ALEXANDER-TYPE INVARIANTS OF HYPERSURFACE COMPLEMENTS

By

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Abstract

Alexander-type invariants of a space X are invariants of certain covering spaces of X, along with the action of the deck group. The spaces we study in this thesis are complex affine hypersurface complements $\mathbb{C}^n \backslash H$, which carry important information about the topology and the singularities of the hypersurface H itself. The Alexander-type invariants that we study are twisted Alexander polynomials and higher order degrees.

In our study of twisted Alexander polynomials, we restrict ourselves to the case where H is a complex hyperplane arrangement. We study the torsion properties of the twisted Alexander modules of the affine complement M of complex essential hyperplane arrangements, as well as those of punctured stratified tubular neighborhoods of complex essential hyperplane arrangements. We investigate divisibility properties between the twisted Alexander polynomials of the two spaces, compute the (first) twisted Alexander polynomial of a punctured stratified tubular neighborhood of an essential line arrangement, and study the possible roots of the twisted Alexander polynomials of both the complement and the punctured stratified tubular neighborhood of an essential hyperplane arrangement in higher dimensions. We compute the twisted Alexander polynomials of the boundary manifold of a pair of line arrangements to show how they can distinguish non-homeomorphic homotopy equivalent arrangement complements. We relate the twisted Alexander polynomials of M with the corresponding twisted homology jump loci.

In our study of higher order degrees, we restrict ourselves to the case of plane curves. In particular, we study finiteness (and vanishing) properties of the higher order degrees associated to complements of complex affine plane curves with mild singularities at infinity. Our results impose new obstructions on the class of groups that can be realized as fundamental groups of affine plane curve complements. We also clarify the relationship between the higher order degrees and the multivariable Alexander polynomial of a non-irreducible plane curve.

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Chapter 1

Introduction

During my time as a graduate student, I have studied Alexander-type invariants of affine hypersurface complements in two of my papers, namely [12] and [13], the latter being joint work with my advisor, Laurentiu Maxim. This thesis is based on the results of both of those papers: Chapter 2 is based on [12], and Chapter 3 is based on [13]. In this introduction, I will try to give a unified view of the invariants studied in each of these two chapters, as well as build up geometric intuition behind the technical definitions that will appear later on.

1.1 What are Alexander-type invariants?

Alexander-type invariants are invariants of the homology of certain covering spaces of a space, which in this thesis will be a complex affine hypersurface complement. The general idea behind wanting to study Alexander-type invariants of affine hypersurface complements is to get information about the topology of an affine hypersurface H in $\mathbb{C}^n$, given by the zeroes of a polynomial function $f : \mathbb{C}^n \longrightarrow \mathbb{C}$, by studying the topology of its complement $U := \mathbb{C}^n \setminus H$. This situation is similar to that of classical knot theory, where the topology of the complement of a knot in S^3 is useful to study the knot itself. This analogy is based on the fact that in both cases, the space studied has real codimension 2 in the ambient space.

The approach of studying hypersurface complements goes back to the work of Zariski ([38]), who observed that the position of the singularities of a plane curve influenced the topology of the curve, and that the fundamental group of the complement of the curve detected this phenomenon. However, fundamental groups of hypersurface complements are, in general, highly non-abelian and are not manageable objects to work with. Alexander-type invariants, are easier to handle, and they are also sensitive to the type and position of singularities. These invariants appeared first in classical knot theory, which is central to the study of the topology of curve complements. Many of these Alexander-type invariants, like the 0-th and first (twisted) Alexander polynomials discussed in Chapter 2, or the higher order degrees discussed in Chapter 3, can be computed from a *presentation* of the fundamental group of the complement, and represent a major advantage to having to deal with the fundamental group itself. By the work of Zariski and van Kampen ([26], [4], [35]), we have an algorithm to compute a presentation of $\pi_1(U)$.

1.1.1 Classical Alexander invariants

The classical (univariable) Alexander invariants are associated to an infinite cyclic cover of U, which is described as follows. The fundamental group of a hypersurface complement U is known to be generated by a choice of positively oriented (with the orientation induced by the complex structure) loops around irreducible components of H, and two positively oriented meridian loops around the same components are known to be conjugate elements in the fundamental group. Therefore, one can define the linking number homomorphism ψ as

$$\begin{aligned}\psi : \pi_1(U) &\longrightarrow \mathbb{Z} \\ \gamma &\longmapsto 1\end{aligned} \qquad \text{for every } \gamma \text{ a positively oriented meridian.}$$

This definition mimics the one in knot theory, where the linking number homomorphism from the fundamental group of the complement of an oriented knot in S^3 to $\mathbb{Z}$ takes every positively oriented meridian to 1. Unlike in the complex affine setting, this positive orientation is not canonical for knot complements, and depends on the fixed orientation of the knot.

The kernel of this epimorphism ψ defines an infinite cyclic cover of U, that is, a connected cover of U whose deck transformations group is $\mathbb{Z}$. We denote the infinite cyclic cover constructed using the linking number homomorphism ψ by $U^\infty \longrightarrow U$. *The classical (univariable) Alexander invariants are invariants of U^∞ along with the action of the deck group.*

The most well-known example of classical (univariable) Alexander invariants are perhaps the (univariable) Alexander polynomial(s), and the reader can find the precise definition in Definition 2.9 and Remark 2.10 (over $\mathbb{Q}$). Roughly, the idea behind the definition is that, if we fix a ring R (which is usually taken to be $\mathbb{Z}$ or $\mathbb{Q}$), $H_*(U^\infty; R)$ has a natural $R[t^{\pm 1}]$-module structure, where t acts as a generator of the deck group. *The (univariable) Alexander polynomials (over R) are elements of $R[t^{\pm 1}]$ that capture some information about the natural $R[t^{\pm 1}]$-module structure of the homology of the infinite cyclic cover induced by the linking number homomorphism.*

If instead of a knot we have a link with r components, it makes sense to fix an orientation of each of the components of the link and look at the epimorphism from the fundamental group of the link complement in S^3 to $\mathbb{Z}^r$ (instead of to $\mathbb{Z}$), where a positively oriented meridian around the i-th component of the link goes to the i-th canonical basis element of $\mathbb{Z}^r$. If we translate this idea to the world of affine hypersurface complements, we get the following epimorphism

$$\begin{aligned} \mathrm{ab} : \pi_1(U) &\longrightarrow \mathbb{Z}^r \\ \gamma_i &\longmapsto e_i \text{ for all } i = 1, \ldots, r \end{aligned}$$

where r is the number of irreducible components of the hypersurface H, γ_i is a positively oriented meridian around the i-th component of H, and e_i is the i-th canonical basis element of $\mathbb{Z}^r$. Note that ab is indeed the abelianization homomorphism, since $H_1(U, \mathbb{Z}) \cong \mathbb{Z}^r$, where the identification is done in the way that the map ab describes.

The kernel of this epimorphism ab defines the universal abelian cover of U, a connected cover of U whose deck transformations group is $\mathbb{Z}^r$. We denote this cover by $U^{\mathrm{ab}} \longrightarrow U$. *The classical (multivariable) Alexander invariants are invariants of* U^{ab} *along with the action of the deck group.*

The most well-known example of classical (multivariable) Alexander invariants are perhaps the (multivariable) Alexander polynomial(s) on r variables, where r is the number of irreducible components of H. The reader can find the precise definition of the (first) multivariable Alexander polynomial (over $\mathbb{Z}$) in Definition 3.36. Roughly, the idea behind the definition is that, if we fix a ring R, $H_*(U^{\mathrm{ab}}; R)$ has a natural $R[t_1^{\pm 1}, \ldots, t_r^{\pm 1}]$-module structure, where t_i acts as the i-th basis element of the deck group $\mathbb{Z}^r$ for all $i = 1, \ldots, r$. *The (multivariable) Alexander polynomials (over* R*) are elements of* $R[t_1^{\pm 1}, \ldots, t_r^{\pm 1}]$ *that capture some information about the natural* $R[t_1^{\pm 1}, \ldots, t_r^{\pm 1}]$*-module structure of the homology of the universal abelian cover.*

Note that we have a tower of coverings

$$U^{\mathrm{ab}} \longrightarrow U^{\infty} \longrightarrow U. \tag{1.1}$$

Generally speaking, Alexander-type invariants associated to U^{ab} are more complicated to compute than those associated to U^{∞}. For example, if we work with field coefficients $\mathbb{F}$, $\mathbb{F}[t^{\pm 1}]$ is a PID, but the multivariable Laurent polynomial ring $\mathbb{F}[t_1^{\pm 1}, \ldots, t_r^{\pm 1}]$ is not if $r \geq 2$. Hence, the $\mathbb{F}[t^{\pm 1}]$-module structure of the finitely generated $\mathbb{F}[t^{\pm 1}]$-module $H_*(U^{\infty}, \mathbb{F})$ is in principle easier to describe than the $\mathbb{F}[t_1^{\pm 1}, \ldots, t_r^{\pm 1}]$-module structure of the finitely generated $\mathbb{F}[t_1^{\pm 1}, \ldots, t_r^{\pm 1}]$-module $H_*(U^{\mathrm{ab}}, \mathbb{F})$. However, the fact that the image of ab distinguishes meridians about different irreducible components of H make Alexander-type invariants associated to U^{ab} stronger than those associated to U^{∞} in some sense.

1.1.2 Possible generalizations

In this section, we discuss how we generalize classical Alexander invariants in this thesis. We do so in two ways: In Chapter 2 we achieve this by “twisting” the coefficients, and in Chapter 3 by looking at covers above U^{ab} in 1.1.

Twisted (univariable) Alexander polynomials

We will describe how to generalize the classical (univariable) Alexander invariants to get twisted invariants.

Suppose that the hypersurface H is defined as the zeroes of a reduced polynomial $f = f_1 \cdot \ldots \cdot f_r$, where $f_i \neq f_j$ for all $i \neq j$, and all of them are monic and irreducible. Then, f induces a map $f : U \longrightarrow \mathbb{C}^*$, which, in fundamental groups, induces

$$f_* : \pi_1(U) \longrightarrow \mathbb{Z}.$$

f_* is in fact the linking number homomorphism ψ. Hence, ψ, which we described geometrically in the previous section, also has an algebraic description as the map induced on fundamental groups by a polynomial map.

However, H is also the zeroes of $f_\varepsilon = (f_1)^{\varepsilon_1} \cdot \ldots \cdot (f_r)^{\varepsilon_r}$, for any $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_r) \in (\mathbb{Z}_{>0})^r$, but the map that f_ε induces in fundamental groups is different from ψ, and induces a different infinite cyclic cover from U^∞. Namely, this map is

$$\begin{array}{rcl} (f_\varepsilon)_* : \pi_1(U) & \longrightarrow & \mathbb{Z} \\ \gamma_i & \longmapsto & \varepsilon_i \end{array},$$

where γ_i is a positively oriented meridian around the component of H defined by $f_i = 0$. We denote the associated infinite cyclic cover by U^ε, and we know that it is connected if and only if $(f_\varepsilon)_*$ is an epimorphism, or equivalently, if and only if $\gcd(\varepsilon_1, \ldots, \varepsilon_r) = 1$.

Considering these other infinite cyclic covers apart from U^∞ is useful if we want to use our invariants to distinguish hypersurface complements for the following reason. Let H_1 and H_2 be two hypersurfaces in $\mathbb{C}^n$. Suppose that we have a map $g : \mathbb{C}^n \backslash H_1 \longrightarrow \mathbb{C}^n \backslash H_2$, and ψ_i is the linking number homomorphism associated to $\mathbb{C}^n \backslash H_i$, for $i = 1, 2$. The pullback of ψ_2 by g is not necessarily ψ_1. Hence, computing Alexander invariants associated to ψ_1 and ψ_2 will not necessarily yield invariants that we can compare by the map g, thus the need to consider different infinite cyclic covers.

Apart from considering alternate infinite cyclic covers, we can further generalize classical (univariable) Alexander invariants by introducing a "twist" in the coefficients, coming from a representation $\rho : \pi_1(U) \longrightarrow \mathrm{GL}(\mathbb{V})$, where $\mathbb{V}$ is a finite dimensional vector space over some base field $\mathbb{F}$. In this way, we can think of the (univariable) twisted Alexander polynomials, the main object of study in Chapter 2, as invariants of the (co)homology of U^ε with local coefficients given by the induced representation of

$\pi_1(U^\varepsilon)$ by ρ, for some ε and some ρ like the ones described. Note that, in the classical case, these local coefficients were in fact constant. Introducing this twist by ρ is also useful to distinguish hypersurface complements. In fact, this point, along with the point made in the previous paragraph, is exemplified in Example 2.40, an application of the results in Chapter 2. In this example, the introduction of generic twists and flexibility with our infinite cyclic covers allows us to distinguish two homotopy equivalent but not homeomorphic line arrangement complements, a problem for which the classical univariable Alexander polynomials are not strong enough.

Higher order degrees

Let $G := \pi_1(U)$ the fundamental group of a hypersurface complement. The higher order degrees that we are about to describe are invariants of the group G, along with the linking number homomorphism ψ. By a Lefschetz-type argument, we know that intersecting U with succesive generic hyperplanes gives us an affine curve complement in $\mathbb{C}^2$ with the same fundamental group, so we will assume that $U = \mathbb{C}^2 \backslash C$ is the complement of an affine curve C in this section.

The higher order degrees are invariants associated to covers that lie above U^{ab} in the tower of coverings 1.1. From G, we can inductively construct the rational derived series of the group

$$G =: G_r^{(0)} \trianglerighteq G_r^{(1)} \trianglerighteq G_r^{(2)} \trianglerighteq \dots,$$

where $G_r^{(1)} = [G, G]$, and the quotient of two succesive elements in this series is torsion free abelian. The precise definition of the rational derived series is given in Definition 3.6. From the rational derived series, we construct $\Gamma_n := G/G_r^{(n+1)}$ for all $n \geq 0$, and the linking number homomorphism factors through Γ_n as follows:

$$\begin{tikzcd}
G \arrow[rr, two heads, "\psi"] \arrow[dr, two heads, "p_n"'] & & \mathbb{Z} \\
& \Gamma_n \arrow[ur, two heads, "\overline{\psi}"'] &
\end{tikzcd}$$

where p_n is the natural projection. Let $U_{\Gamma_n} \longrightarrow U$ be the covering space associated to the kernel of p_n. Note that Γ_0 coincides with the abelianization of G, so $p_0 = \text{ab}$. Also note that Γ_n is a quotient of Γ_{n+1} for all $n \geq 0$. Hence, we have the following tower of covering spaces:

Tower of coverings	Deck groups	$\xrightarrow{\text{acting on}}$	First Homology
$\vdots$	$\vdots$		$\vdots$
$\downarrow$			
U_{Γ_n}	Γ_n		$H_1(U_{\Gamma_n}, \mathbb{Z})$
$\downarrow$ $\vdots$ $\downarrow$	$\vdots$		$\vdots$
U_{Γ_1}	Γ_1		$H_1(U_{\Gamma_1}, \mathbb{Z})$
$\downarrow$			
$U_{\Gamma_0} = U^{\text{ab}}$	$\Gamma_0 \cong \mathbb{Z}^r$		$H_1(U^{\text{ab}}, \mathbb{Z})$
$\downarrow$			
U^{∞}	$\mathbb{Z}$		$H_1(U^{\infty}, \mathbb{Z})$
$\downarrow$			
U			

For every $n \geq 0$, $H_1(U_{\Gamma_n}; \mathbb{Z})$ has a natural right $\mathbb{Z}[\Gamma_n]$-module structure by deck transformations. We call $\overline{\Gamma_n}$ the kernel of $\overline{\psi}$, and construct the right ring of quotients $\mathbb{K}_n := \mathbb{Z}[\overline{\Gamma_n}] \left(\mathbb{Z}[\overline{\Gamma_n}] \backslash \{0\}\right)^{-1}$, which is a skew field. With this notation, we can define the

n-th order degree of the curve C as

$$\delta_n(C) := \mathrm{rk}_{\mathbb{K}_n} H_1(U_{\Gamma_n};\mathbb{Z}) \otimes_{\mathbb{Z}[\overline{\Gamma_n}]} \mathbb{K}_n \in \mathbb{Z} \cup \{\infty\}.$$

Let us try to give an intuitive explanation of what these higher order degrees mean. They can be thought of as the degrees of a generalized version of the Alexander polynomial for higher coverings. In fact, as shown in Theorem 3.45, the definition of $\delta_0(C)$ coincides with the degree of the classical (multivariable) Alexander polynomial associated to U, provided that $\delta_0(C)$ is finite. By Eq. (3.2), these higher order degrees can be also understood as a measure of the "sizes" of consecutive terms in the rational derived series of a group.

The discussion we had on the last paragraph of section Section 1.1.1 applies here. The higher order degrees, as invariants of higher covers, have the potential of detecting information about the topology of an affine curve complement that is invisible to the classical Alexander invariants. This is in fact the case for knots, where δ_0 is the degree of the classical Alexander polynomial of the knot: higher order degrees provide lower bounds for the genus of a knot [3], and δ_1 has been shown to give a better bound than δ_0 for some knots [19]. However, by passing to these higher covers, computations of examples become much more complicated, since the rings involved in the higher order world are non-commutative.

1.2 Outline of the thesis

1.2.1 Chapter 2

In Chapter 2, which is based on [12], we study twisted Alexander polynomials. In this introduction, we have talked about twisted Alexander polynomials of affine hypersurface complements, but in Chapter 2 we restrict ourselves to the case when the hypersurface is a hyperplane arrangement. This improves on the previous work in the literature regarding twisted Alexander polynomials of curve (Cogolludo and Florens, [5]) or hypersurface (Maxim and Wong, [30]) complements, by relaxing the assumption that the curve (resp. hypersurface) be transversal to the line (resp. hyperplane) at infinity, that both Cogolludo-Florens and Maxim-Wong imposed. Hyperplane arrangements are not transversal at infinity in general, and the novelty of our approach comes from using a fibration structure over a circle of a certain punctured neighborhood V of the arrangement to overcome the transversality at infinity assumption that was previously used. This fibration structure was discovered by Kohno and Pajitnov using analytic arguments in [22].

We use Kohno and Pajitnov's work to relate the twisted Alexander polynomials of a hyperplane arrangement complement to those of a different punctured neighborhood of the arrangement, W^*, which, unlike V, is well-suited for computations. One of the applications we give of our results in [12] uses (univariable) twisted Alexander polynomials to distinguish two non-homeomorphic homotopy equivalent line arrangement complements. We also relate the twisted Alexander polynomials of a hyperplane arrangement complement with the corresponding twisted homology jump loci.

For a more detailed historical perspective of twisted Alexander polynomials to study affine hypersurfaces, the reader is referred to the introduction of Chapter 2.

1.2.2 Chapter 3

In Chapter 3, which is based on [13] (joint work with Laurentiu Maxim), we talk about higher order degrees. In particular, we study finiteness (and vanishing) properties of the higher order degrees associated to complements of complex affine plane curves with mild singularities at infinity. We relax in different ways the transversality at infinity assumption of the previous work in the literature regarding higher order degrees of affine curve complements [28]. In relation to an old question of Serre, our results impose new obstructions on the class of groups that can be realized as fundamental groups of affine plane curve complements. Amongst other things, we are able to generalize Leidy and Maxim's uniform upper bound for the higher order degrees of a curve [28] to the case of line arrangements (which are not necesarily transversal to the line at infinity), and to the case when the plane curve is allowed to have simple tangents at infinity. We prove that if the curve C contains a generic smooth curve C'', then all of its higher order degrees vanish. We also clarify the relationship between the higher order degrees and the multivariable Alexander polynomial of a non-irreducible plane curve, and use it to compute the 0-th order degree of every line arrangement.

Leidy and Maxim obtained their bound by computing the induced higher order degrees of the link complement at infinity, and noticing that this space "dominates" the curve complement in some sense. When the curve is transversal at infinity, the link at infinity is just a Hopf link with as many components as the degree of the curve, making computations feasible. In our generalization to line arrangements, we find spaces other than the link complement at infinity that have that same “dominating” behavior to arrive at the same uniform bound by a completely different strategy. In our generalization to curves with mild singularities at infinity, we also study the link complement at infinity. But, unlike in Leidy-Maxim’s case, this link is not well-understood in general, and we find a way to break it up into easier pieces to be able to carry out the computations.

For a more detailed historical perspective of higher order degrees to study plane curves, as well as for a summary of the precise results proved in Chapter 3, the reader is referred to the introduction of that chapter.

1.2.3 Appendix A

Hyperplane arrangements are the object of study of Chapter 2, and line arrangements are discussed separately in Chapter 3 as well. To compute examples of the results outlined in this thesis that have to do with line or hyperplane arrangements, one has to know a presentation of the fundamental group of line arrangement complements. With this in mind, we include Appendix A at the end, where we recall Arvola’s presentation of the fundamental group on a line arrangement complement [1], making the notation match the notation used in this thesis.

Chapter 2

Twisted Alexander Modules of Hyperplane Arrangement Complements

Based on [12].

2.1 Introduction

The twisted Alexander polynomial was first used to study plane algebraic curves by Cogolludo and Florens in [5]. In their paper, they refine Libgober's divisibility results regarding the classical Alexander polynomial ([27]), and use the twisted Alexander polynomials to distinguish Zariski pairs (pairs of plane curves with homeomorphic tubular neighborhoods but non-homeomorphic complements) that the classical Alexander polynomial cannot distinguish.

Cohen and Suciu study the multivariable twisted Alexander polynomials of the boundary manifold of a line arrangement in [6], and use the non-twisted version to obtain a complete description of the first characteristic variety of the fundamental group of the boundary manifold. Hironaka [18] and Florens-Guerville-Marco [15] have studied relationships between the topology of a line arrangement complement and that of the boundary manifold of such an arrangement.

In [30], Maxim and Wong investigated torsion properties for the twisted Alexander modules of the affine complements of complex hypersurfaces in general position at infinity. They did so by using the link (complement) at infinity, which fibers over a circle, and the hypersurface complement can be obtained from it by adding cells of dimension greater or equal than the middle dimension. They were also able to describe a polynomial such that the roots of the (one-variable) twisted Alexander polynomials of the hypersurface complements were roots of it. This polynomial came from studying the twisted Alexander modules of the link at infinity.

Kohno and Pajitnov showed in [22] that complex essential hyperplane arrangements also had a similar structure. Hyperplane arrangements are not necessarily in general position at infinity, but there is a different space that plays a similar role as the one the link at infinity plays in the case of hypersurfaces in general position at infinity studied by Maxim and Wong. This space is the boundary of a certain neighborhood of the arrangement, it fibers over a circle, and the arrangement complement can be obtained from it by adding cells of the middle dimension.

In this chapter, we follow Maxim and Wong's approach to study the torsion properties for the twisted Alexander modules of complex essential hyperplane arrangement complements, using the structure proved by Kohno and Pajitnov. We investigate divisibility properties between the twisted Alexander polynomials of arrangement complements and those of punctured tubular neighborhoods of arrangements, compute the (first) twisted Alexander polynomial of a punctured stratified tubular neighborhood of an essential line arrangement, and study the possible roots of the twisted Alexander polynomials of both the complement and the punctured stratified tubular neighborhood of an essential hyperplane arrangement in higher dimensions.

In the last section we give two applications of our results. The first application (Example 2.40) uses twisted Alexander polynomials to distinguish two non-homeomorphic homotopy equivalent line arrangement complements. The result proved in Example 2.40 was already known: it was first proved by Jiang and Yau in [20] as a corollary of their powerful result that states that homeomorphic complex projective line arrangement complements have isomorphic intersection posets, and later reproved by Cohen and Suciu in [6] using multivariable Alexander polynomials. The interesting part about our proof given in Example 2.40 is that it does not rely on the heavy machinery of Jiang and Yau and it uses an a priori easier invariant than Cohen and Suciu, since multivariable Alexander polynomials are in principle harder to compute than univariable (twisted) ones due to the fact that they do not live in a PID. The second application relates the zeros of twisted Alexander polynomials to the twisted homology jump loci of rank one $\mathbb{C}$-local systems.

2.1.1 Setup

Let H_j be a complex hyperplane in $\mathbb{C}^n$ given by the zero locus of an affine linear map $\xi_j : \mathbb{C}^n \to \mathbb{C}$, where $j = 1, \dots, m$.

Definition 2.1. *The hyperplane arrangement $\mathcal{A} = \{H_1, \dots, H_m\}$ is called essential if the maximal codimension of a non-empty intersection of a subfamily of $\mathcal{A}$ is n.*

Let $\{H_1, \dots, H_m\}$ be an essential hyperplane arrangement, let

$$H = \bigcup_{j=1}^{m} H_j$$

be the union of the hyperplanes, and let

$$M = \mathbb{C}^n \backslash H$$

be the complement in $\mathbb{C}^n$.

Remark 2.2. *Every hyperplane arrangement complement is homotopy equivalent to the complement of an essential one in an affine space of less or equal dimension, so we do not lose information by restricting ourselves to the study of essential hyperplane arrangements.*

Now, we need to identify and name certain loops in $\pi_1(M)$ that will be used throughout this chapter. For a complete algorithm describing a presentation of $\pi_1(M)$, we refer the reader to Appendix A, where we summarize the work in [1]. It is a well-known fact that $\pi_1(M)$ is generated by a choice of meridians a_j around each hyperplane H_j, for $j = 1, \ldots, m$. These meridians $a_1, \ldots, a_m$ have a canonical (positive) orientation induced by the complex structure.

For the rest of Section 2.1.1, we will deal with the case where $\mathcal{A}$ is a line arrangement (that is, $n = 2$). Let $P_1, \ldots, P_s$ be the singular points of H, and let d_k be the number of lines in $\mathcal{A}$ passing through P_k.

Definition 2.3. *We denote by M_k the local complement*

$$M_k := M \cap \mathbb{B}^4_k$$

where $\mathbb{B}^4_k$ is a small 4-ball in $\mathbb{C}^2$ centered at the point P_k, for $k = 1, \ldots, s$.

Note that M_k is homotopy equivalent to $M \cap S^3_k$, where S^3_k is the boundary of $\overline{\mathbb{B}^4_k}$. In fact, $M \cap S^3_k = S^3_k \backslash L_k$, where L_k is a Hopf link on d_k components. Also M_k is homeomorphic to a central line arrangement complement $U_k \subset \mathbb{C}^2$ consisting on d_k distinct lines passing through the origin.

Definition 2.4. *We denote by* β_k *the loop in* $\pi_1(M_k)$ *corresponding via the homeomorphism described above to a meridian about the line at infinity with* ***negative*** *orientation in* U_k*.*

Remark 2.5. *Two meridians about the same line with the same orientation are not necessarily the same elements in* $\pi_1(M)$*, but they are conjugate to one another. The above definition of* β_k *for all* $k = 1, \dots, s$ *is well defined only up to conjugation in* $\pi_1(M_k)$*, but this will suffice for our purposes. Abusing notation and disregarding base points, we will look at the* β_k*'s inside of* $\pi_1(M)$ *via the maps* $\pi_1(M_k) \longrightarrow \pi_1(M)$ *induced by inclusion.*

Remark 2.6. β_k *can be taken to be the composition of* d_k *loops* $\gamma_1 \cdot \ldots \cdot \gamma_{d_k}$*, where each one of these loops is a certain positively oriented meridian about each of the* d_k *lines in* $\mathcal{A}$ *going through* P_k*. If the reader wishes to know what line of* $\mathcal{A}$ *corresponds to each of the* γ*'s in a given example, they can do so using Arvola's presentation for* $\pi_1(M)$ *(*[1]*), which we recall in Appendix A. For the purposes of this chapter, we just need to know that a presentation for the fundamental group of* M_k *(and* $S^3_k \backslash L_k$*) is given by*

$$\langle \beta_k, \gamma_1, \dots, \gamma_{d_k - 1} \mid [\beta_k, \gamma_l] \ \ \textit{for } l = 1, \dots, d_k - 1 \rangle \tag{2.1}$$

(see [30, Lemma 2.7]*).*

2.1.2 General construction of Alexander modules and polynomials

Let $\mathbb{F}$ be a field, and let $\mathbb{V}$ be a finite dimensional $\mathbb{F}$-vector space. Let X be a path-connected finite CW complex, let $\rho : \pi_1(X) \to \mathrm{GL}(\mathbb{V})$ be a linear representation, and let $\varepsilon : \pi_1(X) \to \mathbb{Z}$ be a group homomorphism. Together, ρ and ε define the homological twisted Alexander modules $H_i^{\varepsilon,\rho}(X, \mathbb{F}[t^{\pm 1}])$, as in [30, Section 2.1].

Definition 2.7. *The i-th (homological) twisted Alexander module* $H_i^{\varepsilon,\rho}(X, \mathbb{F}[t^{\pm 1}])$ *of* (X, ε, ρ) *is the i-th homology of the complex of* $\mathbb{F}[t^{\pm 1}]$*-modules*

$$C_*^{\varepsilon,\rho}(X, \mathbb{F}[t^{\pm 1}]) := (\mathbb{F}[t^{\pm 1}] \otimes_{\mathbb{F}} \mathbb{V}) \otimes_{\mathbb{F}[\pi_1(X)]} C_*(\widetilde{X}, \mathbb{F}).$$

Here, $C_*(\widetilde{X}, \mathbb{F})$ *is the cellular homology complex of the universal cover* $\widetilde{X}$ *of* X*, seen as a free left* $\mathbb{F}[\pi_1(X)]$*-module via the action given by deck transformations. We regard* $\mathbb{F}[t^{\pm 1}] \otimes_{\mathbb{F}} \mathbb{V}$ *as a right* $\mathbb{F}[\pi_1(X)]$*-module, with the right action given by*

$$(p(t) \otimes v) \cdot \alpha = (p(t) \cdot t^{\varepsilon(\alpha)}) \otimes (v \cdot \rho(\alpha))$$

for every $p(t) \in \mathbb{F}[t^{\pm 1}]$*,* $v \in \mathbb{V}$ *and* $\alpha \in \pi_1(X)$*, where* v *is regarded as a row vector and* $\rho(\alpha)$ *as a square matrix.*

Together, ε and ρ define a tensor representation

$$\begin{aligned} \varepsilon \otimes \rho : \pi_1(X) &\longrightarrow \mathrm{Aut}_{\mathbb{F}[t^{\pm 1}]}(\mathbb{F}[t^{\pm 1}] \otimes_{\mathbb{F}} \mathbb{V}) \\ \alpha &\longmapsto p(t) \otimes v \mapsto (p(t) \cdot t^{\varepsilon(\alpha)}) \otimes (v \cdot \rho(\alpha)) \end{aligned}$$

which gives rise to a local system of $\mathbb{F}[t^{\pm 1}]$ modules $\mathcal{L}_{\varepsilon,\rho}$.

Remark 2.8. *There is an* $\mathbb{F}[t^{\pm 1}]$*-module isomorphism* $H_i^{\varepsilon,\rho}(X, \mathbb{F}[t^{\pm 1}]) \cong H_i(X, \mathcal{L}_{\varepsilon,\rho})$ *(see* [30, Section 4.4]*). We will use both the chain complex definition and properties of homology with local systems when it is most convenient.*

Since X is a finite CW-complex and $\mathbb{V}$ is finite dimensional over $\mathbb{F}$, $C_*(\widetilde{X},\mathbb{F})$ is a complex of finitely generated free left $\mathbb{F}[\pi_1(X)]$-modules. Thus, the twisted (homological) Alexander modules are finitely generated $\mathbb{F}[t^{\pm 1}]$-modules over the principal ideal domain $\mathbb{F}[t^{\pm 1}]$, and therefore have a direct sum decomposition into cyclic modules.

Definition 2.9. *The i-th (homological) twisted Alexander polynomial of* (X,ε,ρ) *is defined as the order of the torsion part of the i-th twisted Alexander module* $H_i^{\varepsilon,\rho}(X,\mathbb{F}[t^{\pm 1}])$*. We denote this polynomial by* $\Delta_i^{\varepsilon,\rho}(X)$*, and it is an element in* $\mathbb{F}[t^{\pm 1}]$ *that is well defined up to multiplication by a unit of* $\mathbb{F}[t^{\pm 1}]$*.*

Equivalently, $\Delta_i^{\varepsilon,\rho}(X)$ *can be defined as a generator of the first non-zero Fitting ideal of the* $\mathbb{F}[t^{\pm 1}]$*-module* $H_i^{\varepsilon,\rho}(X,\mathbb{F}[t^{\pm 1}])$*.*

Remark 2.10. *The classical Alexander polynomials (with* $\mathbb{Q}$ *coefficients) correspond to* $\varepsilon = \psi$ *the linking number homomorphism (as defined in Chapter 1),* $\mathbb{V} = \mathbb{Q}$*, and* ρ *the trivial representation.*

Let $\varepsilon : \pi_1(M) \to \mathbb{Z}$ be a fixed group homomorphism. Throughout this chapter, we will assume that ε is an **epimorphism**. As we already pointed out, $\pi_1(M)$ is generated by a choice of positively oriented meridians a_j around each hyperplane H_j. In fact, $H_1(M,\mathbb{Z}) \cong \mathbb{Z}^m$ is the free abelian group generated by the classes of those meridians. Hence, ε is completely determined by the value it takes in those oriented meridians. We will denote by $\varepsilon_j := \varepsilon(a_j)$ for all $j = 1,\ldots,m$. In this chapter, we will study the twisted Alexander modules $H_*^{\varepsilon,\rho}(M,\mathbb{F}[t^{\pm 1}])$, and unless stated otherwise, we require ε to be a **positive** epimorphism, that is, $\varepsilon_j > 0$ for all $1 \leq j \leq m$.

Throughout the chapter, we will use the following notation.

Notation 2.11. *Let* $\gamma \in \pi_1(M)$*. We denote by* $\det_{\varepsilon,\rho}(\gamma)$ *the determinant*

$$\det(t^{\varepsilon(\gamma)}\rho(\gamma) - Id) \in \mathbb{F}[t^{\pm 1}].$$

2.1.3 Reidemeister torsion

In Section 2.4.1, we will be using the **torsion** $\tau(C_*)$ of a finite chain complex C_* of finite dimensional vector spaces over a field $\mathbb{K}$, as defined in [32, Section 3] (but we use multiplicative notation instead of additive notation, unlike in [32]). The torsion $\tau(C_*)$ is an element of $\mathbb{K}^*/\{\pm 1\}$, and depends on a choice of bases for both the chain complex and its homology. In particular, if C_* is acyclic, $\tau(C_*)$ only depends on a choice of bases for C_*. The actual definition of the torsion is not going to be relevant in this document. The torsion behaves well with respect to short exact sequences, as exemplified in the following result.

Lemma 2.12 ([32])**.** *Let*

$$0 \longrightarrow C' \longrightarrow C \longrightarrow C'' \longrightarrow 0$$

be a short exact sequence of based finite chain complexes of finite dimensional vector spaces, with compatible bases. Let $\mathcal{H}$ *be the associated long exact sequence in homology, viewed as a based acyclic complex, the bases being the fixed bases of the homology of* C'*,* C*, and* C''*. Then,*

$$\tau(C) = \tau(C')\tau(C'')\tau(\mathcal{H})$$

where the torsion is taken with respect to the fixed bases.

Let (X, ρ, ε) be as in Definition 2.7. By tensoring $C_*^{\varepsilon,\rho}(X, \mathbb{F}[t^{\pm 1}])$ with the field of rational functions $\mathbb{F}(t)$, we construct a finite chain complex of based finite dimensional vector spaces over $\mathbb{F}(t)$, which we call $C_*^{\varepsilon,\rho}(X, \mathbb{F}(t))$.

Definition 2.13 ([21, Section 3])**.** *We denote by $\tau_{\varepsilon,\rho}(X)$ the **twisted Reidemeister torsion** of (X, ε, ρ), which is defined as*

$$\tau_{\varepsilon,\rho}(X) = \tau(C_*^{\varepsilon,\rho}(X, \mathbb{F}(t))).$$

In this definition we have not specified a choice of bases of $C_*^{\varepsilon,\rho}(X, \mathbb{F}(t))$, but we will only consider bases of the form $b \otimes c_i$, where b is a basis of $\mathbb{V}$ as a vector space over $\mathbb{F}$ and c_i is a "geometric" basis of $C_i(\widetilde{X}, \mathbb{F})$ as a free left $\mathbb{F}[\pi_1(X)]$-module, that is, a basis obtained by lifting i-cells of X for all i. We also have not specified a choice of bases of the homology of $C_*^{\varepsilon,\rho}(X, \mathbb{F}(t))$, but we do not have to, as exemplified by the following result.

Lemma 2.14 ([21, Section 3])**.** *Suppose that $C_*^{\varepsilon,\rho}(X, \mathbb{F}(t))$ is acyclic. Then, $\tau_{\varepsilon,\rho}(X)$ is independent of the choice of bases up to multiplication by a unit of $\mathbb{F}[t^{\pm 1}]$.*

In light of this last result, we will always consider $\tau_{\varepsilon,\rho}(X)$ to be an element of $\mathbb{F}(t)$ up to multiplication by a unit of $\mathbb{F}[t^{\pm 1}]$.

We end this section by stating the relation between the twisted Reidemeister torsion and the twisted Alexander polynomials.

Lemma 2.15 ([21, Theorem 3.4])**.** *Suppose that $C_*^{\varepsilon,\rho}(X, \mathbb{F}(t))$ is acyclic, and let $\tau_{\varepsilon,\rho}(X)$ be the twisted Reidemeister torsion of (X, ε, ρ). Then,*

$$\tau_{\varepsilon,\rho}(X) = \frac{\prod_i \Delta_{2i+1}^{\varepsilon,\rho}(X)}{\prod_i \Delta_{2i}^{\varepsilon,\rho}(X)}$$

up to multiplication by a unit of $\mathbb{F}[t^{\pm 1}]$*.*

2.1.4 Overview of the main results

In this chapter, we study the twisted Alexander modules and the twisted Alexander polynomials of both M and a punctured stratified tubular neighborhood W^* of H, as defined explicitly in Definition 2.18.

In Section 2.2, we start by recalling a result from [22] (Theorem 2.16) involving a space V which is the boundary of a certain neighborhood of the arrangement H which fibers over a circle. This result will be very useful for proving the main result in that section, namely Theorem 2.20, which relates the topologies of M and W^*.

Theorem 2.20. *M has the homotopy type of W^* with cells of dimension $\geq n$ attached.*

In Section 2.3 we study the torsion properties of the twisted Alexander modules of both M and W^*, using the space V and the fibration structure we know by Theorem 2.16 to do so. The main result in this section is Theorem 2.27.

Theorem 2.27. *The twisted Alexander modules $H_i^{\varepsilon,\rho}(M, \mathbb{F}[t^{\pm 1}])$ are torsion $\mathbb{F}[t^{\pm 1}]$-modules for every $0 \leq i \leq n-1$, they are trivial modules for $i > n$, and $H_n^{\varepsilon,\rho}(M, \mathbb{F}[t^{\pm 1}])$ is a free $\mathbb{F}[t^{\pm 1}]$-module of rank $(-1)^n \cdot \dim_{\mathbb{F}}(\mathbb{V}) \cdot \chi(M)$*

In the proof of this last result, and making use of Theorem 2.20, we will also arrive at Corollary 2.29, which gives us a divisibility result.

Corollary 2.29. *$H_i^{\varepsilon,\rho}(M, \mathbb{F}[t^{\pm 1}])$ and $H_i^{\varepsilon,\rho}(W^*, \mathbb{F}[t^{\pm 1}])$ are torsion $\mathbb{F}[t^{\pm 1}]$-modules for any $0 \leq i \leq n-1$. Moreover, their twisted Alexander polynomials $\Delta_i^{\varepsilon,\rho}(M)$ and $\Delta_i^{\varepsilon,\rho}(W^*)$ coincide for $0 \leq i < n-1$, and $\Delta_{n-1}^{\varepsilon,\rho}(M)$ divides $\Delta_{n-1}^{\varepsilon,\rho}(W^*)$.*

Finally, in Section 2.4 we study the twisted Alexander polynomials of W^*, which also give us information about the twisted Alexander polynomials of M by the divisibility result of Section 2.3, namely Corollary 2.29. This is easier to do in the case of line arrangements (Section 2.4.1), where we are able to find an explicit formula for $\Delta_1^{\varepsilon,\rho}(W^*)$ as in the following result.

Theorem 2.31. *Let $\mathcal{A} = \{H_1, \dots, H_m\} \subset \mathbb{C}^2$ be an essential line arrangement, $H = \bigcup_{i=1}^{m} H_i$, $M = \mathbb{C}^2 \backslash H$ and let W^* be a punctured stratified tubular neighborhood of H. Let $P_1, \dots, P_s$ be the singular points of H, let s_i be the number of singular points of H on H_i, and let d_k be the number of lines of $\mathcal{A}$ going through the singular point P_k. Let $a_1, \dots, a_m$ be the generators of $\pi_1(M)$ as described in Section 2.1.1, and β_k as described in Definition 2.4. Then, following Notation 2.11, we have*

1. $\Delta_1^{\varepsilon,\rho}(W^*) = \left(\prod_{k=1}^{s} \det_{\varepsilon,\rho}(\beta_k)^{d_k-2} \right) \cdot \left(\prod_{i=1}^{m} \det_{\varepsilon,\rho}(a_i)^{s_i-1} \right) \cdot \Delta_0^{\varepsilon,\rho}(M).$
2. $\Delta_1^{\varepsilon,\rho}(M)$ *divides*

$$\left(\prod_{k=1}^{s} \det_{\varepsilon,\rho}(\beta_k)^{d_k-2} \right) \cdot \left(\prod_{i=1}^{m} \det_{\varepsilon,\rho}(a_i)^{s_i-1} \right) \cdot \gcd_{i=1,\dots,m} \{\det_{\varepsilon,\rho}(a_i)\}.$$

In some cases, we will be able to refine the result for $\Delta_1^{\varepsilon,\rho}(M)$ given by Theorem 2.31, part 2, as shown in the other main result of Section 2.4.1.

Theorem 2.36. *Let $\mathcal{B}$ be the set of lines in $\mathcal{A} = \{H_1, \dots, H_m\}$ such that for each line in $\mathcal{B}$ no other line in $\mathcal{A}$ is parallel to it. Suppose that $\mathcal{B} = \{H_1, \dots, H_l\} \neq \emptyset$. Let s_i be the number of singular points of H in H_i, which we denote by $P_1^i, \dots, P_{s_i}^i$, and let d_k^i be the number of lines of $\mathcal{A}$ going through the singular point P_k^i. Let $a_1, \dots, a_m$ be the generators of $\pi_1(M)$ as described in Section 2.1.1, and let $\beta_{k,i}$ be the resulting loop from* *composing all of the meridian loops around lines in $\mathcal{A}$ going through the singular point P_k^i as in Definition 2.4 (and Remark 2.6). Then, $\Delta_1^{\varepsilon,\rho}(M)$ divides*

$$\left(\gcd_{r=1,\dots,m}\{\det_{\varepsilon,\rho}(a_r)\}\right)\cdot\gcd_{i=1,\dots,l}\left\{\left(\prod_{k=1}^{s_i}\det_{\varepsilon,\rho}(\beta_{k,i})^{d_k^i-2}\right)\cdot\det_{\varepsilon,\rho}(a_i)^{s_i-1}\right\}.$$

In the higher dimensional case discussed in Section 2.4.2, we use the natural stratification of H to obtain an open cover of W^*, namely

$$\{\mathcal{S}_l^k \mid k=0,\dots,n-1; l=1,\dots,s_k\}$$

such that each one of the open sets in the cover fibers over the corresponding stratum S_l^k of dimension k of H. Then, we use the Mayer-Vietoris cohomology spectral sequence for the twisted Alexander modules associated to this open cover to get a bound for the twisted Alexander polynomials $\Delta_i^{\varepsilon,\rho}(M)$, and arrive at the following result, which generalizes Theorem 2.31.

Theorem 2.39. *Let $\mathcal{A}=\{H_1,\dots,H_m\}$ be an essential hyperplane arrangement in $\mathbb{C}^n$, with the natural induced stratification $\{S_l^k \mid k=0,\dots,n-1; l=1,\dots,s_k\}$, and let M be the complement of that arrangement in $\mathbb{C}^n$. For every k and l, let $F_{l,k}$ be the fiber of the fibration $\mathcal{S}_l^k \longrightarrow S_l^k$ and let $\gamma_\infty(F_{l,k})$ be a meridian around the hyperplane at infinity in $\mathbb{CP}^{n-k}$ with positive orientation, where $F_{l,k}$ is naturally seen in $\mathbb{CP}^{n-k}$. Then, for any $i=0,\dots,n-1$, the zeros of the i-th Alexander polynomial of M (i.e. $\Delta_i^{\varepsilon,\rho}(M)$) are among those of*

$$\prod_{k=0}^{n-1}\prod_{l=1}^{s_k}\det_{\varepsilon,\rho}(\gamma_\infty(F_{l,k})).$$

2.2 The homotopy type of M

We will study the topology of M with the help of two functions (f_ε and g_ε) defined from the fixed positive epimorphism $\varepsilon : \pi_1(M) \to \mathbb{Z}$ as follows:

$$\begin{aligned} f_\varepsilon : M &\longrightarrow \mathbb{R} \\ z &\longmapsto \prod_{j=1}^{m} \mid \xi_j(z)|^{\varepsilon_j} \end{aligned}$$

$$\begin{aligned} g_\varepsilon : M &\longrightarrow \mathbb{R}/2\pi\mathbb{Z} \cong S^1 \\ z &\longmapsto \arg\left(\prod_{j=1}^{m} \xi_j(z)^{\varepsilon_j}\right) = \sum_{j=1}^{m} \varepsilon_j \cdot \arg(\xi_j(z)) \end{aligned}$$

Let $\delta > 0$ small enough, let $V := f_\varepsilon^{-1}(\delta)$. The following result can be found in [22, Theorem 2.3]. In the original statement of this theorem in [22], one of the hypotheses is that $(\varepsilon_1, \dots, \varepsilon_m)$ is of "rank 1", that is, that $\{\varepsilon_1, \dots, \varepsilon_m\}$ span a 1-dimensional $\mathbb{Q}$-vector space over $\mathbb{R}$, which is automatically satisfied in our case.

Theorem 2.16. *For every $\delta > 0$ small enough, we have that*

1. *V is a C^∞ manifold of dimension $2n-1$.*
2. *The inclusion $V \hookrightarrow f_\varepsilon^{-1}((0,\delta])$ is a homotopy equivalence.*
3. *The map $g_{\varepsilon|V} : V \to S^1$ is a fiber bundle, and the fiber F has the homotopy type of a finite CW-complex of dimension $n-1$.*
4. *M has the homotopy type of V with $|\chi(M)|$ cells of dimension n attached.*

Remark 2.17. *Note that the space V depends on both δ and the homomorphism ε.*

Theorem 2.16 gives us some good properties of $f_\varepsilon^{-1}((0,\delta])$, which we will be using in Section 2.3. However, those properties alone will not be enough for us to compute possible roots of the twisted Alexander polynomials of M. The rest of this section is devoted to describe a different neighborhood of the arrangement with a nice stratification and prove some properties about it that will come in handy in Section 2.4.

We stratify our hyperplane arrangement in the natural way: two points P and P' in H lie in the same stratum if the collections of hyperplanes in the arrangement containing P and P' coincide. Each stratum is a smooth submanifold of $\mathbb{C}^n$. We define a neighborhood W of H inductively as follows. Let S_k be the union of strata of dimension k in H. For each stratum of dimension 0, we pick a ball of radius δ_0 around it, and call $W(\delta_0)$ the union of those balls. Now, we take a tubular neighborhood of $S_1 \backslash \overline{W\left(\frac{\delta_0}{2}\right)}$ of radius $\delta_1 < \delta_0$, and define $W(\delta_0, \delta_1)$ as the union of $W(\delta_0)$ with this tubular neighborhood that we have just described. Now, we take a tubular neighborhood of $S_2 \backslash \overline{W\left(\frac{\delta_0}{2}, \frac{\delta_1}{2}\right)}$ of radius $\delta_2 < \delta_1$ and create $W(\delta_0, \delta_1, \delta_2)$. We proceed inductively until we reach $W := W(\delta_0, \ldots, \delta_{n-1})$.

Note that, when all of the δ's are small enough, all of these neighborhoods that we have defined are homeomorphic. **From now on, we will assume that all of the δ's are small enough, and will not specify them.**

Definition 2.18. *We call W a stratified tubular neighborhood of H. We let $W^* = W \backslash H$, and call it a* ***punctured stratified tubular neighborhood of*** *H.*

Remark 2.19. *W^* is homotopy equivalent to ∂W, when seeing W as a subset of $\mathbb{C}^n$.*

The following theorem relates the topologies of W^* and M.

Theorem 2.20. *M has the homotopy type of W^* with cells of dimension $\geq n$ attached.*

The proof of this theorem is an immediate consequence of the following proposition.

Proposition 2.21. *Let $j : W^* \hookrightarrow M$ be the inclusion. Then*

1. *$j_* : \pi_i(W^*) \longrightarrow \pi_i(M)$ is an isomorphism for $i < n-1$.*
2. *$j_* : \pi_{n-1}(W^*) \longrightarrow \pi_{n-1}(M)$ is an epimorphism.*

Proof. The outline of the proof is going to be the following. First, we will find two stratified tubular neighborhoods W and W' of H and a $\delta > 0$ such that $W' \subset f_\varepsilon^{-1}([0,\delta]) \subset W$. Then, we will get the result about W^* from the information about $f_\varepsilon^{-1}((0,\delta])$ that we know from Theorem 2.16 and the fact that the inclusion $W' \backslash H \hookrightarrow W \backslash H$ is a homotopy equivalence.

Let us start with a stratified tubular neighborhood $W = W(\delta_0, \ldots, \delta_{n-1})$, and let δ' be the minimum of the δ_i's. We have that every point that is at distance less than δ' of H is contained in W. Also note that the factors defining $f_\varepsilon(z)$ are all proportional to a positive power of the distance of a point z to the hyperplane defined by that factor. Hence, for sufficiently small δ, $f_\varepsilon^{-1}([0,\delta])$ will be contained in the set of points on $\mathbb{C}^n$ that are at distance less than δ' of H, which is in turn contained in W. Thus, we have found δ such that $f_\varepsilon^{-1}([0,\delta]) \subset W$.

Let us find a stratified tubular neighborhood W' of H such that $W' \subset f_\varepsilon^{-1}([0,\delta])$ to complete the first part of our outline of the proof. This W' is constructed by taking the union of tubular neighborhoods of open sets of the strata like in Definition 2.18, but not requiring those tubular neighborhoods to have a fixed radius. These "generalized" stratified tubular neighborhoods are still homotopy equivalent to the ones in Definition 2.18. It is straightforward to see that we can find one such W' inside of $f_\varepsilon^{-1}([0,\delta])$. In particular, we have that $W'\backslash H \subset f_\varepsilon^{-1}\left((0,\delta]\right) \subset W\backslash H = W^*$.

Let us look at the following diagram, where all of the arrows are induced by inclusions.

$$\pi_i(W'\backslash H) \xrightarrow{a_i} \pi_i(f_\varepsilon^{-1}\left((0,\delta]\right)) \xrightarrow{b_i} \pi_i(W^*) \xrightarrow{c_i} \pi_i(M) \tag{2.2}$$

Since the inclusion from $W'\backslash H$ to W^* is a homotopy equivalence, we have that $b_i \circ a_i$ is an isomorphism for all i. In particular, b_i is an epimorphism for all i. Also, by Theorem 2.16, parts 2 and 4, we have that $c_i \circ b_i$ is an isomorphism if $i < n-1$ and an epimorphism if $i = n-1$. In particular, b_i is a monomorphism if $i < n-1$, and c_i is an epimorphism for $i \leq n-1$. This concludes the proof of the second assertion of the proposition.

Since we already know that b_i is an epimorphism for all i and a monomorphism if $i < n-1$, we find that b_i is an isomorphism if $i < n-1$. Since $c_i \circ b_i$ is an isomorphism for $i < n-1$, we get that c_i is an isomorphism for $i < n-1$, and this concludes the proof of the first assertion of the proposition. □

2.3 Torsion properties of twisted Alexander modules

From now on, we fix $\delta > 0$ small enough so that Theorem 2.16 holds. Let $j : V \hookrightarrow M$ be the inclusion, and $j_* : \pi_1(V) \to \pi_1(M)$ be the map it induces on fundamental groups. Abusing notation, we will also denote by ε and ρ the induced maps on $\pi_1(V)$ that we get by composing j_* with ε and ρ respectively.

Proposition 2.22. *Let $n \geq 2$. The inclusion map $j : V \hookrightarrow M$ induces isomorphisms of $\mathbb{F}[t^{\pm 1}]$-modules*

$$H_i^{\varepsilon,\rho}(V, \mathbb{F}[t^{\pm 1}]) \xrightarrow{\cong} H_i^{\varepsilon,\rho}(M, \mathbb{F}[t^{\pm 1}])$$

for any $i < n-1$, and an epimorphism of $\mathbb{F}[t^{\pm 1}]$-modules

$$H_{n-1}^{\varepsilon,\rho}(V, \mathbb{F}[t^{\pm 1}]) \twoheadrightarrow H_{n-1}^{\varepsilon,\rho}(M, \mathbb{F}[t^{\pm 1}]).$$

Proof. We consider two cases: $n > 2$ and $n = 2$.

Suppose that $n > 2$. By Theorem 2.16, part 4, the space M is obtained from V by attaching cells of dimension $n \geq 3$, so j_* is an isomorphism of fundamental groups. Hence, the chain complexes $C_*^{\varepsilon,\rho}(V, \mathbb{F}[t^{\pm 1}])$ and $C_*^{\varepsilon,\rho}(M, \mathbb{F}[t^{\pm 1}])$ are the same from place $n-1$ down, and j induces an inclusion $C_n^{\varepsilon,\rho}(V, \mathbb{F}[t^{\pm 1}]) \hookrightarrow C_n^{\varepsilon,\rho}(M, \mathbb{F}[t^{\pm 1}])$. The result follows from this observation.

Now, let us consider the case $n = 2$. In this case, applying Theorem 2.16, part 4, only tells us that j_* is an epimorphism between the fundamental groups. We have that $\ker j_* \subset \ker \varepsilon \circ j_*$ is a normal subgroup of $\pi_1(V)$. Let $V_{\ker j_*}$ be the covering space associated to $\ker j_*$, and note that $\pi_1(V)/\ker j_* \cong \pi_1(M)$.

We construct the chain complex

$$D_* := (\mathbb{F}[t^{\pm 1}] \otimes_{\mathbb{F}} \mathbb{V})) \otimes_{\mathbb{F}[\pi_1(V)/\ker j_*]} C_*(V_{\ker j_*}, \mathbb{F}).$$

The inclusion $V \hookrightarrow M$ induces a map $V_{\ker j_*} \to \widetilde{M}$, where $\widetilde{M}$ is the universal cover of M. Since the space M is obtained from V by attaching cells of dimension ≥ 2, this map induces isomorphisms

$$D_i \xrightarrow{\cong} C_i^{\varepsilon,\rho}(M, \mathbb{F}[t^{\pm 1}]) = (\mathbb{F}[t^{\pm 1}] \otimes_{\mathbb{F}} \mathbb{V})) \otimes_{\mathbb{F}[\pi_1(M)]} C_i(\widetilde{M}, \mathbb{F})$$

for $i = 0, 1$ and a monomorphism

$$D_2 \hookrightarrow C_2^{\varepsilon,\rho}(M, \mathbb{F}[t^{\pm 1}]).$$

Thus, we have an isomorphism

$$H_0(D_*) \xrightarrow{\cong} H_0^{\varepsilon,\rho}(M, \mathbb{F}[t^{\pm 1}])$$

and an epimorphism

$$H_1(D_*) \twoheadrightarrow H_1^{\varepsilon,\rho}(M, \mathbb{F}[t^{\pm 1}]).$$

By [7, Section 2.5, p. 50], the homology of D_* is the same as the homology of $C_*^{\varepsilon,\rho}(V, \mathbb{F}[t^{\pm 1}])$. The result follows from this observation. □

Remark 2.23. *Using the discussion following diagram (2.2) in the proof of Proposition 2.21, and repeating the same steps in the proof of Proposition 2.22, we can conclude that the same results hold for the maps* $H_i^{\varepsilon,\rho}(V, \mathbb{F}[t^{\pm 1}]) \longrightarrow H_i^{\varepsilon,\rho}(W^*, \mathbb{F}[t^{\pm 1}])$ *and* $H_i^{\varepsilon,\rho}(W^*, \mathbb{F}[t^{\pm 1}]) \longrightarrow H_i^{\varepsilon,\rho}(M, \mathbb{F}[t^{\pm 1}])$ *induced by inclusion.*

The following corollary is a direct consequence of Proposition 2.22 and Remark 2.23.

Corollary 2.24. *Let* $n \geq 2$. *For any* $0 \leq i \leq n-1$, *if* $H_i^{\varepsilon,\rho}(V, \mathbb{F}[t^{\pm 1}])$ *is a torsion* $\mathbb{F}[t^{\pm 1}]$*-module, then so are* $H_i^{\varepsilon,\rho}(M, \mathbb{F}[t^{\pm 1}])$ *and* $H_i^{\varepsilon,\rho}(W^*, \mathbb{F}[t^{\pm 1}])$.

Now, we will show that the hypothesis of Corollary 2.24 is actually satisfied.

Theorem 2.25. *Let* $n \geq 2$. *Then,* $H_i^{\varepsilon,\rho}(V, \mathbb{F}[t^{\pm 1}])$ *is a torsion* $\mathbb{F}[t^{\pm 1}]$*-module for all* i.

Proof. Note that $(g_\varepsilon)_* = \varepsilon$. Let $V^\varepsilon \xrightarrow{p_1} V$ be the covering space induced by $\ker \varepsilon$. Recall that by Theorem 2.16, part 3, the map $(g_\varepsilon)_{|V} : V \to S^1$ is a fiber bundle. We call the fiber F.

The covering space $V^\varepsilon \xrightarrow{p_1} V$ is the pullback by $(g_\varepsilon)_{|V}$ of the universal cover $\mathbb{R} \xrightarrow{p_2} S^1$, and we have the following commutative diagram of the pullback

$$\begin{tikzcd}
V^\varepsilon \arrow[r] \arrow[d, "p_1"] & \mathbb{R} \arrow[d, "p_2"] \\
V \arrow[r, "g_\varepsilon"] & S^1
\end{tikzcd}$$

Note that $V^\varepsilon \to \mathbb{R}$ is a fiber bundle over a contractible space with fiber F, so V^ε is homeomorphic to $F \times \mathbb{R}$, and therefore homotopically equivalent to F.

Let $\mathcal{L}_\rho$ be the local system of $\mathbb{F}$-vector spaces given by the representation of $\pi_1(V^\varepsilon)$ induced by ρ. By [21, Theorem 2.1], we have that

$$H_i^{\varepsilon,\rho}(V, \mathbb{F}[t^{\pm 1}]) \cong H_i(V^\varepsilon, \mathcal{L}_\rho)$$

as $\mathbb{F}[t^{\pm 1}]$-modules for all i. Since V^ε is homotopy equivalent to F, which by Theorem 2.16, part 3, has the homotopy type of a finite CW-complex, we have that the $H_i(V^\varepsilon, \mathcal{L}_\rho)$ are finite dimensional $\mathbb{F}$-vector spaces for all i, and thus $H_i^{\varepsilon,\rho}(V, \mathbb{F}[t^{\pm 1}])$ are torsion $\mathbb{F}[t^{\pm 1}]$-modules for all i. □

Let us recall the following fact, which can be found in [21].

Proposition 2.26. *Let X be a finite CW-complex. If ε is non-trivial, then*

$$H_0^{\varepsilon,\rho}(X, \mathbb{F}[t^{\pm 1}])$$

is a torsion $\mathbb{F}[t^{\pm 1}]$-module.

Now, we are ready to prove the main result in this section.

Theorem 2.27. *The twisted Alexander modules* $H_i^{\varepsilon,\rho}(M,\mathbb{F}[t^{\pm 1}])$ *are torsion* $\mathbb{F}[t^{\pm 1}]$*-modules for every* $0 \leq i \leq n-1$*, they are trivial modules for* $i > n$*, and* $H_n^{\varepsilon,\rho}(M,\mathbb{F}[t^{\pm 1}])$ *is a free* $\mathbb{F}[t^{\pm 1}]$*-module of rank* $(-1)^n \cdot \dim_{\mathbb{F}}(\mathbb{V}) \cdot \chi(M)$

Proof. The space M is an affine variety of complex dimension n, so it is homotopy equivalent to a finite CW complex of real dimension n ([8,31]). Thus $H_i^{\varepsilon,\rho}(M,\mathbb{F}[t^{\pm 1}]) = 0$ for $i > n$. This also implies that $H_n^{\varepsilon,\rho}(M,\mathbb{F}[t^{\pm 1}])$ is a free module, since it is the kernel of a morphism of free $\mathbb{F}[t^{\pm 1}]$-modules.

Now, let us prove that the twisted Alexander modules $H_i^{\varepsilon,\rho}(M,\mathbb{F}[t^{\pm 1}])$ are torsion $\mathbb{F}[t^{\pm 1}]$-modules for every $0 \leq i \leq n-1$. If $n = 1$, this is true by Proposition 2.26. Suppose that $n \geq 2$. In that case, by Corollary 2.24, we just need to show that $H_i^{\varepsilon,\rho}(V,\mathbb{F}[t^{\pm 1}])$ is a torsion $\mathbb{F}[t^{\pm 1}]$-module for every $0 \leq i \leq n-1$, which is true by Theorem 2.25.

Finally, let us compute the rank of $H_n^{\varepsilon,\rho}(M,\mathbb{F}[t^{\pm 1}])$. We abuse notation and call $\mathcal{L}_{\varepsilon,\rho}$ the local system of vector spaces over the field of rational functions $\mathbb{F}(t)$ defined by the tensor representation induced by ε and ρ (instead of the local system of $\mathbb{F}[t^{\pm 1}]$-modules induced by ε and ρ). By [7, Proposition 2.5.4], we have that

$$(-1)^n \operatorname{rank}_{\mathbb{F}[t^{\pm 1}]} H_n^{\varepsilon,\rho}(M,\mathbb{F}[t^{\pm 1}]) = \chi(M,\mathcal{L}_{\varepsilon,\rho}) =$$

$$= \operatorname{rank}_{\mathbb{F}[t^{\pm 1}]}(\mathbb{F}[t^{\pm 1}] \otimes_{\mathbb{F}} \mathbb{V}) \cdot \chi(M) = \dim_{\mathbb{F}}(\mathbb{V}) \cdot \chi(M).$$

Hence,

$$\operatorname{rank}_{\mathbb{F}[t^{\pm 1}]} H_n^{\varepsilon,\rho}(M,\mathbb{F}[t^{\pm 1}]) = (-1)^n \cdot \dim_{\mathbb{F}}(\mathbb{V}) \cdot \chi(M).$$

□

Remark 2.28. *The space* V *depends on the epimorphism* ε*, but* W^* *does not. This dependence on* ε *came in handy in the proof of Theorem 2.27, although it can be proved* *that the Alexander modules* $H_i^{\varepsilon,\rho}(W^*, \mathbb{F}[t^{\pm 1}])$ *are torsion for all* $i \geq 0$ *directly, as we will see in Section 2.4.2.*

We end this section with the result that we will use in Section 2.4, which is a consequence of everything we have discussed in this section.

Corollary 2.29. $H_i^{\varepsilon,\rho}(M, \mathbb{F}[t^{\pm 1}])$ *and* $H_i^{\varepsilon,\rho}(W^*, \mathbb{F}[t^{\pm 1}])$ *are torsion* $\mathbb{F}[t^{\pm 1}]$*-modules for any* $0 \leq i \leq n-1$*. Moreover, their twisted Alexander polynomials* $\Delta_i^{\varepsilon,\rho}(M)$ *and* $\Delta_i^{\varepsilon,\rho}(W^*)$ *coincide for* $0 \leq i < n-1$*, and* $\Delta_{n-1}^{\varepsilon,\rho}(M)$ *divides* $\Delta_{n-1}^{\varepsilon,\rho}(W^*)$*.*

2.4 Roots of twisted Alexander polynomials

2.4.1 Line arrangement case ($n = 2$)

Let $\mathcal{A} = \{H_1, \ldots, H_m\} \subset \mathbb{C}^2$ be an essential line arrangement. Note that, in the line arrangement case, the only two twisted Alexander polynomials that we will be considering are the 0-th and the first ones.

The 0-th case is always easy to compute, not just in dimension 2. The 0-th and first twisted Alexander polynomials of any finite CW complex can be computed from a presentation of the fundamental group using Fox Calculus ([21, Section 4]). In particular, if M is the complement of a complex hyperplane arrangement $\{H_1, \ldots, H_m\}$, after a Lefschetz type argument we can in principle use a presentation of $\pi_1(M)$ to compute the 0-th and first twisted Alexander polynomials of M. Let us consider the map of $\mathbb{F}[t^{\pm 1}]$-modules

$$\partial : (\mathbb{F}[t^{\pm 1}] \otimes \mathbb{V})^m \to \mathbb{F}[t^{\pm 1}] \otimes \mathbb{V}$$

given by the column matrix with entries

$$t^{\varepsilon(a_i)}\rho(a_i) - \mathrm{Id} \in \mathcal{M}_{\dim_{\mathbb{F}}\mathbb{V}\times\dim_{\mathbb{F}}\mathbb{V}}(\mathbb{F}[t^{\pm 1}]), \qquad i = 1, \ldots, m$$

where $a_1, \ldots, a_m$ are the generators of $\pi_1(M)$ as described in Section 2.1.1. The 0-th twisted Alexander polynomial $\Delta_0^{\varepsilon,\rho}(M)$ is just a generator of the Fitting ideal of the cokernel of ∂, so it is the greatest common divisor of the minors of size $\dim_{\mathbb{F}}\mathbb{V}$ of the column matrix we just described (see [21, Section 4]). Hence, we have the following result.

Proposition 2.30. *$\Delta_0^{\varepsilon,\rho}(M)$ is the greatest common divisor of the minors of size* $\dim_{\mathbb{F}}\mathbb{V}$ *of the column matrix with entries*

$$t^{\varepsilon(a_i)}\rho(a_i) - \mathrm{Id} \in \mathcal{M}_{\dim_{\mathbb{F}}\mathbb{V}\times\dim_{\mathbb{F}}\mathbb{V}}(\mathbb{F}[t^{\pm 1}])$$

for $i = 1, \ldots, m$. In particular, using Notation 2.11,

$$\Delta_0^{\varepsilon,\rho}(M) \text{ divides } \gcd_{i=1,\ldots,m}\{\det\nolimits_{\varepsilon,\rho}(a_i)\}$$

Now, let us study the first twisted Alexander polynomials of M. We have the following result.

Theorem 2.31. *Let $\mathcal{A} = \{H_1, \ldots, H_m\} \subset \mathbb{C}^2$ be an essential line arrangement, $H = \bigcup_{i=1}^{m} H_i$, $M = \mathbb{C}^2 \backslash H$ and let W^* be a punctured stratified tubular neighborhood of H. Let $P_1, \ldots, P_s$ be the singular points of H, let s_i be the number of singular points of H on H_i, and let d_k be the number of lines of $\mathcal{A}$ going through the singular point P_k. Let $a_1, \ldots, a_m$ be the generators of $\pi_1(M)$ as described in Section 2.1.1, and β_k as described in Definition 2.4. Then, following Notation 2.11, we have*

1. $\Delta_1^{\varepsilon,\rho}(W^*) = \left(\prod_{k=1}^{s} \det_{\varepsilon,\rho}(\beta_k)^{d_k-2}\right) \cdot \left(\prod_{i=1}^{m} \det_{\varepsilon,\rho}(a_i)^{s_i-1}\right) \cdot \Delta_0^{\varepsilon,\rho}(M)$.

2. $\Delta_1^{\varepsilon,\rho}(M)$ *divides*

$$\left(\prod_{k=1}^{s} \det_{\varepsilon,\rho}(\beta_k)^{d_k-2}\right) \cdot \left(\prod_{i=1}^{m} \det_{\varepsilon,\rho}(a_i)^{s_i-1}\right) \cdot \gcd_{i=1,\ldots,m} \left\{\det_{\varepsilon,\rho}(a_i)\right\}.$$

Proof. We will use techniques coming from [5, Theorem 5.6]. Let $F = H\backslash \bigsqcup_{k=1}^{s}(H \cap \mathbb{B}_k^4)$ be the surface obtained by removing small balls $\mathbb{B}_k^4$ around the singular points P_k. Note that what we are really removing from our surface is a 2-dimensional open disk D_i^k from every line H_i in $\mathcal{A}$ containing P_k.

Let $N = F \times S^1$. N should be thought of as the boundary of a tubular neighborhood around the non-singular part of H. We have that $\partial N = \partial F \times S^1$, and since ∂F is a union of disjoint S^1's (one from every disk D_i^k removed), then ∂N is a union of disjoint tori $\bigsqcup_{k,i} T_i^k$ (again, one from every disk D_i^k removed). Let us fix a point f_k^i in the S^1 corresponding to the boundary of the disk D_i^k for every such disk removed.

Let L_k be the link of the singularity at the point P_k (which is a Hopf link with d_k components), and let S_k^3 be the boundary of $\mathbb{B}_k^4$. We consider the space

$$X = N \cup_{\left(\bigsqcup_{k,i} T_i^k\right)} \left(\bigsqcup_{k=1}^{s} S_k^3 \backslash L_k\right) \subset M$$

where the gluing is done as follows. A meridian around the i-th component of L_k (the one corresponding to the line H_i, which we will denote by L_k^i) is glued to $\{f_k^i\} \times S^1 \subset N$, and L_k^i is glued to the S^1 corresponding to the boundary of D_i^k.

By the definition of the stratified tubular neighborhood W, we have that X is homotopy equivalent to ∂W. By Corollary 2.29, the first twisted Alexander polynomial of the line arrangement complement M divides the first twisted Alexander polynomial of ∂W (which is homotopy equivalent to W^*, see Remark 2.19), so our goal now is to compute $\Delta_1^{\varepsilon,\rho}(X)$.

Notice that N has m connected components, one for every line H_i in our arrangement. That is, if we define $F_i = F \cap H_i$, then $N = \bigsqcup_{i=1}^{m} F_i \times S^1$. Notice that F_i is just a complex line H_i (or a real plane) with s_i disks removed, one for every singular point of H in H_i. Thus, F_i is homotopy equivalent to a wedge sum of s_i circles, and hence $F_i \times S^1$ (and N) is homotopy equivalent to a 2-dimensional CW-complex.

It is also well-known ([26, Lemma 2]) that $S^3_k \backslash L_k$ has the homotopy type of a 2-dimensional CW-complex as well. The space X also has the homotopy type of a 2-dimensional CW-complex by how it is constructed.

We have the following Mayer-Vietoris short exact sequence of complexes with coefficients in $\mathbb{F}(t)$.

$$0 \to \bigoplus_{k,i} C_*^{\varepsilon,\rho}(T_i^k, \mathbb{F}(t)) \to \left(\bigoplus_{k} C_*^{\varepsilon,\rho}(S^3_k \backslash L_k, \mathbb{F}(t)) \right) \oplus C_*^{\varepsilon,\rho}(N, \mathbb{F}(t)) \to C_*^{\varepsilon,\rho}(X, \mathbb{F}(t)) \to 0$$

Let $\mathcal{H}$ be the Mayer-Vietoris long exact sequence of the twisted homology groups (seen as a complex). We will consider the twisted Reidemeister torsion $\tau_{\varepsilon,\rho}$ (as defined in Definition 2.13) of all the pieces involved in this short exact sequence, namely N, $\bigsqcup_{k=1}^{s} S^3_k \backslash L_k$, and their intersection $\bigsqcup_{k,i} T_i^k$.

As pointed out in Lemma 2.14, the twisted Reidemeister torsion for acyclic complexes is independent of the choice of bases up to multiplication by a unit in $\mathbb{F}[t^{\pm 1}]$, and, as we will see in the proof of Proposition 2.32, we only consider the twisted Reidemeister torsion of acyclic complexes in this proof. Since all of those pieces (including X) have the homotopy type of a 2-dimensional CW-complex, then the only non-trivial Alexander polynomials are the 0-th and the first ones for all of those spaces, and by Lemma 2.15, we have that

$$\tau_{\varepsilon,\rho}(\cdot) = \frac{\Delta_1^{\varepsilon,\rho}(\cdot)}{\Delta_0^{\varepsilon,\rho}(\cdot)}$$

for all of the relevant spaces in this problem (X, N, $\bigsqcup_{k,i} T_i^k$ and $\bigsqcup_{k=1}^{s} S_k^3 \backslash L_k$)

By Lemma 2.12, we have that

$$\left(\prod_{k=1}^{s} \tau_{\varepsilon,\rho}(S_k^3 \backslash L_k)\right)(\tau_{\varepsilon,\rho}(N)) = \left(\prod_{k,i} \tau_{\varepsilon,\rho}(T_i^k)\right)\tau_{\varepsilon,\rho}(X)\tau(\mathcal{H}) \tag{2.3}$$

were $\tau(\mathcal{H})$ is the torsion of a complex.

Now, we use the following result.

Proposition 2.32. *$\mathcal{H}$ is the 0 complex. In particular, $\tau(\mathcal{H}) = 1$*

Proof. We need to show that the complexes $C_*^{\varepsilon,\rho}(T_i^k, \mathbb{F}(t))$ (for every k and i), $C_*^{\varepsilon,\rho}(S_k^3 \backslash L_k, \mathbb{F}(t))$ (for every k), $C_*^{\varepsilon,\rho}(N, \mathbb{F}(t))$ and $C_*^{\varepsilon,\rho}(X, \mathbb{F}(t))$ are acyclic. By the long exact sequence in homology, it suffices to show that three out of those four are acyclic.

By [30, Proposition 2.9], since $\mathbb{F}(t)$ is flat over $\mathbb{F}[t^{\pm 1}]$ and $\varepsilon(\beta_k) \neq 0$ (in fact, $\varepsilon(\beta_k) > 0$ by Remark 2.6), we have that

$$C_*^{\varepsilon,\rho}(S_k^3 \backslash L_k, \mathbb{F}(t))$$

is acyclic.

Let us now see that $C_*^{\varepsilon,\rho}(N, \mathbb{F}(t))$ is acyclic, or equivalently, that $H_j^{\varepsilon,\rho}(F_i \times S^1, \mathbb{F}(t)) = 0$ for all $i = 1, \ldots, m$ and $j \geq 0$. We can compute $H_0^{\varepsilon,\rho}(F_i \times S^1, \mathbb{F}(t))$ and $H_1^{\varepsilon,\rho}(F_i \times S^1, \mathbb{F}(t))$ directly using Fox Calculus ([21, Section 4]), a technique that only requires a presentation of the fundamental group. Recall that F_i is homotopy equivalent to a wedge sum of s_i circles, and let $b_1^i, \ldots, b_{s_i}^i$ be loops around the respective circles.

With this notation, we see that

$$\pi_1(F_i \times S^1) = \langle b_1^i, \dots, b_{s_i}^i, a_i \mid [b_j^i, a_i], j = 1, \dots, s_i \rangle. \tag{2.4}$$

In this presentation we are abusing notation, since the base point of a_i is not in $F_i \times S^1$. By a_i in this presentation, we mean a loop contained in $F_i \times S^1$ that is isotopic to a_i in M after a change of base points.

Using that $\varepsilon(a_i)$ is not 0 for any $i = 1, \dots, m$ in a routine Fox Calculus computation using this presentation, we get that $H_j^{\varepsilon,\rho}(F_i \times S^1, \mathbb{F}(t)) = 0$ for $j = 0, 1$.

To finish proving that $H_j^{\varepsilon,\rho}(F_i \times S^1, \mathbb{F}(t)) = 0$ for all j, we just have to show it for $j = 2$, since $F_i \times S^1$ is homotopy equivalent to a 2-dimensional CW-complex. This 2-dimensional CW-complex is the cartesian product of a wedge sum of s_i S^1's and an S^1. Thus, it has one 0-cell, $(s_i + 1)$ 1-cells, and s_i 2-cells. Hence, an Euler characteristic argument tells us that $H_2^{\varepsilon,\rho}(F_i \times S^1, \mathbb{F}(t)) = 0$, concluding our proof of the acyclicity of $C_*^{\varepsilon,\rho}(N, \mathbb{F}(t))$.

The only thing left to prove here is that $C_*^{\varepsilon,\rho}(T_i^k, \mathbb{F}(t))$ is acyclic (for every k and i). This is just a computation that follows the same steps as what we did for $C_*^{\varepsilon,\rho}(F_i \times S^1, \mathbb{F}(t))$, so we will omit it. It also relies on the fact that $\varepsilon(\gamma_i) \neq 0$, for every meridian γ_i around H_i and for all $i = 1, \dots, m$. □

Now, using this result, equation (2.3) becomes

$$\left(\prod_{k=1}^{s} \tau_{\varepsilon,\rho}(S_k^3 \backslash L_k) \right) (\tau_{\varepsilon,\rho}(N)) = \left(\prod_{k,i} \tau_{\varepsilon,\rho}(T_i^k) \right) \tau_{\varepsilon,\rho}(X). \tag{2.5}$$

We want to compute $\Delta_1^{\varepsilon,\rho}(X)$. By Proposition 2.22, we have that $\Delta_0^{\varepsilon,\rho}(X) = \Delta_0^{\varepsilon,\rho}(M)$, and we know $\Delta_0^{\varepsilon,\rho}(M)$ by Proposition 2.30. Hence, to compute $\Delta_1^{\varepsilon,\rho}(X)$, it suffices to compute $\tau_{\varepsilon,\rho}(X) = \frac{\Delta_1^{\varepsilon,\rho}(X)}{\Delta_0^{\varepsilon,\rho}(X)}$. By the equation relating the torsions that we just found, it suffices to compute the twisted Reidemeister torsion for the other pieces.

Proposition 2.33.

1. $\tau_{\varepsilon,\rho}(N) = \prod_{i=1}^{m} \det_{\varepsilon,\rho}(a_i)^{s_i-1}$
2. $\tau_{\varepsilon,\rho}\left(\bigsqcup_{k,i} T_i^k\right) = 1$
3. $\tau_{\varepsilon,\rho}\left(\bigsqcup_{k=1}^{s} S_k^3 \backslash L_k\right) = \prod_{k=1}^{s} \det_{\varepsilon,\rho}(\beta_k)^{d_k-2}$

Proof. First of all, by the multiplicativity of the torsion (which can be inferred from Lemma 2.15), we have that

$$\tau_{\varepsilon,\rho}(N) = \prod_{i=1}^{m} \tau_{\varepsilon,\rho}(F_i \times S^1), \qquad \tau_{\varepsilon,\rho}\left(\bigsqcup_{k,i} T_i^k\right) = \prod_{k,i} \tau_{\varepsilon,\rho}(T_i^k).$$

Using the presentations given in equations (2.1) and (2.4) and Fox Calculus ([21, Section 4]), we can compute the twisted Reidemeister torsion of all the spaces involved, namely

$$\begin{aligned}
&\tau_{\varepsilon,\rho}(F_i \times S^1) = \det_{\varepsilon,\rho}(a_i)^{s_i-1} && \text{for all } i = 1, \ldots, m \\
&\tau_{\varepsilon,\rho}(T_i^k) = 1 && \text{for all } k, i \\
&\tau_{\varepsilon,\rho}(S_k^3 \backslash L_k) = \det_{\varepsilon,\rho}(\beta_k)^{d_k-2} && \text{for all } k = 1, \ldots, s \quad \text{[30, Proposition 2.9].}
\end{aligned}$$

□

Now, we can use Proposition 2.33 and equation (2.5) to get

$$\tau_{\varepsilon,\rho}(X) = \prod_{k=1}^{s} \det_{\varepsilon,\rho}(\beta_k)^{d_k-2} \prod_{i=1}^{m} \det_{\varepsilon,\rho}(a_i)^{s_i-1}$$

where this equality is defined up to multiplication by a unit of $\mathbb{F}[t^{\pm 1}]$.

Hence

$$\Delta_1^{\varepsilon,\rho}(X) = \left(\prod_{k=1}^{s} \det_{\varepsilon,\rho}(\beta_k)^{d_k-2}\right) \cdot \left(\prod_{i=1}^{m} \det_{\varepsilon,\rho}(a_i)^{s_i-1}\right) \cdot \Delta_0^{\varepsilon,\rho}(M)$$

so, $\Delta_1^{\varepsilon,\rho}(X)$ divides

$$\left(\prod_{k=1}^{s} \det{}_{\varepsilon,\rho}(\beta_k)^{d_k-2}\right) \cdot \left(\prod_{i=1}^{m} \det{}_{\varepsilon,\rho}(a_i)^{s_i-1}\right) \cdot \gcd_{i=1,\dots,m} \{\det{}_{\varepsilon,\rho}(a_i)\}.$$

Now, by Corollary 2.29 and the fact that W^* is homotopy equivalent to X, the proof of Theorem 2.31 is complete. □

Remark 2.34 (Twisted Alexander polynomials of the boundary manifold)**.** *Let $\mathcal{A} = \{l_1, \dots, l_m\} \subset \mathbb{C}^2$ be an essential line arrangement, and let $l_0 = \mathbb{CP}^2 \backslash \mathbb{C}^2$ be the line at infinity. We consider the projective line arrangement $\mathcal{A}' = \mathcal{A} \cup \{l_0\} \subset \mathbb{CP}^2$. The boundary manifold B of the affine arrangement $\mathcal{A}$ is the boundary of the manifold obtained by gluing balls around the singular points of the arrangement $\mathcal{A}'$ and tubes around the smooth part of the lines, similar to what we did in the construction of W^*.*

We have that the map induced by inclusion $\pi_0(B) \longrightarrow \pi_0(M)$ is an isomorphism, since both spaces are connected, and $\pi_1(B) \longrightarrow \pi_1(M)$ is an epimorphism, by a Lefschetz type argument. Thus, M is obtained from B by adjoining cells of dimension ≥ 2, which as we have seen in the proof of Proposition 2.22 is enough to show that

$$\Delta_0^{\varepsilon,\rho}(M) = \Delta_0^{\varepsilon,\rho}(B)$$

and that

$$\Delta_1^{\varepsilon,\rho}(M) \text{ divides } \Delta_1^{\varepsilon,\rho}(B),$$

provided that $H_1^{\varepsilon,\rho}(B, \mathbb{F}[t^{\pm 1}])$ is a torsion $\mathbb{F}[t^{\pm 1}]$-module. Moreover, following the proof of Theorem 2.31, we conclude that $H_1^{\varepsilon,\rho}(B, \mathbb{F}[t^{\pm 1}])$ is indeed torsion (because $C_*^{\varepsilon,\rho}(B, \mathbb{F}(t))$ is acyclic) and

$$\frac{\Delta_1^{\varepsilon,\rho}(B)}{\Delta_0^{\varepsilon,\rho}(B)} = \left(\prod_{k=1}^{s} \det{}_{\varepsilon,\rho}(\beta_k)^{d_k-2}\right) \cdot \left(\prod_{i=0}^{m} \det{}_{\varepsilon,\rho}(a_i)^{\widetilde{s}_i-2}\right)$$

where s is the number of singular points of the ***projective*** *arrangement, the β_k's are certain distinguished loops near each of the singular points as in Definition 2.4, $\widetilde{s_i}$ is the number of singular points of the projective arrangement on the line l_i, and a_i is a positively oriented meridian around the line l_i. This agrees with the result obtained in a different way by Cohen and Suciu in* [6, Theorem 5.2] *for multivariable twisted Alexander polynomials.*

Note that, if s_i is the number of singular points of the ***affine*** *arrangement on the line l_i, for $i = 1, \ldots, m$, then $\widetilde{s_i} = s_i + 1$, so we can see that*

$$\Delta_1^{\varepsilon,\rho}(W^*) \text{ divides } \Delta_1^{\varepsilon,\rho}(B),$$

and we conclude that the punctured stratified tubular neighborhood W^* constitutes a better bound than the boundary manifold B for the roots of the first twisted Alexander polynomial of M.

Remark 2.35. *For Proposition 2.30 we do not need that ε be a positive epimorphism, just that it is a non-trivial map. For Theorem 2.31 to hold, we just need that ε takes non-zero values on the distinguished loops that appear in the formula of the twisted Alexander polynomial of W^*, as one can see in the proof.*

In some cases, we can refine the result given by Theorem 2.31 as follows:

Theorem 2.36. *Let $\mathcal{B}$ be the set of lines in $\mathcal{A} = \{H_1, \ldots, H_m\}$ such that for each line in $\mathcal{B}$ no other line in $\mathcal{A}$ is parallel to it. Suppose that $\mathcal{B} = \{H_1, \ldots, H_l\} \neq \emptyset$. Let s_i be the number of singular points of H in H_i, which we denote by $P_1^i, \ldots, P_{s_i}^i$, and let d_k^i be the number of lines of $\mathcal{A}$ going through the singular point P_k^i. Let $a_1, \ldots, a_m$ be the generators of $\pi_1(M)$ as described in Section 2.1.1, and let $\beta_{k,i}$ be the resulting loop from* *composing all of the meridian loops around lines in $\mathcal{A}$ going through the singular point P_k^i as in Definition 2.4 (and Remark 2.6). Then, $\Delta_1^{\varepsilon,\rho}(M)$ divides*

$$\left(\gcd_{r=1,\dots,m}\{\det_{\varepsilon,\rho}(a_r)\}\right)\cdot\gcd_{i=1,\dots,l}\left\{\left(\prod_{k=1}^{s_i}\det_{\varepsilon,\rho}(\beta_{k,i})^{d_k^i-2}\right)\cdot\det_{\varepsilon,\rho}(a_i)^{s_i-1}\right\}.$$

Proof. Let $1\leq i\leq l$. Let $\mathbb{B}^4_{k,i}$ be a small ball around the singular point P_k^i, with boundary $S^3_{k,i}$, and let $L_{k_i}\subset S^3_{k,i}$ be the link of the singularity of H at the point P_k^i. We will follow the proof of Theorem 2.31 and use the notation introduced there, but this time we define X_i (instead of X) as the result of gluing $F_i\times S^1$ and $\bigsqcup_{k=1}^{s_i}S^3_{k,i}\backslash L_{k_i}$ along the correspondig tori.

X_i is connected, so the map induced by inclusion $\pi_0(X_i)\longrightarrow\pi_0(M)$ is an isomorphism. Moreover, since no other line in $\mathcal{A}$ is parallel to H_i, we can see by a Lefschetz type argument that the map $\pi_1(X_i)\longrightarrow\pi_1(M)$ induced by inclusion is an epimorphism. Thus, M is obtained from X_i by adjoining cells of dimension ≥ 2, which as we have seen in the proof of Proposition 2.22, is enough to show that

$$H_1^{\varepsilon,\rho}(X_i,\mathbb{F}[t^{\pm1}])\longrightarrow H_1^{\varepsilon,\rho}(M,\mathbb{F}[t^{\pm1}])$$

is an epimorphism. Moreover, following the proof of Theorem 2.31, we can show that all of the complexes involved in the Mayer-Vietoris short exact sequence of complexes with coefficients in $\mathbb{F}(t)$ except for $C_*^{\varepsilon,\rho}(X_i,\mathbb{F}(t))$ are acyclic, so the long exact sequence in homology will tell us that $C_*^{\varepsilon,\rho}(X_i,\mathbb{F}(t))$ is acyclic as well. In particular, $H_1^{\varepsilon,\rho}(X_i,\mathbb{F}[t^{\pm1}])$ is torsion, and $\Delta_1^{\varepsilon,\rho}(M)$ divides $\Delta_1^{\varepsilon,\rho}(X_i)$ for all $i=1,\dots,l$.

Following the proof of Theorem 2.31, we get that

$$\Delta_1^{\varepsilon,\rho}(X_i)=\left(\prod_{k=1}^{s_i}\det_{\varepsilon,\rho}(\beta_{k,i})^{d_k^i-2}\right)\cdot\det_{\varepsilon,\rho}(a_i)^{s_i-1}\cdot\Delta_0^{\varepsilon,\rho}(M)$$

for all $i = 1, \dots, l$, and the result follows immediately by Proposition 2.30. $\square$

2.4.2 Higher-dimensional case

We will follow the notation of [24, Section 3]. Let $\mathcal{A} = \{H_1, \dots, H_m\}$ be an essential hyperplane arrangement in $\mathbb{C}^n$. We consider the natural stratification of $H = \bigcup_{i=1}^{m} H_i$, the one in which two points P and P' in H lie in the same stratum if the collections of hyperplanes in the arrangement containing P and P' coincide. Let $S_1^k, \dots, S_{s_k}^k$ be the collection of connected strata of (complex) dimension k. For each stratum, we define the multiplicity $m(S_i^k)$ as the number of hyperplanes in $\mathcal{A}$ containing a point from this stratum.

Remark 2.37. *By Corollary 2.29, the zeros of the i-th twisted Alexander polynomials of our arrangement complement M are among the zeros of the i-th twisted Alexander polynomial of a punctured stratified tubular neighborhood W^* of the arrangement, for $i = 0, \dots, n-1$. This observation prompts us to study what the zeros of the twisted Alexander polynomials of W^* could be.*

Let

$$\{\mathcal{S}_l^k \mid k = 0, \dots, n-1; l = 1, \dots, s_k\}$$

be a collection of open subsets of W^*, each of which fibers over the corresponding stratum S_l^k, and chosen so that their union is W^*. These open subsets can be taken to be the tubular neighborhoods of open subsets of the strata that appeared in the construction of W (before Definition 2.18) minus H. The fiber of $\mathcal{S}_l^k \longrightarrow S_l^k$ is a central hyperplane arrangement complement consisting on $m(S_l^k)$ hyperplanes in $\mathbb{C}^{n-k}$. As it is pointed out in [24, p. 5], if $k_1 \geq k_2$, then $\mathcal{S}_{l_1}^{k_1} \cap \mathcal{S}_{l_2}^{k_2}$ is not empty if and only if the stratum $S_{l_2}^{k_2}$ is in the closure of the stratum $S^{k_1}_{l_1}$, and, in this intersection, the fibration that we consider is the one from $\mathcal{S}^{k_1}_{l_1} \to S^{k_1}_{l_1}$ restricted to it.

Let $\mathbb{W} := \mathbb{V}^* = \operatorname{Hom}_{\mathbb{F}}(\mathbb{V}, \mathbb{F})$ be the dual vector space of $\mathbb{V}$, and let

$$\rho^* : \pi_1(M) \longrightarrow \mathrm{GL}(\mathbb{W})$$

be the dual representation of $\rho : \pi_1(M) \longrightarrow \mathrm{GL}(\mathbb{V})$, given by

$$(w \cdot \alpha)(v) = w(v \cdot \alpha^{-1})$$

for every $w \in \mathbb{W}$, $\alpha \in \pi_1(M)$ and $v \in \mathbb{V}$.

We consider the involution given by

$$\begin{aligned} \bar{\cdot} : \mathbb{F}[t^{\pm 1}] &\longrightarrow \mathbb{F}[t^{\pm 1}] \\ t &\longmapsto \bar{t} := t^{-1} \end{aligned}$$

and we define the **conjugate $\mathbb{F}[t^{\pm 1}]$-module structure** of an $\mathbb{F}[t^{\pm 1}]$-module as the one obtained by composing the $\mathbb{F}[t^{\pm 1}]$-module structure with the involution $\bar{\cdot}$. Then, as justified in [30, p. 6 and p. 17], we have that

$$\overline{H}^i(W^*, \mathcal{L}_{\varepsilon,\rho^*}) \cong H^i\left(\operatorname{Hom}_{\mathbb{F}[t^{\pm 1}]}\left(C^{\varepsilon,\rho}_*(W^*, \mathbb{F}[t^{\pm 1}]), \mathbb{F}[t^{\pm 1}]\right)\right)$$

for all i, where $\overline{H}^i(W^*, \mathcal{L}_{\varepsilon,\rho^*})$ means the $\mathbb{F}[t^{\pm 1}]$-module $H^i(W^*, \mathcal{L}_{\varepsilon,\rho^*})$ with the conjugate module structure, and $\mathcal{L}_{\varepsilon,\rho^*}$ is the local system of $\mathbb{F}[t^{\pm 1}]$-modules induced by the tensor representation $\varepsilon \otimes \rho^*$. Therefore, the Universal Coefficient Theorem (UCT, for short) applied to the principal ideal domain $\mathbb{F}[t^{\pm 1}]$ yields

$$\overline{H}^i(W^*, \mathcal{L}_{\varepsilon,\rho^*}) \cong \operatorname{Hom}_{\mathbb{F}[t^{\pm 1}]}(H^{\varepsilon,\rho}_i(W^*, \mathbb{F}[t^{\pm 1}]), \mathbb{F}[t^{\pm 1}]) \oplus \operatorname{Ext}_{\mathbb{F}[t^{\pm 1}]}(H^{\varepsilon,\rho}_{i-1}(W^*, \mathbb{F}[t^{\pm 1}]), \mathbb{F}[t^{\pm 1}]).$$

Now, applying Corollary 2.29 we get that $H^{\varepsilon,\rho}_i(W^*, \mathbb{F}[t^{\pm 1}])$ is a torsion $\mathbb{F}[t^{\pm 1}]$-module for all $i \leq n-1$, so by the UCT, we get that

$$\overline{H}^i(W^*, \mathcal{L}_{\varepsilon,\rho^*}) \cong \operatorname{Ext}_{\mathbb{F}[t^{\pm 1}]}(H^{\varepsilon,\rho}_{i-1}(W^*, \mathbb{F}[t^{\pm 1}]), \mathbb{F}[t^{\pm 1}]) \cong H^{\varepsilon,\rho}_{i-1}(W^*, \mathbb{F}[t^{\pm 1}])$$

for all $i \leq n-1$.

Remark 2.38. *In fact, we will see later on that $H_i^{\varepsilon,\rho}(W^*, \mathbb{F}[t^{\pm 1}])$ is a torsion $\mathbb{F}[t^{\pm 1}]$-module for all i, so*

$$\overline{H}^i(W^*, \mathcal{L}_{\varepsilon,\rho^*}) \cong H_{i-1}^{\varepsilon,\rho}(W^*, \mathbb{F}[t^{\pm 1}])$$

for all i.

Hence, the order of $H^i(W^*, \mathcal{L}_{\varepsilon,\rho^*})$ is $\overline{\Delta_{i-1}^{\varepsilon,\rho}(W^*)(t)} = \Delta_{i-1}^{\varepsilon,\rho}(W^*)(t^{-1})$ for all i. As indicated in Remark 2.37, we are interested in studying the zeros of $\Delta_i^{\varepsilon,\rho}(W^*)$ for all i, or equivalently, the inverses of the zeros of the order of $H^{i+1}(W^*, \mathcal{L}_{\varepsilon,\rho^*})$ for all i.

Let us consider the Mayer-Vietoris spectral sequence of the sheaf $\mathcal{L}_{\varepsilon,\rho^*}$ associated to the open covering

$$\{\mathcal{S}_l^k \mid k = 0, \ldots, n-1; l = 1, \ldots, s_k\}$$

The first page of this spectral sequence is

$$E_1^{p,q} = \bigoplus H^q(\mathcal{S}_{l_1}^{k_1} \cap \ldots \cap \mathcal{S}_{l_{p+1}}^{k_{p+1}}, \mathcal{L}_{\varepsilon,\rho^*}) \tag{2.6}$$

where the direct sum is taken over all the possible non-empty intersections of $p+1$ open sets of our covering. This spectral sequence converges to $H^{p+q}(W^*, \mathcal{L}_{\varepsilon,\rho^*})$.

Let us study the elements in the first page of our Mayer-Vietoris spectral sequence, namely, the elements of the form

$$H^q(\mathcal{S}_{l_1}^{k_1} \cap \ldots \cap \mathcal{S}_{l_{p+1}}^{k_{p+1}}, \mathcal{L}_{\varepsilon,\rho^*}).$$

Without loss of generality, we can assume that $k_1 \geq \ldots \geq k_{p+1}$, and that $S_{l_j}^{k_j} \subset \overline{S_{l_1}^{k_1}}$ for every $j = 2, \ldots, p+1$. Let $f : \mathcal{S}_{l_1}^{k_1} \to S_{l_1}^{k_1}$ be the fibration. We consider a good cover $\mathcal{U}$ of $f(\mathcal{S}_{l_1}^{k_1} \cap \ldots \cap \mathcal{S}_{l_{p+1}}^{k_{p+1}})$, which is an open set of the manifold $S_{l_1}^{k_1}$, and thus it is a manifold.

By good cover we mean an open cover where all open sets and all finite intersections of those open sets are contractible. Let

$$\mathcal{V} := \{f^{-1}(A) \mid A \in \mathcal{U}\}$$

We consider the Mayer Vietoris spectral sequence of the sheaf $\mathcal{L}_{\varepsilon,\rho^*}$ associated to the open covering $\mathcal{V}$. The first page of this spectral sequence is

$$E_1^{p,q} = \bigoplus H^q(B_{r_1} \cap \ldots \cap B_{r_{p+1}}, \mathcal{L}_{\varepsilon,\rho^*}) \tag{2.7}$$

where the direct sum is taken over all the possible non-empty intersections $B_{r_1} \cap \ldots \cap B_{r_{p+1}}$ of $p+1$ open sets of our covering $\mathcal{V}$. This spectral sequence converges to $H^{p+q}(\mathcal{S}_{l_1}^{k_1} \cap \ldots \cap \mathcal{S}_{l_{p+1}}^{k_{p+1}}, \mathcal{L}_{\varepsilon,\rho^*})$.

Note that, since non-empty finite intersections of open sets in $\mathcal{U}$ are contractible, and the map f restricted to $\mathcal{S}_{l_1}^{k_1} \cap \ldots \cap \mathcal{S}_{l_{p+1}}^{k_{p+1}}$ is a locally trivial fibration with fiber F_{l_1,k_1}, which is the complement of a central hyperplane arrangement consisting on $m(S_{l_1}^{k_1})$ hyperplanes in $\mathbb{C}^{n-k_1}$, then $B_{r_1} \cap \ldots \cap B_{r_{p+1}}$ is homeomorphic to the product of F_{l_1,k_1} and a contractible open set, and thus it is homotopy equivalent to F_{l_1,k_1}. Thus,

$$H^q(B_{r_1} \cap \ldots \cap B_{r_{p+1}}, \mathcal{L}_{\varepsilon,\rho^*}) \cong H^q(F_{l_1,k_1}, \mathcal{L}_{\varepsilon,\rho^*}).$$

Note that central hyperplane arrangements are homotopy equivalent to the complement in S^{2n-1} of their links at infinity, so by [30, proof of Theorem 4.1], $H_q^{\varepsilon,\rho}(F_{l_1,k_1}, \mathbb{F}[t^{\pm 1}])$ (and consequently $H^q(F_{l_1,k_1}, \mathcal{L}_{\varepsilon,\rho^*})$) are torsion for all q. This implies that all of the elements in spectral sequence (2.7) are torsion modules. In fact, by [30, Theorem 4.11], the zeros of the order of $H^q(F_{l_1,k_1}, \mathcal{L}_{\varepsilon,\rho^*})$ are among those of the order of the cokernel of the endomorphism

$$t^{-\varepsilon(\gamma_\infty(F_{l_1,k_1}))}\rho^*(\gamma_\infty(F_{l_1,k_1}))^{-1} - \mathrm{Id} \in \mathrm{End}(\mathbb{F}[t^{\pm 1}] \otimes_{\mathbb{F}} \mathbb{V})$$

where $\gamma_\infty(F_{l_1,k_1})$ is a loop around the hyperplane at infinity in $(\mathbb{CP})^{n-k_1}$. Note that ε restricted to $\pi_1(F_{l_1,k_1})$ is not necessarily an epimorphism, but the proof of [30, Theorem 4.11] does not require it.

Let us try to describe these "loops at infinity" that we are using in more detail. Let $H_{t_1},\ldots,H_{t_{m(S^{k_1}_{l_1})}}$ be the hyperplanes going through the stratum $S^{k_1}_{l_1}$ associated to the fiber F_{l_1,k_1}. We have that $\gamma_\infty(F_{l_1,k_1})$ has an expression of the form

$$(\gamma_{t_1}\cdot\ldots\cdot\gamma_{t_{m(S^{k_1}_{l_1})}})^{-1}$$

where $\gamma_{t_1},\ldots,\gamma_{t_{m(S^{k_1}_{l_1})}}$ are an appropriate choice of meridians around each component of the central hyperplane arrangement given by F_{l_1,k_1} in the appropriate order.

Note that $\varepsilon(\gamma_\infty(F_{l_1,k_1}))<0$, so the order of the cokernel of

$$t^{-\varepsilon(\gamma_\infty(F_{l_1,k_1}))}\rho^*(\gamma_\infty(F_{l_1,k_1}))^{-1}-\mathrm{Id}$$

is exactly

$$\det(t^{-\varepsilon(\gamma_\infty(F_{l_1,k_1}))}\rho^*(\gamma_\infty(F_{l_1,k_1}))^{-1}-\mathrm{Id})$$

By the discussion above, the zeros of the order of $H^q(F_{l_1,k_1},\mathcal{L}_{\varepsilon,\rho^*})$ are among those of

$$\det(t^{-\varepsilon(\gamma_\infty(F_{l_1,k_1}))}\rho^*(\gamma_\infty(F_{l_1,k_1}))^{-1}-\mathrm{Id}).$$

Using the spectral sequence (2.7), we get that the zeros of the order of

$$H^q(\mathcal{S}^{k_1}_{l_1}\cap\ldots\cap\mathcal{S}^{k_{p+1}}_{l_{p+1}},\mathcal{L}_{\varepsilon,\rho^*})$$

are among the zeros of $\det(t^{-\varepsilon(\gamma_\infty(F_{l_1,k_1}))}\rho^*(\gamma_\infty(F_{l_1,k_1}))^{-1}-\mathrm{Id})$.

Now, by using spectral sequence (2.6), we see that $H^i(W^*,\mathcal{L}_{\varepsilon,\rho^*})$ is torsion for all i. By the Universal Coefficient Theorem, this means that $H_i^{\varepsilon,\rho}(W^*,\mathbb{F}[t^{\pm1}])$ is also torsion for all i, as anticipated in Remark 2.38. Moreover, the zeros of the order of $H^q(W^*, \mathcal{L}_{\varepsilon,\rho^*})$ are among the zeros of

$$\prod_{k=0}^{n-1}\prod_{l=1}^{s_k}\det(t^{-\varepsilon(\gamma_\infty(F_{l,k}))}\rho^*(\gamma_\infty(F_{l,k}))^{-1} - \mathrm{Id})$$

for all q.

Hence, by Remark 2.37 and Remark 2.38, and the fact that $\rho^*(\alpha)^{-1} = \rho(\alpha)^T$ for every $\alpha \in \pi_1(M)$ (seen as matrices in $\mathrm{GL}_n(\mathbb{F})$), we get that the zeros of $\Delta_i^{\varepsilon,\rho}(M)$ are among the zeros of

$$\prod_{k=0}^{n-1}\prod_{l=1}^{s_k}\det(t^{\varepsilon(\gamma_\infty(F_{l,k}))}\rho(\gamma_\infty(F_{l,k})) - \mathrm{Id}).$$

Thus, we have arrived to the following result.

Theorem 2.39. *Let $\mathcal{A} = \{H_1, \ldots, H_m\}$ be an essential hyperplane arrangement in $\mathbb{C}^n$, with the natural induced stratification $\{S_l^k \mid k = 0, \ldots, n-1; l = 1, \ldots, s_k\}$, and let M be the complement of that arrangement in $\mathbb{C}^n$. For every k and l, let $F_{l,k}$ be the fiber of the fibration $\mathcal{S}_l^k \longrightarrow S_l^k$ and let $\gamma_\infty(F_{l,k})$ be a meridian around the hyperplane at infinity in $\mathbb{CP}^{n-k}$ with positive orientation, where $F_{l,k}$ is naturally seen in $\mathbb{CP}^{n-k}$. Then, for any $i = 0, \ldots, n-1$, the zeros of the i-th Alexander polynomial of M (i.e. $\Delta_i^{\varepsilon,\rho}(M)$) are among those of*

$$\prod_{k=0}^{n-1}\prod_{l=1}^{s_k}\det{}_{\varepsilon,\rho}(\gamma_\infty(F_{l,k})).$$

We can see that this result generalizes the one obtained in the line arrangement case. The meridians around the hyperplane at infinity in the line arrangement case are β_k^{-1} ($k = 1, \ldots, s$), which correspond to the 0-dimensional strata (the singular points); and a_i^{-1} ($i = 1, \ldots, m$), which correspond to the 1-dimensional strata.

2.5 Applications

2.5.1 Topology of a hyperplane arrangement complement via twisted Alexander polynomials.

In the following example, we discuss how twisted Alexander polynomials can give us information about the topology of the complement of a line arrangement. In particular, they can be used to distinguish the homeomorphism type of certain line arrangement complements that are homotopy equivalent.

Example 2.40. *Let us consider a pair of line arrangements (the Falk arrangements)* $\mathcal{A}_1$ *and* $\mathcal{A}_2$ *shown in Fig. 1, which are given by the zeros of* $p_1(x,y)$ *and* $p_2(x,y)$ *respectively, where*

$$p_1(x,y) = (x+1)(x-1)(x+y)y(x-y)$$
$$p_2(x,y) = (x+1)(x-1)(y+1)(y-1)(x-y-1).$$

Figure 1: The real part of the Falk Arrangements $\mathcal{A}_1$ and $\mathcal{A}_2$.

In [14]*, Falk showed that the complements of these two arrangements are homotopy equivalent even though they are combinatorially quite different. These two complements are not homeomorphic, as shown by Jiang and Yau in* [20]*. In* [6]*, Cohen and Suciu reproved that the complements are not homeomorphic by showing that the boundary manifolds of* $\mathcal{A}_1$ *and* $\mathcal{A}_2$ *are not homotopy equivalent, which they did by showing that* *their corresponding multivariable Alexander polynomials had a different number of distinct factors.*

We will show that the boundary manifolds of $\mathcal{A}_1$ *and* $\mathcal{A}_2$ *are not homotopically equivalent by showing that certain (one-variable) twisted Alexander polynomials of their boundary manifolds have a different number of distinct roots with multiplicity, thus reproving the result by Jiang and Yau by using simpler invariants.*

Proof. Let B_j be the boundary manifold of $\mathcal{A}_j$, for $j = 1, 2$. Let M_j be the complement in $\mathbb{C}^2$ of the arrangement $\mathcal{A}_j$, for $j = 1, 2$. We will argue by contradiction. Let us assume that there exists a homotopy equivalence

$$h : B_1 \longrightarrow B_2$$

We denote by h_* the map that h induces on fundamental groups. Let $i_2 : B_2 \hookrightarrow M_2$ be the inclusion and $(i_2)_*$ the map it induces on fundamental groups. Let

$$\varepsilon : \pi_1(M_2) \longrightarrow \mathbb{Z}$$

be an epimorphism, and let

$$\rho : \pi_1(M_2) \longrightarrow \mathbb{C}^*$$

be a one dimensional representation. Restricting ourselves to one dimensional representations makes computing twisted Alexander polynomials so much easier, since they factor through the abelianization of $\pi_1(M_2)$ and we do not have to care about the conjugation of meridians due to the braiding in the fundamental group.

We abuse notation and also call ε and ρ the maps induced by pulling back ε and ρ by $(i_2)_*$ and $(i_2)_* \circ h_*$ on $\pi_1(B_2)$ and $\pi_1(B_1)$ respectively. In this setting, since h is a homotopy equivalence, we have that

$$\frac{\Delta_1^{\varepsilon,\rho}(B_1)}{\Delta_0^{\varepsilon,\rho}(B_1)} = \frac{\Delta_1^{\varepsilon,\rho}(B_2)}{\Delta_0^{\varepsilon,\rho}(B_2)}$$

up to multiplication by a unit of $\mathbb{C}[t^{\pm 1}]$. Thus, the set of non-zero roots with multiplicity corresponding to both sides should be the same. We will show that for some choice of ε and ρ, they are not, which will conclude our proof.

Let a_i^j be a meridian around the line given by the i-th factor of $p_j(x,y)$, and let $a_0^j = \left(\prod_{i=1}^5 a_i^j\right)^{-1}$, for $j=1,2$ and $i=1,\ldots,5$. The loop a_0^j is not necessarily a meridian around the line at infinity (due to the order chosen in the multiplication), but will have the same image by ε and ρ than said meridian. We denote by $a^j_{i_1 i_2 i_3} = a^j_{i_1}\cdot a^j_{i_2}\cdot a^j_{i_3}$. Note that, if the lines l_{i_1}, l_{i_2} and l_{i_3} intersect in a triple point, then $a^j_{i_1 i_2 i_3}$ will have the same image by ε and ρ than the corresponding β_k.

We choose $\varepsilon : \pi_1(M_2) \longrightarrow \mathbb{Z}$ to be an epimorphism such that all of the loops involved in the formulas for $\frac{\Delta_1^{\varepsilon,\rho}(B_2)}{\Delta_0^{\varepsilon,\rho}(B_2)}$ and $\frac{\Delta_1^{\varepsilon,\rho}(B_1)}{\Delta_0^{\varepsilon,\rho}(B_1)}$ given by Remark 2.34 have a non-zero image by ε (recall Remark 2.35). This choice of ε depends on h_*, and generically, this condition on ε is satisfied. In that case, we have that

$$\begin{aligned}\frac{\Delta_1^{\varepsilon,\rho}(B_2)}{\Delta_0^{\varepsilon,\rho}(B_2)} &= \left(\prod_{i=1}^4 \rho(a_i^2)t^{\varepsilon(a_i^2)}-1\right)^2\cdot(\rho(a_5^2)t^{\varepsilon(a_5^2)}-1)^3\cdot\\ &\cdot(\rho(a_0^2)t^{\varepsilon(a_0^2)}-1)\cdot(\rho(a_{034}^2)t^{\varepsilon(a_{034}^2)}-1)\cdot(\rho(a_{012}^2)t^{\varepsilon(a_{012}^2)}-1).\end{aligned} \tag{2.8}$$

so, up to multiplication by a unit of $\mathbb{C}[t^{\pm 1}]$,

$$\begin{aligned}\frac{\Delta_1^{\varepsilon,\rho}(B_2)}{\Delta_0^{\varepsilon,\rho}(B_2)} &= \left(\prod_{i=1}^4 \rho(a_i^2)t^{\varepsilon(a_i^2)}-1\right)^2\cdot(\rho(a_5^2)t^{\varepsilon(a_5^2)}-1)^3\cdot\\ &\cdot(\rho(a_0^2)t^{\varepsilon(a_0^2)}-1)\cdot(\rho(a_{125}^2)t^{\varepsilon(a_{125}^2)}-1)\cdot(\rho(a_{345}^2)t^{\varepsilon(a_{345}^2)}-1).\end{aligned} \tag{2.9}$$

Also, we have that

$$\frac{\Delta_1^{\varepsilon,\rho}(B_1)}{\Delta_0^{\varepsilon,\rho}(B_1)} = \left(\prod_{i=0}^{5}\left(\rho(a_i^1)t^{\varepsilon(a_i^1)}-1\right)\right)^2 (\rho(a_{012}^1)t^{\varepsilon(a_{012}^1)}-1)(\rho(a_{345}^1)t^{\varepsilon(a_{345}^1)}-1).$$

Note that, up to multiplication by a unit of $\mathbb{C}[t^{\pm 1}]$, the last two factors are the same, so

$$\frac{\Delta_1^{\varepsilon,\rho}(B_1)}{\Delta_0^{\varepsilon,\rho}(B_1)} = \left(\prod_{i=0}^{5}\left(\rho(a_i^1)t^{\varepsilon(a_i^1)}-1\right)\right)^2 (\rho(a_{345}^1)t^{\varepsilon(a_{345}^1)}-1)^2. \tag{2.10}$$

Now that we have fixed ε, we can choose ρ so that any root of $(\rho(a)t^{\varepsilon(a)}-1)$ is different than any root of $(\rho(b)t^{\varepsilon(b)}-1)$ for different loops a and b involved in the formula (2.9). That way, if we pick a given root of the term $(\rho(a_5^2)t^{\varepsilon(a_5^2)}-1)^3$, we know it only appears 3 times as a root of $\frac{\Delta_1^{\varepsilon,\rho}(B_2)}{\Delta_0^{\varepsilon,\rho}(B_2)}$. On the other hand, no non-zero root of $\frac{\Delta_1^{\varepsilon,\rho}(B_1)}{\Delta_0^{\varepsilon,\rho}(B_1)}$ can have odd multiplicity, so we have reached a contradiction. □

2.5.2 Twisted jump loci vs. twisted Alexander polynomials

Let $\mathcal{A} = \{H_1, \dots, H_m\}$ be an essential hyperplane arrangement in $\mathbb{C}^n$, let $H = \bigcup_{i=1}^{m} H_i$, and let $M = \mathbb{C}^n \backslash H$ be the arrangement complement in $\mathbb{C}^n$. Let $\mathbb{V}$ be an n-dimensional vector space over $\mathbb{C}$, and let

$$\rho : \pi_1(M) \longrightarrow \mathrm{GL}(\mathbb{V})$$

be a representation. We denote by V_ρ the corresponding $\mathbb{V}$-local system on M.

Definition 2.41. *The* ***rank 1 homology jump loci of*** M ***twisted by*** ρ *are defined to be*

$$\mathcal{V}_i^k(M,\rho) = \{\eta \in Hom(\pi_1(M), \mathbb{C}^*) \mid \dim_{\mathbb{C}} H_i(M, \mathcal{L}_\eta \otimes V_\rho) \geq k\}$$

for all $i, k \geq 0$, *where* $\mathcal{L}_\eta$ *is the rank* 1 *local system on* M *defined by* η.

There exists a natural isomorphism

$$(\mathbb{C}^*)^m \xrightarrow{\cong} \operatorname{Hom}(\pi_1(M), \mathbb{C}^*)$$

that takes any tuple $(z_1, \dots, z_m) \in (\mathbb{C}^*)^m$ to the unique morphism that sends the positively oriented meridians around the line H_i to z_i for all $i = 1, \dots, m$. In this way, we can see the homology jump loci $\mathcal{V}_i^k(M, \rho)$ inside of $(\mathbb{C}^*)^m$.

Let $\varepsilon : \pi_1(M) \longrightarrow \mathbb{Z}$ be an epimorphism. It induces the following map

$$\begin{aligned} \varepsilon^* : \mathbb{C}^* \quad &\longrightarrow \quad \operatorname{Hom}(\pi_1(M), \mathbb{C}^*) \\ a \quad &\longmapsto \quad h_a \circ \varepsilon \end{aligned}$$

where $h_a : \mathbb{Z} \longrightarrow \mathbb{C}^*$ is the only group homomorphism taking 1 to a. Since ε is an epimorphism, we have that the image of ε^* is naturally isomorphic to $\mathbb{C}^*$. With this notation, we have the following result that relates the zeros of twisted Alexander polynomials of M and the twisted rank 1 homology jump loci.

Proposition 2.42.

$$\{t \in \mathbb{C}^* \mid \Delta_i^{\varepsilon,\rho}(M)(t) \cdot \Delta_{i-1}^{\varepsilon,\rho}(M)(t) = 0\} = \mathcal{V}_i^1(M, \rho) \cap \operatorname{Im}(\varepsilon^*)$$

for $0 \leq i \leq n-1$*, and*

$$\{t \in \mathbb{C}^* \mid \Delta_{n-1}^{\varepsilon,\rho}(M)(t) = 0\} = \mathcal{V}_n^{(\dim_{\mathbb{C}} \mathbb{V} \cdot |\chi(M)|+1)}(M, \rho) \cap \operatorname{Im}(\varepsilon^*)$$

where $\mathcal{V}_i^k(M, \rho) \cap \operatorname{Im}(\varepsilon^*)$ *is seen as a subset of* $\mathbb{C}^*$*.*

Proof. We will follow the notation in [10, Theorem 4.5], where the non-twisted case is discussed.

Let $a \in \mathbb{C}^*$. The homomorphism $\varepsilon^*(a)$ defines a 1-dimensional local system, which we will call $\mathcal{L}_a$. We consider the following short exact sequence of vector spaces over $\mathbb{C}$:

$$0 \longrightarrow \mathbb{C}[t^{\pm 1}] \xrightarrow{t-a} \mathbb{C}[t^{\pm 1}] \xrightarrow{t=a} \mathbb{C} \longrightarrow 0$$

Tensoring by $\mathbb{V}$, we obtain the following short exact sequence of vector spaces over $\mathbb{C}$:

$$0 \longrightarrow \mathbb{C}[t^{\pm 1}] \otimes_{\mathbb{C}} \mathbb{V} \xrightarrow{f} \mathbb{C}[t^{\pm 1}] \otimes_{\mathbb{C}} \mathbb{V} \xrightarrow{g} \mathbb{C} \otimes_{\mathbb{C}} \mathbb{V} \longrightarrow 0 \tag{2.11}$$

The vector space $\mathbb{C}[t^{\pm 1}] \otimes_{\mathbb{C}} \mathbb{V}$ can be given the structure of a right $\mathbb{C}[\pi_1(M)]$-module, as we described in Definition 2.7. Moreover, $\mathbb{C} \otimes_{\mathbb{C}} \mathbb{V} \cong \mathbb{V}$ can also be given the structure of a right $\mathbb{C}[\pi_1(M)]$-module, with the right action given by

$$v \cdot \alpha = a^{\varepsilon(\alpha)} v \cdot \rho(\alpha)$$

for every $v \in \mathbb{V}$ and $\alpha \in \pi_1(X)$, where v is regarded as a row vector and $\rho(\alpha)$ as a square matrix.

We can check that both f and g respect the right $\mathbb{C}[\pi_1(M)]$-module structure, so the short exact sequence (2.11) is also a short exact sequence of right $\mathbb{C}[\pi_1(M)]$-modules.

Let $\widetilde{M}$ be the universal cover of M. We have that $C_i(\widetilde{M}, \mathbb{C})$ is a free left $\mathbb{C}[\pi_1(M)]$-module for all $i \in \mathbb{Z}$, as explained in Definition 2.7. In particular, it is flat, so we can tensor (2.11) by $C_i(\widetilde{M}, \mathbb{C})$ to get

$$0 \longrightarrow C_i^{\varepsilon,\rho}(M, \mathbb{C}[t^{\pm 1}]) \longrightarrow C_i^{\varepsilon,\rho}(M, \mathbb{C}[t^{\pm 1}]) \longrightarrow \mathbb{V} \otimes_{\mathbb{C}[\pi_1(M)]} C_i(\widetilde{M}, \mathbb{C}) \longrightarrow 0$$

These short exact sequences for $i \in \mathbb{Z}$ extend to a short exact sequence of complexes (i.e. they are compatible with the differentials), so we get the corresponding long exact sequence in homology, namely

$$\begin{aligned} \ldots \longrightarrow H_i^{\varepsilon,\rho}(M,\mathbb{F}[t^{\pm 1}]) \xrightarrow{t-a} H_i^{\varepsilon,\rho}(M,\mathbb{F}[t^{\pm 1}]) \longrightarrow \\ \longrightarrow H_i(M,\mathcal{L}_a\otimes V_\rho) \longrightarrow H_{i-1}^{\varepsilon,\rho}(M,\mathbb{F}[t^{\pm 1}]) \longrightarrow \ldots \end{aligned} \tag{2.12}$$

By Theorem 2.27 and the fact that $\mathbb{C}[t^{\pm 1}]$ is a principal ideal domain, we get that, for $0\leq i\leq n-1$, the twisted Alexander modules have a primary decomposition of the form

$$H_i^{\varepsilon,\rho}(M,\mathbb{F}[t^{\pm 1}]) \cong \mathbb{C}[t^{\pm 1}]/\left((t-b_1)^{r_1}\right)\oplus\ldots\oplus\mathbb{C}[t^{\pm 1}]/\left((t-b_{l_i})^{r_{l_i}}\right).$$

Let $N(a,i)$ be the number of direct summands in the $(t-a)$-torsion part of $H_i^{\varepsilon,\rho}(M,\mathbb{F}[t^{\pm 1}])$. We have that

$$N(a,i)=\dim_{\mathbb{C}}\ker\left(H_i^{\varepsilon,\rho}(M,\mathbb{F}[t^{\pm 1}]) \xrightarrow{t-a} H_i^{\varepsilon,\rho}(M,\mathbb{F}[t^{\pm 1}])\right).$$

Let us consider (2.12) as a long exact sequence of vector spaces. By a dimension counting argument, we deduce that

$$\dim_{\mathbb{C}} H_i(M,\mathcal{L}_a\otimes V_\rho)=N(a,i)+N(a,i-1)$$

for $0\leq i\leq n-1$. Note that a is a zero of $\Delta_l^{\varepsilon,\rho}(M)$ if and only if $N(a,l)\geq 1$. Thus

$$\{t\in\mathbb{C}^*\mid \Delta_i^{\varepsilon,\rho}(M)(t)\cdot\Delta_{i-1}^{\varepsilon,\rho}(M)(t)=0\}=\mathcal{V}_i^1(M,\rho)\cap\operatorname{Im}(\varepsilon^*)$$

for $0\leq i\leq n-1$.

Taking into account that $H_n^{\varepsilon,\rho}(M,\mathbb{F}[t^{\pm 1}])$ is a free $\mathbb{C}[t^{\pm 1}]$-module of dimension $\dim_{\mathbb{C}}\mathbb{V}\cdot|\chi(M)|$ (Theorem 2.27), by a dimension counting argument in the long exact sequence (2.12), we have that

$$\dim_{\mathbb{C}} H_n(M,\mathcal{L}_a\otimes V_\rho)=\dim_{\mathbb{C}}\mathbb{V}\cdot|\chi(M)|+N(a,n-1).$$

Thus,

$$\{t \in \mathbb{C}^* \mid \Delta^{\varepsilon,\rho}_{n-1}(M)(t) = 0\} = \mathcal{V}_n^{(\dim_{\mathbb{C}} \mathbb{V}\cdot|\chi(M)|+1)}(M,\rho) \cap \operatorname{Im}(\varepsilon^*).$$

□

The result that we just proved, along with the main results of Section 2.4, can give us some information about the rank 1 twisted homology jump loci of M. More specifically, the following corollaries follow from Proposition 2.30, Theorem 2.31, Theorem 2.36 and Theorem 2.39 respectively.

Corollary 2.43. *Using the same notation and under the same assumptions of Proposition 2.30, we have that the points of*

$$\mathcal{V}_0^1(M,\rho) \cap \operatorname{Im}(\varepsilon^*)$$

are in one-to-one correspondence with the common roots of all of the dimension $\dim_{\mathbb{C}} \mathbb{V}$ *minors of the column matrix with entries*

$$t^{\varepsilon(a_i)}\rho(a_i) - \operatorname{Id} \in \mathcal{M}_{(\dim_{\mathbb{C}} \mathbb{V})\times(\dim_{\mathbb{C}} \mathbb{V})}(\mathbb{C}[t^{\pm 1}])$$

for $i = 1, \ldots, m$.

Corollary 2.44. *Using the same notation and under the same assumptions of Theorem 2.31, we have that both*

$$\mathcal{V}_1^1(M,\rho) \cap \operatorname{Im}(\varepsilon^*)$$

and

$$\mathcal{V}_2^{(\dim_{\mathbb{C}} \mathbb{V}\cdot|\chi(M)|+1)}(M,\rho) \cap \operatorname{Im}(\varepsilon^*)$$

are contained in

$$\left\{t \in \mathbb{C}^* \mid \left(\prod_{k=1}^{s} \det{}_{\varepsilon,\rho}(\beta_k)^{d_k-2}\right) \cdot \left(\prod_{i=1}^{m} \det{}_{\varepsilon,\rho}(a_i)^{s_i-1}\right) \cdot \Delta_0^{\varepsilon,\rho}(M)(t) = 0\right\}.$$

Corollary 2.45. *Using the same notation and under the same assumptions of Theorem 2.36, we have that both*

$$\mathcal{V}_1^1(M,\rho)\cap \operatorname{Im}(\varepsilon^*)$$

and

$$\mathcal{V}_2^{(\dim_{\mathbb{C}} \mathbb{V}\cdot|\chi(M)|+1)}(M,\rho)\cap \operatorname{Im}(\varepsilon^*)$$

are contained in

$$\bigcap_{i=1}^{l}\left\{t\in\mathbb{C}^* \mid \Delta_0^{\varepsilon,\rho}(M)(t)\cdot\left(\prod_{k=1}^{s_i}\det{}_{\varepsilon,\rho}(\beta_{k,i})^{d_k^i-2}\right)\cdot \det{}_{\varepsilon,\rho}(a_i)^{s_i-1}=0\right\}.$$

Corollary 2.46. *Using the same notation and under the same assumptions of Theorem 2.39, we have that*

$$\mathcal{V}_i^1(M,\rho)\cap \operatorname{Im}(\varepsilon^*)$$

for $0\leq i\leq n-1$, and

$$\mathcal{V}_n^{(\dim_{\mathbb{C}} \mathbb{V}\cdot|\chi(M)|+1)}(M,\rho)\cap \operatorname{Im}(\varepsilon^*)$$

are all contained in

$$\left\{t\in\mathbb{C}^* \mid \prod_{k=0}^{n-1}\prod_{l=1}^{s_k}\det{}_{\varepsilon,\rho}(\gamma_\infty(F_{l,k}))=0\right\}.$$

Chapter 3

Higher Order Degrees of Affine Plane Curve Complements

Based on joint work with Laurentiu Maxim [13].

3.1 Introduction

In knot theory, a strategy to address problems that the Alexander polynomial is not strong enough to solve is to consider non-abelian invariants (e.g., see [3]). These are Alexander-type invariants of coverings corresponding to terms of the derived series of a knot group, and share most of the properties of the classical Alexander invariants. Despite the difficulties of working with modules over non-commutative rings, there are applications to estimating knot genus, detecting fibered, prime and alternating knots, and to knot concordance. Higher order Alexander invariants can be associated to any finitely presented group $G = \pi_1(X)$, in terms of coverings of X given by the terms in the rational derived series of G. These in turn have striking applications if one considers the fundamental group of a link complement or that of a closed 3-manifold [16]. For example, they can be used to obtain lower bounds for the Thurston norm, and provide new algebraic obstructions to a 4-manifold of the form $M^3 \times S^1$ admitting a symplectic structure.

Motivated by their success in classical knot theory and low-dimensional topology, C. Leidy and L. Maxim initiated in [28] the study of higher order Alexander-type invariants for complex affine plane curve complements. The exploration of topology of complex plane curves and of their complements is a subject that goes back to works of Zariski, Enriques, Hirzebruch, Deligne, or Fulton, and which has flourished in more recent research endeavors by Libgober, Dimca, Suciu, Artal-Bartolo, Cogolludo-Agustín, etc. As the fundamental group of a plane curve complement is in general highly non-abelian, one typically considers invariants of the fundamental group that still capture most of the topology of the curve, but which are more manageable, e.g., Alexander-type invariants (see [5], [23], [28], [29], [33], [37], etc.).

To any affine plane curve $C \subset \mathbb{C}^2$, in [28] one associates a sequence $\{\delta_n(C)\}_n$ of (possibly infinite) integers, called the *higher order degrees* of C. Roughly speaking, these integers measure the "sizes" of quotients of successive terms in the rational derived series $\{G_r^{(n)}\}_{n\geq 0}$ of the fundamental group $G := \pi_1(\mathbb{C}^2 \setminus C)$ of the curve complement (see Definition 3.6). It was also noted in [28] that the higher order degrees of plane curves (at any level n) are sensitive to the "position" of singular points, this being one of the initial motivations for adapting and studying Alexander-type invariants in the context of plane curve complements.

While in theory higher order degrees of a plane curve complement can be computed by Fox free calculus from a presentation of $G = \pi_1(\mathbb{C}^2 \setminus C)$, such calculations are in general tedious, see [29] for some examples. Furthermore, as these integers can also be interpreted as Betti-type invariants associated to the tower of coverings of $\mathbb{C}^2 \setminus C$ corresponding to the subgroups $G_r^{(i)}$ (the first of which is the universal abelian cover), a priori there is no reason to expect that such invariants have any good vanishing or finiteness properties. The main result of [28] proved that for curves in general position at infinity (i.e., whose projective completion is transversal to the line at infinity in $\mathbb{C}P^2$) these higher order degrees are in fact *finite*, and a uniform upper bound was given only in terms of the degree m of the curve by comparing with the corresponding invariants of the (m-Hopf) link at infinity. More precisely, one has the following result:

Theorem 3.1. [28, Corollary 4.8] *If $C \subset \mathbb{C}^2$ is a reduced plane curve of degree m, in general position at infinity, then:*

$$\delta_n(C) \leq m(m-2), \quad \textit{for all } n.$$

One of the goals of this chapter is to provide generalizations of Theorem 3.1 to various contexts in which the assumption of good behavior at infinity is relaxed. As the sequence of higher order degrees of a plane curve is an invariant of the fundamental group of the complement, a better understanding of its properties (such as finiteness) will impose new obstructions on the class of groups that can be realized as fundamental groups of affine plane curve complements.

Our first result generalizes Theorem 3.1 to the context of essential complex line arrangements. (Note that such line arrangements are not necessarily in general position at infinity.) We prove the following (see Theorem 3.14):

Theorem 3.2. *Assume that the complex affine plane curve C defines an essential line arrangement $\mathcal{A} = \{L_1, \ldots, L_m\} \subset \mathbb{C}^2$ (that is, not all lines in $\mathcal{A}$ are parallel). Then,*

$$\delta_n(C) \leq m(m-2), \quad \textit{for all } n.$$

Moreover, the equality holds only in the case where $\mathcal{A}$ consists of m lines going through one point, and in that case the equality holds for all $n \geq 0$.

In the special case when an arrangement contains one line which meets all other lines transversally, we show in Theorem 3.21 the following result (which was asserted without a proof in [29]):

Theorem 3.3. *Assume that the affine plane curve C defines a line arrangement $\mathcal{A} = \{L_1, \dots, L_m\} \subset \mathbb{C}^2$, which is obtained from an essential line arrangement $\mathcal{A}' = \{L_1, \dots, L_{m-1}\}$ by adjoining a line L_m that is transversal to every line in $\mathcal{A}'$ (that is, the singularities of the curve C along the irreducible component L_m consist of $m-1$ nodes). Then*

$$\delta_n(C) = 0, \text{ for all } n \geq 0.$$

At the opposite spectrum, i.e., if the plane curve C defines a line arrangement $\mathcal{A} = \{L_1, \dots, L_m\} \subset \mathbb{C}^2$ consisting of m distinct parallel lines, then an easy calculation shows that (see Proposition 3.15):

$$\delta_n(C) = \begin{cases} \infty, & m > 1, \\ 0, & m = 1, \end{cases}$$

for all $n \geq 0$.

We also prove the following generalization of Theorem 3.1 in the context when the plane curve C is allowed to have mild singularities at infinity. More precisely, we show the following (see Theorem 3.24):

Theorem 3.4. *Let $C \subset \mathbb{C}^2$ be a reduced plane curve of degree m, let $\overline{C}$ be its closure in $\mathbb{C}P^2$ and let L_∞ be the line at infinity. Suppose that the intersections of L_∞ and $\overline{C}$ are* *either transversal or* L_∞ *is the tangent line to* $\overline{C}$ *at a smooth point and it is a simple tangent there (i.e., it has multiplicity* 2*). If any of the following two conditions hold*

(a) $m = 2$;

(b) at least one of the intersections of $\overline{C}$ *and* L_∞ *is transversal,*

then

$$\delta_n(C) \leq m(m-2),$$

for all $n \geq 0$.

Moreover, we generalize Theorem 3.3 to the context of plane curves as follows (see Theorem 3.25):

Theorem 3.5. *Let* $n \geq 0$. *Assume that the affine plane curve* C *is of the form* $C = L \cup C'$, *where* C' *is a curve of degree* $m-1$ *in* $\mathbb{C}^2$ *such that* $\delta_n(C')$ *is finite, and* L *is a line transversal to* C' *such that* $L \cap C'$ *consists of* $m-1$ *distinct points. Then,*

$$\delta_n(C) = 0.$$

A natural question in the context of non-commutative Alexander-type invariants is to relate the higher order degrees of a plane curve complement to the previously studied Alexander-type invariants, such as the Alexander polynomials. Preliminary steps in this direction have already been made in [28], were the authors showed that if the (one-variable) Alexander polynomial of an irreducible plane curve is trivial then *all* higher-order degrees $\{\delta_n\}_n$ vanish. (If the curve is irreducible, then $\delta_0(C)$ is the degree of the Alexander polynomial of C.) In relation with the universal abelian invariants of a curve, it was also noted in [28] that if the codimension (in the character torus) of the first characteristic variety of the plane curve complement is > 1 then $\delta_0(C) = 0$ (this fact was first pointed out by A. Libgober in an informal conversation with the second author, see also Corollary 3.47). However, curves (e.g., in general position at infinity) may have supports of codimension one in the character variety (cf. [25]), and for this boundary case we show here that $\delta_0(C)$ is the degree of the multivariable Alexander polynomial Δ_C of the plane curve C (see Theorem 3.45).

The chapter is structured as follows. In Section 3.2, we recall the definition of higher order degrees of an affine plane curve complement. Section 3.3 supplies proofs of Theorems 3.2 and 3.3. Theorems 3.4 and 3.5 are proved in Section 3.4. Finally, in Section 3.5, we indicate the relation between $\delta_0(C)$ and the degree of the multivariable Alexander polynomial of the plane curve complement in the case when C is not irreducible.

3.2 Higher-order invariants of a plane curve complement

Though most of the background material presented in this section applies to any finitely presented group, we focus mainly on fundamental groups of complex affine plane curve complements.

Let $C = \{f(x, y) = 0\}$ be a reduced curve in $\mathbb{C}^2$ of degree m, with complement

$$U := \mathbb{C}^2 \setminus C,$$

and denote by $G := \pi_1(U)$ the fundamental group of its complement. If C has s irreducible components, then

$$H_1(G; \mathbb{Z}) = H_1(U; \mathbb{Z}) = G/G' = \mathbb{Z}^s, \tag{3.1}$$

generated by meridian loops about the smooth parts of the irreducible components of C.

In this section we recall the definition of the higher-order Alexander-type invariants of the group G. These were originally used in the study of knots and, respectively, 3-manifolds, see e.g., [3,16], and they were ported to the study of plane curve complements in [28], e.g., to show that certain groups cannot be realized as fundamental groups of such complements.

Definition 3.6. *The rational derived series of the group G is defined as follows: $G_r^{(0)} = G$, and for $n \geq 1$,*

$$G_r^{(n)} = \{g \in G_r^{(n-1)} \mid g^k \in [G_r^{(n-1)}, G_r^{(n-1)}], \text{ for some } k \in \mathbb{Z} \setminus \{0\}\}.$$

It is easy to see that $G_r^{(i)} \lhd G_r^{(j)} \lhd G$, if $i \geq j \geq 0$. The successive quotients of the rational derived series are torsion-free abelian groups. In fact (cf. [16, Lemma 3.5]),

$$G_r^{(n)}/G_r^{(n+1)} \cong \left(G_r^{(n)}/[G_r^{(n)}, G_r^{(n)}]\right)/\{\mathbb{Z} - \text{torsion}\}.$$

Therefore, for $G = \pi_1(\mathbb{C}^2 \setminus C)$ we get from (3.1) that $G' = G_r^{(1)}$.

The use of the rational derived series as opposed to the usual derived series is needed in order to avoid zero-divisors in the group ring $\mathbb{Z}\Gamma_n$, where

$$\Gamma_n := G/G_r^{(n+1)}.$$

By construction, Γ_n is a poly-torsion-free-abelian group, in short a PTFA ([16, Corollary 3.6]), i.e., it admits a normal series of subgroups such that each of the successive quotients of the series is torsion-free abelian. Then $\mathbb{Z}\Gamma_n$ is a right and left Ore domain, so it embeds in its classical right ring of quotients $\mathcal{K}_n$, a skew-field. Every module over $\mathcal{K}_n$ is a free module, and such modules have a well-defined rank $\mathrm{rk}_{\mathcal{K}_n}$ which is additive on short exact sequences. (These statements also apply to the right ring of quotients $\mathcal{K}$ of the group ring $\mathbb{Z}\Gamma$ of any PTFA group Γ, e.g., see [28, Remark 2.4] and the references therein.)

Definition 3.7. The *n-th order Alexander module* of (the complement of) the plane curve C is defined as

$$\mathcal{A}_n^{\mathbb{Z}}(C) = H_1(U; \mathbb{Z}\Gamma_n) = H_1(U_{\Gamma_n}; \mathbb{Z}),$$

where U_{Γ_n} is the covering of U corresponding to the subgroup $G_r^{(n+1)}$. That is,

$$\mathcal{A}_n^{\mathbb{Z}}(C) = G_r^{(n+1)}/[G_r^{(n+1)}, G_r^{(n+1)}],$$

viewed as a right $\mathbb{Z}\Gamma_n$-module.

The *n-th order rank* of (the complement of) C is:

$$r_n(C) = \mathrm{rk}_{\mathcal{K}_n} H_1(U; \mathcal{K}_n).$$

Remark 3.8. Note that $\mathcal{A}_0^{\mathbb{Z}}(C) = G_r^{(1)}/[G_r^{(1)}, G_r^{(1)}] = G'/G''$, which is usually referred to as the Alexander invariant of the complement (see also Definition 3.35).

Example 3.9. *If the curve C is in general position at infinity (i.e., the line at infinity in $\mathbb{C}P^2$ is transversal to the projective completion of C), and it is nonsingular or has only nodal singular points (i.e., locally defined by $x^2 - y^2 = 0$), then $G = \pi_1(\mathbb{C}^2 \setminus C)$ is abelian, and therefore $\mathcal{A}_n^{\mathbb{Z}}(C) = 0$ for all n (e.g., see* [28, Remark 3.4]*).*

In [28], one associates to any plane curve C (or, equivalently, to the fundamental group G of its complement) a sequence of non-negative integers $\delta_n(C)$ as follows (it is more convenient to work over a principal ideal domain, or a PID for short, so we look for a "convenient" one): Let $\psi \in H^1(G;\mathbb{Z})$ be the primitive class representing the linking number homomorphism

$$G \xrightarrow{\psi} \mathbb{Z},\ \alpha \mapsto \mathrm{lk}(\alpha, C).$$

Since G' is in the kernel of ψ, we have a well-defined induced epimorphism $\bar{\psi} : \Gamma_n \to \mathbb{Z}$. Let $\bar{\Gamma}_n = \ker \bar{\psi}$. Then $\bar{\Gamma}_n$ is a PTFA group, so $\mathbb{Z}\bar{\Gamma}_n$ has a right ring of quotients

$$\mathbb{K}_n = (\mathbb{Z}\bar{\Gamma}_n)S_n^{-1},$$

where $S_n = \mathbb{Z}\bar{\Gamma}_n \setminus \{0\}$. Set

$$R_n := (\mathbb{Z}\Gamma_n)S_n^{-1}.$$

Then R_n is a flat left $\mathbb{Z}\Gamma_n$-module.

A very important role in what follows is played by the fact that R_n is a PID; in fact, R_n isomorphic to the ring of skew-Laurent polynomials $\mathbb{K}_n[t^{\pm 1}]$. This can be seen as follows: by choosing a $t \in \Gamma_n$ such that $\bar{\psi}(t) = 1$, we get a splitting ϕ of $\bar{\psi}$, and the embedding $\mathbb{Z}\bar{\Gamma}_n \subset \mathbb{K}_n$ extends to an isomorphism $R_n \cong \mathbb{K}_n[t^{\pm 1}]$. However this isomorphism depends in general on the choice of splitting of $\bar{\psi}$.

Definition 3.10. (1) The *n-th order localized Alexander module* of the plane curve C is defined to be

$$\mathcal{A}_n(C) = H_1(U; R_n),$$

viewed as a right R_n-module. If we choose a splitting ϕ to identify R_n with $\mathbb{K}_n[t^{\pm 1}]$, we define $\mathcal{A}_n^{\phi}(C) = H_1(U; \mathbb{K}_n[t^{\pm 1}])$.

(2) The *n-th order degree of* C is defined to be:

$$\delta_n(C) = \mathrm{rk}_{\mathbb{K}_n} \mathcal{A}_n(C) = \mathrm{rk}_{\mathbb{K}_n} \mathcal{A}_n^{\phi}(C).$$

Remark 3.11. Note that $\delta_n(C) < \infty$ if and only if $\mathrm{rk}_{\mathcal{K}_n} H_1(U; \mathcal{K}_n) = 0$, i.e. $\mathcal{A}_n(C)$ is a torsion R_n-module.

Remark 3.12. If the plane curve C is irreducible, then $\delta_0(C)$ is the degree of the Alexander polynomial of C; see [28, Remark 3.9].

The higher order degrees $\delta_n(C)$ are integer invariants of the fundamental group G of the complement (endowed with the linking number homomorphism). Indeed, by [16], one has:

$$\delta_n(C) = \mathrm{rk}_{\mathbb{K}_n} \left(G_r^{(n+1)} / [G_r^{(n+1)}, G_r^{(n+1)}] \otimes_{\mathbb{Z}\bar{\Gamma}_n} \mathbb{K}_n \right). \tag{3.2}$$

In fact, the use of the linking number homomorphism makes these invariants depend on the pair $(\mathbb{C}^2, C)$, rather than on the curve complement U. Note that since the isomorphism between R_n and $\mathbb{K}_n[t^{\pm 1}]$ depends on the choice of splitting, one cannot define in a canonical way a higher-order version of the Alexander polynomial. However, for any choice of splitting, the degree of the associated higher-order Alexander polynomial is the same, hence this yields a well-defined invariant of the group G endowed with the linking number homomorphism, which is exactly the higher-order degree δ_n defined above.

The higher-order degrees of C may be computed by means of Fox free calculus from a presentation of $G = \pi_1(\mathbb{C}^2 \setminus C)$, see [16, Section 6] for details. Such computational techniques will be used freely in this chapter.

It was shown in [28] that if C is an irreducible plane curve, or a curve in general position at infinity (i.e., for which the line at infinity in $\mathbb{C}P^2$ is transversal to the projective completion of C), then the higher-order degrees $\delta_n(C)$ are finite. More precisely, one has the following:

Theorem 3.13. *If $C \subset \mathbb{C}^2$ is a reduced plane curve of degree m, in general position at infinity, then:*

$$\delta_n(C) \leq m(m-2), \quad \textit{for all } n.$$

In particular, the n-th order Alexander module $\mathcal{A}_n^{\mathbb{Z}}(C)$ is a torsion $\mathbb{Z}\Gamma_n$-module, for all n.

One of the goals of this chapter is to provide generalizations of Theorem 3.13 to various contexts in which the assumption of good behavior at infinity is relaxed.

3.3 Complex line arrangements

Our first result, Theorem 3.14 below, generalizes Theorem 3.13 to the context of essential complex line arrangements. In Theorem 3.21 we study the special class of arrangements containing a line with only nodal singularities.

Assume that all irreducible components of the reduced plane curve C are complex lines, i.e., the defining polynomial $f = \prod_{i=1}^{m} \ell_i$ factorizes into a product of affine forms $\ell_i : \mathbb{C}^2 \to \mathbb{C}$, $i = 1, \ldots, m$. Let

$$L_i := \ker(\ell_i),$$

and let

$$\mathcal{A} := \{L_1, \ldots, L_m\} \subset \mathbb{C}^2$$

be the corresponding complex line arrangement, with complement U. As before, we will use the notation $\delta_n(C)$ for the higher-order degrees of the complement $U := \mathbb{C}^2 \setminus C = \mathbb{C}^2 \setminus \mathcal{A}$.

3.3.1 Upper bounds on higher-order degrees

In this section, we prove the following generalization of Theorem 3.13 to the context of essential complex line arrangements.

Theorem 3.14. *Assume that the complex affine plane curve C defines an essential line arrangement $\mathcal{A} = \{L_1, \dots, L_m\} \subset \mathbb{C}^2$ (that is, not all lines in $\mathcal{A}$ are parallel). Then,*

$$\delta_n(C) \leq m(m-2), \quad \textit{for all } n.$$

Moreover, the equality holds only in the case where $\mathcal{A}$ consists of m lines going through one point, and in that case the equality holds for all $n \geq 0$.

At the opposite spectrum (i.e., if the essentiality assumption is dropped), we have the following:

Proposition 3.15. *If the plane curve C defines a line arrangement $\mathcal{A} = \{L_1, \dots, L_m\} \subset \mathbb{C}^2$ consisting of m distinct parallel lines, then:*

$$\delta_n(C) = \begin{cases} \infty, & m > 1, \\ 0, & m = 1, \end{cases}$$

for all $n \geq 0$.

Proof. In this case, $\mathbb{C}^2 \setminus C$ is homotopy equivalent to a wedge sum of m circles. If $m = 1$, we have that $\pi_1(\mathbb{C}^2 \setminus C) \cong \mathbb{Z}$, so it is abelian. It then follows from (3.2) that

$$\delta_n(C) = 0, \text{ for all } n \geq 0.$$

Suppose now that $m > 1$. The chain complex computing $H_*(\mathbb{C}^2 \setminus C; R_n)$ looks like

$$\cdots \to 0 \to (R_n)^m \to R_n \to 0$$

Hence, $H_1(\mathbb{C}^2 \setminus C; R_n)$ is a non-zero free right R_n-module, so

$$\delta_n(C) = \infty, \text{ for all } n \geq 0.$$

□

Theorem 3.14 is a consequence of the following two preparatory lemmas (Lemma 3.16 and Lemma 3.19). In Lemma 3.16 we consider the case when there is a line in the arrangement which has no singularities at infinity, whereas in Lemma 3.19 every line is assumed to have singularities at infinity.

Lemma 3.16. *In the notations of Theorem 3.14, assume that there exists a line in $\mathcal{A}$ such that no other line in $\mathcal{A}$ is parallel to it. Then,*

$$\delta_n(C) \leq m(m-2), \quad \textit{for all } n.$$

Moreover, the equality is achieved only in the case when C consists of m lines going through one point, and in that case the equality holds for all $n \geq 0$.

Proof. Reordering, we can assume that L_1 is not parallel to any other line in $\mathcal{A}$. Let $P_1, \ldots, P_r$ be the singular points of C in L_1. Let

$$F = L_1 \setminus \bigsqcup_{i=1}^{r} (L_1 \cap \mathbb{B}_i^4)$$

be the (real) surface obtained by removing small balls $\mathbb{B}_i^4 \subset \mathbb{C}^2$ around the singular points P_i. Hence F is obtained from L_1 by removing a 2-dimensional open disk D_i around every singular point P_i.

Let

$$N = F \times S^1.$$

Here N should be thought of as the boundary of a tubular neighborhood around the non-singular part of L_1. We have that $\partial N = \partial F \times S^1$, and since ∂F is a union of disjoint S^1's (one from every disk D_i removed), then ∂N is a union of disjoint tori $\bigsqcup_{i=1}^{r} T_i$ (again, one from every disk D_i removed). Let us fix a point Q_i in the circle S^1 corresponding to the boundary of the disk D_i for every such disk removed.

Let d_i be the number of lines in $\mathcal{A}$ going through the singular point P_i, $i = 1, \ldots, r$. Let K_i be the link of the singularity at the point P_i (hence K_i is a Hopf link with d_i components), and let S_i^3 be the boundary of $\mathbb{B}_i^4$. We consider the space

$$X = N \cup_{\left(\bigsqcup_{i=1}^{r} T_i\right)} \left(\bigsqcup_{i=1}^{r} S_i^3 \setminus K_i \right) \subset \mathbb{C}^2 \setminus C,$$

with the neighborhood N assumed small enough, where the gluing is done as follows: a meridian around the component of K_i corresponding to the line L_1 is glued to $\{Q_i\} \times S^1 \subset N$, and a longitude of the component of K_i corresponding to L_1 is glued to the S^1 corresponding to the boundary of D_i.

The homology of the space X with R_n-coefficients can be computed from the corresponding Mayer-Vietoris sequence:

$$\begin{aligned} \cdots \to H_2(N; R_n) \oplus \left(\bigoplus_{i=1}^{r} H_2(S_i^3 \backslash K_i; R_n) \right) \to H_2(X; R_n) \xrightarrow{\alpha} \\ \xrightarrow{\alpha} \bigoplus_{i=1}^{r} H_1(T_i; R_n) \to H_1(N; R_n) \oplus \left(\bigoplus_{i=1}^{r} H_1(S_i^3 \backslash K_i; R_n) \right) \to H_1(X; R_n) \to \\ \to \bigoplus_{i=1}^{r} H_0(T_i; R_n) \to H_0(N; R_n) \oplus \left(\bigoplus_{i=1}^{r} H_0(S_i^3 \backslash K_i; R_n) \right) \to H_0(X; R_n) \to 0 \end{aligned} \tag{3.3}$$

Hence, using the additivity of the rank of $\mathbb{K}_n$-modules (recall that $\mathbb{K}_n$ is the right ring of quotients of the Ore domain $\mathbb{Z}\bar{\Gamma}_n$), we have that:

$$\begin{aligned}\operatorname{rk}_{\mathbb{K}_n} H_1(X;R_n) = \operatorname{rk}_{\mathbb{K}_n} H_1(N;R_n) + \sum_{i=1}^{r} \operatorname{rk}_{\mathbb{K}_n} H_1(S_i^3\backslash K_i;R_n) - \sum_{i=1}^{r} \operatorname{rk}_{\mathbb{K}_n} H_1(T_i;R_n)+ \\ +\operatorname{rk}_{\mathbb{K}_n} \operatorname{Im}(\alpha) + \sum_{i=1}^{r} \operatorname{rk}_{\mathbb{K}_n} H_0(T_i;R_n) - \operatorname{rk}_{\mathbb{K}_n} H_0(N;R_n)- \\ - \sum_{i=1}^{r} \operatorname{rk}_{\mathbb{K}_n} H_0(S_i^3\backslash K_i;R_n) + \operatorname{rk}_{\mathbb{K}_n} H_0(X;R_n).\end{aligned} \tag{3.4}$$

Abusing notation, for any $i = 1, \dots, r$ we denote by $\psi : \pi_1(S_i^3\backslash K_i) \longrightarrow \mathbb{Z}$ the (local) linking number homomorphism induced by $\psi : \pi_1(\mathbb{C}^2\backslash C) \longrightarrow \mathbb{Z}$. Then the infinite cyclic cover of $S_i^3\backslash K_i$ induced by the homomorphism ψ is homeomorphic to $F_i \times \mathbb{R}$, where F_i is the Milnor fiber corresponding to the singular point P_i. The Γ_n-cover of $S_i^3\backslash K_i$ factors through this infinite cyclic cover, so we have the following isomorphism of $\mathbb{K}_n$-modules (e.g., see [21, Section 2.1])

$$H_j(S_i^3\backslash K_i;R_n) \cong H_j(F_i;\mathbb{K}_n), \quad \text{for all } j \geq 0. \tag{3.5}$$

The Milnor fiber F_i has the homotopy type of a wedge sum of μ_i circles, where μ_i is the Milnor number associated to the singular point P_i. Together with (3.5), this yields that $H_2(S_i^3\backslash K_i;R_n) = 0$. Moreover, since the singularity P_i consists of the intersection of d_i lines, one has

$$\mu_i = (d_i - 1)^2$$

and hence, since the Euler characteristic with coefficients on a 1-dimensional local system over a skew field does not depend on the local system, we have that

$$\begin{gathered}\operatorname{rk}_{\mathbb{K}_n} H_2(S_i^3\backslash K_i;R_n) = 0 \\ \operatorname{rk}_{\mathbb{K}_n} H_0(S_i^3\backslash K_i;R_n) - \operatorname{rk}_{\mathbb{K}_n} H_1(S_i^3\backslash K_i;R_n) = \chi(F_i) = 1 - (d_i - 1)^2,\end{gathered} \tag{3.6}$$

for all $i = 1, \dots, r$.

Similarly, abusing notation again, we denote by $\psi : \pi_1(N) \longrightarrow \mathbb{Z}$ the homomorphism induced by the linking number homomorphism $\psi : \pi_1(\mathbb{C}^2 \backslash C) \longrightarrow \mathbb{Z}$. Recall that $N = F \times S^1$, and F is homotopy equivalent to a wedge sum of r circles. From this, we see that the infinite cyclic cover of N associated to ψ is homeomorphic to $F \times \mathbb{R}$, so it is homotopy equivalent to F. Since the Γ_n-cover of N factors through this infinite cyclic cover, we get as before that

$$H_j(N; R_n) \cong H_j(F; \mathbb{K}_n), \quad \text{for all } j \geq 0,$$

and hence, we have that

$$\begin{aligned} \mathrm{rk}_{\mathbb{K}_n} H_2(N; R_n) = 0, \\ \mathrm{rk}_{\mathbb{K}_n} H_0(N; R_n) - \mathrm{rk}_{\mathbb{K}_n} H_1(N; R_n) = \chi(F) = 1 - r. \end{aligned} \tag{3.7}$$

Similarly, the Γ_n-cover of the torus T_i factors through the infinite cyclic cover of T_i corresponding to the homomorphism induced by the linking number homomorphism ψ, and this infinite cyclic cover is homeomorphic to $S^1 \times \mathbb{R}$, hence homotopy equivalent to S^1. Consequently, we have that

$$H_j(T_i; R_n) \cong H_j(S^1; \mathbb{K}_n), \quad \text{for all } j \geq 0, i = 1, \ldots, r, \tag{3.8}$$

and hence

$$\mathrm{rk}_{\mathbb{K}_n} H_0(T_i; R_n) - \mathrm{rk}_{\mathbb{K}_n} H_1(T_i; R_n) = \chi(S^1) = 0 \tag{3.9}$$

for all $i = 1, \ldots, r$.

Note that the above calculation (more precisely, the vanishing of the second homology of N and $S^3_i \backslash K_i$) also implies that that the map α in (3.3) is injective. Thus,

$$\mathrm{rk}_{\mathbb{K}_n} \mathrm{Im}(\alpha) = \mathrm{rk}_{\mathbb{K}_n} H_2(X; R_n).$$

Since X has the homotopy type of a 2-dimensional CW-complex (this can be seen from the way X is constructed), we have that $H_2(X; R_n)$ is a free (right) R_n-module. Thus, $\mathrm{rk}_{\mathbb{K}_n} H_2(X; R_n)$ is either 0 or infinite. But

$$\mathrm{rk}_{\mathbb{K}_n}\mathrm{Im}(\alpha) \leq \sum_{i=1}^{r} \mathrm{rk}_{\mathbb{K}_n} H_1(T_i; R_n),$$

and the right hand side of this inequality is a finite number (by (3.8)). Thus,

$$\mathrm{rk}_{\mathbb{K}_n}\mathrm{Im}(\alpha) = 0. \tag{3.10}$$

Finally, we show that

$$\mathrm{rk}_{\mathbb{K}_n} H_0(X; R_n) = 0 \tag{3.11}$$

by using Fox Calculus (e.g., see [16, Section 6]). Since $\mathcal{A}$ is an essential line arrangement, we have that $m \geq 2$. Let $\gamma_1, \ldots, \gamma_m \in \pi_1(X)$ be positively oriented meridians around $L_1, \ldots, L_m$, respectively. We fix a presentation of $\pi_1(X)$ with $\{\gamma_1, \ldots, \gamma_m\}$ as the first m generators. The complex of right R_n-modules that computes $H_1(X; R_n)$ using this fixed presentation is

$$\cdots \xrightarrow{\partial_2} (R_n)^l \xrightarrow{\partial_1} R_n \xrightarrow{\partial_0} 0,$$

where $l \geq m$, and ∂_1 is given by the row matrix $\overline{A}$, with

$$A = \begin{pmatrix} \gamma_1 - 1 & \gamma_2 - 1 & \cdots & \gamma_m - 1 & \cdots \end{pmatrix}.$$

Here $\overline{A}$ denotes the matrix obtained from A by taking the involution $\overline{\cdot}$ of all of its entries, and the involution in $\mathbb{Z}[\Gamma_n]$ is given by

$$\overline{\sum_\lambda n_\lambda g_\lambda} = \sum_\lambda n_\lambda g_\lambda^{-1}.$$

(The involution is needed here since we are dealing with a complex of right R_n-modules, as opposed to the usual formulation of Fox Calculus, where one works with left modules.) Hence,

$$\overline{A} = \begin{pmatrix} \gamma_1^{-1} - 1 & \gamma_2^{-1} - 1 & \cdots & \gamma_m^{-1} - 1 & \cdots \end{pmatrix}.$$

Let $e_1, \ldots e_l$ be the canonical basis in $(R_n)^l$. We have that

$$\partial_1((e_1 - e_2)\gamma_1) = 1 - \gamma_2^{-1}\gamma_1,$$

which is a unit in R_n for all n, since $\gamma_2^{-1}\gamma_1 \in \bar{\Gamma}_n$ corresponds to a non-zero element in $\Gamma_0 = H_1(\mathbb{C}^2 \setminus C; \mathbb{Z})$. Hence ∂_1 is surjective, so

$$\mathrm{rk}_{\mathbb{K}_n} H_0(X; R_n) = 0,$$

as desired.

Substituting (3.6), (3.7), (3.9), (3.10) and (3.11) in equation (3.4), we get that

$$\mathrm{rk}_{\mathbb{K}_n} H_1(X; R_n) = \sum_{i=1}^{r} \left((d_i - 1)^2 - 1\right) + r - 1 = \sum_{i=1}^{r} (d_i - 1)^2 - 1. \tag{3.12}$$

The next step in our proof is to relate $\mathrm{rk}_{\mathbb{K}_n} H_1(X; R_n)$ to $\delta_n(C)$. Since L_1 is not parallel to any other line in the arrangement $\mathcal{A}$, the inclusion map $X \hookrightarrow \mathbb{C}^2 \setminus C$ induces an epimorphism

$$\pi_1(X) \twoheadrightarrow \pi_1(\mathbb{C}^2 \setminus C).$$

This can be seen as follows. Let T be a tubular neighborhood of L_1 in $\mathbb{C}^2$ such that X is a deformation retract of $T \backslash (T \cap C)$. Since L_1 is not parallel to any other line in the arrangement, there exists a generic line L (a line transversal to every other line in the arrangement) such that all of the intersections with lines in the arrangement happen in the interior of T. Thus, the map

$$\pi_1((\mathbb{C}^2\backslash C)\cap L\cap T)\longrightarrow\pi_1((\mathbb{C}^2\backslash C)\cap L)$$

induced by inclusion is an epimorphism, as one can see a set of generators of $\pi_1((\mathbb{C}^2\backslash C)\cap L)$ inside of $(\mathbb{C}^2\backslash C)\cap L\cap T$. By a Zariski theorem of Lefschetz type ([8, Theorem 6.5, Chapter 1]), we have that the map induced by inclusion

$$\pi_1((\mathbb{C}^2\backslash C)\cap L)\longrightarrow\pi_1(\mathbb{C}^2\backslash C)$$

is an epimorphism. Then the following commutative diagram yields that $\pi_1(X)\longrightarrow\pi_1(\mathbb{C}^2\backslash C)$ is an epimorphism, where all the arrows in the diagram are induced by inclusion maps.

$$\begin{tikzcd}
\pi_1(X) \arrow[d] \arrow[dr] & \\
\pi_1(T\backslash(T\cap C)) \arrow[r] & \pi_1(\mathbb{C}^2\backslash\mathbb{C}) \\
\pi_1((\mathbb{C}^2\backslash C)\cap L\cap T) \arrow[u] \arrow[r, two heads] & \pi_1((\mathbb{C}^2\backslash C)\cap L) \arrow[u, two heads]
\end{tikzcd}$$

We have thus shown that $\pi_1(X)\longrightarrow\pi_1(\mathbb{C}^2\setminus C)$ is an epimorphism. This implies (as in the proof of [28, Theorem 4.1]) that there is an R_n-module epimorphism

$$H_1(X;R_n)\twoheadrightarrow H_1(\mathbb{C}^2\setminus C;R_n),$$

and hence

$$\delta_n(C)\leq\mathrm{rk}_{\mathbb{K}_n}H_1(X;R_n)=\sum_{i=1}^{r}(d_i-1)^2-1. \tag{3.13}$$

Since L_1 is not parallel to any other line in the arrangement, we have that

$$\sum_{i=1}^{r}(d_i-1)=m-1.$$

Furthermore,

$$d_i - 1 \leq m - 1 \quad \text{for all } i = 1, \ldots, r,$$

where the equality is only satisfied in the case where $\mathcal{A}$ consists of m lines going through a single point. Altogether,

$$\begin{aligned} \delta_n(C) &\leq \sum_{i=1}^{r} (d_i - 1)^2 - 1 \leq (m-1) \cdot \left(\sum_{i=1}^{r} (d_i - 1) \right) - 1 \\ &= (m-1)^2 - 1 = m(m-2), \end{aligned}$$

where the second inequality can only be an equality in the case where $\mathcal{A}$ consists of m lines going through a single point. In fact, if $\mathcal{A}$ consists of m lines going through a single point, then X is a deformation retract of $\mathbb{C}^2 \setminus C$, so in that case the first inequality is also an equality (since (3.13) becomes an equality) and $\delta_n(C) = m(m-2)$ for all n. (An alternative proof of the fact that $\delta_n(C) = m(m-2)$ in the case when the arrangement consists of m lines passing through a point was given in [36] by using Fox Calculus.) □

Remark 3.17. *In concrete examples, one can use (3.13) to get a better (combinatorial) upper bound for $\delta_n(C)$. Moreover, if there are several lines in $\mathcal{A}$ such that no other line in $\mathcal{A}$ is parallel to them, we can take the tubes around each of those lines to get different bounds for $\delta_n(C)$ similar to (3.13), and then take the minimum of all of these bounds.*

Example 3.18. *Consider the line arrangement of m lines given by $m-1$ parallel lines $L_2, \ldots, L_m$ and a line L_1 transversal to all of them. In this case, the tube X around L_1 is homotopy equivalent to the arrangement complement, so by (3.12) we have that*

$$\delta_n(C) = rk_{\mathbb{K}_n} H_1(X; R_n) = m - 2$$

for all n.

In view of Lemma 3.16, the following result completes the proof of Theorem 3.14.

Lemma 3.19. *In the notations of Theorem 3.14, assume that for every line in $\mathcal{A}$ there exists a different line in $\mathcal{A}$ that is parallel to it. Then,*

$$\delta_n(C) \leq (m-2)(m-1) - 1, \quad \textit{for all } n.$$

In particular,

$$\delta_n(C) \leq m(m-2), \quad \textit{for all } n.$$

Proof. Reordering, we can assume that the lines $L_1, \ldots, L_k$ are all parallel, with L_j not parallel to L_1 for all $k+1 \leq j \leq m$, and $k \geq 2$.

Let $\overline{L_1}$ be the closure of L_1 in $\mathbb{C}P^2$, and let T be a tubular neighborhood of $\overline{L_1}$ in $\mathbb{C}P^2$ with boundary ∂T, constructed so that $\partial T \setminus (C \cup L_\infty)$ is the space

$$X_\infty = N \cup_{\left(\left(\bigsqcup_{i=1}^{r} T_i\right) \sqcup T_\infty\right)} \left(\left(\bigsqcup_{i=1}^{r} S_i^3 \backslash K_i \right) \sqcup (S_\infty^3 \backslash K_\infty) \right) \subset \mathbb{C}^2 \setminus C$$

defined similarly as the space X from the proof of Lemma 3.16. Here, $L_\infty \subset \mathbb{C}P^2$ is the line at infinity, S_∞^3 is a 3-sphere centered at the intersection point P_∞ of $\overline{L_1}$ with the line at infinity, K_∞ is the link of P_∞, and T_∞ is the torus along which we glue N to $S_\infty^3 \backslash K_\infty$.

By construction, X_∞ is a deformation retract of $T \backslash (C \cup L_\infty)$, and by a similar argument using a Zariski theorem of Lefschetz type (like in the proof of Lemma 3.16), we get that the inclusion map $X_\infty \hookrightarrow \mathbb{C}^2 \backslash C$ induces an epimorphism

$$\pi_1(X_\infty) \twoheadrightarrow \pi_1(\mathbb{C}^2 \setminus C),$$

which in turn implies that

$$\delta_n(C) \leq \operatorname{rk}_{\mathbb{K}_n} H_1(X_\infty; R_n). \tag{3.14}$$

To compute $\operatorname{rk}_{\mathbb{K}_n} H_1(X_\infty; R_n)$, we follow the same steps as in the proof of Lemma 3.16, based on a Mayer-Vietoris argument. The only difference will appear when computing $H_j(S^3_\infty \backslash K_\infty; R_n)$ for $j = 0, 1, 2$, since the linking number homomorphism ψ satisfies that $\psi(\gamma_\infty) = -m$, where γ_∞ is a positively oriented meridian around the line at infinity.

Following the same computation as in the proof of (3.11), we get that

$$H_0(S^3_\infty \backslash K_\infty; R_n) = 0 \tag{3.15}$$

To compute $\operatorname{rk}_{\mathbb{K}_n} H_2(S^3_\infty \backslash K_\infty; R_n)$ and $\operatorname{rk}_{\mathbb{K}_n} H_1(S^3_\infty \backslash K_\infty; R_n)$, we will use Fox Calculus, since we cannot relate these groups to the homology of a Milnor fiber. For this, we first need to find a nice presentation of $\pi_1(S^3_\infty \backslash K_\infty)$.

By the choices made in the first paragraph of our proof, K_∞ is the Hopf link on $k+1$ components, with $k \geq 2$. A presentation of $\pi_1(S^3_\infty \backslash K_\infty)$ is given by (e.g., see [30, Lemma 2.7])

$$\pi_1(S^3_\infty \backslash K_\infty) \quad = \quad \langle \gamma_1, \gamma_2, \dots, \gamma_k, y \mid \gamma_i y \gamma_i^{-1} y^{-1} \text{ for all } i = 1, \dots, k \rangle,$$

where $\gamma_1, \gamma_2, \dots, \gamma_k$ are positively oriented meridians around $L_1, \dots, L_k$ respectively. An equivalent presentation of $\pi_1(S^3_\infty \backslash K_\infty)$ can be given so that y is the product of meridian loops $\gamma_1, \gamma_2, \dots, \gamma_k$, γ_∞ (in a certain order that is not important here), e.g., see [30, Remark 2.8]. In particular, from this second presentation we get that

$$\psi(y) = k - m.$$

For simplicity, let $a_1 = \gamma_1$, and $a_j = \gamma_j \gamma_1^{-1}$ for all $j = 2, \dots, k$. Then, we get

$$\pi_1(S^3_\infty \backslash K_\infty) = \langle a_1, a_2, \dots, a_k, y \mid a_i y a_i^{-1} y^{-1} \text{ for all } i = 1, \dots, k \rangle, \tag{3.16}$$

with

$$\begin{aligned}&\psi(a_1) = 1,\\ &\psi(a_j) = 0, \text{ for all } j = 2, \ldots, k,\\ &\psi(y) = k - m.\end{aligned}$$

We compute $H_1(S^3_\infty \backslash K_\infty; R_n)$ and $H_2(S^3_\infty \backslash K_\infty; R_n)$ as right R_n-modules, by using the presentation of $\pi_1(S^3_\infty \backslash K_\infty)$ given in (3.16). The chain complex computing these groups looks like

$$\ldots \to (R_n)^k \xrightarrow{\partial_2} (R_n)^{k+1} \xrightarrow{\partial_1} R_n \to 0,$$

where ∂_2 is given by the matrix

$$\overline{\begin{pmatrix} 1-y & 0 & \ldots & 0 \\ 0 & 1-y & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1-y \\ a_1 - 1 & a_2 - 1 & \cdots & a_k - 1 \end{pmatrix}} =$$

$$= \begin{pmatrix} 1-y^{-1} & 0 & \ldots & 0 \\ 0 & 1-y^{-1} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1-y^{-1} \\ a_1^{-1} - 1 & a_2^{-1} - 1 & \ldots & a_k^{-1} - 1 \end{pmatrix}$$

Note that $1 - a_j^{-1}$ is not zero in $\mathbb{Z}\bar{\Gamma}_n$, since a_j^{-1} is not the identity in Γ_0, for $j = 2, \ldots k$. Note also that y commutes with $a_1, \ldots, a_k$ in $\pi_1(S^3_\infty \backslash K_\infty)$. Multiply the k-th row by $1 - a_k^{-1}$ on the left (this is a unit in R_n). Add the first row times $1 - a_1^{-1}$, the second row times $1 - a_2^{-1}$, $\cdots$, the $(k-1)$-st row times $1 - a_{k-1}^{-1}$, and the last row times $1 - y^{-1}$

to the k-th row (all the multiplications are on the left) to get

$$\begin{pmatrix} 1-y^{-1} & 0 & \dots & 0 & 0 \\ 0 & 1-y^{-1} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1-y^{-1} & 0 \\ 0 & 0 & \dots & 0 & 0 \\ a_1^{-1}-1 & a_2^{-1}-1 & \dots & a_{k-1}^{-1}-1 & a_k^{-1}-1 \end{pmatrix}$$

Note that $a_k^{-1}-1$ is a unit in R_n, and we multiply the last column by $(a_k^{-1}-1)^{-1}$ on the right. Add the last column times $1-a_j^{-1}$ to the j-th column for all $j=1,\dots,k-1$. We get

$$\begin{pmatrix} 1-y^{-1} & 0 & \dots & 0 & 0 \\ 0 & 1-y^{-1} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1-y^{-1} & 0 \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

This matrix corresponds to ∂_2 after a change of basis in both $(R_n)^k$ and $(R_n)^{k+1}$. Therefore, we get $H_2(S^3_\infty\backslash K_\infty; R_n)=0$ and

$$H_1(S^3_\infty\backslash K_\infty, x_0; R_n) = R_n \oplus \left(R_n/(1-y^{-1})\right)^{\oplus(k-1)},$$

where x_0 is a point in $S^3_\infty\backslash K_\infty$, and there are $k-1$ direct summands of the form $R_n/(1-y^{-1})$. By [16, Proposition 5.6], we get that

$$H_1(S^3_\infty\backslash K_\infty; R_n) = \bigoplus_{k-1 \text{ copies}} R_n/(1-y^{-1}),$$

so

$$\mathrm{rk}_{\mathbb{K}_n} H_1(S^3_\infty \backslash K_\infty; R_n) = (k-1)(m-k).$$

Now, as in the proof of Lemma 3.16, we get that

$$\begin{aligned} \mathrm{rk}_{\mathbb{K}_n} H_1(X_\infty; R_n) &= \sum_{i=1}^{r} \big((d_i-1)^2 - 1\big) + (k-1)(m-k) + r - 1 \\ &= \sum_{i=1}^{r} (d_i-1)^2 + (k-1)(m-k) - 1. \end{aligned} \tag{3.17}$$

Since $k \geq 2$, we have that $m-k \leq m-2$. Also, $d_i \leq m-k+1$ for all $i = 1, \ldots, r$, and $\sum\limits_{i=1}^{r}(d_i - 1) = m-k$. Thus,

$$\begin{aligned} \sum_{i=1}^{r}(d_i-1)^2 + (k-1)(m-k) - 1 &\leq (m-k)\cdot\left(\sum_{i=1}^{r}(d_i-1) + k - 1\right) - 1 \\ &= (m-k)(m-1) - 1 \\ &\leq (m-2)(m-1) - 1, \end{aligned}$$

which completes the proof. □

Remark 3.20. *The higher order degrees $\delta_n(C)$ are not homeomorphism invariants of $\mathbb{C}^2 \setminus C$. For example, if $m \geq 3$, the case discussed in Example 3.18 and that of m lines going through a point have homeomorphic complements. However, as we have discussed, their higher order degrees $\delta_n(C)$ are $m-2$ and $m(m-2)$, respectively, for all $n \geq 0$. This is due to the dependence of higher order degrees on the local system given by the linking number homomorphism.*

3.3.2 Vanishing of higher-order degrees

In the case when an arrangement contains one line which meets all other lines transversally, higher order degrees are particularly simple. In this section, we prove the following result, which was asserted (without proof) in [29, Section 3.1].

Theorem 3.21. *Assume that the affine plane curve C defines a line arrangement $\mathcal{A} = \{L_1, \ldots, L_m\} \subset \mathbb{C}^2$, which is obtained from an essential line arrangement $\mathcal{A}' = \{L_1, \ldots, L_{m-1}\}$ by adjoining a line L_m that is transversal to every line in $\mathcal{A}'$ (that is, the singularities of the curve C along the irreducible component L_m consist of $m-1$ nodes). Then*

$$\delta_n(C) = 0, \text{ for all } n \geq 0.$$

Before proving the theorem, we recall some notation. Let C' be the curve defined by $\mathcal{A}'$, let $U = \mathbb{C}^2 \backslash C$ denote as before the complement of $\mathcal{A}$, and let $U' = \mathbb{C}^2 \setminus C'$ be the complement of $\mathcal{A}'$. Let $u_0 \in U$, which we will take as the base point for the fundamental groups of both $\pi_1(U)$ and $\pi_1(U')$.

By [33, Lemma 2], we have that

$$1 \longrightarrow \mathbb{Z} \xrightarrow{g_1} \pi_1(U) \xrightarrow{g_2} \pi_1(U') \longrightarrow 1 \tag{3.18}$$

is a central extension, where $g_1(1)$ is a positively oriented meridian around L_m, and the map g_2 is induced by inclusion. By the Zariski-Van Kampen theorem (see, e.g., [8, Chapter 4, Section 3]), we find a presentation of $\pi_1(U)$ of the form

$$\pi_1(U) = \langle y_1, \ldots, y_m \mid s_j(y_1, \ldots, y_m) \rangle \tag{3.19}$$

where $y_1, \ldots, y_{m-1}$ are certain positively oriented meridians about irreducible components of C' and y_m is a positively oriented meridian about L_m. All of the y_i's are contained in a generic line section of $\mathbb{C}^2 \setminus C$, and $s_j(y_1, \ldots, y_m)$ are certain words on the generators given by braid monodromy. Note that since y_m, and $g_1(1)$ are both positively oriented meridians about L_m, they must be conjugate, and since $g_1(1)$ is in the center of $\pi_1(U)$, then $y_m = g_1(1)$ in $\pi_1(U)$.

Consider the following splitting of g_1

$$\begin{array}{rcl} h: \ \pi_1(U) & \longrightarrow & \mathbb{Z} \\ y_i & \mapsto & 0 \quad \text{for} \ \ i = 1, \ldots m-1 \\ y_m & \mapsto & 1 \end{array}$$

which is well defined because it factors through the abelianization of $\pi_1(U)$. Hence, $h(s_j(y_1, \ldots, y_m)) = 0$ for all j, which, along with the fact that y_m is in the center of $\pi_1(U)$, allows us to find an equivalent presentation

$$\pi_1(U) = \langle y_1, \ldots, y_m \mid [y_i, y_m] \text{ for all } i = 1, \ldots, m-1; r_j(y_1, \ldots, y_{m-1}) \text{ for } j = 1, \ldots, l \rangle \tag{3.20}$$

where the r_j's are words in the letters $y_1, \ldots y_{m-1}$. By setting $x_m = y_m$ and $x_i = y_i y_m^{-1}$ for $i = 1, \ldots, m-1$, and taking into account that $\psi(r_j(y_1, \ldots, y_{m-1})) = 0$ for all $j = 1, \ldots, l$, we obtain the following equivalent presentation for $\pi_1(U)$

$$\pi_1(U) = \langle x_1, \ldots, x_m \mid [x_i, x_m] \text{ for all } i = 1, \ldots, m-1; r_j(x_1, \ldots, x_{m-1}) \text{ for } j = 1, \ldots, l \rangle. \tag{3.21}$$

Using (3.18) and (3.20), we get the following presentation for $\pi_1(U')$

$$\pi_1(U') = \langle y_1, \ldots, y_{m-1} \mid r_j(y_1, \ldots, y_{m-1}) \text{ for } j = 1, \ldots, l \rangle. \tag{3.22}$$

Note that the following map is an isomorphism

$$\begin{array}{rcl} f: \pi_1(U') \times \mathbb{Z} & \longrightarrow & \pi_1(U) \\ (y_i, t) & \longmapsto & x_i \cdot x_m^t \end{array}.$$

For any $n \geq 0$, we denote by $\Gamma_n(U)$ (resp., $\Gamma_n(U')$) the PTFA group corresponding to $\pi_1(U)$ (resp., $\pi_1(U')$), as in Section 3.2. We then have that f induces an isomorphism

$$f_n : \Gamma_n(U') \times \mathbb{Z} \longrightarrow \Gamma_n(U).$$

Moreover, if $\psi : \pi_1(U) \longrightarrow \mathbb{Z}$ is the linking number homomorphism, then $\bar{\Gamma}_n(U)$ is identified with $\Gamma_n(U')$ via f_n, where

$$\bar{\Gamma}_n(U) = \ker\left(\bar{\psi} : \Gamma_n(U) \longrightarrow \mathbb{Z}\right),$$

with $\bar{\psi}$ induced from ψ. As in Section 3.2, we let $S_n = \mathbb{Z}\left[\bar{\Gamma}_n(U)\right] \setminus \{0\}$, $\mathbb{K}_n = \mathbb{Z}\left[\bar{\Gamma}_n(U)\right] S_n^{-1}$, and $R_n = \mathbb{Z}[\Gamma_n(U)]S_n^{-1}$. Let $\mathcal{K}_n(U')$ denote the (skew) field of quotients of the Ore domain $\mathbb{Z}[\Gamma_n(U')]$.

Remark 3.22. *Notice that f_n identifies $\mathbb{K}_n$ with $\mathcal{K}_n(U')$.*

Consider the matrix of Fox derivatives for $\pi_1(U')$, that is,

$$\left(\frac{\partial r_j(y_1, \ldots, y_{m-1})}{\partial y_i}\right)_{i,j}, \quad 1 \leq i \leq m-1, 1 \leq j \leq l,$$

which has entries in $\mathbb{Z}[\pi_1(U')]$, and we take its involution

$$A = \overline{\left(\frac{\partial r_j(y_1, \ldots, y_{m-1})}{\partial y_i}\right)_{i,j}}.$$

Let $q'_n : \pi_1(U') \longrightarrow \Gamma_n(U')$ be the projection, and let

$$B(n) = A^{q'_n},$$

that is, the matrix formed by the images of the entries of A by q'_n. Since $\mathcal{K}_n(U')$ is flat over $\mathbb{Z}[\Gamma_n(U')]$, we have that $B(n)$ is a presentation matrix for the right $\mathcal{K}_n(U')$-module $H_1(U', u_0; \mathcal{K}_n(U'))$; again, we refer to [16, Section 6] for more details about Fox calculus.

Lemma 3.23. *The rank of the left $\mathcal{K}_n(U')$-module generated by the rows of $B(n)$ is $m-2$.*

Proof. Since U' is the complement of an essential line arrangement, we get by Theorem 3.14 that $\delta_n(C')$ is finite. By [28, Remark 3.8], this means that

$$\mathrm{rk}_{\mathcal{K}_n(U')} H_1(U'; \mathcal{K}_n(U')) = 0,$$

and by [16, Proposition 5.6], we get that

$$\mathrm{rk}_{\mathcal{K}_n(U')} H_1(U', u_0; \mathcal{K}_n(U')) = \mathrm{rk}_{\mathcal{K}_n(U')} H_1(U'; \mathcal{K}_n(U')) + 1 = 1.$$

Since $B(n)$ is an $(m-1) \times l$ matrix, the rank of the left $\mathcal{K}_n(U')$-module generated by the rows of $B(n)$ (which is the same as the rank of the right $\mathcal{K}_n(U')$-module generated by the columns of $B(n)$) must be $m-2$. □

We are now ready to prove Theorem 3.21.

Proof of Theorem 3.21. Let $n \geq 0$. We start by considering the presentation matrix for $H_1(U, u_0; R_n)$ as a right R_n-module given by the involution of the matrix of Fox derivatives corresponding to the presentation of $\pi_1(U)$ from (3.21), which is

$$\left(\begin{array}{cccc|ccc}
1-x_m^{-1} & 0 & \cdots & 0 & & & \\
0 & 1-x_m^{-1} & \cdots & 0 & & B(n) & \\
\vdots & \vdots & \ddots & \vdots & & & \\
0 & 0 & \cdots & 1-x_m^{-1} & & & \\
\hline
x_1^{-1}-1 & x_2^{-1}-1 & \cdots & x_{m-1}^{-1}-1 & 0 & \cdots & 0
\end{array}\right) \tag{3.23}$$

where $B(n)$ is seen as a matrix in $\mathbb{K}_n \subset R_n$ by the identification of $\mathbb{K}_n$ and $\mathcal{K}_n(U')$ given by f_n, and the rest of the entries are seen in R_n. By Lemma 3.23, the rank of the left $\mathbb{K}_n$-module spanned by the rows of $B(n)$ is $m-2$. We denote this by $\mathrm{rk}_{\mathbb{K}_n} B(n) = m-2$.

Note that $(1-x_j^{-1})$ are non-zero elements of S_n for all $j=1,\dots,m-1$. We multiply the first row (on the left) by $(1-x_1^{-1})$, and then add the j-th row times $(1-x_j^{-1})$ to the first row for all $j=2,\dots,m$. Taking into account that x_m commutes with everything else, we get

$$\left(\begin{array}{cccc|ccc}
0 & 0 & \cdots & 0 & & & \\
0 & 1-x_m^{-1} & \cdots & 0 & & B_1(n) & \\
\vdots & \vdots & \ddots & \vdots & & & \\
0 & 0 & \cdots & 1-x_m^{-1} & & & \\
\hline
x_1^{-1}-1 & x_2^{-1}-1 & \cdots & x_{m-1}^{-1}-1 & 0 & \cdots & 0
\end{array}\right)$$

where $\mathrm{rk}_{\mathbb{K}_n} B_1(n) = m-2$, since we just did row operations in $\mathbb{K}_n$ to get from $B(n)$ to $B_1(n)$. Multiplying the first column (on the right) by $(x_1^{-1}-1)^{-1}$ and then doing column operations, we get

$$\left(\begin{array}{cccc|ccc}
0 & 0 & \cdots & 0 & & & \\
0 & 1-x_m^{-1} & \cdots & 0 & & B_1(n) & \\
\vdots & \vdots & \ddots & \vdots & & & \\
0 & 0 & \cdots & 1-x_m^{-1} & & & \\
\hline
1 & 0 & \cdots & 0 & 0 & \cdots & 0
\end{array}\right) \tag{3.24}$$

Performing row and column operations in $\mathbb{K}_n$, and using that $\mathrm{rk}_{\mathbb{K}_n} B_1(n) = m-2$, we get a matrix of the form

$$\left(\begin{array}{cccc|ccc|ccc}
0 & * & \cdots & * & 0 & \dots & 0 & 0 & \cdots & 0 \\
\hline
0 & * & \cdots & * & & & & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & & I_{m-2} & & \vdots & \ddots & \vdots \\
0 & * & \cdots & * & & & & 0 & \cdots & 0 \\
\hline
1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0
\end{array}\right)$$

where I_{m-2} is the identity matrix of dimension $m-2$. Performing column operations we can get this matrix to look like

$$\left(\begin{array}{cccc|ccc|ccc}
0 & * & \cdots & * & 0 & \dots & 0 & 0 & \cdots & 0 \\
\hline
0 & 0 & \cdots & 0 & & & & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \multicolumn{3}{c|}{I_{m-2}} & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & & & & 0 & \cdots & 0 \\
\hline
1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0
\end{array}\right)$$

Permuting the first and last rows, and putting the columns corresponding to I_{m-2} as columns $2, 3, \dots m-1$, we get

$$\left(\begin{array}{ccc|ccc|ccc}
 & & & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\multicolumn{3}{c|}{I_{m-1}} & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
 & & & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\hline
0 & \cdots & 0 & * & \cdots & * & 0 & \cdots & 0
\end{array}\right)$$

Let $\mathcal{K}_n$ be the skew field of quotients of $\mathbb{Z}[\Gamma_n(U)]$. By a similar argument as in the proof of Lemma 3.23, the rank of the left $\mathcal{K}_n$-module spanned by the rows of this matrix must be $m-1$, so the last row is actually identically 0. Hence, $\delta_n(C) = 0$. $\square$

3.4 Plane curves

In this section, we adapt some of the results of Section 3.3 to the context of affine plane curve complements.

3.4.1 Upper bounds on higher-order degrees

The goal of this section is to prove the following generalization of Theorem 3.13, in which the plane curve C is allowed to have *mild* singularities at infinity.

Theorem 3.24. *Let $C \subset \mathbb{C}^2$ be a reduced plane curve of degree m, let $\overline{C}$ be its closure in $\mathbb{C}P^2$ and let L_∞ be the line at infinity. Suppose that the intersections of L_∞ and $\overline{C}$ are either transversal or L_∞ is the tangent line to $\overline{C}$ at a smooth point and it is a simple tangent there (i.e., it has multiplicity 2). If any of the following two conditions hold*

(a) $m = 2$;

(b) at least one of the intersections of $\overline{C}$ and L_∞ is transversal,

then

$$\delta_n(C) \leq m(m-2),$$

for all $n \geq 0$.

Proof. Assume that the intersection $\overline{C} \cap L_\infty$ consists of r distinct points. The case $r = m$ corresponds to the curve C being in general position at infinity, which was already considered in Theorem 3.13. So, without any loss of generality, we may assume that $r \leq m-1$. The proof of the theorem in this case relies on a Mayer-Vietoris argument similar to the one we used in the proofs of Lemma 3.16 and Lemma 3.19.

Let T be a tube in $\mathbb{C}P^2$ around L_∞, and let $\{P_1, \dots, P_r\} = \overline{C} \cap L_\infty$. If $m > 2$, we can assume (after reordering) that the intersection of $\overline{C}$ and L_∞ is transversal at P_{r-1}. Then $T \setminus (\overline{C} \cup L_\infty)$ deformation retracts to the space

$$X = \partial T \backslash C = N \cup_{\left(\bigsqcup_{i=1}^{r} T_i\right)} \left(\bigsqcup_{i=1}^{r} S_i^3 \backslash K_i \right) \subset \mathbb{C}^2 \setminus C$$

where, as in the proof of Lemma 3.16, N is the boundary of a tube around the non-singular part of L_∞, that is, a tube around L_∞ minus a disk around every point P_i, $S_i^3 \backslash K_i$ is the link complement of the singularity of $\overline{C} \cup L_\infty$ at P_i, and T_i is a 2-torus described as in the proof of Lemma 3.16. Note that X is the link (complement) at infinity, which is the space used in [28] for proving Theorem 3.13, with the difference that if C is in general position at infinity, X is just the complement of the Hopf link on m components.

By a Zariski-Lefschetz type theorem again, the inclusion $X \hookrightarrow \mathbb{C}^2 \setminus C$ induces an epimorphism

$$\pi_1(X) \twoheadrightarrow \pi_1(\mathbb{C}^2 \setminus C),$$

which in turn implies that

$$\delta_n(C) \leq \operatorname{rk}_{\mathbb{K}_n} H_1(X; R_n).$$

It thus suffices to show that $\operatorname{rk}_{\mathbb{K}_n} H_1(X; R_n) \leq m(m-2)$.

The Mayer-Vietoris sequence for the homology of X with R_n-coefficients yields the same equality as in formula (3.4) of Lemma 3.16, so it remains to compute (or bound) all of the terms on the right-hand side of (3.4).

We begin by noticing that N is homotopy equivalent to the cartesian product of a wedge sum of $r-1$ circles and S^1 if $r > 1$, and to S^1 if $r = 1$ (and $m = 2$). If $r = 1$, a direct Fox Calculus computation yields that

$$\begin{aligned} H_2(N; R_n) &= 0, \\ H_1(N; R_n) &= 0, \\ \operatorname{rk}_{\mathbb{K}_n} H_0(N; R_n) &= m, \end{aligned} \tag{3.25}$$

where one only uses the fact that a positively oriented meridian γ_∞ around L_∞ generates $\pi_1(N) \cong \mathbb{Z}$ and $\psi(\gamma_\infty) = -m$.

If $r > 1$, a presentation for the fundamental group of $N \simeq \left(\bigvee_{r-1} S^1 \right) \times S^1$ is given as:

$$\pi_1(N) = \langle a_1, \ldots, a_{r-1}, b \mid [a_i, b] \text{ for } i = 1, \ldots, r-1 \rangle,$$

where each a_i corresponds to a circle in the wedge sum, which in turn corresponds to the boundary of a disk centered at P_i, while b is a positively oriented meridian about L_∞. In particular, since the intersection of $\overline{C}$ and L_∞ is transversal at P_{r-1}, the loop a_{r-1} can be chosen to be an oriented meridian about the irreducible component of $\overline{C}$ going through P_{r-1}. Hence, $\psi(a_{r-1}) = 1$, where ψ denotes as before the linking number homomorphism. Setting $x_{r-1} = a_{r-1}$ and $x_j = a_j a_{r-1}^{-\psi(a_j)}$ for all $j = 1, \ldots, r-2$, we get the equivalent presentation

$$\pi_1(N) = \langle x_1, \ldots, x_{r-1}, b \mid [x_i, b] \text{ for } i = 1, \ldots, r-1 \rangle.$$

The involution of the matrix of Fox derivatives looks like the left-hand side of the matrix in equation (3.23) of Section 3.3.2, after changing m for r and x_m for b.

If $r > 2$, we have two possible cases: either there exists $j \in \{1, \ldots, r-2\}$ such that $x_j \neq 0$ in Γ_n (in which case, by reordering, we can assume that $j = 1$), or $x_j = 0$ in Γ_n for all $j = 1, \ldots, r-2$.

In the first case, the same computations as in Section 3.3.2 yield the left-hand side of the matrix in equation (3.24) of Section 3.3.2. Using that $\psi(b) = -m$, we get that

$$H_2(N; R_n) = 0,$$
$$\mathrm{rk}_{\mathbb{K}_n} H_1(N; R_n) = m(r-2).$$

We can also see by using Fox Calculus and the fact that $x_1^{-1} - 1$ is a unit in $\mathbb{K}_n$ that

$\mathrm{rk}_{\mathbb{K}_n} H_0(N;R_n) = 0$, so we get that

$$\begin{aligned} H_2(N;R_n) &= 0, \\ \mathrm{rk}_{\mathbb{K}_n} H_1(N;R_n) - \mathrm{rk}_{\mathbb{K}_n} H_0(N;R_n) &= m(r-2). \end{aligned} \tag{3.26}$$

In fact, these equalities also hold for the case $r = 1$ considered in (3.25).

If $r > 2$ and $x_j = 0$ in Γ_n for all $j = 1, \dots, r-2$, the complex that computes $H_*(N;R_n)$ by Fox Calculus looks like

$$\longrightarrow (R_n)^{r-1} \xrightarrow{\partial_2} (R_n)^r \xrightarrow{\partial_1} R_n \longrightarrow 0$$

where ∂_2 is given by the matrix

$$\begin{pmatrix} 1-b^{-1} & 0 & 0 & \cdots & 0 \\ 0 & 1-b^{-1} & 0 & \cdots & 0 \\ 0 & 0 & 1-b^{-1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1-b^{-1} \\ 0 & 0 & \cdots & 0 & x_{r-1}^{-1}-1 \end{pmatrix}$$

and ∂_1 by

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & x_{r-1}^{-1}-1 & b^{-1}-1 \end{pmatrix},$$

and we can see directly that (3.26) also holds in this case.

Finally, let us analyze the case $r = 2$. In this case, N is homotopy equivalent to a torus, and the Γ_n-cover of N factors through the infinite cyclic cover of N corresponding to ψ, which is homotopy equivalent to S^1 (a similar argument was used in the proof of

Lemma 3.16). Thus, in this case,

$$\begin{aligned}
\mathrm{rk}_{\mathbb{K}_n} H_1(N;R_n) - \mathrm{rk}_{\mathbb{K}_n} H_0(N;R_n) &= \mathrm{rk}_{\mathbb{K}_n} H_1(S^1;\mathbb{K}_n) - \mathrm{rk}_{\mathbb{K}_n}, H_0(S^1;\mathbb{K}_n) \\
&= \chi(S^1) \\
&= 0,
\end{aligned}$$

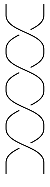
so the equalities in (3.26) also hold.

Let us next compute the local contributions in (3.4), i.e., corresponding to the link complements of the P_i's. Suppose that the intersection of $\overline{C}$ and L_∞ is transversal at P_i. Then the link K_i of P_i is the Hopf link on 2 components, and we can pick meridians a, b around the components of the link to generate $\pi_1(S^3_i \backslash K_i) \cong \mathbb{Z}^2$, and which satisfy $\psi(a) = 1$ and $\psi(b) = -m$. In this case, $S^3_i \backslash K_i$ deformation retracts onto a torus, so using again an argument involving the infinite cyclic cover, we get that

$$\mathrm{rk}_{\mathbb{K}_n} H_1(S^3_i \backslash K_i; R_n) = \mathrm{rk}_{\mathbb{K}_n} H_0(S^3_i \backslash K_i; R_n), \tag{3.27}$$

where both ranks are finite, and $H_2(S^3_i \backslash K_i; R_n) = 0$.

On the other hand, if L_∞ is the tangent line to $\overline{C}$ at P_i, with multiplicity 2, then K_i is a type $(2,4)$ torus link, which corresponds to the following braid



Again, we can find a presentation for $\pi_1(S^3_i \backslash K_i)$ with two generators and one relation, namely

$$\pi_1(S^3_i \backslash K_i) = \langle a, b \mid ababa^{-1}b^{-1}a^{-1}b^{-1} \rangle,$$

where a corresponds to a positively oriented meridian around the irreducible component of $\overline{C}$ going through P_i and b corresponds to a positively oriented meridian around L_∞. We then have that $\psi(a) = 1$ and $\psi(b) = -m$. By setting $x = a$ and $y = ba^m$, we get the equivalent presentation

$$\pi_1(S^3_i \backslash K_i) = \langle x, y \mid xyx^{-m+1}yx^{-1}y^{-1}x^{m-1}y^{-1}\rangle.$$

Let $s = xyx^{-m+1}yx^{-1}y^{-1}x^{m-1}y^{-1}$. When regarded in $\mathbb{K}_n[t^{\pm 1}]$, $\frac{\partial s}{\partial x}$ is a polynomial of degree at most $m-1$, and $\frac{\partial s}{\partial x} = 0$ if only if $y = 1$ in $\mathbb{K}_n[t^{\pm 1}]$. Suppose that $y \neq 1$ in $\mathbb{K}_n[t^{\pm 1}]$. Since there is only one relation, a Fox Calculus computation yields that

$$\mathrm{rk}_{\mathbb{K}_n} H_1(S^3_i \backslash K_i; R_n) \leq m-1,$$

and

$$\mathrm{rk}_{\mathbb{K}_n} H_0(S^3_i \backslash K_i; R_n) = 0.$$

Hence, if $y \neq 1$ in $\mathbb{K}_n[t^{\pm 1}]$, then

$$\mathrm{rk}_{\mathbb{K}_n} H_1(S^3_i \backslash K_i; R_n) - \mathrm{rk}_{\mathbb{K}_n} H_0(S^3_i \backslash K_i; R_n) \leq m-1. \tag{3.28}$$

If $y = 1$ in $\mathbb{K}_n[t^{\pm 1}]$, we note that $\frac{\partial s}{\partial y}$ is a polynomial of degree m, so $\mathrm{rk}_{\mathbb{K}_n} H_1(S^3_i \backslash K_i; R_n) = m$. Moreover, if $\frac{\partial s}{\partial x} = 0$, then a Fox Calculus computation yields that $\mathrm{rk}_{\mathbb{K}_n} H_0(S^3_i \backslash K_i; R_n) = 1$. Hence, if $y = 1$ in $\mathbb{K}_n[t^{\pm 1}]$, the same inequality as in (3.28) holds. In both cases ($y = 1$ and $y \neq 1$ in $\mathbb{K}_n[t^{\pm 1}]$), we see that $H_2(S^3_i \backslash K_i; R_n) = 0$.

Using a presentation of $\pi_1(X)$ in which there is a generator a such that $\psi(a) = 1$ (for example, taking a to be a positively oriented meridian around the irreducible component of $\overline{C}$ going through P_i), we also get that

$$\mathrm{rk}_{\mathbb{K}_n} H_0(X; R_n) \leq 1 \tag{3.29}$$

via a Fox Calculus computation.

Next, we deal with the contributions to formula (3.4) of the tori T_i, for any $i = 1, \dots, r$. We have the following presentation of the fundamental group

$$\pi_1(T_i) = \langle c, b \mid [c, b] \rangle,$$

where b is a positively oriented meridian about L_∞, $\psi(c) = 1$ if the intersection of $\overline{C}$ with L_∞ is transversal at P_i, and $\psi(c) = 2$ otherwise.

If the intersection of $\overline{C}$ with L_∞ is transversal at P_i, then the Γ_n-cover of T_i factors through the infinite cyclic cover corresponding to ψ. Moreover, since $\psi(c) = 1$, this infinite cyclic cover is homeomorphic to $S^1 \times \mathbb{R}$, thus homotopy equivalent to S^1. Therefore,

$$\begin{gathered}\mathrm{rk}_{\mathbb{K}_n} H_0(T_i; R_n) - \mathrm{rk}_{\mathbb{K}_n} H_1(T_i; R_n) = \mathrm{rk}_{\mathbb{K}_n} H_0(S^1; \mathbb{K}_n) - \mathrm{rk}_{\mathbb{K}_n} H_1(S^1; \mathbb{K}_n) = \chi(S^1) = 0, \\ \mathrm{rk}_{\mathbb{K}_n} H_2(T_i; R_n) = \mathrm{rk}_{\mathbb{K}_n} H_2(S^1; \mathbb{K}_n) = 0.\end{gathered} \tag{3.30}$$

If L_∞ is tangent to $\overline{C}$ at P_i (note that there are $m - r$ such P_i's), then we have as before that $\psi(c) = 2$. There is a $\mathbb{K}_n$-module isomorphism (e.g., see [21, Section 2.1])

$$H_*(T_i; R_n) \cong H_*(\widetilde{T_i}; \mathbb{K}_n),$$

where $\widetilde{T_i}$ is the (possibly disconnected) infinite cyclic cover of T_i corresponding to ψ. Note that $\widetilde{T_i}$ is either homeomorphic to $S^1 \times \mathbb{R}$, or to the disjoint union $(S^1 \times \mathbb{R}) \sqcup (S^1 \times \mathbb{R})$, depending on whether m is odd or even, respectively. In both cases, $\widetilde{T_i}$ is homotopy equivalent to a one-dimensional finite CW-complex with vanishing Euler characteristic.

Hence

$$\begin{gathered}\mathrm{rk}_{\mathbb{K}_n} H_0(T_i; R_n) - \mathrm{rk}_{\mathbb{K}_n} H_1(T_i; R_n) = \mathrm{rk}_{\mathbb{K}_n} H_0(\widetilde{T}_i; \mathbb{K}_n) - \mathrm{rk}_{\mathbb{K}_n} H_1(\widetilde{T}_i; \mathbb{K}_n) = \chi(\widetilde{T}_i) = 0,\\ \mathrm{rk}_{\mathbb{K}_n} H_2(T_i; R_n) = \mathrm{rk}_{\mathbb{K}_n} H_2(\widetilde{T}_i; \mathbb{K}_n) = 0.\end{gathered} \tag{3.31}$$

Arguing as in the proof of Lemma 3.16, we also get that

$$\mathrm{rk}_{\mathbb{K}_n} \mathrm{Im}(\alpha) = 0. \tag{3.32}$$

Altogether, substituting (3.26), (3.27), (3.28), (3.29), (3.30), (3.31) and (3.32) into (3.4), we get that

$$\mathrm{rk}_{\mathbb{K}_n} H_1(X; R_n) \leq m(r-2) + (m-r)(m-1) + 1 = m^2 - 3m + r + 1.$$

Moreover, since we assumed that $r \leq m - 1$, we get that

$$\mathrm{rk}_{\mathbb{K}_n} H_1(X; R_n) \leq m^2 - 2m = m(m-2),$$

thus concluding the proof. □

3.4.2 Vanishing of higher-order degrees

In the case when an irreducible component of a plane curve C is a line L which meets all other components of $C \cup L_\infty$ transversally (with L_∞ denoting the line at infinity in $\mathbb{C}P^2$), higher order degrees are particularly simple. This is exemplified in the following generalization of Theorem 3.21.

Theorem 3.25. *Let $n \geq 0$. Assume that the affine plane curve C is of the form $C = L \cup C'$, where C' is a curve of degree $m-1$ in $\mathbb{C}^2$ such that $\delta_n(C')$ is finite, and L is a line transversal to C', such that $L \cap C'$ consists of $m-1$ distinct points. Then,*

$$\delta_n(C) = 0.$$

Proof. The proof follows the same steps as in the proof of Theorem 3.21. Let $U = \mathbb{C}^2 \setminus C$ and $U' = \mathbb{C}^2 \setminus C'$. The Zariski-Van Kampen theorem (see, e.g., [8, Chapter 4, Section 3]) can be used to find a presentation of $\pi_1(U)$ such as the one described in (3.19). Using [33, Lemma 2] and the same arguments as in the proof of Theorem 3.21, we find presentations of $\pi_1(U)$ and $\pi_1(U')$ such as the ones described in (3.20) and (3.22), respectively. Thus, Remark 3.22 still holds in this setting. The rest of the proof follows the same Fox Calculus computation as in the proof of Theorem 3.21, except for the proof of Lemma 3.23, which in this case follows from the hypothesis that $\delta_n(C')$ is finite. $\square$

Remark 3.26. *The fact, used in the proof of Theorem 3.25, that* $\pi_1(\mathbb{C}^2 \setminus C) \cong \pi_1(\mathbb{C}^2 \setminus C') \times \mathbb{Z}$*, can also be deduced from the Oka-Sakamoto theorem* [34]*. Indeed, under our assumptions, we have by* [34] *that*

$$\pi_1(\mathbb{C}^2 \setminus C) \cong \pi_1(\mathbb{C}^2 \setminus C') \times \pi_1(\mathbb{C}^2 \setminus L) \cong \pi_1(\mathbb{C}^2 \setminus C') \times \mathbb{Z}.$$

In particular, this also shows that Theorem 3.25 can be generalized as follows:

Theorem 3.27. *Let* $n \geq 0$*. Assume that the affine plane curve* C *is of the form* $C = C' \cup C''$*, where* C' *is a curve of degree* m' *in* $\mathbb{C}^2$ *such that* $\delta_n(C')$ *is finite, and* C'' *is a curve of degree* m'' *such that* $\pi_1(\mathbb{C}^2 \setminus C'') \cong \mathbb{Z}$ *(for example,* C'' *can be any smooth irreducible curve). Assume that* $C' \cap C''$ *consists of* $m'm''$ *distinct points in* $\mathbb{C}^2$*. Then,*

$$\delta_n(C) = 0.$$

3.5 Relationship with the first characteristic variety

In this section, we relate the higher-order degrees to more classical Alexander-type invariants. We begin by recalling the following result:

Proposition 3.28. ([28, Proposition 5.1]) *If C is an irreducible affine plane curve, then*

$$\delta_0(C) = \deg \Delta_C(t),$$

where $\Delta_C(t)$ denotes the Alexander polynomial of the curve complement. If, moreover, the Alexander polynomial is trivial, then all higher-order degrees vanish.

In what follows, we generalize the above result to non-irreducible affine plane curves. Let us first introduce some notation, following the conventions from [37].

Let R be a Noetherian commutative ring with unit. Assume also that R is a unique factorization domain. Let M be a finitely generated R-module. Then M admits a finite presentation of the form

$$R^q \xrightarrow{\Phi} R^m \longrightarrow M.$$

Definition 3.29. *The i-th elementary ideal of M, denoted $E_i(M)$, is the ideal of R generated by the minors of size $m - i$ of the $m \times q$ matrix Φ, with the convention that $E_i(M) = R$ if $i \geq m$ and $E_i(M) = 0$ if $m - i > q$.*

Remark 3.30. *The i-th elementary ideal (also referred to as the $(i+1)$-st Fitting ideal) does not depend on the choice of representation of M as an R-module.*

Remark 3.31. *It follows immediately from Definition 3.29 that*

$$E_i(M) \subset E_{i+1}(M)$$

for all $i \geq 0$.

Definition 3.32. *Let $i \geq 0$. We define $\Delta_i(M) \in R$ to be the generator of the smallest principal ideal in R containing $E_i(M)$, that is, the greatest common divisor of all elements of $E_i(M)$.*

Remark 3.33. *$\Delta_i(M)$ is well-defined up to multiplication by a unit of R.*

Let $G = \langle x_1, \dots, x_m \mid r_1, \dots, r_q \rangle$ be a finitely presented group, let H be its maximal, torsion free abelian quotient, and let $\pi : G \longrightarrow H$ be the quotient map. The group ring $\mathbb{Z}H$ is a commutative Noetherian ring with unit, which is also a unique factorization domain.

Let F_m be the free group with generators $x_1, \dots, x_m$. For each $1 \leq j \leq m$, there is a linear operator

$$\frac{\partial}{\partial x_j} : \mathbb{Z}F_m \longrightarrow \mathbb{Z}F_m$$

(called the *j-th Fox derivative*) uniquely determined by the following properties:

(a) $\frac{\partial 1}{\partial x_j} = 0$,

(b) $\frac{\partial x_i}{\partial x_j} = \delta_{ij}$,

(c) $\frac{\partial uv}{\partial x_j} = \frac{\partial u}{\partial x_j} + u\frac{\partial v}{\partial x_j}$, for any $u, v \in F_m$.

Let $\phi : F_m \longrightarrow G$ be the presenting homomorphism.

Definition 3.34. *The Alexander matrix of the given presentation of G is*

$$\Phi_G = \left(\frac{\partial r_i}{\partial x_j}\right)^{\pi\circ\phi} : (\mathbb{Z}H)^q \longrightarrow (\mathbb{Z}H)^m.$$

Now, let X be a connected CW-complex with a unique 0-cell x_0, and finitely many 1-cells. Let $G = \pi_1(X, x_0)$ be the fundamental group, and let H be its maximal torsion-free abelian quotient, that is, $H \cong \mathbb{Z}^{b_1(G)}$. The canonical projection $\pi : G \longrightarrow H$ defines a local system of $\mathbb{Z}H$-modules. The long exact sequence for the homology of the pair (X, x_0) with coefficients in $\mathbb{Z}H$ given by this local system is

$$\ldots \to 0 \to H_1(X; \mathbb{Z}H) \to H_1(X, x_0; \mathbb{Z}H) \to H_0(x_0; \mathbb{Z}H) \to H_0(X; \mathbb{Z}H) \to 0,$$

where

$$\ker\left(H_0(x_0;\mathbb{Z}H)=\mathbb{Z}H\to H_0(X;\mathbb{Z}H)\right)$$

can be identified with the augmentation ideal $I_H=\ker(\epsilon:\mathbb{Z}H\to\mathbb{Z})$, and ϵ is defined as

$$\begin{aligned}\epsilon:\mathbb{Z}H &\longrightarrow \mathbb{Z}\\ \sum n_ih_i &\longmapsto \sum n_i\end{aligned}$$

for $n_i\in\mathbb{Z}$, $h_i\in H$.

The $\mathbb{Z}H$-modules $H_1(X;\mathbb{Z}H)$ and $H_1(X,x_0;\mathbb{Z}H)$ depend only on the fundamental group G, so we denote them by B_G and A_G, respectively. From the above discussion, these modules fit into the following short exact sequence of $\mathbb{Z}H$-modules

$$0\to B_G\to A_G\to I_H\to 0.$$

Definition 3.35. *The $\mathbb{Z}H$-module B_G is called the Alexander invariant of X, and A_G is called the Alexander module of X.*

Definition 3.36. *The Alexander polynomial of the group G, denoted by Δ_G, is defined by*

$$\Delta_G=\Delta_1(A_G)=\gcd(E_1(A_G))\in\mathbb{Z}H.$$

Remark 3.37. *Sometimes, the Alexander polynomial of the fundamental group of a CW-complex X is defined as $\Delta_0(B_G)=\gcd(E_0(B_G))$, but the definition using the Alexander module instead of the Alexander invariant is more suitable for our purposes. In the case when $b_1(X)=1$ (e.g., $X=\mathbb{C}^2\setminus C$, where C is an irreducible affine plane curve), the two definitions coincide. Besides, the Alexander matrix Φ_G provides a presentation for A_G, so we can compute Δ_G from it.*

Definition 3.38. *Let X be a connected CW-complex with finite k-skeleton, and let $G=\pi_1(X)$. The homology jump loci of X (over $\mathbb{C}$) are the Zariski closed sets*

$$\mathcal{V}^i_d(X)=\{\rho\in Hom(G,\mathbb{C}^*)\mid \dim_{\mathbb{C}} H_i(X;\mathbb{C}_\rho)\geq d\},$$

where $\mathbb{C}_\rho$ is the rank-one $\mathbb{C}$-local system on X induced by ρ, $0\leq i\leq k$, and $d>0$. When $i=1$, we use the simplified notation $\mathcal{V}_d(X)$ for $\mathcal{V}^1_d(X)$.

Let X be a connected CW complex with a unique 0-cell and finitely many 1-cells. Let $\mathrm{Hom}(G,\mathbb{C}^*)^0$ be the identity component of the algebraic group $\mathrm{Hom}(G,\mathbb{C}^*)$. The projection map $\pi:G\longrightarrow H$ induces an isomorphism $\pi^*:\mathrm{Hom}(H,\mathbb{C}^*)\longrightarrow\mathrm{Hom}(G,\mathbb{C}^*)^0$.

Definition 3.39. *The characteristic varieties of X (over $\mathbb{C}$), denoted by $V_d(X)$, are the subvarieties of $(\mathbb{C}^*)^{b_1(X)}\cong\operatorname{Spec}\mathbb{C}H$ given by*

$$V_d(X)=V(E_{d-1}(B_G\otimes\mathbb{C})).$$

We recall the following result from [17], see also [37, Proposition 4.7].

Proposition 3.40. *Let $\rho:H\longrightarrow\mathbb{C}^*$ be a non-trivial character. Then, for all $d\geq 1$,*

$$\pi^*(\rho)\in\mathcal{V}_d(X)\iff\rho\in V\left(E_d(A_G\otimes\mathbb{C})\right)\iff\rho\in V_d(X).$$

Remark 3.41. *If $X=\mathbb{C}^2\setminus C$ is a plane curve complement, then π^* is the identity, and $Hom(G,\mathbb{C}^*)\cong Hom(H,\mathbb{C}^*)$ can be identified with $(\mathbb{C}^*)^m$, where m is the number of irreducible components of the plane curve. In this case, Proposition 3.40 asserts that, away from $(1,\dots,1)$, $\mathcal{V}_d$ and V_d coincide. Moreover, by Remark 3.37, away from $(1,\dots,1)$, we can compute V_d from the dimension $m-d$ minors of the Alexander matrix Φ_G.*

From now on, let C be a plane curve in $\mathbb{C}^2$ with m irreducible components, with $m \geq 2$, and let $G = \pi_1(U, u_0)$, where $U = \mathbb{C}^2 \backslash C$. We denote by $\Delta_C \in \mathbb{Z}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ the Alexander polynomial Δ_G of Definition 3.36.

Definition 3.42. *Let q be a Laurent polynomial in $\mathbb{C}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$. We can write q as*

$$q = \sum_k a_k t_1^{i_1^k} \cdot \dots \cdot t_m^{i_m^k},$$

for some $a_k \in \mathbb{C}^*$, and $(i_1^k, \dots, i_m^k) \neq (i_1^j, \dots, i_m^j)$ for all $k \neq j$. We define the degree of q as

$$\deg(q) = \max_k \left(\sum_{l=1}^m i_l^k \right) - \min_j \left(\sum_{l=1}^m i_l^j \right)$$

Remark 3.43. *The degree of a Laurent polynomial q is 0 if and only if q is a homogeneous polynomial up to multiplication by a unit of $\mathbb{C}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$. Multiplying by a unit of $\mathbb{C}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ does not change the degree, so $\deg(\Delta_C)$ is well-defined.*

Lemma 3.44.

$$E_0(H_1(U; R_0)) = E_1(H_1(U, u_0; R_0)).$$

Proof. Note that R_0 is a commutative ring. From the long exact sequence of a pair with coefficients in R_0, we get

$$0 \to H_1(U; R_0) \to H_1(U, u_0; R_0) \to H_0(u_0; R_0) \to H_0(U; R_0)$$

If $m \geq 2$, following the same Fox Calculus computation as in the proof of Lemma 3.16 for $H_0(X; R_0)$, we get that $H_0(U; R_0) = 0$, so we have the short exact sequence of R_0-modules

$$0 \to H_1(U; R_0) \to H_1(U, u_0; R_0) \to R_0 \to 0$$

which splits, so the result follows. □

In the above notation, we have the following result relating the multivariate Alexander polynomial and the zero-th higher order degree, thus extending Proposition 3.28 to the non-irreducible case.

Theorem 3.45. *Suppose $m \geq 2$, and assume that $\delta_0(C)$ is finite. Then,*

$$\delta_0(C) = \deg(\Delta_C).$$

Proof. Note that all the rings that we will deal with in this proof are commutative, so we will not distinguish between left and right modules. Let $\gamma_1, \ldots, \gamma_m$ be positively oriented meridians around the different components of C, let $\psi : \pi_1(U) \to \mathbb{Z}$ be the linking number homomorphism. Consider the splitting of ψ given by

$$\begin{aligned} \phi : \mathbb{Z} &\longrightarrow \pi_1(U) \\ 1 &\longmapsto \gamma_1 \end{aligned}$$

which we use to identify R_0 with $\mathbb{K}_0[t^{\pm 1}]$. With this identification, we can think of R_0 as

$$Q\left(\mathbb{Z}\left[\left(\frac{t_2}{t_1}\right)^{\pm 1}, \ldots, \left(\frac{t_m}{t_1}\right)^{\pm 1}\right]\right)[t_1^{\pm 1}]$$

where $\mathbb{K}_0 = Q\left(\mathbb{Z}\left[\left(\frac{t_2}{t_1}\right)^{\pm 1}, \ldots, \left(\frac{t_m}{t_1}\right)^{\pm 1}\right]\right)$ is the field of quotients of the ring $\mathbb{Z}\left[\left(\frac{t_2}{t_1}\right)^{\pm 1}, \ldots, \left(\frac{t_m}{t_1}\right)^{\pm 1}\right]$. We can think of $\mathbb{Z}[\Gamma_0]$ as $\mathbb{Z}[t_1^{\pm 1}, \ldots, t_m^{\pm 1}]$ seen inside of R_0 this way. Note that the degree of any $q \in \mathbb{Z}[t_1^{\pm 1}, \ldots, t_m^{\pm 1}]$ is the same as the degree of q in t seen as an element of $\mathbb{K}_0[t^{\pm 1}]$.

Let $f_1, \ldots, f_r$ be a set of generators of $E_1(A_G)$. We have that

$$\Delta_C = \gcd_{\mathbb{Z}[t_1^{\pm 1}, \ldots, t_m^{\pm 1}]}(f_1, \ldots, f_r).$$

Since R_0 is a flat $\mathbb{Z}[\Gamma_0]$-module, $f_1, \dots, f_r$ are also a set of generators of $E_1(H_1(U, u_0; R_0))$. By assumption, $\delta_0(C)$ is finite, which means that $E_1(H_1(U, u_0; R_0))$ is not the zero ideal. Moreover, $R_0 \cong \mathbb{K}_0[t^{\pm 1}]$ is a PID, so there exists $p \in \mathbb{K}_0[t^{\pm 1}]$ such that

$$E_1(H_1(U, u_0; R_0)) = (p)$$

and, by definition, $\delta_0(C) = \deg_{\mathbb{K}_0[t^{\pm 1}]}(p)$. Moreover, by considering the prime decomposition of $f_1, \dots, f_r$ in $\mathbb{Z}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ and applying Lemma 3.46 (below), we can choose such $p \in \mathbb{Z}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ such that there exists $a \in \mathbb{Z}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$, where a is a unit in R_0, with

$$p \cdot a = \Delta_C$$

But units in R_0 which are contained in $\mathbb{Z}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ all have degree 0 as elements of $\mathbb{Z}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$, therefore

$$\deg_{\mathbb{Z}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]}(\Delta_C) = \deg_{\mathbb{Z}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]}(p) = \delta_0(C).$$

□

Lemma 3.46. *Let* $R = Q\left(\mathbb{Z}\left[\left(\frac{t_2}{t_1}\right)^{\pm 1}, \dots, \left(\frac{t_m}{t_1}\right)^{\pm 1}\right]\right)[t_1^{\pm 1}]$*, and* $S = \mathbb{Z}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$*. Let* q *be a prime element in* S*. Then,* q *is either prime or a unit in* R*.*

Proof. This is an exercise in commutative algebra, which is a direct consequence of [2, Proposition 3.11, iv)], for example. □

As a consequence of Proposition 3.40, Proposition 3.28, Theorem 3.45 and [11, Corollary 3.2], we get the following:

Corollary 3.47.

$$\operatorname{codim} V_1(U) > 1 \Longleftrightarrow \Delta_C \in \mathbb{Z}\backslash 0 \Longrightarrow \delta_0(C) = 0.$$

Moreover, Proposition 3.28 and Theorem 3.45 imply the following:

Corollary 3.48. *$\delta_0(C) = 0$ if and only if Δ_C has a representative in $\mathbb{Z}[t_1^{\pm 1}, \ldots, t_m^{\pm 1}]$ that is a non-zero homogeneous polynomial.*

Remark 3.49. *Note that since $\mathcal{K}_0$ is flat over $\mathbb{Z}[\Gamma_0]$, by* [16, Proposition 5.6] *we have that*

$$\operatorname{codim} V_1(U) = 0 \iff E_1(A_G) = (0) \iff rk_{\mathcal{K}_0} H_1(U, u_0; \mathcal{K}_0) > 1 \iff \delta_0 \text{ is not finite.}$$

Hence, if $\delta_0(C)$ is finite (as is the case for all curves in general position at infinity ([28]*), the ones described in Theorem 3.24, or all line arrangements except the one consisting of m parallel lines, for $m \geq 2$), then* $\operatorname{codim} V_1(U) \geq 1$.

Example 3.50. *Let C be the line arrangement described in Example 3.18, and let t_i the variable corresponding to the component L_i, for $i = 1, \ldots, m$. Then*

$$\Delta_C = (t_1 - 1)^{m-2}.$$

Let us now specialize our results to the case when C is an essential line arrangement. We recall the following result from [37].

Proposition 3.51. *(*[37, Theorem 9.15]*) Let C be an essential line arrangement. Then:*

1. *If C consists of m lines going through a point (a pencil of lines), with $m \geq 3$, then*

$$\Delta_C = (t_1 t_2 \ldots t_m - 1)^{m-2}.$$

2. *If C is as in Example 3.18, then $\Delta_C = (t_1 - 1)^{m-2}$.*

3. *For all other essential arrangements,* $\Delta_C \in \mathbb{Z} \setminus \{0\}$.

Then Theorem 3.45 and Proposition 3.51 have the following consequence:

Corollary 3.52. *Let C be an essential line arrangement. Then:*

1. $\operatorname{codim} V_1(C) > 1 \iff \delta_0(C) = 0$. *That is, the result from Corollary 3.47 is an "if and only if" for line arrangements.*
2. *If C is a pencil of lines, then* $\delta_0(C) = m(m-2)$.
3. *If C is as in Example 3.18, then* $\delta_0(C) = (m-2)$.
4. *If C is not as in items (2) or (3), then* $\delta_0(C) = 0$.

Remark 3.53. *By Lemma 3.16 and Example 3.50,* $\delta_n(C) = \delta_0(C)$ *for all* n *for the cases described in items (2) and (3) of the above Corollary.*

Appendix A

The fundamental group of a hyperplane arrangement complement

In this appendix, we recall a presentation of the fundamental group of a complex affine hyperplane arrangement complement due to Arvola ([1]), although our notation will be more similar to that in [15, Section 3.2]. We do so in quite a lot of detail to fix notation that will be crucial for the reader interested in computing examples of the results in this thesis. By a Lefschetz type argument, it suffices to restrict ourselves to the complex line arrangement case, which we will do from now on. We follow the same notation as in Chapter 2.

A.1 The marked 2-graph

We are going to identify $\mathbb{C}^2$ (with coordinates (z_1, z_2)) with $\mathbb{R}^4$ (with coordinates (x_1, y_1, x_2, y_2)) by

$$z_1 = x_1 + iy_1 \quad , \quad z_2 = x_2 + iy_2$$

Let $\mathcal{A} = \{H_1, \ldots, H_m\} \in \mathbb{C}^2$ be an essential line arrangement, and let $H = \cup_{i=1}^{m} H_i$.

We denote by $\mathcal{P} = \{P_1, \ldots, P_s\}$ the set of singular points of the arrangement $\mathcal{A}$, that is, the set of points that lie on at least two lines of the arrangement. Since our arrangement is essential, the set $\mathcal{P}$ is not empty. Let $\pi : \mathbb{C}^2 \longrightarrow \mathbb{C}$ the projection onto the first coordinate.

With a suitable change of coordinates, we can assume that the following conditions hold:

Assumption A.1. *No line in $\mathcal{A}$ has an equation of the form $z_1 = c$ for any constant $c \in \mathbb{C}$. This implies that $\pi_{|H_i} : H_i \longrightarrow \mathbb{C}$ is a homeomorphism for all $i = 1, \ldots, m$.*

Assumption A.2. *Each pair of distinct points in $\mathcal{P}$ can be distinguished by their x_1 coordinates alone. Reordering, we will assume that*

$$x_1(P_1) < x_1(P_2) < \ldots < x_1(P_s)$$

We pick a point $p \in \mathbb{C} = \operatorname{Im}(\pi)$ such that $x_1(p) < x_1(P_1)$, and a piecewise linear path $h : [0,1] \longrightarrow \mathbb{R}$ such that $\gamma(t) = (t + x_1(p), h(t))$ is a path in $\mathbb{R}^2 = \mathbb{C} = \operatorname{Im}(\pi)$ starting in p and passing through $\pi(P_1), \pi(P_2), \ldots, \pi(P_s)$ in order, satisfying that γ is horizontal in a neighborhood of each $\pi(P_i)$ and p.

Definition A.3. *The 3-graph of $\mathcal{A}$ relative to the map h is*

$$\Gamma_h^3 = \{(t + x_1(p), x_2, y_2) \ \mid \ t \in [0,1], (t + x_1(p), h(t), x_2, y_2) \in H\}$$

Note that Γ_h^3 is naturally homeomorphic to $\pi^{-1}(\operatorname{Im}(\gamma)) \cap H$.

Definition A.4. *Let $\phi : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be the projection onto the first two coordinates The 2-graph of $\mathcal{A}$ relative to the map h is*

$$\Gamma_h^2 := \phi(\Gamma_h^3)$$

Note that $\phi_{|\Gamma_h^3} : \Gamma_h^3 \longrightarrow \Gamma_h^2$ is not one to one necessarily. Those self intersections of Γ_h^2 are called **virtual crossings**, as opposed to the crossings corresponding to points in $\mathcal{P}$, which are called **actual crossings**. After a change of coordinates and a change of map h if necessary, we can (and will, from now on) assume the following:

Assumption A.5. *All virtual crossings of* Γ_h^2 *are transverse.*

Assumption A.6. *All virtual crossings of* Γ_h^2 *can be distinguished from other virtual and actual crossings by their first coordinate* (x_1) *alone.*

Assumption A.7. *All of the corresponding points in the path* γ *to virtual crossings lie in the interior of an open set where* γ *is linear.*

Remark A.8. *We can recover* Γ_h^3 *up to isotopy from* Γ_h^2 *if we mark the virtual crossings in* Γ_h^2 *to indicate whether they represent an under or an overcrossing. We will call this the* ***marked 2-graph****.*

Before we move on to recalling how to compute the fundamental group of M from the marked 2-graph, we will do an example.

Example A.9. *Let* $\mathcal{A} = \{H_1, H_2, H_3, H_4\}$ *be the arrangement of lines in* $\mathbb{C}^2$ *given by equations*

$$H_1 = \{z_1 - z_2 = 1\}$$

$$H_2 = \{z_1 - z_2 = 0\}$$

$$H_3 = \{z_1 - iz_2 = 0\}$$

$$H_4 = \{z_1 + 2z_2 = 0\}$$

We have that $\mathcal{P} = \{P_1, P_2, P_3\}$, *where*

$$P_1 = H_2 \cap H_3 \cap H_4 = (0,0)$$

$$P_2 = H_1 \cap H_3 = \left(\tfrac{1-i}{2}, -\tfrac{1+i}{2}\right)$$

$$P_3 = H_1 \cap H_4 = \left(\tfrac{2}{3}, -\tfrac{1}{3}\right)$$

We pick $p = (-1,0) \in \mathbb{R}^2 = \mathbb{C} = Im(\pi)$, *and the path* $\gamma : [0,1] \longrightarrow \mathbb{C}$ *(depicted in Fig. 2) starting at* p, *where the points* $\pi(P_j)$ *have been marked for all* $j = 1,2,3$.

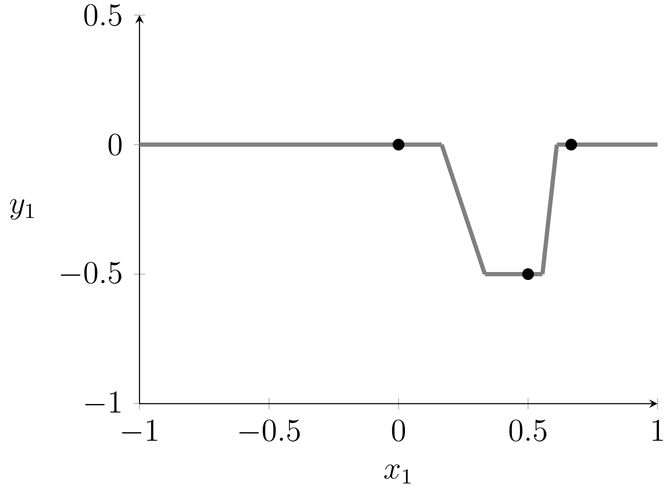


Figure 2: The path γ associated to Example A.9.

The corresponding 3*-graph is depicted in Fig. 3, where the points corresponding to the points in* $\mathcal{P}$ *have also been marked*

We cannot really distinguish the virtual crossings in Fig. 3, so let's take a look at the corresponding marked 2-graph, as depicted in Fig. 4. The letters associated to the strands will make sense later.

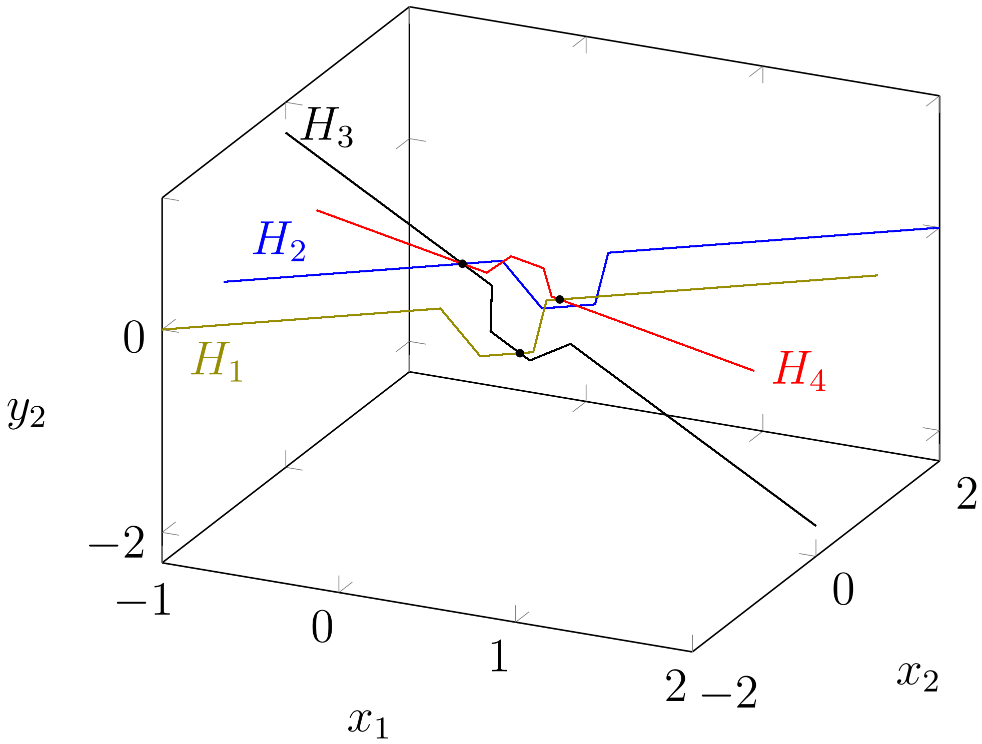


Figure 3: The 3-graph associated to Example A.9.

A.2 Generators and local meridians near the singular points

Once we have possibly done a change of coordinates and picked a suitable map h to construct our marked 2-graph, it is time to use it to compute the fundamental group of the arrangement complement $M := \mathbb{C}^2 \backslash H$. By the work of Zariski and Van Kampen, we know that we can compute $\pi_1(M)$ using the projection π we picked in Appendix A.1 by braid monodromy arguments ([26], [4], [35]).

Since we are considering line arrangements, the link around each singularity is going to be a Hopf link with as many components as lines going through the singularity, so once we know how to express suitable meridians around each component of the Hopf

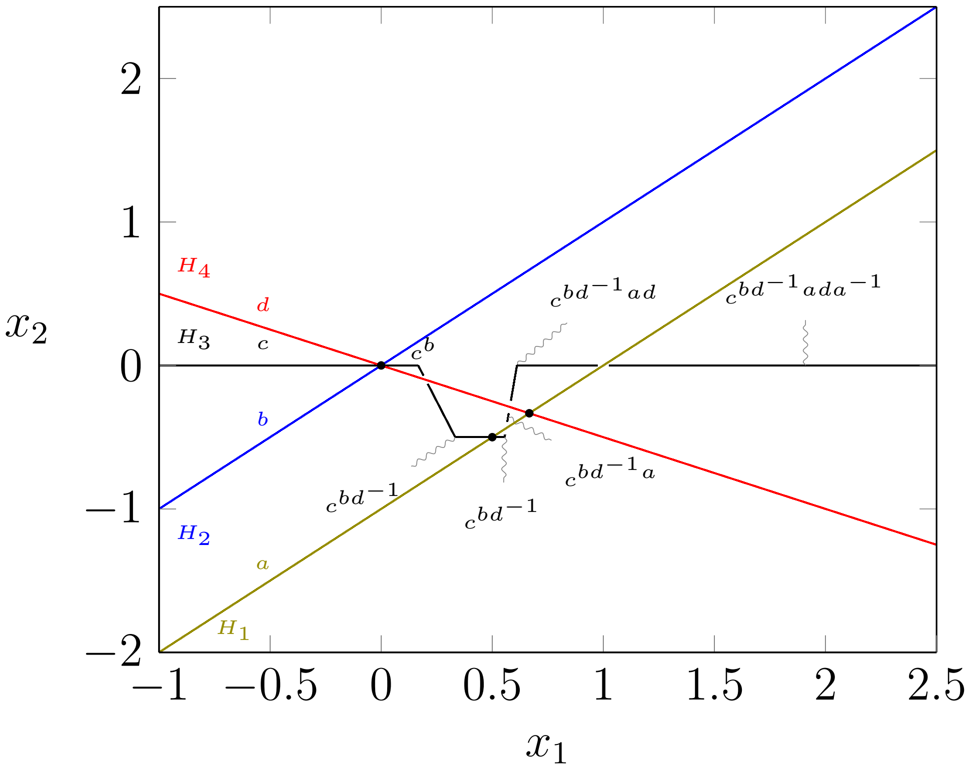


Figure 4: The marked 2-graph associated to Example A.9, with the actual crossings depicted by black dots.

link as words in our fixed generators of the fundamental group of M (which we will fix in a moment), we'll know how to compute a presentation for $\pi_1(M)$. Let's start by fixing a set of generators of $\pi_1(M)$.

Pick a big enough positive real number R such that all of the points in H that lie in the preimage by the projection π of the path γ have their y_2 coordinate smaller than R. In Example A.9, we can pick $R = 2$. We are going to pick as our base point the point $(\gamma(0), iR) \in \mathbb{C}^2$ (which we will omit in our notation for $\pi_1(M)$). Note that $\pi^{-1}(\gamma(0)) \cap H$ consists of m distinct points (one point per line in the arrangement), and all of them can be distinguished by their x_2 coordinate only by Assumption A.7.

Our chosen generators for $\pi_1(M)$ are going to be meridians around each of the lines in the arrangement, are going to be contained entirely in $\pi^{-1}(\gamma(0))$, and will have the orientation induced by the complex structure. These meridians, which we will denote by $a_1, a_2, \ldots, a_m$, will start at our base point, follow a straight line to a point near the intersection of the corresponding line H_j with the fiber $\pi^{-1}(\gamma(0))$, go around that point, and come back to the base point following that same straight line. A good way to visualize this is probably to look at what the chosen meridians would be in Example A.9, where $\gamma(0) = -1 \in \mathbb{C}$. This is depicted in Fig. 5, which shows the fiber $\pi^{-1}(\gamma(0))$ of Example A.9. In that figure we have denoted the meridians by a, b, c, d, where $a = a_1$, $b = a_2$, $c = a_3$ and $d = a_4$. The black dots represent the intersection of the arrangement $\mathcal{A}$ with the fiber $\pi^{-1}(\gamma(0))$.

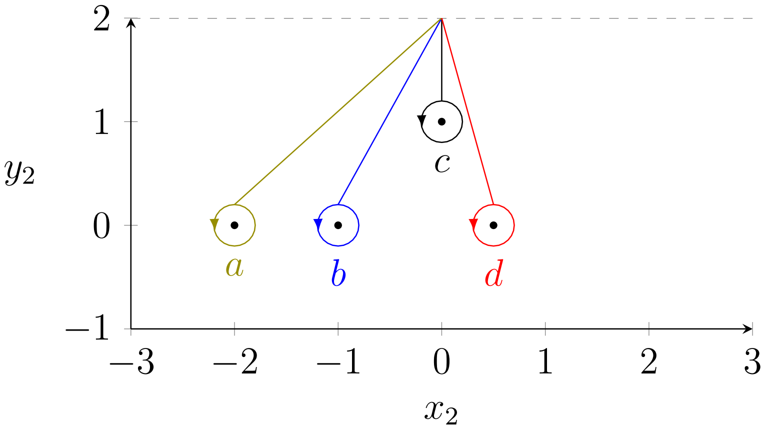


Figure 5: Generators of $\pi_1(M)$ for Example A.9.

From now on, we are going to abuse notation and identify $\pi_1(M)$ with $\pi_1(M, \mathrm{Im}(\gamma) \times \{iR\})$ (we can do so because $\mathrm{Im}(\gamma) \times \{iR\} \subset \mathbb{C}^2$ is contractible). We do this to not have to be as careful with the base point of the local meridians, which we are about to define.

Definition A.10. *Let $t \in [0,1]$ be such that no (actual or virtual) crossing of the marked*

2-graph of $\mathcal{A}$ *associated to the map* h *has their* x_1*-coordinate be* $t + x_1(p) = x_1(\gamma(t))$*, and let* $H_i \in \mathcal{A}$*. We define the* ***local meridian around*** H_i ***at time*** t *to be a path contained in the fiber* $\pi^{-1}(\gamma(t))$ *starting at the point* $(\gamma(t), iR)$ *that goes around* H_i*, defined in the same way as we defined the generators* $\{a_1, \ldots, a_m\}$ *of* $\pi_1(M)$ *in the fiber* $\pi^{-1}(\gamma(0))$ *(looking like the meridians depicted in Fig. 5).*

Remark A.11. *The expression of the local meridians around* H_i *at time* t *as a word in* $\{a_1, \ldots, a_m\}$ *only depends on the strand (the segment in between crossings, actual or virtual) of the unmarked* 2*-graph corresponding to* H_i *at time* t *in which it lies.*

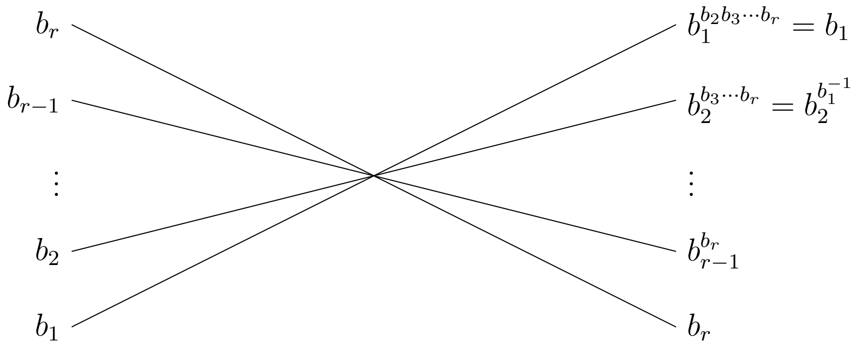
To use braid monodromy arguments to compute a presentation of the fundamental group of $\pi_1(M)$ ([26], [4], [35]), we just have to be able to express the local meridians corresponding to the strands in the 2-graph arriving to a singular point of the arrangement as words in $\{a_1, \ldots, a_m\}$.

Arvola's algorithm does just that in the following way: We start at the left of our marked 2-graph, and assign generators to each of the m-strands (the ones we have called $a_1, \ldots, a_m$). We continue to follow our marked 2-graph to the right, and whenever we encounter a crossing (actual or virtual) we assign words in $\{a_1, \ldots, a_m\}$ to the strands coming after it following the rules in Fig. 6, where we have used the following notation.

Notation A.12. $a^b := b^{-1}ab$

We follow this algorithm until we reach the right end of our marked 2-graph. When we are done, the words in each of the strands correspond to the expression of the corresponding local meridians as a word in our generators $\{a_1, \ldots, a_m\}$.

An example of the computation of Arvola's words for Example A.9 can be seen in Fig. 4. In it, we've only marked the words corresponding to local meridians around the



(a) Actual crossing

(b) Positive virtual crossing (c) Negative virtual crossing

Figure 6: Computation of Arvola's words. These are the consequence of rectifying the braid corresponding to the crossing.

line H_3. All of the local meridians around H_1, H_2 and H_4 are a, b and d respectively, and those don't appear in the picture.

A.3 A presentation for $\pi_1(M)$

We will use the following notation.

Notation A.13. *We denote* $[b_1, \ldots, b_r]$ *the equality of all the cyclic permutations, namely*

$$b_1 \cdot b_2 \cdot \ldots \cdot b_r = b_2 \cdot \ldots \cdot b_r \cdot b_1 = \ldots = b_r \cdot b_1 \cdot \ldots \cdot b_{r-1}$$

Each actual crossing in the marked 2-graph of $\mathcal{A}$ corresponds to a singular point in $\mathcal{P} = \{P_1, \ldots, P_s\}$. For the actual crossing corresponding to P_k, we take a look at the d_k strands just to the left of the crossing, corresponding to the lines in $\mathcal{A}$ that go through P_k. We denote the strands in those words by $b_1(P_k), \dots, b_{d_k}(P_k)$, labeled from the bottom up. Let $R_{P_k} = [b_1(P_k), \dots, b_{d_k}(P_k)]$.

Theorem A.14 (Arvola, [1], Theorem 4.7)**.** *Let $a_1, \dots, a_m$ be the generators of $\pi_1(M)$ described in Appendix A.2, and let $\mathcal{P} = \{P_1, \dots, P_s\}$ be the singular points of $\mathcal{A}$. Then,*

$$\pi_1(M) = \langle a_1, \dots, a_m \quad | \quad R_{P_k} \qquad k = 1, \dots, s\rangle$$

Remark A.15. *The relations R_{P_k} are just those coming from the braid monodromy around the singular points, taking into account that the links of the singular points are Hopf links.*

Remark A.16. *Let $\beta_k = b_1(P_k) \cdot b_2(P_k) \cdot \ldots \cdot b_{d_k}(P_k)$. The relations given by R_{P_k} can be written as the following $d_k - 1$ relations*

$$[\beta_k, b_1(P_k)]$$

$$[\beta_k, b_1(P_k) \cdot b_2(P_k)]$$

$$\vdots$$

$$[\beta_k, b_1(P_k) \cdot b_2(P_k) \cdot \ldots \cdot b_{d_k - 1}(P_k)]$$

which can easily be seen to be equivalent to the following $d_k - 1$ relations

$$[\beta_k, b_j(P_k)] \qquad \textit{for all } j = 1, \dots, d_k - 1$$

The following is a corollary of Theorem A.14 and Remark A.16, and will be useful later on.

Corollary A.17. *Let* $a_1, \dots, a_m$ *be the generators of* $\pi_1(M)$ *described in Appendix A.2, let* $\mathcal{P} = \{P_1, \dots, P_s\}$ *be the singular points of* $\mathcal{A}$*, and let* $\beta_k = b_1(P_k) \cdot b_2(P_k) \cdot \ldots \cdot b_{d_k}(P_k)$ *for all* $k = 1, \dots, s$*. Then,*

$$\pi_1(M) = \langle a_1, \dots, a_m \quad | \quad [\beta_k, b_j(P_k)] \qquad k = 1, \dots, s; \quad j = 1, \dots, d_k - 1 \rangle$$

We conclude this appendix by giving a presentation of the line arrangement described in Example A.9. With the notation used there, using Theorem A.14 we get that

$$\pi_1(M) = \langle a, b, c, d \quad | \quad [b, c, d], [a, c^{bd^{-1}}], [a, d] \rangle$$

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