Tiles, Game, And Coordinate Plane: Exploring Learning Vector Addition Through Concreteness

Fading

by

Yilang Zhao

A dissertation submitted in partial fulfillment of
the requirements for the degree of

Doctor of Philosophy
(Curriculum and Instruction)

at the
UNIVERSITY OF WISCONSIN-MADISON
2023

Date of final oral examination: 05/02/2023

The dissertation is approved by the following members of the Final Oral Committee:
Matthew Berland, Professor, Curriculum and Instruction
Mitchell Nathan, Professor, Educational Psychology
YJ Kim, Assistant Professor, Curriculum and Instruction
Martina Rau, Associate Professor, Educational Psychology
Rosemary Russ, Associate Professor, Curriculum and Instruction
ACKNOWLEDGMENTS

First and foremost, I would like to express my deepest gratitude to Ms. Jamie Gibson of Horizon Science Academy Columbus Middle School. Without your consistent support, there would not have been this dissertation. I am extremely grateful to you for the days and nights spent with me after the regular school hours. I once thought doing my dissertation research in Columbus, Ohio, was like playing a road game that was often much tougher, but your backing made everything easier on my side. I will never forget what you told me: “if no one supports doctoral students’ research, we will have no doctors.”

In addition, this dissertation would not have been possible without the support from my advisors, Dr. Matthew Berland and Dr. Mitchell Nathan. When this project was still a collection of random ideas, it was your help that made it a doable dissertation project. Also, I would like to extend my sincere thanks to my committee faculty members—Dr. YJ Kim, Dr. Martina Rau, and Dr. Rosemary Russ. I had classes with you all, and our communication extensively inspired me.

Furthermore, I am extremely grateful to my parents, Hongyi Zhao and Guizhen Lang. Our regular Friday video chatting provides me with an invaluable sense of belonging and an implacable feeling of being loved.

Last but not least, I could not have undertaken this journey without my partner, Fuyi Feng. When we first met, I was still a second-year doctoral student, and I told you I wanted to be Dr. Zhao someday. With your endless support, I am glad that I made it today! I am overall a pretty boring person, and my life before you were mostly in black and white. Thank you for bringing those delightful colors into my life and making it vivid.
ABSTRACT

This study explores learning vector addition and demonstrating the understanding of vector addition. Vector addition is a key skill in mathematics and science education, but there is a limited amount of research that focuses on this topic. My study investigates how an instructional design framework concreteness fading can be used for learning vector addition and how learners demonstrate their understanding through a constructionist storytelling activity. My research questions are 1) how does “concreteness fading” structure students’ sense-making around vector addition? 2) how do gestures interact with “concreteness fading” in shaping students’ understanding of vector addition? 3) how might the understanding and skills gained from “concreteness fading” be applied to create stories?

In response to these research questions, I designed an intervention that incorporated a physical activity, a computer simulation game, and a worksheet to implement concreteness fading as well as an extra story design and making activity for learning demonstration. I recruited 26 8th graders from a middle school in a large Midwestern city in the United States. By conducting qualitative analysis with grounded theory method and video analysis, I found that 1) concreteness fading fostered students’ mathematical sense-making; 2) gestures could establish common ground and show transitional knowledge states; 3) students could incorporate the knowledge from the intervention and their prior knowledge to create stories to show their understanding of vector addition.

My study extends the current literature by exploring how concreteness fading can be used to teach a complex mathematical topic and showing how concreteness fading learning outcomes can be displayed through a constructionist storytelling activity. Future research may test the replicability of my findings and extend this framework to other domains.
# TABLE OF CONTENTS

**ACKNOWLEDGMENTS** .................................................................................................................. 1

**ABSTRACT** ................................................................................................................................. II

**TABLE OF CONTENTS** .................................................................................................................. III

**LIST OF TABLES** .......................................................................................................................... V

**LIST OF FIGURES** ......................................................................................................................... VI

**CHAPTER ONE: INTRODUCTION** ................................................................................................. 1

  - Background and Research Questions ......................................................................................... 1
  - Methodology Overview ............................................................................................................... 5
  - Significance of the Study ........................................................................................................... 6
  - Chapter Overviews ..................................................................................................................... 6
  - Term Definitions ........................................................................................................................... 7

**CHAPTER TWO: LITERATURE REVIEW** ......................................................................................... 9

  - Underexplored Vector Addition Learning .................................................................................. 9
  - Concreteness Fading as an Instructional Design Framework .................................................. 11
  - Exploring Learning from the Process ......................................................................................... 16
  - Investigating Learning through Gestures .................................................................................. 20
  - Constructionist Storytelling ........................................................................................................ 23
  - Chapter Summary ....................................................................................................................... 26

**CHAPTER THREE: INSTRUCTIONAL DESIGN AND METHODS** ................................................... 27

  - Study Design .............................................................................................................................. 27
  - Participants ................................................................................................................................. 33
  - Data Collection ........................................................................................................................... 33
  - Data Analysis ............................................................................................................................... 34
    - Grounded Theory Method ...................................................................................................... 35
    - Video Analysis on Gestures .................................................................................................... 39
    - Pointing Gesture Types .......................................................................................................... 40
    - Representational Gesture Types ......................................................................................... 43
    - Coding Gestures ...................................................................................................................... 46

**CHAPTER FOUR: MAKING SENSE OF VECTOR ADDITION** ........................................................... 48

  - Results ....................................................................................................................................... 48
    - Connections Across Tasks ..................................................................................................... 48
    - The Affordances of Color ...................................................................................................... 51
    - Direction and Magnitude ....................................................................................................... 53
    - Operations ............................................................................................................................... 57
    - Meaning of Symbols in AR .................................................................................................... 63
    - Structural Understanding ...................................................................................................... 66
  - Discussion ..................................................................................................................................... 69
    - Findings .................................................................................................................................... 69
    - Limitations and Design Principles .......................................................................................... 73

**CHAPTER FIVE: SHOWING UNDERSTANDING THROUGH GESTURES** ....................................... 75

  - Results ....................................................................................................................................... 75
    - Overall Gestures Coding Results .......................................................................................... 75
    - Common Ground and Transitional States in Concreteness Fading Condition ................... 77
    - Common Ground and Transitional States in Concreteness Introduction Condition ........... 93
  - Discussion ..................................................................................................................................... 100
    - Common Ground ................................................................................................................... 100
LIST OF TABLES

TABLE 1 Term Definitions ........................................................................................................................................................................... 8
TABLE 2 McNeill’s (1992) Four Gesture Types .................................................................................................................................................. 20
TABLE 3 Conceptual Story Design Sheet .................................................................................................................................................... 31
TABLE 4 Tasks for Each Condition ............................................................................................................................................................... 34
TABLE 5 Gesture Coding Scheme ................................................................................................................................................................. 39
TABLE 6 Connections Open Codes Counts and Coverage ............................................................................................................................... 50
TABLE 7 Color Affordances Open Codes Counts and Coverage ....................................................................................................................... 52
TABLE 8 Direction Open Codes Counts and Coverage .................................................................................................................................. 54
TABLE 9 Different Directions Across Tasks ................................................................................................................................................ 55
TABLE 10 Magnitude Open Codes Counts and Coverage ............................................................................................................................... 56
TABLE 11 Operations in EP Open Codes Counts and Coverage ..................................................................................................................... 58
TABLE 12 Operations in ID Open Codes Counts and Coverage ................................................................................................................... 60
TABLE 13 Operations in AR Open Codes Counts and Coverage .................................................................................................................. 61
TABLE 14 Meaning of Symbols in AR Open Codes Counts and Coverage ..................................................................................................... 65
TABLE 15 Focused Codes of Structural Understanding for Different Conditions ............................................................................................. 66
TABLE 16 Structural Understanding Open Codes Counts and Coverage ...................................................................................................... 69
TABLE 17 Overall Gesture Coding Results .................................................................................................................................................. 76
TABLE 18 Descriptive Statistics for Gestures Across Tasks ............................................................................................................................. 76
TABLE 19 Learning Objectives and Contexts by Group ................................................................................................................................. 108
TABLE 20 Focused Codes and Categories for Constructionist Storytelling ..................................................................................................... 109
TABLE 21 Vector Understanding Open Codes Counts and Coverage ............................................................................................................. 111
TABLE 22 Apply New Learning Open Codes Counts and Coverage .............................................................................................................. 112
TABLE 23 Coordinates Open Codes Counts and Coverage ......................................................................................................................... 118
LIST OF FIGURES

FIGURE 1 DIFFERENT TYPES OF MATHEMATICAL CONCEPTIONS (SFARD, 1991) .................................................. 18
FIGURE 2 TASK ENACTIVE PHYSICALITY ................................................................. 28
FIGURE 3 SCREENSHOT OF THE FOOTBALL SIMULATION GAME IN TASK ICONIC DEPICTION .......................... 29
FIGURE 4 QUESTIONS IN TASK ABSTRACT REPRESENTATION ............................................. 30
FIGURE 5 PARTICIPANTS IN TASK CONSTRUCTIONIST STORYTELLING .............................. 31
FIGURE 6 CAMPUS MAP IN TASK CONCRETENESS REINFORCEMENT ................................. 32
FIGURE 7 THE WORKFLOW OF GROUNDED THEORY METHOD (ADAPTED FROM TWEED & CHARMAZ, 2011) .......... 36
FIGURE 8 AN EXAMPLE OF USING COLORS TO DIFFERENTIATE SEGMENTS ......................................... 37
FIGURE 9 POINTING REAL-WORLD OBJECTS ............................................................... 41
FIGURE 10 POINTING SCREEN OBJECTS ................................................................. 41
FIGURE 11 POINTING WORKSHEET OBJECTS ........................................................................ 42
FIGURE 12 POINTING GESTURES REFERRING TO VIRTUAL SPACE ........................................ 42
FIGURE 13 POINTING GESTURES TO INDEX REPRESENTATIONS ........................................ 43
FIGURE 14 REPRESENTATIONAL GESTURES TO SIMULATE ACTIONS .................................... 44
FIGURE 15 REPRESENTATIONAL GESTURES TO SIMULATE MATHEMATICAL OBJECTS .................. 44
FIGURE 16 REPRESENTATIONAL GESTURES TO REPLACE SPEECH ........................................ 45
FIGURE 17 REPRESENTATIONAL GESTURES TO ABSTRACT POINTING ................................... 46
FIGURE 18 AN EXAMPLE OF GESTURE CODING SPREADSHEET ............................................. 47
FIGURE 19 CODES FOR CONNECTIONS ................................................................................. 49
FIGURE 20 CODES FOR COLOR AFFORDANCES ......................................................................... 52
FIGURE 21 CODES FOR UNDERSTANDING DIRECTIONS ..................................................... 54
FIGURE 22 CODES FOR UNDERSTANDING MAGNITUDE ..................................................... 56
FIGURE 23 CODES FOR OPERATIONS IN TASK EP ............................................................ 58
FIGURE 24 CODES FOR OPERATIONS IN TASK ID .................................................................. 59
FIGURE 25 CODES FOR OPERATIONS IN TASK AR .................................................................. 60
FIGURE 26 COPY AND PASTE APPROACH FOR Q1 ............................................................. 62
FIGURE 27 CODES FOR MEANING OF SYMBOLS IN TASK AR ............................................... 65
Figure 28 Codes for Other Structural Understanding .......................................................... 68
Figure 29 Chris Pointing at Tiles to Create Common Ground ................................................. 78
Figure 30 Chris Showing a Transitional State While Placing Sticks ........................................ 79
Figure 31 Chris Elaborating the Idea of a Right Angle on Tiles .............................................. 80
Figure 32 Chris Showing Half of the Square on Tiles ............................................................ 81
Figure 33 Chris Counting Physical Objects to Measure Distance on Tiles ................................. 81
Figure 34 Chris Explaining Relationship between Red and Blue Sticks .................................... 82
Figure 35 Chris Using Pointing Gestures to Suggest in the Simulation Game ............................ 83
Figure 36 Chris Referring to Screen Objects ........................................................................ 84
Figure 37 Chris Suggesting Subtracting in the Simulation Game .............................................. 84
Figure 38 Chris Using Gestures to Explain Connections between Tasks EP and ID ................... 85
Figure 39 Chris Using Gestures to Explain the Relationship between the XY Values and the Ball Landing Position ........................................................................................................ 86
Figure 40 Chris Using Gestures to Explain Thoughts about Q1 ............................................... 87
Figure 41 Chris Using Gestures to Explain His Answer to Q1 .................................................. 88
Figure 42 Chris Using Gestures to Show Rough Ideas about Q2 .............................................. 89
Figure 43 Chris Using Gestures to Explain His Thoughts about Q2 .......................................... 90
Figure 44 Chris Using Gestures to Explain His Answer to Q2 .................................................. 91
Figure 45 Chris Explaining the Vector Symbol ........................................................................ 92
Figure 46 Chris Explaining His Arrow for Q2 ........................................................................ 93
Figure 47 Ian Explaining Arrow Symbol in Q1 ......................................................................... 94
Figure 48 Ian’s Metaphor about Q1 .......................................................................................... 94
Figure 49 Ian Explaining His Answer to Q1 ............................................................................ 95
Figure 50 Ian Explaining His Drawing for Q2 ......................................................................... 96
Figure 51 Ian Explaining the Relationship between Two Questions ......................................... 97
Figure 52 Ian Explaining the Relationship between the Red/Blue Arrows and the Yellow Arrow .. 98
Figure 53 Ian Talking about the Connections between Tasks AR and ID .................................. 99
Figure 54 Ian Explaining the Relationship between Sticks ........................................................ 99
FIGURE 55 Ian’s Additional Explanations of Sticks ................................................................. 100
FIGURE 56 Codes for Vector Understanding ............................................................................. 110
FIGURE 57 Codes for Apply New Learning ............................................................................. 112
FIGURE 58 Codes for Participant Knowledge ......................................................................... 114
FIGURE 59 Coding Stripe for CF Group 2.............................................................................. 116
FIGURE 60 Codes for Coordinates in Story ............................................................................ 118
FIGURE 61 Codes for Artifact Making-Aesthetics ................................................................... 119
FIGURE 62 Coding Stripe for CO Group 1’s Knowledge Integration ...................................... 121
CHAPTER ONE: INTRODUCTION

Background and Research Questions

This dissertation is an empirical study that includes an instructional design to teach eighth graders vector addition and let them demonstrate their understanding of vector addition. First, I would like to talk about the value of researching vector addition learning. Vector addition has been identified as a key skill in secondary mathematics and science education. According to the *Principles and Standards for School Mathematics* (National Council of Teachers of Mathematics, 2000), in the section *Number and Operations*, 9-12 graders are expected to understand vectors and conduct vector addition through concrete contexts. Furthermore, in the *Next Generation Science Standards* (National Research Council, 2013), one of the disciplinary core ideas is *motion and stability* under the section *Forces and Interactions*, requiring students to identify the pattern of an object’s movement in various situations; to predict future motion from the pattern of past motion, students need the vector addition skill.

There are studies that cover the topic of vector from a mathematics education perspective (e.g., Mai et al., 2017; Spyrou et al., 2021; Watson et al., 2003) and a physics education perspective (e.g., Shaffer & McDermott, 2005; Van Deventer & Wittmann, 2007; White, 1983), but most of those studies focus on the conceptual understanding of vector rather than the skill of vector addition. Although developing a solid conceptual understanding of vector seems to be a prerequisite for acquiring the vector addition skill, vector addition, as a procedural skill, may be easier for students to begin with. Sfard (1991) states that for a new mathematical concept, operational knowledge usually precedes structural knowledge; in this study’s context, learners might be able to conduct vector addition before they establish a solid conceptual understanding of vector. Thus, this study focuses on how students learn vector addition.
Although vector addition is one of the most important skills in secondary mathematics education, there is a need for more research about vector addition learning in secondary mathematics education. According to Knight (2002), only one-third of college introductory physics students have a working knowledge of vector addition; another one-third have a partial knowledge but make significant errors on simple vector addition questions; the rest one-third have no usable knowledge of vector. Poorly grounded vector addition skills will cause tremendous difficulties in students’ postsecondary STEM courses (Knight, 2002). One possible reason is that current secondary-level mathematics education extensively relies on abstract concepts and symbols, which is criticized as “formalism first” by Nathan (2012). Unfamiliar symbols and notational systems leave students with confusion and impede their future vector-related learning. Thus, how to empower students to make sense of vectors and establish grounded vector addition skills becomes an emerging topic for both secondary educators and educational researchers.

For building students’ solid vector addition skills, concreteness fading (CF), which refers to an instructional approach allowing students to start learning an abstract concept with concrete learning materials in specific learning contexts and gradually transition to abstract learning materials in more general learning contexts (Fyfe et al., 2014; Goldstone & Son, 2005), seems to be a promising candidate for teaching middle school and early high school students vector addition. A CF intervention allows students to start their mathematics learning with physical activities, which they feel more comfortable with compared to unfamiliar mathematical symbols, and gradually helps them transition to more abstract representations of the same concept by providing connections among different tasks. Therefore, in this study, I designed a CF intervention that aimed to teach middle schoolers vector addition.
Although previous CF studies demonstrated the efficacy of teaching the equal sign (McNeil & Fyfe, 2012), the basic circuits (Jaakkola & Veermans, 2018), the complex system thinking (Goldstone & Son, 2005), etc., the underlying learning mechanism in a CF intervention still needs further scrutinization (Fyfe & Nathan, 2019). Also, there are no previous CF studies concentrating on a complex mathematics topic. Thus, in this study, I explored the learning in a CF intervention that taught vector addition from two aspects: mathematics sense-making and the use of gesture. Mathematical sense-making requires learners to understand and interpret mathematical concepts and problems in a meaningful way, be able to reason their problem-solving logic, and use their own words to explain those concepts (Schoenfeld, 1992). Exploring whether students can present their own logic to solve vector addition problems and use their own words to explain their problem-solving strategies can help understand their learning in my CF intervention. Hence, my first research question is:

**RQ1: How does “concreteness fading” structure students’ sense-making around vector addition?**

Besides, observing students’ use of gestures in the CF intervention can also be a good approach to understanding their learning since people use gestures to express those ideas that cannot be fully delivered through language (Hostetter & Alibali, 2019). Gestures can reveal learners’ mental representations of a concept (Nemirovsky & Ferrara, 2009) and their way of grounding unfamiliar concepts and formalisms (Koedinger et al., 2008). More specifically, when learners are in the process of familiarizing themselves with new concepts, their gestures only imply their partial knowledge, multiple hypotheses for problem-solving (Goldin-Meadow et al.,
1993), and shared understanding when in a group learning setting (Alibali et al., 2013). By exploring learners’ pointing, representational, and metaphorical gestures (a taxonomy for researching gestures in mathematics instruction established by Alibali & Nathan, 2012), my second research question is

**RQ2: How do gestures interact with “concreteness fading” in shaping students’ understanding of vector addition?**

With concrete and abstract learning materials embedded, there is supposed to be learning taking place during the process of a CF intervention. I am also interested in the transferability of knowledge gained from a CF intervention. Previous CF studies often use knowledge tests (e.g., Jaakkola & Veermans, 2018; McNeil & Fyfe, 2012) to examine the learning outcomes of a CF intervention. Knowledge tests are good at estimating structural mathematical knowledge but may not be effective at evaluating procedural mathematical knowledge, because multiple choices and blank-filling questions limit the space for learners to demonstrate their understanding. Aiming at the limitation of knowledge tests and inspired by Papert's (1980) constructionism, I proposed a storytelling activity in which learners would design, make, and tell their own story to demonstrate their understanding of vector addition gained from the CF intervention. Thus, my third research question is:

**RQ3: How might the understanding and skills gained from “concreteness fading” be applied to create stories?**
Methodology Overview

To answer the above three research questions, I designed a three-task CF intervention with an additional constructionist storytelling activity. The intervention had three tasks (Enactive Physicality, Iconic Depiction, and Abstract Representation), all in a football context: in the task Enactive Physicality, the participants were involved in a physical activity in which they delivered and passed the football on a tiled floor by using informal vector addition knowledge; in the task Iconic Depiction, the participants played a football passing simulation game that included some formal mathematical symbols; in the task Abstract Representation, the participants worked on a worksheet with textbook-style vector addition questions. After the three tasks, the participants were provided with crafting materials to design and make a story to demonstrate their learning. There were two conditions: concreteness fading (CF) and concreteness introduction (CI). The participants in the CF condition followed the order of Enactive Physicality, Iconic Depiction, and Abstract Representation to complete the learning tasks, while the participants in the CI condition followed a reversed order. Both conditions took the following constructionist storytelling activity. The reason for setting up two different conditions was both concrete-to-abstract and abstract-to-concrete sequences could lead to optimum learning (Suh et al., 2020). Thus, I was also interested in how the order of progression might influence learning vector addition, yet due to the limited sample size, I would avoid comparing two conditions or producing generalizable statistical significance.

In this study, two qualitative data analysis methods were used. Since there were no previous studies exploring the vector addition learning in a CF intervention, grounded theory method was used to analyze discourse data to answer RQ1 and RQ3. Grounded theory method used data to generate theories (Glaser & Strauss, 1967), which was appropriate for explorative
research questions. For RQ2, I used video analysis to inspect the use of gestures by first segmenting gesture video clips by different types of gestures (Alibali & Nathan, 2012) and then conducting an inductive coding to analyze those gestures.

**Significance of the Study**

Since vector addition learning is underexplored, this study can expand the literature on vector addition education by investigating how middle schoolers understand vector addition through a progressive instructional activity in which they are exposed to different formats of vector components and how they demonstrate their understanding of vector addition through story-making and storytelling. Besides, this study can also expand the literature of concreteness fading research by scrutinizing the learning process of concreteness fading (e.g., the transformation of representations) and providing an innovative approach to enable learners to display their knowledge gained from a CF intervention.

In addition, practitioners can also benefit from this study. The instructional design in this study can inspire mathematics educators to rethink the approach they deliver mathematics knowledge to their students and the way they understand their students’ acquisition of new mathematics concepts. Furthermore, the instructional design of this study can excite them to utilize concreteness fading as an instructional design framework to develop lessons on other STEM topics.

**Chapter Overviews**

Chapter one provides an overview of the study background, raises research questions, synopsizes the methods, and defines terms. Chapter two reviews the literature about vector
addition learning, concreteness fading, gesture use in mathematics learning, and constructionism
to establish a theoretical foundation for this study. Chapter three introduces the instructional
design of this study by presenting details about the three learning tasks and the constructionist
storytelling activity as well as the process of data collection and the data analysis methods.
Chapter four presents the analysis of the discourse during the learning tasks to examine the
participants’ mathematics sense-making in response to RQ1. Chapter five also focuses on the
learning tasks by displaying the analysis on the use of gestures to explore how the participants
establish common grounds and show transitional knowledge. Chapter six is about the
constructionist storytelling activity and demonstrates how the participants demonstrate their
learning from the intervention by examining their story-making process. The last Chapter seven
concludes the study, and proposes questions and suggestions for future studies.

**Term Definitions**

There are six acronyms used in this dissertation (see Table 1), all of which are the
shortened names of conditions or tasks in the instructional design of this study.

<table>
<thead>
<tr>
<th>Term</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>CF</td>
<td>Concreteness Fading, the instructional design framework and also the condition that has learning tasks following the order of Enactive Physicality, Iconic Depiction, and Abstract Representation.</td>
</tr>
<tr>
<td>CI</td>
<td>Concreteness Introduction, the condition that has learning tasks following the order of Abstract Representation, Iconic Depiction, and Enactive Physicality.</td>
</tr>
<tr>
<td>EP</td>
<td>Enactive Physicality, the learning task in which the participants take part in the activity on the tiled floor to experience informal vector addition.</td>
</tr>
<tr>
<td>ID</td>
<td>Iconic Depiction, the learning task in which the participants play a computer simulation that requires them to use informal vector addition knowledge to solve problems.</td>
</tr>
<tr>
<td>AR</td>
<td>Abstract Representation, the learning task in which the participants work on textbook style questions.</td>
</tr>
<tr>
<td>CS</td>
<td>Constructionist Storytelling, the task that allows the participants to create and construct a story to demonstrate their understanding of vector addition gained from previous learning tasks.</td>
</tr>
<tr>
<td>CRReinf</td>
<td>Concreteness Reinforcement, the learning task in which the participants work on food delivery problems that relate to the movement on the coordinate plane.</td>
</tr>
<tr>
<td>--------</td>
<td>---------------------------------------------------------------------------------------------------------------------------------</td>
</tr>
<tr>
<td>ARReinf</td>
<td>Abstraction Reinforcement, the learning task in which the participants watch the lecture videos that teach vector addition and work on exercise questions.</td>
</tr>
</tbody>
</table>

Table 1 Term Definitions
CHAPTER TWO: LITERATURE REVIEW

In this chapter, there are five questions to answer: 1) how did previous studies research vector addition learning? 2) What is concreteness fading, and why can it be a promising instructional design framework? 3) Why should the process of learning be researched? 4) How can we inspect learning through observing gestures? 5) Why can constructionism be a good design framework for a learning demonstration activity? I explored relevant theories and empirical studies to answer the five questions above and identified the gap in the literature.

Underexplored Vector Addition Learning

Unlike the conception of numbers which is ubiquitous in daily life, vector addition seems more intangible to middle schoolers. Although students might have informal vector experience, such as experiencing the combination of forces, as an advanced topic in secondary mathematics education, vector addition may cause students ample difficulties and will stem their learning of advanced mathematics and physics courses in the future if they lack solid understanding (Aguirre & Erickson, 1984). However, only a limited amount of previous literature concentrates on vector addition learning, and several typical misconceptions are identified. By investigating how undergraduate students perform vector addition tasks in introductory-level physics classes, Nguyen and Meltzer's (2003) study reveals that students use the “tip-to-tail” method and the parallelogram addition rule by rote to conduct vector addition and often mistakenly apply the Pythagorean theorem to deal with vector addition problems. Also, in an introductory level college physics course, Knight's (1995) research discovers that when encountering vector addition problems, students have difficulties in understanding vector components and fail to connect vector’s algebraic representations with its geometric content. These two studies indicate
that lacking a solid understanding of vector addition causes problems in college-level mathematics and physics courses, and there are also studies with different age groups. Since vector addition is often introduced in secondary mathematics classes, Gubrud and Novak (1973) recruit eighth, ninth, and tenth graders’ to investigate how they meaningfully intertwine their previously learned and new information to perform vector addition. Their study reveals that although all three graders can learn the original materials, compared to ninth and eighth graders, tenth graders are better at stabilizing and structuralizing vector addition as new knowledge to their existing knowledge during the instruction and have better performance six weeks later.

Furthermore, another study conducted by Aguirre and Erickson (1984) also invites ten graders to work on a vector addition worksheet to estimate their performance on vector addition in a physics context; they discover that those students have the awareness that the direction and the magnitude of the resultant are influenced by the two contributing components by making inferences in a concrete setting, which is a boat’s movement is influenced by currents in the river.

The above studies cover the topic of vector addition learning, but they have limitations to some extent. Although Nguyen and Meltzer’s and Knight’s studies provide insights about students’ difficulties in performing vector addition, their subjects are college students in introductory-level physics classes. There are two potential problems with this subject group: 1) students begin their vector addition learning in their 9th grade, so their mathematical developmental levels vary from college students at that age; 2) their audience is students in the physics classes, who might already have basic vector addition knowledge. The knowledge base of their subjects prevents the researchers from discovering real problems with learning vector addition for first-time learners. Furthermore, Gubrud and Novak’s and Aguirre and Erickson’s
studies have appropriate subjects, but Gubrud and Novak’s study focuses on comparing the performance of different graders and does not have deep scrutiny on the underlying mechanism of how the students gain vector addition skills, and meanwhile Aguirre and Erickson’s study focuses more on vector’s conceptual learning and its impact on vector addition. In addition, all these studies rely on abstract learning materials (textbook style materials and subsequent knowledge tests) to investigate learning phenomena on vector addition and there is a need to examine learning vector addition in a multimodal way that incorporates both abstract and concrete learning.

Overall, there is a limited amount of literature exploring vector addition learning and the materials that those studies used are mostly textbook-style worksheets, which are more appropriate for investigating students’ deficits in vector addition skills but less apt for understanding the development of their mathematical thinking and sense-making on this topic. Thus, designing an intervention that includes multiple types of learning materials for students to learn vector addition and to comprehend their underlying learning process during the intervention is necessary.

**Concreteness Fading as an Instructional Design Framework**

To understand why CF can be a promising instructional design framework for vector addition learning, it is essential to probe the advantages and disadvantages of abstract and concrete learning as well as the rationale and necessity of combining these two modes of learning.

When teaching students abstract mathematical concepts such as vector addition, a common practice in the current secondary-level mathematics class is learning through abstract
materials. Abstract learning often requires students to build abstract representations of the concepts they are supposed to learn. Since abstraction can exclude unnecessary perceptual details of an object (Fyfe et al., 2014), using abstract learning materials enables students to generalize their gained knowledge, carry it into new contexts (Kaminski et al., 2009), and focus more on underlying structural and representational aspects instead of superficial characteristics (Kaminski et al., 2009; Uttal et al., 1997). Nathan (2012), however, holds the perspective that against abstract learning and claims that abstraction is often associated with a notational system that lacks meaning to those who are not familiar with the convention. The absence of learner’s personal meaning will plausibly fail to let the learner connect the learning content with something familiar, to fulfill the learner’s goals to make the learning experience useful, or to stimulate the learner’s identity to foster self-motivated learning (Priniski et al., 2018). In addition, relevant research discovers that abstract representations often lead to ineffective and more error-prone problem-solving (Koedinger & Nathan, 2004) and rigid application of learned procedures (McNeil & Alibali, 2005). In mathematics teaching and learning, teachers may face a dilemma that students, with abstract knowledge, are able to identify a similar problem and select the apt procedure, yet they lack comprehension on how the procedure works and often make illogical mistakes (Fyfe & Nathan, 2019). This phenomenon is described by Wertheimer (2020) as “unproductive thinking” that refers to students finding the malfunctioning of those standardized procedures; one possible reason for having unproductive thinking is the students’ knowledge to produce standardized procedures is rote memorized instead of meaningfully grounded. Moreover, abstraction-oriented mathematics is based on information-processing learning theories, such as Clements and Battista's (1992) theories about spatial reasoning, Gutiérrez et al.'s (1991) evaluation paradigm, and Hiele's (1986) levels of geometric reasoning.
Although information-processing theories provide a scientific lens to explore how mathematical thinking develops, those theories have the limitations similar to the drawbacks of stimulus-response behaviorism: 1) regarding learning as a passive, atomistic, and mechanical process; 2) neglecting the contexts that actual learning, as a cognitive process, takes place; and 3) researching learning in an artificial laboratory setting and ignoring learner’s affective, social, biological, and metacognitive aspects of cognition (Mayer, 1996).

Due to the disadvantages abstract learning has, many may think that incorporating concrete materials, such as Diene’s blocks (Marlow, 2023), in mathematics education is an alternative. One important thing to clarify is the concept of “concreteness:” what kind of materials can be considered concrete? Koedinger et al. (2008) use the term “grounded” to describe the representations that relate to physical objects and everyday events. Specifying Koedinger et al.'s (2008) notion of “grounded” representations, Fyfe and Nathan (2019) defined four types of information to determine the level of concreteness: 1) physicality (whether the object is two-dimensional or three-dimensional), 2) perceptual richness (the visual surface features of an object), 3) dependency on existing knowledge (how conceptual familiarity can help with understanding a representation), and 4) the narrative context to which the representation belongs (how linguistic familiarity can help with understanding a representation). With these four types of information embedded, concrete learning materials have at least four potential benefits in mathematics learning: 1) concrete materials allow learners to activate their real-world knowledge in their learning experiences (Schliemann & Carraher, 2002); 2) concrete materials enable learners to construct their own knowledge of abstract concepts (Brown et al., 2009); 3) concrete materials can encourage learners to involve physical actions that improve understanding and knowledge retention (Martin & Schwartz, 2005); and 4) concrete materials
avoid lengthy procedural instruction and prevent students from quickly getting disengaged (Nathan, 2012). On the other hand, concrete materials may cause students the trouble of being distracted by irrelevant perceptual details (Kaminski et al., 2008), being attracted by the materials themselves rather than their referents (Uttal et al., 1997), and failing to transfer knowledge from the current context to new settings (Goldstone & Sakamoto, 2003).

Given that abstract and concrete learning both have their own pros and cons, concreteness fading (CF) can be an instructional approach that leverages those advantages and alleviates those disadvantages. CF is inspired by Bruner's (1966) three progressive forms of understanding new concepts—enactive form, iconic form, and symbolic form—and has multiple versions of definition. Bruner (1966) declares that learners first establish representations through actions—like learning tennis skills by observing and mimicking their tennis coach (enactive form)—then attempt to recognize and utilize a visualizable path or pattern that is from their perceptions (iconic form), and lastly form representations in generative words or language (symbolic form). Considering Bruner’s notion of three forms, Goldstone and Son (2005) define CF as “the process of successively decreasing the concreteness of a simulation with the intent of eventually attaining a relatively idealized and decontextualized representation that is still clearly connected to the physical situation that it models.” However, the initial enactive form of a concept may not always be a simulation. A broader definition is proposed by Fyfe and Nathan (2019) as “concreteness fading is the three-step progression by which a concrete representation of a concept is explicitly faded into a generic and idealized representation of that same concept.” According to this definition, CF always starts with a concrete presentation which can be a physical manipulative or a simulation with rich contextual information and eventually transforms into more idealized and transferrable representations by implementing the fading process in
which extraneous perceptual and irrelevant conceptual information is gradually removed from the learning context and mutual references across different representations are established. It is worth noting that CF aims to let learners think about those representations each as an instantiation of the concept at one time instead of keeping all the representations in mind simultaneously and holds an idealized representation of the new concept resided in an invariant relationship across various contexts of learning (Fyfe & Nathan, 2019).

In terms of the benefits of implementing CF, each stage has its own advantages, so does the entire progression. The beginning stage allows students to start comprehending the equivocal abstract representations with concrete objects of which they have plentiful knowledge (Goldstone & Son, 2005; Son et al., 2012) and provides students with embodied learning experiences to connect abstract representations with perceptual and physical processes (Fyfe & Nathan, 2019), which helps them construct conceptual metaphors from motor-sensory experiences to make sense of ambiguous mathematical ideas (Lakoff & Núñez, 2000). Moreover, concepts and representations are not fixed in people’s brains, so people often think about their past experiences and imaginatively reconstruct a perceptual simulation that grounds their comprehension (Barsalou, 2008). Thus, the concrete stage is capable of building learners’ images that can be retrieved when learners are facing symbols that are disconnected from their underlying concepts (Fyfe et al., 2014).

As discussed previously, sole concrete learning has limitations. Hence, besides the first concrete learning stage, the other two stages are equally important. In the second stage, learners are introduced to some formal symbols in a concrete context and unrelated perceptual information is gradually removed. The exclusion of extraneous perceptual information prevents learners from paying attention to the material itself (Kaminski et al., 2008) and the learning
material in this stage plays a scaffolding role that helps learners go through their zone of proximal development (Vygotsky, 1980) and provides them with a smooth process from understanding what they are familiar with to think about something they are not familiar with in a comfortable way, so the second stage can explicitly link the first stage and the final stage by providing mutual referents (Fyfe et al., 2015). The last stage, in which abstract representations are presented, decontextualizes the learning content from concreteness and enables learners to focus on the structural patterns and representational features to promote their generation of portable knowledge and development of alternative representations for the same concept (Fyfe et al., 2014), which may contribute to their preparation for future learning.

Therefore, CF reinforces the advantages of both concrete learning and abstract learning, mitigates the disadvantages of sole concrete or abstract learning, and offers students chances to refer to previously established representations and construct new knowledge in a progressive way. Previous studies explore the efficacy of the CF technique in teaching the equal sign (McNeil & Fyfe, 2012), basic circuits (Jaakkola & Veermans, 2018), and complex systems thinking (Goldstone & Son, 2005). However, there were no previous studies applying CF to vector addition among secondary students, but it has the potential.

**Exploring Learning from the Process**

Although designing a vector addition learning intervention with CF seems promising, understanding the development of mathematical knowledge during the learning process is the subsequent step. Both previous vector addition studies and CF studies concentrate more on learning outcomes rather than the process of learning. For those vector addition studies, Knight's (1995) study incorporates a vector knowledge test that consists of nine abstract questions and
measures the participant’s performance in this test. Similarly, Gubrud and Novak's (1973) study includes a two-part test—one part for testing student’s ability of recalling and using the rules for vector addition, and the other for testing students’ utility of vector addition rules in solving physics problems. Abstract questions are also used in Nguyen and Meltzer's study (2003) and the participants’ performance was measured by their quiz scores. Although Aguirre and Erickson's (1984) study conducts a qualitative analysis to inquire students’ mental models of vector addition, they still provide students with abstract questions in a concrete setting and ask them to explain how they solve the problems. Their study focuses on those answers but lacks an exploration of the process of answer production. For those CF studies, the knowledge test is also the dominant approach to examine learning. Fyfe et al.'s (2014) study encompasses two sets of transfer test problems in which all the questions include a numeric equation with a number blank in one side of the equation to test student’s understanding of the equal sign and their ability to equalize the equation. In Jaakkola and Veermans's (2018) CF study for teaching circuit basics, the participants were asked to complete a knowledge assessment questionnaire that is consisted of multiple choice and blank filing questions to show their learning outcomes. Similarly, Goldstone and Son (2005) utilize a multiple-choice quiz that closely associates with the learning content to assess the participant’s complex thinking ability developed during their CF intervention. Testing knowledge after an intervention seems to be the most direct and effective way to measure learning, but it is also valuable to investigate the knowledge acquisition in the process of learning, especially when designing an intervention to teach a complex mathematics topic.

The value of exploring the process of learning is based on Sfard's (1991) notion that there are two ways of conceiving abstract mathematical concepts—structurally and operationally: a
structural conception is regarding mathematical objects as a static structure as if existing in space and time; on the contrary, an operational conception is treating mathematical concepts as processes, algorithms, and actions. The below figure (Figure 1) is the example she used to explain these two types of mathematical conceptions. The left graph of the parabola is the structural representation of the middle algebraic expression, and the right computer program is the operational representation of the same algebraic expression.

<table>
<thead>
<tr>
<th>Graph</th>
<th>Algebraic expression</th>
<th>Computer program</th>
</tr>
</thead>
</table>
| ![Graph](image) | $y = 3x^4$ | 10 INPUT X  
20 Y = 1  
30 FOR I = 1 TO 4  
40 Y = Y * X  
50 NEXT I  
60 Y = 3 * Y |

Figure 1 Different Types of Mathematical Conceptions (Sfard, 1991)

Sfard (1991) contends that when forming a mathematical concept, operational conceptions often precede structural conceptions. Given the above example, it is easier for students to understand the procedures of the computer program when their teacher helps them go through the program step by step; on the other hand, the graphical representation—the parabola—requires students’ geometrical and algebraical understanding to comprehend the mapping between x and y. Furthermore, Sfard counters the statement that geometric ideas, such as the idea of the circle, are often conceived structurally before the emergence of their procedural descriptions; she declares that geometricians first looked at things that were proximately round and produced the law of roundness before they expressed roundness as a new concept. Therefore,
when learning a new mathematical concept, students often start with operational knowledge including playing with the concept in some concrete contexts rather than immediately recognizing it as a mathematical object with properties.

Sfard’s view on mathematical conception is well aligned with the idea of CF. As mentioned beforehand, a CF intervention starts with concrete learning materials that associate with learners’ previous knowledge and experience, which allows learners to play with the idea using their naïve operational knowledge (e.g., solving problems with physical movement). Gradually, the learners will encounter more structural mathematical conceptions, such as symbols and formal mathematical language. Therefore, investigating the process of learning in a CF intervention can enlighten learner’s processive conversion from the operational representation of a mathematical concept to the structural representation of the same concept.

Besides, researching the process of learning can enable us to understand students’ mathematical sense-making in a CF intervention. Mathematical sense-making refers to the process of understanding and interpreting mathematical concepts and problems in a meaningful way, which involves making connections between different mathematical ideas, using logical reasoning to solve problems, and being able to explain mathematical concepts in one's own words (Schoenfeld, 1992). It can be very hard to make students connect different ideas and present logical problem-solving reasoning in a knowledge test. Therefore, understanding mathematical sense-making in a CF intervention requires researchers to pay more attention to the process of learning.
Investigating Learning through Gestures

When learners are in a process of learning, they often attempt to connect new ideas with their previous perceptual or physical experiences. Regarding how to investigate the underlying mechanism of these connections, observing gestural use can be a good approach since people embody their mathematical ideas in their gestures (Swart et al., 2017) and express their developing ideas through their gestures (Hostetter & Alibali, 2019). Gesture has a close relationship with language. From a linguistic perspective, McNeill (1992) identifies three narrative levels: narrative, metanarrative, and paranarrative, and meanwhile categorizes gestures into four types (see Table 2) that will appear at these levels accordingly. According to McNeill (1992), when someone is telling a story, at the narrative level, events are ordered and have referents from the world of the story, so there are iconic gestures exhibiting the events at this level. At the metanarrative level, instead of referring to events, references are to the story structure and unconstrained by the order of events. Correspondingly, deictic gestures that show the relationship of the original content to the gesture space and metaphoric gestures that view the structure as an object, or a space appear at this level. There are often deictic gestures or even no gestures at the paranarrative level since people’s talk often avoids the narrator’s role and makes references to their own experiences at this level. Thus, gestures are accompanied by languages, and different types of gestures appear at different narrative levels. When observing gestures, it is vital to inspect the narrative content before determining the roles of different gestures.

<table>
<thead>
<tr>
<th>Gesture Type</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>Iconic</td>
<td>A gesture that displays the semantic content of speech</td>
</tr>
<tr>
<td>Metaphoric</td>
<td>A gesture that presents an image of an abstract concept</td>
</tr>
<tr>
<td>Deictic</td>
<td>A gesture that points and refers to concrete entities</td>
</tr>
<tr>
<td>Beat</td>
<td>A gesture that has no perceptible meaning</td>
</tr>
</tbody>
</table>

Table 2 McNeill’s (1992) Four Gesture Types
From a cognitive scientist’s perspective, Kendon (2000) asserts that gestural expressions are involved in the cognitive action of linguistic expression and as bodily visible actions, gestures also include people’s willing expression. Also, when people use gestures to express, representations of concrete entities or abstract symbols are archived in gestures in a way different from spoken languages. To be specific, gestures can add meaning to spoken language, overcome spoken language’s limitations of temporal linearity, and provide more precise and clear expressions to precipitants.

In the field of education, focusing on gestures in an instructional intervention can reveal learners’ mental representations of a concept (Nemirovsky & Ferrara, 2009) and the way that learners ground unfamiliar concepts and formalisms (Koedinger et al., 2008). Adapting McNeill’s gesture types, Alibali and Nathan (2012) categorize gestures into three types in a mathematical learning setting: pointing gestures, which reflect grounded cognition in a physical environment; representational gestures, which demonstrate mental simulations of actions and perceptions; and metaphoric gestures, as a subcategory of representational gesture, which indicates body-based conceptual metaphors.

Previous studies have also examined the functions of pointing and representational gestures. Before exploring the use of gestures, a key concept, common ground, needs to be introduced. Clark and Schaefer (1989) define common ground as the knowledge, beliefs, and assumptions that are mutually shared by people in a conversation or dialogue. Since learning often involves student-teacher and student-student interactions, lacking a common ground (Clark & Brennan, 1991) may cause learning difficulties, because establishing shared understanding in learning activities is key to new concept acquisition and new knowledge building (e.g., Blake & Pope, 2008). Although establishing and keeping a common ground in a mathematics classroom
can be arduous due to the presence of the representations of abstract ideas (Alibali et al., 2013), from an embodied perspective, it is still possible to capture the common ground because mathematical cognition is from people’s perception and action and also rooted in the physical environment (Lakoff & Núñez, 2000). Gestures are often used to create and maintain the common ground between different subjects by depicting shared referents and connecting new representations to familiar representations (Nathan & Alibali, 2011). To be more specific, Alibali and Nathan (2012) find that pointing gestures ground mathematical thinking in the physical environment and representational gestures evince grounded mathematical thinking by simulating actions and reflecting perceptions. Thus, using gestures to create mutually shared references that are grounded from the physical environment and show common perceptual states and actions that can be understood by a group of learners is key to creating and maintaining common ground (Alibali et al., 2013). To better understand learning in a group-based CF intervention, investigating how learners use gestures to create common ground is important.

Regarding acquiring new concepts, learning has been identified as a process of accumulation or replacement: Accumulation theorists consider learning as the acquisition of an increasing amount of information or skills, whereas replacement theorists think learning is qualitative reorganization or change in representations and rules (Mazur & Hastie, 1978). Taking on the replacement learning theory, Goldin-Meadow and her colleagues (1993) assert that there must be a period in which an old concept is gradually replaced by a new idea and they name this period as a transitional state. There are multiple studies examining the transitional state or its equivalence. The transitional state may indicate that 1) learners are ready to acquire a new concept (readiness to learn) after exposing to instruction (Brainerd, 1977); 2) learners have gained partial knowledge and attempt to use a consistent but incorrect strategy to solve problems.
(Siegler, 1981); 3) learners may hold multiple problem-solving hypotheses simultaneously in their mind and activate them when solving and reasoning about a problem (Keil, 1984; Wilkinson, 1982). Since gestures can convey a considerable amount of information when solving tasks (Kendon, 1994; McNeill, 1992), gestures in the transitional state can disclose how a new concept is gradually grasped when learners start to use partial knowledge to strategize and demonstrate the co-existence of multiple hypotheses. One possible way is to observe the mismatch between learner’s gesture and their speech. Goldin-Meadow et al. (1993) argue that when learners are in the process of learning a new concept, the gesture-speech discordance is integral to the transitional state and caused by the transitional knowledge, so the discordance can reflect the process of reasoning and provide insights about the internal process in learner’s mind. Specifically, if learners have multiple hypotheses or representations in their minds, they are likely to have a gesture that doesn’t align with what they are talking about due to the increased cognitive demand in their transitional state.

Therefore, to better explore learning in a CF intervention, a feasible approach is to scrutinize gesture-based common ground that learners created to reflect their shared understanding during the intervention, and also the gestures learners used, especially those mismatched with speech to show their transitional states, because the transitional states can be a significant index of the progressive concept acquisition.

**Constructionist Storytelling**

“Concreteness fading” has been proven as an effective instructional design framework, but a major concern is its transferability (Suh et al., 2020). Few previous studies focus on whether and how the entire “concreteness fading” process can support students’ future
application of their constructed understanding and learned skills. Regarding how to understand student learning in mathematics education, the National Council of Teachers of Mathematics (NCTM) (2000) provides Standards and Principles for School Mathematics that emphasize the role of conceptual understanding and promote building new knowledge from experience and previous knowledge. NCTM (2000) also proposes that mathematics should be regarded as an integrated field instead of collective strands and standards, and mathematics education should encourage students to establish connections between mathematical topics with other related contexts of their interests and experiences. What NCTM addresses is not only the acquisition of mathematics content knowledge but also the utility of mathematics, which aligns with Seymour Papert’s (1980) idea of constructionism and makes constructionism an option to comprehend student’s mathematical understanding.

Papert's (1980) theory of constructionism contends that deep learning emerges when learners explore and play with ideas to learn by engaging in personal self-driven projects. Papert emphasizes learning through making because in the process of making, learners would be able to learn with their favorite representations, artifacts, and, more importantly, their object-to-think-with (Ackermann, 2001). Metaphorically, the object-to-think-with is like soap sculptures that learners are making; instead of making something ‘correct,’ they intend to build what they have in mind (Harel & Papert, 1991). This object-to-think-with helps learners establish connections between sensory and abstract knowledge as well as between the individual and the context, which shapes them in a personalized way to demonstrate their understanding. Given utilizing the object-to-think-with to demonstrate mathematical understanding, Kafai and her colleagues (1998) conducted a study to explore how students applied their knowledge of fractions by asking students to design an educational game to teach others fraction. They discovered that when
designing an educational game, students embedded fraction representations in their games as quiz-style puzzles that were detached from their game context in the beginning. By providing them with some conceptual design ideas, students were able to integrate fractions into their game mechanics as an intrinsic part. Kafai et al.’s study showed that a constructionist learning activity had the capacity of providing students with an opportunity to demonstrate their mathematical understanding.

When designing a constructionist learning environment, Resnick et al. (2009) suggest that constructionist learning is supposed to have “low-floor,” “high-ceiling,” and “wide-walls.” To be specific, a constructionist learning experience can ensure that learners can start with something easy (low-floor) but also have space to increase the complexity of their construction (high-ceiling). Also, there ought to be possibilities for learners to build various types of projects that fit their interests and learning types (wide-walls). Moreover, constructionist learning ought to be tinkerable, so learners can always make in-time adjustments to their constructs and dive into an iterative and incremental design process for their knowledge construction. In light of Resnick et al.’s design suggestions, designing a mathematical storytelling activity can be apt for students to demonstrate their vector addition understanding. Engaging in a storytelling activity enables students not only to generate and elaborate their mathematical ideas but also to move their role from passive receiver of mathematical knowledge to active interpreter of mathematical texts and contexts (Borasi et al., 1990; Rosenblatt, 1994). A storytelling activity to demonstrate learning aligns with a CF intervention well. Bruner (2009) states that there are two modes of thought, logico-scientific and narrative. The logico-scientific mode of thought focuses on establishing testable and generalizable knowledge about the world, whereas narrative thought is rooted in human intention and action that describes particular experiences and stresses
meaningfulness over truth. Thus, a storytelling activity that makes students show their understanding of vector addition in a narrative mode of thought promotes the authenticity of learning and matches the ideal of constructionism.

Chapter Summary

In summary, in this chapter, I first reviewed relevant studies about vector addition learning and discovered that previous studies about vector addition learning concentrated more on vector addition content knowledge acquisition. Then, after reviewing CF research, I found that CF could be a promising theoretical framework for designing an intervention to teach vector addition. Also, I noticed that the process of learning was ignored by previous CF studies, and by referring to Sfard’s notion of two ways of mathematical conception, I believed that exploring the process of learning in a CF intervention was key to understanding students’ mathematics sense-making. Regarding the methods to investigate the process of CF learning, I reviewed relevant gesture studies and identified that it was essential to probe learners’ common ground establishment and transition states by observing their use of gestures. Last, aiming at the absence of learning manifestation in previous CF studies, I presented the theory of constructionism and proposed that there could be an additional stage that incorporates the ideas of constructionism and storytelling to the typical three-stage CF intervention for learners to demonstrate their learning.
CHAPTER THREE: INSTRUCTIONAL DESIGN AND METHODS

In this chapter, I introduced the instructional design for this study that was based on CF framework and constructionism. In addition, I presented the data collection process and the two analysis methods–grounded theory method and video analysis.

**Study Design**

There were four distinctive tasks in my study design: Enactive Physicality (EP), Iconic Depiction (ID), Abstract Representation (AR), and Constructionist Storytelling (CS). EP, ID, and AR were both learning tasks, and CS was a learning demonstration task. The CF condition followed the order of EP-ID-AR-CS, and the CI condition followed the order of AR-ID-EP-CS. Each condition had groups of three or four participants.

In the task Enactive Physicality, the researcher first provided the participants with 48 pieces of 2 feet*2 feet cardboard tiles in two colors (light green and dark green, 24 pieces each) and asked them to tile the floor. Then, two participants stepped onto the tiles and randomly selected two points to stand on. After they picked two points, the researcher asked the participants to figure out a way to describe their positions. With the participants coming up with an approach to describe positions on the tiles, the researcher gave one participant a foam football and ask the participant to walk along the tiles to deliver the football the other participant. After the other participant received the football, the researcher asked to measure the tiles they walked to their left/right and front/back. Meanwhile, the researcher asked a third participant to use red and blue sticks to path the distance walked. Then, the researcher asked the participant who delivered the football to directly pass the football to the other participant and asked the third
participant to measure the distance with yellow sticks (see Figure 2). This task included two rounds, and the participants switched their roles in each round.

In the task Iconic Depiction, the participants played a football passing simulation game developed by the researcher. In this game (See Figure 3), the participants needed to decide the units to add along the X and Y axes (red and blue arrows) to make the quarterback (red dot) pass the football to the wide receiver (blue dot) on a football field with a grid on it. If the ball was successfully delivered to the wide receiver, the wide receiver would move to a random place towards the end zone, and the quarterback would move to the place where the wide receiver stayed before moving. If the pass failed, the ball would be brought back to the quarterback, and the participants needed to determine the units along the X and Y axes again. Also, there was a button by clicking which the participants could see the projected trajectory (yellow arrow). The winning condition of this simulation game was to have a passing touchdown, the quarterback successfully passing the ball to the wide receiver in the end zone. All the participants played this simulation game as a group for three rounds. In round one, they got familiar with the simulation
game. In round two, the yellow arrow, which indicated the trajectory of the football, was revealed to the participants. In round three, the yellow arrow was hidden.

![Figure 3 Screenshot of the Football Simulation Game in Task Iconic Depiction](image)

In the task Abstract Representation, the participants had two questions in a football context (see Figure 4), in which formal vector notations and the Cartesian coordinate plane were presented. The participants first worked separately on these questions and then had a group discussion to share their answers and thoughts. The researcher told the participants that there were no correct answers to those questions, and they just needed to show their thoughts.
In the task Constructionist Storytelling, the participants built a story together to demonstrate what they learned today (See Figure 5 as an example). Since constructionism emphasized “low-floor,” “high-ceiling,” and “wide-walls” (Resnick et al., 2009), the participants started with any rough ideas by considering one of the learning objectives in the conceptual story design sheet provided (see Table 3) and then used the materials–crafting sticks, IKEA artist’s figures, color markers, and blank paper–to build up their story (low-floor). When the participants made a story and built it with materials, the researcher asked some prompt questions to encourage the iteration of the story by incorporating vector-related concepts (high-ceiling). There were five contexts to choose from, and the story design was in a freestyle (wide-walls). The reason for preparing predefined contexts was the limited time for this task. Because with predefined contexts, the participants were able to start their story constructing quickly rather than spending time figuring out a context.
There were also two other conditions—Concreteness Only (CO) and Abstraction Only (AO)—in this study. Before the task CS, the participants in the CO condition had the task EP and another task called Concreteness Reinforcement. In the task Concreteness Reinforcement, the participants were asked to solve problems in another concrete context. In this task, the researcher introduced the food delivery rover first and told the participants that, assuming they were in the engineering team, they needed to design routes for the food delivery rovers on a university campus. Then the participants received a map with a grid on it (see Figure 6) and some food delivery questions (see Appendix A). The researcher asked two participants to design routes for
the rover to make sure the food would be delivered to the hungry students as soon as possible and asked the other participant to log the routes on a sketchbook. After completing the route design, the participants were requested to describe their routes and explain why they produced those routes. After that, the researcher modified the destination in one of the questions to a tall building and asked the participants to redesign the route with a food delivery drone. With the food delivery drone available, the researcher asked the participants to redesign the route and explained why they decided on that route.

Figure 6 Campus Map in Task Concreteness Reinforcement

The participants in the AO condition also had two tasks prior to the task CS. Before they had the task AR, they had a task called Abstraction Reinforcement. This task started with two lecture-style videos which taught the participants the concept of vectors, vector components, vector notation, vector addition, etc. After watching a video, the participants had 5 minutes to discuss and took some notes on a sketchbook about what they thought was important and what was confusing. When the participants finished the videos and notetaking, the researcher asked them to solve two textbook-style vector addition questions (see Appendix B). The participants had 10 minutes to discuss the questions. At the same time, the researcher encouraged the participants to recall the content of the videos, make use of their notes, and communicate with
each other actively. Failing to solve the questions in this task was acceptable, and the task ended when the time ran up.

**Participants**

For this study, I recruited 26 eighth graders from a charter school in a large Midwestern city in the U.S. 52% of students in this school were female and 48% were male. Students’ math performance in this school was somewhat above expectations, according to U.S. News & World Report. All the participants had learned the coordinate plane in their 7th-grade mathematics class and never learned vectors before. According to my pre-study screening, none of them could conduct vector addition correctly. They were randomly divided into eight groups of three or four for this study. Considering there were no previous studies investigating vector addition learning in a CF intervention and this study was exploratory, a small number of participants was acceptable due to limited resources.

**Data Collection**

Eight study sessions were conducted from February to March 2022 at that charter school. Each study session lasted approximately 2 hours after regular school hours. Before each study session, I obtained consent from the participants’ parents and assent from the participants. Each study session included three or four participants as a group (except CO group 2 that only had two participants). There were three CF groups (CF Group 1, CF Group 2, and CF Group 3), two CI groups (CI Group 1 and CI Group 2), two CO groups (CO Group 1 and CO Group 2), and one AO group (AO Group). Table 4 shows what tasks the participants in each condition took and the duration of each task.
At the beginning of each task, the researcher asked the participants to propose some rough ideas about how to solve the problem, and then the participants worked on the task as a group. During the study session, the researcher asked prompt questions (see Appendix C) to facilitate the participants’ discussion. Two video recorders were set up from different angles to capture video and audio data.

**Data Analysis**

For the three research questions raised in Chapter one, RQ1 and RQ2 focused on the three learning tasks (EP, ID, and AR), and RQ3 concentrated on the Task Constructionist Storytelling. For RQ1 and RQ2, I only analyzed the participants in the conditions CF and CI. The reasons for not including the participants in CO and AO conditions were 1) RQ1 and RQ2 explored the learning process in a CF intervention: both Bruner’s initial framework (Bruner, 1966) and other research (e.g., Fyfe & Nathan, 2019) emphasized the three learning phases in a CF intervention and upheld the necessity of the in-between fading stage (Suh et al., 2020); 2) CO and AO conditions did not have variation on the learning tasks and I discussed the flaws of sole concrete or abstract learning in Chapter two, so the analysis on these two conditions did not help answer RQ1 and RQ2; 3) there were problems with CO and AO data collection: three of the four participants in CO Group 1 were in the robotics team and had been to a national robotics competition before the study, so they were somewhat advanced in mathematics and already seemed to have some ideas about vector and vector addition; CO Group 2 only had two

<table>
<thead>
<tr>
<th>Condition</th>
<th>Task 1</th>
<th>Task 2</th>
<th>Task 3</th>
<th>Task 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Concreteness Fading</td>
<td>EP (20 mins)</td>
<td>ID (20 mins)</td>
<td>AR (20 mins)</td>
<td>CS (30 mins)</td>
</tr>
<tr>
<td>Concreteness Introduction</td>
<td>AR (20 mins)</td>
<td>ID (20 mins)</td>
<td>EP (20 mins)</td>
<td>CS (30 mins)</td>
</tr>
<tr>
<td>Concreteness Only</td>
<td>EP (20 mins)</td>
<td>CReif (40 mins)</td>
<td>–</td>
<td>CS (30 mins)</td>
</tr>
<tr>
<td>Abstraction Only</td>
<td>ARReif (40 mins)</td>
<td>AR (20 mins)</td>
<td>–</td>
<td>CS (30 mins)</td>
</tr>
</tbody>
</table>

Table 4 Tasks for Each Condition
participants who were both quiet so their discussion during the learning tasks was not as rich as other groups’; AO Group participants were very struggled with the boring lecture videos, one of whom even fell asleep while watching the video. For RQ3, all the groups created and built their own story and including CO and AO groups was reasonable, so the qualitative data from all the groups was analyzed. In response to RQ1 and RQ3, grounded theory method was used to analyze the discourse data. In response to RQ2, video analysis was used to analyze the gestures.

Grounded Theory Method

For RQ1 and RQ3, all the audio data was transcribed, and a grounded theory method was applied to analyze the transcripts. I used Nvivo 12 to code the dataset. Only a limited number of previous research focused on vector addition learning and no previous study concentrated on the learner’s process of learning during a CF intervention or the learning demonstration after a CF intervention. Given the absence of an available theoretical framework and the infeasibility of conducting deductive coding, I used grounded theory method to analyze the discursive data. As a qualitative research method, grounded theory method uses data to generate theories, and the type of data suitable for grounded theory is rich-detailed and can be placed in its relevant situational contexts (Glaser & Strauss, 1967). With an iterative process of analysis, grounded theory method emphasizes theory construction by analyzing the actions and processes from the data (Charmaz, 2014). The below chart (see Figure 7) shows the process of utilizing grounded theory method to analyze my qualitative data.
For my RQ1, the discursive data collected in the three learning tasks—EP, ID, and AR—was structured by the researcher’s prompt questions (see Appendix C). After the researcher introduced the process of a specific task, the participants received the learning materials and responded to the instruction and questions from the researcher. Each group was working on the learning tasks together as a group and only proceeded the task when all the group members came up with a group idea, so the unit of analysis for RQ1 was one group. Besides, the conversations between the participants and the researcher were structured by those prompt questions. Thus, I segmented the discursive data by the researcher’s prompt questions; each time when the researcher had a prompt question or a follow-up question, there was a segment of discourse. During the data segmentation, I used three colors—yellow, blue, and grey—to highlight different segments for the following coding (see Figure 8).
Unlike the structured discourse in the three learning tasks, the discursive data I collected for RQ3 was in a more naturalistic setting. When designing and making stories, the participants frequently interacted with and talked to each other, so their modes of communication varied. Even though the group was not responding to the researcher together, they were working on the same story, so the unit of analysis for RQ3 was still one group. In addition, since the participants did not have structured conversations, arbitrarily segmenting their discursive data by utterance or sentence could not provide meaningful units for my analysis. Alternatively, I segmented the discursive data by identifying *topic shifting* for the following coding. Below is an example of how a topic shifting was identified. In this short excerpt, this group of participants were attempting to name their characters: P3 first proposed two characters, himself and LeBron James, and then P2 expressed her dislike of Kobe Bryant and was told by P3 and P1 not to hate. Clearly, prior to P2’s utterance, they were naming the characters, and P2’s utterance shifted the topic (see the red texts).
[00:03:26] Researcher: We have two characters. Who are they?
[00:03:31] P2: One of us.
[00:03:32] P3: You gotta name them?
[00:03:34] Researcher: You want to name them?
[00:03:35] P3: That's me, P3.
[00:03:36] Researcher: Oh, P3. Okay. Oh, you can get rid of your name tag and put it here. Another one. LeBron James?
[00:03:45] Researcher: Oh, P3 and LeBron James.
[00:03:46] P2: I hate Kobe Bryant.
[00:03:47] P3: No, no, no, no, no. Don't hate.
[00:03:49] P1: You're not hating.
[00:03:53] P2: Yeah.

Following the data segmentation, the next steps of grounded theory method were to conduct inductive coding, iterate the coding process to seek connections among codes, and eventually form categories for the following theory building. The inductive coding process included three levels: open coding, focused coding, and theoretical coding (Charmaz, 2014; Strauss & Corbin, 1990). Beginning from the start of data collection, open codes are open, short, and precise, staying close to data; focused coding, on the other hand, is more conceptual than open coding and attempts to synthesize the codes from open coding (Glaser, 1978). With a set of more focused codes, the last step was to seek relationships between those codes, integrate them analytically, and elevate the abstract level of those codes to construct theories (Glaser, 2008). I coded every discursive segment in the open coding and found that some discursive segments had multiple open codes. In the result parts of Chapter four and six–that responded to RQ1 and RQ3–I would present the counts of each open code and their coverage percentage. While reviewing a code in Nvivo 12, the coverage percentage showed the percentage of characters coded at a specific node as well as the percentage of the page area coded when the file was a PDF (Nvivo 12, n.d.). The counts of nodes and the coverage percentage implied the weight of a code in a document. After the open codes were produced, since some discursive segments might have
multiple open codes, my focused coding was comparative and sought similarities and differences between different codes. After focused coding, I established connections between those focused codes and generated findings in response to the research questions.

**Video Analysis on Gestures**

In response to RQ2, I conducted a video analysis of the participants’ speech and gestures. I first captured every gesture in the videos. To segment gesture data, McNeill (2007) defined the smallest unit of the imagery-language dialectic as a *growth point* that was an empirically recoverable idea unit from which the meaning of the speech-gesture synchrony could be inferred. Thus, this study segmented the videos into “clips” that included growth points. If the participants had a couple of gestures when talking about one topic, these gestures were grouped but would be analyzed separately, as they were at the same growth point. Based on Alibali and Nathan's (2012) taxonomy, I developed a coding scheme (see Table 5) to code gestures. I first determined whether a video clip contained a pointing or representational gesture. For pointing gestures, I documented the object that the participant was pointing to. For representational gestures, I examined whether it was metaphoric or iconic. If there was no clear metaphor involved, I identified what the gesture depicted. If there was a metaphor, I identified the body-based concept this gesture represented.

<table>
<thead>
<tr>
<th>Gesture Type</th>
<th>Pointing Gesture Target</th>
<th>Representation Target</th>
<th>Representation Type</th>
<th>Metaphoric</th>
<th>Metaphor</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pointing or Representative</td>
<td>Specific target</td>
<td>Specific representations</td>
<td>Inductive types</td>
<td>Yes/No</td>
<td>Specific metaphors</td>
</tr>
</tbody>
</table>

Table 5 Gesture Coding Scheme
The unit of analysis was an individual. I selected two specific students (Chris from CF Group 2 and Ian from CI Group 2) from two groups in different conditions via intensify sampling (Creswell, 2018). The examples selected were rich with the phenomenon that I was interested in and were not extreme in any sense. I conducted a fine-grained video analysis to the gestures of these two students. To answer RQ2, my analysis was from two aspects. The first was how they attempted to establish and maintain the common ground within their group via gesturing. Even though the common ground they created was for the entire group, I would only focus on their way of creating the common ground via gesturing instead of examining the contribution of their common ground. The other was how they demonstrated their transitional state by their use of gestures to show the partial knowledge, the possible consistent but incorrect problem-solving strategies, and the activation of multiple hypotheses. Before interpreting their gestures for RQ2, I used an inductive bottom-up method (e.g., Braun & Clarke, 2006) to categorize gestures used in the learning tasks and found three types of pointing gestures and four types of representational gestures.

**Pointing Gesture Types**

Pointing gestures were more frequent than representational gestures. There were three types of pointing gestures identified. The first type was deictic gestures. The participants used deictic gestures to refer to real-world, screen, and worksheet objects. The below excerpts showed the objects the participants were using deictic gestures to refer to. In Figure 9, the participant was pointing at the black dot and counting the rows she was at and used light green and dark green tiles to locate herself. In Figure 10, since the participants were playing the simulation game, their pointing target was screen items, yet they were able to point at the X/Y to instruct their
teammates to add units along the axes to pass the ball. In Figure 11, they became more comfortable with the symbols and could point at the symbols (e.g., the Y axis and the point on the coordinate plane) to help them express. Therefore, the objects which the participants’ pointing gestures were targeting associated with the tasks—physical objects in the task Enactive Physicality, virtual items in the task Iconic Depiction, and symbols in the task Abstract Representation—and were consistent between the two conditions.

[00:00:15] **Researcher:** For you all, I would like you to figure out a way to describe your position here. Any rough ideas?

[00:00:43] **P1:** My position is like in the fifth row.

[00:00:58] **P1:** Yes. I’m on the third [00:01:00] dot in this way.

[00:01:03] **Researcher:** Okay.

[00:01:03] **P1:** And I’m sitting like between light green tiles and dark green tiles.

Figure 9 Pointing Real-world Objects

[00:05:09] **P3:** I need more this one.

[00:05:11] **P3:** Wait, add to this.

Figure 10 Pointing Screen Objects
The second type of pointing gesture was to refer to virtual space. For instance, in Figure 12, the participant first used her pen to point at the red arrow and moved to the location on the coordinate plane where she wanted the red arrow to move to and did the same thing to the blue arrow.

The third type was to index representations. The participants sometimes pointing at something did not refer to it but served as a prerequisite for an upcoming representation. For example, in Figure 13, the participant pointed at 6 on X-axis and then traced the grid line to point
(6,2). In this example, his pointing at 6 was not indexing the number on the X-axis but helping him locate the coordinate point (6,2) as he knew that the number on X-axis was a component of the coordinate space (6,2).

![Image of pointing gesture]

**Figure 13 Pointing Gestures to Index Representations**

**Representational Gesture Types**

Four types of representational gestures were identified. The first type was to simulate actions. Hostetter and Alibali (2019) stated that people automatically used gestures to reflect the motor activity that they were thinking about and speaking about, which was defined as Gesture as Simulated Action (GSA). For example, in Figure 14, the participant was lifting her left upper arm and having a throwing gesture to talk about her understanding of the question.
The second type was to simulate a mathematical object. The topic in this study was fresh new to 8th graders, so they lacked sufficient mathematical language to express thoughts and needed gestures as auxiliaries to help them precisely deliver their thoughts. For example, in Figure 15, the participant was being asked about the relationship between the yellow arrow and the red and blue arrows in the game. He waved his hand and said it was the diagonal line.

P3: It's the diagonal line. The yellow one.
The third was to replace speech. Sometimes, when the participants omitted their words but had a gesture instead, this gesture delivered a clear message to others about what this participant intended to say. For example, in Figure 16, when being asked about the relationship between the two vectors on the worksheet, this participant used his pen to trace the right-angle symbol he had drawn before between the two vectors and said, “it looks like...” His tracing the right-angle symbol implied he wanted to say there was a right angle between the two vectors.

[00:10:01] **Researcher:** What do you think? Why you drew a line like this?

[00:10:04] **P2:** Because I said it looks like...so I made it.

![Figure 16 Representational Gestures to Replace Speech]

The fourth type was abstract pointing. The participants used this type of representational gesture to refer to what they had encountered before in this intervention but did not exist in the current context. For example, in Figure 17, the participant was pointing at the table and saying the question on the worksheet was like the simulation game they did.
After identifying that there were three types of pointing gestures and four types of representational gestures, I used Excel spreadsheets to code every single gesture captured in the videos. Figure 18 was a screenshot of my gesture coding spreadsheet. There were seventeen columns in the spreadsheet. The first five columns (#, Date, Task, Participant, Time) were for locating a specific gesture in the video; the next four columns (Pointing (Y/N), Pointing Type, Pointing Target, Pointing Cog Fun) were for recording the information of a gesture if it was a pointing gesture by determining its pointing gesture type, pointing target, and possible cognitive function in the context where that gesture emerged; similarly, the next three columns stored the information of any possible representational gestures by documenting the representation that the gesture indicated and its possible cognitive function in the context; the next three columns for metaphoric gestures were same to the representational gesture columns; the last column was notes that helped me retrieve the context of a specific gesture after I finished all the gesture coding. I had an Excel spreadsheet for each CF and CI group, and these spreadsheets recorded every pointing, representational, and metaphoric gesture that appeared in the study session. With
all the gestures coded, I selected two participants to analyze their gestures for understanding their common ground and transitional states.

Figure 18 an Example of Gesture Coding Spreadsheet
CHAPTER FOUR: MAKING SENSE OF VECTOR ADDITION

In this chapter, the results and discussion were presented in response to RQ1: How does “concreteness fading” structure students’ sense-making around vector addition? As introduced in Chapter three, a grounded theory method was utilized to understand vector addition sense-making in the three-stage learning tasks. The results part showed what focused codes I generated from open coding and the findings from categorizing focused codes; the discussion part demonstrated the findings and discussed the insights and implications.

Results

Connections Across Tasks

One of the most important features of a CF intervention was to establish connections between different learning tasks. Only if the learners recognized that the learning tasks are interrelated would they have a progressive learning experience. There were 14 open codes, 5 focused codes, and 2 categories; below Figure 19 shows the connections that the participants in different groups established across the three learning tasks. Table 6 shows the counts and the coverage percentage of each open code for connections.
<table>
<thead>
<tr>
<th>Category</th>
<th>Focused Code</th>
<th>Open Code</th>
<th>Counts</th>
<th>Coverage</th>
</tr>
</thead>
<tbody>
<tr>
<td>CONCEPTUAL CONNECTIONS</td>
<td>Coordinate Plane</td>
<td>coor. plane between ID EP</td>
<td>1</td>
<td>0.33%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>coor. plane between AR ID</td>
<td>1</td>
<td>0.46%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>coor. plane in all</td>
<td>2</td>
<td>0.94%</td>
</tr>
<tr>
<td>Similar Task Elements</td>
<td>sticks show lines in game</td>
<td></td>
<td>3</td>
<td>1.45%</td>
</tr>
<tr>
<td></td>
<td>sticks and XY</td>
<td></td>
<td>2</td>
<td>0.88%</td>
</tr>
<tr>
<td></td>
<td>wind and velocity in common</td>
<td></td>
<td>2</td>
<td>0.74%</td>
</tr>
<tr>
<td></td>
<td>forces in common</td>
<td></td>
<td>2</td>
<td>1.11%</td>
</tr>
<tr>
<td></td>
<td>yellow arrow and stick</td>
<td></td>
<td>1</td>
<td>0.56%</td>
</tr>
</tbody>
</table>

Figure 19: Codes for Connections
There were two types of connections—CONCEPTUAL CONNECTIONS and PERCEPTUAL CONNECTIONS. Most of the connections were conceptual connections. The first was the recognition on the coordinate plane. Interestingly, the participants from one CF group (CF Group 3) and one CI group (CF group 2) told the researcher that there was a coordinate plane in all three learning tasks, though there were no explicit coordinate planes in either task ID or task EP. Possibly, some design elements, such as the grid or the X/Y symbols, in the tasks ID and EP made the participants feel like there was a coordinate plane. Furthermore, the participants were also able to identify similar elements across the three learning tasks. They realized that the sticks used in the task EP were like the arrows in the game and the number of sticks used was similar to the X/Y values in the task ID. Besides noticing the similarities among the sticks, the arrows, and the X/Y values, the participants also found the yellow stick in the task EP represented the yellow arrow in the task ID, both showing the distance between the original point and the final landing point. In addition, the participants from CI Group 2 claimed that there were wind speed and initial velocity as well as forces applied to the ball in all the tasks. However, the wind speed and initial velocity were only in the task AR’s worksheet and there were no force concepts involved in any learning materials. Another connection was the football context. Both CF and CI participants identified that all three learning tasks were in a football context.

<table>
<thead>
<tr>
<th>Same Football Contexts</th>
<th>same football context</th>
<th>3</th>
<th>1.91%</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>football throwing</td>
<td>1</td>
<td>0.23%</td>
</tr>
<tr>
<td>General Sense of Connections</td>
<td>similar to game</td>
<td>3</td>
<td>1.40%</td>
</tr>
<tr>
<td></td>
<td>similar to worksheet</td>
<td>2</td>
<td>0.36%</td>
</tr>
<tr>
<td>PERCEPTUAL CONNECTIONS</td>
<td>Color Connections</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>same tile color</td>
<td>2</td>
<td>1.00%</td>
</tr>
<tr>
<td></td>
<td>same red and blue</td>
<td>4</td>
<td>3.19%</td>
</tr>
</tbody>
</table>

Table 6 Connections Open Codes Counts and Coverage
context. The last conceptual connection was group specific. The CI groups stated that the ID task was like the worksheet questions they had done in the previous AR task; the CF groups claimed that the worksheet questions were like the game they played in the previous task ID. In addition to these conceptual connections, there was only one type of perceptual connection—*Color Connections*. These were only identified in two CF groups (CF Group 2 and CF group 3). They realized that the dark green, and light green grass-like color was used in all three learning tasks and the red and blue colors were used in sticks/arrows/lines in different tasks. However, color was not only for perceptually connecting different tasks but had more affordances.

**The Affordances of Color**

Previous CF studies barely paid attention to the use of color in the CF learning process. By inspecting the participants’ mentioning of color in their discourse, I produced 8 open codes and 3 focused codes to find that there were three affordances of color in this intervention. The first was to establish perceptual connections between tasks, which was discussed beforehand. The second was to *Replace Variables*, and the third was to *Represent Objects*. Figure 20 shows the affordances of color in this intervention. Table 7 shows the counts and the coverage percentage of each open code for color affordances.
In the task ID, the participants frequently used the red/blue color to refer to the X/Y variables (e.g., “the red is X. And then subtract one more…”), though the simulation game in the task ID never indicated this mapping. It was probably because in the simulation game, the horizontal arrow was in red, and the vertical arrow was in blue. Also, the participants in the CI groups regarded the yellow arrow in the simulation game as the resultant, which was the “answer to the X/Y button.” This linkage implied that the participants in the CI groups realized there was
a clear relationship between the X/Y values and the yellow arrow. As a consequence, in the following task EP, one participant declared that the yellow sticks showed the direction in which the ball was going and the red and blue sticks were the forces applied to the ball. Another finding related to the color was that the participants in both conditions used colors to represent some objects in the tasks EP and ID. In the task EP, the red and blue colors were referred to the sticks that the participants placed on the tiles (e.g., “the blue and red is the distance [on the tile].”); similarly, in the task ID, the red and blue colors represented the arrows or the axes in the simulation game (e.g., “take some off the blue”).

Direction and Magnitude

Since the two defining characteristics of a vector were its direction and its magnitude, the way that the participants contemplated the concepts of direction and magnitude across the three learning tasks could evince their understanding of a vector. Since the direction and the magnitude (the term “magnitude” was substituted by “distance” or “length” in the intervention to avoid terminological difficulties) were separate ideas, below Figure 21 presented the participants’ understanding of direction and Figure 22 showed their views on magnitude across learning tasks. There were 9 open codes and 3 focused codes related to the participants’ comprehension on the direction of a vector in the intervention, and 6 open codes and 3 focused codes about the understanding of vector’s magnitude. Table 8 and Table 10 showed the counts and the coverage percentage of each open code for direction and magnitude-related discourse.
The use of *Real-world Directions* was common in the intervention. There were four kinds of real-world directions observed: left/right (e.g., “it's going right”), up/down (e.g., “Red is facing up.”), forward/backward (e.g., “I think one is like the friction because it's holding it back the A. And then the B is like how far we'll go forward.”), and north/south/east/west (e.g., “She's facing north.”). In addition, *Geometric Directions* refers to the directions that incorporated the participants’ geometric knowledge, including diagonal (e.g., “Researcher: Is there any direction..."
the yellow arrow is facing? P1: Yeah. It's facing diagonal.”), from start to end point (e.g., “P3: I counted the tiles of how far it was and it was 10. 10 being the tiles from the start to the point.”), and along axis (e.g., “P1: It is down along the X.”). The geometric directions were not as straightforward as the real-world directions, so the participants’ use of geometric directions was less frequent. The Combined Directions was more complex as it included both real-world directions and geometric knowledge. There were two types of combined directions; one was the number plus direction (e.g., “P1: I think the red vector is 12 miles to the right. And the blue vector is 20 miles, 20 units up.”), and the other was positive/negative plus direction (e.g., “P3: Yeah, that would be positive. Then downwards and left would be negative.”). I also counted the times that CF and CI groups mentioned real-world directions, geometric directions, and combined directions in different tasks (see Table 9). The CF participants used the real-world directions and geometric directions in both tasks EP and ID and had combined directions only in task AR. On the other hand, the CI participants had a number of real-world directions in their first task AR and a couple of the geometric directions, and the combined directions in the following two tasks. The possible reason for the pattern of different directions used will be discussed in the subsequent discussion part of this chapter.

<table>
<thead>
<tr>
<th></th>
<th>Real-world Directions</th>
<th>Geometric Directions</th>
<th>Combined Directions</th>
</tr>
</thead>
<tbody>
<tr>
<td>CF Groups</td>
<td>2</td>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>CI Groups</td>
<td>1</td>
<td>1</td>
<td>7</td>
</tr>
</tbody>
</table>

Table 9 Different Directions Across Tasks
Figure 22 Codes for Understanding Magnitude

<table>
<thead>
<tr>
<th>Category</th>
<th>Focused Code</th>
<th>Open Code</th>
<th>Counts</th>
<th>Coverage</th>
</tr>
</thead>
<tbody>
<tr>
<td>UNDERSTAND MAGNITUDE</td>
<td>Physical Obj Measurement</td>
<td>stick measurement</td>
<td>13</td>
<td>11.2%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>title measurement</td>
<td>6</td>
<td>5.58%</td>
</tr>
<tr>
<td></td>
<td>XY Value Measurement</td>
<td>x/y distance</td>
<td>6</td>
<td>5.54%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>sum of xy</td>
<td>2</td>
<td>1.39%</td>
</tr>
<tr>
<td></td>
<td>Length Comparison</td>
<td>more time to walk</td>
<td>1</td>
<td>0.58%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>shortest long yellow</td>
<td>1</td>
<td>0.30%</td>
</tr>
</tbody>
</table>

Table 10 Magnitude Open Codes Counts and Coverage

As for the magnitude, in the task EP, the participants used Physical Objects as Measures including the sticks and tiles. The walked distances were easier to measure with those placed red/blue sticks, but the ball flying path which was estimated by yellow sticks was measured by either sticks or the number of tiles that the placed yellow sticks went through. In the next task ID, the only measurement for the participants was the X/Y values in the game. There were two primary ways to use those values: one was to regard the Y value as the height the result arrow could possibly reach, and the X value determined the degree that the result arrow went down; the
other was to add the X and Y values to get the sum as the result arrow’s length. Noticeably, the participants in CF Group 3 also compared the lengths in the task EP: they thought even though the yellow sticks seemed to be the longest, but it was the shortest one. They explained that the red and blue sticks represented the distances they walked on the tiles, yet the yellow sticks were the football’s flying path, which was shorter because it took less time to toss the ball to the other participant on the tiles.

**Operations**

Recalling Sfard's (1991) notion of structural and operational knowledge, when facing a new mathematical concept, operational knowledge might be easier for students to get compared to structural knowledge. In other words, in a mathematical learning activity, students would possibly know how to produce solutions to the given problems before they realized the rationale of doing that. Therefore, I examined the operations that the participants had in each task of this intervention. I had 10 open codes and 3 focused codes for the operations in EP, 2 open codes and 1 focused code for the operations in ID, and 11 open codes and 6 focused codes for the operations in AR. Table 11, Table 12, and Table 13 showed the counts and the coverage percentage of each open code for operations in tasks EP, ID, and AR.
Figure 23 Codes for Operations in Task EP

<table>
<thead>
<tr>
<th>Category</th>
<th>Focused Codes</th>
<th>Open Codes</th>
<th>Counts</th>
<th>Coverage</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>OPERATIONS IN EP</strong></td>
<td>Body Movement</td>
<td>step to align</td>
<td>1</td>
<td>0.59%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>ask for turn</td>
<td>1</td>
<td>1.84%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>turn to make a</td>
<td>1</td>
<td>0.50%</td>
</tr>
<tr>
<td></td>
<td>Relative Positions</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Define Coordinate Plane</td>
<td>define axis</td>
<td>4</td>
<td>2.08%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>define origin</td>
<td>4</td>
<td>5.12%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>define sections</td>
<td>4</td>
<td>3.05%</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Category: Body Movement</strong></td>
<td>Define: Learners’ body movement on the tiles</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Open Codes: step to align</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Open Codes: ask for turn</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Open Codes: turn to make a right angle</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Category: Define Coordinate Plane</strong></td>
<td>Define: Learners try to define the tiled floor as a coordinate plane</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Open Codes: define axis</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Open Codes: define origin</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Open Codes: define sections</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Category: Relative Positions</strong></td>
<td>Definition: Learners use other objects to help locate themselves on the tiles</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Open Codes: relative positions between persons</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Open Codes: row dot position</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Open Codes: away from side</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Open Codes: between tiles</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 11 Operations in EP Open Codes Counts and Coverage

Figure 23 shows that there were three focused codes for the operations in EP—Body Movement, Define Coordinate Plane, and Relative Positions. Since all the participants were
asked to figure out two ways to describe a way to describe their position on the tiles in the task EP, all the groups discovered that they could make the tiled floor a coordinate plane, so they all defined the origin and the axes, and one group (CI Group 1) even defined the four sections on their coordinate plane. Besides regarding the tiled floor as a large coordinate plane, the participants used relative positions to locate themselves on the tiles. Specifically, they realized that their locations could be described with the relative positions between persons (e.g., “my position is six right and five down of his.”), the row and dot on the tiles (e.g., “my position is like in the fifth row. I'm on the third dot in this way.”), the distance from a side (e.g., “two tiles away from that side”), and the places between different tiles (e.g., “I'm sitting like between light green tiles and dark green tiles.”). After the participants decided their ways to describe the positions, they started to move on the tiles to deliver football, during which three actions were observed. The first was one participant to step to the same line to align with the other participant. The reason for stepping to align was they found it was hard to figure out the distance between them if they were not in the same line on the tiles. The second was to ask for a turn: when being asked to deliver the football, the participant P3 in CF group 1 asked the researcher if he could turn on the tiles and the researcher gave him a positive answer. The last action was observed in CF group 2. When they were delivering the football, they realized that there was always a right angle in the place where they needed to turn.
Table 12 Operations in ID Open Codes Counts and Coverage

<table>
<thead>
<tr>
<th>Category</th>
<th>Focused Code</th>
<th>Open Code</th>
<th>Counts</th>
<th>Coverage</th>
</tr>
</thead>
<tbody>
<tr>
<td>OPERATIONS IN ID</td>
<td>Add/Subtract X/Y</td>
<td>add/subtract xy, add/subtract color</td>
<td>15</td>
<td>6.09%</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>3</td>
<td>1.33%</td>
</tr>
</tbody>
</table>

There was only one type of operation in task ID, which was to *add or subtract X/Y values* in the game. As shown in Figure 24, X/Y values in game was either directly referred (e.g., “Try to add one more to X.”) or denoted by the red/blue colors (e.g., “Add more to the blue one.”).
<table>
<thead>
<tr>
<th>Category</th>
<th>Focused Code</th>
<th>Open Code</th>
<th>Counts</th>
<th>Coverage</th>
</tr>
</thead>
<tbody>
<tr>
<td>OPERATIONS IN AR</td>
<td>Count Blocks to Get Length</td>
<td>count tiles</td>
<td>3</td>
<td>1.11%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>count units</td>
<td>6</td>
<td>2.18%</td>
</tr>
<tr>
<td>Trace Line to Find Point</td>
<td>go to find a dot</td>
<td></td>
<td>2</td>
<td>0.93%</td>
</tr>
<tr>
<td></td>
<td>find a connected dot</td>
<td></td>
<td>2</td>
<td>1.41%</td>
</tr>
<tr>
<td>Copy and Paste Approach</td>
<td>copy and paste</td>
<td></td>
<td>4</td>
<td>2.81%</td>
</tr>
<tr>
<td></td>
<td>square approach</td>
<td></td>
<td>3</td>
<td>2.02%</td>
</tr>
<tr>
<td>Origin Match Approach</td>
<td>origin matching</td>
<td></td>
<td>3</td>
<td>3.1%</td>
</tr>
<tr>
<td>Locate Points</td>
<td>locate x and y on the coordinate plane</td>
<td></td>
<td>3</td>
<td>1.62%</td>
</tr>
<tr>
<td>Add Equations</td>
<td>add a and b</td>
<td></td>
<td>2</td>
<td>0.64%</td>
</tr>
<tr>
<td></td>
<td>add wind speed and initial velocity</td>
<td></td>
<td>1</td>
<td>0.19%</td>
</tr>
<tr>
<td></td>
<td>add aX plus bY</td>
<td></td>
<td>13</td>
<td>6.28%</td>
</tr>
</tbody>
</table>

Table 13 Operations in AR Open Codes Counts and Coverage

The task AR included two questions, and the participants had different operations while solving these two questions. For Question 1, the participants in both conditions *Counted Blocks to Get Lengths* on the coordinate plane. They either counted the *tiles* on the grid (e.g., “I counted the tiles of how far it was and it was 10.”) or the *units* on the axes (e.g., “I am counting the units.”). The rest three operations were only observed in CF groups. First, the CF participants either traced the line to *find a dot* in the end of the line they drew as the answer to Question 1 or to find a dot that *connected to the line* they drew. This operation indicated that they thought the answer to this question would be a dot. The second operation was to use a *copy-and-paste* approach to find the answer. Taking P1 in CF group 1 as an example, she moved the red vector to the tip of the blue vector and the blue vector to the tip of the red vector to form a rectangle on the coordinate plane (see Figure 26), and she name her method as “copy-and-paste.” This was also observed in CF Group 2: P3 told the researcher that there was a “*square*” on the coordinate plane and used his fingers to trace the possible sides of his square on the coordinate plane. In
addition, P2 in CF Group 1 told the researcher that the origins (one was the blue dot that represented the quarterback and the other was the point (24,26).) were matching up when he showed his answer to his group.

For Question 2 in the task AR, the participants in both conditions had similar operations; they located points on the coordinate plane and added up the equations given in the Question 2. To locate the points on the coordinate plane, the participants found the number on the X/Y axis first and then traced the gridline to find the exact points on the coordinate plane. As for the sum of the two equations, the participants did exactly the same thing but expressed in three different ways—adding \( a \) and \( b \), adding wind speed and initial velocity, and adding \( aX + bY \). All these
three forms of expression appeared in Question 2, so the participants had different ways to express the same operation, which was adding the given equations.

**Meaning of Symbols in AR**

The meaning of symbols in the task AR was arduous to all the participants. In specific, Figure 27 showed that there were 3 main focused codes coded: *Arrow Meaning* (with 6 open codes), *Q1 Result Format* (with 3 open codes), and *Q2 Result Format* (with 4 open codes). In the task AR, the researcher asked the participants the meaning of those arrows in the questions. *Table 14* shows the counts and the coverage percentage of each open code for the meaning of symbols in AR.

Only one group (CI Group 1) told the researcher that the *arrow meant direction*, which aligned with the definition of vector (i.e., as a displacement, the arrow of a vector expresses its direction.). Other participants in both CF and CI conditions had various thoughts about the meaning of arrow: some thought the *arrow meant keeping going* (e.g., “You can't change your mid arrow, so it's gonna keep you going this way.”), whereas others thought it *meant stop* (e.g., “I think that it would be stopping out like the arrow where it stops.”); other comprehension included regarding arrow as *speed* (e.g., “I think the arrow means how many miles per hour are going right now.”) or *distance* (e.g., “Because it's showing the distance.”). The participants also compared the magnitudes of different arrows and believed that the longer arrow had a stronger impact on the movement.

Besides the arrow symbol that was provided in the questions, the participants also had various symbols for their own answers to the questions. For Q1, the CF participants all had a *straight line*, yet the CI participants had either a *curve* or a *straight line then a curve*. The
differences in the answer format of Q1 might be because Q1 was similar to the stick placing activity on the tiled floor in the task EP and the arrows puzzle in the task ID, which the CF participants had experienced before the task AR. As a result, their answer—a line—was close to the yellow stick line in the task EP and the yellow arrow in the task ID. On the other hand, the CI participants encountered this question as their first activity, so they assumed that the answer would be similar to the trajectory of a ball flying in the wind, which was like a curve.

As for the answer formats to Question 2, the participants in both conditions had their own understanding. Only did the participants in CF Group 2 believe that the answer should be an arrow. After coming up with the answer equation “4x+6y,” some participant believed that the answer was a dot because 4x+6y was similar to the coordinate (4,6); some noticed that it was not coordinates, so they thought it would be a line that connected the origin and the point (4,6). In addition, there were also participants who conjectured that the “4x + 6y” was the answer. The reason for the diversity in Question 2’s answer was probably the abstractness of Question 2. Even CF participants had not solved a question as abstract as Question 2 in their previous learning tasks.
Figure 27 Codes for Meaning of Symbols in Task AR

<table>
<thead>
<tr>
<th>Category</th>
<th>Focused Code</th>
<th>Open Code</th>
<th>Counts</th>
<th>Coverage</th>
</tr>
</thead>
<tbody>
<tr>
<td>SYMBOLS IN AR</td>
<td>Arrow Meaning</td>
<td>arrow means direction</td>
<td>2</td>
<td>1.23%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>arrow means keep going</td>
<td>5</td>
<td>2.66%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>arrow means speed</td>
<td>2</td>
<td>1.10%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>arrow means stop</td>
<td>6</td>
<td>3.49%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>arrow means distance</td>
<td>1</td>
<td>0.75%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>arrow length impact</td>
<td>2</td>
<td>0.90%</td>
</tr>
<tr>
<td>Q1 Result Format</td>
<td>as a line</td>
<td>1</td>
<td>0.34%</td>
<td></td>
</tr>
<tr>
<td></td>
<td>as a curve</td>
<td>2</td>
<td>1.13%</td>
<td></td>
</tr>
<tr>
<td></td>
<td>as a straight line then curve</td>
<td>1</td>
<td>0.61%</td>
<td></td>
</tr>
<tr>
<td>Q2 Result Format</td>
<td>as a line</td>
<td>3</td>
<td>2.35%</td>
<td></td>
</tr>
<tr>
<td></td>
<td>as a point</td>
<td>10</td>
<td>6.19%</td>
<td></td>
</tr>
<tr>
<td></td>
<td>as an arrow</td>
<td>4</td>
<td>1.62%</td>
<td></td>
</tr>
<tr>
<td></td>
<td>as aX + bY</td>
<td>1</td>
<td>0.36%</td>
<td></td>
</tr>
</tbody>
</table>

Table 14 Meaning of Symbols in AR Open Codes Counts and Coverage
Structural Understanding

Compared to the rich operational understanding, there was less structural understanding in the intervention. Except for the meanings of symbols in the task AR discussed in the previous part, I categorized the other structural understanding by the learning tasks (see Figure 28). There were 6 focused codes (2 per category) and 16 open codes. Table 16 shows the counts and the coverage percentage of each open code for structural understanding across tasks. It is worth noting that some of those codes were condition-specific, as shown in Table 15.

<table>
<thead>
<tr>
<th>Category</th>
<th>Focused Code</th>
<th>CF Condition</th>
<th>CI Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>STRUCTURAL UNDERSTANDING IN EP</td>
<td>Mathematical Sense in EP</td>
<td>X</td>
<td>–</td>
</tr>
<tr>
<td></td>
<td>Red Blue Stick Relationship</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td>STRUCTURAL UNDERSTANDING IN ID</td>
<td>Red Blue Arrow Relationship</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td></td>
<td>Result in ID</td>
<td>–</td>
<td>X</td>
</tr>
<tr>
<td>STRUCTURAL UNDERSTANDING IN AR</td>
<td>Force Metaphors</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td></td>
<td>Q1 Factors</td>
<td>–</td>
<td>X</td>
</tr>
</tbody>
</table>

Table 15 Focused Codes of Structural Understanding for Different Conditions

In the task EP, the Mathematical Sense was observed only in the participants in the CF condition. The participants in CF Group 2 discovered that the shape formed by the sticks they placed on the tiles seemed like a right triangle, and the participants in CF Group 1 thought the diagonal yellow sticks were like a vector because they thought the vector should be diagonal. In response to the researcher’s question about the relationship between the red and blue sticks on the tiles, there were different opinions: the participants in CF group 2 said the red and blue sticks were connected; on the other hand, the participants in CI Group 2 thought the red and blue sticks showed the direction of the ball and the force applied to it as well as the syntonicity between the
red, blue, and yellow sticks (e.g., “you add a red stick, you are going to have to add another yellow stick because it's going to move left and middle. So you have to add another blue stick.”).

In the task ID, the participants answered a similar question: what the relationship between the red and the blue arrow in the simulation game is. The participants in CF Group 3 thought the red/blue arrows were the X/Y values, and the yellow arrow was the distance from the quarterback to the ball landing position in the game. On the other hand, the participants in CI groups either thought the red and blue arrows worked together to make a movement or assumed that one arrow was the force attracting the other arrow. This difference might be caused by the sequence of the learning tasks. The CF participants used the red/blue sticks on the tiles, so they quickly recognized the similar red/blue arrows. The CI participants, however, relied more on their intuition to guess the relationship. Additionally, the participants from CI Group 2 expressed their thoughts about the result yellow arrow; they thought the yellow arrow was indicating the landing position of the football in the game and meanwhile was the answer to the X/Y button.

In the task AR, three groups (CF Group 1, CI Group 1, and CI Group 2) used a force metaphor to help understand the questions. The metaphor that CF Group 1 used was more analytical, which compared two forces with different magnitudes that faced opposite directions. The participants from CI Group 1 had a vague feeling that in the question, one vector was holding back the other, which was like friction. The participant P2 from CI Group 2 had an embodied force metaphor to explain the question: he pretended to throw a pen towards one of his teammates and showed how the trajectory of that pen would be affected by the wind. The wind effect was also pointed out as a factor that might influence the final landing position in Question 1 when the group started to explore their rough ideas for that question. Besides, the CI participants believed that the spiral shape of a football and the throw speed of the quarterback
would also impact on the movement of the football in the Question 1. The rough thoughts from the CI participants manifested that facing an abstract problem, they needed to use some concrete ideas to help comprehend the abstract question.

Figure 28 Codes for Other Structural Understanding
<table>
<thead>
<tr>
<th>Category</th>
<th>Focused Code</th>
<th>Open Code</th>
<th>Counts</th>
<th>Coverage</th>
</tr>
</thead>
<tbody>
<tr>
<td>STRUCTURAL UNDERSTANDING</td>
<td>Mathematical Sense in EP</td>
<td>right triangle</td>
<td>5</td>
<td>2.42%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>diagonal vector</td>
<td>1</td>
<td>1.21%</td>
</tr>
<tr>
<td></td>
<td>Red Blue Stick Relationship</td>
<td>connected</td>
<td>1</td>
<td>0.69%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>as dir and force</td>
<td>1</td>
<td>0.66%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>syntonicity</td>
<td>1</td>
<td>0.37%</td>
</tr>
<tr>
<td>STRUCTURAL UNDERSTANDING</td>
<td>Red Blue Arrow Relationship</td>
<td>red x blue y</td>
<td>4</td>
<td>2.65%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>yellow distance</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>movement</td>
<td>1</td>
<td>0.22%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>attracting force</td>
<td>1</td>
<td>1.24%</td>
</tr>
<tr>
<td></td>
<td>Result in ID</td>
<td>as landing position</td>
<td>1</td>
<td>0.35%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>as the answer to</td>
<td>3</td>
<td>0.85%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>the XY button</td>
<td></td>
<td></td>
</tr>
<tr>
<td>STRUCTURAL UNDERSTANDING</td>
<td>Forces Metaphors</td>
<td>Newtonian forces</td>
<td>2</td>
<td>2.33%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>friction</td>
<td>1</td>
<td>0.71%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>pen throwing</td>
<td>1</td>
<td>0.47%</td>
</tr>
<tr>
<td></td>
<td>Q1 Factors</td>
<td>wind effect</td>
<td>10</td>
<td>4.88%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>ball shape</td>
<td>1</td>
<td>0.61%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>throw speed</td>
<td>3</td>
<td>1.89%</td>
</tr>
</tbody>
</table>

Table 16 Structural Understanding Open Codes Counts and Coverage

Discussion

In this part, in response to RQ1 (How does “concreteness fading” structure students’ sense-making around vector addition?), I synthesized the results to generate findings about the participants’ sense-making in the CF intervention, discussed the limitations, and proposed three suggestions for designing a successful CF intervention. My findings were from three perspectives: connections across tasks, operational and structural knowledge, and the use of symbols.

Findings

Finding 1: The same question context, identical colors, homogenous problems, and familiar previous knowledge can establish connections across learning tasks.
A successful CF intervention ought to have mutual referents to let learners recognize explicit connections and realize that the representations at each stage are interconnected (Suh et al., 2020). Previous CF studies concentrated more on the similarity of questions across different stages (e.g., the equal sign questions in McNeil and Fyfe's study, 2012), and they strived to reduce the perceptual richness of the questions in order to improve the transferability of the knowledge gained from the third symbolic stage. This strategy might work when the topic was simple, but for a more complex mathematical topic like vector addition in my study, the same question context could help with students offloading. As shown in Figure 19, the participants realized that the contexts in the three learning tasks were all about football, which became a mutual referent they established. Also, keeping the colors of some key elements–sticks, arrows, and vectors–could help learners quickly recognize the connections and explore the different representations of the same concept. It is because color as a graphical device can reduce visual search and support quicker information access (Keller & Grimm, 2005), and learners can use color as a clue to retrieve information (Hanna & Remington, 1996). My results about the participants’ use of colors (see Figure 20) reflected that they recognized that the same colors might be a clue across tasks; for instance, the red and blue colors were used for sticks, arrows, and vectors in different learning tasks, and the participants realized these three objects were alike, which became another mutual referent. In addition, the activation of their previous knowledge–the coordinate plane and algebraic arithmetic in this study–across different learning tasks allowed them to engage in the learning tasks that introduced new concepts. According to Myhill and Brackley (2004), students with prior knowledge would have less cognitive load and better learning engagement. The purpose of setting up similar contexts for all three learning tasks
was also to help students offload when switching learning tasks and enable them to engage in the current learning task. In my case, the prior knowledge that the participants brought into the study included their prior mathematical knowledge and the knowledge about the study that was developed in the early tasks and carried over to the late tasks. Figure 19 showed that the participants noticed that there was a coordinate plane across tasks, even though there was no clear indication of a coordinate plane in tasks EP and ID. Their prior mathematical knowledge about coordinate plane made them recognize that a plane with grid could be a coordinate plane, so they thought there was a coordinate plane when they saw the tiled floor and the grid in the simulation game. As for the participants’ knowledge about the study, a good example was that the participants’ operations across tasks (see Figure 23, Figure 24, and Figure 25) and their corresponding understanding of magnitude (see Figure 22), the participants used sticks to measure distance on the tiles, then the x/y value to determine the length of arrow in the game, and the counts of blocks or the units along axes to find the lengths of a vector on the worksheet. Similar task settings allowed the participants to leverage their knowledge and helped them establish connections across tasks.

**Finding 2: Structural knowledge building can be intertwined with operational knowledge construction.**

As presented in the results section in this chapter, the operational knowledge construction (see Figure 23, Figure 24, and Figure 25) was much richer than the structural knowledge building (see Figure 28). Therefore, I contemplated that these two processes were intertwined rather than sequential, and the possible stage when students mix up their operational and
structural understanding was the in-between task Iconic Depiction. There was a good example from CF Group 2 when they were asking about the relationship between the red and blue stick/arrow. In the task EP, the participant P3 in this group told the researcher, “Well, there's a relationship with the red and blue one, cuz they connect. But the yellow one, they go in the straight.” Then in the task ID, P3 said “We have to use the sticks to make, to show what we're going, like the X and Y integers.” He noticed that the relationship between the red and blue arrows (though he used the term “sticks”) in the simulation was not simply connected but had some connections with the X and Y integers. The task ID demonstrated the process in which the X and Y values were associated with the red and blue arrows, so when he was playing the simulation game by adding/subtracting X/Y values, he realized that the relationship between the red and blue arrows was not as simple as the relationship between the red and blue sticks in the task EP. This change enabled him to enter the final AR task in which he proposed the “square” approach, which was close to the formal “parallelogram method” for vector addition.

**Finding 3: Unfamiliar symbols led to naïve understanding but enabled mathematical sense-making.**

My findings on the arrow meaning and the answer formats for the two abstract questions revealed that the participants were struggled with those unfamiliar symbols (see the codes in Figure 27), which was understandable. It is because symbols are often from a heavily regulated notational system that conveyed no meaning to those who are not familiar with the system (Nathan, 2012). Thus, the participants developed various understandings of the symbols, such as different answer formats for the two questions in task AR (see Figure 27). However, from a
mathematical sense-making perspective, the participants were still gaining, because when the participants were explaining those symbols, they were in the process of reasoning their problem-solving rationale and tried to express the unfamiliar mathematical ideas in their own words. Inviting students to think about their problem-solving process and allowing multiple problem-solving strategies can build a student-centered mathematical learning experience and provide a supportive mathematical learning experience (Schoenfeld, 1992).

**Limitations and Design Principles**

There were three limitations in this part of the study. The first was the sample size. I only had five groups, three in CF condition and two in CI condition. Consequently, I deliberately avoid the comparison between conditions, because even though I could find some differences between these two conditions, the small sample size cannot produce any generalizable statistical significance. The second was the omission of the Concreteness Only and Abstraction Only conditions. The reason for excluding these two conditions was they were not related to RQ1. However, the learning process in these two conditions was also valuable to explore. Lastly, I explored learning at a group level. This was unconventional for a CF study, as most previous CF studies explored learning at an individual level. However, due to limited resources and scheduling issues, my study could only be a group study, but I believed it would be worthwhile to explore the individual learning process in a CF intervention.

Despite the limitations, I suggested three design principles for the future CF intervention design: when designing a CF intervention for a complex mathematical topic, 1) keep the design elements, such as colors and question contexts, consistent and incorporate some space for utilizing previous knowledge; 2) track the development of similar ideas across stages to discover
the moments when operational and structural knowledge mixes up; 3) allow naïve understanding of unfamiliar symbols to foster mathematical sense-making and provide in-time lectures to teach those symbols afterward.
CHAPTER FIVE: SHOWING UNDERSTANDING THROUGH GESTURES

By watching all the video clips that included gestures, I deduced three types of pointing gestures and four types of representational gestures as presented in Chapter three. All these gestures were used to identify the common ground that Chris and Ian created and the transitional states that they entered. In this chapter, I first presented the overall coding results and then had an in-depth exploration of the two selected participants’ common grounds and transitional states by following their learning tasks order. Later, I discussed how their gesture-based common grounds and transitional states responded to RQ2: How do gestures interact with “concreteness fading” in shaping student understanding of vector addition?

Results

Overall Gestures Coding Results

Table 17 shows the counts of each group’s pointing gestures, representational gestures, and metaphoric gestures sorted by the types and the learning tasks. Table 18 displays the descriptive statistics for each type of gesture in different tasks. In general, the participants had the most gestures in task AR and had the least gestures in task ID. It might be because the abstract questions AR demanded the participants to have more gestures to help express their thoughts as the participants were unfamiliar with the topic and lacked the appropriate mathematical vocabulary to express themselves. Hence, gesturing became a viable approach to communicating ideas. As for the participants’ gesture use in task ID, the simulation game was already a space for them to communicate by referring to the exact in-game elements such as X/Y values, plus/minus buttons, and arrows. The gesture use in task EP was moderate because the participants possibly needed to refer to physical items and the space with their gestures.
Noticeably, the tendency of gesture use was not aligned with the decline of the learning materials’ level of concreteness; instead, it was in a “V” shape. Also, metaphoric gestures were barely used compared to pointing and representational gestures, and most metaphoric gestures were observed in task AR. Probably, those abstract questions led the participants to use a metaphor to understand and share their opinions with their teammates.

<table>
<thead>
<tr>
<th>Group</th>
<th>Task</th>
<th>Pointing Gesture Counts</th>
<th>Representational Gesture Counts</th>
<th>Metaphoric Gesture Counts</th>
<th>Grand total</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>delictic</td>
<td>virtual</td>
<td>space</td>
<td>index representations</td>
</tr>
<tr>
<td>CF</td>
<td>EP</td>
<td>9</td>
<td>5</td>
<td>3</td>
<td>17</td>
</tr>
<tr>
<td>Group 1</td>
<td>ID</td>
<td>5</td>
<td>2</td>
<td>0</td>
<td>7</td>
</tr>
<tr>
<td></td>
<td>AR</td>
<td>24</td>
<td>9</td>
<td>15</td>
<td>48</td>
</tr>
<tr>
<td>CF</td>
<td>EP</td>
<td>15</td>
<td>7</td>
<td>6</td>
<td>28</td>
</tr>
<tr>
<td>Group 2</td>
<td>ID</td>
<td>16</td>
<td>4</td>
<td>5</td>
<td>25</td>
</tr>
<tr>
<td></td>
<td>AR</td>
<td>36</td>
<td>6</td>
<td>17</td>
<td>59</td>
</tr>
<tr>
<td>CF</td>
<td>EP</td>
<td>13</td>
<td>6</td>
<td>7</td>
<td>26</td>
</tr>
<tr>
<td>Group 3</td>
<td>ID</td>
<td>8</td>
<td>1</td>
<td>1</td>
<td>10</td>
</tr>
<tr>
<td></td>
<td>AR</td>
<td>18</td>
<td>4</td>
<td>10</td>
<td>32</td>
</tr>
<tr>
<td>CI</td>
<td>EP</td>
<td>25</td>
<td>10</td>
<td>7</td>
<td>42</td>
</tr>
<tr>
<td>Group 1</td>
<td>ID</td>
<td>5</td>
<td>3</td>
<td>1</td>
<td>9</td>
</tr>
<tr>
<td></td>
<td>AR</td>
<td>21</td>
<td>2</td>
<td>7</td>
<td>30</td>
</tr>
<tr>
<td>CI</td>
<td>EP</td>
<td>25</td>
<td>21</td>
<td>35</td>
<td>81</td>
</tr>
<tr>
<td>Group 2</td>
<td>ID</td>
<td>3</td>
<td>1</td>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td></td>
<td>EP</td>
<td>14</td>
<td>3</td>
<td>7</td>
<td>24</td>
</tr>
</tbody>
</table>

Table 17 Overall Gesture Coding Results

<table>
<thead>
<tr>
<th>Task</th>
<th>Task EP</th>
<th>Task ID</th>
<th>Task AR</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>mean</td>
<td>std</td>
<td>mean</td>
</tr>
<tr>
<td>Pointing Gestures</td>
<td>35.20</td>
<td>25.94</td>
<td>11.40</td>
</tr>
<tr>
<td>Representational Gestures</td>
<td>17.40</td>
<td>15.36</td>
<td>7.40</td>
</tr>
<tr>
<td>Metaphoric Gestures</td>
<td>0.20</td>
<td>0.45</td>
<td>0.80</td>
</tr>
</tbody>
</table>

As the total amount of gestures observed was too many to present here, I selected two participants for an in-depth analysis of their gestural use. For the selected two participants, Chris from CF Group 2 had 18 pointing gestures in task EP, 20 pointing gestures in task ID, and 26 pointing gestures in task AR. As for his representational gestures, there were 11 in task EP, 6 in
task ID, and 17 in task AR. He had 3 metaphoric gestures, 1 in task EP and 2 in task AR. Ian, from CI Group 2, had 10 pointing gestures (5 in task AR and 5 in task EP), 27 representational gestures (15 in task AR, 4 in task ID, and 8 in task EP), and 3 metaphoric gestures (all in task ID). The reasons for selecting these two participants were 1) they had very rich verbal expressions while gesturing, which allowed me to better understand their gestures, and 2) they both demonstrated a clear train of thought across learning tasks, which made it viable to trace their common ground building and transitional knowledge state development. In addition, when I was watching those video clips, I noticed that there was a clear when the students attempted to initiate their common ground: when they were gesturing, they intended to include the pronoun “we” in their utterance and faced their teammates with talking. On the other hand, when they were about to show their transitional state, they usually turned their face to me to gesture and talk to me.

Common Ground and Transitional States in Concreteness Fading Condition

For the CF group, in task EP, the observed participant Chris was the notetaker, but after the group decided to use real-world direction as a part of their way to describe their position, he participated in the discussion and used a pointing gesture to create the common ground about using the length of a tile as the unit to measure the distance between his teammates on the tiles. From Figure 29, P3 pointed at the tiles and counted to measure the distance, and this pointing gesture created the common ground, so the group then knew how to describe the distance on the tiles.
The first transitional state presented was during Chris placing yellow sticks to indicate the ball passing path. Figure 30 shows he was thinking the sticks could not go through the tiles, but then with the prompt from the researcher, he corrected himself. This excerpt showed he was in a transitional state, and had at least two hypotheses activated at the same time: one was the ball path could not go through the tiles and the other was the ball path was actually diagonal. Also, there was a gesture-speech mismatch as he waved his arm as a representation of a diagonal line to indicate the ball path should be a diagonal line but said he could not make a diagonal line.
Then the group had another round of play and Chris stepped on the tiles. When asked to use another way to describe their positions on the tiles, Chris teammate mentioned there was a right angle. Chris quickly got her idea and then used his finger to count the black dots and point at the dot where he would make a turn (see Figure 31). This common ground was not established by Chris, but Chris used a pointing gesture to elaborate on the idea of an invisible right angle on the tiles.
After the group had done football passing and placed sticks on the tiles, the researcher asked the participants what the sticks looked like. Chris’s teammate thought it looked like half of a square, and Chris then used a pointing gesture to refer to a virtual space and a representational gesture to air sketch the two absent sides of that square (see Figure 32). Chris’s gestures created the common ground and confirmed their thought about the shape made by the sticks—half of a square and this idea was carried over during the entire intervention.
Then when asked about how long the yellow stick line was, Chris demonstrated his transitional state about measuring distance on the tiles. He pointed at the three black dots that the yellow stick line went through and indexed the representation which was the distance between the three dots (see Figure 33). In this excerpt, he used the same strategy—using objects on the tiles to measure the distance—that he used before, which indicated he was in the transitional state as he attempted to use a consistent (his strategy was not inherently incorrect) strategy to solve different problems.
The next question was whether there was a relationship between the red/blue and the yellow stick line. Chris used representational gestures that simulated the connectivity between red and blue lines and the yellow line’s state of going straight alone (see Figure 34). His gestures created the partial common ground that the relationship between lines was connectivity, because he thought the yellow line was separated from the red and blue lines, which was ignored by his teammates. This was also a transitional state because he attempted to use partial knowledge—the ball-delivering and passing activity he experienced in this stage—to respond to the researcher’s question.

[00:15:47] **Researcher:** Is there any relationship between this line and the red one and the blue one?

[00:15:52] **P3:** well, there's a relationship with the red and blue one, cuz they connect. But the yellow one, they go in the straight.

[00:16:01] **Researcher:** What do you think? Any relationships?

[00:16:06] **P2:** They connect with each other

[00:16:07] **P1:** They are connected with the yellow things.

Then the group entered the Iconic stage and started to play the simulation game. The first common ground created in this stage was Chris used a pointing gesture to revoice the suggestion from his teammate (see Figure 35). This is a typical practice in this stage as they worked on the football passing problem together in the simulation.
game. His pointing gesture might also imply that he made some connection between the X/Y numbers and the length of the red/blue arrows.

When the participants were playing the simulation game, they extensively relied on screen objects to create their common ground for problem-solving. Figure 36 shows Chris attempted to use a representational gesture to tell his teammate that they needed to make the yellow arrow longer in order to make the touchdown pass. Besides creating the common ground, he also had a gesture-speech mismatch here: he said, “we have to touch this,” but instead of pointing at the target wide receiver in the game (blue dot), he extended the yellow arrow. This implied he knew the length of the yellow arrow determined the passing distance, and his problem strategy relied on motion.
In the next round of play, the researcher asked the participants to hide the yellow arrow and try to pass the ball only with red/blue arrows. Chris attempted to create the common ground by pointing at the point where the blue arrow was supposed to be (see Figure 37). At the same time, there was also a gesture-speech mismatch: Chris wanted to subtract the value of Y, but he pointed at the blue arrow rather than the Y value. This mismatch implied that Chris was in a transitional state that made him gradually connect the length of the arrow and the value of X/Y.
After the group completed the game, the researcher asked about the similarities between the simulation game and the previous physical activity. Chris made two representational gestures and a pointing gesture (see Figure 38)–the left top was to simulate the thin stick, the left bottom was to simulate going along a direction, and the right was to point at the X/Y values–to explain the similarities. This excerpt showed Chris had two hypotheses of how to go along a direction on a tiled plane: one was to use sticks to path and the other was to use X/Y values. His transitional knowledge was the X/Y values were similar to the sticks he placed on the tiled floor before.

![Figure 38 Chris Using Gestures to Explain Connections between tasks EP and ID](image)

The last question in task ID was the relationship between the X/Y value and the position where the football would land in the game. Chris demonstrated his transitional knowledge with three representational gestures (see Figure 39). He first extended his arm to show the X determined how long the football would go horizontally and then lifted the other arm to show the Y decided how high it would reach. Then he waved his arm to explain how the X would curve the trajectory. This excerpt showed that Chris had two hypotheses in mind: one was the X/Y value determined the movement of the football, which was from his observation in the simulation game; the other was that the upward movement was impacted by the X, so the
trajectory curved, which was inferred by him. These two hypotheses implied that he was in a transitional state to understand how the X/Y value would influence the trajectory.

When facing the first question in task AR, Chris used a set of gestures to explain his thoughts (see Figure 40), which displayed his transitional state. He first had a representational gesture to conduct abstract pointing—by pointing at the table where they just played the simulation game, he thought the question was similar to the game. Then he moved his finger diagonally from the blue dot, saying where we went should have a relationship. Then he pointed at the X and Y axis and had a representational gesture making a cutting shape with his hand to simulate the possible line. This excerpt shows he had two hypotheses about how to solve the problem. The first was to refer to the task they did before and describe the possible ball movement. The second was the X and Y axes were necessary to make a touchdown, and the axes would influence the potential trajectory. His second hypothesis was also consistent with the one he explained in the previous task.
Then the researcher asked the group to solve the problem and then explained their answer to the first question on the worksheet. Chris had very rich gestures to create the common ground and show his transitional knowledge. In Figure 41, Chris first used a finger to trace the line he drew and another finger to indicate the location where his line actually stopped (left top). By using this representational gesture that simulated the mathematical object, he attempted to let his teammates know that he believed the answer was not an infinite line but a line with a fixed length. He then traced the blue and red arrow with his finger and pen (left middle and bottom) to state that the line was influenced by the Y and X axis. Although the arrows were parallel to the axes, he had a gesture-speech mismatch here—tracing the arrows but mentioning axes—and this mismatch implied he considered the arrows paralleled to the axes had the same impacts on the in-between trajectory. He simulated the trajectory by waving his paw (right top) and claimed the Y determined the height and X made the up-and-down change to the trajectory. This simulated gesture demonstrated that he had a consistent strategy developed in the Iconic stage, which is a symbol of being in a transitional state. He then attempted to reinforce the common ground he
established before by tracing the line he drew and the arrows. His tracing was representational as he used his tracing to replace his speech which was supposed to be the line that was impacted by these two arrows.

![Image of Chris using gestures to explain](image)

Figure 41 Chris Using Gestures to Explain His Answer to Q1

Regarding the second question on the worksheet, Chris’s rough idea was to find the dots from the X and Y values from the question. He started to use a representational gesture to trace the grid line to find $6x+2y$ and then used his pen to tap the arrowheads of $\vec{a}$ and $\vec{b}$ on the coordinate plane (see Figure 42). This excerpt showed that Chris was in a transitional state as he attempted to use partial knowledge to develop his problem-solving strategy. To him, $6x+2y$ seemed like (6,2), which was a coordinate that they learned before in their mathematics class. Therefore, he ignored $\vec{a} = 6x + 2y$ in the question but focused on the numbers, thinking he should look for the points first.
After the group worked on the second question for a while, the researcher asked the group to explain their answer. Chris’s explanation characterized his transitional state. Figure 43 demonstrated that he first waved his paw (left top) to indicate that the trajectory would be impacted by the axis and then traced (left middle) the Y axis that determined the height, which showed his consistent strategy of deciding the trajectory. He then pointed at the number 4 on the Y axis and number -2 on the X axis, saying it was higher than how far (left bottom), followed by tracing $\vec{a}$ (right top). Although he was pointing at the numbers on the axis, he did not mean to make references to those numbers. Instead, he compared those numbers and believed $\vec{a}$ was impacted by the numbers. He showed two hypotheses here: one was concluded from previous tasks that Y and X axes would influence the in-between arrow; the other was the values of Y and X could also be the determinant. In addition, he had another two representational gestures: he pointed at the number -2 he wrote (right middle) and moved the pen he was holding along the Y axis (right bottom). These two gestures indicated he attempted to make connections between the two hypotheses.
Figure 43 Chris Using Gestures to Explain His Thoughts about Q2

The group then realized they needed to add the equations in the question, and all got the answer $4x+6y$. The researcher then asked them to put the result on the coordinate plane. Chris pointed at the result he got first and then pointed at $\vec{a}$ and $\vec{b}$ (see Figure 44). This excerpt showed that Chris noticed that the result $4x+6y$ was the combination of vector $\vec{a}$ and $\vec{b}$. However, he later put a dot on the coordinate plane to represent $4x+6y$, which indicated he was using an incorrect but consistent strategy to plot a potential vector on the coordinate plane.
Then the researcher prompted the group to explore the symbol of the vector by asking them what they thought the tiny arrows above the letter a and b meant. Chris had two representational gestures: he extended his arm to indicate the arrow meant it would keep going in a direction and put another hand above to imply the direction was non-changeable (see Figure 45). He showed his partial knowledge of the arrow symbol, which was his intuition, and explained it in an embodied way.
Around the end of this task, the researcher asked the group to try to draw an arrow on the coordinate plane and then explain why they drew that arrow. Chris used a couple of pointing and representational gestures to explain his arrow (see Figure 46): he first pointed at the arrowhead of the arrow he drew (left top) and then pointed at the space between the arrowhead of $\vec{a}$ and the arrowhead of his arrow, saying $\vec{b}$ should be put there (left bottom). He then pointed at the equations that he added together (right top) and pointed at the arrowhead of his arrow (right bottom), stating he added the equations, and the result dot was at the arrowhead of his arrow. From this excerpt, Chris showed his progress in his transitional state. He still had multiple hypotheses: one was from the very first Physical task, which made him think he should move $\vec{b}$ above to make a square; the other was from this task, which was by adding the equations, he would be able to get a dot that he regarded as the combination of vector. In addition, he still held the consistent strategy that developed before: finding the result of combining vectors was to look
for a dot on the coordinate plane. The multiple hypotheses and consistent strategy implied that Chris was in a transitional state of understanding vector addition.

Common Ground and Transitional States in Concreteness Introduction Condition

The selected student in CI condition, Ian, had fewer gestures than Chris did. As aforementioned, CI groups experienced the same tasks in a reversed order–AR, ID, and then EP.

Regarding the arrow symbol in the first question on the worksheet, Ian started with representational gestures (see Figure 47) to demonstrate his understanding: he used his pen to trace the blue arrow, air sketched a line perpendicular to the blue arrow mimicking the wind, and air sketched a curve in-between to represent the ball trajectory. This excerpt showed Ian’s initial understanding and would be compared with his later thoughts to see his transitional state.
Ian then used a metaphor to explain his thoughts to his teammates to create the common ground within the group. In Figure 48, he started with a representational gesture pretending to throw his pen toward one of his teammates (left top) and swung the other hand to simulate the wind (left bottom). He then put the hand that simulated the wind on the pen-holding hand (right top) as if the pen flying was influenced by the wind and eventually off the target (right bottom). Ian attempted to use this metaphor and his representational gestures to create the common ground within the group to show his understanding of the first question on the worksheet.

Then the researcher asked the group to solve the first question and explain their answers once they had done. Ian used representational gestures to explain his answer (see Figure 49) Ian first traced the curve he drew as if it was the ball path (left top) and then traced the blue arrow
(left bottom) and red arrow (right top), stating those were the forces applied from different directions. He then traced his curve again, saying the ball would go in the middle when the wind was pushing it east and the ball was being thrown to the north (right bottom). Ian demonstrated that he used a consistent hypothesis to solve the problem here: the ball was being thrown and meanwhile impacted by the wind. He also had a gesture-speech mismatch—tracing the in-between curve but talking about the wind and the thrown direction—that showed he was in a transitional state.

Ian Explaining His Answer to Q1

Then the group started to work on the second question. After they completed the second question, the researcher asked them to explain what they drew on the coordinate plane. Ian pointed the equations first saying he added the equations, then located the point he dotted on the coordinate plane and traced the line he drew from the origin to the point (see Figure 49). This excerpt showed that Ian regarded the result of his addition as a point on the coordinate plane and thought the result would be a movement from the origin to the point. He attempted to use his partial knowledge to comprehend and solve the question here, and his problem-solving strategy was consistent with his explanation of his answer to the first question, which indicated he was in a transitional state.
The last question in AR was to talk about the relationship between the two questions on the worksheet. Ian used pointing and representational gestures (see Figure 51): he pointed at the worksheet, saying both questions were about football, and then claimed that both questions had wind speed and someone throwing the ball with a throwing gesture. This excerpt showed that Ian only captured the surface structure of the questions rather than any deep thinking about vector addition.
In task ID, when being asked about the relationship between the red/blue arrows and the yellow arrow in the simulation game, Ian used a couple of representational gestures to demonstrate his thinking (see Figure 52): he first having a gesture of closing both his hands (left top) and then having a pulling gesture (left bottom), but saying not pulling. He then had two hands lift as mimicking the two arrows (right top) and used his pen to air sketch an arrow in-between (right bottom). This excerpt displayed Ian’s transitional state from his gesture-speech mismatch and two hypotheses. He had a pulling gesture but said it was not pulling, implying his idea was not fully developed. The first closing hand gesture suggested that he thought there might be two forces applied to the object, but he later thought it was a movement attracted by another force, which was two hypotheses simultaneously developed in his mind.
The next question was to talk about the relationship between the two tasks in two stages. Ian’s representational gestures established the common ground and showed his transitional knowledge (see Figure 53): he first pointed at the table with his pen, which was abstract pointing to refer to the coordinate plane or the football field context on the worksheet; he then waved his arms to show that there were two forces applied that balanced the ball to reach the destination. This excerpt created the common ground for his group that both questions were in the football context, which later was mentioned by his teammates, and also showed his consistent two-force hypothesis.
The final task was EP. After the group placed the sticks, the researcher asked Ian to talk about the relationship between the red/blue and yellow sticks. He was the notetaker for the first round. Ian used his pen to trace the ball path in the air and then pointed at the blue/red sticks (see Figure 54). This excerpt showed that Ian attempted to connect the current problem with the previous activity, but he failed to explain the relationship clearly as he thought the red/blue sticks were the distance and did not link to the yellow sticks.

[00:09:46] **P2:** Like we did in the game. He's throwing the ball at her. The yellow sticks are the path in which the ball takes.

[00:10:06] **Researcher:** How about the blue and red?

[00:10:10] **P2:** Blue and red is the distance. Some distance is similar.
In the second round, Ian was on the tiles, and the researcher asked the same question. Ian used his foot to point at different sticks and explained the yellow sticks showed the direction that the object was going, and the blue/red sticks were the direction of forces applied to it (see Figure 55). His foot pointing created the common ground between the researcher and him to make the referents in his explanation clear to the researcher. Also, he applied his consistent hypothesis here—the yellow sticks implied the object direction and the red/blue sticks were the forces applied to it—which indicated his transitional state.

Figure 55 Ian’s Additional Explanations of Sticks

Discussion

Common Ground

The use of gestures to create common ground was associated with the tasks the participants were doing. Plausibly, creating common ground required more mental effort. When human beings are learning, they could only hold a small amount of information during learning (Sweller, 2011), so their gestures have the affordance of offloading. Offloading refers to the use of physical action to reduce the information processing demand in human memory (Risko &
Gilbert, 2016). In this study, both pointing and representational gestures had this affordance. For pointing gestures, during the task EP, participants were pointing to the items of high physicality and perceptual richness (e.g., tiles and sticks they used) that were highly contextualized in their conversation. When they were playing the football simulation game during the task ID, they started to point at the arrows or the parameters of X and Y on the screen, which they realized had a relationship with the ball movement in the game. The targets the participants pointed to in task AR were mostly formalisms such as lines, axes, and dots on the coordinate plane. These gesture referents revealed that participants got more comfortable with idealized symbols on a coordinate plane across the intervention stages. The similar contexts and problems across stages helped them transition from tangible physical learning to conceptual abstract reasoning. For representational gestures, a signal that the participants might be creating the common ground was they had abstract pointing (pointing at the space where something they had encountered was there) and direct consequence (using a gesture to replace their speech, for the definition of direct consequence, see Alibali & Nathan, 2012). Since the intervention in both conditions was sequential, abstract pointing helped students make reference to something that they had experienced, and this shared representation could build a possible solution to more abstract questions that could be understood by the group (e.g., making reference to the game in Error! Reference source not found.). Furthermore, the direct consequence implied the concept that this gesture represented had been known by the group. If it was a new concept, simple gesturing could not make sense to others, so when the direct consequence gesture appeared, the gesturing person intended to use their gesture to bring up a concept known by their group and created the common ground for the group.
In addition, there was an interesting phenomenon that happened in both conditions: the participants relied on the representational gestures that represent motion to create common ground. A good example was Ian’s dual explanations on the first question of the worksheet (see Figure 47 and Figure 48). Since the arrow was an unfamiliar symbol to him, he first attempted to explain it in the context of the question. However, he then used a more concrete way to further explain his thoughts by making the analogy of the wind affecting the pen he threw. As mathematical ideas were embodied (Lakoff & Núñez, 2000), Ian’s analogy showed that incorporating motion into explaining mathematical ideas could easily create the common ground for the group compared to solely relying on the idealized symbols.

In this study, two primary purposes for establishing common ground were identified. First, the participants were using both pointing and representational gestures to clarify the problems they faced and ensure the whole group was on the same page for the following problems solving (e.g., in Figure 36 Error! Reference source not found., Chris explained to his teammates what they needed to do was to pass the ball to get a touchdown). Also, when the group came up with a shared idea, the participants intended to have representational gestures to both visualize their thought and elaborate the group’s shared idea. For instance, in Figure 34, Chris visualized his idea that the relationship between the red and blue sticks was they were connected and by tracing the red and blue sticks, his gesture reinforced the common ground from his speech and invited his teammates into the common ground.

**Transitional State**

According to Goldin-Meadow et al. (1993), a signal that the learner might be in a transitional state was s/he had multiple hypotheses and those hypotheses could be activated
simultaneously. In my study, I discovered the sequence of concreteness fading (enactive-iconic-abstract) yielded a clear process of being in a transitional state. By observing their use of gestures, I found that compared to Ian who was in the CI condition, Chris had more consistent hypotheses across the different stages in the entire intervention and more evident activation of multiple hypotheses, showing that he entered the transitional states.

In the Enactive stage, Chris had four initial hypotheses: 1) ball path was diagonal; 2) there was a half of square shape; 3) the red and blue were connected but the yellow went alone; 4) physical objects could be used to measure distance on the tiles. Among these four hypotheses developed in the task EP, the first two were more abstract as they were less related to the physical objects that they were using, but all four were highly contextualized in the task environment. In the task ID, when the X/Y values appeared, Chris developed another four hypotheses between the length of arrows and the X/Y numbers: 1) the X/Y values determined the red/blue arrow length and the yellow arrow was the passing distance; 2) the X/Y values in the task ID were same as the sticks in the task EP; 3) the X/Y values together determine the ball path; 4) the Y value determined the height of the ball and the X value curved the ball path. With numbers added in task ID, Chris was able to think about the movement of the ball in terms of its moving direction and length. However, there were co-existing contradictory hypotheses—whether the X/Y values simultaneously determined the ball path or the X value curved the path the Y value initially determined—which suggested Chris was in a transitional state in which he developed a couple of immature ideas to guide his problem-solving. Also, in this task, Chris started to connect different elements in two tasks (sticks used in the task EP and the X/Y values in the task ID). In the task AR, the contradictory hypotheses remained. On the one hand, Chris thought the X/Y values made a potential diagonal trajectory, which was consistent with the
hypothesis from the very first task EP; on the other hand, he believed the Y value was the height and the X value made upward and downward movement, which was one of the two main hypotheses in the second task ID. Furthermore, when solving the second question on the worksheet, he developed a new hypothesis: adding the X/Y values was to combine vectors and the result of the addition was a point. This hypothesis seemed to originate from his previous problem-solving, yet unfortunately, no further questions were asked. The hypotheses that Chris developed during the intervention could tell that the procedural CF tasks guided Chris to transition from highly contextualized hypotheses to abstract symbolic hypotheses.

CI tasks could also lead to a transitional state, but this transitional state was relatively stable and had fewer hypotheses. In the first task AR, Ian developed two hypotheses: 1) the arrows were forces applied—one as ball being thrown and the other as the wind; 2) by adding equations given in the question, the result was a point and they needed to move from the origin to the point. His first hypothesis was very concrete. His second hypothesis implied he had some idea about what a vector meant, but this idea was quickly abandoned when working on the next task. In the second task ID, he hypothesized that 1) the arrows were the forces applied and 2) the ball movement was made by one force and attracted by the other force. The same hypothesis appeared in the third task EP, too. He regarded the red/blue sticks as forces applied to the ball and the yellow sticks as the ball trajectory. Thus, Ian had a dominant hypothesis—the arrows were force—that was developed through his embodied analogy in the early task AR. The reversed sequence of tasks in the CI condition stabilized and reinforced his initial hypothesis. As a result, he might be in an early stage of his transitional state as all his hypotheses were concrete and from his previous embodied experiences.
**Insights and Limitations**

Since there were no previous studies investigating gesturing in CF/CI intervention, this study provided a new lens to explore learning in CF/CI intervention. Compared to knowledge testing (e.g., the knowledge questionnaire in Jaakkola & Veermans, 2018), scrutinizing the gesture use during the intervention could better understand how students formed their transitional knowledge during the intervention and how they carried over different hypotheses in different tasks to solve problems. Also, inspecting how students use gestures to create the common ground could help understand how they attempt to create shared cognition within their group when solving problems that involve unfamiliar concepts.

The sequence of CF tasks enabled students to develop rough contextualized hypotheses first and gradually transition to abstract symbolic hypotheses. Although their hypotheses might be unstable or even contradictory, their capability of establishing and utilizing multiple hypotheses indicated they were in the transitional state and could be ready for more formal vector addition learning. In contrast, in the CI condition, students likely relied on their naïve hypothesis developed from the first AR task and carried over through the entire intervention. Therefore, the following instruction might not be suitable for CI condition students.

This study also had several limitations. First, it was somewhat hard to capture every gesture in video analysis. Also, even though I conducted a very focused video analysis, some gestures were not captured due to the angle of the cameras. Second, there was great individual variability of gestural production. This variability, to some extent, affected the analysis as students might be shy to express themselves with gestures when facing the researcher whom they are not familiar with, and some of their real train of reasoning became impossible to find by only observing their gestures. Third, due to the limited resources, this study only had five groups and
sixteen participants, making it hard to generalize a quantitative pattern to determine the relationship between the use of gestures and the creation and maintaining of the common ground as well as the development of transitional states.
CHAPTER SIX: DEMONSTRATING LEARNING IN STORIES

In this chapter, I presented the results from my grounded theory method in response to RQ3: How might the understanding and skills gained from “concreteness fading” be applied to create stories? Since the CS task, as a learning demonstration task, was independent from the previous learning tasks, the data from all groups was analyzed. The discussion part showed the findings from my grounded theory method as well as the insights and implications.

Results

Below Table 19 shows the learning objective and context each group chose for their story. The context for the participants’ stories came with uniformity: except for one group (CO Group 1) that picked robotics as their story context, all other groups chose either soccer or basketball as their story context. Group CI Group 2 first wanted to make a story in a robotics context but changed their mind when building their story, and eventually ended up with a soccer context. The learning objective they chose for their story varied a lot. Four groups intended to use their story to describe the movement in terms of $x$ and $y$; two groups planned to use their story to demonstrate important things about vectors; one group decided to show their knowledge of the coordinate system. CO Group 1 had four participants, two of whom wanted to describe the movement with the story, whereas the other two wanted to use their story to show their understanding of the coordinate system.
<table>
<thead>
<tr>
<th>Group</th>
<th>Learning Objective</th>
<th>Context</th>
</tr>
</thead>
<tbody>
<tr>
<td>CF Group 1</td>
<td>Describe the movement (resultant) in terms of x and y.</td>
<td>soccer</td>
</tr>
<tr>
<td>CF Group 2</td>
<td>Describe the movement (resultant) in terms of x and y.</td>
<td>basketball</td>
</tr>
<tr>
<td>CF Group 3</td>
<td>Describe the movement (resultant) in terms of x and y.</td>
<td>basketball</td>
</tr>
<tr>
<td>CI Group 1</td>
<td>Describe any locations on a coordinate plane.</td>
<td>soccer</td>
</tr>
<tr>
<td>CI Group 2</td>
<td>Describe the most important things about vectors.</td>
<td>robotics  -&gt; soccer</td>
</tr>
<tr>
<td>CO Group 1</td>
<td>Two choose: Describe the movement (resultant) in terms of x and y. The other two choose: Describe any locations on a coordinate plane.</td>
<td>robotics</td>
</tr>
<tr>
<td>CO Group 2</td>
<td>Describe the movement (resultant) in terms of x and y.</td>
<td>basketball</td>
</tr>
<tr>
<td>AO Group</td>
<td>Describe the most important things about vectors.</td>
<td>basketball</td>
</tr>
</tbody>
</table>

Table 19 Learning Objectives and Contexts by Group

I coded the transcripts from the audio tracks when the participants were making the story. My open coding included 40 codes, and then I conducted focused coding by synthesizing those open codes and got 12 codes. Considering the theoretical relevance among those 12 focused codes, I eventually categorized them into 5 categories, 4 of which were labeled as categories used for generating theories. Table 20 shows all the focused codes and categories.

<table>
<thead>
<tr>
<th>Category</th>
<th>Definition</th>
<th>Focused codes</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>APPLYING NEW LEARNING</td>
<td>Students are applying the information they got from the previous learning tasks into their story creating and constructing.</td>
<td>Connectivity</td>
<td>Students are attempting to connect the current storytelling task to a previous task(s).</td>
</tr>
<tr>
<td></td>
<td></td>
<td>General learning design</td>
<td>Students are discussing what knowledge they need in their story.</td>
</tr>
<tr>
<td>COORDINATES</td>
<td>Students are applying coordinates knowledge which they are familiar with prior to the study in the process</td>
<td>Artifact Making-Grid</td>
<td>Students are making a grid or a coordinate plane for their story.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Coordinates-position</td>
<td>Students are using knowledge about locations on the coordinate plane.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Coordinates-units</td>
<td>Students are using the concept of unit on the coordinate plane.</td>
</tr>
</tbody>
</table>
Content: Vector Understanding Gaining

When making their stories, six out of eight groups attempted to incorporate their understanding of vector in their stories. The below chart (Figure 56) shows how the participants embedded their understanding of vector into their stories. Table 21 shows the counts and the coverage percentage of each open code for vector understanding. The focused codes displayed that there were three major types of understanding that students had while creating and developing their stories: movement, vector components, and other vector perceptions. For the first type of understanding–movement–the participants perceived a vector as a movement in two different ways. One way was to regard the movement as distance in a real-world direction (e.g., “I think I messed up. Didn't she go three yards to the left? And scored?”), and the other was along an axis (e.g., “He went from the origin, then he moved five spaces to the X axis. And then he went up by four to the Y axis to get to the bank”). The second type of understanding was comprehending vector in the form of vector components. There are also two forms of understanding: one was more abstract that expresses the units plus axis (e.g., “Oh, like four x, five x plus seven y”), and the other had the idea of units on axis but relied on the motion (e.g., “If
we throw from the two, X-axis when the ball reaches him, it's probably like 9 X axis and 9 Y axis”). The third type of understanding included three kinds of perception of a vector. One was purely symbolic terms from their previous learning tasks (e.g., medium or resultant). Another was a combination of direction, length, and initial speed (e.g., “the three vectors [components] are direction going west, the length is 20 feet, and how hard he kicked is 25 miles per hour”).

The third was more concrete in the context, regarding vector as a line that is a motion trajectory in their story (e.g., “it (the vector line) goes upward and down”).

**Figure 56 Codes for Vector Understanding**
Context: What Determines the Story

There were two factors that contributed to students setting up the context of their stories. The first factor was the learning activities in the CF/CI/CO/AO intervention. As the two focused codes showed in Figure 57, the participants had two ways to apply what they had learned in the previous intervention—by connecting to previous learning tasks and by proposing concepts that they wanted to incorporate into their story. Table 22 shows the counts and the coverage percentage of each open code for applying new learning. To establish the connections to previous learning activities in the intervention, the participants explicitly talked about what they had done previously (e.g., the ball passing activity), what elements they had encountered before (e.g., the arrow in the simulation), or the content knowledge (e.g., the movement of X and Y). Another way was from a meta-learning perspective. Since the participants had never learned vector or vector components before, some ideas they came up with for designing their story were likely from the intervention. Observing the open codes under the focused code General Learning Design revealed that when designing what to include in their stories, the participants 1) applied the concept of combining direction and length into their story (e.g., “two important things, a direction and length. We're going to put that into robotics. A punch, this direction and length”),

<table>
<thead>
<tr>
<th>Category</th>
<th>Focused Code</th>
<th>Open Code</th>
<th>Counts</th>
<th>Coverage</th>
</tr>
</thead>
<tbody>
<tr>
<td>VECTOR UNDERSTANDING</td>
<td>Movement</td>
<td>distance in direction</td>
<td>4</td>
<td>12.04%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>movement along axes</td>
<td>3</td>
<td>13.26%</td>
</tr>
<tr>
<td>Vector Components</td>
<td>vector component</td>
<td></td>
<td>2</td>
<td>4.94%</td>
</tr>
<tr>
<td></td>
<td>vector components as movement</td>
<td></td>
<td>4</td>
<td>13.32%</td>
</tr>
<tr>
<td>Other Vector Perception</td>
<td>abstract form</td>
<td></td>
<td>2</td>
<td>4.14%</td>
</tr>
<tr>
<td></td>
<td>vector as dir len and speed</td>
<td></td>
<td>2</td>
<td>4.83%</td>
</tr>
<tr>
<td></td>
<td>vector as line</td>
<td></td>
<td>9</td>
<td>21.59%</td>
</tr>
</tbody>
</table>

Table 21 Vector Understanding Open Codes Counts and Coverage
2) suggested content knowledge related learning outcome (e.g., “…I can tell them how to get the answer at the end. So they don't have to like have so many lines and solve them. All they gotta do is A equals B plus… No, it's Ax equals…”), 3) asked content knowledge question like “what are vectors?” and 4) determined what target knowledge they need for their stories (e.g., planning to teach distance on a coordinate plane).

![Figure 57 Codes for Apply New Learning](image)

<table>
<thead>
<tr>
<th>Category</th>
<th>Focused Code</th>
<th>Open Code</th>
<th>Counts</th>
<th>Coverage</th>
</tr>
</thead>
<tbody>
<tr>
<td>APPLY NEW LEARNING</td>
<td>Connectivity</td>
<td>connect to pa</td>
<td>7</td>
<td>18.24%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>connect to sim</td>
<td>1</td>
<td>0.39%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>mention learning</td>
<td>2</td>
<td>3.28%</td>
</tr>
<tr>
<td>General Learning Design</td>
<td>applying dir and len</td>
<td>1</td>
<td></td>
<td>1.47%</td>
</tr>
<tr>
<td></td>
<td>outcome</td>
<td>1</td>
<td></td>
<td>2.02%</td>
</tr>
<tr>
<td></td>
<td>question about</td>
<td>1</td>
<td></td>
<td>0.84%</td>
</tr>
<tr>
<td></td>
<td>target knowledge</td>
<td>1</td>
<td></td>
<td>0.94%</td>
</tr>
</tbody>
</table>

Table 22 Apply New Learning Open Codes Counts and Coverage
The second factor was the participants’ prior knowledge. Their prior knowledge includes four types: schooling knowledge, sports knowledge, preference, and outside knowledge (see Figure 58). The schooling knowledge was from the participants’ school life (e.g., some schools might have art crafting classes), and the participants added these kinds of experiences to their stories to construct the context of their stories. Sports knowledge, such as athletes and sports teams, was the most common participant knowledge observed in our study, possibly because of the football context used in the intervention. Participants also expressed their preferences (e.g., their dislike of a basketball player) when discussing the setting of their story. Lastly, the outside knowledge was the information that could not be specified, and the participants knew before the study (e.g., the news about the robotics team of their school going to a competition).
Although these two factors set up the context of the participants’ stories, they did not occur simultaneously. For example, below is the coding stripe graph for CF Group 2 (see Figure 59). At the beginning of their story design, they had some conversations that mentioned previous learning, coded as APPLY NEW LEARNING (blue stripe). Below is an excerpt from the transcript showing how they wanted to create a story similar to what they had done in the EP stage.

[00:02:58] P1: I think that this person will be getting involved from this person.  
[00:03:03] P3: Like the way look in the passing.  
[00:03:05] P1: Yeah.  
[00:03:06] P2: Like how we do the football.
[00:03:07] P3: Yeah. Because A, it says how to describe the movement so we can do it by the movement of the basketball.
[00:03:13] Researcher: The movement of the basketball? What did you say?
[00:03:15] P2: Like how we do the football when we were spending on the square.
[00:03:18] Researcher: Okay, so we can do the same thing in our basketball context.
[00:03:22] P3: And we can use the sticks, like make the tiles again.

On the other hand, their participant knowledge (purple stripe) came a little bit later when they were designing the plot for their story. For example, the below excerpt showed that this group of participants wanted to set an NBA all-star game in their story, and P3 explained to the researcher how NBA all-star game teams were named.

[00:05:53] Researcher: Do they still have the team like west Allstar and east Allstar, or they have different teams this year?
[00:05:58] P3: No, the names of the teams, like the name of the captains. They didn't do west or east.
[00:06:02] Researcher: Okay. What's are the teams this year?
[00:06:06] P3: Team LeBron and Team Kevin Durant.
[00:06:08] Researcher: Okay.

Therefore, although the participants’ prior knowledge and the gaining from the learning activities in the intervention determined the context of their stories, these two factors played different roles: the gaining from the intervention often provided the participants with a blueprint of what their story would look like, and once they had the blueprint, their prior knowledge contributed to the following plot design. Knowing how the participants set up the story was not telling enough, so I would look at how the participants constructed their stories in the next part.
Design: How Stories are Made

As discussed in the previous part, the participants utilized their gaining from the intervention and prior knowledge to set up the context for their story. After having a story context, the next step was to add a plot and actually construct the story. I found that the participants 1) incorporated the knowledge of the coordinate plane extensively, 2) enriched their story by adding aesthetic elements, and 3) presented their understanding of vectors by showing the plot that took place on the coordinate plane in their story.

As previously mentioned in Chapter three, the participants learned the coordinate plane in their 7th-grade math class. Presumably, they were more familiar with the coordinate
plane compared to vector and vector addition. As a result, the coordinate plane became an important component of their story. Figure 60 shows three important uses of the coordinate plane. Table 23 shows the counts and the coverage percentage of each open code for coordinates used in the story.

First, the participants either made a grid or a coordinate plane for their story. Noticeably, some groups explicitly stated that they were making a coordinate plane (e.g., “if they draw a coordinate plane showing everywhere either Ronaldo or Morgan arrives before they score a goal”), while other groups mentioned they would make a grid (e.g., “I’ll make the grid”), though they all had it as a coordinate plane. Second, they utilized a coordinate plane to describe the location of the characters in their story. The most common use was to use coordinates to describe the positions of their characters, but the participants also used negativity/positivity to refer to the direction of the movement and described the movement along an axis, only mentioning the target number. The third use of the coordinate plane was to use the units on the coordinate plane to measure the distance in their story either by referring to the units or by placing crafting sticks.
The participants did not only include a coordinate system in their story but also added aesthetic elements to their story. Figure 61 shows the ways that the participants added...
those aesthetic elements to enrich their stories. The most common practice was to name their characters so they could easily refer to the figures in the story, so did they add specific places to their grid and symbols to represent those places. To make the story more vivid, the participants also colored some symbols to make it more realistic, decorated their characters, made actual sports items, and manipulated the figures to give them poses that aligned with the plot in their story. The participants not just added aesthetic elements to fulfill their aesthetic taste but also to visualize their story.

Figure 61 Codes for Artifact Making-Aesthetics
Besides including coordinates and aesthetic elements in their story, the participants also demonstrated the ability to integrate their understanding of vectors, the utility of the coordinate plane, and the story plot, when telling their story. The below graph (see Figure 62) shows there were two times when the participants in CO Group 1 attempted to integrate their understanding of vectors and knowledge of coordinate plane into their storytelling (see circled coding stripes). The below excerpts were the corresponding discourse. In these two excerpts, the participants (P3 and P4) displayed the same way of articulating the movement of their character on the coordinate plane. P3 described the movement of the character with spaces along the axes and the real-world space directions (left/right/up/down). Similarly, P4 had the same strategy—using the units in real-world space directions with coordinates as destinations. Both of their descriptions of the movement were rooted in their storytelling (e.g., the character broke his arm or felt hungry) and based on the coordinate plane they created, and meanwhile, their storytelling involved the length along a direction, which was a key property of vector, showing their understanding of vector.

Excerpt 1
[00:19:40] P3: He went from the origin, then he moved five spaces to the X axis. And then he went up by four to the Y axis to get to the bank. Then after he went to the bank, he moved one space to the right and four spaces to get to the mall. After getting hungry, he went to the...
[00:20:10] P4: Pizzeria
[00:20:11] P3: He went the pizzeria, which is two spaces over here. One space up. And then he just went to school. He went down all the way over to two, three. He just went to the left, all the way down there. But then after breaking his arm, the police who were at (3,1), they went down by one and then go to the right by two.

Excerpt 2
[00:23:29] P4: Because he's moving around a lot. When he is at the origin (0,0), he goes all the way, five units to the right and then goes up four units. And then after that, he wants to go shopping. So he goes to the left once and then he goes
up four times. And then he gets hungry. So he moves to the left twice, and then

goes up once. But then he has to go to school and fix his robot. So it comes all the

way back down to (2,3), and then he falls and breaks his arm. And then the police

at (3,1) have to move from their spot and come all the way to his school, which is

(3,2). And then take him all the way to the hospital, which is (6,10).

Discussion

In this part, I showed three findings from my grounded theory method in three aspects:

content knowledge, the context of the story, and the progress of manifesting learning with

storytelling. Then, I discussed how my findings could contribute to learning vector and vector

addition. Last, I examined the limitations and proposed suggestions for future studies.

Findings

Finding 1: The participants attempted to understand vector addition in two formats: vector

components and vector motion.

The first finding focused on the content knowledge that the participants demonstrated.

Either vector or vector addition was a fresh new concept for the participants. However, after

saturating in the idea of using vector components to conduct vector addition during the learning

tasks, the participants could establish their own understanding of vectors, which were vector
components and vector motion (see Figure 56). Using vector components to conduct vector addition was the target skill of our intervention, but it could be hard for all the groups to develop this skill in a two-hour intervention. Only two groups (CF Group 1 and CO Group 2) showed this ability in their story by adding the units along X or Y axis to reach a target position on a coordinate plane. Most participants held the idea of using units along an axis or a direction or viewed vector in the form of motion (vector as movement on the coordinate plane or vector as a velocity line). Although this idea might be primitive in regard to learning vector addition, it was promising as learners recognized that a vector could be a relationship between two positions on a coordinate plane and distinguished it from the coordinates. The participants showing their partial understanding of vectors implied that the intervention in this study enabled students to enter a transitional state (see Goldin-Meadow et al., 1993) in which they demonstrated partial knowledge and became ready to acquire new concepts by following instructions.

**Finding 2: The context in the learning tasks and the participants’ prior knowledge have separate influence on the context of the story made.**

The second finding concentrated on the context of the stories the participants made. It was not a surprise that most groups chose a sports context (see Table 19) and brought sports knowledge into their story because the football context and the physical football passing activity in the previous learning tasks played a significant role in helping the participants understand the mathematics content. Also, the schooling knowledge and the outside personal knowledge the participants brought into the story (see Figure 58) seem to be a positive indication that the participants were not just completing a “task” but actually engaging in this form of learning
demonstration. When learners connected the current learning experience with their prior knowledge, their interests, and the real world, it was likely that they were establishing personal and epistemological connections which could promote learning engagement for sustainable active learning (Kafai & Resnick, 1996).

**Finding 3: The previous learning tasks enable the participants to construct stories on a coordinate plane, and the storytelling activity provides a ground for manifesting an integrated understanding of coordinates and vector addition.**

The third finding highlighted the process of creating, constructing, and telling stories. As mentioned in the method section, the participants had learned the coordinate system before the study. Therefore, their familiarity with the coordinate system as well as the settings in the learning tasks (e.g., the tiled floor in the physical activity and the grid in the simulation) enabled them to construct a coordinate plane for their story, which provided a valuable tool for showing vector addition (see Figure 60). Next, when constructing their story on the coordinate plane, adding aesthetic elements (see Figure 61) showed that the participants were engaged in the process of story-making, which implied that the learners were experiencing “thick authenticity” (Shaffer & Resnick, 1999) in the constructionist storytelling activity. Furthermore, during the storytelling, the participants were able to integrate their knowledge of the coordinate plane and their preliminary understanding of vectors as inherent parts of their story, which aligned with Kafai et al.'s (1998) definition of *intrinsic integration*. Intrinsic integration requires learners to embed their understanding of a concept as an inherent component into their narrative or artifacts.
Thus, the constructionist storytelling activity provided learners with an opportunity to create intrinsic integration of their vector understanding and the authentic story they made.

**Insights and Limitations**

Concluding from the previous discussion, I proposed two suggestions for designing a constructionist activity for students to show their understanding of vector addition after a CF intervention. First, *a constructionist storytelling activity should allow students to demonstrate their naïve understanding of the content knowledge*. As I stated above, previous CF studies often used knowledge tests to evaluate learning outcomes. Knowledge tests could be a good measurement of the degree that students master the content knowledge. However, simply focusing on the outcome omitted an important aspect of the development of mathematical thinking. Sfard (1991) declared that there were two different ways of perceiving mathematical conception structurally (viewing mathematical concepts as objects) and operationally (viewing mathematical concepts as processes). Knowledge tests might fail to show a learner’s operational understanding of a mathematical concept, but a constructionist storytelling activity could show both sides. In this study, understanding vector addition as the operation on vector components was structural, while viewing vector addition as movement was more operational. The dual sides of perceiving vector addition could prepare learners’ future learning of the same concept.

Second, *a constructionist storytelling activity should provide space for students to make connections to the previous learning activities and bring in their own knowledge*. As a subsequent activity, it was important for instructors to help students establish connections to the previous learning experience, such as proving a similar context. Also, constructionism emphasizes personal meaningfulness and requires the object-to-think-with for learners
(Ackermann, 2001; Papert, 1980), so when designing a constructionist storytelling activity, instructors should think about what kind of context could promote students bring their own knowledge, which demands instructors to understand their students and communicate with them before and during the intervention.

I also identified some limitations in this part of the study. As mentioned in the previous method section, the nature of a grounded theory method was to generate findings from data to answer research questions and to form theories later. Therefore, the findings generated from this analysis were deeply rooted in this study’s qualitative data and might be hard to replicate. Besides, I attempted to minimalize the impact of different group conditions because producing any generalizable theories could be risky with such a small sample size. For future studies, I would suggest 1) having more groups for every condition; 2) having other analysis methods to examine if the findings from this grounded theory method analysis were replicable.
CHAPTER SEVEN: CONCLUSION AND FUTURE STUDIES

The final chapter of this dissertation restated the research problems, reviewed the methods used in the study, discussed findings, and, more importantly, proposed possible future studies.

About This Study

As introduced in Chapters one and two, vector addition is a key skill for a successful STEM career, but vector addition education is an underexplored topic in the field of secondary-level mathematical education. To design an effective vector addition learning experience, I used “concreteness fading” (CF) as a design framework and my CF instructional design was introduced in Chapter three. Aiming at the absence of learning demonstration in a CF intervention, my design added an additional task for learners to manifest their gaining from the intervention in a form of story-making and storytelling.

In regard to the research on CF, I realized that most previous CF studies paid no attention to the learning process in a CF intervention and focused more on the learning outcomes. Exploring the process of learning, however, was valuable, because learners often developed procedural knowledge in the learning process before they were able to develop a structural understanding of a mathematical concept, recalling Sfard's (1991) notion that operational knowledge often preceded structural knowledge in mathematics learning, which was discussed in-depth in Chapter two. Therefore, two of my research questions (RQ1: How does “concreteness fading” structure students’ sense-making around vector addition? and RQ2: How do gestures interact with “concreteness fading” in shaping student understanding of vector addition?) related to the learning process in a CF intervention from two different aspects—mathematical sense-making and embodied cognition. Besides, I also noticed that previous CF
studies only had three learning tasks and did not provide ample opportunities for learners to
demonstrate their learning, so I designed an extra constructionist storytelling task for my
instructional design in response to my RQ3 (How might the understanding and skills gained from
“concreteness fading” be applied to create stories?).

Then, my methods, including the study design, data collection, and data analysis
approaches, were showed in Chapter three. Since there were no previous studies exploring vector
addition learning under a CF framework, by examining the qualitative data, a grounded theory
method was used to inspect the processes of learning in a CF intervention and demonstrating
learning through constructionist storytelling. Chapters four and six showed the results from using
this method to respond to RQ1 and RQ3. Besides, I also used a video analysis approach to
inspect how two participants used their gestures to create common grounds for their group and
how their gestures implied their transitional knowledge states to answer RQ2, which was
presented in Chapter five. Since my three research questions are from different perspectives, the
results, findings, and limitations were discussed separately in each chapter.

**Future Questions and Future Studies**

One of the limitations of this dissertation was that the results and findings in Chapters
four and six did not present theories. Due to limited resources, I only had three CF groups, two
CI groups, two CO groups, and one AO groups. With a relatively small sample size, I kept my
findings more grounded in my dataset. In the spirit of grounded theory method (Glaser &
Strauss, 1967), this section proposes new questions to be answered through further study.
Question 1. How do similar design elements and problem settings in a concreteness fading intervention build learners mutual referents between tasks?

Fyfe and Nathan (2019) hypothesize that each CF stage should have explicit references to the previous stage(s) for more effective learning in a CF intervention. In light of their hypothesis, I believe a practicable way was to embed similar design elements, such as same colors, and to make problem settings same. In my study, consistent design elements and questions can create learners a sense of familiarity and make them realize that they are not tackling three separate tasks. Instead, they will possibly explore some other similarities, which seems to be a chance for them to explore the underlying mathematical concepts across tasks.

Question 2. How is structural knowledge simultaneously developed when learners construct their operational knowledge in the fading stage (2nd stage) of a concreteness fading intervention?

Although building structural knowledge was harder than mastering operational knowledge (Sfard, 1991), learners can possibly have the opportunities to build both at the same time in the fading stage during a CF learning course. Observing the CF participants’ learning in the task ID in this study, I noticed that the participants spontaneously recognized the arrows in the game as the sticks in the physical activity. The arrows were tied to the X/Y value, which was considered as a more formal element, but they were not intimidated by this formalism. Instead, the participants seemed to accept the idea that the X/Y value determined the length of the red/blue arrow, and the direction of the arrow was the direction of the axis. Adding/subtracting
the X/Y value in task ID was obviously operational knowledge for winning the game, but the linkage between the X/Y value and the arrow was structural knowledge, which concurrently emerged when the participants were using their operational knowledge to solve the problems in the simulation game. Thus, future studies can scrutinize the fading task more deeply to uncover the magic of simultaneous operational and structural knowledge construction.

**Question 3. How do unfamiliar symbols in the abstract stage of a concreteness fading intervention enhance mathematical sense-making?**

Unfamiliar symbols are often considered as a major burden that impedes mathematical learning (Nathan, 2012). However, when asking the participants to explicate the formats of their symbols for those abstract questions, I realized that their explanations evinced rich mathematical sense-making. Schoenfeld (2016) states that a key ability of mathematical sense-making is to use learners’ own words to explain their mathematical understanding. In a CF intervention, the first two stages allowed learners to develop their relevant mathematical understanding, and in the third stage, they felt more comfortable showing their understanding of those unfamiliar symbols that seemed to be connected with what they had encountered in the previous stages. Future studies can consider including more complex mathematical symbols to test whether learners were still able to display their sense-making on those symbols.

**Question 4. How does the development of conceptual knowledge in a concreteness fading learning experience associate with the form of learning activities?**
Although previous studies (e.g., Watson et al., 2003) explored the conceptual understanding of a vector, the conceptual understanding of vector addition was barely researched. In this study, the stories that the participants created indicated that they regarded vector addition either as the addition of vector components or a process of moving along a direction. It was probably because there were two specific activities that helped the participants construct their understanding: football throwing that involved motion and the simulation game that had the X/Y components of a vector. Vector addition can also be performed through a more traditional palatogram approach, so future studies can test if learners can conduct this approach when they need to consistently move vectors from tip to toe in a concreteness fading intervention.

Besides the above questions related to vector addition, I believe concreteness fading also has the potential in other STEM education domains. For example, a possible study in the field of computer science education can be teaching new programmers the concept of recursion. “Recursion in computer science is a method where the solution to a problem depends on solutions to smaller instances of the same problem (as opposed to iteration)” (Recursion, n.d.). A physical activity for recursion can be playing nesting dolls and the abstract question can be understanding a function that solves a recursive problem. The fading stage may present learners with graphical codes showing the recursive process of playing nesting dolls. Many STEM ideas are eligible for a concreteness fading intervention, and for effective concreteness intervention learning, the instructor needs to know the learners and design tailored activities. In addition, I hold the opinion that concreteness fading learning may be more suitable for small lessons rather than big curriculum. A small lesson with a specific topic can leverage the advantages of
concreteness fading because learners can quickly establish train of thoughts on the relevant concepts by building mutual referents between the ideas that they are familiar with and the concepts that they need to understand. On the other hand, in a semester-long course, if learners stay in the enactive stage, the iconic stage, and the symbolic stage for three weeks, respectively, they may feel it hard to make connections because they are saturated in each stage and likely treat the learning experience in each stage as an independent learning process.

Overall, concreteness fading, as an instructional framework, has the capability of teaching complex mathematical topics. The traditional abstraction-dominated mathematical learning is counterintuitive and makes mathematical learning less welcoming by intimidating new learners with confusing symbols and notations; on the contrary, the progression of concreteness fading—from enactive, through iconic, and to symbolic—can not only provide learners with a smooth, effective, engaging, and meaningful learning experience, but also creates a more inclusive and propitious mathematical learning environment.
APPENDIX

Appendix A. Food Delivery Questions in Task Concreteness Reinforcement

i. There is an order from the Traditions at Scott, a dining hall at Scott House (H5), to Jameson Crane Sports (D2). Assuming the rover will not return to Scott House, please plan your Yandex rover’s route and determine whether it needs a backup battery.

ii. There is another order from the Traditions at Kennedy, a dining hall at Kennedy Commons (H7), to Heffner Wetland Research and Education Building (G1). Assuming the over will not return to Kennedy Commons, please plan your Yandex rover’s route.

iii. There is a final order from Ohio Union (I7) to the Agricultural Administration Building (E5). Please plan your Yandex drone’s route and report how much battery it will consume after the order.

Appendix B. Exercise Questions in Task Abstraction Reinforcement

i. There are two vectors 2x-3y and -3x+5y. What is the resultant (sum) of these two vectors?

ii. A force of -3x+4y newtons is being applied to an object. What other force should be applied to this object in order to achieve a total force of 2x-y?
Appendix C. Researcher’s Prompt Questions

Task EP:

1. Could you explain to your teammates how you describe your position? And why do you think your position should be XXX (a form of two numbers, coordinates, or something else)?
   On tile students explain to each other.

2. [After placing blue and red sticks, for the stick placing student]
   Could you explain to your teammates why you put sticks in this way?

   [For on tile students]
   Do you agree with him/her? Why do you think it is a good way to connect points like this?

   [For the group]
   How would you describe the directions of these red and blue sticks?

3. [After placing yellow sticks, for the stick placing student]
   How long do you think this line is in terms of sticks you have placed?

   [For on tile students]
   How would you like to describe the direction of these yellow sticks?

4. [For each student, head to tail]
   Could you explain to [NEXT STUDENT] what you think the relationship is between those yellow sticks and those red and blue sticks in terms of their lengths and directions?
5. [For CI students]
   Did you find anything in this question that relates to what we have done before? What is it?

Task ID:

1. [For the manipulating student, before click go]
   Could you explain how you get the x and y units to your teammates?

2. [To the rest students]
   Do you agree with him/her? Why do you think this is the correct answer?

3. [For each student, head to tail, after playing]
   Could you explain to [NEXT STUDENT] what you think the relationship is between the red and blue arrows and the yellow arrow in terms of their lengths and directions?

4. [For each student, head to tail, last question]
   Did you find any connections between this simulation and our previous activity? What are they?

Task AR:

1. [For each student, in the beginning, for each question]
   Do you have any rough ideas about how to solve this?

2. [For CF students]
   Did you find anything in this question that relates to what we have done before? What is it?
3. [For AO students]

Think about the videos we just watched. Did you find anything in this question that relates to the previous videos? What is it?

4. [After solving the question, head to tail]

Could you explain to [NEXT STUDENT] how you get your answer?

5. [For CF and AO students, head to tail, last question]

Could you explain to [Next STUDENT] what kind of connections you find between the questions you solved and [the previous activities we have done]/[the previous videos we have watched]?

Task CS:

1. [After storytelling, head to tail]

Could you explain how your story is going to teach one of your friends about [Their Learning Objective]?

Task Concreteness Reinforcement:

1. [For the route planning student]

Could you explain to your teammates why you have this path?

2. [For each student, head to tail]

Did you find anything that relates to what we have done in the previous activity? What is it?
Task Abstraction Reinforcement:

1. [After watching the video, head to tail]
   Did you find anything new to you?
   Did you find anything that confused you?

2. [After solving the problem, head to tail]
   Did you find anything in these questions that relate to anything from the videos we just watched?
REFERENCES

Ackermann, E. (2001). Piaget’s constructivism, Papert’s constructionism: What’s the difference? 


https://doi.org/10.1037/10096-006


https://search.library.wisc.edu/catalog/9912312120702121


https://doi.org/10.1016/j.learninstruc.2014.10.004


Mai, T., Feudel, F., & Biehler, R. (2017). A vector is a line segment between two points?

Students’ concept definitions of a vector during the transition from school to university. *CERME 10*. The Tenth Congress of the European Society for Research in Mathematics Education.


https://doi.org/10.1016/j.learninstruc.2012.05.001

University of Chicago Press.


learning: A project of the National Council of Teachers of Mathematics (pp. 334–370).

Macmillan Publishing Co, Inc.


https://doi.org/10.2307/1165995


