inverse-gamma model for all choices of parameters), $\psi_2^{f^1}(a_1)$ is negative and $\psi_2^{f^2}(a_2)$ is positive (gamma and beta models), or $\psi_2^{f^1}(a_1)$ is positive and $\psi_2^{f^2}(a_2)$ is non-positive (inverse-beta model with the remaining choices of parameters). By flipping the x and y axes in the second case, we only need to consider the first and third cases.

For the case where $\psi_2^{f^1}(a_1)$ and $\psi_2^{f^2}(a_2)$ are both positive define $A_N = m - \widetilde{m}$ and $B_N = \widetilde{n} - n$. This case is illustrated in Figure 12. By (2.6.1) and increasing N_0 if necessary, there exist positive constants c_1, c_2, C_1, C_2 such that for $N \geqslant N_0$,

$$c_1 b N^{2/3} \leqslant A_N \leqslant C_1 b N^{2/3},$$
$$c_2 b N^{2/3} \leqslant B_N \leqslant C_2 b N^{2/3}.$$

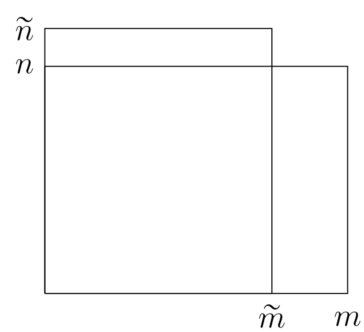

Figure 12: Case 1: $\psi_2^{f_1}$ and $\psi_2^{f_2}$ are both positive.

In the case where $\psi_2^{f^1}(a_1) > 0$ and $\psi_2^{f^2}(a_2) \leqslant 0$ we define $c := \frac{1}{2}(\frac{m}{\widetilde{m}} + \frac{n}{\widetilde{n}})$ and let $\overline{m} = c\widetilde{m}$, $\overline{n} = c\widetilde{n}$. This case is illustrated in Figure 13. This $(\overline{m}, \overline{n})$ will satisfy

$$|\overline{m} - M\psi_1^{f^2}(a_2 - \lambda)| \vee |\overline{n} - M\psi_1^{f^1}(a_1 + \lambda)| \leqslant \gamma_0 c^{1/3} M^{2/3}$$

where $M = cN$. A Taylor expansion gives

$$c = 1 + \left(\frac{\psi_2^{f^2}(a_2)}{\psi_1^{f^2}(a_2)} - \frac{\psi_2^{f^1}(a_1)}{\psi_1^{f^1}(a_1)} \right) \frac{\lambda}{2} + o(\lambda)$$

and thus

$$m - \overline{m} = \frac{\lambda}{2}\left(\frac{\psi_2^{f^2}(a_2)}{\psi_1^{f^2}(a_2)} + \frac{\psi_2^{f^1}(a_1)}{\psi_1^{f^1}(a_1)}\right) m + o(N^{2/3}),$$
$$\overline{n} - n = \frac{\lambda}{2}\left(\frac{\psi_2^{f^2}(a_2)}{\psi_1^{f^2}(a_2)} + \frac{\psi_2^{f^1}(a_1)}{\psi_1^{f^1}(a_1)}\right) n + o(N^{2/3}).$$

The quantity $\frac{\psi_2^{f^2}(a_2)}{\psi_1^{f^2}(a_2)}+\frac{\psi_2^{f^1}(a_1)}{\psi_1^{f^1}(a_1)}$ is positive since $\psi_1^{f^1}$ and $\psi_1^{f^2}$ are both positive and $\psi_1^{f^2}(a_2)\psi_2^{f^1}(a_1)+\psi_1^{f^1}(a_1)\psi_2^{f^2}(a_2) > 0$ by Lemma C.2 in the Appendix. Letting $\overline{A} = m-\overline{m}$ and $\overline{B} = \overline{n}-n$, there exist positive constants c_1', c_2', C_1', C_2' such that

$$c_1' bM^{2/3} \leqslant \overline{A}_M \leqslant C_1' bM^{2/3},$$
$$c_2' bM^{2/3} \leqslant \overline{B}_M \leqslant C_2' bM^{2/3}.$$

Recall that $\mathbb{P}^{(a_1,a_2)}$ is used to denote the probability measure on the polymer environment

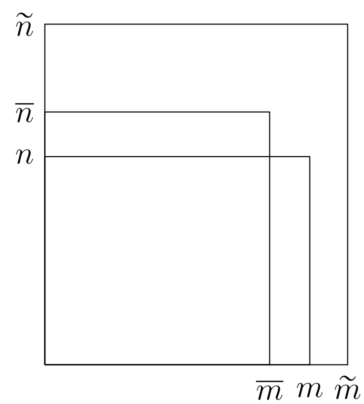

Figure 13: Case 2: $\psi_2^{f_1} > 0$ and $\psi_2^{f_2} \leqslant 0$.

with parameters a_1 and a_2. Let $(\widetilde{a}_1, \widetilde{a}_2) = (a_1+\lambda, a_2-\lambda)$. Our goal is to show that

$$\mathbb{P}^{(a_1,a_2)}\big(\log Z_{m,n} \geqslant \mathbb{E}[\log Z_{m,n}] + c_0 N^{1/3}\big) \geqslant \epsilon_0.$$

We will do so by making estimates using the $(\widetilde{a}_1, \widetilde{a}_2)$ environment and then use a coupling of the two environments to transfer the results to the (a_1, a_2) environment.

We would first like to show that in the $(\tilde{a}_1, \tilde{a}_2)$ environment, with high probability the quenched probability gives most of the weight to paths which exit the x-axis at a point of order $bN^{2/3}$. That is: there exist positive constants C_3, C such that, given any $\varepsilon > 0$,

$$\mathbb{P}^{(\tilde{a}_1,\tilde{a}_2)}\big\{Q_{m,n}(c_1 bN^{2/3} \leqslant t_1 \leqslant C_3 bN^{2/3}) \geqslant 1-\varepsilon\big\} \geqslant 1 - \frac{C}{b^3} \tag{2.6.2}$$

holds for all sufficiently large N.

We start by using Lemma 2.25 to relate an upper bound on t_1 to a lower bound on t_2.

$$Q_{m,n}(t_1 \leqslant A_N) \stackrel{d}{=} Q_{m,\tilde{n}}(v_1(B_N) \leqslant A_N) = Q_{m,\tilde{n}}(w_1(A_N) > B_N) \stackrel{d}{=} Q_{\tilde{m},\tilde{n}}(t_2 > B_N).$$

Using this and Corollary 2.23, there exists $\delta > 0$ such that

$$\begin{aligned}
\mathbb{P}^{(\tilde{a}_1,\tilde{a}_2)}&\big\{Q_{m,n}(t_1 > c_1 bN^{2/3}) \geqslant 1 - e^{-\frac{\delta}{N}B_N^2}\big\} \\
&\geqslant \mathbb{P}^{(\tilde{a}_1,\tilde{a}_2)}\big\{Q_{m,n}(t_1 > A_N) \geqslant 1 - e^{-\frac{\delta}{N}B_N^2}\big\} \\
&= \mathbb{P}^{(\tilde{a}_1,\tilde{a}_2)}\big\{Q_{m,n}(t_1 \leqslant A_N) \leqslant e^{-\frac{\delta}{N}B_N^2}\big\} \\
&= \mathbb{P}^{(\tilde{a}_1,\tilde{a}_2)}\big\{Q_{\tilde{m},\tilde{n}}(t_2 > B_N) \leqslant e^{-\frac{\delta}{N}B_N^2}\big\} \\
&\geqslant 1 - Cb^{-3}.
\end{aligned}$$

This implies that

$$\mathbb{P}^{(\tilde{a}_1,\tilde{a}_2)}\big\{Q_{m,n}(t_1 \leqslant c_1 bN^{2/3}) \geqslant e^{-\frac{\delta}{N}B_N^2}\big\} \leqslant Cb^{-3}.$$

Applying the upper bound (2.4.17) directly with $C_3 > C_2$, we obtain

$$\mathbb{P}^{(\tilde{a}_1,\tilde{a}_2)}\big\{Q_{m,n}(t_1 > C_3 bN^{2/3}) \geqslant e^{-\frac{\delta}{N}B_N^2}\big\} \leqslant Cb^{-3}$$

for another positive constant C. Taking a union bound we put the two bounds together and get

$$\mathbb{P}^{(\tilde{a}_1,\tilde{a}_2)}\big\{Q_{m,n}(c_1bN^{2/3} \leqslant t_1 \leqslant C_3bN^{2/3}) \geqslant 1 - 2e^{-\frac{\delta}{N}B_N^2}\big\} \geqslant 1 - Cb^{-3}.$$

Taking N large enough, we get (2.6.2).

The argument for the case where we use $(\overline{m}, \overline{n})$ and $\overline{A}, \overline{B}$ is unchanged, with the exception of using the scaling parameter M rather than N. This difference can be absorbed into the constants.

In order to make use of the bound (2.6.2) for the system with the original (a_1, a_2) environment we create a new measure $\breve{\mathbb{P}}$ which has both a_1 and $\tilde{a}_1$ distributed weights along the x-axis and estimate the Radon-Nikodym derivative of the (a_1, a_2) environment with respect to this new environment.

Let $\breve{\omega}$ denote the environment that has the same weights as the (a_1, a_2) environment except for the weights $R^1_{i,0}$ for $1 \leqslant i \leqslant \lfloor C_3bN^{2/3} \rfloor$, which will be distributed with parameter $\tilde{a}_1$. Let $\breve{\mathbb{P}}$ denote the probability measure of this environment. Then for each path $x_\centerdot$ with $c_1bN^{2/3} \leqslant t_1(x_\centerdot) \leqslant C_3bN^{2/3}$, the weight of the path in the $(\tilde{a}_1, \tilde{a}_2)$ environment and the weight of the path in the $\breve{\omega}$ environment agree. Thus, defining $Z_{m,n}(A) := \sum_{x_\centerdot \in A} \prod_{k=1}^{m+n} \omega_{(x_{k-1},x_k)}$,

$$Z_{m,n}(c_1bN^{2/3} \leqslant t_1 \leqslant C_3bN^{2/3}) \tag{2.6.3}$$

is the same in distribution under $\mathbb{P}^{(\tilde{a}_1,\tilde{a}_2)}$ and $\breve{\mathbb{P}}$. We can now make use of the bound (2.6.2).

Using a third-order Taylor expansion, the same series of calculations which leads to inequality (2.4.11) in the proof of Proposition 2.20 gives the existence of a constant

$C' > 0$ such that:

$$\begin{aligned}\mathbb{E}^{(\tilde{a}_1,\tilde{a}_2)}[\log Z_{m,n}] - \mathbb{E}^{(a_1,a_2)}[\log Z_{m,n}] &= m\left(\Psi_0^{f_1}(\tilde{a}_1) - \Psi_0^{f_1}(a_1)\right) \\ &\quad + n\left(\Psi_0^{f_2}(\tilde{a}_2) - \Psi_0^{f_2}(a_2)\right) \\ &\geqslant -\gamma b N^{1/3} C' + 4c_4 b^2 N^{1/3} - b^3 C' \\ &\geqslant c_4 b^2 N^{1/3}\end{aligned} \tag{2.6.4}$$

where $c_4 := \frac{1}{8}\Big(\psi_1^{f^2}(a_2)\psi_2^{f^1}(a_1) + \psi_1^{f^1}(a_1)\psi_2^{f^2}(a_2)\Big)$ is positive by Lemma C.2 in the Appendix. The last inequality is obtained by first fixing b large enough then increasing N_0 if necessary.

We now split the probability

$$\begin{aligned}&\mathbb{P}^{(\tilde{a}_1,\tilde{a}_2)}\big\{Q_{m,n}(c_1 b N^{2/3} \leqslant t_1 \leqslant C_3 b N^{2/3}) \geqslant 1-\varepsilon\big\} \\ &= \mathbb{P}^{(\tilde{a}_1,\tilde{a}_2)}\left\{\frac{1}{Z_{m,n}} Z_{m,n}(c_1 b N^{2/3} \leqslant t_1 \leqslant C_3 b N^{2/3}) \geqslant 1-\varepsilon\right\} \\ &\leqslant \breve{\mathbb{P}}\left\{Z_{m,n}(c_1 b N^{2/3} \leqslant t_1 \leqslant C_3 b N^{2/3}) \geqslant (1-\varepsilon)e^{\mathbb{E}^{(\tilde{a}_1,\tilde{a}_2)}[\log Z_{m,n}] - \frac{1}{2}c_4 b^2 N^{1/3}}\right\} \qquad (2.6.5) \\ &\qquad + \mathbb{P}^{(\tilde{a}_1,\tilde{a}_2)}\left\{Z_{m,n} \leqslant e^{\mathbb{E}^{(\tilde{a}_1,\tilde{a}_2)}[\log Z_{m,n}] - \frac{1}{2}c_4 b^2 N^{1/3}}\right\} \\ &\leqslant \breve{\mathbb{P}}\left\{Z_{m,n}(c_1 b N^{2/3} \leqslant t_1 \leqslant C_3 b N^{2/3}) \geqslant (1-\varepsilon)e^{\mathbb{E}^{(a_1,a_2)}[\log Z_{m,n}] + \frac{1}{2}c_4 b^2 N^{1/3}}\right\} \qquad (2.6.6) \\ &\qquad + \mathbb{P}^{(\tilde{a}_1,\tilde{a}_2)}\left\{Z_{m,n} \leqslant e^{\mathbb{E}^{(\tilde{a}_1,\tilde{a}_2)}[\log Z_{m,n}] - \frac{1}{2}c_4 b^2 N^{1/3}}\right\}. \qquad (2.6.7)\end{aligned}$$

The transition from $\mathbb{P}^{(\tilde{a}_1,\tilde{a}_2)}$ to $\breve{\mathbb{P}}$ in (2.6.5) is due to the equality in distribution of (2.6.3) under these measures. Inequality (2.6.6) comes from (2.6.4).

For (2.6.7) we can use Chebyshev's inequality then the upper bound of the variance to get

$$(2.6.7) \leqslant \frac{C}{b^3}.$$

Thus (2.6.6) $\geqslant 1 - \frac{C}{b^3}$ for some new positive constant C. Let g be the Radon-Nikodym derivative $d\breve{\mathbb{P}}/d\mathbb{P}^{(a_1,a_2)}$. Recall that the distributions differ only on the weights along the x-axis up until site $\lfloor C_3 b N^{2/3} \rfloor$. Thus

$$g(\omega) = \left(\frac{M_{f^1}(a_1)}{M_{f^1}(\tilde{a}_1)} \right)^{\lfloor C_3 b N^{2/3} \rfloor} \prod_{i=1}^{\lfloor C_3 b N^{2/3} \rfloor} \omega_{i,0}^{\lambda}.$$

We can evaluate $\mathbb{E}^{(a_1,a_2)}[g^2]$ explicitly. Increasing N_0, if necessary, so that $2\lambda \leqslant \epsilon$,

$$\mathbb{E}^{(a_1,a_2)}[\omega_{i,0}^{2\lambda}] = \frac{1}{M_{f^1}(a_1)} \int_0^\infty x^{2\lambda} x^{a_1 - 1} f^1(x) dx = \frac{M_{f^1}(a_1 + 2\lambda)}{M_{f^1}(a_1)}.$$

Now

$$\begin{aligned} \mathbb{E}^{(a_1,a_2)}[g^2] &= \left(\frac{M_{f^1}(a_1)}{M_{f^1}(\tilde{a}_1)} \right)^{2\lfloor C_3 b N^{2/3} \rfloor} \prod_{i=1}^{\lfloor C_3 b N^{2/3} \rfloor} \mathbb{E}^{(a_1,a_2)}[\omega_{i,0}^{2\lambda}] \\ &= \left(\frac{M_{f^1}(a_1) M_{f^1}(a_1 + 2\lambda)}{M_{f^1}(a_1 + \lambda)^2} \right)^{\lfloor C_3 b N^{2/3} \rfloor}. \end{aligned}$$

Taking logarithms of both sides,

$$\begin{aligned} &\log \mathbb{E}^{(a_1,a_2)}[g^2] \\ &\quad = \lfloor C_3 b N^{2/3} \rfloor \Big(\log M_{f^1}(a_1) + \log M_{f^1}(a_1 + 2bN^{-1/3}) - 2 \log M_{f^1}(a_1 + bN^{-1/3}) \Big) \end{aligned}$$

Recall that $\frac{\partial^2}{\partial a^2} \log M_{f^1}(a) = \psi_1^{f^1}(a) > 0$. Then

$$\begin{aligned} &\lim_{N\to\infty} \log \mathbb{E}^{(a_1,a_2)}[g^2] \\ &\qquad = C_3 b \lim_{N\to\infty} \frac{\log M_{f^1}(a_1) + \log M_{f^1}(a_1 + 2bN^{-1/3}) - 2 \log M_{f^1}(a_1 + bN^{-1/3})}{N^{-2/3}} \\ &\qquad = C_3 b^2 \lim_{N\to\infty} \frac{\psi_0^{f^1}(a_1 + 2bN^{-1/3}) - \psi_0^{f^1}(a_1 + bN^{-1/3})}{N^{-1/3}} \\ &\qquad = C_3 b^3 \psi_1^{f^1}(a_1) > 0 \end{aligned}$$

Increase N_0 if necessary so that for all $N \geqslant N_0$,

$$\mathbb{E}^{(a_1,a_2)}[g^2] \leqslant e^{2C_3 b^3}.$$

Defining the event

$$D = \left\{ Z_{m,n}(c_1 b N^{2/3} \leqslant t_1 \leqslant C_3 b N^{2/3}) \geqslant (1-\epsilon) e^{\mathbb{E}^{(a_1,a_2)}[\log Z_{m,n}] + \frac{1}{2} c_4 b^2 N^{1/3}} \right\},$$

we get

$$\begin{aligned} 1 - \frac{C}{b^3} \leqslant (2.6.6) &= \breve{\mathbb{P}}(D) \\ &= \mathbb{E}^{(a_1,a_2)}[g \mathbb{1}_D] \\ &\leqslant \left(\mathbb{E}^{(a_1,a_2)}[g^2]\right)^{1/2} \left(\mathbb{P}^{(a_1,a_2)}(D)\right)^{1/2} \\ &\leqslant e^{C_3 b^3} \left(\mathbb{P}^{(a_1,a_2)}(D)\right)^{1/2}. \end{aligned}$$

Thus

$$\epsilon_0 := (1 - \frac{C}{b^3})^2 e^{-2C_3 b^3} \leqslant \mathbb{P}^{(a_1,a_2)}(D).$$

Finally we have that

$$\begin{aligned} \epsilon_0 \leqslant \mathbb{P}^{(a_1,a_2)}(D) \leqslant \mathbb{P}^{(a_1,a_2)} &\left(Z_{m,n} \geqslant (1-\varepsilon) e^{\mathbb{E}^{(a_1,a_2)}[\log Z_{m,n}] + \frac{1}{2} c_4 b^2 N^{1/3}} \right) \\ = \mathbb{P}^{(a_1,a_2)} &\Big(\log Z_{m,n} \geqslant \log(1-\varepsilon) + \mathbb{E}^{(a_1,a_2)}[\log Z_{m,n}] + \tfrac{c_4 b^2 N^{1/3}}{2} \Big) \\ \leqslant \mathbb{P}^{(a_1,a_2)} &\big(\log Z_{m,n} \geqslant \mathbb{E}^{(a_1,a_2)}[\log Z_{m,n}] + c_0 N^{1/3} \big). \end{aligned}$$

Increasing N_0 if necessary and taking $c_0 = \frac{1}{4} c_4 b^2$ the final inequality holds for all $N \geqslant N_0$. This concludes the proof. □

We can use the variance lower bound to obtain a lower bound on the exit points of the path from the horizontal and vertical axes.

Corollary 2.27. *Assume that the polymer environment is distributed as in* (2.3.2) *and the sequence* $(m,n) = (m_N, n_N)_{N=1}^{\infty}$ *satisfies* (2.1.5) *for some positive constant* γ. *Then* *there exist positive constants* c_0, c_1, N_0 *depending only on* (a_1, a_2), β *and* γ *such that for all* $N \geqslant N_0$,

$$c_0 \leqslant P_{m,n}(t_1 > c_1 N^{2/3} \text{ or } t_2 > c_1 N^{2/3}).$$

Proof. Averaging (2.1.3) and (2.1.4) of Proposition 2.1 then applying Lemma 2.19 followed by the lower bound of Theorem 2.2 gives the existence of positive constants c, C, N_0 such that for all $N \geqslant N_0$

$$\begin{aligned} cN^{2/3} \leqslant \mathbb{V}\mathrm{ar}[\log Z_{m,n}] &= E_{m,n}[\sum_{i=1}^{t_1} L_{R^1}(R^1_{i,0})] + E_{m,n}[\sum_{j=1}^{t_2} L_{R^2}(R^2_{0,j})] \\ &\leqslant C(E_{m,n}[t_1 + t_2] + 2). \end{aligned}$$

Letting $c_1 := c/6C$ and increasing N_0 if necessary followed by an application of the Cauchy-Schwartz inequality along with Lemma 2.24 gives

$$\begin{aligned} 3c_1 \leqslant E_{m,n}[\frac{t_1 + t_2}{N^{2/3}}] &\leqslant 2c_1 + E_{m,n}[\frac{t_1 + t_2}{N^{2/3}} : t_1 + t_2 > 2c_1 N^{2/3}] \\ &\leqslant 2c_1 + C' P_{m,n}(t_1 + t_2 > 2c_1 N^{2/3})^{\frac{1}{2}} \end{aligned}$$

for some positive constant C'. Thus

$$c_0 := (\frac{c_1}{C'})^2 \leqslant P_{m,n}(t_1 + t_2 > 2c_1 N^{2/3}) \leqslant P_{m,n}(t_1 > c_1 N^{2/3} \text{ or } t_2 > c_1 N^{2/3}),$$

which completes the proof. $\square$

We now prove the path fluctuation lower bound.

Proof of (2.1.10). If $\tau = 0$, this reduces to Corollary 2.27. If $\tau \in (0, 1)$ put $(k, l) = (\lfloor \tau m \rfloor, \lfloor \tau n \rfloor)$. Then the sequence $(m - k, n - l)$ satisfies (2.1.5) with a new scaling parameter $M = (1-\tau)N$. By the down-right property and Lemma 2.25

$$\begin{aligned} Q_{m-k,n-l}(t_1 > u \text{ or } t_2 > u) &\stackrel{d}{=} Q^{(k,l)}_{m,n}(t_1^{(k,l)} > u \text{ or } t_2^{(k,l)} > u) \\ &= Q_{m,n}(v_1(l) > k+u \text{ or } w_1(k) > l+u) \\ &\leqslant Q_{m,n}(v_1(l) > \tau m + \frac{u}{2} \text{ or } w_1(k) > \tau n + \frac{u}{2}) \end{aligned}$$

provided that $u \geqslant 2$. Corollary 2.27 applied to the sequence $(m-k, n-l)$ completes the proof. $\square$

Chapter 3

Characterization

The content of this chapter is joint work with Christian Noack and is a modified form of an article which has been published in the *Electronic Journal of Probability* [13].

3.1 Introduction

One method which has been used to study certain models of percolation and polymers is to introduce a version of the model with boundary conditions that possesses a stationarity property. This stationarity property allows for the exact computation of some quantities of interest, such as the free energy. In [41] O'Connell and Yor introduce a model for a directed polymer in a Brownian environment with a Burke-type stationarity property. In [46] Seppäläinen and Valkó use this stationarity to find bounds on the fluctuation exponents of the free energy and the fluctuation of the paths. In [12] Cator and Groeneboom relate a stationary version of the Hammersley process to the location of a second class particle and determine the order of the variance of the longest weakly north-east path. In [4] Balázs, Cator, and Seppäläinen use a stationary version of the last passage growth model with exponential weights to study the variance of the last passage time and transversal fluctuations of the maximal path.

We define the integrability property $T^{h,Y}$-invariance (Definition 3.1) which encapsulates this stationarity in the setting of lattice directed polymers. This property implies a preservation in distribution of ratios of partition functions. The first model discovered possessing this property is the log-gamma model, introduced by Seppäläinen in [45]. In his paper $T^{h,Y}$-invariance is used to prove the conjectured values for the fluctuation exponents of the free energy and the polymer path in the stationary point-to-point case and to prove upper bounds for the exponents in the point-to-point and point-to-line cases without boundary conditions. In [25] Georgiou and Seppäläinen use $T^{h,Y}$-invariance to obtain large deviation results for the log-gamma polymer. In the setting of directed polymer models, this is the first instance where precise large deviation rate functions for the free energy were derived.

Thereafter three additional models admitting $T^{h,Y}$-invariant versions were found: the strict-weak model, introduced simultaneously by Corwin, Seppäläinen, and Shen in [20] and O'Connell and Ortmann in [38], the beta model, introduced by Barraquand and Corwin in [7] as the beta RWRE, and the inverse-beta model, introduced by Thiery and Le Doussal in [50]. The stationary versions of these models were given by Balázs, Rassoul-Agha, and Seppäläinen in [6] for the beta model, Thiery in [49] for the inverse-beta model, and by Corwin, Seppäläinen, and Shen in [20] for the strict-weak model.

In this work we present a uniqueness result for $T^{h,Y}$-invariant models. That is, under some regularity assumptions and up to the two natural modifications of reflection and scaling, the log-gamma, strict-weak, beta, and inverse-beta are the only $T^{h,Y}$-invariant models.

This work is similar in spirit to the physics works of Evans, Majumdar, and Zia ([22], [51], and [23]), who consider mass transport models on graphs and provide a characterization of the models which have a product form stationary measure. The work of Povolotsky [42] uses the framework of Evans, Majumdar, and Zia and obtains a three parameter family of zero range mass transfer models which are integrable via Bethe ansatz. In fact, both the beta and the inverse-beta models were obtained as limits of Povolotsky's family of models.

In the paper [14] we use $T^{h,Y}$-invariance along with a Mellin transform framework to simultaneously prove the conjectured values for the fluctuation exponents of the free energy and polymer path in the stationary point-to-point version of these four models.

3.1.1 The polymer model

The directed polymer in a random environment, first introduced by Huse and Henley [27], models a long chain of molecules in the presence of random impurities. Imbrie and Spencer [28] formulated this model as a random walk in a random environment. See the lectures by Comets [15] for a survey of results on directed polymers. We consider a class of 1+1-dimensional directed polymers on the integer lattice.

On each edge e of the $\mathbb{Z}_+^2$ lattice we place a positive random weight. For $x \in \mathbb{N}^2$, let u_x and v_x denote the horizontal and vertical incoming edge weights. We assume that the collection of pairs $\{(u_x, v_x)\}_{x \in \mathbb{N}^2}$ is independent and identically distributed, but do not insist that u_x is independent of v_x (in fact we will later assume v_x is a function of u_x). Call this collection the *bulk weights*. For $x \in \mathbb{N} \times \{0\}$, let R_x^1 denote the horizontal incoming edge weight, and for $x \in \{0\} \times \mathbb{N}$, let R_x^2 denote the vertical incoming edge weight. We assume the collections $\{R_x^1\}_{x \in \mathbb{N} \times \{0\}}$ and $\{R_y^2\}_{y \in \{0\} \times \mathbb{N}}$ are independent and identically distributed, and refer to them as the *horizontal* and *vertical boundary* *weights*, respectively. We further assume that the horizontal boundary weights, the vertical boundary weights, and the bulk weights are independent of each other. This assignment of edge weights is illustrated in Figure 14.

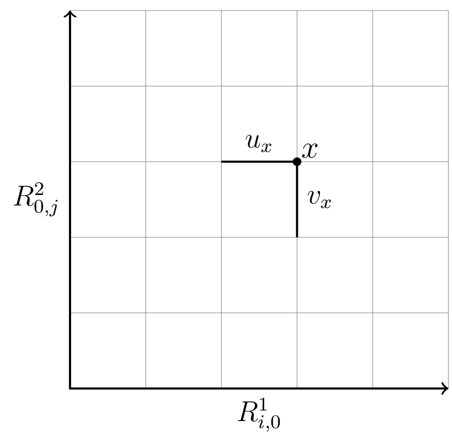

Figure 14: Assignment of edge weights.

For $(m,n) \in \mathbb{Z}^2_+ \backslash \{(0,0)\}$, let $\Pi_{m,n}$ be the collection of all up-right paths from $(0,0)$ to (m,n). See Figure 15 for an example of such a path. We identify paths $x_\bullet = (x_0, x_1, \ldots, x_{m+n})$ by their sequence of vertices, but also associate to paths their sequence of edges $(e_1, \ldots, e_{m+n})$, where $e_i = \{x_{i-1}, x_i\}$. The point-to-point partition function for the directed polymer is defined as

$$Z_{m,n} := \sum_{x_\bullet \in \Pi_{m,n}} \prod_{i=1}^{m+n} \omega_{e_i} \qquad \text{for } (m,n) \in Z^2_+ \backslash \{(0,0)\},$$

where ω_e is the weight associated to the edge e. At the origin, define $Z_{0,0} := 1$. Write $\alpha_1 = (1,0)$, $\alpha_2 = (0,1)$. The partition functions satisfy the recurrence relation

$$Z_x = u_x Z_{x-\alpha_1} + v_x Z_{x-\alpha_2} \qquad \text{for } x \in \mathbb{N}^2. \tag{3.1.1}$$

For $k = 1, 2$ define ratios of partition functions

$$R^k_x := \frac{Z_x}{Z_{x-\alpha_k}} \qquad \text{for all } x \text{ such that } x - \alpha_k \in \mathbb{Z}^2_+.$$

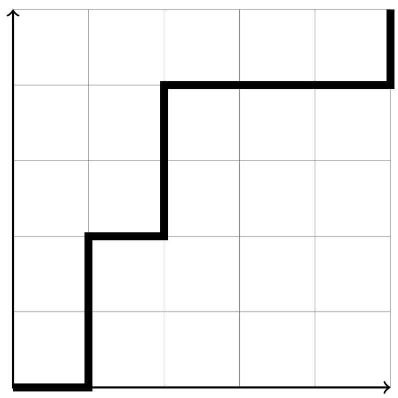

Figure 15: An up-right path from $(0,0)$ to $(5,5)$.

Note that these extend the definitions of $R^1_{i,0}$ and $R^2_{0,j}$, since for example $Z_{i,0} = \prod_{k=1}^{i} R^1_{k,0}$. The recurrence relation (3.1.1) yields the recursions

$$\begin{aligned} R^1_x &= u_x + v_x \frac{R^1_{x-\alpha_2}}{R^2_{x-\alpha_1}} \\ R^2_x &= u_x \frac{R^2_{x-\alpha_1}}{R^1_{x-\alpha_2}} + v_x \end{aligned} \qquad \text{for } x \in \mathbb{N}^2. \tag{3.1.2}$$

We look to exploit these recursions to obtain more structure of the ratios R^1_x and R^2_x, which in turn allows us to analyze quantities of interest such as the free energy, $\log Z_{m,n}$. The notation $X \overset{d}{=} Y$ is used to specify that random vectors X and Y have the same distribution. We look for cases where $(R^1_x, R^2_x) \overset{d}{=} (R^1_{x-\alpha_2}, R^2_{x-\alpha_1})$, under the assumption that u_x and v_x have a functional dependence of the form $(u_x, v_x) = \big(Y_x, h(Y_x)\big)$ for some positive random variable Y_x and positive function h. We further assume that there exist positive random variables R^1, R^2, Y such that the horizontal boundary weights, the vertical boundary weights, and the bulk weights are distributed as R^1, R^2, and $\big(Y, h(Y)\big)$, respectively.

When Y is a random variable taking values in the domain of h and (R^1, R^2) is a random vector taking values in $(0,\infty)^2$, define the random vector

$$T^{h,Y}(R^1,R^2) := \left(Y + h(Y)\frac{R^1}{R^2},\, Y\frac{R^2}{R^1} + h(Y)\right). \tag{3.1.3}$$

Note that with $(u_x, v_x) = \big(Y_x, h(Y_x)\big)$, the recursive equations (3.1.2) imply

$$\big(R^1_x, R^2_x\big) = T^{h,Y_x}\big(R^1_{x-\alpha_2}, R^2_{x-\alpha_1}\big) \qquad \text{for all } x \in \mathbb{N}^2. \tag{3.1.4}$$

Definition 3.1. *Let $O_3 \subset (0,\infty)$, $h : O_3 \to (0,\infty)$, and assume the random variable Y takes values in O_3. Let (R^1, R^2) be a random vector taking values in $(0,\infty)^2$ that is independent of Y. We say that (R^1,R^2) is $T^{h,Y}$-*invariant *if $T^{h,Y}(R^1,R^2) \overset{d}{=} (R^1,R^2)$.*

Definition 3.1, while stated in terms of the random variables (R^1,R^2) and Y, is really a property of the distributions of (R^1,R^2) and Y.

If (R^1,R^2) is $T^{h,Y}$-invariant with R^1 independent of R^2, then (3.1.4) and an induction argument imply that the polymer model possesses a form of stationarity:

$$\big(R^1_x, R^2_x\big) \overset{d}{=} \big(R^1, R^2\big) \qquad \text{for all } x \in \mathbb{N}^2. \tag{3.1.5}$$

Although our two main theorems require R^1 and R^2 to be independent, the results in Section 3.2 hold without this independence.

3.1.2 Main results

Our first main result, Theorem 3.2, consists of showing that, under some regularity assumptions, $T^{h,Y}$-invariance can only occur if h is of the form $h(y) = a + by$ for real numbers a, b satisfying $a \vee b > 0$. Our second main result, Theorem 3.4, consists of showing that if h has this form, then $T^{h,Y}$-invariance only arises as a modification of the four known invariant models (described in (3.1.7) through (3.1.10)). In the case of

$h(y) = y$, which is equivalent to vertex disorder, the uniqueness of the vertex weight distributions was already shown (Lemma 3.2 of [45]).

Given a real valued function f we call $\{x : f(x) \neq 0\}$ the support of f. Note that we do not insist on taking the closure of this set. Define the non-random analogue of (3.1.3),

$$T^{h,y}(r_1, r_2) := \big(y + h(y)\tfrac{r_1}{r_2}, y\tfrac{r_2}{r_1} + h(y)\big). \tag{3.1.6}$$

Theorem 3.2. *Let R^1, R^2, Y be positive, independent random variables with respective densities f_1, f_2, f_3. Assume that the support of f_j is $O_j \subset (0,\infty)$ for $j = 1, 2, 3$, where each O_j is open and O_3 is connected. Assume f_1, f_2 are twice differentiable on O_1 and O_2 respectively and that f_3 is three times differentiable on O_3. Suppose $h : O_3 \to (0,\infty)$ is four times differentiable, the mapping $O_1 \times O_2 \times O_3 \ni (r_1, r_2, y) \mapsto T^{h,y}(r_1, r_2)$ surjects onto $O_1 \times O_2$, and $\frac{r_2}{r_1} + h'(y) \neq 0$ for all $(r_1, r_2, y) \in O_1 \times O_2 \times O_3$. If (R^1, R^2) is $T^{h,Y}$-invariant, then h must be of the form $h(y) = a + by$, where a, b are real numbers satisfying $a \vee b > 0$.*

Remark 3.3. *If (R^1, R^2, Y) has support $O_1 \times O_2 \times O_3$ and (R^1, R^2) is $T^{h,Y}$-invariant, then the surjectivity condition is a natural assumption. The assumption $\frac{r_2}{r_1} + h'(y) \neq 0$ is a convenience used in Lemma 3.11 which allows us to extend the preservation of distribution of the pair (R^1, R^2) to the triple (R^1, R^2, Y) (see Definition 3.6). However, this assumption can be removed by an application of Sard's theorem (see Lemma 3.12) at the expense of making a.e. statements throughout Section 3.2. As an example for when the assumption $\frac{r_2}{r_1} + h'(y) \neq 0$ is satisfied, we can take h to be any differentiable increasing function. Note that the assumptions do not require O_1 or O_2 to be connected.*

Before giving the second main result we give the form of each of the four known invariant models.

The notation $X \sim \mathrm{Ga}(\alpha,\beta)$ is used to denote that a random variable is gamma(α,β) distributed, i.e. has density $\Gamma(\alpha)^{-1}\beta^{\alpha}x^{\alpha-1}e^{-\beta x}$ supported on $(0,\infty)$, where $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}dx$ is the gamma function. $X \sim \mathrm{Be}(\alpha,\beta)$ is used to say that X is beta(α,β) distributed, i.e. has density $\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$ supported on $(0,1)$. We then use $X \sim \mathrm{Ga}^{-1}(\alpha,\beta)$ and $X \sim \mathrm{Be}^{-1}(\alpha,\beta)$ to denote that $X^{-1} \sim \mathrm{Ga}(\alpha,\beta)$ and $X^{-1} \sim \mathrm{Be}(\alpha,\beta)$, respectively. We also use $X \sim \big(\mathrm{Be}^{-1}(\alpha,\beta)-1\big)$ to denote that $X+1 \sim \mathrm{Be}^{-1}(\alpha,\beta)$. The symbol $\otimes$ is used to denote (independent) product distribution.

- **Inverse-gamma**: This is also known as the log-gamma model. Assume $\mu > \lambda > 0$, $\beta > 0$ and

$$\big(R^1, R^2, Y\big) \sim \mathrm{Ga}^{-1}(\mu-\lambda,\beta) \otimes \mathrm{Ga}^{-1}(\lambda,\beta) \otimes \mathrm{Ga}^{-1}(\mu,\beta). \tag{3.1.7}$$

  Then (R^1, R^2) is $T^{h,Y}$-invariant, where $h(y) = y$. (See Lemma 3.2 of [45].)

- **Gamma**: This is also known as the strict-weak model. Assume λ, μ, $\beta > 0$ and

$$\big(R^1, R^2, Y\big) \sim \mathrm{Ga}(\mu+\lambda,\beta) \otimes \mathrm{Be}^{-1}(\lambda,\mu) \otimes \mathrm{Ga}(\mu,\beta). \tag{3.1.8}$$

  Then (R^1, R^2) is $T^{h,Y}$-invariant, where $h(y) = 1$. (See Lemma 6.3 of [20].)

- **Beta**: Assume λ, μ, $\beta > 0$ and

$$\big(R^1, R^2, Y\big) \sim \mathrm{Be}(\mu+\lambda,\beta) \otimes \mathrm{Be}^{-1}(\lambda,\mu) \otimes \mathrm{Be}(\mu,\beta). \tag{3.1.9}$$

  Then (R^1, R^2) is $T^{h,Y}$-invariant, where $h(y) = 1-y$. (See Lemma 3.1 of [6].)

- **Inverse-beta**: Assume $\mu > \lambda > 0$, $\beta > 0$ and

$$\big(R^1, R^2, Y\big) \sim \mathrm{Be}^{-1}(\mu-\lambda,\beta) \otimes \big(\mathrm{Be}^{-1}(\lambda,\beta+\mu-\lambda)-1\big) \otimes \mathrm{Be}^{-1}(\mu,\beta). \tag{3.1.10}$$

Then (R^1, R^2) is $T^{h,Y}$-invariant, where $h(y) = y - 1$. (See Proposition 3.1 of [49].)

The name of each model refers to the distribution of the bulk weights. We call these models the **four basic beta-gamma models**.

Theorem 3.4. *Let $O_j \subset (0,\infty)$ for $j = 1, 2, 3$ and assume $h : O_3 \to (0,\infty)$ has the form $h(y) = a + by$, where a, b are real numbers satisfying $a \vee b > 0$. Assume the mapping $O_1 \times O_2 \times O_3 \ni (r_1, r_2, y) \mapsto T^{h,y}(r_1, r_2)$ surjects onto $O_1 \times O_2$, and R^1, R^2, Y are non-degenerate, independent random variables taking values in O_1, O_2, O_3 respectively.*

(a) If $a = 0$ and $b > 0$, then (R^1, R^2) is $T^{h,Y}$-invariant if and only if $\left(R^1, \frac{1}{b}R^2, Y\right)$ is distributed as in (3.1.7)*.*

(b) If $a > 0$ and $b = 0$, then (R^1, R^2) is $T^{h,Y}$-invariant if and only if $\left(R^1, \frac{1}{a}R^2, Y\right)$ is distributed as in (3.1.8)*.*

(c) If $a > 0$, $b < 0$, and $-b \notin \{\frac{y}{x} : (x, y) \in O_1 \times O_2\}$, then (R^1, R^2) is $T^{h,Y}$-invariant if and only if either $\left(-\frac{b}{a}R^1, \frac{1}{a}R^2, -\frac{b}{a}Y\right)$ or $\left(\frac{1}{a}R^2, -\frac{b}{a}R^1, 1 + \frac{b}{a}Y\right)$ is distributed as in (3.1.9)*.*

(d) If $a < 0$ and $b > 0$, then (R^1, R^2) is $T^{h,Y}$-invariant if and only if $\left(-\frac{b}{a}R^1, -\frac{1}{a}R^2, -\frac{b}{a}Y\right)$ is distributed as in (3.1.10)*.*

(e) If $a, b > 0$, then (R^1, R^2) is $T^{h,Y}$-invariant if and only if $\left(\frac{1}{a}R^2, \frac{b}{a}R^1, 1 + \frac{b}{a}Y\right)$ is distributed as in (3.1.10)*.*

Figure 16 illustrates which one of the four basic beta-gamma models corresponds to each choice of parameters a, b.

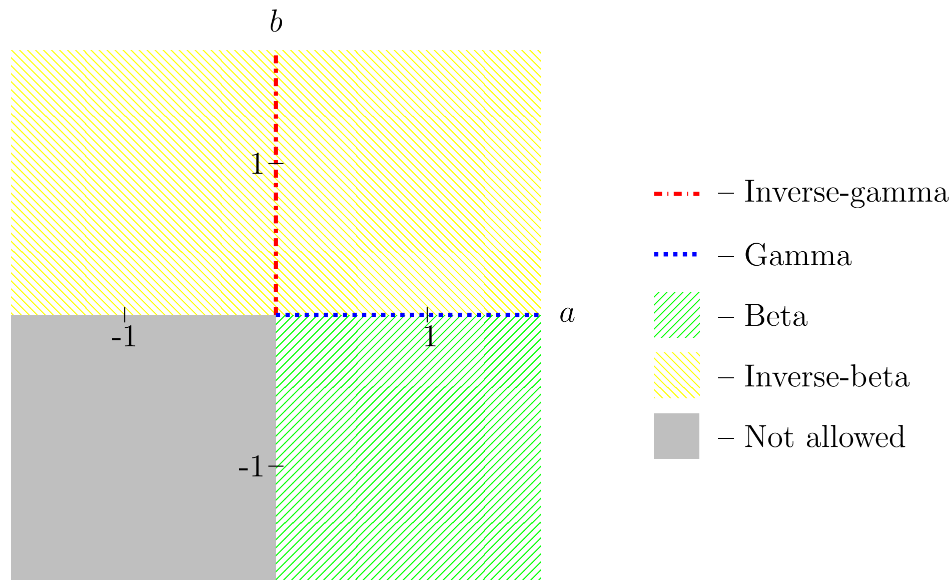

Figure 16: Modifications of the four beta-gamma models.

In the related physics paper [50], Thiery and Le Doussal study the implications of Bethe ansatz solvability in the context of $1+1$-dimensional lattice directed polymers. In their work, they consider the model without boundary and do not impose the additional assumption that the weights on incoming horizontal and vertical edges, u_x and v_x, have a functional dependence. Making the assumption of coordinate Bethe ansatz solvability (for a precise definition see II.B in [50]), they arrive at a formula for the joint moments of u_x and v_x. This is carried out under the assumption that all joint moments of u_x and v_x are finite. Ignoring the finiteness of these moments, they consider the implications of the joint moment conditions in an attempt to classify all weights u_x and v_x leading to coordinate Bethe ansatz solvability. From this classification they are able to retrieve the four basic beta-gamma models. This suggests a direct connection between the integrability properties of Bethe ansatz solvability and stationarity. In the current work we do not further explore this connection, but consider it an interesting direction for future research.

Structure of the chapter: In Section 3.2 we define the stronger property T^h-invariance, and give conditions for when $T^{h,Y}$-invariance is equivalent to T^h-invariance. T^h-invariance will be used as a tool in proving our main theorems. The proof of Theorem 3.2 is then given in Section 3.3. In Section 3.4 we describe the natural modifications of reflection and scaling. The proof of Theorem 3.4 is given in Section 3.5.

3.2 Equivalences between $T^{h,Y}$-invariance and T^h-invariance

First define

$$T_1^h(r_1, r_2, y) := y + h(y)\frac{r_1}{r_2} \qquad\qquad T_2^h(r_1, r_2, y) := y\frac{r_2}{r_1} + h(y). \tag{3.2.1}$$

Notice that (R^1, R^2) is $T^{h,Y}$-invariant if and only if

$$\big(T_1^h, T_2^h\big)\big(R^1, R^2, Y\big) := \big(T_1^h(R^1, R^2, Y), T_2^h(R^1, R^2, Y)\big) \stackrel{d}{=} \big(R^1, R^2\big).$$

In this section we determine conditions which allow us to construct a function T_3^h such that $\big(T_1^h, T_2^h, T_3^h\big)(R^1, R^2, Y) \stackrel{d}{=} (R^1, R^2, Y)$. Moreover, T_3^h will be such that $T := \big(T_1^h, T_2^h, T_3^h\big)$ is an involution. Recall that a function T is an involution if $T \circ T$ is the identity function. This augmentation of our mapping $T^{h,Y}$ to an involution T encapsulates a form of reversibility of the polymer model.

Definition 3.5. *Let $O \subset (0,\infty)^2$, $O_3 \subset (0,\infty)$, and $h : O_3 \to (0,\infty)$. We say that an involution $T : O \times O_3 \to O \times O_3$ is a* polymer involution adapted to h *if its first two coordinates are as in* (3.2.1).

Existence and uniqueness of polymer involutions is addressed in Lemma 3.8. When the polymer involution adapted to h is unique we write T^h. In our two main theorems we assume that R^1 and R^2 are independent and therefore take $O = O_1 \times O_2$. We allow for arbitrary $O \subset (0,\infty)^2$ since the results in this section allow for dependence between R^1 and R^2.

Definition 3.6. *Suppose (R^1, R^2, Y) is a random vector taking values in $O \times O_3$, where $O \subset (0,\infty)^2$, $O_3 \subset (0,\infty)$, and Y is independent of (R^1, R^2). Let $h : O_3 \to (0,\infty)$. If there exists a polymer involution T on $O \times O_3$ adapted to h such that $T(R^1, R^2, Y) \stackrel{d}{=} (R^1, R^2, Y)$, then we say (R^1, R^2, Y) is* T-invariant (with respect to h).

If (R^1, R^2, Y) is T-invariant, the polymer model with weight distributions (R^1, R^2, Y) not only has property (3.1.5), but possesses a stronger form of stationarity called the Burke property (see Theorem 3.3 of [45]), named after Burke's theorem on the output distribution of M/M/1 queues (see the reference [31]). In Definition 3.5 we restrict our attention to involutions, as T-invariance not only implies stationarity, but also a form of reversibility: the construction of a dual measure (see Section 3.2 of [45] and Proposition III.3 of [49] for more details).

The four basic beta-gamma models are not only $T^{h,Y}$-invariant, but are in fact T^h-invariant as well. The rest of this section is dedicated to relating the properties of $T^{h,Y}$-invariance and T^h-invariance, as given in the following proposition.

Proposition 3.7. *Let $O \subset (0,\infty)^2$, $O_3 \subset (0,\infty)$, and $h : O_3 \to (0,\infty)$. Assume (R^1, R^2, Y) is a random vector taking values in $O \times O_3$ and that Y is independent of (R^1, R^2). Then the following two conditions are equivalent.*

(a) The mapping $O \times O_3 \ni (r_1, r_2, y) \mapsto T^{h,y}(r_1, r_2)$ surjects onto O, for every $(r_1, r_2) \in O$ the function $O_3 \ni y \mapsto y\frac{r_2}{r_1} + h(y)$ is injective, and (R^1, R^2) is $T^{h,Y}$-invariant.

(b) There exists a unique polymer involution T^h adapted to h on $O \times O_3$ and (R^1, R^2, Y) is T^h-invariant.

The proof of Proposition 3.7 follows from combining Lemmas 3.8, 3.10, and Remark 3.9 below.

We use the notation $\pi_j : (0,\infty)^2 \to (0,\infty)$ to denote the projection onto the j-th coordinate for $j = 1,2$. Given $O \subset (0,\infty)^2$, $Q(O)$ will denote the set $\left\{\frac{y}{x} : (x,y) \in O\right\}$. When $O = O_1 \times O_2$ we will write $\frac{O_2}{O_1}$ for $Q(O)$.

When T is a polymer involution adapted to h we will often use the following notation

$$(\widetilde{r}_1, \widetilde{r}_2, \widetilde{y}) := T(r_1, r_2, y). \tag{3.2.2}$$

More precisely, by equations (3.2.1)

$$\widetilde{r}_1 := y + h(y)\frac{r_1}{r_2}, \qquad \widetilde{r}_2 := y\frac{r_2}{r_1} + h(y), \qquad \widetilde{y} := T_3^h(r_1, r_2, y).$$

Note that these definitions imply that

$$\frac{\widetilde{r}_2}{\widetilde{r}_1} = \frac{r_2}{r_1}. \tag{3.2.3}$$

This equality of ratios will turn out to be quite useful.

The following lemma gives an equivalence to the existence of a unique polymer involution.

Lemma 3.8. *Let $O \subset (0,\infty)^2$, $O_3 \subset (0,\infty)$, $h : O_3 \to (0,\infty)$, and T_1^h, T_2^h be as in* (3.2.1)*. Then the following are equivalent:*

(a) $\left(T_1^h, T_2^h\right)(O \times O_3) = O$ and for every $(r_1, r_2) \in O$ the function $O_3 \ni y \mapsto T_2^h(r_1, r_2, y) = y\frac{r_2}{r_1} + h(y)$ is injective.

(b) $G(s,y) := \Big(y + \frac{h(y)}{s},\, ys + h(y)\Big)$ *is a bijection between* $Q(O) \times O_3$ *and* O.

(c) There exists a unique polymer involution T^h *on* $O \times O_3$ *adapted to* h. *Moreover,*

$$T^h = (G \otimes id) \circ \psi_{2,3} \circ (G \otimes id)^{-1}, \tag{3.2.4}$$

where $\psi_{2,3}(a,b,c) = (a,c,b)$ *and* $(G \otimes id)(a,b,c) := (G(a,b),c)$.

(d) There exists a polymer involution on $O \times O_3$ *adapted to* h *such that* T_3^h *has no* y*-dependence.*

Proof. $(a) \Rightarrow (b)$: Note that

$$G\big(\tfrac{r_2}{r_1}, y\big) = \big(T_1^h, T_2^h\big)(r_1, r_2, y) \tag{3.2.5}$$

implies $G\big(Q(O) \times O_3\big) = O$. Injectivity of G follows from $\frac{\pi_2 \circ G(s,y)}{\pi_1 \circ G(s,y)} = s$ and the injectivity condition on T_2^h.

$(b) \Rightarrow (c)$: We first show uniqueness. Suppose $T = \big(T_1^h, T_2^h, T_3^h\big)$ is a polymer involution on $O \times O_3$ adapted to h. For fixed $(r_1, r_2, y) \in O \times O_3$, with notation as in (3.2.2), we have $T(\widetilde{r}_1, \widetilde{r}_2, \widetilde{y}) = (r_1, r_2, y)$ since T is an involution. Using (3.2.3) we have

$$(r_1, r_2) = \big(T_1^h, T_2^h\big)(\widetilde{r}_1, \widetilde{r}_2, \widetilde{y}) = G\big(\tfrac{r_2}{r_1}, \widetilde{y}\big).$$

Therefore

$$G^{-1}(r_1, r_2) = \big(\tfrac{r_2}{r_1},\, T_3^h(r_1, r_2, y)\big). \tag{3.2.6}$$

Since G^{-1} has no y-dependence, neither does T_3^h. One can now check that

$$T = (G \otimes \mathrm{id}) \circ \psi_{2,3} \circ (G \otimes \mathrm{id})^{-1} \tag{3.2.7}$$

proving uniqueness. Existence follows by simply setting $T_3^h(r_1, r_2, y) = \pi_2 \circ G^{-1}(r_1, r_2)$. This forces equality (3.2.7), the right side of which is indeed a polymer involution adapted to h.

$(c) \Rightarrow (d)$ is clear.

$(d) \Rightarrow (a)$: Let T be a polymer involution on $O \times O_3$ adapted to h for which T_3^h has no y-dependence. Clearly the first two components of T, $\big(T_1^h, T_2^h\big)$, surject onto O. Now fix $(r_1, r_2) \in O$. Since $T_1^h(r_1, r_2, y) = \frac{r_1}{r_2} T_2^h(r_1, r_2, y)$ and T is itself injective, we have injectivity of $y \mapsto T_2^h(r_1, r_2, y)$. $\square$

Remark 3.9. *Note that the conditions in part (a) of Lemma 3.8 depend only on the sets O, O_3, and the function h. The condition $\big(T_1^h, T_2^h\big)(O \times O_3) = O$ in part (a) is equivalent to the condition that the mapping $O \times O_3 \ni (r_1, r_2, y) \mapsto T^{h,y}(r_1, r_2)$ surjects onto O (recall definition* (3.1.6)*).*

When the polymer involution T is such that T_3^h has no y-dependence, we will simply write $T_3^h(r_1, r_2)$. The following lemma gives conditions for when $T^{h,Y}$-invariance is equivalent to T^h-invariance.

Lemma 3.10. *Suppose O, O_3, and h satisfy one of the equivalent conditions in Lemma 3.8. Let (R^1, R^2, Y) be a random vector taking values in $O \times O_3$ and assume that Y is independent of the pair (R^1, R^2). Let T^h be the unique polymer involution adapted to h, defined by* (3.2.4)*, and write $\widetilde{Y} = T_3^h(R^1, R^2)$. Then the following are equivalent:*

(a) (R^1, R^2) is $T^{h,Y}$-invariant.

(b) R^2/R^1 is independent of $\widetilde{Y}$ and $\widetilde{Y} \overset{d}{=} Y$.

(c) (R^1, R^2, Y) is T^h-invariant.

Proof. $(a) \Leftrightarrow (b)$: Put $\big(\widetilde{R}^1, \widetilde{R}^2\big) = \big(T_1^h, T_2^h\big)(R^1, R^2, Y)$. Using equations (3.2.5) and (3.2.6),

$$\begin{aligned}
&G\big(R^2/R^1, Y\big) = \big(\widetilde{R}^1, \widetilde{R}^2\big) \stackrel{d}{=} \big(R^1, R^2\big) \\
\Leftrightarrow\ &\big(R^2/R^1, Y\big) \stackrel{d}{=} G^{-1}\big(R^1, R^2\big) = \big(R^2/R^1, \widetilde{Y}\big) \\
\Leftrightarrow\ &R^2/R^1 \text{ is independent of } \widetilde{Y} \text{ and } Y \stackrel{d}{=} \widetilde{Y}.
\end{aligned}$$

$(c) \Rightarrow (a)$ is clear. We now show that (a) and (b) imply (c). Since T_3^h has no y-dependence, Y is independent of the pair $(R^2/R^1, \widetilde{Y})$. Therefore the triple $(R^2/R^1, Y, \widetilde{Y})$ is independent. Thus $\big(\widetilde{R}^1, \widetilde{R}^2\big) = G(R^2/R^1, Y)$ is independent of $\widetilde{Y}$. Now combining (a) and $\widetilde{Y} \stackrel{d}{=} Y$ we get $\big(\widetilde{R}^1, \widetilde{R}^2, \widetilde{Y}\big) \stackrel{d}{=} (R^1, R^2, Y)$. □

We now give an analogue of Lemma 3.8 in which h and T^h are continuously differentiable. We compute the Jacobian matrix and determinant of T^h in order to later give an explicit form for the density of $T^h(R^1, R^2, Y)$ in terms of the density of (R^1, R^2, Y) (see proof of Proposition 3.13).

Given a differentiable transformation $F : U \to \mathbb{R}^m$, where $U \subset \mathbb{R}^n$ is open, use the notations $DF(u)$ and $D[F](u)$ to denote the Jacobian matrix of F evaluated at the point $u \in U$. When $m = n$ we say F is a C^1-diffeomorphism if F is injective, continuously differentiable, and its Jacobian matrix is invertible throughout U.

Lemma 3.11. *Let $O \subset (0, \infty)^2$, $O_3 \subset (0, \infty)$, $h : O_3 \to (0, \infty)$, and T_1^h, T_2^h be as in* (3.2.1)*. Further assume O and O_3 are open, O_3 is connected, and h is continuously differentiable. Then the following are equivalent:*

(a) $\big(T_1^h, T_2^h\big)(O \times O_3) = O$ *and the following function does not vanish on* $Q(O) \times O_3$

$$L(s, y) := s + h'(y). \tag{3.2.8}$$

(b) $G(s,y) := \left(y + \frac{h(y)}{s},\, ys + h(y)\right)$ *is a* C^1*-diffeomorphism between* $Q(O) \times O_3$ *and* O*. Moreover its Jacobian matrix and determinant are given by*

$$DG(s,y) = \begin{bmatrix} -h(y)/s^2 & L(s,y)/s \\ y & L(s,y) \end{bmatrix}, \quad \det DG(s,y) = -\frac{L(s,y)}{s}\left(y + \frac{h(y)}{s}\right). \tag{3.2.9}$$

(c) There exists a unique C^1*-diffeomorphic polymer involution* T^h *on* $O \times O_3$ *adapted to* h*. Moreover* T^h_3 *has no* y *dependence and the Jacobian matrix and determinant of* T^h *are given by*

$$DT^h(r_1,r_2,y) = \frac{1}{r_1}\begin{bmatrix} h(y)/s & -h(y)/s^2 & L(s,y)r_1/s \\ -ys & y & L(s,y)r_1 \\ \widetilde{y}s/L(s,\widetilde{y}) & h(\widetilde{y})/\big(sL(s,\widetilde{y})\big) & 0 \end{bmatrix}, \tag{3.2.10}$$

$$\det DT^h(r_1,r_2,y) = -\left(\frac{y}{r_1} + \frac{h(y)}{r_2}\right)\frac{L(s,y)}{L(s,\widetilde{y})},$$

where $s = \frac{r_2}{r_1}$ *and* $\widetilde{y} = T^h_3(r_1,r_2)$*.*

(d) There exists a differentiable polymer involution on $O \times O_3$ *adapted to* h*.*

Proof. $(a) \Rightarrow (b)$: For fixed $(r_1,r_2) \in O$, since $y \mapsto \frac{\partial T^h_2}{\partial y}(r_1,r_2,y) = L\big(\frac{r_2}{r_1},y\big)$ does not vanish on the connected set O_3, the conditions of Lemma 3.8-(a) are satisfied. Therefore G is a bijection. The continuous differentiability of h now implies that G is continuously differentiable. The Jacobian matrix and determinant of G can now be calculated. Notice that for all $(s,y) \in Q(O) \times O_3$, $y + h(y)/s = \pi_1 \circ G(s,y) \in \pi_1(O) \subset (0,\infty)$. Thus the Jacobian determinant of G does not vanish on $Q(O) \times O_3$, which shows it is a C^1-diffeomorphism.

$(b) \Rightarrow (c)$: Since G is a bijection, Lemma 3.8 gives existence and uniqueness of the polymer involution $T^h = (G \otimes \mathrm{id}) \circ \psi_{2,3} \circ (G \otimes \mathrm{id})^{-1}$. Since G is a C^1-diffeomorphism, the inverse function theorem tells us T^h is a C^1-diffeomorphism as well. Now fix $(r_1, r_2, y) \in O \times O_3$ and put $(s, \widetilde{y}) = \left(\frac{r_2}{r_1},\ T_3^h(r_1, r_2)\right)$. By (3.2.6)

$$(s, \widetilde{y}) = G^{-1}(r_1, r_2). \tag{3.2.11}$$

$DG^{-1}(r_1, r_2)$ is now the inverse of the matrix $DG\big(G^{-1}(r_1, r_2)\big) = DG(s, \widetilde{y})$. (3.2.11) implies $(r_1, r_2) = G(s, \widetilde{y}) = \big(\widetilde{y} + h(\widetilde{y})/s,\ \widetilde{y}s + h(\widetilde{y})\big)$. Using this one can show that

$$DG^{-1}(r_1, r_2) = \frac{1}{r_1}\begin{bmatrix} -s & 1 \\ \frac{s\widetilde{y}}{L(s,\widetilde{y})} & \frac{h(\widetilde{y})}{sL(s,\widetilde{y})} \end{bmatrix} \quad \text{and} \quad \det DG^{-1}(r_1, r_2) = -\frac{s}{r_1 L(s, \widetilde{y})}. \tag{3.2.12}$$

Using equations (3.2.9), (3.2.11), and (3.2.12) we can compute

$$\begin{aligned} DT^h(r_1, r_2, y) &= \Big[D(G \otimes \mathrm{id})\left(\psi_{2,3} \circ (G^{-1} \otimes \mathrm{id})(r_1, r_2, y)\right)\Big] \cdot \Big[D\psi_{2,3}\left((G^{-1} \otimes \mathrm{id})(r_1, r_2, y)\right)\Big] \\ &\quad \cdot \Big[D\big(G^{-1} \otimes \mathrm{id}\big)(r_1, r_2, y)\Big] \\ &= [DG(s, y) \otimes 1] \cdot [D\psi_{2,3}] \cdot \big[DG^{-1}(r_1, r_2) \otimes 1\big] \\ &= \begin{bmatrix} \frac{-h(y)}{s^2} & \frac{L(s,y)}{s} & 0 \\ y & L(s, y) & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \cdot \frac{1}{r_1}\begin{bmatrix} -s & 1 & 0 \\ \frac{s\widetilde{y}}{L(s,\widetilde{y})} & \frac{h(\widetilde{y})}{sL(s,\widetilde{y})} & 0 \\ 0 & 0 & r_1 \end{bmatrix} \\ &= \frac{1}{r_1}\begin{bmatrix} h(y)/s & -h(y)/s^2 & L(s, y)r_1/s \\ -ys & y & L(s, y)r_1 \\ \widetilde{y}s/L(s, \widetilde{y}) & h(\widetilde{y})/\big(sL(s, \widetilde{y})\big) & 0 \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned}\det\left(DT^h(r_1,r_2,y)\right) &= \det\left(DG(s,y)\right)\det\left(D\psi_{2,3}\right)\det\left(DG^{-1}(r_1,r_2)\right)\\ &= -\frac{L(s,y)}{s}\left(y+\frac{h(y)}{s}\right)(-1)\left(-\frac{s}{r_1 L(s,\widetilde{y})}\right)\\ &= -\left(\frac{y}{r_1}+\frac{h(y)}{r_2}\right)\frac{L(s,y)}{L(s,\widetilde{y})}.\end{aligned}$$

$(c) \Rightarrow (d)$ is clear.

$(d) \Rightarrow (a)$: If T is a differentiable polymer involution adapted to h, then its Jacobian matrix has the same entries as the 2×3 upper portion of (3.2.10), as $\left(T_1^h, T_2^h\right)$ are completely determined. Therefore the determinant of the top-left 2×2 minor of the Jacobian matrix of T is zero. Thus L vanishing at a point $(s,y) \in Q(O) \times O_3$ would imply the Jacobian determinant of T vanishes at any point $(r_1,r_2,y) \in O \times O_3$ such that $\frac{r_2}{r_1} = s$. Since $T \circ T$ is the identity function, the Jacobian determinant of T cannot vanish on $O \times O_3$. Thus L cannot vanish on $Q(O) \times O_3$. $\square$

Let $B := \left\{(r_1,r_2,y) \in O \times O_3 : \frac{r_2}{r_1} + h'(y) = 0\right\}$. The following Lemma shows that when h is twice continuously differentiable, the image of B under the mapping $\left(T_1^h, T_2^h\right)$ has Lebesgue measure zero.

Lemma 3.12. *Take all assumptions from Lemma 3.11 with the addition that h is twice continuously differentiable. Then $\left(T_1^h, T_2^h\right)(B)$ has Lebesgue measure zero.*

Proof. For convenience define $H(r_1,r_2,y) := \left(T_1^h.T_2^h\right)(r_1.r_2.y)$. The Jacobian matrix of H is given by the top 2×3 portion of the matrix (3.2.10). Therefore $(r_1,r_2.y)$ is a critical point of H, meaning the rank of $DH(r_1,r_2,y) < 2$, if and only if $L\left(\frac{r_2}{r_1},y\right) = 0$ $\left(\text{since } y\frac{r_2}{r_1} + h(y) = \widetilde{r}_2 > 0\right)$, which occurs if and only if $(r_1,r_2,y) \in B$. Sard's theorem [44] yields the desired result. $\square$

3.3 Proof of Theorem 3.2

We begin by using Lemma 3.11 to give another useful equivalence to T-invariance under some regularity assumptions. In the appendix of [49], Thiery uses a specific case of the following proposition to prove the invariance of the inverse-beta model. It can also be used to prove invariance of the other three basic beta-gamma models.

Proposition 3.13. *Let (R^1, R^2, Y) be a random vector with density ρ and assume Y is independent of (R^1, R^2). Suppose the support of ρ equals $O \times O_3$ where $O \subset (0,\infty)^2$ is open and $O_3 \subset (0,\infty)$ is open and connected. Let $h : O_3 \to (0,\infty)$ be continuously differentiable and T be a differentiable polymer involution adapted to h on $O \times O_3$. Then (R^1, R^2, Y) is T-invariant if and only if*

$$q \circ T(x) = q(x) \qquad \text{for a.e. } x \in O \times O_3$$

where $q(r_1, r_2, y) := \frac{r_2}{|L(r_2/r_1, y)|}\rho(r_1, r_2, y)$ and $L(s, y) = s + h'(y)$, as given in (3.2.8).

Proof of Proposition 3.13. Recall the notation (3.2.2). By Lemma 3.11, L does not vanish on $Q(O) \times O_3$ and T is in fact a C^1-diffeomorphism with

$$\det DT(r_1, r_2, y) = -\left(\frac{y}{r_1} + \frac{h(y)}{r_2}\right)\frac{L(r_2/r_1, y)}{L(r_2/r_1, \widetilde{y})} = -\frac{\widetilde{r}_2 L(r_2/r_1, y)}{r_2 L(r_2/r_1, \widetilde{y})}.$$

Therefore $T(R^1, R^2, Y)$ has density

$$\widehat{\rho}(x) := \rho\big(T^{-1}(x)\big)\left|\det DT^{-1}(x)\right| = \rho\big(T(x)\big)\left|\det DT(x)\right|$$

supported on $x \in O \times O_3$. Thus T-invariance of (R^1, R^2, Y) is equivalent to $\rho(x) = \widehat{\rho}(x)$ a.e. on $O \times O_3$.

Using (3.2.3) we can explicitly write $\widehat{\rho}(r_1, r_2, y) = \rho(\widetilde{r}_1, \widetilde{r}_2, \widetilde{y})\left|\frac{\widetilde{r}_2 L(r_2/r_1, y)}{r_2 L(\widetilde{r}_2/\widetilde{r}_1, \widetilde{y})}\right|$. After rearranging terms, the condition $\rho(x) = \widehat{\rho}(x)$ for a.e. $x \in O \times O_3$ yields the desired result. $\square$

We now prove the first main result.

Proof of Theorem 3.2. (R^1, R^2, Y) has density $\rho\big(r_1, r_2, y\big) = f_1(r_1)f_2(r_2)f_3(y)$. By Lemma 3.11, there exists a unique differentiable polymer involution T^h on $O_1 \times O_2 \times O_3$ adapted to h and the function $L(s, y) = s + h'(y)$ does not vanish on the set $\frac{O_2}{O_1} \times O_3$. By Lemma 3.10, (R^1, R^2, Y) is T^h-invariant. Applying Proposition 3.13 gives $q \circ T^h = q$ a.e. on $O_1 \times O_2 \times O_3$. Since f_1, f_2, f_3, and T^h are continuous, this equality holds everywhere on $O_1 \times O_2 \times O_3$. Since the support of f_j equals O_j, we can further assume $f_j(x) = \exp(\eta_j(x))$ for $x \in O_j$, $j = 1, 2, 3$. Note that η_j has the same differentiability properties as f_j. Set $s = \frac{r_2}{r_1}$ and recall the notation (3.2.2). Taking logarithms of the equality $q \circ T^h = q$ then computing the total derivative we obtain

$$D[\log q](\widetilde{r}_1, \widetilde{r}_2, \widetilde{y}) \cdot DT^h(r_1, r_2, y) = D[\log q](r_1, r_2, y), \tag{3.3.1}$$

where DT^h is given in (3.2.10) and

$$D[\log q](r_1, r_2, y) = \left[\frac{r_2}{r_1^2 L(s, y)} + \eta_1'(r_1), \frac{h'(y)}{r_2 L(s, y)} + \eta_2'(r_2), -\frac{h''(y)}{L(s, y)} + \eta_3'(y)\right].$$

Using the fact that T^h is an involution and (3.2.3), $r_2 = T_2^h(\widetilde{r}_1, \widetilde{r}_2, \widetilde{y}) = \widetilde{y}s + h(\widetilde{y})$. One can then check that

$$DT^h(r_1, r_2, y) \cdot [r_1,\, r_2,\, 0]^T = [0,\, 0,\, r_2/L(s, \widetilde{y})]^T.$$

Thus multiplying both sides of equation (3.3.1) on the right by $[r_1,\, r_2,\, 0]^T$ gives

$$1 + r_1\eta_1'(r_1) + r_2\eta_2'(r_2) = r_2 g(s, \widetilde{y}), \tag{3.3.2}$$

where

$$g(s, y) := \frac{\eta_3'(y)}{L(s, y)} - \frac{h''(y)}{L(s, y)^2}.$$

Applying the operator $\frac{\partial^2}{\partial r_1 \partial r_2}$ to the left-hand side of (3.3.2) gives zero. We now exploit the fact that $\frac{\partial^2}{\partial r_1 \partial r_2}$ applied to the right hand side must equal zero to ultimately arrive at the conclusion that $h''(y) = 0$.

Note that if f is differentiable then for all non-negative integers k and n,

$$D\left[\frac{s^k f(y)}{L(s,y)^n}\right](s,y) = s^{k-1}\left[\frac{1}{L(s,y)^n}, \frac{-nf(y)}{L(s,y)^{n+1}}\right] \cdot \begin{bmatrix} kf(y) & sf'(y) \\ s & sh''(y) \end{bmatrix}. \tag{3.3.3}$$

First calculate, using (3.2.10) and (3.3.3),

$$\begin{aligned}
\frac{\partial}{\partial r_1}\big(r_2 g(s,\widetilde{y})\big) &= r_2 Dg(s,\widetilde{y}) \cdot \left[\frac{\partial s}{\partial r_1}, \frac{\partial \widetilde{y}}{\partial r_1}\right]^T \\
&= s^2 Dg(s,\widetilde{y}) \cdot \left[-1, \frac{\widetilde{y}}{L(s,\widetilde{y})}\right]^T \\
&= s\left[\frac{1}{L(s,\widetilde{y})}, -\frac{\eta_3'(\widetilde{y})}{L(s,\widetilde{y})^2}\right] \cdot \begin{bmatrix} 0 & s\eta_3''(\widetilde{y}) \\ s & sh''(\widetilde{y}) \end{bmatrix} \cdot \left[-1, \frac{\widetilde{y}}{L(s,\widetilde{y})}\right]^T \\
&\quad - s\left[\frac{1}{L(s,\widetilde{y})^2}, -\frac{2h'(\widetilde{y})}{L(s,\widetilde{y})^3}\right] \cdot \begin{bmatrix} 0 & sh''(\widetilde{y}) \\ s & sh''(\widetilde{y}) \end{bmatrix} \cdot \left[-1, \frac{\widetilde{y}}{L(s,\widetilde{y})}\right]^T \\
&= t(s,\widetilde{y}) := \sum_{j=2}^{4} \frac{s^2 \kappa_j(\widetilde{y})}{L(s,\widetilde{y})^j},
\end{aligned}$$

where

$$\begin{aligned}
\kappa_2(y) &= y\eta_3''(y) + \eta_3'(y) \\
\kappa_3(y) &= -yh''(y)\eta_3'(y) - yh'''(y) - 2h''(y) \\
\kappa_4(y) &= 2yh''(y)^2.
\end{aligned}$$

Taking an r_2 partial derivative and multiplying by r_1, by (3.2.10)

$$
\begin{aligned}
0 &= r_1 \frac{\partial^2}{\partial r_2 \partial r_1}\big(r_2 g(s,\widetilde{y})\big) = r_1 \frac{\partial}{\partial r_2} t(s,\widetilde{y}) \\
&= r_1 Dt(s,\widetilde{y}) \cdot \left[\frac{\partial s}{\partial r_2}, \frac{\partial \widetilde{y}}{\partial r_2}\right]^T \\
&= Dt(s,\widetilde{y}) \cdot \left[1, \frac{h(\widetilde{y})}{sL(s,\widetilde{y})}\right]^T .
\end{aligned}
$$

This equality holds for all $(r_1, r_2, y) \in O_1 \times O_2 \times O_3$. Since T^h is an involution on $O_1 \times O_2 \times O_3$, it also holds after interchanging $(r_1, r_2, y) \leftrightarrow (\widetilde{r}_1, \widetilde{r}_2, \widetilde{y})$. Notice that, by (3.2.3), $s = \frac{r_2}{r_1}$ is unaffected by this interchange. Therefore, applying this interchange and using (3.3.3)

$$
\begin{aligned}
0 &= Dt(s,y) \cdot \left[1, \frac{h(y)}{sL(s,y)}\right]^T \\
&= \sum_{j=2}^{4} s \left[\frac{1}{L(s,y)^j}, \frac{-j\kappa_j(y)}{L(s,y)^{j+1}}\right] \cdot \begin{bmatrix} 2\kappa_j(y) & s\kappa_j'(y) \\ s & sh''(y) \end{bmatrix} \cdot \left[1, \frac{h(y)}{sL(s,y)}\right]^T
\end{aligned}
$$

for all $(s, y) \in \frac{O_2}{O_1} \times O_3$. Multiplying by $L(s, y)^6/s$ gives

$$
0 = \sum_{j=2}^{4} \left[L(s,y)^{5-j}, -j\kappa_j(y)L(s,y)^{4-j}\right] \cdot \begin{bmatrix} 2\kappa_j(y) & \kappa_j'(y) \\ s & h''(y) \end{bmatrix} \cdot \left[L(s,y),\, h(y)\right]^T . \tag{3.3.4}
$$

Now fix $y \in O_3$. The right hand side is now a fourth degree polynomial in s which vanishes on the open set $\frac{O_2}{O_1}$. It must therefore vanish at all values $s \in \mathbb{R}$. Taking $s = -h'(y)$ so that $L(s, y) = 0$, (3.3.4) gives

$$
0 = -4\kappa_4(y)h(y)h''(y) = -8yh(y)h''(y)^3.
$$

The fact that y and $h(y)$ are positive implies $h''(y) = 0$. Since this holds for all $y \in O_3$, which we assumed to be connected, h has the form $h(y) = a + by$ where a, b are real numbers. The condition $a \vee b > 0$ follows from the fact that h maps a subset of $(0, \infty)$ into $(0, \infty)$. $\square$

3.4 Reflection and scaling

We describe two procedures which preserve T-invariance. By applying these procedures to the four basic beta-gamma models, we can obtain a T-invariant model corresponding to $h(y) = a + by$ for each choice of a, b such that $a \vee b > 0$.

We first define the reflection procedure. Let T be a polymer involution adapted to h on $O_1 \times O_2 \times O_3$ and assume that h is injective so that $h : O_3 \to h(O_3)$ is a bijection. Define the mapping $\rho(r_1, r_2, y) := (r_2, r_1, h(y))$. Define the mapping and the random vector

$$\widehat{T} := \rho \circ T \circ \rho^{-1} \quad \text{and} \quad \big(\widehat{R}^1, \widehat{R}^2, \widehat{Y}\big) := \big(R^2, R^1, h(Y)\big). \tag{3.4.1}$$

One can then check that $\widehat{T}$ is a polymer involution adapted to h^{-1} on $O_2 \times O_1 \times h(O_3)$. Furthermore, (R^1, R^2, Y) is T-invariant with respect to h if and only if $\big(\widehat{R}^1, \widehat{R}^2, \widehat{Y}\big)$ is $\widehat{T}$-invariant with respect to h^{-1}.

In the directed polymer setting, this procedure of mapping

$$h \mapsto h^{-1} \qquad \big(R^1, R^2, Y\big) \mapsto \big(\widehat{R}^1, \widehat{R}^2, \widehat{Y}\big) \qquad T \mapsto \widehat{T}$$

corresponds to interchanging the horizontal and vertical coordinates while remaining in the same framework. This is illustrated in Figure 17.

We now define the scaling procedure. If $O \subset (0, \infty)$ and c is a positive constant, define $cO := \{cx : x \in O\}$. Note that $cO \subset (0, \infty)$. Let c_1, c_2 be positive constants. Let T be a polymer involution adapted to h on $O_1 \times O_2 \times O_3$. Define the mapping $\sigma(r_1, r_2, y) := (c_1 r_1, c_2 r_2, c_1 y)$. Define the two mappings and the random vector

$$\check{T} := \sigma \circ T \circ \sigma^{-1} \qquad \check{h}(y) := c_2 h\big(\tfrac{y}{c_1}\big) \qquad \big(\check{R}^1, \check{R}^2, \check{Y}\big) := \big(c_1 R^1, c_2 R^2, c_1 Y\big). \tag{3.4.2}$$

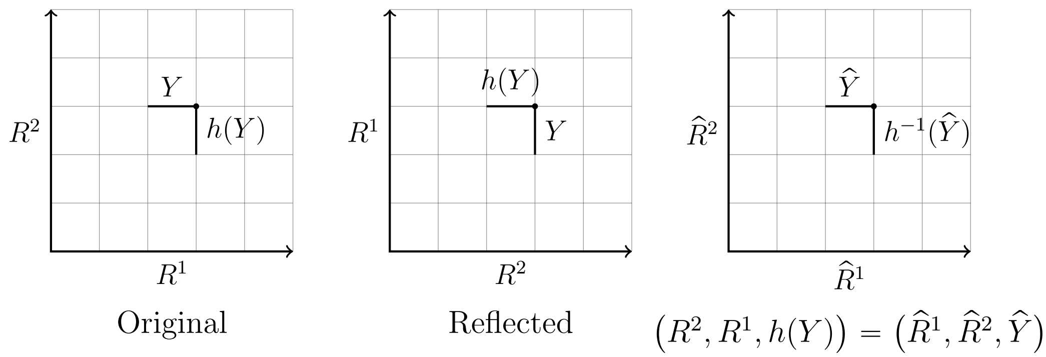

Figure 17: Reflection

One can check that $\breve{T}$ is a polymer involution adapted to $\breve{h}$ on $c_1O_1 \times c_2O_2 \times c_1O_3$. Furthermore, (R^1, R^2, Y) is T-invariant with respect to h if and only if $\big(\breve{R}^1, \breve{R}^2, \breve{Y}\big)$ is $\breve{T}$-invariant with respect to $\breve{h}$.

In the directed polymer setting, this procedure of mapping

$$h \mapsto \breve{h} \qquad\qquad \big(R^1, R^2, Y\big) \mapsto \big(\breve{R}^1, \breve{R}^2, \breve{Y}\big) \qquad\qquad T \mapsto \breve{T}$$

corresponds to scaling the horizontal axis weights by c_1 and the vertical axis weights by c_2 while remaining in the same framework. This procedure is illustrated in Figure 18.

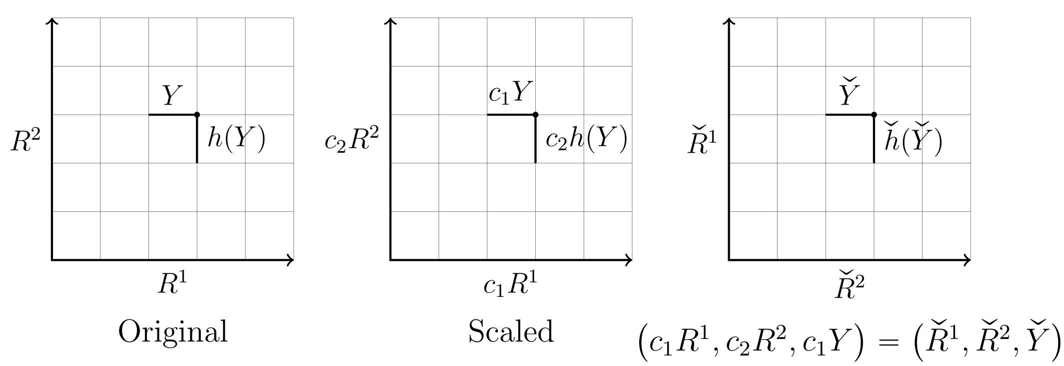

Figure 18: Scaling

One can also check that the reflection and scaling procedures commute. By using the reflection and scaling procedures, the following lemma reduces the existence and uniqueness of T-invariant models corresponding to $h(y) = a + by$ where $a \vee b > 0$ to the existence and uniqueness for values $(a, b) = (0, 1), (1, 0), (1, -1)$, and $(-1, 1)$.

For real numbers a, b such that $a \vee b > 0$, define

$$T^{(a,b)}(r_1, r_2, y) := \Big(y + (a + by)\tfrac{r_1}{r_2},\ y\tfrac{r_2}{r_1} + (a + by),\ \tfrac{r_1(r_2-a)}{r_2+br_1}\Big). \tag{3.4.3}$$

One can check that when $h(y) = a + by$, (3.2.4) implies that $T^h = T^{(a,b)}$. The domain of $T^{(a,b)}$ is discussed prior to Lemma 3.18.

Lemma 3.14. *Let a, b be real numbers satisfying $a \vee b > 0$, $h(y) = a+by$, and $T = T^{(a,b)}$ as defined in* (3.4.3)*. Let R^1, R^2, and Y be random variables.*

(a) If $a = 0$ and $b > 0$, then (R^1, R^2, Y) is T-invariant with respect to h if and only if $\big(R^1, \frac{1}{b}R^2, Y\big)$ is $T^{(0,1)}$-invariant with respect to $\check{h}(y) = y$.

(b) If $a > 0$ and $b = 0$, then (R^1, R^2, Y) is T-invariant with respect to h if and only if $\big(R^1, \frac{1}{a}R^2, Y\big)$ is $T^{(1,0)}$-invariant with respect to $\check{h}(y) = 1$

(c) If $a > 0$ and $b < 0$, then (R^1, R^2, Y) is T-invariant with respect to h if and only if $\big(-\frac{b}{a}R^1, \frac{1}{a}R^2, -\frac{b}{a}Y\big)$ is $T^{(1,-1)}$-invariant with respect to $\check{h}(y) = 1 - y$.

(d) If $a < 0$ and $b > 0$, then (R^1, R^2, Y) is T-invariant with respect to h if and only if $\big(-\frac{b}{a}R^1, -\frac{1}{a}R^2, -\frac{b}{a}Y\big)$ is $T^{(-1,1)}$-invariant with respect to $\check{h}(y) = y - 1$.

(e) If $a > 0$ and $b > 0$, then (R^1, R^2, Y) is T-invariant with respect to h if and only if $\big(\frac{b}{a}R^1, \frac{1}{a}R^2, \frac{b}{a}Y\big)$ is $T^{(1,1)}$-invariant with respect to $\check{h}(y) = y + 1$.

(f) If $a = 1$ and $b = 1$, then (R^1, R^2, Y) is T-invariant with respect to h if and only if $(R^2, R^1, 1 + Y)$ is $T^{(-1,1)}$-invariant with respect to $h^{-1}(y) = y - 1$.

Proof. Let c_1, c_2 be positive constants. After applying the scaling procedure with (c_1, c_2), with notation as in (3.4.2), one can check that

$$\check{h}(y) = ac_2 + \tfrac{bc_2}{c_1} y \quad \text{and} \quad \check{T} = T^{(ac_2, bc_2/c_1)}.$$

Recall that $\left(\check{R}^1, \check{R}^2, \check{Y}\right) = \left(c_1 R^1, c_2 R^2, c_1 Y\right)$ is $\check{T}$-invariant with respect to $\check{h}$ if and only if $\left(R^1, R^2, Y\right)$ is T-invariant with respect to h. Now (a) through (e) follow by taking

$$(c_1, c_2) = \left(1, \tfrac{1}{b}\right), \left(1, \tfrac{1}{a}\right), \left(-\tfrac{b}{a}, \tfrac{1}{a}\right), \left(-\tfrac{b}{a}, -\tfrac{1}{a}\right), \left(\tfrac{b}{a}, \tfrac{1}{a}\right)$$

respectively.

For part (f), after applying the reflection procedure, with notation as in (3.4.1), one can check that $\widehat{T} = T^{(-1,1)}$. Since $\left(\widehat{R}^1, \widehat{R}^2, \widehat{Y}\right) = (R^2, R^1, 1 + Y)$ is $\widehat{T}$-invariant with respect to $h^{-1}(y) = y - 1$ if and only if (R^1, R^2, Y) is T-invariant with respect to $h(y) = y + 1$, the result follows. $\square$

3.5 Proof of Theorem 3.4

The following two theorems, due to Seshadri and Wesołowski (2003) and Lukacs (1955) give characterizations of gamma and beta random variables, which will be used in the sequel.

Theorem 3.15 ([47])**.** *Let A and B be non-degenerate independent random variables taking values in $(0, 1)$. Then the pair $(C, D) := \left(\frac{1-B}{1-AB}, 1 - AB\right)$ is independent if and only if there exist positive constants p, q, r such that $(A, B) \sim Be(p, q) \otimes Be(p + q, r)$, in which case $(C, D) \sim Be(r, q) \otimes Be(r + q, p)$.*

Theorem 3.16 ([33]). *Let A and B be non-degenerate independent positive random variables. Then the pair $(C, D) := \big(A + B,\, \frac{A}{A+B}\big)$ is independent if and only if there exist positive constants $\lambda_A, \lambda_B, \beta$ such that $(A, B) \sim Ga(\lambda_A, \beta) \otimes Ga(\lambda_B, \beta)$, in which case $(C, D) \sim Ga(\lambda_A + \lambda_B, \beta) \otimes Be(\lambda_A, \lambda_B)$.*

Notice that the mapping $(A, B) \mapsto \big(A + B,\, A/(A + B)\big)$ has the inverse $(A, B) \mapsto \big(AB,\, A(1 - B)\big)$. The following statement is a corollary of Theorem 3.16.

Corollary 3.17. *Let A and B be non-degenerate independent random variables. Further assume that A is positive and B takes values in $(0, 1)$. Then the pair $(C, D) := \big(AB,\, A(1-B)\big)$ is independent if and only if there exist positive constants $\lambda_A, \lambda_B, \beta$ such that $(A, B) \sim Ga(\lambda_A+\lambda_B, \beta)\otimes Be(\lambda_A, \lambda_B)$ in which case $(C, D) \sim Ga(\lambda_A, \beta)\otimes Ga(\lambda_B, \beta)$.*

The next lemma constrains the sets on which $T^{(a,b)}$ (as defined by (3.4.3)) can be a polymer involution. To specify this constraint, we define the following sets. For real numbers (a, b) such that $a \vee b > 0$,

$$V_a^{\pm} := \{x > 0 : \pm(x - a) > 0\}, \qquad W_{a,b}^{\pm} := \{x > 0 : \pm(a + bx) > 0\}$$
$$D_{a,b}^{\pm} := W_{a,b}^{\pm} \times V_a^{\pm} \times W_{a,b}^{+}.$$

Lemma 3.18. *Let a, b be real numbers satisfying $a\vee b > 0$. Let $O_j \subset (0,\infty)$ for $j = 1, 2, 3$ such that O_3 is not a singleton. If $T^{(a,b)}$, as defined in (3.4.3), is a polymer involution on $O_1 \times O_2 \times O_3$ with respect to h of the form $h(y) = a + by$ then $O_1 \times O_2 \times O_3 \subset D_{a,b}^{+}$ or $O_1 \times O_2 \times O_3 \subset D_{a,b}^{-}$ assuming $D_{a,b}^{\pm}$ is non-empty.*

Proof. We first show the following holds:

(i) For all $(r_1, r_2) \in O_1 \times O_2$, the three numbers $a + br_1, \frac{r_2}{r_1} + b, r_2 - a$ are all either strictly positive, strictly negative, or equal to zero.

Fix $(r_1, r_2, y) \in O_1 \times O_2 \times O_3$ and put $\widetilde{y} = T_3^{(a,b)}(r_1, r_2) = \frac{r_1(r_2-a)}{r_2+br_1}$. Then the following two equalities hold

$$r_2 - a = \widetilde{y}\big(\tfrac{r_2}{r_1} + b\big), \qquad a + br_1 = \tfrac{r_1}{r_2}\big(a + b\widetilde{y}\big)\big(\tfrac{r_2}{r_1} + b\big). \tag{3.5.1}$$

Since $T^{(a,b)}$ is an involution on $O_1 \times O_2 \times O_3$, $\widetilde{y} \in O_3$. Recall that, by Definition 3.5, h maps $O_3 \to (0, \infty)$. Therefore $O_3 \subset W^+_{a,b}$ and the four numbers r_1, r_2, $\widetilde{y}$, and $h(\widetilde{y}) = a+b\widetilde{y}$ are all positive. (3.5.1) now gives (i).

By Lemma 3.8, for all $(r_1, r_2) \in O_1 \times O_2$ the mapping $O_3 \ni y \mapsto T_2^{(a,b)}(r_1, r_2, y) = y(\frac{r_2}{r_1} + b) + a$ is injective. Therefore $\frac{r_2}{r_1} + b$ does not vanish for any $(r_1, r_2) \in O_1 \times O_2$. Thus, by (i)

$$O_1 \times O_2 \subset \big(W^+_{a,b} \times V^+_a\big) \cup \big(W^-_{a,b} \times V^-_a\big). \tag{3.5.2}$$

If $O_1 \cap W^+_{a,b} = \varnothing$, then by (3.5.2) $O_1 \times O_2 \subset W^-_{a,b} \times V^-_a$. In this case $O_1 \times O_2 \times O_3 \subset D^-_{a,b}$. On the other hand, if $O_1 \cap W^+_{a,b} \neq \varnothing$ then there exists $r_1 \in O_1$ such that $a + br_1 > 0$. By (i), $r_2 - a > 0$ for all $r_2 \in O_2$. Thus $O_2 \subset V^+_a$. Now (3.5.2) implies that $O_1 \times O_2 \subset W^+_{a,b} \times V^+_a$ which gives $O_1 \times O_2 \times O_3 \subset D^+_{a,b}$, completing the proof.

□

Using (3.5.1) one can in fact check that $T^{(a,b)}$ is an involution on both $D^+_{a,b}$ and $D^-_{a,b}$ assuming they are non-empty.

The following proposition characterizes T^h-invariant models corresponding to $h(y) = a + by$ when $(a, b) = (0, 1)$, $(1, 0)$, $(1, -1)$, and $(-1, 1)$.

Proposition 3.19. *For a, b real numbers, let $h(y) = a + by$ and assume $T^{(a,b)}$, as defined in (3.4.3), is a polymer involution adapted to h on $O_1 \times O_2 \times O_3 \subset (0, \infty)^3$. Assume that (R^1, R^2, Y) are non-degenerate independent random variables taking values in $O_1 \times O_2 \times O_3$.*

(a) If $(a, b) = (0, 1)$, then (R^1, R^2, Y) is $T^{(0,1)}$-invariant if and only if $\left(R^1, R^2, Y\right)$ is distributed as in (3.1.7)

(b) If $(a, b) = (1, 0)$, then (R^1, R^2, Y) is $T^{(1,0)}$-invariant if and only if $\left(R^1, R^2, Y\right)$ is distributed as in (3.1.8)

(c) If $(a, b) = (1, -1)$, then (R^1, R^2, Y) is $T^{(1,-1)}$-invariant if and only if either $\left(R^1, R^2, Y\right)$ or $(R^2, R^1, 1 - Y)$ is distributed as in (3.1.9)

(d) If $(a, b) = (-1, 1)$, then (R^1, R^2, Y) is $T^{(-1,1)}$-invariant if and only if $\left(R^1, R^2, Y\right)$ is distributed as in (3.1.10).

Proof. Observe that $T_3^{(a,b)}$ has no y-dependence. Thus, by Lemma 3.8, $T^{(a,b)}$ is the unique polymer involution adapted to h on $O_1 \times O_2 \times O_3$. By Lemma 3.10, (R^1, R^2, Y) is $T^{(a,b)}$-invariant if and only if the following two properties hold:

(i) $\frac{R^2}{R^1}$ is independent of $T_3^{(a,b)}(R^1, R^2)$.

(ii) $Y \stackrel{d}{=} T_3^{(a,b)}(R^1, R^2)$.

Recall that

$$T_3^{(a,b)}(R^1, R^2) = \frac{R^1(R^2 - a)}{R^2 + bR^1}.$$

We now prove (a). Put $(A,B) := ((R^1)^{-1},(R^2)^{-1})$. Then (A,B) are non-degenerate independent positive random variables. Now

$$\frac{R^2}{R^1} = \frac{A}{B} \quad \text{and} \quad T_3^{(0,1)}(R^1,R^2) = (A+B)^{-1}.$$

So (i) holds if and only if $A/(A+B) = (1+B/A)^{-1}$ is independent of $A+B$. By Theorem 3.16 this occurs if and only if there exist positive constants $\lambda_A, \lambda_B, \beta$ such that $(A,B) \sim \mathrm{Ga}(\lambda_A,\beta) \otimes \mathrm{Ga}(\lambda_B,\beta)$. In such a case, $A+B = C \sim \mathrm{Ga}(\lambda_A+\lambda_B,\beta)$. Thus $T_3^{(0,1)}(R^1,R^2) = (A+B)^{-1} \sim \mathrm{Ga}^{-1}(\lambda_A+\lambda_B,\beta)$. Now put $(\mu,\lambda) = (\lambda_A+\lambda_B,\lambda_B)$ and use (ii) to get $(R^1,R^2,Y) \sim \mathrm{Ga}^{-1}(\mu-\lambda,\beta) \otimes \mathrm{Ga}^{-1}(\lambda,\beta) \otimes \mathrm{Ga}^{-1}(\mu,\beta)$. This completes the proof of (a).

We now prove (b). Notice that $D^-_{1,0} = \varnothing$. Therefore by Lemma 3.18 we have that (R^1,R^2,Y) takes values in $D^+_{1,0} = (0,\infty) \times (1,\infty) \times (0,\infty)$. Put $(A,B) := \big(R^1,(R^2)^{-1}\big)$. Then (A,B) are non-degenerate independent random variables taking values in $(0,\infty) \times (0,1)$. Now

$$\frac{R^2}{R^1} = \frac{1}{AB} \quad \text{and} \quad T_3^{(1,0)}(R^1,R^2) = A(1-B).$$

So (i) holds if and only if AB is independent of $A(1-B)$. By Corollary 3.17, this occurs if and only if there exist positive constants $\lambda_A, \lambda_B, \beta$ such that $(A,B) \sim \mathrm{Ga}(\lambda_A+\lambda_B,\beta) \otimes \mathrm{Be}(\lambda_A,\lambda_B)$. In such a case, $T_3^{(1,0)}(R^1,R^2) = A(1-B) = D \sim \mathrm{Ga}(\lambda_B,\beta)$. Now put $(\mu,\lambda) = (\lambda_B,\lambda_A)$ and use (ii) to get $(R^1,R^2,Y) \sim \mathrm{Ga}(\mu+\lambda,\beta) \otimes \mathrm{Be}^{-1}(\lambda,\mu) \otimes \mathrm{Ga}(\mu,\beta)$. This completes the proof of (b).

We now prove (c). By Lemma 3.18, (R^1,R^2,Y) either takes values in

$$D^+_{1,-1} = (0,1) \times (1,\infty) \times (0,1) \quad \text{or} \quad D^-_{1,-1} = (1,\infty) \times (0,1) \times (0,1).$$

First consider the case when (R^1,R^2,Y) takes values in $D^+_{1,-1}$. Put $(A,B) := \big((R^2)^{-1},R^1\big)$. Then (A,B) are non-degenerate independent random variables, both taking values in

$(0,1)$. Now
$$\frac{R^2}{R^1} = \frac{1}{AB} \quad \text{and} \quad T_3^{(1,-1)}(R^1, R^2) = 1 - \frac{1-B}{1-AB}.$$
So (i) holds if and only if $1-AB$ is independent of $(1-B)/(1-AB)$. By Theorem 3.15, this occurs if and only if there exist positive constants p, q, r such that $(A, B) \sim \text{Be}(p, q) \otimes \text{Be}(p+q, r)$. In such a case, $1 - T_3^{(1,-1)}(R^1, R^2) = (1-B)/(1-AB) = C \sim \text{Be}(r, q)$. Thus $T_3^{(1,-1)}(R^1, R^2) \sim 1 - \text{Be}(r, q) = \text{Be}(q, r)$. Now put $(\mu, \lambda, \beta) = (q, p, r)$ and use (ii) to get $(R^1, R^2, Y) \sim \text{Be}(\mu + \lambda, \beta) \otimes \text{Be}^{-1}(\lambda, \mu) \otimes \text{Be}(\mu, \beta)$.

In the case where (R^1, R^2, Y) takes values in $D^-_{1,-1}$, applying the reflection procedure as in (3.4.1), one can check that $\widehat{T} = T^{(1,-1)}$ and the resulting random variables $\left(\widehat{R}^1, \widehat{R}^2, \widehat{Y}\right) = (R^2, R^1, 1-Y)$ take values in $D^+_{1,-1}$. By the first case, we are done. This completes the proof of (c).

We now prove (d). Notice that $D^-_{-1,1} = \varnothing$. Therefore by Lemma 3.18 (R^1, R^2, Y) must take values in $D^+_{-1,1} = (1,\infty) \times (0,\infty) \times (1,\infty)$. Put $(A, B) := (1 - (R^1)^{-1}, 1 - (R^2+1)^{-1})$. Then (A, B) are non-degenerate independent random variables, both taking values in $(0,1)$. Therefore
$$\left(1 + \frac{R^2}{R^1}\right)^{-1} = \frac{1-B}{1-AB} \quad \text{and} \quad T_3^{(-1,1)}(R^1, R^2) = \frac{1}{1-AB}.$$
So (i) holds if and only if $(1-B)/(1-AB)$ is independent of $1-AB$. By Theorem 3.15, this occurs if and only if there exist positive constants p, q, r such that $(A, B) \sim \text{Be}(p, q) \otimes \text{Be}(p+q, r)$. In such a case, $T_3^{(-1,1)}(R^1, R^2) = (1-AB)^{-1} = D^{-1} \sim \text{Be}^{-1}(r+q, p)$. Now put $(\mu, \lambda, \beta) = (r+q, r, p)$ and use (ii) to get $(R^1, R^2, Y) \sim \text{Be}^{-1}(\mu - \lambda, \beta) \otimes (\text{Be}^{-1}(\lambda, \beta + \mu - \lambda) - 1) \otimes \text{Be}^{-1}(\mu, \beta)$. This completes the proof of (d). □

We now prove the second main result.

Proof of Theorem 3.4. When $h(y) = a + by$, for all fixed $(r_1, r_2) \in O_1 \times O_2$ the mapping $y \mapsto T_2^h(r_1, r_2, y) = y\left(\frac{r_2}{r_1} + b\right) + a$ is injective whenever $b \geqslant 0$. In the case $b < 0$ and $a > 0$ this injectivity follows from the assumption $-b \notin \left\{\frac{y}{x} : (x, y) \in O_1 \times O_2\right\}$. Therefore the conditions of Proposition 3.7-(a) are satisfied in all cases, which gives the existence of a unique polymer involution T^h adapted to $h(y) = a + by$ such that (R^1, R^2, Y) is T^h-invariant. By (3.2.4), $T^h = T^{(a,b)}$ as defined in (3.4.3). Now applying Lemma 3.14 then Proposition 3.19 completes the proof. $\square$

Chapter 4

Stationary multi-path inverse-gamma polymer

4.1 Introduction

In this chapter we further study the inverse-gamma (log-gamma) directed polymer, which is exactly solvable using the geometric Robinson-Schensted-Knuth (gRSK) correspondence. We consider multi-path polymers, which are represented as tuples of non-intersecting up-right lattice paths.

The classical RSK correspondence is a combinatorial mapping between matrices with entries in $\mathbb{Z}_+$ and pairs of semi-standard Young tableaux of the same shape. See the texts by Fulton [24] and Stanley [48] for constructions of the correspondence and some of its applications. The RSK correspondence is the combinatorial structure that lies behind the solvability of the problem of the length of the longest increasing subsequence of a random permutation [3] and the solvability of directed last passage percolation with geometric or exponential weights [29].

The geometric RSK correspondence (gRSK) is a mapping from matrices with positive real entries to pairs of triangular arrays. The mapping was first introduced by Kirillov [32]. In [19], Corwin, O'Connell, Seppäläinen, and Zygouras use the matrix product formulation for gRSK developed by Noumi and Yamada [37] to apply the gRSK correspondence to the study of directed polymers. If the entries of the input matrix to gRSK are used as weights for a polymer environment, the output triangular arrays can be expressed in terms of multi-path polymer partition functions. In the case where the input matrix has inverse-gamma distributed entries with certain parameters, the pushforward probability measure on triangular arrays has an explicit form in terms of Whittaker functions.

In the paper [39], O'Connell, Seppäläinen, and Zygouras give an alternative formulation of the gRSK correspondence as a correspondence between matrices by using sequences of "local moves" acting on matrix entries and use this formulation to study an inverse-gamma polymer in a symmetric environment. Nguyen and Zygouras [35] extend the gRSK correspondence to polygonal input arrays and obtain formulas for the Laplace transform of the joint distribution of inverse-gamma polymer partition functions at different space-time points. Bisi and Zygouras [8] use the gRSK correspondence to study the point-to-line inverse-gamma polymer.

In the papers [36] and [18], Nica and Corwin-Nica study models of multi-path directed polymers and in the intermediate disorder regime and show convergence of the partition functions to the multi-layer continuum polymer introduced by O'Connell and Warren [40].

The first result of this chapter utilizes a theorem from [19] about the invariant distributions of a certain Markov process on triangular arrays to specify an inverse-gamma polymer environment for which ratios of multi-path polymer partition functions have a form of stationarity similar to the down-right property of Definition 1.1. From this stationarity we can easily compute the expectation of the multi-path free energy.

The other main result of this chapter applies the stationary setup and the coupling method used in Section 2.3 of Chapter 2 to derive a formula for the variance of the two-path partition function. This variance formula involves the multi-path exit points from the boundaries layers and could potentially be used as a starting point for the analysis of the fluctuation exponents of the two-path polymer free energy.

4.2 Model and results

We define the multi-path polymer model. Let $(Y_{i,j})_{(i,j)\in\mathbb{N}^2}$ be a collection of independent positive random weights. These weights are the *polymer environment.* Let $\Pi^{(\ell)}_{m,n}$ denote the collection of all ℓ-tuples $\pi = (\pi_1, \dots, \pi_\ell)$ of non-intersecting up-right lattice paths in $\mathbb{N}^2$ such that the ith path π_i starts at $(1, i)$ and ends at $(m, n - \ell + i)$. An example 3-tuple is given in Figure 19. Given $\pi = (\pi_1, \dots, \pi_\ell) \in \Pi^{(\ell)}_{m,n}$, we define its weight to be the product of all the random weights associated to the vertices that the paths traverse. That is,

$$wt(\pi) = \prod_{i=1}^{\ell} \prod_{x\in\pi_i} Y_x.$$

As with the single path case, we can then construct a quenched polymer measure on $\Pi^{(\ell)}_{m,n}$,

$$Q^{(\ell)}_{m,n}(\pi) := \frac{wt(\pi)}{Z^{(\ell)}_{m,n}},$$

where

$$Z^{(\ell)}_{m,n} := \sum_{\pi\in\Pi^{(\ell)}_{m,n}} wt(\pi)$$

is the ℓ-path partition function. Let $E^{Q^{(\ell)}_{m,n}}$ denote the quenched expectation.

For $m, n \geqslant \ell$, we define the ratios

$$U^{(\ell)}_{m,n} = \frac{Z^{(\ell)}_{m,n}}{Z^{(\ell)}_{m-1,n}} \quad \text{and} \quad V^{(\ell)}_{m,n} = \frac{Z^{(\ell)}_{m,n}}{Z^{(\ell)}_{m,n-1}}.$$

We would like these ratios to possess a stationarity property analogous to the down-right property (Definition 1.1) formulated in Chapter 1.

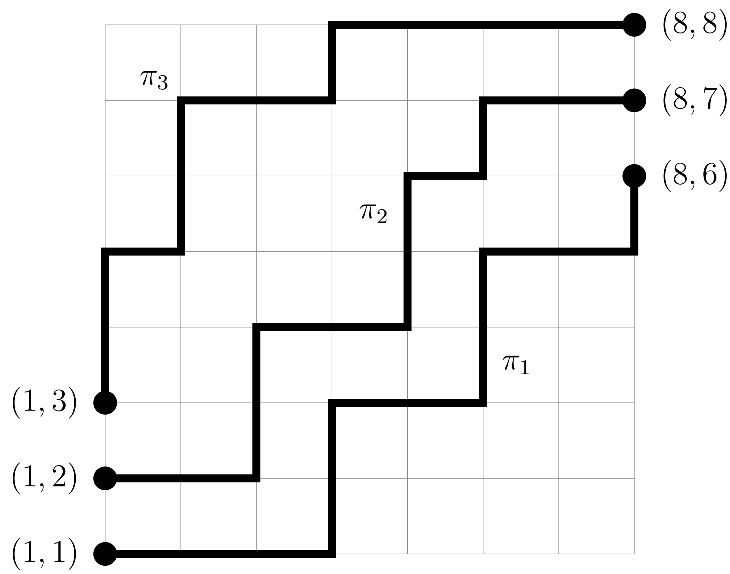

Figure 19: An example 3-tuple $\pi = (\pi_1, \pi_2, \pi_3) \in \Pi^{(3)}_{8,8}$.

Fix $\ell \in \mathbb{N}$ and parameters $0 < \theta_1 < \cdots < \theta_\ell < \mu$. We now define the ℓ-path stationary inverse-gamma polymer environment.

- Bulk weights: $Y_{i,j} \sim \mathrm{Ga}^{-1}(\mu)$ for $i, j \geqslant \ell + 1$.
- Lower boundary layers: $Y_{i,j} \sim \mathrm{Ga}^{-1}(\theta_j)$ for $i \geqslant \ell + 1$ and $j \in \{1, \ldots, \ell\}$.
- Upper boundary layers: $Y_{i,j} \sim \mathrm{Ga}^{-1}(\mu - \theta_{\ell-i+1})$ for $i \in \{1, \ldots, \ell\}$ and $j \geqslant \ell + 1$.
- Corner boundary weights: $Y_{i,j} \sim \mathrm{Ga}^{-1}(\theta_j - \theta_{\ell-i+1})$ for $(i, j) \in \{1, \ldots, \ell\}^2$ such that $i + j \geqslant \ell + 2$. Let the remaining weights be constant 1.

For the stationary polymer, we insist that the polymer paths start in the cut-off corner. That is, for $1 \leqslant k \leqslant \ell$ and $m \geqslant \ell$, $n \geqslant k$, redefine the sets $\Pi^{(k)}_{m,n}$ to be the collection of all k-tuples $(\pi_1, \ldots, \pi_k)$ of up-right lattice paths such that for $1 \leqslant i \leqslant k$ the path π_i starts at $(\ell - i + 1, i)$ and ends at $(m, n - k + i)$. We let $Z^{(k)}_{m,n}$ and ratios $U^{(k)}_{m,n}, V^{(k)}_{m,n}$ be defined using these sets.

Figure 20 illustrates this setup for the $\ell = 2$ and $\ell = 3$ cases.

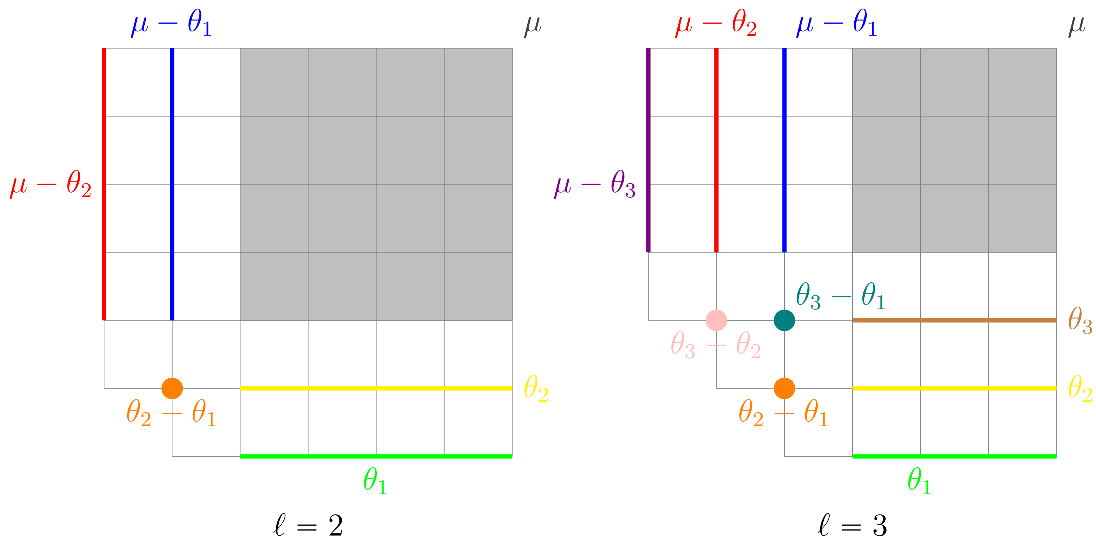

Figure 20: Stationary ℓ-path inverse-gamma polymer environment for $\ell = 2, 3$.

We now give the form of the stationarity.

Lemma 4.1. *Let $\ell \in \mathbb{N}$ and suppose that $\{Y_{i,j}\}_{(i,j)\in\mathbb{N}^2}$ is distributed as the ℓ-path stationary inverse-gamma environment and let paths start in the cut-off corner. Let $m, n \geqslant \ell + 1$. Then the collections of ratios*

- $\bigcup_{i=\ell+1}^{m} \{U^{(1)}_{i,n}, U^{(2)}_{i,n}/U^{(1)}_{i,n}, \ldots, U^{(\ell)}_{i,n}/U^{(\ell-1)}_{i,n}\}$

- $\bigcup_{j=\ell+1}^{n} \{V^{(1)}_{m,j}, V^{(2)}_{m,j}/V^{(1)}_{m,j}, \ldots, V^{(\ell)}_{m,j}/V^{(\ell-1)}_{m,j}\}$

are independent, independent of each other, and have marginal distributions

$$\begin{aligned} U^{(1)}_{i,n} &\sim Ga^{-1}(\theta_1), \\ U^{(k)}_{i,n}/U^{(k-1)}_{i,n} &\sim Ga^{-1}(\theta_k) \quad \text{for } 2 \leqslant k \leqslant \ell, \\ V^{(1)}_{m,k} &\sim Ga^{-1}(\mu-\theta_1), \\ V^{(k)}_{m,j}/V^{(k-1)}_{m,j} &\sim Ga^{-1}(\mu-\theta_k) \quad \text{for } 2 \leqslant k \leqslant \ell. \end{aligned}$$

As an application of the above lemma, we can compute the expectation of multi-path polymer free energy.

Lemma 4.2. *Under the assumptions of Lemma 4.1, the expectation of the ℓ-path polymer free energy is*

$$\mathbb{E}[\log Z^{(\ell)}_{m,n}] = \mathbb{E}[Z^{(\ell)}_{\ell,\ell}] - (m-\ell)\sum_{k=1}^{\ell}\psi_0(\theta_k) - (n-\ell)\sum_{k=1}^{\ell}\psi_0(\mu-\theta_k), \tag{4.2.1}$$

where

$$\mathbb{E}[Z^{(\ell)}_{\ell,\ell}] = -\sum_{d=1}^{\ell-1}\sum_{k=d+1}^{\ell}\psi_0(\theta_k-\theta_{k-d}), \tag{4.2.2}$$

and ψ_0 is the digamma function.

The starting point for the analysis of the free energy and polymer path fluctuation exponents in Chapter 2 was the variance formula (2.1.3) relating the variance of the free energy to the exit point of a polymer path from the boundary. The following result is a variance formula for the two-path case.

Given an up-right lattice path π, define $t_j(\pi) := \max\{i : (i,j) \in \pi\}$ (the x-coordinate of the exit point of π from level j).

Proposition 4.3. *Suppose that $\{Y_{i,j}\}_{(i,j)\in\mathbb{Z}_+^2}$ is distributed as the 2-path stationary inverse-gamma environment. Let $m,n \geqslant 3$. Define $A \subset \Pi_{m,n}^{(2)}$ to be the collection of all pairs (π_1,π_2) that pass through the point $(2,2)$. Then*

$$\begin{aligned}\mathbb{V}ar \log Z_{m,n}^{(2)} &= \psi_1(\theta_2-\theta_1) + (n-2)(\psi_1(\mu-\theta_1)+\psi_1(\mu-\theta_2)) - (m-2)(\psi_1(\theta_1)+\psi_1(\theta_2))\\ &\quad + 2\mathbb{E}E^{Q_{m,n}^{(2)}}\Bigg[\mathbb{1}((\pi_1,\pi_2)\in A)L(-\theta_2+\theta_1, Y_{2,2}) + \sum_{i=3}^{t_1(\pi_1)} L(-\theta_1, Y_{i,1})\\ &\quad + \sum_{j=t_1(\pi_1)\vee 3}^{t_2(\pi_1)} L(-\theta_2, Y_{j,2}) + \sum_{k=3}^{t_2(\pi_2)} L(-\theta_2, Y_{k,2})\Bigg],\end{aligned}$$

where ψ_1 is the trigamma function, and

$$L(a,x) := x^{-a}e^{1/x}\int_0^x(-\psi_0(-a)-\log y)y^{a-1}e^{-1/y}dy.$$

4.3 Proofs

We begin by collecting the necessary notation used in [19].

For $N \in \mathbb{N}$, define the space of triangular arrays with positive real entries, $\mathbb{T}_N = \{z_{k,\ell} : 1 \leqslant \ell \leqslant k \leqslant N,\ z_{k,\ell} \in (0,\infty)\}$. Figure 21 illustrates an element of $\mathbb{T}_4$. In Section 2.1 of [19], the geometric RSK correspondence is defined in terms of geometric row insertion, a process in which a word $b = (b_1,\dots,b_N) \in (0,\infty)^N$ is inserted into a triangular array $z \in \mathbb{T}_N$ to obtain an updated triangular array $z' \in \mathbb{T}_N$ (denoted $z' = z \leftarrow b$). The geometric row insertion procedure is summarized by the following equations ((2.5) of

[19]):

$$
\begin{aligned}
a_{k,1} &= b_k && \text{for } 1 \leqslant k \leqslant N \\
a_{k+1,\ell+1} &= a_{k+1,\ell}\frac{z_{k+1,\ell}z'_{k,\ell}}{z'_{k+1,\ell}z_{k,\ell}} && \text{for } 1 \leqslant \ell \leqslant k < N \\
z'_{k,\ell} &= a_{k,\ell}(z_{k,\ell} + z'_{k-1,\ell}) && \text{for } 1 \leqslant \ell < k \leqslant N \\
z'_{k,k} &= a_{k,k}z_{k,k} && \text{for } 1 \leqslant k \leqslant N.
\end{aligned}
$$

$$
\begin{array}{ccccccc}
 & & & z_{11} & & & \\
 & & z_{22} & & z_{21} & & \\
 & z_{33} & & z_{32} & & z_{31} & \\
z_{44} & & z_{43} & & z_{42} & & z_{41}
\end{array}
$$

Figure 21: A triangular array in $\mathbb{T}_4$.

For a semi-infinite matrix of positive weights $d = \{d_{t,j} : t \geqslant 1, 1 \leqslant j \leqslant N\}$, we write $d^{[t]} = (d_{t,1}, \ldots, d_{t,N})$ for the t-th row of d. Given such a matrix d and an initial state $z(0) \in \mathbb{T}_N$, iteratively inserting the rows of d defines a temporal evolution:

$$z(t) := \left[z(t-1) \leftarrow d^{[t]}\right] = \left[z(0) \leftarrow d^{[1]} \leftarrow \cdots \leftarrow d^{[t]}\right] \quad \text{for } t \geqslant 1.$$

The relation between the process $z(t)$ defined above and directed polymers is explained in Section 2.2 of [19]. Starting with an empty array (the insertion into an empty or partially empty array is defined in Section 2.1) and inserting the first t columns of d to obtain $z(t) \in \mathbb{T}_N$, the entries of $z(t)$ are given by

$$
\begin{aligned}
z_{k,1}(t) &= Z^{(1)}_{t,k} \quad \text{for } 1 \leqslant k \leqslant N, \\
z_{k,\ell}(t) &= Z^{(\ell)}_{t,k}/Z^{(\ell-1)}_{t,k} \quad \text{for } 2 \leqslant \ell \leqslant k \leqslant N,
\end{aligned}
$$

where the partition functions are defined using the polymer environment weights $Y_{i,j} = d_{i,j}$.

A semi-infinite random matrix $d = \{d_{i,j} : i \geqslant 1, 1 \leqslant j \leqslant N\}$ is a *solvable inverse-gamma weight matrix* if its entries are independent and there are parameters $(\widehat{\theta}_i : i \geqslant 1)$ and $(\theta_j : 1 \leqslant j \leqslant N)$ such that entry $d_{i,j} \sim \mathrm{Ga}^{-1}(\widehat{\theta}_i + \theta_j)$. Given a (possibly random) initial state $z(0) \in \mathbb{T}_N$, successive insertion of the rows of a solvable inverse-gamma weight matrix defines a Markov process $(z(t))_{t \in \mathbb{Z}_+}$ (see Section 3.1 of [19] for details). The following theorem describes the invariant distributions of a process defined through ratios of entries of $z(t)$.

Theorem 4.4 (Theorem 3.10 of Corwin-O'Connell-Seppäläinen-Zygouras [19])**.** *Let $z(t)$ evolve on the space $\mathbb{T}_N$ according to the Markovian dynamics governed by a solvable inverse-gamma weight matrix with parameters $(\widehat{\theta}_i : i \geqslant 1)$ and $(\theta_j : 1 \leqslant j \leqslant N)$.*

(a) The process $\eta(t)$, defined for $1 \leqslant \ell < k \leqslant N$ by

$$\eta_{k,\ell}(t) = \frac{z_{k,l}(t)}{z_{k-1,\ell}(t)},$$

is a Markov chain in its own filtration.

(b) Let $1 \leqslant \ell < N$ and assume $\theta_1 < \theta_2 < \cdots < \theta_\ell < \min\{\theta_{\ell+1}, \ldots, \theta_N\}$. Then the process $(\eta_1(t), \ldots, \eta_\ell(t))$ has an invariant distribution where the variables $\{\eta_{k,j} : 1 \leqslant j \leqslant \ell, j < k \leqslant N\}$ are independent with marginal distributions $\eta_{k,j} \sim Ga^{-1}(\theta_k - \theta_j)$. If the process is started with this distribution, then the following statement holds for all times $t \geqslant 1$: the variables $\{\eta_{k,j}(t) : 1 \leqslant j \leqslant \ell, j < k \leqslant N\} \cup \{z_{N,j}(m)/z_{N,j}(m-1) : 1 \leqslant m \leqslant t, 1 \leqslant j \leqslant \ell\}$ are independent with marginals $\eta_{k,j} \sim Ga^{-1}(\theta_k - \theta_j)$ and $z_{N,j}(m)/z_{N,j}(m-1) \sim Ga^{-1}(\widehat{\theta}_m + \theta_j)$.

The stationarity of the ℓ-path stationary polymer environment is a consequence of the above theorem.

Proof of Lemma 4.1. Define the initial state $z(0)$ using the partition functions in the first ℓ columns of the $m \times n$ cut-off rectangle. That is,

$$z_{k,1}(0) := Z^{(1)}_{\ell,k} \quad \text{for } 1 \leqslant k \leqslant n$$
$$z_{k,j}(0) := Z^{(j)}_{\ell,k}/Z^{(j-1)}_{\ell,k} \quad \text{for } 2 \leqslant j \leqslant \ell \leqslant n.$$

For $\ell + 1 \leqslant k \leqslant n$, define $\theta_k = \mu$. By construction, the ratios $\eta(0)$ defined below are independent and have marginal distributions

$$\eta_{k,1}(0) = V^{(1)}_{\ell,k} \sim \mathrm{Ga}^{-1}(\theta_k - \theta_1) \qquad \text{for } 1 < k \leqslant N$$
$$\eta_{k,j}(0) = V^{(j)}_{\ell,k}/V^{(j-1)}_{\ell,k} \sim \mathrm{Ga}^{-1}(\theta_k - \theta_j) \qquad \text{for } 2 \leqslant j < k \leqslant n.$$

The weights $\{Y_{i,j} : \ell + 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n\}$ form a solvable inverse-gamma weight matrix with parameters $\widehat{\theta}_i = 0$ and θ_j as defined in the stationary setup. Applying part (b) of Theorem 4.4 with $N = n$, $t = m - \ell$ and weight matrix $d_{i,j} = Y_{\ell+i,j}$, the variables $\{\eta_{k,j}(m-\ell) : 1 \leqslant j \leqslant \ell, j < k \leqslant n\} \cup \{z_{n,j}(i)/z_{n,j}(i-1) : 1 \leqslant i \leqslant m-\ell, 1 \leqslant j \leqslant \ell\}$ are independent with marginals $\eta_{k,j} \sim \mathrm{Ga}^{-1}(\theta_k - \theta_j)$ and $z_{n,j}(i)/z_{n,j}(i-1) \sim \mathrm{Ga}^{-1}(\theta_j)$. Furthermore, since these random variables were obtained through the geometric row insertion of the remaining columns of weights, their interpretation as ratios of polymer partition functions is extended to polymer partition functions using weights in the $m \times n$ section of the lattice. That is, $\eta_{k,j}(m-\ell) = V^{(j)}_{m,k}/V^{(j-1)}_{m,k}$ and $z_{n,j}(i)/z_{n,j}(i-1) = U^{(j)}_{i,n}/U^{(j-1)}_{i,n}$. $\square$

The computation of the expectation of the free energy is a direct application of the stationarity lemma.

Proof of Lemma 4.2. We expand

$$\begin{aligned}\log Z^{(\ell)}_{m,n} &= \log Z^{(\ell)}_{\ell,\ell} + \sum_{i=1}^{m-\ell}\big(\log Z^{(\ell)}_{\ell+i,\ell} - \log Z^{(\ell)}_{\ell+i-1,\ell}\big) + \sum_{j=1}^{n-\ell}\big(\log Z^{(\ell)}_{m,\ell+j} - \log Z^{(\ell)}_{m,\ell+j-1}\big)\\ &= \log Z^{(\ell)}_{\ell,\ell} + \sum_{i=1}^{m-\ell}\log U^{(\ell)}_{\ell+i,\ell} + \sum_{j=1}^{n-\ell} V^{(\ell)}_{m,\ell+j}.\end{aligned}$$

We further expanding the U and V terms,

$$\begin{aligned}\log U^{(\ell)}_{\ell+i,\ell} &= \log U^{(1)}_{\ell+i,\ell} + \sum_{k=2}^{\ell}\log\frac{U^{(k)}_{\ell+i,\ell}}{U^{(k-1)}_{\ell+i,\ell}},\\ \log V^{(\ell)}_{m,\ell+j} &= \log V^{(1)}_{m,\ell+j} + \sum_{k=2}^{\ell}\log\frac{V^{(k)}_{m,\ell+j}}{V^{(k-1)}_{m,\ell+j}}.\end{aligned}$$

Recall that if $X \sim \mathrm{Ga}^{-1}(\theta)$, then $\mathbb{E}[\log X] = -\psi_0(\theta)$. Applying Lemma 4.1, we obtain

$$\mathbb{E}[\log U^{(\ell)}_{\ell+i,\ell}] = -\sum_{k=1}^{\ell}\psi_0(\theta_k), \qquad \mathbb{E}[\log V^{(\ell)}_{m,\ell+j}] = -\sum_{k=1}^{\ell}\psi_0(\mu-\theta_k),$$

from which (4.2.1) follows. Since $\Pi^{(\ell)}_{\ell,\ell}$ contains only the single ℓ-tuple of paths that passes through all corner sites, $Z^{(\ell)}_{\ell,\ell}$ is the product of the corner boundary weights. Thus equation (4.2.2) follows from the assumption on the distributions of the corner boundary weights. $\square$

The two-path variance formula utilizes the stationarity lemma and a coupling argument similar to that given in Section 2.3 of Chapter 2.

Proof of Proposition 4.3. Define the sums

$$\begin{aligned}S_S &= \log Z^{(2)}_{2,2} + \sum_{i=3}^{m}\log U^{(2)}_{i,2}, & S_N &= \sum_{i=3}^{m}\log U^{(2)}_{i,n},\\ S_W &= \log Z^{(2)}_{2,2} + \sum_{j=3}^{n}\log V^{(2)}_{2,j}, & S_E &= \sum_{j=3}^{n}\log V^{(2)}_{m,j}.\end{aligned}$$

We can express $\log Z^{(2)}_{m,n} = S_S + S_E = S_W + S_N$. By Lemma 4.1, S_N is independent of S_E and so

$$
\begin{aligned}
\mathbb{V}\mathrm{ar}(\log Z^{(2)}_{m,n}) &= \mathbb{V}\mathrm{ar}(S_W + S_N) = \mathbb{V}\mathrm{ar}(S_W) + \mathbb{V}\mathrm{ar}(S_N) + 2\mathbb{C}\mathrm{ov}(S_W, S_N) \\
&= \mathbb{V}\mathrm{ar}(S_W) + \mathbb{V}\mathrm{ar}(S_N) + 2\mathbb{C}\mathrm{ov}(S_S + S_E - S_N, S_N) \\
&= \mathbb{V}\mathrm{ar}(S_W) - \mathbb{V}\mathrm{ar}(S_N) + 2\mathbb{C}\mathrm{ov}(S_S, S_N). \qquad (4.3.1)
\end{aligned}
$$

By Lemma 4.1, we can express S_W and S_N as sums of independent random variables. Recall that if $X \sim \mathrm{Ga}^{-1}(\theta)$, then $\mathbb{V}\mathrm{ar}(\log(X)) = \psi_1(\theta)$. It then follows that

$$
\begin{aligned}
\mathbb{V}\mathrm{ar}(S_W) &= \psi_1(\theta_2 - \theta_1) + (n-2)\big(\psi_1(\mu - \theta_1) + \psi_1(\mu - \theta_2)\big), \\
\mathbb{V}\mathrm{ar}(S_N) &= (m-2)\big(\psi_1(\theta_1) + \psi_1(\theta_2)\big).
\end{aligned} \qquad (4.3.2)
$$

Next define

$$
S^{(1)}_S = \sum_{i=3}^{m} \log U^{(1)}_{i,2}, \qquad\qquad S^{(2)}_S = \sum_{i=3}^{m} \log U^{(2)}_{i,2}.
$$

Then since $Z^{(2)}_{2,2} = Y_{2,2}$, $S_S = \log Y_{2,2} + S^{(1)}_S + S^{(2)}_S$ and $\log Z^{(2)}_{m,n} = \log Y_{2,2} + S^{(1)}_S + S^{(2)}_S + S_N$.

We now use a coupling argument similar to that of Section 2.3 of Chapter 2. We express the horizontal boundary weights and corner weight as functions of uniform random variables so that we can perturb the parameters θ_1 and θ_2. Let $\{\zeta^1_i, \zeta^2_i : i \geqslant 2\}$ be i.i.d. uniform(0,1) random variables that are independent of the weights $\{Y_{i,j} : i \geqslant 1, j \geqslant 3\}$. Using the definition (2.3.3) of function H^f from Chapter 2 we have that

$$
\begin{aligned}
H^f(-\theta_1, \zeta^1_i) &\overset{d}{=} Y_{i,1} \sim \mathrm{Ga}^{-1}(\theta_1) \quad \text{for } i \geqslant 3, \\
H^f(-\theta_2, \zeta^2_i) &\overset{d}{=} Y_{i,2} \sim \mathrm{Ga}^{-1}(\theta_2) \quad \text{for } i \geqslant 3, \\
H^f(-\theta_2 + \theta_1, \zeta^2_2) &\overset{d}{=} Y_{2,2} \sim \mathrm{Ga}^{-1}(\theta_2 - \theta_1),
\end{aligned}
$$

where $f := e^{-1/x}$ (see Table 10). Suppressing the f superscript of H, define the coupled environment

$$\begin{aligned}\omega(\theta_1,\theta_2) =& \{\widehat{Y}_{2,2}(\theta_1,\theta_2) := H(-\theta_2+\theta_1,\zeta_2^2)\} \cup \{\widehat{Y}_{i,1}(\theta_1) := H(-\theta_1,\zeta_i^1), i \geqslant 3\} \\ &\cup \{\widehat{Y}_{i,2}(\theta_2) := H(-\theta_2,\zeta_i^2) : i \geqslant 3\} \cup \{Y_{i,j} : i \geqslant 1, j \geqslant 3\}\end{aligned}$$

and note that it is equal in distribution to the original polymer environment. Let $\widehat{\mathbb{P}}$ and $\widehat{\mathbb{E}}$ denote the probability measure and corresponding expectation of the uniform random variables ζ_i^ℓ and the weights above the horizontal boundary, $\{Y_{i,j} : i \geqslant 1, j \geqslant 3\}$. For $\ell = 1, 2$ let $Z^{(\ell)}_{m,n}(\theta_1,\theta_2)$ and $Q^{(\ell)}_{m,n}(\theta_1,\theta_2)$ denote the partition functions and quenched polymer measures under the $\omega(\theta_1,\theta_2)$ environment. By utilizing Lemma B.2 of the appendix, we can show that

$$-\frac{\partial}{\partial\theta_1}\widehat{\mathbb{E}}[S_N(\theta_1,\theta_2)] = \widehat{\mathbb{C}\mathrm{ov}}\big(S_S^{(1)}(\theta_1,\theta_2) - \log\widehat{Y}_{2,2}(\theta_1,\theta_2), S_N(\theta_1,\theta_2)\big), \tag{4.3.3}$$

$$-\frac{\partial}{\partial\theta_2}\widehat{\mathbb{E}}[S_N(\theta_1,\theta_2)] = \widehat{\mathbb{C}\mathrm{ov}}\big(S_S^{(2)}(\theta_1,\theta_2) + \log\widehat{Y}_{2,2}(\theta_1,\theta_2), S_N(\theta_1,\theta_2)\big) \tag{4.3.4}$$

where $S_S^{(i)}(\theta_1,\theta_2)$ and $S_N(\theta_1,\theta_2)$ are the sums defined using the coupled environment. Furthermore, there are open intervals $I_1 \ni \theta_1$ and $I_2 \ni \theta_2$ on which the covariances (4.3.3) and (4.3.4) change continuously under perturbations of θ_1 and θ_2.

For a pair $(\pi_1,\pi_2) \in \Pi^{(2)}_{m,n}$, we decompose the weight of the path to isolate the coupled weights. With the convention that an empty product has value one,

$$\begin{aligned}wt(\pi_1,\pi_2)(\theta_1,\theta_2) = &\left(\prod_{i=3}^{t_1(\pi_1)}\widehat{Y}_{i,1}(\theta_1)\prod_{j=t_1(\pi_1)\vee 3}^{t_2(\pi_1)}\widehat{Y}_{j,2}(\theta_2)\prod_{k=3}^{t_2(\pi_2)}\widehat{Y}_{k,2}(\theta_2)\right) \\ &\times\left(\prod_{(2,2)\in(\pi_1,\pi_2)}\widehat{Y}_{2,2}(\theta_1,\theta_2)\right)\overline{wt}(\pi_1,\pi_2),\end{aligned} \tag{4.3.5}$$

where $\overline{wt}(\pi_1, \pi_2)$ is the portion of the weight picked up in the rectangle $\{1, \ldots, m\} \times \{3, \ldots, n\}$. Recall from (2.3.7) that

$$\frac{\partial}{\partial a} H^f(a,p) = H^f(a,p) L^f(a, H^f(a,p)), \tag{4.3.6}$$

where $L^f(a,x)$ is defined in (2.3.8).

Using the decomposition (4.3.5) and derivative (4.3.6), we now compute the derivatives of the free energy with respect to the two parameters, θ_1 and θ_2. Let $A \subset \Pi^{(2)}_{m,n}$ be the set of all pairs (π_1, π_2) that contain the point $(2,2)$. Suppressing the superscript f of L^f,

$$\begin{aligned}
\frac{\partial}{\partial \theta_1} \log Z^{(2)}_{m,n}(\theta_1, \theta_2) &= \frac{1}{Z^{(2)}_{m,n}} \sum_{(\pi_1,\pi_2) \in \Pi_{m,n}} wt(\pi_1, \pi_2) \Bigg(\mathbb{1}((\pi_1,\pi_2) \in A) L(\theta_1 - \theta_2, \widehat{Y}_{2,2}) \\
&\qquad + \sum_{i=3}^{t_1(\pi_1)} -L(-\theta_1, \widehat{Y}_{i,1}) \Bigg) \\
&= L(\theta_1 - \theta_2, \widehat{Y}_{2,2}) Q^{(2)}_{m,n}(A) - E^{Q^{(2)}_{m,n}} \sum_{i=3}^{t_1(\pi_1)} L(-\theta_1, \widehat{Y}_{i,1}) \qquad (4.3.7) \\
\frac{\partial}{\partial \theta_2} \log Z^{(2)}_{m,n}(\theta_1, \theta_2) &= \frac{1}{Z^{(2)}_{m,n}} \sum_{(\pi_1,\pi_2) \in \Pi_{m,n}} wt(\pi_1, \pi_2) \Bigg(- \mathbb{1}((\pi_1,\pi_2) \in A) L(\theta_1 - \theta_2, \widehat{Y}_{2,2}) \\
&\qquad + \sum_{j=t_1(\pi_1) \vee 3}^{t_2(\pi_1)} -L(-\theta_2, \widehat{Y}_{j,2}) + \sum_{k=3}^{t_2(\pi_2)} -L(-\theta_2, \widehat{Y}_{k,2}) \Bigg) \\
&= -L(\theta_1 - \theta_2, \widehat{Y}_{2,2}) Q^{(2)}_{m,n}(A) - E^{Q^{(2)}_{m,n}} \Bigg[\sum_{j=t_1(\pi_1) \vee 3}^{t_1(\pi_1)} L(-\theta_2, \widehat{Y}_{j,2}) \\
&\qquad + \sum_{k=3}^{t_2(\pi_2)} L(-\theta_2, \widehat{Y}_{k,2}) \Bigg]. \qquad (4.3.8)
\end{aligned}$$

On the other hand, using the decomposition

$$\log Z^{(2)}_{m,n}(\theta_1, \theta_2) = \log \widehat{Y}_{2,2}(\theta_1, \theta_2) + (\log Z^{(2)}_{2,n} - \log Z^{(2)}_{2,2}) + S_N(\theta_1, \theta_2)$$

and the fact that $(\log Z_{2,n}^{(2)} - \log Z_{2,2}^{(2)})$ has no dependence on θ_1 or θ_2, the derivatives of the free energy can also be expressed as

$$\frac{\partial}{\partial\theta_1} \log Z_{m,n}^{(2)}(\theta_1, \theta_2) = L(\theta_1 - \theta_2, \widehat{Y}_{2,2}) + \frac{\partial}{\partial\theta_1} S_N(\theta_1, \theta_2), \tag{4.3.9}$$

$$\frac{\partial}{\partial\theta_2} \log Z_{m,n}^{(2)}(\theta_1, \theta_2) = -L(\theta_1 - \theta_2, \widehat{Y}_{2,2}) + \frac{\partial}{\partial\theta_2} S_N(\theta_1, \theta_2). \tag{4.3.10}$$

Using the equality in distribution of the original and the coupled environments, equations (4.3.3), (4.3.4), (4.3.9), (4.3.10), (4.3.7), (4.3.8), and a justification similar to that in the proof of Lemma 2.17 of Chapter 2, we then pass the derivative through the expectation to obtain:

$$\begin{aligned}
\mathbb{C}\mathrm{ov}(S_S, S_N) &= -\frac{\partial}{\partial\theta_1}\left(\widehat{\mathbb{E}} S_N(\theta_1, \theta_2)\right) - \frac{\partial}{\partial\theta_2}\left(\widehat{\mathbb{E}} S_N(\theta_1, \theta_2)\right) \\
&= -\widehat{\mathbb{E}}\left[\frac{\partial}{\partial\theta_1} \log Z_{m,n}^{(2)} + \frac{\partial}{\partial\theta_2} \log Z_{m,n}^{(2)}\right] \\
&= \mathbb{E} E^{Q_{m,n}^{(2)}}\left[\mathbb{1}((\pi_1, \pi_2) \in A) L(\theta_1 - \theta_2, Y_{2,2}) + \sum_{i=3}^{t_1(\pi_1)} L(-\theta_1, Y_{i,1})\right. \\
&\qquad \left. + \sum_{j=t_1(\pi_1)\vee 3}^{t_2(\pi_1)} L(-\theta_2, Y_{j,2}) + \sum_{k=3}^{t_2(\pi_2)} L(-\theta_2, Y_{k,2})\right].
\end{aligned} \tag{4.3.11}$$

Combining equation (4.3.1) with equations (4.3.2) and (4.3.11) completes the proof. □

Appendix A

Verification of Hypothesis 2.15

Lemma A.1. *If the function f satisfies the conditions of Hypothesis 2.15 and $g(x) := f(\frac{1}{x})$ for $x \in (0, \infty)$, then g also satisfies the conditions of Hypothesis 2.15.*

Proof. Note that $\mathrm{supp}(g) = \mathrm{supp}(f)^{-1}$. Fix a compact $K \subset D(M_g)$ and let $a \in K$. By parts (c) and (b) of Remark 2.12, $\psi_0^g(a) = -\psi_0^f(-a)$ and $-K \subset D(M_f)$. Thus there exists a positive constant C depending only $-K$ such that for all $b \in -K$, (2.3.9) and (2.3.10) hold. It therefore suffices to show the following two relations hold:

$$L^g(a,x) = L^f(-a, \frac{1}{x}) \qquad \text{for all } x \in \mathrm{supp}(g) \tag{A.0.1}$$

$$\int_0^1 \Big|\frac{\partial}{\partial a} L^g(a, H^g(a,p))\Big| dp = \int_0^1 \Big|\frac{\partial}{\partial b} L^f(b, H^f(b,p))\Big| dp \tag{A.0.2}$$

where the right hand side of (A.0.2) is evaluated at $b = -a$.

(A.0.1) can be proven by using $\psi_0^g(a) = -\psi_0^f(-a)$ and making the substitution $y \mapsto \frac{1}{y}$ in the first integral appearing in (2.3.8).

(A.0.2) will now follow from (A.0.1) and

$$H^g(a, 1-p) = \frac{1}{H^f(-a,p)} \qquad \text{for all } p \in (0,1).$$

To see that this equality holds, let $X \sim m_g(a)$ and $x > 0$. Using part (a) of Remark

2.12

$$F^g(a,x) = \mathbb{P}(X \leqslant x) = \mathbb{P}(X^{-1} \geqslant x^{-1}) = 1 - \mathbb{P}(X^{-1} < x^{-1}) = 1 - F^f(-a, x^{-1}). \tag{A.0.3}$$

Fix $p \in (0,1)$ and recall the definition of $H^\bullet$, (2.3.3). Note that $H^f(-a,p)$ and $H^g(a,1-p)$ lie in $\mathrm{supp}(f)$ and $\mathrm{supp}(g) = \mathrm{supp}(f)^{-1}$ respectively. Plugging $x = H^g(a, 1-p)$ into (A.0.3) gives

$$1 - p = F^g(a, H^g(a, 1-p)) = 1 - F^f\big(-a, \frac{1}{H^g(a,1-p)}\big).$$

Rearranging yields

$$F^f\big(-a, \frac{1}{H^g(a,1-p)}\big) = p = F^f(-a, H^f(-a,p)).$$

Since $x \mapsto F^f(-a,x)$ is one-to-one on $\mathrm{supp}(f)$ we have the desired result. $\square$

Lemma A.2. *Each of the functions f in Table 10 satisfy Hypothesis 2.15.*

Proof. Fix $b > 0$. By Lemma A.1 it suffices to show the three functions

$$f(x) = e^{-bx}, \qquad f(x) = (1-x)^{b-1}\mathbb{1}_{\{0<x<1\}}, \qquad f(x) = \big(\frac{x}{x+1}\big)^b$$

satisfy the conditions of Hypothesis 2.15. In [45] (equation 3.30 and the computation following equation 4.7), Seppäläinen showed that the function $f(x) = e^{-x}$ satisfies these conditions. A simple rescaling then shows that these conditions are also satisfied for $f(x) = e^{-bx}$.

We will write $C_0(a), C_1(a), \ldots$ to indicate the positive constants $C_k(a)$ have a continuous dependence on a. We claim it is sufficient to show that if $f(x) = (1-x)^{b-1}\mathbb{1}_{\{0<x<1\}}$

or $f(x) = (\frac{x}{x+1})^b$, then for all $x \in \operatorname{supp}(f)$ the following three bounds hold:

$$L^f(a,x) \leqslant C_0(a)(1+|\log x|) \tag{A.0.4}$$

$$|x\frac{f'(x)}{f(x)}|L^f(a,x) \leqslant C_1(a)(1+|\log x|) \tag{A.0.5}$$

$$|G^f(a,x)| \leqslant C_2(a)(1+(\log x)^2) \tag{A.0.6}$$

where

$$\begin{aligned} G^f(a,x) :&= \frac{x^{-a}}{f(x)}\int_0^x (\psi_1^f(a) + \psi_0^f(a)\log y - (\log y)^2)y^{a-1}f(y)dy \\ &= -\frac{x^{-a}}{f(x)}\int_x^\infty (\psi_1^f(a) + \psi_0^f(a)\log y - (\log y)^2)y^{a-1}f(y)dy. \end{aligned} \tag{A.0.7}$$

Note that the second equality in the definition of $G^f(a,x)$ follows from the definitions of $\psi_0^f(a)$ and $\psi_1^f(a)$ in part (c) of Remark 2.11. (A.0.4) clearly implies (2.3.9). To show (2.3.10) is satisfied, using (2.3.7), we calculate

$$\begin{aligned} \frac{\partial}{\partial a}L^f(a,H^f(a,p)) &= \frac{\partial L^f}{\partial a}(a,H^f(a,p)) + \frac{\partial}{\partial a}H^f(a,p)\frac{\partial L^f}{\partial x}(a,H^f(a,p)) \\ &= \left(\frac{\partial L^f}{\partial a}(a,x) + xL^f(a,x)\frac{\partial L^f}{\partial x}(a,x)\right)\Big|_{x=H^f(a,p)}. \end{aligned}$$

Since

$$\begin{aligned} \frac{\partial L^f}{\partial a}(a,x) + xL^f(a,x)\frac{\partial L^f}{\partial x}(a,x) =&(\psi_0^f(a) - 2\log x)L^f(a,x) - aL^f(a,x)^2 \\ &+ G^f(a,x) - x\frac{f'(x)}{f(x)}L^f(a,x)^2, \end{aligned}$$

the conditions (A.0.4), (A.0.5), and (A.0.6) imply the existence of a positive constant $C_3(a)$ such that for all $x \in \operatorname{supp}(f)$,

$$\left|\frac{\partial L^f}{\partial a}(a,x) + xL^f(a,x)\frac{\partial L^f}{\partial x}(a,x)\right| \leqslant C_3(a)\big(1+(\log x)^2\big).$$

Condition (2.3.10) now follows from

$$\int_0^1 \left|\frac{\partial}{\partial a} L^f(a, H^f(a,p))\right| dp \leqslant C_3(a) \int_0^1 \big(1 + (\log H^f(a,p))^2\big) dp$$
$$= C_3(a)\big(1 + \psi_1^f(a) + (\psi_0^f(a))^2\big) < \infty.$$

The last equality is justified by parts (a) and (c) of Remark 2.11 along with the fact that $H^f(a,\eta) \sim m_f(a)$ when η is uniformly distributed on $(0,1)$.

We first show (A.0.4), (A.0.5) and (A.0.6) for the case $f(x) = (1-x)^{b-1}\mathbb{1}_{\{0<x<1\}}$. Let $a \in D(M_f) = (0,\infty)$. Then there exists some positive constant $C_4(a)$ such that the following two inequalities hold:

$$\big|\psi_0^f(a) - \log y\big| y^{a-1} f(y) \leqslant \begin{cases} C_4(a)(1-\log y)y^{a-1} & \text{if } 0 < y < \frac{1}{2} \\ C_4(a)(1-y)^{b-1} & \text{if } \frac{1}{2} \leqslant y < 1 \end{cases}$$

$$\big|\psi_1^f(a) + \psi_0^f(a)\log y - (\log y)^2\big| y^{a-1} f(y) \leqslant \begin{cases} C_4(a)\big(1+(\log y)^2\big)y^{a-1} & \text{if } 0 < y < \frac{1}{2} \\ C_4(a)(1-y)^{b-1} & \text{if } \frac{1}{2} \leqslant y < 1. \end{cases}$$

Since $a > 0$, (2.3.8) and (A.0.7) give: for $0 < x < \frac{1}{2}$,

$$L^f(a,x) \leqslant \frac{2^b C_4(a)}{x^a} \int_0^x (1-\log y)y^{a-1}dy \leqslant C_0(a)(1+|\log x|) \tag{A.0.8}$$
$$\big|G^f(a,x)\big| \leqslant \frac{2^b C_4(a)}{x^a} \int_0^x \big(1+(\log y)^2\big)y^{a-1}dy \leqslant C_2(a)\big(1+(\log x)^2\big).$$

Similarly, the secondary expressions in (2.3.8) and (A.0.7) give: for $1/2 \leqslant x < 1$,

$$L^f(a,x) \leqslant \frac{2^a C_4(a)}{(1-x)^{b-1}} \int_x^1 (1-y)^{b-1}dy \leqslant C_0(a)(1-x) \tag{A.0.9}$$
$$\big|G^f(a,x)\big| \leqslant \frac{2^a C_4(a)}{(1-x)^{b-1}} \int_x^1 (1-y)^{b-1}dy \leqslant C_2(a)(1-x)$$

where we increased $C_0(a)$ and $C_2(a)$ if necessary. Thus the bounds (A.0.4) and (A.0.6) hold. Moreover, by (A.0.8) and (A.0.9),

$$\left|x\frac{f'(x)}{f(x)}\right|L^f(a,x) = |b-1|\frac{x}{1-x}L^f(a,x) \leqslant \begin{cases} C_1(a)(1+|\log x|) & \text{if } 0 \leqslant x < \frac{1}{2} \\ C_1(a) & \text{if } \frac{1}{2} \leqslant x < 1 \end{cases}$$

proving the bound (A.0.5).

We now consider the case $f(x) = (\frac{x}{x+1})^b$. Let $a \in D(M_f) = (-b, 0)$. Then

$$\left|\psi_0^f(a) - \log y\right| y^{a-1} f(y) \leqslant \begin{cases} C_4(a)(1-\log y)y^{a+b-1} & \text{if } 0 < y < 1 \\ C_4(a)(1+\log y)y^{a-1} & \text{if } y \geqslant 1 \end{cases}$$

$$\left|\psi_1^f(a) + \psi_0^f(a)\log y - (\log y)^2\right| y^{a-1} f(y) \leqslant \begin{cases} C_4(a)\big(1+(\log y)^2\big)y^{a+b-1} & \text{if } 0 < y < 1 \\ C_4(a)(1+(\log y)^2)y^{a-1} & \text{if } y \geqslant 1. \end{cases}$$

Since $a+b > 0$, (2.3.8) and (A.0.7) give: for $0 < x < 1$,

$$\begin{aligned} L^f(a,x) &\leqslant \frac{2^b C_4(a)}{x^{a+b}} \int_0^x (1-\log y)y^{a+b-1}dy \leqslant C_0(a)(1+|\log x|) \\ \left|G^f(a,x)\right| &\leqslant \frac{2^b C_4(a)}{x^{a+b}} \int_0^x \big(1+(\log y)^2\big)y^{a+b-1}dy \leqslant C_2(a)\big(1+(\log x)^2\big). \end{aligned}$$

Similarly, since $a < 0$, the secondary expressions in (2.3.8) and (A.0.7) give: for $x \geqslant 1$,

$$\begin{aligned} L^f(a,x) &\leqslant \frac{2^b C_4(a)}{x^a} \int_x^\infty (1+\log y)y^{a-1}dy \leqslant C_0(a)(1+|\log x|) \\ \left|G^f(a,x)\right| &\leqslant \frac{2^b C_4(a)}{x^a} \int_x^\infty (1+(\log y)^2)y^{a-1}dy \leqslant C_2(a)(1+(\log x)^2) \end{aligned}$$

where we increased $C_0(a)$ and $C_2(a)$ if necessary. Thus the bounds (A.0.4) and (A.0.6) hold. Since $|x\frac{f'(x)}{f(x)}| = b\frac{1}{x+1} \leqslant b$, (A.0.4) implies (A.0.5) completing the proof. $\square$

Appendix B

Lemmas used in Section 2.3 and Section 2.4

Lemma B.1. *Assume the polymer environment is such that* $\log R^1$, $\log R^2$, $\log Y^1$, *and* $\log Y^2$ *have finite second moments. Then* $\mathbb{E}[(\log Z_x)^2] < \infty$ *for any* $x \in \mathbb{Z}_+^2$.

Proof. Since $\log Z_{k,0} = \sum_{i=1}^{k} R_{i,0}^1$ and $\log Z_{0,\ell} = \sum_{j=1}^{\ell} \log R_{0,j}^2$, $\log Z_x$ has finite second moment for each $x \in \mathbb{Z}_+^2 \backslash \mathbb{N}^2$. If $x \in \mathbb{N}^2$, the recursion (2.2.1) implies that

$$\begin{aligned}(\log Y_x^1 + \log Z_{x-\alpha_1}) &\wedge (\log Y_x^2 + \log Z_{x-\alpha_2}) \\ &\leqslant \frac{\log Z_x}{2} \leqslant (\log Y_x^1 + \log Z_{x-\alpha_1}) \vee (\log Y_x^2 + \log Z_{x-\alpha_2}).\end{aligned}$$

Thus

$$(\log Z_x)^2 \leqslant 4(\log Y_x^1 + \log Z_{x-\alpha_1})^2 + 4(\log Y_x^2 + \log Z_{x-\alpha_2})^2.$$

Since $\log Y^1$ and $\log Y^2$ have finite second moments, an inductive argument finishes the proof. □

Lemma B.2. *Suppose* $f_k : (0,\infty) \to [0,\infty)$ *for* $k = 1, \ldots, r$ *and* $a_0 < a < a_1$ *are real numbers such that* $[a_0, a_1] \subset \bigcap_{k=1}^{r} D(M_{f_k})$. *Suppose we have a collection of independent random variables* $\{X_k\}_{k=1}^r$ *where* $X_k \sim m_{f_k}(a)$ *for all* $1 \leqslant k \leqslant r$. *Let* $\mathbb{E}^a$ *be the expectation corresponding to the product measure induced by* $\{X_k\}_{k=1}^r$.

Let $S = \sum_{k=1}^r \log X_k$ *and* $A : \mathbb{R}^r \to \mathbb{R}$ *be a measurable function such that* $\mathbb{E}^a[A(X_1, \ldots, X_r)^2] < \infty$ *for all* $a \in [a_0, a_1]$. *Then*

$$\frac{\partial}{\partial a}\mathbb{E}^a[A(X_1, \ldots, X_r)] = \mathbb{C}ov^a(A(X_1, \ldots, X_r), S) \quad \textit{for all} \quad a \in (a_0, a_1)$$

and $(a_0, a_1) \ni a \mapsto \frac{\partial}{\partial a}\mathbb{E}^a[A(X_1, \ldots, X_r)]$ *is continuous.*

Proof. The joint density of $(\log X_1, \log X_2, \ldots, \log X_r)$ is given by

$$g(x_1, \ldots, x_r) = \frac{e^{a\sum_{k=1}^r x_k}}{\prod_{k=1}^r M_{f_k}(a)} \prod_{k=1}^r f_k(e^{x_k}).$$

Thus the density of S is given by

$$h_a(s) = \frac{e^{as}}{\prod_{k=1}^r M_{f_k}(a)} \int_{\mathbb{R}^{r-1}} f_1(e^{x_1}) f_2(e^{x_2 - x_1}) \ldots f_r(e^{s - x_{r-1}}) dx_1, \ldots, x_{r-1} \tag{B.0.1}$$

Therefore the joint density of $(\log X_1, \log X_2, \ldots, \log X_r)$ given that $S = s$ is

$$\frac{g(x_1, \ldots, x_r)\mathbb{1}_{\{\sum_{k=1}^r x_k = s\}}}{h_a(s)} = \frac{\prod_{k=1}^r f_k(e^{x_k})\mathbb{1}_{\{\sum_{k=1}^r x_k = s\}}}{\int_{\mathbb{R}^{r-1}} f_1(e^{x_1}) f_2(e^{x_2 - x_1}) \ldots f_r(e^{s - x_{r-1}}) dx_1, \ldots, x_{r-1}},$$

which has no a dependence. Thus

$$\begin{aligned}
\tfrac{\partial}{\partial a}\mathbb{E}^a[A(X_1, \ldots, X_r)] &= \tfrac{\partial}{\partial a}\int_{\mathbb{R}} \mathbb{E}^a[A(X_1, \ldots, X_r)|S = s] h_a(s) ds \\
&= \int_{\mathbb{R}} \mathbb{E}^a[A(X_1, \ldots, X_r)|S = s] \tfrac{\partial}{\partial a} h_a(s) ds \\
&= \int_{\mathbb{R}} \mathbb{E}^a[A(X_1, \ldots, X_r)|S = s] h_a(s)\big(s - \sum_{k=1}^r \tfrac{\partial}{\partial a}\log M_{f_k}(a)\big) ds \\
&= \mathrm{Cov}^a(A(X_1, \ldots, X_r), S).
\end{aligned}$$

The last equality comes from $\mathbb{E}[S] = \sum_{k=1}^r \mathbb{E}[\log X_k] = \sum_{k=1}^r \frac{\partial}{\partial a}\log M_{f_k}(a)$, by part (a) of Remark 2.11. The interchanging of the derivative and the integral is justified by the bound

$$\int_{\mathbb{R}} \mathbb{E}[|A(X_1, \ldots, X_r)|\,|S = s] \sup_{a \in [a_0, a_1]} \left|\frac{\partial}{\partial a} h_a(s)\right| ds < \infty. \tag{B.0.2}$$

Once we show that there is a constant C depending only on a_0 and a_1 such that

$$\sup_{a\in[a_0,a_1]}\left|\frac{\partial}{\partial a}h_a(s)\right| \leqslant C(1+|s|)(h_{a_0}(s)+h_{a_1}(s)) \tag{B.0.3}$$

we will have the bound (B.0.2) since

$$\begin{aligned}\int_{\mathbb{R}} \mathbb{E}[|A(X_1,\ldots,X_r)|\,|S=s](1+|s|)h_{a_j}(s)ds &= \mathbb{E}^{a_j}[|A(X_1,\ldots,X_r)|(1+|S|)]\\ &\leqslant \mathbb{E}^{a_j}[A(X_1,\ldots,X_r)^2]^{\frac{1}{2}}\mathbb{E}^{a_j}[(1+|S|)^2]^{\frac{1}{2}}.\end{aligned}$$

The last expression is finite since $\mathbb{E}^{a_j}[A(X_1,\ldots,X_r)^2]<\infty$ by assumption, and S is a finite sum of independent random variables each of which has finite exponential moments, by part (a) of Remark 2.11. Notice that the bound (B.0.2) also implies that $a\mapsto \frac{\partial}{\partial a}\mathbb{E}^a[A(X_1,\ldots,X_r)]$ is continuous. All that is left to do is verify the bound (B.0.3). To accomplish this, notice that equation (B.0.1) implies that $\frac{\partial}{\partial a}\log h_a(s) = s-\mathbb{E}^a[S]$. So

$$\sup_{a\in[a_0,a_1]}\left|\frac{\partial}{\partial a}h_a(s)\right| \leqslant C_1(1+|s|)\sup_{a\in[a_0,a_1]}h_a(s)$$

where $C_1 := 1\vee \sup_{a\in[a_0,a_1]}|E^a[S]|$. Thus it suffices to show that $\sup_{a\in[a_0,a_1]}h_a(s)\leqslant C_2\Big(h_{a_0}(s)+h_{a_1}(s)\Big)$ for some constant C_2 independent of s. By part (c) of Remark 2.11, $a\mapsto\mathbb{E}^a[S]$ is an increasing function. Therefore, for all $s\leqslant\mathbb{E}^{a_0}[S]$ the function $a\mapsto h_a(s)$ is non-increasing on $[a_0,a_1]$. Thus

$$\sup_{a\in[a_0,a_1]}h_a(s)\leqslant h_{a_0}(s) \text{ for all } s\leqslant\mathbb{E}^{a_0}[S].$$

On the other hand, if $s>\mathbb{E}^{a_0}[S]$, then $\frac{\partial}{\partial a}\log\Big(h_a(s)\exp\left(a(\mathbb{E}^{a_1}[S]-\mathbb{E}^{a_0}[S])\right)\Big) = s-\mathbb{E}^a[S]+\mathbb{E}^{a_1}[S]-\mathbb{E}^{a_0}[S]>0$ for all $a\in[a_0,a_1]$. Thus for all $s>\mathbb{E}^{a_0}[S]$, $a\mapsto h_a(s)\exp\Big(a(\mathbb{E}^{a_1}[S]-\mathbb{E}^{a_0}[S])\Big)$ is increasing on the interval $[a_0,a_1]$. Therefore,

$$\sup_{a\in[a_0,a_1]}h_a(s)\leqslant C_3h_{a_1}(s) \text{ for all } s>\mathbb{E}^{a_0}[S]$$

where $C_3 = \exp\Big((a_1 - a_0)(\mathbb{E}^{a_1}[S] - \mathbb{E}^{a_0}[S])\Big)$. We now get the desired result with $C_2 = 1 + C_3$. □

Lemma B.3. *Assume that the polymer environment is distributed as in* (2.3.2) *and let* ϵ *be small enough such that for all* $|\lambda| \leqslant \epsilon$, $a_1 + \lambda \in D(M_{f^1})$ *and* $a_2 - \lambda \in D(M_{f^2})$. *Let* $(m, n) \in \mathbb{N}^2$ *and* $k \in \mathbb{N}$. *Then, with notation as in* (2.3.5), $Q^{(a_1+\lambda, a_2-\lambda)}_{m,n}(t_1 \geqslant k)$ *is stochastically non-decreasing in* λ *and* $Q^{(a_1+\lambda, a_2-\lambda)}_{m,n}(t_2 \geqslant k)$ *is stochastically non-increasing in* λ.

Proof.

$$\frac{\partial}{\partial b_i} Q^{(b_1,b_2)}_{m,n}(t_j \geqslant k) = \frac{\partial}{\partial b_i}\left(\frac{1}{Z_{m,n}(b_1,b_2)} \sum_{x_\bullet \in \Pi_{m,n}} \mathbb{1}_{\{t_j \geqslant k\}} W(b_1,b_2)(x_\bullet)\right). \tag{B.0.4}$$

If $i \neq j$, the sum in (B.0.4) has no b_i dependence, so

$$\frac{\partial}{\partial b_i} Q^{(b_1,b_2)}_{m,n}(t_j \geqslant k) = \frac{-1}{(Z_{m,n}(b_1,b_2))^2}\left(\frac{\partial}{\partial b_i} Z_{m,n}(b_1,b_2)\right) \sum_{x_\bullet \in \Pi_{m,n}} \mathbb{1}_{\{t_j \geqslant k\}} W(b_1,b_2)(x_\bullet),$$

which is non-positive by (2.3.18). If $i = j$, then by (2.3.17) and (2.3.18),

$$\begin{aligned}
\frac{\partial}{\partial b_i} Q^{(b_1,b_2)}_{m,n}(t_i \geqslant k) &= \frac{\sum_{x_\bullet \in \Pi_{m,n}} \mathbb{1}_{\{t_i \geqslant k\}} \frac{\partial}{\partial b_i} W(b_1,b_2)(x_\bullet)}{Z_{m,n}(b_1,b_2)} \\
&\quad - \left(\frac{\partial}{\partial b_i} \log Z_{m,n}(b_1,b_2)\right) \frac{\sum_{x_\bullet \in \Pi_{m,n}} \mathbb{1}_{\{t_i \geqslant k\}} W(b_1,b_2)(x_\bullet)}{Z_{m,n}(b_1,b_2)} \\
&= \mathbb{C}\mathrm{ov}^{Q^{(b_1,b_2)}_{m,n}}\Big(\sum_{k=1}^{t_i} L^{f^i}(b_i, H^{f^i}(b_i, \eta^i_k)), \mathbb{1}_{\{t_i \geqslant k\}}\Big),
\end{aligned}$$

which is non-negative. □

Appendix C

Properties of ψ_n^f

Model	$\psi_n^{f^1}(a_1)$	$\psi_n^{f^2}(a_2)$
IG	$(-1)^{n+1}(\Psi_n(\mu-\theta)-\delta_{n,0}\log\beta)$	$(-1)^{n+1}(\Psi_n(\theta)-\delta_{n,0}\log\beta)$
G	$\Psi_n(\mu+\theta)-\delta_{n,0}\log\beta$	$(-1)^{n+1}(\Psi_n(\theta)-\Psi_n(\mu+\theta))$
B	$\Psi_n(\mu+\theta)-\Psi_n(\mu+\theta+\beta)$	$(-1)^{n+1}(\Psi_n(\theta)-\Psi_n(\mu+\theta))$
IB	$(-1)^{n+1}(\Psi_n(\mu-\theta)-\Psi_n(\mu-\theta+\beta))$	$\Psi_n(\mu-\theta+\beta)+(-1)^{n+1}\Psi_n(\theta)$

Figure 22: ψ_n^f functions for each of the four basic beta-gamma models.

By [1] (p.260 line 6.4.1) the polygamma function of order n, $\Psi_n(x) = \frac{\partial^{n+1}}{\partial x^{n+1}}\log\Gamma(x)$, has integral representation

$$\Psi_n(x) = (-1)^{n+1}\int_0^\infty \frac{t^n e^{-xt}}{1-e^{-t}}dt. \tag{C.0.1}$$

Lemma C.1. *For any $n \in \mathbb{N}$, the map $a \mapsto \frac{\Psi_{n+1}(a)}{\Psi_n(a)}$ is strictly increasing on $(0,\infty)$.*

Proof. Fix $n \in \mathbb{N}$ and $a \in (0,\infty)$. We will show that $\frac{\partial^2}{\partial a^2}\log|\Psi_n(a)| > 0$.

After substituting $y = e^{-t}$ in (C.0.1) we get

$$|\Psi_n(a)| = \int_0^\infty y^{a-1}f(y)dy = M_f(a)$$

where $f(y) := \frac{(-\log y)^n}{1-y}\mathbb{1}_{\{0<y<1\}}$. Note that $D(M_f) = (0,\infty)$. Now given a random variable $X \sim m_f(a)$, by part (c) of Remark 2.11,

$$\frac{\partial^2}{\partial a^2}\log|\Psi_n(a)| = \frac{\partial^2}{\partial a^2}\log M_f(a) = \mathbb{V}\mathrm{ar}[\log X] > 0,$$

since X is non-degenerate. $\square$

Lemma C.2. *Assume the polymer environment is distributed as in* (2.3.2)*. Then*

$$\psi_1^{f^1}(a_1)\psi_2^{f^2}(a_2) + \psi_1^{f^2}(a_2)\psi_2^{f^1}(a_1) > 0.$$

Proof. Recall that $\psi_1^{f^j}$ are always positive and by (C.0.1) Ψ_n has sign $(-1)^{n+1}$ throughout $(0, \infty)$.

For the inverse-gamma model (1.2.2) with fixed constants $\beta > 0$ and $\mu > \theta > 0$, Table 22 implies that $\psi_2^{f^j}(a_j) > 0$ for $j = 1, 2$. The conclusion follows immediately.

For the gamma model (1.2.3) with fixed positive constants β, μ, and θ, by Table 22

$$\psi_1^{f^1}(a_1)\psi_2^{f^2}(a_2) + \psi_1^{f^2}(a_2)\psi_2^{f^1}(a_1) = -\Psi_1(\theta + \mu)\Psi_2(\theta) + \Psi_1(\theta)\Psi_2(\theta + \mu).$$

The quantity on the right hand side is positive if and only if

$$\frac{\Psi_2(\theta + \mu)}{\Psi_1(\theta + \mu)} > \frac{\Psi_2(\theta)}{\Psi_1(\theta)}$$

which holds true by Lemma C.1 with $n = 1$.

For the beta model (1.2.4) with fixed positive constants β, μ, and θ, using Table 22

$$\begin{aligned} \psi_1^{f^1}(a_1)\psi_2^{f^2}(a_2) + \psi_1^{f^2}(a_2)\psi_2^{f^1}(a_1) > 0 & \qquad \Leftrightarrow \\ \frac{\psi_2^{f^1}(a_1)}{\psi_1^{f^1}(a_1)} > -\frac{\psi_2^{f^2}(a_2)}{\psi_1^{f^2}(a_2)} & \qquad \Leftrightarrow \\ \frac{\Psi_2(\theta + \mu + \beta) - \Psi_2(\theta + \mu)}{\Psi_1(\theta + \mu + \beta) - \Psi_1(\theta + \mu)} > \frac{\Psi_2(\theta + \mu) - \Psi_2(\theta)}{\Psi_1(\theta + \mu) - \Psi_1(\theta)}. & \end{aligned} \tag{C.0.2}$$

By Cauchy's mean value theorem there exist constants $\theta < \xi_1 < \theta + \mu < \xi_2 < \theta + \mu + \beta$ such that the left and right-hand sides of (C.0.2) equal $\frac{\Psi_3(\xi_2)}{\Psi_2(\xi_2)}$ and $\frac{\Psi_3(\xi_1)}{\Psi_2(\xi_1)}$ respectively. Lemma C.1 with $n = 2$ now gives (C.0.2).

For the inverse-beta model (1.2.5) with fixed constants $\beta > 0$ and $\mu > \theta > 0$, by Table 22, $\psi_2^{f^1}(a_1) > 0$, $\psi_1^{f^2}(a_2) > \Psi_1(-\theta + \mu + \beta)$, and $\psi_2^{f^2}(a_2) > \Psi_2(-\theta + \mu + \beta)$. Therefore

$$\begin{aligned}\psi_1^{f^1}(a_1)\psi_2^{f^2}(a_2)+&\psi_1^{f^2}(a_2)\psi_2^{f^1}(a_1)\\ &> \psi_1^{f^1}(a_1)\Psi_2(-\theta + \mu + \beta) + \Psi_1(-\theta + \mu + \beta)\psi_2^{f^1}(a_1)\\ &= \Psi_1(-\theta + \mu)\Psi_2(-\theta + \mu + \beta) - \Psi_1(-\theta + \mu + \beta)\Psi_2(-\theta + \mu).\end{aligned}$$

Letting $x = -\theta + \mu$, the last line is positive if and only if

$$\frac{\Psi_2(x + \beta)}{\Psi_1(x + \beta)} > \frac{\Psi_2(x)}{\Psi_1(x)}$$

which holds true by Lemma C.1 with $n = 1$. $\square$

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