Categorical Enumerative Invariants of Elliptic Curves

by

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Abstract

Categorical enumerative invariants (CEI) constitute a specific class of invariants associated with a smooth, proper, and cyclic $\mathcal{A}_\infty$-algebra and a splitting of its non-commutative Hodge filtration. It is conjectured that they encompass all currently known enumerative invariants in both symplectic and complex geometry.

Each project in this thesis is centered around the exploration of CEI, with a particular focus on elliptic curves.

The first project concerns a conjecture regarding the Taylor expansion of the j-function around the hexagonal and square points.

In the second project we address a crucial property of Gromov-Witten invariants: the holomorphic anomaly equation. In this project, we demonstrate that, under certain modularity assumptions, this equation is also satisfied by CEI invariants.

The third project studies the degeneration of the Hodge to de Rham spectral sequence of the nodal cubic curve. Within this project we classify all liftable Hochschild classes. This classification is important in the computation of the $(2, 1)$-CEI invariant of the nodal cubic curve.

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Chapter 1

Introduction

1.1 Descendent Gromov-Witten invariants

1.1.1. The genus g Gromov-Witten invariants of a smooth symplectic manifold $\check{X}$ are numerical invariants constructed via the moduli space/stack $\overline{M}_{g,n}(\check{X},\beta)$ of stable maps into $\check{X}$. In essence, they are designed to count the "number" of pseudo-holomorphic curves intersecting n chosen submanifolds within $\check{X}$.

These invariants are defined by constructing the moduli space $\overline{M}_{g,n}(\check{X},\beta)$, parameterizing stable maps f from curves C of genus g with n marked points $p_1,\ldots,p_n$ to $\check{X}$, satisfying $f_*[C]=\beta\in H_2(\check{X})$:

$$\overline{M}_{g,n}(\check{X},\beta)=\left\{f:(C,p_1,\ldots,p_n)\to\check{X}\right\}.$$

This space naturally maps via a forgetful map π to $\overline{M}_{g,n}$ which forgets the map f; it also maps via evaluation maps ev_i to X, sending f to $f(p_i)$:

$$\begin{array}{ccc}\overline{M}_{g,n}(\check{X},\beta) & \xrightarrow{\mathrm{ev_i}} & X\\ \downarrow{\scriptstyle\pi} & & \\ \overline{M}_{g,n} & & \end{array}$$

Next one can construct a virtual fundamental class $[\overline{M}_{g,n}(\check{X},\beta)]^{\mathrm{vir}} \in H_*(\overline{M}_{g,n}(\check{X},\beta))$ for this moduli space and compute integrals on it. Specifically, one defines the *genus g descendent Gromov-Witten invariant* by the formula:

$$\langle \tau_{k_1}(\gamma_1)\dots\tau_{k_n}(\gamma_n)\rangle_{g,n}^{\check{X},\beta} := \int_{[\overline{M}_{g,n}(\check{X},\beta)]^{\mathrm{vir}}} \prod_{i=1}^{n} \mathrm{ev}_i^*(\gamma_i)\cdot \pi^*(\psi_i^{k_i}) \in \mathbb{Q}.$$

Here, $\gamma_i \in H^*(\check{X},\Lambda)$ represent cohomology classes, and $\psi_i \in H^2(\overline{M}_{g,n})$ denote cotangent line classes, where Λ is an appropriate Novikov field.

1.1.2. We begin by introducing some notations to facilitate the packaging of the aforementioned invariants into a single invariant.

Let u be a formal variable of homological degree -2, utilized for tracking ψ-class insertions. We denote H_-, H_+, and H_{Tate} as the graded vector spaces

$$H_- = H[u^{-1}],\quad H_+ = H[\![u]\!],\quad H_{\mathsf{Tate}} = H(\!(u)\!),$$

associated to the graded vector space $H = H^*(\check{X},\Lambda)$. We define the *residue pairing* $\langle -,-\rangle_{\mathsf{res}} : H_{\mathsf{Tate}} \otimes H_{\mathsf{Tate}} \to \Lambda$ as follows:

$$\langle x\cdot u^k, y\cdot u^l\rangle_{\mathsf{res}} = \begin{cases} (-1)^l\langle x,y\rangle_{\mathsf{Poincare}} & \text{if } k+l=0 \\ 0 & \text{otherwise.}\end{cases}$$

For a fixed pair (g,n), the descendant invariants mentioned earlier are consolidated into a single invariant $F_{g,n}^{\check{X}} \in \mathrm{Sym}^n(H_-)$ defined by the requirement:

$$\langle F_{g,n}^{\check{X}}, (\gamma_1 u^{k_1})\cdots(\gamma_n u^{k_n})\rangle_{\mathsf{res}} = \sum_{d=0}^{\infty} \langle \tau_{k_1}(\gamma_1)\dots\tau_{k_n}(\gamma_n)\rangle_{g,n}^{\check{X},d}.$$

Additionally, we sometimes employ the notation:

$$F_{g,n}^{\check{X}}(\gamma_1 u^{k_1},\dots,\gamma_n u^{k_n}) := \langle F_{g,n}^{\check{X}}, (\gamma_1 u^{k_1})\cdots(\gamma_n u^{k_n})\rangle_{\mathsf{res}}$$

to represent the pairing with insertions.

1.1.3. One noteworthy feature of Gromov-Witten invariants is their characterization as (real) symplectic deformation invariants. This means that although a specific almost complex structure is required for their computation, the outcome remains independent of this choice. Moreover, these invariants establish a link to algebraic geometry through the concept of mirror symmetry.

1.2 Mirror symmetry

The initial formulation of enumerative mirror symmetry, as proposed in [COGP91], introduced a groundbreaking conjecture. It conjectured that the genus-zero Gromov-Witten invariants of a quintic threefold $\check{X}$ could be determined by solving a differential equation linked to the variation of Hodge structure associated with another space known as the *mirror quintic* X. Subsequently physicists discovered additional mirror pairs $(X, \check{X})$, all of which exhibited similar connections between the genus-zero Gromov-Witten invariants of $\check{X}$ and the variation of Hodge structure of X. Notably, these connections relied solely on the symplectic structure of $\check{X}$ and the complex structure of X.

A broader conceptualization of enumerative mirror symmetry was proposed by Kontsevich in his work [Kon95]. In this generalization, he conjectured a deeper connection between the spaces X and $\check{X}$ within a mirror pair. Specifically, he suggested the existence of a derived equivalence of categories between the derived category $D^b_{\text{coh}}(X)$ of coherent sheaves and the Fukaya category $\text{Fuk}(\check{X})$. This assertion is famously known as *homological mirror symmetry.*

1.3 Categorical enumerative invariants

1.3.1. Kontsevich's proposal implied that enumerative mirror symmetry should follow from homological mirror symmetry. To achieve this, he conjectured the existence of some invariants which are now known as *categorical enumerative invariants.* These invariants are associated with an enriched triangulated category $\mathcal{C}$ and possess the property that when the Fukaya category $\mathrm{Fuk}(\check{X})$ is used as input, they are expected to recover the Gromov-Witten invariants of $\check{X}$. Similarly, when the derived category $D^b_{\mathrm{coh}}(X)$ of a Calabi-Yau variety X is provided as input, they should yield a new set of invariants associated with X, known as the *B-model Gromov-Witten invariants.*

It's worth noting that these invariants are defined for all genera, not just genus zero. However, it is expected that the genus zero B-model Gromov-Witten invariants will align with the invariants obtained from the variation of Hodge structures. Consequently, when $(X, \check{X})$ forms a mirror pair, the B-model invariants of X correspond to the Gromov-Witten invariants of $\check{X}$, thereby implying classical enumerative mirror symmetry.

1.3.2. Let's now explore the historical context surrounding the development of categorical enumerative invariants. In genus zero, categorical Gromov-Witten invariants satisfying the aforementioned properties were first introduced by Ganatra, Perutz, and Sheridan in their work [GPS15]. Their approach built upon the insights of Saito [Sai83a; Sai83b] and Barannikov-Kontsevich [Bar01; BK98]. However, it's important to note that their construction does not extend to higher genera. Separately Costello [Cos09] introduced a definition of categorical invariants for arbitrary genera, drawing inspiration from the work of Kontsevich and Soibelman [KS09]. Nevertheless, computing explicit examples using Cestello's definition remains exceptionally challenging.

1.3.3. In a subsequent development, Căldăraru and Tu [CT20] successfully computed B-model categorical invariants for the case where $g = 1$ and $n = 1$ on an elliptic curve E_τ. They accomplished this computation by employing Costello's definition and using the explicit $\mathcal{A}_\infty$-model of $D^b_{\mathrm{coh}}(E_\tau)$ provided by Polishchuk [Pol11] as their input. In their later work [CT24], Căldăraru and Tu provided explicit and computable formulas for B-model invariants associated with an arbitrary cyclic $\mathcal{A}_\infty$-algebra A and a splitting of the Hodge filtration s on its cyclic homology. These invariants are referred to as *categorical enumerative invariants* (CEI).

More precisely, given a pair (A, s) with A being a smooth proper and cyclic $\mathcal{A}_\infty$-algebra and $s : \mathrm{HH}_*(A) \to \mathrm{HC}^-_*(A)$ a splitting of its Hodge filtration, they constructed categorical enumerative invariants

$$F^{A,s}_{g,n} \in \mathrm{Sym}^n(\mathrm{HH}_*(A)[u^{-1}]),$$

for pairs (g, n) satisfying $g \geqslant 0$, $n \geqslant 1$, $2g-2+n > 0$. These CEI invariants are defined by explicit Feynman sums over a specific type of graphs, known as partially directed stable graphs. The CEI invariants $F^{A,s}_{g,n}$ are expected to play a role in the non-commutative setting similar to that of the invariants $F^X_{g,n}$ in Gromov-Witten theory. In particular, basic CEI invariants with insertions $\gamma_1, \ldots, \gamma_n \in \mathrm{HH}_*(A)$ can be defined from $F^{A,s}_{g,n}$ by

$$\langle \tau_{k_1}(\gamma_1), \ldots, \tau_{k_n}(\gamma_n)\rangle^{A,s}_g = \langle F^{A,s}_{g,n}, (\gamma_1 u^{i_1}) \cdots (\gamma_n u^{i_n})\rangle_{\mathsf{res}} \in \mathbb{C},$$

where the residue pairing is defined on Hoschschild homology of A using the Mukai pairing.

1.3.4. It's important to note that, as of now, the conjecture asserting the equality of categorical enumerative invariants and classical Gromov-Witten invariants remains unproved. Nonetheless, existing evidence overwhelmingly suggests that when the algebra A is Morita equivalent to the Fukaya category $\mathrm{Fuk}(\check{X})$ of a symplectic manifold $\check{X}$, these invariants do align: the CEI potential $F^{A,s}_{g,n}$ recovers the GW potential $F^{\check{X}}_{g,n}$ via a specific procedure. Here s is a specific splitting that is naturally attached to a Fukaya category.

For instance, in the case where $A = \mathbb{C}$ corresponds to the ground field, Tu proved in [Tu21] that the categorical enumerative invariants coincide with the Gromov-Witten invariants of a point.

1.4 A (1,1)-calculation for elliptic curves

1.4.1. In this section, we review the calculation of $(1, 1)$ Gromov-Witten invariants and CEI invariants of elliptic curves. The details are explained in [CT20].

1.4.2. Consider the mirror pair $(X, \check{X})$ of a two-torus $\check{X}$ and the family of elliptic curves

$X = E_\tau$ around the cusp $\tau = i\infty$. A well-known result shows that the $(1,1)$ Gromov-Witten invariant of degree $d \geqslant 1$, with the insertion the Poincaré dual class $[pt]^{\mathrm{PD}}$ of a point (with no ψ-class), corresponds to the number of isogeny classes of degree d of a fixed elliptic curve. Explicitly, it is given by

$$\langle [pt]^{\mathrm{PD}} \rangle_{1,1}^{\check{X},d} = \sum_{k|d} k,$$

hence the generating series of these invariants (including $d = 0$ case) is

$$F_{1,1}^{\check{X}}([pt]^{\mathrm{PD}}) = \langle F_{1,1}^{\check{X}}, [pt]^{\mathrm{PD}} \rangle_{\mathsf{res}} = -\frac{1}{24} + \sum_{d \geqslant 1} \left(\sum_{k|d} k \right) Q^d = -\frac{1}{24} E_2(Q), \tag{1.1}$$

where E_2 is the holomorphic quasi-modular Eisenstein form of weight two.

1.4.3. On the B-side, for an elliptic curve $X = E_\tau$, the $\mathcal{A}_\infty$-algebra we choose is a holomorphic modification of Polishchuk's $\mathcal{A}_\infty$-model A_τ [Pol11]. This algebra is still Morita equivalent to $D^b_{\mathsf{coh}}(X)$, but its structure constants vary holomorphically with τ. See [CT20] for details.

The splitting of the Hodge filtration on negative cyclic homology $\mathrm{HC}_*^-(A_\tau) = \mathrm{HC}_*^-(E_\tau)$ can be described geometrically as a splitting of the following short exact sequence

$$0 \to H^0(\Omega_{E_\tau}) \to H^1_{\mathrm{dR}}(E_\tau) \to H^1(\mathcal{O}_{E_\tau}) \to 0. \tag{1.2}$$

Here $H^0(\Omega_{E_\tau})$ is generated by the class $[dz]$, and $H^1(\mathcal{O}_{E_\tau})$ is generated by a class $[\xi]$. We use the splitting that sends $[\xi]$ to $\frac{1}{\tau - \overline{\tau}}([d\overline{z}] - [dz])$. This splitting is invariant under the monodromy obtained from the Gauss-Manin connection around the cusp $\tau = i\infty$. Hence we will denote it as s^{MI}.

With this $\mathcal{A}_\infty$-algebra A_τ and this specific monodromy invariant splitting s^{MI} as input, one can compute the B-model potential $F_{1,1}^{A_\tau, s^{\mathsf{MI}}}$. It depends on the choice of τ, hence if we denote $q = \exp(2\pi i \tau)$, and let q vary in a neighbourhood of $q = 0$, the main result of

[CT20] is that

$$F_{1,1}^{A_\tau, s^{\mathsf{MI}}}([\xi]) = \langle F_{1,1}^{A_\tau, s^{\mathsf{MI}}}, ([\xi]) \rangle_{\mathsf{res}} = -\frac{1}{24} E_2(q).$$

This result matches with the classical $(1,1)$ Gromov-Witten invariants in the sense that we describe in 1.5.4.

1.5 Flat Coordinates

1.5.1. Historically, after computation of Gromov-Witten invariants $F_{g,n}^{\check{X}}$ for a given symplectic manifold $\check{X}$, researchers found a method to package these invariants at all genera into a single comprehensive generating series $\mathcal{D}^{\check{X}}$, known as the *total descendant Gromov-Witten potential* or the *A-model potential* [Giv01a; Coa08]. This allows one to consider the Gromov-Witten invariants as the Taylor coefficients of the formal function $\mathcal{D}^{\check{X}}$ defined on the so called Kähler moduli space. The formal variable used (typically denoted as Q) gives a coordinate on this moduli space.

1.5.2. In a parallel manner, for a family of complex manifolds X_q, one can construct a Hodge-theoretic function (referred to as the *period*), derived from the variation of Hodge structures over the moduli space of complex structures M^{cx}. However, this construction is well-defined only for genus 0. For higher genera, we will use the CEI potentials $F_{g,n}^{A,s}$ in a manner analogous to the GW invariants. In this way one packages CEI potentials $F_{g,n}^{A,s}$ of all genera into a single generating series $\mathcal{D}^{A,s}$, called the *total descendant CEI potential* or the *B-model potential*.

1.5.3. It's crucial to emphasize that the two potentials are expressed in different coordinates: the A-model potential employs a formal variable Q, while the B-model potential is formulated in terms of a flat coordinate q. To obtain a meaningful comparison between these two potentials, the variables q and Q are identified through an invertible map denoted as the *mirror map* ψ.

In physics the formal coordinate Q is viewed as a flat coordinate on the complexified

Kähler moduli space $M^{\mathsf{Käh}}$ (a concept not mathematically rigorously defined), and the mirror map is interpreted as an isomorphism

$$\psi : M^{\mathsf{cx}} \to M^{\mathsf{Käh}}$$

between germs of M^{cx} and $M^{\mathsf{Käh}}$ around specific points. Traditionally, these special points are the large volume and the large complex structure limit points, respectively.

1.5.4. In section 1.4, for the explicit computation for elliptic curves, we implicitly used two facts:

1. $q = \exp(2\pi i\tau)$ is the flat coordinate around the cusp on the moduli space of elliptic curves,
2. the mirror map ψ for elliptic curves identifies q with Q.

The frist results of this thesis, joint with Căldăraru and Huang, involve a conjecture regarding the potential flat coordinates around hexagonal or square points. This conjecture, in turn, led to the surprising Conjecture 2.1.6 predicting the so-called elliptic expansion of the j-function around these hexagonal and square points. Remarkably, this conjecture has been later proved in the work [HMOZ22].

1.6 Holomorphic Anomaly Equation for CEI

Starting from now we will focus exclusively on the case where the mirror pair consists of a two-torus $\check{X}$ and an elliptic curve X.

1.6.1. One important property of the Gromov-Witten potentials is that they are quasi-modular forms. To be more precise, the genus g Gromov-Witten potential $F^{\check{X}}_{g,1}(\alpha)$ of the two-torus with one appropriate insertion α is a holomorphic quasi-modular Eisenstein modular form of weight $2g-2+\deg\,\alpha$. Hence by a theorem of Zagier [Zag08], $F^{\check{X}}_{g,1}$ belongs to the degree $(2g-2+\deg\,\alpha)$ part of the grade vector space $\mathbb{C}[E_2(Q), E_4(Q), E_6(Q)]$.

As an example, we have seen in Equation 1.1 that the $g = 1$ Gromov-Witten potential of the two-torus with one insertion being the Poincaré dual class of a point yields $F_{1,1}^{\check{X}}([pt]^{\mathrm{PD}})(Q) = -\frac{1}{24}E_2(Q)$, which is a holomorphic quasi-modular form of weight 2.

1.6.2. The *holomorphic anomaly equation*, as introduced by Bershadsky, Cecotti, Ooguri, and Vafa [BCOV94], and proved by Oberdieck and Pixton [OP18] in the case of GW invariants of elliptic curves, describes part of $F_{g,n}^{\check{X}}(\gamma_1 u^{k_1}, \ldots, \gamma_n u^{k_n})(Q)$ that includes $E_2(Q)$. Specifically, their theorem states that

Theorem 1.6.3. *Let $\mathcal{C}_g(\gamma_1, \ldots, \gamma_n)$ be the generating series of Gromov-Witten classes, and regard it as a cycle-valued quasimodular form, i.e., a polynomial in C_2, C_4, C_6 with coefficients in $H^*(\overline{M}_{g,n})$, we have*

$$\begin{aligned}\frac{d}{dC_2}\mathcal{C}_g(\gamma_1, \ldots, \gamma_n) =& \iota_* \mathcal{C}_{g-1}(\gamma_1, \ldots, \gamma_n, 1, 1) \\ &+ \sum_{\substack{g=g_1+g_2 \\ \{1,\ldots,n\}=S_1 \sqcup S_2}} j_*(\mathcal{C}_{g_1}(\gamma_{S_1}, 1) \boxtimes \mathcal{C}_{g_2}(\gamma_{S_2}, 1)) \\ &- 2\sum_{i=1}^{n}\left(\int_E \gamma_i\right)\psi_i \cdot \mathcal{C}_g(\gamma_1, \ldots, \gamma_{i-1}, 1, \gamma_{i+1}, \ldots, \gamma_n),\end{aligned} \tag{1.3}$$

where $\gamma_{S_i} = (\gamma_k)_{k \in S_i}$ and $1 \in H^*(E)$ is the unit.

On the B-model side the analogous conjecture is the following statement:

Conjecture 1.6.4. *The CEI invariants $F_{g,n}^{A_\tau, s^{\mathsf{MI}}}(\gamma_1 u^{k_1}, \ldots, \gamma_n u^{k_n})(q)$ of the family $\{E_q\}$ of elliptic curves for Im $q \gg 0$, with appropriate insertions $\gamma_i u^{k_i} \in \mathrm{HH}_*(A_\tau)[\![u]\!]$ are holomorphic quasi-modular forms, i.e., they belong to the polynomial ring $\mathbb{C}[E_2(q), E_4(q), E_6(q)]$.*

To prove this conjecture, we would need to show that CEI invariants satisfy the following three properties:

1. $F_{g,n}^{A_\tau, s^{MI}}(\gamma_1 u^{k_1}, \ldots, \gamma_n u^{k_n})(q)$ is holomorphic.
2. $F_{g,n}^{A_\tau, s^{MI}}(\gamma_1 u^{k_1}, \ldots, \gamma_n u^{k_n})(q)$ behaves well with respect to the $\mathrm{SL}_2(\mathbb{Z})$-action.
3. $F_{g,n}^{A_\tau, s^{MI}}(\gamma_1 u^{k_1}, \ldots, \gamma_n u^{k_n})(q)$ has finite limit at $q = 0$.

Unfortunately we could not prove the last property in general, so Conejcture 1.6.4 remains open.

However, if we assume this conjecture is true, then we can prove the following result (Theorem 3.5.6) for CEI, which is a direct analogues of Theorem 1.6.3 of Oberdieck-Pixton.

Theorem 1.6.5. *If Conjecture 1.6.4 is valid, then the B-model potential*

$$F_{g,n}^{A_\tau, s^{\mathsf{MI}}}(\gamma_1 u^{k_1}, \ldots, \gamma_n u^{k_n})(q)$$

satisfies the holomorphic anomaly equation.

The key idea of the proof is to study the so-called *Givental group action* [Giv01a; Giv01b; Giv04] on CEI potentials. We briefly review it here.

1.6.6. Given a vector space H with a nondegenerate symmetric bilinear form $\langle -, - \rangle$ we constructed vector spaces $H_{\mathsf{Tate}} = H(\!(u)\!)$ (endowed with the residue pairing $\langle -, - \rangle_{\mathsf{res}}$) and $H_+ = H[\![u]\!]$.

The Givental group Giv associated to this data is the subgroup of the group of automorphisms of the symplectic vector space $(H_{\mathsf{Tate}}, \langle -, - \rangle_{\mathsf{res}})$ preserving the Lagrangian subspace H_+ and acting as identity on H. Explicitly, an element of Giv is of the form

$$\sigma = id + \sigma_1 \cdot u + \sigma_2 \cdot u^2 + \cdots$$

with each $\sigma_j \in \mathrm{End}(H)$, and σ is required to satisfy

$$\langle \sigma \cdot x, \sigma \cdot y \rangle_{\mathsf{res}} = \langle x, y \rangle_{\mathsf{res}} \quad \text{for any } x, y \in H_{\mathsf{Tate}}.$$

In the CEI setting, for an $\mathcal{A}_\infty$-algebra A, we will let H be $\mathrm{HH}_*(A)$, and $\langle -, - \rangle$ be the Mukai pairing.

1.6.7. When A is smooth and proper, the set of splittings of the non-commutative Hodge filtration is nonempty, and it is a left torsor over the Givental group, by letting an element σ act on a splitting $s : H_+ = HH_*(A) \to HC_*^-(A)$ via pre-composing with σ^{-1}:

$$\sigma \cdot s : H_+ \xrightarrow{\sigma^{-1}} H_+ \xrightarrow{s} HC_*^-(A).$$

1.6.8. So far we have only been using one splitting s^{MI} when computing the CEI invariants of the family of elliptic curves. However, there exists another interesting splitting s^{CC}, called the *complex conjugate splitting*. Explicitly, it is given by the splitting of the short exact sequence

$$0 \to H^0(\Omega_{E_\tau}) \to H^1_{\mathrm{dR}}(E_\tau) \to H^1(\mathcal{O}_{E_\tau}) \to 0,$$

that sends $[\xi] \in H^1(\mathcal{O}_{E_\tau})$ to $\frac{1}{\tau - \overline{\tau}}[d\overline{z}]$.

Since the set of splittings is a torsor over Giv, there exists a Givental group element σ that sends the splitting s^{MI} to the splitting s^{CC}, i.e.,

$$\sigma \cdot s^{\mathsf{MI}} = s^{\mathsf{CC}}.$$

The detailed construction of σ can be found in Section 3.4.

1.6.9. The Givental group acts on the set of splittings, hence in turn it acts on the *total descendent potential* $\mathcal{D}^{A,s}$. This is described in the work of Căldăraru and Tu [CT24]. The action on potentials of an element $\sigma \in \mathsf{Giv}$ will be denoted by $\hat{\sigma}$:

$$\mathcal{D}^{A,\sigma\cdot s} = \hat{\sigma}(\mathcal{D}^{A,s}).$$

This action can be described using the combinatorial model of stable graphs, as explained in Givental [Giv01a] and Pandharipande-Pixton-Zvonkine [PPZ15]. Explicitly, one can compute $\hat{\sigma}(F^{A,s}_{g,n})$ using $\left\{F^{A,s}_{g,n}\right\}$ as follows:

$$\hat{\sigma}(F^{A,s}_{g,n}) = \sum_{G \text{ stable of type } (g,n)} \frac{1}{|\mathrm{Aut}(G)|} \prod \mathsf{Cont}(v) \prod \mathsf{Cont}(e) \prod \mathsf{Cont}(l),$$

where each vertex v is decorated by $F^{A,s}_{g,n}$'s, each edge e is decorated by an operator H^σ that is constructed from σ, and each leg l is decorated by σ. Here the decorations H^σ and σ will be explained in Section 3.4.

The key idea of the proof of Theorem 1.6.5 is to analyse the difference between $\hat{\sigma}(F^{A_\tau,s^{\mathsf{MI}}}_{g,n})$ and $F^{A_\tau,s^{\mathsf{MI}}}_{g,n}$ in two different ways: one is the action of the Givental group, another is the Kaneko-Zagier operator applied to the potential $F^{A_\tau,s^{\mathsf{MI}}}_{g,n}$.

1.6.10. Theorem 1.6.5 holds significant importance in the computation of the categorical enumerative invariants for elliptic curves, as it reduces the computation of these invariants for any elliptic curve, when $g \leqslant 5$, to a computation of the invariants for a *single* elliptic curve target. Since $F^{A_\tau,s^{\mathsf{MI}}}_{g,n}$ are expanded around the cusp, we would like to choose the nodal cubic curve (at cusp $\tau = i\infty$) as the special point. However the nodal cubic is singular, hence it does not admit a splitting of Hodge filtration. We will explain how to deal with this problem in Section 1.7 and Chapter 4.

1.7 Liftable Hochschild classes

1.7.1. In the influential work of Deligne and Illusie [DI87], they proved that for a smooth projective variety Y over a field of characteristic 0, the Hodge to de Rham (HdR) spectral sequence ${}^1E^{p,q}_Y = H^p(Y, \Omega^q_Y) \Longrightarrow H^{p+q}_{\mathrm{dR}}(Y)$ degenerates at first page.

1.7.2. The groups on both sides of the HdR spectral sequence are related to Hochschild homology and negative cyclic homology. Therefore, the HdR spectral sequence can be generalized to the world of non-commutative geometry. The resulting spectral sequence

$$\mathrm{HH}_*(A)[\![u]\!] \Longrightarrow \mathrm{HC}^-_*(A)$$

is called the *Hochschild to cyclic spectral sequence.* Kontsevich and Soibelman [KS09] conjectured that for any smooth and proper $\mathcal{A}_\infty$-algebra A the Hochschild to cyclic spectral sequence also degenerates at ${}^1E$. This was later proved by Kaledin [Kal08; Kal17].

1.7.3. By contract, the spectral sequence does not need to degenerate at ${}^1E$ for proper but not smooth varieties. We study a specific singular case in Chapter 4, where X is the projective nodal cubic curve over $\mathbb{C}$. There is a similar *derived Hodge to de Rham spectral sequence*

$$ {}^1E_X^{p,q} = H^p(X, \bigwedge^q \mathbb{L}_X^\bullet) \Longrightarrow H_{\mathrm{dR}}^{p+q}(X), \tag{1.4}$$

where $\mathbb{L}_X^\bullet$ is the cotangent complex of X. Our main result (Theorem 4.3.4) is

Theorem 1.7.4. *The above spectral sequence (1.4) degenerates at the second page.*

Our proof relies on comparison with the affine nodal cubic curve and the projective resolution of singularities. Moreover, our result of the nodal cubic implies the Hochschild to cyclic spectral sequence also degenerates on the second page.

1.7.5. In the smooth case, the degeneration of Hochschild to cyclic spectral sequence on the first page implies that all Hochschild classes can be lifted to negative cyclic homology, i.e., the natural map $\mathrm{HC}_n^-(X) \to \mathrm{HH}_n(X)$ is surjective for all n. This fails to be true in the singular case in general, as we have already seen that the HdR spectral sequence for nodal cubic curve X degenerates only on the second page. Nevertheless, in Section 4.5, we study the map $\mathrm{HC}_n^-(X) \to \mathrm{HH}_n(X)$. We prove that it is an isomorphism for $n \in 2\mathbb{N} \cup \{-1\}$ and it is 0 for other n.

In particular, the generator $[\xi]$ of $\mathrm{HH}_{-1}(X)$ lifts to $\mathrm{HC}_{-1}^-(X)$ uniquely. This is crucial for the definition of the $(2,1)$ CEI invariant for the nodal cubic curve. The current definition of CEI requires the $\mathcal{A}_\infty$-algebra to be smooth and proper, hence can not be applied to the case of nodal curve directly. However, since $[\xi]$ is mirror to the class $[pt]^{\mathsf{PD}}$, it is the only class that yields nontrivial calculations. The liftablity of $[\xi]$ guarantees that we can insert the class $[\xi]u^2$ into the computation of the $(2,1)$ CEI invariant. Furthermore, in the calculation process of $F_{2,1}^{A_\tau, s^{\mathsf{MI}}}([\xi]u^2)$, non-liftable Hochschild classes are expected not to arise. This enables us to fully compute the $(2,1)$ CEI invariant.

1.8 Summary

In this section, we explain a potential application uniting the three different projects under a common theme: the computation of the $(2,1)$ CEI invariant $F_{2,1}^{A_\tau, s^{\mathsf{MI}}}$. We summarize our approach as follows:

1. The only insertion that yields nontrivial computation is $[\xi]u^2$. Assuming the resulting invariant extends to the cusp[1], we can prove that the resulting CEI is a holomorphic quasi-modular form of weight 4. Hence it is of the form

$$F_{2,1}^{A_\tau, s^{\mathsf{MI}}}([\xi]u^2) = aE_2^2(q) + bE_4(q).$$

2. The coefficient a is determined by the holomorphic anomaly equation.
3. To determine the coefficient b, we only need to evaluate $F_{2,1}^{A_\tau, s^{\mathsf{MI}}}([\xi]u^2)$ at one specific value of q. Our choice is $q = 0$.
4. The CEI invariants are not completely defined for proper but non-smooth $\mathcal{A}_\infty$-algebras. However, we conjecture that when the insertions are liftable Hochschild classes, then we can perform the computations in the same way as before. In particular, the class $[\xi]$ does lift to HC_*^- from our study of the degeneration of Hochschild to cyclic spectral sequence for the nodal cubic curve. Thus we are able to compute the $(2,1)$ CEI at cusp, which corresponds to $q = 0$. This determines the coefficient b.

1.9 Outline

In Chapter 2 we discuss the work in [CHH21] which describes our the conjecture about the elliptic expansion of the j-function around special points in $\overline{M}_{1,1}$.

In Chapter 3 we discuss the work in [CHT] about holomorphic anomaly equation. We show

[1]During a private conversation with Andrei Căldăraru, he claimed that he is able to prove this.

that with certain assumptions, CEI invariants satisfy the holomorphic anomaly equation.

In Chapter 4 we discuss the work in [He23] about the degeneration of the Hodge to de Rham spectral sequence for the nodal cubic curve. Moreover, we classify all the liftable Hochschild classes.

In Chapter 5 we discuss some open questions and further directions.

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Chapter 2

Moonshine at Landau-Ginzburg points

2.1 The conjecture

2.1.1. The Monstrous Moonshine conjecture describes a surprising relationship, discovered in the late 1970s, between the coefficients of the Fourier expansion of Klein's j-function around the cusp

$$j(\tau) = \frac{1}{q} + 744 + 196884q + 21493760q^2 + 864299970q^3 + 20235856256q^4 + \cdots$$

and dimensions of irreducible representations of the Monster group. Fourier expansions of other modular forms around the cusp are critically important in number theory and algebraic geometry. In particular such expansions appear directly in computations of Gromov-Witten invariants of elliptic curves [Dij95].

2.1.2. In this note we study the elliptic expansion of the j-function around the hexagonal point $j = 0$ and the square point $j = 1728$, instead of around the cusp $j = \infty$. At $j = 0$ the elliptic curve is the Fermat cubic, cut out in $\mathbb{P}^2$ by $x^3 + y^3 + z^3 = 0$, while at $j = 1728$

it is given by $x^4 + y^4 + z^2 = 0$ in the weighted projective space $\mathbb{P}^2_{1,1,2}$.

From an enumerative geometry perspective the fact that we work around the hexagonal and square points instead of around the cusp suggests that we are working with Fan-Jarvis-Ruan-Witten (FJRW) invariants instead of Gromov-Witten invariants. See (2.2.7) for details.

2.1.3. Let $\mathbb{H}$ and $\mathbb{D}$ denote the upper half plane and the unit disk in the complex plane, respectively. Fix $\tau_* = e^{\pi \mathtt{i}/3}$ or $\tau_* = \mathtt{i}$ as the points* in $\mathbb{H}$ around which to carry out the expansion.

The uniformizing map S around τ_* is the map

$$S : \mathbb{H} \to \mathbb{D}, \qquad S(\tau) = \frac{\tau - \tau_*}{\tau - \bar{\tau}_*},$$

with inverse

$$S^{-1} : \mathbb{D} \to \mathbb{H}, \qquad S^{-1}(w) = \frac{\tau_* - \bar{\tau}_* w}{1 - w}.$$

The *elliptic* expansion of j around τ_* is simply the Taylor expansion of $j \circ S^{-1}$ around $w = 0$. Its coefficients are closely related [Zag08, Proposition 17] to the values of the higher modular derivatives $\partial^n j(\tau_*)$,

$$j\left(S^{-1}(w)\right) = \sum_{n=0}^{\infty} \frac{(4\pi \mathrm{Im}\tau_*)^n \partial^n j(\tau_*)}{n!} w^n.$$

2.1.4. The values of the higher modular derivatives of j can be computed term-by-term by a well-known recursive procedure. The results are rational multiples of products of powers of the Chowla-Selberg period† Ω and of π.

Let $s(w) = 2\pi\Omega^2 \cdot S(w)$ denote the rescaling of S by the factor $2\pi\Omega^2$. Then around

*Any other point in the $\mathrm{SL}(2, \mathbb{Z})$ orbit of τ_* works equally well, with only minor changes in the constants below.

†The exact value of Ω is unimportant, but in this case $\Omega = 1/\sqrt{6\pi}\,(\Gamma(1/3)/\Gamma(2/3))^{3/2}$ for the hexagonal point and $\Omega = 1/\sqrt{8\pi}\,(\Gamma(1/4)/\Gamma(3/4))$ for the square point.

$\tau_* = \exp(\pi \mathtt{i}/3)$ we have

$$j\left(s^{-1}(w)\right) = 13824w^3 - 39744w^6 + \frac{1920024}{35}w^9 - \frac{1736613}{35}w^{12} + \cdots,$$

while around $\tau_* = \mathtt{i}$ we have

$$j\left(s^{-1}(w)\right) = 1728 + 20736w^2 + 105984w^4 + \frac{1594112}{5}w^6 + \frac{3398656}{5}w^8 + \cdots.$$

2.1.5. The following power series have been introduced independently by Shen-Zhou [SZ18, (3.41), (3.45)] in their study of the LG/CY correspondence for elliptic curves, and by Tu [Tu21, Section 4] in his study of categorical Saito theory of Fermat cubics:

$$g(t) = \sum_{n=0}^{\infty} (-1)^n \frac{((3n-2)!!!)^3}{(3n)!} t^{3n},$$
$$h(t) = \sum_{n=0}^{\infty} (-1)^n \frac{((3n-1)!!!)^3}{(3n+1)!} t^{3n+1}.$$

In both cases it was argued that the ratio $h(t)/g(t)$ gives a flat coordinate on the moduli space of versal deformations $x^3 + y^3 + z^3 + 3txyz = 0$ of the Fermat cubic.

Similarly, for the elliptic quartic we introduce the two power series below

$$g(t) = \sum_{n=0}^{\infty} \frac{((4n-3)!!!!)^2}{(2n)!} t^{2n},$$
$$h(t) = \sum_{n=0}^{\infty} \frac{((4n-1)!!!!)^2}{(2n+1)!} t^{2n+1}.$$

Even though the notation g, h appears overloaded, it should be evident from context which power series we refer to.

Our main result is the following conjecture.

Conjecture 2.1.6. *(a) Around the hexagonal point the elliptic expansion of the j-function* *satisfies*

$$j\left(s^{-1}\left(\frac{h(t)}{g(t)}\right)\right) = 27t^3\left(\frac{8-t^3}{1+t^3}\right)^3$$
$$= 13824t^3 - 46656t^6 + 99144t^9 - 171315t^{12} + 263169t^{15} - \cdots .$$

(b) Around the square point the elliptic expansion of the j-function satisfies

$$j\left(s^{-1}\left(\frac{h(t)}{g(t)}\right)\right) = (192+256t^2)\left(\frac{3+4t^2}{1-4t^2}\right)^2$$
$$= 1728 + 20736t^2 + 147456t^4 + 851968t^6 + 4456448t^8 + \cdots .$$

2.1.7. Notes. It is remarkable that the coefficients in the above power series are all integers, despite $j(s^{-1}(w))$ only having rational coefficients. (For the expansion at the hexagonal point the integrality of the coefficients follows from [SZ18].) Our attempts to find other modular forms with this integrality property, using other combinations of the Eisenstein modular forms E_2, E_4, and E_6 have been unsuccessful.

The validity of the formulas above has been verified by computer up to t^{24}. A recent proof of Conjecture 2.1.6 was announced, after our paper was made public, in [HMOZ22].

2.2 Mirror symmetry origin of the conjecture

2.2.1. The original statement of mirror symmetry is formulated as the equality of two power series associated with a pair $(X, \check{X})$ of mirror symmetric families of Calabi-Yau varieties. These two power series are

(a) the generating series, in a formal variable Q, of the enumerative invariants of the family X (the A-model potential);

(b) the Taylor expansion of a Hodge-theoretic function (the period) on the moduli space of complex structures M^{cx} of the mirror family $\check{X}$, with respect to a flat coordinate

q on this moduli space (the B-model potential).

In order to compare the two power series, the variables q and Q are identified via an invertible map ψ called the mirror map.

In physics, the formal variable Q is viewed as a flat coordinate on the (ill-defined mathematically) complexified Kähler moduli space $M^{\mathsf{K\ddot{a}h}}$, and the mirror map is interpreted as an isomorphism

$$\psi : M^{\mathsf{cx}} \to M^{\mathsf{K\ddot{a}h}}$$

between germs of M^{cx} and $M^{\mathsf{K\ddot{a}h}}$ around special points. Traditionally these special points are the large complex limit point and the large volume point, respectively.

2.2.2. The original mirror symmetry computation of [COGP91] follows this pattern. It predicts a formula for the generating series of genus zero Gromov-Witten invariants of the quintic X, by equating it to the expansion of a period (solution of the Picard-Fuchs equation) for the family of mirror quintics $\check{X}$. The equality of the two sides allows one to calculate the genus zero Gromov-Witten invariants, by expanding the period map of the family $\check{X}$ with respect to a certain flat coordinate on the moduli space of complex structures of mirror quintics.

As another example consider a two-torus X (elliptic curve with arbitrary choice of complex structure). The $g = 1, n = 1$ Gromov-Witten invariant of degree $d \geqslant 1$ with insertion the Poincaré dual class of a point counts in this case the number of isogenies of degree d to a fixed elliptic curve. As such it satisfies

$$\langle [\mathsf{pt}]^{\mathsf{PD}} \rangle_{1,1}^{X,d} = \sum_{k|d} k = \sigma_1(d),$$

and hence the generating series of these invariants (including the $d = 0$ case) is $-\frac{1}{24}E_2(Q)$ where E_2 denotes the quasi-modular Eisenstein form of weight two (see [Dij95] for the the original derivation of this calculation). The main result of ([CT20]) is that this equals the expansion in $q = \exp(2\pi \mathtt{i} \tau)$, around $q = 0$, of the function of categorical enumerative

$(1,1)$ invariants for the corresponding family $\check{X}$ of mirror elliptic curves.

2.2.3. Implicit in the above calculation for elliptic curves are the two facts that

(a) q is the flat coordinate, around the cusp, on the moduli space of elliptic curves;

(b) the mirror map ψ for elliptic curves identifies q with Q.

The main intuition behind Conjecture 2.1.6 is a similar set of assumptions but for the flat coordinates around the hexagonal or square points instead of around the cusp. Below we will give precise conjectural descriptions of the flat coordinates q and Q around the hexagonal point $\check{F} \in M^{\mathsf{cx}}$ and its mirror $F \in M^{\mathsf{K\ddot{a}h}}$. The analysis for the square point is entirely similar.

2.2.4. To understand these flat coordinates we need good descriptions of $M^{\mathsf{K\ddot{a}h}}$ and M^{cx} around F and $\check{F}$. We will review first the classical situation (around the cusp) described in the work of Polishchuk-Zaslow [PZ98].

Polishchuk-Zaslow takes the space $M^{\mathsf{K\ddot{a}h}}$ on a two-torus to be the quotient of $\mathbb{H}$, with coordinate ρ, by $\rho \sim \rho + 1$. For each $\rho \in M^{\mathsf{K\ddot{a}h}}$ they construct a Fukaya category $\mathcal{F}^0(X^\rho)$ on the two-torus X^ρ endowed with this structure. The quotient above is precisely the same as the neighborhood of the cusp on the moduli space M^{cx} of complex structures on a two-torus[‡]. For Polishchuk-Zaslow the mirror map is simply the identity $\tau \leftrightarrow \rho$: the complex elliptic curve $\check{X}^\tau$ with modular parameter τ corresponds to the two-torus X^ρ with complexified Kähler structure $\rho = \tau$.

2.2.5. Even without explicitly constructing $M^{\mathsf{K\ddot{a}h}}$ as a moduli space of geometric objects, we could have understood its structure around the large volume limit point through mirror symmetry. Indeed, we could have simply taken $M^{\mathsf{K\ddot{a}h}}$ to *be* the neighborhood of the large complex structure limit point in M^{cx}, a space we understand. With this point of view, the mirror map is always the identity.

2.2.6. We would like to understand a similar picture around the hexagonal point $\check{F} \in M^{\mathsf{cx}}$.

[‡]We ignore the stack structure of M^{cx}, which only adds an extra $\mathbb{Z}/2\mathbb{Z}$ stabilizer.

Even though the results of Polishchuk-Zaslow do not extend to $\check{F}$, we can still conjecture that there is a larger moduli space $M^{\mathsf{K\ddot{a}h}}$ of "extended Kähler structures" (which no longer parametrizes just classical Kähler classes as before) and a point $F \in M^{\mathsf{K\ddot{a}h}}$ such that F corresponds under mirror symmetry to $\check{F}$. Then the point of view in (2.2.5) allows us to understand the local structure of $M^{\mathsf{K\ddot{a}h}}$ around F: it should be the same as M^{cx} around $\check{F}$.

The germ of M^{cx} around $\check{F}$ is the quotient of $\mathbb{H}$ by

$$\tau \sim \frac{\tau - 1}{\tau},$$

exhibiting the germ of $\mathbb{H}$ around τ_* as a triple cover of M^{cx} branched over $\check{F}$. We will *define* the germ of $M^{\mathsf{K\ddot{a}h}}$ around F to be the quotient of $\mathbb{H}$ (with coordinate ρ) by $\rho \sim (\rho - 1)/\rho$. We think of $\rho \in \mathbb{H}$ as giving an (extended type) "complexified Kähler class" on the two torus, and write X^ρ for this (fictitious) symplectic geometry object. We emphasize that we do not attempt to give a rigorous mathematical definition of X^ρ, though it would be natural to suggest that the non-commutative geometric object associated to it should be the Fukaya-Seidel category one sees at this point. Despite this, the mirror map is, as before, $\tau \leftrightarrow \rho$.

2.2.7. The natural question is then what is the flat coordinate on $M^{\mathsf{K\ddot{a}h}}$ (as defined above) around F. We conjecture that this flat coordinate is $Q = s(\rho)^3$. The justification for this comes from work of Li-Shen-Zhou [LSZ23], where the authors suggest that the natural way to interpret the generating series of FJRW invariants for two-tori as a function of ρ is via the map s (with a different rescaling from ours). It would be natural to guess from their work that $s(\rho)$ is the flat coordinate. However, since ρ is only defined up to the equivalence $\rho \sim (\rho - 1)/\rho$, the equality

$$s\left(\frac{\rho - 1}{\rho}\right)^3 = s(\rho)^3$$

implies that Q descends[§] to a coordinate on $M^{\mathsf{Käh}}$, which we conjecture to be the flat coordinate around F.

2.2.8. In the B-model we have seen ([SZ18], [Tu21]) that $h(t)/g(t)$ gives a flat coordinate on the base $\mathbb{A}^1_t$ of the Hesse pencil of elliptic curves,

$$E_t : \qquad x^3 + y^3 + z^3 + 3txyz = 0.$$

In particular, Tu's work was motivated by a study of categories of graded matrix factorizations, but via Orlov's correspondence [Orl06] these are equivalent to the derived categories of the above elliptic curves.

Again, $h(t)/g(t)$ does not give a coordinate on M^{cx} because locally $\mathbb{A}^1_t$ is a branched triple cover of M^{cx} around $\check{F}$. Its replacement $q = (h(t)/g(t))^3$ does descend to a coordinate on M^{cx} around $\check{F}$, and we conjecture it is flat.

2.2.9. By our construction of $M^{\mathsf{Käh}}$ the mirror map ψ is the identity, so the mirror of the complex curve $\check{X}^\tau$ with modular parameter τ is the symplectic object X^ρ with $\rho = \psi(\tau) = \tau$. (Despite being equal we prefer to keep ρ and τ distinct since they represent different geometric objects.)

Flat coordinates are unique up to multiplication by a scalar when the moduli spaces $M^{\mathsf{Käh}}$ and M^{cx} are one-dimensional. (The rescaling factor $2\pi\Omega^2$ in (2.1.4) was chosen so that this constant equals one.) It follows that the flat coordinates of X^ρ and $\check{X}^\tau$ are equal for $\rho = \tau$.

Consider a Hesse elliptic curve E_t for some value of t. It can be written as $\check{X}^\tau$ for some (non-unique) modular parameter $\tau \in \mathbb{H}$. The mirror of this curve is X^ρ for $\rho = \tau$. (We think of $\rho \in M^{\mathsf{Käh}}$, so the ambiguity in τ disappears.) It follows that

$$\left(\frac{h(t)}{g(t)}\right)^3 = q(\check{X}^\tau) = Q(X^\rho) = s(\rho)^3,$$

[§]This is not the only modification of $s(\tau)$ that descends to a coordinate on $M^{\mathsf{Käh}}$, which in general will not be flat. The same issue appears in the B-model.

or, using the fact that s is invertible,

$$s^{-1}\left(\frac{h(t)}{g(t)}\right) \sim \rho$$

where $\sim$ is the equivalence relation used to define M^{cx} in (2.2.5). Applying the j-function to both sides and noting that it is $\sim$-invariant we get

$$j\left(s^{-1}\left(\frac{h(t)}{g(t)}\right)\right) = j(\rho) = j(E_t).$$

For the Hesse pencil the j-function can be computed easily [AD09] and the result is

$$j(E_t) = 27t^3\left(\frac{8-t^3}{1+t^3}\right)^3.$$

This is the statement of the conjecture.

Chapter 3

The holomorphic anomaly equation for categorical enumerative invariants of elliptic curves

3.1 Introduction

3.1.1. An important aspect of Gromov-Witten theory is the fact that generating series of these invariants yield (quasi) modular forms. Classically, modular forms arise as counting functions for points, representing zero-dimensional objects. However, in Gromov-Witten theory, the generating function serves as a counting mechanism for the virtual number of holomorphic curves, which are one-dimensional objects. Hence it is natural to speculate whether modular forms also play a role in this context. One might attempt to compute these forms explicitly and hope that the results can be organized as modular forms. This approach has been successfully applied to elliptic curves [OP06] and the reduced Gromov-Witten theory of K3 surfaces [MPT10]. However, it's important to note that both steps in this strategy require significant effort and expertise. Computing Gromov-Witten invariants is generally a challenging task, and even when computations are feasible, organizing them into modular forms remains a challenge. Unlike the case of counting points, where patterns can often be discerned by examining a large number of coefficients, the organization of Gromov-Witten invariants into modular forms is not straightforward.

The idea of resolving this issue originates from mirror symmetry.

3.1.2. The classical mirror symmetry for $g = 0$ asserts that counting rational curves in a Calabi-Yau threefold $\check{X}$ (A-model) is equivalent to studying variation of Hodge structures of its mirror Calabi-Yau threefold X (B-model). Higher genus mirror symmetry extends this equivalence to counting higher genus curves in a Calabi–Yau threefold. While Gromov–Witten theory provides a rigorous mathematical framework for counting curves of any genus and thus naturally extends to higher genus A-models, the corresponding higher genus B-model, which generalizes the theory of variation of Hodge structures, has been far more enigmatic.

A potential candidate for the higher genus B-model was proposed by Bershadsky, Cecotti, Ooguri, and Vafa in seminal papers [BCOV93; BCOV94] (BCOV theory). By exploring this physical B-model of Gromov-Witten theory, BCOV conjectured boldly that the Gromov-Witten generating functions for any Calabi-Yau manifolds are, in fact, quasi-modular forms. A central concept in [BCOV94] is that a natural B-model Gromov-Witten potential should exhibit modularity but be non-holomorphic. Moreover, its anti-holomorphic dependence should be governed by an equation known as the *holomorphic anomaly equations*. Over the past decade, Klemm and collaborators have produced a series of papers aimed at solving the holomorphic anomaly equations [ABK08; HKQ09]. Mathematically, some of this work has been understood in recent years in the work of Guo, Janda and Ruan [GJR18]. One significant outcome has been the remarkable prediction of Gromov-Witten invariants for the quintic 3-fold up to genus 51. This represents a significant achievement, given that for the quintic mathematicians can only compute

Gromov-Witten invariants for genus zero and one.

For elliptic curves, building on this physical insight, Milano, Ruan, and Shen [MRS18] proved the modularity and holomorphic anomaly equation for elliptic orbifolds, leaving the case of elliptic curves as a conjecture. This conjecture was later proved by Oberdieck and Pixton [OP18].

3.1.3. However, another candidate for the higher genus B-model emerged from the work of Căldăraru and Tu in their paper [CT24]. They offered explicit and computable formulas for certain types of invariants associated with a (smooth, proper and cyclic) $\mathcal{A}_\infty$-algebra A and a splitting of the Hodge filtration s on its cyclic homology. These invariants are known as *categorical enumerative invariants* (CEI).

The major conjecture in this area posits that when the Fukaya category $\mathrm{Fuk}(\check{X})$ of a space X is inputted, these CEI invariants coincide with classical Gromov-Witten invariants. Consequently, when the derived category $D^b_{\mathsf{coh}}(X)$ of coherent sheaves on the mirror space X is inputted, these CEI invariants are expected to provide the B-model Gromov-Witten invariants.

Naturally, one might expect these CEI invariants to satisfy the holomorphic anomaly equation. In this chapter, we focus on the case of elliptic curves.

3.2 Convention

In this section, we fix some notations.

3.2.1. Let A be a smooth unital $\mathcal{A}_\infty$-algebra of Calabi-Yau dimension d with a cyclic pairing $\langle -, - \rangle$. Denote L its shifted reduced Hochschild chains $(C_*(A)[d], b)$, where b is the differential.

3.2.2. L admits a circle action, which is given by the Connes' operator B. Associated to this chain complex with the circle action, one can set

$$L_{\mathsf{Tate}} = (L(\!(u)\!), b + uB),$$

$$L_{+} = (L[\![u]\!], b + uB),$$

$$L_{-} = (L[u^{-1}], b + uB),$$

where u is a formal variable of degree -2.

3.2.3. L also admits a chain-level Mukai pairing

$$\langle -, - \rangle_{\mathsf{Muk}} : L \otimes L \to \mathbb{C},$$

such that B is self-adjoint, i.e.,

$$\langle Bx, y \rangle_{\mathsf{Muk}} = (-1)^{|x|} \langle x, By \rangle_{\mathsf{Muk}} \quad \text{for all } x, y \in L.$$

This paring induces a *higher residue pairing* on the associated Tate complex, with value in $\mathbb{C}(\!(u)\!)$, defined as

$$\langle x, y \rangle_{\mathsf{hres}} = \langle \sum_k x_k \cdot u^k, \sum_l y_l \cdot u^l \rangle_{\mathsf{hres}} := \sum_{k,l} (-1)^l \langle x_k, y_l \rangle_{\mathsf{Muk}} \cdot u^{k+l}.$$

3.2.4. Denote by

$$H := H_*(L) = \mathrm{HH}_*(A)[d]$$

the shifted Hochschild homology of A. Similar as above, one can set

$$H_{\mathsf{Tate}} = H_*(L_{\mathsf{Tate}}) = H(\!(u)\!),$$

$$H_{+} = H_*(L_{+}) = H[\![u]\!],$$

$$H_{-} = H_*(L_{-}) = H[u^{-1}].$$

3.3 Two infinity models and two splittings

3.3.1. In [CCT20; CT20], Căldăraru, Cestello and Tu proposed a construction of categorical enumerative invariant associated to a pair (A, s) consisting of a smooth, proper and cyclic $\mathcal{A}_\infty$-algebra A of Calabi-Yau dimension d, and a splitting $s : \mathrm{HH}_*(A) \to \mathrm{HC}_*^-(A)$ of the Hodge filtration on its cyclic homology. From this data they defined the categorical enumerative potential

$$F_{g,n}^{A,s} \in \mathsf{Sym}^n(H_-),$$

for any pair (g, n) satisfying $2g - 2 + n > 0, n > 0$. Additionally, in [CT24] Căldăraru and Tu provided an explicitly computable formula for the image $\bar{\iota}(F_{g,n}^{A,s})$ under the embedding

$$\bar{\iota} : \mathsf{Sym}^n\left(H_-\right) \to \mathrm{Hom}(H_+, \mathsf{Sym}^{n-1}(H_-)) .$$

Since $\bar{\iota}$ is injective, $\bar{\iota}(F_{g,n}^{A,s})$ uniquely determines $F_{g,n}^{A,s}$.

3.3.2. Some explicit results have been obtained for small g, n. In [CT20], the authors computed the $(1,1)$-invariant $F_{1,1}^{A_\tau^{\mathsf{hol}}, s^{\mathsf{MI}}}$ of $(A_\tau^{\mathsf{hol}}, s^{\mathsf{MI}})$, where A_τ^{hol} is the holomorphic $\mathcal{A}_\infty$-algebra model for the derived category $\mathsf{D}^b_{\mathsf{coh}}(E_\tau)$ of an elliptic curve E_τ (which is slightly different from the A_∞-model A_τ^{mod} constructed by Polishchuk [Pol11]), and $s^{\mathsf{MI}} : \mathrm{HH}_*(E_\tau) \to \mathrm{HC}_*^-(E_\tau)$ is the monodromy invariant splitting of the non-commutative Hodge filtration. The data of such a splitting is equivalent to a splitting of the short exact sequence

$$0 \to \mathrm{HH}_1(E_\tau) \to H^1_{\mathsf{dR}}(E_\tau) \to \mathrm{HH}_{-1}(E_\tau) \to 0,$$

and the monodromy invariant splitting is the one that sends the generator $[\xi] \in \mathrm{HH}_{-1}(E_\tau) \simeq H^1(E_\tau, \mathcal{O}_{E_\tau})$ to $\frac{1}{\tau-\bar{\tau}}([d\bar{z}] - [dz]) \in H^1_{\mathsf{dR}}(E_\tau)$. The result of their computation is

$$\bar{\iota}\left(F_{1,1}^{A_\tau^{\mathsf{hol}}, s^{\mathsf{MI}}}\right)([\xi]) = -\frac{1}{24}E_2(q).$$

Here we regard $\bar{\iota}\left(F_{1,1}^{A_\tau^{\mathsf{hol}}, s^{\mathsf{MI}}}\right)([\xi])$ as a function on $q = \exp(2\pi i\tau)$ in the neighborhood of

$\tau = i\infty$.

3.3.3. In the same paper, the authors proved that if we use Polishchuk's original model A_τ^{mod} whose structure constants vary in a modular fashion with q, and replace the splitting by the complex conjugate splitting s^{CC}, which maps $[\xi]$ to $\frac{1}{\tau-\bar{\tau}}[d\bar{z}]$, then the categorical enumerative invariant one gets is

$$\bar{\iota}\left(F_{1,1}^{A_\tau^{\mathsf{mod}},s^{\mathsf{CC}}}\right)([\xi]) = -\frac{1}{24}E_2^*(\tau),$$

where $E_2^*(\tau)$ is the modular completion of $E_2(\tau)$, defined as

$$E_2^*(\tau) = E_2(\tau) - \frac{3}{\pi^2}\frac{2\pi i}{\tau-\bar{\tau}}.$$

3.3.4. We will only use the splitting s^{MI} for the holomorphic model and the splitting s^{CC} for Polishchuk's model, and we will just use $F_{g,n}^{\mathsf{hol}}(\tau)$ to denote $F_{g,n}^{A_\tau^{\mathsf{hol}},s^{\mathsf{MI}}}$ and $F_{g,n}^{\mathsf{mod}}(\tau)$ to denote $F_{g,n}^{A_\tau^{\mathsf{mod}},s^{\mathsf{CC}}}$, as it won't make confusion.

3.3.5. We would like to generalize the above two computations to arbitrary (g,n)-invariants for elliptic curves. Before stating the result, we first summarize some useful facts from the theory of quasi-modular holomorphic forms and almost holomorphic modular forms.

3.3.6. Denote by $\tilde{M}(\Gamma) = \mathbb{C}[C_2, C_4, C_6]$ the ring of quasi-modular holomorphic forms for the group $\Gamma = \mathsf{SL}(2,\mathbb{Z})$. Denote by $\hat{M}(\Gamma) = \mathbb{C}[C_2^*, C_4, C_6]$ the ring of almost holomorphic modular forms. (See [KZ95] for definition of these.)

In terms of $q = e^{2\pi i\tau}$, these are defined by explicit formulas:

$$C_k(q) = -\frac{B_k}{k\cdot k!} + \frac{2}{k!}\sum_{n\geqslant 1}\sum_{d|n} d^{k-1}q^n = -\frac{B_k}{k\cdot k!}E_k(q),$$

where B_k are the Bernoulli numbers. In particular,

$$C_2(q) = -\frac{1}{24}E_2(\tau), \quad C_2^*(q) = -\frac{1}{24}E_2^*(\tau).$$

$\tilde{M}(\Gamma)$ and $\hat{M}(\Gamma)$ are closed under the differential operators ∂_τ and $\hat{\partial}_\tau = \partial_\tau + \frac{\mathsf{wt}}{\tau-\bar{\tau}}$ respectively. There is a differential ring isomorphism $\mathsf{KZ} : \tilde{M}(\Gamma) \to \hat{M}(\Gamma)$ called the Kaneko-Zagier map, defined as

$$\begin{aligned} \mathsf{KZ} : \tilde{M}(\Gamma) &\to \hat{M}(\Gamma), \\ C_2 &\mapsto C_2^* \\ C_4 &\mapsto C_4 \\ C_6 &\mapsto C_6. \end{aligned}$$

Conjecture 3.3.7. *The invariants $F^{\mathsf{hol}}_{g,n}(\tau)$ and $F^{\mathsf{mod}}_{g,n}(\tau)$ with appropriate insertions belong to $\tilde{M}(\Gamma)$ and $\hat{M}(\Gamma)$ respectively.*

To prove this conjecture, we would need to show that CEI satisfy the following three properties:

1. $F^{\mathsf{hol}}_{g,n}(\tau)$ is holomorphic.
2. $F^{\mathsf{hol}}_{g,n}(\tau)$ behaves well with respect to the $\mathrm{SL}_2(\mathbb{Z})$-action.
3. $F^{\mathsf{hol}}_{g,n}(\tau)$ has finite limit at $\tau = i\infty$.

Conditions 1 and 2 are easy to prove, as they follow from the fact that all the structures involved in the computations are holomorphic, respectively modular. However, condition 3 is difficult to prove and we were unable to prove it.

If we assume this conjecture to be true, then we can prove that the two categorical enumerative potentials are related by the this Kaneko-Zagier map as follows:

Lemma 3.3.8. *Assume Conjecture 3.3.7 is valid. Then for any $g, n \geqslant 0$,*

$$F^{\mathsf{mod}}_{g,n}(\tau) = \mathsf{KZ}(F^{\mathsf{hol}}_{g,n}(\tau)).$$

Proof. We briefly review how the categorical enumerative invariants are defined. In [CT24, Theorem 9.1], Căldăraru and Tu defined $\bar{\iota}(F_{g,n}^{A,s})$ as a Feynman sum over *partially directed stable graphs*

$$\bar{\iota}(F_{g,n}^{A,s}) = \sum_{m\geqslant 1} \sum_{G\in\Gamma(\!(g,1,n-1)\!)_m} (-1)^{m-1}\frac{\mathsf{wt}(G)}{|\mathrm{Aut}(G)|}\prod_{v\in V_G}\mathsf{Cont}(v)\prod_{e\in E_G}\mathsf{Cont}(e)\prod_{l\in L_G}\mathsf{Cont}(l).$$

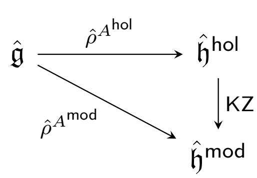
Here, $\Gamma(\!(g,1,n-1)\!)_m$ denotes partially directly stable graphs with genus g, 1 input leg, $n-1$ output legs, and m vertices (as defined in [CT24]).

The vertex contributions $\mathsf{Cont}(v)$ are given by the tensors $\hat{\beta}^A_{g,k,l} = \rho^A(\hat{\mathcal{V}}^{\mathsf{comb}}_{g,k,l})$ obtained from the combinatorial string vertices depending only on A. The contribution of edges and leaves involve the choice of the splitting s, and the weight $\mathsf{wt}(G)$ is some rational number.

We prove the lemma by showing that the contributions of each partially directed stable graph are related by the Kaneko-Zagier map.

First we review the comparison of the two $\mathcal{A}_\infty$-models A_τ^{hol} and A_τ^{mod}. They have the same underlying vector space and the same basis. The coefficients of nontrivial higher multiplications are related by Kaneko-Zagier map, see [CT20, p. 5.12] for details and see [Pol11, Theorem 2.5.1] for a complete list of nontrivial higher multiplications. Hence it makes sense to denote $A_\tau^{\mathsf{mod}} = \mathsf{KZ}(A_\tau^{\mathsf{hol}})$.

Then consider the following diagram



where

$$\hat{\mathfrak{g}} = \left(\bigoplus_{g,k\geqslant 1,l} C_*(M^{\mathrm{fr}}_{g,k,l}, \underline{\mathsf{sgn}})_{\mathsf{hS}}\right)[\![\hbar,\lambda]\!][2], \quad \text{(defined in [CT24, Section 5.3])}$$

$\hat{\mathfrak{h}}^{\mathsf{hol}}$ and $\hat{\mathfrak{h}}^{\mathsf{mod}}$ are

$$\hat{\mathfrak{h}} = \left(\bigoplus_{k \geqslant 1, l} \mathrm{Hom}^{\mathsf{c}}(\mathrm{Sym}^k(L_+[1]), \mathrm{Sym}^l(L_-))_{\mathsf{hS}} \right) [\![\hbar, \lambda]\!], \quad \text{(defined in [CT24, Section 4.2])}$$

associated to $L^{\mathsf{hol}} = C_*(A_\tau^{\mathsf{hol}})[d]$ and $L^{\mathsf{mod}} = C_*(A_\tau^{\mathsf{mod}})[d]$ respectively. The vertical arrow KZ is defined as replacing every C_2 by C_2^* in the computation.

It is worth noting that the definition of the map $\hat{\rho}^A$ only involves the $\mathcal{A}_\infty$-algebra structure, it dose not depend on the splittings. Since the two algebras satisfy $A_\tau^{\mathsf{mod}} = \mathsf{KZ}(A_\tau^{\mathsf{hol}})$, we conclude that the above diagram commutes. In particular, for the decoration string vertices $\hat{\mathcal{V}}_{g,k,l}^{\mathsf{comb}}$, we get

$$\hat{\rho}^{A^{\mathsf{mod}}}(\hat{\mathcal{V}}_{g,k,l}^{\mathsf{comb}}) = \mathsf{KZ}(\hat{\rho}^{A^{\mathsf{hol}}}(\hat{\mathcal{V}}_{g,k,l}^{\mathsf{comb}})).$$

This implies that the contribution of vertices are related by the Kaneko-Zagier map as we expected.

Next we show that the contributions of edges and leaves are related by the Kaneko-Zagier map as well. In [CT24, Theorem 9.1], the authors proved that the incoming leaves are decorated by S and the outgoing leaves are decorated by R, where $S : (L, b) \to (L_+, b+uB)$ is a chain level lift of the splitting $s : H_*(L) \to H_*(L_+)$ of the form

$$S = \mathsf{id} + S_1 u + S_2 u^2 + \cdots, S_j \in \mathsf{End}(L).$$

In the elliptic curve case, the class $[\Omega] \in \mathrm{HH}_1(A_\tau)$ has a canonical lift to cyclic homology, therefore to specify the above chain map is equivalent to choosing a lift $[S(\xi)] \in \mathrm{HC}^-_{-1}(A_\tau)$ of the class $[\xi] \in \mathrm{HH}_{-1}(A_\tau)$. Such a lift is of the form

$$S(\xi) = \xi + \alpha_1 \cdot u + \alpha_2 \cdot u^2 + \cdots.$$

In [CT20, Section 9], the authors proved that for elliptic curves, it is enough to determine

α_1, which satisfies the set of two equations

$$\begin{cases} b(\alpha_1) = -1 \otimes \xi \\ b^{1|1}(\partial_\tau \mu^* \mid \alpha_1) = 0. \end{cases}$$

Here, $b^{1|1}$ is the operator defined in [She19, Section 3.14]. The higher liftings $\alpha_2, \alpha_3, \ldots$ are uniquely determined by α_1 up to homology. Using the two $\mathcal{A}_\infty$-models with corresponding splittings, we obtain two liftings $S^{\mathsf{hol}}(\xi)$ and $S^{\mathsf{mod}}(\xi)$ of ξ. (Recall that the basis of A_τ^{hol} and A_τ^{mod} are the same, so we use the same ξ to denote the class in $\mathrm{HH}_{-1}(A_\tau^{\mathsf{hol}})$ and $\mathrm{HH}_{-1}(A_\tau^{\mathsf{mod}})$.) Their first-order terms α_1^{hol} and α_1^{mod} are characterized by the following two sets of equations:

$$\begin{cases} b^{\mathsf{hol}}(\alpha_1^{\mathsf{hol}}) = -1 \otimes \xi \\ b^{1|1,\mathsf{hol}}(\partial_\tau \mu^{*,\mathsf{hol}} \mid \alpha_1^{\mathsf{hol}}) = 0, \end{cases} \qquad \begin{cases} b^{\mathsf{mod}}(\alpha_1^{\mathsf{mod}}) = -1 \otimes \xi \\ b^{1|1,\mathsf{mod}}(\hat{\partial}_\tau \mu^{*,\mathsf{mod}} \mid \alpha_1^{\mathsf{mod}}) = 0. \end{cases}$$

Here, $\hat{\partial}_\tau = \partial_\tau + \frac{\mathsf{wt}}{\tau - \bar{\tau}}$ is the modular differential appearing in section 3.3.6. These two sets of equations are two large systems of linear equations, applying Kaneko-Zagier map to the holomorphic one will give the modular one, as KZ maps b^{hol} to b^{mod}, $b^{1|1,\mathsf{hol}}$ to $b^{1|1,\mathsf{mod}}$, $\mu^{*,\mathsf{hol}}$ to $\mu^{*,\mathsf{mod}}$, ∂_τ to $\hat{\partial}_\tau$, and $\xi \in \mathrm{HH}_{-1}(A_\tau^{\mathsf{hol}})$ to $\xi \in \mathrm{HH}_{-1}(A_\tau^{\mathsf{mod}})$. So KZ also maps the solution α_1^{hol} of the first system to the solution α_1^{mod} of the second system. As the higer liftings are determined by α_1 up to homology, we have shown that the edge contributions are related by the Kaneko-Zagier map as wanted.

Combining with the argument above about the vertices contribution, we conclude that for any partially directed stable graph G, the contributions are related by the Kaneko-Zagier map. So summing over all the graphs, we proved $F_{g,n}^{\mathsf{mod}}(\tau) = \mathsf{KZ}(F_{g,n}^{\mathsf{hol}}(\tau))$. □

Remark 3.3.9. It's worth noting that the two $\mathcal{A}_\infty$-algebras A_τ^{hol} and A_τ^{mod} are quasi-isomorphic. Furthermore, our construction of categorical enumerative invariants is Morita invariant. Therefore, the two different potentials F^{hol} and F^{mod} depend solely on their corresponding splittings. This underscores the importance of understanding the relationship between these splittings in determining the resulting invariants.

Corollary 3.3.10. *Assuming the Conjecture 3.3.7, we have*

$$F_{g,n}^{\mathsf{mod}}(\tau) - F_{g,n}^{\mathsf{hol}}(\tau) = -\frac{1}{2}\frac{1}{2\pi i}\frac{dF_{g,n}^{\mathsf{hol}}(\tau)}{dC_2} \cdot \frac{1}{\tau - \bar{\tau}} + O\left(\frac{1}{(\tau - \bar{\tau})^2}\right).$$

Proof. This can be proved by induction on the weight of $F_{g,n}^{\mathsf{hol}}(\tau)$ and direct calculation. □

This corollary implies that to study $\frac{dF_{g,n}^{\mathsf{hol}}(\tau)}{dC_2}$, it suffices to analyze the difference on the left-hand side, focusing on the terms that do not contain higher powers of $\frac{1}{\tau-\bar{\tau}}$. Understanding this difference hinges on the concept of Givental group action.

3.4 Givental group action

3.4.1. Given a vector space H with a nondegenerate symmetric bilinear form $\langle -, - \rangle$ we constructed the vector spaces $H_{\mathsf{Tate}} = H(\!(u)\!)$ (endowed with the residue pairing $\langle -, - \rangle_{\mathsf{res}}$) and $H_+ = H[\![u]\!]$.

The Givental group Giv associated to this data is the subgroup of the group of automorphisms of the symplectic vector space $(H_{\mathsf{Tate}}, \langle -, - \rangle_{\mathsf{res}})$ preserving the Lagrangian subspace H_+ and acting as identity on H. Explicitly, an element of Giv is of the form

$$\sigma = id + \sigma_1 \cdot u + \sigma_2 \cdot u^2 + \cdots$$

with each $\sigma_j \in \mathrm{End}(H)$, and σ is required to satisfy

$$\langle \sigma \cdot x, \sigma \cdot y \rangle_{\mathsf{res}} = \langle x, y \rangle_{\mathsf{res}} \quad \text{for any } x, y \in H_{\mathsf{Tate}}.$$

In the CEI of elliptic curves setting, for the $\mathcal{A}_\infty$-algebra A_τ, we will let H be $\mathrm{HH}_*(A_\tau)$, and $\langle -, - \rangle$ be the Mukai pairing.

3.4.2. Given any two splittings s_1 and s_2 of the noncommutative Hodge filtration on $\mathrm{HC}_*^-(E_\tau)$, there exists an element σ in the Givental group Giv that maps s_1 to s_2. To define σ, we first extend s_i u-linearly to get a map $\tilde{s}_i : \mathrm{HH}_*(E_\tau)[\![u]\!] \to \mathrm{HC}_*^-(E_\tau)$. Since E_τ is smooth, the Hochschild to cyclic spectral sequence degenerates at the first page, making $\tilde{s}_i$ an isomorphism. We then define σ as

$$\sigma := \tilde{s}_1^{-1} \circ \tilde{s}_2 : \mathrm{HH}_*(E_\tau)[\![u]\!] \to \mathrm{HH}_*(E_\tau)[\![u]\!].$$

By definition, $\sigma \cdot \tilde{s}_1 = \tilde{s}_2$. Since σ is u-linear, we can restrict σ to $\mathrm{HH}_*(E_\tau)$ and write it explicitly as

$$\sigma = \sigma_0 + \sigma_1 \cdot u + \sigma_2 \cdot u^2 + \dots,$$

where each σ_i is a map $\mathrm{HH}_*(E_\tau) \to \mathrm{HH}_*(E_\tau)$.

Lemma 3.4.3. $\sigma_1 = s_{21} - s_{11}$, *where* s_{11} *and* s_{21} *are the* u^1*-part of the splittings,*

$$s_1 = \mathsf{id} + s_{11} \cdot u + s_{12} \cdot u^2 + \dots$$

$$s_2 = \mathsf{id} + s_{21} \cdot u + s_{22} \cdot u^2 + \dots$$

Proof. By direct calculation, $s_1^{-1} = \mathsf{id} - s_{11} \cdot u + O(u^2)$, then $\sigma = s_1^{-1} \circ s_2 = \mathsf{id} + (s_{21} - s_{11}) \cdot u + O(u^2)$. □

3.4.4. In [CT24, Lemma 7.4], it is demonstrated that for any splitting $s : H_*(L) \to H_*(L_+)$ at the homology level, there exists a corresponding chain-level splitting $S : L \to L_+$, given by a map of the form

$$S = \mathsf{id} + S_1 \cdot u + S_2 \cdot u^2 + \dots,$$

where $S_i \in \mathsf{End}(L)$, such that the induced map on homology is precisely s.

3.4.5. Given a chain-level splitting S, similar to before, extend it by linearity to obtain a quasi-isomorphism $S : (L_+, b) \to (L_+, b + uB)$. As the u^0 component of S is the identity, we can express its inverse R as

$$R = \mathsf{id} + R_1 \cdot u + R_2 \cdot u^2 + \dots.$$

Using R and S, we define maps (see [CT24] for details)

$$H_{i,j} : u^{-i}L \otimes u^{-j}L \to \mathbb{C}, \qquad u^{-i}x \otimes u^{-j}y \mapsto \langle (-1)^j \sum_{l=0}^{j} S_l R_{i+j+1-l}x, y\rangle,$$

associated with the chain-level splitting $S : L \to L_+$.

In the case of $A = A_\tau^{\mathsf{hol}}$, for degree reasons, the only nontrivial $H_{i,j}$ is given by

$$H_{0,0} : L \otimes L \to \mathbb{C}, \qquad x \otimes y \mapsto \langle x, R_1 y\rangle.$$

3.4.6. Similarly, we define maps $H^\sigma_{i,j} : u^{-i}H \otimes u^{-j}H \to \mathbb{C}$ associated with a Givental group element σ. For degree reasons again, the only nontrivial $H^\sigma_{i,j}$ is

$$H^\sigma_{0,0} : H \otimes H \to \mathbb{C}, \qquad \alpha \otimes \beta \mapsto \langle \alpha, \sigma_1 \beta\rangle.$$

Then Lemma 3.4.3 implies

Lemma 3.4.7. *Consider the two splittings $s_1 = s^{\mathsf{MI}}$ and $s_2 = s^{\mathsf{CC}}$. Let σ be the Givental group element that maps s_1 to s_2, and define the maps $H^\sigma_{i,j}$ as above. Then*

$$H^\sigma_{i,j}(\alpha, \beta) = 0,$$

unless $i = j = 0$ and α, β are constant multiples of $[\xi]$. In this case

$$H^\sigma_{0,0}([\xi], [\xi]) = \frac{1}{2\pi i} \frac{1}{\tau - \bar{\tau}}.$$

Proof. If $H^{\sigma}_{i,j}(u^{-i}\alpha, u^{-j}\beta) \neq 0$, then $\deg(u^{-i}\alpha) + \deg(u^{-j}\beta) = -2$, so $\deg(\alpha) + \deg(\beta) = -2(i+j+1)$. However, for elliptic curves, $\deg(\alpha) \geqslant -1, \deg(\beta) \geqslant -1$. So the only nontrivial case is $i = j = 0$, and $\deg(\alpha) = \deg(\beta) = -1$. Hence $\alpha, \beta \in \mathrm{HH}_{-1}(E_\tau)$, so they are constant multiple of $[\xi]$.

Moreover, by a direct computation, we have

$$\begin{aligned} H^{\sigma}_{0,0}([\xi],[\xi]) &= \langle [\xi], \sigma_1([\xi]) \rangle \\ &= \langle [\xi], \alpha_1^{\mathsf{mod}} - \alpha_1^{\mathsf{hol}} \rangle \\ &= \langle [\xi], \frac{1}{2\pi i}\frac{1}{\tau - \bar{\tau}}[\Omega] \rangle \\ &= \frac{1}{2\pi i}\frac{1}{\tau - \bar{\tau}} \end{aligned}$$

where the third equality is obtained from [CT20, Proposition 10.15]. □

3.5 Holomorphic anomaly equation

3.5.1. One important property of classical Gromov-Witten invariants is that they are conjectured to satisfy the holomorphic anomaly equation. In [MRS18], Milanov, Ruan, and Shen proved a holomorphic anomaly equation for some elliptic orbifolds, such as $\mathbb{P}^1$, which are stack quotients of an elliptic curve by a nontrivial finite group. However, the specific case of the elliptic curve was left as a conjecture. This conjecture was later proved by Oberdieck and Pixton [OP18], who actually established a more general result about Gromov-Witten classes.

3.5.2. We briefly revisit the case of the elliptic curve E_τ. Consider the generating series of Gromov-Witten classes:

$$\mathcal{C}_g(\gamma_1, \ldots, \gamma_n) = \sum_{d=0}^{\infty} \mathcal{C}_{g,d}(\gamma_1, \ldots, \gamma_n) q^d$$

where the cohomology classes $\gamma_i \in H^*(E_\tau)$, and $\mathcal{C}_{g,d}(\gamma_1, \ldots, \gamma_n)$ are cohomology classes of the moduli space $\overline{M}_{g,n}$. Hence this series can be regarded as an element in $H^*(\overline{M}_{g,n}) \otimes \mathbb{Q}[\![q]\!]$.

Remark 3.5.3. When integrating against the cotangent line classes $\psi_i \in H^2(\overline{M}_{g,n})$, we recover the Gromov-Witten invariants:

$$\sum_{d=0}^{\infty} \langle \tau_{k_1}(\gamma_1), \ldots, \tau_{k_n}(\gamma_n) \rangle_{g,d}^{E_\tau} q^d = \int_{\overline{M}_{g,n}} \psi^{k_1} \cdots \psi^{k_n} \cdot \mathcal{C}_g(\gamma_1, \ldots, \gamma_n).$$

One of the main result of [OP18] is the following analogue of Conjecture 3.3.7.

Proposition 3.5.4. For any $\gamma_1, \ldots, \gamma_n \in H^*(E_\tau)$, the series $\mathcal{C}_g(\gamma_1, \ldots, \gamma_n)$ is a cycle-valued quasimodular form:

$$\mathcal{C}_g(\gamma_1, \ldots, \gamma_n) \in H^*(\overline{M}_{g,n}) \otimes \tilde{M}(\Gamma).$$

We now introduce two maps that are needed to state the holomorphic anomaly equation. Let $\iota : \overline{M}_{g-1,n+2} \to \overline{M}_{g,n}$ be the gluing map along the last two marked points, and for any $g = g_1 + g_2$ and $\{1, \ldots, n\} = S_1 \sqcup S_2$, let

$$j : \overline{M}_{g_1, S_1 \sqcup \{\bullet\}} \times \overline{M}_{g_2, S_2 \sqcup \{\bullet\}} \to \overline{M}_{g,n}$$

be the gluing map along the marked points $\{\bullet\}$, where $\overline{M}_{g_i,S_i}$ is the moduli space of stable curves with marking in the set S_i.

Theorem 3.5.5. *Considering $\mathcal{C}_g(\gamma_1, \ldots, \gamma_n)$ as a polynomial in C_2, C_4, C_6 with coefficients* *in* $H^*(\overline{M}_{g,n})$, *we have*

$$\begin{aligned}\frac{d}{dC_2}\mathcal{C}_g(\gamma_1,\dots,\gamma_n) =& \iota_*\mathcal{C}_{g-1}(\gamma_1,\dots,\gamma_n,1,1)\\ &+ \sum_{\substack{g=g_1+g_2\\ \{1,\dots,n\}=S_1\sqcup S_2}} j_*(\mathcal{C}_{g_1}(\gamma_{S_1},1)\boxtimes\mathcal{C}_{g_2}(\gamma_{S_2},1))\\ &- 2\sum_{i=1}^{n}\left(\int_E\gamma_i\right)\psi_i\cdot\mathcal{C}_g(\gamma_1,\dots,\gamma_{i-1},1,\gamma_{i+1},\dots,\gamma_n),\end{aligned} \tag{3.1}$$

where $\gamma_{S_i} = (\gamma_k)_{k\in S_i}$ *and* $1\in H^*(E)$ *is the unit.*

Roughly speaking, equation 3.1 measures the dependence of the modular completion of $\mathcal{C}_g(\dots)$ on the non-holomorphic parameter and is thus called a *holomorphic anomaly equation*. Practically, it determines the quasi-modular form from lower weight data up to a purely modular part (involving only C_4 and C_6).

Our main theorem is

Theorem 3.5.6. *Assuming Conjecture 3.3.7, then the categorical enumerative potential* $F^{\mathsf{hol}}_{g,n}(\tau)$ *of an elliptic curve* E_τ *satisfies the holomorphic anomaly equation*

$$\begin{aligned}\frac{d}{dC_2}F^{\mathsf{hol}}_{g,n}(\tau) =& \Delta F^{\mathsf{hol}}_{g-1,n+2}(\tau)\\ &+ \sum_{\substack{g=g_1+g_2\\ n=n_1+n_2}}\frac{1}{2}\left\{F^{\mathsf{hol}}_{g_1,n_1+1}(\tau),F^{\mathsf{hol}}_{g_2,n_2+1}(\tau)\right\}\\ &+ F^{\mathsf{hol}}_{g,n}(\tau)\lrcorner\,\Xi,\end{aligned} \tag{3.2}$$

where

$$\Delta : C_*(M_{g,n})\to C_{*+1}(M_{g,n+2})$$

is the self twist-sewing operator,

$$\{-,-\} : C_*(M_{g_1,n_1})\otimes C_*(M_{g_2,n_2})\to C_{*+1}(M_{g_1+g_2,n_1+n_2-2})$$

is the twist-sewing operator,

$$\Xi : H_- \to H_- \text{ is a map defined as } \quad x \mapsto \langle x, [\xi] \rangle_{\mathsf{res}} \cdot u^{-1}[\xi],$$

and $\lrcorner$ *is the truncation operator, defined as*

$$\alpha_1 \otimes \cdots \alpha_n \lrcorner \beta := \sum_{i=1}^{n} \alpha_1 \otimes \cdots \otimes \alpha_{i-1} \otimes \beta(\alpha_i) \otimes \alpha_{i+1} \otimes \cdots \alpha_n.$$

Remark 3.5.7. It is easier to understand those operators in equation 3.2 using ribbon graphs. Recall that $F^{\mathsf{hol}}_{g,n}(\tau)$ can be expressed as a sum of ribbon graphs. Then Δ corresponds to gluing two output legs of a ribbon graph. $-,-$ corresponds to gluing one output of two different ribbon graphs together to form a single ribbon graph.

Proof. The key idea of the proof is to study the Givental group action on CEI potentials $F_{g,n}$. Given a group element σ and a splitting s, suppose we have computed $F^{A,s}_{g,n}$ for all g and n. Then we can compute $F^{A,\sigma\cdot s}_{g,n}$ using $\left\{F^{A,s}_{g,n}\right\}$ as follows:

$$F^{A,\sigma\cdot s}_{g,n} = \sum_{G \text{ stable of type } (g,n)} \frac{1}{|\mathrm{Aut}(G)|} \prod \mathsf{Cont}(v) \prod \mathsf{Cont}(e) \prod \mathsf{Cont}(l),$$

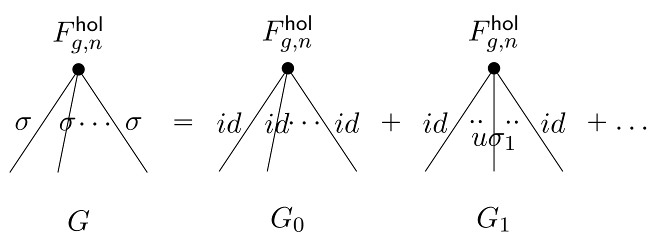
where each vertex v is decorated by $F^{A,s}_{g(v),n(v)}$, each edge e is decorated by H^{σ}, and each leg l is decorated by σ.

In the elliptic curve case, let $s = s^{\mathsf{MI}}$ be the monodromy invariant splitting, and σ be the group element such that $\sigma \cdot s = s^{\mathsf{CC}}$. Consequently, we have $F^{A_\tau,s}_{g,n} = F^{\mathsf{hol}}_{g,n}(\tau)$ and $F^{A_\tau,\sigma\cdot s}_{g,n} = F^{\mathsf{mod}}_{g,n}(\tau)$. Furthermore, according to Corollary 3.3.10, computing $\frac{d}{dC_2}F^{\mathsf{hol}}_{g,n}(\tau)$ involves studying the difference $F^{\mathsf{mod}}_{g,n}(\tau) - F^{\mathsf{hol}}_{g,n}(\tau)$, and focusing on terms with coefficients of the first order, specifically those with $\frac{1}{\tau-\bar{\tau}}$.

Lemma 3.4.7 reveals that the only nonzero H^{σ} is $H^{\sigma}_{0,0}([\xi],[\xi]) = \frac{1}{2\pi i}\frac{1}{\tau-\bar{\tau}}$, already possesses a factor of $\frac{1}{\tau-\bar{\tau}}$. Consequently, terms of the first order cannot contain more than one edge. Hence, our analysis will concentrate on graphs with either no edges or with precisely one edge.

- Consider the graphs G without edges. They must have exactly one vertex, denoted by v. This single vertex is decorated by $F^{\mathsf{hol}}_{g,n}$, while each of the n legs is decorated by σ. Since $F^{\mathsf{hol}}_{g,n}$ is multi-linear in its outputs, and $\sigma = id + \sigma_1 \cdot u$ is linear, we can express the graph G as a sum of three types of graphs.

  Firstly, let's denote the first type as G_0, where all legs are decorated by id. Secondly, we have the second type, denoted as G_1, featuring exactly one leg decorated by $u\sigma_1$. Finally, there's the third type, labeled by G_2, characterized by having more than one leg decorated by $u\sigma_1$.



  Let's examine the contributions from each type of graph in more detail. Firstly, consider G_0, which corresponds to the term $F^{\mathsf{hol}}_{g,n}(\tau)$. This represents the base contribution of the CEI potential without any additional factors introduced by the Givental group action.

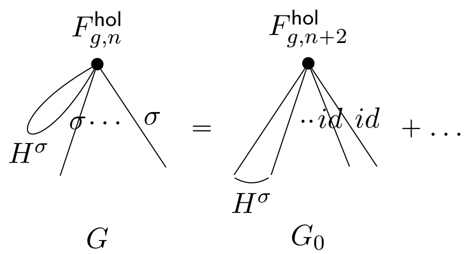
  Moving on to G_1, we observe that it generates terms with exactly one factor of $\frac{1}{\tau-\bar{\tau}}$. This is due to the fact that the only non-zero term in σ_1 is $\sigma_1([\xi]) = \frac{1}{2\pi i}\frac{1}{\tau-\bar{\tau}}[\Omega]$. Remarkably, graphs of type G_1 precisely correspond to the third part of the right-hand side in equation 3.2.

  As for G_2, since it contributes higher powers of $\frac{1}{\tau-\bar{\tau}}$, we can disregard these terms.

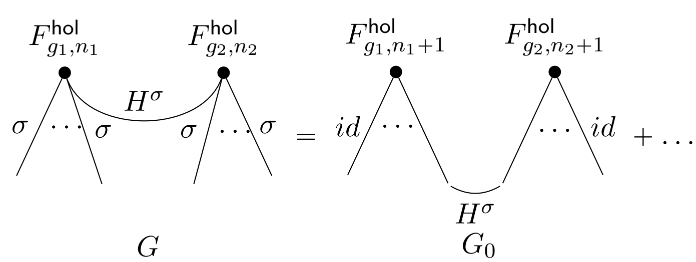
- Let's analyze the graphs with exactly one edge, distinguishing between the scenarios where the edge is a loop and where it is not.

  In the case where the edge is a loop, the graph G has a single vertex v of type

$(g-1, n+2)$, with all legs decorated by σ. Leveraging the same approach as previously outlined, we decompose σ into $\sigma = id + u\sigma_1$. Notably, since the decoration $H^{\sigma}_{0,0}$ on the loop edge e already introduces a factor of $\frac{1}{\tau-\bar{\tau}}$, we do not need σ_1 on the legs. Consequently, we can simplify graph G to a type $(g-1, n+2)$ graph G_0 with all legs decorated by id, followed by the twist-sewing of two legs. This simplification yields the first part on the right-hand side of equation 3.2.



In the case where the edge e is not a loop, the graph G consists of two vertices v_1 and v_2, of types (g_1, n_1) and (g_2, n_2), respectively. These vertices satisfy $g_1 + g_2 = g$ and $n_1 + n_2 = n + 2$. Again, we focus on the simplified graph G_0, where all legs are decorated by id. This simplification corresponds to the second part on the right-hand side of equation 3.2.



Summing these simplified graphs results in the expression on the right-hand side of equation 3.2. This completes the analysis of the different graphs contributing to the holomorphic anomaly equation. □

Chapter 4

Hodge to de Rham degeneration for the nodal cubic curve

4.1 Introduction

4.1.1. The study of algebraic de Rham cohomology goes back to Grothendieck. In [Gro66], he shows that for a smooth scheme X over $\mathbb{C}$ there is an associated complex of sheaves of differentials $\Omega^\bullet_X$ whose hypercohomology $H^*_{\mathsf{dR}}(X) := \mathbb{H}^*(\Omega^\bullet_X)$ computes the singular cohomology of the analytification of X. This complex is known as the *algebraic de Rham complex*. Furthermore, the algebraic de Rham complex admits a filtration by naive truncations, which leads to a spectral sequence ${}^1E^{p,q} = H^p(X, \Omega^q_X) \implies H^{p+q}_{\mathsf{dR}}(X)$. This spectral sequence is called the *Hodge to de Rham (HdR) spectral sequence*. Grothendieck also shows that when X is smooth over $\mathbb{C}$, this spectral sequence degenerates at first page. Deligne and Illusie [DI87] generalize this degeneration result to the case when X is smooth and proper over any field k of characteristic 0, using the method of reduction to positive characteristic.

4.1.2. The groups on both sides of the HdR spectral sequence are related to Hochschild homology and negative cyclic homology, thus the HdR spectral sequence can be generalized to the world of noncommutative geometry. The resulting spectral sequence $\mathrm{HH}_*(X)[\![u]\!] \Longrightarrow \mathrm{HC}^-_*(X)$ is called the *Hochschild to cyclic spectral sequence.* This can be further generalized to a spectral sequence associated to any $\mathcal{A}_\infty$-algebra A. Konstevich and Soibelman [KS09] conjectured that for any smooth and proper $\mathcal{A}_\infty$-algebra A over a field of characteristic 0, the Hochschild to cyclic spectral sequence also degenerates at ${}^1E$. This was later proved by Kaledin [Kal08; Kal17].

In this paper we study what happens when X is not smooth. In particular we investigate the projective nodal cubic curve. Our main theorem is

Theorem 4.1.3. *The Hodge to de Rham spectral sequence of the projective nodal cubic curve degenerates at* ${}^2E$*.*

The key to proving this theorem is in comparing the HdR spectral sequence for the projective nodal cubic with the ones for the affine nodal cubic as well as for the desingularization $\mathbb{P}^1$ of the nodal cubic.

4.1.4. In the smooth projective case, the Hochschild to cyclic spectral sequence degenerates at ${}^1E$, so every Hochschild homology class can be lifted to negative cyclic homology, i.e., the map $\mathrm{HC}^-_n \to \mathrm{HH}_n$ is surjective for any n. This no longer holds for singular X. However, by computing $\mathrm{HH}_*(X)$ and $\mathrm{HC}^-_*(X)$, we can prove that the Hochschild to cyclic spectral sequence for nodal curve also degenerates at ${}^2E$. This enables us to classify those Hochschild classes which lift, and to understand the map $\mathrm{HC}^-_*(X) \to \mathrm{HH}_*(X)$.

4.1.5. This liftability of Hochschild classes, especially the class in $\mathrm{HH}_{-1}(X)$, is crucial in the computation of categorical enumerative invariants [CT24]. We will discuss this in section 4.5.

4.1.6. In Section 4.2 we will compute the Hochschild homology $\mathrm{HH}_*(X)$ of the nodal curve X. In Section 4.3 we study the degeneration of the HdR spectral sequence of X. In Section 4.4 we compute the negative cyclic homology $\mathrm{HC}^-_*(X)$ of X. In Section 4.5 we characterize all the liftable Hochschild classes. Finally, in Section 4.6 we include some results for the cuspidal curve case.

4.2 Hochschild homology

Let $X \subset \mathbb{P}^2$ be the nodal cubic curve. Explicitly, X is cut out by the equation $y^2z = x^3 - x^2z$. In this section we will compute $\mathrm{HH}_*(X)$.

Let L_X denote the cotangent sheaf of X.

Lemma 4.2.1. *There are two descriptions of L_X.*

1. *L_X admits a resolution $0 \to \mathcal{O}_X(-3) \to \Omega^1_{\mathbb{P}^2}|_X \to L_X \to 0$,*
2. *L_X fits into a nontrivial extension $0 \to \mathcal{O}_P \to L_X \to \mathcal{I}_P \to 0$, here $P = [0:0:1]$ is the node in X, and $\mathcal{I}_P$ is the ideal sheaf.*

Proof. The first part follows directly from the fact that $X \subset \mathbb{P}^2$ is a local complete intersection, and the conormal sheaf is $\mathcal{I}_X/\mathcal{I}_X^2 \cong \mathcal{O}_{\mathbb{P}^2}(-3)|_X \cong \mathcal{O}_X(-3)$.

To prove the second part, recall that for any coherent sheaf $\mathcal{F}$, there is a short exact sequence

$$0 \to \mathcal{T} \to \mathcal{F} \to \mathcal{G} \to 0,$$

where $\mathcal{T}$ is a torsion subsheaf of $\mathcal{F}$ and $\mathcal{G}$ is torsion free. In the case $\mathcal{F} = L_X$, a local calculation will show that $\mathcal{T} = \mathcal{O}_P$ and $\mathcal{G}$ is locally isomorphic to $\mathcal{I}_P$. So $\mathcal{G} = \mathcal{I}_P \otimes \mathcal{K}$ for some line bundle $\mathcal{K}$ on X. Then using part (1), it's easy to compute the Euler characteristic $\chi(X, L_X) = 0$, thus the line bundle $\mathcal{K} \cong \mathcal{O}_X$. Hence $\mathcal{G} \cong \mathcal{I}_P$, and L_X fits into a short exact sequence

$$0 \to \mathcal{O}_P \to L_X \to \mathcal{I}_P \to 0.$$

□

Remark 4.2.2. We will call the two term complex $\mathcal{O}_X(-3) \to \Omega^1_{\mathbb{P}_2}$ with amplitude $[-1, 0]$ the *cotangent complex* of X, and denote it $\mathbb{L}^\bullet_X$. Because X is a complete intersection, this is a valid description of the cotangent complex of X in the sense of Illusie.

With these two descriptions of L_X, we have

Corollary 4.2.3. *The cohomology of L_X is given by*

$$H^i(X, L_X) = \begin{cases} \mathbb{C} & i = 0, 1 \\ 0 & \text{otherwise} \end{cases}.$$

Proof. We have computed the Euler characteristic $\chi(X, L_X) = 0$. Combining with the long exact sequence of cohomology obtained from the second description, we get the result. □

Lemma 4.2.4. *The hypercohomology of $\bigwedge^2 \mathbb{L}_X^\bullet$ is given by*

$$\mathbb{H}^i(X, \bigwedge^2 \mathbb{L}_X^\bullet) = \begin{cases} \mathbb{C} & i = 0, -1 \\ 0 & \text{otherwise} \end{cases}.$$

(Here by $\bigwedge^2$ we mean the derived exterior product.)

Proof. A direct calculation shows that

$$\begin{aligned} \bigwedge^2 \mathbb{L}_X^\bullet = \mathrm{Sym}^2(\mathbb{L}_X^\bullet[1])[-2] &\simeq \Big(0 \to \mathcal{O}_X(-6) \to \Omega^1_{\mathbb{P}^2}|_X(-3) \to \mathcal{O}_X(-3) \to 0\Big)[0] \\ &\simeq \mathcal{O}_X(-3) \otimes \Big(0 \to \mathcal{O}_X(-3) \to \Omega^1_{\mathbb{P}^2}|_X \to \mathcal{O}_X \to 0\Big)[0]. \end{aligned}$$

If we reduce to an affine open subset $\mathcal{D}(z) \cap X$, the above short exact sequence $0 \to \mathcal{O}_X(-3) \to \Omega^1_{\mathbb{P}^2}|_X \to \mathcal{O}_X \to 0$ becomes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R & \longrightarrow & R^{\oplus 2} & \longrightarrow & R & \longrightarrow & 0 \\ & & 1 & \longmapsto & \begin{pmatrix} 3x^2 + 2x \\ -2y \end{pmatrix} & & & & \\ & & & & \begin{pmatrix} f \\ g \end{pmatrix} & \longmapsto & 2yf + (3x^2 + 2x)g & & \end{array}$$

where $R = \mathbb{C}[x,y]/(x^3+x^2-y^2)$. By direct algebraic calculation we find that $H^0 = \mathbb{C}$ generated by 1, and $H^{-1} = \mathbb{C}$ generated by $\begin{pmatrix}(3x+2)y\\ 2(x^2+x)\end{pmatrix}$. Notice that $\begin{pmatrix}(3x+2)y\\ 2(x^2+x)\end{pmatrix}$ is annihilated by x, y, hence the (-1) cohomology sheaf is supported at the node P. So $0 \to \mathcal{O}_X(-3) \to \Omega^1_{\mathbb{P}^2}|_X \to \mathcal{O}_X \to 0$ is quasi-isomorphic to $0 \to 0 \to \mathcal{O}_P \to \mathcal{O}_P \to 0$. Twisting this complex by $\mathcal{O}_X(-3)$ will not affect hypercohomology. □

With Lemma 4.2.4, it is easy to compute hypercohomology of higher exterior powers of $\mathbb{L}^\bullet_X$.

Corollary 4.2.5. *The hypercohomology of $\bigwedge^k \mathbb{L}^\bullet_X$ is given by*

$$\mathbb{H}^i(\bigwedge^k \mathbb{L}^\bullet_X) = \begin{cases} \mathbb{C}, & \text{if } i = -k+2, -k+1 \\ 0, & \text{otherwise} \end{cases}.$$

Proof. A direct calculation gives

$$\begin{aligned} \bigwedge^k \mathbb{L}^\bullet_X &= \Big(0 \to \mathcal{O}_X(-3k) \to \Omega^1_{\mathbb{P}^2}|_X(-3k+3) \to \mathcal{O}_X(-3k+3) \to 0\Big)[k-2] \\ &\simeq \mathcal{O}_X(-3k+3) \otimes \Big(0 \to \mathcal{O}_X(-3) \to \Omega^1_{\mathbb{P}^2}|_X \to \mathcal{O}_X \to 0\Big)[k-2] \\ &\simeq \mathcal{O}_X(-3k+3) \otimes \Big(0 \to 0 \to \mathcal{O}_P \to \mathcal{O}_P \to 0\Big)[k-2] \end{aligned}$$

The rest of the argument is the same as in Lemma 4.2.4. □

Combining the above calculations we have

Theorem 4.2.6. *The Hochschild homology for the projective nodal cubic curve is given by*

$$\mathrm{HH}_n(X) = \begin{cases} \mathbb{C} & \text{if } n = -1 \\ \mathbb{C}^2 & \text{if } n = 0 \\ \mathbb{C} & \text{if } n \geqslant 1 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The key idea we are going to use is from the work [BF08], where the authors prove that there exists a decomposition of the Hochschild complex

$$\mathbb{H}_{X/Y} = \mathrm{Sym}^\bullet(\mathbb{L}^\bullet_{X/Y}[1]) \cong \bigoplus_n (\bigwedge^n \mathbb{L}^\bullet_{X/Y})[n].$$

We apply to our case for $\mathbb{H}_X = \mathbb{H}_{X/k} = \mathrm{Sym}^\bullet(\mathbb{L}^\bullet_X[1])$, and take hypercohomology to obtain $\mathrm{HH}_{-*}(X) := \mathrm{R}^*\Gamma(\mathrm{Sym}^\bullet_{\mathcal{O}_X}(\mathbb{L}^\bullet_X[1]))$. Then the theorem follows from above calculation and the fact that $H^0(X, \mathcal{O}_X) = H^1(X, \mathcal{O}_X) = \mathbb{C}$. $\square$

4.3 Hodge to de Rham spectral sequence

4.3.1. Recall that for a smooth projective variety Y of dimension n there exists a *de Rham complex* of sheaves

$$0 \to \mathcal{O}_Y \to \Omega^1_Y \to \Omega^2_Y \to \Omega^3_Y \to \cdots \to \Omega^n_Y \to 0,$$

where Ω^i_Y is the sheaf of Kähler differential i-forms on Y.

4.3.2. For a singular variety X (in particular when X is the nodal cubic curve), there is an analogous *derived de Rham complex*

$$\hat{\mathrm{dR}}^\bullet : 0 \to \mathcal{O}_X \to \mathbb{L}^\bullet_X \to \bigwedge^2 \mathbb{L}^\bullet_X \to \bigwedge^3 \mathbb{L}^\bullet_X \to \dots.$$

Notice that this derived de Rham complex is usually unbounded for singular varieties. In general, we must consider completion, but this is not necessary for the nodal cubic curve X, as $\mathbb{L}^\bullet_X$ is bounded.

4.3.3. Using the Hodge filtration on $\hat{\mathrm{dR}}^\bullet$ (the filtration by powers of $\mathbb{L}^\bullet_X$) we obtain a spectral sequence whose first page is

$${}^1E^{p,q} = \mathrm{R}^{p+q}\Gamma(\hat{\mathrm{dR}}^p[-p]) = \mathrm{R}^q\Gamma(\hat{\mathrm{dR}}^p) = H^q(X, \bigwedge^p \mathbb{L}^\bullet_X).$$

This spectral sequence converges to $H^{p+q}(X, \hat{\mathrm{dR}}^\bullet) = H^{p+q}_{\mathsf{sing}}(X, \mathbb{C})$, see [Bha12]. We will call this spectral sequence the *Hodge to de Rham spectral sequence*. Our main theorem is the following:

Theorem 4.3.4. *The Hodge to de Rham spectral sequence for X degenerates at page ${}^2E$.*

Before the proof, we can write down a few terms in the first page, given explicitly as

$$\begin{array}{llll}
1 & H^1(\mathcal{O}_X) \xrightarrow{\sigma} H^1(L_X) & & \\
0 & H^0(\mathcal{O}_X) \xrightarrow{\alpha} H^0(L_X) \xrightarrow{\gamma} H^0(\bigwedge^2 \mathbb{L}_X^\bullet) & & \\
-1 & & H^{-1}(\bigwedge^2 \mathbb{L}_X^\bullet) \xrightarrow{\beta_1} H^{-1}(\bigwedge^3 \mathbb{L}_X^\bullet) & \\
-2 & & & H^{-2}(\bigwedge^3 \mathbb{L}_X^\bullet) \xrightarrow{\beta_2} H^{-2}(\bigwedge^4 \mathbb{L}_X^\bullet) \\
\vdots & & & \quad \ddots \qquad \ddots
\end{array}$$

Notice that from our calculation in last section, all these terms are 1-dimensional.

We outline our proof in four steps, established in a series of lemmas.

- Step 1: $\alpha = 0$, Lemma 4.3.5.
- Step 2: β_k is an isomorphism for all $k \geqslant 1$, Lemma 4.3.6.
- Step 3: γ is an isomorphism, Lemma 4.3.7.
- Step 4: $\sigma = 0$, Lemma 4.3.8.

Lemma 4.3.5. *The map $\alpha : H^0(\mathcal{O}_X) \to H^0(L_X)$ is 0.*

Proof. This is trivial, since $H^0(\mathcal{O}_X)$ is the only nontrivial term in the 0-diagonal, and the spectral sequence converges to the singular homology with $H^0(X, \mathbb{C}) = \mathbb{C}$, hence it must survive until the ${}^\infty E$-page. $\square$

Lemma 4.3.6. *The maps $\beta_k : H^{-k}(\bigwedge^{k+1} \mathbb{L}_X^\bullet) \to H^{-k}(\bigwedge^{k+2} \mathbb{L}_X^\bullet)$ are all isomorphisms for $k \geqslant 1$.*

Proof. First observation is that $\bigwedge^{k+1} \mathbb{L}^\bullet_X$ is supported at the node when $k \geqslant 1$, so the computation of the maps β_k is local. Thus we can use a local affine model around the node, i.e., the affine nodal curve Y.

Let $V = \mathbb{C}\langle e_x, e_y, e_\epsilon\rangle$ be a graded $\mathbb{C}$-vector space with $\deg e_x = \deg e_y = 0$ and $\deg e_\epsilon = -1$. Then $S := \mathrm{Sym}^* V \cong \mathbb{C}[x,y] \otimes \mathbb{C}[\epsilon]/(\epsilon^2)$ forms a graded $\mathbb{C}$-algebra with $\deg x = \deg y = 0, \deg \epsilon = -1$. When endowing it with a $\mathbb{C}[x,y]$-linear differential δ on S as follows:

$$\mathbb{C}[x,y]\cdot \epsilon \xrightarrow{\delta:\epsilon\mapsto x^3+x^2-y^2} \mathbb{C}[x,y],$$

(S,δ) forms a differential graded algebra that is quasi-isomorphic to $R = \mathbb{C}[x,y]/(x^3 + x^2 - y^2)$, i.e., S is a differential graded resolution of R. Thus to compute $\bigwedge^k \mathbb{L}^\bullet_X \cong \bigwedge^k \mathbb{L}^\bullet_Y$ when $k \geqslant 2$, we can use the dg model $\bigwedge^k L_{S/\mathbb{C}}$.

Recall that $L_{S/\mathbb{C}} \cong S \otimes_{\mathbb{C}} V$ is a $\mathbb{C}[x,y]$-module. We claim that there is a way to construct a $\mathbb{C}[x,y]$-linear differential Δ such that $(L_{S/k}, \Delta)$ forms a differential graded S-module. Explicitly, $(L_{S/\mathbb{C}}, \Delta)$ is defined as follows:

$$\mathbb{C}[x,y]\cdot\epsilon\otimes d\epsilon \xrightarrow{\Delta_2} \begin{matrix}\mathbb{C}[x,y]\cdot 1\otimes d\epsilon\\ \oplus \\ \mathbb{C}[x,y]\cdot \epsilon\otimes dx \\ \oplus \\ \mathbb{C}[x,y]\cdot\epsilon\otimes dy\end{matrix} \xrightarrow{\Delta_1} \begin{matrix}\mathbb{C}[x,y]\cdot 1\otimes dx\\ \oplus \\ \mathbb{C}[x,y]\cdot 1\otimes dy\end{matrix}$$

Here we use $dx, dy, d\epsilon$ to denote e_x, e_y, e_ϵ respectively. The differentials Δ_1, Δ_2 are defined as:

$$\Delta_2(\epsilon\otimes d\epsilon) := (x^3+x^2-y^2)\cdot 1\otimes d\epsilon - (3x^2+2x)\cdot\epsilon\otimes dx + 2y\cdot\epsilon\otimes dy,$$

$$\Delta_1(1\otimes d\epsilon) := (3x^2+2x)\cdot 1\otimes dx - 2y\cdot 1\otimes dy,$$

$$\Delta_1(\epsilon\otimes dx) := \epsilon\cdot 1\otimes dx,$$

$$\Delta_1(\epsilon \otimes dy) := \epsilon \cdot 1 \otimes dy.$$

With the above definitions it is easy to check that $(L_{S/\mathbb{C}}, \Delta)$ is a differential graded (S, δ)-module and there exists a naturally defined chain map $d^{dR} : S \to L_{S/\mathbb{C}}$ (the de Rham differential) as follows:

$$\begin{array}{ccccc}
 & & \mathbb{C}[x,y]\cdot\epsilon & \xrightarrow{\delta} & \mathbb{C}[x,y] \\
 & & \downarrow d_1^{dR} & & \downarrow d_0^{dR} \\
 & & \mathbb{C}[x,y]\cdot 1\otimes d\epsilon & & \\
 & & \oplus & & \mathbb{C}[x,y]\cdot 1\otimes dx \\
\mathbb{C}[x,y]\cdot\epsilon\otimes d\epsilon & \xrightarrow{\Delta_2} & \mathbb{C}[x,y]\cdot\epsilon\otimes dx & \xrightarrow{\Delta_1} & \oplus \\
 & & \oplus & & \mathbb{C}[x,y]\cdot 1\otimes dy \\
 & & \mathbb{C}[x,y]\cdot\epsilon\otimes dy & &
\end{array}$$

Here,

$$d_0^{\mathsf{dR}}(f) := \frac{\partial f}{\partial x} \cdot 1 \otimes dx + \frac{\partial f}{\partial y} \cdot 1 \otimes dy,$$

and

$$d_1^{\mathsf{dR}}(g \cdot \epsilon) := g \cdot 1 \otimes d\epsilon + \frac{\partial g}{\partial x} \cdot \epsilon \otimes dx + \frac{\partial g}{\partial y} \cdot \epsilon \otimes dy.$$

With the above definition of $(L_{S/\mathbb{C}}, \Delta)$ one can compute higher exterior powers $\bigwedge^k L_{S/\mathbb{C}}$, and they will still be dg S-modules. One can extend the definition of d^{dR} to higher exterior powers to form the de Rham complex $(\bigwedge^\bullet L_{S/\mathbb{C}}, d^{\mathsf{dR}})$.

For our purpose it will be enough to show that the maps $H^{-k}(\bigwedge^{k+1} L_{S/\mathbb{C}}) \to H^{-k}(\bigwedge^{k+2} L_{S/\mathbb{C}})$ are surjective. However, it is actually enough to prove this just for $k = 1$, since $\bigwedge^{k+1} L_{S/\mathbb{C}}$ is basically a shift of $\bigwedge^2 L_{S/\mathbb{C}}$ when $k \geqslant 1$.

When $k = 1$ we only need to consider the degree -1 part of the morphism $d^{\mathsf{dR}} : \bigwedge^2 L_{S/\mathbb{C}} \to$

$\bigwedge^3 L_{S/\mathbb{C}}$. By an explicit calculation we have

$$\begin{array}{c} \mathbb{C}[x,y]\cdot 1\otimes(dx\otimes d\epsilon) \\ \oplus \\ \mathbb{C}[x,y]\cdot 1\otimes(dy\otimes d\epsilon) \xrightarrow{\Delta_1^{\wedge 2}} \mathbb{C}[x,y]\cdot 1\otimes(dx\wedge dy) \\ \oplus \\ \mathbb{C}[x,y]\cdot \epsilon\otimes(dx\wedge dy) \\ \big\downarrow d^{dR} \\ \mathbb{C}[x,y]\cdot 1\otimes(dx\wedge dy)\otimes d\epsilon \end{array}$$

Here we use $\Delta^{\wedge 2}$ to denote the differential in $\bigwedge^2 L_{S/\mathbb{C}}$. In matrix form it can be written as $\Delta_1^{\wedge 2} = \begin{pmatrix} -2y & -3x^2-2x & x^3+x^2-y^2 \end{pmatrix}$. The de Rham differential is given as

$$d^{\mathsf{dR}} : \begin{pmatrix} f \\ g \\ h \end{pmatrix} \mapsto \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} + h\right)\cdot 1\otimes(dx\wedge dy)\otimes d\epsilon.$$

In particular, $\begin{pmatrix} -3xy-2y \\ 2x^2+2x \\ 6x+4 \end{pmatrix} \in \ker(\Delta_1^{\wedge 2})$ and $d^{\mathsf{dR}}(\begin{pmatrix} -3xy-2y \\ 2x^2+2x \\ 6x+4 \end{pmatrix}) = (13x+8)\cdot 1\otimes(dx\wedge dy)\otimes d\epsilon$ is a generator for $H^{-1}(\bigwedge^3 L_{S/\mathbb{C}}) \cong \mathbb{C}\cdot 1\otimes(dx\wedge dy)\otimes d\epsilon$. Thus $H^{-1}(\bigwedge^2 L_{S/\mathbb{C}}) \to H^{-1}(\bigwedge^3 L_{S/\mathbb{C}})$ is surjective, and the same holds for $H^{-1}(\bigwedge^2 \mathbb{L}_X^{\bullet}) \to H^{-1}(\bigwedge^3 \mathbb{L}_X^{\bullet})$. □

Lemma 4.3.7. *The map $\gamma : H^0(L_X) \to H^0(\bigwedge^2 \mathbb{L}_X^{\bullet})$ is an isomorphism.*

Proof. It suffices to show that γ is surjective. Recall our second description of the cotangent sheaf L_X: it is a coherent sheaf that fits into a short exact sequence $0 \to \mathcal{O}_P \to L_X \to \mathcal{I}_P \to 0$, where P is the node, and $\mathcal{O}_P$ is the skyscraper sheaf and $\mathcal{I}_P$ is the ideal sheaf. Notice that $\dim H^0(\mathcal{O}_P) = \dim H^0(L_X) = 1$, so they are isomorphic. Meanwhile, there exists a commutative diagram

$$\begin{array}{ccccc}
H^0(\mathcal{O}_P) & \longrightarrow & H^0(L_X) & \xrightarrow{\gamma} & H^0(\bigwedge^2 \mathbb{L}^\bullet_X) \\
\| & & \downarrow & & \downarrow \wr \\
H^0(\mathcal{O}_P) & \longrightarrow & H^0(L_X|_U) & \longrightarrow & H^0(\bigwedge^2 \mathbb{L}^\bullet_X|_U) \\
\| & & \mid\wr & & \mid\wr \\
H^0(\mathcal{O}_P) & \longrightarrow & H^0(L_Z|_V) & \longrightarrow & H^0(\bigwedge^2 \mathbb{L}^\bullet_Z|_V) \\
\| & & \uparrow & & \uparrow \wr \\
H^0(\mathcal{O}_P) & \longrightarrow & H^0(L_Z) & \longrightarrow & H^0(\bigwedge^2 \mathbb{L}^\bullet_Z)
\end{array}$$

where Z is the affine scheme Spec $\mathbb{C}[x, y]/(xy)$, U and V are the formal neighborhoods of the node P in X and Z respectively. Notice that since $\bigwedge^2 \mathbb{L}^\bullet_X$ is supported at the node P, restricting to the formal neighborhood U will induce an isomorphism on global sections. The only nontrivial isomorphism in the diagram is $H^0(L_X|_U) \simeq H^0(L_Z|_V)$, which can be obtained by $L_X|_U \simeq L_U \simeq L_V \simeq L_Z|_V$, see [Pér16, Proposition 3.9].

To show that the two maps in the first row compose to an isomorphism, it suffices to show the same holds for the last row, which is easy by direct calculation. Similar to L_X, L_Z also admits a resolution

$$0 \to T \xrightarrow{1 \mapsto xdy + ydx} T \cdot dx \oplus T \cdot dy \to L_Z \to 0,$$

where $T = \mathbb{C}[x, y]/(xy)$. Hence the last row can be written as

$$\begin{array}{ccccc}
T/(x, y) & \longrightarrow & \frac{T \cdot dx \oplus T \cdot dy}{xdy + ydx} & \longrightarrow & T/(x, y) \cdot dx \wedge dy \\
1 & \longmapsto & xdy & \longmapsto & 1 \cdot dx \wedge dy
\end{array}$$

This shows $H^0(\mathcal{O}_P) \to H^0(L_Z) \to H^0(\bigwedge^2 \mathbb{L}^\bullet_Z)$ is an isomorphism, and the same holds for X. So $\gamma : H^0(L_X) \to H^0(\bigwedge^2 \mathbb{L}^\bullet_X)$ is surjective, thus an isomorphism. □

Lemma 4.3.8. *The map $\sigma : H^1(\mathcal{O}_X) \to H^1(L_X)$ is 0.*

Proof. The key idea, due to Benjamin Antieau, is to compare the spectral sequence to the spectral sequence associated to the normalization $\tilde{X}$ of X. More explicitly, consider the resolution of singularities $\pi : \tilde{X} \to X$, where $\tilde{X}$ is isomorphic to $\mathbb{P}^1$. Then consider the Hodge to de Rham spectral sequence associated to $\tilde{X}$, where the first page is given by ${}^1E^{p,q}_{\tilde{X}} := H^q(\tilde{X}, \Omega^p_{\tilde{X}})$. The map π induces a morphism $E_X \to E_{\tilde{X}}$ of spectral sequences. In particular, we obtain a commutative diagram

$$\begin{array}{ccc} H^1(X, \mathcal{O}_X) & \xrightarrow{\sigma} & H^1(X, L_X) \\ \downarrow & & \downarrow \\ H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) & \longrightarrow & H^1(\tilde{X}, \Omega^1_{\tilde{X}}) \end{array} \tag{4.1}$$

Since $\tilde{X}$ is isomorphic to $\mathbb{P}^1$, we know $H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) = 0$, so to show σ is 0, it suffices to show that the right vertical map $H^1(X, L_X) \to H^1(\tilde{X}, \Omega^1_{\tilde{X}})$ is an isomorphism. This map factors through

$$H^1(X, L_X) \xrightarrow{\varphi} H^1(\tilde{X}, \pi^* L_X) \xrightarrow{\psi} H^1(\tilde{X}, \Omega^1_{\tilde{X}}),$$

where φ is induced from the unit $L_X \to \pi_* \pi^* L_X$. (These functors are derived functors.)

We know $H^1(X, L_X) \cong \mathbb{C}$, and $H^1(\tilde{X}, \Omega^1_{\tilde{X}}) \cong \mathbb{C}$ as $\tilde{X} \cong \mathbb{P}^1$. We will show that the middle term $H^1(\tilde{X}, \pi^* L_X)$ is also $\mathbb{C}$ by first investigating the derived pullback of cotangent sheaf $\pi^* L_X$.

Notice that $L_X \to \pi_* \pi^* L_X$ is an isomorphism outside of the node, hence the computation of $\pi^* L_X$ is a local computation. So we can take an open neighborhood around the node and use the local model, i.e., a crossing of two lines for X. Then the local model for $\tilde{X}$ is just a disjoint union of two lines, and $T = \mathbb{C}[x, y]/(xy)$ is the coordinate ring for X. Then $L_X = T < dx, dy > /(xdy + ydx)$ and L_X admits a resolution

$$0 \to T \xrightarrow{1 \mapsto \begin{pmatrix} y \\ x \end{pmatrix}} T^2 \xrightarrow{\begin{pmatrix} f \\ g \end{pmatrix} \mapsto f+g} L_X \to 0.$$

The pullback of this exact sequence to $\tilde{X}$ gives

$$0 \to T' \xrightarrow{1 \mapsto \begin{pmatrix} (x,0) \\ (0,y) \end{pmatrix}} T'^2 \to \pi^* L_X \to 0,$$

where $T' = \mathbb{C}[x] \oplus \mathbb{C}[y]$, and the map $\tilde{X} \to X$ corresponds to the map of $\mathbb{C}$-algebras $T \xrightarrow{f \mapsto (f/y, f/x)} T'$. By direct calculation, we see that $\pi^* L_X \cong \mathbb{C}^2 \oplus T'$.

The above local calculation implies that

$$\pi^* L_X \cong \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_p \oplus \mathcal{O}_q,$$

with p, q corresponding to the preimages of the node P. In order to determine the degree of the line bundle $\mathcal{O}_{\mathbb{P}^1}(a)$ we compute the Euler characteristic $\chi(\pi^* L_X)$.

Notice that the resolution of singularities morphism π can be written explicitly as factoring through the twisted cubic $C \subset \mathbb{P}^3$,

$$\begin{array}{ccccc} \mathbb{P}^1 = \tilde{X} & \xrightarrow{\cong} & C & \hookrightarrow & \mathbb{P}^3 \\ & \searrow^{\pi} & \downarrow & & \downarrow \\ & & X & \longrightarrow & \mathbb{P}^2 \end{array} .$$

Then it is easy to see that π has degree 3, hence $\pi^* \mathcal{O}_X(-3) = \mathcal{O}_{\mathbb{P}^1}(-9)$. Now consider the Euler exact sequence on $\mathbb{P}^2$:

$$0 \to \Omega^1_{\mathbb{P}^2} \to \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 3} \to \mathcal{O}_{\mathbb{P}^2} \to 0.$$

Restricting to X and pulling back to $\tilde{X}$ under the map π, we obtain a short exact sequence

$$0 \to \pi^* \Omega^1_{\mathbb{P}^2}|_X \to \mathcal{O}_{\mathbb{P}^1}(-3)^{\oplus 3} \to \mathcal{O}_{\mathbb{P}^1} \to 0.$$

It is still exact since all terms are locally free. So

$$\chi(\pi^*\Omega^1_{\mathbb{P}^2}|_X) = 3\chi(\mathcal{O}_{\mathbb{P}^1}(-3)) - \chi(\mathcal{O}_{P^1}) = -7.$$

Since L_X admits a resolution by $0 \to \mathcal{O}_{\mathbb{P}^2}(-3)|_X \to \Omega^1_{\mathbb{P}^2}|_X \to L_X \to 0$, π^*L_X admits a resolution by $0 \to \mathcal{O}_{\mathbb{P}^1}(-9) \to \pi^*\Omega^1_{\mathbb{P}^2}|_X \to \pi^*L_X \to 0$. Hence

$$\chi(\pi^*L_X) = \chi(\pi^*\Omega^1_{\mathbb{P}^2}|_X) - \chi(\mathcal{O}_{\mathbb{P}^1}(-9)) = 1.$$

So $\chi(\mathcal{O}_{\mathbb{P}^1}(a)) = \chi(\pi^*L_X) - \chi(\mathcal{O}_p \oplus \mathcal{O}_q) = -1$, hence $a = -2$, and we conclude that $\pi^*L_X \cong \mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_p \oplus \mathcal{O}_q$. In particular, this shows that $H^1(\tilde{X}, \pi^*L_X) = \mathbb{C}$.

Lastly, we show that the two maps φ, ψ in

$$H^1(X, \mathbb{L}^\bullet_X) \xrightarrow{\varphi} H^1(X, \pi_*\pi^*\mathbb{L}^\bullet_X) = H^1(\tilde{X}, \pi^*\mathbb{L}^\bullet_X) \xrightarrow{\psi} H^1(\tilde{X}, \Omega^1_{\tilde{X}})$$

are both isomorphisms.

- For the first map φ, notice that using derived projection formula, $\pi_*\pi^*L_X$ can be written as

  $$\pi_*\pi^*L_X \cong \pi_*(\pi^*L_X \otimes_{\mathcal{O}_{\tilde{X}}} \mathcal{O}_{\tilde{X}}) \cong L_X \otimes_{\mathcal{O}_X} \pi_*\mathcal{O}_{\tilde{X}}.$$

  Since $\mathcal{O}_X$ and $\pi_*\mathcal{O}_{\tilde{X}}$ are isomorphic except at the node P, they fit into a short exact sequence

  $$0 \to \mathcal{O}_X \to \pi_*\mathcal{O}_{\tilde{X}} \to \mathcal{O}_P \to 0.$$

  Tensoring with L_X produces a right exact sequence:

  $$L_X \otimes \mathcal{O}_X \to L_X \otimes \pi_*\mathcal{O}_{\tilde{X}} \to L_X \otimes \mathcal{O}_P \to 0.$$

  Denoting the kernel of the first map by $\mathcal{K}$, we obtain a long exact sequence

  $$0 \to \mathcal{K} \to L_X \otimes \mathcal{O}_X \to L_X \otimes \pi_*\mathcal{O}_{\tilde{X}} \to L_X \otimes \mathcal{O}_P \to 0.$$

It splits into two short exact sequences

$$
\begin{array}{ccccccccc}
0 \longrightarrow \mathcal{K} \longrightarrow L_X \otimes \mathcal{O}_X & \longrightarrow & L_X \otimes \pi_* \mathcal{O}_{\tilde{X}} \longrightarrow L_X \otimes \mathcal{O}_P \longrightarrow 0 \\
& \searrow \quad \nearrow & \\
& \mathcal{G} & \\
& \nearrow \quad \searrow & \\
0 & & 0
\end{array}
$$

Since $\mathcal{O}_X$ and $\pi_* \mathcal{O}_{\tilde{X}}$ are isomorphic except at node, we know $\mathcal{K}$ and $L_X \otimes \mathcal{O}_P$ are supported at the node P, so the long exact sequence of cohomology gives $H^1(X, L_X) \cong H^1(X, \mathcal{G})$ and $H^1(X, \mathcal{G}) \twoheadrightarrow H^1(X, L_X \otimes \pi_* \mathcal{O}_{\tilde{X}}) = H^1(X, \pi_* \pi^* L_X)$. Combined with the above computation that $H^1(X, L_X)$ and $H^1(X, \pi_* \pi^* L_X)$ are both isomorphic to $\mathbb{C}$, we know that φ is an isomorphism.

- For the morphism ψ, notice that it is induced from the natural pullback differential map $\pi^* L_X \to \Omega^1_{\tilde{X}}$, i.e., a map of sheaves $\mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_p \oplus \mathcal{O}_q \to \mathcal{O}_{\mathbb{P}^1}(-2)$. Since ψ is a map on H^1, the $\mathcal{O}_p \oplus \mathcal{O}_q$ part is not important here. So this is actually a map $\mathcal{O}_{\mathbb{P}^1}(-2) \to \mathcal{O}_{\mathbb{P}^1}(-2)$, thus to show ψ is an isomorphism, it suffices to show the map $\mathcal{O}_{\mathbb{P}^1}(-2) \to \mathcal{O}_{\mathbb{P}^1}(-2)$ is nonzero, which can be checked along any affine open subset. Since $\pi : \tilde{X} \to X$ is an isomorphism outside of the node, the pullback differential map is an isomorphism outside of the node, which shows that the map $\mathcal{O}_{\mathbb{P}^1}(-2) \to \mathcal{O}_{\mathbb{P}^1}(-2)$ is nonzero as desired.

Hence the right vertical map in diagram 4.1 is an isomorphism, and the map $\sigma : H^1(\mathcal{O}_X) \to H^1(\mathbb{L}^\bullet_X)$ is 0. □

4.3.9. Combining Lemmas 4.3.5, 4.3.6, 4.3.7 and 4.3.8 we obtain a proof of Theorem 4.3.4.

Explicitly the first page looks like

$$\begin{array}{llll}
\boxed{H^1(\mathcal{O}_X)} \xrightarrow{0} \boxed{H^1(\mathbb{L}^\bullet)} & & & \\
\boxed{H^0(\mathcal{O}_X)} \xrightarrow{0} H^0(\mathbb{L}^\bullet) \xrightarrow{\simeq} H^0(\bigwedge^2 \mathbb{L}^\bullet) & & & \\
 & H^{-1}(\bigwedge^2 \mathbb{L}^\bullet) \xrightarrow{\simeq} H^{-1}(\bigwedge^3 \mathbb{L}^\bullet) & & \\
 & & H^{-2}(\bigwedge^3 \mathbb{L}^\bullet) \xrightarrow{\simeq} H^{-2}(\bigwedge^4 \mathbb{L}^\bullet) & \\
 & & & \ddots \qquad \ddots
\end{array} \tag{4.2}$$

where the three boxed terms $H^1(\mathcal{O}_X)$, $H^1(\mathbb{L}^\bullet_X)$ and $H^0(\mathcal{O}_X)$ remain unchanged till page ${}^\infty E$.

4.4 Negative cyclic homology

With Theorem 4.3.4 we are able to compute the negative cyclic homology $\mathrm{HC}^-_*(X)$.

In [Ant19], Antieau proves that there is a decreasing filtration on $\mathrm{HC}^-_*(X)$ with graded pieces given by

$$gr^n \mathrm{HC}^-_{-*}(X) \cong \mathrm{R}^*\Gamma(X, \mathbf{F}^n_H \mathrm{d}\hat{\mathrm{R}}^\bullet[2n]),$$

where $\mathbf{F}^\bullet_H \mathrm{d}\hat{\mathrm{R}}^\bullet$ is the Hodge filtration (stupid filtration) of the derived de Rham complex $\mathrm{d}\hat{\mathrm{R}}^\bullet$. Thus we can compute $\mathrm{HC}^-_*(X)$ in two steps:

1. when $n = 0$, $\mathbf{F}^0_H \mathrm{d}\hat{\mathrm{R}}^\bullet[0] \cong \mathrm{d}\hat{\mathrm{R}}^\bullet$. Hence from [Bha12], we know

   $$gr^0 \mathrm{HC}^-_{-*}(X) = \mathrm{R}^*\Gamma(\mathrm{d}\hat{\mathrm{R}}^\bullet) \cong H^*(X, \mathbb{C}).$$

   This can be generalized to any $n < 0$ with a shift of degree by $2n$.

2. when $n = 1$, we need to compute the hypercohomology of the truncated complex $\mathbf{F}^1_H(\mathrm{d}\hat{\mathrm{R}}^\bullet[2])$. However the Hodge filtration on $\mathbf{F}^1_H(\mathrm{d}\hat{\mathrm{R}}^\bullet)$ also induces a spectral sequence, whose first page is just the first page of Hodge to de Rham spectral sequence 4.2 for X, but only has columns $\geqslant 1$. Moreover, the differentials on this spectral

sequence are also the same as in 4.2. Thus it is easy to see that the only nontrivial cohomology is in degree 2, so

$$gr^1\mathrm{HC}^-_{-*}(X) = \begin{cases} \mathbb{C} & \text{in degree } 0 \\ 0 & \text{in other degrees} \end{cases}.$$

This can be generalized to any $n \geqslant 2$, noticing that every $\mathbf{F}^n_H \hat{\mathrm{dR}}^\bullet$ only has nontrivial cohomology in degree 2, and we only need to take care of the $2n$ shifting of degrees.

To summarize, we have the following chart of dimensions of vector spaces

$*$	-4	-3	-2	-1	0	1	2	3	4	5	6
$gr^{-2}\mathrm{HC}^-_{-*}$									1	1	1
$gr^{-1}\mathrm{HC}^-_{-*}$							1	1	1		
$gr^{0}\mathrm{HC}^-_{-*}$					1	1	1				
$gr^{1}\mathrm{HC}^-_{-*}$					1						
$gr^{2}\mathrm{HC}^-_{-*}$			1								
$gr^{3}\mathrm{HC}^-_{-*}$	1										

Here the first row $*$ is the cohomological degree. After switching to homological degree, we conclude that

Theorem 4.4.1. *The negative cyclic homology for projective nodal cubic curve X is given by*

$$\mathrm{HC}^-_n(X) = \begin{cases} \mathbb{C}^2 & \textit{if } n \leqslant 0 \textit{ and even} \\ \mathbb{C} & \textit{if } n \leqslant 0 \textit{ and odd} \\ \mathbb{C} & \textit{if } n > 0 \textit{ and even} \\ 0 & \textit{otherwise.} \end{cases}$$

4.5 Liftable classes

In this section we will study the natural map $\mathrm{HC}^-_*(X) \to \mathrm{HH}_*(X)$. Before this, let's introduce the *Hochschild to cyclic spectral sequence.* Recall for any variety Y there exists a spectral sequence whose first page is given by

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
\cdots & & 0 & \longrightarrow & \mathrm{HH}_{-1} & \xrightarrow{uB} & u\mathrm{HH}_0 & \xrightarrow{uB} & \cdots \\
\cdots & & 0 & \longrightarrow & \mathrm{HH}_{0} & \xrightarrow{uB} & u\mathrm{HH}_1 & \xrightarrow{uB} & \cdots \\
\cdots & & 0 & \longrightarrow & \mathrm{HH}_{1} & \xrightarrow{uB} & u\mathrm{HH}_2 & \xrightarrow{uB} & \cdots \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

where u is a formal variable of homological degree -2, and B is Connes' operator. It converges to the negative cyclic homology $\mathrm{HC}_*^-(Y)$ at ${}^\infty E$.

Theorem 4.5.1. *The Hochschild to cyclic spectral sequence for the nodal cubic curve X degenerates at page ${}^2E$.*

Proof. The proof is straightforward once we apply the Hochschild-Kostant-Rosenberg (HKR) isomorphism to the terms in the first page.

Recall we have the HKR isomorphism [BF08]

$$\mathrm{HH}_k(X) \simeq \prod_{q-p=k} H^p(X, \bigwedge^q \mathbb{L}_X^\bullet).$$

Moreover the map $uB : \prod_{q-p=k} H^p(X, \bigwedge^q \mathbb{L}_X^\bullet) \to \prod_{q-p=k} H^p(X, \bigwedge^{q+1} \mathbb{L}_X^\bullet)$ is induced from the de Rham differential $d : \bigwedge^q \mathbb{L}_X^\bullet \to \bigwedge^{q+1} \mathbb{L}_X^\bullet$, i.e, the map is a direct product of maps

$H^p(X, \bigwedge^q \mathbb{L}^\bullet_X) \to H^p(X, \bigwedge^{q+1} \mathbb{L}^\bullet_X)$. Thus we can rewrite the first page as

$$\begin{array}{cccccccccc}
 & \vdots & & \vdots & & \vdots & & & & \\
\cdots & 0 & \longrightarrow & H^1(\mathcal{O}) & \xrightarrow{0} & u\begin{pmatrix} H^0(\mathcal{O}) \\ \oplus \\ H^1(L) \end{pmatrix} & \xrightarrow{0} & u^2 H^0(L) & \xrightarrow{\simeq} & \cdots \\
\cdots & 0 & \longrightarrow & \begin{matrix} H^0(\mathcal{O}) \\ \oplus \\ H^1(L) \end{matrix} & \xrightarrow{0} & uH^0(L) & \xrightarrow{\simeq} & u^2 H^0(\bigwedge^2 \mathbb{L}^\bullet_X) & \xrightarrow{0} & \cdots \\
\cdots & 0 & \longrightarrow & H^0(L) & \xrightarrow{\simeq} & uH^0(\bigwedge^2 \mathbb{L}^\bullet) & \xrightarrow{0} & u^2 H^{-1}(\bigwedge^2 \mathbb{L}^\bullet) & \xrightarrow{\simeq} & \cdots \\
\cdots & 0 & \longrightarrow & H^0(\bigwedge^2 \mathbb{L}^\bullet) & \xrightarrow{0} & uH^{-1}(\bigwedge^2 \mathbb{L}^\bullet) & \xrightarrow{\simeq} & u^2 H^{-1}(\bigwedge^3 \mathbb{L}^\bullet) & \xrightarrow{0} & \cdots \\
 & \vdots & & \vdots & & \vdots & & & &
\end{array}$$

where the information of differentials in this page comes from our study of HdR spectral sequence. Thus the ${}^2E$ page looks like

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & & \\
\cdots & 0 & H^1(\mathcal{O}) \arrow[rrd] & u\begin{pmatrix} H^0(\mathcal{O}) \\ \oplus \\ H^1(L) \end{pmatrix} & 0 & \cdots \\
\cdots & 0 & \begin{matrix} H^0(\mathcal{O}) \\ \oplus \\ H^1(L) \end{matrix} \arrow[rrd] & 0 & 0 & \cdots \\
\cdots & 0 & 0 & 0 & 0 & \cdots \\
\cdots & 0 & H^0(\bigwedge^2 \mathbb{L}^\bullet) & 0 & 0 & \cdots \\
 & \vdots & \vdots & \vdots & &
\end{tikzcd}$$

It is easy to see that this spectral sequence already degenerates. □

Corollary 4.5.2. *The natural map* $\mathrm{HC}^-_n(X) \to \mathrm{HH}_n(X)$ *is*

- *an isomorphism, if* $n = -1$ *or* $n \geqslant 0$ *and even.*

- 0, *otherwise.*

Proof. This follows directly from the above spectral sequence, since in pages ${}^2E = {}^\infty E$, along the n-th diagonal we should get a filtration of $\mathrm{HC}^-_n(X)$. But this filtration is either 0, or it only contains one term, that is $\mathrm{HC}^-_n(X) \cong \mathrm{HH}_n(X)$. □

Classifying all the liftable Hochschild classes is important for computations of categorical enumerative invariants [CT24]. Roughly speaking, CEI are invariants associated to an $\mathcal{A}_\infty$-algebra A and extra data (usually given as a splitting of the Hodge filtration). Given Hochschild classes of $\mathrm{HH}_*(A)$ as input, a CEI computation outputs complex numbers. Originally, such computations were only defined for smooth and proper $\mathcal{A}_\infty$-algebras, so we can't use that formalism to compute CEI of the nodal cubic curve. However, Căldăraru and Tu conjecture that for a nonsmooth but proper $\mathcal{A}_\infty$-algebra A, one should be able to perform such computations as well, provided that the inserted classes are all liftable to $\mathrm{HC}^-_*(A)$.

In particular, our Corollary 4.5.2 implies that we should be able to compute CEI of the nodal cubic curve, with insertion classes in $\mathrm{HH}_{-1}(X)$. Combining with our observation that CEIs satisfy holomorphic anomaly equation, we should be able to reduce computation of genus $\leqslant 5$ CEIs for any elliptic curves to genus $\leqslant 5$ CEIs for the special nodal cubic curve, which are more approachable from a numerical computation point of view.

Remark 4.5.3. We have tried to apply the same method to study what happens for the degenerate quintic $x_0 x_1 \cdots x_4 = 0$ in $\mathbb{P}^4$, which is also interesting for computation of CEI. However its Hodge to de Rham spectral sequence does not degenerate at ${}^2E$.

4.6 Appendix: cuspidal curve

Our study of the nodal cubic curve has a strong motivation from enumerative geometry, but we can apply the same ideas to study the projective cuspidal curve C. The proof will be easier than the nodal curve case. We just outline some of the results we get, and sketch the proofs.

Theorem 4.6.1. *For the cuspidal curve C,*

1. *its Hochschild homology is given by*

$$\mathrm{HH}_n(C) = \begin{cases} \mathbb{C} & n = -1 \\ \mathbb{C}^3 & n = 0 \\ \mathbb{C}^2 & n > 0 \end{cases}$$

2. *its Hodge to de Rham spectral sequence degenerates at page ${}^2E$;*

3. *its negative cyclic homology is given by*

$$\mathrm{HC}^-_n(C) = \begin{cases} \mathbb{C}^3 & n = 0 \\ \mathbb{C}^2 & n \neq 0 \text{ and even} \\ 0 & \text{otherwise} \end{cases}$$

4. *The natural map $\mathrm{HC}^-_n(C) \to \mathrm{HH}_n(C)$ is*
   - *an isomorphism, if $n \geqslant 0$ and even,*
   - *0, otherwise.*

Proof. As before the cotangent sheaf L_C also admits a resolution

$$0 \to \mathcal{O}_C(-3) \to \Omega^1_{\mathbb{P}^2}|_C \to L_C \to 0.$$

We can still compute the derived exterior powers of $\mathbb{L}^\bullet_C$, for example

$$\bigwedge^2 \mathbb{L}^\bullet_C = \mathcal{O}_C(-3) \otimes \big(0 \to \mathcal{O}_C(-3) \to \Omega^1_{\mathbb{P}^2}|_C \to \mathcal{O}_C \to 0\big)\,[0],$$

and this will be a local calculation since $\bigwedge^2 \mathbb{L}^\bullet_C$ supports at the singular point. Using the affine local model, we can compute the cohomology of the above chain complex $H^0(\bigwedge^2 \mathbb{L}_C^\bullet) = H^{-1}(\bigwedge^2 \mathbb{L}_C^\bullet) = \mathbb{C}^2$. Then the remaining computations are similar to the nodal curve case. $\square$

Chapter 5

Open questions and further direction

In this chapter we list some open questions suggested by our work.

1. This is already stated as Conjecture 1.6.4 in Chapter 1: we would like to prove that the CEI invariants of nodal cubic have finite limit at cusp (hence they are quasi-modular forms).

2. Currently, the CEI invariants are only defined for smooth and proper $\mathcal{A}_\infty$-algebras. However, Gromov-Witten invariants are well-defined for non-smooth varieties. If we believe homological mirror symmetry, then there should exists a version of CEI invariants that is indeed defined for non-smooth $\mathcal{A}_\infty$-algebras. So we would like to extend the current definition to non-smooth but proper $\mathcal{A}_\infty$-algebras.

3. In some sense, a non-smooth but proper $\mathcal{A}_\infty$-algebra and a non-proper but smooth $\mathcal{A}_\infty$-algebra are dual to each other. We would like to know if there is any "dual" version of CEI invariants that is well-defined for non-proper but smooth $\mathcal{A}_\infty$-algebras.

4. There are other special points other than the cusp on moduli space of elliptic curves. In particular, there is the hexagonal point, corresponding to $\tau = \exp(2\pi i/3)$. Around

it, one can define the so called FJRW invariants [FJR11; FJR13] of the Fermat cubic $W = x^3 + y^3 + z^3$.

Meanwhile, the natural A-model category associated to a point near the hexagonal point is the wrapped Fukaya category. We conjecture that using this category as input of CEI construction, one should recover FJRW invariants. On the other hand, on the B-side, one could use the matrix factorization category $\mathrm{MF}(X, W)$ as input. It is derived equivalent to the wrapped Fukaya category by homological mirror symmetry conjecture. Hence if we believe homological mirror symmetry, using matrix factorization category as input of CEI will allow us to construct the so called *higher genus B-model FJRW invariants.*

5. We would like to know how the Hodge to de Rham spectral sequence degenerates at the large complex structure limit point in some other moduli spaces, such as the degenerate quintic threefold in the moduli of quintics. In particular, we would like to classify all the liftable Hochschild classes for the mirror quintic.

6. Comparing the nodal cubic curve X and the cuspidal curve C, they have different singularity types, but their Hodge to de Rham spectral sequences degenerate at the same page. This suggests that the degeneration of HdR spectral sequence can not distinguish different singularity types. However, the liftable classes in $\mathrm{HH}_*(X)$ and $\mathrm{HH}_*(C)$ are different. In particular, the class of $\mathrm{HH}_{-1}(X)$ is liftable while the class of $\mathrm{HH}_{-1}(C)$ is not. We would like to know if this observation can be generalized, namely if we could distinguish different singularity type by studying their liftable Hochschild classes.

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