HOMOLOGICAL INVARIANTS OF FI-MODULES AND FI_G-MODULES

By

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Abstract

In this work we approach the theory of finitely generated FI_G-modules through the language of abelian categories. In the first chapter, we define various homological invariants of finitely generated FI_G-modules, and show that they encode certain previously observed phenomena. More specifically, we show that the derived functors of the derivative encode the so-called Nagpal number of the module. We also provide a theory of depth for finitely generated FI_G-modules, which generalizes previous work of Sam and Snowden. In the second chapter, joint with Liping Li, we study a theory of local cohomology for FI_G-modules. Using this theory we refine the results of the first chapter, while expanding it in many ways. It is shown that these local cohomology modules encode a plethora of significant properties, including the regularity of the module. Finally, the third chapter deals with removing the assumption of finite generation from the first two. We prove that the weaker assumption of coherence is sufficient for much of the theory to continue to work. As an application, we prove a kind of local duality for coherent FI_G-modules.

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Preface

In this preface, we give a very brief summary of the work which follows. Each chapter in the body of the work has its own introduction and background section, making them all self-contained. The purpose of this preface is to give a brief sense of the topics in the body of the work. For a completely rigorous treatment, one should read through the chapters which follow.

The philosophy of asymptotic algebra can be stated as follows. If a family of objects displays some kind of asymptotically regular behavior, the entire family can be encoded into a single object which is finitely generated in some abelian category. For example, let $\mathcal{M}$ be an orientable manifold of dimension at least 2, which can be realized as the interior of a manifold with boundary. The n-strand (ordered) configuration space of $\mathcal{M}$ is defined to be

$$\mathrm{Conf}_n(\mathcal{M}) := \{(x_1, \ldots, x_n) \mid x_i \neq x_j\}$$

The n-strand unordered configuration space of $\mathcal{M}$ is defined to be the quotient space of $\mathrm{Conf}_n(\mathcal{M})$ by the natural symmetric group action. It was classically observed that for each i the homology groups $H_i(\mathrm{UConf}_n(\mathcal{M}))$ are eventually constant in n. This phenomenon was explained by McDuff [McD], who showed that the abelian group $\bigoplus_{n\geq 0} H_i(\mathrm{UConf}_n(\mathcal{M}))$ can be endowed with an action by the ring $\mathbb{Z}[x]$, which turns it into a finitely generated graded module over this ring.

Despite the success of McDuff's result, it was noted that the ordered configuration spaces did not share this kind of stability in their homology groups. Indeed, the homology groups of these spaces would have to be treated in a considerably more subtle way. Observe first that for each n the homology group $H_i(\mathrm{Conf}_n(\mathcal{M}))$ carries the structure of a $\mathbb{Z}[\mathfrak{S}_n]$-module. The key insight of Church [Chu], was that these homology groups will stabilize after accounting for this symmetric group action. Of course, while it doesn't make literal sense to say that some $\mathfrak{S}_n$ representation is "the same as" some $\mathfrak{S}_{n+1}$ representation, one would never object to someone saying the trivial representation of $\mathfrak{S}_n$ is the same as the trivial representation of $\mathfrak{S}_{n+1}$. The work of Church, and later Church, Ellenberg, and Farb, made this intuition rigorous through the use of FI-modules.

Let FI denote the category of finite sets and injections. An FI-module over $\mathbb{Z}$ is a functor from the category FI to the category of abelian groups. The work of Church shows that the group $\bigoplus_{n\geq 0} H_i(\mathrm{Conf}_n(\mathcal{M}))$ can be endowed with the structure of a finitely generated FI-module over $\mathbb{Z}$, thereby providing a kind of generalization to the original approach to McDuff. Since the work of Church, Ellenberg, and Farb, there has been an explosion of interest in FI-modules, as well as modules over related categories. In the following work, we will be concerned with understanding modules over FI_G for a given group G, a category which generalizes FI. Put briefly, one may think of FI_G as the category whose acting groups are no longer the symmetric groups, but rather the wreath product of the symmetric group with G.

The following work is a compilation of three previous papers [R, LR, R2]. The second of these [LR] was also coauthored by Liping Li. The purposes of these three works were in understanding FI_G-modules from the perspective of abelian categories. More specifically, I hoped to prove facts about FI_G-modules by applying techniques most commonly used in commutative algebra.

In the first chapter, we introduce the concept of depth for FI_G-modules. Depth is a very well studied invariant from commutative algebra, and the notion we introduce for FI_G-modules shares numerous similarities with this classical invariant. Using this concept, we are able to provide effective bounds on a celebrated theorem of Nagpal [N], and clarify previous work of Church and Ellenberg [CE]. Such effective bounds have been shown by Wiltshire-Gordon [W-G] to have many natural applications in topology and beyond.

In the second chapter, Liping Li and I expand upon the first work. It is explained that much of [R] actually derives from a kind of local cohomology theory for FI_G-modules. Just as with depth, local cohomology is a very well studied subject from commutative algebra. It is discovered that the local cohomology theory for FI_G-modules shares an almost alarming amount of similarity with the classical case. For instance, the aforementioned result of Nagpal, which is itself a kind of Hilbert polynomial analogue for FI_G-modules, is shown to fall out of the theory. The main conjecture of [LR] was recently proven to be the case by Nagpal, Sam and Snowden in [NSS].

In the third chapter, we seek to answer the question: "when is finite generation necessary?" Inspired by work of Church and Ellenberg [CE], the author introduces a definition of coherence for FI_G-modules, and proves that the category of coherent FI_G-modules is always abelian. This result allows one to study FI_G-modules in the absence of finite generation, and therefore allows one to consider cases wherein the group G is not finite. The chapter concludes by applying previously established technical results to prove a kind of local duality for FI_G-modules.

Contents

Chapter 1

Homological Invariants of FI-modules and FI_G-modules

1.1 Introduction

Let FI denote the category whose objects are the sets $[n] := \{1, \ldots, n\}$, and whose morphisms are injections. An FI**-module** over a commutative ring k is a functor V : FI $\to$ k-Mod. These objects were introduced by Church, Ellenberg, and Farb in [CEF], and were shown to have a plethora of applications to topology and number theory due to their strong connection to representation stability theory, introduced in [CF]. Following the work of Church, Ellenberg, and Farb, Wilson studied modules over a very similar category, FI_{BC} [W]. The category FI_{BC} is that whose objects are the sets $[n]$, while morphisms are pairs $(f, g) : [n] \to [m]$ of an injection of sets with a map of sets $g : [n] \to \mathbb{Z}/2\mathbb{Z}$. Composition in this category is defined to be,

$$(f, g) \circ (f', g') = (f \circ f', h), \qquad h(x) = g'(x) + g(f'(x)).$$

Wilson shows in [W] that FI_{BC}-modules also naturally arise in studying stability phenomena in topology. It is shown in [W] that many well known properties of FI-modules are also present in FI_{BC}-modules. These many similarities are explained by the fact that both are specific cases of a more general object.

Let G be a group. The category FI_G, introduced in [SS2], is that whose objects are the sets $[n]$, and whose morphisms are pairs $(f, g) : [n] \to [m]$ such that f is an injection, and $g : [n] \to G$ is a map of sets. Composition in this category is defined in the analogous way to FI_{BC}-modules. If $G = 1$ is the trivial group, then FI_G is equivalent to the category FI of finite sets and injections. If $G = \mathbb{Z}/2\mathbb{Z}$, then FI_G is equivalent to the category FI_{BC}, discussed above. For any commutative ring k, an FI_G-module over k is a covariant functor $V : \mathrm{FI}_G \to k\text{-Mod}$. We will often write $V_n := V([n])$.

In the present paper we study various homological invariants of FI_G-modules, and show how they relate to concrete questions about stability. In particular, we generalize the bounds on Castelnuovo-Mumford regularity in [CE, Theorem A], and provide explicit bounds on results from [CEFN, Theorem B] and [NS]. If V is an FI_G-module, then we define $H_0(V)$ on any finite set $[n]$ to be the quotient of V_n by the images of all maps $V(f, g)$ where $(f, g) \in \mathrm{Hom}_{\mathrm{FI}_G}([m], [n])$ and $m < n$. The functor $V \mapsto H_0(V)$ is right exact, and we define its right derived functors, H_i, to be the **homology functors**. The paper [CE] studied these functors in the case of FI-modules, and showed various applications to the homology of congruence subgroups.

We say that V is **generated in degree** $\leq m$ if $\deg(H_0(V)) \leq m$ (See Definition 1.7), where the **degree** of an FI_G-module $\deg(V)$ is the largest n such that $V_n \neq 0$. Similarly, we say that V has **first homological degree** $\leq r$ if $\deg(H_1(V)) \leq r$ (See Definition 1.7, and Remark 1.14 for more on this definition). If a module has finite generating and first homological degrees, then it is said to be **presented in finite degree**.

The main tool in much of the paper is the use of the derivative functor. Given an FI_G-module, we define its **first shift** $\mathrm{S}\,V$ to be the module defined on points by $\mathrm{S}\,V_n = V_{n+1}$. For any $(f,g):[n]\to[m]$, the map $\mathrm{S}\,V(f,g): V_{n+1}\to V_{m+1}$ will be the map $V(f_+,g_+)$, where f_+ agrees with f on $[n]$ and maps $n+1$ to $r+1$, while g_+ agrees with G on $[n]$ and maps $n+1$ to the identity (see Definition 1.17). We will write S_b to denote the b-th iterate of S. This functor was first introduced in [CEFN, Definition 2.8], and has since seen use in many papers (e.g. [N], [NS], [L], [GL]). If V is any FI_G-module, then the map induced by the natural inclusion $f^n:[n]\to[n+1]$ (i.e. that which sends i to i for all i), paired with the trivial map into G, induces a map of FI_G-modules $V\to \mathrm{S}\,V$. The **derivative** of V, denoted DV, is defined to be the cokernel of this map (see Definition 2.28). As with the shift functor, we set D^a to be the a-th iterate of D. Because this functor is right exact, we can consider its left derived functors, which we denote $H_i^{D^a}$.

The connection between the derivative functor and homology was established for FI-modules in [CE]. Church and Ellenberg show that many properties of the derivative are encoded in the combinatorics of FI, which they then compute and relate to the regularity. One of the main objectives of the latter part of this paper is to argue that, in fact, the homological properties of the derivative functor provide deeper insights than were previously noted. Two invariants related to the derivative functors that we introduce and study in this work are the *depth* and the *derived regularity* of a module. The primary goal of this paper is to explore how these two new invariants can be used to refine our understanding of the homology and structure of FI_G-modules.

If V is an FI_G-module, then we set its **depth** to be the smallest non-negative value a for which $H_1^{D^{a+1}}(V) \neq 0$. While this definition does not at first seem related to more classical notions of depth, we will find that it satisfies many desirable properties. This is explored deeply in Section 2.31. Using the properties of depth discussed in Section 2.31, we will be able to prove the following theorem.

Theorem A. *Let V be a FI_G-module which is generated in finite degree over a commutative ring k. Then the following are equivalent:*

1. *V admits a filtration $0 = V^{(0)} \subseteq V^{(1)} \subseteq \ldots \subseteq V^{(n)} = V$, such that the cofactors are relatively projective (see Definition 1.5);*
2. *There is a series of surjections $Q^{(n)} = V \twoheadrightarrow Q^{(n-1)} \twoheadrightarrow \ldots \twoheadrightarrow Q^{(0)} = 0$ whose successive kernels are relatively projective;*
3. *V is homology acyclic;*
4. *$H_1(V) = 0$;*
5. *V admits a finite resolution by homology acyclic objects which are generated in finite degree.*

If, in addition, V is presented in finite degree, then the condition $H_i(V) = 0$ for some $i > 0$ is also equivalent to the above.

Remark 1.1. *A very recent preprint of Li and Yu [LY] has overlap with this paper. Namely, they prove a weaker version of Theorem 3.9 as their Theorem 1.3. While earlier versions of this work were otherwise independent, we use arguments inspired by [LY,* *Section 3] in Section 2.31 to generalize the results of these previous versions.*

Relatively projective FI_G-modules will be defined and expanded upon in later sections. For now, one can imagine these objects as being projective in the traditional sense. In fact, we will later see that all projective modules are relatively projective, although relatively projective FI_G-modules need not be projective. However, they turn out to be acyclic with respect to many natural functors on the category of FI_G-modules. If k is a field of characteristic zero, then the above theorem implies that every non-projective object in the category of finitely generated FI_G-modules has infinite projective dimension. This fact was proven by Sam and Snowden in the case of FI-modules in [SS3, Section 0.1]. We say that an FI_G module V is **$\sharp$-filtered** whenever it satisfies any of the conditions in the above theorem.

Our second application of depth will be to establish a firm connection between homological properties of the category of finitely generated FI_G-modules, and the phenomenon of the stable range.

If G is a finite group, and V is a finitely generated FI_G-module over a field k, we define its **Hilbert function** to be $H_V(n) = \dim_k(V_n)$. In [CEF, Theorem 3.3.4], Church, Ellenberg, and Farb prove that if k is a field of characteristic 0, and V is a finitely generated FI-module, then there is a polynomial $P_V(x) \in \mathbb{Q}[x]$ such that $H_V(n) = P_V(n)$ for all $n \gg 0$. They go on to show that this equality holds for $n \geq r + d$, where r is the first homological degree of V, and d is the generating degree of V. This was was also proven by Sam and Snowden in [SS3], although their bound is stated in terms of a kind of local cohomology theory [SS3, Theorem 5.1.3 and Remark 7.4.6]. Later, Church, Ellenberg, Farb, and Nagpal [CEFN, Theorem B] prove that if k is any field, then $H_V(n)$ agrees with a polynomial for $n \gg 0$. In this case, the authors do not provide bounds on when this stabilization occurs. These same theorems were later proven by Wilson in the case where $G = \mathbb{Z}/2\mathbb{Z}$ [W, Theorems 4.16 and 4.20]. Later, Sam and Snowden proved that the Hilbert function is eventually polynomial for an arbitrary finite group, although they did not provide bounds on when the equality begins [SS, Theorem 10.1.2].

The question of how big n has to be before this stability begins is known as the stable range problem. We say that a finitely generated FI_G-module V over a field k has **stable range** $\geq m$ if there is a polynomial $P_V(x) \in \mathbb{Q}[x]$ such that for any $n \geq m$, $H_V(n) = P_V(n)$.

Theorem B. *Let G be a finite group, and let V be a finitely generated FI_G-module over a field k. Then the stable range of V is at least $r + \min\{r, d\}$ where r is the first homological degree of V, and d is the generating degree.*

The work in this paper therefore provides a new proof of the bounds given in [CEF, Theorem 3.3.4] and [W, Theorem 4.16], while providing a novel bound in the cases where k is a field of positive characteristic or where $G \neq 1, \mathbb{Z}/2\mathbb{Z}$.

One of the major insights of Nagpal in [N, Theorem A] is that the aforementioned phenomenon of the stable range is actually a simple consequence of a much deeper structural theorem. This theorem was later generalized by Nagpal and Snowden in [NS].

Theorem ([NS]). *Assume that G is a polycyclic-by-finite group, and let V be a finitely generated FI_G-module over a Noetherian ring k. Then for $b \gg 0$, $\mathrm{S}_b V$ is $\sharp$-filtered.*

Note that $\sharp$-filtered modules have a polynomial Hilbert function for all n whenever G is finite. The above theorem is therefore a generalization of the stable range phenomenon, as previously stated.

We call the smallest $b \geq 0$ such that $\mathrm{S}_b V$ is $\sharp$-filtered the **Nagpal number** of V, $N(V)$. Neither [N] nor [NS] provide bounds on $N(V)$ In this paper, we show that the Nagpal number of a module V is actually encoded by its derived regularity. The **derived regularity** of V, denoted $\partial\mathrm{reg}(V)$, is the maximum across all integers a of the degree of $H_1^{D^a}(V)$.

Theorem C. *Let V be an FI_G-module which is presented in finite degree over a commutative ring k. Then $\mathrm{S}_b V$ is $\sharp$-filtered for $b \gg 0$. Moreover, in this case $\mathrm{S}_b V$ is $\sharp$-filtered if and only if $b > \partial\mathrm{reg}(V)$. In particular, if V is not $\sharp$-filtered then $N(V) = \partial\mathrm{reg}(V) + 1$.*

The major content of Theorem C is that the language of depth and derived regularity allows us to reinterpret the Nagpal number in terms of the finiteness of a particular homological invariant. Moreover, the generality of this language allows us to prove the result of Nagpal-Snowden in a more general context.

It was observed by Church and Ellenberg in [CE] that the derivative functors of a module could be used to bound its Castelnuovo-Mumford regularity. The **Castelnuovo-Mumford regularity** of an FI_G-module V is the smallest integer N such that

$$\deg(H_i(V)) - i \leq N$$

for all $i \geq 1$ (see Definition 1.13). It was proven by Sam and Snowden in [SS3, Corollary 6.3.5] that finitely generated FI-modules in characteristic zero have finite regularity. Following this, Church and Ellenberg proved that FI-modules which are presented in finite degree have finite regularity over any ring, and they provided a bound on this regularity [CE, Theorem A]. More recently, Li and Yu have provided different bounds on the regularity of FI-modules [LY, Theorem 1.8]. One of the early goals of this paper is to prove that similar bounds exist for FI_G-modules. Indeed, we will find that the regularity of a module V, which is presented in finite degree, can be bounded in terms of the generating degree and the relation degree of V.

Theorem D. *Let V be an FI_G-module over a commutative ring k with generating degree $\leq d$ and first homological degree $\leq r$. Then $\mathrm{reg}(V) \leq r + \min\{r, d\} - 1$.*

Note that the above bound exactly agrees with the bound given by Church and Ellenberg for FI-modules in [CE, Theorem A]. Indeed, much of the work done in proving the above theorem will involve generalizing the techniques used in that paper.

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1.2 FI_G-Modules

1.2.1 Basic Definitions

For the remainder of this paper we fix a commutative ring k, and a group G. We will use $[n]$ to denote the set $[n] = \{1, \ldots, n\}$. By convention, $[0] = \emptyset$.

Definition 1.2. *We define the category* FI_G *to be that whose objects are finite sets, and whose morphisms are pairs* $(f, g) : S \to T$*, of an injection of sets* $f : S \hookrightarrow T$*, and a map of sets* $g : S \to G$*. Given two composable morphisms in* FI_G*,* $(f, g), (f', g')$*, we define* $(f, g) \circ (f', g') = (f \circ f', h)$*, where* $h(x) = g'(x) \cdot g(f'(x))$*.*

Note that for any n*,* $\mathrm{Aut}_{\mathrm{FI}_G}([n]) = k[\mathfrak{S}_n \wr G]$*. For the remainder of this paper we shall write* G_n *to denote the group* $\mathfrak{S}_n \wr G$*.*

One immediately observes that the full subcategory of FI_G whose objects are the sets of the form $[n]$ is equivalent to FI_G. For convenience of exposition, we will from this point on refer to this category as FI_G. In the case where G is the trivial group, one sees that the category FI_G is naturally equivalent to the category FI of finite sets and injections. If instead we specialize to $G = \mathbb{Z}/2\mathbb{Z}$, then FI_G is naturally equivalent to the category FI_{BC} discussed in [W].

Definition 1.3. *An* FI_G***-module*** *over k is a covariant functor $V : \mathrm{FI}_G \to Mod_k$ from FI_G to the category of k-modules. We will often use the shorthand $V_n := V([n])$, and write $(f, g)_* : V_n \to V_m$ to denote the map induced by an arrow $(f, g) \in \mathrm{Hom}_{\mathrm{FI}_G}([n], [m])$. The collection of morphisms $(f, g)_*$ are known as the* ***induced maps*** *of V, while the maps $(f, g)_*$, with $n < m$, are called the* ***transition maps*** *of V. The collection of* FI_G*-modules over k, along with natural transformations, form a category, which we denote* FI_G-Mod.

Many constructions from the category -Mod_k will continue to work in FI_G-Mod, so long as one applies the construction "point-wise." For example, there is a natural notion of direct sum of two FI_G-modules V, W, where we set $(V \oplus W)_n = V_n \oplus W_n$. The induced maps of the sum are defined in the obvious way. One may similarly define point-wise notions of kernel and cokernel, which make FI_G-Mod an abelian category.

One should observe that for any fixed n, and any FI_G-module V, the module V_n carries the action of an $k[\mathrm{Aut}_{\mathrm{FI}_G}([n])] = k[G_n]$ module. One may therefore think of an

FI_G-module as a single object which encodes a collection of compatible G_n representations, where the compatibility is given by the transition maps. This was the original motivation for Church, Ellenberg, and Farb [CEF] studying FI-modules and their relationship with Church and Farb's representation stability found in [CF].

Definition 1.4. *We use FB_G to denote the subcategory of FI_G whose objects are the sets $[n]$, and whose morphisms are pairs (f, g) such that f is a bijection. An* FB_G***-module over*** k *is a functor $V : \mathrm{FB}_G \to Mod_k$. We denote the category of FB_G-modules over k by FB_G-Mod.*

One can think of FB_G-modules as sequences of $k[G_n]$-modules, with n increasing. We see that FB_G-Mod can be thought of as a subcategory of FI_G-Mod, the subcategory of modules with trivial transition maps. Because of this, we will often use terms and definitions from the theory of FI_G-modules when describing FB_G-modules.

Definition 1.5. *For any non-negative integer n, we define the* ***free*** FI_G***-module of degree*** n *$M(n)$ by the following assignments: $M(n)_m := k[\mathrm{Hom}_{\mathrm{FI}_G}([n], [m])]$ is the free k-module spanned by vectors $\{e_{(f,g)}\}$ indexed by the members of $\mathrm{Hom}_{\mathrm{FI}_G}([n], [m])$, while induced maps act on the natural basis by composition. We will also refer to direct sums of free modules as being free.*

If W is a $k[G_n]$-module, then we define the ***relatively projective*** *FI_G-module over W by the following assignments: $M(W)_m = W \otimes_{k[G_n]} k[\mathrm{Hom}_{\mathrm{FI}_G}([n], [m])]$, while induced* *maps act by composition in the second coordinate. More generally, if W is an* FB_G*-module, then the rule* $M(W) = \bigoplus_{n\geq 0} M(W_n)$ *makes M into a functor from* FB_G*-Mod to* FI_G*-Mod. Modules in the image of this functor will also be referred to as relatively projective. We observe that* $M(n) = M(k[G_n])$.

Remark 1.6. *Note that the terminology for the above definitions is not consistent in the literature. Relatively projective modules are the same as those denoted* $\mathrm{FI}\sharp$*-modules in [CEF], and those denoted free in [CE]. Free modules are the same as those denoted principally projective in [SS].*

Proposition 1.3. *If W is a $k[G_n]$-module, and V is any* FI_G*-module, then*

$$\mathrm{Hom}_{\mathrm{FI}_G\text{-}Mod}(M(W), V) = \mathrm{Hom}_{G_n}(W, V_n). \tag{1.1}$$

Proof

Given any map ϕ_n from the right hand side, we can extend it to a map ϕ of FI_G-modules by just insisting it commute with transition maps. For example, for any $m > n$, the module $M(W)_m = W \otimes_{G_n} k[\mathrm{Hom}_{\mathrm{FI}_G}([n], [m])]$ is generated by pure tensors $w \otimes (f, g)$, where $w \in W$ and $(f, g) \in \mathrm{Hom}_{\mathrm{FI}_G}([n], [m])$. We therefore define

$$\phi_m : M(W)_m \to V_m, \ \phi_m(w \otimes (f, g)) := (f, g)_*(\phi_n(w \otimes id)).$$

One can quickly see that this defines a well defined morphism of FI_G-modules.

□

The adjunction (1.1) immediately implies that $M(W)$ is projective whenever W is a projective $k[G_n]$-module. In fact, we will see in section 2.4 that all projective modules are relatively projective. Observe that this implies that the free FI_G-modules are actually projective, and therefore FI_G-Mod has sufficiently many projective objects.

Note that in the special case where $W = k[G_n]$, the adjunction (1.1) becomes

$$\mathrm{Hom}_{\mathrm{FI}_G\text{-Mod}}(M(n), V) = V_n.$$

In other words, a map from the free object of degree n is equivalent to a choice of an element of V_n. More precisely, the map sending a homomorphism $\phi : M(n) \to V$ to $\phi(id_{[n]})$ is an isomorphism.

We will prove other important properties of the M functor in Section 1.5.1.

Definition 1.7. *Given a non-negative integer m, we say that an FI_G-module V is* ***generated in degree*** $\leq m$ *if there exists a surjection*

$$\bigoplus_{i \in I} M(n_i) \twoheadrightarrow V,$$

where I is some index set and $n_i \leq m$ for all $i \in I$. If the index set I can be taken to be finite, then we say that V is ***finitely generated****. We denote the category of finitely generated FI_G-modules by FI_G-mod. By convention, the trivial FI_G-module is said to be generated in degree ≤ -1.*

We say that V has ***relation degree*** $\leq r$ *if there is an exact sequence*

$$0 \to K \to M \to V \to 0,$$

with M relatively projective, such that K is generated in degree $\leq r$. An exact sequence of the above form is known as a ***presentation*** *for the module V, and we say that V is*

presented in finite degree *if* V *has finite relation and generating degrees.*

The M***-functor*** $M : \mathrm{FB}_G\text{-Mod} \to \mathrm{FI}_G\text{-Mod}$ *is that given by*

$$M(W) = \bigoplus_i M(W_i).$$

Let V be an FI_G-module, and let S be any subset of $\sqcup_n V_n$. Then we define the span of S to be the FI_G-module defined on objects by

$$\mathrm{span}_k(S)_m = \{w \in V_m \mid w = \sum_i \lambda_i (f_i, g_i)_*(x_i) \text{ with } x_i \in S,\ \lambda_i \in k, \text{ and } (f_i, g_i) \in \mathrm{Hom}_{\mathrm{FI}_G}([n_i], [m])\},$$

with induced maps restricted from V. Then V is generated in degree $\leq n$ if and only if $\mathrm{span}_k(\sqcup_{i \leq n} V_i) = V$. From the remark about maps from free objects, one can also see that V is finitely generated if and only if it is the span of a finite set of elements.

Definition 1.8. *Given an* FI_G*-module* V *we define* $\deg(V)$*, the* ***degree*** *of* V*, to be the supremum* $\sup\{n \mid V_n \neq 0\} \in \mathbb{N} \cup \{-\infty, \infty\}$*, where we use the convention that the supremum of the empty set is* $-\infty$*. We say that* V *has* ***finite degree*** *if and only if* $\deg(V) < \infty$*.*

It is an immediate consequence of the relevant definitions that all FB_G-modules with finite generating degree have finite degree.

One very non-obvious fact about the category FI_G-mod is that it can be abelian. While finite generation is clearly preserved by quotients, it is not obvious that submodules of finitely generated objects are also finitely generated. We have the following theorem, usually called the **Noetherian property**.

Theorem 1.9 (SS2, Corollary 1.2.2)**.** *If k is a Noetherian ring and G is a polycyclic-by-finite group, then the category FI_G-mod is abelian. That is, submodules of finitely generated FI_G-modules are also finitely generated.*

Historically, the Noetherian property was proven for FI-modules over a field of characteristic 0 in [S, Theorem 2.3] and independently in [CEF, Theorem 1.3]. It was proven for FI-modules over a general Noetherian ring in [CEFN, Theorem A]. The case $G = \mathbb{Z}/2\mathbb{Z}$ was proven in [W, Theorem 4.21]. The paper [SS2] proves the theorem for all polycyclic-by-finite groups G.

Remark 1.10. *Note that the above theorem requires that the group G be polycyclic-by-finite. In this paper we will not need this assumption on G. In particular, our results will be independent of the Noetherian property.*

1.3.1 The Homology Functors and Nakayama's Lemma

Definition 1.11. *Let V be an FI_G-module, and let n be a non-negative integer. We use $V_{<n} \subseteq V_n$ to denote the submodule of V_n spanned by the images of all transition maps.*

Put another way, $V_{<n}$ is the submodule of V_n generated by the elements

$$\cup_{i<n} \cup_{(f,g)\in \text{Hom}_{\text{FI}_G}([i],[n])} (f,g)_*(V_i).$$

The ***zeroth homology*** *of V is the FB_G-module defined by $H_0(V)_n = V/V_{<n}$.*

This notion was first introduced for FI-modules in [CEF], and later expanded upon in [CEFN] and [CE]. This functor was also considered in [GL] and [GL2], albeit in a slightly different language.

Proposition 1.4. *The zeroth homology functor H_0 enjoys the following properties:*

1. *for any $k[G_n]$-module W, $H_0(M(W))_n = W$, while $H_0(M(W))_m = 0$ for all $m \neq n$;*
2. *if $\{v_i\}_{i\in I} \subseteq \sqcup_n H_0(V)_n$ is a generating set for $H_0(V)$, and w_i is a lift of v_i for each $i \in I$, then $\{w_i\}_{i\in I}$ is a generating set for V. Equivalently, $H_0(V) = 0$ if and only if $V = 0$ (Nakayama's Lemma);*
3. *$H_0(V)$ is generated in degree $\leq n$ (resp. finitely generated) if and only if V is generated in degree $\leq n$ (resp. finitely generated);*
4. *H_0 is left adjoint to the inclusion functor FB_G-Mod $\to$ FI_G-Mod;*
5. *$H_0(V)$ is right exact, and maps projective modules to projective modules.*

Proof

The first non-zero entry in $M(W)$ is $M(W)_n = W$. On the other hand, one sees immediately from definition that $M(W)$ is generated in this degree. In other words, all other elements in $M(W)$ are linear combinations of transition maps applied to elements of $M(W)_n$. This implies the first statement.

Let v_i and w_i be as in the second statement. Let j be the least index such that $V_j \neq 0$. Then $H_0(V)_j = V_j$, and therefore V_j is generated by the w_i by assumption. To finish the proof we proceed by induction. If $n > j$, and $v \in V_n$, then the image of v in $H_0(V)_n$ can be expressed as a linear combination of the v_i. In particular, v is a linear combination of the w_i, as well as images of elements from lesser degrees. Applying the inductive hypothesis completes the proof.

The third statement is an immediate consequence of Nakayama's Lemma.

Let V be an FI_G-module, and let W be a FB_G-module. If $\phi : H_0(V) \to W$ is any map of FB_G-modules, then for each n we define a map of $k[G_n]$-modules $\widetilde{\phi}_n : V_n \to W_n$ via $\widetilde{\phi}_n(v) = \phi_n(\pi(v))$, where $\pi : V_n \to H_0(V)_n$ is the quotient map. We claim that $\widetilde{\phi}$ is actually a morphism of FI_G-modules. If $(f,g)_*$ is any transition map, and $v \in V_m$, then

$$\widetilde{\phi}_n((f,g)_*(v)) = \phi_n(\pi((f,g)_*(v))) = 0 = (f,g)_*(\widetilde{\phi}_n(v)).$$

On the other hand, if $\sigma \in k[G_n]$, then

$$\widetilde{\phi}_n(\sigma(v)) = \phi_n(\pi(\sigma(v))) = \phi_n(\sigma(\pi(v))) = \sigma(\phi_n(\pi(v))) = \sigma(\widetilde{\phi}(v)).$$

Conversely, let $\phi : V \to W$ be a morphism of FI_G-modules. Because ϕ respects transition maps, and because W has trivial transition maps, it follows that ϕ_n vanishes on the images of the transition maps into V_n. In particular, the map $\widetilde{\phi} : H_0(V) \to W$ given by $\widetilde{\phi}_n(v) = \phi_n(v)$ is well defined. The two constructions give above are clearly inverses of one another, proving the adjunction.

The last statement is a consequence of standard homological algebra. Left adjoints are always right exact, and any left adjoint to an exact functor must preserve projectives.

□

As a quick application of the above proposition, we prove that all projective modules are relatively projective.

Proposition 1.5. *Let V be an FI_G-module. Then V is projective if and only if $V = \bigoplus_i M(W_i)$, where each W_i is some projective $k[G_i]$-module. In particular, all projective FI_G-modules are relatively projective.*

Proof

We have already seen that modules of the form $\bigoplus_i M(W_i)$, with W_i projective, are projective. Conversely, let V be a projective FI_G-module. Then part 5 of Proposition 1.4 implies that $H_0(V)$ is a projective FB_G-module. It follows that the quotient map $q_n : V_n \to H_0(V)_n$ admits a section $\iota_n : H_0(V)_n \to V_n$. Proposition 2.3 implies there is a map $\bigoplus_i M(H_0(V)_i) \to V$ induced by the collection of ι_n. Nakayama's lemma implies that the map is a surjection. We claim that this map is actually injective as well. Let K be its kernel, and apply H_0 to the exact sequence

$$0 \to K \to \bigoplus M(H_0(V)_i) \to V \to 0.$$

Because V is projective, and because H_0 is right exact by the previous proposition, it follows that there is an exact sequence

$$0 \to H_0(K) \to H_0(V) \to H_0(V) \to 0$$

where we have used part 2 to simplify the second term. It is easy to see from construction that the right most non-trivial morphism in this sequence is an isomorphism, and therefore $H_0(K) = 0$. Nakayama's lemma now implies that $K = 0$, as desired.

□

Definition 1.12. *We write H_i for the i-th derived functor of H_0. We call the collection of these functors the **homology functors**.*

The following nomenclature is used in [L].

Definition 1.13. *If V is an FI_G-module, then for each i we define its **i-th homological degree** to be $hd_i(V) = \deg(H_i(V))$. For some non-negative constant N, we say that V has **regularity** $\leq N$ if $hd_i(V) - i \leq N$ for each $i \geq 1$. We write $\mathrm{reg}(V)$ for the smallest value N for which V has regularity $\leq N$.*

Remark 1.14. *Nakayama's lemma tells us that the zeroth homological degree is an optimal bound on the the generating degree of V. One would hope that the first homological degree would be an optimal bound on the relation degree of V. Indeed, if*

$$0 \to K \to M \to V \to 0$$

is a presentation of V with M generated in degree $\leq d$ and K generated in degree $\leq r$, then an application of the H_0 functor implies that

$$hd_1(V) \leq hd_0(K) = r \leq \max\{d, hd_1(V)\}. \tag{1.2}$$

Despite this, it is not clear whether V admits a presentation whose kernel is generated in degree $\leq hd_1(V)$.

The bounds (1.2) imply that $r = hd_1(V)$ whenever $hd_1(V) \geq hd_0(V)$. This is the typical case. In fact, if instead we assume that $hd_1(V) < hd_0(V)$, then Li and Yu show [LY, Corollary 3.4] there exists an exact sequence

$$0 \to V' \to V \to Q \to 0$$

where Q is relatively projective and $hd_0(V') = hd_0(V) - 1$. One immediately notes from this, and Theorem 3.9, that $H_i(V') = H_i(V)$ for $i \geq 1$. In the paper [CE, Theorem A], Church and Ellenberg show that

$$\text{reg}(V) \leq r + \min\{r, d\} - 1$$

where r is the relation degree of V and d is the generating degree. The observations made in this remark will allow us to convert this bound to a bound in terms of the first homological degree $hd_1(V)$. If it is the case that $hd_1(V) \geq d$, then $r = hd_1(V) \geq d$, and the above bound becomes

$$\text{reg}(V) \leq r + d - 1 = hd_1(V) + \min hd_1(V), d - 1.$$

Otherwise, we may apply the lemma of Li and Yu, as well as induction, to conclude there is some submodule $V'' \subseteq V$ which is generated in degree $\leq hd_1(V)$ and

$$\text{reg}(V) = \text{reg}(V'') \leq 2hd_1(V'') - 1 = hd_1(V) + \min\{hd_1(V), d\} - 1.$$

We may therefore conclude that the bounds of Church and Ellenberg remain true when the relation degree is replaced with the first homological degree. In this paper, we will prove bounds using methods of Church and Ellenberg. To conclude the bounds promised in the introduction, one simply applies the methods in this remark.

The main result of [CE] is a bound on the regularity of an FI-module in terms of its generating and relation degrees. Later, [L, Theorem 1.5] used different methods to prove conditional bounds on the regularity of finitely generated FI_G-modules whenever G is finite. Li also gives non-conditional bounds in the case of FI, and where k is a field of characteristic 0 [L, Theorem 1.17]

1.5.1 The Category $\mathrm{FI}_G\sharp$ and the M functor

Definition 1.15. *We define the category* $\mathrm{FI}_G\sharp$ *as follows. The objects of the category* $\mathrm{FI}_G\sharp$ *are once again the sets* $[n]$*, while the morphisms are triples* $(A, f, g) : [n] \to [m]$ *such that* $A \subseteq [n]$*,* $f : A \to [m]$ *is an injection, and* $g : A \to G$ *is a map of sets. Composition in this category is defined in the following way. If* (A, f, g) *and* (B, f', g') *are two morphisms which can be composed then* $(B, f', g') \circ (A, f, g) = (A \cap f^{-1}(B), f' \circ f, h)$ *where* $h(x) = g(x)g'(f(x))$*, as before. An* $\mathrm{FI}_G\sharp$*-module over* k *is a covariant functor* $\mathrm{FI}_G\sharp \to Mod_k$*.*

This category has been studied in the case where G is the trivial group [CEF], as well as the case where $G = \mathbb{Z}/2\mathbb{Z}$ [W]. One sees that there is a natural inclusion $\mathrm{FI}_G \to \mathrm{FI}_G\sharp$, which induces a forgetful functor $\mathrm{FI}_G\sharp$-Mod $\to \mathrm{FI}_G$-Mod. For this reason we may consider a kind of homology functor $H_0 : \mathrm{FI}_G\sharp\text{-Mod} \to \mathrm{FB}_G\text{-Mod}$ which is defined as the composition of the forgetful map, and the usual zeroth homology functor.

The first example of an $\mathrm{FI}_G\sharp$-module is the free module $M(m)$. Indeed, We endow $M(m)$ with the structure of an $\mathrm{FI}_G\sharp$-module as follows. let $e_{(f,g)} \in M(m)_n$ be one of the canonical basis vectors, and let $(A, f', g') : [n] \to [r]$ be a morphism in $\mathrm{FI}_G\sharp$. Then we set

$$(A, f', g')_* e_{(f,g)} = \begin{cases} 0 & \text{if } f([m]) \not\subseteq A \\ e_{(f'\circ f,h)} & \text{otherwise,} \end{cases}$$

where $h : [m] \to G$ is the function $h(x) = g(x)g'(f(x))$. This same argument shows that $M(W)$ is an $\mathrm{FI}_G\sharp$-module for any FB_G-module W.

The above discussion shows that we may consider M as being valued in $\mathrm{FI}_G\sharp$-Mod.

Proposition 1.6. *The functor $M : \mathrm{FB}_G$-Mod $\to \mathrm{FI}_G$-Mod enjoys the following properties*

1. *The composition $H_0 \circ M$ is isomorphic to the identity;*
2. *M is exact;*
3. *for all $i \geq 1$, $H_i \circ M = 0$.*

Proof

The first statement follows immediately from the first part of Proposition 1.4 and the definition of M.

For the second statement, it suffices to show that the functor preserves exactness of sequences of the form

$$0 \to W' \to W \to W'' \to 0$$

where W', W and W'' are $k[G_m]$-modules for some m. For any n, we have that $M(W)_n = W \otimes_{k[G_m]} k[\mathrm{Hom}_{\mathrm{FI}_G}([m],[n])]$. Because kernels and cokernels are computed point-wise, it suffices to show that $k[\mathrm{Hom}_{\mathrm{FI}_G}([m],[n])]$ is a flat $k[G_m]$-module. Fix a representative from each orbit of the G_m action on $\mathrm{Hom}_{\mathrm{FI}_G}([m],[n])$. If we set I to be the collection of these maps, then let $\mathfrak{B} = \{e_{(f,g)}\}_{(f,g)\in I}$ be the associated set of canonical basis vectors of $k[\mathrm{Hom}_{\mathrm{FI}_G}([m],[n])]$. We claim that $k[\mathrm{Hom}_{\mathrm{FI}_G}([m],[n])]$ is a free $k[G_m]$-module with basis $\mathfrak{B}$. Because the orbits partition the whole of $\mathrm{Hom}_{\mathrm{FI}_G}([m],[n])$, it follows that this set is spanning. On the other hand, assume that one has an equation $\sum_{(f,g)\in\mathfrak{B}} e_{(f,g)}x_{(f,g)} = 0$, for some $x_{(f,g)} \in k[G_m]$. We may write $x_{(f,g)} = \sum_{\sigma\in G_m} a_{(f,g),\sigma}\sigma$, and therefore

$$\sum_{(f,g),\sigma} a_{(f,g),\sigma} e_{(f,g)\circ\sigma} = 0.$$

We observe that for distinct $(f,g),(f',g') \in \mathfrak{B}$ and any $\sigma,\tau \in G_m$, the elements $(f,g)\circ\sigma$ and $(f',g')\circ\tau$ must be distinct, as they are in different orbits by construction. If we fix (f,g) and vary σ, then $(f,g)\circ\sigma = (f,g)\circ\tau$ implies that $\sigma = \tau$ because (f,g) is monic. In particular, the above sum can be written

$$\sum_{(f,g)\circ\sigma=(f',g')\in\mathrm{Hom}_{\mathrm{FI}_G}([m],[n])} a_{(f,g),\sigma} e_{(f',g')} = 0$$

with each (f',g') appearing at most once. This implies that $a_{(f,g),\sigma} = 0$ for all f, g, and σ, as desired.

The final statement follows from the first two. Because M maps projective objects to projective objects by Proposition 1.5, The derived functor of the composition $H_0 \circ M$ can be computed using the Grothendieck spectral sequence. This spectral sequence will only have one row because M is exact. It therefore degenerates and we find that the derived functors of $H_0 \circ M$ are isomorphic to $H_i \circ M$. On the other hand, statement 2 tells us that $H_0 \circ M$ is the identity functor, which is clearly exact. This completes the proof.

□

We note that the composition $M \circ H_0$ is not isomorphic the identity functor if we consider M as being valued in FI_G-Mod. If we instead consider M as being valued in $\mathrm{FI}_G\,\sharp$-Mod, then this composition is isomorphic to the identity as the following theorem shows.

Theorem 1.16 ([CEF],[W])**.** *The functor* $M : \mathrm{FB}_G$ *-Mod* $\rightarrow \mathrm{FI}_G\,\sharp$*-Mod is an equivalence of categories with inverse* $H_0 : \mathrm{FI}_G\,\sharp$*-Mod* $\rightarrow \mathrm{FB}_G$ *-Mod.*

The two citations given prove the theorem in the cases where G is the trivial group, and where $G = \mathbb{Z}/2\mathbb{Z}$, respectively. The proofs go through essentially word for word to prove Theorem 1.16 in the general case.

Theorem 1.16 can be considered the justification for the terminology $\sharp$-filtered from Theorem 3.9.

1.6.1 The Shift Functor and Torsion

The final piece we need from the basic theory of FI_G-modules is the shift functor. This functor was heavily featured in both [GL] and [L], and will be of great use to us in what follows.

Definition 1.17. *Let Σ denote the endofunctor of FI_G, which sends $[n]$ to $[n+1]$, and takes a map $(f,g):[n]\to[m]$ to the map $(f_+,g_+):[n+1]\to[m+1]$ defined by*

$$f_+(x)=\begin{cases} f(x) & \text{if } x\neq n+1\\ m+1 & \text{otherwise}\end{cases}\qquad g_+(x)=\begin{cases} g(x) & \text{if } x\neq n+1\\ 1 & \text{otherwise.}\end{cases}$$

*We define the **shift functor** S with respect to Σ to be the endofunctor of FI_G-Mod*

$$\mathrm{S}\,V := V\circ\Sigma.$$

For any integer $b\geq 1$ we set S_b to be the b-th iterate of S.

Shift functors were originally introduced in [CEFN] in the case of FI-modules, and have since seen use in various papers in the field (e.g. [N], [GL], [NS], [L]). The following proposition collects many of the important properties of the shift functor.

Proposition 1.7. *The shift functor S enjoys the following properties:*

1. S *is exact;*

2. if V *is generated in degree* $\leq n$*, then so is* $\mathrm{S}\,V$*;*

3. If W *is any* $k[G_n]$*-module, then* $\mathrm{S}(M(W)) = M(\mathrm{Res}^{k[G_n]}_{k[G_{n-1}]} W) \oplus M(W)$.

Proof

Kernels and cokernels are computed point-wise by definition. It follows immediately from this that S is exact.

If $0 \to K \to F \to V \to 0$ is a presentation for V, with F free, then exactness of the shift functor implies that $0 \to \mathrm{S}\,K \to \mathrm{S}\,F \to \mathrm{S}\,V \to 0$ is exact as well. It therefore suffices to show that $\mathrm{S}(M(m))$ is generated in degree $\leq m$. Let $n > m+1$, and let $e_{(f,g)}$ be a canonical basis vector in $M(m)_n$. Let $h : [m] \to [m+1]$ be the injection which sends $f^{-1}(n)$ to $m+1$, if it exists, and is the identity elsewhere, and let $\widetilde{h} : [m] \to [n-1]$ be the injection which agrees with f away from $f^{-1}(n)$, and sends $f^{-1}(n)$ to something outside the image of f. Finally, we let $\mathbf{1} : [m] \to G$ be the trivial map into G. Then we have

$$\Sigma(\widetilde{h}, \mathbf{1}) \circ (h, g) = (f, g)$$

This shows that $\mathrm{S}\,M(m)$ is generated in degree m, as desired.

For the first part of final statement, Theorem 1.16 implies that to show that $\mathrm{S}\,M(W)$ is relatively projective, it will suffice to show that it is an $\mathrm{FI}_G\sharp$-module. Let $(A, f, g) : [m] \to [n]$ be a morphism in $\mathrm{FI}_G\sharp$. Then we may define an endofunctor $\Sigma_\sharp$ of $\mathrm{FI}_G\sharp$, which maps $[m]$ to $[m+1]$ and $\Sigma_\sharp(A, f, g) = (A \cup \{m+1\}, f_+, g_+)$ where f_+ agrees with f on A, and sends $m+1$ to $n+1$, and g_+ agrees with g on A and sends $m+1$ to the identity. Observe that $\Sigma_\sharp$ restricts to Σ on $\mathrm{FI}_G \subseteq \mathrm{FI}_G\sharp$. In particular, the functor S can be extended naturally to a functor on $\mathrm{FI}_G\sharp$. This shows that $\mathrm{S}\,M(W)$ is relatively projective.

Once again applying Theorem 1.16, it remains to compute $H_0(\mathrm{S}\,M(W))$. The second part of this proposition implies that it suffices to compute $H_0(\mathrm{S}\,M(W))$ in degrees $m-1$ and m. It is clear from definition that $H_0(\mathrm{S}\,M(W))_{m-1} = \mathrm{Res}^{k[G_m]}_{k[G_{m-1}]} W$. A direct computation also shows that the transition maps originating from $\mathrm{S}\,M(W)_{m-1}$ will hit all pure tensors in $\mathrm{S}\,M(W)_m = M(W)_{m+1}$ except for those of the form $w \otimes (f, g)$ where $f^{-1}(m+1) = \emptyset$. The group G_m now acts on these pure tensors in precisely the way it acts on W. In particular, $H_0(W)_m = W$, which concludes the proof.

□

Remark 1.18. *If G is an infinite group, then shifts do not need to preserve finite generation. Indeed,*

$$\mathrm{S}\,M(m) = M(m-1)^{m\cdot|G|} \oplus M(m)$$

by the above proposition.

Note that the last two properties were proven for FI-modules in [CEFN, Lemma 2.12] and [N, Lemma 2.2]. The next property of the shift functor which is important to us is its connection with torsion.

Definition 1.19. *Let V be an FI_G-module. Fix $b \geq 0$, and let $(f_b^n, \mathbf{1}) : [n] \to [n+b]$ denote the morphism in FI_G whose injection is the standard inclusion (j maps to j for* *all j), and whose G-map is the trivial map. Then the collection of the induced maps $(f_b^n, \mathbf{1})_* : V_n \to V_{n+b}$ define a morphism of FI_G-modules $\iota_b : V \to \mathrm{S}_b V$. We say that V is **torsion free** if ι_b is injective for all b. Any element of $\sqcup_n V_n$ which appears in the kernel of some ι_b is called a **torsion** element of V. If every element of V is torsion, then we say the module V is itself torsion.*

The fact that every FI_G-module maps into its shift will be used throughout this paper. One should observe that an FI_G-module V is torsion free if and only if $\iota := \iota_1$ is injective. Indeed, if $v \in V_n$ is in the kernel of some $(f_b^n, \mathbf{1})_*$, then we write

$$0 = (f_b^n, \mathbf{1})_*(v) = (f_1^{n+b-1}, \mathbf{1})_*(f_{b-1}^n, \mathbf{1})_*(v)$$

If $(f_{b-1}^n, \mathbf{1})_*(v) = 0$, then we repeat the above until we find a non-trivial element in the kernel of $(f_1^a, \mathbf{1})_*$ for some $a \geq n$. Also note that if $v \in V_n$ is in the kernel of some transition map, then it must in fact be in the kernel of some ι as well. Indeed, this follows from the fact that the action of G_n on $\mathrm{Hom}_{\mathrm{FI}_G}([m],[n])$ is transitive.

Lemma 1.20. *Let V be an FI_G-module, which is generated in degree $\leq m$ and related in degree $\leq r$. Then for any b, $\mathrm{coker}(V \to \mathrm{S}_b V)$ is generated in degree $< m$ and related in degree $< r$.*

Proof

Looking through the proof of the second part of Proposition 1.7, one finds that the inclusion

$$M(W) \hookrightarrow M(\mathrm{Res}_{k[G_{n-1}]}^{k[G_n]} W) \oplus M(W) = \mathrm{S}\, M(W)$$

is exactly ι. Let F be a free module generated in degree $\leq m$ which surjects onto V. Then exactness of the shift functor implies we have the following commutative diagram with exact rows,

$$\begin{array}{ccccccc}
\operatorname{coker}(K \to \mathrm{S}_b K) & \longrightarrow & \operatorname{coker}(F \to \mathrm{S}_b F) & \longrightarrow & \operatorname{coker}(V \to \mathrm{S}_b V) & \longrightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \\
\mathrm{S}_b K & \longrightarrow & \mathrm{S}_b F & \longrightarrow & \mathrm{S}_b V & \longrightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \\
K & \longrightarrow & F & \longrightarrow & V & \longrightarrow & 0
\end{array}$$

The middle column is a split exact sequence $0 \to F \to Q \oplus F \to Q \to 0$, for some free module Q generated in degree $< m$, by the previous remarks. This shows that $\operatorname{coker}(V \to \mathrm{S}_b V)$ is generated in degree $< m$. Therefore, $\operatorname{coker}(K \to \mathrm{S}_b K)$ is generated in degree $< r$. Because the rows of the above diagram are exact, we conclude that the relation degree of $\operatorname{coker}(V \to \mathrm{S}_b V)$ is $< r$.

□

1.7.1 ♯-Filtered Objects and the First Half of Theorem 3.9

Definition 1.21. *We say that an* FI_G*-module* V *is* ***♯-filtered*** *if it admits a filtration*

$$0 = V^{(0)} \subseteq \ldots \subseteq V^{(n-1)} \subseteq V^{(n)} = V$$

whose cofactors are relatively projective.

If k is a field, and G is a finite group, the dimension data of a finitely generated ♯-filtered object is described by a single polynomial for all n. That is to say, the Hilbert function

$$n \mapsto \dim_k V_n$$

is a polynomial in n for all n. Indeed, a direct computation verifies that for any finite dimensional $k[G_m]$-module W,

$$\dim_k M(W)_n = \binom{n}{m} \dim_k W$$

for all $n \geq 0$.

Theorem 1.22 ([NS])**.** *Assume that G is a polycyclic-by-finite group, and let V be a finitely generated FI_G-module over a Noetherian ring k. Then for $b \gg 0$, $\mathrm{S}_b V$ is $\sharp$-filtered.*

Definition 1.23. *The* ***Nagpal number****, $N(V) \in \mathbb{N} \cup \{\infty\}$, of an FI_G-module V is the smallest value b such that $\mathrm{S}_b V$ is $\sharp$-filtered.*

Note that in the context of this paper, it is not clear whether $N(V)$ is finite. The Nagpal-Snowden theorem tells us that this will be the case whenever G is polycyclic-by-finite, V is finitely generated, and k is a Noetherian ring. One of the main results of this paper will be to show that $N(V)$ is finite in many other cases as well (see Theorem C). The above discussion implies the following immediate corollary.

Corollary 1.24. *If G is a finite group, and V is a finitely generated FI_G-module over a field k, then there is a polynomial $P_V(x) \in \mathbb{Q}[x]$ such that the Hilbert function $H_V(n) = \dim_k V_n$ is equal to $P_V(n)$ for $n \geq N(V)$.*

Remark 1.25. *Keeping in mind Theorems C and B, the above corollary provides a parallel between* FI_G*-modules and graded modules over a polynomial ring. Namely, it is a consequence of the Hilbert Syzygy Theorem that the regularity of a graded module* M *over a polynomial ring provides a bound to the obstruction of the Hilbert polynomial (See [E] or [E2]). The results of this paper therefore imply a similar relationship between the regularity of an* FI_G*-module and bounds on its stable range. This might come as somewhat of a surprise, as Theorem 3.9 implies that all non-*$\sharp$*-filtered modules require infinite resolutions; a stark contrast to the Hilbert Syzygy Theorem.*

This corollary was proven for FI-modules over a field of characteristic 0 in [CEF, Theorem 1.5] and [SS3, Theorem 5.1.3], and over an arbitrary field in [CEFN, Theorem B]. Following this, polynomial stability was proven in the case where $G = \mathbb{Z}/2\mathbb{Z}$ in [W, Theorem 4.20]. It was proven for general FI_G-modules in [SS, Theorem 10.1.2]. None of these sources used the Nagpal-Snowden Theorem in their work. Using the new homological invariants defined in this paper, we will be able to replace $n \geq N(V)$ in the above corollary with an explicit lower bound on n.

Remark 1.26. *Although it is not proven in [NS], Theorem 1.22 actually implies that the Grothendieck group* $\mathcal{K}_0(\mathrm{FI}_G\text{-mod})$ *is generated by the classes of torsion modules and relatively projective modules whenever* k *is a Noetherian ring and* G *is polycyclic-by-finite. Indeed, if* V *is an* FI_G*-module we have the exact sequence*

$$0 \to T(V) \to V \to V' \to 0$$

where V' *is torsion free. Because* V' *is torsion free, it embeds into all of its shifts. The*

Nagpal-Snowden theorem therefore implies that V' embeds into a $\sharp$-filtered object, and Lemma 1.20 shows that the cokernel of this embedding is generated in strictly lower degree than V. Induction implies the desired result. This fact was proven for FI*-modules over a field of characteristic 0 by Sam and Snowden in [SS3, Proposition 4.9.1]. Note that we may also view this presentation of the Grothendieck group as a consequence of the classification theorem from Section 2.31.*

At this point in the paper, we are ready to prove the first collection of equivalences guaranteed by Theorem 3.9.

Theorem 1.27. *For an FI_G-module V which is generated in finite degree, the following are equivalent:*

1. *V is $\sharp$-filtered;*
2. *There is a series of surjections $Q^{(n)} = V \twoheadrightarrow Q^{(n-1)} \twoheadrightarrow \ldots \twoheadrightarrow Q^{(0)} = 0$ whose successive kernels are relatively projective;*
3. *V is homology acyclic;*
4. *$H_1(V) = 0$;*

Proof

The third part of Proposition 1.6 shows that the first two statement imply the third, and clearly the third implies the fourth.

Assume that $H_1(V) = 0$, and let i be the least index such that $V_i \neq 0$. Then we may construct a map $M(V_i) \to V$, which is an isomorphism in degree i. Denote the kernel of this map by $K^{(n)}$, and its image by $I^{(n)}$. This leaves us with a pair of exact sequences,

$$0 \to K^{(n)} \to M(V_i) \to I^{(n)} \to 0 \tag{1.3}$$

$$0 \to I^{(n)} \to V \to Q^{(n-1)} \to 0 \tag{1.4}$$

Applying H_0 to (1.3), we find that $H_0(M(V_i))$ surjects onto $H_0(I^{(n)})$ and therefore these must be isomorphic. Indeed, $H_0(M(V_i))$ is zero everywhere but in degree i, where it is V_i, and $H_0(I^{(n)})$ must be V_i in degree i by construction. This shows that the map $H_0(I^{(n)}) \to H_0(V)$ is an injection. Applying H_0 to (1.4) and using our assumption we obtain the exact sequence

$$H_2(Q^{(n-1)}) \to H_1(I^{(n)}) \to 0 \to H_1(Q^{(n-1)}) \to H_0(I^{(n)}) \to H_0(V) \tag{1.5}$$

By what was just discussed we may conclude that $H_1(Q^{(n-1)}) = 0$. We observe that the first degree j for which $Q_j^{(n-1)} \neq 0$ will be strictly larger than i. This allows us to iterate the above process. Moreover, because V was generated in finite degree, we know that the same is true about Q. This shows that $H_0(Q)$ is supported in precisely one less degree than $H_0(V)$. It follows from this that this process will eventually terminate. To finish the proof, it suffices by induction to show that $K^{(n)} = 0$. Once again looking at H_0 applied to (1.3) we find

$$0 \to H_1(I^{(n)}) \to H_0(K^{(n)}) \to H_0(M(V_i)) \to H_0(I^{(n)}) \to 0$$

We have already discussed that the last map is an isomorphism, so $H_1(I^{(n)}) = 0$ if and only if $H_0(K^{(n)}) = 0$. In this case Nakayama's lemma would imply that $K^{(n)} = 0$. It therefore remains to show that $H_1(I^{(n)}) = 0$.

Note that at the final step in this construction we will be left with a sequence of the form

$$0 \to K^{(1)} \to M(W') \to Q^{(1)} \to 0,$$

where W' is a $k[G_j]$-module for some j. Indeed, $H_0(Q^{(1)})$ is only supported in a single degree by assumption, and therefore the map $M(W') \to Q^{(1)}$ must actually be surjective by Nakayama's lemma. Applying H_0, and using the assumptions that $H_0(M(W')) \to H_0(Q^{(1)})$ is an isomorphism and $H_1(Q^{(1)}) = 0$, we conclude that $H_0(K^{(1)}) = 0$. It follows that $K^{(1)} = 0$, and therefore $Q^{(1)} = M(W')$ is homology acyclic by part three of Proposition 1.6. The first two terms in (1.5) now imply that $H_1(I^{(2)}) = 0$. Proceeding inductively, we eventually reach the conclusion that $H_1(I^{(n)}) = 0$, as desired.

We have thus far shown that the second statement is equivalent to the third and fourth, and that the first statement implies these. It only remains to show that the second statement implies the first. Assume that V admits a cofiltration as in the third statement of the theorem, and assume that the factors of this cofiltration are given by the collection $\{M(W_i)\}_{i=1}^{g}$, with W_i a $k[G_i]$-module. We first observe that $H_0(V)$ is the FB_G-module which is W_i in degree i, and zero elsewhere. Indeed, this follows from how the W_i were constructed above. For each i, let $\{w_{i,j}\}_{j=1}^{\kappa_1}$ be a generating set for W_i. Applying Nakayama's lemma we obtain a surjection

$$\bigoplus_{i=1}^{g} M(i)^{\kappa_i} \twoheadrightarrow V.$$

Set V' to be the submodule of V generated by lifts of the $w_{i,j}$ with $i < g$. It remains to show that $V/V' = M(W_g)$.

Call $Q := V/V'$, and apply H_0 to the exact sequence

$$0 \to V' \to V \to Q \to 0.$$

By the previously proven equivalences, we are left with the sequence

$$0 \to H_1(Q) \to H_0(V') \to H_0(V) \to H_0(Q) \to 0$$

By construction, $H_0(V')$ is the module $H_0(V)$ with the term W_g set to zero, and the map $H_0(V') \to H_0(V)$ is the obvious inclusion. This implies two things: $H_1(Q) = 0$, and $H_0(Q)$ is the module which is W_g in degree g, and zero elsewhere.

The structure of $H_0(Q)$ implies that Q is zero up to degree g, where it is W_g. The identity map on W_g induces a surjection

$$M(W_g) \twoheadrightarrow Q,$$

which we claim is an isomorphism. Letting K be the kernel of this map, and using the fact that $H_1(Q) = 0$, we are obtain an exact sequence

$$0 \to H_0(K) \to H_0(M(W_g)) \to H_0(Q) \to 0.$$

The final map is an isomorphism by construction, and so $H_0(K) = 0$. This concludes the proof.

□

Remark 1.28. *As was noted during the proof, we again observe that the filtration constructed above has the property that the $k[G_n]$-modules which appear in the cofactors*

$M(W)$ are precisely the non-trivial terms of $H_0(V)$.

Remark 1.29. *The theorem just proven is the first, and largest, part of Theorem 3.9. One may have noted that very little about the structure of FI_G specifically was used in the previous proof. Indeed, this theorem will hold for modules over many other categories. Examples of these categories include FI_d, of finite sets with injections decorated by a d-coloring of the complement of their image, and VI, of finite vector spaces over a fixed finite field with injective linear maps. The interested reader should see [GL][L][SS][PS] for more on modules over these categories.*

It is natural for one to ask if we can prove the second half of Theorem 3.9 in a more general context. The answer to this question no, and it is most easily illustrated by the following example of Jordan Ellenberg. Let $\mathcal{C}$ be the natural numbers, viewed as a poset category. The above theorem will hold in this category. One immediately finds that the $\mathcal{C}$-module $M(0)$ is the object which is k in every degree, while $M(1)$ is the object which is 0 in degree 0, and k in all other degrees. In particular, there is a natural embedding $M(1) \hookrightarrow M(0)$, whose cokernel is the object which is k in degree 0, and 0 elsewhere. It is clear that this cokernel is not sharp filtered, and therefore we have a non $\sharp$-filtered object which admits a finite resolution by $\sharp$-filtered objects. It is an interesting question to ask for which categories one has the latter two equivalences of Theorem 3.9.

One technical corollary to Theorem 1.27 is the following.

Lemma 1.30. *Given an exact sequence*

$$0 \to V' \to V \to V'' \to 0,$$

of FI_G*-modules which are generated in finite degree such that* V'' *is* $\sharp$*-filtered,* V' *is* $\sharp$*-filtered if and only if* V *is* $\sharp$*-filtered.*

Proof

One applies the zeroth homology functor, and uses the fact that V'' is $\sharp$-filtered, to conclude that $H_i(V) = H_i(V')$ for all $i \geq 1$. Theorem 1.27 now implies the lemma.

□

This fact was first proven in [D, Proposition A.6] in a much more general context. More recently, it was also proven in [LY, Corollary 3.6] for FI-modules. In fact, the result proven in these papers is slightly stronger than that given above, as it includes the case where V' and V are known to be $\sharp$-filtered. We will provide a different proof of this strengthening as a consequence of the depth classification theorem in Section 2.31.

Remark 1.31. *The work thus far completed in this paper seems to indicate that* $\sharp$*-filtered objects are a fundamentally important class in* FI_G*-mod. One observes that there is a chain of classes*

$$\textit{Projective Objects} \subseteq \textit{Relatively Projective Objects} \subseteq \sharp\textit{-Filtered Objects}$$

In the case where k *is a field of characteristic 0, the above inclusions are equalities by Proposition 1.5. However, in general the inclusions can be proper. For example, if* k *is a field of characteristic* $p > 0$*, then an example of Nagpal [N, Example 3.35], which was* *independently discovered by Gan and Li [GL, Section 3], shows that there are $\sharp$-filtered objects which are not relatively projective. On the other hand, if W is a non-projective $k[G_n]$-module for some n, then $M(W)$ is not projective. It becomes an interesting question to ask whether there is some homological criterion which separates $\sharp$-filtered from relatively projective.*

1.8 The Church-Ellenberg Approach to Regularity

1.8.1 The Derivative and its Basic Properties

Definition 1.32. *Let V be an FI_G-module, and let ι denote the natural map $\iota : V \to \mathrm{S}\, V$. The **derivative** of V is the FI_G-module*

$$DV = \mathrm{coker}(\iota).$$

For any $a \geq 0$, we define D^a to be a-th iterate of D.

Because S is exact, and because DV is defined as a cokernel, it follows immediately that D is a right exact functor.

Definition 1.33. *We will follow [CE] and write $H_i^{D^a}$ to denote the i-th left derived functor of D^a for any $a \geq 1$.*

Proposition 1.9 ([CE], Proposition 3.5 and Lemma 3.6)**.** *The derivative functor D enjoys the following properties:*

1. *for any* $k[G_n]$*-module* W,

$$D(M(W)) = M(\mathrm{Res}^{G_n}_{G_{n-1}} W);$$

2. *if* V *is* $\sharp$*-filtered, then it is acyclic with respect to* D^a *for all* $a \geq 1$*;*

3. *if* V *is generated in degree* $\leq m$*, then* DV *is generated in degree* $\leq m-1$*. Conversely, if* $\deg(D^aV) \leq m$ *for some* a, m*, then* V *is generated in degree* $\leq a+m$*;*

4. *for any* FI_G*-module* V *there is an exact sequence*

$$0 \to H_1^D(V) \to V \xrightarrow{\iota} SV \to DV \to 0;$$

5. $H_1^D(V) = 0$ *if and only if* V *is torsion free;*

6. *for any* FI_G*-module* V*,* $H_i^D(V) = 0$ *for all* $i > 1$*;*

7. *if* $\deg(V) \leq n$ *then* $\deg(DV) \leq n-1$*, and* $\deg(H_1^D(V)) \leq \deg(V)$*.*

Remark 1.34. *In the cited paper, the authors only prove that relatively projective objects are acyclic with respect to* D^a*. Part 2 of the previous proposition actually follows immediately from this. Also note that the provided source only proves these statements for* FI*-modules. The proofs are exactly the same.*

One of the main results of the cited paper was to prove that, in the case of FI, the functors $H_i^{D^a}$ all had finite degree. They did this by providing an explicit bound on the degree in terms of certain invariants of V [CE, Theorem 3.8]. We will eventually be able to do this as well in the case of FI_G-modules.

Observe that Propositions 3.3, 1.7, and 1.5 imply that both S and D preserve projective objects. This will allow us to call upon the Grothendieck spectral sequence in the following lemma.

Lemma 1.35. *There is a natural isomorphism of functors*

$$\mathrm{S}_b \circ D^a \cong D^a \circ \mathrm{S}_b$$

for all $a, b \geq 1$. More generally, there are natural isomorphisms of functors

$$\mathrm{S}_b \circ H_i^{D^a} \cong H_i^{D^a} \circ \mathrm{S}_b$$

for all $b, a \geq 1$ and all $i \geq 0$.

Proof

We begin with the first claim. It clearly suffices to show the statement in the case where $a = b = 1$. Let V be an FI_G-module, and let $\tau_n : V_{n+2} \to V_{n+2}$ denote the isomorphism induced by the transposition $(n+2, n+1)$ paired with the trivial map into G. We claim that this map induces an isomorphism $\mathrm{S}\, D(V)_n \cong D(\mathrm{S}\, V)_n$.

Reviewing how everything is defined, we see that on points

$$\begin{aligned} \mathrm{S}\, D(V)_n &= V_{n+2}/\operatorname{im}((f^{n+1}, \mathbf{1})_*), \\ D(\mathrm{S}\, V)_n &= V_{n+2}/\operatorname{im}((f_+^n, \mathbf{1})_*) \end{aligned}$$

where $f^{n+1} : [n+1] \to [n+2]$ is the standard inclusion, $f^n_+ : [n+1] \to [n+2]$ is the map which is the identity on $[n]$ and sends $n+1$ to $n+2$, and where $\mathbf{1}$ is the trivial map into G. In particular,

$$\tau_n(f^{n+1}, \mathbf{1})_* = ((n+2, n+1)f^{n+1}, \mathbf{1})_* = (f^n_+, \mathbf{1})_*.$$

This shows that τ_n induces an isomorphism between $\mathrm{S}\, D(V)_n$ and $D(\mathrm{S}\, V)_n$, as desired. We claim that τ is actually a map of FI_G-modules. Let $(f, h) : [n] \to [m]$ be an map in FI_G. Then the map induced in $D(\mathrm{S}\, V)$ will be the image under the quotient of $(f_{++}, h_{++})_* : V_{n+2} \to V_{m+2}$, where f_{++} agrees with f on $[n]$ and sends $n+i$ to $m+i$ for $1 \leq i \leq 2$, and h_{++} agrees with h on $[n]$ and sends both $n+1$ and $n+2$ to 1. On the other hand, the map induced on $\mathrm{S}\, D(V)$ will also be the image in the quotient of $(f_{++}, h_{++})_*$. Then,

$$(f_{++}, h_{++})_*\tau_n = (f_{++}(n+1, n+2), h_{++})_* = ((m+1, m+2)f_{++}, h_{++})_* = \tau_m(f_{++}, h_{++})_*.$$

The fact that the collection of τ gives us a map of functors is easily checked.

The second statement is largely homological formalism. Let $H_i^{D^a \circ \mathrm{S}_b}$ denote the i-th left derived functor of $D^a \circ \mathrm{S}_b$, and similarly define $H_i^{\mathrm{S}_b \circ D^a}$. Because S_b is exact, the Grothendieck spectral sequences for both of these derived functors have a single row, or column, respectively. In particular,

$$H_i^{D^a} \circ \mathrm{S}_b = H_i^{D^a \circ \mathrm{S}_b} = H_i^{\mathrm{S}_b \circ D^a} = \mathrm{S}_b \circ H_i^{D^a}.$$

This concludes the proof.

□

We have already discussed the fact that the modules $H_i^{D^a}(V)$ have finite degree for all i and a whenever V is finitely generated. The previous lemma implies $\mathrm{S}_b V$ will be acyclic with respect to all derivative functors for b sufficiently large. We will reinterpret this later in terms of depth.

Before we finish this section, we take a moment to point out the exact sequence relating the derived functors of varying derivatives. In particular, if one writes $D^a = D \circ D^{a-1}$, then $H_p^{D^a}$ can be computed using the Grothendieck spectral sequence. By part 6 of Proposition 3.3, we know that this spectral sequence only has two columns. Thus,

$$0 \to DH_i^{D^a}(V) \to H_i^{D^{a+1}}(V) \to H_1^D(H_{i-1}^{D^a}(V)) \to 0. \tag{1.6}$$

1.9.1 The Relationship Between the Derivative and Regularity

Definition 1.36. *Let V be an FI_G-module. We define the* ***derived regularity*** *of V to be the quantity $\partial\mathrm{reg}(V) := \sup\{\deg(H_1^{D^a}(V))\}_{a=1}^{\infty} \in \mathbb{N} \cup \{-\infty, \infty\}$. If V is acyclic with respect to D^a for all a, then we set $\partial\mathrm{reg}(V) = -\infty$.*

We also define the ***derived width*** *of V to be the quantity $\partial\mathrm{width}(V) = \sup\{\deg(H_1^{D^a}(V)) + a\}_{a=1}^{\infty} \in \mathbb{N} \cup \{-\infty, \infty\}$.*

We will find the derived width and regularity of a module to be of great importance in what follows. Our first goal will be to show that both of these quantities are bounded relative to one another.

Proposition 1.10. *Let V be an FI_G-module which is presented in finite degree and generated in degree $\leq d$. Then,*

$$\partial\mathrm{reg}(V) + 1 \leq \partial\mathrm{width}(V) \leq \partial\mathrm{reg}(V) + \max\{hd_1(V), d\}$$

Proof

The proposition is clear if $\partial\mathrm{reg}(V) = \infty$, so we assume not. By assumption there is a presentation

$$0 \to K \to M \to V \to 0$$

such that M is generated in degree $\leq d$, and K is generated in degree $\leq \max\{hd_1(V), d\}$ (See Remark 1.14). Applying D^a to the above sequence, with $a > \max\{hd_1(V), d\}$, and applying the second and third part of Proposition 3.3, it follows that $H_1^{D^a}(V) = 0$. Therefore, any a for which $H_1^{D^a}(V) \neq 0$ must satisfy $H_1^{D^a}(V) + a \leq \partial\mathrm{reg}(V) + \max\{hd_1(V), d\}$.

It follows from the above work that we may find some a such that $\deg(H_1^{D^a}(V)) = \partial\mathrm{reg}(V)$. Then,

$$\partial\mathrm{reg}(V) + 1 = \deg(H_1^{D^a}(V)) + 1 \leq \deg(H_1^{D^a}(V)) + a \leq \partial\mathrm{width}(V).$$

This completes the proof.

□

For the purposes of relating the derived regularity to the usual regularity of a module, we will need to know how the degrees of $H_i^{D^a}(V)$ depend on i and a.

Proposition 1.11. *Let V be an FI_G-module which is presented in finite degree. Then, for all a and $i \geq 1$,*

$$\deg(H_i^{D^a}(V)) \leq \partial\mathrm{width}(V) - 1 + i - a,$$

Proof

This proposition was essentially proven in [CE, Theorem 3.8] for FI-modules. We follow their general strategy here.

The exact sequence (1.6) implies that the first a for which V is not acyclic with respect to D^a must have $H_1^{D^a}(V) \neq 0$. If it is the case that $\partial\mathrm{reg}(V) = -\infty$, this implies that V is acyclic with respect to D^a for all a, and the above inequality holds. We may therefore assume that $\partial\mathrm{reg}(V)$ is finite. We proceed by induction on a. If $a = 1$, then Proposition 3.3 implies that $H_i^D(V) = 0$ for all $i > 1$, and the bound holds trivially in this case. If $i = 1$, then the bound follows from the definition of derived width.

Assume that the statements hold up to some a. We first note that the bound holds when $i = 1$ by the definition of derived width. We once again call upon (1.6) and write,

$$0 \to DH_i^{D^a}(V) \to H_i^{D^{a+1}}(V) \to H_1^D(H_{i-1}^{D^a}(V)) \to 0.$$

Assuming that $i > 1$, we know that

$$\begin{aligned} \deg(H_i^{D^a}(V)) &\leq \partial\mathrm{width}(V) - 1 + i - a \\ \deg(H_{i-1}^{D^a}(V)) &\leq \partial\mathrm{width}(V) - 1 + i - a - 1. \end{aligned}$$

Applying part 7 of Proposition 3.3 to both of these inequalities, we find that

$$\begin{aligned}\deg(DH_i^{D^a}(V)) &\leq \partial\text{width}(V) - 1 + i - a - 1\\ \deg(H_1^D(H_{i-1}^{D^a}(V))) &\leq \partial\text{width}(V) - 1 + i - a - 1.\end{aligned}$$

This proves the claim.

□

We finish this section by showing the connection between derived regularity, and the previously mentioned notion of regularity .

Proposition 1.12. *Let V be an FI_G-module which is presented in finite degree and has $\leq r$. Then,*

$$hd_i(V) \leq \max\{\partial\text{width}(V) - 1, \max\{hd_1(V), d\} - 1\} + i,$$

for all $i \geq 1$. In particular,

$$hd_i(V) \leq \partial\text{reg}(V) + \max\{hd_1(V), d\} - 1.$$

Proof

As with the previous proposition, this statement largely follows from the work in [CE, Theorem 3.9]. For the remainder of the proof, we fix $N := \max\{\partial\text{width}(V)-1, \max\{hd_1(V), d\}-1\}$. We begin with a projective resolution of V,

$$\ldots \to M_1 \to M_0 \to V \to 0.$$

Writing this as a collection of short exact sequences, we have

$$0 \to X_{i+1} \to M_i \to X_i \to 0$$

for some modules X_i, with $X_0 = V$. By repeatedly applying Nakayama's lemma, we may assume without loss of generality that $\deg(H_0(M_i)) = \deg(H_0(X_i))$ for all i Our first objective will be to prove the following claim:

$$\deg(H_0(X_i)) \leq N + i$$

for all $i \geq 1$. We prove this by induction on i. If $i = 1$, then

$$\deg(H_0(X_1)) \leq \max\{hd_1(V), d\} + 1 \leq N + 1.$$

Assume that $\deg(H_0(X_i)) \leq N + i$ for some $i \geq 1$, and consider the exact sequence

$$0 \to X_{i+1} \to M_i \to X_i \to 0. \tag{1.7}$$

By the assumption made at the beginning of the proof, we know that M_i is generated in degree $\leq N + i$. Part 3 of Proposition 3.3 implies that $D^{N+i+1} M_i = 0$. Applying this functor to (1.7), we therefore find

$$0 \to H_1^{D^{N+i+1}}(X_i) \to D^{N+i+1} X_{i+1} \to 0,$$

where we have used the fact that M_i is projective to imply the leading zero. We know that $H_1^{D^{N+i+1}}(X_i) = H_{i+1}^{D^{N+i+1}}(V)$, and Proposition 2.61 implies

$$0 \geq \partial\text{width}(V) - 1 + i + 1 - N - i - 1 \geq \deg(H_{i+1}^{D^{N+i+1}}(V)) = \deg(H_1^{D^{N+i+1}}(X_i)) = \deg(D^{N+i+1} X_{i+1}).$$

The third part of Proposition 3.3 implies that X_{i+1} is generated in degree $\leq N + i + 1$, as desired. To finish the proof, one applies the zeroth homology functor to (1.7) to obtain,

$$0 \to H_1(X_i) \to H_0(X_{i+1}).$$

This shows that $\deg(H_1(X_i)) \leq N + i + 1$. We know that $H_1(X_i) = H_{i+1}(V)$, finishing the proof of the proposition.

□

Remark 1.37. *One should note the following consequence of the above proof. If V is an* FI_G*-module which is presented in finite degree, and which has finite derived width, then V admits a projective resolution whose every member is generated in finite degree.*

One application of this result is in bounding the regularity of FI_G-modules with finite degree.

Corollary 1.38. *Let V be an* FI_G*-module with* $\deg(V) < \infty$*. Then,*

$$hd_i(V) \leq i + \deg(V)$$

Proof

To begin, we claim that $\partial\mathrm{width}(V) - 1 \leq \deg(V)$. In particular, for all $a \geq 1$

$$\deg(H_1^{D^a}(V)) \leq \deg(V) + 1 - a. \tag{1.8}$$

We prove this claim by induction on a. If $a = 1$, then the bound holds by the fourth part of Proposition 3.3. Assume we have proven the bound for some $a \geq 1$. The sequence (1.6) implies

$$0 \to DH_1^{D^a}(V) \to H_1^{D^{a+1}}(V) \to H_1^D(D^aV) \to 0.$$

The module D^aV has degree at most $\deg(V) - a$, and therefore $H_1^D(D^aV)$ also has degree at most $\deg(V) - a$ by the last part of Proposition 3.3. On the other hand, induction tells us that $\deg(H_1^{D^a}(V)) \leq \deg(V) - a + 1$, and therefore $\deg(DH_1^{D^a}(V)) \leq \deg(V) - a$ by the third part of Proposition 3.3. The above exact sequence implies the claim.

To finish the proof, one simply notes that $hd_1(V) \leq \deg(V)$, and applies Proposition 1.12.

□

This same bound is found in [L, Theorem 1.5] using different methods.

1.12.1 Bounding the Derived Width and the Proof of Theorem D

The purpose of this section is to generalize the methods of Church and Ellenberg to provide explicit bounds on the regularity in terms of the generating and relation degrees of the module V. More specifically, we will provide bounds on the derived width of V and apply Proposition 1.12.

The notation used in this section traces its origins to [CE]. Indeed, one may consider this section as an expository account of [CE, Section 2], where we are careful in generalizing relevant definitions to FI_G-modules.

Definition 1.39. *Let V be an FI_G-module, and fix a pair of integers $i \leq n$. Then we write $V_{n-\{i\}}$ to denote the submodule of V_n generated by images of induced maps of the* *form* $(f_i, \mathbf{1})_*$*, where* $\mathbf{1} : [n-1] \to G$ *is the trivial map and* $f_i : [n-1] \to [n]$ *is the map*

$$f_i(x) = \begin{cases} x & \text{if } x < i \\ x+1 & \text{otherwise.} \end{cases}.$$

Remark 1.40. *Because it will be important later, we note that, in fact,* $V_{n-\{i\}}$ *is the submodule of* V_n *generated by images of maps of the form* (f_i, g)*, where* $g : [n-1] \to G$ *is any map. Indeed, this follows from the identity*

$$(f_i, g)_*(v) = (f_i, \mathbf{1})_*((id, g)_*(v)).$$

With this notation it is immediate that for any $a \geq 1$,

$$D^a V_n = V_{n+a} / \sum_{i=1}^{a} V_{n+a-\{n+i\}}.$$

Let $0 \to K \to M \to V \to 0$ be a presentation for V. Then applying D^a it follows that

$$H_1^{D^a}(V) = \ker(K_{n+a} / \sum_{i=1}^{a} K_{n+a-\{n+i\}} \to M_{n+a} / \sum_{i=1}^{a} M_{n+a-\{n+i\}}).$$

This implies the following proposition.

Proposition 1.13. *The degree of* $H_1^{D^a}(V)$ *is the smallest integer* m *such that*

$$K_{n+a} \cap \sum_{i=1}^{a} M_{n+a-\{n+i\}} = \sum_{i=1}^{a} K_{n+a-\{n+i\}}$$

for all $n > m$.

This reformulation was first observed by Church and Ellenberg in [CE]. In that paper, it is shown that bounding the value m in the above proposition is deeply rooted in the combinatorics of FI-modules. For the remainder of this section, we show that the techniques of [CE] can be applied to FI_G.

Definition 1.41. *Fix a non-negative integer n, and let $i \neq j$ be elements of $[n]$. Then we set*

$$J_i^j = (id, \mathbf{1}) - ((i,j), \mathbf{1}) \in \mathbb{Z}[G_n],$$

where $\mathbf{1}$ is the trivial map into G. For any non-negative integer m, we define I_m to be the ideal of $\mathbb{Z}[G_n]$ generated products of the form

$$J_{i_1}^{j_1} \cdots J_{i_m}^{j_m}$$

where all of the indices are distinct elements of $[n]$.

For any non-negative integer b, we write $\Sigma(b)$ to denote the collection of b-element subsets $S \subseteq [2b]$ such that the i-th smallest element of S is at most $2i-1$. For any $1 \leq a \leq b$ we write $\Sigma(a,b) = \{S \in \Sigma(b) \mid [a] \subseteq S\}$.

For any $S \in \Sigma(b)$, we may write its entries in increasing order as $s_1 < \ldots < s_b$, and we may write the entries of its compliment in increasing order as $t_1 < \ldots < t_b$. Then define

$$J_S := \prod_i J_{s_i}^{t_i}.$$

The paper [CE] spends some time discussing the interesting combinatorial properties of $\Sigma(a,b)$, and its connection with the Catalan numbers. One observation they make is the following.

Lemma 1.42 ([CE], Section 2.1)**.** *Let $S \subseteq [2b]$ have b elements, and write its elements in increasing order as $s_1 < \ldots < s_b$. Assume that $U \subseteq [n]$ is such that $S \subseteq U$. For any b distinct elements $i_1 < \ldots < i_b$ in $[n] - U$, write H for the subgroup of $\mathfrak{S}_n$ generated by the disjoint transpositions (i_p, s_p). Then,*

$$S \in \Sigma(b) \implies U \text{ is lexicographically first among } \{\sigma \cdot U \mid \sigma \in H\}.$$

We note that the in the case of FI, defining J_i^j does not require a choice of map $[n] \to G$. In the above definition we have chosen the trivial map for the following reason. If $(f, g) : [m] \to [n]$ is any map in FI_G, and $(\sigma, \mathbf{1}) \in G_n$, then $(\sigma, \mathbf{1}) \circ (f, g) = (\sigma \circ f, g)$. In other words, choosing the map into G to be trivial grants us the ability to often times ignore the map g. This choice will allow us to use the arguments of [CE].

For the remainder of this section we fix integers $r \leq n$, and write $F := \mathbb{Z}[\mathrm{Hom}_{\mathrm{FI}_G}([r], [n])]$.

Definition 1.43. *Let $1 \leq a \leq b$ be integers, and assume $S \in \Sigma(b)$. Then we define the following submodules of F,*

1. $F^{\neq S} := ((f, g) \in F \mid S \not\subseteq \operatorname{im} f)$;

2. $F^b := ((f, g) \in F \mid \forall S \in \Sigma(b), S \not\subseteq \operatorname{im} f) = \cap_{S \in \Sigma(b)} F^{\neq S}$;

3. $F^{a,b} := ((f,g) \in F \mid \forall S \in \Sigma(a,b), S \not\subseteq \operatorname{im} f) = \cap_{S\in\Sigma(a,b)} F^{\neq S}$;

4. $F_{=S} := ((f,g) \in F \mid [2b] \cap \operatorname{im} f = S)$.

Proposition 1.14 ([CE] Propositions 2.3, 2.4, 2.6, and Lemma 2.5). *Let a, b, m, and p be non-negative integers.*

1. *If $n \geq b + r$, then*

$$F = I_b \cdot F + F^b.$$

2. *If $a \leq b$ and $2b \leq n$, then*

$$F^{a,b+1} \subseteq F^{a,b} + \sum_{S\in\Sigma(a,b)} J_S \cdot F_{=S}.$$

3. *If V is an FI_G-module which is generated in degree $\leq r$, then $I_{r+1} \cdot M_n = 0$ for all $n \geq 0$.*

4. *Given $(f,g) : [r] \to [n]$ and $\{i_1, j_1, \ldots, i_m, j_m\} \subseteq [n]$, if $\operatorname{im} f \cap \{i_p, j_p\} = \emptyset$ for some p, then $J_{i_1}^{j_1} \cdots J_{i_m}^{j_m}(f,g) = 0$.*

Proof

The cited source proves all of these claims for FI-modules. As stated previously, by our choice in defining the elements J_i^j all of these arguments will work almost verbatim. We work through the proof of the first statement as an example of this.

It is clear that $F = I_b \cdot F + F^b$ if and only if the latter group contains all the canonical basis vectors. Assume that this is not the case, and pick $(f,g) \notin I_b \cdot F + F^b$

so that the image of f is lexicographically largest among all basis vectors missing from $I_b \cdot F + F^b$. Note that the image of f has size r, and there are therefore at least b elements in the compliment of this image, by assumption. Write $i_1 < i_2 < \ldots < i_b$ for some sequence of elements in $[n] - \operatorname{im} f$. By assumption, $(f, g) \notin F^b$, which implies that there is some $S \in \Sigma(b)$ such that S is contained in the image of f. Write the elements of S in increasing order as $s_1 < \ldots < s_b$, and consider $((s_p, i_p), \mathbf{1}) \cdot (f, g) = ((s_p, i_p)f, g)$. Applying Lemma 1.42 with U being the image of f, we conclude that the image of $(s_p, i_p)f$ is lexicographically larger than the image of f. By our assumption on f, it must be the case that $((s_p, i_p)f, g) \in I_b \cdot F + F^b$. Calling $J := J_{i_1}^{s_1} \cdots J_{i_p}^{s_p}$, it follows from definition that

$$J - (id, \mathbf{1}) = \sum_{id \neq \sigma \in H} (-1)^{\sigma} (\sigma, \mathbf{1})$$

where H is the subgroup of $\mathfrak{S}_n$ generated by the transpositions (s_p, i_p). In particular, $(J - (id, \mathbf{1}))(f, g) \in I_b \cdot F + F^b$ by the previous computation. On the other hand, $J \cdot (f, g) \in I_b \cdot F \subseteq I_b \cdot F + F^b$ by definition. This shows that $(f, g) \in I_b \cdot F + F^b$, which is a contradiction.

$\square$

We are now ready to state and prove the main theorem of this section. As with all the previous statements, the proof of the following proceeds in precisely the same way it did in the case of FI-modules.

Theorem 1.44 ([CE], Theorem A)**.** *Let $K \subseteq M$ be torsion free* FI_G*-modules, and assume that M is generated in degree $\leq d$ and K is generated in degree $\leq r$. Then for* *all* $n \geq \min\{r, d\} + r + 1$, *and all* $a \leq n$,

$$K_n \cap \sum_{i=1}^{a} M_{n-\{i\}} = \sum_{i=1}^{a} K_{n-\{i\}}.$$

In particular, if V *is an* FI_G*-module which is generated in degree* $\leq d$ *and related in degree* $\leq r$ *then* $\partial\mathrm{width}(V) \leq \min\{r, d\} + r$.

Proof

We first note that the second statement follows as an immediate consequence of the first statement, Proposition 1.13, and the definition of derived regularity. It therefore suffices to prove the first statement. Due to its similarity with the proof in the provided source, we only give an outline here for the convenience of the reader.

Our first reduction will be to assume that K and M are FI_G-modules over $\mathbb{Z}$. Observe that if V is an FI_G-module over a ring k, then it can also be considered as an FI_G-module over $\mathbb{Z}$. It is clear that doing this does not change whether V is generated in finite degree.

Because K is generated in degree $\leq r$, the map $F \otimes K_r \to K_n$ is surjective for all $n > r$. Let a be as in the statement of the theorem, and let $b \geq a$ be an integer. We define the following submodules of K_n

$$K^b := \mathrm{im}(F^b \otimes K_r \to K_n),\ K^{a,b} := \mathrm{im}(F^{a,b} \otimes K_r \to K_n)$$

The remainder of the proof proceeds in the following steps. One first shows that $K^{a,\min\{r,d\}+r+1} = K_n$, and then that $K^{a,b+1} \cap \sum_i M_{n-\{i\}} \subseteq K^{a,b}$. At this point induction on b, beginning at $\min\{r, d\}+r+1$ and ending at a, implies that $K_n \cap \sum_i M_{n-\{i\}} \subseteq K^{a,a}$.

Note that $K^{a,a}$ is, by definition, the submodule of K_n generated by images of induced maps (f,g) such that $[a]$ is not contained in the image of f. In other words, using the remark at the beginning of the section,

$$K^{a,a} = \sum_{i=1}^{a} K_{n-\{i\}}.$$

This proves the theorem.

Applying the first part of Proposition 1.14, we find

$$K_n = \mathrm{im}(F \otimes K_r) = \mathrm{im}(I_{\min\{r,d\}+r+1}F + F^{\min\{r,d\}+r+1} \otimes K_r) = I_{\min\{r,d\}+r+1} \cdot K_n + K^{\min\{r,d\}+r+1}$$

Applying the third part of Proposition 1.14, we have that $I_{\min\{r,d\}+r+1} \cdot K_n = 0$, and therefore

$$K_n = K^{\min\{r,d\}+r+1} \subseteq K^{a,\min\{r,d\}+r+1} \subseteq K_n.$$

The second claim - that $K^{a,b+1} \cap \sum_i M_{n-\{i\}} \subseteq K^{a,b}$ - is considerably more subtle. We direct the reader to the original source for the details.

□

Corollary 1.45. *Let V be an FI_G-module which is presented in finite degree, and which is generated in degree $\leq d$. Then*

$$\partial\mathrm{width}(V) \leq hd_1(V) + \min\{d, hd_1(V)\}.$$

Proof

This follows from the techniques discussed in Remark 1.14. One simply notes that the functors D^a are right exact, and that $\sharp$-filtered objects are acyclic with respect to these functors.

□

Using all we have thus far learned, we can finally prove Theorem D.

Proof of Theorem D

One applies Corollary 1.45, and Proposition 1.12.

□

1.15 Depth

1.15.1 Definition and the Classification Theorem

The “usual” first definition of depth in classical commutative algebra is stated in terms of regular sequences (see [E, Chapter 18] for the classical theory). One then proves a relationship between this definition and the Koszul complex. It is not immediately obvious what one would mean by a regular sequence in an FI_G-module, however. Another approach one might consider is defining depth through some kind of local cohomology theory. This may not work in this setting, as FI_G-mod over a field of characteristic p, for example, will not have sufficiently many injectives. It is therefore not even clear that a local cohomology theory exists in this case. Note that if k is a field of characteristic 0, then Sam and Snowden have developed a theory of local cohomology and depth for FI-modules [SS3]. The definition we give now seems completely divorced from the classical theory. We hope that through the proofs that follow, one can develop a better idea of why this is the right definition. In Section 1.16.2, we explore a more classically motivated definition, and prove that it is equivalent.

Definition 1.46. *Let V be an FI_G-module. Then we define the* ***depth*** $\mathrm{depth}(V)$ *of V to be the infimum*

$$\mathrm{depth}(V) := \inf\{a \mid H_1^{D^{a+1}}(V) \neq 0\} \in \mathbb{N} \cup \{\infty\}\},$$

where we use the convention that the infimum of the empty set is ∞.

Lemma 1.47. *Let V be an FI_G-module which is presented in finite degree, and for which there is some $\delta \geq 0$ such that V is acyclic with respect to D^a for all $a \leq \delta$, while V is not acyclic with respect to $D^{\delta+1}$. Then for all $l \geq 1$, $H_l^{D^{\delta+l}}(V) \neq 0$, while $H_i^{D^{\delta+l}}(V) = 0$ for $i > l$.*

Proof

We proceed by induction on l. If $l = 1$. We know that $H_i^{D^{\delta+1}}(V) \neq 0$ for some i. On the other hand, if one plugs in any $i > 1$ and $a = \delta + 1$ into (1.6), then one finds $H_i^{D^{\delta+1}}(V) = 0$. This shows that $H_1^{D^{\delta+1}}(V) \neq 0$, while $H_i^{D^{\delta+1}}(V) = 0$ for all $i > 1$.

Assume that $H_l^{D^{\delta+l}}(V) \neq 0$. Then the sequence (1.6) shows

$$0 \to DH_{l+1}^{D^{\delta+l}}(V) \to H_{l+1}^{D^{\delta+l+1}}(V) \to H_1^D(H_l^{D^{\delta+l}}(V)) \to 0.$$

By assumption $H_l^{D^{\delta+l}}(V) \neq 0$. Proposition 2.61 and Theorem 1.44 tell us that $H_l^{D^{\delta+l}}(V)$ has finite degree, and therefore has nontrivial torsion. The fifth part of Proposition 3.3 implies that $H_1^D(H_i^{D^{\delta}}(V)) \neq 0$, whence $H_{l+1}^{D^{\delta+l+1}}(V) \neq 0$. If $i > l$, then the above sequence, and induction, implies our desired vanishing.

□

Lemma 1.47 implies the following alternative characterization of depth.

Proposition 1.16. *Let V be presented in finite degrees. Then,*

$$\mathrm{depth}(V) = \sup\{a \mid V \textit{ is acyclic with respect to } D^a\}.$$

Our next goal will be to show the following incremental property of depth. It shows how the depth of a module relates to the depth of its syzygies.

Lemma 1.48. *Given an exact sequence of* FI_G*-modules which are presented in finite degree,*

$$0 \to V' \to X \to V \to 0,$$

such that X is $\sharp$-filtered, $\mathrm{depth}(V') = \mathrm{depth}(V) + 1$.

Proof

We begin by recalling that X is acyclic with respect to D^a for all $a \geq 1$ (Proposition 3.3). Applying D^a to the given exact sequence, it therefore follows that $H_i^{D^a}(V') = H_{i+1}^{D^a}(V)$ for all $i, a \geq 1$. If $\mathrm{depth}(V) = \infty$, then this immediately implies the same about $\mathrm{depth}(V')$. If $\mathrm{depth}(V) = \delta < \infty$, then these equalities imply that $H_i^{D^a}(V') = 0$ at least up to $a = \delta$. Moreover, Lemma 1.47 tells us that $H_1^{D^{\delta+1}}(V') = H_2^{D^{\delta+1}}(V) = 0$, while $H_1^{D^{\delta+2}}(V') = H_2^{D^{\delta+2}}(V) \neq 0$, showing that $\mathrm{depth}(V') = \delta + 1$.

□

To prove the classification theorem, we will need a collection of technical lemmas. The following lemma was originally proven by Nagpal in [N, Lemma 2.2] for FI-modules, and was later reproven by Li and Yu in [LY, Lemma 3.3].

Lemma 1.49. *Let W be a $k[G_m]$-module for some m, and assume that there is an exact sequence*

$$0 \to U \to M(W) \to V \to 0,$$

such that U is generated in degree $\leq m$. Then U and V are both relatively projective.

Proof

Applying H_0 to the given exact sequence, we find that

$$0 \to H_1(V) \to H_0(U) \to H_0(M(W)) \to H_0(V) \to 0$$

Because U was generated in degree m, it follows that $H_0(U)$ is only supported in degree m. Because U is a submodule of $M(W)$, it follows that $H_0(U)_m = U_m$, and therefore the map $H_0(U) \to H_0(M(W))_m$ is injective. We conclude that $H_1(V) = 0$. Our result now follows by Theorem 1.27.

□

The following lemma was proven in a slightly stronger form in [LY, Lemma 3.11] using different methods.

Lemma 1.50. *Let V be an* FI_G*-module which is generated in degree $\leq m$, and assume* *that* $H_1^{D^{m+1}}(V) = 0$. *Then the relation degree of* V *is* $\leq m$. *In particular, if* V *is generated in finite degree and has infinite depth, then* V *is presented in finite degree.*

Proof

By assumption there is an exact sequence,

$$0 \to K \to M \to V \to 0$$

where M is free and generated in degree $\leq m$. Applying the functor D^{m+1} to this sequence, and using Proposition 3.3 as well as our assumption, we find that $D^{m+1}K = 0$. Proposition 3.3 implies that K is generated in degree $\leq m$.

□

Theorem 1.51 (The Depth Classification Theorem)**.** *Let* V *be an* FI_G*-module which is presented in finite degree. Then:*

1. $\mathrm{depth}(V) = 0$ *if and only if* V *has torsion;*

2. $\mathrm{depth}(V) = \infty$ *if and only if* V *is* $\sharp$*-filtered;*

3. $\mathrm{depth}(V) = \delta$ *is a positive integer if and only if there is an exact sequence*

$$0 \to V \to X_{\delta-1} \to \ldots \to X_0 \to V' \to 0$$

where X_i *is sharp filtered for all* i, $hd_0(V) = hd_0(X_{\delta-1})$, $hd_0(X_i) > hd_0(X_{i-1}) > hd_0(V')$ *for all* i, *and* V' *is some* FI_G*-module with torsion.*

Proof

The first statement is just a rephrasing of the fifth part of Proposition 3.3.

The backwards direction of the second statement is a rephrasing of part 2 of Proposition 3.3. We will prove the forward direction by induction on the generating degree. If $V = 0$, then it is $\sharp$-filtered by definition. Assume that V is generated in non-negative degree $\leq m$, and let V' be the submodule of V generated by the elements of V_i with $i < m$. Then we have an exact sequence,

$$0 \to V' \to V \to Q \to 0$$

where Q is generated in degree m. Applying the H_0 functor we find that

$$H_1(V) \to H_1(Q) \to H_0(V')$$

By assumption, $\deg(H_0(V')) \leq m - 1$, while $\deg(H_1(V))$ is at most the relation degree of V, which is at most m by Lemma 1.50. It follows that $\deg(H_1(Q)) \leq m$.

If,

$$0 \to K \to M(W) \to Q \to 0$$

is an exact sequence, with W a $k[G_m]$-module, then applying H_0 we find

$$0 \to H_1(Q) \to H_0(K) \to H_0(M(W))$$

It follows that

$$hd_0(K) \leq \max\{hd_0(M(W)), hd_1(Q)\} \leq m.$$

Nakayama's lemma and Lemma 3.15 imply that Q is $\sharp$-filtered. In particular, it has infinite depth and therefore V' also has infinite depth. Applying Lemma 1.50 we know that

V' is presented in finite degree, and therefore induction tells us that it is also $\sharp$-filtered. Lemma 1.30 implies that V is $\sharp$-filtered, as desired.

The backwards direction of the third statement follows from Lemma 1.48. Conversely, assume that V is presented in finite degree, and $0 < \text{depth}(V) = \delta < \infty$. Theorems 1.44 and C (proven in the next section) imply that we may find some $b \gg 0$ such that $\mathrm{S}_b V$ is $\sharp$-filtered. Because V has positive depth, it is torsion free, and therefore the map $V \to \mathrm{S}_b V$ is an embedding. The cokernel of this map is also presented in finite degree by Lemma 1.20, and its relation and generating degrees are strictly smaller. The claim now follows from an application of Lemma 1.48 and induction on the generating degree.

□

Remark 1.52. *Lemma 1.50 guarantees that any infinite depth module which is generated in finite degree must actually be related in finite degree as well. The second equivalence of the classification theorem may therefore be stated under the assumption that V is generated in finite degree.*

The proof that infinite depth modules are equivalent to $\sharp$-filtered modules was significantly different in the earlier versions of this paper. In these versions, the techniques used in the proof only worked if k was a Noetherian ring, G was finite, and V was finitely generated. The recent preprint [LY] proves various new properties of the derivative for FI*-modules. The proof of Theorem 2.31 given here has adapted certain techniques of the paper [LY], which has allowed us to generalize our original result.*

One immediate consequence of Theorem 2.31 is a bound on $\mathrm{depth}(V)$ in terms of the generating degree of V.

Corollary 1.53. *Let V be an FI_G-module which is presented in finite degree, and is of finite depth. Then* $\mathrm{depth}(V) \leq hd_0(V)$.

As promised earlier, the classification theorem implies the following strengthening of Lemma 1.30.

Corollary 1.54. *Let*

$$0 \to V' \to V \to V'' \to 0$$

be an exact sequence of FI_G*-modules which are generated in finite degree. Then any two of V', V, or V'' are $\sharp$-filtered if and only if the third is as well.*

Proof

Lemma 1.30 deals with all cases except that in which V' and V are $\sharp$-filtered. In this case Lemma 1.48 implies that V'' must have infinite depth, and the classification theorem then implies that V'' is $\sharp$-filtered, as desired.

□

This lemma is the last piece needed to prove Theorem 3.9.

Proof of Theorem 3.9

The equivalence of the first four statements was proven in Theorem 1.27. It is clear that any of these four implies the fifth statement. Conversely, if V admits a finite resolution by $\sharp$-filtered modules, then we conclude that V must be $\sharp$-filtered through repeated applications of Corollary 3.8.

Assume now that V is presented in finite degree. It is clear that the first five statements imply that $H_i(V) = 0$ for some $i \geq 0$. Conversely, fix $i > 0$ such that $H_i(V) = 0$. If $i = 1$, then the previously proven equivalences imply that V is $\sharp$-filtered. If $i > 1$, we may truncate a free resolution of V at the i-th step, and write,

$$0 \to X_{i-1} \to M_{i-2} \to \ldots \to M_0 \to V \to 0 \tag{1.9}$$

where the M_j modules are free. Note that the proof of Proposition 1.12 as well as Theorem 1.44 imply that we may assume all M_j, as well as X_{i-1}, are generated in finite degree. Using the fact that $H_1(X_{i-1}) = H_i(V) = 0$, we conclude that X_{i-1} is $\sharp$-filtered.

□

1.16.1 The Proofs of Theorems C and B

The classification theorem will now allow us to prove the connection between regularity and depth.

Proof of Theorem C

Lemma 1.35 tells us that there are isomorphisms for all a and b,

$$\mathrm{S}_b\, H_1^{D^a}(V) = H_1^{D^a}(\mathrm{S}_b\, V).$$

The depth classification theorem implies that $\mathrm{S}_b\, V$ is sharp filtered if and only if $\mathrm{S}_b\, H_1^{D^a}(V) = 0$ for all $a \geq 1$. This is equivalent to saying that $\deg(H_1^{D^a}(V)) < b$.

□

At this point we are ready to put all the pieces together. This theorem tells us that the question of when a module becomes sharp filtered is equivalent to bounding its derived regularity. Proposition 1.12 then implies that bounds on the derived regularity leads to bounds on the homological degrees of the module. Moreover, we notice that this theorem can be used in both directions. Known bounds on how far a module must be shifted to become sharp filtered give bounds on the regularity of V. Conversely, known bounds on the derived regularity of V, such as those given in Theorem 1.44, give bounds on how far a module must be shifted to become $\sharp$-filtered. In particular, we have the following.

Corollary 1.55. *Let V be an FI_G-module which presented in finite degree, and is generated in degree $\leq d$. Then $\mathrm{S}_b V$ is sharp filtered for $b \geq hd_1(V) + \min\{hd_1(V), d\}$.*

This gives us all we need to bound the stable range when G is a finite group.

The Proof of Theorem B

We recall that $\sharp$-filtered modules have a dimension polynomial for all n (see the discussion immediately following Definition 2.8). Corollary 1.55, and the techniques of Remark 1.14, immediately imply Theorem B.

□

Remark 1.56. *In [L, Theorem 1.5], Li proves that if G is a finite group, and if b is such that $hd_i(\mathrm{S}_b V) \leq hd_0(\mathrm{S}_b V) + i$ for all i, then $hd_i(V) \leq hd_0(V) + b + i$ for all i. Theorems 3.9 and 1.22, along with this theorem of Li, imply Theorem D. Indeed, by the* *results in this paper, Li's result implies*

$$hd_i(V) \leq hd_0(V) + \partial\text{reg}(V) + 1 + i.$$

This is related to the bound discussed in Theorem D, and is better in certain cases.

1.16.2 An Alternative Definition of Depth

In this section we explore an alternative definition for depth, which is more classically rooted. For convenience of exposition, we assume throughout this section that k is a field of characteristic 0, and that G is a finite group. It is the belief of the author that all that follows can be done in the generality of the rest of the paper. The main result of this section will show that this alternative definition is equivalent to that given above.

If R is a commutative ring with ideal $I \subseteq R$, it is classically known that for any R-module M,

$$\text{depth}(I, M) = \min\{i \mid \text{Ext}^i(R/I, M) \neq 0\}.$$

See [E, Proposition 18.4] for the proof of this equality. This is the definition of depth which we will now emulate for our setting.

In [GL], the Gan and Li define the ring $k\,\text{FI}_G = \bigoplus_{n \leq m} k[\text{Hom}_{\text{FI}_G}([n],[m])]$, whose multiplication is defined by

$$(f,g)\cdot(f',g') = \begin{cases} (f,g)\circ(f',g') & \text{if the codomain of } f' \text{ is equal to the domain of } f \\ 0 & \text{otherwise.} \end{cases}$$

Writing e_n for the identity morphism in $\text{Hom}_{\text{FI}_G}([n],[n])$, we say that a $k\,\text{FI}_G$-module V is graded if $V = \bigoplus_n e_n \cdot V$. In the paper [GL], Gan and Li show that the category of graded $k\,\mathrm{FI}_G$-modules is equivalent to the category of FI_G-modules.

One observes that the ring $k\,\mathrm{FI}_G$ has a natural two sided ideal generated by the upwards facing maps, $\mathfrak{m} = ((f,g) : [n] \to [m] \mid n < m)$. The quotient $\mathcal{G} = k\,\mathrm{FI}_G/\mathfrak{m}$ is the direct sum $\bigoplus_n k[G_n]$ where all upwards facing maps act trivially.

Definition 1.57. *We write $\mathcal{G}$ to denote the FI_G-module which is defined on points by $\mathcal{G}_n = k[G_n]$, and whose transition maps are trivial. For any FI_G-module V, The k-module $\mathrm{Hom}_{\mathrm{FI}_G\text{-}Mod}(\mathcal{G}, V)$ carries the structure of an FB_G-module via*

$$\mathrm{Hom}_{\mathrm{FI}_G\text{-}Mod}(\mathcal{G}, V)_n = \mathrm{Hom}_{\mathrm{FI}_G\text{-}Mod}(k[G_n], V)$$

We write $\mathrm{Hom}(\mathcal{G}, \bullet) : \mathrm{FI}_G\text{-Mod} \to \mathrm{FB}_G\text{-Mod}$ to denote this functor.

Proposition 1.17. *The functor $\mathrm{Hom}(\mathcal{G}, \bullet)$ enjoys the following properties:*

1. $\mathrm{Hom}(\mathcal{G}, V) \neq 0$ *if and only if V has torsion;*
2. $\mathrm{Hom}(\mathcal{G}, \bullet)$ *takes finitely generated FI_G-modules to finitely generated FB_G-modules;*
3. $\mathrm{Hom}(\mathcal{G}, \bullet)$ *is right adjoint to the inclusion functor FB_G-mod $\to \mathrm{FI}_G$-mod. In particular, $\mathrm{Hom}(\mathcal{G}, \bullet)$ is left exact, and maps injective objects of FI_G-mod to injective objects of FB_G-mod.*

Proof

For the first statement, a map $\mathrm{Hom}_{\mathrm{FI}_G\text{-mod}}(k[G_n], V)$ is just a map of $k[G_n]$-modules

$f : k[G_n] \to V_n$, whose image is in the kernel of every transition map out of V_n. Therefore if $\mathrm{Hom}(\mathcal{G}, V) \neq 0$, V must have torsion. Conversely, if $v \in V_n$ is a torsion element, then there is some transition map $(f, g)_* : V_n \to V_m$ such that $(f, g)_*(v) = 0$. We may write the map (f, g) as

$$(f, g) = (f_1, g_1) \circ (f_2, g_2)$$

where $(f_1, g_1) : [m - 1] \to [m]$ and $(f_2, g_2) : [n] \to [m - 1]$. If $(f_2, g_2)_*(v) = 0$, then we may repeat this decomposition until it is not. Assuming that this is not the case, then $(f_2, g_2)_*(v) \in V_{m-1}$ is in the kernel of all transition facing maps. In particular, $\mathrm{Hom}(\mathcal{G}, V)_{m-1} \neq 0$.

Any element of $\mathrm{Hom}(\mathcal{G}, V)_n$ will naturally correspond to a torsion element of V_n by the previous discussion. In particular, we may consider $\mathrm{Hom}(\mathcal{G}, V)$ as a submodule of V. The Noetherian Property implies that $\mathrm{Hom}(\mathcal{G}, V)$ is finitely generated.

Let W be a finitely generated FB_G-module, and let V be a finitely generated FI_G-module. If we consider W as being an FI_G-module with trivial transition maps, then a morphism $W \to V$ must send elements of W to elements of V which are in the kernel of all transition maps. By the previous discussion, these correspond precisely to the elements of $\mathrm{Hom}(\mathcal{G}, V)$. Conversely, if we have a map $\phi : W \to \mathrm{Hom}(\mathcal{G}, V)$, then we set $\widetilde{\phi} : W \to V$ by $\widetilde{\phi}_n(w) = \phi_n(w)(id)$.

The final two statements are general facts from homological algebra. Any functor which is right adjoint has the first property, and if its left adjoint is exact then it has the second property.

□

We have already discussed the fact that the category FI_G-mod might not have sufficiently many injective objects if k is a field of characteristic $p > 0$. When k is a field of characteristic 0, and G is a finite group, however, this is not an issue. In fact, Sam and Snowden [SS3, Theorem 4.3.1], as well as Gan and Li [GL, Theorem 1.7], have shown that every object has finite injective dimension in this case. Moreover, the cited papers prove that all projective objects are also injective, and these are precisely the torsion free injective objects. In other words, one has the following chain of equalities

$$\{\sharp\text{-Filtered Modules}\} = \{\text{Projective Modules}\} = \{\text{Torsion Free Injective Modules}\}$$

Definition 1.58. *We will denote the right derived functors of* $\mathrm{Hom}(\mathcal{G}, \bullet)$ *by* $\mathrm{Ext}^i(\mathcal{G}, \bullet)$*. Given a finitely generated* FI_G*-module* V*, we define its* ***classical depth*** *to be the quantity* $\mathrm{depth}^{\mathrm{class}}(V) := \min\{i \mid \mathrm{Ext}^i(\mathcal{G}, \bullet) \neq 0\}$.

As was the case with the other notion of depth, the first property one needs to prove to justify using the name depth is the increment property.

Lemma 1.59. *If there is an exact sequence of finitely generated* FI_G*-modules*

$$0 \to V' \to X \to V \to 0$$

such that X *is* $\sharp$*-filtered, then* $\mathrm{depth}^{\mathrm{class}}(V') = \mathrm{depth}^{\mathrm{class}}(V) + 1$.

Proof

Applying the functor $\mathrm{Hom}(\mathcal{G}, \bullet)$ to the above sequence, and using the fact that X is injective and torsion free, we find that $\mathrm{Ext}^{i+1}(\mathcal{G}, V') = \mathrm{Ext}^{i}(\mathcal{G}, V)$ for all $i \geq 0$. This immediately implies the desired result.

□

The first step in showing that the two notions of depth coincide is to show that they are the same at the extremes. Proposition 1.17 implies that depth zero and classical depth zero are equivalent. We next prove that objects of infinite classical depth are precisely the $\sharp$-filtered modules.

Proposition 1.18. *For any finitely generated* FI_G*-module* V*,* $\mathrm{depth}^{\mathrm{class}}(V) = \infty$ *if and only if* V *is* $\sharp$*-filtered.*

Proof

Because $\mathrm{depth}^{\mathrm{class}}(V)$ is defined in terms of the vanishing of Ext functors, it follows that $\mathrm{depth}^{\mathrm{class}}(V) = \infty$ if V is injective and torsion free. Conversely assume that V has infinite classical depth. This implies, in particular, that V is torsion free, and therefore embeds into all its shifts. Using Theorem 1.22, we may find some $b >> 0$ such that S_b is $\sharp$-filtered. Lemma 1.20 implies the cokernel of $V \hookrightarrow \mathrm{S}_b V$ is generated in strictly smaller degree, while Lemma 1.59 implies that it has infinite classical depth. Proceeding inductively, we will eventually be left with a module V' such that $\mathrm{S}_{b'} V'$ is sharp filtered for some $b' >> 0$ and $\mathrm{S}_{b'} V'/V'$ is $\sharp$-filtered. Corollary 3.8 implies that V was $\sharp$-filtered to begin with.

□

This is all we need to prove the equivalence.

Theorem 1.60. *Let V be a finitely generated* FI_G*-module. Then* $\mathrm{depth}^{\mathrm{class}}(V) = \mathrm{depth}(V)$*.*

Proof

If $\mathrm{depth}^{\mathrm{class}}(V) = 0$ or $\mathrm{depth}^{\mathrm{class}}(V) = \infty$, then we have already seen $\mathrm{depth}(V)$ agrees with $\mathrm{depth}^{\mathrm{class}}(V)$. Assume that $\mathrm{depth}^{\mathrm{class}}(V) = \delta > 0$, and assume for the sake of contradiction that $\mathrm{depth}(V) = \delta' < \delta$. Because δ is positive, it must be the case that δ' is positive as well. We may therefore find some $b >> 0$ such that $\mathrm{S}_b V$ is $\sharp$-filtered. This gives us an embedding

$$0 \to V \to \mathrm{S}_b V \to Q \to 0$$

for some Q with $\mathrm{depth}^{\mathrm{class}}(Q) = \delta - 1$, and $\mathrm{depth}(Q) = \delta' - 1$. Proceeding inductively, we would eventually we left with a module with $\mathrm{depth}(V') = 0$ and $\mathrm{depth}^{\mathrm{class}}(V) > 0$. This is a contradiction.

□

Remark 1.61. *In [SS3], Sam and Snowden provide a definition of the depth of an* FI*-module in characteristic 0. This definition is formulated in terms of a kind of Auslander-Buchsbaum formula. It can be shown that this definition of depth is also equivalent to the definitions given in this paper.*

Chapter 2

Depth and the local cohomology of FI_G-modules (joint with Liping Li)

2.1 Introduction

2.1.1 Motivation

Since the fundamental work of Church, Ellenberg, and Farb in [CEF], the representation theory of the category FI, whose objects the finite sets $[n] = \{1, \ldots, n\}$ and whose morphisms are injections, has played a central role in Church and Farb's representation stability theory [CF]. The methodology of Church, Ellenberg, and Farb has since been generalized to account for many other well known facts in asymptotic algebra. The main idea behind these generalizations is that stability properties of sequences of group representations can be converted to considerations in the representation theory of certain infinite categories, usually equipped with nice combinatorial structures. This philosophy was carried out by a series of works in this area; see [CEF, CEFN, GL, N, R, SS, SS2, W]. In this paper, we will be concerned with representations of the category FI_G, where G is a finite group, which was introduced in [SS2].

The homological aspect of the representation theory of FI and its generalization FI_G originates from the papers [CEFN], and [CE] by Church, Ellenberg, Farb and Nagpal. In these papers, the notion of FI-module homology was introduced, as well as the Castelnuovo-Mumford regularity (See Definitions 2.12 and 2.13). The paper [CE] was the first to provide explicit bounds on this regularity [CE, Theorem A]. Soon afterward, the techniques of Church and Ellenberg were expanded through two different approaches. One approach was rooted in classical representation theory, and pursued by the first author and Yu in [LY, L2]. The second approach applied certain important ideas in commutative algebra, and was studied by the second author in [R]. Both approaches established a strong relationship between homological properties of FI_G and its representation stability phenomena, which was accomplished by obtaining upper bounds on certain homological invariants; see [L2, Theorem 1.3] and [R, Theorems C and D]. So far this project is still under active exploration.

This paper has two major goals. Firstly, it is a known fact that most finitely generated FI_G-modules have infinite projective dimension (see [LY, Theorem 1.5]). Moreover, it can be shown that the category of finitely generated FI_G-modules over an arbitrary Noetherian ring does not usually have sufficiently many injective objects (see Theorem 2.46). As a result of this, it becomes important for one to develop different machinery for computing certain homological invariants. The first half of this paper is concerned with creating such machinery, which will be built around the homological properties of $\sharp$-*filtered* modules (see Definition 2.8). One may think of these modules as acting as both the projective and torsion free injective objects of the category of finitely generated FI_G-modules (See Theorems C and D)

Secondly, we apply the homological machinery developed in the first half of the paper to the concepts of *depth* (See Definition 2.30) and *local cohomology*. For instance, the depth and *classical depth* (See Definition 2.32) of an FI_G-module were shown to be equivalent by the second author in [R, Theorem 4.15] for fields of characteristic 0. It is therefore natural to wonder whether they also coincide for arbitrary commutative Noetherian rings. Moreover, Sam and Snowden described a local cohomology theory for FI in [SS3] for fields of characteristic 0, so we ask whether a generalized theory for FI_G can be developed in the much wider framework of commutative Noetherian rings. The main goal of the second half of the paper is to give affirmative answers to the above concerns.

2.1.2 An inductive method

An extremely useful combinatorial property of the category FI_G is that it is equipped with a self-embedding functor, which induces a shift functor Σ in the category of FI_G-modules (See definition 3.2.3). This functor has seen heavy use in the literature. Examples of this include [CEF, CEFN, CE, GL, L, LY, N, R, L2]. The importance of this functor lies in the following key facts: it preserves both left and right projective modules; and for every finitely generated FI_G-module V, after applying the shift functor Σ enough times, V becomes a $\sharp$-filtered module [N, Theorem A].

From the perspective of more classically rooted representation theory, the shift functor Σ is a kind of restriction. It has a left adjoint functor (called *induction*) and a right adjoint functor (called *coinduction*), denoted by us L and R. The coinduction functor R was explicitly constructed in [GL]. In this paper we systematically consider these functors, showing that their behaviors perfectly adapt to $\sharp$-filtered modules. Moreover, the Eckmann-Shapiro lemma holds in the context of FI_G-modules, which allows us to reduce general questions to their simplest cases.

Theorem A. *Let k be a commutative Noetherian ring, and let V_n be a finitely generated kG_n-module, where $G_n = G \wr \mathfrak{S}_n$. We have:*

1. *the functors Σ, L, and R are all exact;*
2. *the functors Σ, L, and R preserve $\sharp$-filtered modules. In particular (see Definition 2.8):*

$$\Sigma M(V_n) \cong M(V_n) \oplus M(\mathrm{Res}^{G_n}_{G_{n-1}} V_n);$$
$$R(M(V_n)) \cong M(V_n) \oplus M(\mathrm{Ind}^{G_{n+1}}_{G_n} V_n);$$
$$L(M(V_n)) \cong M(\mathrm{Ind}^{G_{n+1}}_{G_n} V_n).$$

3. *For two finitely generated FI_G-modules V and V' and $i \geqslant 0$, one has*

$$\mathrm{Ext}^i_{k\,\mathrm{FI}_G}(L(V), V') \cong \mathrm{Ext}^i_{k\,\mathrm{FI}_G}(V, \Sigma V');$$
$$\mathrm{Ext}^i_{k\,\mathrm{FI}_G}(\Sigma V, V') \cong \mathrm{Ext}^i_{k\,\mathrm{FI}_G}(V, R(V')).$$

As an example of these induction style arguments, we prove the aforementioned equivalence of depth and classical depth. If V is an FI_G-module, there is always a natural map $V \to \Sigma V$ (See Definition 2.28 for the explicit definition). The cokernel of this map defines a functor, which we call the *derivative* DV. The derivative functor was first introduced in [CE], and was later used by Yu and the authors in [LY], and [R]. The paper [R] used the derived functors of the derivative and its iterates, $H_i^{D^a}(\bullet)$, to develop a theory of depth for FI_G-modules. It was also noted in that paper that there was a more classically rooted definition of depth called the classical depth of the module. Using Theorem A we will be able prove the following.

Theorem B. *Let V be a finitely generated FI_G-module over a Noetherian ring k. Then the depth and classical depth of V coincide. That is:*

$$\inf\{a \mid H_1^{D^{a+1}}(V) \neq 0\} = \inf\{i \mid \mathrm{Ext}^i_{\mathscr{C}\text{-Mod}}(k\mathscr{C}/\mathfrak{m}, V) \neq 0\}.$$

2.1.3 A machinery for homological calculations

In practice, to compute the homology (resp. cohomology) groups of representations of a ring, one usually approximates these representations by suitable projective (resp. injective) resolutions. However, we have already discussed that this strategy does not work very well in the context of FI_G-modules. To overcome this obstacle, one has to find suitable objects which satisfy the following two requirements: they must be acyclic with respect to certain important functors; and finitely generated FI_G-modules must be approximated by finite complexes of such objects.

The work of Nagpal, Yu, and the authors in [N, Theorem A], [LY, Theorem 1.3], and [R, Theorem B], strongly suggest that $\sharp$-filtered modules are best candidates. These objects were introduced as modules which were "almost" projective, and were later proven to be acyclic with respect to many natural right exact functors. One of the major realizations of this paper is that $\sharp$-filtered objects are also acyclic with respect to many left exact functors. In particular, the following results convince us that $\sharp$-filtered modules do play the role of both projective objects and injective objects for many homological computations.

Theorem C (Homological characterizations of $\sharp$-filtered modules)**.** *Let k be a commutative Noetherian ring, and let V be a finitely generated FI_G-module over k. Denote the endomorphism group of object $[s]$ in FI_G by G_s, $s \geqslant 0$. Then V is $\sharp$-filtered if and only if it satisfies one of the following equivalent conditions:*

1. $\mathrm{Tor}_i^{k\,\mathrm{FI}_G}(kG_s, V) = 0$ *for* $i \geqslant 1$ *and* $s \geqslant 0$*;*
2. $\mathrm{Tor}_1^{k\,\mathrm{FI}_G}(kG_s, V) = 0$ *for* $s \geqslant 0$*;*
3. $\mathrm{Tor}_i^{k\,\mathrm{FI}_G}(kG_s, V) = 0$ *for* $s \geqslant 0$ *and a certain* $i \geqslant 1$*;*
4. $\mathrm{Ext}^i_{k\,\mathrm{FI}_G}(kG_s, V) = 0$ *for* $i, s \geqslant 0$*;*

5. $\mathrm{Ext}^i_{k\,\mathrm{FI}_G}(T, V) = 0$ *for* $i \geqslant 0$ *and all finitely generated torsion modules (See Definition 2.36).*

Remark 2.1. *The first three homological characterizations of* $\sharp$*-filtered modules are not new. They have been described in [LY, Theorem 1.3] and [R, Theorem B].*

The Tor functors in the above theorem are strongly related the the notion of FI_G*-module homology, which we will formally define later. We state the above theorem using the language of Tor so that the relationship between the five given statements is more clear.*

Theorem D (Homological orthogonal relations)**.** *Let* k *be a commutative Noetherian ring, and let* V *be a finitely generated* FI_G*-module over* k*. Then*

1. T *is a torsion module if and only if* $\mathrm{Ext}^i_{k\,\mathrm{FI}_G}(T, V) = 0$ *for all* $\sharp$*-filtered modules* V *and all* $i \geqslant 0$.

2. V *is an injective module if and only if* $\mathrm{Ext}^1_{k\,\mathrm{FI}_G}(W, V) = 0$ *whenever* W *is a* $\sharp$*-filtered module or* W *is a finitely generated torsion module.*

3. $\mathrm{Ext}^i_{k\,\mathrm{FI}_G}(V, F) = 0$ *for all* $\sharp$*-filtered modules* F *and all* $i \geqslant 1$ *if and only if* $\Sigma_N V$ *is a projective module for* $N \gg 0$.

4. $\mathrm{Ext}^i_{k\,\mathrm{FI}_G}(T, V) = 0$ *for* $i \geqslant 1$ *and all finitely generated torsion modules* T *if and only if* V *is a direct sum of an injective torsion module and a* $\sharp$*-filtered module.*

This should convince the reader that $\sharp$-filtered objects are acyclic with respect to many natural left and right exact functors. To prove that $\sharp$-filtered objects can be used to approximate arbitrary modules, we call upon the following theorem.

Theorem 2.2 ([L2], Theorem 1.3). *Let k be a commutative Noetherian ring, and let V be a finitely generated FI_G-module over k. Then there is a complex*

$$\mathcal{C}^\bullet V : 0 \to V \to F^0 \to \ldots \to F^n \to 0$$

enjoying the following properties:

1. *every F^i is a $\sharp$-filtered module;*
2. $\mathrm{gd}(F^i) \leq \mathrm{gd}(V) - i$ *(see Definition 2.13). Therefore, $n \leq \mathrm{gd}(V)$;*
3. *the cohomologies $H^i(\mathcal{C}^\bullet V)$ are finitely supported (See Definition 2.13), and*

$$\begin{cases} \mathrm{td}(H^i(V)) = \mathrm{td}(V) & \text{if } i = -1 \\ \mathrm{td}(H^i(V)) \leq 2\,\mathrm{gd}(V) - 2i - 2 & \text{if } 0 \leq i \leq n. \end{cases}$$

We will find this theorem vital in our studies of the local cohomology of FI_G-modules.

2.1.4 A local cohomology theory

Another application of the above machinery and inductive method is in developing a local cohomology theory of FI_G-modules over arbitrary commutative Noetherian rings, generalizing the corresponding work in [SS3].

Given a finitely generated FI_G-module V, there is a natural exact sequence

$$0 \to V_T \to V \to V_F \to 0$$

where V_T is torsion and V_F is torsion free (see Definition 2.13). The assignment $V \to V_T$ gives rise to a functor, which we denote $H^0_{\mathfrak{m}}(\bullet)$. This is a left exact functor, and its right derived functors $H^i_{\mathfrak{m}}(\bullet)$ are called the *local cohomology functors*. Considering the profound impact that local cohomology has in more classical settings, it is natural for one to wonder whether the same is true for FI_G-modules.

Before one considers applying local cohomology modules in bounding various homological invariants, it is important that we develop some way of computing the modules in a systematic way. Using the above homological computation machinery and inductive method, we can provide such a computational tool.

Theorem E. *Let k be a commutative Noetherian ring, V be a finitely generated* FI_G-*module, and let $\mathcal{C}^\bullet V$ be the complex in Theorem 2.2. Then, for $i \geqslant -1$, there are isomorphisms*

$$H^i(\mathcal{C}^\bullet V) \cong H^{i+1}_{\mathfrak{m}}(V).$$

Consequently, $H^i_{\mathfrak{m}}(V)$ is a finitely generated, torsion $\mathscr{C}$-module. Moreover,

$$\begin{cases} \mathrm{td}(H^i_{\mathfrak{m}}(V)) = \mathrm{td}(V) & \text{if } i = 0 \\ \mathrm{td}(H^i_{\mathfrak{m}}(V)) \leq 2\,\mathrm{gd}(V) - 2i - 2 & \text{if } 1 \leq i \leq \mathrm{gd}(V). \end{cases}$$

Using this theorem, we can show that local cohomology groups are related to certain important homological invariants such as the depth, *Nagpal number* (see Definition 2.20), and *regularity* of a module (see Definition 2.13).

Theorem F. *Let k be a commutative Noetherian ring, and let V be a finitely generated* FI_G*-module. Then:*

1. *the depth of V is the smallest integer i such that $H^i_{\mathfrak{m}}(V) \neq 0$;*
2. *the Nagpal number of V, $N(V)$ satisfies the bounds*

$$N(V) = \max\{\mathrm{td}(H^i_{\mathfrak{m}}(V)) \mid i \geqslant 0\} + 1 \leq \max\{\mathrm{td}(V), 2\,\mathrm{gd}(V) - 2\} + 1.$$

3. *The regularity of V,* $\mathrm{reg}(V)$ *satisfies the bounds*

$$\mathrm{reg}(V) \leqslant \max\{\mathrm{td}(H^i_{\mathfrak{m}}(V)) + i\} \leqslant \max\{2\,\mathrm{gd}(V) - 1, \mathrm{td}(V)\}.$$

Motivated by the classical result for polynomial rings, we also state the following conjecture.

Conjecture 2.3. *Let k be a commutative Noetherian ring, and let V be a finitely generated* FI_G*-module. Then*

$$\mathrm{reg}(V) = \max\{\mathrm{td}(H^i_{\mathfrak{m}}(V)) + i\}.$$

The previous theorem tells us that the right hand side of our conjectured identity is an upper bound on the left hand side. It therefore only remains to show the opposite inequality. We note that the natural way one might try to accomplish this, namely through some induction argument on the projective dimension, cannot work in this case for reasons we have already stated. If this conjecture proves to be false, it would provide evidence that the Hilbert Syzygy theorem is actually vital to the statement being true in the classical case.

2.1.5 Organization

This paper is organized as follows. In Section 2 we describe some elementary definitions and results, which will be used throughout this paper. In Section 3 we study the shift functor and its adjoint functors, prove a few crucial technical tools, and use them to show that the depth and classical depth coincide. In Section 4 we use the inductive tools developed in Section 4 to prove a variety of homological structure theorems about $\sharp$-filtered modules. Finally, in the last section we develop a local cohomology theory of FI_G-modules and discuss its applications. We also present the above conjecture and its useful consequences.

2.2 Preliminaries

2.2.1 Elementary Definitions

For the remainder of this paper we fix a finite group G, and a commutative Noetherian ring k. The category FI_G is that whose objects are the finite sets $[n] := \{1, \ldots, n\}$ and whose morphisms are pairs, $(f, g) : [n] \to [m]$, of an injection $f : [n] \hookrightarrow [m]$ with a map of sets $g : [n] \to G$. Given two composable morphisms $(f, g), (f', g')$, composition in this category is defined by

$$(f, g) \circ (f', g') = (f \circ f', h)$$

where $h(x) = g'(x)g(f'(x))$. It follows immediately from this that for any $[n]$, the endomorphisms of $[n]$ form the group $\mathrm{End}_{\mathrm{FI}_G}([n]) = G \wr \mathfrak{S}_n$, where $\mathfrak{S}_n$ is the symmetric group on n letters. We will write G_n as a shorthand for this group.

Two important special cases of FI_G are those wherein $G = 1$ is the trivial group, and $G = \mathbb{Z}/2\mathbb{Z}$. In the first case, FI_G is equivalent to the category FI of finite sets and injective morphisms. In the second case we have $\mathrm{FI}_G = \mathrm{FI}_{BC}$, which was studied in [W].

Definition 2.4. *An* FI_G***-module*** *is a covariant functor* V *from* FI_G *to the category of* k*-modules. We write* V_n *for the module* $V([n])$*, and, given any map* $(f, g) : [n] \to [m]$*, we write* $(f, g)_*$ *for* $V(f, g)$*. We call the morphisms* $(f, g)_*$ *the* ***induced maps*** *of* V*. In the specific case where* $n < m$*, we call* $(f, g)_*$ *a* ***transition map*** *of* V*.*

The collection of FI_G*-modules, with natural transformations, form an abelian category which we denote* $\mathrm{FI}_G\text{-Mod}$.

The definition for FI_G-module given above was introduced by Church, Ellenberg, and Farb in [CEF]. This was followed by the work of Wilson [W], as well as that of Sam and Snowden [SS][SS2]. More recently, a new approach to the subject has been considered, which is more rooted in classical representation theory. This can be seen in the works of Gan, the first author, and Yu [GL] [L] [LY]

Definition 2.5. *Let* $k\,\mathrm{FI}_G$ *denote the* ***category algebra*** *whose additive group is given by*

$$k\,\mathrm{FI}_G := \bigoplus_{n\leq m} k[\mathrm{Hom}_{\mathrm{FI}_G}([n],[m])],$$

where $k[\mathrm{Hom}_{\mathrm{FI}_G}([n],[m])]$ *is the free* k*-module with a basis indexed by the set* $\mathrm{Hom}_{\mathrm{FI}_G}([n],[m])$*. Multiplication in* $k\,\mathrm{FI}_G$ *is defined on basis vectors* $(f,g):[n]\to[m], (f',g'):[r]\to[s]$ *by*

$$(f,g)\cdot(f',g') = \begin{cases} (f,g)\circ(f',g') & \text{if } n=s \\ 0 & \text{otherwise.} \end{cases}$$

Write $e_n \in \mathrm{End}_{\mathrm{FI}_G}([n])$ *for the identity on* $[n]$ *paired with the trivial map into* G*. Then we say that a module* V *over* $k\,\mathrm{FI}_G$ *is* ***graded*** *if* $V = \bigoplus_n e_n\cdot V$*. In this case we write* $V_n := e_n \cdot V$*.*

Remark 2.6. *Because all* $k\,\mathrm{FI}_G$*-modules considered in this paper are graded, we will simply refer to them as* $k\,\mathrm{FI}_G$*-modules.*

If V is a $k\,\mathrm{FI}_G$-module, then we obtain an FI_G-module by setting $V_n := e_n \cdot V$, and defining the induced maps in the obvious way. It is clear that this defines an equivalence between the category of FI_G-modules, and the category of (graded) $k\,\mathrm{FI}_G$-modules. We will use both definitions interchangeably during the course of this paper.

Remark 2.7. *In everything that follows, our results will not depend on the finite group G. Therefore, to clarify notation, we will write $\mathscr{C} := \mathrm{FI}_G$.*

Definition 2.8. *Let W be a kG_n-module for some $n \geqslant 0$. Then the* ***basic filtered*** *module over W, is the $\mathscr{C}$-module $M(W)$ defined by the assignments*

$$M(W)_m = k[\mathrm{Hom}_{\mathscr{C}}([n],[m])] \otimes_{kG_n} W.$$

The induced maps of $M(W)$ are defined by composition on the first coordinate. In the special case where $W = kG_n$, we write $M(n) := M(W)$, and refer to direct sums of these modules as being ***free****.*

Since kG_n can be viewed as a subalgebra of $k\mathscr{C}$, we see that M is nothing but the induction functor $k\mathscr{C} \otimes_{kG_n} \bullet$.

We say that a $\mathscr{C}$-module V is ***♯-filtered*** *if it admits a filtration*

$$0 = V^{(-1)} \subseteq V^{(0)} \subseteq V^{(1)} \subseteq \ldots \subseteq V^{(n)} = V$$

such that $V^{(i)}/V^{(i-1)} = M(W^{(i)})$, for some kG_i-module $W^{(i)}$, for each $i \geqslant 0$.

It was shown in [LY], as well as [R], that $\sharp$-filtered objects are of a fundamental importance to the study of homological properties of $\mathscr{C}$-modules. For instance, $\sharp$-filtered objects are precisely the acyclic objects with respect to certain natural right exact functors. These will be discussed in the coming sections (See Theorems 3.9 and 2.31). One of the interesting consequences of the results in this paper is that $\sharp$-filtered objects are also acyclic with respect to certain left exact functors.

The following proposition follows easily from the relevant definitions.

Proposition 2.3. *Let W be a kG_n-module. Then there is a natural adjunction,*

$$\mathrm{Hom}_{\mathscr{C}\text{-Mod}}(M(W), V) \cong \mathrm{Hom}_{kG_n\text{-Mod}}(W, V_n),$$

given by

$$\phi \mapsto \phi_n.tor$$

One observes that for a kG_n-module W, $M(W)$ is projective if and only if W is projective. In fact, it can be shown that all projective $\mathscr{C}$-modules are direct sums of basic filtered modules [R, Proposition 2.13]. Therefore, free $\mathscr{C}$-modules are always projective.

Definition 2.9. *Given a finitely generated kG_n-module W, we say that $M(W)$ is* ***finitely generated****. We say that a $\sharp$-filtered object is* ***finitely generated*** *if the cofactors in its defining filtration are finitely generated. Finally, we say that a $\mathscr{C}$-module is finitely generated if and only if it is a quotient of a finitely generated $\sharp$-filtered object.*

We denote the category of finitely generated $\mathscr{C}$-modules by $\mathscr{C}$-mod.

Remark 2.10. *One immediately remarks that the free module $M(n)$ is finitely generated for all n. In fact, it is an easily seen consequence of Nakayama's Lemma (Lemma 2.4) that a module V is finitely generated if and only if there is a finite set I, and a finite collection of non-negative integers $\{n_i, m_i\}_{i \in I}$ such that V is a quotient of $\bigoplus_{i \in I} M(n_i)^{m_i}$.*

Proposition 2.3 informs us that a map $M(n) \to V$ is equivalent to a choice of an element of V_n. Putting everything together, we can conclude that V is finitely generated if and only if there is some finite set of elements in $\sqcup_n V_n$ which are not contained in any proper submodule. We call such a set a ***generating set of elements*** *for V. While a definition of this sort may seem more natural, we use the above definition to be more in line with the philosophy of Theorem 3.9.*

One remarkable fact about the category $\mathscr{C}$-mod is that it is also abelian. Given any morphism of finitely generated $\mathscr{C}$-modules, it is clear that its image and cokernel must also be finitely generated. The significance of saying that $\mathscr{C}$-mod is abelian therefore comes from the fact that the kernel of this morphism must also be finitely generated. Put another way, the category $\mathscr{C}$-mod is Noetherian.

This fact was proven by Sam and Snowden in [SS2, Corollary 1.2.2] for general $\mathscr{C}$-modules, although it had been proven in certain specific cases earlier [W, Theorem 4.21][CEF, Theorem 1.3][CEFN, Theorem A][S, Theorem 2.3]. We often refer to the following result as the **Noetherian property**.

Theorem 2.11 ([SS2], Corollary 1.2.2)**.** *If V is a finitely generated $\mathscr{C}$-module over a Noetherian ring k, then all submodules of V are also finitely generated.*

Much of the remainder of the paper will be concerned with various homological properties of the category $\mathscr{C}$-mod.

2.3.1 The Homology Functors

For the remainder of this paper, we write $\mathfrak{m} \subseteq k\mathscr{C}$ to denote the ideal

$$\mathfrak{m} := \bigoplus_{n<m} k[\mathrm{Hom}_{\mathscr{C}}([n],[m])].$$

Definition 2.12. *For any $k\mathscr{C}$-module V, we use $H_0(V)$ to denote $k\mathscr{C}/\mathfrak{m} \otimes_{k\mathscr{C}} V$. In the language of $\mathscr{C}$-modules, $H_0 : \mathscr{C}\text{-mod} \to \mathscr{C}\text{-mod}$ is the functor defined by*

$$H_0(V)_n = V_n/V_{<n}$$

where $V_{<n}$ is the submodule of V_n spanned by the images of transition maps originating from V_m with $m < n$. We use H_i to denote the derived functors

$$H_i(V) := \mathrm{Tor}_i^{k\mathscr{C}}(k\mathscr{C}/\mathfrak{m}, V)$$

*We call the functors H_i the **homology functors**.*

Proposition 2.4 (Nakayama's Lemma)**.** *Let V be a $\mathscr{C}$-module, let $\{\widetilde{v}_i\} \subseteq \sqcup_n H_0(V)_n$ be a collection of elements which generate $H_0(V)$, and let v_i be a lift of $\widetilde{v}_i$ for each i. Then $\{v_i\}$ is a generating set for V.*

Proof

Let j be the least index such that $V_j \neq 0$. Then $V_j = H_0(V)_j$, and it is clear that every element of V_j is a linear combination of those $\widetilde{v}_i$ in $H_0(V)_j$. Next, let $n > j$, and let $v \in V_n$. Then the image of v in $H_0(V)_n$ can be expressed as some linear combination of elements of $\{\widetilde{v}_i\}$. By definition, this implies that v is a linear combination of elements of $\{v_i\}$, as well as images of elements from lower degrees. The result now follows by induction.

□

One immediate consequence of Nakayama's Lemma is that if V is finitely generated, then $H_0(V)$ is supported in finitely many degrees.

Definition 2.13. *Given a $\mathscr{C}$-module V, we define its* ***support*** *to be the smallest integer N for which $V_n = 0$ for all $n > N$, if such an integer exists. We define the* ***torsion degree*** *of V by*

$$\mathrm{td}(V) := \sup\{n \mid \mathrm{Hom}_{k\mathscr{C}}(kG_n, V) \neq 0\} \in \mathbb{N} \cup \{-\infty, \infty\},$$

where $\mathrm{td}(V) = -\infty$ if and only if V is ***torsion free****. Note that if V is finitely supported, then $\mathrm{td}(V)$ is precisely its support.*

*The i**-th homological degree** of V is defined to be the quantity,*

$$\mathrm{hd}_i(V) := \mathrm{td}(H_i(V)).$$

The zeroth homological degree is often referred to as the ***generating degree*** *of V and written $\mathrm{gd}(V)$.*

The ***regularity*** *of* V*, denoted* $\operatorname{reg}(V)$*, is the smallest integer* N *such that*

$$\operatorname{hd}_i(V) \leq N + i$$

for all $i \geqslant 1$*. We say that* $\operatorname{reg}(V) = \infty$ *if no such* N *exists, and we say* $\operatorname{reg}(V) = -\infty$ *if* V *is acyclic with respect to the homology functors.*

Remark 2.14. *It is a subtle but important point that the regularity of a module is defined by only bounding the higher homologies. From the perspective of classical commutative algebra this might seem a bit strange. We stress, however, that Theorem F is false if the definition of regularity is altered to include* $\operatorname{td}(H_0(V))$*. From the perspective of local cohomology this can be explained by the fact that* $\sharp$*-filtered objects are local cohomology acyclic, despite also being those objects which one might consider as substitutes for projectives (see Corollary 2.54).*

It is a remarkable fact that the regularity of any finitely generated $\mathscr{C}$-module is not ∞. This was proven in the case of FI-modules in characteristic 0 by Sam and Snowden [SS3, Corollary 6.3.5], general FI-modules by Church and Ellenberg [CE, Theorem A], and for $\mathscr{C}$-modules by the authors and Yu [R, Theorem A] [LY, Theorem 1.8]. Most of these papers also provide explicit bounds on the regularity of a $\mathscr{C}$-module in terms of its first two homological degrees. Later, we will provide new bounds on the regularity of a module in terms of its local cohomology modules.

As previously stated, $\sharp$-filtered modules are precisely those which are acyclic with respect to the homology functors.

Theorem 2.15 ([LY] Theorem 1.3, [R] Theorem B)**.** *Let V be a finitely generated $\mathscr{C}$-module. Then the following are equivalent:*

1. *V is $\sharp$-filtered;*
2. *V is homology acyclic;*
3. *$H_1(V) = 0$;*
4. *$H_i(V) = 0$ for some $i \geqslant 1$.*

Remark 2.16. *Note that this theorem implies the first three conditions of Theorem C.*

It follows as a consequence of this theorem that the only finitely generated modules which can have finite projective dimension are $\sharp$-filtered objects. The question of what $\sharp$-filtered modules can have finite projective dimension is considered in [LY, Theorem 1.5].

2.5 The Shift Functor and its Adjoints

2.5.1 The shift functor

Definition 2.17. *For any morphism $(f, g) : [n] \to [m]$ in $\mathscr{C}$, we define $(f_+, g_+) : [n+1] \to [m+1]$ to be the morphism*

$$f_+(x) = \begin{cases} f(x) & \text{if } x < n+1 \\ m+1 & \text{otherwise} \end{cases}, \quad g_+(x) = \begin{cases} g(x) & \text{if } x < n+1 \\ 1 & \text{otherwise.} \end{cases}$$

Let $\iota : \mathscr{C} \to \mathscr{C}$ be the functor defined by

$$\iota([n]) = [n+1], \ \iota(f, g) = (f_+, g_+).$$

*Then we define the **shift functor**, or the **restriction functor**, $\Sigma : \mathscr{C}\text{-Mod} \to \mathscr{C}\text{-Mod}$ by $\Sigma V := V \circ \iota$. We write Σ_b for the b-th iterate of Σ.*

*We note that the map ι induces a proper injective map of algebras $k\mathscr{C} \to k\mathscr{C}$, which we call the **self-embedding** of $k\mathscr{C}$.*

One of the most important properties of the shift functor is that it preserves $\sharp$-filtered objects. This was first observed by Nagpal in [N, Lemma 2.2].

Proposition 2.6. *Let W be a kG_n-module. Then there is an isomorphism of $\mathscr{C}$-modules*

$$\Sigma M(W) \cong M(\mathrm{Res}^{G_n}_{G_{n-1}} W) \oplus M(W).$$

In particular, if X is a $\sharp$-filtered $\mathscr{C}$-module, then ΣV is as well.

Proof

We will construct the isomorphism here, and direct the reader to [R, Proposition 2.21] for a proof that the map we construct is actually an isomorphism.

Proposition 2.3 implies that any map $M(\operatorname{Res}^{G_n}_{G_{n-1}} W) \oplus M(W) \to \Sigma M(W)$ is determined by two maps, $\phi_1 : \operatorname{Res}^{G_n}_{G_{n-1}} W \to \Sigma M(W)_{n-1}$ and $\phi_2 : W \to \Sigma M(W)_n$. It is clear from definition that $M(W)_{n-1} = \operatorname{Res}^{G_n}_{G_{n-1}} W$, and so we may choose ϕ_1 to be the identity. Once again applying the definition of the shift functor, we note that the only pure tensors $w \otimes (f, g) \in \Sigma M(W)_n$ which are not in the image of a transition map are those for which $f^{-1}(n+1) = \emptyset$. Looking at the collection of all such pure tensors, we find that they form a copy of W in $\Sigma M(W)_n$. We define

$$\phi_2(w) = w \otimes (f_n, \mathbf{1})$$

where $f_n : [n] \to [n+1]$ is the standard inclusion, and $\mathbf{1}$ is the trivial map into G.

$\square$

One remarkable fact about the shift functor, first observed by Nagpal in [N, Theorem A], is that all finitely generated $\mathscr{C}$-modules are "eventually" $\sharp$-filtered.

Theorem 2.18 ([N], Theorem A)**.** *Let V be a finitely generated $\mathscr{C}$-module. Then $\Sigma_b V$ is $\sharp$-filtered for $b \gg 0$.*

One major consequence of this theorem is the phenomenon of the stable range. If k is a field, and W is a finite dimensional kG_n-module, then one may easily compute

$$\dim_k(M(W)_m) = \binom{m}{n} \dim_k W.$$

Nagpal's theorem therefore implies the following.

Corollary 2.19. *Let V be a finitely generated $\mathscr{C}$-module over a field k. Then there is a polynomial $P_V \in \mathbb{Q}[x]$ such that for all $n \gg 0$, $\dim_k V_n = P_V(n)$.*

Definition 2.20. *Let V be a finitely generated $\mathscr{C}$-module. The smallest b for which $\Sigma_b V$ is $\sharp$-filtered is known as the* ***Nagpal number*** *of V, and is denoted $N(V)$.*

The paper [R, Theorem C] examines the Nagpal number from the perspective of a theory of depth. In this work, the second author provides bounds on $N(V)$ in terms of the generating degree and the first homological degree. Similar bounds were later found by the first author in [L2, Theorem 1.3] using different means. We will examine the notion of depth in the coming sections, and its connection to local cohomology.

2.6.1 The coinduction functor

Definition 2.21. *If V is a $\mathscr{C}$-module, then we define the* ***coinduction functor*** *$R : \mathscr{C}\text{-Mod} \to \mathscr{C}\text{-Mod}$*

$$R(V)_n := \mathrm{Hom}_{k\mathscr{C}}(\Sigma M(n), V).$$

If $(f,g) : [n] \to [m]$ is a morphism in $\mathscr{C}$, and $\phi : \Sigma M(n) \to V$ is a morphism of $\mathscr{C}$-modules, then we define $(f,g)_*\phi \in \mathrm{Hom}_{k\mathscr{C}}(\Sigma M(m), V)$ by

$$((f,g)_*\phi)_r(f',g') = \phi_r((f',g') \circ (f,g))$$

where $(f', g') : [m] \to [r+1]$*.*

Remark 2.22. *Because it will be useful to us later, we note that the coinduction functor is exact. Indeed, we have already seen that the shift functor preserves projective objects, and that* $M(n)$ *is projective for all* n*. This implies that* $\mathrm{Hom}_{k\mathscr{C}}(\Sigma M(n), \bullet)$ *is exact for all* n*, whence* R *is exact.*

Proposition 2.7 ([GL], Lemma 4.2)**.** *The coinduction functor is right adjoint to the shift functor.*

The coinduction functor was introduced by Gan and the first author in [GL]. Theorem 2.24 generalizes Theorem 1.3 of that paper. We will eventually use this more general result to prove that depth and classical depth agree for $\mathscr{C}$-modules over any Noetherian ring (see Theorem B).

Lemma 2.23. *Let* W *be a finitely generated* kG_n*-module. Then* $M(W)$ *is a summand of* $R(M(W))$*.*

Proof

We will construct a split injection $M(W) \to R(M(W))$. Proposition 2.3 tells us that such a map is equivalent to a map $W \to R(M(W))_n = \mathrm{Hom}_{k\mathscr{C}}(\Sigma M(n), M(W))$. Proposition 2.6 tells us that

$$\Sigma M(n) \cong M(n-1)^{|G| \cdot n} \oplus M(n),$$

and therefore

$$\mathrm{Hom}_{k\mathscr{C}}(\Sigma M(n), M(W)) \cong \mathrm{Hom}_{k\mathscr{C}}(M(n), M(W)) \cong W.$$

Being explicit, the isomorphism $\mathrm{Hom}_{k\mathscr{C}}(\Sigma M(n), M(W)) \cong W$ is given by

$$\phi \mapsto \phi_n(f_n, \mathbf{1})$$

where $f_n : [n] \to [n+1]$ is the standard inclusion, and $\mathbf{1}$ is the trivial map into G. We claim that the map $M(W) \to R(M(W))$ induced by the identity on W is a split injection.

To prove the claim, we will construct a splitting $\psi : R(M(W)) \to M(W)$. Indeed, for a morphism $\phi : \Sigma M(r) \to M(W)$ we set

$$\psi_r(\phi) = \phi_r(f_r, \mathbf{1}),$$

where $f_r : [r] \to [r+1]$ is the standard inclusion, and $\mathbf{1}$ is the trivial map into G. The fact that ψ defines a morphism of $\mathscr{C}$-modules is routinely checked. In fact, if $(f, g) : [r] \to [s]$ is any map in $\mathscr{C}$ then,

$$(f,g)_*(\psi_r(\phi)) = (f,g)_*\phi_r(f_r,\mathbf{1}) = \phi_s((f_+,g_+)\circ(f_r,\mathbf{1})) = \phi_s((f_s,\mathbf{1})\circ(f,g)) = ((f,g)_*\phi)_s(f_s,\mathbf{1}) = \psi_s((f,$$

By the discussion in the previous paragraph, it is clear that ψ is a splitting of our map.

□

This is all we need to prove the main theorem of this section.

Theorem 2.24. *Let W be a finitely generated kG_n-module. Then $R(M(W))$ is a direct sum of basic filtered modules. More specifically,*

$$R(M(W)) \cong M(W) \oplus M(\mathrm{Ind}_{G_n}^{G_{n+1}} W).$$

Proof

We may find some integer m such that there is an exact sequence

$$0 \to M(Z) \to M(n)^m \to M(W) \to 0$$

where Z is some kG_n-module. Applying the exact coinduction functor, and using [GL, Theorem 1.3], we obtain an exact sequence

$$0 \to R(M(Z)) \to M(n)^m \oplus M(n+1)^m \to R(M(W)) \to 0$$

It follows from this that $\mathrm{hd}_1(R(M(W))) \leq n+1$, and that $R(M(W))$ is generated in degrees n and $n+1$.

Lemma 2.23 tells us that the submodule of $R(M(W))$ generated by $R(M(W))_n \cong W$ is precisely $M(W)$. That is, there is a split exact sequence

$$0 \to M(W) \to R(M(W)) \to Q \to 0$$

for some module Q generated in degree exactly $n+1$. Applying the H_0 functor to this sequence, we find that

$$H_1(R(M(W))) \to H_1(Q) \to H_0(M(W))$$

from which it follows that $\mathrm{hd}_1(Q) \leq n+1$. The argument of [LY, Corollary 3.4] now implies that Q is actually a basic filtered module $Q \cong M(U)$ for some kG_{n+1}-module U.

Now we complete the proof by showing that $U \cong \mathrm{Ind}_{G_n}^{G_{n+1}} W$. By considering the value of $R(M(W))$ on the object $n+1$, we have

$$R((M(W))_{n+1} = \mathrm{Hom}_{k\mathscr{C}}(\Sigma(M(n+1)), M(W))$$
$$\cong \mathrm{Hom}_{k\mathscr{C}}(M(n+1) \oplus M(n)^{(n+1)|G|}, M(W))$$
$$\cong M(W)_{n+1} \oplus M(W)_n^{(n+1)|G|}$$

The endomorphism group G_{n+1} acts transitively on the $(n+1)|G|$ copies of $M(W)_n \cong W$. Therefore, as a left kG_{n+1}-module, $R(W)_{n+1}$ is a direct sum of W_{n+1} and $\mathrm{Ind}_{G_n}^{G_{n+1}} W_n$. Note that these two direct summands are actually isomorphic since

$$M(W)_{n+1} \cong k[\mathrm{Hom}_{\mathscr{C}}(n, n+1)] \otimes_{kG_n} W \cong kG_{n+1} \otimes_{kG_n} W,$$

where $k[\mathrm{Hom}_{\mathscr{C}}(n, n+1)]$ as a (kG_{n+1}, kG_n)-bimodule is isomorphic to kG_{n+1}. But we have

$$R(W)_{n+1} \cong W_{n+1} \oplus U.$$

Since we already know that $R(W)_{n+1}$ is a direct sum of two copies of the induced module, it forces U to be isomorphic to the induced module.

□

2.7.1 The induction functor and the proof of Theorem A

We now spend some time considering the left adjoint of the shift functor. For our purposes, its most important property will be related to the Eckmann-Shapiro Lemma (Proposition 2.9).

Unlike the coinduction functor, we will see that the shift functor's left adjoint cannot be easily expressed in the language of $\mathscr{C}$-modules. We will therefore present this functor entirely in the language of $k\mathscr{C}$-modules.

Definition 2.25. *Let V be a $k\mathscr{C}$-module. Set $\mathscr{C}_+$ to be the full subcategory of $\mathscr{C}$ whose objects are the sets $[n]$ with $n > 0$, and let $k\mathscr{C}_+$ be the corresponding subalgebra of $k\mathscr{C}$. Then we define the **Induction functor** L as the $k\mathscr{C}$-module,*

$$L(V) := k\mathscr{C}_+ \otimes_{k\mathscr{C}} V$$

where here we consider $k\mathscr{C}_+$ as a $k\mathscr{C}$-bimodule via normal multiplication on the left, and via the self-embedding on the right.

Proposition 2.8. *The induction functor is left adjoint to the shift functor.*

Proof

Let $k\mathscr{C}_+$ be as in the definition of the induction functor. We will prove that there is a natural isomorphism of functors

$$\Sigma V \cong \mathrm{Hom}_{k\mathscr{C}}(k\mathscr{C}_+, V).$$

Keeping this in mind, the proposition is just the usual Tensor-Hom adjunction.

For this proof only, set $A := \mathrm{Hom}_{k\mathscr{C}}(k\mathscr{C}_+, V)$. Then A is a $k\mathscr{C}$-module. Recall that we use e_n to denote the morphism of $\mathscr{C}$ defined by the pair of the identity on $[n]$ and the trivial map into G. For any $\phi \in A_n$, define

$$\psi_V(\phi) = \phi(e_{n+1}) \in V_{n+1}.$$

We claim that $\psi : A \to \Sigma V$ is a morphism of $k\mathscr{C}$-modules. Indeed, if $(f,g) : [n] \to [m]$ is any morphism in $\mathscr{C}$, and $\phi \in A_n$, then

$$\psi_V((f,g)_*\phi) = ((f,g)_*\phi)(e_{m+1}) = \phi(e_{m+1} \circ (f_+, g_+)) = (f_+, g_+)_*\phi(e_{n+1}) = (f,g)_*(\psi_V(\phi)).$$

The fact that the collection of ψ_V, with V varying, define a natural transformation of functors is easily checked.

□

Remark 2.26. *The induction and coinduction functors of $\mathscr{C}$-modules are not isomorphic. Indeed, we have already seen that $R(M(n)) \cong M(n) \oplus M(n+1)$, while Theorem A tells us that $L(M(n)) \cong M(n+1)$.*

We have already seen that the shift functor preserves projective $\mathscr{C}$-modules (Proposition 2.6). For the purposes of the Eckmann-Shapiro lemma, however, we will need to know whether it preserves right projective modules. This is indeed the case.

Lemma 2.27. *Let P be a projective right $k\mathscr{C}$-module. Then ΣP is also a projective right $k\mathscr{C}$-module.*

Proof

It suffices to show that the projective modules of the form $e_i \cdot k\mathscr{C}$ remain projective once shifted. In fact, it is the case that

$$\Sigma(e_n \cdot k\mathscr{C}) \cong (e_{n-1} \cdot k\mathscr{C})^n.$$

This is easily checked by explicitly computing the action.

□

Putting everything in the previous two sections together, we can finally prove Theorem A.

The proof of Theorem A

We have already seen that Σ and R are exact. The fact that L is also exact follows immediately from its definition. Indeed, it is a simple computation to show that

$$L(V)_{n+1} = \mathrm{Ind}_{G_n}^{G_{n+1}} V_n$$

for any $n \geq 0$. Because kernels and cokernels are computed pointwise, exactness of L follows from the exactness of the classical induction functor.

Proposition 2.6, and Theorem 2.24 imply the first two claims of the second statement. Let W be a kG_n-module. Then by Proposition 2.8, if V is a $\mathscr{C}$-module

$$\mathrm{Hom}_{k\mathscr{C}}(L(M(W)), V) \cong \mathrm{Hom}_{k\mathscr{C}}(M(W), \Sigma V) \cong \mathrm{Hom}_{kG_n}(W, \mathrm{Res}_{G_n}^{G_{n+1}} V_{n+1}) \cong \mathrm{Hom}_{kG_{n+1}}(\mathrm{Ind}_{G_n}^{G_{n+1}} W, V_{n+1})$$

Proposition 2.3 implies that $L(M(W)) \cong M(\mathrm{Ind}_{G_n}^{G_{n+1}} W)$, as desired.

The final part of the theorem is just the Eckmann-Shapiro lemma from representation theory, and is proven in the same way in this context.

□

For ease of reference later, we take a moment to state the Eckmann-Shapiro lemma.

Proposition 2.9 (The Eckmann-Shapiro lemma). *Let V and V' be finitely generated $\mathscr{C}$-modules. Then there are isomorphisms for all $i \geqslant 0$*

$$\operatorname{Ext}^i_{k\mathscr{C}}(L(V), V') \cong \operatorname{Ext}^i_{k\mathscr{C}}(V, \Sigma V'); \tag{2.1}$$

$$\operatorname{Ext}^i_{k\mathscr{C}}(\Sigma V, V') \cong \operatorname{Ext}^i_{k\mathscr{C}}(V, R(V')). \tag{2.2}$$

2.10 Homological computations

2.10.1 Depth and the proof of Theorem B

In this section we will review the concept of depth first introduced in [R]. Following this, we will consider an invariant we call classical depth, and prove that it is equivalent to the depth of [R].

Definition 2.28. *Let V be a $\mathscr{C}$-module, and let $\tau_V : V \to \Sigma V$ be the map induced by the pairs $(f_n, \mathbf{1})$, where $f_n : [n] \to [n+1]$ is the standard inclusion and the trivial map into G. We define the* ***derivative functor*** *to be the cokernel*

$$DV := \operatorname{coker}(\tau_V).$$

We write D^a for the a-th iterate of D.

Note that the derivative functor is right exact, and so we write $H_i^{D^a}$ to denote the i-th derived functor of D^a. By [CE, Lemma 3.6], V is torsion free if and only if τ_V is *injective.*

Remark 2.29. *It follows from the proof of Proposition 2.6 that* $\tau_{M(W)} : M(W) \to \Sigma M(W)$ *is a split injection. Moreover, the compliment of* $M(W)$ *in* $\Sigma M(W)$ *is generated in strictly smaller degrees. Therefore, if* V *is any finitely generated* $\mathscr{C}$*-module with* $\mathrm{gd}(V) = d$*, then* $\mathrm{gd}(DV) \leq d - 1$*. In fact, it was shown by the first author and Yu that* $\mathrm{gd}(DV) = d - 1$*, so long as* $DV \neq 0$ *[LY, Proposition 2.4]. Using the same proofs, we can actually say something a bit more general. For any positive integer* b*, Let* τ_b *be the map*

$$\tau_b : V \to \Sigma_b V$$

induced by the pair $(f_n^b : \mathbf{1})$*, where* $f_n^b : [n] \to [n+b]$ *is the standard inclusion, and* $\mathbf{1}$ *is the trivial map into* G*. Then* $\mathrm{gd}(\mathrm{coker}(\tau_b)) < \mathrm{gd}(V)$*.*

The derivative functor was introduced as a means of bounding the regularity of FI-modules by Church and Ellenberg in [CE, Theorem A]. In that paper, Church and Ellenberg use the derivative as a convenient homological tool for approximating the homological degrees. Following this, the second author discovered that the derivative functor held much higher homological significance than was previously observed. Namely, it was shown that the derivative was critical in developing a theory of depth for $\mathscr{C}$-modules [R]. At the same time as this, the first author and Yu also used the derivative in proving many homological facts about $\mathscr{C}$-modules [LY].

Definition 2.30. *Let V be a finitely generated $\mathscr{C}$-module. We define the **depth** of V to be the quantity,*

$$\operatorname{depth}(V) := \inf\{a \mid H_1^{D^{a+1}}(V) \neq 0\} \in \mathbb{N} \cup \{\infty\},$$

where we use the convention that the infimum of the empty set is ∞.

Theorem 2.31 ([R], Theorem 4.4)**.** *Let V be a finitely generated $\mathscr{C}$-module. Then,*

1. $\operatorname{depth}(V) = 0$ *if and only if V is not torsion free;*

2. $\operatorname{depth}(V) = \infty$ *if and only if V is $\sharp$-filtered;*

3. $\operatorname{depth}(V) = a > 0$ *is finite if and only if there is an exact sequence*

$$0 \to V \to X_{a-1} \to \ldots \to X_0 \to V' \to 0$$

where X_i is $\sharp$-filtered for each i, $\operatorname{gd}(X_i) > \operatorname{gd}(X_{i-1})$, and V' is not torsion free.

While the above theorem justifies the use of the terminology depth, it might still be unclear at this point where this definition of depth comes from. In the remainder of this section, we will use the classical nature of the language of $k\mathscr{C}$-modules to provide an alternative definition of depth, which we prove is equivalent to the above.

Definition 2.32. *Let V be a finitely generated $k\mathscr{C}$-module, and recall that $\mathfrak{m} \subseteq k\mathscr{C}$ is the ideal generated by all non-permutations. Then we define the **classical depth** of V to be the quantity*

$$\operatorname{depth}^{\text{class}}(V) := \inf\{i \mid \operatorname{Ext}^i_{\mathscr{C}\text{-Mod}}(k\mathscr{C}/\mathfrak{m}, V) \neq 0\}$$

where we use the convention that the infimum of the empty set is ∞.

Remark 2.33. *As a slight technical point, one should note that the category of finitely generated $\mathscr{C}$-modules may not have sufficiently many injectives if k is not a field of characteristic 0 (see Theorem 2.46). This is of course not problematic here, as we may compute these* Ext*-groups through a projective resolution of $k\mathscr{C}/\mathfrak{m}$. Later, when we discuss cohomology, this will become more of an issue.*

In the paper [R, Theorem 4.17], the second author shows that $\text{depth}^{\text{class}}(V) = \text{depth}(V)$ whenever k is a field of characteristic 0. This proof, however, is highly dependent on the properties of $\mathscr{C}$-modules over a field of characteristic 0. The proof we provide in this section will work over any Noetherian ring.

We will proceed with a collection of reductions, which end with us only needing to show a particular collection of Ext-groups vanish. We begin with the following lemma.

Lemma 2.34. *Depth and classical depth are equivalent if and only if $\sharp$-filtered objects have infinite classical depth.*

Proof

This follows from the techniques used in the proof of [R, Theorem 4.17].

□

Our second major reduction is to note that

$$\mathrm{Ext}^i_{k\mathscr{C}}(k\mathscr{C}/\mathfrak{m}, V) \cong \prod_n \mathrm{Ext}^i_{k\mathscr{C}}(kG_n, V).$$

It therefore suffices to show that $\mathrm{Ext}^i_{k\mathscr{C}}(kG_s, V) = 0$ for all s, whenever V is $\sharp$-filtered. Moreover, a simple homological argument shows that it actually suffices to assume that $V = M(W)$ for some kG_n-module W.

Lemma 2.35. *The modules* $\mathrm{Ext}^i_{k\mathscr{C}}(kG_s, V)$ *are zero for all* $s, i \geqslant 0$ *and all basic filtered modules* V *if and only if the modules* $\mathrm{Ext}^i_{k\mathscr{C}}(kG_0, V)$ *are zero for all* $i \geqslant 0$ *and all basic filtered modules* V*.*

Proof

The forward direction is clear. Assume that $\mathrm{Ext}^i_{k\mathscr{C}}(kG_0, V) = 0$ for all $i \geqslant 0$ and all basic filtered modules V.

A straight forward computation verifies that $L(kG_s) \cong kG_{s+1}$ for all $s \geq 0$. Therefore, if V is any basic filtered module, and $i, s \geqslant 0$

$$\mathrm{Ext}^i_{k\mathscr{C}}(kG_s, V) \cong \mathrm{Ext}^i_{k\mathscr{C}}(L_s(kG_0), V) \cong \mathrm{Ext}^i_{k\mathscr{C}}(kG_0, \Sigma_s V) = 0.$$

Note that the second isomorphism follows from the Eckmann-Shapiro lemma and our assumption as well as Proposition 2.6.

□

We have now reduced the problem enough to be solvable.

The proof of Theorem B

By the previous lemmas it suffices to prove that $\mathrm{Ext}^i_{k\mathscr{C}}(kG_0, V) = 0$, where $V = M(W)$ for some kG_n-module W.

We first note for any $\mathscr{C}$-module V', elements of $\mathrm{Hom}_{k\mathscr{C}}(kG_0, V')$ can be thought of as elements of V'_0 which are in the kernel of all transition maps out of V'_0. Since basic filtered modules are torsion free, it follows that our desired result holds for $i = 0$. Next, consider the exact sequence

$$0 \to J_0 \to M(0) \to kG_0 \to 0.$$

If V is generated in degree $\geqslant 2$, then it is clear that $\mathrm{Hom}_{k\mathscr{C}}(J_0, V) = 0$. If V is generated in degree 1, then an element of $\mathrm{Hom}_{k\mathscr{C}}(J_0, V)$ is a choice of an element of V_1 whose image under all transition maps is invariant with respect to the G_m-action. It is easily seen that no such elements exist in our case, and so once again $\mathrm{Hom}_{k\mathscr{C}}(J_0, V) = 0$. Finally, if V is generated in degree 0, then $\mathrm{Hom}_{k\mathscr{C}}(M(0), V) \cong \mathrm{Hom}_{k\mathscr{C}}(J_0, V)$. In all cases, we conclude $\mathrm{Ext}^1_{k\mathscr{C}}(kG_0, V) = 0$.

Using Theorem 2.24, we may write $RV \cong V \oplus V'$, where $V' = M(U)$ for some kG_{n+1}-module U. In particular, it must be the case that V' is generated in degree $\geqslant 1$, and therefore $\mathrm{Hom}_{k\mathscr{C}}(J_0, W) = 0$ by the previous paragraph's discussion. Applying the functor $\mathrm{Hom}_{k\mathscr{C}}(J_0, \bullet)$ one gets

$$\mathrm{Ext}^1_{k\mathscr{C}}(J_0, V) \subseteq \mathrm{Ext}^1_{k\mathscr{C}}(J_0, R(V)).$$

But the Eckmann-Shapiro lemma implies

$$\mathrm{Ext}^1_{k\mathscr{C}}(J_0, R(V)) \cong \mathrm{Ext}^1_{k\mathscr{C}}(\Sigma J_0, V) \cong \mathrm{Ext}^1_{k\mathscr{C}}(M(0), V) = 0.$$

This forces

$$0 = \mathrm{Ext}^1_{k\mathscr{C}}(J_0, V) \cong \mathrm{Ext}^2_{k\mathscr{C}}(kG_0, V).$$

Now suppose that the conclusion holds for some $i \geqslant 2$, and consider $\mathrm{Ext}^{i+1}_{k\mathscr{C}}(kG_0, V)$. Then

$$\mathrm{Ext}^{i+1}_{k\mathscr{C}}(kG_0, V) \cong \mathrm{Ext}^i_{k\mathscr{C}}(J_0, V).$$

Applying $\mathrm{Hom}_{k\mathscr{C}}(J_0, \bullet)$ to the exact sequence

$$0 \to V \to RV \to V' \to 0$$

one gets

$$\mathrm{Ext}^{i-1}_{k\mathscr{C}}(J_0, V') \to \mathrm{Ext}^i_{k\mathscr{C}}(J_0, V) \to \mathrm{Ext}^i_{k\mathscr{C}}(J_0, RV).$$

The first term is 0 by induction hypothesis, and the last term is 0 as well by using the Eckmann-Shapiro lemma, as we did previously. This proves the claim.

□

As an interesting consequence of our theorem, we obtain the following vanishing theorem for Ext-modules.

Proposition 2.11. *Let V be a finitely generated $\mathscr{C}$-module. Then for all $i \geqslant 0$, and for all $n \geqslant N(V)$,*

$$\mathrm{Ext}^i_{k\mathscr{C}}(kG_n, V) = 0.$$

Proof

Take N to be the Nagpal number of V. Then for all $n \geqslant N$,

$$\mathrm{Ext}^i_{k\mathscr{C}}(kG_n, V) \cong \mathrm{Ext}^i_{k\mathscr{C}}(L_n(kG_0), V) \cong \mathrm{Ext}^i_{k\mathscr{C}}(kG_0, \Sigma_n V) = 0$$

since $\Sigma_n V$ is $\sharp$-filtered. This completes the proof.

$\square$

2.11.1 Homological Orthogonal Relations

In this section, we more closely examine the relationship between torsion and $\sharp$-filtered modules. In particular, over the course of this section we will be proving Theorems C and D.

Definition 2.36. *Let V be a $\mathscr{C}$-module. We say that an element $v \in V_n$ is* ***torsion*** *if it is in the kernel of some transition map out of n. In the language of $k\mathscr{C}$-modules, we say that an element $v \in V_n$ is torsion, if there is some map $(f, g) : [n] \to [m]$ such that $(f, g) \cdot v = 0$. We say that V is* ***torsion*** *if its every element is torsion.*

If V is a $\mathscr{C}$-module, then there is always an exact sequence

$$0 \to V_T \to V \to V_F \to 0$$

where V_T is a torsion module called the ***torsion part*** *of V, and V_F is a torsion free module called the* ***torsion free part*** *of V.*

While the following lemma might seem tautological, and indeed we will find it not difficult to prove, it is not immediate from the definitions thus far provided. Recall that we say that V is torsion free whenever τ_V is injective.

Lemma 2.37. *A module V is torsion free if and only if it contains no torsion elements.*

Proof

The key observation of this proof is that an element $v \in V_n$ is in the kernel of some $(f,g)_* : V_n \to V_m$ if and only if it is in the kernel of every transition map from n to m. Indeed, this follows from the fact that the left action of G_m on $\mathrm{Hom}_{\mathscr{C}}([n],[m])$ is transitive. Moreover, we can produce, from v, a torsion element in V_{m-1} by viewing (f,g) as a composition of a map from $[n]$ to $[m-1]$ and a map from $[m-1]$ to $[m]$. It follows immediately from this that V has a torsion element if and only if it has an element which is in the kernel of a transition map of the form $(f_n, \mathbf{1})$, where $f_n : [n] \to [n+1]$ is the standard inclusion and $\mathbf{1}$ is the trivial map into G. We recall that the map $\tau : V \to \Sigma V$ is induced by these transition maps, and therefore the proposition follows.

□

Lemma 2.38. *Let V be a finitely generated torsion module. Then $\mathrm{td}(V) < \infty$.*

Proof

By assumption, there is a finite set $\{v_i\} \subseteq \sqcup V_n$ which generates V. By the observations in the proof of the previous lemma, for each i we may find some n_i such that v_i is in the kernel of every transition map into V_n with $n \geqslant n_i$. Because there are finitely many generators, we may find some N such that all generators are in the kernel of transition maps into V_n whenever $n \geqslant N$. It follows that $\mathrm{td}(V) \leq N$.

□

We begin with the following theorem, which expands upon some of the work in previous sections.

Theorem 2.39. *Let V be a finitely generated $\mathscr{C}$-module. Then*

1. *T is a torsion module if and only if $\mathrm{Ext}^i_{k\mathscr{C}}(T, V) = 0$ for all basic filtered modules V and all $i \geqslant 0$.*

2. *V is a $\sharp$-filtered module if and only if $\mathrm{Ext}^i_{k\mathscr{C}}(kG_s, V) = 0$ for all $s, i \geqslant 0$*

3. *V is an injective module if and only if $\mathrm{Ext}^1_{k\mathscr{C}}(W, V) = 0$ whenever W is a basic filtered module or W is a finitely generated torsion module.*

Proof

Let T be a finitely generated torsion module. Note that $\mathrm{td}(T) < \infty$ by Lemma 2.38. Let $N = \mathrm{td}(T)$, and let T' be the submodule of T generated by T_N. Then we have an exact sequence

$$0 \to T' \to T \to T'' \to 0$$

where $\mathrm{td}(T'') < \mathrm{td}(T)$. Applying the functor $\mathrm{Hom}_{k\mathscr{C}}(\bullet, V)$, we find that the proposition follows if we can show $\mathrm{Ext}^i_{k\mathscr{C}}(T', V) = \mathrm{Ext}^i_{k\mathscr{C}}(T'', V) = 0$. By induction on the torsion degree, we only need to show that $\mathrm{Ext}^i_{k\mathscr{C}}(T', V) = 0$.

Because T was finitely generated, it follows that T' is finitely generated by the Noetherian property. Therefore there is an exact sequence

$$0 \to \Omega \to kG_N^m \to T' \to 0$$

for some module Ω which is also supported exclusively in degree N. Applying the functor $\mathrm{Hom}_{k\mathscr{C}}(\bullet, V)$, and using the facts that V is torsion free and Theorem B, we find that

$$0 = \mathrm{Hom}_{k\mathscr{C}}(\Omega, V) \to \mathrm{Ext}^1_{k\mathscr{C}}(T', V) \to 0 = \mathrm{Ext}^1_{k\mathscr{C}}(kG^m_N, V).$$

Therefore $\mathrm{Ext}^1_{k\mathscr{C}}(T', V) = 0$. Looking further along the above exact sequence, we also conclude that $\mathrm{Ext}^{i+1}_{k\mathscr{C}}(T', V) \cong \mathrm{Ext}^i_{k\mathscr{C}}(\Omega, V)$ for all $i \geqslant 1$. Using the fact that T' was arbitrary, and that $\mathrm{Ext}^1_{k\mathscr{C}}(T', V) = 0$, the only if direction of the first statement now follows by induction.

Now we prove the if direction of the first statement; that is, $\mathrm{Ext}^i_{k\mathscr{C}}(T, V) = 0$ for all $\sharp$-filtered modules V implies that T is torsion. But this is clear. Indeed, if T is not torsion, its torsionless part $T_F \neq 0$. Then $\Sigma_N T_F$ is a nonzero $\sharp$-filtered module for $N \gg 0$, and we get a nonzero map $T \to T_F \to \Sigma_N T_F$ where the first component is surjective and the second component is injective.

The second statement is simply Theorem B, along with Theorem 2.31.

One direction of the third is trivial. To prove the other direction, Theorem 2.2 implies that for every finitely generated $k\mathscr{C}$-module W there is a finite complex of $\sharp$-filtered modules

$$0 \to W \to F^0 \to F^1 \to \ldots \to F^n \to 0.$$

Theorem 2.2 also tells us that all homologies in this complex are finitely generated torsion modules. This complex gives rise to a few short exact sequences

$$0 \to W_T \to W \to W_F \to 0,$$
$$0 \to W_F \to F^0 \to W^{(1)} \to 0,$$
$$\ldots,$$
$$0 \to W_F^{(n)} \to F^n \to W^{(n+1)} \to 0,$$

where $W^{(n+1)}$ is torsion. By assumption, one recursively deduces that $\mathrm{Ext}^i_{k\mathscr{C}}(W^{(s)}, V) = 0$ for all $i \geqslant 0$ and $0 \leqslant s \leqslant n+1$, where $W^{(0)} = W$. Therefore, V is injective since W is arbitrarily chosen.

□

If we only consider extension groups with positive degrees, we have:

Theorem 2.40. *Let V be a finitely generated $\mathscr{C}$-module. Then*

1. $\mathrm{Ext}^i_{k\mathscr{C}}(V, F) = 0$ *for all basic filtered modules F and all $i \geqslant 1$ if and only if $\Sigma_N V$ is a projective module for $N \gg 0$.*

2. $\mathrm{Ext}^i_{k\mathscr{C}}(T, V) = 0$ *for $i \geqslant 1$ and all finitely generated torsion modules T if and only if V is a direct sum of an injective torsion module and a $\sharp$-filtered module.*

Proof

If $\Sigma_N V$ is a projective module, then one has

$$0 = \mathrm{Ext}^i_{k\mathscr{C}}(\Sigma_N V, F) \cong \mathrm{Ext}^i_{k\mathscr{C}}(V, R_N(F)).$$

But F is isomorphic to a direct summand of $R_N(F)$ by Theorem 2.24. Consequently, $\mathrm{Ext}^i_{k\mathscr{C}}(V, F) = 0$ for all $i \geqslant 1$ and all $\sharp$-filtered modules F.

Conversely, for $N \gg 0$, we know that $\tilde{V} = \Sigma_N V$ is a $\sharp$-filtered module. We show that if it is not projective, then there exists a $\sharp$-filtered module F such that $\mathrm{Ext}^1_{k\mathscr{C}}(V, F) \neq 0$. Suppose that $\mathrm{gd}(\tilde{V}) = n \geqslant 0$. Since $\tilde{V}$ is filtered, there is a short exact sequence

$$0 \to V' \to \tilde{V} \to V'' \to 0$$

such that V' is the submodule generated by $\bigoplus_{i<n} V_i$ and V'' is a basic filtered module generated in n. Without loss of generality we can assume that V'' is not projective since otherwise we can replace $\tilde{V}$ by V' and repeat the above process.

Now consider V'' which is isomorphic to $M(V''_n)$. Since V'' is not projective, V''_n as a kG_n-module cannot be projective. Therefore, we can find a finitely generated kG_n-module W_n such that $\mathrm{Ext}^1_{kG_n}(V''_n, W_n) \neq 0$. That is, there is a non-split exact sequence

$$0 \to W_n \to U_n \to V''_n \to 0.$$

Applying the exact functor $M(\bullet)$ we get a non-split exact sequence

$$0 \to M(W_n) \to M(U_n) \to M(V''_n) \to 0.$$

Consequently, $\mathrm{Ext}^1_{k\mathscr{C}}(M(V''_n), F) \neq 0$, where $F = M(W_n)$. Moreover, applying the functor $\mathrm{Hom}_{k\mathscr{C}}(\bullet, F)$ to the exact sequence

$$0 \to V' \to \tilde{V} \to V'' \to 0$$

we deduce that $\mathrm{Ext}^1_{k\mathscr{C}}(\tilde{V}, F) \neq 0$ since $\mathrm{Ext}^1_{k\mathscr{C}}(V'', F) \neq 0$ and $\mathrm{Hom}_{k\mathscr{C}}(V', F) = 0$ because $\mathrm{gd}(V') < \mathrm{gd}(F) = n$. But $\tilde{V} = \Sigma_N V$. Using the adjunction, one deduces that

$\mathrm{Ext}^1_{k\mathscr{C}}(V, R_N(F)) \neq 0$. However, $R_N(F)$ is still filtered. This contradicts the given condition. In this way we prove the first statement.

Now we turn to the second statement. The if direction is clear. For the other direction, we consider the short exact sequence

$$0 \to V_T \to V \to V_F \to 0.$$

Applying $\mathrm{Hom}_{k\mathscr{C}}(T, \bullet)$ and noting that $\mathrm{Hom}_{k\mathscr{C}}(T, V_F) = 0$, by the given assumption, we deduce that $\mathrm{Ext}^1_{k\mathscr{C}}(T, V_T) = 0$. But the finitely generated torsion module V_T is injective if and only if it viewed as a representation of the finite full subcategory with objects $n \leqslant \mathrm{td}(V_T)$ is still injective (see [GL, Section 2,4]), if and only if $\mathrm{Ext}^1_{k\mathscr{C}}(T, V_T) = 0$ for all torsion modules. From this observation we conclude that V_T is injective, so $V \cong V_T \oplus V_F$. Moreover, from the long exact sequence we conclude that $\mathrm{Ext}^1_{k\mathscr{C}}(T, V_F) = 0$ for all $i \geqslant 0$ and all torsion modules. By the second statement of the previous theorem, V_F is a $\sharp$-filtered module.

□

These two results give another classification of finitely generated injective modules when k is a field of characteristic 0.

Corollary 2.41. *If k is a field of characteristic 0, then every finitely generated projective module is injective as well, and every finitely generated injective module is a direct sum of a finite dimensional injective module and a projective module.*

Proof

Let P be a finitely generated projective module. We know that $\mathrm{Ext}^i_{k\mathscr{C}}(kG_s, P) = 0$ for all $i, s \geqslant 0$ since projective modules and $\sharp$-filtered modules coincide in this case. Indeed, this follows from the fact that $M(W)$ is projective if and only if W is projective. For the same reason, clearly one has $\mathrm{Ext}^i_{k\mathscr{C}}(F, P) = 0$ for all finitely generated $\sharp$-filtered modules F. By (3) of Theorem 2.39, P is injective as well.

If I is an injective module, then by the second statement of the previous theorem, I is a direct sum of a finite dimensional injective module and a $\sharp$-filtered module, which in this case is projective.

□

This fact was proven for FI-modules by Sam and Snowden in [SS3, Theorem 4.2.5]. The result was generalized to $\mathscr{C}$-modules by the Gan and the first author in [GL, Theorem 1.7].

Remark 2.42. *Note that the results in this section seem to indicate that $\mathscr{C}$-modules are a natural candidate for a tilting theory. We do not pursue this further here, although we note that it is a possible area for further research.*

2.11.2 Injective objects in the category $\mathscr{C}$-mod

In this section we substantiate the claim made throughout the paper that the category $\mathscr{C}$-mod over a Noetherian ring will often times not have sufficiently many injective objects. In fact, we will prove the stronger statement that in many cases the category does not have any torsion free injective objects. In [GL, Theorem 1.7], as well as [SS3, Theorem 4.3.1], It is shown that the category $\mathscr{C}$-mod does have sufficiently many injective objects whenever k is a field of characteristic 0. The main theorem of this section therefore represents a drastic departure from the cases which were previously studied.

We begin with some homological lemmas, which follow from the work of the previous sections.

Lemma 2.43. *Let V be a finitely generated, torsion free injective $\mathscr{C}$-module. Then V is $\sharp$-filtered.*

Proof

This follows at once from Theorem D.

□

Lemma 2.44. *The shift functor Σ preserves injective objects. The derivative functor D preserves torsion free injective objects. That is to say, if V is a torsion free injective object, then so is DV.*

Proof

We have already seen that Σ is right adjoint to the exact induction functor. This implies that it must preserve injective objects. On the other hand, if V is a torsion free injective object, then there is a split exact sequence

$$0 \to V \to \Sigma V \to DV \to 0.$$

The fact that ΣV is injective implies that DV must be as well. Moreover, DV must be torsion free, as the same is true of ΣV.

□

Lemma 2.45. *If V is an injective $\mathscr{C}$-module, then V_n is an injective kG_n-module for all $n \geq 0$.*

Proof

Let $U \subseteq W$ be kG_n-modules, and assume we have a map $\phi : U \to V_n$. Then Proposition 2.3 implies that there is a map $\overline{\phi} : M(U) \to V$ such that $\overline{\phi}_n = \phi$. Because V is injective, this will lift to a map $\psi : M(W) \to V$. Looking at this map in degree n, we obtain the desired lift of ϕ.

□

This is all we need to prove the main theorem of this section.

Theorem 2.46. *Let k be a Noetherian ring, and assume that either k is a field of characteristic $p > 0$, or that there are no non-trivial finitely generated injective k-modules. Then the category $\mathscr{C}$-mod does not admit any torsion free injective modules over k. In particular, under either of the above hypotheses, the category $\mathscr{C}$-mod does not have sufficiently many injective objects.*

Proof

Let V be a finitely generated, torsion free injective $\mathscr{C}$-module. Then V is $\sharp$-filtered by Lemma 2.43. Moreover, [LY, Proposition 2.4] tells us that $\mathrm{gd}(DV) = \mathrm{gd}(V) - 1$, so long as $\mathrm{gd}(V) \neq 0$. In particular, Lemma 2.44 implies that if we apply the derivative functor enough times to V, we will be left with a torsion free, injective module, which is generated in degree 0. All such modules take the form $M(W)$, where W is a finitely generated module over $kG_0 \cong k$. Note that Lemma 2.45 implies that W is injective as a k-module.

Now assume that k is a field of characteristic $p > 0$. Then $M(0)$ will be a summand of $M(W)$, and therefore will be injective as well. This is a contradiction of Lemma 2.45, as $M(0)_n$ is the trivial module for all n, and the trivial module is not injective for $n \geqslant p$.

If, on the other hand, k satisfies the second of our two conditions, then we reach a contradiction with the fact that W must be a finitely generated injective module.

□

Remark 2.47. *The above theorem was proven by Changchang Xi and the first author during the latter's visit to Capital Normal University in December of 2015. It was independently proven by the second author a short time later. The first author would like to thank Prof. Xi for hosting him during this visit, and both authors thank Prof. Xi for kindly allowing us to include this result in the paper.*

Remark 2.48. *In [SS3, Theorem 2.5.1], Sam and Snowden prove that the category* $\mathscr{C}\text{-mod}^{tor}$ *and the Serre quotient category* $\mathscr{C}\text{-mod}/\mathscr{C}\text{-mod}^{tor}$ *are equivalent whenever* k *is a field of characteristic 0. In [GL], it is shown that* $\mathscr{C}\text{-mod}^{tor}$ *has sufficiently many injective objects whenever* k *is a field. The above theorem therefore seems to indicate* *that the equivalence of Sam and Snowden will fail whenever k has positive characteristic.*

2.12 Local Cohomology

In this final portion of the paper, we aim to develop a theory of local cohomology for finitely generated $\mathscr{C}$-modules. This problem was first considered by Sam and Snowden in [SS3]. Their work only applies to the case where k is a field of characteristic 0. Much of the difficulty of treating the theory over a general Noetherian ring is that it is unclear whether the category $\mathscr{C}$-mod has sufficiently many injective objects. Despite this fact, we will discover that $\sharp$-filtered objects can play the role of injective modules in many computations. The reader is encouraged to compare the theorems in this part of the paper to theorems in the local cohomology of modules over a polynomial ring.

2.12.1 The Torsion Functor

Definition 2.49. *We write $\mathscr{C}\text{-Mod}^{tor}$ for the category of torsion $\mathscr{C}$-modules. We will also used $\mathscr{C}\text{-mod}^{tor}$ to denote the category of finitely generated torsion modules. By Lemma 2.38, $\mathscr{C}\text{-mod}^{tor}$ is equivalent to the category of finitely generated $\mathscr{C}$-modules with finite support.*

*The **torsion functor** $\widetilde{H^0_{\mathfrak{m}}} : \mathscr{C}\text{-Mod} \to \mathscr{C}\text{-Mod}^{tor}$ is defined by setting $H^0_{\mathfrak{m}}(V)$ to be the maximal torsion submodule of V. We will use $H^0_{\mathfrak{m}} : \mathscr{C}\text{-mod} \to \mathscr{C}\text{-mod}^{tor}$ to denote the restriction of $\widetilde{H^0_{\mathfrak{m}}}$ to $\mathscr{C}$ mod .*

Our goal for the remainder of this paper is to study the torsion functor and its derived functors. Unfortunately, it is not yet clear that this functor actually has derived functors. When k is a field of characteristic 0 Sam and Snowden [SS3, Theorem 4.3.1], as well as Gan and the first author [GL, Theorem 1.7], have shown that the category $\mathscr{C}$-mod has sufficiently many injective objects. When k is not a field of characteristic 0 this is no longer the case. Luckily, we can easily show that the larger category $\mathscr{C}$-Mod does have sufficiently many injective objects.

Proposition 2.13. *The category* $\mathscr{C}$-Mod *has sufficiently many injective objects.*

Proof

It suffices to show that $\mathscr{C}$-Mod satisfies Grothendieck's AB Criterion [G, Theorem 1.10.1]. This follows from the fact that it is a functor category from a small category ($\mathscr{C}$) into a category which satisfies this criterion (k-Mod).

□

Because of this proposition, we know that we may at least make sense of the derived functors of $\widetilde{H^0_\mathfrak{m}}$.

Definition 2.50. *We write* $\widetilde{H^i_\mathfrak{m}}$ *to denote the* i*-th right derived functor of* $\widetilde{H^0_\mathfrak{m}}$*. We refer to these as the* ***local cohomology functors****.*

Our goal for much of what follows will be to show that if V is a finitely generated $\mathscr{C}$-module, then the modules $\widetilde{H}^i_{\mathfrak{m}}(V)$ are also finitely generated. Once this is accomplished, we will spend the remainder of the paper showing how this result applies to invariants such as the Nagpal number and regularity.

2.13.1 Some Acyclics

In this section we will classify two important families of finitely generated modules which are acyclic with respect to local cohomology: $\sharp$-filtered objects and torsion modules. To accomplish this, we will need to view the torsion functor from a slightly different perspective.

Definition 2.51. *For each $n \geqslant 1$, and each $r \geqslant 0$, we define the $\mathscr{C}$-module $M(r)/\mathfrak{m}^n$ as the quotient of $M(r)$ by the submodule generated by $M(r)_{r+n}$. We also use $\mathscr{H}om(k\mathscr{C}/\mathfrak{m}^n, \bullet) : \mathscr{C}\text{-Mod} \to \mathscr{C}\text{-Mod}$ to denote the functor*

$$\mathscr{H}om(k\mathscr{C}/\mathfrak{m}^n, V) := \bigoplus_r \mathrm{Hom}_{k\mathscr{C}}(M(r)/\mathfrak{m}^n, V).$$

We set $\mathscr{E}xt^i(k\mathscr{C}/\mathfrak{m}^n, \bullet) : \mathscr{C}\text{-Mod} \to \mathscr{C}\text{-Mod}$ to be the i-th derived functor of $\mathscr{H}om(k\mathscr{C}/\mathfrak{m}^n, \bullet)$. Explicitly,

$$\mathscr{E}xt^i(k\mathscr{C}/\mathfrak{m}^n, V) := \bigoplus_r \mathrm{Ext}^i_{k\mathscr{C}}(M(r)/\mathfrak{m}^n, V)$$

Remark 2.52. *We note that the $\mathscr{C}$-modules $\mathrm{Hom}_{k\mathscr{C}}(k\mathscr{C}/\mathfrak{m}^n, V)$ and $\mathscr{H}om(k\mathscr{C}/\mathfrak{m}^n, V)$ are not necessarily isomorphic. Indeed, morphisms which appear as elements in the latter module are necessarily supported in finitely many degrees, while this is not the case* *for the prior module.*

Also note that if we restrict ourselves to working with finitely generated modules, then $\mathscr{H}om(k\mathscr{C}/\mathfrak{m}^n, \bullet)$ *and* $\mathrm{Hom}_{k\mathscr{C}}(k\mathscr{C}/\mathfrak{m}^n, \bullet)$ *are isomorphic. This follows from Lemma 2.53, as well as the fact that the torsion parts of finitely generated modules are finitely supported.*

Lemma 2.53. *Let* $r \geqslant 0$ *and* $n \geqslant 1$ *be integers, and let* V *be a* $\mathscr{C}$*-module. Then* $\mathscr{H}om(k\mathscr{C}/\mathfrak{m}^n, V)$ *is naturally isomorphic to a submodule of* V*. Namely,*

$$\mathscr{H}om(k\mathscr{C}/\mathfrak{m}^n, V)_r \;=\; \text{those elements of } V_r \text{ which are in the kernel of every transition map } (f,g) : [r] \to [s] \text{ with } s - r \geqslant n.$$

Proof

By definition, $\mathscr{H}om(k\mathscr{C}/\mathfrak{m}^n, V)_r = \mathrm{Hom}_{k\mathscr{C}}(M(r)/\mathfrak{m}^n, V)$. Proposition 2.3 implies that any morphism $M(r) \to V$ is determined by the image of the identity in degree r. It follows that a morphism $M(r)/\mathfrak{m}^n \to V$ is determined by a choice of element in V_r, with the added restriction that it is in the kernel of all transition maps into V_{r+n}. This completes the proof.

□

Proposition 2.14. *There is a natural isomorphism of functors,*

$$\widetilde{H^0_{\mathfrak{m}}}(\bullet) \cong \lim_{\to} \mathscr{H}om(k\mathscr{C}/\mathfrak{m}^n, \bullet).$$

More generally, there are natural isomorphisms

$$\widetilde{H^i_{\mathfrak{m}}}(\bullet) \cong \lim_{\to} \mathscr{E}xt^i(k\mathscr{C}/\mathfrak{m}^n, \bullet)$$

for all $i \geqslant 0$*.*

Proof

The fact that $\lim_{\to} \mathscr{E}xt^i(k\mathscr{C}/\mathfrak{m}^n, \bullet)$ are the derived functors of $\lim_{\to} \mathscr{H}om(k\mathscr{C}/\mathfrak{m}^n, \bullet)$ follows from the fact that filtered colimits are exact, as well as the relevant definitions. It therefore suffices to prove the first statement. This statement follows immediately from the previous lemma.

□

This new perspective on the torsion functor will allow us to use what we already proved about ♯-filtered modules to conclude that they are acyclic with respect to local cohomology.

Corollary 2.54. *If* V *is a ♯-filtered* $\mathscr{C}$*-module, then* $\widetilde{H^i_{\mathfrak{m}}}(V) = 0$ *for all* i*.*

Proof

Follows immediately from the previous proposition, as well as the first part of Theorem 2.39.

□

Proposition 2.14 also implies that finitely generated torsion modules are acyclic with respect to $\widetilde{H^0_{\mathfrak{m}}}$. Indeed, while it is not the case that $\mathscr{E}xt^i(k\mathscr{C}/\mathfrak{m}^n, V) = 0$ for all n and all torsion modules V, this is the case for n sufficiently large.

Corollary 2.55. *Let V be a finitely generated, torsion $\mathscr{C}$-module. Then for all $n \gg 0$ and all $i \geqslant 1$, $\mathscr{E}xt^i(k\mathscr{C}/\mathfrak{m}^n, V) = 0$. In particular, $\widetilde{H^i_{\mathfrak{m}}}(V) = 0$ for $i \geqslant 1$.*

Proof

Fix some integer $r \geqslant 0$ and $i \geqslant 1$. It suffices to show that $\mathrm{Ext}^i_{k\mathscr{C}}(M(r)/\mathfrak{m}^n, V) = 0$ for all $n \gg 0$. Write $K^{(r,n)}$ for the submodule of $M(r)$ generated by $M(r)_{n+r}$. Then by definition there is an exact sequence,

$$0 \to K^{(r,n)} \to M(r) \to M(r)/\mathfrak{m}^n \to 0.$$

Using the fact that $M(r)$ is projective, we conclude that for all $i \geqslant 1$ there is an exact sequence

$$\mathrm{Ext}^{i-1}_{k\mathscr{C}}(K^{(r,n)}, V) \to \mathrm{Ext}^i_{k\mathscr{C}}(M(r)/\mathfrak{m}^n, V) \to 0.$$

Choose n such that $r + n > \mathrm{td}(V)$. We may construct a projective resolution of $K^{(r,n)}$, say $F^\bullet \to K^{(r,n)} \to 0$, such that for all j, and all $m < r + n$, $F^j_m = 0$. The module $\mathrm{Ext}^{i-1}_{k\mathscr{C}}(K^{(r,n)}, V)$ is a subquotient of the module $\mathrm{Hom}_{k\mathscr{C}}(F^{i-1}, V)$, which is zero by Proposition 2.3 and our choice of n. This completes the proof of the first statement. The second statement is an immediate consequence of Proposition 2.14.

□

2.14.1 Computing Local Cohomology

In this section we present a complex $\mathcal{C}^\bullet V$, associated to a finitely generated $\mathscr{C}$-module V, whose cohomology modules are precisely the local cohomology modules of V. This complex will allow us to show that the local cohomology modules of V are always finitely generated, and will allow us to relate local cohomology to the Nagpal number and the regularity of V.

Let V be a finitely generated $\mathscr{C}$-module. By Nagpal's Theorem, we may find an integer b_{-1} such that $\Sigma_{b_{-1}}V$ is $\sharp$-filtered. This yields an exact sequence

$$V \stackrel{\tau_{b_{-1}}}{\to} \Sigma_{b_{-1}}V \to Q^{(0)} \to 0$$

where $\tau_{b_{-1}}$ is the map defined in remark 3.3. Call $F^0 := \Sigma_{b_{-1}}V$, and recall from Remark 3.3 that $\mathrm{gd}(Q^{(0)}) < \mathrm{gd}(V)$. Proceeding inductively, we may find an integer b_i for which $F^{i+1} := \Sigma_{b_i}Q^{(i)}$ is $\sharp$-filtered. We also have maps $\partial^i : F^i \to F^{i+1}$ defined by the composition of the quotient map $F^i \to Q^{(i)}$, and the map $\tau_{b_{i+1}} : Q^{(i)} \to F^{i+1}$. Putting it all together, we obtain a complex

$$\mathcal{C}^{\bullet}V : 0 \to V \to F^0 \to \ldots \to F^n \to 0$$

Note that this complex is necessarily bounded by our observation that the generating degree of $Q^{(i)}$ is always strictly less than that of $Q^{(i-1)}$. If we define $Q^{(-1)} := V$, then one also notes that

$$H^i(\mathcal{C}^{\bullet}V) = \widetilde{H^0_{\mathfrak{m}}}(Q^i)$$

The complex $\mathcal{C}^{\bullet}V$ was first introduced by Nagpal in [N, Theorem A] and was rediscovered by the first author and Yu in [LY, Theorem 1.7]. Following this, the first author used this complex to prove bounds on the regularity and the Nagpal number of V, which we saw in Theorem 2.2.

Remark 2.56. *One may have noted that the construction of the complex $\mathcal{C}^{\bullet}V$ depended on the integers b_i. Indeed, the assignment $V \mapsto \mathcal{C}^{\bullet}V$ is non-functoral in the category* *of chain complexes of $\mathscr{C}$-modules. However, this assignment is functoral in the derived category. That is to say, choosing different values for the integers b_i yields a complex which is quasi-isomorphic to the original complex.*

One thing that is important for the present work, is that this complex actually computes the local cohomology modules of V.

Proof of Theorem E

Recall the modules $Q^{(i)}$ defined during the construction of $\mathcal{C}^\bullet V$. We have already noted that $H^i(\mathcal{C}^\bullet V) \cong \widetilde{H^0_\mathfrak{m}}(Q^{(i)})$. We will prove that

$$\widetilde{H^0_\mathfrak{m}}(Q^{(i)}) \cong \widetilde{H^{i+1}_\mathfrak{m}}(V).$$

The claim is clear when $i = -1$. Otherwise, we have an exact sequence

$$0 \to Q^{(i)}_T \to Q^{(i)} \to Q^{(i)}_F \to 0$$

where $Q^{(i)}_T$ is the torsion part of $Q^{(i)}$ and $Q^{(i)}_F$ is the torsion free part. Corollary 2.55 implies $\widetilde{H^i_\mathfrak{m}}(Q^{(i)}_F) \cong \widetilde{H^i_\mathfrak{m}}(Q^{(i)})$ for $i \geqslant 1$. Next, we look at the exact sequence

$$0 \to Q^{(i)}_F \to F^{i+1} \to Q^{(i+1)} \to 0$$

and apply Corollary 2.54 to conclude

$$\widetilde{H^0_\mathfrak{m}}(Q^{(i+1)}) \cong \widetilde{H^1_\mathfrak{m}}(Q^{(i)}_F) \cong \widetilde{H^1_\mathfrak{m}}(Q^{(i)}).$$

We reach our desired conclusion by induction.

The bounds given in the theorem follow immediately from Theorem 2.2.

□

We are now free for the remainder of the paper to consider all local cohomology modules as existing inside the category $\mathscr{C}$-mod^{tor}. In particular, it is no longer necessarily to distinguish $H^0_{\mathfrak{m}}$ from $\widetilde{H^0_{\mathfrak{m}}}$.

Definition 2.57. *We write $H^i_{\mathfrak{m}} : \mathscr{C}\text{-mod} \to \mathscr{C}\text{-mod}^{tor}$ to denote the i-th derived functor of $H^0_{\mathfrak{m}}$.*

2.14.2 Applications of Theorem E

We spend this section exploring the plethora of applications of Theorem E. In particular, we will prove Theorem F, along with many other results. To begin, we obtain a new homological characterization of $\sharp$-filtered modules.

Proposition 2.15. *Let V be a finitely generated $\mathscr{C}$-module. Then $H^i_{\mathfrak{m}}(V) = 0$ for all $i \geqslant 0$ if and only if V is $\sharp$-filtered. In particular, V is $\sharp$-filtered if and only if $\mathcal{C}^{\bullet}V$ is exact.*

Proof

The backwards direction follows from Corollary 2.54. If $H^i_{\mathfrak{m}}(V) = 0$ for all i, then $\mathcal{C}^{\bullet}V$ is exact in all degrees by Theorem E. This implies V is $\sharp$-filtered by Theorem 3.9.

□

This will allow us to classify all modules which are acyclic with respect to the torsion functor.

Proposition 2.16. *Let V be a finitely generated $\mathscr{C}$-module. Then V is acyclic with respect to the torsion functor if and only if its torsion free part V_F is $\sharp$-filtered.*

Proof

The exact sequence

$$0 \to V_T \to V \to V_F \to 0$$

implies that $H^i_{\mathfrak{m}}(V) \cong H^i_{\mathfrak{m}}(V_F)$ for all $i \geqslant 1$ by Corollary 2.55. The previous proposition now implies our result.

□

The fact that the complex $\mathcal{C}^\bullet V$ is bounded from above implies that $H^i_{\mathfrak{m}}(V) = 0$ for all $i \gg 0$. We may therefore make sense of the following definition.

Definition 2.58. *Let V be a finitely generated $\mathscr{C}$-module which is not $\sharp$-filtered. Then we define its **cohomological dimension** to be the quantity*

$$\operatorname{cd}(V) := \sup\{i \mid H^i_{\mathfrak{m}}(V) \neq 0\} \in \mathbb{N}.$$

Proposition 2.17. *If V is a finitely generated $\mathscr{C}$-module which is not $\sharp$-filtered, Then any non-trivial local cohomology modules $H^i_{\mathfrak{m}}(V)$ must have*

$$\operatorname{depth}(V) \leq i \leq \operatorname{cd}(V).$$

Moreover, $H^i_{\mathfrak{m}}(V) \neq 0$ *at each of the two extremes.*

Proof

We only need to show that $H^i_{\mathfrak{m}}(V) = 0$ for all $i < \operatorname{depth}(V)$, and that $H^i_{\mathfrak{m}}(V) \neq 0$ for $i = \operatorname{depth}(V)$. If δ is the smallest value for which $H^i_{\mathfrak{m}}(V) \neq 0$, then Theorem E implies that there is an exact sequence,

$$0 \to V \to F^0 \to \ldots \to F^{\delta-1} \to V' \to 0$$

where F^i is $\sharp$-filtered, and V' has torsion. This implies that $\operatorname{depth}(V) = \delta$ by Theorem 2.31.

□

Corollary 2.59. *Let* V *be a finitely generated* $\mathscr{C}$*-module which is not* $\sharp$*-filtered. Then* $\operatorname{cd}(V) \leq \operatorname{gd}(V)$*.*

We next turn our attention to the relationship between local cohomology and the Nagpal number and regularity. We begin with the following observation.

Theorem 2.60. *Let* V *be a finitely generated* $\mathscr{C}$*-module which is not* $\sharp$*-filtered. Then*

$$N(V) = \max\{\operatorname{td}(H^i_{\mathfrak{m}}(V)) \mid i \geqslant 0\} + 1.$$

In particular,

$$N(V) \leq \max\{\operatorname{td}(V), 2\operatorname{gd}(V) - 2\} + 1.$$

Proof

Proposition 2.15 tells us that V is $\sharp$-filtered if and only if $\mathcal{C}^\bullet V$ is exact. Using the fact that the shift functor is exact, and the construction of the complex $\mathcal{C}^\bullet V$, it follows that $\Sigma_b V$ is filtered whenever

$$b > \max\{\mathrm{td}(H^i(\mathcal{C}^\bullet V)) \mid i \geqslant -1\} = \max\{\mathrm{td}(H^i_{\mathfrak{m}}(V)) \mid i \geqslant 0\}.$$

The desired bound follows immediately from Theorem 2.2.

□

Note that this bound on the Nagpal number is not new. Indeed, it was proven by the first author in [L2, Theorem 1.3]. A similar bound was found by the second author in [R, Theorem D].

Theorem 2.61. *Let V be a finitely generated $\mathscr{C}$-module. Then,*

$$\mathrm{reg}(V) \leqslant \max\{\mathrm{td}(H^i_{\mathfrak{m}}(V)) + i\}.$$

In particular,

$$\mathrm{reg}(V) \leqslant \max\{2\,\mathrm{gd}(V) - 1, \mathrm{td}(V)\}.$$

Proof

We proceed by induction on the generating degree of V. The bound is vacuously true if $V = 0$. Assume that $V \neq 0$, and note we have an exact sequence

$$0 \to V_T \to V \to V_F \to 0.$$

Applying Corollary 2.55 we find that $H^i_{\mathfrak{m}}(V) \cong H^i_{\mathfrak{m}}(V_F)$ for all $i \geqslant 1$. Because V_F is torsion free, Remark 3.3 implies we have an exact sequence

$$0 \to V_F \to F \to C \to 0$$

where F is $\sharp$-filtered, and C is generated in strictly smaller degree. Applying the homology functor, and using Theorem 3.9 we find by induction that for all $i \geqslant 1$

$$\mathrm{td}(H_i(V_F)) - i = \mathrm{td}(H_{i+1}(C)) - (i+1) + 1 \leq \mathrm{reg}(C) + 1 \leq \max\{\mathrm{td}(H^s_{\mathfrak{m}}(C)) + s \mid s \geqslant 0\} + 1.$$

If we instead apply the torsion functor to this exact sequence, we find that $H^s_{\mathfrak{m}}(C) \cong H^{s+1}_{\mathfrak{m}}(V)$ for all $s \geqslant 0$. Therefore, recalling that regularity only requires we bound the higher homologies (Definition 2.13),

$$\mathrm{reg}(V_F) \leq \max\{\mathrm{td}(H^i_{\mathfrak{m}}(V)) + i \mid i > 0\}.$$

On the other hand, V_T is a torsion module and therefore [L, Theorem 1.5] [R, Corollary 3.11] imply that $\mathrm{reg}(V_T) \leq \mathrm{td}(V_T) = \mathrm{td}(V)$. Putting everything together we obtain our desired bound.

The second bound follows immediately from the first and Theorem 2.2.

□

As with the bounds on the Nagpal number, the second bound in the above theorem is not new. The first author had discovered this bound earlier in [L2, Theorem 1.3].

2.17.1 A Conjecture and its Consequences

In this section we state our primary conjecture, which firmly establishes the relationship between local cohomology and regularity. Following this, we take time to illustrate some interesting consequences of the conjecture.

Conjecture 2.62. *Let V be a finitely generated $\mathscr{C}$-module which is not $\sharp$-filtered. Then,*

$$\text{reg}(V) = \max\{\text{td}(H^i_{\mathfrak{m}}(V)) + i\}.$$

We have already seen that $\max\{\text{td}(H^i_{\mathfrak{m}}(V)) + i\}$ is an upper bound on the regularity of V (Theorem 2.61). The opposite inequality seems to be much harder to prove. The reader familiar with classical local cohomology theory may recognize a similar statement from local cohomology of modules over a polynomial ring [E2, Theorem 4.3]. What is interesting is that the proofs that $\text{reg}(V) \geqslant \max\{\text{td}(H^i_{\mathfrak{m}}(V)) + i\}$ in that context often proceed by induction on the projective dimension. Theorem 3.9 suggests that such an approach will not work.

We now spend the remainder of this section detailing some corollaries to the above.

Corollary 2.63. *Let V be a finitely generated torsion module, and assume Conjecture 2.62. Then*

$$\text{reg}(V) = \text{td}(V)$$

Proof

This follows immediately from Conjecture 2.62 and Corollary 2.55.

□

What is perhaps more interesting, is what the conjecture implies about the relationship between regularity and the shift functor. To see this, first note that for all $i \geqslant 0$, $H^i_{\mathfrak{m}}(\Sigma V) \cong \Sigma H^i_{\mathfrak{m}}(V)$. Indeed, this follows from Theorem E, as well as how the complex $\mathcal{C}^\bullet V$ was constructed. We therefore conclude the following.

Corollary 2.64. *Let V be a finitely generated $\mathscr{C}$-module such that ΣV is not $\sharp$-filtered, and assume Conjecture 2.62. Then*

$$\operatorname{reg}(\Sigma V) = \operatorname{reg}(V) - 1$$

Proof

Assuming the conjecture we have,

$$\operatorname{reg}(\Sigma V) = \max\{\operatorname{td}(H^i_{\mathfrak{m}}(\Sigma V))+i\} = \max\{\operatorname{td}(\Sigma H^i_{\mathfrak{m}}(V))+i\} = \max\{\operatorname{td}(H^i_{\mathfrak{m}}(V))+i\}-1 = \operatorname{reg}(V)-1.$$

□

Chapter 3

On the degree-wise coherence of FI_G-modules

3.1 Introduction

Let FI be the category whose objects are the sets $[n] := \{1, \ldots, n\}$, and whose morphisms are injections. An FI-**module** over a commutative ring k is a functor from the category FI to the category of k-modules. FI-modules were first introduced by Church, Ellenberg, and Farb as a way to study stability phenomena common throughout mathematics [CEF]. Following this work, representations of various other categories were studied by a large collection of authors. See [W], [SS], [SS2], [PS], for examples of this work. In this paper, we will be concerned with modules over a category which naturally generalizes FI, FI_G.

Let G be a group. Then the category FI_G is that whose objects are the sets $[n]$, and whose morphisms $(f, g) : [n] \to [m]$ are pairs of an injection f with a map of sets $g : [n] \to G$. If (f, g) and (f', g') are two composable morphisms in FI_G, then we define

$$(f, g) \circ (f', g') := (f \circ f', h), \qquad h(x) = g'(x) \cdot g(f'(x))$$

If $G = 1$ is the trivial group, then it is easily seen that FI_G is equivalent to the category

FI. If, instead, we assume that $G = \mathbb{Z}/2\mathbb{Z}$, then FI_G is equivalent to the category FI_{BC} first introduced by Wilson in [W]. An FI_G**-module** over a commutative ring k is defined in the same way as it was for FI-modules. FI_G-modules were first introduced by Sam and Snowden in [SS2].

For much of this paper, we will be concerned with the category FI_G-Mod of FI_G-modules. It is immediate that FI_G-Mod is an abelian category with the usual abelian operations being computed on points. Because of its close connections with the category k-Mod, one may define many properties of FI_G-modules which are analogous to properties of k-modules. One such property, which is most important to us, is finite generation. We say that an FI_G-module V is **finitely generated** if there exists a finite set $\{v_i\} \subseteq \sqcup_{n\geq 0} V([n])$, which no proper submodule contains. Perhaps the most significant fact about finitely generated FI_G-modules is that they are often times Noetherian.

Theorem 3.1 (Corollary 1.2.2 [SS2])**.** *Let G be a polycyclic-by-finite group, and let k be a Noetherian ring. Then submodules of finitely generated FI_G-modules are themselves finitely generated.*

Note that another way of thinking of the above theorem is that the category FI_G-mod of finitely generated FI_G-modules is abelian under sufficient restrictions on k and G. The hypotheses of the above theorem are currently the most general known. It is conjectured that G being polycyclic-by-finite is also necessary for the Noetherian property to hold [SS2]. One of the main goals of this paper is to argue that many theoretical constructions in the theory of FI_G-modules can actually be done independent of the Noetherian property. Instead, we argue that **degree-wise coherence** is often sufficient.

We say that an FI_G-module is degree-wise coherent if there is a set (not necessarily finite) $\{v_i\} \subseteq \sqcup_{n\geq 0} V([n])$ such that:

1. no proper submodule contains $\{v_i\}$, and there is some $N \gg 0$ such that $\{v_i\} \subseteq \sqcup_{n=0}^{N} V([n])$. In this case we say that V is **generated in finite degree**;
2. the module of relations between the elements $\{v_i\}$ is itself generated in finite degree (see Definition 3.5).

One can think about the above definition in the following way. Instead of requiring that our module have finitely many generators, we only require that it admits a generating set whose elements appear in at most finitely many degrees. In addition, we also require that these generators have relations which are bounded in a similar sense. The significance of this condition traces its origins to the paper [CE], although they do not use the same terminology. Following this work, degree-wise coherent modules were studied more deeply by the author in [R]. The first goal of this paper will be to understand the connection between being degree-wise coherent, and having finite torsion.

We say an element $v \in V([n])$ is **torsion** if there is some morphisms $(f,g) : [n] \to [m]$ in FI_G, such that $V(f,g)(v) = 0$. The **torsion degree** of an FI_G-module is the quantity,

$$\mathrm{td}(V) := \sup\{n \mid V_n \text{ contains a torsion element.}\}$$

It was first observed by Church and Ellenberg that degree-wise coherent FI-modules will necessarily have finite torsion degree [CE, Theorem D]. It was then later shown by the author that the same statement was true for FI_G-modules [R, Theorem 3.19]. More recently, Li has conjectured that the converse of this statement was true as well [L3]. In this paper, we will prove this conjecture in the affirmative.

Theorem A. *Let G be a group, and let k be a commutative ring. If V is an FI_G-module which is generated in finite degree, then V is degree-wise coherent if and only if $\mathrm{td}(V) < \infty$.*

As a first application of the above technical theorem, we will be able to show that degree-wise coherent modules form an abelian category.

Theorem B. *Let G be a group, and k a commutative ring. Then the category $\mathrm{FI}_G\text{-Mod}^{coh}$ of degree-wise coherent modules is abelian.*

This theorem was recently proven independently by Li in his note [L3, Proposition 3.4]. One immediately sees that the above theorem is independent of the ring k, as well as the group G. As stated previously, working in the category $\mathrm{FI}_G\text{-Mod}^{coh}$ often has benefits which the category $\mathrm{FI}_G\text{-mod}$ does not permit. Perhaps the most explicit of these benefits is the existence of infinite shifts, which we discuss below. Of course, one should note that there are also benefits which are exclusive to finitely generated modules. The most obvious of these is the ability to do explicit computations.

Much of the remainder of the paper is dedicated to showing how well known theorems about finitely generated FI_G-modules will continue to hold in the category $\mathrm{FI}_G\text{-Mod}^{coh}$. In particular, we focus on generalizing the local cohomology theory of FI_G-modules, introduced by Li and the author in [LR].

If V is an FI_G-module, then the **0-th local cohomology functor** is defined by

$$H^0_{\mathfrak{m}}(V) := \text{ the maximal torsion submodule of } V.$$

$H^0_{\mathfrak{m}}$ is a left exact functor, and we denote its derived functors by $H^i_{\mathfrak{m}}$. Section 3.11.1 is largely dedicated to arguing that the theorems of [LR] will continue to hold in $\mathrm{FI}_G\text{-Mod}^{coh}$. One of the main results of [LR], is that whenever V is finitely generated there is a complex $\mathscr{C}^\bullet V$ which computes $H^i_{\mathfrak{m}}$ (see Definition 3.18). One problem with this complex, is that it's not functoral in V. Allowing ourselves to work in the category $\mathrm{FI}_G\text{-Mod}^{coh}$, we can fix this issue using the infinite shift.

Let $\iota : \mathrm{FI}_G \to \mathrm{FI}_G$ be the functor defined by the assignments,

$$\iota([n]) = [n+1], \qquad \iota((f,g) : [n] \to [m]) = (f_+, g_+)$$

where

$$f_+(x) = \begin{cases} f(x) & \text{if } x < n+1 \\ m+1 & \text{otherwise} \end{cases}, \qquad g_+(x) = \begin{cases} g(x) & \text{if } x < n+1 \\ 1 & \text{otherwise.} \end{cases}$$

The shift functor Σ is defined to be

$$\Sigma(V) := V \circ \iota.$$

We write Σ_b to denote the b-th iterate of Σ. In Section 3.2.3, it is show that there is a commutative diagram for all $b \geq 1$,

$$\begin{array}{ccc} V & \longrightarrow & \Sigma_{b+1} \\ \| & & \uparrow \\ V & \longrightarrow & \Sigma_b \end{array}$$

The **infinite shift** Σ_∞ is the directed limit of the right column of this diagram. That is,

$$\Sigma_\infty V := \varinjlim \Sigma_b V.$$

The collection of maps $V \to \Sigma_b$ in the above diagram induce a morphism $V \to \Sigma_\infty V$. The **infinite derivative** is defined to be the cokernel of this map

$$D_\infty V := \mathrm{coker}(V \to \Sigma_\infty V).$$

One should observe that is rarely ever the case that the infinite derivative or the infinite shift are finitely generated. We will see, however, that if V is degree-wise coherent, then the same is true of both $\Sigma_\infty V$ and $D_\infty V$. It is shown in Section 3.9.1 that the infinite derivative functor is right exact. We use $H_i^{D_\infty^b}$ to denote the i-th left derived functor of the b-th iterate of D_∞. The main result of the final section of the paper is the following.

Theorem C. *Let V be a degree-wise coherent FI_G-module of dimension $d < \infty$ (see Definition 3.34). Then there are isomorphisms for all $i \geq 1$,*

$$H_i^{D_\infty^{d+1}}(V) \cong H_{\mathfrak{m}}^{d+1-i}(V).$$

One can think of the above theorem as a kind of local duality for FI_G-modules, in so far as it describes the equivalence of local cohomology with the derived functors of some right exact functor. We have already discussed the fact that the functor D_∞ does not exist within the category of finitely generated modules, and therefore the above represents a means of uniformly describing local cohomology modules in a way which is inaccessible by simply working with finitely generated modules.

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3.2 Preliminaries

3.2.1 Elementary Definitions

Let G be a group, and let k be a commutative ring.

Definition 3.2. *The category* FI_G *is that whose objects are the finite sets* $[n] := \{1, \ldots, n\}$*, and whose morphisms are pairs* $(f, g) : [n] \to [m]$*, where* $f : [n] \to [m]$ *is an injection of sets and* $g : [n] \to G$ *is a map of sets. For two composable morphisms* $(f, g), (h, g')$*, we define*

$$(f, g) \circ (h, g') := (f \circ h, g'')$$

where $g''(x) = g'(x) \cdot g(h(x))$*. For each non-negative integer* n*, we denote the group of endomorphisms* $\mathrm{End}_{\mathrm{FI}_G}([n]) = \mathfrak{S}_n \wr G$ *by* G_n*.*

An FI_G***-module*** *over* k *is a covariant functor* $V : \mathrm{FI}_G \to k\text{-Mod}$*. We use* V_n *to denote the* k*-module* $V([n])$*. For any* FI_G*-morphism* $(f, g) : [n] \to [m]$ *we write* $(f, g)_*$ *for the map* $V(f, g)$*. We call these maps the* ***induced maps*** *of* V*, and in the case where* $n < m$ *we say that* $(f, g)_*$ *is a* ***transition map*** *of* V*.*

Given any FI_G*-module* V*, its* ***degree*** *is the quantity,*

$$\deg(V) := \sup\{n \mid V_n \neq 0\} \in \mathbb{N} \cup \{\pm\infty\}$$

where we use the convention that the supremum of the empty set is $-\infty$*.*

We note that the category of FI_G-modules and natural transformations FI-Mod is abelian. Indeed, one computes kernels and cokernels in a pointwise fashion. One nice feature of FI_G-modules is that many properties of k-modules have natural analogs. Perhaps the most significant of these properties is finite generation.

Definition 3.3. *Let V be an FI_G-module. We say that V is* ***finitely generated*** *if there is a finite collection $S \subseteq \sqcup_{n\geq 0} V_n$ which no proper submodule of V contains. We denote the category of finitely generated FI_G-modules by $\mathrm{FI}_G\text{-mod}$.*

Finitely generated FI_G-modules were first studied by Sam and Snowden in [SS2]. Prior to this, the case wherein $G = 1$ was studied by Church, Ellenberg, Farb, and Nagpal in [CEF], and [CEFN]. This case was also featured prominently in the work of Sam and Snowden [SS3]. We note that Church, Ellenberg, Farb, and Nagpal refer to these modules as being FI-modules. The case wherein $G = \mathbb{Z}/2\mathbb{Z}$ was studied by Wilson in [W]. Wilson refers to these modules as being FI_{BC}-modules.

Theorem 3.4 (Corollary 1.2.2 [SS2])**.** *Assume that G is a polycyclic-by-finite group, and that k is a Noetherian ring. Then the category $\mathrm{FI}_G\text{-mod}$ is abelian. That is, submodules of finitely generated modules are finitely generated.*

One should observe the two hypotheses of the above theorem. In this paper we will not be studying finitely generated FI_G-modules, instead focusing on degree-wise coherent modules (see Definition 3.5). Working with these more general modules will allow us to prove many theorems without needing to restrict the ring k or the group G. One goal of this paper is to argue that degree-wise coherence is a more natural condition than finite generation in many contexts.

Definition 3.5. *Let $r \geq 0$ be an integer. The* ***principal projective*** FI_G***-module generated in degree*** r $M(r)$ *is defined on points by*

$$M(r)_n := k[\mathrm{Hom}_{\mathrm{FI}_G}([r],[n])],$$

where $k[\mathrm{Hom}_{\mathrm{FI}_G}([r],[n])]$ is the free k-module with basis labeled by the set $\mathrm{Hom}_{\mathrm{FI}_G}([r],[n])$. The induced maps of this module act by composition on the basis vectors. More generally, if W is a kG_r-module, then we define the ***free*** FI_G***-module relative to*** W $M(W)$ *by the assignments*

$$M(W)_n := k[\mathrm{Hom}_{\mathrm{FI}_G}([r],[n])] \otimes_{kG_r} W.$$

The induced maps of this module act by composition in the first component. In this case, we say that $M(W)$ is generated in degree r. Direct sums of modules of either of these two types will generally be referred to as ***free modules****. The* ***generating degree*** *of a free module is the supremum of the generating degrees of its free summands.*

We say that a module V is ***♯-filtered*** *if it admits a finite filtration*

$$0 = V^{(-1)} \subseteq \ldots \subseteq V^{(n)} = V.$$

such that $V^{(i)}/V^{(i-1)}$ is a free module for each i. In this case, the integer n is called the generating degree of V.

A ***presentation*** *for a module V is an exact sequence of the form,*

$$0 \to K \to F \to V \to 0,$$

where F is a free-module. If F is ♯-filtered with generating degree n, then we say that V is ***generated in degree*** $\leq n$*. If, in addition, K is generated in finite degree, then* *we say that V is **degree-wise coherent**. We denote the category of modules which are generated in finite degree by* $\mathrm{FI}_G\text{-Mod}^{coh}$.

Note that free modules are not always projective, although projective modules are always free. Indeed, it can be shown that for a kG_r-module W, $M(W)$ is projective as an FI_G-module if and only if W is projective as a kG_r-module. Proofs of these facts can be found in [R].

3.2.2 The homology functors and regularity

Definition 3.6. *Let V be an FI_G-module. Then the **0-th homology functor** is defined on points by*

$$H_0(V)_n := V_n/V_{<n},$$

where $V_{<n}$ is the submodule of V_n spanned by the images of all transition maps into V_n. We write H_i to denote the i-th derived functor of H_0.

The i-th homological degree of a module V is the quantity

$$\mathrm{hd}_i(V) := \deg(H_i(V)) \in \mathbb{N} \cup \{\pm\infty\}.$$

the 0-th homological degree $\mathrm{hd}_0(V)$ *will be referred to as the **generating degree** of the module, and is denoted by* $\mathrm{gd}(V)$*. The **regularity** of a module V is*

$$\mathrm{reg}(V) := \inf\{N \mid \mathrm{hd}_i(V) - i \leq N \forall i \geq 1\} \in \mathbb{N} \cup \{\pm\infty\}.$$

Remark 3.7. *Note that in the above definition, regularity is computed using strictly positive homological degrees. This is slightly different from how regularity is defined in classical commutative algebra. When we discuss local cohomology later in this paper, it will be explained why the above definition was chosen.*

It is an easy check to show that the definition of $\mathrm{gd}(V)$ *given above agrees with the notion of generating degree given in Definition 3.5. It is also important that one notes the connection between the module of relations of* V*, and the first homological degree* $hd_1(V)$*. Given a presentation,*

$$0 \to K \to F \to V \to 0$$

we may apply the homology functor to find,

$$hd_1(V) \leq gd(K) \leq \max\{gd(V), hd_1(V)\}.$$

In particular, V *is degree-wise coherent if and only if both* $gd(V)$ *and* $hd_1(V)$ *are finite.*

If V *is acyclic with respect to the homology functors, then we define its regularity to be* $-\infty$*.*

The regularity of FI-modules was first studied by Sam and Snowden in [SS3, Corollary 6.3.5], in the case where k is a field of characteristic 0. Following this, Church and Ellenberg provided explicit bounds on the regularity of FI-modules over any commutative ring k [CE, Theorem A]. The author then adapted the techniques of Church and Ellenberg to work for general FI_G-modules [R, Theorem D].

Theorem 3.8 ([CE],[R]). *Let V be an FI_G-module. Then,*

$$\mathrm{reg}(V) \leq \mathrm{hd}_1(V) + \min\{\mathrm{hd}_1(V), \mathrm{gd}(V)\} - 1.$$

In particular, if V is degree-wise coherent, then V has finite regularity.

One notable takeaway from the work of Church and Ellenberg is that their bound is only dependent on the generating degree and first homological degree of the module. In particular, their work entirely takes place in the category $\mathrm{FI}\text{-}\mathrm{Mod}^{coh}$. This philosophy was also heavily featured in [R]. One goal of the present work is to develop an understanding of the category $\mathrm{FI}_G\text{-}\mathrm{Mod}^{coh}$.

Following this work, regularity was studied Liang Gan, Li, and the author in [Ga], [L], [L2], and [LR]. The paper [LR] studied the connection between regularity and a local cohomology theory for FI_G-modules, in the case where G is a finite group. We will later rediscover this connection in the more general context of the current work.

To conclude this section, we state the theorem which classifies the homology acyclic modules.

Theorem 3.9 (Theorem 1.3 [LY], Theorem A [R]). *Let V be a degree-wise coherent module. Then the following are equivalent:*

1. *V is acylic with respect to the homology functors;*

2. *$H_1(V) = 0$;*

3. $H_i(V) = 0$ *for some* $i \geq 1$*;*

4. V *is* $\sharp$*-filtered.*

3.2.3 The shift and derivative functors

Definition 3.10. *Let* $\iota : \mathrm{FI}_G \to \mathrm{FI}_G$ *be the functor which is defined on objects by* $\iota([n]) = [n+1]$*, while for each morphism* $(f, g) : [n] \to [m]$ *we set* $\iota(f, g) = (f_+, g_+))$ *where,*

$$f_+(x) := \begin{cases} f(x) & \text{if } x \leq n \\ m+1 & \text{otherwise} \end{cases}, \qquad g_+(x) := \begin{cases} g(x) & \text{if } x \leq n \\ 1 & \text{otherwise.} \end{cases}$$

The ***shift functor*** *is defined as the composition*

$$\Sigma V := V \circ \iota.$$

We write Σ_a *for the* a*-th iterate of* V*.*

For each positive integer a*, there is a natural map of* FI_G*-modules* $\tau_a : V \to \Sigma_a V$ *defined on each point by the transition map* $(f_a^n, \mathbf{1})_*$*, where* $f_a^n : [n] \to [n+a]$ *is the natural inclusion while* $\mathbf{1}$ *is the trivial map into* G*. The* ***length*** a ***derivative functor*** *is the cokernel of this map*

$$D_a V := \mathrm{coker}(\tau_a)$$

We write D_a^b *for the* b*-th iterate of* D_a*. In the case where* $a = 1$*, we will write* $D := D_1$*.*

The derivative functors were introduced by Church and Ellenberg in [CE], and have since seen use in [R] and [LY]. Later, we will consider the direct limit of all derivative functors, which we call the infinite derivative (see Definition 3.26). We record some useful properties of the derivative and shift functors below. Proofs of these facts can be found in [R, Proposition 3.3] and [CE, Proposition 3.5].

Proposition 3.3. *Fix an integer $a \geq 1$. The length a derivative functor and the shift functor enjoy the following properties:*

1. *If V is an FI_G-module which is degree-wise coherent, then the same is true of D_aV and ΣV;*
2. *If $\mathrm{gd}(V) \leq d$, then $\mathrm{gd}(\Sigma V) \leq d$ and $\mathrm{gd}(D_aV) < d$;*
3. *D_a is right exact, and Σ_a is exact;*
4. *For any kG_r-module W, both $\Sigma M(W)$ and $D_aM(W)$ are free modules. In fact,*

$$\Sigma M(W) \cong M(W) \oplus M(\mathrm{Res}^{G_r}_{G_{r-1}} W), \qquad DM(W) \cong M(\mathrm{Res}^{G_r}_{G_{r-1}} W). \tag{3.1}$$

In particular, Σ and D_a preserve $\sharp$-filtered modules.

Remark 3.11. *Note that if G is a finite group, then Σ and D_a both preserve finitely generated FI_G-modules. This is no longer the case if G is infinite. It is always the case that these functors preserve being degree-wise coherent.*

Part 3 of Proposition 3.3 implies that the functors D_a have left derived functors. We will follow the notation of [CE] and [R] and write $H_i^{D_a^b}$ for the i-th derived functor of D_a^b. One of the main insights of [CE] was that the properties of the modules $H_i^{D^b}(V)$ are critical in bounding the regularity of V. Later, the author [R] showed that the functors $H_1^{D^b}$ could be used to define a theory of depth for FI_G-modules. Proofs for the following facts can be found in [CE] and [R].

Proposition 3.4. *Fix integers $a, b, i \geq 1$. The functors $H_i^{D_a^b}$ enjoy the following properties:*

1. *If V is degree-wise coherent, then $\deg(H_i^{D_a^b}) < \infty$;*

2. *For any module V, there is an exact sequence*

$$0 \to H_1^{D_a}(V) \to V \stackrel{\tau_a}{\to} \Sigma_a V \to D_a V \to 0.$$

3. *If $i > b$, then $H_i^{D_a^b} = 0$.*

Remark 3.12. *The cited sources prove these facts in the case where $a = 1$. The proofs are identical for arbitrary a.*

Note that the exact sequence in the second part of Proposition 3.4 is strongly related to torsion. This will be explored in the next section.

Definition 3.13. *Let V be a degree-wise coherent module. Then we define its* ***depth*** *to be the quantity,*

$$\operatorname{depth}(V) := \inf\{b \mid H_1^{D^{b+1}}(V) \neq 0\} \in \mathbb{N} \cup \{\infty\}.$$

Remark 3.14. *In [LR] an alternative notion of depth is provided, which is defined in terms of the vanishing of particular* Ext *groups. It is shown in that paper that both notions agree with one another. Due to the emphasis on the derivative functors in this paper, we will use the above definition.*

Perhaps the most significant property of the shift functor is the following structural theorem. Note that this theorem was proven by Nagpal [N, Theorem A] in the case where G is a finite group, k is a Noetherian ring, and V is finitely generated. It was then generalized by Nagpal and Snowden [NS] to the case where G is a polycyclic-by-finite group. Finally, the author [R] proved the theorem to the level of generality presented here.

Theorem 3.15. *Let V be an FI_G-module which is degree-wise coherent. Then for $b \gg 0$, $\Sigma_b V$ is $\sharp$-filtered.*

Definition 3.16. *We denote the smallest b for which $\Sigma_b V$ is $\sharp$-filtered by $N(V)$.*

It is natural for one to ask whether it is possible bound $N(V)$. Indeed, this was accomplished by the author in [R, Theorem C].

Theorem 3.17. *Let V be an FI_G-module which is degree-wise coherent. If V is not $\sharp$-filtered, then $H_1^{D^b}(V) = 0$ for $b \gg 0$, and*

$$N(V) = \max_b \{\deg(H_1^{D^b}(V))\}$$

One of the many consequences of Theorem 3.15 is the construction of the following complex, which we will see play a major part in the local cohomology of FI_G-modules.

Definition 3.18. *Let V be an FI_G-module which is degree-wise coherent. Setting $b_{-1} := N(V)$, there is an exact sequence*

$$V \stackrel{\tau_{b_{-1}}}{\to} F^0 := \Sigma_b V \to D_{b_{-1}} V \to 0$$

By Proposition 3.3, the module $D_{b_{-1}} V$ is degree-wise coherent and is generated in strictly smaller degree than V. We may therefore repeat this process finitely many times to obtain the complex

$$\mathscr{C}^\bullet V : 0 \to V \to F^0 \to \ldots \to F^n \to 0.$$

The complex $\mathscr{C}^\bullet V$ was introduced by Nagpal in [N, Theorem A]. It was subsequently studied by Li in [L2], and by Li and the author in [LR]. Note that the assignment $V \mapsto \mathscr{C}^\bullet V$ is not functoral. Later, we will construct a uniform version of the complex $\mathscr{C}^\bullet V$ which is functoral in V (see Definition 3.28).

3.5 Degree-wise coherence

3.5.1 Connections with torsion

Definition 3.19. *Let V be an FI_G-module. An element $v \in V_n$ is **torsion** if it is in the kernel of some - and therefore all - transition maps out of V_n. We say that a module V is **torsion** if its every element is torsion.*

Note that every FI_G-module fits into an exact sequence of the form

$$0 \to V_T \to V \to V_F \to 0$$

where V_T is a torsion module, and V_F is torsion free.

*The **torsion degree** of an FI_G-module is the quantity*

$$\mathrm{td}(V) := \deg(V_T).$$

The exact sequence of Proposition 3.4 implies that $\mathrm{td}(V) = \deg(H_1^D(V))$. Proposition 3.4 also tells us that $\deg(H_1^D(V))$ is finite. We therefore obtain the following corollary.

Lemma 3.20. *Let V be a degree-wise coherent module. Then $\mathrm{td}(V) < \infty$. In particular, a degree-wise coherent module V is torsion if and only if $\deg(V) < \infty$.*

We will see later that a converse of this statement is true as well. That is, if V is generated in finite degree, and $\mathrm{td}(V) < \infty$, then V is degree-wise coherent. To prove this fact, we will need the following proposition. It is, in some sense, a rephrasing of [CE, Theorem D]. Church and Ellenberg proved this for FI-modules, and it was generalized to FI_G-modules by the author in [R, Theorem 3.19].

Proposition 3.6. *Let $V \subseteq M$ be torsion-free FI_G-modules which are generated in finite degree. Then $\mathrm{td}(M/V) < \infty$.*

Proof

We have an exact sequence,

$$0 \to V \to M \to M/V \to 0$$

Applying the functor D, we obtain an exact sequence

$$H_1^D(M) \to H_1^D(M/V) \to DV \to DM.$$

By assumption M is torsion-free, and therefore $H_1^D(M) = 0$. This implies that $H_1^D(M/V) \cong \ker(DV \to DM)$. Unpacking definitions, [CE, Theorem D] and [R, Theorem 3.19] imply that this kernel is only non-zero in finitely many degrees.

□

We are now able to prove the main theorem of this section.

Theorem 3.21. *Let V be an* FI_G*-module which is generated in finite degree. Then V is degree-wise coherent if and only if* $\mathrm{td}(V) < \infty$.

Proof

We have already seen the forward direction. Conversely, assume that $\mathrm{gd}(V) < \infty$ and $\mathrm{td}(V) < \infty$. Then we have an exact sequence

$$0 \to V_T \to V \to V_F \to 0$$

where V_T is torsion, and V_F is torsion free. Applying the homology functor, it follows that

$$\deg(H_1(V)) \leq \max\{\deg(H_1(V_T)), \deg(H_1(V_F))\}$$

It is easily seen that $\deg(H_1(V_T)) < \infty$, and therefore it suffices to show that $\deg(H_1(V_F))$ is finite. In particular, we may assume without loss of generality that V is torsion free.

Assuming that V is torsion free, we have an exact sequence

$$0 \to V \to \Sigma V \to DV \to 0$$

where ΣV is also torsion free. Proposition 3.6 now implies that $\mathrm{td}(DV) < \infty$. We also know, however, that $\mathrm{gd}(DV) < \mathrm{gd}(V) < \infty$ by Proposition 3.3. Applying induction on the generating degree, we may assume that DV is degree-wise coherent. Proposition 3.4 implies that $\deg(H_i^{D^b}(DV)) < \infty$ for all i, b.

Next, we claim that for all $i, b \geq 1$, $H_i^{D^b}(DV) \cong H_i^{D^{b+1}}(V)$. To see this, we compute the derived functors of D^{b+1}, when viewed as the composition $D^b \circ D$. Proposition 3.4 implies the Grothendieck spectral sequence associated to this composition only has two rows. It therefore degenerates to the long exact sequence

$$\ldots \to H_{i-1}^{D^b}(H_1^D(V)) \to H_i^{D^b}(DV) \to H_i^{D^{b+1}}(V) \xrightarrow{\partial} H_{i-1}^{D^b}(H_1^D(V)) \to \ldots$$

The fact that V is torsion-free implies $H_1^D(V) = 0$, and therefore $H_i^{D^b}(DV) \cong H_i^{D^{b+1}}(V)$ for all i.

Recall that we have shown that DV is degree-wise coherent. The above isomorphisms therefore imply that $\deg(H_i^{D^b}(V)) < \infty$ for all i, b. Theorem 3.16 now implies that $\Sigma_b V$ is $\sharp$-filtered for $b \gg 0$. In particular, we have an exact sequence

$$0 \to V \to \Sigma_{N(V)} V \to D_{N(V)} V \to 0$$

By assumption V is generated in finite degree, and therefore $D_{N(V)}V$ is degree-wise coherent. Applying the homology functor, and using Theorems 3.9 and 3.8, we conclude that $\deg(H_1(V)) < \infty$, as desired.

□

Remark 3.22. *The author's interest in proving the above theorem was heavily influenced by recent work of Li [L3]. In that work, Li argues the forward direction of the theorem, and leaves the converse as a conjecture. The author would like to thank Professor Li for* *pointing him in the direction of this problem.*

Remark 3.23. *It is important that one develop an intuition for why one would suspect Theorem 3.21 is true. In the work of Li and the author [LR, Theorem F], it is shown that the regularity of a finitely generated* FI_G*-module, where* G *is finite and* k *is Noetherian, can be bound in terms of the torsion degrees of its local cohomology modules (see Definition 3.29). Li has shown that the higher local cohomology modules can be bounded entirely in terms of the generating degree [L2]. Put together, it follows that the regularity of a finitely generated* FI_G*-module is bounded by a constant depending only on its torsion degree and its generating degree. Theorem 3.21 implies that these bounds on regularity will continue to hold even if we do not assume that the module is finitely generated.*

3.6.1 The category $\mathrm{FI}_G\text{-Mod}^{coh}$

In this section we consider the category of degree-wise coherent modules, and examine some of its technical properties. The main result of this section will be to show that $\mathrm{FI}_G\text{-Mod}^{coh}$ is abelian. We once again note that the category of finitely generated FI_G-modules is only known to be abelian when k is Noetherian, and G is polycyclic-by-finite. This would seem to indicate that the property of being degree-wise coherent is often times better suited for homologically flavored questions about FI_G-modules.

One recurring theme throughout the proofs in this section is Theorem 3.21. This theorem tells us that the property of being degree-wise coherent can be partially checked on the maximal torsion submodule. This will allow us to prove non-obvious facts about submodules of degree-wise coherent submodules. One example of this is the following.

Proposition 3.7. *Let V be a degree-wise coherent* FI_G*-module, and let $V' \subseteq V$ be a submodule which is generated in finite degree. Then V' is also degree-wise coherent.*

Proof

Because V' is a submodule of V, we must have $\mathrm{td}(V') \leq \mathrm{td}(V)$. Theorem 3.21 now implies the proposition.

□

Note that the above proposition justifies the terminology of coherence. Recall that a module M over a commutative ring R is said to be coherent if it is finitely presented, and every finitely generated submodule of M is also finitely presented. It is well known that a module over a coherent ring is finitely presented if and only if it is coherent. When k is a field of characteristic 0, Sam and Snowden's language of twisted commutative algebras imply that the category of FI-modules is equivalent to the category of GL_∞-equivariant modules over a polynomial ring in infinitely many variables [SS3]. A polynomial ring in infinitely many variables over a field is coherent, and therefore Proposition 3.7 can be heuristically thought of as a consequence of this.

Proposition 3.8. *Let,*

$$0 \to V' \to V \to V'' \to 0$$

be an exact sequence of FI_G*-modules. Then any two of V', V, or V'' are degree-wise* *coherent only if the third is as well.*

Proof

The above exact sequence induces the exact sequence,

$$H_2(V'') \to H_1(V') \to H_1(V) \to H_1(V'') \to H_0(V') \to H_0(V) \to H_0(V'') \to 0.$$

This implies the collection of bounds,

$$\mathrm{hd}_1(V) \leq \max\{\mathrm{hd}_1(V'), \mathrm{hd}_1(V'')\}, \qquad \mathrm{gd}(V) \leq \max\{\mathrm{gd}(V'), \mathrm{gd}(V'')\} \tag{3.2}$$

$$\mathrm{hd}_1(V') \leq \max\{\mathrm{hd}_2(V''), \mathrm{hd}_1(V)\}, \qquad \mathrm{gd}(V') \leq \max\{\mathrm{hd}_1(V''), \mathrm{gd}(V)\} \tag{3.3}$$

$$\mathrm{hd}_1(V'') \leq \max\{\mathrm{gd}(V'), \mathrm{hd}_1(V)\}, \qquad \mathrm{gd}(V'') \leq \mathrm{gd}(V). \tag{3.4}$$

If V' and V'' are degree-wise coherent, then the first pair of bounds immediately implies the same about V. If we instead assume that V'' and V' are degree-wise coherent, then Theorem 3.8 implies that $\mathrm{hd}_2(V'') < \infty$. The second pair of bounds now imply that V' is degree-wise coherent. Finally, if V' and V are degree-wise coherent then the third pair of bounds imply that V'' must be as well.

□

This is all we need to prove the main theorem of this section.

Theorem 3.24. *The category* $\mathrm{FI}_G\text{-Mod}^{coh}$ *is abelian.*

Proof

The only thing that needs to be checked is that $\mathrm{FI}_G\text{-Mod}^{coh}$ permits images, kernels and cokernels. That is, if $\phi : V \to V'$ is a morphism of degree-wise coherent modules, then we must show that $\ker(\phi), \operatorname{im}(\phi)$ and $\operatorname{coker}(\phi)$ are all degree-wise coherent. We have a pair of exact sequences

$$0 \to \ker(\phi) \to V \to \operatorname{im}(\phi) \to 0$$

$$0 \to \operatorname{im}(\phi) \to V' \to \operatorname{coker}(\phi) \to 0$$

The module $\operatorname{im}(\phi)$ is generated in finite degree because it is a quotient of V, and $\operatorname{td}(\operatorname{im}(\phi)) < \infty$ because it is a submodule of V'. Theorem 3.21 implies that $\operatorname{im}(\phi)$ is degree-wise coherent, whence $\ker(\phi)$ and $\operatorname{coker}(\phi)$ are as well by Proposition 3.8.

□

Remark 3.25. *Li has also independently proven this theorem in his work [L3, Proposition 3.4]. His methods do not use Theorem 3.21.*

3.9 Applications

In this half of the paper, we consider applications of the machinery developed in previous sections. To start, we will define the infinite shift and derivative functors. Using these functors, we will describe a local cohomology theory for degree-wise coherent FI_G-modules. Finally, we finish by proving a kind of local duality theorem for FI_G-modules.

3.9.1 The infinite shift and derivative functors

Definition 3.26. *Let V be an FI_G-module. For each positive integer a, the transition map $(f^{n+a}, \mathbf{1})_*$, induced by the pair of the standard inclusion $f^{n+a} : [n+a] \to [n+a+1]$ and the trivial map into G, gives a map $\Sigma_a V \to \Sigma_{a+1} V$. The **infinite shift** of V is the direct limit*

$$\Sigma_\infty V := \lim_{\to} \Sigma_a V$$

*The maps $(f^{n+a}, \mathbf{1})_*$ also induce maps $D_a V \to D_{a+1} V$. The **infinite derivative** of the module V is the direct limit*

$$D_\infty V := \lim_{\to} D_a V$$

One should immediately note that if V is finitely generated, then neither $\Sigma_\infty V$, nor $D_\infty V$ are necessarily finitely generated. These functors do preserve degree-wise coherence, as we shall now prove.

Proposition 3.10. *The infinite shift and derivative functors enjoy the following properties:*

1. *Σ_∞ is exact, and D_∞ is right exact;*

2. *for all FI_G-modules V, there is an exact sequence*

$$V \to \Sigma_\infty V \to D_\infty V \to 0.$$

   V is torsion-free if and only if the map $V \to \Sigma_\infty V$ is injective;

3. *for any kG_n-module W, $\Sigma_\infty M(W) \cong M(W) \oplus Q$ where Q is some free-module generated in degree $< r$, while $D_\infty M(W) \cong Q$. In particular, both the infinite shift and derivative functors preserve $\sharp$-filtered objects;*

4. *if* $\mathrm{gd}(V) \le d$ *is finite, then* $\mathrm{gd}(\Sigma_\infty V) \le d$ *and* $\mathrm{gd}(D_\infty V) < d$*;*

5. *if* V *is degree-wise coherent, then* $\Sigma_\infty V$ *is* $\sharp$*-filtered, and* $D_\infty V$ *is degree-wise coherent.*

Proof

The first statement follows from Proposition 3.3, as well as the exactness of filtered colimits.

Write ω for the poset category of the natural numbers. We define the functors $F_i : \omega \to \mathrm{FI}_G\text{-Mod}$, $i = 1, 2, 3$ as follows:

$$F_1(a) = V, \qquad F_2(a) = \Sigma_a V, \qquad F_3(a) = D_a V.$$

Note that F_1 maps all morphisms of ω to the identity, while F_2 and F_3 map the morphisms of ω to the previously discussed maps $\Sigma_a V \to \Sigma_{a+1} V$ and $D_a V \to D_{a+1} V$. Then all relevant definitions imply there is an exact sequence

$$F_1 \to F_2 \to F_3 \to 0$$

Applying the exact direct limit functor to this exact sequence implies the first half of the second claim. If V is torsion free, then the map $F_1 \to F_2$ is exact by definition of torsion, and this will be preserved after taking direct limits. Conversely, assume that V has torsion. In particular, there is an element $v \in V_n$ for some n, such that v is in the kernel of some transition map $V_n \to V_m$. In this case, every transition map to V_r, with $r \ge m$, will also contain v in its kernel. In particular, v will be an element in the kernel of the maps $V \to \Sigma_a V$ for all $a > 0$. This implies that the element v is in the kernel of the map $V \to \Sigma_\infty V$.

The fact that $\Sigma_\infty M(W)$ takes the prescribed form follows immediately from Proposition 3.3 and (3.1). The statement about the infinite derivative follows from the second part of this proposition.

The fourth statement follows from the first statement and the third.

The fifth statement follows from the fourth, as well as Theorem 3.15.

□

While the infinite shift and derivative functors may be harder to compute than their finite counter-parts, they allow us to more uniformly state certain theorems. For instance, we will see that infinite shifts can be used to fix the issue of functoriality of the complex $\mathscr{C}^\bullet V$. We will also see that the infinite derivative functor can be used to prove a kind of local duality for FI_G-modules.

The above proposition implies that the functors Σ_∞ and D_∞ can be considered as endofunctors of the abelian category $\mathrm{FI}_G\text{-Mod}^{coh}$. This proposition also tells us that D_∞ admits left derived functors in this category.

Definition 3.27. *For each $b \geq 1$, we will write $H_i^{D_\infty^b} : \mathrm{FI}_G\text{-Mod}^{coh} \to \mathrm{FI}_G\text{-Mod}^{coh}$ to denote the i-th derived functor of D_∞^b.*

Proposition 3.11. *The functors $H_i^{D_\infty^b}$ enjoy the following properties:*

1. *for all degree-wise coherent modules V, there is an exact sequence*

$$0 \to H_1^{D_\infty}(V) \to V \to \Sigma_\infty V \to D_\infty V \to 0;$$

   In particular, if V is torsion free, then $H_1^{D_\infty}(V) = 0$;

2. *If V is $\sharp$-filtered, then $H_i^{D_\infty^b}(V) = 0$ for all $i, b \geq 1$;*

3. *for all degree-wise coherent modules V, and all $b, i \geq 1$, $\deg(H_i^{D_\infty^b}(V)) < \infty$.*

Proof

Let

$$0 \to K \to F \to V \to 0$$

be a presentation for V. Then we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} & & D_\infty^b(K) & \longrightarrow & \Sigma_\infty D_\infty^b(K) & \longrightarrow & D_\infty^{b+1}(K) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & D_\infty^b(F) & \longrightarrow & \Sigma_\infty D_\infty^b(F) & \longrightarrow & D_\infty^{b+1}(F) & \longrightarrow & 0 \end{array}$$

Note that the second row is exact on the left, as $D_\infty^b(F)$ is $\sharp$-filtered, and therefore it is torsion free. Applying the snake lemma, we obtain a long exact sequence

$$H_1^{D_\infty^b}(V) \to \Sigma_\infty H_1^{D_\infty^b}(V) \to H_1^{D_\infty^{b+1}}(V) \to D_\infty^b(V) \to \Sigma_\infty D_\infty^b(V) \to D_\infty^{b+1}(V) \to 0 \quad (3.5)$$

Now assume that $b = 0$. In this case the above becomes the claimed exact sequence of the first part of the proposition.

We can prove the second statement by induction on b. Note that Theorem 3.9 implies that any presentation of a $\sharp$-filtered module will necessarily have a $\sharp$-filtered first syzygy. It follows that it suffices to prove the second claim in the proposition for $i = 1$. Because $\sharp$-filtered objects are torsion free, the first part of this proposition implies the claim for $b = 1$. Otherwise, the exact sequence (3.5) degenerates to,

$$0 \to H_1^{D_\infty^{b+1}}(V) \to D_\infty^b(V) \to \Sigma_\infty D_\infty^b(V)$$

Using the fact that the infinite derivative of a $\sharp$-filtered object is still $\sharp$-filtered, as well as the fact that $\sharp$-filtered objects are torsion free, we obtain our desired vanishing.

Straight forward homological dimension shifting arguments imply that it suffices to prove the third claim for $i = 1$. We proceed by induction on b. If $b = 1$, then the first statement along with Theorem 3.21 imply that $H_1^{D_\infty}(V)$ has finite degree. Assume that the statement is true for some integer $b \geq 1$, and consider the sequence (3.5). By induction we know that $H_1^{D_\infty^b}(V)$ has finite degree, and therefore $\Sigma_\infty H_1^{D_\infty^b}(V) = 0$. The above sequence will simplify to

$$0 \to H_1^{D_\infty^{b+1}}(V) \to D_\infty^b V \to \Sigma_\infty D_\infty^b V \to D_\infty^{b+1} V \to 0.$$

Proposition 3.10 implies that $D_\infty^b(V)$ is degree-wise coherent, and therefore it has finite torsion degree by Theorem 3.21. We conclude that $H_1^{D_\infty^{b+1}}(V)$ has finite degree, as desired.

$\square$

To finish this section, we define an improved version of the complex $\mathscr{C}^\bullet V$. This new complex will share almost all of $\mathscr{C}^\bullet V$'s most important properties, while having the advantage of being functoral in V.

Definition 3.28. *Let V be a degree-wise coherent FI_G-module. Then Theorem 3.15 and Proposition 3.10 imply that $\Sigma_\infty V$ is $\sharp$-filtered, and that there is an exact sequence*

$$V \to \Sigma_\infty V = F^0 \to D_\infty V \to 0$$

where $D_\infty V$ is also degree-wise coherent with strictly smaller generating degree. Repeating this process, we obtain a complex

$$\mathscr{C}^\bullet_\infty V : 0 \to V \to F^0 \to F^1 \to \ldots \to F^n \to 0.$$

Note that by construction,

$$H^i(\mathscr{C}^\bullet_\infty V) \cong \ker(D^{i+1}_\infty V \to \Sigma_\infty D^{i+1}_\infty V) \cong H^{D_\infty}_1(D^{i+1}_\infty V)$$

where D^0_∞ is the identity functor by convention. In particular, the cohomology modules of $\mathscr{C}^\bullet_\infty$ all have finite degree.

3.11.1 Local Cohomology

In this section, we record results about local the local cohomology of the modules in FI-Mod^{coh}. These facts were proven about finitely generated modules in [LR], and the proofs from that paper will work in this context as well, thanks to Theorems 3.24 and 3.15. The fact that these two results imply that the work of [LR] will hold for degree-wise coherent modules was also noted by Li in [L3].

Definition 3.29. *Recall that every FI_G-module V fits into an exact sequence*

$$0 \to V_T \to V \to V_F \to 0$$

*where V_T is torsion, and V_F is torsion free. The **0-th local cohomology functor** is defined by*

$$H^0_{\mathfrak{m}}(V) := V_T$$

*The category FI_G-Mod is Grothendieck, and therefore we can define the right derived functors of $H^0_{\mathfrak{m}}$. The i-th derived functor of $H^0_{\mathfrak{m}}$ is denoted by $H^i_{\mathfrak{m}}$, and is known as the **i-th local cohomology functor**.*

One of the main results of the paper [LR, Theorem E], is that, when working over a Noetherian ring, $H^i_{\mathfrak{m}}(V)$ is finitely generated whenever V is. In this work we will show that $H^i_{\mathfrak{m}}(V)$ is degree-wise coherent whenever V is. To do so, we first record the following alternative definition of local cohomology.

Definition 3.30. *For each integer $r \geq 0$, and each integer $n \geq 1$, we define the module $M(r)/\mathfrak{m}^n M(r)$ to be the quotient of $M(r)$ by the submodule generated by $M(r)_{r+n}$. Then we define the functor $\mathscr{H}om(k\,\mathrm{FI}_G/\mathfrak{m}^n, \bullet) : \mathrm{FI}_G\text{-Mod} \to \mathrm{FI}_G\text{-Mod}$ by*

$$\mathscr{H}om(k\,\mathrm{FI}_G/\mathfrak{m}^n, V)_r := \mathrm{Hom}_{\mathrm{FI}_G\text{-Mod}}(M(r)/\mathfrak{m}^n M(r), V)$$

Note that a map $M(r)/\mathfrak{m}^n M(r) \to V$ is determined by a choice of element V_r, which is in the kernel of all transition maps into V_{r+n}. Given such a map $\phi : M(r)/\mathfrak{m}^n M(r) \to V$, and a morphism $(f,g) : [r] \to [m]$ in FI_G, we define $(f,g)_*\phi$ to be the map $M(m)/\mathfrak{m}^n M(m) \to V$ which sends the identity in degree m to $(f,g)_*(\phi(id_r))$. This defines an FI_G-module stucture on $\mathscr{H}om(k\,\mathrm{FI}_G/\mathfrak{m}^n, V)$. We use $\mathscr{E}xt^i(k\,\mathrm{FI}_G/\mathfrak{m}^n, \bullet)$ to denote the i-th derived functor of $\mathscr{H}om(k\,\mathrm{FI}_G/\mathfrak{m}^n, \bullet)$.

One important observation is that for each $r \geq 0$ and $n \geq 1$, there is a map

$$M(r)/\mathfrak{m}^{n+1}M(r) \to M(r)/\mathfrak{m}^n M(r).$$

This induces maps $\mathrm{Hom}_{\mathrm{FI}_G\text{-Mod}}(M(r)/\mathfrak{m}^n M(r), V) \to \mathrm{Hom}_{\mathrm{FI}_G\text{-Mod}}(M(r)/\mathfrak{m}^{n+1}M(r), V)$, which one may check are compatible with the induced maps of $\mathscr{H}om(k\,\mathrm{FI}_G/\mathfrak{m}^n, V)$. In particular, for any V we obtain a morphism of FI_G-modules

$$\mathscr{H}om(k\,\mathrm{FI}_G/\mathfrak{m}^n, V) \to \mathscr{H}om(k\,\mathrm{FI}_G/\mathfrak{m}^{n+1}, V).$$

This also gives us maps

$$\mathscr{E}xt^i(k\,\mathrm{FI}_G/\mathfrak{m}^n, V) \to \mathscr{E}xt^i(k\,\mathrm{FI}_G/\mathfrak{m}^{n+1}, V)$$

for each $i \geq 0$. This justifies the following proposition.

Proposition 3.12. *There is an isomorphism of functors,*

$$H^0_{\mathfrak{m}}(\bullet) \cong \varinjlim \mathscr{H}om(k\,\mathrm{FI}_G/\mathfrak{m}^n, V),$$

inducing isomorphisms of derived functors

$$H^i_{\mathfrak{m}}(\bullet) \cong \varinjlim \mathscr{E}xt(k\,\mathrm{FI}_G/\mathfrak{m}^n, V)$$

Using this alternative description, one then goes on to prove the following acyclicity results.

Proposition 3.13. *Let V be degree-wise coherent. If V is either a torsion module, or a $\sharp$-filtered module, then*

$$H^i_{\mathfrak{m}}(V) = 0$$

for all $i \geq 1$.

Next, we recall the complex $\mathscr{C}^\bullet_\infty V$. By construction this complex is comprised of $\sharp$-filtered modules in its positive degrees, and its cohomologies are all degree-wise coherent torsion modules. The above proposition can therefore be used to prove the following.

Theorem 3.31. *Let V be a degree-wise coherent module. Then there are isomorphisms for all $i \geq 0$,*

$$H^i_{\mathfrak{m}}(V) \cong H^{i-1}(\mathscr{C}^\bullet_\infty V)$$

In particular, if V is degree-wise coherent, then the same is true of its local cohomology modules.

This theorem has a long list of consequences, some of which we list now.

Corollary 3.32. *Let V be a degree-wise coherent module. Then V is acyclic with respect to local cohomology if and only if there is an exact sequence*

$$0 \to V_T \to V \to V_F \to 0$$

where V_T is a torsion module, and V_F is $\sharp$-filtered.

Corollary 3.33. *Let V be a degree-wise coherent module. Then $H^i_{\mathfrak{m}}(V) = 0$ for $i \gg 0$, while*

$$\mathrm{depth}(V) = \inf\{i \mid H^i_{\mathfrak{m}}(V) \neq 0\}$$

Definition 3.34. *Let V be a degree-wise coherent module which is not $\sharp$-filtered. Then Corollary 3.33 implies that there is a largest i for which $H^i_{\mathfrak{m}}(V) \neq 0$. We define the **dimension** of the module V to be the quantity,*

$$\dim_{\mathrm{FI}_G}(V) := \sup\{i \mid H^i_{\mathfrak{m}}(V) \neq 0\}.$$

If V is $\sharp$-filtered, then we set $\dim_{\mathrm{FI}_G}(V) = \infty$.

Corollary 3.35. *Let V be a degree-wise coherent module. Then,*

$$N(V) = \max_i\{\deg(H^i_{\mathfrak{m}}(V))\} + 1,$$

whenever V is not $\sharp$-filtered.

Corollary 3.36. *Let V be a degree-wise coherent module. Then,*

$$\mathrm{reg}(V) \leq \max_i\{\deg(H^i_{\mathfrak{m}}(V)) + i\}.$$

The reader might have noticed that Corollary 3.36 looks very similar to a classic result from the local cohomology theory of the polynomial ring. Indeed, it is the belief of the author that the following is true.

Conjecture 3.37. *Let V be a degree-wise coherent module. Then,*

$$\operatorname{reg}(V) = \max_i \{\deg(H^i_{\mathfrak{m}}(V)) + i\} \tag{3.6}$$

Remark 3.38. *Note that the above conjecture would be false if our definition of* $\operatorname{reg}(V)$ *included the* 0*-th homological degree. Indeed, Proposition 3.13 implies that* $\sharp$*-filtered modules are torsion free acyclics with respect to local cohomology, and therefore the right hand side of (3.6) is always* $-\infty$ *for* $\sharp$*-filtered modules.*

Note that as of the writing of this paper, not much is known about this conjecture. It was shown to be true for torsion modules by Liang Gan, and Li in their paper [GL2].

Remark 3.39. *The conjecture has, since the original publication of this paper, been proven in full generality by Nagpal, Sam and Snowden in [NSS].*

To finish this section, we more closely examine the relationship between the infinite derivative and local cohomology.

Proposition 3.14. *Let* $H_i^{D_\infty^b} : \mathrm{FI}_G\text{-Mod}^{coh} \to \mathrm{FI}_G\text{-Mod}^{coh}$ *denote the* i*-th derived functor of* D_∞^b*. Then for all* $i, b \geq 1$ *there are natural isomorphisms of functors*

$$H_i^{D_\infty^b} \cong H_{i-1}^{D_\infty^{b-1}} \cong \ldots \cong H_1^{D_\infty^{b-i+1}} \cong H_1^{D_\infty} \circ D_\infty^{b-i}.$$

Proof

Consider the Grothendieck spectral sequence associated to the composition $D_\infty \circ D_\infty^{b-1}$. Note that Proposition 3.11 implies that $H_i^{D_\infty}(V) = 0$ for all $i > 1$, and all degree-wise coherent modules V, and therefore this spectral sequence only has two columns. The spectral sequence will therefore degenerate to the collection of short exact sequences

$$0 \to D_\infty H_i^{D_\infty^{b-1}}(V) \to H_i^{D_\infty^b}(V) \to H_1^{D_\infty}(H_{i-1}^{D_\infty^{b-1}}(V)) \to 0.$$

Proposition 3.11 tells us that $H_i^{D_\infty^{b-1}}(V)$ has finite degree, and therefore the left most term in these exact sequences is always zero. This same proposition also implies that $H_1^{D_\infty}(H_{i-1}^{D_\infty^{b-1}}(V)) \cong H_{i-1}^{D_\infty^{b-1}}(V)$ whenever $i > 1$. Naturality of the isomorphisms $H_i^{D_\infty^b}(V) \cong H_{i-1}^{D_\infty^{b-1}}(V)$ follows from the naturality of the Grothendieck spectral sequence. The result now follows by induction.

□

Theorem 3.31 tells us that the local cohomology modules of a degree-wise coherent module V can be computed as the torsion submodules of the infinite derivatives of V. Proposition 3.14 directly relates these torsion modules to the derived functors of these infinite derivatives. Putting everything together, we have proven the following theorem. One may think of this as a kind of “local duality,” as it relates the local cohomology functors to the derived functors of some right exact functor.

Theorem 3.40. *Let V be a degree-wise coherent module of dimension d. Then there are isomorphisms for all $i \geq 1$,*

$$H_i^{D_\infty^{d+1}}(V) \cong H_{\mathfrak{m}}^{d+1-i}(V).$$

Proof

Theorem 3.31 and Proposition 3.14 imply

$$H_i^{D_\infty^{d+1}}(V) \cong H_1^{D_\infty}(D_\infty^{d+1-i}V) \cong H^{d-i}(\mathscr{C}_\infty^\bullet V) \cong H_{\mathfrak{m}}^{d+1-i}(V).$$

□

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