Applications of automatic transversality in contact homology

By

Joanna Nelson

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The dissertation is approved by the following members of the Final Oral Committee:

Professor Gloria Mari-Beffa, Professor, Mathematics

Professor Mohammed Abouzaid, Associate Professor, Mathematics

Professor Jordan Ellenberg, Professor, Mathematics

Professor Alex Nagel, Professor, Mathematics

Professor Lindsay Stovall, Assistant Professor, Mathematics

Abstract

In 2000, Eliashberg, Givental, and Hofer [EGH00] sketched a new approach called symplectic field theory to construct invariants of contact and symplectic manifolds. Despite an extensive literature [AM12], [Bo02], [Bo09], [BEE11], [BEE12], [BCE07], [BC05], [BEHWZ], [BO09], [Us99], [MLY04] even cylindrical contact homology, the least complex of these invariants, has yet to be rigorously defined or computed in any non-trivial situation. This paper establishes how the heuristic arguments sketched in the aforementioned literature are not sufficient to define a homology theory. After introducing a class of contact forms which we call *dynamically separated,* see Definition 1.2, we provide a rigorous foundation for cylindrical contact homology in dimension 3, reliant only upon established analytic techniques [ADfloer], [CFHW], [Dr04], [H93], [H99], [HK99], [HWZI], [HWZ02], [MSbig$J$], [Sa99], [Sc95], [We10]. We then provide a new aproach to compute cylindrical contact homology for a large class of examples. The issue of invariance under the choice of nondegenerate dynamically separated contact form or choice of compatible almost complex structure remains unresolved.

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Chapter 1

Introduction and results

Symplectic field theory started with an impressionistic outline in 2000 by Eliashberg, Givental, and Hofer in [EGH00]. As stated in the introduction of their paper, it contained practically no proofs and was meant only as a "very sketchy overview" of what contact homology and other related invariants should be if all the analytic difficulties could be resolved. For many years the issues in understanding compactness and transversality were not discussed in conjunction with cylindrical contact homology. See [AM12], [Bo02], [Bo09], [BEE11], [BEE12], [BEHWZ], [BCE07], [BO09], [Us99], [MLY04]. Transversality issues with multiply covered curves and their branched covers cannot be avoided by merely excluding contractible Reeb orbits of a particular index, as has been asserted in existing literature. Without transversality one has no way of guaranteeing that what has been computed is an invariant or even a homology.

The ongoing polyfold project by Hofer, Wysocki, and Zehnder [H06], [HWZ10a], [HWZ10b], [HWZgw] has been fruitful in overcoming various issues arising in pseudo-holomorphic curve dependent theories, but the analytic foundations of contact homology are still a work in progress. The main obstacle is due to multiply covered pseudoholomorphic curves and their branched covers, as these yield nonempty moduli spaces of nonpositive virtual dimension. Standard methods of perturbing J as in [FHS95] fail to do away with these moduli spaces because the chain complexes for contact homology have been set up with an unavoidable S^1-symmetry. All known perturbation schemes rely on breaking the underlying S^1-symmetry, and hence are not appropriate for contact homology. As a result a completely general definition of cylindrical contact homology still awaits an appropriate theory of so-called *abstract perturbations*.

While linearized contact homology will not be discussed in this paper, we remark that the presence of these same moduli spaces of multiply covered pseudoholomorphic curves also prevents this theory from being a well-defined homology theory. It has been claimed in the literature that linearized contact homology, a more general theory, can be defined even when cylindrical contact homology cannot be [Bo09], [BO09] but the lack of transversality results suggests otherwise. More details on transversality and regularity can be found in Section 5.3 and 6. However, by restricting our attention to a class of contact forms that satisfy a uniform growth in their Conley-Zehnder indices, we will see that in dimension 3 we can construct a well-defined cylindrical contact homology, though unresolved issues in proving invariance still remain. We call such contact forms dynamically separated and give the precise definition in Definition 1.2.

For the uninitiated we begin with a brief recollection of how one aims to construct contact homology, with more details contained in later chapters. We denote by (M, ξ) a co-oriented contact manifold of dimension $2n-1$ and denote a globally defining 1-form for the contact structure by α, so that ker $\alpha = \xi$. The general philosophy in these pseudoholomorphic curve homology theories is to transfer the finite dimensional Morse theory picture of critical points and negative gradient flow to the infinite dimensional world of closed periodic orbits and pseudoholomorphic curves by making use of a Fredholm theory that is well-behaved under certain transversality assumptions.

Cylindrical contact homology aims to be a $\mathbb{Z}_2$-graded object defined over $\mathbb{Z}_2$-coefficients after an appropriate selection of a nondegenerate contact form α, used to define the contact structure ξ. When using the canonical absolute $\mathbb{Z}_2$ grading, one assigns a Reeb orbit an odd grading when it is positive hyperbolic and an even grading when it is elliptic or negative hyperbolic. In certain cases, such as when $c_1(\xi) = 0$ we can use the existence of a volume form to upgrade to a $\mathbb{Z}$-grading. For this to be well-defined, we stipulate that $H^1(M) = 0$ to ensure that there is only one homotopy class of trivializations associated to the complex line bundle that is the canonical representation of $-c_1(\xi)$. Otherwise the $\mathbb{Z}$-grading will not be independent of the choice of volume form. Further details may be found in [Se00] and [Se06]. In this paper we assume that $c_1(\xi) = 0$ and $H^1(M) = 0$.

For the purposes of this paper it suffices to restrict ourselves to $\mathbb{Z}_2$-coefficients, though we mention that there is a notion of coherent orientations which would allow one to define contact homology over $\mathbb{Z}$-coefficients or $\mathbb{Q}$-coefficients; an introduction to these conventions may be found in [BM04] as adapted from Floer theory in [FH93]. The chain complex C_* is comprised of all nondegenerate closed Reeb orbits and their iterates, associated to a contact manifold $(M, \xi = \ker \alpha)$, which appear naturally as the critical points of the **symplectic action functional**

$$\begin{aligned} \mathcal{A}: \quad C^\infty(S^1, M) &\to \mathbb{R}, \\ \gamma &\mapsto \int_\gamma \alpha. \end{aligned}$$

The Reeb vector field R_α is uniquely determined by

$$\iota(R_\alpha) d\alpha = 0, \quad \alpha(R_\alpha) = 1.$$

The differential ∂ depends on the choice of a compatible almost complex structure J and counts rigid[1] pseudoholomorphic curves interpolating between closed Reeb orbits.

[1]These are elements of a moduli space with virtual dimension 0, that is these curves connect Reeb

The pseudoholomorphic curves in this context are solutions of the Cauchy-Riemann equation, specifically maps

$$u : (\mathbb{R} \times S^1, j) \to (\mathbb{R} \times M, J)$$

satisfying

$$\bar{\partial}_{j,J} u := du + J \circ du \circ j \equiv 0$$

subject to finite energy condition, which implies that the infinite ends of the domain converge to closed Reeb orbits at $\pm\infty$ in the symplectization. Such curves are called asymptotically cylindrical or finite energy pseudoholomorphic curves. The grading in contact homology of a Reeb orbit is given by

$$|\gamma| = \mu_{CZ}(\gamma) + n - 3$$

where n appears in the dimension of the contact manifold M^{2n-1}, and μ_{CZ} is the Conley-Zehnder index of a path of symplectic matrices obtained from the linearization of the flow along γ, restricted to ξ. This index is a Maslov type index for arcs of symplectic matrices which is a generalized winding number that controls embedding properties of pseudoholomorphic curves. Understanding the behavior of pseudoholomorphic curves and demonstrating that the counts of such objects is independent of all the choices one made along the way and invariant of the underlying contact manifold requires that all moduli spaces be cut out transversally.

The initial motivation for concocting such a homology theory is so that one can qualitatively understand the behavior of any Reeb vector field associated to ξ. This is in general hard because there are many different contact forms defining the same contact

orbits of Conley-Zehnder index difference 1 in the symplectization.

structure, whose corresponding Reeb vector fields could have wildly different flows. If it were well-defined, cylindrical contact homology would serve as a powerful qualitative invariant, providing concrete relationships between topological aspects of a contact manifold and any Reeb dynamics associated to the contact structure ξ. Applications of such a theory include determining if every Reeb vector field associated to a particular contact structure ξ admit a closed characteristic (the Weinstein Conjecture) or distinguishing different contact structures. These questions are very much analogous to those found in the world of symplectic topology and Hamiltonian dynamics.

The conjecture from the original [EGH00] paper of Eliashberg, Givental, and Hofer that has only been presented with a sketch of a proof, is as follows.

Conjecture 1.1. *Let (M^{2n-1}, ξ) be a co-oriented contact manifold*[2]*. Further assume that all closed orbits of the Reeb vector field associated to α are non-degenerate and that there are no contractible orbits of grading $|\gamma| = -1, 0, 1$. Then for every free homotopy class $\bar{a}$*

(i) $\partial^2 = 0$

(ii) $H_*(C_*^{\bar{a}}, \partial)$ *is independent of the contact form α for ξ, and the compatible almost complex structure $\mathfrak{J}$.*

We will see in Sections 5 and 6 that an *abstract perturbation package* is required to show that $\partial^2 = 0$ and to prove invariance under these general assumptions. This is because the exclusion of orbits with grading -1, 0, 1 is not sufficient to preclude the breaking phenomenon exhibited by pseudoholomorphic cylinders due to the presence of nonempty moduli spaces of nonpositive virtual dimension.

[2] Co-oriented means there exits a global $\alpha \in \Omega^1(M)$ such that $\xi = \ker \alpha$

Moreover, we need to demonstrate that the rigid pseudoholomorphic cylinders over Reeb orbits are regular so that one can modify gluing arguments from Floer homology to the world of contact homology. These regularity results are necessary to prove that (C_*, ∂) forms a chain complex and to construct the the chain homotopy equation. A thorough discussion of transversality and regularity requirements can be found in in Sections 4, 5, and 6, and details on gluing arguments and the geometry of the moduli spaces may be found in in Section 7.

By restricting ourselves to the following class of contact forms, which we term **dynamically separated**, we will be able to exclude the presence of moduli spaces of nonpositive virtual dimension. We also obtain regularity results for all asymptotically cylindrical pseudoholomorphic cylinders after a generic choice of J in the symplectization and for all asymptotically cylindrical pseudoholomorphic cylinders which do not limit on positive hyperbolic orbits of the same index in a cobordism. These are defined as follows.

Definition 1.2. We call a contact form associated to (M^3, ξ) **dynamically separated** whenever the following two conditions are satisfied.

(i) $3 \leq \mu_{CZ}(\gamma) \leq 5,$ for all closed simple contractible Reeb orbits γ.

(ii) $\mu_{CZ}(\gamma^k) = \mu_{CZ}(\gamma^{k-1}) + 4,$ where γ^j is the j-th iterate of a simple orbit γ. (1.1)

Implicit in this definition is the assumption imposed earlier; that there exists an absolute integral grading of the Conley-Zehnder indices of our Reeb orbits. Hence we must assume that $c_1(\xi) = 0$ and $H^1(M) = 0$. We explain the full details of this in Section 9, and as a result we see that this gives a integrally graded cylindrical contact homology.

In order to exclude moduli spaces of nonpositive virtual dimension we use the uniform increase in Conley-Zehnder index in conjunction with the assumption that $3 \leq \mu_{CZ}(\gamma) \leq 5$, for all closed simple contractible Reeb orbits γ. We can then appeal to the work of Wendl [We10] and Dragnev [Dr04] to surmount the aforementioned difficulties. However, Wendl's automatic transversality results restrict us to work only with three dimensional contact manifolds.

The basic ideas in these arguments as follows. First we appeal to Dragnev's regularity results in [Dr04], which apply to moduli spaces consisting of somewhere injective pseudoholomorphic curves. After a generic choice of J, these results allow us to exclude the existence of nonpositive simple pseudoholomorphic cylinders associated to dynamically separated contact forms, as simple pseudoholomorphic curves are somewhere injective. The uniform increase in Conley-Zehnder allows us to appeal to Wendl's automatic transversality results to obtain the requisite regularity results for multiply covered pseudoholomorphic cylinders in symplectizations. In addition, the restriction to the class of nondegenerate dynamically separated contact forms enables us to exclude all nonpositive moduli spaces of pseudoholomorphic curves that could appear in the breaking phenomenon of asymptotically cylindrical pseudoholomorphic curves, obstructing $\partial^2 = 0$.

Together these results allow us to conclude that the boundary of the set of pseudoholomorphic cylinders with index difference 2 consists only of broken trajectories, which are cylinders of index difference 1 glued along an intermediary orbit. We use these results to prove an analogue of Floer's gluing theorem in Section 7. The multiplicities of our Reeb orbits and finite energy pseudoholomorphic cylinders are encoded in the structure of the graph that can be associated to the compactification, which allows us to prove that $\partial^2 = 0$.

We remark that a uniform increase which is larger than 4 would also work, provided item (i) in the definition of dynamically separated remains. In the case of simple non-contractible orbits there is more flexibility on the lower bound of the Conley-Zehnder index, but one needs either $1 \leq \mu_{CZ}(\gamma) \leq 3$ or $3 \leq \mu_{CZ}(\gamma) \leq 5$ for all Reeb orbits in the same free homotopy class in order for the regularity results of Section 6 to be applicable to cylinders of index difference 1 or 2. The reasons why a uniform increase of 3 or 2 fails to yield the desired results can be seen in Section 6.2.

By only considering contact manifolds equipped with nondegenerate dynamically separated contact forms we obtain a cylindrical contact homology. When the theorem is stated as below, the proof relies only upon established analytic techniques, as in [ADfloer], [CFHW], [Dr04], [H93], [H99], [HK99], [HWZI], [HWZ02], [MSbig$J$], [Sa99], and [We10].

Theorem 1.3. *Let (M^3, ξ) be a co-oriented contact manifold*[3] *with a nondegenerate dynamically separated contact form α defining ξ and J a generic compatible almost complex structure. The vector space $C_*(\alpha)$ generated by the closed Reeb orbits of α admits the linear map ∂, as defined in (8.4) satisfying $\partial^2 = 0$, thus (C_*, ∂) forms a chain complex.*

Remark 1.4. At this time, we are unable to remove the dependence of the homology on the choice of nondegenerate dynamically separated contact form defining ξ or the compatible almost complex structure J. This is because the automatic transversality are results are inconclusive for a cylinder with both ends at positive hyperbolic orbits of

[3] Co-oriented means there exits a global $\alpha \in \Omega^1(M)$ such that $\xi = \ker \alpha$.

the same Conley-Zehnder index in a cobordism.

After establishing that a meaningful formulation of cylindrical contact homology exists, it is desirable to demonstrate it can be computed in a rigorous fashion for a set of non-empty examples. As we point out later in Example 5.1, even the typically noiseless ellipsoid fails to satisfy the stringent requirements in the definition of dynamically separated and gives rise to a moduli space of nonpositive dimension. Adding to the difficulty in computing cylindrical contact homology is the requirement that the Reeb vector field associated to α have only nondegenerate closed Reeb orbits. The understandable symmetric Reeb dynamics are typically associated to symmetric contact forms, which are highly degenerate.

A generic perturbation can be used to obtain a nondegenerate contact form, but this turns easy to understand dynamics into ones which are frequently imperceptible and results in a perturbed Cauchy-Riemann equation, which tends to be impossible to solve in practice. A potential way to avoid breaking nice symmetry is to use a Morse-Bott approach to compute (cylindrical[4]) contact homology, which relaxes the condition of non-degeneracy on α and permits the use of a larger class of admissible contact forms. This framework was sketched by Bourgeois for specialized settings in his thesis [Bo02], but never published.

We construct a new means of computing cylindrical contact homology, coming from an explicit perturbation of the canonical contact form associated to a prequantization space, satisfying a proportionality between the index and the action of the persisting orbits in Section 10. This is accomplished by directly perturbing the critical manifolds realized as Reeb orbits and establishing a natural filtration on the action of the Reeb

[4]This means we count cylindrical curves only.

orbits, leading to a formal version of filtered homology. Similar methods were employed in [CFHW] to determine the stability of the action spectrum of the contact type boundary of symplectic manifolds in the world of symplectic cohomology. By appealing to the realization of (S^3, ξ_{std}) as a prequantization space and making use of this approach, we are able to rigorously compute the cylindrical contact homology for the standard sphere. This approach does not rely on those ideas of Bourgeois, nor on those of [AM12], [vK08], [Mo11], [Pa09], [Us99], [Va12], [MLY04].

The methods of this paper can be used to compute cylindrical contact homology of certain circle bundles over symplectic manifolds, namely prequantization spaces. The author is currently extending these methods to circle bundles over certain symplectic orbifolds, such as the Seifert fibered spaces. We are able to directly show via Conley-Zehnder index considerations that the differential vanishes in the computation for cylindrical contact homology of (S^3, ξ_{std}). This agrees with what has been conjectured, should an abstract perturbation package exist. The geometric details of these methods are sketched in the following section.

1.1 Prequantization

It is well known that one can realize the contact 3-sphere $(S^3, \xi_{std} = \ker \lambda_0)$ as the Hopf fibration

$$S^1 \hookrightarrow S^3 \xrightarrow{h} S^2$$

$$h(u, v) = (2u\bar{v}, |u|^2 - |v|^2), \ (u, v) \in S^3 \subset \mathbb{C}^2$$

over the standard symplectic 2-sphere (S^2, ω_0). This setup generalizes to the contact $(2n+1)$-sphere, obtained as a circle bundle $h : S^{2n+1} \to \mathbb{CP}^n$ over complex projective space. Alternatively one may think of this as the restriction of the tautological line bundle over $\mathbb{CP}^n$ to the unit sphere in $\mathbb{C}^{n+1}$. Each one of these constructions is a canonical example of a prequantization space, whose definition we recall as follows.

Take (Σ^{2n-2}, ω) to be a symplectic manifold and suppose that the cohomology class $-[\omega]/(2\pi) \in H^2(\Sigma; \mathbb{R})$ is the image of an integral class $e \in H^2(\Sigma; \mathbb{Z})$. Let $h : V^{2n-1} \to \Sigma$ be the principal S^1 bundle with first Chern class e. This means that S^1 acts freely on V with quotient Σ and that the primary obstruction to finding a section $\Sigma \to V$ is $e \in H^2(\Sigma; \mathbb{Z})$. The derivative of the S^1 action, denoted R, is a vector field on V tangent to the fibers. Since ω is a closed form in the cohomology class $-2\pi e$, there exists a real-valued connection 1-form λ on V whose curvature is ω. These conditions mean λ is invariant under the S^1 action, $\lambda(R) = 1$, and $d\lambda = h^*\omega$. It follows that λ is a contact form on V whose associated Reeb vector field is none other than R. This framework means that the Reeb orbits are comprised of the fibers of this bundle, by design of period 2π, and their iterates.

This construction lends itself to a natural perturbation of (S^3, λ_0) and holds for any prequantization space. It is comprised of adding a small lift of a Morse-Smale function on (S^2, ω_0), or on (Σ^{2n-2}, ω) in the more general setting, to the original contact form

$$\lambda_\varepsilon = (1 + \varepsilon h^* H)\lambda_0. \tag{1.2}$$

Since $(1 + \varepsilon h^* H) > 0$ for small $\varepsilon > 0$, the contact structure remains unchanged as ker $\lambda_\varepsilon =$ ker $\lambda_0 = \xi_{std}$. The perturbed Reeb dynamics are given by

$$R_\varepsilon = \frac{R}{1 + \varepsilon h^* H} + \frac{\varepsilon \tilde{X}_H}{(1 + \varepsilon h^* H)^2}. \tag{1.3}$$

Here X_H is a Hamiltonian vector field[5] on S^2 and $\tilde{X}_H$ its horizontal lift,

$$\text{i.e. } dh(q)\tilde{X}_H(q) = X_{\varepsilon H}(h(q)) \quad \text{and} \quad \lambda_0(\tilde{X}_H) = 0.$$

The only fibers that remain Reeb orbits of this perturbed contact form are the fibers over the critical points of H. For small enough ε we will be able to show that these surviving orbits are non-degenerate. However we obtain additional Reeb orbits that cover closed orbits of X_H. Since εH and εdH are small, these Reeb orbits all have periods much greater than $2k\pi$, for some sufficiently small choice of $\varepsilon(k)$. In Sections 9 and 10 we demonstrate that by letting $\varepsilon \to 0$ these Reeb orbits become increasingly long and that they ultimately do not contribute to cylindrical contact homology. This is accomplished by appealing to a formal construction of a filtration on the action and index and showing that up to a given action[6] level $\mathcal{T}$ there is a choice of ε such that the dynamically separated condition holds for the perturbed contact form λ_ε.

Here is a precise formulation of the needed results.

Lemma 1.5. *For all actions $\mathcal{T}$, we can choose $\varepsilon_0 > 0$ sufficiently small so that for all ε such that $0 < \varepsilon < \varepsilon_0$ all periodic orbits of R_ε in (1.3) of action $\mathcal{T}' \leq \mathcal{T}$ are nondegenerate and all simple orbits of action $\mathcal{T}' \leq \mathcal{T}$ are in one-to-one correspondence with the critical points of H.*

For p a critical point of H, we denote by γ_p^k the k-fold cover of the simple Reeb orbit γ over p. The Morse index of H at p is denoted by $\text{index}_p(H)$. The following formula gives the Conley-Zehnder index of closed Reeb orbits of $(S^3, \lambda_\varepsilon)$ over critical points p of H.

[5] We use the convention $\omega(X_H, \cdot) = dH$.

[6] Note that the action of a Reeb orbit is synonymous with length.

Theorem 1.6. *Let ε_0 be chosen such that Lemma 1.5 holds so that γ_p is a nondegenerate orbit over a critical point p of H and all k-fold covers of γ_p of action $\mathcal{T}' \leq \mathcal{T}$ are also nondegenerate. Then the formula for their Conley-Zehnder indices is given by*

$$\mu_{CZ}(\gamma_p^k) = 4k - 1 + \mathrm{index}_p(H). \tag{1.4}$$

Thus the grading[7] for cylindrical contact homology is

$$\begin{aligned} |\gamma_p^k| &= \mu_{CZ}(\gamma_p^k) - 1 \\ &= 4k - 2 + index_p(H). \end{aligned} \tag{1.5}$$

The computations of Sections 9 and 10 tell us that for some choice of ε_0 the only closed Reeb orbits of the perturbed Reeb vector field R_ε, where $\varepsilon < \varepsilon_0$, of action less than

$$\mathcal{T}_k := 2\pi k + 1$$

must lie in one fiber and occur as a k-fold multiple cover of a simple Reeb orbit lying over a critical point of the Morse-Smale function in the base. We denoted these Reeb orbits by γ_p^k.

Theorem 9.7 establishes a proportionality between the action and index of the Reeb orbits γ_p^k as we obtain

$$\mu_{CZ}(\gamma_p^k) = 4k - 1 + \mathrm{index}_p H.$$

This natural filtration on both the action and the index allow us to compute a formal version of filtered cylindrical contact homology. The proportionality between the action and index of the Reeb orbits, permits the use of direct limits to recover the full cylindrical

[7] The grading in contact homology of a Reeb orbit is $|\gamma| = \mu_{CZ}(\gamma) + n - 3$ where n appears in the dimension of the contact manifold M^{2n-1}.

contact homology from the truncated chain groups, consisting of

$$C_*^{<\mathfrak{T}_k}(S^3, \lambda_{\varepsilon_k}, H) = \{\gamma_p^j \mid j \in [1,k] \text{ and } p \in \mathrm{Crit}(H)\}.$$

This process is analogous to the approach taken in symplectic cohomology, however we will not make use of continuation maps and will instead appeal directly to the proportionality that has been established between the action and index of the Reeb orbits under consideration.

One useful byproduct of this approach is that we can compute a well-defined cylindrical contact homology of standard contact 3-sphere by taking $H = z$, the height function on S^2 (see also Figure 1). We obtain a maximum at the north pole (index 2) and a minimum at the south pole (index 0). Thus, because index increases by 4 each time we wrap around a closed orbit we are able to apply Theorem 1.3 and obtain the expected result. The truncated chain complexes will be generated only by the fibers of the Hopf fibration and their k-fold covers, lying over the critical points of H. In the case that H is the height function, we obtain Reeb orbits in $C_*^{<\mathfrak{T}_k}(S^3, \lambda_{\varepsilon_k})$ which have only odd Conley-Zehnder index, so the differential vanishes. We are presently working on demonstrating that the resulting differential for different choices of H behaves analogously to the the Morse-Smale differential on S^2 and the Morse orbifold differential in the more general setting.

Given the ambiguity in regards to invariance we cannot conclude that the homology is invariant of H or the almost complex structure J. We obtain

Theorem 1.7. *The direct limit of the homology of the truncated chain complexes,*

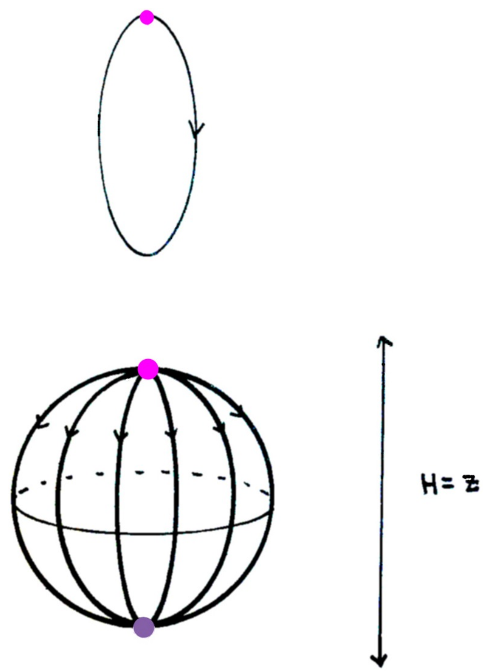

Figure 1: $-\nabla H$ for $H = z$ with a fiber over S^2

$H_*(C_*^{<\mathfrak{T}_k}(M, \lambda_\varepsilon))$ for the sphere (S^3, λ_0, H), with $H = z$ is the height function is defined after a generic choice of J and given by

$$\varinjlim_{\varepsilon\to 0} \varinjlim_{k\to\infty} H_*(C_*^{<\mathfrak{T}_k}(M, \lambda_\varepsilon)) = \varinjlim_{k\to\infty} \varinjlim_{\varepsilon\to 0} H_*(C_*^{<\mathfrak{T}_k}(M, \lambda_\varepsilon)) = \begin{cases} \mathbb{Z}_2 & * \geq 2 \text{ and even,} \\ 0 & * \text{ else.} \end{cases}$$

Remark 1.8. We clarify that the above theorem is not meant to suggest that we have computed cylindrical contact homology for the standard contact form λ_0, but rather indicates that we have chosen a specific Morse-Smale function H. Then we use the filtration on the action and index to recover the limit of the truncated cylindrical contact homology for this choice, as λ_ε is dynamically separated up to a given action level, which is inversely proportional to ε.

We conjecture that we should be able to obtain invariance for other choices of H, other choices of generic compatible almost complex structures J, and other dynamically separated contact forms associated to (S^3, ξ_{std}) in regards to the above theorem. These issues of invariance will be addressed in future work.

Remark 1.9. We point out that these methods do not generalize in the obvious way to the standard contact $(2n+1)$-sphere, due to the lack of transversality and regularity results for symplectizations of contact manifolds of dimensions greater than 3. By obvious we refer to the fact that the contact $(2n+1)$-sphere can be obtained as a circle bundle over $\mathbb{CP}^n$ and taking $H([z_0 : z_1 : ... : z_n]) = \sum_{j=1}^n j|z_j|^2$ as the Hamiltonian on the base.

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Organization of the article. The necessary notions from contact and symplectic geometry are found in Chapter 2. The basics of pseudoholomorphic curves are given in Chapter 3 and the preliminaries of the moduli spaces constructed is provided in Chapters 4. Issues of transversality for multiply covered curves and their branched covers and the geometry of moduli spaces of pseudoholomorphic curves in symplectizations is discussed in Chapters 5 and 6. Gluing and its implications on the structure of the moduli spaces is discussed in Chapter 7, as well .

With the analytic foundations in place, we are able to give an overview of cylindrical contact homology and the proof of Theorem 1.3 in Section 8. Conley-Zehnder index computations are carried out in Chapter 9. Natural filtrations on both the action and index give rise to a formal version of filtered homology, allowing us to recover cylindrical contact homology of bundles equipped with a perturbed contact form which is dynamically separated. This is established in Chapter 10, and combined with the Conley-Zehnder index results yields Theorem 1.7.

Chapter 2

Contact considerations

A **contact structure** ξ equipped to a compact manifold M^{2n-1} is a maximally non-integrable hyperplane distribution. This means that one can always locally (and globally in the case of co-orientable structures) write ξ as the kernel of a 1-form α such that $\alpha \wedge (d\alpha)^{n-1}$ is a volume form for M. This is equivalent to the condition that $d\alpha$ be nondegenerate on ξ. Note that the contact structure is unaffected when we multiply the contact form α by any strictly positive or negative function on M.

We say that two contact structures $\xi_0 = \ker\,\alpha_0$ and $\xi_1 = \ker\,\alpha_1$ on a manifold M are **contactomorphic** whenever there is a diffeomorphism $\psi : M \to M$ such that ψ sends ξ_0 to ξ_1:

$$\psi_*(\xi_0) = \xi_1$$

Note that the diffeomorphism $\psi : M \to M$ being a contactomorphism is equivalent to the existence of a non-zero function $g : M \to \mathbb{R}$ such that $\psi^*\alpha_1 = g\alpha_0$. Finding an explicit contactomorphism often proves to be a rather difficult and messy task, but an application of Moser's argument yields Gray's stability theorem, which essentially states that there are no non-trivial deformations of contact structures on a fixed closed manifold.

Theorem 2.1 (Gray's stability theorem)**.** *Let $\xi_t,\ t \in [0, 1]$, be a smooth family of contact* *structures on a closed manifold M. Then there is an isotopy $(\psi_t)_{t\in[0,1]}$ of M such that*

$$\psi_{t*}(\xi_0) = \xi_t \quad \textit{for each } t \in [0,1]$$

A proof of Gray's stability theorem can be found in [Ge08].

Associated to a contact form α there is a Reeb vector field, transverse to ξ.

Definition 2.2. For any contact manifold (M, ξ), with α a contact form for ξ, the **Reeb vector field** is defined as the unique vector field determined by α such that

$$\iota(R_\alpha)d\alpha = 0, \quad \alpha(R_\alpha) = 1.$$

The first condition says that R_α points along the unique null direction of the form $d\alpha$ and the second condition normalizes R_α. Because

$$\mathcal{L}_{R_\alpha}\alpha = d\iota_{R_\alpha}\alpha + \iota_{R_\alpha}d\alpha,$$

the flow of R_α preserves the form α and hence the contact structure ξ. Note that if one chooses a different contact form $f\alpha$, the corresponding vector field $R_{f\alpha}$ is potentially very different from R_α, and its flow may have wildly different properties. However one might expect that the dynamics be qualitatively determined by ξ, providing the initial motivation for constructing a homology theory.

We will primarily be interested in studying closed orbits of the Reeb vector field so we review the associated terminology. We will refer to closed orbits of the Reeb vector field as **Reeb orbits**. Moreover, two Reeb orbits, each of period T

$$\gamma,\ \gamma' \colon \mathbb{R}/T\mathbb{Z} \to M$$

are considered equivalent if they differ by reparametrization, i.e. precomposition with a translation of $\mathbb{R}/T\mathbb{Z}$. If $\gamma\colon \mathbb{R}/T\mathbb{Z} \to M$ is a Reeb orbit and k a positive integer, then the k**-fold cover** or iterate of γ is the composition of γ with $\mathbb{R}/kT\mathbb{Z} \to \mathbb{R}/T\mathbb{Z}$. We call a Reeb orbit γ **simple** if and only if it is not the k-fold iterate of another Reeb orbit where $k > 1$. Simple orbits are sometimes called embedded orbits.

Next we will define what it means for a Reeb orbit $\gamma\colon \mathbb{R}/T\mathbb{Z}$ to be **nondegenerate**, which requires us to explain the notion of a linearized return map. Let $\varphi_T\colon M \to M$ denote the diffeomorphism obtained by flowing along the Reeb vector field for time T. From above, we know that this preserves the contact form, thus for any $t \in \mathbb{R}/T\mathbb{Z}$ we obtain a symplectic linear map

$$\Phi_\gamma := d\varphi_T : (\xi_{\gamma(t)}, d\alpha) \to (\xi_{\gamma(t)}, d\alpha),$$

which is known as the **linearized return map** associated to the Reeb orbit γ. A more geometric interpretation of this map may be realized as follows. Let D be a small embedded disk in M centered at $\gamma(t)$ and transverse to γ, such that

$$T_{\gamma(t)}D = \xi_{\gamma(t)}.$$

For $x \in D$ close to the center, there is a unique point in D which is reached by following the Reeb flow for a time close to T, yielding a partially defined "return map" $\phi : D \to D$ defined near the origin. The derivative of this map at the origin is the linearized return map Φ_γ.

Definition 2.3. We say that the Reeb orbit γ is **nondegenerate** whenever Φ_γ does not have 1 as an eigenvalue. Alternatively one says that 1 is not a Floquet multiplier associated to γ.

We remark that the nondegeneracy condition does not depend on the choice of $t \in \mathbb{R}/T\mathbb{Z}$ because the linearized return maps for different t are conjugate to each other. If the Reeb orbit γ is nondegenerate, then it is isolated, because Reeb orbits close to γ give rise to fixed points of the map ϕ and the condition that $1 - d\phi$ be invertible at the origin implies that ϕ has no fixed points near the origin.

Furthermore, the Reeb vector field gives us a natural splitting of the tangent bundle of M,

$$TM = \langle R_\alpha \rangle \oplus \xi. \tag{2.1}$$

This follows from the fact that $d\alpha$ is nondegenerate on ξ and since $\xi = \ker \alpha$, no nonzero vector can simultaneously annihilate $d\alpha$ and α.

Next we briefly review the canonical contact form on S^3 and its Reeb dynamics.

Example 2.4 (Canonical Reeb dynamics on the 3-sphere)**.** If we define the following function $f\colon \mathbb{R}^4 \to \mathbb{R}$

$$f(x_1, y_1, x_2, y_2) = x_1^2 + y_1^2 + x_2^2 + y_2^2,$$

then $S^3 = f^{-1}(1)$. Recall that the canonical contact form on $S^3 \subset \mathbb{R}^4$ is given to be

$$\lambda_0 := -\frac{1}{2} df \circ J = (x_1 dy_1 - y_1 dx_1 + x_2 dy_2 - y_2 dx_2)\,|_{S^3}. \tag{2.2}$$

The Reeb vector field is given by

$$R = \left(x_1 \frac{\partial}{\partial y_1} - y_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial y_2} - y_2 \frac{\partial}{\partial x_2} \right).$$

Equivalently we may reformulate these using complex coordinates by identifying $\mathbb{R}^4$ with $\mathbb{C}^2$ via

$$u = x_1 + iy_1, \quad v = x_2 + iy_2.$$

We obtain

$$\lambda_0 = \frac{i}{2}(ud\bar{u} - \bar{u}du + vd\bar{v} - \bar{v}dv)|_{S^3},$$

and

$$\begin{aligned} R &= (ix_1 - y_1, ix_2 - y_2) \\ &= (iu, iv) \\ &= i\left(u\frac{\partial}{\partial u} - \bar{u}\frac{\partial}{\partial \bar{u}} + v\frac{\partial}{\partial v} - \bar{v}\frac{\partial}{\partial \bar{v}}\right) \end{aligned} \tag{2.3}$$

To see that the orbits of R define the fibers of the Hopf fibration recall that a fiber through a point

$$(u, v) = (x_1 + iy_1, x_2 + iy_2) \in S^3 \subset \mathbb{C}^2,$$

can be parameterized as

$$\varphi(t) = (e^{it}u, e^{it}v),\ t \in \mathbb{R}. \tag{2.4}$$

We compute the time derivative of the fiber

$$\dot{\varphi}(0) = (iu, iv) = (ix_1 - y_1, ix_2 - y_2).$$

Expressed as a real vector field on $\mathbb{R}^4$, which is tangent to S^3, this is the Reeb vector field R as it appears in Equation 2.3, so the Reeb flow does indeed define the Hopf fibration.

2.1 Hypersurfaces of contact type and symplectizations

As contact geometry is the odd-dimensional sibling of symplectic geometry, one expects a natural setting where we might observe an interdependence between them. The most useful constructions relating the two arise when we consider hypersurfaces in symplectic manifolds, which admit a natural contact form. To understand this geometry, we first need to give the definition of a Liouville vector field. References to these constructions include the textbooks [MSintro] by McDuff and Salamon as well as [Ge08] by Geiges.

Definition 2.5. A **Liouville vector field** Y on a symplectic manifold (W, ω) is a vector field satisfying

$$\mathcal{L}_Y \omega = \omega$$

The flow ψ_t of such a vector field is conformal symplectic, i.e. $\psi_t^*(\omega) = e^t \omega$. Note that the flow of these fields are volume expanding, so such fields may only exist locally on compact manifolds.

We say that a hypersurface $Q \subset (W, \omega)$ is of **contact type**, whenever Q is a codimension 1 submanifold of W which admits a contact form α that agrees with the symplectic form, i.e.

$$d\alpha = \omega|_Q.$$

Note that for a hypersurface Q of a symplectic manifold, whenever there exists a Liouville vector field Y defined in a neighborhood of Q, which is transverse to Q, we can define a 1-form on Q by the formula.

$$\alpha := \iota_Y \omega.$$

In the following proposition we see that this is a contact form on any hypersurface Q transverse to Y, which agrees with the symplectic form, i.e. $d\alpha = \omega|_Q$.

Proposition 2.6. *Assume (W, ω) admits a Liouville vector field Y, defined in a neighborhood of a hypersurface Q with Y transverse to Q. Then Q is of contact type.*

Proof. The Cartan formula

$$\mathcal{L}_Y = d \circ \iota_Y + \iota_Y \circ d$$

combined with the fact that ω is closed allows us to write the Liouville condition on Y as $d(\iota_Y \omega) = \omega$. Assuming W to be of dimension $2n$, we compute:

$$\begin{aligned} \alpha \wedge (d\alpha)^{n-1} &= \iota_Y \omega \wedge (d(\iota_Y \omega))^{n-1} \\ &= \iota_Y \omega \wedge \omega^{n-1} \\ &= \tfrac{1}{n} \iota_Y (\omega^n) \end{aligned}$$

Since ω^n is a volume form on W, it follows that $\alpha \wedge (d\alpha)^{n-1}$ is a volume form when restricted to the tangent bundle of any hypersurface transverse to Y in W. □

The following is a useful result demonstrating that the existence of Liouville vector fields transverse to a hypersurface $Q \subset (W, \omega)$ is equivalent to the existence of a contact form on Q which is compatible with ω.

Proposition 2.7. *Let (W, ω) be a symplectic manifold and $Q \subset W$ a compact hypersurface. Then the following are equivalent*

(i) There exists a contact form α on Q such that $d\alpha = \omega|_Q$.

(ii) There exists a Liouville vector field $Y : U \to TW$ defined in a neighborhood U of Q, which is transverse to Q.

Proof. First assume that (ii) is satisfied and define $\alpha = \iota_Y \omega$. Then

$$d\alpha = d(\iota_Y \omega) = \omega$$

Since $T_q Q$ is odd dimensional, there exists a nonzero $\tilde{v} \in T_q Q$ such that $\omega_q(\tilde{v}, v) = 0$ for all $v \in T_q Q$. Since ω is nondegenerate we have $\alpha_q(\tilde{v}) = \omega_q(Y(q), \tilde{v}) \neq 0$. Hence

$$\xi_q = \{v \in T_q Q \mid \omega_q(Y(q), v) = 0\}$$

is a hyperplane field on Q and $\tilde{v}$ is transversal to ξ_q. In fact, ξ_q is the symplectic complement of $\mathrm{span}\{Y(q), \tilde{v}\}$. This implies $\omega = d\alpha$ is nondegenerate on ξ_q and hence α restricts to a contact form on Q.

Conversely suppose that $\alpha \in \Omega^1(Q)$ is a contact form such that $d\alpha = \omega|_Q$. Let $R_\alpha \in \chi(Q)$ be the Reeb vector field of α:

$$\iota_{R_\alpha} d\alpha = 0, \qquad \iota_{R_\alpha} \alpha = 1$$

Choose a vector field $Y \in \chi(W)$ such that

$$\omega(Y, R_\alpha) = 1, \qquad \omega(Y, \xi) = 0$$

on Q. This can be done by picking any vector field Y_0 such that $\omega(Y_0, R_\alpha) = 1$ on Q. Then for every $q \in Q$ there exists a unique vector $Y_1(q) \in \xi_q$ such that $\omega(Y_0 + Y_1, v) = 0$ for all $v \in \xi_q$. Define $Y = Y_0 + Y_1$ on Q and extend to a vector field on W. Next we define $\phi : Q \times \mathbb{R} \to W$ by

$$\phi(q, t) = exp_q(tY(q))$$

Then

$$\begin{aligned} \phi^*\omega|_{Q\times\{0\}} &= \phi^* d\alpha|_{Q\times\{0\}} \\ &= d(\phi^*\alpha)|_{Q\times\{0\}} \\ &= d(e^t\alpha)|_{Q\times\{0\}} \\ &= d\alpha - \alpha \wedge dt \end{aligned}$$

Now by Moser's argument there exists a local diffeomorphism $\psi : Q \times (-\epsilon, \epsilon) \to M$ such that

$$\psi(q, 0) = q, \qquad \psi^*\omega = e^t(d\alpha - \alpha \wedge dt).$$

So the required Liouville vector-field is $\psi_*(\frac{\partial}{\partial t})$.

□

Example 2.8. The radial vector field

$$X_0 = \frac{1}{2}\sum_{j=1}^{n} x_j \frac{\partial}{\partial x_j} + y_j \frac{\partial}{\partial y_j}$$

is a Liouville vector field on $\mathbb{R}^{2n} \setminus \{\mathbf{0}\}$. It is transverse along the unit sphere S^{2n-1}. The corresponding 1-form

$$\lambda_0 = \iota_{X_0}\omega_0 = \frac{1}{2}\sum_{j=1}^{n}(x_j dy_j - y_j dx_j) \tag{2.5}$$

is the canonical contact form for S^{2n-1}.

There are 2 equivalent means of defining the symplectization in the case of a co-oriented contact structure[1], each endowing the symplectization with the structure of a principal bundle with structure group $\mathbb{R}$ or $\mathbb{R}^+$. In discussions involving pseudoholomorphic curves, one typically uses the construction with structure group $\mathbb{R}$. We take $(M, \alpha = \ker \xi)$ to be a contact manifold, whose **symplectization** is given by the manifold $\mathbb{R} \times M$, to which we may associate the following symplectic form

$$\omega = e^{\tau}(d\alpha - \alpha \wedge dt) = d(e^{\tau}\alpha).$$

Here τ is the coordinate on $\mathbb{R}$, and it should be noted that α is interpreted as a 1-form on $\mathbb{R} \times M$, as we identify α with its pullback under the projection $\mathbb{R} \times M \to M$. A simple calculation demonstrates that

$$Y = \frac{\partial}{\partial \tau}$$

gives the Liouville vector field, enabling one to realize (M, α) as a hypersurface of contact type inside its symplectization $(\mathbb{R} \times M, d(e^{\tau}\alpha))$. The symplectization $(\mathbb{R} \times M, d(e^{\tau}\alpha))$ is a fiber bundle over M whose fibers are precisely the orbits of the Liouville vector

[1] If (M, ξ) is not co-oriented then the symplectization has structure group $\mathbb{R}^*$.

field $\frac{\partial}{\partial\tau}$. Thus we see that in fact the symplectization is a principal bundle over M with structure group $\mathbb{R}$.

2.2 Almost complex structures

The other required component of a pseudoholomorphic curve theory is the notion of an almost complex structure. Recall that any contact structure ξ may be equipped with a complex structure J such that (ξ, J) is a complex vector bundle.

We denote the set of **compatible almost complex structures** on ξ by

$$\mathcal{J} = \{J : \xi \to \xi \mid J^2 = -\mathbb{1},\ d\alpha(J\cdot, J\cdot) = d\alpha(\cdot,\cdot),\ d\alpha(\cdot, J\cdot) > 0\}.$$

This set is nonempty and contractible, as in the symplectic case which is discussed in [MSintro]. We can consider $(\xi, d\alpha, J)$ as a symplectic vector bundle with a Hermitian structure. Isomorphism classes of symplectic vector bundles are in a 1-1 correspondence with complex vector bundles. As a result (ξ, J) is frequently said to be a *complex vector bundle*, and one suppresses the 'almost' in almost complex structure despite the fact that we do not require elements of $\mathcal{J}$ to be integrable.

Next we describe the canonical extension of the almost complex structure J to $\mathbb{R}\times M$, which we will call $\tilde{J}$. This is possible from the aforementioned splitting of the tangent bundle of M; see (2.1). This splitting allows us to realize the tangent bundle of the symplectization as

$$T(\mathbb{R} \times M) = \mathbb{R}\frac{\partial}{\partial\tau} \oplus \mathbb{R}R_\alpha \oplus \xi. \tag{2.6}$$

Definition 2.9 (Canonical extension of J to $\tilde{J}$ on $\mathbb{R} \times M$)**.** Writing a tangent vector as $[a, b; v]$ where $a,\ b \in \mathbb{R}$ and $v \in \xi$ we may define the extended almost complex structure

$\tilde{J}$ on $\mathbb{R} \times M$ by

$$\tilde{J}[a, b; v] = [-b, a, Jv].$$

Hence $\tilde{J}|_\xi = J$ and $\tilde{J}$ acts on $\mathbb{R}\frac{\partial}{\partial \tau} \oplus \mathbb{R}R_\alpha$ in the same manner as multiplication by i acts on $\mathbb{C}$, namely $\tilde{J}\frac{\partial}{\partial \tau} = R_\alpha$. Note that this procedure uniquely determines the extension of J to $\tilde{J}$.

The naturality in the way we have defined this complex structure is illustrated in the following example.

Example 2.10. Consider $S^3 \subset \mathbb{C}^2$ with its standard contact form, see Example 2.4. Recall that for a point $p \in S^3$ we can equivalently define ξ_p as the set of complex tangencies at p, namely

$$\xi_p = T_pS^3 \cap J(T_pS^3).$$

This is the unique complex subspace of $\mathbb{C}^2$ that is contained in T_pS^3. Hence the restriction of the standard complex structure

$$\begin{aligned} i: \quad T\mathbb{C}^2 &\to T\mathbb{C}^2 \\ (p, v) &\mapsto (p, iv) \end{aligned}$$

yields a compatible almost complex structure J on ξ.

In addition, we note that the following map is a diffeomorphism

$$\begin{aligned} \varphi: \quad \mathbb{R} \times S^3 &\to \mathbb{C}^2 \setminus \{\mathbf{0}\} \\ (\tau, p) &\mapsto e^{2\tau} p \end{aligned}$$

satisfying $D\varphi \circ \tilde{J} = i \circ D\varphi$. Its inverse is given by

$$\begin{aligned} \psi: \quad \mathbb{C}^2 \setminus \{\mathbf{0}\} &\to \mathbb{R} \times S^3 \\ z &\mapsto \left(\frac{1}{2} \ln |z|, \frac{z}{|z|}\right) \end{aligned}$$

Thus we see that the symplectization of S^3 admits the "same" complex structure as well.

Chapter 3

The letter J is for pseudoholomorphic

Pseudoholomorphic curves are defined in symplectic manifolds after selecting a compatible almost complex structure, but prove to be considerably more analytically challenging in non-compact symplectic manifolds. In the world of contact geometry we consider them in the symplectization of a contact manifold (see Section 2.1). They were first used by Hofer [H93] in this context to prove the Weinstein conjecture for S^3. The statement of the Weinstein Conjecture is as follows, and was originally formulated in [W79].

Conjecture 3.1 (The Weinstein Conjecture)**.** *Let ξ be a contact structure on M. Then for any contact form defining ξ the associated Reeb vector field has at least one periodic orbit.*

We note that for contact manifolds of dimension 3 the Weinstein Conjecture was proven by Taubes in [T07]. A nice summary of the history and development of the Weinstein Conjecture, as well as an outline of Taubes' proof is given by Hutchings in [Hu10].

We will cover the necessary notions in the following sections, but the interested reader may wish to supplement our discussion with [H99], [HK99], [HWZI].

3.1 Pseudoholomorphic curves in symplectizations

Given a Riemann surface (which is not necessarily assumed to be compact) (Σ, j) and a symplectic manifold (W, J) equipped with a compatible almost complex structure, a map $u : (\Sigma, j) \to (W, J)$ is called a **pseudoholomorphic curve** whenever

$$\bar{\partial}_{j,J} u := du + J \circ du \circ j \equiv 0, \tag{3.1}$$

or equivalently,

$$du \circ j = J \circ du. \tag{3.2}$$

In words, a pseudoholomorphic curve must satisfy the Cauchy-Riemann equation (3.1) which is equivalent to the condition that it has a complex-linear differential (3.2).

The curve u is considered **equivalent** to another pseudoholomorphic curve $u' : (\Sigma', j') \to (W, J)$ if there exists a holomorphic bijection $\phi : (\Sigma, j) \to (\Sigma', j')$ such that $u' \circ \phi = u$. For the sake of brevity, we often refer to a "pseudoholomorphic curve" as u, although we are actually formally considering a equivalence class of triples (Σ, j, u) satisfying the above conditions, which should be written as $[\Sigma, j, u]$.

If we take (W, J) to be the symplectization with its standard complex structure $(\mathbb{R} \times M, \tilde{J})$, as discussed in Section 2.2, then one can reformulate the **complex linearity** condition of a tangent map $D_p u$ for $\alpha + i\beta \in \mathbb{C}$ and $z \in T_p\Sigma$ as

$$D_p u((\alpha + i\beta)z) = \alpha D_p u(z) + \beta \tilde{J} D_p u(z).$$

Remark 3.2. A pseudoholomorphic curve u defined from a closed surface Σ into the symplectization is necessarily constant. We will prove this later in Section 3.3 as Proposition 3.13.

In light of the above remark, we will specifically consider pseudoholomorphic curves from a multiply punctured Riemann sphere to the symplectization $(\mathbb{R} \times M, \omega := d(e^t\alpha))$ with almost complex structure $\tilde{J}$. We will frequently write a pseudoholomorphic curve in a symplectization as $u = (a, f)$ where $a \in \mathbb{R}$ and $f \in M$.

For cylindrical contact homology we will be interested in counting rigid[1] pseudoholomorphic curves whose domain is homotopic to a cylinder. These pseudoholomorphic curves will have the twice punctured sphere as their domain, but since $(S^2 \setminus \{x, y\}, j_0)$ is holomorphic to $(\mathbb{R} \times S^1, j_{cyl})$, we can alternatively use the infinite cylinder as our domain. In addition, j_0 is taken to be the standard complex structure on S^2, restricted to the punctured sphere, and as such j_0 will typically be suppressed in the notation.

Here are two baby examples to give the flavor of the sorts of pseudoholomorphic curves we will be interested in studying.

Example 3.3. Let $\gamma : \mathbb{R} \to M$ be a closed orbit of the Reeb flow. Then

$$\begin{aligned} u: \quad (\mathbb{C}, j_0) &\to (\mathbb{R} \times M, \tilde{J}) \\ x + iy &\mapsto (x, \gamma(y)) \end{aligned}$$

is a pseudoholomorphic curve.

Example 3.4. If γ is a T-periodic orbit of the Reeb vector field then

$$\begin{aligned} v: \quad (\mathbb{R} \times S^1, j) &\to (\mathbb{R} \times M, \tilde{J}) \\ (s, e^{2\pi it}) &\mapsto (Ts, \gamma(Tt)) \end{aligned}$$

[1] These are elements of a moduli space with virtual dimension 0, that is these curves connect Reeb orbits of Conley-Zehnder index difference 1 in the symplectization. Explanations of this jargon will follow in subsequent chapters.

is pseudoholomorphic as well. Note that u is a pseudoholomorphic cylinder mapping to the Reeb orbit γ under the natural projection $p : \mathbb{R} \times M \to M$. Often one writes u as $\mathbb{R} \times \gamma$.

The types of pseudoholomorphic curves we will be most interested in studying are those which asymptotically limit on closed nondegenerate orbits of the Reeb vector field as we approach a puncture of $\dot{\Sigma}$. This behavior is exhibited by a certain subclass of pseudoholomorphic curves, which was first noticed and discussed at length by Hofer, Wysocki, and Zehnder in [H93], [HWZI], [HWZII]. These will be pseudoholomorphic curves which have a specific finite energy associated to them.

Before we can describe the asymptotics and define the energy, we must study the local behavior of solutions to the Cauchy-Riemann equations. Then we can give the appropriate definitions of area and Hofer energy of a pseudoholomorphic curve. We will revisit the pseudoholomorphic curves of Examples 3.3 and 3.4 in Section 3.3, to better understand behavior exhibited by the subclass of pseudoholomorphic curves which have finite positive energy and area.

3.2 Local behavior

It will be useful to understand the local behavior of pseudoholomorphic curves. By working in local coordinates, we can obtain a maximum principle as well as some other identities which will be helpful in the following sections. We denote by $u := (a, f) : (\dot{\Sigma}, j) \to (\mathbb{R} \times M, \tilde{J})$ a pseudoholomorphic curve in the symplectization of (M, α). Let X be a local nowhere vanishing vector field on $\dot{\Sigma}$ and define $Y = J \circ X$, yielding a local frame.

Remark 3.5. Note that while X and Y are non-vanishing globally because there exists a global Hermitian trivialization of $T\dot{\Sigma}$, we cannot a priori conclude that globally they come from a coordinate system on $\mathbb{R}^{2n}$. However since $\{X, Y\}$ are a global frame, we can obtain a global 2-form by taking the dual of the polyvector field $X \wedge Y$, which we denote by

$$\Omega_{\{X,Y\}} = (X \wedge Y)^*$$

For the purposes of this section it is preferable to work locally with coordinate $(s + it) \in \mathbb{C}$ by associating X with $\frac{\partial}{\partial s}$ and Y with $\frac{\partial}{\partial t}$. However in later sections, the above remark will allow us to work with a general global frame.

We will be interested in considering closed Riemannian surfaces with a finite number of points removed, thus we can find a Hermitian trivialization of $T\Sigma$. As a result we may think in terms of a global frame $\{\frac{\partial}{\partial s}, \frac{\partial}{\partial t}\}$. Recall that there is a projection π of the tangent bundle of M along the Reeb vector field

$$\pi : TM \to \xi.$$

Then

$$u_s := Du \circ \frac{\partial}{\partial s} : \Sigma \to T(\mathbb{R} \times M) \cong \mathbb{R}\frac{\partial}{\partial \tau} \oplus \mathbb{R}R_\alpha \oplus \xi$$

can be written as

$$u_s(z) = [a_s(z), \alpha f_s(z); \pi f_s(z)],$$

where $f_s = Df \circ \frac{\partial}{\partial s}$. Similarly we have

$$u_t(z) := Du \circ \frac{\partial}{\partial t} = [a_t, \alpha f_t; \pi f_t].$$

Now we can check that $u = (a, f)$ is pseudoholomorphic if and only if

$$u_s + \tilde{J} u_t = 0. \tag{3.3}$$

We have that (3.3) is equivalent to

$$\begin{cases} a_s &= \alpha(u_t), \\ a_t &= -\alpha(u_s) \\ \pi u_t &= J\pi u_s \end{cases} \tag{3.4}$$

The first two equations of (3.4) can be written as

$$f^*\alpha = -da \circ j = *d\alpha, \text{ where } * \text{ denotes the Hodge star operator.} \tag{3.5}$$

The third equation of (3.4) means that the map $\pi \circ df : T\Sigma \to \xi$ is complex linear on each fiber, hence $\pi \circ Df(z)$ is either zero or an isomorphism. Differentiating (3.6) yields

$$f^*d\alpha = d(da \circ j) = \Delta a \; ds \wedge dt, \quad \text{where } \Delta a = a_{ss} + a_{tt}. \tag{3.6}$$

Therefore,

$$\Delta a = f^*d\alpha\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial t}\right) = d\alpha(f_s, f_t) = d\alpha(\pi f_s, J\pi f_t) = |\pi u_s|^2 = |\pi u_t|^2,$$

as $d\alpha(\cdot, J\cdot)$ defines a metric on ξ. Hence the function a is subharmonic. As a result of the strong maximum principle, see for example [HK99], we obtain the following maximum principle, applicable to pseudoholomorphic curves in symplectizations.

Proposition 3.6 (Maximum principle). *If the real valued portion a of a pseudoholomorphic curve $u := (a, f)$ assumes a local maximum in the interior of $\dot{\Sigma}$ then u is the constant map.*

3.3 Energy and area

The following energy and area estimates will prove extremely useful in understanding the behavior of pseudoholomorphic curves in noncompact symplectic manifolds with cylindrical ends. To provide some motivation for the importance of such notions, we first visit the concept of area in the case of pseudoholomorphic curves whose images live in closed symplectic manifolds. In this case, we recall that the crucial assumption in Gromov's compactness theorems is that the area of the curve be finite, defined as follows. Let $\nu : (\Sigma, j) \to (N, J)$, with (Σ, j) be a pseudoholomorphic curve defined on a closed Riemann surface into a closed symplectic manifold (N, ω) equipped with a compatible almost complex structure J. Then the area of ν is given by

$$A(\nu) = \int_\Sigma \nu^* \omega. \tag{3.7}$$

Since the almost complex structure J has been chosen to be compatible with the symplectic form ω, this means that $A(\nu)$ is essentially the area of the image of the curve ν measured in terms of the Riemannian metric $g = \omega(\cdot, J\cdot)$. In addition, because ω is closed, we know that this quantity $A(\nu)$ is actually a topological invariant of the map ν. This means that the areas of pseudoholomorphic curves are controlled by straightforward topological data.

When one is only concerned with closed symplectic manifolds, the area is often referred to as the **energy of a pseudoholomorphic curve** and denoted by $E(\nu)$ in the literature. In the setting that we are interested in we will need to modify the usual notion of energy, as the area of the image of a non-compact proper pseudoholomorphic curve in an open symplectic manifold with cylindrical ends is never finite with respect to any complete metric. Moreover in the case of symplectizations, we will see that there exist no non-constant compact pseudoholomorphic curves.

In [HWZI] and [HWZ02], Hofer, Wysocki, and Zehnder introduced quantities commonly referred to as the area and (Hofer) energy of a pseudoholomorphic curve. The first serves as a substitute for the notion of area as above in (3.7) and the finiteness of the latter provides a relationship between the asymptotic behavior of the pseudoholomorphic curve and the dynamics of a nondegenerate Reeb vector field of the contact manifold. We will call these the **area** and **energy** of a pseudoholomorphic curve respectively, and they are defined as follows.

As before, we take (Σ, j) to be a closed connected Riemann surface and $\Gamma \subset \text{int}\ \Sigma$ a finite set of interior punctures, with $\dot{\Sigma} = \Sigma \setminus \Gamma$. The target of interest is $(\mathbb{R} \times M, d(e^t\alpha))$, the symplectization of a contact manifold (M, α), equipped with a compatible almost complex structure $\tilde{J}$, as defined in Definition 2.9. Let $u := (a, f) : (\dot{\Sigma}, j) \to (\mathbb{R} \times M, \tilde{J})$ be a pseudoholomorphic curve.

Definition 3.7. The **area** of the pseudoholomorphic curve u is given by the formula

$$A(u) := \int_{\dot{\Sigma}} u^* d\alpha = \int_{\dot{\Sigma}} f^* d\alpha. \tag{3.8}$$

In some literature the area is called the ω-energy or $d\alpha$-energy where $\omega = d(e^\tau \alpha)$ is the symplectic form on the symplectization of (M, α). In early literature on contact homology this was referred to simply as energy. We will not use these conventions and refer to it as the area and denote it by $A(u)$. Note that the area depends only on the M-component, f, of the curve u. From the discussion of the local behavior in the preceding section we have the following result regarding the non-negativity of area.

Proposition 3.8 (Non-negativity of area)**.** *For any finite area pseudoholomorphic curve u we have*

$$A(u) := \int_{\dot{\Sigma}} f^* d\alpha \geq 0$$

Proof. The local computations (3.6) allow us to write

$$f^* d\alpha \geq 0 = |\pi u_s|^2 ds \wedge dt = |\pi u_t|^2 ds \wedge dt,$$

where π is the projection of the tangent bundle of M along the Reeb vector field

$$\pi : TM \to \xi.$$

We can go through the same process in global coordinates by making use of the global frame $\{X, Y\}$ as discussed in Remark 3.5. Namely, we may write

$$u_X := Du \circ X : \dot{\Sigma} \to T(\mathbb{R} \times M) \cong \mathbb{R}\frac{\partial}{\partial \tau} \oplus \mathbb{R}R_\alpha \oplus \xi$$

as

$$u_X(z) = [a_X(z), \alpha f_X(z); \pi f_X(z)],$$

where $f_X = Df \circ X$. Similarly we have

$$u_Y(z) := Du \circ Y = [a_Y, \alpha f_Y; \pi f_Y],$$

as well as the obvious analogue of (3.4) and (3.6). Recall that we obtain a non-vanishing 2-form from the dual of the polyvector field $X \wedge Y$, which we denote by

$$\Omega_{\{X,Y\}} = (X \wedge Y)^*.$$

Then globally we obtain that

$$A(u) := \int_{\dot{\Sigma}} f^* d\alpha = \int_{\dot{\Sigma}} |\pi u_X|^2 \Omega_{\{X,Y\}} = \int_{\dot{\Sigma}} |\pi u_Y|^2 \Omega_{\{X,Y\}} \geq 0$$

as desired.

□

Combined with the local computations of (3.4) we obtain the following corollary.

Corollary 3.9. *If a pseudoholomorphic cylinder $u := (a, f)$ has $A(u) = 0$ then the image $f(\dot{\Sigma})$ is contained in a trajectory of the Reeb vector field R_α.*

In order to define the **Hofer energy** of a pseudoholomorphic curve u, which we denote by $E(u)$, we need to use a class of smooth maps to extend the contact form α on M to a 1-form on $\mathbb{R} \times M$. This is done as follows. Let

$$\mathscr{S} = \{\phi \in C^\infty(\mathbb{R}, [0, 1]) \mid \phi' \geq 0\}$$

and define for $\phi \in \mathscr{S}$ the 1-form α_ϕ on $\mathbb{R} \times M$ by:

$$\alpha_\phi(\tau, p)(\rho, v) := \phi(\tau)\alpha_p(v) \text{ for } (\rho, v) \in T_{(\tau,p)}(\mathbb{R} \times M).$$

Definition 3.10. The **Hofer energy** of u is given by

$$E(u) := \sup_{\phi \in \mathscr{S}} \int_{\dot{\Sigma}} u^* d\alpha_\phi.$$

The Hofer energy of a pseudoholomorphic curve is also referred to as α**-energy**. In this paper we will refer to it simply as **energy**, which is also typical.

Proposition 3.11 (Non-negativity of energy)**.** *For any finite energy pseudoholomorphic curve u we have*

$$E(u) := \sup_{\phi \in \mathscr{S}} \int_{\dot{\Sigma}} u^* d\alpha_\phi \geq 0.$$

Proof. One may compute the integrand in light of the local computations of Section 3.2 with respect to the local frame $\{\partial_s, \partial_t\}$,

$$u^* d\alpha_\phi = (\phi'(a)|\nabla a|^2 + \phi(a)\Delta a) ds \wedge dt = (\phi'(a)|\nabla a|^2 + \phi(\tau)|\pi f_s|^2) ds \wedge dt.$$

As in the proof of Proposition 3.11 we can convert this into the following global non-negative expression,

$$u^* d\alpha_\phi = (\phi'(a)|\nabla a|^2 + \phi(\tau)|\pi f_X|^2)\Omega_{\{X,Y\}} \geq 0. \tag{3.9}$$

□

As a corollary we see that $E(u) > 0$ implies that u is non-constant, based on the expression we obtain for the integrand, (3.9).

Corollary 3.12. *$E(u) = 0$ if and only if u is constant.*

This energy condition sorts out normalizations of algebraic curves among all pseudoholomorphic curves in the case of S^3 with its standard form and is discussed in [HK99]. This suggests that one should view finite energy pseudoholomorphic curves as nonintegrable generalizations of algebraic curves. In a way they form the subclass of pseudoalgebraic curves in the class of pseudoholomorphic curves. The interested reader may find more details on this in [HK99] as will not discuss this matter further here.

This corollary allows us to prove that any pseudoholomorphic curve defined on a closed Riemannian surface is constant by Stokes' theorem.

Proposition 3.13. *A pseudoholomorphic curve u defined on a closed surface Σ into the symplectization of a contact manifold, i.e. $(\mathbb{R} \times M, d(e^\tau \alpha))$ is constant.*

Proof. Stokes' theorem yields

$$\int_\Sigma u^* d\alpha_\phi = \int_{\partial\Sigma} u^* \alpha_\phi = 0.$$

for all $\phi \in \mathscr{S}$ hence $E(u) = 0$. □

Next we revisit the examples we began with, to better understand the different controls that area and energy have on a pseudoholomorphic curve. In the next section we will introduce the asymptotics of [HWZI] associated to finite energy pseudoholomorphic curves and delve deeper into the implications that finite energy has on the behavior of pseudoholomorphic curves.

If the area of a curve is zero, we see from the expression for the area of a pseudoholomorphic curve as given in Proposition 3.11 that this is equivalent to

$$\pi u_X = \pi u_Y = 0.$$

Next we revisit the pseudoholomorphic curve discussed in Example 3.4 and compute its area and energy. We will see in the next section that the behavior of the pseudoholomorphic curve in this example is the model for how general pseudoholomorphic curves of finite energy behave near a puncture, namely Theorem 3.15 of [HWZI]. A corollary of their theorem is that if we can prove the existence of a pseudoholomorphic curve with finite energy in the symplectization of (M, α) then there exist periodic orbits of the Reeb vector field associated to α.

Example 3.14. Recall that Example 3.4 consisted of the pseudoholomorphic cylinder over a periodic orbit of the Reeb vector field, defined as

$$\begin{aligned} v: \quad (\mathbb{R} \times S^1, j) &\rightarrow (\mathbb{R} \times M, \tilde{J}) \\ (s, e^{2\pi i t}) &\mapsto (Ts, \gamma(Tt)), \end{aligned}$$

where the Reeb orbit γ is T-periodic.

The computations done in local coordinates yield

$$u_s = T\frac{\partial}{\partial \tau}, \quad u_t = TR_\alpha = \tilde{J}u_s.$$

Thus the we see that the area vanishes, $A(u) = 0$. However in this example the Hofer energy of u satisfies

$$E(u) = T,$$

as Stokes' theorem yields

$$E(u) = \sup_{\phi \in \mathscr{S}} \lim_{R\to\infty} \int_{[-R,R]\times S^1} u^* d\alpha_\phi = \sup_{\phi \in \mathscr{S}} \lim_{R\to\infty} (\phi(-R)T - \phi(R)T) = T.$$

Moreover we note that

$$\begin{aligned}\lim_{s\to\infty} f(s,t) &= \gamma(tT) \quad \text{in } C^\infty(M),\\ \lim_{s\to\infty} \frac{a(s,t)}{s} &= \gamma(tT) \quad \text{in } C^\infty(\mathbb{R}).\end{aligned}$$

In other words, the M-part of u converges to a periodic orbit of the Reeb vector field of period T, while the $\mathbb{R}$-part is asymptotic to $(s,t) \to Ts$ as $s \to \infty$.

On the other hand, we want to understand an example of a curve with $E(u) = \infty$. This is the case if we consider the trivial solution of Example 3.3. Recall that this pseudoholomorphic curve was defined by

$$\begin{aligned} u: \ (\mathbb{C}, j_0) &\to (\mathbb{R}\times M, \tilde{J})\\ x+iy &\mapsto (x, \gamma(y))\end{aligned}$$

where $\gamma : \mathbb{R} \to M$ was a closed orbit of the Reeb flow. Then if we take a function $\phi \in \mathscr{S}$ with $\phi \neq 0$ we compute

$$\int_{\mathbb{C}} u^* d\alpha_\phi = (\phi(\infty) - \phi(-\infty)) \int_{\mathbb{R}} dt = \infty.$$

In literature, a pseudoholomorphic curve u is said to be a **finite energy curve**[2] whenever

$$0 < E(u) < \infty.$$

Note that we are excluding all constant pseudoholomorphic curves as by required $E(u) > 0$. In later literature the terminology textbfasymptotically cylindrical pseudoholomorphic curves replaced finite energy surfaces. As this is more descriptive of the behavior of the curves, we will use asymptotically cylindrical to describe pseudoholomorphic curves u with $0 < E(u) < \infty$.

[2]In earlier literature these were sometimes referred to as finite energy surfaces.

We devote the entirety of the following section to the important applications and consequences of the finiteness of Hofer energy, as established in [H93], [HK99], [HWZI], [HWZII], [HWZ02], [HWZ03]. In particular we will recall in Theorems 3.15 and 3.18 the significance of the concept of **finite energy surfaces** and their implication on the existence of periodic orbits of the Reeb vector field, related to the asymptotic behavior of the pseudoholomorphic curve.

3.4 Hofer energy and asymptotics

The finiteness of Hofer energy is an extremely important distinguishing characteristic of pseudoholomorphic curves. In this section we will explore the relationship between finiteness of energy and asymptotic behavior near the punctures of a pseudoholomorphic curve to closed nondegenerate periodic orbit of the Reeb vector field R_α. The nature of this asymptotic behavior will be made precise in Theorems 3.15 and 3.18. This phenomenon and its applications first appeared in [H93] and [HWZI]. As a result, finite energy pseudoholomorphic curves are often referred to as **asymptotically cylindrical pseudoholomorphic curves.** Recall that Hofer energy was defined in Definition 3.10 of the previous section.

The following two theorems give the relationship between pseudoholomorphic curves with finite Hofer energy and an asymptotic convergence in C^∞ to nondegenerate periodic orbits of the Reeb vector field. In addition, it tells us that if one can prove the existence of a finite energy surface, then there exists a nondegenerate periodic orbit of the Reeb vector field.

Theorem 3.15 (Hofer-Wysocki-Zehnder [HWZI])**.** *Let* $v = (a, f) : ([0, \infty) \times S^1, j) \to$

$(\mathbb{R}\times M, \tilde{J})$ be a pseudoholomorphic curve having finite Hofer energy,

$$0 < E(v) < \infty$$

and having its $\mathbb{R}$ component unbounded from above. Then the following quantity exists and is positive,

$$T := \lim_{s\to\infty} \int_0^1 f^*\alpha > 0.$$

Moreover there exists a sequence $\rho_k \to \infty$ such that

$$\lim_{k\to\infty} f(\rho_k e^{it}) = \gamma(tT) \text{ in } C^\infty(\mathbb{R})$$

for a T-periodic solution $\gamma(t)$ of the Reeb vector field R_α. If this solution is nondegenerate then

$$\lim_{\rho\to\infty} f(\rho e^{2\pi i t}) = \gamma(tT),$$

with convergence in $C^\infty(\mathbb{R})$. In the case that $E(v) = 0$ we have that

$$T := \lim_{s\to\infty} \int_0^1 f^*\alpha = 0,$$

then the pseudoholomorphic curve $w : D^2 \setminus \{0\} \to \mathbb{R}\times M$ defined by

$$w(e^{-2\pi z}) = v(z),$$

can be extended smoothly to the unit disk D^2.

Before we state the next theorem, pertaining to a precise description of the behavior of pseudoholomorphic curves having finite Hofer energy near their punctures in local coordinates, we make some remarks about the behavior of pseudoholomorphic cylinders in relation to Theorem 3.15. Let $u = (a, f) : (\mathbb{R}\times S^1, j) \to (\mathbb{R}\times M, \tilde{J})$ be a pseudoholomorphic cylinder. Then at each end u has either a removable puncture or it converges to a Reeb cylinder in the contact manifold M component at the $+\infty$ or $-\infty$ end of the symplectization.

Since a pseudoholomorphic curve defined on a closed surface is necessarily constant as discussed in Remark 3.13, we know that both ends cannot be removable punctures unless u is the constant curve. In addition we know that at least one of the ends must tend towards $+\infty$ otherwise we obtain a contradiction with the maximum principle in Proposition 3.6.

An alternative way of understanding Theorem 3.15, is to say that the pseudoholomorphic curve $u : (S^2 \setminus \{x, y_1, ...y_s\}, j) \to (\mathbb{R} \times M, \tilde{J})$ converges to vertical cylinders over closed Reeb orbits at $t = \pm\infty$. We illustrate this in Figure 2.

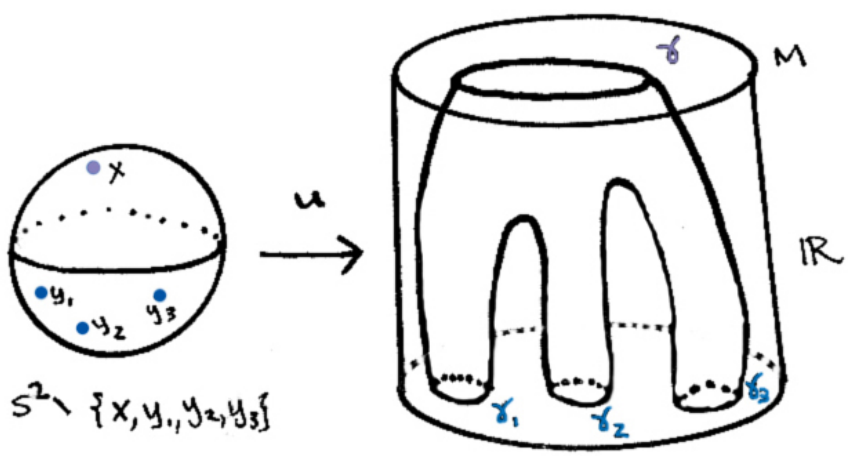


Figure 2: A pseudoholomorphic curve u in $\mathbb{R} \times M$ with $s = 3$.

Moreover, we obtain the following lemma in regard to how the symplectic action

$$\mathcal{A}(\gamma) := \int_\gamma \alpha = \int_{S^1} \gamma^* \alpha$$

decreases along the pseudoholomorphic cylinders u which converge to the Reeb trajectories γ_+ at the positive end and γ_- at the negative end. In the language of moduli spaces, which we explain in Section 4, one writes $u \in \mathcal{M}(\gamma_+; \gamma_-)$.

Lemma 3.16. *Suppose there exists a pseudoholomorphic cylinder u which converge to the Reeb trajectory γ_+ at the positive end and to γ_- at the negative end. Then*

$$\mathcal{A}(\gamma_+) \geq \mathcal{A}(\gamma_-),$$

with equality if and only if $\gamma_+ = \gamma_-$ and the image of u is an $\mathbb{R}$-invariant pseudoholomorphic cylinder.

Proof. Let $u : (\mathbb{R} \times S^1, j) \to (\mathbb{R} \times M, \tilde{J})$ be a pseudoholomorphic cylinder which converges to the Reeb trajectory γ_+ at the positive end and to γ_- at the negative end. By Stokes' theorem,

$$\mathcal{A}(\gamma_+) - \mathcal{A}(\gamma_-) = \int_{\mathbb{R} \times S^1} u^* d\alpha$$

We know that the integral on the right hand side converges because of the asymptotics of u. By condition that $\tilde{J}$ is a compatible almost complex structure we know that $u^* d\alpha \geq 0$ in $\mathbb{R} \times S^1$ with equality only when u is tangent to $\mathbb{R}$ cross the Reeb direction, i.e. when $\gamma_+ = \gamma_-$ and u is as in Example 3.4. □

In other words, when u converges to the trajectories γ_+ at the positive end and to γ_- at the negative end of the symplectization symplectic area may be expressed as

$$A(u) = \int_{\mathbb{R} \times S^1} u^* d\alpha = \int_{\gamma_+} \alpha - \int_{\gamma_-} \alpha = \mathcal{A}(\gamma_+) - \mathcal{A}(\gamma_-).$$

The next theorem allows us to study for large ρ, a pseudoholomorphic curve in a tubular neighborhood of its limit $\gamma(t)$. Before stating this theorem we will need to explain some notation and construct local coordinates for particular pseudoholomorphic curves. For the purposes of this paper we will restrict to contact manifolds of dimension

3, but note that the results of this section can be extended for a contact manifold of higher dimension.

We consider a pseudoholomorphic cylinder

$$u = v \circ \varphi = (a, f),$$

with the biholomorphic map

$$\begin{aligned} \varphi : \ \mathbb{R} \times S^1 &\to \mathbb{C} \setminus \{0\} \\ (s,t) &\mapsto e^{2\pi(s+it)}. \end{aligned} \tag{3.10}$$

Note that

$$\lim_{s \to \infty} f(s,t) = \gamma(Tt) \text{ in } C^\infty(S^1).$$

In the case that u converges to γ we can introduce suitable coordinates near γ, as in the following lemma, proven in [HWZI]. As usual, we let (M, α) be a 3-dimensional contact manifold and $\gamma(t)$ a T-periodic orbit of the Reeb vector field R_α. We denote T_0 to be the minimal period so that $T = kT_0$ for some positive integer k.

Lemma 3.17. *There is an open neighborhood $U \subset S^1 \times \mathbb{R}^2$ of $S^1 \times \{0\}$ and an open neighborhood $V \subset M$ of*

$$\wp = \{\gamma(t) \mid t \in \mathbb{R}\}$$

and a diffeomorphism $\varphi : U \to V$ mapping $S^1 \times \{0\}$ onto $\wp$ such that

$$\varphi^* \alpha = f \alpha_0.$$

Here

$$\alpha_0 = xdy - dz$$

is the standard contact form on $\mathbb{R}^3$*, and* f *is a smooth positive function* $f : U \to \mathbb{R}$ *satisfying*

$$\begin{aligned} f(\theta,0,0) &= T_0 \\ df(\theta,0,0) &= 0 \end{aligned}$$

for all $\theta \in S^1$*.*

Using the coordinates from the lemma we may write

$$(a, f) = (a, (\theta, s, t)) = (a, (\theta, (z)) = (a, \varphi^{-1} \circ f). \tag{3.11}$$

Working in the universal cover of $S^1 \times \mathbb{R}^3$ we may view

$$(a(s,t), \theta(s,t), z(s,t)) : [s_0, \infty) \times \mathbb{R} \to \mathbb{R}^4.$$

where

$$\theta(s, t+1) = \theta(s,t) + k.$$

Here we assume that γ is nondegenerate and $T_0 = T/k$ is its minimal period.

In the case that u converges to γ the following theorem gives the asymptotic description of a nondegenerate finite energy plane. In the statement of this theorem we have that

$$\mu : [s_0, \infty) \to \mathbb{R}$$

is a smooth function satisfying

$$\lim_{s\to\infty} \mu(s) = \lambda < 0.$$

The number λ is an eigenvalue of a self adjoint operator A in $L^2(S^1, \mathbb{R})$ related to the linearized Reeb flow φ_t along the orbit $\gamma(t)$ that we are limiting on. The operator is defined by

$$A = -J_0 \frac{\partial}{\partial t} - S_\infty(t),$$

with

$$S_\infty(t) = S_\infty(t + 2\pi)$$

a symmetric, 1-periodic, smooth 2×2 matrix function defined by

$$S_\infty(t) = -J_0 \pi_m dR(m) \pi_m$$

where $m = (kt, 0) \in \mathbb{R} \times \mathbb{R}^2$. Moreover,

$$\vec{e}(t) = \vec{e}(t+1) \neq 0$$

is an eigenvector of A corresponding to the eigenvalue $\lambda < 0$. We are now ready to state the following theorem.

Theorem 3.18 (Theorem 1.4 in [HWZI]). *There exist constants $c \in \mathbb{R}$ and $d > 0$ such that*

$$|\partial^\beta [a(s,t) - Ts - c]| \leq N e^{-ds}$$

$$|\partial^\beta [\theta(s,t) - kt]| \leq N e^{-ds}$$

for all multi-indices β, with constants $N = N_\beta$. Moreover, we have the asymptotic formula for the transversal approach to $\gamma(t)$:

$$z(s,t) = e^{\int_{s_0}^{s} \mu(\tau) d\tau} [\vec{e}(t) + r(s,t)] \in \mathbb{R}^2,$$

with

$$\lim_{s \to \infty} \partial^\beta r(s,t) = 0 \text{ uniformly in } t \text{ for all derivatives.}$$

These notions are important in our study of the linearization of the $\bar{\partial}_{\tilde{J}}$ equation, as it allows us to introduce a suitable system of weights on the Sobolev spaces. This will be explained in the next chapter.

Chapter 4

Fredholm foundations

In this chapter we begin our study of moduli spaces of asymptotically cylindrical pseudoholomorphic curves from a punctured Riemann surface into the symplectization of a contact manifold. This provides us with the functional analytic backbone to obtain the virtual dimension of moduli spaces of such maps in terms of the Riemann surface and the asymptotic data given by the periodic solutions of the Reeb vector field associated to the contact form. We proceed by reviewing Dragnev's transversality results from [Dr04], which apply to moduli spaces consisting of somewhere injective asymptotically cylindrical pseudoholomorphic curves. Dragnev's results for this class of somewhere injective pseudoholomorphic curves allow us to conclude that these moduli spaces are cut out transversally after a generic choice of J, and hence smooth manifolds whose dimension coincides with the Fredholm index. Moduli spaces consisting of multiply covered pseudoholomorphic curves are discussed later, in Chapters 5 and 6.

At the end of this chapter we state the folk theorem that a non-constant finite energy surface factors through a somewhere injective one. This is an extension of an analogous theorem regarding finite energy planes, proven in the appendix of Hofer, Wysocki, and Zehnder's [HWZII] as well as the theorem for closed curves in the book [MSbig$J$] by McDuff and Salamon. A proof of this fact follows from taking the argument given in McDuff-Salamon for the closed case and appealing to the behavior of finite energy punctured holomorphic curves near a puncture, as described in Siefrings work [Si08], and will be given separate from this thesis.

4.1 Fredholm theory setup

We will not review the basic definitions of Fredholm operators in this paper. A lovely exposition may be found Appendix A of [MSbig$J$]. This section provides a sketch of the necessary functional analysis that will be used to glean information in regards to the space of finite energy solutions of our favorite nonlinear elliptic partial differential equation, that is maps $u : \dot{\Sigma} := (\Sigma \setminus \{x, y_1, ..., y_s\}, j) \to (\mathbb{R} \times M, J)$ satisfying the Cauchy Riemann equation

$$\bar{\partial}_{j,J} u := du + J \circ du \circ j \equiv 0.$$

We further assume that u is **asymptotically cylindrical**. This means that after partitioning the punctures into positive and negative subsets

$$\Gamma = \Gamma^+ \cup \Gamma^-,$$

we can make a choice of a biholomorphic identification of a punctured neighborhood of each $z \in \Gamma^\pm$ with the half-cylinder $Z_\pm$ which is subject to an asymptotic formula, which we will precisely explain. Namely, we write

$$Z_+ = [0, \infty) \times S^1 \quad \text{and} \quad Z_- = (-\infty, 0] \times S^1,$$

and after choosing cylindrical coordinates (s, t) for u near the puncture, we have that for $|s|$ sufficiently large, the following asymptotic formula is satisfied

$$u \circ \phi(s, t) = \exp_{(Ts, \gamma(Tt))} h(s, t) \in E_\pm.$$

Here $(E_-, J) \cong ((-\infty, 0] \times M, \tilde{J})$ and $(E_+, J) \cong ([0, \infty) \times M, \tilde{J})$

As before, $T > 0$ is a constant, $\gamma : \mathbb{R} \to M_\pm$ is a T-periodic orbit of $X_\pm$, and the exponential map is defined with respect ot any $\mathbb{R}$-invariant metric on $\mathbb{R} \times M_\omega$, $h(s,t) \in \xi|_{\gamma(Tt)}$ goes to 0 uniformly in t as $s \to \pm\infty$, and $\phi : Z_\pm \to Z_\pm$ is a smooth embedding such that

$$\phi(s,t) - (s + s_0, t + t_0) \to 0 \text{ as } s \to \pm\infty$$

for some constants $s_0 \in \mathbb{R}$ and $t_0 \in S^1$. We denote by γ_z the T-periodic orbit parametrized by γ and call it the **asymptotic orbit** of u at the puncture z. With this asymptotic behavior in mind, it has become common to think of $(\dot{\Sigma}, j)$ as a Riemann surface with cylindrical ends, and as such neighborhoods of the punctures are often called **ends** of $\dot{\Sigma}$. Asymptotically cylindrical pseudoholomorphic curves were studied in the work of Hofer, Wysocki, and Zehnder [H93], [HWZI], [HWZII], [HWZIII], [HWZIV], [HWZ03]. Their work guarantees that the asymptotically cylindrical pseudoholomorphic curves are those which satisfy a useful finite energy condition. The details of this was discussed in Section 3.4.

While ultimately we are only interested in counting cylindrical pseudoholomorphic curves, i.e. ones whose domain is $S^2 \setminus \{x, y\}$, we still need to understand the more general situation and take the domain of our pseudoholomorphic curves to be a multiply punctured sphere $(\dot{\Sigma}, j) := (S^2 \setminus \{x, y_1, ...y_s\}, j_0)$. However, the following construction is given generally and works for multiply punctured arbitrary Riemann surfaces, as well as for those with more than one positive puncture. We review the main points of the Fredholm theory associated to moduli spaces of **asymptotically cylindrical pseudoholomorphic curves.** The full details of the Fredholm theory construction can be found in [Dr04] and [Sc95], and we will make references to these sources as appropriate.

The functional analysis required for studying partial differential equations problems is often quite subtle and can seem largely unmotivated. The broad strokes of these theories consists of abstractly recasting the partial differential equations as operators acting on appropriate linear spaces. We write this symbolically as

$$A : X \to Y,$$

where the operator A encodes the structure of the partial differential equation, including possible boundary conditions or certain asymptotics, and X and Y are spaces of functions. After making appropriate choices in regard to what the correct abstract operators and function spaces are, we can ascertain the solvability of various equantions involving A by invoking our favorite and now applicable theorems from functional analysis.

In the setting of interest to us, A is the linearized Cauchy-Riemann operator. The difficulties in finding the correct spaces of functions X and Y arise from the lack of analytic estimates necessary to demonstrating that the solutions we constructed actually belong to spaces of functions that are neither "too smooth" nor "too ill-behaved." As a result, we must work in the setting of weighted Sobolev spaces. We will need an even more souped-up version of Sobolev spaces as compared with the symplectic setting, e.g. [MSbig$J$], to surmount the difficulties in working with linearizations of Cauchy-Riemann operators ($\bar{\partial}_{\tilde{J}}$) in open symplectic manifolds which are asymptotically cylindrical. This data will be encoded in weights, which will describe the asymptotics near the punctures. However the ultimate goal remains the same; we wish to demonstrate the same sorts of geometric results may be prescribed to the space of solutions.

We begin by describing the weighted Sobolev spaces and how they can be used to account for the behavior exhibited by asymptotically cylindrical pseudoholomorphic curves near their punctures, as precisely described in Theorems 3.15, 3.18. These weights are derived from the nondegeneracy properties of the periodic orbits of the Reeb vector field, which we will explain as follows. We warn the reader that this is quite involved to do precisely and will take a bit of time.

Throughout we assume that we have selected a contact form α for our contact manifold M, which is nondegenerate, meaning that all the Reeb orbits of R_α are nondegenerate. Recall that we may separate finite energy asymptotically cylindrical pseudoholomorphic curves $u = (a, f)$ in symplectizations into an $\mathbb{R}$-component denoted by a and an M-component denoted by f. We further assume that f converges to nondegenerate periodic orbits of the Reeb vector field as $s \to \pm\infty$ in $C^\infty(\mathbb{R})$. We use the coordinates described immediately before (3.11) and in Theorem 3.18 to define the notion of $(\delta, 1, p)$**-convergence**.

To do this we introduce cylindrical coordinates near the puncture

$$u : [R, \infty) \to \mathbb{R} \times M$$

where $u(s, t) \to \gamma(Tt + c)$ uniformly for $t \in S^1$. here γ is a T-periodic orbit for the Reeb vector field. Then we can write with ϕ as given in (3.10), and $\theta,\ z$ as given in (3.11)

$$f(s, t) = \phi(\theta(s, t), z(s, t)).$$

For constants $c \in \mathbb{R}$ and $k \in \mathbb{Z}_{>0}$ we define

$$\begin{aligned} a_c(s, t) &= a(s, t) - Ts - c \\ \theta_k(s, t) &= \theta(s, t) - kt \end{aligned}$$

Here $T = kT_0$, where T_0 is the minimal period of γ. Later we will frequently need cylindrical coordinates for d-different punctures of a pseudoholomorphic curve and this will be denoted by $\{\sigma_i\}_{i=1}^d$.

Definition 4.1 ($(\delta, 1, p)$-convergence)**.** Let $0 < \delta < \infty$ and $p > 2$. We say that u is $(\delta, 1, p)$-convergent to a periodic orbit (γ, T) of the Reeb vector field whenever

$$(s,t) \to \big(e^{\delta s} a_c(s,t),\ e^{\delta s}\theta_k(s,t)\big) \tag{4.1}$$

are in $W^{1,p}([R,\infty) \times S^1, \mathbb{R}^2)$ for some $c,\ R \in \mathbb{R}$, $k \in \mathbb{Z}_{>0}$, and

$$(s,t) \to e^{\delta s} z(s,t) \tag{4.2}$$

is in $W^{1,p}([R,\infty) \times S^1, \mathbb{R})$.

Remark 4.2. If we were to work with contact manifold of dimension $2n-1$ we would have that (4.1) is in $W^{1,p}([R,\infty) \times S^1, \mathbb{R}^{2n-2})$ instead of in $W^{1,p}([R,\infty) \times S^1, \mathbb{R}^2)$. Otherwise the formulation of the above definition remains the same in the setting of higher dimensional contact manifolds.

A priori, this definition depends on the choice of φ as given in (3.10). The definition of $(\delta, 1, p)$-convergence is independent of this choice. We refer the reader to [Dr04] Lemma 2 for a proof of this useful fact.

Lemma 4.3. *The definition of $(\delta, 1, p)$-convergence is independent of the choice of φ.*

The notion of $(\delta, 1, p)$-convergence will be important in defining the appropriate function spaces used to account for the behavior exhibited by asymptotically cylindrical pseudoholomorphic curves near their punctures, as precisely described in Theorems 3.15, 3.18. This will allow us to later understand the geometric structure of the moduli spaces of asymptotically cylindrical pseudoholomorphic curves.

Before continuing with the definitions of the appropriate function spaces involved in our later analysis, we will recall some notation from before. We will take Σ to be a closed Riemann surface with a finite set of punctures $\Gamma = \Gamma^+ \cup \Gamma^-$ and $\dot{\Sigma} = \Sigma \setminus \Gamma$. Near each puncture we have cylindrical coordinates, $\{\sigma_i\}_{i=1}^d$, where $d := \#\Gamma$, corresponding to $\gamma_1, ..., \gamma_d$ smooth nondegenerate periodic orbits of the Reeb vector field. We will denote the symplectization $(\mathbb{R} \times M, d(e^\tau \alpha))$ by W. We define the space

$$C^\infty_{\gamma_1,...,\gamma_d}(\dot{\Sigma}, W) = \Bigg\{ h := (b, g) \in C^\infty(\dot{\Sigma}, W) \mid \lim_{\epsilon_i s \to \infty} (g \circ \sigma_i)(s,t) = \gamma_i(T_i t + e_i), \\ \lim_{\epsilon_i s \to \infty} \tfrac{1}{s}[(a \circ \sigma_i)(s,t) - T_i s] = 0, e_i \in \mathbb{R},\ i = 1, ..., d \Bigg\}. \tag{4.3}$$

Here in the limit as $\epsilon_i s \to \infty$ we take $\epsilon_i = +1$ for a positive puncture, i.e. one in Γ^+, and $\epsilon = -1$ for a negative puncture. Next we must find an appropriate completion of this space $C^\infty_{\gamma_1,...,\gamma_d}(\dot{\Sigma}, W)$ to a Banach manifold of maps of a certain Sobolev type class, which will involve the notion of $(\delta, 1, p)$-convergence.

By using the projection π onto the contact structure along the Reeb vector field, we can define the metrics g_J on M and $\tilde{g}_{\tilde{J}}$ on W as follows:

$$g_J(X, Y) = \alpha(X)\alpha(Y) + d\alpha(\pi(X), J\pi(Y)) \tag{4.4}$$

$$\tilde{g}_{\tilde{J}}((\rho, X), (\varrho, Y)) = \rho\varrho + \alpha(X)\alpha(Y) + d\alpha(\pi(X), J\pi(Y)) \tag{4.5}$$

Denote by ∇ and $\tilde{\nabla}$ the Levi-Civita connections associated to g_J and $\tilde{g}_{\tilde{J}}$ respectively, along with the respective exponential maps exp and $\widetilde{\exp}$. We obtain the following lemma.

Lemma 4.4. *Let $\tilde{\nabla}$ and g_J be defined as above, Y a section of TM, X a section of the contact structure ξ and R_α the Reeb vector field. Then the following holds.*

1. $\tilde{\nabla}_{R_\alpha} R_\alpha = 0$

2. $\tilde{\nabla}_Y R_\alpha$ *and* $\tilde{\nabla}_{R_\alpha} X$ *are sections of* ξ*.*

For a proof, see [Dr04], Lemma 3.

Next we would like to define Sobolev space structures on the pullback bundles u^*TW for maps u such that the M-part of u converges at the punctures to the periodic orbits of the Reeb vector field, as described in Theorems 3.15, and 3.18.

Take $u_0 = (a_0, f_0) : [R, \infty) \times S^1 \to W$ to be $(\delta, 1, p)$-convergent to a periodic orbit $(\gamma, T = kT_0)$ of the Reeb vector field R_α. Let

$$\eta : [R, \infty) \times S^1 \to TW$$

be of class $W^{1,p}$ such that $\eta(s,t) \in T_{u_0(s,t)}TW$. We can write

$$\eta(s,t) = (b(s,t), g(s,t)R_\alpha(u_0(s,t)) + Q(s,t)) \tag{4.6}$$

where $Q(s,t) \in \xi_{u_0(s,t)}$.

Remark 4.5. Recall that one may define local analogues of the $W^{k,p}(\Omega)$ spaces, denoted by $W^{k,p}_{\text{loc}}(\Omega)$, to consist of functions belonging to $W^{k,p}(\Omega')$ for all Ω' compactly supported in Ω. From functional analysis (see for instance Chapter 7 of Gilbarg and Trudinger [GT]) we know that functions in $W^{k,p}_{\text{loc}}(\Omega)$ of compact support will in fact belong to $W^{k,p}_0(\Omega)$, the closure of $C^\infty_c(\Omega)$ in $W^{k,p}(\Omega)$.

Definition 4.6. We say that $\eta \in W^{1,p}_\delta(u_0^*TW)$ whenever $\eta \in W^{1,p}_{\text{loc}}(u_0^*TW)$ with

$$e^{\delta s}(b, g) \in W^{1,p}([R, \infty) \times S^1, \mathbb{R}^2), \tag{4.7}$$

and

$$e^{\delta s} Q \in W^{1,p}(u_0^* \xi). \tag{4.8}$$

That $e^{\delta s} Q \in W^{1,p}(f_0^* \xi)$ means

$$\int |e^{\delta s} Q|^p + \int |\nabla_s e^{\delta s} Q|^p + \int |\nabla_t e^{\delta s} Q|^p < \infty,$$

with $|Q|^2 = g_J(Q, Q) = d\alpha(Q, Q)$ for $Q \in \xi$ and the integration taken over $[R, \infty) \times S^1$, see Lemma 4.4

We define $u = (a, f)$ when $\eta \in C^\infty_{\gamma_1,\ldots,\gamma_d}((u_0^* TW))$ and $c, d \in \mathbb{R}$, assuming $|\eta(s,t)|$ and $|c|$ are sufficently small, by

$$\begin{aligned} a(s,t) &= b(s,t) + d \\ u(s,t) &= \exp_{u_0(s,t)}\left((g(s,t) + c) R_\alpha(u_0(s,t)) + Q(s,t)\right), \end{aligned}$$

where $\eta = (b, hR_\alpha + Q)$ and again, $Q \in \xi$. We have the following proposition.

Proposition 4.7. *Assume u_0 is $(\delta, 1, p)$-convergent to a periodic orbit (γ, T) and η, c, d are described as above. Then u is $(\delta, 1, p)$-convergent to (γ, T). Moreover if u is $(\delta, 1, p)$-convergent to (γ, T) for some η then $\eta \in W^{1,p}_\delta(u_0^* TW)$.*

For a proof, see [Dr04], Proposition 1.

Next we will use this Sobolev space structure to complete the space $C^\infty_{\gamma_1,\ldots,\gamma_d}(\dot{\Sigma}, W)$, recall (4.3), by completing it with maps from $\dot{\Sigma}$ to W which are $(\delta, 1, p)$-convergent at the punctures to periodic orbits $\{\gamma_i\}_{i=1}^d$ of the Reeb vector field. This will require some more notational set up. On W we take the metric $\tilde{g}_{\tilde{J}}$ as in (4.5) and $\tilde{\nabla}$, the associated Levi-Civita connection. Denote by $\mathcal{D} \subset TW$ the associated injectivity neighborhood of the zero section.

Pick $\varepsilon > 0$ such that 2ε is less than the injectivity radius of the zero section. We use this to define $\mathcal{D}_\varepsilon \subset \mathcal{D}$ by

$$\mathcal{D}_\varepsilon = \{(w, \zeta) \mid w \in W,\ \zeta \in T_w W,\ ||\zeta|| < \varepsilon\}.$$

Denote by $B^2_\varepsilon(0) \subset \mathbb{R}^2$ the disk of radius ε and center at $\mathbf{0} \in \mathbb{R}^2$ and by $D^{2d}_\varepsilon(0)$ the polydisk

$$D^{2d}_\varepsilon(0) = B^2_\varepsilon(0) \times ... \times B^2_\varepsilon(0).$$

Let $(x, y) = (x_1, y_1, ..., x_d, y_d) \in D^{2d}_\varepsilon(0)$. Take R to be sufficently large and consider a smooth function

$$\kappa : \mathbb{R} \to [0, 1]$$

$$\kappa(s) = \begin{cases} 0 & |s| \leq R + 1/2, \\ 1 & |s| \geq R + 1 \end{cases}$$

Let Z_R be the positive end of the infinite cylinder, $[R, \infty) \times S^1$ and Z_{-R} the negative end, $(-\infty, -R] \times S^1$. Now for $h = (b, g) \in C^\infty_{\gamma_1,...,\gamma_d}(\dot{S}, W)$, we define $h_{(x,y)}$ as follows, using the coordinates from Lemma 3.17 and Theorem 3.18:

$$h_{(x,y)}(s,t) = \begin{cases} h(s,t) & \text{on } \Sigma \setminus \bigcup_{i=1}^d \sigma_i(Z_{\epsilon_i R}) \\ (a(s,t) + \kappa(s) y_i, \theta(s,t) + \kappa(s) x_i, (1 - \kappa(s)) z(s,t)) & \text{on } \sigma_i(Z_{\epsilon_i R}) \end{cases}$$

As before, we take $\epsilon_i = +1$ for a positive puncture, i.e. one in Γ^+, and $\epsilon = -1$ for a negative puncture. Now we are finally able to define the desired completion of the space $C^\infty_{\gamma_1,...,\gamma_d}(\dot{\Sigma}, W)$.

Definition 4.8. Given periodic orbits $\{\gamma_i\}_{i=1}^s$ of the Reeb vector field we define

$$\mathcal{B} = \mathcal{P}^{1,p,\delta}_{\gamma_1,..,\gamma_d}(\dot{\Sigma}, W) = \{\tilde{\exp} \circ \eta \mid \eta \in W^{1,p}_\delta\left(h^*_{(x,y)}\mathcal{D}_\varepsilon\right)\}$$

where $h \in C^\infty_{\gamma_1,\ldots,\gamma_d}(\dot\Sigma, W)$, $(x,y) \in D^{2d}_\varepsilon(0)$, and

$$W^{1,p}_\delta\left(h^*_{(x,y)}\mathcal{D}_\varepsilon\right) = \{\eta \in W^{1,p}_\delta(h^*_{(x,y)}TW) \mid \eta(z) \in \mathcal{D}_\varepsilon,\ z \in \dot\Sigma\}.$$

Here we have abused notation slightly, and we point out that the less cumbersome $\tilde{\exp} \circ \eta$ should be written $\tilde{\exp}_{h_{(x,y)}(z)}\eta(z)$.

We have the following theorems, whose proofs are the same as in Theorem 2.1.7 and 2.2.1 from [Sc95] and are therefore omitted.

Theorem 4.9 ([Sc95])**.** *The space $\mathcal{B} = \mathcal{P}^{1,p,\delta}_{\gamma_1,..,\gamma_d}(\dot\Sigma, W)$ is endowed with the differentiable structure of an infinite dimensional, separable Banach manifold.*

Remark 4.10. We point out that with the definition of $\mathcal{B}$ above, a Banach neighborhood U of a map $u \in \mathcal{B}$ is described as a bundle over the polydisk $D^{2d}_\varepsilon(0)$,

$$U = \bigcup_{(x,y)\in D^{2d}_\varepsilon(0)} \{\tilde{\exp}_{u_{(x,y)}}\eta(z) \mid \eta \in W^{1,p}_\delta(u^*_{(x,y)}TW)\}.$$

We may identify U with $W^{1,p}_\delta(u^*_{(x,y)}TW) \times D^{2d}_\varepsilon(0)$ by choosing a suitable trivialization as follows. Let

$$\Pi_{(x,y)} : u^*TW \to u^*_{(x,y)}TW$$

denote parallel transport along the shortest geodesic from a point of U to a point of $u_{(x,y)}$. Then we identify $(\eta,(x,y)) \in W^{1,p}_\delta(u^*_{(x,y)}TW) \times D^{2d}_\varepsilon(0)$ with $\tilde{\exp}_{u_{(x,y)}}\Pi_{(x,y)}\eta$.

Theorem 4.11 ([Sc95])**.** *The vector spaces $W^{1,p}_\delta(u^*TW)$ and $L^p_\delta(u^*TW)$ are well-defined for every $u \in \mathcal{B}$. Moreover*

$$W^{1,p}_\delta(\mathcal{B}^*TW) = \bigcup_{h\in\mathcal{B}} W^{1,p}_\delta(u^*TW)$$

$$L^p_\delta(\mathcal{B}^*TW) = \bigcup_{u\in\mathcal{B}} L^p_\delta(u^*TW)$$

are smooth vector bundles over $\mathcal{B}$. There is a natural identification

$$T_h\mathcal{B} \cong W^{1,p}_\delta(u^*TW) \oplus \mathbb{R}^{2d}.$$

Our next construction is of the bundle $X^{\tilde{J}}$ over $\dot{\Sigma} \times W$. It is defined as follows

$$X^{\tilde{J}} = \Lambda^{0,1}\dot{\Sigma} \oplus_{\tilde{J}} TW$$

$$X^{\tilde{J}}_{(z,w)} = \{\phi \in \mathrm{Hom}(T_z\dot{\Sigma}, T_wW) \mid \phi \circ j(z) = -\tilde{J}(z,w) \circ \phi\}.$$

Now we can define the following Banach space bundle $\mathcal{E}$ over the Banach manifold $\mathcal{B}$ by

$$\mathcal{E} = L^p_\delta(\mathcal{B}^*X^{\tilde{J}}) = \bigcup_{u\in\mathcal{B}} \{u\} \times L^p_\delta(u^*X^{\tilde{J}}),$$

where

$$\mathcal{E}_u = L^p_\delta\left(\Lambda^{0,1}\dot{\Sigma} \oplus_{\tilde{J}} u^*TW\right)$$

4.2 The linearized operator

First we will fix some notation and spaces. We will sketch the details of the relevant constructions here, following the work of [Dr04] and [Sc95], as usual.

The $\bar{\partial}_{\tilde{J}}$ operator may be defined as a smooth section of a Banach bundle over a Banach manifold as follows

$$\bar{\partial}_{\tilde{J}} : \mathcal{B} \to \mathcal{E}$$

$$\bar{\partial}_{\tilde{J}}(u) = du + \tilde{J} \circ du \circ j$$

Next we will want to understand the linearization of $\bar{\partial}_{\tilde{J}}$ at a solution $u \in \bar{\partial}^{-1}(0)$. We will obtain this linearization by projecting the tangent space at a point of the bundle on its vertical subspace, which is identified with the fiber of the bundle. This is well-defined as the point is contained in the zero section of the bundle. In formulas we express this as

$$D\bar{\partial}_{\tilde{J}}(u) : T_u\mathcal{B} \to T_{(u,0)}\mathcal{E}$$

$$T_{(u,0)}\mathcal{E} = T_u\mathcal{B} \oplus \mathcal{E}_u.$$

We denote the projection by $\Pi : T_{(u,0)}\mathcal{E} \to \mathcal{E}_u$ and define the **linearization of** $\bar{\partial}_{\tilde{J}}$ **at the solution** u by the following map,

$$\mathcal{F}_u : T_u\mathcal{B} \to \mathcal{E}_u$$

$$\mathcal{F}_u = \Pi \circ D\bar{\partial}_{\tilde{J}}(u).$$

We note that the linearization $\mathcal{F}_u$ is determined for each pseudoholomorphic curve, but its definition for a general u depends on the choice of connection. In this setting we will work with the Levi-Civita connection ∇ of the metric $g_{\tilde{J}} = \omega(\cdot, \tilde{J}\cdot)$, as it will allow us to give an explicit formula for the linearization, $\mathcal{F}_u$.

Using the identification provided by Theorem 4.11 and Remark 4.10 the parallel transport for $\zeta \in u^*_{(x,y)}TW$ can be written as

$$\Phi_u^{(x,y)}(\zeta) : T_{u_{(x,y)}}W \to T_{\exp_{u_{(x,y)}}\zeta}W,$$

and we can define the following map

$$P_u(\eta, (x,y)) = \Pi^{-1}_{(x,y)} \circ \Phi_u^{(x,y)}\left(\Pi_{(x,y)}\eta\right)^{-1} \circ \bar{\partial}_{\tilde{J}}\ \tilde{\exp}_{u_{(x,y)}}(\Pi_{(x,y)}\eta)$$

This construction allows us to succinctly express the linearization $\mathcal{F}_u$ as an operator

$$\mathcal{F}_u : W^{1,p}_\delta\left(u^*TW\right) \oplus \mathbb{R}^{2d} \to L^p_\delta\left(u^*X^{\tilde{J}}\right)$$

as

$$\mathcal{F}_u(\eta, (x,y)) = \frac{d}{dt}\bigg|_{t=0} P_u(t\eta, t(x,y)).$$

From this expression we obtain

$$\mathcal{F}_u(\eta, (x,y)) = D_u(\eta, (0,0)) + K_u(0, (x,y)), \tag{4.9}$$

and we will shortly explain what D_u and K_u are.

D_u may be explicitly defined as follows, when viewed as an operator from $W^{1,p}_\delta(u^*TW)$ to $L^p_\delta\left(u^*X^{\tilde{J}}\right)$

Proposition 4.12. *The operator*

$$D_u : W^{1,p}_\delta(u^*TW) \to L^p_\delta\left(u^*X^{\tilde{J}}\right),$$

has the following expression

$$D_u\zeta = \nabla\zeta + \tilde{J}(u) \circ \nabla\zeta \circ j + \nabla_\zeta \tilde{J}(u) \circ d(u) \circ j.$$

A proof of this is given for Proposition 2 of [Dr04], following from computations standard in the world of Riemannian geometry.

The operator K_u is a finite dimensional operator with compact support. Notice that $K_u = 0$ on $\sigma_i(Z_{\varepsilon_i(R+1)})$ for $i = 1, ...d$ because of how $u_{(x,y)}$ has been constructed. Therefore by homotoping K_u to 0 we may conclude that the Fredholm property and transversality for the operator $\mathcal{F}_u$ is satisfied provided it can be established for the operator D_u. In this case, we can relate the Fredholm indices of $\mathcal{F}_u$ and D_u as follows:

$$\text{index}(\mathcal{F}_u) = \text{index}(D_u) + 2d \tag{4.10}$$

As a result, notation is frequently abused and D_u is often referred to as the linearization of $\bar{\partial}_{\tilde{J}}$.

4.3 Moduli spaces of somewhere injective pseudoholomorphic curves

We will be interested in studying the geometric properties of moduli space consisting of asymptotically cylindrical pseudoholomorphic curves in symplectizations interpolating between fixed nondegenerate Reeb orbits. For nondegenereate closed Reeb orbits $\gamma, \gamma_1, ...\gamma_s$ of periods $T, T_1, ..., T_s$, we denote by

$$\mathcal{M}(\gamma; \gamma_1, ...\gamma_s),$$

to be the **moduli space of equivalence classes of unparametrized asymptotically cylindrical pseudoholomorphic curves**, with one positive puncture and s negative punctures. Recall that asymptotically cylindrical refers to the prescribed asymptotic conditions as discussed in Theorem 3.15.

We define the equivalence relation between asymptotically cylindrical pseudoholomorphic curves as follows.

Two asymptotically cylindrical pseudoholomorphic curves

$$u : (\Sigma \setminus \{x, y_1, ..., y_s\}, j) \to (\mathbb{R} \times M, J),$$

$$u' : (\Sigma \setminus \{x', y_1', ..., y_s'\}, j) \to (\mathbb{R} \times M, J)$$

are **equivalent** if and only if there exists a biholomorphism ϕ of Σ such that

(i) $\phi(x) = x,' \quad \phi(y_i) = y_i'$ for $i = 1, ..., s$

(ii) $u = u' \circ \phi$

In this case we see that $u = u' \circ \phi$.

Frequently for brevity we denote the finite set of punctures by $\Gamma := \{x, y_1, ..., y_s\}$ which we assume has been ordered. In this case we can define the equivalence classes of data $[(\Sigma, j, \Gamma, u)]$,

$$(\Sigma, j, \Gamma, u) \sim (\Sigma', j', \Gamma', u')$$

whenever there exists a biholomorphic map

$$\phi : (\Sigma, j) \to (\Sigma', j')$$

taking Γ to Γ' with the ordering preserved such that

$$u = u' \circ \phi.$$

When talking about asymptotically cylindrical pseudoholomorphic curves belonging to a particular moduli space $\mathcal{M}$ one frequently writes $u \in \mathcal{M}$. It is more precise to write $(\Sigma, j, \Gamma, u) \in \mathcal{M}$ since technically one is referring to the equivalence classes represented by this pseudoholomorphic curve, $[(\Sigma, j, \Gamma, u)] \in \mathcal{M}$, but since u determines Σ and Γ uniquely we will stick to the simpler notation.

Since $\tilde{J}$ is $\mathbb{R}$-invariant, $\mathbb{R}$ acts on these moduli spaces by "external" translations

$$u = (a, f) \to (a + \rho, f),$$

and we denote the quotient by

$$\hat{\mathcal{M}} := \hat{\mathcal{M}}(\gamma, \gamma_1, ...\gamma_s) = \mathcal{M}(\gamma, \gamma_1, ...\gamma_s)/\mathbb{R}.$$

We will wait to discuss the implications of this external action until later.

In this section we are primarily interested in studying the equivalence classes of somewhere injective asymptotically cylindrical pseudoholomorphic curves. We denote the set of all somewhere injective pseudoholomorphic curves by $\mathcal{N}(\gamma, \gamma_1, ...\gamma_s)$, and $\mathcal{N}(\gamma, \gamma_1, ...\gamma_s) \subset \mathcal{M}(\gamma, \gamma_1, ...\gamma_s)$. We define **somewhere injective** pseudoholomorphic curves as follows. As before, we take (Σ, j) to be a compact connected Riemann surface without boundary and $\Gamma \subset \text{int } \Sigma$ is a finite set of interior punctures, with $\dot{\Sigma} = \Sigma \backslash \Gamma$ and (W, J) to be a suitable almost complex manifold. The notation (Σ', j'), Γ', $\dot{\Sigma}'$, designates other examples of such objects. A pseudoholomorphic curve $u : \dot{\Sigma} \to W$ is said to be **multiply covered** whenever there exists a a pseudoholomorphic curve $v : \dot{\Sigma}' \to W$, and a holomorphic map $\varphi : \Sigma \to \Sigma'$ with $\Gamma' = \varphi(\Gamma)$ such that

$$u = v \circ \varphi, \quad \deg(\varphi) > 1.$$

The pseudoholomorphic curve u is called **simple** whenever it is not multiply covered. We will see shortly that simple pseudoholomorphic curves (in a given homology class) form a smooth finite dimensional manifold for generic J. This is equivalent to understanding the multiply covered curves as the exceptional case, and as such they are often singular points in the moduli space of pseudoholomorphic curves. The proof of this result is based on the observation that every simple pseudoholomorphic curve is **somewhere injective**, which means that for some $z \in \dot{\Sigma}$

$$du(z) \neq 0 \quad u^{-1}(u(z)) = \{z\}.$$

A point $z \in \dot{\Sigma}$ with this property is called an **injective point** of u.

Before we can state the following results of Dragnev in regards to somewhere injective finite energy pseudoholomorphic curves, we must define a suitable Banach space on which we can vary J and $\tilde{J}$. To accomplish this we follow the approach of Floer, as in [Fl88] and introduce Floer's C_ϵ-space. First we will need to fix some notation. Let $J_0 : \xi \to \xi$

be a compatible almost complex structure associated to the defining contact form α. Let $\tilde{J}_0$ be the corresponding extension over the symplectization W, as explained in Section 2.2. Consider the space of all smooth maps $\Psi(p) : \xi_p \to \xi_p$ satisfying

$$\begin{aligned} \Psi(p)J_0(p) + J_0(p)\Psi(p) &= 0 \\ d\alpha(\Psi X, Y) + d\alpha(X, \Psi Y) &= 0 \quad \text{for } X, Y \in \xi. \end{aligned} \tag{4.11}$$

Let $\epsilon = \{\epsilon_n\}_{n=1}^{\infty}$ be a sequence of positive numbers such that $\lim_{n\to\infty} \epsilon_n = 0$. The C_ϵ-space consists of C^∞ homomorphisms of ξ whose sums of weighted C^k norms decay sufficiently fast. It is defined as

$$C_\epsilon = \left\{ \Psi \in \mathrm{Hom}_{\mathbb{R}}(\xi),\ \Psi \in C^\infty \mid ||\Psi||_\epsilon = \sum_{n=1}^{\infty} \epsilon_n ||\Psi||_n < \infty \right\},$$

where $||\Psi||_k$ is the C^k norm with respect to a metric on W. If $\epsilon \to 0$ sufficiently fast then $(C_\epsilon, ||\cdot||_\epsilon)$ is a separable Banach space, which is dense in C^∞. For $\Delta > 0$, denote by

$$U_\Delta = \left\{ \tilde{J} \mid J = J_0 \exp(-J_0 \Psi),\ \Psi \in C_\epsilon,\ ||\Psi||_\epsilon < \Delta \right\}.$$

The map $\Psi \to \tilde{J} \in U_\Delta$ provides a global chart for U_Δ, equipped with a separable Banach manifold structure.

Now we are ready to state the results of Dragnev. These appear as Theorem 4, and Corollaries 1 and 2 in Dragnev, [Dr04].

Theorem 4.13 (Dragnev [Dr04])**.** *The set of all such somewhere injective curves* $\mathcal{N}(\gamma, \gamma_1, ...\gamma_s)$ *carries the structure of a separable manifold. The projection map* p

$$p : \mathcal{N}(\gamma, \gamma_1, ...\gamma_s) \to U_\Delta$$

$$p(\mathrm{im}(u), \tilde{J}) = \tilde{J}$$

is a Fredholm map with Fredholm index near im(u)

$$\begin{aligned} \text{ind } u &= (n-3)(1-s) + \mu_{CZ}(\gamma) - \sum_{i=1}^{s} \mu_{CZ}(\gamma_i) \\ &= |\gamma| - \sum_{i=1}^{s} |\gamma_i| \end{aligned} \tag{4.12}$$

Since we are working with 3 dimensional contact manifolds note that formula 4.12 reduces to

$$\text{ind } u = (s-1) + \mu_{CZ}(\gamma) - \sum_{i=1}^{s} \mu_{CZ}(\gamma_i).$$

For the rest of this paper we will use the 3 dimensional formula.

As a result of the above theorem we obtain several important corollaries.

Corollary 4.14. *For regular values $\tilde{J}$ of p, $p^{-1}(\tilde{J})$ is a smooth finite dimensional manifold whose dimension agrees with the Fredholm index.*

Corollary 4.15. *There exists a dense subset $\mathcal{S} \subset U_\Delta$ such that for every $\tilde{J} \in \mathcal{S}$ if $u : (S^2 \setminus \{x', y_1', ..., y_s'\}, j) \to (\mathbb{R} \times M, \tilde{J})$ is a somewhere injective finite energy surface for $\tilde{J}$ then*

$$\text{ind } u = (s-1) + \mu_{CZ}(\gamma) - \sum_{i=1}^{s} \mu_{CZ}(\gamma_i) \geq 1$$

provided that $\pi \circ Du$ does not vanish identically. Recall that $\pi : TS^3 \to \xi$ is the projection along the Reeb vector field R_α.

As a result of this Corollary, we obtain Theorem 2.1 of [HWZ03]. In the future when we say something holds for generic J, this refers to the assumption that we have selected J from the dense subset $\mathcal{S} \subset U_\Delta$ such that Corollary 4.15 holds.

Remark 4.16. Our results remain valid if we consider symplectic cobordisms. The only difference is that the inequality in Corollary 4.15 reduces to

$$\text{index } u = (s-1) + \mu_{CZ}(\gamma) - \sum_{i=1}^{s} \mu_{CZ}(\gamma_i) \geq 0,$$

due to the fact that in this case the almost complex structure $\tilde{J}$ is not $\mathbb{R}$-invariant.

Next we provide a precise statement of the folk theorem that a non-constant finite energy pseudoholomorphic curve whose domain is a punctured Riemann surface, factors through a somewhere injective one. A proof of such a result when the domain is a closed Riemann surface can be found in [MSbig$J$]. A proof in the case of finite energy planes[1] is proven by Hofer, Wysocki, and Zehnder in the appendix of [HWZII]. There is some debate as to whether or not the result of Hofer, Wysocki, and Zehnder extends in an obvious manner to the types of pseudoholomorphic curves we are considering, e.g. ones whose domain are punctured more than once. Instead we expect that one would need to combine the argument found in [MSbig$J$] with results regarding the behavior of non-constant finite energy curves near a puncture, which are provided by Siefring in [Si08].

As before, we take (Σ, j) to be a compact connected Riemann surface without boundary and $\Gamma \subset \text{int}\ \Sigma$ is a finite set of interior punctures, with $\dot{\Sigma} = \Sigma \setminus \Gamma$ and (W, J) to be a suitable almost complex manifold. Recall that the pseudoholomorphic curve u is called **simple** whenever it is not multiply covered. We will see shortly that simple pseudoholomorphic curves (in a given homology class) form a smooth finite dimensional manifold for generic J.

If a curve is not somewhere injective then it is necessarily a branched cover.

Theorem 4.17. *Let $u : \dot{\Sigma} \to W$ be a non-constant finite energy pseudoholomorphic curve. Then there exists a compact Riemann surface Σ' with a finite set of interior*

[1] Recall these are asymptotically cylindrical pseudoholomorphic curves whose domains are once punctured spheres.

punctures Γ' *and a holomorphic branched covering* $\varphi : \Sigma \to \Sigma'$ *with* $\Gamma' = \varphi(\Gamma)$ φ *and a pseudoholomorphic curve* $v : \dot{\Sigma}' \to W$*, and a holomorphic covering such that*

$$u = v \circ \varphi.$$

Remark. Note that somewhere injective curves automatically exclude those which are a multiply covered cylinder or a branched cover of a multiply covered cylinder.

Chapter 5

Traversing transversality troubles

Rigorous descriptions of the moduli spaces necessary to the conjectures of [EGH00] have only been given under specialized circumstances in Dragnev [Dr04] and Wendl [We10]. Despite the wealth of literature, no comprehensive attempts have been made to clarify issues of transversality of moduli spaces of pseudoholomorphic curves in symplectizations arising in the construction of contact homology. The following two chapters detail the difficulties presented by multiply covered cylinders and their branched covers in defining both a chain complex and invariant. While our attention is restricted to the cylindrical contact homology setting, these same issues must be dealt with in order to realize linearized contact homology as a homology.

The automatic transversality results of Wendl in [We10] describe conditions which are sufficient for punctured pseudoholomorphic curves on a 4-dimensional symplectic cobordism W to be transversally cut out by the Cauchy-Riemann equations, without genericity assumptions on J. These are applicable to arbitrary pseudoholomorphic curves with totally real boundary and cylindrical ends in a 4-dimensional cobordism of two symplectized contact manifolds. One can achieve regularity results even for multiply covered curves in special circumstances by exploiting the "niceness" of the behavior of pseudoholomorphic curves in dimension 4 combined with intersection theory. This allows us to give these geometrically natural moduli spaces the structure of globally smooth orbifolds.

We begin our discussion with an outline of the troubles in defining contact homology and an overview of Wendl's automatic transversality results in [We10]. We also provide examples of Reeb orbits which give rise to moduli spaces of nonpositive virtual dimension. The next chapter is devoted to providing the full numerical details of how the Conley-Zehnder index computations associated to nondegenerate Reeb orbits of dynamically separated forms allow us to avoid the breaking off phenomenon and appeal to Wendl's automatic transversality results. These computations necessitate the strong conditions required of dynamically separated contact forms, as they allow us to obtain a well-defined chain complex without constructing a theory of virtual chains.

5.1 Quandaries of the multiply covered

The idea that one can make moduli spaces consisting of asymptotically cylindrical multiply covered pseudoholomorphic curves and their branched covers non-singular after selecting a compatible almost complex structure J generically, does not work as desired in this setting. Even after a generic choice of J, moduli spaces of such curves have the unfortunate property that even those of nonpositive virtual dimension are not necessarily empty. The presence of such curves must be excluded so as to avoid the breaking phenomenon, which can preclude $\partial^2 = 0$. Furthermore, a multiply covered cylinder may be of smaller index than the cylinder it covers leading to a failure of compactness and the inability to properly define the differential ∂ or continuation maps and chain homotopies.

The main reason that these moduli spaces may be nonempty even after a generic choice of J is because multiply covered pseudoholomorphic curves and their branched covers are not somewhere injective. Consequently the results of Dragnev [Dr04] as described in previous chapter are not applicable, so moduli spaces of curves of nonpositive virtual dimension could exist, even after a generic choice of J. This occurs in the following example with the ellipsoid.

Example 5.1 (Ellipsoid)**.** The 3-dimensional ellipsoid can be described by $E := f^{-1}(1)$, with

$$\begin{array}{rccc} f: & \mathbb{C}^2 & \to & \mathbb{R} \\ & (u,v) & \mapsto & \dfrac{|u|^2}{a} + \dfrac{|v|^2}{b} \end{array}$$

and $a, b \in \mathbb{R}_{>0}$. We obtain a contact structure for the ellipsoid by taking its set of complex tangencies,

$$\xi_p = T_pE \cap J_0(T_pE),$$

which may be described as the kernel of the 1-form

$$\alpha = -\frac{1}{2} df \circ J_0.$$

The Reeb vector field is given by

$$R_\alpha = \frac{1}{a}\left(u\frac{\partial}{\partial u} - \bar{u}\frac{\partial}{\partial \bar{u}}\right) + \frac{1}{b}\left(v\frac{\partial}{\partial v} - \bar{v}\frac{\partial}{\partial \bar{v}}\right).$$

This vector field rotates the u-plane at angular speed $\frac{1}{a}$ and the v-plane at angular speed $\frac{1}{b}$. In the case that a/b is irrational, we can check that there are only two nondegenerate simple Reeb orbits associated to the Reeb vector field R_α. These are determined by the circles $u = 0$ and $v = 0$ respectively. We denote these by γ_1 and γ_2 respectively.

One can check that their Conley-Zehnder indices are described by

$$\mu_{CZ}(\gamma_i^k) = 2\lfloor k(1+\phi_i)\rfloor + 1, \tag{5.1}$$

where $\phi_1 = a/b$ and $\phi_2 = b/a$, see [Lo02]. Moreover one can deduce that (5.1) spans all odd positive numbers for all multiples of γ_1 and γ_2 and that their Conley-Zehnder indices never coincide.

Since either $\phi_1 < 1/2$ or $\phi_2 < 1/2$ we have for this i that

$$\begin{aligned} \mu_{CZ}(\gamma_i) &= 3 \\ \mu_{CZ}(\gamma_i^2) &= 5. \end{aligned}$$

This means that any asymptotically cylindrical pseudoholomorphic curve $u \in \mathcal{M}(\gamma_i^2; \gamma_i, \gamma_i)$, has

$$\mathrm{ind}(u) = 0$$

Thus the virtual dimension of $\mathcal{M}(\gamma_i^2; \gamma_i, \gamma_i)$ is 0. However this moduli space is never nonempty, since it contains the double branched covers of the trivial cylinder over γ_i, which form a 2-dimensional family. As a result transversality can never be achieved for this moduli space.

A detailed explanation of why these curves obstruct the construction of a well-defined homological invariant of a contact manifold is given in Section 6.3.

In addition, one needs to achieve regularity for all finite energy pseudoholomorphic curves under consideration, so that one can appeal to Theorem 5.5 to obtain some of the "numerous not entirely innocent subtleties entailed" in the conjectures of Section 1.7 of [EGH00]. The most important of these is as follows.

Conjecture 5.2. *For a generic choice of J,the Cauchy-Riemann equation gives rise to a section $\bar{\partial}_{\tilde{J}}$ of a certain Banach bundle $\mathcal{E} \to \mathcal{B}^A$, where A is the homology class represented by the curves. Its vertical differential has the form of a linearized Cauchy -*

Riemann operator and is a Fredholm operator with index

$$\begin{aligned} \dim \mathcal{M}^A(\gamma;\gamma_1,...,\gamma_s) &= (n-3)(1-s)+\mu_{CZ}(\gamma)-\sum_{i=1}^{s}\mu_{CZ}(\gamma_i)+2\langle c_1(\xi),A\rangle \\ &= |\gamma|-\sum_{i=1}^{s}|\gamma_i|-2\langle c_1(\xi),A\rangle \end{aligned} \tag{5.2}$$

Here $c_1 \in H^2(W)$ *is the first Chern class of* (W, J)*.*

The geometric structure and smoothness of these moduli spaces is difficult to ascertain in this setting. In Section 1.7 of [EGH00] they note that making the moduli spaces non-singular by picking generic J is needed for the purpose of curve counting but does not always work properly. It is therefore crucial that the moduli spaces of stable J-holomorphic curves are non-singular virtually. Transversality cannot be achieved by perturbing J alone for multiply covered curves and their branched covers. Standard approaches as given by Floer, Hofer, and Salamon in [FHS95] to obtaining transversality via a perturbation of J do not help with the issues of multiply covered cylinders due to the presence of the S^1-symmetry.

In previous literature [Us99], [Bo02], [Bo09] it was stated that when transversality could not be achieved by perturbing the almost complex structure that the difficulty could still be resolved via a *delicate* virtual cycle technique involving multivalued perturbations. However full details were never given and recent literature by McDuff and Wehrheim [MW] suggests that this procedure is even more delicate than previously indicated. This would *in theory* permit one to equip these moduli spaces with some additional canonical structure, thereby functioning *in theory* the same way as if they were orbifolds with boundary of dimension prescribed by the Fredholm index. Thus a completely general definition of cylindrical contact homology still awaits an appropriate theory of so-called *abstract perturbations*.

These abstract perturbations would allow any singular J moduli spaces, to be equipped with some canonical structures that makes them function in the theory the same way as if they were orbifolds with boundary and had the dimension prescribed by the Fredholm index. The analytic difficulties in obtaining virtual smoothness for the moduli spaces described in Conjecture 5.2 are severe. It should be noted that [EGH00] was "meant only as an informal exposition whose role was just to illustrate the involved ideas, rather than to give complete rigorous arguments."

The polyfold theory developed by Hofer, Wysocki, and Zehnder [H06], [HWZ10a], [HWZ10b], [HWZgw] hopes to resolve these severe transversality issues once the existence of an abstract perturbation theorem and an implicit function theorem can be established. This would resolve transversality troubles in a completely abstract functional-analytic framework once a moduli space problem could be recast in the formal language of polyfolds. We emphasize that this is still work in progress, and the necessary theorems for cylindrical and contact homology will require reworking even after a "full SFT transversality package" has been developed. For the reader interested in a further discussion of traversing transversality troubles, especially in relation to the polyfold framework, we suggest [H06] and [FFGW]. Those uninitiated to Fredholm theory or the regularity and transversality of pseudoholomorphic curves should consult [MSbig$J$], though many of the main definitions and results can be found in Section 3.

The only results available are those proven by Dragnev as discussed earlier, applicable only to moduli spaces of somewhere injective curves, and Wendl's automatic transversality results, applicable to more general curves but only in dimension 4. Wendl's results are still quite restrictive, as even if a multiply covered cylinder is of dimension 0, we cannot always guarantee that it is regular. As a result cannot formulate the necessary gluing results needed to prove invariance. However, in the case of nondegenerate dynamically separated contact forms Wendl's results provide us with a practical means of determining that the curves u of interest achieve regularity so that we can appeal to Theorem 5.5, which corresponds to the above conjecture. Combined with the results of the following chapter we will be able to conclude that the chain complex $(C_*, \partial; \alpha)$ associated to a nondegenerate dynamically separated contact form is well-defined after choosing J generically. We summarize and state the pertinent results from [We10] in the next section.

5.2 Notation and setting

Wendl's work and theorems are applicable to 4 dimensional almost complex manifold with noncompact cylindrical ends approaching 3-manifolds $M_\pm$ equipped with stable Hamiltonian structures. This is more general than we need, as we are only interested in symplectizations and cobordisms of a given contact manifold. However, even for cobordisms his results are limited as we cannot use them to obtain regularity for asymptotically cylindrical (e.g. finite energy) pseudoholomorphic cylinders which limit on positive hyperbolic orbits of the same index in a cobordism. As a result we cannot prove the chain homotopy equation or obtain the continuation maps necessary for any proof of invariance for cylindrical contact homology.

Before giving the statements of the theorems of interest to us, we will briefly review the setting and notation used by Wendl in [We10] after restricting to symplectizations or cylindrical cobordisms of contact manifolds of dimension 3. In this and the following section we will only state his results in this restricted setting which is of interest to us.

We will consider pseudoholomorphic curves

$$u : (\dot{\Sigma}, j) \to (W, J),$$

where (Σ, j) is a closed connected Riemann surface and $\Gamma \subset \text{int } \Sigma$ is a finite set of interior punctures, with $\dot{\Sigma} = \Sigma \setminus \Gamma$. For the purposes of this paper we need only consider $\Sigma = S^2$, but Wendl's results hold in the more general setting. Here (W, J) will be the symplectization of a contact 3-manifold (M, α) or a cylindrical cobordism between (M, α_1, J_1) and (M, α_2, J_2) with ker $\alpha_1 =$ ker α_2.

By definition we require u to satisfy the nonlinear Cauchy-Riemann equation

$$du \circ j = J \circ du$$

We further assume that u a finite energy pseudoholomorphic curve. We denote

$$\mathcal{M} := \mathcal{M}(J)$$

to be the moduli space of equivalence classes of asymptotically cylindrical pseudoholomorphic curves in W. Recall an equivalence class is defined by the data (Σ, j, Γ, u) where Γ is considered to be an ordered set, and we define

$$(\Sigma, j, \Gamma, u) \sim (\Sigma', j', \Gamma', u')$$

whenever there exists a biholomorphic map

$$\phi : (\Sigma, j) \to (\Sigma', j')$$

taking Γ to Γ' with the ordering preserved such that

$$u = u' \circ \phi$$

For any $u \in \mathcal{M}$, we denote by $\mathcal{M}_u$ **the connected component of $\mathcal{M}$ containing u.**

By imposing constraints on the asymptotic behavior at some of the punctures we will be able to consider subspaces of $\mathcal{M}$. We make this precise in the following definition,

Definition 5.3. For a given punctured surface $\dot{\Sigma} = \Sigma \setminus (\Gamma^+ \cup \Gamma^-)$ let c denote a choice of periodic orbit γ_z in $M_\pm$ for some subset of punctures $z \in \Gamma^\pm$. We call c a choice of **asymptotic constraints**, and refer to each puncture z for which c specifies an orbit γ_z as a **constrained puncture**.

For any choice of domain $\dot{\Sigma}$ and asymptotic constraints c, we can consider the **constrained moduli space**

$$\mathcal{M}^c \subset \mathcal{M}.$$

The constrained moduli space $\mathcal{M}^c$ consists of curves $u : \dot{\Sigma} \to W$ that approach the specified orbit γ_z^c at each of the constrained punctures $z \in \Gamma$ and arbitrary orbits at the unconstrained punctures. If the asymptotic orbits of a pseudoholomorphic curve u are all nondegenerate, then the **virtual dimension** of $\mathcal{M}_u^c$ is given by the **Fredholm index**

$$\mathrm{ind}(u; c) = (n-3)\chi(\dot{\Sigma}) + 2c_1^\Phi(u^*TW) + \mu^\Phi(u; c), \tag{5.3}$$

as in [We10]. Wendl expresses the Fredholm index for the general setting, where $c_1^\Phi(u^*TW)$ is the relative first Chern number of $(u^*TW, J) \to \dot{\Sigma}$ with respect to a suitable choice of trivialization Φ along the ends and boundary. Given the assumptions and notation used in this paper, we can reduce the expression of the index in (5.3) to something more familiar. Recall that we assumed that $c_1(\xi)$ vanishes, thus in the case of symplectizations when $W = \mathbb{R} \times M$ we have that

$$c_1(\xi) = c_1(\mathbb{R} \times M) = 0,$$

so the term $2c_1^\Phi(u^*TW)$ is 0 in (5.3). In the case when W is a cylindrical cobordism of (M, ξ) with $c_1(\xi)$, we have that $2c_1^\Phi(u^*TW)$ is still 0. This requires some work, but as we are not considering with cobordisms or invariance in this paper we will not give the details.

The term $\mu^\Phi(u; c)$ in (5.3) for the settings of interest to us in this paper is none other than the difference of the Conley-Zehnder indices of the constrained orbits. Precisely, if we take $\Gamma^+ = \{x\}$ and $\Gamma^- = \{y_1, ..., y_s\}$ and $u : (S^2 \setminus \{x, y_1, ..., y_s\}, j) \to (\mathbb{R} \times M, \tilde{J})$ a pseudoholomorphic curve, asymptotically cylindrical to Reeb orbits $\gamma, \gamma_1, ..., \gamma_s$, then we have that

$$\mathcal{M}_u^c = \mathcal{M}(\gamma; \gamma_1, ..., \gamma_s).$$

As a result we obtain

$$\mu^\Phi(u; c) = \mu_{CZ}(\gamma) - \sum_{i=1}^{s} \mu_{CZ}(\gamma_i).$$

In addition, the term $\chi(\dot{\Sigma})$ is simply $(2 - 2g - \#\Gamma^+ - \#\Gamma^-)$, where g is the genus of Σ, and since we are restricted to dimension 4, we get that $(n - 3) = -1$. Thus we obtain the following familiar formula for the Fredholm index of a pseudoholomorphic curve $u : S^2 \setminus \{x, y_1, ..., y_s\} \to \mathbb{R} \times M$, asymptotically cylindrical to Reeb orbits $\gamma, \gamma_1, ..., \gamma_s$, with $s > 1$:

$$\operatorname{ind}(u; c) = -(1 - s) + \mu_{CZ}(\gamma) - \sum_{i=1}^{s} \mu_{CZ}(\gamma_i). \tag{5.4}$$

Remark 5.4. Otherwise if $s = 1$ and the curve u is asmyptotically cylindrical to Reeb orbits γ_+ and γ_- we obtain

$$\operatorname{ind}(u; c) = 2 + \mu_{CZ}(\gamma_+) - \mu_{CZ}(\gamma_-).$$

When we consider the moduli space of equivalence classes of curves u, we have $\mathcal{M}^c :=$

$\mathcal{M}(\gamma_+;\gamma_-)$ and we obtain the usual formula for the virtual dimension

$$\dim(\mathcal{M}(\gamma_+;\gamma_-)) = \mu_{CZ}(\gamma_+) - \mu_{CZ}(\gamma_-).$$

This is because we must subtract the dimension of the group of automorphisms of the domain, which is a cylinder, which is 2. However, if we consider the space of parametrized[1] pseudoholomorphic solutions, which we denote by $\mathcal{S}(\gamma_+;\gamma_-)$, we obtain

$$\dim(\mathcal{S}(\gamma_+;\gamma_-)) = 2 + \mu_{CZ}(\gamma_+) - \mu_{CZ}(\gamma_-).$$

In order to be clear in the statement of the following theorems, we note that when $\dot{\Sigma}$ has underlying symmetry the virtual dimension of $\mathcal{M}^c$ will be less than the Fredholm index of u, as discussed in the above Remark. To account for these cases we will use the notation $\widetilde{\text{ind}}(u;c)$, where

$$\widetilde{\text{ind}}(u;c) = \text{ind}(u;c) - \dim(\text{Aut}(\dot{\Sigma})) \tag{5.5}$$

In other words $\widetilde{\text{ind}}(u;c)$ is the **reduced Fredholm index** of u, which agrees with the virtual dimension of $\mathcal{M}^c$, the moduli space of unparametrized asymptotically cylindrical pseudoholomorphic curves.

Recall, as in Section 3 that the $\bar{\partial}_{\tilde{J}}$ operator, can be expressed as a smooth section of a Banach space bundle. We have

$$\bar{\partial}_{\tilde{J}} : \mathcal{B} \to \mathcal{E}$$

$$\bar{\partial}_{\tilde{J}}(u) = du + \tilde{J} \circ du \circ j.$$

Any neighborhood of any non-constant (Σ, j, Γ, u) in $\mathcal{M}^c$ is in one-to-one correspondence with $\bar{\partial}_{\tilde{J}}^{-1}(0)/\text{Aut}(\dot{\Sigma}, j)$, where the group $\text{Aut}(\dot{\Sigma}, j)$ of biholomorphic maps $(\Sigma, j) \to (\Sigma, j)$

[1] This means we do not mod out by the previously discussed equivalence relation.

fixing Γ acts on pairs $(j', u') \in \bar{\partial}_{\tilde{J}}^{-1}(0)$ by

$$\varphi \cdot (j', u') = (\varphi^* j', u' \circ \varphi).$$

In the case when $\Sigma = S^2$ and Γ consists of 2 points, then $\dot{\Sigma}$ is biholomorphic to a cylinder. The group of automorphisms of a cylinder consists of rotations and translations.

One says that $(\Sigma, j, \Gamma, u) \in \mathcal{M}$ is **regular** whenever it represents a transverse intersection with the zero-section. This is equivalent to requiring that the linearization

$$D\bar{\partial}_{\tilde{J}}(u) : T_u\mathcal{B} \to T_{(u,0)}\mathcal{E}$$

be surjective. Observe that if u is a non-constant pseudoholomorphic curve, then the action of $\mathrm{Aut}(\dot{\Sigma}, j)$ induces a natural inclusion of its Lie algebra $\mathfrak{aut}(\dot{\Sigma}, j)$ into $\ker D\bar{\partial}_{\tilde{J}}(u)$.

The first theorem in [We10] is the following standard folk theorem, which we have restricted to symplectizations[2] of 3-dimensional contact manifolds (M, ξ) with $c_1(\xi) = 0$. We will also only consider moduli spaces of asymptotically pseudoholomorphic curves limiting on nondegenerate orbits, though we remark that this can be relaxed to Morse-Bott orbits.

Theorem 5.5 (Theorem 0 [We10]). *Assume that $u\colon (\dot{\Sigma}, j) \to (W, J)$ is a non-constant curve in $\mathcal{M}^c$ with only nondegenerate asymptotic orbits. If u is regular, then a neighborhood of u in $\mathcal{M}^c$ naturally admits the structure of a smooth orbifold of dimension*

$$\widetilde{\mathrm{ind}}(u; c) = -(1 - s) + \mu_{CZ}(\gamma) - \sum_{i=1}^{s} \mu_{CZ}(\gamma_i),$$

whose isotropy group at u is given by

$$\mathrm{Aut} := \{\varphi \in \mathrm{Aut}(\dot{\Sigma}, j) \mid u = u \circ \varphi\}.$$

[2] This theorem also holds for cylindrical cobordisms of (M, ξ).

Moreover, there is a natural isomorphism

$$T_u\mathcal{M}^c = \ker D\bar{\partial}_{\tilde{J}}(j,u)/\mathfrak{aut}(\dot{\Sigma},j).$$

In particular, regularity implies that $\mathcal{M}^c$ is a manifold near u if u is somewhere injective. Such a result has been demonstrated by Dragnev in the case of symplectizations in [Dr04]. However, of particular interest to us is the case when u is multiply covered. In this general setting, we see that the isotropy group for an orbifold singularity has order bounded by the covering number of u. This is in contrast to the standard theory of pseudoholomorphic curves, as in [MSbig$J$], as we are interested in cases where the curve u achieves regularity despite being multiply covered, which is why the moduli space, while still smooth, may be an orbifold instead of a manifold. A more precise description of this structure is given in Section 7.4, and will be important later in proving $\partial^2 = 0$.

5.3 Automatic transversality results in dimension 3

Wendl expresses the criterion for automatic transversality of a curve in terms of its boundary and asymptotic data, homological properties, and the number of critical points in [We10]. His work generalizes results from Hofer, Lizan, and Sikorav [HLS97] and Ivashkovich and Shevchishin [IS99] to punctured curves with boundary that need not be somewhere injective or immersed. This work cannot be generalized to higher dimensions; it is inherent to the special intersection properties exhibited by pseudoholomorphic curves in dimension 4.

Automatic transversality results are extremely helpful, as these are results that enable us to determine the regularity of a pseudoholomorphic curve.[3] Before we can state these results, we must review a few numbers that encode certain topological and geometric data. These are the **normal first Chern number** and a quantity encoding the total order of critical points of u.

We denote the normal first Chern number by $c_N(u;c)$ and this is a half integer. The simplest means of defining it is via the following formula

$$2c_N(u;c) = \widetilde{\operatorname{ind}}(u;c) - 2 + \#\Gamma_0(c) \tag{5.6}$$

as we are only interested in closed Riemann surfaces Σ whose genus is 0. The subset $\Gamma_0(c) \subset \Gamma$ consists of the **punctures for which the asymptotic orbit has even Conley-Zehnder index**, which is only correct if all orbits are nondegenerate. In the Morse-Bott case, the definition is more complicated and refer the reader to [We10] as we are only interested in nondegenerate orbits. A related quantity is $\Gamma_1(c) := \Gamma \setminus \Gamma_0(c)$, which consists of the **punctures for which the asymptotic orbit has odd Conley-Zehnder index**.

To better illustrate the use of normal first Chern numbers consider the case where Σ is closed and there are no punctures. Then a combination of (5.3) and (5.6) yields the relation

$$c_N(u;c) = c_1(u^*TW) - \chi(\Sigma).$$

This tells us that if u is immersed then $c_N(u;c)$ is the first Chern number of the normal bundle.

[3]For example, if we were wandering about in a forest or thesis and happened upon a pseudoholomorphic curve, we would have some criterion allowing us to ascertain its regularity and could conclude if Theorem 5.5 was applicable.

The other ingredient that we need relates to the number of critical points of a pseudoholomorphic curve $u : \dot{\Sigma} \to W$. Note that since a non-constant curve u is necessarily immersed near the ends, that as a result it can only have at most finitely many critical points. In fact, the bundle

$$u^*TW \to \dot{\Sigma}$$

admits a natural holomorphic structure such that the section

$$du \in \Gamma(\mathrm{Hom}_{\mathbb{C}}(T\dot{\Sigma}, u^*TW))$$

is holomorphic. Thus its critical points are isolated and have positive order, which we will denote by $\mathrm{ord}(du; z)$ for any $z \in \mathrm{Crit}(u)$. This leads us to define the quantity

$$Z(du) := \sum_{z \in du^{-1}(0) \cap \mathrm{int}\dot{\Sigma}} \mathrm{ord}(du; z) \tag{5.7}$$

which is an integer as we are working with closed Riemann surfaces.

Remark 5.6. Note that $Z(du) = 0$ if and only if u is immersed.

The last bit of notation that we will need is a convenient piece of shorthand notation. For given constants $r \in \mathbb{R}$ and $G \geq 0$, we define the nonnegative integer

$$K(r, G) = \min\{k + \ell \mid k \in \mathbb{Z}_{\geq 0},\ \ell \in 2\mathbb{Z}_{\geq 0},\ k \leq G,\ \text{and } 2k + \ell > 2r\}. \tag{5.8}$$

In most pseudoholomorphic world applications, it turns out that $r < 0$, so $K(r, G) = 0$.

Now that we have the proper set up we can finally state Wendl's automatic transversality results, which again we have restricted appropriately for the considerations of this paper.

Theorem 5.7 (Theorem 1 [We10]). *Suppose that* $\dim W = 4$ *and* $(\Sigma, j, \Gamma, u) \in \mathcal{M}^c$ *is a non-constant pseudoholomorphic curve with only nondegenerate asymptotic orbits. If*

$$\widetilde{\mathrm{ind}}(u;c) > c_N(u;c) + Z(du), \tag{5.9}$$

then u *is regular. Moreover, when this condition is not satisfied, we have the following bounds on the dimension of* $\ker D\bar{\partial}_{\tilde{J}}(j,u)$*. If* $\widetilde{\mathrm{ind}}(u;c) \leq 2Z(du)$*, then*

$$\begin{aligned} 2Z(du) &\leq \dim(\ker D\bar{\partial}_{\tilde{J}}(j,u))/\mathfrak{aut}(\dot{\Sigma}, j) \\ &\leq 2Z(du) + K(c_N(u;c) - Z(du), \#\Gamma_0(c)), \end{aligned}$$

and if $2Z(du) \leq \widetilde{\mathrm{ind}}(u;c)$*, then*

$$\begin{aligned} \widetilde{\mathrm{ind}}(u;c) &\leq \dim(\ker D\bar{\partial}_{\tilde{J}}(j,u))/\mathfrak{aut}(\dot{\Sigma}, j) \\ &\leq \widetilde{\mathrm{ind}}(u;c) + K(c_N(u;c) + Z(du) - \widetilde{\mathrm{ind}}(u;c), \#\Gamma_0(c)). \end{aligned}$$

Remark 5.8. Note that if we plug in the definition of the first Chern number of the normal bundle, $c_N(u;c)$, and the index formula, then condition (5.9) is equivalent to

$$\widetilde{\mathrm{ind}}(u;c) > 2g + \#\Gamma_0(c) - 2 + 2Z(du), \tag{5.10}$$

or

$$2c_1^{\Phi}(u^*TW) + \mu^{\Phi}(u;c) + \#\Gamma_1(c) > 2Z(du),$$

where $\Gamma_1(c) := \Gamma \setminus \Gamma_0(c)$, which consists of the punctures for which the asymptotic orbit has odd Conley-Zehnder index. These are direct generalizations of the criteria given in [HLS97], [IS99], and [We05].

Remark 5.9. There is an important special case of the dimension bound which we will use in the application. Namely, if $c_N(u;c) < Z(du)$, then

$$K(c_N(u;c) - Z(du), \#\Gamma_0(c)) = 0,$$

and $\dim\ker(D\bar{\partial}_{\tilde{J}}(u))$ becomes $2Z(du)$, which is its smallest possible size.

In the following section we demonstrate how these results can be applied to settings in which nondegenerate Reeb orbits are obtained from dynamically separated contact forms and why the the dynamically separated condition is needed.

Chapter 6

Requisite regularity results

In this chapter we obtain regularity results for multiply covered finite energy pseudo-holomorphic cylinders[1] in symplectizations of contact manifolds equipped with nondegenerate dynamically separated contact forms. This is accomplished by appealing to Wendl's automatic transversality results in [We10] and Dragnev's results for somewhere injective curves in [Dr04]. These regularity results are the first step to constructing a meaningful cylindrical contact chain complex.

In addition, we will demonstrate that moduli spaces of multiply covered asymptotically cylindrical pseudoholomorphic curves and their branched covers associated to nondegenerate dynamically separated contact forms have dimension at least that of the dimension of the group of automorphisms acting on the space. This allows us to exclude moduli spaces of nonpositive index which would otherwise obstruct $\partial^2 = 0$ in cylindrical contact homology. Combined with the applicable automatic transversality results, we are able to apply the motto, *"the moduli space is smooth if the index is sufficiently large."* As a result we can rule out the breaking phenomenon and the failure of compactness needed to show that the cylindrical contact homology differential ∂ is well defined.

[1] Asymptotically cylindrical pseudoholomorphic cylinders sounds awkward, so we use the phrase finite energy pseudoholomorphic cylinders instead.

These results are obtained via an explicit numerical description of the index of multiply covered asymptotically cylindrical pseudoholomorphic curves and their branched covers associated to dynamically separated contact forms. These computations are given in Sections 6.1 and 6.2. We begin with a precise formulation of these results. Throughout we let x and z be nondegenerate Reeb orbits associated to a dynamically separated contact form in the same free homotopy class with $\mu_{CZ}(x) - \mu_{CZ}(z) = 2$. We denote the set of all Reeb orbits associated to a dynamically separated contact form in the same free homotopy class by $\mathscr{P}$.

Theorem 6.1. *The compactification of $\hat{\mathcal{M}}(x,z)$ is obtained by including all **broken cylinders**, which are pairs of curves $(\mathcal{C}_u, \mathcal{C}_v) \in \hat{\mathcal{M}}(x,y) \times \hat{\mathcal{M}}(y,z)$. We denote the compactification by $\overline{\mathcal{M}}(x;z)$ and obtain after a generic choice of J,*

$$\overline{\mathcal{M}}(x;z) := \hat{\mathcal{M}}(x;z) \ \cup \bigcup_{\substack{y \in \mathscr{P} \\ \mu_{CZ}(y) = \mu_{CZ}(x) - 1}} \hat{\mathcal{M}}(x;y) \times \hat{\mathcal{M}}(y;z).$$

From Theorem 6.1 we obtain the following corollary, which is instrumental in the proof that $\partial^2 = 0$.

Corollary 6.2. *Let x and z be nondegenerate Reeb orbits associated to a dynamically separated contact form with $\mu_{CZ}(x) - \mu_{CZ}(z) = 2$. Then after a generic choice of J,*

$$\partial\overline{\mathcal{M}}(x;z) \ \subseteq \bigcup_{\substack{y \in \mathscr{P} \\ \mu_{CZ}(y) = \mu_{CZ}(x) - 1}} \hat{\mathcal{M}}(x;y) \times \hat{\mathcal{M}}(y;z).$$

The precise geometry in terms of how boundary of the compactified moduli space of finite energy pseudoholomorphic cylinders interpolating between orbits of index difference two is actually equal to the product of rigid finite energy pseudoholomorphic cylinders, i.e. that

$$\partial\overline{\mathcal{M}}(x,z) = \bigcup_{\substack{y\in\mathscr{P} \\ \mu_{CZ}(y)=\mu_{CZ}(x)-1}} \hat{\mathcal{M}}(x,y)\times\hat{\mathcal{M}}(y,z).$$

will be discussed in Chapter 7. The specifics in regards to why $\overline{\mathcal{M}}(x,z)$ may be realized as a graph can be found in Section 7.4. There is some subtlety due to multiply covered Reeb orbits, manifest in the expression for the differential, and will be addressed later. Together the results of this chapter and the next allow us to demonstrate that one can define a cylindrical contact homology, dependent on the choice of nondegenerate dynamically separated contact form.

The following preliminary propositions will be used to prove Theorem 6.1, with the full details of the proof of the main theorem to follow in Section 6.3.

Proposition 6.3. *After a generic choice of J, the only index 0 finite energy asymptotically cylindrical pseudoholomorphic cylinders that exist in the symplectization are trivial.*

Before stating the next proposition, we review the set up used in its statement. Let γ_+ and γ_- be distinct nondegenerate simple Reeb orbits associated to a dynamically separated contact form, with $\mu_{CZ}(\gamma_+)-\mu_{CZ}(\gamma_-)=1,$ or 2. In the case that $\mu_{CZ}(\gamma_+)-\mu_{CZ}(\gamma_-)=0$ the above Proposition implies that γ_+ and γ_- are the same orbit. We denote by $\mathcal{M}(\gamma_+^\ell;\gamma_-^d)$ the moduli space consisting of finite energy pseudoholomorphic curves,

$$u:(S^2\setminus\{x,y\},j)\to(\mathbb{R}\times M,\tilde{J})$$

with γ_+ the orbit corresponding to the puncture $\Gamma^+ = \{x\}$ and γ_- the orbit corresponding to the puncture $\Gamma^- = \{y\}$. When we restrict ourselves to finite energy pseudoholomorphic cylinders we may equivalently take the domain of u to be $(\mathbb{R} \times S^1, j)$ with γ_+ corresponding to the $+\infty$ direction and γ_- corresponding to the $-\infty$ direction of the domain.

Proposition 6.4. *After a generic choice of J the following holds for $\mathcal{M}(\gamma_+^\ell; \gamma_-^d) \neq \emptyset$.*

1. *We have $\ell \geq d \geq 1$ and $\mu_{CZ}(\gamma_+^\ell) - \mu_{CZ}(\gamma_-^d) \geq 1$*
2. *All curves $u \in \mathcal{M}(\gamma_+^\ell; \gamma_-^d)$ are regular.*
3. *The moduli space $\mathcal{M}(\gamma_+^\ell; \gamma_-^d)$ is a smooth orbifold of dimension $\mu_{CZ}(\gamma_+^\ell) - \mu_{CZ}(\gamma_-^d)$.*

The specifics of the orbifold structure associated to $\mathcal{M}(\gamma_+; \gamma_-)$ will be discussed in Section 7.4. In Proposition 6.4 we must choose J generically so that we can appeal to Dragnev's results for underlying simple cylinders, as in Theorem 4.15. This stipulation ensures that $\ell \geq d \geq 1$ when combined with the conditions on behavior of the Conley-Zehnder indices of iterated orbits associated to nondegenerate dynamically separated contact forms. Namely, the dynamically separated condition rules out the possibility of multiply covered pseudoholomorphic cylinders to have smaller index than the cylinders they cover. Further details as well as the proof of this theorem can be found in Section 6.1.

Aside from the case when γ_+ and γ_- represent the same multiply covered Reeb orbit of even Conley-Zehnder index, this Proposition 6.4 follows immediately from the criterion of Theorem 5.7 and Theorem 5.5 of Section 5. We can directly obtain regularity for the trivial cylinder in a symplectization by appealing to Lemma 2.4 in [Sa99]. We may appeal to the methods of Salamon in this setting because all compatible almost complex structures associated to a symplectization are $\mathbb{R}$-invariant.

6.1 Dynamics associated to dynamically separated contact forms

In this section we study finite energy pseudoholomorphic cylinders of both the multiply covered and somewhere injective variety. We are interested in the cylinders which interpolate between closed nondegenerate Reeb orbits associated to dynamically separated contact forms, of index difference 1 or 2. We will show that once we have chosen J generically, these satisfy the automatic transversality requirements of Theorem 5.7. As a result we can conclude that these cylinders will be regular and that Theorem 5.5 is applicable, prescribing the necessary structure to these moduli spaces. This gives us the proofs of Propositions 6.3 and 6.4 and is the first step in ensuring that $\partial^2 = 0$.

We begin with a few words on terminology and notation. From now on we will refer to such finite energy pseudoholomorphic entities merely as multiply covered cylinders or simple cylinders of index 1 or 2. Technically these cylinders do not have Fredholm index 1 or 2, though the (virtual) dimension of $\mathcal{M}(\gamma_+;\gamma_-)$ will be 1 or 2 respectively. To avoid an arithmetic headache we will use the terminology unreduced (Fredholm) index of a curve, which we denoted by $\widetilde{\text{ind}}(u;c)$. Recall that the unreduced index was defined in (5.5) by

$$\widetilde{\text{ind}}(u;c) = \text{ind}(u;c) - \dim(\text{Aut}(\dot{\Sigma}))$$

and it agrees with the virtual dimension of the moduli space of unparametrized asmyptotically cylindrical pseudoholomorphic curves, $\mathcal{M}^c$. See also the discussion in Remark 5.4 in regard to this technicality.

The finite energy condition is the one discussed in Section 3.3, and we refer to asymptotically cylindrical pseudoholomorphic cylinders as finite energy cylinders. Unless otherwise specified an orbit will be assumed to be nondegenerate and associated to a Reeb vector field generated by a dynamically separated contact form. By index difference 1 or 2 we mean that the difference in the Conley-Zehnder index of the orbit corresponding to $+\infty$ and the Conley-Zehnder index of the orbit corresponding to $-\infty$ is 1 or 2.

The precise correspondence of the orbits to $\pm\infty$ is discussed at length in Section 3.4, but we briefly recall the basics. Let $u \in \mathcal{M}(\gamma_+;\gamma_-)$. We denote its components in the symplectization $(\mathbb{R}\times M, d(e^\tau\alpha), \tilde{J})$ by

$$u := (a,f) : (\mathbb{R}\times S^1, j) \to (\mathbb{R}\times M, \tilde{J}).$$

Additionally in the C^∞ topology, we require

$$\lim_{s\to+\infty} a(s,t) = +\infty, \quad \lim_{s\to-\infty} a(s,t) = -\infty,$$

and

$$\lim_{s\to+\infty} f(s,t) = \gamma_+(T_+t), \quad \lim_{s\to-\infty} f(s,t) = \gamma_-(T_-t),$$

where T_+ is the period of the Reeb orbit γ_+ and T_- is the period of the Reeb orbit γ_-.

Recall that we defined a contact form α to be **dynamically separated** in Definition 1.2; namely that the following conditions are satisfied for any closed Reeb orbits associated to R_α:

(i) $3 \le \mu_{CZ}(\gamma) \le 5,$ for all closed simple contractible Reeb orbits γ.

(ii) $\mu_{CZ}(\gamma^k) = \mu_{CZ}(\gamma^{k-1}) + 4,$ where γ^j is the j-th iterate of a simple orbit γ.

Remark 6.5. One can relax condition (i) in the case of noncontractible orbits to allow for simple noncontractible orbits γ_a in the same free homotopy class a to satisfy either $1 \leq \mu_{CZ}(\gamma_a) \leq 3$ or $3 \leq \mu_{CZ}(\gamma_a) \leq 5$. The results of the following section are only needed for contractible orbits as they involve excluding the breaking phenomenon, which is why we can relax (i), provided the results of this section still hold.

We include the following illustrations, Figures 3 and 4 to help us keep track of the multiplicities of the ends in relation to an underlying simple cylinder. Note that depending on the multiplicities of the orbits and existence of a covering map, a finite energy cylinder interpolating between γ_+^ℓ and γ_-^d may or may not be a multiply covered pseudoholomorphic curve.

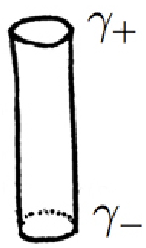


Figure 3: A simple cylinder.

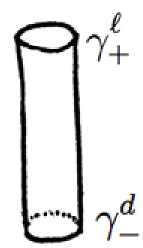


Figure 4: The plot thickens.

With the notation and terminology established, we proceed in our demonstration that the first two main items of Proposition 6.4 follow from imposing the condition that the contact form be dynamically separated. Recall that these are

1. We have $\ell \geq d \geq 1$ and $\widetilde{\text{ind}}(u) = \mu_{CZ}(\gamma_+^\ell) - \mu_{CZ}(\gamma_-^d) \geq 0$

2. All curves $u \in \mathcal{M}(\gamma_+^\ell; \gamma_-^d)$ are regular.

Note that Item (3) of Theorem 6.4 follows immediately from (1) and (2) with the help of Theorem 5.5. As a reminder, the inequality we are interested in establishing is that of (5.10),

$$\widetilde{\text{ind}}(u;c) > \#\Gamma_0(c) - 2 + 2Z(du).$$

For the remainder of this section we will assume that the assumptions of Proposition 6.4 are in effect.

First we recall Corollary 3.17 from Wendl in regard to the quantity $Z(du)$ which has been defined in (5.7). It may be thought of as a simple version of the folk theorem which states that generically, spaces of pseudoholomorphic curves with at least a certain number of criticial points have positive codimension.

Corollary 6.6. *For generic J, all somewhere injective curves $u \in \mathcal{M}$ satisfy*

$$2Z(du) \leq \widetilde{\text{ind}}(u;c).$$

Thus in our setting somewhere injective pseudoholomorphic curves of reduced index 0 or 1 are necessarily immersed for generic J, as $Z(du) = 0$. However, we will need to consider what can happen to finite energy pseudoholomorphic cylinders of index 2 separately. In the case where the finite energy cylinder is somewhere injective, we immediately obtain the following as a result of Corollary 6.6.

Lemma 6.7. *For generic J, a somewhere injective finite energy pseudoholomorphic cylinder of index 2 satisfies*

$$2Z(du) \leq 2.$$

This lemma applies to all somewhere injective curves, which includes the case where the orbits may be multiply covered, provided the cylinder is still simple, i.e. there exists no biholomorphism of the source such that the cylinder factors through a covering of simple cylinder.

The following lemma details the relationship between $Z(du)$ of a multiple cover of a asymptotically cylindrical pseudoholomorphic curve and the underlying simple pseudoholomorphic curve. Let u be a multiply covered pseudoholomorphic curve. We can write it as a composition

$$u = v \circ \varphi,$$

where v is somewhere injective and φ is a holomorphic covering such that $\varphi : (\mathbb{R} \times S^1, j) \to (\mathbb{R} \times S^1, j)$ with $\deg(\varphi) > 1$. We will see that this expression allows us to realize $Z(du)$ as the ramification number of φ.

Lemma 6.8. *Let u be a multiply covered embedded pseudoholomorphic cylinder, as above. Then*

$$Z(du) = Z(d\varphi).$$

Proof. Recall that we can write u as the composition

$$u = v \circ \varphi,$$

where v is embedded and φ is a holomorphic covering of the source of u. Since v is embedded we have that $Z(dv) = 0$. Then it follows from the chain rule that the critical points of u can only arise from branch points of the cover φ, hence $Z(du) = Z(d\varphi)$. □

This interpretation of $Z(du)$ in relation to multiply covered curves useful, as it yields the following result in regard to the quantity $Z(du)$ associated to any multiple cover of a finite energy cylinder of reduced index 2.

Lemma 6.9. *For generic J, any finite energy pseudoholomorphic cylinder u associated to a dynamically separated contact form of reduced index 2 satisfies*

$$2Z(du) \leq 2.$$

Proof. We know that u is either somewhere injective or the multiple cover of a somewhere injective cylinder of reduced index 2. We will proceed to prove the lemma by considering these two cases. If u is somewhere injective this follows from Lemma 6.7.

If u is not somewhere injective, then u is the multiple cover of a somewhere injective finite energy cylinder v, i.e.

$$u = v \circ \varphi,$$

with φ as before.

We note that a holomorphic covering of a cylinder $(\mathbb{R} \times S^1, j)$ cannot have any critical points. This is because in order to obtain a k-fold cover of a simple finite energy cylinder, where $k > 1$ the covering φ of the source must be of the form $\varphi(t, \theta) = (t, \theta/k)$ for $(t, e^{\theta}) \in \mathbb{R} \times S^1$ up to reparametrization of S^1. Here the θ values 0 and 2π are identified to obtain the circle. Then from Lemma 6.8 we know that

$$Z(du) = Z(d\varphi) = 0.$$

Moreover, we have that v has at most 1 critical point of order 1 because of Lemma 6.7 and how the quantity $Z(du)$ has been defined. The chain rule, combined with the fact that φ must be of the described form tells us that the critical points of u and their order must coincide with those of v. Thus $2Z(du) \leq 2$. □

Remark 6.10. We note that if all the finite energy pseudoholomorphic cylinders that are not somewhere injective are multiple covers of embedded finite energy cylinders, then the proof of the above lemma can be greatly simplified. This is because the dynamically separated condition forces u to be a multiple cover of a finite energy cylinder interpolating between simple orbits. To see this, recall that the dynamically separated assumption means that for any simple orbit γ we have $3 \leq \mu_{CZ}(\gamma) \leq 5$ and the Conley-Zehnder index increases uniformly by 4 each time we cover γ. Thus the only way to obtain

$$\mu_{CZ}(\gamma_+^\ell) - \mu_{CZ}(\gamma_-^d) = 2$$

is for $\ell = d$ because of the uniform increase by 4 of the Conley-Zehnder index under iteration and for $\mu_{CZ}(\gamma_+) = 3$ and $\mu_{CZ}(\gamma_-) = 5$. Provided this finite energy cylinder is not somewhere injective, it must be a multiple cover of a finite energy cylinder in $\mathcal{M}(\gamma_+, \gamma_-)$ where γ_+ and γ_- are simple.

This allows us to appeal to Proposition 3.7 of [Hu2] to conclude that a finite energy pseudoholomorphic cylinder of reduced index 2 which limits only on simple Reeb orbits is embedded. However this cannot be guaranteed to be the case in general, but may be of interest in specialized geometric settings.

The next step is to demonstrate that after selecting J generically we can exclude all cylinders of negative index, *including cylinders which are multiply covered.* This is possible because of the behavior of Conley-Zehnder indices prescribed to orbits associated to dynamically separated contact forms. We may think of the following lemma as saying that no matter how many times we travel around in circles, the only multiply covered cylinders encountered continue to be of reduced index 0, 1, or 2 respective to the index difference of underlying somewhere injective cylinder. This is an important result, as it means there are no badly iterated multiply covered cylinders, and is a crucial ingredient in proving Theorem 6.1.

We remark that this lemma yields a proof of Proposition 6.3, which stated that after a generic choice of J, the only reduced index 0 cylinders that remain are necessarily trivial.

Lemma 6.11. *For a generic choice of J, all finite energy nontrivial J-holomorphic cylinders in the symplectization of a dynamically separated contact manifold have positive reduced index. Moreover, the only reduced index 0 cylinders are trivial cylinders.*

Proof. We begin by noting that for simple orbits γ_+ and γ_-,

$$\mu_{CZ}(\gamma_+^\ell) = \mu_{CZ}(\gamma_+) + 4(\ell - 1), \tag{6.1}$$

and likewise,

$$\mu_{CZ}(\gamma_-^d) = \mu_{CZ}(\gamma_-) + 4(d - 1). \tag{6.2}$$

If we are only considering contractible simple orbits, we have

$$3 \leq \mu_{CZ}(\gamma_+), \mu_{CZ}(\gamma_-) \leq 5. \tag{6.3}$$

We include a remark after this proof to clarify the allowable numerics for noncontractible orbits.

Since we have started with γ_+ and γ_- as simple orbits we know that any finite energy $u \in \mathcal{M}(\gamma_+; \gamma_-)$ is simple, provided γ_+ and γ_- are distinct. Moreover for p and q coprime, any $u \in \mathcal{M}(\gamma_+^p; \gamma_-^q)$ is simple. Then Dragnev's results in [Dr04], specifically Corollary 4.15, allow us to conclude that $\mathrm{ind}(u) \geq 1$.

In particular this means for $p \geq q$, we have

$$\mu_{CZ}(\gamma_+^p) - \mu_{CZ}(\gamma_-^q) = \mu_{CZ}(\gamma_+) - \mu_{CZ}(\gamma_-) + 4(p - q), \tag{6.4}$$

and that (6.6) tells us $\sup\left(\mu_{CZ}(\gamma_+) - \mu_{CZ}(\gamma_-)\right) = 2$. Thus $\mu_{CZ}(\gamma_+^p) - \mu_{CZ}(\gamma_-^q) \geq 0$ if and only if $p \geq q$.

Moreover, from the above (6.4) we see that

$$\mu_{CZ}(\gamma_+^p) - \mu_{CZ}(\gamma_-^q) = 0$$

if and only if

$$\mu_{CZ}(\gamma_+) - \mu_{CZ}(\gamma_-) + 4(p - q) = 0.$$

Thus we would need $\mu_{CZ}(\gamma_+) = \mu_{CZ}(\gamma_-)$ and $p = q$. Next we explain why the equality of the Conley-Zehnder indicies implies that after a generic choice of J the only $u \in \mathcal{M}(\gamma_+^p; \gamma_-^q)$ must be trivial, provided $\mu_{CZ}(\gamma_+^p) - \mu_{CZ}(\gamma_-^q) = 0$.

Note that γ_+ and γ_- are simple, so any $u \in \mathcal{M}(\gamma_+; \gamma_-)$ must be somewhere injective, thus Dragnev's results are applicable. Thus after a generic choice of J, the only way to have $\mu_{CZ}(\gamma_+) = \mu_{CZ}(\gamma_-)$ and $\mathcal{M}(\gamma_+; \gamma_-) \neq \emptyset$ is for γ_+ and γ_- to be the same orbit. We see after a generic choice of J that $p = q$ and $\mu_{CZ}(\gamma_+) = \mu_{CZ}(\gamma_-)$. Thus after a generic choice of J the only finite energy cylinders which may persist must be trivial cylinders, as γ_+ and γ_- are the same orbit and $p = q$. Note that this argument completes the proof of Proposition 6.3.

To prove that all multiply covered cylinders have non-negative reduced index we appeal to the uniformity of the increase of the Conley-Zehnder of Reeb orbits associated to dynamically separated contact forms. The Reeb orbits associated to multiply covered cylinders must be a multiple cover a simple orbit. This means that any orbits can be expressed as γ_+^ℓ and γ_-^d for some $\ell, d > 1$.

From the prime factorization of integers we know that any cylinder can be obtained as the multiple cover of some simple cylinder, requires $\ell = ap$ and $d = aq$ for some $a \in \mathbb{N}$. Since we have chosen J generically, we know that $p \geq q$. Note that if we are interested in nontrivial cylinders only, if $p = q$ then $\mu_{CZ}(\gamma_+) \neq \mu_{CZ}(\gamma_-)$. Appealing again to the formulas (6.1) and (6.2) in the same fashion of (6.4) we obtain that any multiply covered cylinder u has $\widetilde{\text{ind}}(u) \geq 0$, with $\widetilde{\text{ind}}(u) = 0$ only in the case that u is a trivial cylinder. $\square$

Remark 6.12. The analysis for noncontractible orbits follows identically, provided that we are in an analogous situation with

$$1 \leq \mu_{CZ}(\gamma_a) \leq 3. \tag{6.5}$$

or

$$3 \leq \mu_{CZ}(\gamma_a) \leq 5. \tag{6.6}$$

This is the reason we can somewhat relax the requirement in the definition of dynamically separated for noncontractible orbits.

With the preceding two lemmas in place, we can finish proving Proposition 6.4 as follows.

Proof. As previously mentioned, it suffices to demonstrate for all multiply covered cylinders of index difference 1 or 2 the following inequality (5.10) holds,

$$\widetilde{\operatorname{ind}}(u; c) > \#\Gamma_0(c) - 2 + 2Z(du).$$

Recall that this is equivalent to the curve u satisfying the conditions for automatic transversality as given in Theorem 5.7, thus we may conclude it is regular even though it may be multiply covered.

If J has been chosen generically, then Corollary 6.6 implies that the quantity $Z(du) = 0$ for all index 1 cylinders and Lemma 6.9 implies that $Z(du)$ is at most 1 for finite energy cylinders of index 2.

We are only interested in finite energy asymptotically cylindrical pseudoholomorphic curves limiting on nondegenerate Reeb orbits, so the subset $\Gamma_0(c) \subset \Gamma$ consists of punctures for which the asymptotic orbit has even Conley-Zehnder index. Condition (i) in the definition of dynamically separated tells us that for all simple closed contractible Reeb orbits γ, the Conley-Zehnder index satisfies $3 \leq \mu_{CZ}(\gamma) \leq 5$. Each time we iterate we add 4 so for any nontrivial cylinder we have at most 1 puncture of even Conley-Zehnder index associated to a finite energy cylinder of index 1 and no punctures of even Conley-Zehnder index associated to an finite energy cylinder of index 2.

Recall that J has been chosen generically, so Lemma 6.11 tells us that for all nontrivial cylinders u,

$$\widetilde{\text{ind}}(u; c) \geq 1.$$

The proof of Proposition 6.4 is then complete, as $1 > -1$ and $2 > -2 + 2 = 0$. □

6.2 Numerics of branched covers

In this section we will exclude all branched covers of multiply covered asymptotically cylindrical pseudoholomorphic curves from having nonpositive index that are be an obstruction to obtaining a homology theory from the contact chain groups. This is accomplished by appealing to the uniform increase in Conley-Zehnder index of dynamically separated contact forms and the bounds on the index of the simple orbits. These results will allow us to prove Theorem 6.1 in the following section.

Before we delve into the numerics of all possible branched covers of a multiply covered cylinder we discuss the numerics of a "pair of pants" in detail first. We will use the same notation as before, and the setup is given as follows. Denote by γ_+^ℓ the ℓ-fold cover of a simple orbit γ_+ and γ_-^d the d-fold cover of a simple orbit γ_- with $\mu_{CZ}(\gamma_+) \geq \mu_{CZ}(\gamma_-)$. An arbitrary branched cover with two negative punctures of a curve $u \in \mathcal{M}(\gamma_+^\ell; \gamma_-^d)$

belongs to the moduli space

$$\mathcal{M}\left(\gamma_+^{\ell(k_1+k_2)};\gamma_-^{dk_1},\gamma_-^{dk_2}\right),$$

where $k_1,\ k_2 \in \mathbb{Z}_{>0}$. The following illustration is helpful to visualize the geometry of such a configuration.

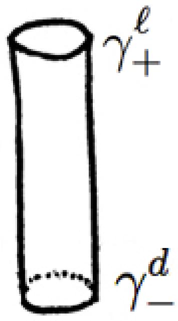


Figure 5: The underlying cylinder.

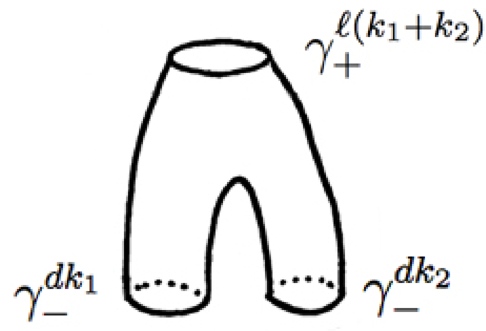


Figure 6: A branched cover.

Proposition 6.13. *Let α be a nondegenerate dynamically separated contact form associated to a 3-manifold M, with J a generic compatible almost complex structure. Let γ_+ and γ_- be simple distinct closed contractible Reeb orbits associated to α. Then*

$$\begin{aligned}\dim \mathcal{M}(\gamma_+^{\ell(k_1+k_2)},\gamma_-^{dk_1},\gamma_-^{dk_2}) &= 1+\mu_{CZ}\left(\gamma_+^{\ell(k_1+k_2)}\right)-\mu_{CZ}\left(\gamma_-^{dk_1}\right)-\mu_{CZ}\left(\gamma_-^{dk_2}\right)\\ &\geq 0,\end{aligned}$$

with equality only in the case that $\gamma_+ = \gamma_-$ and $\mu_{CZ}(\gamma_+) = \mu_{CZ}(\gamma_-) = 5$.

The proof of this crucial fact involves some tedious arithmetic, which we present in full detail. We proceed by unraveling the expressions for the Conley-Zehnder indices in terms of the number of iterations and the Conley-Zehnder index of the underlying simple Reeb orbits, remembering that $\mu_{CZ}(\gamma^k) = \mu_{CZ}(\gamma^{k-1}) + 4$.

Remark 6.14. In the case that the underlying simple closed orbits γ_+ and γ_- are not distinct then it is possible for dim $\mathcal{M}(\gamma_+^{\ell(k_1+k_2)}, \gamma_-^{dk_1}, \gamma_-^{dk_2}) = 0$. We saw this in the ellipsoid. However, we will not have to worry about such configurations obstructing in the proof of Theorem 6.1 due to the uniform increase in index increase by 4. More details are given in Section 6.3.

Proof. Let us label

$$\mu_{CZ}\left(\gamma_+^{\ell(k_1+k_2)}\right) - \mu_{CZ}\left(\gamma_-^{dk_1}\right) - \mu_{CZ}\left(\gamma_-^{dk_2}\right). \tag{6.7}$$

We begin by noting that

$$\mu_{CZ}(\gamma_+^{\ell}) \;=\; \mu_{CZ}(\gamma_+) + 4(\ell - 1).$$

Since

$$\mu_{CZ}\left(\gamma_+^{\ell(k_1+k_2)}\right) \;=\; \mu_{CZ}(\gamma_+) + 4(\ell(k_1+k_2) - 1),$$

we obtain

$$\mu_{CZ}\left(\gamma_+^{\ell(k_1+k_2)}\right) \;=\; \mu_{CZ}(\gamma_+^{\ell}) + 4\ell(k_1+k_2-1). \tag{6.8}$$

Also,

$$\begin{aligned} \mu_{CZ}(\gamma_-^{dk_i}) &= \mu_{CZ}(\gamma_-) + 4(dk_i - 1) \\ &= \mu_{CZ}(\gamma_-^{d}) + 4d(k_i - 1). \end{aligned} \tag{6.9}$$

So we can write

$$\begin{aligned} (6.7) &= 1 + \mu_{CZ}(\gamma_+^{\ell}) + 4\ell(k_1+k_2-1) - 2\mu_{CZ}(\gamma_-^{d}) - 4d(k_1-1) - 4d(k_2-1). \\ &= 1 + 4(\ell - d)(k_1+k_2-1) + 4d + \mu_{CZ}(\gamma_+^{\ell}) - 2\mu_{CZ}(\gamma_-^{d}). \end{aligned}$$

As a result of Proposition 6.4 we have that

$$\mu_{CZ}(\gamma_+^{\ell}) \;\geq\; \mu_{CZ}(\gamma_-^{d}),$$

We have that (6.7) is the least when $\ell = d$, since $\ell \geq d$ (see Proposition 6.4). So we need to make sure that

$$1 + 4d + \mu_{CZ}(\gamma_+^d) - 2\mu_{CZ}(\gamma_-^d) \quad > \quad 0. \tag{6.10}$$

If γ_+^d and γ_-^d are distinct and their CZ indices agree, then generically the original underlying cylinder connecting γ_+ and γ_- cannot exist for a generic choice of J by Dragnev's results (see Proposition 6.3). Thus in checking (6.10) holds we can take $\mu_{CZ}(\gamma_+^d) - \mu_{CZ}(\gamma_-^d) = 1$. Here we take the infimum and supremum of the Conley-Zehnder indices over all contractible Reeb orbits, remembering they all have underlying simple orbits with Conley-Zehnder index3, 4, or 5 as in my definition of dynamically separated. Keeping (6.9) in mind, we obtain

$$\begin{aligned}
\inf(6.7) &= 1 + 4d + 1 - \sup(\mu_{CZ}(\gamma_-^d)) \\
&= 1 + 4d + 1 - (5 + 4(d-1)) \\
&= 1 \\
&> 0.
\end{aligned}$$

In the case that γ_- and γ_+ are the same orbit and $\mu_{CZ}(\gamma_-) = \mu_{CZ}(\gamma_+) = 5$ we see that we obtain

$$1 + 4d + \mu_{CZ}(\gamma_+^d) - 2\mu_{CZ}(\gamma_-^d) = 0.$$

In the case that $\mu_{CZ}(\gamma_-) = \mu_{CZ}(\gamma_+) = 3$ or 4, it is easy to see that

$$1 + 4d + \mu_{CZ}(\gamma_+^d) - 2\mu_{CZ}(\gamma_-^d) > 0.$$

Thus the proposition is proven.

□

Now we can move onto considering more general branched covers, where there may be an arbitrary number of negative ends.

Proposition 6.15. *After a generic choice of J, any branched cover of an asymptotically cylindrical pseudoholomorphic curve of positive index with one positive puncture associated to a dynamically separated contact form has positive index.*

Next we will want to consider the case when we take a branched cover with only one positive puncture of a pseudoholomorphic curve coming from a moduli space of dimension 0. From Proposition 6.13 we know that this means the underlying pseudoholomorphic curve associated to this branched cover must be an element of

$$\mathcal{M}(\gamma^{\ell(k_1+k_2)}, \gamma^{dk_1}, \gamma^{dk_2}),$$

with $\mu_{CZ}(\gamma) = 5$. This situation is dealt with the following corollary, whose proof follows by combining the preceding proofs of Propositions 6.13 and 6.15.

Corollary 6.16. *After a generic choice of J any branched cover of a pseudoholomorphic curve of index 0 associated to a dynamically separated contact form belongs to the following moduli space*

$$\mathcal{M}(\gamma^{\ell(k_1+...+k_s)}; \gamma^{d_1k_1}, ..., \gamma^{d_sk_s})$$

with $\mu_{CZ}(\gamma) = 5$.

We will see in the following section that these "exceptional" asymptotically cylindrical pseudoholomorphic curves do not interfere with the proof of Theorem 6.1 because of the uniform increase by 4 of the Conley-Zehnder index in the definition of dynamically separated.

First we set up some notation for the proof. Let the following moduli space have positive dimension,

$$\mathcal{M}_{\text{pos}} = \mathcal{M}(\gamma_+^\ell; \gamma_{\beta_1}^{d_1}, ..., \gamma_{\beta_r}^{d_r}).$$

We are interested in the moduli space consisting of branched covers of $\mathcal{M}_{\text{pos}}$, which we denote by

$$\mathcal{M}_{\text{branch}} := \mathcal{M}(\gamma_+^{\ell(k_1+...+k_s)}; \gamma_{\beta_1}^{d_1k_1}, ..., \gamma_{\beta_s}^{d_sk_s}).$$

Here $\gamma_+, \gamma_{\beta_i}$ are all simple Reeb orbits, which are not necessarily distinct. Note that $s > r$ and that $\#(\{\text{distinct } \beta_i\}_{i\in\{1,..s\}}) \leq \#(\{\text{distinct } \beta_i\}_{i\in\{1,..r\}})$, as the only simple Reeb orbits appearing in $\mathcal{M}_{\text{branch}}$ must have come from $\mathcal{M}_{\text{pos}}$.

A slightly more horrible numerical argument identical to the methods used to prove Proposition 6.13 yields the desired results. We will still go through the details, so as to not unduly burden the skeptical reader.

Proof. We begin by noting that

$$\dim \mathcal{M}_{\text{pos}} = (-1)(1-r) + \mu_{CZ}(\gamma_+^\ell) - \sum_{i=1}^{r} \mu_{CZ}(\gamma_{\beta_i}^{d_i})$$

and

$$\dim \mathcal{M}_{\text{branch}} = (-1)(1-s) + \mu_{CZ}(\gamma_+^{\ell(k_1+...+k_s)}) - \sum_{i=1}^{r} \mu_{CZ}(\gamma_{\beta_i}^{d_ik_i}).$$

Then the equations (6.8) and (6.9) from the proof of Proposition 6.13 allow us to express $\dim \mathcal{M}_{\text{branch}}$ as

$$\begin{aligned} \dim \mathcal{M}_{\text{branch}} \;=\; & (s-1) \;+\; \mu_{CZ}(\gamma_+^\ell) - \sum_{i=1}^{s} \mu_{CZ}(\gamma_{\beta_i}^{d_i}) + \\ & 4\ell(k_1 + ... + k_s - 1) \;-\; 4\sum_{i=1}^{s} \; d_i(k_i - 1). \end{aligned} \tag{6.11}$$

Note that for all i, $\ell \geq d_i$ as this follows from the reasoning given in Proposition 6.13 and the assumptions that the underlying branched cover is positive and that J has been chosen generically. Thus it suffices to consider the case where $\ell = d_i$.

Note that when $\ell = d_i$, we have $-4\ell + 4\sum_{i=1}^{s} d_i = 4\ell(s-1)$. Thus we need to make sure that

$$s - 1 + \mu_{CZ}(\gamma_+^\ell) - \sum_{i=1}^{s} \mu_{CZ}(\gamma_{\beta_i}^\ell) + 4\ell(s-1) > 0. \tag{6.12}$$

There are two cases to consider. The first is where there is a γ_{β_i} distinct from γ_+. In this case we can assume that for some j, $\mu_{CZ}(\gamma_+^\ell) - \mu_{CZ}(\gamma_{\beta_j}^\ell) = 1$. Now checking (6.12) amounts to observing that

$$\begin{aligned}
\inf(6.11) &= (s-1) + 1 + (1-s)\sum_{i=1}^{s-1} \sup(\mu_{CZ}(\gamma_{\beta_i}^\ell)) + 4\ell(s-1) \\
&= (s-1) + 1 + (1-s)(5 + 4(\ell - 1)) + 4\ell s - 4 \\
&= (s-1) + 1 + 5 + 4\ell - 4 - 5s - 4\ell s + 4s + 4\ell s - 4\ell \\
&= 5 - 4 \\
&= 1 \\
&> 0
\end{aligned}$$

The other case to consider is when all of the γ_{β_i} and γ_+ are actually the same orbit, which we will denote by γ. Since the underlying curve must belong to $\mathcal{M}_{\text{pos}}$ we know that the only possibilites for $\mu_{CZ}(\gamma)$ are 3 or 4 from the previous proposition. In either of these cases the above argument shows that $\inf(6.11) \geq 1$.

□

Remark 6.17. We note that a uniform increase of 2 or 3 fails to yield the proofs of Propositions 6.13 and 6.15, which can be checked directly via the above numerics. On the other hand, a uniform increase by any integer larger than 4 also yields proofs of Propositions 6.13 and 6.15. Even in these cases we must still restrict the simple contractible orbits γ to have $\mu_{CZ}(\gamma) = 3$, 4, or 5 so that the regularity results of Section 6.1 are still applicable.

6.3 Overcoming obstructions

In this section we complete the proof of Theorem 6.1. Recall that we denote x and z and to be nondegenerate Reeb orbits associated to a dynamically separated contact form, with $\mu_{CZ}(x) - \mu_{CZ}(z) = 2$. We denote the set of all Reeb orbits associated to a dynamically separated contact form in the same free homotopy class by $\mathscr{P}$. To prove Theorem 6.1 we need to show that the only possible configurations for breaking of a finite energy pseudoholomorphic cylinder limiting on Reeb orbits of index difference 2 are into two broken cylinders, as in Figure 7.

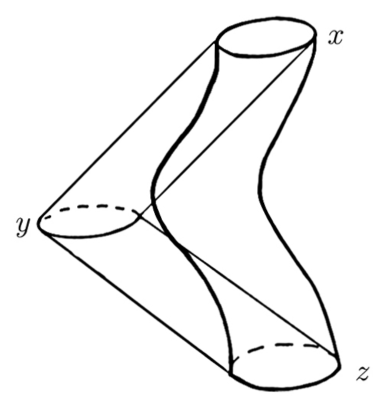


Figure 7: Desired limiting behavior for $u \in \hat{\mathcal{M}}(x; z)$

We will show that the results of the previous section, Propositions 6.13 and 6.15 allow us to appeal to the Symplectic Field Theory compactness results of [**?**] to conclude that the compactification of $\hat{\mathcal{M}}(x; z)$ is obtained by including only the "broken cylinders," namely pairs of curves $(\mathcal{C}_u, \mathcal{C}_v) \in \hat{\mathcal{M}}(x;y) \times \hat{\mathcal{M}}(y;z)$. We denote this space as follows

$$\overline{\mathcal{M}}(x;z) := \hat{\mathcal{M}}(x;z) \ \cup \bigcup_{\substack{y \in \mathscr{P} \\ \mu_{CZ}(y)=\mu_{CZ}(x)-1}} \hat{\mathcal{M}}(x;y) \times \hat{\mathcal{M}}(y;z).$$

As an immediate consequence, we obtain the following inclusion of moduli spaces of finite energy pseudoholomorphic cylinders interpolating between these orbits,

$$\partial\overline{\mathcal{M}}(x;z) \ \subset \bigcup_{\substack{y \in \mathscr{P} \\ \mu_{CZ}(y)=\mu_{CZ}(x)-1}} \hat{\mathcal{M}}(x;y) \times \hat{\mathcal{M}}(y,z). \tag{6.13}$$

The inclusion of moduli spaces necessary to the construction of a Morse or Floer homology theory, as it is a crucial ingredient in demonstrating the $\partial^2 = 0$ or obtaining the chain homotopy equation in the proof of invariance. The presence of moduli spaces of nonpositive virtual dimension obstructs the construction of cylindrical contact homology, as we can no longer obtain the inclusion in (6.13) of compactified moduli spaces. While we will not discuss invariance in this paper, we note that in this case we will be interested in understanding the degenerations of a moduli space whose curves interpolate between nondegenerate Reeb orbits of index difference 1.

We can explicitly see how (6.13) can fail to hold as follows. The compactness arguments of Bourgeois, Eliashberg, Hofer, Wysocki and Zehnder, allow for finite energy pseudoholomorphic cylinders interpolating between Reeb orbits of index difference 1 or 2 to break into buildings of heigh 2 which could consist of a pair of pants, plane, and cylinder, as in Figure 8. In this setting the pseudoholomorphic building can only have one top level puncture because of the maximum principle as described in Proposition 3.6. However as we cannot rule out the possibility of a minimum appearing in the $\mathbb{R}$ component of the symplectization, as in Figure 9. This is the reason that the language

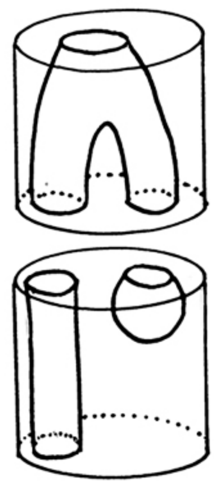


Figure 8: Building of height 2 in a symplectization.

of pseudoholomorphic buildings was introduced to explain the structure of compactified asymptotically cylindrical moduli spaces.

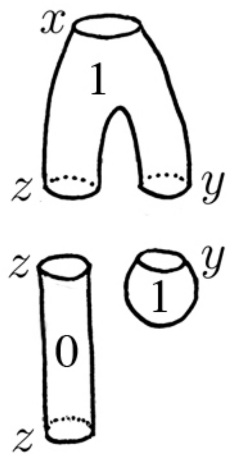
Figure 9: Developing a minimum

It is important to note that this breaking behavior can occur in the compactification of $\hat{\mathcal{M}}(x, z)$ even after excluding contractible orbits of degree -1, 0, 1, as in the assumptions of Conjecture 1.1. This requirement is not sufficient for the purposes of constructing a chain complex without the availability of an abstract perturbation package. We will explain this pictorially, as a dense block of text may obscure the elementary properties of addition under consideration. We begin by explaing the necessity of the assumption that there be no contractible orbits of grading $|\gamma| = -1, 0, 1$, working in a 4 dimensional symplectization.

It can be hard to keep of the -1 when discussing the boundary of a compactified moduli space $\hat{\mathcal{M}}$ that is 1-dimensional, as one may have numerous building components associated to the degeneration of such a moduli space. We will discuss the numerics of boundary of the compactification in terms of the numerics associated to the virtual dimension of $\mathcal{M}$ so that we do not need to keep track of the extra -1, which would be associated to each component in a building of asymptotically cylindrical pseudoholomorphic curves. In order to compactify the moduli space $\hat{\mathcal{M}}(x, z) = \mathcal{M}(x, z)/\mathbb{R}$ we need to include buildings of height 2 consisting of a pair of pants, cylinder, and a plane. The sum of their unreduced virtual dimensions must be 2, as it must be equal to the index of $\mathcal{M}(x, z)$, which is 2.

Note that this is permissible because there is no fundamental geometric difference aside from the change to the virtual dimension formula. We refer to the **unreduced virtual dimension** of a moduli space to when we are working through virtual dimension computations before modding about by the external $\mathbb{R}$-action, and our computations are of the virtual dimension of $\mathcal{M}$ instead of $\hat{\mathcal{M}}$.

The degeneration in Figure 10 demonstrates how the presence of a Reeb orbit y with $\mu_{CZ}(y) = 2$, i.e. $|y| = 1$, since we are working with 3-dimensional contact manifolds, precludes $\partial^2 = 0$.



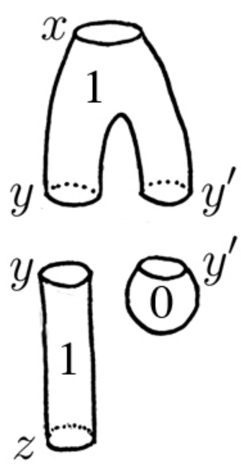
$$\dim \mathcal{M}(x; z, y) = 1 + 2 - \mu_{CZ}(y)$$

$$\dim \mathcal{M}(z; z) = 0$$

$$\dim \mathcal{M}(y; \emptyset) = -1 + \mu_{CZ}(y) = |y|$$

Figure 10: $\mu_{CZ}(y) = 2 \Rightarrow$ ☹

The degeneration in Figure 11 demonstrates how the presence of a Reeb orbit y with $\mu_{CZ}(y) = 1$, i.e. $|y| = 0$, precludes $\partial^2 = 0$.



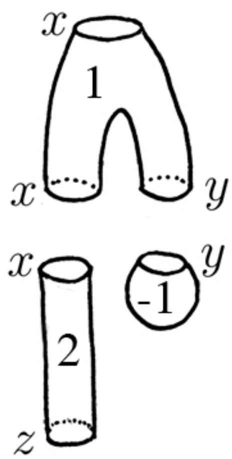
$$\dim \mathcal{M}(x; y, y') = 1 + 1 - \mu_{CZ}(y')$$

$$\dim \mathcal{M}(y; z) = 1$$

$$\dim \mathcal{M}(y'; \emptyset) = -1 + \mu_{CZ}(y') = |y'|$$

Figure 11: $\mu_{CZ}(y) = 1 \Rightarrow$ ☹

The degeneration in Figure 12 demonstrates how the presence of a Reeb orbit y with $\mu_{CZ}(y) = 0$, i.e. $|y| = -1$, precludes $\partial^2 = 0$.



$$\dim \mathcal{M}(x; x, y) = 1 - \mu_{CZ}(y) = |y|$$

$$\dim \mathcal{M}(x; z) = 2$$

$$\dim \mathcal{M}(y; \emptyset) = -1 + \mu_{CZ}(y) = |y|$$

Figure 12: $\mu_{CZ}(y) = 0 \Rightarrow$ ☹.

In addition, we see that we run into trouble if the reduced virtual dimension of any of the building components is negative. If the virtual dimension of $\mathcal{M}^c$ is 0 then this can only obstruct the desired inclusion when $\mathcal{M}^c$ is of the form $\mathcal{M}(x; y, y)$ such that the Reeb orbit y satisfies $\mu_{CZ}(y) = 3$, i.e. $|y| = 2$. This is precisely the sort of situation encountered in the following Example 5.1 with the ellipsoid. To ensure that all planes are of positive reduced virtual dimension, we must require $\mu_{CZ}(y) \geq 3$ for all nondegenerate contractible Reeb orbits as otherwise we cannot ensure that $\partial^2 = 0$ even if all other moduli spaces $\mathcal{M}(x; y_1, ..., y_s)$ have positive virtual dimension.

In addition, one also needs to consider the possibility of buildings which include moduli space of the form $\mathcal{M}(x; y_1, ..., y_s)$. This is because there is no reason branched covers with an arbitrary number of negative ends must also be excluded from having nonpositive virtual dimension. The following illustration of Figure 13 demonstrates a rather complicated compactification of a cylinder, which might arise if the cylinder developed several minima. Note that it is still homotopic to a cylinder.

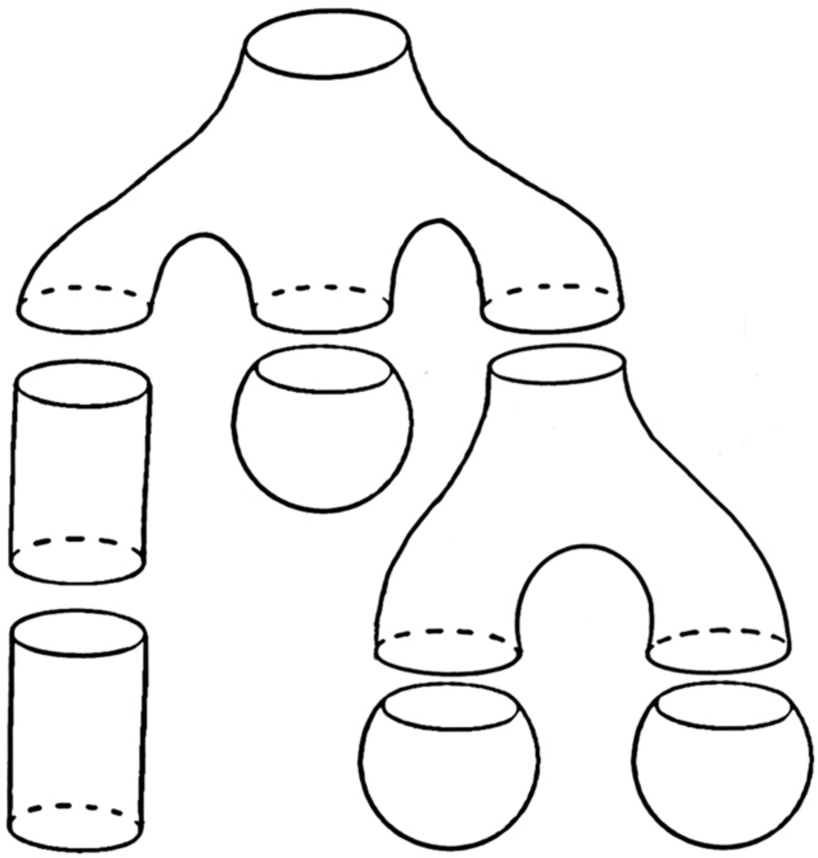


Figure 13: Egads!

Now that we understand the breaking phenomenon and what is entailed in proving the inclusion of (6.13), we can give the proof of Theorem 6.1 as follows.

Proof. We begin by noting that asymptotically cylindrical pseudoholomorphic curves with only one positive puncture at the top can be glued to obtain the buildings which would appear in the compactification of $\hat{\mathcal{M}}(x; z)$. The reasoning for is because we are only interested in counting cylindrical curves, and gluing together curves with multiple positive punctures at the top would give rise to a building of genus greater than 0. Pictorially we have the following figures to illustrate this.

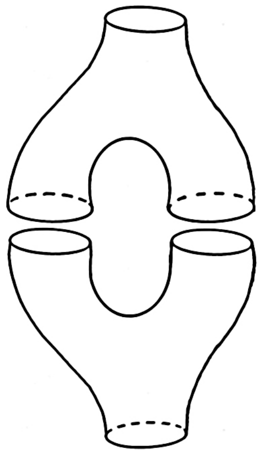


Figure 14: What if?

Figure 15: No donuts.

We see that Propositions 6.13 and 6.15 when combined with the requirement that all contractible closed Reeb orbits have Conley-Zehnder index at least 3 rule out the possibilities of a configuration as illustrated in Figures 10-12 or Figure 13 from occurring in the compactification. This is because we know the total unreduced virtual dimension of such a building sums to at least $2m$ where m is the number of of capping holomorphic planes. Thus there can only be at most one holomorphic plane and we would be in the configuration described in Figure 10. To see that this is not possible we appeal to Proposition 6.13. This tells us that the pair of pants has reduced dimension greater than 0 when $3 \leq \mu_{CZ}(\gamma) \leq 4$ and equal to 0 when $\mu_{CZ}(\gamma) = 5$. But now we have that $|\gamma| = 4$, which is also impossible. Thus we obtain the proof of Theorem 6.1.

□

Remark 6.18. In order to establish a cylindrical contact homology theory which includes noncontractible nondegenerate Reeb orbits as well, we only have to worry about excluding the existence of negative dimensional moduli spaces coming from branched covers of multiply covered curves which interpolate between the closed contractible Reeb orbits only. This is because the pair of pants breaking configuration, as in Figure 8, would only be homotopic to a cylinder when the capping discs are bounded by contractible Reeb orbits. As explained earlier, we still need the regularity results of Section 6.1 for all pseudoholomorphic cylinders which interpolate between Reeb orbits of index difference 1 or 2 in the same free homotopy class.

Chapter 7

A gallimaufry of gluings

The purpose of this chapter is to prove an analogue of Floer's gluing theorem theorem. Gluing, a converse to Gromov compactness, allows us to approximate a broken curve, consisting of two rigid pseudoholomorphic curves sharing a common orbit y with a nearby honest pseudoholomorphic curve. This procedure allows us to prove that the ends of the 1-dimensional compact moduli space $\overline{\mathcal{M}}(x;z)$ correspond exactly to pairs of rigid cylinders in $\hat{\mathcal{M}}(x;y) \times \hat{\mathcal{M}}(y;z)$. Such a theorem is crucial to proving that $\partial^2 = 0$.

In the case of closed pseudoholomorphic curves such a construction is well documented in Chapter 10 of [MSbig$J$], and is rigorously treated in Chapter 9 of [ADfloer] and Chapter 4 of [Sc95] for Hamiltonian Floer homology. Our methods and proof structure will be modeled after the arguments presented by Audin and Damian in [ADfloer], by adapting the analytic set up in regards to weighted Sobolev spaces from Schwarz [Sc95] as in Dragnev [Dr04]. Aside from the surjectivity of gluing, the arguments in the gluing construction follow immediately from the Hamiltonian Floer setting after minor necessary modifications, which are explained in Section 7.2.

The surjectivity arguments are the major difference between gluing in Hamiltonian Floer theory and contact homology, as the gluing construction in Hamiltonian Floer theory yields a unique honest pseudoholomorphic curve, whereas in contact homology it does not. This is due to the inclusion of multiply covered orbits, which allows us to change the parametrizations by rotation of the pseudoholomorphic solutions corresponding to the underlying cylinders in the pregluing construction. This gives rise to additional non-equivalent approximately pseudoholomorphic solutions in the pregluing construction, and yields the same number of non-equivalent honest pseudoholomorphic curves as well. The work of Hofer, Wysocki, and Zehnder [HWZ02] on pseudoholomorphic cylinders of small area provides the proper notions of convergence to obtain these results pertaining to the surjectivity for gluing. This will be extensively discussed in Section 7.3, which will lead to the final details in the proof of Theorem 7.3.

We remark that the lack of uniqueness due to reparametrization is the reason for the nonstandard coefficient in the expression of the differential for contact homology. This also appears in the geometry associated to the moduli spaces of interest and will be discussed in Section 7.4.

7.1 Outline of the gluing construction

Before stating the analogue of Floer's gluing theorem we fix some notation for the moduli spaces and pseudoholomorphic curves under consideration. Let γ_+ and γ_- be two closed Reeb orbits of index difference one or two, that is $\mu_{CZ}(\gamma_+) - \mu_{CZ}(\gamma_-) = 1,\ 2$. We denote the moduli space of pseudoholomorphic curves limiting on these orbits by $\mathcal{S}(\gamma_+;\gamma_-)$, and will refer to this as the space of **parametrized solutions**. We will typically denote pseudoholomorphic curves living in $\mathcal{S}(\gamma_+;\gamma_-)$ by the letters u and v.

Remark 7.1 (Splitting over free homotopy classes of Reeb orbits)**.** There is a splitting on the chain level of cylindrical contact homology over the free homotopy classes of loops. Throughout we will assume that we only consider moduli spaces of curves interpolating between Reeb orbits of the same free homotopy class.

The source of these pseudoholomorphic curves is a twice punctured sphere, or cylinder whose space of biholomorphisms is generated by rotations and translations, which is equivalent to an $S^1 \times \mathbb{R}$'s worth of action. Thus in order to obtain the space of **unparametrized trajectories**, which we will denote by $\mathcal{M}(\gamma_+;\gamma_-)$, we take

$$\mathcal{M}(\gamma_+;\gamma_-) := \mathcal{S}(\gamma_+;\gamma_-)/(S^1 \times \mathbb{R}).$$

We will use the notation $[u]$ to denote the equivalence class of a pseudoholomorphic curves $u \in \mathcal{S}(\gamma_+;\gamma_-)$, under biholomorphisms of the source. Recall that this equivalence relation was defined at the beginning of Section 4.3.

We note that in Hamiltonian Floer homology and symplectic homology one has typically already perturbed the $\bar{\partial}_{\tilde{J}}$ equation for the purposes of achieving transversality, resulting in an almost complex structure which is no longer invariant under the S^1 action. As a result there is only the $\mathbb{R}$ action coming from translations on the source and gluing is unique.

Recall that the target of our pseudoholomorphic curves is the symplectization of a contact manifold, thus the almost complex structure J on the target is by construction $\mathbb{R}$ invariant. There is an additional $\mathbb{R}$ action on $\mathcal{M}(\gamma_+;\gamma_-)$ by external translations $u = (a, f) \to (a + \rho, f)$, where $\rho \in \mathbb{R}$. After modding out by this $\mathbb{R}$ action we decorate a moduli space with a $\hat{}$, as follows

$$\hat{\mathcal{M}}(\gamma_+;\gamma_-) := \mathcal{M}(\gamma_+;\gamma_-)/\mathbb{R}.$$

We denote the equivalence class of pseudoholomorphic curves originally representing $u \in \mathcal{S}(\gamma_+;\gamma_-)$ by $\mathcal{C}_u$.

In order to compactify the space, $\hat{\mathcal{M}}(\gamma_+;\gamma_-)$ we must add in the breaking of pseudoholomorphic curves. By the numerology of Section 6 we were able to ensure that the only possible configurations for breaking of a pseudoholomorphic curve limiting on Reeb orbits of index difference two was into two broken cylinders, each of index difference 1. To be precise we take x, y, and z to be Reeb orbits associated to a nondegenerate dynamically separated contact form, each representing the same free homotopy class, such that $\mu_{CZ}(x) - \mu_{CZ}(z) = 2$. We denote the set of all Reeb orbits in the free homotopy class by $\mathscr{P}$. Then we compactify $\hat{\mathcal{M}}(x;z)$ by including all the broken cylinders, which are pairs of curves $(\mathcal{C}_u, \mathcal{C}_v) \in \hat{\mathcal{M}}(x;y) \times \hat{\mathcal{M}}(y;z)$. That these are the only curves needed to be included in the compactification was proven in Section 6. The compactified space is given by

$$\overline{\mathcal{M}}(x;z) := \hat{\mathcal{M}}(x;z) \ \cup \bigcup_{\substack{y\in\mathscr{P} \\ \mu_{CZ}(y)=\mu_{CZ}(x)-1}} \hat{\mathcal{M}}(x;y) \times \hat{\mathcal{M}}(y;z).$$

The computations of the previous section gave us the following inclusion,

$$\partial\overline{\mathcal{M}}(x;z) \ \subseteq \bigcup_{\substack{y\in\mathscr{P} \\ \mu_{CZ}(y)=\mu_{CZ}(x)-1}} \hat{\mathcal{M}}(x;y) \times \hat{\mathcal{M}}(y;z). \tag{7.1}$$

The object in proving an analogue of Floer's gluing theorem is to demonstrate that the reverse inclusion also holds. The formal statement is as follows.

Theorem 7.2 (Gluing)**.** *Let x, y, and z be three nondegenerate closed Reeb orbits representing the same free homotopy class a of consecutive index*

$$\mu_{CZ}(x) = \mu_{CZ}(y) + 1 = \mu_{CZ}(z) + 2,$$

and let $(u, v) \in \mathcal{S}(x;y) \times \mathcal{S}(y;z)$ be a parametrized solution representing the trajectory $(\mathcal{C}_u, \mathcal{C}_v) \in \hat{\mathcal{M}}(x;y) \times \hat{\mathcal{M}}(y;z)$. Then there exists a differentiable map

$$\psi : [R_0, \infty) \to \mathcal{S}(x, z),$$

for a particular value R_0, such that $\hat{\psi} := \pi \circ \psi$ is an embedding

$$\hat{\psi} : [R_0, \infty) \to \hat{\mathcal{M}}(x; z)$$

satisfying

$$\lim_{R \to +\infty} \hat{\psi}(R) = (\mathcal{C}_u, \mathcal{C}_v) \in \overline{\mathcal{M}}(x; z) \supset \hat{\mathcal{M}}(x; z).$$

Before we state the second theorem which pertains to the surjectivity of gluing, we must introduce some notation to handle the multiplicites of the pseudoholomorphic cylinders and Reeb orbits. We denote by $\mathfrak{m}(\gamma)$ the **multiplicity** of a Reeb orbit γ, and $\mathfrak{m}(\mathcal{C})$ of a finite energy pseudoholomorphic cylinder $\mathcal{C}$. Specifically, if the finite energy pseudoholomorphic cylinder $\mathcal{C} \in \hat{\mathcal{M}}(\gamma_+; \gamma_-)$ represented by $u \in \mathcal{M}(\gamma_+; \gamma_-)$ may be written as the composition of a holomorphic map $\varphi : (\mathbb{R} \times S^1, j) \to (\mathbb{R} \times S^1, j)$ with $\pm\infty = \varphi(\pm\infty)$ where

$$u = v \circ \varphi, \qquad \deg(\varphi) > 1.$$

and v is a simple cylinder, then

$$\mathfrak{m}(\mathcal{C}) = \deg(\varphi).$$

When $\mathcal{C}$ is not multiply covered, then $\mathfrak{m}(\mathcal{C}) = 1$.

By multiplicity of a Reeb orbit, we mean that $\mathfrak{m}(\gamma)$ is the unique positive integer such that γ is the $\mathfrak{m}_\gamma$-fold iterate of a simple Reeb orbit. Note that $\mathfrak{m}(\mathcal{C}) | \mathfrak{m}(\gamma_+)$ and $\mathfrak{m}(\mathcal{C}) | \mathfrak{m}(\gamma_-)$.

For u and v given parametrizations of $\mathcal{C}_u$ and $\mathcal{C}_v$, we can rotate the initial point chosen on the orbit $y(t)$ by $T/\mathfrak{m}(y)$, where T is the period of the orbit y, to obtain

$$k := \frac{\mathfrak{m}(y)}{\operatorname{lcm}(\mathfrak{m}(\mathcal{C}_u), \mathfrak{m}(\mathcal{C}_v))}$$

non-equivalent different broken parametric solutions in $\mathcal{S}(x;y)\times\mathcal{S}(y;z)$. Each represents the same broken trajectory given by $(\mathcal{C}_u, \mathcal{C}_v) \in \hat{\mathcal{M}}(x;y) \times \hat{\mathcal{M}}(y;z)$. and gives rise to k different approximately pseudoholomorphic cylinders $u\#_R v$, and hence k different honest pseudoholomorphic solutions $\{\psi_i\}_{i\in\{1,\dots,k\}}$.

As a result we obtain the following non-uniqueness result in regards to the surjectivity of gluing.

Theorem 7.3 (Surjectivity of Gluing)**.** *If there exists a sequence $(\hat{w}_n)$ converging to $(\mathcal{C}_u, \mathcal{C}_v) \in \hat{\mathcal{M}}(x;y) \times \hat{\mathcal{M}}(y;z)$ then $(\hat{w}_n)$ must lie in the image of one of the $\hat{\psi}_i$'s for $i \in \{1, ..., k\}$ when n is sufficiently large.*

In order to obtain the honest pseudoholomorphic curve $\hat{\psi}$ we use a process known as gluing, which uses two connecting parametrized solutions,

$$u \in \mathcal{S}(x, y), \quad v \in \mathcal{S}(y, z)$$

whose corresponding Fredholm operators D_u and D_v are surjective. We then use these parametrized solutions to construct a one parameter family of **approximate solutions**,

$$w_R = u\#_R v, \tag{7.2}$$

which interpolate between the Reeb orbits x and z, where R is the gluing parameter. This procedure is known as **pregluing**. Since the notation in literature varies, we want to clearly indicate that w_R refers to a one parameter family of approximate solutions and *not the honest pseudoholomorphic curve*. Frequently one designates the approximate solutions by $u\#_R v$, but this makes for a cumbersome subscript, so we have abbreviated it by w_R.

Next we will want to apply an appropriate infinite dimensional implicit function theorem to establish the existence of an honest pseudoholomorphic curve sufficiently close to the family of approximate solutions. This follows after using special cutoff functions to piece together objects very close to the solution we are looking for and use a Newton-type iteration to obtain an actual solution of the Cauchy-Riemann equations.

In order to make use of this refined implicit function theorem, we must prove that the linearized operator D_{w_R} is surjective for sufficiently large R, introduce appropriate weighted Sobolev norms for the vector fields and 1-forms along the preglued curves w_R as defined in (7.2), and demonstrate that D_{w_R} has a right inverse satisfying a uniform bound independent of R. These details of technicalities can be found in [Sc95] and [ADfloer].

We denote by ψ the **honest pseudoholomorphic solution**, which is approximated by w_R and it lives in $\mathcal{S}(x;z)$. Moreover, we will demonstrate that it is of the following form

$$\psi(R) = \exp_{w_R}(\eta(R)),$$

where $\eta(R) \in W^{1,p}_{\delta}(w_R^* TW)$.

Due to the choices involved in picking a starting point on the Reeb orbit y, we obtain

$$k := \frac{\mathfrak{m}(y)}{\operatorname{lcm}(\mathfrak{m}(\mathcal{C}_u), \mathfrak{m}(\mathcal{C}_v))}$$

non-equivalent pregluings w_R via reparametrization. Near each choice of pregluing, w_R, the honest solution ψ is determined uniquely. This is the reason for the statement of Theorem 7.3, as well as for the coefficient appearing in the expression (8.4) for the cylindrical contact homology differential ∂. This is in contrast to the setting in Hamiltonian Floer homology, and requires notions of convergence from [HWZ02]. More details on this will be given in Section 7.3.

7.2 Pregluing and construction of the honest pseudoholomorphic curve.

In this section we provide more details on the construction of the one parameter family of approximate solutions w_R. We give details on the specialized cut off functions used to piece together the cylinders that are very close to the solution we are looking for.

Pseudoholomorphic curves in symplectizations or cobordisms can be decomposed into two components. Writing u and v as the two parametrized solutions which we will be gluing together, we denote their components in the symplectization $(\mathbb{R} \times M, d(e^\tau \alpha), \tilde{J})$ by

$$u := (a, f), \quad v := (b, g).$$

These parametrized elements $u \in \mathcal{S}(x; y)$ and $v \in \mathcal{S}(y; z)$ have been chosen after we fixing an initial point on the Reeb orbit $y(t)$ and ensuring that they satisfy

$$\lim_{s \to -\infty} f(s,t) = \lim_{s \to +\infty} g(s,t) = y(Tt), \tag{7.3}$$

where T is the period of the Reeb orbit y.

Our approximately holomorphic cylinder also decomposes in this manner as well. We denote these components as follows

$$w_R = (c_R, h_R).$$

In order to precisely define w_R, we select two cutoff functions $\beta^+(s)$ and $\beta^-(s)$ in

$\mathbb{C}^\infty(\mathbb{R}, [0,1])$, satisfying for fixed $0 < \varepsilon < 1/2$,

$$\beta^-(s) = \begin{cases} 1 & s \leq -1, \\ 0 & s \geq -\varepsilon \end{cases} \qquad \beta^+(s) = \begin{cases} 1 & s \geq 1, \\ 0 & s \leq \varepsilon. \end{cases}$$

The approximate solution component in the $\mathbb{R}$-direction is defined by

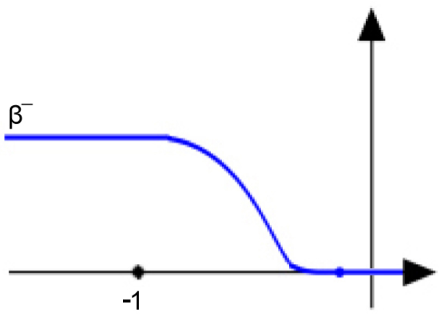


Figure 16: Model for the cutoff function $\beta^-(s)$.

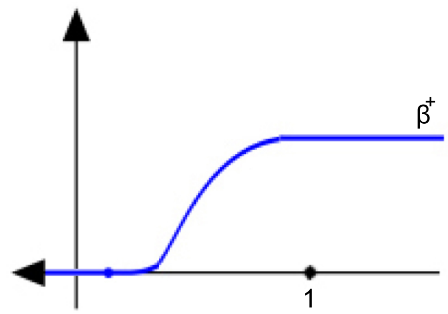


Figure 17: Model for the cutoff function $\beta^+(s)$.

$$c_R(s) = \begin{cases} b(s+R,t) & s \leq -1, \\ \beta^-(s)(b(s+R,t) - b(R,0)) + \beta^+(s)(a(s-R,t) - a(-R,0)) & s \in (-1,1), \\ a(s-R,t) & s \geq 1, \end{cases}$$

and in the contact component M we have

$$h_R(s) = \begin{cases} g(s+R,t) & s \leq -1, \\ \exp_{y(t)}\left(\beta^-(s)\exp^{-1}_{y(t)}(f(s+R,t)) + \beta^+(s)\exp^{-1}_{y(t)}(g(s-R,t))\right) & s \in (-1,1), \\ f(s-R,t) & s \geq 1. \end{cases}$$

An illustration of this construction in the contact manifold compnent is given in Figure 18. It appears here with the kind permission of Dietmar Salamon, with a slight modification from its original form in [Sa99].

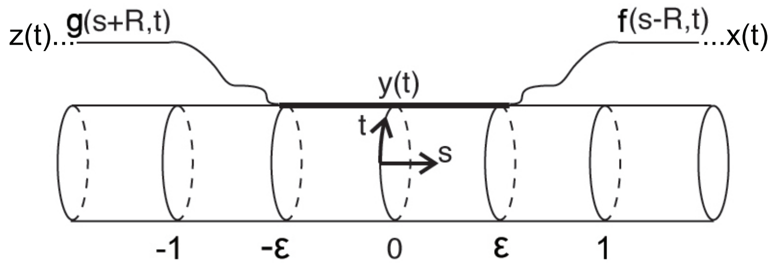


Figure 18: Pregluing construction

The parametrizations u and v have been chosen such that for $|s| \leq 1$ that $u(s + R, t)$ and $v(s - R, t)$ lie in the image of the exponential map of a solution $Y(t)$ to the linearized $\bar{\partial}_{\tilde{J}}$ equation. We will refer to solutions to the linearized pseudoholomorphic curve equation, such as these, as **pseudoholomorphic vector fields** along a solution. Namely, they are contained in the set

$$\left\{ \exp_{y(t)} Y(t) \mid \sup_{t \in S^1} ||Y(t)|| \leq r_0 \right\}.$$

Given the asymptotics described in Equation 7.3,

$$\lim_{s \to -\infty} f(s,t) = \lim_{s \to +\infty} g(s,t) = y(Tt),$$

we know that the above will be true when R is chosen to be sufficently large. The exact value of R_0 is not important, and when we write $R \geq R_0$ signifies that we are only considering those R which are sufficently large.

The interpolation w_R that we have constructed satisfies the following properties.

1. The approximate solution w_R is an element of $C^\infty(x, z)$.

2. For $s \in [-\varepsilon, \varepsilon]$, we have $h_R(s,t) = y(t)$.

3. For $s \leq R - 1$ we have $h_R(s - R, t) = g(s,t)$ and

$$\lim_{R \to +\infty} h_R(s - R, t) = g(s,t) \text{ in } C^\infty_{\text{loc}}.$$

Likewise we have for $s \geq 1 - R$ we have $h_R(s+R,t) = f(s,t)$ and

$$\lim_{R \to +\infty} h_R(s+R,t) = f(s,t) \text{ in } C^\infty_{\text{loc}}.$$

4. The approximate solution w_R is a differentiable function with respect to R.

5. h_R tends to $y(t)$ in C^∞_{loc} as $R \to +\infty$.

For $R \geq R_0$ we constructed an approximately pseudoholomorphic curve w_R in $C^\infty(x,y)$. The next step is to construct for $R \geq R_0$ an honest pseudoholomorphic curve, $\psi \in \mathcal{S}(x;y)$. As previously mentioned, it will be of the following form

$$\psi(R) = \exp_{w_R}(\eta(R)),$$

where $\eta(R) \in W^{1,p}_\delta(w_R^* TW)$. By honest, we meant that it will satisfy the Cauchy-Riemann equation,

$$\bar{\partial}_{\tilde{J}}(\psi) = 0.$$

Note that for $p > 2$, $\eta(R)$ is continuous, so because of elliptic regularity ψ will be of class C^∞.

The full details of the gluing methods may be found in [ADfloer] and [Sc95], and generalize immediately to the contact homology setting, aside from the surjectivity of gluing as in Theorm 7.3, which we explain in the following section. The idea of these arguments, which yield the proof of the analogue of Floer's gluing theorem, Theorem 7.2 are as follows. In order to appeal to Newton's method, one first utilizes a stabilization technique which eliminates the finite dimensional cokernel. This gives us a subspace $W_R^\perp$ where we can find a right inverse to $L_R|_{W_R^\perp}$. The right inverse can be found once we have demonstrated that the linearizations associated to u and v are surjective and that there is a uniformly bounded family of right inverses associated to Dw_R for R sufficiently large in the $W^{1,p}$-norm. With the right inverse and estimate in place, we can appeal to Floer's Picard Lemma to obtain a unique solution of the linearized operator $D\bar{\partial}_{\tilde{J},R}(\eta) = 0$, starting at 0, obtaining the honest pseudoholomorphic curve as desired by exponentiating along w_R as desired. Here $\eta \in W^{2,p}_\delta(\mathbb{R} \times S^1, {w_R}^*T(\mathbb{R} \times M))$.

7.3 Surjectivity of gluing

The purpose of this section is to investigate the lack of uniqueness involved in the gluing construction and prove Theorem 7.3. Recall that this states if there exists a sequence $(\hat{w}_n) \in \hat{\mathcal{M}}(x,z)$ converging to $(\mathcal{C}_u, \mathcal{C}_v) \in \hat{\mathcal{M}}(x,y) \times \hat{\mathcal{M}}(y,z)$ then $(\hat{w}_n)$ must lie in the image of one of the $\hat{\psi}_i$'s for $i \in \{1, ..., k\}$, where

$$k := \frac{\mathfrak{m}(y)}{\operatorname{lcm}(\mathfrak{m}(\mathcal{C}_u), \mathfrak{m}(\mathcal{C}_v))}.$$

Geometrically this is due to he fact that if y is a multiply covered orbit of multiplicity $\mathfrak{m}(y)$ then we can change the parametrizations of u and v of the cylinders $\mathcal{C}_u$ and $\mathcal{C}_v$ by rotating the initial point chosen on the orbit y thru $\dfrac{T}{\mathfrak{m}(y)}$, where T is the period of y. This gives rise to k unique pregluings, which we will denote by $(u\#_R v)_i$ for $i \in \{1, ..., k\}$. These k approximately holomorphic cylinders give rise to k honest pseudoholomorphic cylinders ψ_i. Such a result is important because it tells us that the multiplicities of our Reeb orbits and finite energy pseudoholomorphic cylinders will be encoded in the structure of the graph given by the compactification of $\hat{\mathcal{M}}(x,z)$. We discuss this geometry in the following section, Section 7.4 and will use it to prove $\partial^2 = 0$ in Section 8.2.

Before we can prove this theorem we will need several lemmas and propositions and will recall the necessary notions of convergence and results of Section 4 in the work of

Hofer, Wysocki, and Zehnder in [HWZ02] on pseudoholomorphic cylinders of small area. We begin by introduce the language of asymptotic and directional convergence modulo $\mathbb{R}$ of [HWZ02], which requires a few concepts. Let S_+ and S_- be two compact disk-like Riemann surfaces with smooth boundaries. For the purposes of our discussion we will take the same complex structure j on both. Let $o_\pm$ be interior points of $S_\pm$. Then we can obtain a **noded surface** by identifying o_- and o_+ in the disjoint union $S_- \sqcup S_+$. The noded surface is denoted by (S, o).

A **deformation** of a compact Riemann surface (Σ, j) of annulus type is a continuous surjection

$$\phi : \Sigma \to S$$

onto the nodal surface (S, o) such that $\phi^{-1}(o)$ is a smooth embedded circle, and

$$\phi\colon \Sigma \setminus \{\phi^{-1}(o)\} \to S \setminus \{o\}$$

is an orientation-preserving diffeomorphism. We push forward the complex structure j to $S \setminus \{o\}$, obtaining $\phi_* j$.

We consider a sequence of compact Riemann surfaces (S_n, j_n) of annulus type whose moduli converge to ∞,

$$\operatorname{mod}(S_n, j_n) \to \infty$$

On this sequence of surfaces we consider a sequence

$$w_n = (a_n, f_n) : (S_n, j_n) \to (\mathbb{R} \times M, \tilde{J})$$

of pseudoholomorphic finite energy maps. Let

$$w = (a, f)\colon (S \setminus \{o\}, j) \to (\mathbb{R} \times M, \tilde{J})$$

be a pseudoholomorphic finite energy map having negative puncture at $o_+ \in S_+$ and positive puncture $o_- \in S_-$ and assume that their asymptotic limits are the same periodic orbit of the Reeb vector field. For our purposes we will think of this node point as asymptotically limiting on the intermediary Reeb orbit y in our pregluing construction.

With these ideas in place we can define **convergence modulo** $\mathbb{R}$ for a sequence

$$w_n \colon (S_n, j_n) \to (\mathbb{R} \times M, \tilde{J})$$

of pseudoholomorphic curves is said to converge to

$$w : (S \setminus \{o\}, j) \to (\mathbb{R} \times M, \tilde{J})$$

whenever there exists a sequence of deformations

$$\phi_n : (S_n, j_n) \to (S, j)$$

satisfying in $C^\infty_{\text{loc}}(S_\pm \setminus \{o_\pm\})$ the following two conditions

1. $w_n \circ \phi_n^{-1} \to w$;

2. $(\phi_n)_* j_n \to j$.

Next we introduce an example of how this construction is applicable to our setting.

Example 7.4. Let

$$w : (S \setminus \{o\}, j) \to (\mathbb{R} \times M, \tilde{J})$$

be defined so that

$$w = (u, v)|_{\{(-\infty,0)\cup(0,\infty)\}\times S^1}$$

and

$$(S_n, j_n) = (\{(-R_n, 0) \cup (0, R_n)\} \times S^1, j_n)$$

where j_n is the usual complex structure on $\mathbb{R} \times S^1$ restricted to S_n. We take

$$w_n = (u, v)|_{S_n}$$

The sequence of deformations is

$$\phi_n : (\mathbb{R} \times S^1, j) \to S$$

in which we pinch $(0, e^{it})$ to o. Then w_n converges modulo $\mathbb{R}$ to w.

Next we need to introduce a second notion of convergence, directional convergence, related to the notion of asymptotically marked points. This language will allow us to relate the choice of underlying parametrization for $(\mathcal{C}_u, \mathcal{C}_v) \in \hat{\mathcal{M}}(x; y) \times \hat{\mathcal{M}}(y; z)$ to the choice of asymptotic markers. As a result we will be able to see if two parametrized solutions (u, v) and $(v', u') \in \mathcal{S}(x; y) \times \mathcal{S}(y; z)$ represent the same broken cylinder $(\mathcal{C}_u, \mathcal{C}_v)$ then they differ by rotation.

Let Σ be a Riemann surface and let $r \in \Sigma$ be an interior point. An asymptotic marker for r consists of a choice of an oriented real line $\vec{r} \subset T_r\Sigma$ in the tangent space at r. The oriented line $\vec{r}$ together with the underlying point r will be called an **asymptotically marked point**.

Given the Riemann surface Σ with an asymptotically marked point $\vec{r}$ there is a distinguished class of holomorphic coordinate systems around r which are said to be **compatible** with the asymptotic marker. They are defined as follows. We take any compact disk-like neighborhood $\mathcal{D} \subset \Sigma$ with smooth boundary around r. Then we take the unique biholomorphic map σ from the closed unit disk D onto $\mathcal{D}$ mapping 0 to r so that the tangent $T_{\sigma(0)}$ sends $1 \in \mathbb{R}$ to an (oriented) basis vector in $\vec{r}$. We note that for two such coordinate systems σ and τ which are compatible with the asymptotic marker, the linearized transition map at 0

$$D(\sigma^{-1} \circ \tau)(0) : \mathbb{C} \to \mathbb{C}$$

acts by multiplication by a positive real number.

The next step is to introduce a special class of holomorphic polar coordinates around an asymptotically marked point r, which will be of use later. We note that these coordinates exist for an arbitrary finite energy pseudoholomorphic map, which will will still denote by

$$w = (a, f) : (\Sigma \setminus \{r\}, j) \to (\mathbb{R} \times M, \tilde{J}).$$

After we explain this construction, we will demonstrate how to specialize it to finite energy pseudoholomorphic maps whose domain is $(S \setminus \{o\}, j)$ arising from the noded surface (S, o).

Remark 7.5. If we take r to be a non-removable puncture in the sense of Theorem 3.15 then we may assume that associated asymptotic limit is a nondegenerate periodic orbit of the Reeb vector field, which we denote by γ. We will only be interested in the case when r is a non-removable puncture.

Let $\vec{r}$ be an asymptotic marker associated to a positive puncture r. Then there is a special class of holomorphic polar coordinates around r which is compatible with the asymptotic marker and is defined as follows. We take a holomorphic coordinate system $\sigma : D \to \mathcal{D}$ around r which is compatible with the asymptotic marker as described above and define the following holomorphic map, when r is a positive puncture,

$$\begin{array}{rrcl} \nu_- : & \mathbb{R}^+ \times S^1 & \to & \mathcal{D} \setminus \{r\}, \\ & (s, t) & \mapsto & \sigma(e^{-2\pi(s+it)}). \end{array}$$

If the puncture r is negative, we take the holomorphic polar coordinates as before, but with

$$\begin{aligned} \nu_+ : \ \mathbb{R}^- \times S^1 &\to \mathcal{D} \setminus \{r\}, \\ (s,t) &\mapsto \sigma(e^{2\pi(s+it)}). \end{aligned}$$

Remark 7.6. In either situation (in regards to the charge of the puncture), if one considers the composition $g \circ \nu_\pm$, then $g \circ \nu_\pm(s,t)$ will converge in $C^\infty(\mathbb{R})$ as $s \to \infty$ to a parametrization of the asymptotic limit. The limiting loop

$$[t \mapsto \gamma(t)]$$

is independent of the choice of σ as long as σ is compatible with the asymptotic marker.

We refer to a finite energy pseudoholomorphic curve equipped with the asymptotically marked point $\vec{r}$ and the special holomorphic polar coordinates $\nu_\pm$ as being **compatible with the asymptotic markers.**

Next we consider the previously discussed class of finite energy pseudoholomorphic maps whose target is $(S \setminus \{o\}, j)$. We explain how one may realize the node o as an asymptotically marked point $\vec{o}$, specialize the construction of the special holomorphic polar coordinates $\nu_\pm$ to such maps, and state what it means for these maps to be compatible with the obvious asymptotic marker, $\vec{o}$, as defined above.

Example 7.7. Let

$$w = (a, f) : (S \setminus \{o\}, j) \to (\mathbb{R} \times M, \tilde{J}),$$

where (S, o) is the noded surface introduced earlier. Recall that we took o_+ to be a negative puncture and o_- to be a positive puncture. Moreover, the negative asymptotic limit of

$$w|_{S_+ \setminus \{o_+\}}$$

coincides with the positive asymptotic limit of

$$w|_{S_-\setminus\{o_-\}}.$$

In addition, we assume that $\vec{o_\pm}$ are asymptotic marked points. If ν_+ are negative holomorphic polar coordinates at o_+ compatible with $\vec{o_+}$ and ν_- are positive holomorphic polar coordinates at o_- compatible with $\vec{o_-}$, we finally require for all $t \in S^1$ that

$$\lim_{s\to\infty} f\circ\nu_-(s,t) = \lim_{s\to-\infty} f\circ\nu_+(s,t).$$

A finite energy pseudoholomorphic curve $w=(a,f):(S\setminus\{o\},j)\to(\mathbb{R}\times M,\tilde{J})$ having all these properties is said to be **compatible with the asymptotic markers**.

With these concepts in place we can define **directional convergence modulo** $\mathbb{R}$ as follows .

Definition 7.8. Assume that the finite energy surface

$$w=(a,f)\colon(S\setminus\{o\},j)\to(\mathbb{R}\times M,\tilde{J})$$

is compatible with the asymptotic markers as described above. The sequence

$$w_n:(S_n,j_n)\to(\mathbb{R}\times M,\tilde{J})$$

is said to be **directionally convergent** to w for the given asymptotic marked points $\vec{o}_\pm$ if there exists a sequence

$$\phi_n : S_n \to S$$

of deformations onto the nodal surface S and a sequence

$$\Psi_n : [-R_n, R_n]\times S^1 \to S_n$$

of biholomorphic maps where $2R_n = \mathrm{mod}(S_n,j_n)\to\infty$,satisfying in $C^\infty_{\mathrm{loc}}(S_\pm\setminus\{o_\pm\})$ the following conditions as before

1. $f_n \circ \phi_n^{-1} \to f$,

2. $(\phi_n)_* j_n \to j$,

as well as that the sequences of mappings defined by

$$\begin{aligned} e_n^+(z) &:= \Psi_n^{-1} \circ \phi_n^{-1} - (R_n, 0) \quad \text{for } z \in S_+ \setminus \{o_+\} \\ e_n^-(z) &:= \Psi_n^{-1} \circ \phi_n^{-1} + (R_n, 0) \quad \text{for } z \in S_- \setminus \{o_-\} \end{aligned}$$

converge as $n \to \infty$ to some limit maps $e^\pm$ in the sense that

$$\begin{aligned} e_n^+ \to e^+ &\quad \text{in } C^\infty_{\text{loc}}(S_+ \setminus \{o_+\}, \mathbb{R}^+ \times S^1) \\ e_n^- \to e^- &\quad \text{in } C^\infty_{\text{loc}}(S_- \setminus \{o_-\}, \mathbb{R}^- \times S^1) \end{aligned}$$

By construction the the limit maps $e^\pm$ are necessarily biholomorphic and are inverse maps of holomorphic polar coordinates, which can be explicitly defined as follows.

We can define the associated holomorphic coordinate systems as

$$\sigma_\pm : D \to S_\pm$$

by

$$\begin{aligned} \sigma_\pm(0) &= o_\pm, \\ (e^\pm)^{-1}(s,t) &= \sigma_\pm(e^{\pm 2\pi(s+it)}) \text{ on } \mathbb{R}^\mp \times S^1 \end{aligned}$$

and also imposing that $\sigma_\pm$ be compatible with the asymptotic markers.

Since the $e_n^\pm$ are convergent, it follows from the properties of holomorphic mappings that the limiting maps $e^\pm$ are biholomorphic. This implies that given any $\delta > 0$ the preimage $\Psi_n^{-1} \circ \phi_n^{-1}(o)$ of the node, which is a priori a circle in $[-R_n, R_n] \times S^1$, is actually contained in

$$[-R_n + \delta, R_n - \delta] \times S^1$$

if n is sufficiently large.

As a consequence, the composition $\phi_n \circ \Psi_n$ is defined on the complement of $[-R_n + \delta, R_n - \delta] \times S^1$ if n is sufficiently large. Consequently

$$f_n \circ \Psi_n(-R_n + s, t) = f_n \circ \phi_n^{-1} \circ (\phi_n \circ \Psi(-R_n + s, t))$$

is well-defined for every $s \geq 0$ and for sufficiently large n. Moreover the right hand side converges to the map

$$f \circ (e^{\pm})^{-1} : \mathbb{R}^{\mp} \times S^1 \to M.$$

Hence, introducing the translations

$$\mathcal{T}_\rho(s, t) = (s + \rho, t),$$

we have

$$f_n \circ \Psi_n \circ \mathcal{T}_{-R_n} \to f \circ (e^-)^{-1} \text{ in } C^\infty_{\text{loc}}(\mathbb{R}^+ \times S^1, M).$$

Likewise we obtain

$$f_n \circ \Psi_n \circ \mathcal{T}_{R_n} \to f \circ (e^-)^{-1} \text{ in } C^\infty_{\text{loc}}(\mathbb{R}^- \times S^1, M).$$

Based on the construction of asymptotic markers and discussion of finite energy pseudoholomorphic maps whose domain is $(S \setminus \{o\}, j)$ being compatible with the asymptotic markers restricted we obtain the following proposition.

Next we recall the following proposition from [HWZ02] in regard to the following uniqueness statement regarding directional convergence.

Proposition 7.9 (Proposition 4.4 of [HWZ02])**.** *If w_n converges directionally to w and to w' then*

$$w = w' \circ \varphi \quad \text{on } S_\pm \setminus \{o\}$$

for two biholomorphic mappings $\varphi_\pm : S_\pm \to S_\pm$ *satisfying*

$$\begin{aligned} \varphi_\pm(o_\pm) &= o_\pm \\ D\varphi_\pm(o_\pm)\vec{o}_\pm &= e^{\pm i\theta} \cdot \vec{o}_\pm \end{aligned}$$

for some $\theta \in \mathbb{R}$*.*

We will use this in the proof of the following proposition.

Proposition 7.10. *The choice of compatible asymptotic markers on* $(u, v)\big|_{(-\infty,0)\cup(0,\infty)}$ *is equivalent to the choice of parametrized solutions for* (u, v) *which represents the broken curve* $(\mathcal{C}_u, \mathcal{C}_v)$*.*

Proof. First we prove that the choice of compatible asymptotic markers implies the choice of parametrized solutions. This follows immediately from the above, Proposition 7.9 and Remark 7.6 since the parametrized solutions u and v can only be chosen after we fix an initial point on the Reeb orbit $y(t)$. However, there are $\mathfrak{m}(y)$ equivalent choices of initial point which differ by rotation through $\frac{T}{\mathfrak{m}(y)}$, where T is the period of y. □

The next step is to understand how many unique pairs of parametrizations (u, v) we obtain via rotation. In fact, the multiplicities of the cylinders $\mathcal{C}_u$ and $\mathcal{C}_v$ will cause some of the parametrizations to coincide, since some of the cylinders may multiply covered and $\mathfrak{m}(\mathcal{C}_u)$ and $\mathfrak{m}(\mathcal{C}_v)$ must be divisors of $\mathfrak{m}(y)$.

Proposition 7.11. *There are*

$$k := \frac{\mathfrak{m}(y)}{lcm(\mathfrak{m}(\mathcal{C}_u), \mathfrak{m}(\mathcal{C}_v))}$$

unique pairs of parametrizations $(u, v)_i$ *with* $i \in \{1, ...k\}$ *such that*

$$\pi((u, v)_i) = (\mathcal{C}_u, \mathcal{C}_v)$$

where $\pi : \mathcal{S}(x; y) \times \mathcal{S}(y; z) \to \hat{\mathcal{M}}(x; y) \times \hat{\mathcal{M}}(y; z)$*.*

Proof. We know that there are at most $\mathfrak{m}(y)$ pairs of parametrizations for $(\mathcal{C}_u, \mathcal{C}_v)$. We know that these each differ by rotating the initial point on the Reeb orbit $y(t)$ through $\frac{T}{\mathfrak{m}(y)}$, where T. To be precise, after fixing an initial point t_0 on the Reeb orbit $y(t)$, we obtain $\mathfrak{m}(y)$ equivalent starting points given by

$$t_0 + \frac{T}{\ell} \text{ for } \ell \in \{1, ...\mathfrak{m}(y)\}.$$

We denote the parametrizations obtained in this manner by $(u, v)_\ell$, with $\ell \in \{1, ...\mathfrak{m}(y)\}$.. Now we need to see if two such parametrizations are indistinguishable. Note that if we rotate the initial point t_0 on $y(t)$ by $\frac{T}{\mathrm{lcm}(\mathfrak{m}(\mathcal{C}_u), \mathfrak{m}(\mathcal{C}_v))}$ the parametrizations chosen for $(\mathcal{C}_u, \mathcal{C}_v)$ are identical. This is because the common periodicity of the multiply covered cylinders implies the same periodicity for the broken curve $(\mathcal{C}_u, \mathcal{C}_v)$ and hence also for any underlying parametrization of $(\mathcal{C}_u, \mathcal{C}_v)$. Thus there are exactly

$$k := \frac{\mathfrak{m}(y)}{\mathrm{lcm}(\mathfrak{m}(\mathcal{C}_u), \mathfrak{m}(\mathcal{C}_v))}$$

unique pairs of parametrizations for $(\mathcal{C}_u, \mathcal{C}_v)$. □

As a result we obtain k unique honest pseudoholomorphic curves, via the gluing procedure explained in the previous sections

Corollary 7.12. *There are k unique honest pseudoholomorphic curves ψ which satisfy Theorem 7.2.*

The next lemma is so that we can reformulate the statement of convergence in the gluing theorem in terms of convergence modulo $\mathbb{R}$.

Lemma 7.13. *If there exists a sequence $(\hat{w}_n) \in \hat{\mathcal{M}}(x; z)$ converging to $(\mathcal{C}_u, \mathcal{C}_v) \in \hat{\mathcal{M}}(x; y) \times \hat{\mathcal{M}}(y; z)$ then $(\hat{w}_n)$ converges modulo $\mathbb{R}$.*

The final ingredient in the proof of the surjectivty of gluing, Theorem 7.3 is the following main theorem from [HWZ02].

Theorem 7.14 (Theorem 4.5 of [HWZ02]). *A convergence sequence w_n has a directionally convergent subsequence.*

As a result we obtain that $(\hat{w}_n)$ has a directionally convergent subsequence. However directional convergence is equivalent to the choice of underlying parametrization for $\hat{w}_n$, and since there are k unique such choices we can only conclude that $(\hat{w}_n)$ must lie in the image of one of the $\hat{\psi}_i$'s for $i \in \{1, ..., k\}$.

7.4 Geometric structure of $\overline{\mathcal{M}}(x; z)$

The purpose of this section is to combine the results of Section 6 in regards to the inclusion of moduli spaces and the gluing results of Section 7 to understand the ends of the 1-dimensional manifold $\hat{\mathcal{M}}(x; z)$. Throughout this section we assume that x and z are nondegenerate Reeb orbits associated to a dynamically separated contact form with $\mu_{CZ}(x) - \mu_{CZ}(z) = 2$ in the same free homotopy class.

The gluing maps of the previous section described the ends of the 1-dimensional manifold $\hat{\mathcal{M}}(x, z)$ associated to the symplectization of a nondegenerate dynamically separated contact manifold. These $\hat{\psi}_i' s$ converge to the broken cylinders of the following form,

$$\mathcal{C} = (\mathcal{C}_u, \mathcal{C}_v)$$

where $\mathcal{C}_u \in \hat{\mathcal{M}}(x; y)$ and $\mathcal{C}_v \in \hat{\mathcal{M}}(y; z)$ with $\mu_{CZ}(y) = \mu_{CZ}(x) - 1$. As a result we obtain the following preliminary theorem.

Theorem 7.15. *The boundary of* $\overline{\mathcal{M}}(x,z)$ *is given by*

$$\partial\overline{\mathcal{M}}(x;z) \;=\; \bigcup_{\substack{y\in\mathscr{P}\\ \mu_{CZ}(y)=\mu_{CZ}(x)-1}} \hat{\mathcal{M}}(x;y)\times\hat{\mathcal{M}}(y;z).$$

Proof. The numerology of Section 6 ensures the ends of the compactified moduli space $\overline{\mathcal{M}}(x,z)$ can only converge to 2 broken rigid pseudoholomorphic cylinders. Namely, from Corollary 6.2 we have that

$$\partial\overline{\mathcal{M}}(x;z) \;\subseteq\; \bigcup_{\substack{y\in\mathscr{P}\\ \mu_{CZ}(y)=\mu_{CZ}(x)-1}} \hat{\mathcal{M}}(x;y)\times\hat{\mathcal{M}}(y;z).$$

The gluing theorem of Section 7 gave us the reverse inclusion, thus we could conclude the following in regard to the boundary of the compactified space $\overline{\mathcal{M}}(x,z)$,

$$\partial\overline{\mathcal{M}}(x;z) \;\supseteq\; \bigcup_{\substack{y\in\mathscr{P}\\ \mu_{CZ}(y)=\mu_{CZ}(x)-1}} \hat{\mathcal{M}}(x;y)\times\hat{\mathcal{M}}(y;z).$$

□

Next we want to endow the compactification of $\hat{\mathcal{M}}(x;z)$ with the structure of a compact labelled graph whose vertices correspond to the broken cylinders and edges correspond to the connected components of $\hat{\mathcal{M}}(x;z)$, which will allow us to demonstrate that $\partial^2=0$ in the following section. Stated as a theorem, we have the following.

Theorem 7.16 (Structure)**.** *The compactification of* $\hat{\mathcal{M}}(x;z)$ *has the structure of a graph whose vertices correspond to the broken cylinders denoted by* $v_{(\mathcal{C}_u,\mathcal{C}_v)}$ *and edges correspond to the connected components of* $\hat{\mathcal{M}}(x;z)$. *Each vertex* $v_{(\mathcal{C}_u,\mathcal{C}_v)}$ *belongs to*

$$k := \frac{\mathfrak{m}(y)}{\operatorname{lcm}(\mathfrak{m}(\mathcal{C}_u),\mathfrak{m}(\mathcal{C}_v))}$$

edges, each corresponding to cylinders of multiplicity given by a divisors of

$$\gcd\left(\mathfrak{m}(\mathcal{C}_u),\mathfrak{m}(\mathcal{C}_v)\right).$$

Remark 7.17. We clarify that the covering multiplicity associated to the all adjacent edges connecting at a single vertex is the same.

Note that the first part of Theorem 7.16 follows from the Theorem 7.15. The remainder of the proof relies on understanding how to encode the multiplicities of the Reeb orbits and finite energy pseudoholomorphic in the structure of the graph given by the compactification of $\hat{\mathcal{M}}(x;z)$. The multiplicity factor is because of the different of the k different gluings ψ_i for $i \in \{1, ..., k\}$ obtained in Theorem 7.3.

We begin by proving the following proposition. Let

$$\mathcal{U} \subseteq \hat{\mathcal{M}}(x;y) \times \hat{\mathcal{M}}(y;z) \subseteq \overline{\mathcal{M}}(x;z).$$

Proposition 7.18. *There exists a neighborhood of the vertex $v_{(\mathcal{C}_u,\mathcal{C}_v)}$ in $\mathcal{U}$ that is homeomorphic to a k-valent graph.*

Proof. Based on surjectivity of gluing, Theorem 7.3 nothing can lie outside of these branches. Thus they must meet locally at a vertex and the number is determined by the gluing construction. All these pieces are embedded and nonintersecting from Theorem 7.2. □

As a result, we obtain the following corollary in regard to the multiplicities of cylinders, which we associate as labels on the edges of $\overline{\mathcal{M}}(x;z)$. For a fixed $r \in \mathbb{N}$ let $\overline{\mathcal{M}}_r(x;z)$ be the subgraph of $\overline{\mathcal{M}}(x;z)$ consisting of the edges labelled with r. Here r is a divisor of $\gcd\left(\mathfrak{m}(\mathcal{C}_u), \mathfrak{m}(\mathcal{C}_v)\right)$. The subgraph $\overline{\mathcal{M}}_r(x;z)$ is a union of connected components of $\overline{\mathcal{M}}(x;z)$. Furthermore, the following proposition tells us that on each connected component of $\overline{\mathcal{M}}_r(x;z)$ all edges are labelled the same.

Proposition 7.19. *All branches adjacent to a given vertex $v_{(\mathcal{C}_u,\mathcal{C}_v)}$ must have the same covering multiplicity, given by the divisors of*

$$\gcd\left(\mathfrak{m}(\mathcal{C}_u), \mathfrak{m}(\mathcal{C}_v)\right).$$

Proof. The isotropy associated to each branch is the number corresponding to the multiplicity of the underlying cylinder. Since we know all curves $w \in \hat{\mathcal{M}}(x;z)$ are regular by Theorem 5.5 there exists a neighborhood $\mathcal{V}$ of $w \in \hat{\mathcal{M}}(x;z)$ such that the isotropy of w is constant on $\mathcal{V}$. Next we need to show that a locally constant isotropy implies a constant isotropy on connected components of $\overline{\mathcal{M}}(x,z)$, i.e. that all branches w_i for $i = \{1, ..., k\}$ adjacent to a given vertex $v_{(\mathcal{C}_u,\mathcal{C}_v)}$ have the same covering multiplicity.

Suppose not. Then we can construct sequences $\mathcal{C}_{a_k}$ and $\mathcal{C}_{b_k}$ in $\hat{\mathcal{M}}(x;z)$ such that $\mathcal{C}_{a_0}$ and $\mathcal{C}_{b_0}$ are branches both adjacent to a given vertex $v_{(\mathcal{C}_u,\mathcal{C}_v)}$, with $\mathfrak{m}(\mathcal{C}_{a_0}) = a$ and $\mathfrak{m}(\mathcal{C}_{b_0}) = b$ with $a \neq b \neq 1$ and

$$\lim_{k\to\infty} \mathcal{C}_{a_k} = (\mathcal{C}_u, \mathcal{C}_v) = \lim_{k\to\infty} \mathcal{C}_{b_k}.$$

Locally by Theorem 5.5 there exist neighborhoods A_k and B_k of $\mathcal{C}_{a_k}$ and $\mathcal{C}_{b_k}$ respectively such that all curves in A_k have multiplicity a and all curves in $\cap B_k$ have multiplicity b. Since $(\mathcal{C}_u, \mathcal{C}_v)$ is a limit point of both these sequences we know there exists a $K \in \mathbb{N}$ such that for all $k \geq K$ we have $A_k \cap B_k \neq \emptyset$. Thus the multiplicities of $\mathcal{C}_{a_k}$ and $\mathcal{C}_{b_k}$ must agree for $k \geq K$. But this contradicts $a \neq b$. Thus

$$\mathfrak{m}(\mathcal{C}_{a_0}) = \mathfrak{m}(\mathcal{C}_{b_0}).$$

□

The above two propositions yield the following corollaries.

Corollary 7.20. *The number of ends of* $\mathcal{M}_r(x;z)$ *are in correspondence with the components* $\operatorname{int}(\overline{\mathcal{M}}_r(x;z))$.

Let $\overline{\mathcal{M}}_1(\gamma_-;\gamma_+)$ be the compactified space of all somewhere injective finite energy pseudoholomorphic cylinders interpolating between the Reeb orbits γ_- and γ_+. In other words, none of the $u \in \overline{\mathcal{M}}_1(\gamma_-;\gamma_+)$ are multiply covered.

Corollary 7.21. *As graphs*

$$\operatorname{int}\left(\overline{\mathcal{M}}_r(\gamma_-^r;\gamma_+^r)\right) = \operatorname{int}\left(\overline{\mathcal{M}}_1(\gamma_-;\gamma_+)\right),$$

As a result, we have the following decomposition for the compactified moduli spaces

$$\overline{\mathcal{M}}(x;z) = \bigsqcup_{r\in\mathbb{N}} \overline{\mathcal{M}}_r(x;z),$$

where r is the covering multiplicity of the pseudoholomorphic cylinders in a given $\overline{\mathcal{M}}_r$, after allowing for the possibility that some of the $\overline{\mathcal{M}}_r(x;z) = \emptyset$.

Chapter 8

Constructing cylindrical contact homology

Cylindrical contact homology is defined as an analogue of Morse theory on the loop space of a co-oriented contact manifold (M, ξ). If there exists a nondegenerate dynamically separated contact form α such that $\ker \alpha = \xi$, the chain complex C_* is generated by all the nondegenerate Reeb orbits associated to the Reeb vector field R_α. These arise as critical points of the following symplectic action functional

$$\begin{aligned} \mathcal{A}: \ C^\infty(S^1, M) &\rightarrow \mathbb{R} \\ \gamma &\mapsto \int_\gamma \alpha. \end{aligned} \tag{8.1}$$

The appropriate notion of "gradient flow lines" between critical points of $\mathcal{A}$ is realized, after selecting an almost complex structure J, by finite energy pseudoholomorphic cylinders

$$u : (\mathbb{R} \times S^1, j) \rightarrow (\mathbb{R} \times M, \tilde{J}) \in \hat{\mathcal{M}}(\gamma_+; \gamma_-),$$

interpolating between closed Reeb orbits γ_+ and γ_- with $\mu_{CZ}(\gamma_+) - \mu_{CZ}(\gamma_-) = 1$. These pseudoholomorphic curves are called **rigid** pseudoholomorphic cylinders, as they are elements of the 0-dimensional moduli space $\hat{\mathcal{M}}(\gamma_+; \gamma_-)$. The differential ∂ provides a weighted count of the number of these rigid pseudoholomorphic cylinders. Currently, we can only show that (C_*, ∂) forms a chain complex for cylindrical contact homology if we restrict ourselves to dimension 3 and if a nondegenerate dynamically separated contact form can be associated to the contact structure ξ.

In Morse theory one uses the Morse index, a count of the negative eigenvalues of the Hessian associated to the critical points, as a grading for the chain complexes of critical points. In contact homology the analogue of the Hessian has infinitely many negative and infinitely many positive eigenvalues, so we instead use a Maslov type index for arcs of symplectic matrices, known as the Conley-Zehnder index. This is a generalized winding number that controls embedding properties of pseudoholomorphic curves, however one does not always obtain a well-defined absolute $\mathbb{Z}$-grading. A canonical absolute $\mathbb{Z}_2$-grading does exist, and in this setting one assigns a Reeb orbit an odd grading when it is positive hyperbolic and an even grading when it is elliptic or negative hyperbolic. There is also a notion of a relative grading, but we will not discuss this here and the interested reader may find details in Section 6.5 of [Hu10].

When $c_1(\xi) = 0$[1] and $H^1(M) = 0$ the Conley-Zehnder indices may be computed in a globally well-defined way and one obtains an absolute $\mathbb{Z}$-grading, arising from the unique existence of a complex volume form on $(\mathbb{R} \times M, \tilde{J})$. If $c_1(\xi) = 0$ but $H^1(M) \neq 0$ then we can still obtain an absolute $\mathbb{Z}$-grading, but this grading is dependent on the choice of volume form which will be parametrized by $H^1(M)$. This agrees with conventions in symplectic homology, see [Se06] and [Se00]. When the notion of a $\mathbb{Z}$-grading exists,

[1] It turns out that $2c_1(\xi) = 0$ is the sufficient and necessary condition to obtain an absolute $\mathbb{Z}$-grading; see [Se00]. However we will restrict ourselves to the simpler setting to avoid the subtleties arising from the fact that the grading is only defined up to a choice of homotopy class of trivializations associated to the complex line bundle that is the canonical representation of $-c_1(\xi)$. Otherwise the $\mathbb{Z}$-grading is parametrized by choice of volume form.

grading in contact homology of a Reeb orbit is

$$|\gamma| = \mu_{CZ}(\gamma) + n - 3.$$

Here n appears in the dimension of the contact manifold M^{2n-1}, and μ_{CZ} is the Conley-Zehnder index of a path of symplectic matrices obtained from the linearization of the flow along γ, restricted to ξ. Since we will only be interested in 3-dimensional contact manifolds, we obtain

$$|\gamma| = \mu_{CZ}(\gamma) - 1.$$

We provide more details on the Conley-Zehnder index and how it may be computed in Section 9.

Before continuing our discussion of the chain complex and differential used in defining cylindrical contact homology, we review our choice of coefficients. In our formulation of cylindrical contact homology we will work with $\mathbb{Z}_2$-coefficients so that we do not have to worry about orienting the moduli spaces under consideration. In the classical setting [FH93], working over $\mathbb{Z}$ or $\mathbb{Q}$-coefficients requires one to prove that the moduli spaces connecting index difference 1 orbits are orientable and then choose a system of *coherent orientations* under which an analogue of Floer's gluing maps are orientation preserving. Once one has demonstrated such orientations exist and selects a choice of them one can define a number $\varepsilon(u) \in \{\pm 1\}$, where u is a rigid pseudoholomorphic cylinder, by comparing this coherent orientation of the index 0 moduli spaces $\hat{\mathcal{M}}$ with the obvious flow orientation.

Ideally cylindrical contact homology should encode the qualitative characteristics of the Reeb vector field associated to any contact form defining the contact structure ξ. At this time we are unable to provide a proof of invariance, due to the regularity difficulties in cylindrical cobordisms of a contact manifold presented by rigid pseudoholomorphic cylinders, which interpolate between orbits of the same even Conley-Zehnder index in a cobordism. However, we are still able to demonstrate that one can define cylindrical contact homology rigorously given a choice of nondegenerate dynamically separated contact form and generic J. We will revisit the issue of invariance in later work.

8.1 Reeb orbits and an action functional

The closed trajectories of the Reeb vector field appear naturally in the study of the following action functional as its critical points.

$$\begin{aligned} \mathcal{A}: \ C^\infty(S^1, M) &\rightarrow \mathbb{R} \\ \gamma &\mapsto \int_\gamma \alpha \end{aligned} \tag{8.2}$$

The set of critical values of the action functional is called the **action spectrum**. A standard exercise in symplectic topology tells us that the critical points of $\mathcal{A}$ are in one to one correspondence with closed Reeb orbits of α. This justifies the analogy between cylindrical contact homology and an infinite dimensional variant of Morse theory by making use of the previously discussed Fredholm theory under certain transversality assumptions.

Before allowing Reeb orbits to be contained in the chain group we must first impose a non-degeneracy condition on the critical points of our action functional, which are the periodic orbits of the Reeb vector field associated to α. In the finite dimensional setting of Morse theory one requires the Hessian matrix of $\mathcal{A}$ evaluated at critical points to be non-singular, which further implies that the critical points are isolated. In our setting we say that a periodic orbit γ is **nondegenerate** when the linearized return map along

γ has no eigenvalue equal to 1. See also the discussion in Section 2.

To make sense of this stipulation we recall some basic facts pertaining to linearizing the flow of a Reeb vector field along a closed orbit and see what transpires in the case of S^3 equipped with its standard contact form λ_0 as in Example 2.4. We saw in this example that the Reeb flow associated to λ_0 defines the Hopf fibration and in particular none of the closed Reeb orbits are isolated.

Example 8.1. In the case of the (S^3, λ_0) we can parametrize the Reeb flow using the ambient coordinates in $\mathbb{C}^2$ by

$$\varphi(t) \quad = \quad (e^{it}u, e^{it}v),$$

as

$$R_{\lambda_0} := \dot{\varphi}(0) \quad = \quad (ix_1 - y_1, ix_2 - y_2)$$

Thus

$$d\varphi(t) = \begin{pmatrix} e^{it} & 0 \\ 0 & e^{it} \end{pmatrix} \quad \text{and} \quad d\varphi(2\pi) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This shows that all of the Reeb orbits associated to the standard contact form are degenerate and as a result we will need to perturb this contact form.

The following result is contained in the appendix of [ABW10], applicable to any co-oriented contact manifold (M, α). It yields a perturbation that preserves a given Morse-Bott submanifold and makes α nondegenerate everywhere else. To accomplish this, it suffices to show that one can perturb α in some precompact subset and demonstrate that all orbits that pass through this subset are nondegenerate.

Proposition 8.2. *Let M be a manifold of dimension $2n-1$ equipped with a smooth contact form α, and let $\mathcal{U} \subset M$ be an open subset with compact closure. Then there* *exists a Baire*[2] *subset*

$$\Lambda_{reg}(\mathcal{U}) \subset \{f \in C^\infty(M) \mid f > 0 \text{ and } f|_{M\setminus\mathcal{U}} \equiv 1\}$$

such that for each $f \in \Lambda_{reg}(\mathcal{U})$*, every periodic orbit of* $R_{f\alpha}$ *passing through* $\mathcal{U}$ *is nondegenerate.*

This means that for a given contact structure ξ, one can find a generic contact form α such that all Reeb orbits are nondegenerate. The issue is that Reeb dynamics associated to generic contact forms are typically not easy to understand, nor can we guarantee that the Conley-Zehnder index has the desired properties such that the work of the previous sections will be applicable. The condition that α be dynamically separated is not a generic condition. We will see that there is a natural way to equip prequantization spaces with a contact form that is nondegenerate and dynamically separated up to a given action level. More details on this construction are given in the next chapter.

The **chain groups** C_* are comprised of all nondegenerate closed orbits $\gamma \in \text{Crit}(\mathcal{A})$ of action $\mathcal{A}(\gamma)$[3]. Simple orbits along with all of their multiple covers are each generators of the chain group. Recall that we say that a closed Reeb orbit γ is **simple** provided it is not a nontrivial multiple cover of another Reeb orbit. The grading is given by the aforementioned Conley-Zehnder index, a Maslov type index for arcs of symplectic matrices obtained by linearizing the Reeb flow along an orbit, restricted to the contact structure. This is possible when $c_1(\xi) = 0$ as we can choose a volume form for $\mathbb{R} \times M$ to carry out the computations, though the choice of volume form will be parametrized by $H^1(M)$. Since we restrict ourselves to dimension three, we define the grading of any

[2] This means that a countable intersection of dense open sets is dense.

[3] Note that action is synonymous with length.

closed Reeb orbit in this setting by

$$|\gamma| = \mu_{CZ}(\gamma) - 1. \tag{8.3}$$

We remark that the chain complex admits a splitting over free homotopy classes of Reeb orbits. Namely if we denote by the free homotopy classes by $a \in [\Sigma M] = [C^0(S^1, M)/S^1]$ then

$$C_* = \bigoplus_{a \in [\Sigma M]} C_*^a,$$

where C_*^a consists of all nondegenerate closed Reeb orbits representing the free homotopy class a of degree $*$. We will not discuss further details, as we are primarily interested in the case of S^3, which only has contractible Reeb orbits.

In order to prove that we can obtain a homology out of this chain complex construction, we additionally stipulate that the chain groups may only consist of Reeb orbits associated to a nondegenerate dynamically separated contact form. The next section provides the remaining details on how to construct the differential $\partial : C_* \to C_{*-1}$ and the proof that (C_*, ∂) is a chain complex.

8.2 Homological considerations

Throughout the entirety of this section we will assume that (α, J) is a **regular dynamically separated pair**, meaning that J is a generic compatible almost complex structure for α, a nondegenerate dynamically separatedcontact form associated to (M, ξ). We take C_* to be the vector space generated by the closed Reeb orbits of R_α. The set of all closed Reeb orbits associated to α will be denoted by $\mathscr{P}$, and we will implicitly assume that we always work with orbits in the same free homotopy class. For simplicity, we will work with $\mathbb{Z}_2$-coefficients instead of with $\mathbb{Q}$, so that we do not have to worry about the issue of coherent orientations.

In order to define the linear map $\partial : C_* \to C_{*-1}$, we will need to keep track of the multiplicity of orbits and cylinders. We denote by $\mathfrak{m}(\gamma)$ the **multiplicity** of a Reeb orbit γ, and $\mathfrak{m}(\mathcal{C})$ of a finite energy pseudoholomorphic cylinder $\mathcal{C}$. Specifically, if the finite energy pseudoholomorphic cylinder $\mathcal{C} \in \hat{\mathcal{M}}(\gamma_+;\gamma_-)$ represented by $u \in \mathcal{M}(\gamma_+;\gamma_-)$ may be written as the composition of a holomorphic map $\varphi : (\mathbb{R} \times S^1, j) \to (\mathbb{R} \times S^1, j)$ with $\pm\infty = \varphi(\pm\infty)$ where

$$u = v \circ \varphi, \quad \deg(\varphi) > 1.$$

and v is a simple cylinder, then

$$\mathfrak{m}(\mathcal{C}) = \deg(\varphi).$$

When $\mathcal{C}$ is not multiply covered, then $\mathfrak{m}(\mathcal{C}) = 1$.

By multiplicity of a Reeb orbit, we mean that $\mathfrak{m}(\gamma)$ is the unique positive integer such that γ is the $\mathfrak{m}_\gamma$-fold iterate of a simple Reeb orbit. The multiplicity factor is necessary to later demonstrate that $\partial^2 = 0$ because when one glues two somewhere injective (i.e. not multiply covered) finite energy pseudoholomorphic cylinders along a Reeb orbit γ which is the k-fold iterate of a simple Reeb orbit, there are k equivalent ways to fix a parametrization of γ . Recall also the results of Chapter 7. More details will be discussed as needed as we proceed with the proof that $\partial^2 = 0$.

Let $x,\ y$ be a pair of closed nondegenerate Reeb orbits, associated to a dynamically separated contact form in the same free homotopy class a such that $\mu_{CZ}(x) - \mu_{CZ}(y) = 1$.

We define the linear map

$$\begin{aligned} \partial: \quad C_*(M,\alpha) & \to C_{*-1}(M,\alpha) \\ x & \mapsto \sum_{\substack{\mu_{CZ}(y)=\mu_{CZ}(x)-1 \\ \mathcal{C}\in\hat{\mathcal{M}}(x;y)}} \left(\frac{\mathfrak{m}(y)}{\mathfrak{m}(\mathcal{C})} \bmod 2\right) y. \end{aligned} \tag{8.4}$$

Thus ∂ provides a weighted count of rigid pseudoholomorphic cylinders interpolating between the closed Reeb orbits x and y.

We write

$$\langle \partial x, y\rangle = \sum_{\substack{\mu_{CZ}(y)=\mu_{CZ}(x)-1 \\ \mathcal{C}\in\hat{\mathcal{M}}(x,y)}} \left(\frac{\mathfrak{m}(y)}{\mathfrak{m}(\mathcal{C})} \bmod 2\right).$$

Based on the preceding discussion of multiplicities of finite energy pseudoholomorphic cylinders and Reeb orbits we see that $\mathfrak{m}(\mathcal{C})$ divides $\mathfrak{m}(y)$, as well as $\mathfrak{m}(x)$, so the above expression for ∂ is well-defined over $\mathbb{Z}_2$-coefficients.

Remark 8.3 (Conventions on defining ∂)**.** There are a few ways in which earlier literature defines the differential. Our convention is as above so that we may directly appeal to the results of the gluing and structure theorems, Theorems 7.2, 7.3, and 7.16, which gives the compactified moduli space $\overline{\mathcal{M}}(x;z)$ the structure of a graph. We will use all these results to prove that $\partial^2=0$. Our convention in (8.4) agrees with Hutchings [Hu10]. In Ustilovsky [Us99] the multiplicity of finite energy pseudoholomorphic cylinders was not appropriately defined, but otherwise the conventions agree.

In a footnote 7 on page 599 of Eliashberg, Givental, and Hofer [EGH00] their convention (modulo sign) is to instead take

$$\langle \partial x, y\rangle = \sum_{\substack{\mu_{CZ}(y)=\mu_{CZ}(x)-1 \\ \mathcal{C}\in\hat{\mathcal{M}}(x,y)}} \left(\frac{n_{x,y}}{\mathfrak{m}(y)}\right),$$

where $n_{x,y}$ is indicated to count equivalence classes of pseudoholomorphic curves with asymptotic markers, which are elements of $\hat{\mathcal{M}}(x;y)$. They state $n_{x,y} = \mathfrak{m}(x)\mathfrak{m}(y)$ when $\mathcal{C} \in \hat{\mathcal{M}}(x;y)$ is not multiply covered, with no explanation of what to do in this case.

In Bourgeois [Bo09] and Yau [MLY04] the convention modulo sign is

$$\langle \partial x, y \rangle = \sum_{\substack{\mu_{CZ}(y)=\mu_{CZ}(x)-1 \\ \mathcal{C} \in \hat{\mathcal{M}}(x,y)}} \left(\frac{\mathfrak{m}(x)}{\mathfrak{m}(\mathcal{C})} \right).$$

However no analogue of the gluing or structure theorems appearing in Chapter 7 are stated. It is conceivable that there is a homological relation between these conventions and ours, should the appropriate gluing and structure theorems be established.

The reason that we take a weighted count is due to the fact that when we glue two non-multiply covered pseudoholomorphic cylinders along a Reeb orbit y which is the $\mathfrak{m}(y)$-fold iterate of an simple Reeb orbit, there are $\mathfrak{m}(y)$ different ways to glue. When one connects two multiply covered pseudoholomorphic cylinders $\mathcal{C}_i \in \hat{\mathcal{M}}(x;y)$ and $\mathcal{C}_j \in \hat{\mathcal{M}}(y;z)$, the results from Sections 7 and 7.4, gives us

$$k := \frac{\mathfrak{m}(y)}{\operatorname{lcm}(\mathfrak{m}(\mathcal{C}_i), \mathfrak{m}(\mathcal{C}_j))}$$

non-equivalent solutions via reparametrization in Theorem 7.2, the analogue of Floer's gluing theorem. Thus we must take into account the covering multiplicity of all pseudoholomorphic cylinders $\mathcal{C} \in \hat{\mathcal{M}}(x;y)$ in the coefficient appearing in the expression (8.4) for the cylindrical contact homology differential ∂. This will be apparent in the proof that (C_*, ∂) forms a chain complex.

The next step is to prove that after a generic choice of J, that the linear map ∂ defines an honest boundary operator under the assumption that the chain group C_* is generated by Reeb orbits associated to a nondegenerate dynamically separatedcontact form. Before proving Theorem 1.3 we restate it.

Theorem 8.4. *Let (M^3, ξ) be a co-oriented contact manifold with a nondegenerate dynamically separated contact form α defining ξ and J a generic compatible almost complex structure. The vector space $C_*(\alpha)$ generated by the closed Reeb orbits of α admits the linear map ∂, as defined in (8.4) by*

$$\begin{array}{rccl} \partial: & C_*(M,\alpha) & \to & C_{*-1}(M,\alpha) \\ & x & \mapsto & \displaystyle\sum_{\substack{\mu_{CZ}(y)=\mu_{CZ}(x)-1 \\ \mathcal{C}\in\hat{\mathcal{M}}(x;y)}} \left(\frac{\mathfrak{m}(y)}{\mathfrak{m}(\mathcal{C})} \ mod\ 2\right) y. \end{array}$$

Then $\partial^2 = 0$, thus (C_*, ∂) forms a chain complex.

As a result of the above theorem we can now define the homology of the complex $(C_*(\alpha), \partial)$ to be

$$HC_*(M, \alpha, \tilde{J}; \mathbb{Z}_2) = \frac{\ker\,\partial}{\mathrm{im}\,\partial},$$

which is called the **cylindrical contact homology** associated to the regular dynamically separated pair $(\alpha, \tilde{J})$.

The proof that we obtain a homology will be similar to those in Floer homology. However it does not follow immediately and the issues of multiply covered cylinders require some care and explanation. The assumption that J has ben chosen generically and that α is a nondegenerate dynamically separated contact form are crucial, as this allows us to exclude all the moduli spaces of nonpositive dimension which obstruct the inclusion of moduli spaces in Theorem 6.1). Moreover these assumptions enable us to appeal to the automatic transversality results of Wendl for finite energy cylinders of index 0, 1 and 2, as discussed in Chapter 6

In addition, the restriction to regular dynamically separated pairs $(\alpha, \tilde{J})$ allows us to appeal to the results of Chapter 7 which gave analogue of Floer's gluing theorem in Theorems 7.2 and 7.3. This was used to demonstrate that the reverse inclusion of moduli spaces holds, allowing us to obtain (8.5). The final component of the proof that $(C_*(\alpha), \partial)$ is a chain complex will be to use Section 7.4 to understand the structure associated to the compactified moduli space $\overline{\mathcal{M}}(x, z)$ as in Theorem 7.16.

Proof. The key is to understand the ends of the compactified 2-dimensional moduli space $\hat{\mathcal{M}}(x; z)$, which is the 1-dimensional moduli space $\overline{\mathcal{M}}(x; z)$. This is accomplished by proving $\overline{\mathcal{M}}(x; z)$ is a compact weighted graph whose boundary is given by

$$\partial\overline{\mathcal{M}}(x; z) \ = \bigcup_{\substack{y \in \mathscr{P} \\ \mu_{CZ}(y) = \mu_{CZ}(x) - 1}} \hat{\mathcal{M}}(x; y) \times \hat{\mathcal{M}}(y; z). \tag{8.5}$$

We combine this with the results from Sections 7 and 7.4, pertaining to gluing and the geometry of these moduli spaces, yielding

$$k := \frac{\mathfrak{m}(y)}{\operatorname{lcm}(\mathfrak{m}(\mathcal{C}_u), \mathfrak{m}(\mathcal{C}_v))}$$

non-equivalent solutions in Theorem 7.2, the analogue of Floer's gluing theorem. This geometric relation enables the following demonstration that the differential, which counts asymptotically cylindrical pseudoholomorphic curves interpolating between index difference 1 orbits, squares to 0.

Here x and z are closed Reeb orbits in the same free homotopy class a satisfying

$\mu_{CZ}(x) = \mu_{CZ}(z) + 2$. We will need to show

$$\begin{aligned}
\langle \partial^2 x, z \rangle &= \sum_{\substack{y \in \mathscr{P} \\ \mu_{CZ}(y) = \mu_{CZ}(x) - 1}} \langle \partial x, y \rangle \langle \partial y, z \rangle \\
&= \sum_{\substack{y \in \mathscr{P} \\ \mu_{CZ}(y) = \mu_{CZ}(x) - 1}} \sum_{\substack{\mathcal{C}_i \in \hat{\mathcal{M}}(x;y) \\ \mathcal{C}_j \in \hat{\mathcal{M}}(y;z)}} \frac{\mathfrak{m}(y)\mathfrak{m}(z)}{\mathfrak{m}(\mathcal{C}_i)\mathfrak{m}(\mathcal{C}_j)} \\
&\equiv 0 \bmod 2.
\end{aligned}$$

We know that $\overline{\mathcal{M}}(x;z)$ can be thought of as a labelled graph whose vertices correspond to the broken cylinders and edges correspond to the connected components of $\hat{\mathcal{M}}(x;z)$. From the surjectivity of gluing, Theorem 7.3, and our discussion in regard to parametrizations of multiply covered curves, Theorem 7.16, we know that each vertex representing $(\mathcal{C}_i, \mathcal{C}_j)$ belongs to

$$k := \frac{\mathfrak{m}(y)}{\operatorname{lcm}(\mathfrak{m}(\mathcal{C}_i), \mathfrak{m}(\mathcal{C}_j))}$$

edges, each corresponding to cylinders of multiplicity determined by the divisors of

$$\gcd\left(\mathfrak{m}(\mathcal{C}_i), \mathfrak{m}(\mathcal{C}_j)\right).$$

We can think of these multiplicities as labels on the edges of $\overline{\mathcal{M}}(x;z)$. For a fixed $r \in \mathbb{N}$ let $\overline{\mathcal{M}}_r(x;z)$ be the subgraph of $\overline{\mathcal{M}}(x;z)$ consisting of the edges labelled with r. The subgraph $\overline{\mathcal{M}}_r(x;z)$ is a union of connected components of $\overline{\mathcal{M}}(x;z)$. Since the number of ends of $\overline{\mathcal{M}}_r(x;z)$ is in correspondence with the components $\operatorname{int}(\overline{\mathcal{M}}_r(x;z))$, we obtain after adding up the indices of its vertices $(\mathcal{C}_i, \mathcal{C}_j)$, for a fixed subgraph labelled by a particular value of r,

$$\sum_{(\mathcal{C}_i,\mathcal{C}_j)\in\overline{\mathcal{M}}_r(x;z)} \frac{\mathfrak{m}(y)}{\operatorname{lcm}(\mathfrak{m}(\mathcal{C}_i),\mathfrak{m}(\mathcal{C}_j))} \equiv 0 \bmod 2.$$

Since r divides $\mathfrak{m}(z)$, we obtain

$$\sum_{(\mathcal{C}_i,\mathcal{C}_j)\in\overline{\mathcal{M}}_r(x;z)} \frac{\mathfrak{m}(y)}{\operatorname{lcm}(\mathfrak{m}(\mathcal{C}_i),\mathfrak{m}(\mathcal{C}_j))} \cdot \frac{\mathfrak{m}(z)}{r} \equiv 0 \bmod 2$$

Then summing up over all $r \in \mathbb{N}$ we obtain

$$\sum_{\substack{y\in\mathscr{P} \\ \mu_{CZ}(y)=\mu_{CZ}(x)-1}} \sum_{\substack{\mathcal{C}_i\in\hat{\mathcal{M}}(x;y) \\ \mathcal{C}_j\in\hat{\mathcal{M}}(y;z)}} \left(\frac{\mathfrak{m}(y)\mathfrak{m}(z)}{\mathfrak{m}(\mathcal{C}_i)\mathfrak{m}(\mathcal{C}_j)} \bmod 2\right) \equiv 0, \tag{8.6}$$

as desired.

□

Chapter 9

Grinding through gradings

This chapter provides the details on the computation Conley-Zehnder index of the Reeb orbits associated to $(S^3, \lambda_\varepsilon)$. The Conley-Zehnder index, a Maslov type index, is an integer assigned to a path of symplectic matrices. It controls the embedding behavior of asymptotically cylindrical pseudoholomorphic curves interpolating between nondegenerate Reeb orbits and provides the absolute $\mathbb{Z}$-grading in pseudoholomorphic curve homology theories. This is because the Fredholm index may be realized as the spectral flow of certain families of elliptic operators. In addition, the path of operators under consideration turn out to have the same crossings as the symplectic path associated to the Reeb orbits, and moreover these crossing forms are isomorphic. As a result we can express the Fredholm index of the linearized $\bar{\partial}_{\tilde{J}}$-operator in terms of the differences of Conley-Zehnder indices. More details can be found in [RS95], [HK99], [Sa99], and [SZ92].

Calculating this index requires some care as one must obtain a path of symplectic matrices from the flow of a Reeb or Hamiltonian vector field in a globally consistent way. When $c_1(\xi) = 0$ there exists a choice complex volume form on the symplectization $(\mathbb{R} \times M, \tilde{J})$, parametrized by $H^1(M)$, which can be used to linearize the Reeb flow along the periodic orbit, restricted to ξ. This yields a path of symplectic matrices from a nondegenerate Reeb orbit and the resulting computation of the Conley-Zehnder index is well-defined, and as a result one makes a slight abuse of language and refers to the Conley-Zehnder index of a Reeb orbit. The following discussion occurs in 3 dimensions, but can be generalized without much difficulty to $2n-1$ dimensions.

9.1 The Conley-Zehnder index

The Conley-Zehnder index is a generalized winding number which assigns an integer $\mu_{CZ}(\Psi)$ to every path of symplectic matrices $\Phi : [0,T] \to \mathrm{Sp}(n)$, with $\Phi(0) = \mathbb{1}$. One typically also stipulates that 1 is not an eigenvalue of the endpoint of this path of matrices, i.e. $\det(\mathbb{1} - \Psi(T)) \neq 0$, to ensure that the Conley-Zehnder index assigns the same integer to homotopic arcs (see [HK99]). This is precisely the situation encountered when linearizing the Hamiltonian or Reeb flow along a nondegenerate periodic orbit.

To obtain a path of symplectic matrices from a closed T-periodic Reeb orbit (γ, T), we must first fix a symplectic trivialization of ξ along γ. In general, $\mu_{CZ}(\gamma, T)$ *will depend on the choice made in the extension* $\sigma : D^2 \to M$ of γ. It will however be the same *for contractible orbits* if each pair of extensions can be homotoped into each other, which will be the case if $\pi_2(M) = 0$. This notion of an extension only makes sense when considering contractible orbits.

In the case of noncontractible orbits, the Conley-Zehnder index $\mu_{CZ}(\gamma, T)$ will be well-defined whenever $\xi \to M$ is a trivial vector bundle. It should be noted that one does not use the global symplectic trivialization of ξ, but rather a complex volume form on $(\mathbb{R} \times M, \tilde{J})$ in linearizing the Reeb flow along an orbit. This is because $c_1(\mathbb{R} \times M, \tilde{J})$ is the obstruction to the existence of a volume form on the symplectization and $c_1(\xi, J) = c_1(\mathbb{R} \times M, \tilde{J})$. We note that the choice of a complex volume form is parametrized by

$H^1(\mathbb{R} \times M)$, so the absolute integral grading is only determined up to the choice of volume form.

To go back to the situation of linearizing the flow along a contractible orbit we will need to use an auxiliary extension, which we call σ. For simplicity, we discuss the situation related to 3-dimensional contact manifolds, though the setting generalizes in the obvious manner to contact manifolds of arbitrary dimension. This is a smooth map $\sigma : D^2 \to M$ on the closed 2-disk in $\mathbb{C}$ which extends (γ, T), meaning it satisfies

$$\sigma(e^{2\pi i \frac{t}{T}}) = \gamma(t).$$

Then we consider the pullback bundle $\sigma^*\xi$ of the contact structure ξ associated to α. Note that the symplectic form $d\alpha$ on ξ induces a symplectic form $\omega = \sigma^* d\alpha$ on $\sigma^*\xi$, and there is a unique trivialization up to homotopy of $\sigma^*\xi$. From above this means that we pick a bundle isomorphism

$$\Upsilon : \sigma^*\xi \to D \times \mathbb{R}^2$$

such that the isomorphism Υ_p between the fibers over $p \in D$ satisfies $\Upsilon_p^* \omega_0 = \omega_p$. Here ω_0 denotes the standard symplectic form on $\mathbb{R}^2$.

Now we can consider the Reeb flow, which we will denote by (φ_t). Its linearization for each $t \in [0, T]$, is a symplectic map given by

$$A_t := d\varphi_t(\gamma(0))|_{\xi_{x(0)}} : \xi_{\gamma(0)} \to \xi_{\gamma(t)}.$$

This can be used to define a symplectic arc

$$\begin{aligned} \Psi : \quad [0, T] \quad &\to \quad \mathrm{Sp}(1) \\ t \quad &\mapsto \quad \Upsilon(e^{2\pi i \frac{t}{T}}) \circ A_t \circ \Upsilon^{-1}(1), \end{aligned} \tag{9.1}$$

where Sp(1) is the symplectic group of $\mathbb{R}^2$, i.e. symplectic 2×2 matrices, and Υ is the bundle isomorphism as discussed above. To each such arc Ψ there is an associated Conley Zehnder index $\mu_{CZ}(\Psi)$. We define the **Conley-Zehnder index** of γ by

$$\mu_{CZ}(\gamma, T) = \mu_{CZ}(\Psi).$$

Note that if Φ° is a closed loop then we have the following relation between the Conley-Zehnder index and the Maslov index μ

$$\mu_{CZ}(\Phi^\circ) = 2\mu(\Phi^\circ).$$

For details on how to define the Conley-Zehnder index of a path of symplectic matrices in terms of the Maslov index we refer the reader to [Sa99] or [RS93]. We will explain how to compute the Conley-Zehnder index in terms of crossing forms in the following section. Before doing this, we will need to discuss the properties associated to the Conley-Zehnder index.

The Conley-Zehnder index does not assign the same integer to all homotopic paths of symplectic matrices in

$$\Sigma(n) = \{\Psi : [0, T] \to \mathrm{Sp}(n) \ : \ \Psi \text{ is continuous, } T > 0 \text{ and } \Psi(0) = \mathbb{1}\}.$$

This issue can be overcome if we restrict to arcs in $\Sigma(1)$ which end at time T in the following open and dense set of all symplectic matrices which do not have 1 as an eigenvalue,

$$\mathrm{Sp}^*(n) := \{\Psi \in \mathrm{Sp}(n) \mid \det(\mathbb{1} - \Psi) \neq 0\}.$$

Such arcs of symplectic matrices will arise precisely when we linearize the flow of non-degenerate periodic orbits. We denote this set of symplectic matrices by

$$\Sigma^*(n) = \{\Psi \in \Sigma(n) \mid \Psi(T) \in \mathrm{Sp}^*(n)\}.$$

Proposition 9.1. *The space* $\mathrm{Sp}^*(n)$ *has two connected components,*

$$\begin{aligned}\mathrm{Sp}^*_{pos}(n) &= \{\Psi \in \mathrm{Sp}^*(n) \mid \Psi \text{ has positive, real eigenvalues } \}\\ &= \{\Psi \in \mathrm{Sp}^*(n) \mid \ \det(\mathbb{1} - \Phi) > 0\}\end{aligned}$$

and

$$\begin{aligned}\mathrm{Sp}^*_{neg}(n) &= \{\Psi \in \mathrm{Sp}^*(n) \mid \Psi \text{ has either complex or negative, real eigenvalues } \}\\ &= \{\Psi \in \mathrm{Sp}^*(n) \mid \ \det(\mathbb{1} - \Psi) < 0\}.\end{aligned}$$

The following theorem lists the various properties of the Conley-Zehnder index, and proofs may be found in [SZ92] and [HK99]. That the homotopy, loop and signature properties uniquely determine the Conley-Zehnder index is proven in [SZ92].

Theorem 9.2. *There exists a unique functor* μ_{CZ}*, called the Conley-Zehnder index, which assigns an integer* $\mu_{CZ}(\Psi)$ *to every path* $\Psi \in \Sigma^*(n)$ *and satisfies the following axioms.*

(Naturality) *For any path* $\Phi : [0, 1] \to \mathrm{Sp}(2n)$*,* $\mu_{CZ}(\Phi\Psi\Phi^{-1}) = \mu_{CZ}(\Psi)$*.*

(Homotopy) *The Conley-Zehnder index is constant on the components of* $\Sigma^*(n)$*. Equivalently, if* Φ^τ *is a homotopy of arcs in* $\Sigma^*(n)$ *then* $\mu_{CZ}(\Phi^\tau)$ *does not depend on* τ*.*

(Zero) *If* $\Psi(s)$ *has no eigenvalue on the unit circle for* $s > 0$ *then* $\mu_{CZ}(\Psi) = 0$*.*

(Product) *If* $n = n' + n''$ *identify* $\mathrm{Sp}(n') \oplus \mathrm{Sp}(n'')$ *in the obvious way with a subgroup of* $\mathrm{Sp}(n)$*. Then*

$$\mu(\Phi' \oplus \Phi'') = \mu(\Phi') + \mu(\Phi'').$$

(Loop) *If* $\Phi : [0, T] \to \mathrm{Sp}(n, \mathbb{R})$ *is a loop with* $\Phi(0) = \Phi(1) = \mathbb{1}$ *then*

$$\mu_{CZ}(\Phi\Psi) = \mu_{CZ}(\Psi) + 2\mu(\Phi).$$

(Signature) *If* $S = S^T \in \mathbb{R}^{2n\times 2n}$ *is a symmetric matrix with* $||S|| < 2\pi$ *and* $\Psi(t) = exp(J_0 St)$ *then*

$$\mu_{CZ}(\Psi) = \frac{1}{2}\mathrm{sign}(S),$$

where $\mathrm{sign}(S)$ *is the signature of* S*, i.e. the number of positive eigenvalues minus the number of negative eigenvalues.*

(Determinant) $(-1)^{\mu_{CZ}(\Psi)-n} = \mathrm{sign}(\det(\mathbb{1} - \Psi(1)))$.

(Inverse) $\mu_{CZ}(\Psi^{-1}) = \mu_{CZ}(\Psi^T) = -\mu_{CZ}(\Psi)$.

(Normalization) *For* $\Phi_{\frac{1}{2}}(t) = e^{\pi i t}$ *on* $[0, 1]$*, we have*

$$\mu_{CZ}(\Phi_{\frac{1}{2}}) = 1.$$

9.2 The beloved crossing form of Robbin and Salamon

We may alternately realize the Conley-Zehnder index in terms of crossing forms, and that both definitions agree is proven in [RS93]. Using crossing forms to compute the Conley-Zehnder is arguably more practical and extends our ability to compute the Conley-Zehnder index of arbitrary paths of symplectic matrices $\Psi(t) \in \Sigma(n)$. Robbin and Salamon use the crossing form to associate with every periodic solution a half integer μ_{RS} which agrees with μ_{CZ} in the nondegenerate case, i.e. when $\Psi(t) \in \Sigma^*(n)$.

To accomplish this we must realize $\Psi(t)$ as a smooth path of Lagrangian subspaces. To do this we review the construction of μ_{CZ} via the index of the Lagrangian path

$$\mathrm{Graph}(\Psi(t)) := \{(x, \Psi(t)x) \mid x \in \mathbb{R}^n\}$$

in $(\mathbb{R}^{2n} \times \mathbb{R}^{2n}, ((-\omega_0) \oplus \omega_0))$ relative to the diagonal

$$\Delta := \{(X, X) \mid X \in \mathbb{R}^{2n}\}.$$

Here ω_0 is the standard symplectic form on $\mathbb{R}^{2n}$. Assuming $\Psi(a) = \mathbb{1}$ and $\det(\mathbb{1} - \Psi(b)) \neq 0$ then the index of this Lagrangian path may be defined as follows,

$$\mu_{CZ}(\Psi) := \mu(\mathrm{Graph}(\Psi), \Delta).$$

This index is an integer and satisfies

$$(-1)^{\mu(\Psi)-n} = \operatorname{sign}\det(\mathbb{1} - \Psi(b)).$$

The above number is the parity of the Lagrangian frame $(\mathbb{1}, \Psi(b))$ for the graph of $\Psi(b)$. Next we explain how to compute this index via quadratic forms defined at crossing numbers.

As before we will take γ to be a nondegenerate closed Hamiltonian (or Reeb) orbit of period T. We fix a symplectic trivialization of ξ along γ as in the previous section so that the linearized flow

$$d\varphi_t : \xi_t \to \xi_{\phi_t(t)}$$

for $t \in [0, T]$ is represented by a path $\Psi_\gamma(t)$ of symplectic matrices, as in (9.1), such that $\Psi_\gamma(0) = \mathbb{1}$ and $\det(\Psi_\gamma(T) - \mathbb{1}) \neq 0$ In the case that γ is degenerate then we would obtain $\det(\Psi_\gamma(T) - \mathbb{1}) = 0$.

A number $t \in [0, T]$ is called a **crossing** if $\det(\Psi_\gamma(t) - \mathbb{1}) = 0$. We denote the set of crossings by

$$E_t := \ker(\Psi_\gamma(t) - \mathbb{1}).$$

For a crossing $t \in [0, T]$, the crossing form $\Gamma(\Psi_\gamma, t)$ is the quadratic form on E_t defined by:

$$\Gamma(\Psi_\gamma, t)(v) := d\alpha(v, \dot{\Psi}_\gamma v) \quad \text{for } v \in E_t.$$

If we are working strictly in $(\mathbb{R}^{2n}, \omega_0)$ we note the following in regard to the expression of the crossing form. Since any path in $\mathrm{Sp}(2n, \mathbb{R})$ is a solution to a differential equation $\dot{\Psi}(t) = J_0 S(t)\Psi(t)$, with $S(t)$ a symmetric matrix we can write the crossing form in $\mathbb{R}^{2n}$ as

$$\Gamma_0(\Psi(t), t)(v) = \langle v, S(t)v \rangle \tag{9.2}$$

A crossing t is **regular** whenever the crossing form at t is nonsingular. Note that regular crossings are necessarily isolated. Any path Ψ is homotopic with fixed end points to a path having only regular crossings. Recall that the **signature** of a nondegenerate quadratic form is the difference between the number of its positive eigenvalues and the number of its negative eigenvalues.

Robbin and Salamon define the index $\mu_{RS}(\Psi_\gamma)$ of the path Ψ_γ having only regular crossings to be

$$\mu_{RS}(\Psi_\gamma) := \frac{1}{2}\mathrm{sign}(\Gamma(\Psi_\gamma, 0)) + \sum_{0 < \text{all crossings } t < T} \mathrm{sign}(\Gamma(\Psi_\gamma, t)) + \frac{1}{2}\mathrm{sign}(\Gamma(\Psi_\gamma, T)).$$

In the case that we have taken the linearized flow of a nondegenerate Reeb orbit to obtain our path of symplectic matrices, i.e. $\Psi(t) \in \Sigma^*(1)$, we obtain

$$\mu_{RS}(\Psi_\gamma) := \frac{1}{2}\mathrm{sign}(\Gamma(\Psi_\gamma, 0)) + \sum_{0 < \text{all crossings } t \leq T} \mathrm{sign}(\Gamma(\Psi_\gamma, t)).$$

This is because $t = T$ is no longer a crossing as $\det(\Psi_\gamma(t) - \mathbb{1}) \neq 0$.

That both the Robbin-Salamon index and the Conley-Zehnder index agree for a path of symplectic matrices $\Psi(t) \in \Sigma^*(n)$ is proven in Robbin-Salamon [RS93]. The main features of the Robbin-Salamon index are the following.

Proposition 9.3. *The Robbin-Salamon index has the following properties.*

(i) *The Robbin-Salamon index satisfies additivity under concatenations of paths,*

$$\mu_{RS}\left(\Psi|_{[a,b]}\right)+\mu_{RS}\left(\Psi|_{[b,c]}\right)=\mu_{RS}\left(\Psi|_{[a,c]}\right)$$

(ii) *The Robbin-Salamon index characterizes paths up to homotopy with fixed end points.*

(iii) *The Robbin-Salamon index satisfies additivity under products,*

$$\mu_{RS}(\Psi'\oplus\Psi'')=\mu_{RS}(\Psi')+\mu_{RS}(\Psi'').$$

As an example of the usefulness of the crossing form expression for the Robbin-Salamon index we compute it for the symplectic path of matrices arising from the flow given by $\varphi_t(z)=e^{it}z$ on $\mathbb{C}$, equipped with the standard symplectic structure. Note that if we take $t\in[0,2\pi n]$ we do not obtain a path of symplectic matrices in $\Sigma^*(1)$ but we may still make use of crossing forms to compute the Robbin-Salamon index for this path.

Example 9.4. Note that the linearization is given by $d\varphi_t(z)\cdot v=e^{it}v$, so we will use

$$\Psi(t)=e^{it}$$

We obtain crossings for $t=2\pi n$ for every $n\in\mathbb{Z}_{\geq 0}$. Note that this is the one dimensional case in which we can express the symmetric matrix $S(t)$ which solves

$$\dot{\Psi}(t)=J_0S(t)\Psi(t)$$

as

$$S(t)=1$$

Then from (9.2) we know that the crossing form may be written as

$$\Gamma_0(\Psi, t)(v) = \langle v, v \rangle$$

For $t = 2\pi n$ with $n \in \mathbb{Z}_{\geq 0}$ we see that Γ_0 is nondegenerate and we obtain

$$\Gamma_0(\Psi, t)(v) = v\bar{v} = a^2 + b^2,$$

as we may write $v = a + ib$. This has signature $+2$, and thus on $[0, 2\pi n]$ with $n \in \mathbb{Z}_{>0}$ we have

$$\mu_{RS}(\Psi(t)) = 2n.$$

Note that if we take $\Psi(t)$ to be defined on the interval $[0, 2\pi n + \varepsilon]$ with $0 < \varepsilon < 2\pi$ then this would be a path of symplectic matrices in $\Sigma^*(1)$ and we would obtain

$$\mu_{CZ}(\Psi(t)) = \mu_{RS}(\Psi(t)) = 2n$$

Next we prepare a lemma giving the computation of the Robbin-Salamon index associated to Reeb orbits coming from the degenerate Hopf flow associated to λ_0 on S^3. This will be useful in the following section.

Lemma 9.5. *For a closed Reeb orbit γ_p^k associated to the degenerate Reeb flow on S^3 generated by the standard contact form λ_0, we have*

$$\mu_{RS}(\gamma_q^k) = 4k.$$

Proof. Recall that (2.2) as in Example 2.4 gives the standard contact form on S^3,

$$\lambda_0 = (x_1 dy_1 - y_1 dx_1 + x_2 dy_2 - y_2 dx_2)\,|_{S^3}. \tag{9.3}$$

The Reeb vector field associated to λ_0 is then given by

$$R = \left(x_1 \frac{\partial}{\partial y_1} - y_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial y_2} - y_2 \frac{\partial}{\partial x_2} \right).$$

For the purposes of computing the Robbin-Salamon index it will be more practical to reformulate the above expressions using complex coordinates. We obtain

$$\lambda_0 = \frac{i}{2}(ud\bar{u} - \bar{u}du + vd\bar{v} - \bar{v}dv)|_{S^3},$$

and

$$\begin{aligned} R &= (ix_1 - y_1, ix_2 - y_2) \\ &= (iu, iv) \\ &= i\left(u\frac{\partial}{\partial u} - \bar{u}\frac{\partial}{\partial \bar{u}} + v\frac{\partial}{\partial v} - \bar{v}\frac{\partial}{\partial \bar{v}}\right) \end{aligned} \tag{9.4}$$

Recall that

$$\varphi_t(u,v) = (e^{it}u, e^{it}v).$$

gives the flow of the Reeb vector field of (9.4). It also gives rise to a symplectomorphism of $\mathbb{C}^2 \setminus \{\mathbf{0}\}$, thereby allowing us to obtain a global trivialization which extends the trivialization around the closed orbits to the closed disks spanned by the orbits.

In this manner we have realized $S^3 \subset \mathbb{C}^2$. In fact, the standard contact 3-sphere sitting inside of $\mathbb{C}^2$ is an example of a strictly Levi pseudoconvex hypersurfaces. These carry a natural contact structure arising from the set of complex tangencies to their boundary. This can be seen as follows. Define the following function $f : \mathbb{R}^2 \to \mathbb{R}$

$$f(x_1, y_1, x_2, y_2) = x_1^2 + y_1^2 + x_2^2 + y_2^2,$$

then $S^3 = f^{-1}(1)$. Moreover at a point (x_1, y_1, x_2, y_2) in S^3 the tangent space is given by

$$T_{(x_1,y_1,x_2,y_2)}S^3 = \text{ker } df_{(x_1,y_1,x_2,y_2)} = \text{ker } (2x_1dx_1 + 2y_1dy_1 + 2x_2dx_2 + 2y_2dy_2).$$

Identifying $\mathbb{R}^4$ with $\mathbb{C}^2$ gives us the standard complex structure

$$J_0 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

We have $J_0 x_i = y_i, \;\; J_0 y_i = -x_i$ for $i = 1, 2$. The complex structure J_0 induces a complex structure on each tangent space in the obvious way, which we will also denote by J_0. Namely, we have for $i = 1, 2$

$$\begin{aligned} J_0 \frac{\partial}{\partial x_i} &= \frac{\partial}{\partial y_i} \\ J_0 \frac{\partial}{\partial y_i} &= -\frac{\partial}{\partial x_i} \end{aligned}$$

We now claim that $\xi = \mathrm{ker}\lambda_0$ is also equal to the set of complex tangencies. By this we mean

$$\xi = \mathrm{ker}\lambda_0 = TS^3 \cap J_0(TS^3). \tag{9.5}$$

Since

$$J_0(T_{(x_1,y_1,x_2,y_2)}S^3) = \ker\,(df_{(x_1,y_1,x_2,y_2)} \circ J)$$

and

$$df_{(x_1,y_1,x_2,y_2)} \circ J = -2x_1dy_1 + 2y_1dx_1 - 2x_2dy_2 + 2y_2dx_2,$$

we see that $\lambda_0 = -\frac{1}{2}(df \circ J)|_{S^3}$ and (9.5) holds as claimed.

Note that

$$T_p(\mathbb{C}^2 \setminus \{0\}) = \mathbb{C}^2$$

and

$$T_pS^3 = \xi_p \oplus \langle R(p) \rangle.$$

As a result of the above computations we obtain the following natural splitting of $\mathbb{C}^2$,

$$\mathbb{C}^2 \cong \xi_p \oplus \xi_p^\omega.$$

Here ξ_p^ω is the symplectic complement of ξ_p, defined as follows

$$\xi_p^\omega = \{v \in T_pS^3 \mid \omega(v,w) = 0 \text{ for all } w \in \xi_p\}.$$

On $\mathbb{C}^2 \setminus \{0\}$ we use the symplectic form $d(e^\tau\lambda_0)$ pulled back under the biholomorphism,

$$\begin{array}{rrcl} \psi : & \mathbb{C}^2 \setminus \{\mathbf{0}\} & \to & \mathbb{R} \times S^3 \\ & z & \mapsto & \left(\frac{1}{2}\ln|z|, \dfrac{z}{|z|}\right) \end{array}$$

which we denote by

$$\omega_0 = \omega_{\mathbb{C}^2\setminus\{\mathbf{0}\}} = \psi^*(d(e^\tau\lambda_0))$$

Note that we may write $\xi_p^{\omega_0}$ as the span of the following vector fields evaluated at p:

$$\begin{array}{rclcl} X & = & -i(u,v) & = & -i\left(u\dfrac{\partial}{\partial u} - \bar{u}\dfrac{\partial}{\partial \bar{u}} + v\dfrac{\partial}{\partial v} - \bar{v}\dfrac{\partial}{\partial \bar{v}}\right), \\ Y & = & (u,v) & = & \left(u\dfrac{\partial}{\partial u} - \bar{u}\dfrac{\partial}{\partial \bar{u}} + v\dfrac{\partial}{\partial v} - \bar{v}\dfrac{\partial}{\partial \bar{v}}\right). \end{array} \tag{9.6}$$

This is because we saw that

$$\xi_p = T_pS^3 \cap J_0(T_pS^3)$$

and if $v \in \xi_p$ then $J_0v \in \xi_p$. A similar result holds for vectors living in $\xi_p^{\omega_0}$. This will be instrumental in computing the Conley-Zehnder index in a moment.

Continuing in our calculation we check that X and Y as in (9.6) yield a standard symplectic or Darboux basis for the symplectic vector space $\xi_p^{\omega_0}$. Recall that this is equivalent to computing

$$\begin{array}{rclcl} \omega_0(X,Y) & = & -\omega_0(Y,X) & = & 1 \\ \omega_0(X,X) & = & \omega_0(Y,Y) & = & 0, \end{array}$$

which we easily obtain this in light of the above as $Y = \tilde{J}X = -\tilde{J}R_{\lambda_0} = \frac{\partial}{\partial \tau}$ with respect to the inclusion $\xi_p^{\omega_0} \subset T_p(\mathbb{R} \times M)$.

Here we have omitted the excessive decoration by p, although the reader should realize that we are evaluating the symplectic form at a point p on the vectors $X(p)$ and $Y(p)$. Alternatively one writes that ω_0 on $\xi_p^{\omega_0}$ is given by

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

As a result we know that ξ^{ω_0} is symplectically trivial, and that we are working thus far with an appropriate trivialization. This gives a means to check that ξ is symplectically trivial as well, as

$$T\mathbb{C}^2 \cong \xi \oplus \xi^{\omega}$$

and we know $c_1(T\mathbb{C}_2) = 0$, thus $c_1(\xi) = 0$.

As remarked earlier, we know that the Reeb flow may be extended to give rise to a symplectomorphism of $\mathbb{C}^2 \setminus \{\mathbf{0}\}$. As a result we can compute how the linearized flow acts on $\xi_p^{\omega_0}$. We obtain

$$\begin{aligned} d\varphi_t(X(p)) &= X(\varphi_t(p)), \\ d\varphi_t(Y(p)) &= Y(\varphi_t(p)). \end{aligned}$$

Note that a trivialization of ξ over any disc in M followed by the above trivialization of ξ^{ω_0} gives a trivialization of $T_p(\mathbb{C}^2 \setminus \{\mathbf{0}\})$ which is homotopic to the standard one.

As a result we may finally conclude that $d\varphi_t$ on $T_p(\mathbb{C}^2 \setminus \{\mathbf{0}\})$ is given by the "standard" differential of φ_t on $\mathbb{C}^2$, namely

$$d\varphi_t = \begin{pmatrix} e^{it} & 0 \\ 0 & e^{it} \end{pmatrix}.$$

Denote a closed Reeb orbit of period T associated to the flow φ, of interest to us by

$$\gamma_p(t) = \{\varphi_t(p) \mid t \in [0,T]\}$$

for $p \in \mathrm{Crit}(h^*F)$.

Then we can write

$$\Phi_{\mathbb{C}^2}(t) := d\varphi_t(p)|_{\mathbb{C}^2}$$

as the path of symplectic matrices associated to the linearized Reeb flow of γ_p extended to $\mathbb{C}^2 \setminus \{\mathbf{0}\}$ for $T \in [0,T]$. Similarly, denote by

$$\Phi_{\xi^{\omega_0}}(t)$$

to be the path of symplectic matrices associated to the linearized Reeb flow of γ_p for $T \in [0,T]$ restricted on the symplectic complement of ξ. Then the naturality, homotopy, and product properties of the Conley-Zehnder index yield

$$\mu_{CZ}(\gamma_p(t)) := \mu_{CZ}\left(d\varphi(t)|_\xi\right) = \mu_{CZ}\left(\Phi_{\mathbb{C}^2}(t)\right) - \mu_{CZ}\left(\Phi_{\xi^{\omega_0}}(t)\right).$$

Since

$$\begin{aligned} X(\varphi_t) &= -i(e^{it}u, e^{it}v) \\ Y(\varphi_t) &= (e^{it}u, e^{it}v) \end{aligned}$$

and

$$\begin{aligned} d\varphi_{2k\pi}(X(p)) &= -i(u,v) &= X(p) \\ d\varphi_{2k\pi}(Y(p)) &= (u,v) &= Y(p) \end{aligned}$$

we obtain

$$\Phi_{\xi^{\omega_0}}(2k\pi) = \mathbb{1},$$

Thus $\mu_{CZ}(\Phi_{\xi^{\omega_0}}(2k\pi)) = 0$. With the help of Example 9.5 we obtain

$$\mu_{CZ}(\gamma_p(t)) := \mu_{CZ}\left(d\varphi(t)|_\xi\right) = \mu_{CZ}\left(\Phi_{\mathbb{C}^2}(t)\right) = 4k,$$

as desired $\square$

9.3 The promised computation of μ_{CZ}

The purpose of this section will be to prove Theorem 9.7, which gives us the formula for the Conley-Zehnder index of closed Reeb orbits of R_ε over critical points p of H. We begin by studying the dynamics of R_ε. Recall that we can realize the contact 3-sphere $(S^3, \xi_{std} = \ker \lambda_0)$ as the Hopf fibration $S^1 \hookrightarrow S^3 \xrightarrow{h} S^2$, which is an example of a prequantization space, as explained in Section 1.1. We will perturb the contact form λ_0 by

$$\lambda_\varepsilon = (1 + \varepsilon h^* H)\lambda_0, \tag{9.7}$$

where H is a Morse-Smale function on (S^2, ω). With ε chosen sufficiently small we can gurantee that $\ker \lambda_\varepsilon = \ker \lambda_0 = \xi$ as $(1 + \varepsilon h^* H) > 0$. While the contact structure is unaffected by this perturbation, the associated Reeb dynamics will be affected. The perturbation of (9.7) alters the Reeb dynamics of R_{λ_0} by perturbing the entire critical S^2's worth of Reeb orbits of R_{λ_0} via H on S^2, which by construction is invariant under the S^1-action of the bundle.

Proposition 9.6. *The perturbed Reeb vector field associated to λ_ε is given by*

$$R_\varepsilon = \frac{R}{1 + \varepsilon h^* H} + \frac{\varepsilon \tilde{X}_H}{(1 + \varepsilon h^* H)^2}. \tag{9.8}$$

where X_H is a Hamiltonian vector field[1] *on S^2 and $\tilde{X}_H$ its horizontal lift,*

$$i.e. \quad dh(q)\tilde{X}_H(q) = X_{\varepsilon H}(h(q)) \quad \text{and} \quad \lambda_0(\tilde{X}_H) = 0.$$

Proof. We have the following splitting of TM with respect to the contact form λ_0,

$$T_pM = \langle R(p) \rangle \oplus \xi_p.$$

[1] We use the convention $\omega(X_H, \cdot) = dH$.

Thus we know that there exists $a, b \in \mathbb{R}$ and Y where $\lambda_0(Y) = 0$ such that

$$R_\varepsilon = aR + bY.$$

We will show that $a = \dfrac{1}{1+\varepsilon h^* H}$, $b = \dfrac{\varepsilon}{(1+\varepsilon h^* H)^2}$ and $Y = \tilde{X}_H$.

We know that R_ε is uniquely determined by the equations

$$\begin{aligned} \lambda_\varepsilon(R_\varepsilon) &= 1, \\ \iota(R_\varepsilon) d\lambda_\varepsilon &= 0. \end{aligned} \tag{9.9}$$

That a is of the desired form follows immediately from the first line of (9.9) as

$$\begin{aligned} \lambda_\varepsilon(R_\varepsilon) &= (1+\varepsilon h^* H)\lambda_0(aR) + (1+\varepsilon h^* H)\lambda_0(bY) \\ &= (1+\varepsilon h^* H)\lambda_0(aR) + 0. \end{aligned}$$

We compute to find

$$d\lambda_\varepsilon = (1+\varepsilon h^* H) d\lambda_\varepsilon + \varepsilon h^* dH \wedge \lambda_0.$$

Then

$$\begin{aligned} d\lambda_\varepsilon(R_\varepsilon, \cdot) &= (1+\varepsilon h^* H)\left(d\lambda_0(aR, \cdot) + d\lambda_0(bY, \cdot)\right) \\ &\quad + \varepsilon h^* dH(aR)\lambda_0(\cdot) - \varepsilon h^* dH(\cdot)\lambda_0(aR) \\ &\quad + \varepsilon h^* dH(bY)\lambda_0(\cdot) - \varepsilon h^* dH(\cdot)\lambda_0(bY), \end{aligned}$$

which reduces to

$$\begin{aligned} d\lambda_\varepsilon(R_\varepsilon, \cdot) &= (1+\varepsilon h^* H) d\lambda_0(bY, \cdot) + \varepsilon h^* dH(aR)\lambda_0(\cdot) \\ &\quad - \tfrac{\varepsilon}{(1+\varepsilon h^* H)} h^* dH(\cdot) + \varepsilon h^* dH(bY)\lambda_0(\cdot). \end{aligned} \tag{9.10}$$

Lest we forget about the symplectic form downstairs, recall

$$d\lambda_0 = h^* \omega$$

and

$$\omega(X_H, \cdot) = dH.$$

Also we have that

$$h^*dH(\cdot) = h^*\omega(X_H, \cdot) = d\lambda_0(\tilde{X}_H, \cdot)$$

and

$$h^*dH(\cdot) \wedge \lambda_0(\cdot) = d\lambda_0(\tilde{X}_H, \cdot) \wedge \lambda_0(\cdot).$$

Thus (9.10) becomes

$$\begin{aligned} d\lambda_\varepsilon(R_\varepsilon, \cdot) &= (1 + \varepsilon h^*H)d\lambda_0(bY, \cdot) + \varepsilon d\lambda_0(\tilde{X}_H, aR)\lambda_0(\cdot) \\ &\quad - \tfrac{\varepsilon}{(1+\varepsilon h^*H)} d\lambda_0(\tilde{X}_H, \cdot) + \varepsilon d\lambda_0(\tilde{X}_H, bY)\lambda_0(\cdot) \\ &= (1 + \varepsilon h^*H)d\lambda_0(bY, \cdot) - \tfrac{\varepsilon}{(1+\varepsilon h^*H)} d\lambda_0(\tilde{X}_H, \cdot) + \varepsilon d\lambda_0(\tilde{X}_H, bY)\lambda_0(\cdot). \end{aligned}$$

Now we see that

$$d\lambda_\varepsilon(R_\varepsilon, \cdot) = 0$$

precisely when $b = \dfrac{\varepsilon}{(1 + h^*H)^2}$ and $Y = \tilde{X}_H$ as desired. □

Theorem 9.7 gives us the formula for the Conley-Zehnder index of closed Reeb orbits of R_ε over critical points p of H. We will denote such Reeb orbits by γ_p and their $k-$fold cover by γ_p^k. The statement is as follows.

Theorem 9.7. *If ε_0 is chosen such that Proposition 9.11 holds and γ_p is a nondegenerate orbit over a critical point p of H, then all k-fold covers of γ_p associated to R_ε for all positive $\varepsilon \leq \varepsilon_0$ of action $T \leq 2\pi k$ are nondegenerate.*

We obtain the following formula for their Conley-Zehnder indices

$$\mu_{CZ}(\gamma_p^k) = 4k - 1 + \mathrm{index}_p(H). \tag{9.11}$$

Thus the grading[2] *for cylindrical contact homology is*

$$\begin{aligned} |\gamma_p^k| &= \mu_{CZ}(\gamma_p^k) - 1 \\ &= 4k - 2 + \text{index}_p(H). \end{aligned} \tag{9.12}$$

We remark that the contribution of $-1 + \text{index}_p(H)$ in the above theorem, relates to half the dimension of the base S^2 and the Morse index of H at a critical point, $\text{index}_p(H)$. The $4k$ is the contribution in the fiber direction to the Conley-Zehnder index. This will be made precise later in the proof.

We organize our work as follows. First we prove a few technical propositions in regards to the perturbed Reeb dynamics associated to R_ε. Namely, we will show that after a choice of sufficiently small ε_0, for all ε such that $0 < \varepsilon \leq \varepsilon_0$, the perturbation λ_ε yields that the only closed Reeb orbits of R_ε, which are of length $\leq 2\pi k$ remain in one fiber and must lie over a critical point of H. In other words these orbits are a multiple cover of a Hopf fiber of length 2π. Then we will prove that all these orbits of length $\leq 2\pi k$ are all nondegenerate. These will be crucial to the filtration set up in the following section and in obtaining a result for the cylindrical contact homology of $(S^3, \lambda_\varepsilon)$.

Lemma 9.8. *The closed orbits of R_ε which do not lie over critical points of H must cover orbits of X_H. Moreover, the periodicity of these orbits is proportional to $\frac{1}{\varepsilon}$.*

Proof. From (9.8) it is clear that the only fibers of the Hopf fibration that remain Reeb orbits associated to the perturbed contact form λ_ε are the fibers over the critical points

[2] Recall that the grading in contact homology of a Reeb orbit is $|\gamma| = \mu_{CZ}(\gamma) + n - 3$ where n appears in the dimension of the contact manifold M^{2n-1}.

of H. Since we have that the Reeb vector field associated to λ_ε is given by

$$R_\varepsilon = \frac{R}{1+\varepsilon h^* H} + \frac{\varepsilon \tilde{X}_H}{(1+\varepsilon h^* H)^2}.$$

The horizontal lift $\tilde{X}_H$ is determined by

$$dh(q)\tilde{X}_H(q) = X_{\varepsilon H}(h(q)) \quad \text{and} \quad \lambda_0(\tilde{X}_H) = 0.$$

where X_H is the Hamiltonian vector field defined by $\omega(X_H, \cdot) = dH$. The flow (φ_t^ε) of R_ε is determined by

$$\dot{\varphi_t^\varepsilon} = R_\varepsilon(\varphi_t^\varepsilon) \tag{9.13}$$

We know that that any additional solutions to (9.13) must be a lifts of a closed orbits of the Hamiltonian vector field X_H, however not all orbits of X_H may lift to closed orbits of R_ε. This is because the orbits of R_ε must close in both the fiber 'R'-component and base 'X_H'-component. Orbits which close in the fiber must have length at least $2\pi k$ for some $k \in \mathbb{Z}_{>0}$. Without loss of generality we can assume that the Morse-Smale function H is bounded between -1 and 1. Thus we have that

$$\frac{\varepsilon}{(1+\varepsilon)^2} < \frac{\varepsilon}{(1+\varepsilon h^* H)^2} \leq \frac{\varepsilon}{(1-\varepsilon)^2}$$

For $\varepsilon < 1$ we can use Taylor series to obtain that

$$\frac{\varepsilon}{(1-\varepsilon)^2} = \varepsilon + 2\varepsilon^2 + o(\varepsilon^2)$$

and

$$\frac{\varepsilon}{(1+\varepsilon)^2} = \varepsilon - 2\varepsilon^2 + o(\varepsilon^2).$$

Thus orbits of X_H which are m-periodic can only give rise to orbits of $\frac{\varepsilon \tilde{X}_H}{(1+\varepsilon h^* H)^2}$ which are $C\frac{m}{\varepsilon}$-periodic for some C. We note that C and m must be bounded away from 0 since X_H is time autonomous and the flow of the Hamiltonian vector field preserves the level sets of H. Since $\lambda_0(\tilde{X}_H) = 0$, we know that any periodic orbits of R_ε which cover periodic orbits of X_H must also be $C'\frac{m}{\varepsilon}$-periodic, for some $C' \geq C$. □

As a corollary we obtain that the orbits covering X_H must be much longer than the surviving orbits of R associated to R_ε. We will denote these surviving orbits by γ_p^k where p is a critical point of a smooth Morse-Smale function H. The k indicates that this is the k-fold cover of an underlying simple orbit, realized by the Hopf fiber over a critical point of H. In other words, the orbit γ_p^k has traversed the Hopf fiber over p exactly k times.

Remark 9.9. Note that the action of a Reeb orbit of γ_p^k R_ε over a critical point p of H is proportional by the length of the fiber, namely

$$\mathcal{A}(\gamma_p^k) = \int_{\gamma_p^k} \lambda_\varepsilon = 2k\pi.$$

Furthermore since h^*H is constant on critical points of p we have that γ_p^k is $(2\pi(1 + h^*H(p))$-periodic.

Corollary 9.10. *There exists a choice of ε_0 sufficiently small, i.e. $0 < \varepsilon_0 < \frac{1}{2}$ such that for all ε with $0 < \varepsilon < \varepsilon_0$ the Reeb orbits of R_ε covering closed orbits of X_H, have action greater than $2\pi k + 1 = \mathcal{A}(\gamma_p^k)$ for each $k \in \mathbb{Z}_{>0}$.*

Proof. An orbit which is a lift of X_H winds around on torus, which may be represented as a rectangle whose vertical length is given by the length of the fibers of the Hopf fibration 2π and whose horizontal width is determined by the length of the periodic orbit associated to X_H. This is illustrated in Figure 19 with a hypothetical orbit in blue.

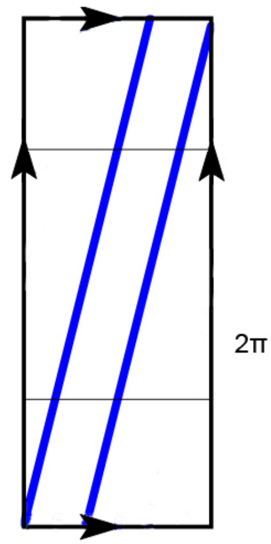


Figure 19: Hypothetical Reeb orbit of R_ε

We also know that orbits which close in the fiber are are T-periodic with

$$2\pi(1-\varepsilon) \leq T \leq 2\pi(1+\varepsilon).$$

Since the above lemma tells us that the periodicity of any orbits which do not lie over critical points of H is proportional to $\frac{1}{\varepsilon}$, the result follows as $\frac{C}{\varepsilon} \gg T$, once ε_0 has been chosen to be sufficiently small as C is bounded away from 0. Thus we can find ε_0, dependent on k, such that these orbits of R_ε are at least $2k(3\pi)$-periodic, and hence have action at least $4k\pi$ for all ε with $0 < \varepsilon < \varepsilon_0$. □

The following proposition shows that for small enough ε the surviving orbits of R, which lie over critical points of H in the R_ε are nondegenerate.

Proposition 9.11. *If H is chosen to be a smooth Morse-Smale function on S^2 there* *exists a choice of* $\varepsilon_0 > 0$ *such that for all* ε *with* $0 < \varepsilon < \varepsilon_0$ *the orbits* γ_p^k *associated to* R_ε *are also nondegenerate.*

Proof. Recall that we may write

$$T_q S^3 = \langle R_\varepsilon \rangle_q \oplus \xi_q,$$

and the Reeb vector field associated to λ_ε is given by

$$R_\varepsilon = \frac{R}{1+\varepsilon h^* H} + \frac{\varepsilon \tilde{X}_H}{(1+\varepsilon h^* H)^2}..$$

Throughout the point q will be chosen such that $h(q) = p$ is a critical point of H, where h is the Hopf map $S^1 \hookrightarrow S^3 \xrightarrow{h} S^2$. To prove that γ_p is nondegenerate we will need to demonstrate that $2\pi(1 + \varepsilon H(p))$-return map of the linearized flow (φ_t^ε) at q restricted to ξ, given by,

$$d\varphi_t^\varepsilon(q) : (\xi_q, d\lambda_\varepsilon) \to (\xi_{\varphi_t(q)}, d\lambda_\varepsilon)$$

does not have 1 as an eigenvalue.

We will want to consider the behavior of the linearized flow under the projection

$$h_* : T_q S^3 \to T_q S^2,$$

induced by the Hopf map h. The following computation,

$$\begin{aligned} d\lambda_\varepsilon &:= d\left((1+\varepsilon h^* H)\lambda_{S^3}\right)|_\xi \\ &= \left(d(\varepsilon h^* H) \wedge \lambda + \varepsilon h^* H d\lambda\right)|_\xi \\ &= (\varepsilon h^* H d\lambda)\,|_\xi. \end{aligned}$$

demonstrates that we can equip ξ with the standard symplectic form $h^*\omega_{S^3} = d\lambda_0$ because h^*H is constant along Hopf fibers over critical points, which are our Reeb orbits of interest. Thus we need only demonstrate that $h_* d\varphi_{2\pi}(q)$ has no eigenvalue equal to 1 in order to prove that γ_p is nondegenerate.

Note that the orbits of the unperturbed degenerate Reeb vector field R define the fibers of the Hopf fibration, see Example 2.4 we know that $R(h(q))$ will be normal to T_pS^2. Thus understanding $h_* d\varphi_{2\pi}(q)$ reduces to understanding the linearized flow ψ associated to the Hamiltonian vector field

$$\frac{X_{\varepsilon H}}{(1+\varepsilon H)^2}.$$

Recall from calculus that for

$$|\varepsilon h^* H| < 1$$

the following Taylor series centered at 0 is given by

$$\frac{1}{(1+\varepsilon H)^2} := \left(1 - \varepsilon H + (\varepsilon H)^2 + o(\varepsilon^2)\right)^2$$

Then we can express

$$\frac{X_{\varepsilon H}}{(1+\varepsilon H)^2} = \left(1 - 2\varepsilon H + o(\varepsilon)\right) X_H$$

and it suffices to consider the linearized flow of

$$(1 - 2\varepsilon H)\, X_{\varepsilon H}.$$

Moreover since X_H is a Hamiltonian vector field we know that it preserves the level sets of H, that is to say

$$X_H(H) = dH(X_H) = 0,$$

thus flows of X_H and HX_H commute, i.e.

$$[X_H, HX_H] = 0.$$

As a result if we can demonstrate that the time 2π-return map of the linearized flow associated to X_H does not have 1 as an eigenvalue then neither will the linearized flow associated to

$$(1 - 2\varepsilon H)\, X_{\varepsilon H}.$$

Since

$$-X_{\varepsilon H}(z) = \varepsilon J_0 \nabla H(z)$$

for all $z \in S^2$ then the linearization of the flow at a critical point p of H is a solution of the differential equation

$$d\dot{\psi}_t = D(-X_{\varepsilon H}(p)) \cdot d\psi_t.$$

Thus the linearization at a critical point must be of the following form,

$$d\psi_t(p) = e^{-\varepsilon t J_0 \nabla^2 H(p)}.$$

However since H was chosen to be a Morse function, its Hessian $\nabla^2 H$ must be nondegenerate at the critical point p. Thus for sufficiently small choice positive ε we see that $d\psi_t(p)$ is nondegenerate for the 2π-periodic orbits γ_p.

We note that the above arguments work for any multiple cover of the orbits over critical points of H, so the result follows.

□

With these details in place we can finish the proof of Theorem 9.7 in regard to the Conley-Zehnder indices of γ_p^k, which we will prove are given by

$$\mu_{CZ}(\gamma_p^k) = 4k - 1 + \text{index}_p(H),$$

To do this we employ an argument similar to the one found in [CFHW].

Proof. Note that we may use $d\lambda_{S^3}$ instead of $d\left((1+\varepsilon h^*H)\lambda_{S^3}\right)$ in computing the Conley-Zehnder indices for closed Reeb orbits over critical points of H as a result of the following computation. We have

$$\begin{aligned} d\left((1+\varepsilon h^*H)\lambda_{S^3}\right)|_\xi &= \left(d(\varepsilon h^*H)\wedge\lambda+\varepsilon h^*H d\lambda\right)|_\xi \\ &= \left(\varepsilon h^*H d\lambda\right)|_\xi. \end{aligned}$$

This tells us that h^*H is constant along Hopf fibers over critical points of H, which are precisely the nondegenerate Reeb orbits of interest to us. This justifies the use of equipping ξ with the standard symplectic form in the computation of the indices of the nondegenerate Reeb orbits over critical points of H, which we will do throughout this proof.

Consider the decomposition

$$T_{\tilde{q}}(\mathbb{R}\times S^3)=\mathbb{R}\oplus\langle R_\varepsilon(p)\rangle\oplus\xi_q,$$

where $\tilde{q}$ is the lift of q under the projection map $\pi:\mathbb{R}\times S^3\to S^3$. Since $p=h(q)$ is a critical point of H we see that the matrix of the linearization at $\tilde{q}$ with respect to this decomposition is given by

$$d\varphi_t^\varepsilon(\tilde{q})=\begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \\ & d\varphi_t^\varepsilon|_{\xi_q} \end{pmatrix}.$$

We denote by

$$\Phi_\varepsilon(t)=d\varphi_t^\varepsilon|_{\xi_q},$$

which is the linearization of the perturbed flow R_ε restricted to ξ_q. Note that when

$h(q) = p$ is a critical point of H then the Reeb orbits associated to R_ε are $2k\pi(1+\varepsilon H(p))$-periodic. We denote

$$T_k := 2k\pi(1 + \varepsilon H(p)).$$

We also denote by

$$\Phi(t) = \begin{pmatrix} e^{it} & 0 \\ 0 & e^{it} \end{pmatrix},$$

the linearization of the flow of the Hopf fibration restricted to ξ_q, and

$$\Psi_\varepsilon(t) = d\tilde{\psi}_t|_{\xi_q},$$

the linearization of the flow associated to $\tilde{X}_H$.

The homotopy

$$L(s,t) = \Phi_{s\varepsilon}(t)\Psi_{(1-s)\varepsilon}(t)$$

connects with fixed end points the path $\Phi_\varepsilon(t)$ to $\Phi(t)\Psi_\varepsilon(t)$. For small ε we know that these paths have ends in $\mathrm{Sp}^*(2)$, which is the set of symplectic matrices with eigenvalues not equal to 1.

Next we use the homotopy

$$K_0(s,t) = \begin{cases} L(s, \frac{2t}{s+1}) & \text{if } t \leq T_k \cdot \frac{s+1}{2} \\ L(2\frac{t}{T_k} - 1, T_k) & \text{if } t \geq T_k \cdot \frac{s+1}{2} \end{cases}$$

together with the fact that $L(s, T_k) \in Sp^*(2)$ for $s \in [0,1]$ and the aforementioned properties of the Conley-Zehnder index and Robbin-Salamon index we obtain

$$\mu_{RS}(\Phi_\varepsilon) = \mu_{RS}(\Phi\Psi_\varepsilon).$$

Another homotopy,

$$K_1(s,t) = \begin{cases} \Phi(\frac{2t}{s+1})\Psi_\varepsilon(st) & \text{if } t \leq T_k \cdot \frac{s+1}{2} \\ \Phi(T_k)\Psi_\varepsilon((s+2)t - (s+1)) & \text{if } t \geq T_k \cdot \frac{s+1}{2} \end{cases}$$

for $(s,t) \in [0,1] \times [0,T_k]$ combined with the aforementioned properties of the Conley-Zehnder index and Robbin-Salamon index implies that

$$\mu_{RS}(\Phi\Psi_\varepsilon) = \mu_{RS}(\Phi) + \mu_{RS}(\Phi(T_k)\Psi_\varepsilon).$$

As a result we obtain

$$\mu_{RS}(\Phi_\varepsilon) = \mu_{RS}(\Phi) + \mu_{RS}(\Phi(T_k)\Psi_\varepsilon).$$

In the proof of Proposition 9.11, we saw that since

$$-X_{\varepsilon H}(z) = \varepsilon J_0 \nabla H(z)$$

for all $z \in S^2$ then the linearization of the flow associated to X_H at a critical point p of H is a solution of the differential equation

$$\dot{d\psi_t} = D(-X_{\varepsilon H}(p)) \cdot d\psi_t.$$

Thus the linearization at a critical point must be of the following form,

$$d\psi_t(p) = e^{\varepsilon t J_0 \nabla^2 H(p)}.$$

We have the decomposition

$$T_q S^3 = \langle R \rangle_q \oplus \xi_q$$

Under the Hopf map, where again $h(q) = p$ is a critical point of H we further have that

$$h_*(T_q S^3) = h_*(\xi_q) = T_p S^2$$

Thus if we extend the flow ψ_t of X_H to the symplectization at a critical point we obtain the following expression for the Hessian of H in the decomposition $T_q(\mathbb{R} \times S^3) =$

$\mathbb{R}^2\langle p\rangle \oplus T_pS^2$, where h^*H has been extended in the obvious way to $\mathbb{R}\times S^3$ to $\tilde{H}$.

$$\nabla^2\tilde{H}(\tilde{q}) = \begin{pmatrix} 0 & 0 \\ 0 & \nabla^2 H(p) \end{pmatrix}$$

This means that $\ker d\psi_t = \{0\}$ for $t>0$ and one is reduced to compute the intersection form only for $t=0$ to obtain the contribution from (the lift of) $-X_H$ to the Robbin-Salamon index. We have

$$\ker(\mathbb{1} - \Psi_\varepsilon(0)) = T_pS^2$$

where Ψ_ε is the symplectic matrix associated to the linearized flow of $-X_H$ at p. Thus we obtain

$$\text{sign }\Gamma(\Psi_\varepsilon,0) = \text{sign }\Gamma_0(\Psi_\varepsilon,0) = \text{sign }\varepsilon\nabla^2 H(p).$$

Recalling the following shift identity from Morse theory, where f is a Morse function on M with critical point p,

$$-\frac{1}{2}\text{sign Hess }f(p) = \text{index}_p f - \frac{\dim M}{2},$$

we obtain

$$\mu_{RS}(\Phi(T_k)\Psi_\varepsilon(t)) = -1 + \text{index}_p H.$$

By Lemma 9.5 we obtain for $0\le t\le T_k := 2k\pi(1+\varepsilon H(p))$

$$\mu_{RS}(\Phi) = 4k.$$

Thus since γ_p^k is nondegenerate we have

$$\mu_{CZ}(\Psi_\varepsilon) = \mu_{RS}(\Psi_\varepsilon) = 4k - 1 + \text{index}_p H.$$

□

Chapter 10

Fun with filtrations

In this chapter we provide the details on the construction of a filtered chain complex by the action and index. We can compute the truncated cylindrical homology for $(S^3, \lambda_\varepsilon)$, and use the filtration to obtain a direct limit argument to recover the full cylindrical contact homology. Since the issue of invariance remains unresolved we cannot make a more meaningful statement as far as the qualitative implications this limit of the truncated contact homology groups has on the contact structure. We conjecture that this limit should be independent of other choices of Morse-Smale functions H or choices of dynamically separated contact forms.

These methods may be generalized to apply to other prequantization spaces and S^1-bundles over symplectic orbifolds. In future work we will use these methods to compute cylindrical contact homology for the lens spaces $(L(n+1,n), \lambda_{std})$. We begin with the same perturbation of the contact form obtained via prequantization of Section 1.1, namely

$$\lambda_\varepsilon = (1 + \varepsilon h^* H)\lambda_0.$$

Here h is the Hopf fibration $S^1 \hookrightarrow S^3 \xrightarrow{h} S^2$ and H is a Morse-Smale function on the base S^2. Recall that the perturbed Reeb dynamics associated to λ_ε are given by

$$R_\varepsilon = \frac{R}{1 + \varepsilon h^* H} + \frac{\varepsilon \tilde{X}_H}{(1 + \varepsilon h^* H)^2}.$$

Here X_H is a Hamiltonian vector field[1] on S^2 and $\tilde{X}_H$ its horizontal lift,

$$\text{i.e. } dh(q)\tilde{X}_H(q) = X_H(h(q)) \quad \text{and} \quad \lambda_0(\tilde{X}_H) = 0.$$

This results in a perturbation of the critical manifold of Reeb orbits associated to the unperturbed contact form λ_0 via the Morse-Smale function H on the base S^2. By construction this perturbation is invariant under the S^1-action of the bundle, and further details as to the dynamics associated perturbed Reeb vector field R_ε were discussed in the previous section. The computations of Propositions 9.8 and 9.11 tell us that there exists a sufficiently small ε_0 such that for all positive ε with $\varepsilon \leq \varepsilon_0$ the only closed Reeb orbits of the perturbed Reeb vector field R_ε of action less than

$$\mathcal{T}_k := 2\pi k + 1$$

must lie in one fiber and occur as a k-fold multiple cover of a simple Reeb orbit lying over a critical point of the Morse-Smale function in the base. We denoted these Reeb orbits by γ_p^k.

Theorem 9.7 establishes a proportionality between the action and index of the Reeb orbits γ_p^k as we obtain

$$\mu_{CZ}(\gamma_p^k) = 4k - 1 + \text{index}_p H.$$

This natural filtration on both the action and the index and allow us to compute a formal version of filtered cylindrical contact homology. The proportionality between the action and index of the Reeb orbits, permits the use of direct limits to recover the full cylindrical contact homology from the truncated chain groups. This process is analogous to the approach taken in symplectic cohomology, however we will not make

[1] We use the convention $\omega(X_H, \cdot) = dH$.

use of continuation maps and will instead appeal directly to the proportionality that has been established between the action and index of the Reeb orbits.

10.1 The truncated chain complex

The truncated chain complexes, consist of all nondegenerate Reeb orbits of action less than $\mathcal{T}_k := 2\pi k + 1$. We will denote these by

$$C_*^{<\mathcal{T}_k}(M, \lambda_\varepsilon) := \{\gamma \mid \gamma \text{ is a closed Reeb orbit and } \mathcal{A}(\gamma) < \mathcal{T}_k\}.$$

Recall that $\mathcal{A}(\gamma)$ is the action functional given by

$$\begin{aligned} \mathcal{A}: \quad C^\infty(S^1, M) &\to \mathbb{R}, \\ \gamma &\mapsto \int_\gamma \alpha. \end{aligned}$$

Given the proportionality between the choice of ε in the equation for the perturbed contact form and the action of the Reeb orbits we can further index the truncated chain complexes by the choice of ε in the equation for the perturbed contact form.

$$\lambda_\varepsilon = (1 + \varepsilon h^* H)\lambda_0.$$

From Proposition 9.8 we know that for every $\mathcal{T}_k$ there exists a choice of ε_k, dependent on k such that the only Reeb orbits of

$$R_\varepsilon = \frac{R}{1 + \varepsilon h^* H} + \frac{\varepsilon \tilde{X}_H}{(1 + \varepsilon h^* H)^2}.$$

of action less than $\mathcal{T}_k$ must be a k-fold cover of a Reeb orbit lying over a critical point p of H. Note that this is also true for any $\varepsilon < \varepsilon_k$. We denoted these orbits by γ_p^k. We will further decorate these truncated chain complexes with the choice of ε

$$C_*^{<\mathcal{T}_k}(S^3, \lambda_{\varepsilon_j}),$$

provided that $\varepsilon_j \leq \varepsilon_k$.

First we show this construction yields an isomorphism between the filtered chain complexes $C_*^{<\mathcal{T}_k}(S^3, \lambda_{\varepsilon_i})$ and $C_*^{\mathcal{T}_{k+1}}(S^3, \lambda_{\varepsilon_{i+1}})$, where $\varepsilon_{i+1} < \varepsilon_i < \varepsilon_k$. Throughout when we we say that $\varepsilon_k > 0$ has been chosen sufficiently small we mean that Lemma 9.8 and Proposition 9.11 hold for all Reeb orbits of period $< \mathcal{T}_k$, thus we have

$$C_*^{<\mathcal{T}_k}(S^3, \lambda_{\varepsilon_k}) = \{\gamma_p^j \mid j \in [1, k] \text{ and } p \in \mathrm{Crit}(H)\}. \tag{10.1}$$

Proposition 10.1. *For fixed k, if $\varepsilon_k > 0$ is sufficiently small and if $\varepsilon_i > \varepsilon_{i+1} > \varepsilon_k > 0$ there exists a chain map*

$$\phi : C_*^{<\mathcal{T}_k}(S^3, \lambda_{\varepsilon_i}) \to C_*^{<\mathcal{T}_k}(S^3, \lambda_{\varepsilon_{i+1}}).$$

Proof. This follows from the truncation on the action and (10.1) since the results of Propositions 9.8 and 9.11 hold, combined with Theorem 9.7. This is because we know that both $C_*^{<\mathcal{T}_k}(S^3, \lambda_{\varepsilon_i})$ and $C_*^{<\mathcal{T}_k}(S^3, \lambda_{\varepsilon_{i+1}})$ are only supported up to degree

$$\sup |\gamma_p^k| = 4k - 1 + \sup \mathrm{index}_p H - 1 = 4k$$

as $0 \leq \mathrm{index}_p H \leq 2 < 4$. □

Next we show the inclusion of chain complexes after choosing a fixed ε_0 as we allow the truncation level to increase to $\mathcal{T}_0$. Here $\mathcal{T}_0$ is the maximal action such that the results of Propositions 9.8 and 9.11 continue to hold.

Proposition 10.2. *For fixed sufficently small $\varepsilon_0 > 0$ there exists the following inclusion of chain complexes for any $\varepsilon_i \leq \varepsilon_0$ and $\mathcal{T}_{k+1} \leq \mathcal{T}_0$*

$$\iota : C_*^{<\mathcal{T}_k}(S^3, \lambda_{\varepsilon_i}) \hookrightarrow C_*^{<\mathcal{T}_{k+1}}(S^3, \lambda_{\varepsilon_{i+1}}).$$

provided all Reeb orbits of action

Proof. This follows immediately from the fact that the action of permitted Reeb orbits in the chain complex increases as ε_i decreases. □

The Conley-Zehnder index considerations of the previous section establish that this perturbed contact form indeed satsifies the dynamically separated condition for all Reeb orbits of action less than $2\pi k + 1$ for some choice of ε. In view of the truncation we can establish that the perturbed contact form is dynamically separated, as all Reeb orbits whose action is less than the truncated action satisfy the uniform growth of Conley-Zehnder index, with simple orbits starting with a Conley-Zehnder indexof 3, 4, or 5.

From the previous section, we saw that the following contact form with a sufficiently small choice of ε and with H the height function on S^2 yields that all the Conley-Zehnder indices of the Reeb orbits over critical points of H are odd. As a result we are able to conclude that the differential vanishes for each of the truncated cylindrical contact homologies, $H_*(C_*^{<\mathcal{T}_k}(M, \lambda_{\varepsilon_k}))$

We have established the existence of the chain map between the truncated chain complexes and inclusion between the truncated chain complexes associated to λ_{ε_i} and λ_{ε_j} for $\varepsilon_i > \varepsilon_j$ chosen sufficiently small in Propositions 10.1 and 10.2. Since these morphisms are compatible with the filtration on the action and index we can take direct limits to formally recover the limit of the truncated cylindrical contact homology of S^3, that is dependent on the choice of H and J. We obtain

$$\varinjlim_{\varepsilon\to 0}\varinjlim_{k\to\infty} H_*(C_*^{<\mathcal{T}_k}(M,\lambda_\varepsilon)) = \varinjlim_{k\to\infty}\varinjlim_{\varepsilon\to 0} H_*(C_*^{<\mathcal{T}_k}(M,\lambda_\varepsilon))$$
$$= \langle\gamma_{\max}^k\rangle_{k\in[1,\infty)} \oplus \langle\gamma_{\min}^k\rangle_{k\in[1,\infty)})$$
$$= \begin{cases} \mathbb{Z}_2 & * \geq 2, \text{ even} \\ 0 & * \text{ else.} \end{cases}.$$

This yields the proof of Theorem 1.7.

We conjecture that we should be able to obtain invariance for other choices of H and other dynamically separated contact forms associated to (S^3, ξ_{std}) in regards to the above theorem. These issues of invariance will be addressed in future work.

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