Schrodinger Scattering:

Perturbative theory and Menchov-Rademacher Theorem Application

By

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Abstract

In this dissertation, we study two aspects of the scattering behavior of Schrödinger operators with slowly decaying and oscillating potentials. One thread starts in the introduction, which develops an approach for demonstrating that the absolutely continuous spectrum of the free Laplacian is preserved by square-summable perturbations that are divergences of square-summable functions. This approach requires the use of standard tools in the analysis of Schrödinger operators: radiation conditions, absorption principle, and precise formulae for the asymptotic decay of the Green's function of the resolvent operators. We discuss the validity of these tools and the forms they take on the integer lattice and the Cayely tree. The other thread starts with a generalization of the Menchov-Rademacher theorem. We apply this generalization to show the existence of wave operators for Schrödinger operators on $\mathbb{R}^+$ where the potential is the sum of a function in L^1 and an L^∞ function that is the derivative of an element of a weighted L^2 space.

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Contents

Chapter 1

Introduction

Since its introduction in 1926, the Schrödinger equation has been the object of intense study because of the central role it plays in the theory of quantum mechanics. The simplest (non-relativistic) formulation of this theory is captured in the Dirac-Von Neumann axioms:

Definition 1.0.1 (Dirac-von Neumann axioms)**.** Let $\mathbb{H}$ be a complex Hilbert space of countable dimension.

1. The *state* of a quantum system is an element of the unit sphere in $\mathbb{H}$.

2. The *observables* of a quantum system are the self-adjoint operators on $\mathbb{H}$.

3. The *expectation value* of an observable A in a quantum system in state ψ is given by $\langle \psi, A\psi \rangle$.

One feature of this set of definitions is that the probability of observing a quantum object in state ψ in a region E in space is given by $\int_E |\psi|^2$. In this formulation, we define the Hamiltonian H by:

$$H_0 \stackrel{\text{def}}{=} -\Delta, \quad H \stackrel{\text{def}}{=} H_0 + V\,,$$

where V represents, abusing notation, the multiplication operator by the function V in the space variable. The time-independent Schrödinger equation states that the dynamics of the quantum system are given by the differential equation:

$$\frac{\partial}{\partial t}\psi(t) = -iH\psi(t)\,. \tag{1.1}$$

Solutions to this system are of the form

$$\psi(t) = e^{-itH}\psi(0)\,, \tag{1.2}$$

so it becomes important to understand the behavior of e^{-itH} as it acts on states ψ. In the case where ψ is an eigenfunction of H, (1.1) and (1.2) become $\psi(t) = e^{-it\lambda}\psi(0)$, where the eigenvalue λ is referred to as the *energy level*.

Historically, the most important Hamiltonians have been the free Hamiltonian (with trivial potential) and those seen in the quantum harmonic oscillator and the hydrogen atom. Consider first the behavior of states in $\mathbb{H} = L^2(\mathbb{R})$ subject to the free Hamiltonian $H_0 = -\frac{\partial^2}{\partial x^2}$. The Schrödinger equation is in this case a dispersive PDE; for example, it satisfies the dispersive inequality:

$$\|e^{-itH_0}u\|_{L^\infty_x(\mathbb{R}^d)} \lesssim t^{-d/2}\|u\|_{L^1_x(\mathbb{R}^d)}\,.$$

For motivated reasons, we will also consider the spectrum of this operator. The operator H_0 is unitarily equivalent under the Fourier transform to the multiplication operator $f(\xi) \to \xi^2 f(\xi)$ on $\mathbb{R}$, and so the spectrum of H_0 is purely absolutely continuous and fills the interval $[0, \infty)$.

The (one-dimensional) quantum harmonic oscillator is the quantum system of states in $\mathbb{H} = L^2(\mathbb{R})$ where the Hamiltonian is given by $H = -\Delta + \frac{1}{2}x^2$. In this case, solutions to the Schrödinger equation are of a different character (see [27]): there is a basis of eigenfunctions

$$\psi_n = \frac{\pi^{-1/4}}{\sqrt{2^n n!}}e^{-x^2/2}H_n(x)\,, \tag{1.3}$$

with $n \in 0, 1, 2, \ldots$ and where H_n are the Hermite polynomials

$$H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2}\,.$$

The nth energy level is given by $\lambda_n = (2n + 1)/2$. The spectrum, correspondingly, has only pure point part, and consists in the set $\sigma(H) = \{(2n + 1)/2 : n \in 0, 1, 2, \ldots\}$.

The behavior of solutions to the Schrödinger equation under the hydrogen atom Hamiltonian exhibits behavior somewhere between the two previous examples. Let $\mathbb{H} = L^2(\mathbb{R}^3)$ and let $V(x) = -1/|x|$. In this case, there is absolutely continuous spectrum filling $[0,\infty)$, but also an infinite sequence of eigenvalues at energies given by $-(2n^2)^{-1}$ for $n = 1, 2, 3, \ldots$. This is in keeping with the physically observed behavior of scattering for positive energies and a discrete spectrum of negative energies corresponding to bound states (see [11, 51]).

The correspondence this set of examples suggests is between confinement in space for eigenvalues and scattering for states with only absolutely continous spectrum. The recurrence for eigenvalues is clear with a simple argument. Suppose that $H\psi = \lambda\psi$. From (1.2), $\psi(t) = e^{-it\lambda}\psi$. This is a periodic function of time, with $\int_E |\psi|^2$ constant; the probability of observing the object in any region is unchanging and so the object is stationary.

Conversely, suppose that the spectral measure corresponding to ψ is purely absolutely continuous. Then, by the spectral theorem, there is a spectral family $E(\lambda)$ such that

$$\begin{aligned} \langle H\psi, \psi\rangle &= \int_{\mathbb{R}} \lambda \, d\langle E(\lambda)\psi, \psi\rangle \,, \\ \langle e^{-itH}\psi, \psi\rangle &= \int_{\mathbb{R}} e^{-it\lambda} \, d\langle E(\lambda)\psi, \psi\rangle \,, \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ denotes the L^2 inner product and where $d\langle E(\lambda)\psi, \psi\rangle$ is finite and absolutely continuous with respect to Lebesgue measure. In other words, this is the Fourier transform of an element of L^1 evaluated at t, so approaches 0 as $t \to \infty$ by the Riemann-Lebesgue Lemma.

It should be stated explicitly that absolute continuity of the spectral measure is not a guarantee of wave propagation. If $\psi \in L^2(\mathbb{R})$ and, for some $x \in \mathbb{R}$, $x \notin \operatorname{ess\,supp}\psi$, then concentration of $\psi(t)$ around x isn't in conflict with the preceding considerations. One approach to making the correspondence between absolutely continuous spectrum and scattering more rigorous is demonstrated in [7], though the model discussed in that paper is physically different to the systems that concern us in this one. An additional complication is that there can be, under some circumstances, non-trivial singular continuous spectrum. This is typically not physical and much effort has gone to excluding this case under various assumptions [47].

Nevertheless, one of the main projects in the study of the Schrödinger equation is the development of a theory that allows the determining of the spectral structure of H for arbitrary V. Here are some of the ideas that have been explored:

1. For radial potentials where $|V| < C(|x|+1)^{-\alpha}$ for some $\alpha > 1$, the spectrum on $[0,\infty)$ is purely absolutely continuous [56].

2. Potentials V with the property that $-\Delta + V$ has a positive eigenvalue embedded in the absolutely continuous spectrum are referred to as *von Neumann-Wigner potentials*. The first example was a radially-symmetric potential on $\mathbb{R}^3$ provided in [52] as

$$V(r) = -\frac{8\sin(2r)}{r} + O(r^{-2}) \quad \text{where } r \to \infty\,.$$

3. Let $V : \mathbb{R}^d \to \mathbb{R}$ satisfy

$$\int |x|^{-d+1} V(x)^2\, dx < \infty\,.$$

   Simon has conjectured that $-\Delta+V$ has absolutely continuous spectrum of infinite multiplicity filling $[0,\infty)$ [49].

In general, the one-dimensional case is understood much better than the multi-dimensional case; it is a major effort to extend results to higher dimensions. One success in this direction is the argument in [16], with which Denisov demonstrates preservation of the absolutely continuous spectrum of the free Laplacian under perturbations $V = \operatorname{div} Q$ with $Q \in C^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$ and $V \in L^2(\mathbb{R}^3)$. The central theme of this thesis is the re-creation of the tools required for the application of this argument in various discrete spaces. In order to better understand how these tools fit together in the larger picture, it is instructive to reproduce the argument for preservation of the absolutely continuous spectrum under potentials that are square-summable divergences in one dimension; this follows in the next section.

1.1 Square-summable Potentials that Oscillate

Let

$$H_0 \stackrel{\text{def}}{=} -\frac{d^2}{d^2x}, \quad H \stackrel{\text{def}}{=} H_0 + V, \quad x \in \mathbb{R},$$

where the potential $V(x)$ is real-valued. The method we are about to explain works in the case when the potential V decays slowly and oscillates. To model this behavior, we assume that

$$V = Q',$$

where $Q \in C^1(\mathbb{R})$ and $Q, Q' \in L^2(\mathbb{R})$. Under these assumptions, $V \in L^2(\mathbb{R})$ and the operator H is self-adjoint. Since V decays at infinity, the Weyl's Theorem on essential spectrum (see, e.g., [46]) gives us $\sigma_{\text{ess}}(H) = [0, \infty)$. We are interested in studying its absolutely continuous component.

Theorem 1.1.1. *If $Q \in C^1(\mathbb{R})$ and $Q, Q' \in L^2(\mathbb{R})$, then the absolutely continuous spectrum of H is equal to $[0, \infty)$, i.e., $\sigma_{\text{ac}}(H) = [0, \infty)$.*

Take $\phi(x)$, a smooth even function, which is supported on the interval $[-1, 1]$ and satisfies: $\phi(x) = 1$ for $x \in [-0.5, 0.5]$ and $0 \leqslant \phi \leqslant 1$. When proving this theorem, it will be convenient to recall the Birman-Kuroda Theorem (see, e.g., [47]) which states that the a.c. spectrum is stable under relative trace class perturbations. In particular, this general result gives

$$\sigma_{\text{ac}}\left(-\frac{d^2}{dx^2} + V\right) = \sigma_{\text{ac}}\left(-\frac{d^2}{dx^2} + V_L\right),$$

where $V_L \stackrel{\text{def}}{=} Q'_L, Q_L \stackrel{\text{def}}{=} Q(1 - \phi(x/L))$ and L is an arbitrary positive number. The function $(Q\phi(x/L))'$ is bounded and compactly supported, so it is a relative trace class perturbation of H_0 for every finite L and the Birman-Kuroda Theorem is applicable.

Since

$$\lim_{L\to\infty} \|V_L\|_2 = 0, \quad \lim_{L\to\infty} \|Q_L\|_2 = 0,$$

it suffices to prove the result for Q and V with arbitrarily small L^2-norms.

Define $\Pi^+ \stackrel{\text{def}}{=} \{k \in \mathbb{C} : \operatorname{Im} k > 0 \text{ and } \operatorname{Re} k > 0\}$ and start with the following observation. Take $f \in L^2(\mathbb{R})$ and assume that it has compact support, i.e., $\operatorname{supp} f \subset [-T, T]$ for some $T > 0$. We also assume it is not identically equal to zero.

For $k \in \Pi^+$ and $z = k^2 \in \mathbb{C}^+$, denote the Green's function of H by $G_{k^2}(x, y)$ so that

$$R_z f \stackrel{\text{def}}{=} (H - k^2)^{-1} f = \int_{\mathbb{R}} G_{k^2}(x, y) f(y) dy\,.$$

The spectral measure of f relative to H is denoted by μ_f and its Cauchy transform is defined by the Spectral Theorem as follows:

$$(R_{k^2} f, f) = \int_{\mathbb{R}} \frac{d\mu_f}{\lambda - k^2}\,.$$

We will prove Theorem 1.1.1 by taking arbitrary $I \subset \mathbb{R}^+$ and showing that

$$\int_I \log \mu_f' d\lambda > -\infty\,, \tag{1.4}$$

provided that $\|Q\|_2$ and $\|Q'\|_2$ are small enough.

If true, (1.4) shows that $\mu_f' \geqslant 0$ a.e. on I. On the other hand, since I is chosen arbitrarily and the a.c. spectrum is stable under relative trace class perturbations, the statement of the theorem follows.

To prove (1.4), we take $R > 0$ and define the following truncated potential:

$$V_R(x) \stackrel{\text{def}}{=} Q_R', \quad Q_R = Q \cdot \phi(x/R)\,.$$

Given our assumptions on Q and ϕ, it is easy to see that

$$\|V_R - V\|_{L^2(\mathbb{R})} \to 0, \quad \|Q_R - Q\|_{L^2(\mathbb{R})} \to 0$$

as $R \to \infty$. If we define $H_R \stackrel{\text{def}}{=} -\partial^2/d^2r + V_R$, the standard perturbation theory gives

$$\langle (H_R - k^2)^{-1} f, f\rangle \to \langle (H - k^2)^{-1} f, f\rangle$$

when $k \in \Pi^+$ and $R \to \infty$. This implies, in particular, that $\mu_f^{(R)} \stackrel{*}{\to} \mu$ when $R \to \infty$, where $\mu_f^{(R)}$ is the spectral measure of f relative to H_R. The logarithmic integral is stable with respect to weak convergence (see [32]), i.e.,:

If the sequence of measures $\{\mu_R\}$ satisfies $\mu_R \stackrel{*}{\to} \mu$ for some interval I, then

$$\int_I \log \mu' d\lambda \geqslant \liminf_{R\to\infty} \int_I \log \mu_R' d\lambda \,.$$

Thus, to establish (1.4), we only need to prove that

$$\int_I \log \mu_f' d\lambda > C_{(I,\|Q\|_2,\|Q'\|_2)} \tag{1.5}$$

holds for compactly supported Q with arbitrarily small values of $\|Q\|_2$ and $\|V\|_2$. From now on, we assume that $\operatorname{supp} Q \subset [-R, R]$.

We will show now that (1.5) is in fact the consequence of some simple facts in complex analysis and a straightforward a priori estimate on the solution $u \stackrel{\text{def}}{=} R_{k^2} f$. Notice that $u(x,k)$ solves

$$-u'' + Vu = k^2 u + f \tag{1.6}$$

and thus satisfies

$$u(x,k) = \frac{e^{ik|x|}}{2ik}\Big(a^{\pm}(k) + o(1)\Big), \quad x \to \pm\infty \,, \tag{1.7}$$

where $a^{\pm}(k)$ will be called *amplitudes*. We also recall that the free Green's function, i.e., the Green's function for $V = 0$, is given by

$$G^0_{k^2}(x,y,k^2) = \frac{e^{ik|x-y|}}{2ik} \,. \tag{1.8}$$

We will need some properties of $a^{\pm}$. First, we notice that

$$u(x,k) = \int_{-T}^{T} G_{k^2}(x,y) f(y) dy, \quad a^{\pm}(k) = 2ik \lim_{x\to\pm\infty} \left(u(x,k) e^{-ik|x|} \right).$$

Since V is compactly supported, $G_{k^2}(x,y)$ is analytic in $k \in \Pi^+$ and is continuous in k up to the boundary $(0,\infty)$; this in turn implies:

1. $a^{\pm}$ is analytic in k in Π^+,

2. $a^{\pm}$ is continuous in k up to $\mathbb{R}^+$.

Lemma 1.1.2 (**Factorization identity**). *We have*

$$\mu_f'(\xi^2) = \frac{|a^+(\xi)|^2 + |a^-(\xi)|^2}{4\pi\xi}, \quad \xi \in \mathbb{R}^+ .$$

Proof. Take $k \in \Pi^+$, multiply (1.6) by $\overline{u}$, and integrate from $-l$ to l, and take the imaginary part to get

$$-\frac{1}{2i} \int_{-l}^{l} \left(u''\bar{u} - \bar{u}''u \right) dx = \left(\operatorname{Im} k^2 \right) \int_{-l}^{l} |u|^2 dx + \operatorname{Im}\langle f, u \rangle . \tag{1.9}$$

We integrate the left hand side by parts to get

$$\int_{-l}^{l} \left(u''\bar{u} - \bar{u}''u \right) dx = \left(u'(l,ik)\bar{u}(l,ik) - u'(-l,ik)\bar{u}(-l,ik) \right) - \left(\bar{u}'(l,ik)u(l,ik) - \bar{u}'(-l,ik)u(-l,ik) \right). \tag{1.10}$$

In (1.9), we take $k \to \xi$ where $\xi \in \mathbb{R}^+$ and use the Spectral Theorem and properties of Poisson integral to conclude that

$$\lim_{k\to\xi} \operatorname{Im}\langle f, u \rangle = -\lim_{k\to\xi} \operatorname{Im} \int \frac{d\mu_f(\lambda)}{\lambda - k^2} = -\pi \mu_f'(\xi^2) .$$

For $u(\pm l, \xi)$, we have

$$u(x,\xi) = \frac{e^{i\xi x}}{2i\xi} a^+(\xi), \quad x > l$$

and

$$u(x,\xi)=\frac{e^{-i\xi x}}{2i\xi}a^-(\xi),\quad x<-l\,.$$

Substitution into (1.10) gives the required formula. □

We can write

$$\log \mu_f'(\xi^2)\geqslant \log|a^+(\xi)|+\log|a^-(\xi)|-\log(2\pi\xi)$$

and proving the bounds

$$\int_I \log|a^+(\xi)|d\xi > C_{(\|Q\|_2,\|Q'\|_2,f)},\quad \int_I \log|a^-(\xi)|d\xi > C_{(\|Q\|_2,\|Q'\|_2,f)} \tag{1.11}$$

is enough to justify (1.5).

The crucial observation is that the functions $\log|a^\pm|$ are subharmonic in Π^+. We take advantage of that in the next lemma. Let $\mathfrak{R}_{[\alpha,\beta],h}$ be the rectangle of height $h\geqslant 1$ with base $[\alpha,\beta]\subset\mathbb{R}^+$ (see Figure 1.1). Denote the corresponding harmonic measure with reference point at $\eta\in\mathfrak{R}_{[\alpha,\beta],h}$ by $\omega_\eta(k)$, $k\in\partial\mathfrak{R}_{[\alpha,\beta],h}$.

Lemma 1.1.3. *We have the bound*

$$\int_{\partial\mathfrak{R}_{[\alpha,\beta],h}} \log|a^\pm(k)|d\omega_\eta(k)\geqslant \log|a^\pm(\eta)|\,. \tag{1.12}$$

Proof. The functions $\log|a^\pm(\eta)|$ are continuous in $\mathfrak{R}_{[\alpha,\beta],h}$ up to its boundary, so the mean value inequality for subharminic functions gives (1.12). □

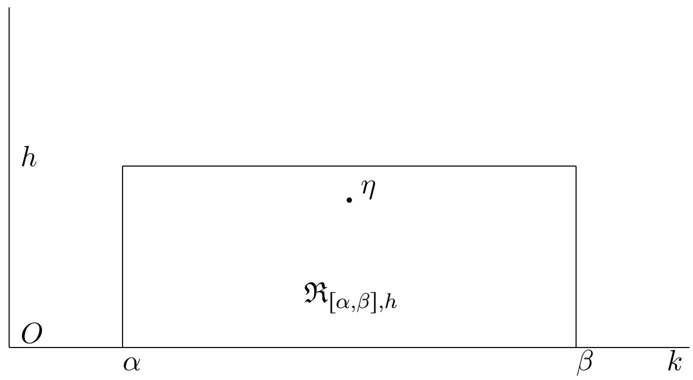


Figure 1.1

The last Lemma gives us an estimate

$$\int_\alpha^\beta \log|a^\pm(\xi)|d\omega_\eta(\xi) \geqslant \log|a^\pm(\eta)| - \int_\Gamma \log^+|a^\pm(k)|d\omega_\eta(k)\,, \tag{1.13}$$

where $\log^+ t \stackrel{\text{def}}{=} \max\{\log t, 0\}$ and $\Gamma \stackrel{\text{def}}{=} \partial\mathfrak{R}_{[\alpha,\beta],h}\backslash[\alpha,\beta]$.

Consider the harmonic measure ω_η first. We get simple bounds:

$$\omega'_\eta(\xi) \sim_\eta \min(|\xi-\alpha|, |\xi-\beta|) \tag{1.14}$$

and, similarly,

$$\omega'_\eta(\alpha+iy) \sim_\eta y, \quad \omega'_\eta(\beta+iy) \sim_\eta y \tag{1.15}$$

for all $y \in (0,1)$. Throughout all of Γ, we have a simple rough bound $\omega'_\eta(k) \lesssim_\eta 1$.

These estimates from Complex Analysis suggest the bounds one needs to get for $a^\pm$ away from the spectrum. We establish these estimates in the following two Lemmas.

Lemma 1.1.4 (Rough bounds). *For $k \in \mathfrak{R}_{[\alpha,\beta],h}$, we have*

$$\|u\|_\infty < \frac{C_{(\|V\|_2, f, \alpha, \beta, h)}}{\operatorname{Im} k}, \qquad \|u'\|_\infty < \frac{C_{(\|V\|_2, f, \alpha, \beta, h)}}{\operatorname{Im} k}. \tag{1.16}$$

Proof. From the Spectral Theorem, we have

$$\|u\|_2 \lesssim \frac{\|f\|_2}{\operatorname{Im} k}.$$

The second resolvent identity yields

$$u = R^0 f - R^0 V u = \int \frac{e^{ik|x-y|}}{2ik} f(y)dy - \int \frac{e^{ik|x-y|}}{2ik} V(u)u(y,k)dy. \tag{1.17}$$

It is enough to use Cauchy-Schwarz to get the bound for u. The proof of the bound for derivative is analogous. □

In what follows, we will focus on proving the bound (1.11) for a^+ only. The estimate for a^- can be obtained similarly. We recall that $f = 0$ outside $[-T, T]$. Thus, if we introduce $A(x,k)$ as

$$u(x,k) = \frac{e^{ikx}}{2ik} A(x,k), \quad x > T.$$

In other words,

$$A(x,k) = 2ike^{-ikx} u(x,k).$$

Lemma 1.1.5. *Suppose $k \in \Pi^+$ and x are fixed. If we let $\|Q\|_2, \|V\|_2 \to 0$, then*

$$A(x,k) \to A_0(x,k), \quad A'(x,k) \to A_0'(x,k),$$

where

$$A_0(x,k) = 2ike^{-ikx} u_0(x,k) = e^{-ikx} \int e^{ik|x-y|} f(y)dy$$

and

$$A_0'(x,k) = \left(e^{-ikx}\int e^{ik|x-y|}f(y)dy\right)_x' .$$

Proof. The proof follows immediately from the identity (1.17). $\square$

For $A(T,k)$ and $A'(T,k)$, we can use the rough bounds (1.16) to get

$$|A(T,k)| \leqslant \frac{C(f,\alpha,\beta,h,\|V\|_2)}{\operatorname{Im} k}, \quad |A'(T,k)| \leqslant \frac{C(f,\alpha,\beta,h,\|V\|_2)}{\operatorname{Im} k} . \tag{1.18}$$

For A, we have an equation

$$A''(x,k) + 2ikA'(x,k) = Q'A(x,k), \quad x > T . \tag{1.19}$$

Next, we will use this equation to obtain some apriori bounds on A. In what follows, constants C will depend only on α,β,h,f. Again, we assume that $\|Q\|_2 + \|V\|_2 \leqslant \delta$ where δ is arbitrarily small and depends only on parameters α,β,h and f.

Lemma 1.1.6 **(Upper bound).** *There is a δ_0 which depends only on α,β,h and f so that*

$$|a^+(k)| \leqslant \frac{C}{\operatorname{Im}^{1.5} k} \tag{1.20}$$

for all $k \in \mathfrak{R}_{[\alpha,\beta],h}$ as long as $\delta \overset{\text{def}}{=} \|Q\|_2 + \|V\|_2 \leqslant \delta_0$.

Proof. We introduce two quantities now:

$$M_0 \overset{\text{def}}{=} \|A\|_{L^\infty[T,\infty)}, \quad M_1 = \|A'\|_{L^2[T,\infty)} .$$

Multiply (1.19) by $\bar{A}'$, integrate from s to t, and take the real part to get

$$|A'(t,k)|^2 \leqslant |A'(s,k)|^2 + C\int_s^t |A'|^2 du + C\int_s^t |QAA'| du .$$

Integrating in s from t to $t+1$ and taking supremum in t after that gives us

$$\sup_{t>T} |A'(t,k)|^2 \lesssim M_1^2 + \delta M_1 M_0\,, \quad \sup_{t>T} |A'(s,k)| \lesssim M_1 + \delta M_0\,. \tag{1.21}$$

Rewrite (1.19) in the form

$$\frac{A''}{2ik} + A' = \frac{Q'A}{2ik}. \tag{1.22}$$

Multiply (1.22) by $\bar{A}$, integrate from T to τ, and take real part of both sides to get

$$\frac{\operatorname{Im} k}{2|k|^2}\int_T^\tau |A'(x,k)|^2dx + \frac{|A(\tau,k)|^2}{2} \leqslant \tag{1.23}$$
$$\frac{|A(T,k)|^2}{2} + \frac{\operatorname{Im} k}{2|k|^2}\int_T^\tau Q'|A|^2dx + \frac{1}{2|k|}\Big(|A'(\tau,k)A(\tau,k)| + |A'(T,k)A(T,k)|\Big)\,.$$

Integrating by parts, we get

$$\int_T^\tau Q'|A|^2dx = Q(\tau)|A(\tau,k)|^2 - Q(T)|A(T,k)|^2 - \int_T^\tau Q'(A'\bar{A} + A\bar{A}')dx\,.$$

For $\|Q\|_\infty$, one can write

$$\|Q\|_\infty \lesssim \|Q\|_2 + \|V\|_2 \leqslant \delta\,.$$

First, send $\tau \to \infty$ in (1.23) and use (1.18) to get

$$M_1^2 \leqslant \frac{C_1}{\operatorname{Im}^3 k} + C_2\delta M_1 M_0\,.$$

Solving this inequality, one has

$$M_1 \leqslant \frac{C_1}{\operatorname{Im}^{1.5} k} + C_2\delta M_0\,. \tag{1.24}$$

Next, we drop the first term in the left hand side of (1.23) and take supremum in τ. Combining the bounds, we get

$$M_0^2 \leqslant \frac{C}{\operatorname{Im}^2 k} + CM_0(M_1 + \delta M_0) + \operatorname{Im} k\Big(\delta M_0^2 + \frac{\delta}{\operatorname{Im}^2 k} + \delta M_0 M_1\Big).$$

Substitute (1.24) into this bound and take δ small enough to get

$$M_0^2 \leqslant \frac{C}{\operatorname{Im}^2 k} + \frac{CM_0}{\operatorname{Im}^{1.5} k}\,.$$

Finally, solving this inequality, we get the bound

$$|a^+(k)| = \lim_{x\to+\infty} |A(x,k)| \leqslant \sup_{x\geqslant T} |A(x,k)| = M_0 \leqslant \frac{C}{\operatorname{Im}^{1.5} k}\,. \tag{1.25}$$

□

In the next Lemma, we obtain a lower bound.

Lemma 1.1.7 (Lower bound). *There is a point* $\eta \in \mathfrak{R}_{[\alpha,\beta],h}$ *such that*

$$|a^+(\eta)| > C > 0\,, \tag{1.26}$$

provided that $\delta = \|Q\|_2 + \|V\|_2 < \delta_0(\alpha, \beta, h, f)$.

Proof. Integrate (1.22) from T to ∞. This gives

$$a^+(k) = A(T,k) + \frac{A'(T,k)}{2ik} + \frac{1}{2ik}\left(\int_T^\infty Q' A dx\right)\,. \tag{1.27}$$

Integrate by parts to get

$$\int_T^\infty Q' A dx = -Q(T)A(T,k) - \int_T^\infty QA' dx\,.$$

We combine these inequalities as

$$\left| a^+(k) - \left(A(T,k) + \frac{A'(T,k)}{2ik}\right)\right| \leqslant C\Big(|Q(T)A(T,k)| + \|Q\|_2 M_1\Big) \tag{1.28}$$

after we apply Cauchy-Schwarz. For $Q(T)$, we write $|Q(T)| \lesssim \|Q\|_2 + \|V\|_2 = \delta$ and we can use

(1.24) and (1.25) for M_1. In the end, we get

$$|a^+(k) - \Psi(k)| \lesssim \frac{C\delta}{\mathrm{Im}^{1.5}\, k}\,,$$

where we introduced

$$\Psi(k) \stackrel{\text{def}}{=} A(T,k) - \frac{A'(T,k)}{2ik}$$

for shorthand. Notice that

$$\Psi(k) \to \Psi_0(k) = A_0(T,k) - \frac{A_0'(T,k)}{2ik} = \int_{-T}^{T} e^{-iky} f(y)dy = \widehat{f}(k)$$

when $\|Q\|_2 \to 0$ and $\|V\|_2 \to 0$ by Lemma 1.1.5. Thus,

$$|a^+(k) - \Psi^0(k)| \to 0$$

over compact subsets in $\mathfrak{R}_{[\alpha,\beta],h}$ when $\delta \to 0$.

Next, consider the function $\Psi_0(k)$. It is analytic in k and is not identically equal to zero. Thus, we can find a point $\eta \in \mathfrak{R}_{[\alpha,\beta],h}$ such that $\Psi_0(\eta) \neq 0$. Fixing that η and taking δ small enough, we get $|a^+(\eta)| > 0$. □

Now, we are ready to finish the proof of the theorem.

Proof of Theorem 1.1.1. We start by fixing an interval I and taking α and β such that $I \subset (\alpha, \beta)$. As discussed before, we can assume that Q is compactly supported and $\|Q\|_2 \leqslant \delta, \|V\|_2 \leqslant \delta$ where δ is an arbitrarily small parameter which depends only of α, β, h and f. Under these assumptions, we have to show

$$\int_I \log |a^\pm(\xi)| d\xi > C(\delta, f, I)\,. \tag{1.29}$$

We will only handle a^+, as the bound for a^- is analogous. The inequality (1.13) and estimate

(1.14) on harmonic measure give us

$$\int_I \log|a^+(\xi)|d\xi \gtrsim \log|a^+(\eta)| - \int_\Gamma \log^+|a^+(k)|d\omega_\eta(k)\,.$$

We need a lower bound for the first term in the right hand side and an upper bound for the second one. We take η from Lemma 1.1.7 and apply this lemma to get the lower bound provided that δ is small enough. To get an upper bound for the estimate, we employ (1.15) and (1.20). In the end, we have (1.29). □

Remark. The method explained above is robust in several ways. First, the use of the Birman-Kuroda Theorem allows one always to assume that the norm of the potential V is arbitrarily small. Secondly, the method requires getting only very rough uniform upper bounds on $|a^\pm|$. Thirdly, we intentionally never exploited ODE techniques. Instead, we used the method of a priori estimates, which turns out to be applicable in higher-dimensional case.

1.2 Organization of this paper

As briefly mentioned in the first section, one of the major challenges in scattering theory is the extension of results that have been proven in the one-dimensional case to higher dimensions. Partly as a suite of toy models to facilitate this translation in the continuous case, but also partly as models for quantum objects in crystals and other lattice-like structures and partly in an attempt to understand how discretization affects computational simulation of quantum mechanics, the study of the Schrodinger operator on discrete spaces has been a major adjunct to its study in Euclidean space.

The question that originally guided the research that appears in this manuscript was whether the arguments presented in the previous section could be translated to any discrete settings. Those arguments depend on several preliminary facts about solutions to the Schrödinger equation. The first of these is the *absorption principle*, which allows us to express the solution u to

$$(H - \xi I)u = f$$

for $\xi \in \mathbb{R}$ as the limit of solutions to the same equation with ξ replaced by k as $k \to \xi$ for $k^2 \in \mathbb{C}^+$. The second of the preliminaries is the radiation conditions, appearing in (1.7), for k on and off the spectrum of H; the third is the *factorization identity* appearing in Lemma 1.1.2.

Another perturbative-theoretic tool we investigate is the *Born series*, produced by recursing on u in the second resolvent identity in (1.17). The nth order Born expansion is

$$Rf = R^0 f - R^0 V R f = \left(\sum_{j=0}^{n-1} (-1)^j (R^0 V)^j R^0 f \right) + (-1)^n (R^0 V)^n R f \,, \tag{1.30}$$

where we abuse notation to identify V with it associated multiplication operator. In the case where $\|R_0 V\|_{2\to 2} < 1$, the norm of the difference between the nth order and mth order Born expansions is

$$\left\| \left(\sum_{j=n}^{m-1} (-1)^j (R^0 V)^j R^0 \right) + (-1)^m (R^0 V)^m R - (-1)^n (R^0 V)^n R \right\| \lesssim \frac{\|R^0 V\|^n}{1 - \|R^0 V\|} (1 + \|R\|)$$

for $n < m$. The sequence of Born expansions of increasing order is Cauchy in the operator norm topology so converges in that topology; we write the full Born series as

$$Rf = \sum_{j=0}^{\infty} (-1)^j (R^0 V)^j R^0 f \,. \tag{1.31}$$

Chapters 3 and 4 discuss the state of what is known for each of these analytical tools for the integer lattice $\mathbb{Z}^d$ and the Cayley tree T_d, with a new formulation for Green's function asymptotics given in Chapter 3. Before those latter chapters, we will take a detour through some results about the asymptotic dynamics of the Schrödinger operator using another approach to formalization of the notion of oscillation and decay. That is what concerns Chapter 2.

Chapter 2

Generalizations of Menchov-Rademacher Theorem and Existence of Wave Operators in Schrödinger Evolution

2.1 Introduction

The Menchov-Rademacher theorem is a classical theorem on the convergence of series of orthogonal functions. Originally proved independently by Rademacher [45] in 1922 and Menchov [42] in 1923 (see also [29]), it states:

Theorem 2.1.1 (Menchov-Rademacher). *Suppose $\{\phi_n(x)\}, n \in \mathbb{N}$ is orthonormal system in $L^2(0,1)$ and the sequence $\{a_n\}$ satisfies*

$$l \overset{\text{def}}{=} \sum_{n=1}^{\infty} a_n^2 \log^2(n+1) < \infty\,.$$

Then, the series $\sum_{n=1}^{\infty} a_n \phi_n(x)$ converges for a.e. $x \in (0,1)$. Moreover, if

$$m(x) \overset{\text{def}}{=} \sup_{n\in\mathbb{N}} \left|\sum_{j=1}^{n} a_j \phi_j(x)\right|$$

defines a maximal function, then

$$\|m\|_{L^2(0,1)} \leqslant Cl^{1/2}$$

with some absolute constant C.

This result can be easily modified to cover orthonormal systems in $L^2_\mu(0,1)$ where μ is a measure on $(0,1)$; most of the literature around the Menchov-Rademacher theorem concerns finding sharp bounds for the constant (e.g. [4], [8], [41]). This chapter proceeds by generalizing the Menchov-Rademacher theorem to a setting of continuous coefficients, then applying it to show existence of wave operators for Schrödinger evolution.

This work is based on a joint paper written with Denisov [19]. My contribution to the paper was mostly focused on the generalization of the Menchov-Rademacher Theorem.

We start with the following definitions.

Definition. We say that $f \in L^2_{\text{loc}}(\mathbb{R}^+)$ if

$$\int_0^a |f(r)|^2 dr < \infty \tag{2.1}$$

for all $a > 0$.

Definition. Let a pair (P, σ) consist of a function $P(r,k) : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{C}$ and a measure σ on $\mathbb{R}$. We say that (P, σ) is a continuous orthonormal system if

(a) for σ-a.e. $k \in \mathbb{R}$, $P(r,k) \in L^2_{\text{loc}}(\mathbb{R}^+)$,

(b) for every $f \in L^2(\mathbb{R}^+)$ and every $a > 0$, we have

$$\int_{\mathbb{R}} \left| \int_0^a f(r)P(r,k)dr \right|^2 d\sigma(k) = \int_0^a |f(r)|^2 dr \,.$$

Our first result is the following theorem.

Theorem 2.1.2. *Suppose (P, σ) is continuous orthonormal system and*

$$L \stackrel{\text{def}}{=} \int_{\mathbb{R}^+} |f(r)|^2 \log^2(2+r) dr.$$

Then, the sequence $\left\{\int_0^n f(r)P(r,k)dr\right\}$ *converges for* σ*-a.e.* $k\in\mathbb{R}$*. Moreover, if*

$$M(k)\stackrel{\text{def}}{=}\sup_{n\in\mathbb{N}}\left|\int_0^n f(r)P(r,k)dr\right|,$$

then $\|M\|_{L^2_\sigma(\mathbb{R})}\leqslant CL^{1/2}$ *with some absolute constant* C*.*

Definition. We will call continuous orthonormal system (P,σ) normalized if there is a continuous positive function κ defined on $\mathbb{R}$ such that

$$\kappa^{-1}\in L^\infty(\mathbb{R}),\quad K\stackrel{\text{def}}{=}\sup_{r\geqslant 0}\int_{\mathbb{R}}\frac{|P(r,k)|^2}{\kappa(k)}d\sigma<\infty\,. \tag{2.2}$$

For the normalized systems, the previous theorem can be improved in the following way.

Theorem 2.1.3. *Consider the normalized continuous orthonormal system* (P,σ,κ) *and suppose that* $f\log(2+r)\in L^2(\mathbb{R}^+)$*, then*

$$\int_{\mathbb{R}}\sup_{t>0}\left|\int_0^t f(r)P(r,k)dr\right|^2\frac{d\sigma}{\kappa(k)}\lesssim\left(\|\kappa^{-1}\|_{L^\infty(\mathbb{R})}+K\right)\int_0^\infty|f(r)|^2\log^2(2+r)dr\,. \tag{2.3}$$

Moreover, as $R\to\infty$*,*

$$\int_0^R f(r)P(r,k)dr\to\int_0^\infty f(r)P(r,k)dr \tag{2.4}$$

for a.e. k *with respect to measure* σ*.*

One example of continuous orthonormal system is given by solutions $\{P(r,k)\}$ to the Krein system [14, 37]. The Krein system is the following linear system of differential equations

$$\begin{cases} P'(r,k)=ikP(r,k)-\overline{A(r)}P_*(r,k), & P(0,k)=1\\ P_*'(r,k)=-A(r)P(r,k), & P_*(0,k)=1\end{cases},\quad k\in\mathbb{C},\quad r\geqslant 0\,. \tag{2.5}$$

In this paper, we will always assume that the coefficient $A\in L^2_{\text{loc}}(\mathbb{R}^+)$. The Cauchy problem (2.5) has the unique solution $(P(r,k),P_*(r,k))$. In [37] (see also, e.g., [13]), Krein showed that $\{P(r,k)\}$

with $r \geqslant 0$ and $k \in \mathbb{R}$ can be viewed as continuous analogs of polynomials, orthogonal on the unit circle. In particular, there is a measure σ on $\mathbb{R}$, which satisfies

$$\int_{\mathbb{R}} \frac{d\sigma(k)}{1+k^2} < \infty\,,$$

and the property

$$\int_{\mathbb{R}} \left| \int_0^a f(r)P(r,k)dr \right|^2 d\sigma = \int_0^a |f(r)|^2 dr \tag{2.6}$$

holds for every $f \in L^2(\mathbb{R}^+)$. In other words, a pair (P, σ) gives an example of continuous orthonormal system. Notice that (2.6) allows us to define the generalized Fourier transform,

$$\int_0^\infty f(r)P(r,k)dr\,,$$

as an element of $L^2_\sigma(\mathbb{R})$.

Under a mild extra assumption on coefficient A, the system (P, σ) becomes normalized and the previous theorem can be applied. More precisely, the following lemma holds.

Lemma 2.1.4. *Suppose the coefficient A in Krein system belongs to the Stummel class, i.e.,*

$$\|A\|_{\mathrm{St}} \stackrel{\mathrm{def}}{=} \sup_{r \geqslant 0} \left(\int_r^{r+1} |A(\rho)|^2 d\rho \right)^{1/2} < \infty\,. \tag{2.7}$$

Then,

$$\sup_{r>0} \int_{\mathbb{R}} \frac{|P(r,k)|^2}{1+k^2} d\sigma \lesssim 1 + \|A\|^2_{\mathrm{St}}\,. \tag{2.8}$$

Moreover, we have (2.3) *and* (2.4) *with* $\kappa(k) = 1 + k^2$ *and* $K \lesssim 1 + \|A\|^2_{\mathrm{St}}$.

The proof of this Lemma is given in Appendix.

Another application of our general results to the Krein systems is given in the following Lemma.

Lemma 2.1.5. *Suppose the coefficient in Krein system satisfies* $A(r)\log(2+r) \in L^2(\mathbb{R}^+)$. *Then*

$$\int_{\mathbb{R}} \left(\sup_{\rho<r_1<r_2} \left| \int_{r_1}^{r_2} A(x)P(x,k)dx \right| \right)^2 \frac{d\sigma}{1+k^2} = \tag{2.9}$$

$$\int_{\mathbb{R}} \left(\sup_{\rho<r_1<r_2} |P_*(r_2,k) - P_*(r_1,k)| \right)^2 \frac{d\sigma}{1+k^2} \lesssim (1+\|A\|_2^2) \int_\rho^\infty |A(r)|^2 \log^2(2+r)dr\,,\ \rho > 0\,.$$

Moreover, for Lebesgue a.e. $k \in \mathbb{R}$, *there is a limit* $\Pi(k) = \lim_{r\to\infty} P_*(r,k)$.

Theorem 2.1.2, Theorem 2.1.3 and Lemma 2.1.5 are proved in the second section. In section 3, we apply Lemma 2.1.5 to show existence of wave operators for Schrödinger evolution which is our central result. Consider

$$H = -\partial^2_{xx} + v$$

on $\mathbb{R}^+$ with Dirichlet boundary condition at zero and denote by $H_0 = -\partial^2_{xx}$ the free Schrödinger operator with the same Dirichlet condition at zero. The Moller wave operators (see, e.g., [57]) are defined by

$$W^\pm(H,H_0) \stackrel{\text{def}}{=} \lim_{t\to\pm\infty} e^{itH}e^{-itH_0}\,,$$

where the limit is the strong limit in $L^2(\mathbb{R}^+)$. The main result of our paper is the following theorem.

Theorem 2.1.6. *Suppose* $v = a' + q$ *where* $q \in L^1(\mathbb{R}^+)$, *a is absolutely continuous on* $\mathbb{R}^+$, *and*

$$a' \in L^\infty(\mathbb{R}^+), \quad a\log(2+r) \in L^2(\mathbb{R}^+)\,. \tag{2.10}$$

Then, the wave operators $W^\pm(H,H_0)$ *exist.*

The existence of wave and modified wave operators for Schrödinger and Dirac equations was extensively studied in the scattering theory of wave propagation, see, e.g., the classical papers by Agmon [1], Hörmander [28], and a book by T. Kato [30] on the subject. The case $v \in L^p(\mathbb{R}^+), 1 \leqslant p < 2$ was considered in [9] where the existence of modified wave operators was proved. See [15] for later developments. In [13], the presence of wave operators was established for Dirac equation with potential in $L^2(\mathbb{R}^+)$. This result is optimal on $L^p(\mathbb{R}^+)$ scale. For more general potentials in Dirac equation and connection to Szegő condition on measure σ, see [5]. Some related recent results, including the multidimensional setting, can be found in, e.g., [21, 22, 39].

Notation

1. If f is defined on $\mathbb{R}$, $\widehat{f}$ denotes its Fourier transform:

$$\widehat{f}(k) \stackrel{\text{def}}{=} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-ikx} dx\,.$$

   The inverse Fourier transform is defined as

$$\check{f}(k) = f^{\vee}(k) \stackrel{\text{def}}{=} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{ikx} dx\,.$$

2. Symbol $C^\infty(\mathbb{R})$ stands for infinitely smooth functions defined on the real line and $C_c^\infty(\mathbb{R})$ denotes the space of smooth functions with compact support.

3. We will use the symbol $C_{(a_1,\ldots,a_k)}$ to indicate a nonnegative function which depends on parameters $(a_1, \ldots, a_k)$. The actual value of C can change from one formula to another.

4. If E is a set on the real line, E^c denotes its complement.

5. For two non-negative functions $f_{1(2)}$, we write $f_1 \lesssim f_2$ if there is an absolute constant C such that

$$f_1 \leqslant C f_2$$

   for all values of the arguments of $f_{1(2)}$. We define $\gtrsim$ similarly and say that $f_1 \sim f_2$ if $f_1 \lesssim f_2$ and $f_2 \lesssim f_1$ simultaneously.

6. If f_2 is non-negative function and $|f_1| \lesssim f_2$, we write $f_1 = O(f_2)$.

2.2 Menchov-Rademacher Theorem for continuous orthogonal systems

We start by giving the proof to Theorem 2.1.2. It is a direct adaptation of the proof of Menchov-Rademacher Theorem in [29].

Proof of Theorem 2.1.2. For $j \in \mathbb{N}$, let $P_j(k) = \int_{2^{j-1}}^{2^j} f(r)P(r,k)dr$ and

$$S'_j(k) = \sum_{l=1}^{j} P_l(k) = \int_1^{2^j} f(r)P(r,k)dr\,.$$

Now,

$$\|P_j\|^2_{L^2_\sigma(\mathbb{R})} = \int_{\mathbb{R}} \left| \int_{2^{j-1}}^{2^j} f(r)P(r,k)dr \right|^2 d\sigma(k) = \int_{2^{j-1}}^{2^j} |f(r)|^2 dr$$

and so

$$\sum_{j\in\mathbb{N}} j^2 \|P_j\|^2_{L^2_\sigma(\mathbb{R})} \sim \int_1^\infty |f(r)|^2 \log^2(2+r)dr\,. \tag{2.11}$$

For any $a > 0$, we have

$$\sum_{j\in\mathbb{N}} \int_{-a}^{a} |P_j(k)| d\sigma(k) \leqslant \sum_{j\in\mathbb{N}} \left(\int_{-a}^{a} |P_j(k)|^2 d\sigma(k) \right)^{1/2} \left(\int_{-a}^{a} d\sigma(k) \right)^{1/2} \leqslant$$

$$\sqrt{\sigma([-a,a])} \sum_{j\in\mathbb{N}} \|P_j\|_{L^2_\sigma(\mathbb{R})} j j^{-1} \leqslant \sqrt{\sigma([-a,a])} \left(\sum_{j\in\mathbb{N}} j^2 \|P_j\|^2_{L^2_\sigma(\mathbb{R})} \right)^{1/2} \left(\sum_{j\in\mathbb{N}} j^{-2} \right)^{1/2}$$

$$\lesssim \sqrt{\sigma([-a,a])} \left(\int_{\mathbb{R}^+} |f(r)|^2 \log^2(2+r)dr \right)^{1/2} = \sqrt{\sigma([-a,a])} L^{1/2}\,.$$

Since a is arbitrary large, by the theorem of Beppo Levi, $\sum_{j\in\mathbb{N}} |P_j(k)|$ converges for σ-a.e. k, as does $\{S'_j(k)\}$.

Let $S'(k) \overset{\text{def}}{=} \sup_{j\in\mathbb{N}} |S'_j(k)|$ be the maximal function over dyadic partial sums. Since $S'(k) \leqslant \sum_{j\in\mathbb{N}} |P_j(k)|$, we have

$$\|S'\|_{L^2_\sigma(\mathbb{R})} \leqslant \left\| \sum_{j\in\mathbb{N}} |P_j| \right\|_{L^2_\sigma(\mathbb{R})} \leqslant \sum_{j\in\mathbb{N}} \|P_j\|_{L^2_\sigma(\mathbb{R})} = \sum_{j\in\mathbb{N}} j^{-1} j \|P_j\|_{L^2_\sigma(\mathbb{R})} \lesssim L^{1/2} \tag{2.12}$$

after applying Cauchy-Schwarz inequality and (2.11).

For $n \in \{0, 1, 2, \ldots, 2^N\}$, we can write $n = \sum_{m=0}^{N} \epsilon_m(n) 2^{N-m}$ with $\epsilon_m(n) \in \{0, 1\}$. For $j \in \{0, 1, \ldots, N\}$, let $n_j = \sum_{m=0}^{j} \epsilon_m(n) 2^{N-m}$.

Noting that $\left|\sum_{j=1}^{N} x_j\right|^2 \leqslant N \sum_{j=1}^{N} |x_j|^2$, we have:

$$\left|\int_{2^N}^{2^N+n} f(r)P(r,k)dr\right|^2 = \left|\sum_{j=1}^{N} \int_{2^N+n_{j-1}}^{2^N+n_j} f(r)P(r,k)dr\right|^2 \leqslant$$

$$N \sum_{j=1}^{N} \left|\int_{2^N+n_{j-1}}^{2^N+n_j} f(r)P(r,k)dr\right|^2 \leqslant N \sum_{j=1}^{N} \sum_{p=0}^{2^j-1} \left|\int_{2^N+p2^{N-j}}^{2^N+(p+1)2^{N-j}} f(r)P(r,k)dr\right|^2$$

and the last expression does not depend on n. Let

$$S_j''(k) \stackrel{\text{def}}{=} \sup_{0 \leqslant n \leqslant 2^j} \left|\int_{2^j}^{2^j+n} f(r)P(r,k)dr\right| .$$

Denote the maximal function over dyadic interval $[2^j, 2^{j+1}]$. We apply the above estimate to get

$$\begin{aligned} \|S_N''\|_{L^2_\sigma(\mathbb{R})}^2 &= \int_{\mathbb{R}} \sup_{0 \leqslant n \leqslant 2^N} \left|\int_{2^N}^{2^N+n} f(r)P(r,k)dr\right|^2 d\sigma(k) \\ &\leqslant \int_{\mathbb{R}} N \sum_{j=1}^{N} \sum_{p=0}^{2^j-1} \left|\int_{2^N+p2^{N-j}}^{2^N+(p+1)2^{N-j}} f(r)P(r,k)dr\right|^2 d\sigma(k) \\ &= N \sum_{j=1}^{N} \sum_{p=0}^{2^j-1} \int_{\mathbb{R}} \left|\int_{2^N+p2^{N-j}}^{2^N+(p+1)2^{N-j}} f(r)P(r,k)dr\right|^2 d\sigma(k) \\ &= N \sum_{j=1}^{N} \sum_{p=0}^{2^j-1} \int_{2^N+p2^{N-j}}^{2^N+(p+1)2^{N-j}} |f(r)|^2 dr = N^2 \int_{2^N}^{2^{N+1}} |f(r)|^2 dr . \end{aligned} \tag{2.13}$$

Taking $S'' = \sup_{j \in \mathbb{N}} S_j''$, we note that $S'' \leqslant \left(\sum_{j \in \mathbb{N}} |S_j''|^2\right)^{1/2}$ so

$$\|S''\|_{L^2_\sigma(\mathbb{R})} \lesssim \left(\sum_{j \in \mathbb{N}} j^2 \int_{2^j}^{2^{j+1}} |f(r)|^2 dr\right)^{1/2} \lesssim L^{1/2}.$$

Finally, we have

$$\|M\|^2_{L^2_\sigma(\mathbb{R})} \lesssim \int_0^1 |f(r)|^2 dr + \int_{\mathbb{R}} \sup_{j\in\mathbb{N}} \left| \int_1^{2^j} f(r)P(r,k)dr \right|^2 d\sigma(k) +$$

$$\int_{\mathbb{R}} \sup_{j\in\mathbb{N}} \sup_{2^j \leqslant n \leqslant 2^{j+1}} \left| \int_{2^j}^n f(r)P(r,k)dr \right|^2 d\sigma(k) = \int_0^1 |f(r)|^2 dr + \|S'\|^2 + \|S''\|^2 \lesssim L\,.$$

Convergence of the sequence $\left\{ \int_0^n f(r)P(r,k)dr \right\}$ for σ-a.e. k follows from convergence of $\{S'_j(k)\}$ established above and the estimate $\int_{\mathbb{R}} \sum_{j\in\mathbb{N}} |S''_j|^2 d\sigma \lesssim L$ which yields convergence of $\sum_{j\in\mathbb{N}} |S''_j|^2$ for σ-a.e. k.

□

Proof of Theorem 2.1.3. We have

$$\int_{\mathbb{R}} \sup_{t\in\mathbb{R}^+} \left| \int_0^t f(r)P(r,k)dr \right|^2 \frac{d\sigma(k)}{\kappa(k)} = \int_{\mathbb{R}} \sup_{t\in\mathbb{R}^+} \left| \int_0^{[t]} f(r)P(r,k)dr + \int_{[t]}^t f(r)P(r,k)dr \right|^2 \frac{d\sigma(k)}{\kappa(k)}$$

$$\lesssim \|\kappa^{-1}\|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} \sup_{n\in\mathbb{N}} \left| \int_0^n f(r)P(r,k)dr \right|^2 d\sigma(k) + \int_{\mathbb{R}} \sup_{t\in\mathbb{R}^+} \left| \int_{[t]}^t f(r)P(r,k)dr \right|^2 \frac{d\sigma(k)}{\kappa(k)}\,.$$

The first integral was controlled in Theorem 2.1.2. The second one can be estimated as follows

$$\int_{\mathbb{R}} \sup_{t\in\mathbb{R}^+} \left| \int_{[t]}^t f(r)P(r,k)dr \right|^2 \frac{d\sigma(k)}{\kappa(k)} \leqslant \int_{\mathbb{R}} \sup_{n\in\mathbb{Z}^+} \left(\int_n^{n+1} |f(r)P(r,k)|dr \right)^2 \frac{d\sigma(k)}{\kappa(k)} \leqslant \tag{2.14}$$

$$\int_{\mathbb{R}} \sup_{n\in\mathbb{Z}^+} \left(\left(\int_n^{n+1} |f|^2 dr \right) \left(\int_n^{n+1} |P(r,k)|^2 dr \right) \right) \frac{d\sigma(k)}{\kappa(k)} \leqslant$$

$$\int_{\mathbb{R}} \sum_{n=0}^{\infty} \left(\left(\int_n^{n+1} |f|^2 dr \right) \left(\int_n^{n+1} |P(r,k)|^2 dr \right) \right) \frac{d\sigma(k)}{\kappa(k)} \leqslant$$

$$\sum_{n=0}^{\infty} \left(\int_n^{n+1} |f|^2 dr \right) \left(\int_n^{n+1} \left(\int_{\mathbb{R}} \frac{|P(r,k)|^2}{\kappa(k)} d\sigma(k) \right) dr \right) \overset{(2.2)}{\leqslant} K\|f\|_2^2\,,$$

which proves (2.3).

To establish (2.4), we notice that

$$\int_0^r f(\rho)P(\rho,k)d\rho = \int_0^{[r]} f(\rho)P(\rho,k)d\rho + \int_{[r]}^r f(\rho)P(\rho,k)d\rho\,.$$

The first term has a limit as $r \to \infty$ for σ-a.e. k as follows from Theorem 2.1.2. For the second one, we can write

$$\left|\int_{[r]}^r f(\rho)P(\rho,k)d\rho\right| \leqslant \int_{[r]}^{[r]+1} |f(\rho)P(\rho,k)|d\rho$$

and the last expression goes to 0 for σ-a.e. k since the series

$$\sum_{n\in\mathbb{N}} \left(\int_n^{n+1} |f(r)P(r,k)|dr\right)^2$$

converges for σ-a.e. k. This convergence follows from the following bound

$$\int_{\mathbb{R}} \sum_{n\in\mathbb{N}} \left(\int_n^{n+1} |f(r)P(r,k)|dr\right)^2 \frac{d\sigma}{\kappa} \leqslant \int_{\mathbb{R}} \sum_{n\in\mathbb{N}} \left(\left(\int_n^{n+1} |f(r)|^2 dr\right)\left(\int_n^{n+1} |P(r,k)|^2 dr\right)\right) \frac{d\sigma}{\kappa} \leqslant$$
$$\left(\sup_{r\geqslant 0}\int_{\mathbb{R}} \frac{|P(r,k)|^2}{\kappa} d\sigma\right) \sum_{n\in\mathbb{N}} \int_n^{n+1} |f(r)|^2 dr \overset{(2.2)}{<} \infty\,.$$

□

Before giving the proof of the Lemma 2.1.5, we list some basic properties of Krein systems which will be needed later in the text. We start by making a remark that

$$P(r,k) = e^{irk}\overline{P_*(r,k)}, \tag{2.15}$$

provided that $k \in \mathbb{R}$. This identity follows directly from (2.5) and can be found in, e.g., [14].

Next, we consider an important case when $A \in L^2(\mathbb{R}^+)$. In [13] (see also original Krein's paper [37]), it was shown that the following properties hold under this condition:

- There is a function $\Pi(k), k \in \mathbb{C}^+$ such that

$$\lim_{r\to\infty} P_*(r,k) = \Pi(k) \tag{2.16}$$

uniformly over compact sets in $\mathbb{C}^+$. This Π is outer and the orthogonality measure σ can be written as follows

$$d\sigma = \frac{dk}{2\pi|\Pi(k)|^2} + d\sigma_s, \tag{2.17}$$

where σ_s is its singular part.

- Integrating the second equation in (2.5), we have

$$P_*(r,k) = 1 - \int_0^r A(\rho)P(\rho,k)d\rho\,. \tag{2.18}$$

Therefore

$$1 - P_*(r,k) = \int_0^r A(\rho)P(\rho,k)d\rho \to \widetilde{A}(k) \stackrel{\text{def}}{=} \int_0^\infty A(\rho)P(\rho,k)d\rho$$

when $r \to \infty$ and convergence is in $L^2(\mathbb{R},\sigma)$ norm. On the other hand, the formula (12.37) in [13] gives

$$\widetilde{A}(k) = 1 - \Pi(k)\cdot \chi_{E_s^c}\,,$$

where E_s^c denotes the complement to E_s, the support of σ_s. Therefore,

$$\lim_{r\to\infty} \|P_*(r,k) - \Pi(k)\cdot\chi_{E_s^c}\|_{2,\sigma} = 0\,. \tag{2.19}$$

- From (2.18) and orthogonality, we get

$$\int_{\mathbb{R}} |P_*(r,k)-1|^2 d\sigma = \int_0^r |A(\rho)|^2 d\rho\,.$$

Proof of Lemma 2.1.5. The second equation in (2.5) gives

$$P_*(r_2,k) - P_*(r_1,k) = -\int_{r_1}^{r_2} A(r)P(r,k)dr\,. \tag{2.20}$$

Theorem 2.1.3 yields necessary estimate on the maximal function and convergence of $P_*(r,k)$ σ-a.e. The limit is equal to Π from (2.16) due to (2.19). $\square$

2.3 Wave operators for Schrödinger evolution: proof of Theorem 2.1.6

We start this section by describing a connection between Krein systems and Dirac and Schrödinger operators on $\mathbb{R}^+$. Consider the Krein system with coefficient $A \in L^2_{\text{loc}}(\mathbb{R}^+)$. It corresponds to Dirac operator

$$\mathcal{D} = \begin{pmatrix} -b & \partial_x - a \\ -\partial_x - a & b \end{pmatrix} \tag{2.21}$$

defined on Hilbert space $(f_1, f_2) \in L^2(\mathbb{R}^+) \times L^2(\mathbb{R}^+)$, where $a(x) = 2\operatorname{Re} A(2x), b(x) = 2\operatorname{Im} A(2x)$ with the boundary condition $f_2(0) = 0$. Indeed, define real-valued functions ϕ and ψ by writing $\phi(x,k) + i\psi(x,k) \overset{\text{def}}{=} P(2x,k)e^{-ikx}$. It can be checked [14, 37] that (ϕ,ψ) are generalized eigenfunctions for Dirac operator (2.21) and that 2σ is its spectral measure. Define $\{\mathcal{E}(x,k)\}, x \geqslant 0$ by

$$\mathcal{E}(x,k) \overset{\text{def}}{=} P(2x,k)e^{-ixk}. \tag{2.22}$$

It turns out that this is also continuous orthonormal system with respect to σ, i.e.,

$$\int_{\mathbb{R}} \left| \int_0^\infty f(x)\mathcal{E}(x,k)dx \right|^2 d\sigma = \|f\|_2^2 \tag{2.23}$$

for every $f \in L^2(\mathbb{R}^+)$ (see [16, 37]). Making an extra assumption that A is real-valued, i.e., that $b = 0$, and absolutely continuous on $\mathbb{R}^+$ and taking the square of $\mathcal{D}$ reveals the connections between Dirac and Schrödinger operators. Indeed,

$$\mathcal{D}^2 = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix}, \tag{2.24}$$

where $H_1 f = -\partial^2_{xx} f + q_1 f, f'(0) + a(0)f(0) = 0$, $H_2 f = -\partial^2_{xx} f + q_2 f, f(0) = 0$,

$$q_1 = a^2 - a', q_2 = a^2 + a'\,.$$

Later in the proof, we will use the spectral decomposition for Dirac $\mathcal{D}$ and the formula (2.24) to write a suitable expression for e^{itH_2}.

The following result implies Theorem 2.1.6 thanks to Lemma 2.1.5.

Theorem 2.3.1. *Suppose the coefficient A in the Krein system is real and absolutely continuous, $A \in L^2(\mathbb{R}^+), A' \in L^\infty(\mathbb{R}^+)$, and*

$$\lim_{\rho\to\infty} \int_{\mathbb{R}} \left(\sup_{\rho<r_1<r_2} \left| \int_{r_1}^{r_2} A(r)P(r,k)dx \right| \right)^2 \frac{d\sigma}{1+k^2} = 0\,. \tag{2.25}$$

Let $a(x) = 2A(2x)$ and let q be real-valued function on $\mathbb{R}^+$ satisfying $q \in L^1(\mathbb{R}^+)$. Then, taking two operators $H = -\partial^2_{xx} + a' + q$ and $H_0 = -\partial^2_{xx}$ both with Dirichlet boundary condition at zero, we get existence of wave operators $W^\pm(H, H_0)$.

This Theorem is the central technical result of our paper. Before giving its proof, we state the following Lemma.

Lemma 2.3.2. *Suppose $t \geqslant 0$, μ is a measure on $\mathbb{R}$, and $p(k), p_t(k) \in L^2_\mu(\mathbb{R})$. Let $\|p\|_{2,\mu} = 1$ and*

$$\lim_{t\to\infty} \|p_t\|_{2,\mu} = 1, \quad \lim_{t\to\infty} \int_\Delta |p - p_t|^2 d\mu = 0 \tag{2.26}$$

for every interval $\Delta \subset \mathbb{R}$. Then, $\lim\limits_{t\to\infty} \|p - p_t\|_{2,\mu} = 0$.

Proof. The proof is based on a standard exhaustion principle. For every $\epsilon \in (0,1)$, we can choose $L > 0$ such that $\int_{\Delta^c} |p|^2 d\mu \leqslant \epsilon$ where $\Delta \overset{\text{def}}{=} [-L, L]$. By (2.26), there is T so that

$$|1 - \|p_t\|^2_{2,\mu}| < \epsilon, \quad \int_\Delta |p - p_t|^2 d\mu < \epsilon$$

for $t > T$. Thus, for $t > T$, we also have

$$\int_{\Delta^c} |p_t|^2 d\mu = \|p_t\|_{2,\mu}^2 - \int_{\Delta} |p_t|^2 d\mu =$$
$$\|p_t\|_{2,\mu}^2 - \left(1 - \int_{\Delta^c} |p|^2 d\mu - \int_{\Delta} (|p|^2 - |p_t|^2) d\mu\right) \leqslant$$
$$|\|p_t\|_{2,\mu}^2 - 1| + \int_{\Delta^c} |p|^2 d\mu + \left|\int_{\Delta} (|p|^2 - |p_t|^2) d\mu\right| \lesssim$$
$$\epsilon + \sqrt{\epsilon},$$

where we used triangle inequality to estimate

$$\left|\int_{\Delta} (|p|^2 - |p_t|^2) d\mu\right| = \left|\|p\|_{L^2_\mu(\Delta)}^2 - \|p_t\|_{L^2_\mu(\Delta)}^2\right| =$$
$$\left(\|p\|_{L^2_\mu(\Delta)} + \|p_t\|_{L^2_\mu(\Delta)}\right) \cdot \left|\|p\|_{L^2_\mu(\Delta)} - \|p_t\|_{L^2_\mu(\Delta)}\right| \lesssim \|p - p_t\|_{L^2_\mu(\Delta)} \leqslant \sqrt{\epsilon}\,.$$

Thus,

$$\int_{\mathbb{R}} |p - p_t|^2 d\mu = \int_{\Delta} |p - p_t|^2 d\mu + \int_{\Delta^c} |p - p_t|^2 d\mu \leqslant \epsilon + 2\int_{\Delta^c} |p|^2 d\mu + 2\int_{\Delta^c} |p_t|^2 d\mu \lesssim \sqrt{\epsilon}$$

for $t > T$ and the proof is finished. □

Proof of Theorem 2.3.1. Since $a^2, q \in L^1(\mathbb{R}^+)$ and relative trace class perturbations do not change existence of wave operators (Birman-Kuroda Theorem, [47], p. 27), it is enough to consider $H = H_2 = a' + a^2$. Take $f \in L^2(\mathbb{R}^+)$. We need to prove existence of

$$\lim_{t \to \pm\infty} e^{itH} e^{-itH_0} f\,, \tag{2.27}$$

where the limit is understood in $L^2(\mathbb{R}^+)$ topology. Notice that, since both groups e^{itH} and e^{-itH_0} preserve $L^2(\mathbb{R}^+)$ norm, it is enough to prove existence of the limit for every $f \in \mathcal{T}$ where $\mathcal{T}$ is any dense subset in $L^2(\mathbb{R}^+)$. We define $\mathcal{T}$ as follows: $\mathcal{T} \stackrel{\text{def}}{=} \{f : \widehat{f_o} \in C_c^\infty(\mathbb{R}), 0 \notin \operatorname{supp}\widehat{f_o}\}$, where f_o denotes the odd extension of f to $\mathbb{R}$. From now on, we assume that $f \in \mathcal{T}, \|f\|_2 = 1$ and that $t \to +\infty$ in (2.27) (the case $t \to -\infty$ can be handled similarly). Denote $f_+ \stackrel{\text{def}}{=} (\widehat{f_o} \cdot \chi_{\xi > 0})^\vee$,

$f_- \stackrel{\text{def}}{=} (\widehat{f}_o \cdot \chi_{\xi<0})^\vee$. Working on the Fourier side, we get

$$e^{-itH_0}f = \frac{1}{\pi}\int_{\mathbb{R}} e^{-it\xi^2}\left(\int_{\mathbb{R}^+} f(u)\sin(\xi u)du\right)\sin(\xi x)d\xi = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-it\xi^2}\left(\int_{\mathbb{R}} f_o(u)e^{-i\xi u}du\right)e^{i\xi x}d\xi\,.$$

The last expression is equal to the restriction of $e^{it\partial^2_{xx}}f_o$ to $\mathbb{R}^+$, where ∂^2_{xx} is considered on all of $\mathbb{R}$. The large time asymptotics of $e^{it\partial^2_{xx}}h$ for $h \in L^2(\mathbb{R})$ is known and given in Lemma A.1.1 from Appendix. Since $\widehat{f}_o(\xi) = \widehat{f}_+(\xi)$ for $\xi > 0$, it is enough to show that

$$I \stackrel{\text{def}}{=} \frac{e^{itk^2}}{1+i}\int_0^\infty \frac{e^{ix^2/(4t)}}{\sqrt{t}}\widehat{f}_+(x/(2t))\psi(x,k)dx \tag{2.28}$$

has a limit in $L^2(\mathbb{R}, 2\sigma)$ when $t \to +\infty$. Indeed, the spectral measure for Dirac operator $\mathcal{D}$ is equal to 2σ, the generalized eigenfunctions are (ϕ, ψ), and the Schrödinger operator is related to Dirac by (2.24) so we can use spectral decomposition for Dirac operator to compute e^{itH} where $H = H_2$. To this end, we will use the following generalized Fourier transform

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \to \mathcal{F} = \int_0^\infty f_1(x)\phi(x,k)dx + \int_0^\infty f_2(x)\psi(x,k)dx$$

and the analog of Plancherel's Theorem

$$\|f_1\|_2^2 + \|f_2\|_2^2 = \|\mathcal{F}\|^2_{2,2\sigma}\,.$$

Since $f \in \mathcal{T}$, $\widehat{f}_+$ is supported on some interval $[a, b]$ and $a > 0$. Use (2.15) and substitute

$$\psi(x,k) = \frac{\overline{P_*}(2x,k)e^{ikx} - P_*(2x,k)e^{-ikx}}{2i}$$

into (2.28) to get

$$I = I_1 - I_2\,, \tag{2.29}$$

where

$$I_1 = \frac{e^{itk^2}}{2i(1+i)} \int_{2at}^{2bt} \frac{e^{ix^2/(4t)}}{\sqrt{t}} \widehat{f}_+(x/(2t)) \overline{P_*}(2x,k) e^{ikx} dx,$$

$$I_2 = \frac{e^{itk^2}}{2i(1+i)} \int_{2at}^{2bt} \frac{e^{ix^2/(4t)}}{\sqrt{t}} \widehat{f}_+(x/(2t)) P_*(2x,k) e^{-ikx} dx\,.$$

Consider I_2; the analysis of I_1 is similar. Integrating by parts, we get

$$\int_{2at}^{2bt} P_*(2x,k) \left(\int_{2at}^{x} \frac{e^{iu^2/(4t)}}{\sqrt{t}} \widehat{f}_+(u/(2t)) e^{-iku} du \right)' dx =$$

$$= P_*(4bt,k) \int_{2at}^{2bt} \frac{e^{iu^2/(4t)}}{\sqrt{t}} \widehat{f}_+(u/(2t)) e^{-iku} du - J_2\,,$$

where, thanks to the second equation in (2.5),

$$J_2 = \int_{2at}^{2bt} 2A(2x) P(2x,k) \left(\int_{2at}^{x} \frac{e^{iu^2/(4t)}}{\sqrt{t}} \widehat{f}_+(u/(2t)) e^{-iku} du \right) dx\,.$$

For the first term, we can write

$$P_*(4bt,k) \int_{2at}^{2bt} \frac{e^{iu^2/(4t)}}{\sqrt{t}} \widehat{f}_+(u/(2t)) e^{-iku} du =$$

$$(P_*(4bt,k) - \Pi(k) \cdot \chi_{E_s^c}) \int_{2at}^{2bt} \frac{e^{iu^2/(4t)}}{\sqrt{t}} \widehat{f}_+(u/(2t)) e^{-iku} du$$

$$+\Pi(k) \cdot \chi_{E_s^c} \int_{2at}^{2bt} \frac{e^{iu^2/(4t)}}{\sqrt{t}} \widehat{f}_+(u/(2t)) e^{-iku} du\,.$$

From (A.12), we get

$$\sup_{t>1} \left\| \int_{2at}^{2bt} \frac{e^{iu^2/(4t)}}{\sqrt{t}} \widehat{f}_+(u/(2t)) e^{-iku} du \right\|_{L^\infty(\mathbb{R})} < C_{(f)}$$

and (2.19) implies

$$\lim_{t\to+\infty} \left\| (P_*(4bt,k) - \Pi(k) \cdot \chi_{E_s^c}) \int_{2at}^{2bt} \frac{e^{iu^2/(4t)}}{\sqrt{t}} \widehat{f}_+(u/(2t)) e^{-iku} du \right\|_{2,\sigma} = 0\,.$$

From (2.17) and (A.11), we obtain

$$\lim_{t\to\infty}\left\| \frac{e^{itk^2}\Pi(k)}{2i(1+i)}\cdot\chi_{E_s^c}\cdot\int_{2at}^{2bt}\frac{e^{iu^2/(4t)}}{\sqrt{t}}\widehat{f}_+(u/(2t))e^{-iku}du-\frac{\sqrt{2\pi}\Pi(k)}{2i}\cdot\chi_{E_s^c}\widehat{f}_+(k)\right\|_{2,\sigma}=0\,. \qquad (2.30)$$

The analysis for I_1 is analogous - it also gives the main term converging to

$$\frac{\sqrt{2\pi}\cdot\overline{\Pi(k)}}{2i}\cdot\chi_{E_s^c}\widehat{f}_+(-k)$$

and a correction which we call J_1. Consider J_1 and J_2. We claim that if we show that

$$\lim_{t\to\infty}\int_\Delta|J_1|^2d\sigma=0,\quad \lim_{t\to\infty}\int_\Delta|J_2|^2d\sigma=0 \qquad (2.31)$$

for every interval $\Delta\subset\mathbb{R}$, then the proof of Theorem 2.3.1 will be finished after application of Lemma 2.3.2. Indeed, in this lemma, we set $\mu=2\sigma$, $p_t=I$ and the limiting function p is

$$p=\chi_{E_s^c}\cdot\sqrt{2\pi}\frac{\overline{\Pi(k)}\widehat{f}_+(-k)-\Pi(k)\widehat{f}_+(k)}{2i}\,.$$

To apply Lemma 2.3.2, we notice that $\|I\|_{2,2\sigma}\to 1$ by Lemma A.1.1. Moreover, (2.17) gives $\|p\|_{2,2\sigma}=\|f\|_2=1$.

We will prove the second identity in (2.31); the first one can be obtained similarly. For J_2, we have

$$J_2=-2\int_{2at}^{2bt}A(2x)P(2x,k)\left(\int_{2at}^{x}\frac{e^{i(u^2/(4t)-ku)}}{\sqrt{t}}\widehat{f}_+(u/(2t))du\right)dx\,.$$

One can write

$$\int_{2at}^{x}\frac{e^{i(u^2/(4t)-ku)}}{\sqrt{t}}\widehat{f}_+(u/(2t))du=\int_{2at}^{0}\frac{e^{i(u^2/(4t)-ku)}}{\sqrt{t}}\widehat{f}_+(u/(2t))du+\int_{0}^{x}\frac{e^{i(u^2/(4t)-ku)}}{\sqrt{t}}\widehat{f}_+(u/(2t))du\,.$$

The first term does not depend on x and we can use (A.12) and (2.6) to write

$$\left\|\int_{2at}^{2bt}A(2x)P(2x,k)\left(\int_{2at}^{0}\frac{e^{i(u^2/(4t)-ku)}}{\sqrt{t}}\widehat{f}_+(u/(2t))du\right)dx\right\|_{2,\sigma}\leqslant C_{(f)}\int_{at}^{bt}|A(x)|^2dx\,, \qquad (2.32)$$

where the last expression converges to zero as $t \to \infty$. For the other term, we have

$$\int_0^x \frac{e^{i(u^2/(4t)-ku)}}{\sqrt{t}} \widehat{f}_+(u/(2t))du = e^{-itk^2} \int_0^x \frac{e^{i(u/(2\sqrt{t})-k\sqrt{t})^2}}{\sqrt{t}} \widehat{f}_+(u/(2t))du\,.$$

The integral can be rewritten as

$$\int_0^x \frac{e^{i(u/(2\sqrt{t})-k\sqrt{t})^2}}{\sqrt{t}} \widehat{f}_+(u/(2t))du =$$

$$\int_{-\infty}^x \frac{e^{i(u/(2\sqrt{t})-k\sqrt{t})^2}}{\sqrt{t}} \widehat{f}_+(u/(2t))du - \int_{-\infty}^0 \frac{e^{i(u/(2\sqrt{t})-k\sqrt{t})^2}}{\sqrt{t}} \widehat{f}_+(u/(2t))du\,.$$

The second term is x–independent so its contribution is negligible by the argument identical to (2.32). For the first one, we change variables and write, using the same variable u,

$$\int_{-\infty}^x \frac{e^{i(u/(2\sqrt{t})-k\sqrt{t})^2}}{\sqrt{t}} \widehat{f}_+(u/(2t))du = 2\int_{-\infty}^{(x-2kt)/2\sqrt{t}} e^{iu^2} \widehat{f}_+(k+u/\sqrt{t})du \tag{2.33}$$

$$= 2\int_{-\infty}^{(x-2kt)/2\sqrt{t}} e^{iu^2}\Big(\widehat{f}_+(k+u/\sqrt{t}) - \widehat{f}_+(k)\Big)du + 2\widehat{f}_+(k)\int_{-\infty}^{(x-2kt)/2\sqrt{t}} e^{iu^2}du\,.$$

We can continue as follows

$$\int_{-\infty}^{(x-2kt)/2\sqrt{t}} e^{iu^2}\Big(\widehat{f}_+(k+u/\sqrt{t}) - \widehat{f}_+(k)\Big)du = \int_{-\infty}^{0} e^{iu^2}\Big(\widehat{f}_+(k+u/\sqrt{t}) - \widehat{f}_+(k)\Big)du + \int_{0}^{(x-2kt)/2\sqrt{t}} e^{iu^2}\Big(\widehat{f}_+(k+u/\sqrt{t}) - \widehat{f}_+(k)\Big)du\,.$$

The first term in the right-hand side does not depend on x and it is uniformly bounded in $k \in \mathbb{R}$ and $t \geqslant 1$ as can be seen by integrating by parts. Thus, its contribution to $\|J_2\|_{L^2_\sigma(\Delta)}$ is also negligible.

We want to apply Lemma A.1.2 from Appendix to the second term. Since we are interested in $k \in \Delta$ and $x \in [at, bt]$, then $|(x-2kt)/2t| < C_{(a,b,\Delta)}$. Hence, the Lemma is applicable with $\epsilon = 1/\sqrt{t}, g(u) = \widehat{f}_+(k+u) - \widehat{f}_+(k)$ which gives

$$\left|\int_{0}^{(x-2kt)/2\sqrt{t}} e^{iu^2}\Big(\widehat{f}_+(k+u/\sqrt{t}) - \widehat{f}_+(k)\Big)du\right| \leqslant C_{a,b,\Delta,f}/\sqrt{t}\,.$$

The proof of Lemma A.1.2 shows that this bound is uniform in $k \in \Delta$. We substitute it and apply (2.8) along with generalized Minkowski inequality to get

$$\left(\int_\Delta \left|\frac{1}{\sqrt{t}}\int_{2at}^{2bt}|A(2x)P(2x,k)|\right|^2 d\sigma\right)^{1/2} \lesssim$$
$$\frac{1}{\sqrt{t}}\int_{2at}^{2bt}|A(2x)|\cdot\left(\int_\Delta |P(2x,k)|^2 d\sigma\right)^{1/2} dx \overset{(2.8)}{\lesssim}$$
$$\frac{C_{(\Delta,\|A\|_{\mathrm{St}})}}{\sqrt{t}}\int_{2at}^{2bt}|A(2x)|dx \leqslant C_{(\Delta,a,b,\|A\|_{\mathrm{St}})}\left(\int_{at}^{bt}|A(x)|^2dx\right)^{1/2}.$$

The last expression converges to zero when $t \to +\infty$. We are only left with controlling the contribution from the last term in (2.33), i.e.,

$$\widehat{f}_+(k)\int_{2at}^{2bt}A(2x)P(2x,k)\left(\int_0^{(x-2kt)/(2\sqrt{t})}e^{iu^2}du\right)dx\,.$$

Let us write partition of unity

$$1 = \mu_- + \mu_0 + \mu_+, \tag{2.34}$$

where μ_0 is even, smooth, supported in $(-2,2)$ and

$$0 \leqslant \mu_0 \leqslant 1, \quad \mu_0 = 1 \text{ if } |x| < 1\,.$$

Function μ_+ is supported on $(1,\infty)$ and is non-decreasing, $\mu_-(x) \overset{\text{def}}{=} \mu_+(-x)$. Then,

$$\int_0^{(x-2kt)/(2\sqrt{t})}e^{iu^2}du = \left(\int_0^{(x-2kt)/(2\sqrt{t})}e^{iu^2}du\right)\Big(\mu_-((x-2kt)/(2\sqrt{t})) + \mu_0(\cdot) + \mu_+(\cdot)\Big)\,.$$

We will apply the following trick several times. Notice that the function $F(x) \overset{\text{def}}{=} (\int_0^x e^{iu^2}du)\mu_0(x) \in C_c^\infty(\mathbb{R})$ thus $\widehat{F} \in L^1(\mathbb{R})$ and we can write

$$F((x-2kt)/(2\sqrt{t})) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\widehat{F}(\xi)\exp(i\xi(x-2kt)/(2\sqrt{t}))d\xi\,.$$

Then,

$$\widehat{f}_+(k)\int_{2at}^{2bt} A(2x)P(2x,k)\left(\mu_0((x-2kt)/(2\sqrt{t}))\int_0^{(x-2kt)/(2\sqrt{t})} e^{iu^2}du\right)dx =$$

$$\widehat{f}_+(k)\int_{2at}^{2bt} A(2x)P(2x,k)F((x-2kt)/(2\sqrt{t}))dx =$$

$$\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \widehat{F}(\xi)\left(\widehat{f}_+(k)e^{-i\xi k\sqrt{t}}\int_{2at}^{2bt} A(2x)P(2x,k)\exp(i\xi x/(2\sqrt{t}))dx\right)d\xi\,.$$

We use generalized Minkowski inequality and (2.6) to estimate the last quantity as follows

$$\left\|\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \widehat{F}(\xi)\left(\widehat{f}_+(k)e^{-i\xi k\sqrt{t}}\int_{2at}^{2bt} A(2x)P(2x,k)\exp(i\xi x/(2\sqrt{t}))dx\right)d\xi\right\|_{2,\sigma} \lesssim$$

$$\left(\int_{\mathbb{R}}|\widehat{F}(\xi)|d\xi\right)\|\widehat{f}_+\|_\infty\left(\int_{at}^{bt}|A(x)|^2dx\right)^{1/2}$$

and the last quantity converges to zero when $t\to\infty$. We apply similar strategy to other terms.

$$\left(\int_0^{(x-2kt)/(2\sqrt{t})} e^{iu^2}du\right)\mu_+((x-2kt)/(2\sqrt{t})) = C\mu_+((x-2kt)/(2\sqrt{t}))$$

$$-\left(\int_{(x-2kt)/(2\sqrt{t})}^{\infty} e^{iu^2}du\right)\mu_+((x-2kt)/(2\sqrt{t}))\,,$$

where $C \stackrel{\text{def}}{=} \int_0^\infty e^{iu^2}du$. Consider

$$\int_{2at}^{2bt} A(2x)P(2x,k)\mu_+((x-2kt)/(2\sqrt{t}))dx = \int_{2at}^{2bt}\left(\int_{2at}^{x} A(2u)P(2u,k)du\right)'\mu_+((x-2kt)/(2\sqrt{t}))dx$$

$$= \left(\int_{2at}^{2bt} A(2u)P(2u,k)du\right)\mu_+((b-k)\sqrt{t}) - \int_{2at}^{2bt}\left(\int_{2at}^{x} A(2u)P(2u,k)du\right)\frac{\mu_+'((x-2kt)/(2\sqrt{t}))}{2\sqrt{t}}dx\,.$$

The first term gives contribution

$$\int_{\mathbb{R}}\left|\widehat{f}_+(k)\left(\int_{2at}^{2bt} A(2u)P(2u,k)du\right)\mu_+((b-k)\sqrt{t})\right|^2 d\sigma \lesssim \|\widehat{f}_+\|_\infty^2\int_{2at}^{2bt}|A(2u)|^2du$$

and the last quantity converges to zero when $t \to \infty$. For the second one, we can write an estimate

$$\left| \int_{2at}^{2bt} \left(\int_{2at}^{x} A(2u)P(2u,k)du \right) \frac{\mu_+'((x-2kt)/(2\sqrt{t}))}{2\sqrt{t}} dx \right| \leqslant \tag{2.35}$$
$$\left(\sup_{2at<r_1<r_2} \left| \int_{r_1}^{r_2} A(2u)P(2u,k)du \right| \right) \cdot \int_{2at}^{2bt} \left| \frac{\mu_+'((x-2kt)/(2\sqrt{t}))}{2\sqrt{t}} \right| dx \,.$$

Since μ_+ was chosen to be non-decreasing, one obtains

$$\int_{2at}^{2bt} \left| \frac{\mu_+'((x-2kt)/(2\sqrt{t}))}{2\sqrt{t}} \right| dx \lesssim 1 \,.$$

Under the assumptions of the theorem, we get

$$\left\| |\widehat{f}_+| \cdot \sup_{2at<r_1<r_2} \left| \int_{r_1}^{r_2} A(2u)P(2u,k) \right| \right\|_{L^2_\sigma(\Delta)} \to 0$$

when $t \to \infty$. Consider the expression

$$\left(\int_{(x-2kt)/(2\sqrt{t})}^{\infty} e^{iu^2} du \right) \mu_+((x-2kt)/(2\sqrt{t}))$$

and apply Lemma A.1.3 from Appendix to write it as

$$\left(\int_{(x-2kt)/(2\sqrt{t})}^{\infty} e^{iu^2} du \right) \mu_+((x-2kt)/(2\sqrt{t})) =$$

$$(2\pi)^{-1/2} e^{ix^2/(4t)} e^{-ixk} e^{ik^2 t} \int_{\mathbb{R}} e^{i\xi(x-2kt)/(2\sqrt{t})} \Psi(\xi) d\xi \,,$$

where $\Psi \in L^1(\mathbb{R})$. Then,

$$\int_{2at}^{2bt} A(2x)P(2x,k) e^{ix^2/(4t)} e^{-ixk} e^{ik^2 t} \left(\int_{\mathbb{R}} e^{i\xi(x-2kt)/(2\sqrt{t})} \Psi(\xi) d\xi \right) dx$$
$$= e^{ik^2 t} \int_{\mathbb{R}} \Psi(\xi) e^{-i\xi k\sqrt{t}} \left(\int_{2at}^{2bt} A(2x) e^{ix^2/(4t)} e^{i\xi x/(2\sqrt{t})} \mathcal{E}(x,k) dx \right) d\xi \,,$$

where $\mathcal{E}(x,k) = P(2x,k)e^{-ikx}$ was introduced in (2.22). Using generalized Minkowski inequality and (2.23), we get

$$\left\| \widehat{f}_+(k) \cdot e^{ik^2 t} \int_{\mathbb{R}} \Psi(\xi) e^{-i\xi k\sqrt{t}} \left(\int_{2at}^{2bt} A(2x) e^{ix^2/(4t)} e^{i\xi x/(2\sqrt{t})} \mathcal{E}(x,k) dx \right) d\xi \right\|_{2,\sigma} \lesssim$$
$$\|\widehat{f}_+\|_\infty \cdot \left(\int_{\mathbb{R}} |\Psi(\xi)| d\xi \right) \cdot \left(\int_{2at}^{2bt} |A(2x)|^2 dx \right)^{1/2}$$

and the last quantity converges to zero when $t \to \infty$.

The contribution from the term

$$\left(\int_0^{(x-2kt)/(2\sqrt{t})} e^{iu^2} du \right) \mu_-((x-2kt)/(2\sqrt{t}))$$

can be handled in the same way. Thus,

$$\lim_{t\to\infty} \int_\Delta |J_2|^2 d\sigma = 0$$

and our Theorem is proved. □

Remark. Notice that we had to use an additional assumption about the maximal function (2.25) only when handling (2.35). It is an intriguing question whether this extra hypothesis can be dropped.

Chapter 3

Off-Spectrum Decay of the Integer Lattice Laplacian Green's Function

3.1 Introduction

As discussed in the first chapter of this thesis, weare interested in several properties of the Schrödinger operator on discrete spaces: absorption principle, radiation conditions, decay of the resolvent for complex λ, and convergence of the Born series.

In this chapter, we consider the operator

$$H_0 \stackrel{\text{def}}{=} -\Delta, \quad x \in \mathbb{Z}^d$$

and the equation

$$(H_0 - \lambda)\psi = f\,, \tag{3.1}$$

where f and V are in $C_0(\mathbb{Z}^d)$, the set of functions with bounded support, and the Laplacian is given by:

$$(\Delta u)(x) = \frac{1}{2} \sum_{\{y:|x-y|=1\}} u(y)\,.$$

Conjugation by the Fourier transform allows us to represent Δ as a multiplication operator on $\mathbb{T}^d$, the dual to $\mathbb{Z}^d$. We identify $\mathbb{T}^d$ with $[-\pi,\pi]^d \subset \mathbb{R}^d$ for $\lambda > 0$ and with $[0, 2\pi] \subset \mathbb{R}^d$ for $\lambda < 0$. Let $\mathcal{F}$ be the Fourier transform on $\mathbb{Z}^d$ and $\mathcal{F}u = \hat{u}$. Let $e_j = (0, \ldots, 0, 1, 0, \ldots, 0)$, where the nonzero entry appears in the jth coordinate, and let $u_{e_j}(x) = u(x + e_j)$. Then we have:

$$\Delta u = \frac{1}{2} \sum_{0<j\leqslant d} (u_{e_j} + u_{-e_j}), \tag{3.2}$$

$$\mathcal{F}^{-1}\mathcal{F}\Delta\mathcal{F}^{-1}\hat{u} = \mathcal{F}^{-1}\left(\mathcal{F}\frac{1}{2} \sum_{0<j\leqslant d} (u_{e_j} + u_{-e_j})\right). \tag{3.3}$$

On the right hand side, this is

$$\begin{aligned}\left(\mathcal{F} \sum_{0<j\leqslant d} u_{e_j} + u_{-e_j}\right)(\xi) &= \sum_{y\in\mathbb{Z}^d} \sum_{0<j\leqslant d} \frac{1}{2}(u(y+e_j) + u(y-e_j))e^{-i\xi\cdot y} \\ &= \sum_{y\in\mathbb{Z}^d} \sum_{0<j\leqslant d} \frac{1}{2}u(y)\left(e^{-i\xi\cdot(y+e_j)} + e^{-i\xi\cdot(y-e_j)}\right) \\ &= \sum_{0<j\leqslant d} \frac{1}{2}\left(e^{i\xi\cdot e_j} + e^{-i\xi\cdot e_j}\right) \sum_{y\in\mathbb{Z}^d} u(y)e^{-i\xi\cdot y} \\ &= \sum_{0<j\leqslant d} \cos(\xi_j)\hat{u}(\xi).\end{aligned}$$

So Δ is given by the multiplication operator ϕ on $\mathbb{T}^d$ conjugated by $\mathcal{F}$, where

$$\phi(\xi) = \sum_{0<j\leqslant d} \cos\xi_j. \tag{3.4}$$

Because the spectrum $\sigma(H)$ is invariant under unitary transformations, the spectrum of Δ is purely absolutely continuous and is given by

$$\sigma(\Delta) = \operatorname{range}\phi = [-d, d].$$

The spectrum $\sigma(\Delta)$ is contained entirely in $\mathbb{R}$ so, for $\eta \in \mathbb{C}\backslash\mathbb{R}$, define the resolvent operator

$$R^0_\eta = (H_0 - \eta)^{-1}.$$

We can express R^0_η in the following form:

$$
\begin{aligned}
(R^0_\eta u)(x) &= \mathcal{F}^{-1}\left(\frac{1}{\sum_{0<j\leqslant d}\cos(\cdot_j)-\eta}\hat{u}\right)(x) \\
&= \int_{\mathbb{T}^d}\frac{\hat{u}(\xi)e^{i\xi\cdot x}}{\sum_{0<j\leqslant d}\cos(\xi_j)-\eta}\,\mathrm{d}\xi \\
&= \int_{\mathbb{T}^d}\frac{e^{i\xi\cdot x}}{\sum_{0<j\leqslant d}\cos(\xi_j)-\eta}\sum_{y\in\mathbb{Z}^d}u(y)e^{-i\xi\cdot y}\,\mathrm{d}\xi \\
&= \sum_{y\in\mathbb{Z}^d}u(y)\int_{\mathbb{T}^d}\frac{e^{i(x-y)\cdot\xi}}{\sum_{0<j\leqslant d}\cos(\xi_j)-\eta}\,\mathrm{d}\xi\,.
\end{aligned}
$$

In other words, application of R^0_η to u is nothing more than integration of u against the Green's function $G(x, y, \eta)$:

$$
G_\eta(x,y) = \int_{\mathbb{T}^d}\frac{e^{i(x-y)\cdot\xi}}{\sum_{0<j\leqslant d}\cos(\xi_j)-\eta}\,\mathrm{d}\xi\,. \tag{3.5}
$$

Eskina in [23] and [24] provides justification for the absorption principle and radiation conditions for the general difference operator

$$
(A+q-\lambda)u = f\,, \tag{3.6}
$$

where A is an operator whose representation after conjugation with the Fourier transform is multiplication by a smooth, real valued $a(k)$. In particular, Eskina's result is valid for λ such that $\nabla a(k) \neq 0$ on the surface

$$
\Gamma(\lambda) = \{k \in \mathbb{T}^d | a(k) = \lambda\}\,.
$$

However, the condition $\nabla a(k) \neq 0$ on $\Gamma(\lambda)$ is only true for λ in $(d-2, d)$ (see [48]).

In [48], Shaban and Vainberg introduce the following notation. Note that their spectrum is scaled by a factor of two in order to accommodate their scaling of the Laplacian.

$$
S_0 = \{n \in \mathbb{Z} : d-n \equiv 0 \bmod 2 \text{ and } n \leqslant d\}\,,
$$

$$
S = \sigma(\Delta)\backslash S_0 = [-d, d]\backslash S_0\,.
$$

Further, if $\lambda \in S$, we define

$$\Gamma(\lambda) = \{k \in \mathbb{T}^d | \phi(k) = \lambda\},$$

ϕ serves the same role for (3.1) as does a for (3.6). However, it is not the case that $\nabla\phi(k) \neq 0$ on Γ for (3.1). This is only true for $\lambda \in (-d, -d+2) \cup (d-2, d)$. Instead, the picture is more complicated for other λ.

For $k \in \Gamma(\lambda)$, set the outward normal to be $n = \nabla\phi/|\nabla\phi| \in S^{d-1}$. $\nabla\phi(k) = -\sum_{0<j\leqslant d} \sin(k_j)\vec{e}_j$, so this is well-defined for $k \notin \{(0,0,\dots,0),(\pi,\pi,\dots,\pi)\}$, hence $\lambda \neq \pm d$.

For $\xi \in \mathbb{Z}^d$, associate to it the direction vector $\omega = \xi/|\xi| \in S^{d-1}$. Let $k(\omega, \lambda; s)$ be the collection of points $k \in \Gamma(\lambda)$ at which $\nabla\phi(k(\omega,\lambda;s)) = \omega$, indexed by $s = 1, 2, \dots, m$. This collection is finite because every point in the collection has non-vanishing Gaussian curvature, so $\{k(\omega,\lambda)\}$ consists of isolated points, and because $\Gamma(\lambda)$ has bounded curvature. Say that ω is singular if there is a $j \in \{1, 2, \dots, m\}$ such that the Gaussian curvature of $\Gamma(\lambda)$ is 0 at $k(\omega, \lambda, j)$.

It is clear from a stationary phase argument that the behavior of ϕ at the points $k(\xi/|\xi|, \lambda; s)$ determines the dominant asymptotic behavior of solutions to (3.1) in nonsingular directions, producing decay like $|\xi|^{(d-1)/2}$. However, this decay rate is destroyed by the presence of vanishing curvature at any point $k(\xi/|\xi|, \lambda; j)$. Collect the set of singular directions in

$$\Omega_0 = \{\omega \in S^{d-1} | \omega \text{ is singular}\}.$$

The set $\Omega\backslash\Omega_0$ is open, so is a collection of open connected components; call these components non-singular domains. Let $V \subset \Omega$ be a non-singular domain. Since V is connected and the multivalued function $\omega \to \{k(\omega,\lambda;s)\}$ is smooth, the number of $k(\omega,\lambda;s)$ mapped to by ω is constant on V. Call the largest s obtained on V by m_V.

Finally, let $\mu(\omega,\lambda;s) = k(\omega,\lambda;s)\cdot\omega$, where the dot product is assessed after inverting our original embedding of the unit ball in $\mathbb{R}^d$ into S^{d-1}. This prepares us for the radiation conditions:

Definition 3.1.1. Let $\psi_\pm$ be in $l^2(\mathbb{Z}^d)$. We say that $\psi_\pm \in W_\pm$ if the following two conditions are met:

i. there is a C such that, for every integer $R > 0$, $\psi_\pm$ satisfies

$$\frac{1}{R} \sum_{\xi \in B_{2R} \setminus B_R} |\psi_\pm(\xi)|^2 < C \,.$$

ii. for any non-singular domain $V \subset \Omega$ and $\omega = \xi/|\xi| \in V$,

$$\psi_\pm(\xi) = \sum_{s=1}^{m_V} \frac{e^{\pm i\mu(\omega,\lambda;s)|\xi|}}{|\xi|^{(d-1)/2}} a_\pm(\omega, \lambda; s) + O\left(\frac{1}{|\xi|^{(d+1)/2}}\right) \text{ as } |\xi| \to \infty \,.$$

The paper of Shaban and Vainberg culminates in the following theorem:

Theorem 3.1.1 (Absorption Principle)**.** *For any $f \in C_0(\mathbb{Z}^d)$, any $q \in C_0(\mathbb{Z}^d)$ and $\lambda \in S$, the limits $\psi_\pm$ of $\psi_\eta = R_\eta f$ as $\eta \to \lambda \pm i0$ exist. Moreover, for each $\lambda \in S$, the equation*

$$(\Delta + q - \lambda)\psi = f$$

admits unique solutions in W_+ and in W_- and these solutions are unique.

However, solutions obtained by taking the limit $\eta \to \lambda \pm i0$ for $\lambda \in S_0$ generally fail to have pointwise limits or grow at infinity.

A complete characterization of the asymptotic behavior of the Green's function for the Laplacian is absent from the literature. In [40], Martin presents several formulations for the Green's function of the Laplacian on the spectrum with this adjacency rule. He restricts his attention to the case where λ is on the spectrum, and provides an explicit formula only for the diagonal $G_\lambda((0,0),(m,n))$. Bhat and Osting [6] build on this by implementing a recursion relation found in [44] to numerically compute the values of the Green's function everywhere in the plane, still restricted to real λ.

The rest of this chapter is devoted to providing a closed form expression for the first order asymmptotics of the Green's function of the Laplacian for arbitrary pairs of elements of $\mathbb{Z}^2$ and any $\lambda \in \mathbb{C}\backslash\mathbb{R}$. For the two-dimensional case, the surface $\Gamma(\lambda)$ is strictly convex for $\lambda \in S$ and so every direction is non-singular.

3.2 Integral Representation of the Green's Function

We will require a more convenient form for the integral representation of the Green's function in order to apply the method of steepest descents, discussed in the next section.

Lemma 3.2.1. *Let $N = n + m$ and $\theta = \frac{m-n}{m+n}$. For $\theta \in [0, 1]$, the following identity holds:*

$$G_\lambda((0,0),(m,n)) = -\frac{2}{\pi\lambda}\int_0^{\pi/2} \frac{\cos(N\theta t)}{\sqrt{1-4\lambda^{-2}\cos^2 t}} \left(\frac{2\lambda^{-1}\cos t}{1+\sqrt{1-4\lambda^{-2}\cos^2 t}}\right)^N dt\,.$$

Proof. For $(m, n) \in \mathbb{Z}^2$, we have an integral representation for the value of the Green's function

$$G_\lambda((0,0),(m,n)) = \int_{\mathbb{T}^2} \frac{1}{\cos\xi + \cos\eta - \lambda} e^{-i(m\xi+n\eta)}\, d\xi\, d\eta\,.$$

Following Martin [40, equation 14] who is in turn following Koster [36], we note, for $\delta > 0$,

$$\frac{1}{\sigma + i\delta} = -i\int_0^\infty e^{i(\sigma+i\delta)\zeta}\, d\zeta\,.$$

Note also that ([40], just before equation 30, but standard)

$$\int_{-\pi}^{\pi} e^{im\xi} e^{2i\zeta\cos\xi}\, d\xi = 2\pi i^m J_m(2\zeta)\,.$$

Combining these equations, we see that

$$G_\lambda((0,0),(m,n)) = i^{m+n-1}\int_0^\infty e^{-i\lambda\zeta} J_m(\zeta) J_n(\zeta)\, d\zeta\,.$$

Now we use Neumann's formula for positive integer orders, [53, §5.43]

$$J_m(\zeta)J_n(\zeta) = \frac{2}{\pi}\int_0^{\pi/2} J_{m+n}(2\zeta\cos t)\cos((m-n)t)\,dt\,.$$

and switch the order of integration to get

$$G_\lambda((0,0),(m,n)) = \frac{2i^{m+n-1}}{\pi}\int_0^{\pi/2}\cos((m-n)t)\int_0^\infty e^{-i\lambda\zeta}J_{m+n}(2\zeta\cos t)\,d\zeta\,dt\,.$$

We apply the following formula [20, Eq. 10.22.49],

$$\int_0^\infty e^{-at}J_\nu(bt)\,dt = \frac{(\frac{b}{2})^\nu}{a^{\nu+1}}F\left(\frac{\nu+1}{2},\frac{\nu+2}{2};\nu+1;-\frac{b^2}{a^2}\right)\,,$$

which holds for $\mathrm{Re}(\nu) > -1$ and $\mathrm{Re}(a+ib) > 0$, and where F is the hypergeometric function. Now,

$$\begin{aligned} G_\lambda((0,0),(m,n)) = \frac{2i^{m+n-1}}{\pi}&\int_0^{\pi/2}\cos((m-n)t)\frac{(\cos t)^{m+n}}{(i\lambda)^{m+n+1}} \\ &\times F\left(\frac{m+n+1}{2},\frac{m+n+2}{2};m+n+1;4\lambda^{-2}\cos^2 t\right)\,dt \\ = -\frac{2}{\pi\lambda^{m+n+1}}&\int_0^{\pi/2}\cos((m-n)t)(\cos t)^{m+n} \\ &\times F\left(\frac{m+n+1}{2},\frac{m+n+2}{2};m+n+1;4\lambda^{-2}\cos^2 t\right)\,dt\,. \end{aligned}$$

We apply the formula at [20, Eq. 15.4.18], which is

$$F\left(a,a+\frac{1}{2};2a;z\right) = \frac{1}{\sqrt{1-z}}\left(\frac{1}{2}+\frac{1}{2}\sqrt{1-z}\right)^{1-2a}\,.$$

This identity holds for the principal branch when $|z| < 1$ and by analytic continuation elsewhere.

$$\begin{aligned} G_\lambda((0,0),(m,n)) = -\frac{2}{\pi\lambda}&\int_0^{\pi/2}\lambda^{-m-n}\cos((m-n)t)(\cos t)^{m+n} \\ &\times\frac{1}{\sqrt{1-4\lambda^{-2}\cos^2 t}}\left(\frac{1}{2}+\frac{1}{2}\sqrt{1-4\lambda^{-2}\cos^2 t}\right)^{-m-n}\,dt \\ = -\frac{2}{\pi\lambda}&\int_0^{\pi/2}\frac{\cos(N\theta t)}{\sqrt{1-4\lambda^{-2}\cos^2 t}}\left(\frac{2\lambda^{-1}\cos t}{1+\sqrt{1-4\lambda^{-2}\cos^2 t}}\right)^N\,dt\,. \end{aligned}$$

□

3.3 Steepest Descent Method and First-Order Asymptotics

Let $g : D \to \mathbb{C}$ and $p : D \to \mathbb{C}$ be holomorphic functions on some $D \subset \mathbb{C}^n$ and let $\mathcal{C} \subset D$ be a curve. Consider the intergral

$$J = \int_{\mathcal{C}} g(z) e^{rp(z)}\, dz \tag{3.7}$$

with $r \in \mathbb{R}$. We are interested in the asymptotics of this integral as r goes to $+\infty$.

Dominant contributions to this integral come from regions of D where p has large real part. However, oscillations coming from variations in the imagingary part of p can damp these contributions. By the Cauchy integral theorem, we can deform $\mathcal{C}$ to some $\mathcal{C}'$ sharing the same endpoints. By choosing a contour $\mathcal{C}'$ such that the imaginary part of p is constant, the integral can be transformed to one that does not oscillate, leading to more straightforward evaluation.

The method of steepest descent originates in a unpublished notes of Riemann, and was first published by Debye [12] in 1909. A particularly clear exposition can be found in [55]; the following treatment is also influenced by [10].

The main tools in this are the two following results:

Lemma 3.3.1 (Watson's Lemma)**.** *Suppose that* $\mu > -1$ *and that* $h : \mathbb{C} \to \mathbb{C}$ *is smooth with* $h(0) \neq 0$*. Suppose also that, for some* $T > 0$*,*

$$\int_0^T |t^\mu h(t)|\, dt < \infty\,.$$

Then, the following integral converges for all $r > 0$ *and has the asymptotic behavior*

$$\int_0^T t^\mu h(t) e^{-rt}\, dt = \left(\sum_{j=0}^{\infty} \frac{h^{(j)}(0)\Gamma(\mu + j + 1)}{j!\, r^{\mu+j+1}} \right) (1 + o(1))$$

as $r \to +\infty$

Theorem 3.3.2 (Lagrange-Bürmann formula)**.** *Suppose $z = f(w)$ is analytic in a neighborhood of a and $f'(a) \neq 0$. Then we can express $w = g(z)$, where $g(z)$ is given by*

$$g(z) = a + \sum_{n=1}^{\infty} g_n \frac{(z - f(a))^n}{n!} \tag{3.8}$$

and where

$$g_n = \lim_{w \to a} \frac{d^{n-1}}{dw^{n-1}} \left(\frac{w - a}{f(w) - f(a)} \right)^n .$$

Write $p(z) = u(z) + iv(z)$ for real-valued functions u and v. Suppose that $\mathcal{C}'$ is a smooth curve such that v is constant on $\mathcal{C}'$ and such that u is strictly monotonic on $\mathcal{C}'$. Let z_0 be the endpoint of $\mathcal{C}'$ where u attains its maximum. Suppose that u maps $\mathcal{C}'$ onto an interval I. Then for $\tau \in I$, τ parametrizes $\mathcal{C}'$, and we can write

$$J = \int_{\mathcal{C}'} g(z) e^{rp(z)} \, dz \tag{3.9}$$

$$= e^{irp(z_0)} \int_I g(z(\tau)) \frac{dz}{d\tau} e^{r\tau} \, d\tau \, . \tag{3.10}$$

After making the substitution $t = u(z_0) - \tau$, we have the expression

$$J = e^{rp(z_0)} \int_{u(z_0) - I} g(z(t)) \frac{dz}{dt} e^{-rt} \, dt \, ,$$

which is of the form treated by Watson's lemma. Write $t^s h(t) = g(z(t)) \frac{\mathrm{dz}}{\mathrm{dt}}$, with $h(t)$ smooth and $h(0) \neq 0$, so that we find

$$J \approx e^{rp(z_0)} \sum_{j=0}^{\infty} \frac{h^{(j)}(z_0) \Gamma(s + j + 1)}{j! r^{s+j+1}}$$

as $r \to \infty$.

The standard example of this process is for the approximation of Ai, the Airy function. $Ai(z)$ is defined as the analytic extension to $z \in \mathbb{C}$ of the solution to $y'' - xy = 0$ for $x \in \mathbb{R}$. This function is given by the integral

$$Ai(z) = \lim_{R\to\infty} \frac{1}{\pi} \int_0^R \cos\left(\frac{1}{3}s^3 + zs\right) ds\,. \tag{3.11}$$

Corollary 3.3.3. *The Airy function defined in (3.11) has the asymptotic expansion:*

$$Ai(z) = \frac{e^{-2z^{3/2}}/3}{2\pi z^{1/4}} \sum_{m=0}^{\infty} \frac{(-1)^m}{3^{2m}} \frac{\Gamma(3m+\frac{1}{2})}{(2m)!} z^{-3m/2}\,. \tag{3.12}$$

Proof. Make the substitutions $s = z^{1/2}t$ and $r = z^{3/2}$ and rescale R to preserve the apparent limits of integration so (3.11) becomes

$$\begin{aligned} &\lim_{R\to\infty} \frac{z^{1/2}}{\pi} \int_0^R \cos\left(\frac{z^{3/2}t^3}{3} + z^{3/2}t\right) dt \\ &= \lim_{R\to\infty} \frac{r^{1/3}}{2\pi} \int_{-R}^R e^{ir\left(\frac{t^3}{3}+t\right)} dt\,. \end{aligned}$$

Define

$$\begin{aligned} p(t) &= i\left(\frac{t^3}{3} + t\right), \\ p(x+iy) = -x^2y + \frac{y^3}{3} - y + i\left(\frac{x^3}{3} - xy^2 + x\right)&, \\ p'(t) &= i(t^2+1)\,, \end{aligned}$$

so critical points of $p(t)$ occur at $t = \pm i$. $\operatorname{Im} p(\pm i) = 0$, and curves on which $\operatorname{Im} p = 0$ are given by

$$0 = \frac{x^3}{3} - xy^2 + x = x(x^2 - 3y^2 + 3)\,.$$

By the Cauchy integral theorem, for $r >> 1$, we can deform the contour of integration to the half of the hyperbola $H = x^2 - 3y^2 + 3 = 0$ where $y > 0$, oriented from $\infty e^{i\frac{5\pi}{6}}$ to $\infty e^{i\frac{\pi}{6}}$ for $|z| < r$, with added contours at distance R from the origin, parametrized by the argument of the point on the curve, connecting H to the real line. Let H_+ be the portion of H with $x > 0$ and H_-

be the portion of H with $x < 0$. If $x + iy$ is a point in H_+, then $-x + iy$ is a point in H_- with $p(-x + iy) = \overline{p(x + iy)}z$ and the dz element along H at $-x + iy$ is the conjugate of the dz element along H at $x + iy$. Let B_R be the ball of radius R centered at the origin. This allows us to write

$$\begin{aligned} Ai(z) &= \lim_{R\to\infty} \frac{r^{1/3}}{2\pi} \left(\int_{C_-} e^{ir\left(\frac{t^3}{3}+t\right)}\,dt + \int_{H_-\cap B_R} e^{ir\left(\frac{t^3}{3}+t\right)}\,dt + \int_{H_+\cap B_R} e^{ir\left(\frac{t^3}{3}+t\right)}\,dt + \int_{C_+} e^{ir\left(\frac{t^3}{3}+t\right)}\,dt \right) \\ &= \lim_{R\to\infty} \frac{r^{1/3}}{2\pi} \left(\overline{\int_{H_+\cap B_R} e^{ir\left(\frac{t^3}{3}+t\right)}\,dt} + \int_{H_+\cap B_R} e^{ir\left(\frac{t^3}{3}+t\right)}\,dt + \overline{\int_{C_-} e^{ir\left(\frac{t^3}{3}+t\right)}\,dt} + \int_{C_+} e^{ir\left(\frac{t^3}{3}+t\right)}\,dt \right) \\ &= \lim_{R\to\infty} \frac{r^{1/3}}{\pi} \left(\operatorname{Re} \int_{H_+\cap B_R} e^{ir\left(\frac{t^3}{3}+t\right)} + \operatorname{Re} \int_{C_+} e^{ir\left(\frac{t^3}{3}+t\right)}\,dt \right). \end{aligned}$$

The contribution from the curves $C_\pm$ is small:

$$\begin{aligned} \left| \lim_{R\to\infty} \frac{r^{1/3}}{\pi} \operatorname{Re} \int_{C_+} e^{ir\left(\frac{t^3}{3}+t\right)}\,dt \right| &\leqslant \lim_{R\to\infty} \frac{r^{1/3}}{\pi} \operatorname{Re} \int_{C_+} \left| e^{-\frac{r}{3}R^3\sin(3\theta)} e^{-rR\sin\theta} \right| dt \\ &\leqslant \lim_{R\to\infty} \frac{r^{1/3}}{\pi} \int_0^{\pi/4} Re^{-\frac{Cr}{3}R^3\theta} e^{-CrR\theta}\,d\theta \\ &\lesssim \lim_{R\to\infty} \frac{r^{1/3}}{\pi} \int_0^{\infty} R^{-2} e^{-u}\,du = 0\,. \end{aligned}$$

Write

$$\begin{aligned} u(z) &= i\left(\frac{i^3}{3} + i\right) - i\left(\frac{t^3}{3} + t\right) \\ &= (t-i)^2 - \frac{i}{3}(t-i)^3 \\ &= (t-i)^2 \left(1 - \frac{i}{3}(t-i)\right) \\ u^{1/2}(z) &= (t-i)\left(1 - \frac{i}{3}(t-i)\right)^{1/2}. \end{aligned}$$

For $t \in H_+$, $u \in [0, \infty)$. $u^{1/2}$ is analytic around $t = i$, and $\frac{d}{dz}u^{1/2} \neq 0$, so we can use the Lagrange-Bürmann formula to produce a series expansion for the inverse function:

$$t - i = \sum_{n=1}^{\infty} \lim_{t\to i} \left[\frac{d^{n-1}}{dt^{n-1}} \left(\frac{t-i}{u^{1/2}(t) - u^{1/2}(i)} \right) \right] \frac{u^{n/2}}{n!}$$

$$= \sum_{n=1}^{\infty} \lim_{t\to i} \left[\frac{d^{n-1}}{dt^{n-1}} \left(1 - \frac{i}{3}(t-i)\right)^{-n/2} \right] \frac{u^{n/2}}{n!}$$

$$= \sum_{n=1}^{\infty} \left(\frac{i}{3}\right)^{n-1} \frac{n}{2}\left(\frac{n}{2}+1\right)\left(\frac{n}{2}\right)\cdots\left(\frac{n}{2}+n-2\right)\frac{u^{n/2}}{n!}$$

$$= \sum_{n=1}^{\infty} \left(\frac{i}{3}\right)^{n-1} \frac{\Gamma(\frac{3n}{2}-1)}{\Gamma(\frac{n}{2})}\frac{u^{n/2}}{n!}\,.$$

Now we have

$$Ai(z) = \frac{r^{1/3}}{\pi} e^{-2z^{3/2}/3}\,\mathrm{Re}\int_0^{\infty} e^{-ru}\frac{dt}{du}\,du$$

$$= \frac{r^{1/3}}{\pi} e^{-2z^{3/2}/3}\,\mathrm{Re}\int_0^{\infty} e^{-ru}\sum_{n=1}^{\infty}\left(\frac{i}{3}\right)^{n-1}\frac{\Gamma(\frac{3n}{2}-1)}{2\Gamma(\frac{n}{2})}\frac{u^{n/2-1}}{(n-1)!}\,du$$

$$= \frac{r^{1/3}}{\pi} e^{-2z^{3/2}/3}\,\mathrm{Re}\sum_{n=1}^{\infty}\left(\frac{i}{3}\right)^{n-1}\frac{\Gamma(\frac{3n}{2}-1)}{2\Gamma(\frac{n}{2})}\frac{1}{(n-1)!}\int_0^{\infty} u^{n/2-1}e^{-ru}\,du$$

$$= \frac{r^{1/3}}{\pi} e^{-2z^{3/2}/3}\,\mathrm{Re}\sum_{n=1}^{\infty}\left(\frac{i}{3}\right)^{n-1}\frac{\Gamma(\frac{3n}{2}-1)}{2(n-1)!}r^{-n/2}$$

$$= \frac{e^{-2z^{3/2}}/3}{2\pi z^{1/4}}\sum_{m=0}^{\infty}\frac{(-1)^m}{3^{2m}}\frac{\Gamma(3m+\frac{1}{2})}{(2m)!}z^{-3m/2}\,.$$

□

In general, it is neither easy to parametrize the curve $\mathrm{Im}\, p(z) = c$ nor to perform the series inversion via the Lagrange-Bürmann formula. In the case where it is possible to obtain the parametrization, Wojdylo [54] provides an explicit formula for the coefficients of the asymptotic expansion of J in terms of partial ordinary Bell polynomials.

On the other hand, if the asymptotic behavior is only desired to first order, it is not very difficult to compute. Return to the setting of (3.9) with $\mathcal{C}'$ a curve that traces a steepest descent curve of p emanating from a saddle, z_0.

Corollary 3.3.4. *Let $p : \mathbb{C} \to \mathbb{C}$ and z_0 a point at which $p'(z_0) = 0$. Let $\mathcal{C}'$ a curve with endpoint and maximum of* $\mathrm{Re}\, p$ *at z_0. Suppose that $\mu > -1$ and that $h : \mathbb{C} \to \mathbb{C}$ is smooth with $h(z_0) \neq 0$.*

Then, to first order, the integral

$$J = \int_{C'} (z - z_0)^\mu h(z) e^{rp(z)}\, dz \tag{3.13}$$

has the asymptotic expansion

$$J \approx \Gamma\left(\mu + \frac{1}{n+1}\right) \frac{h(z_0)}{n+1} \left(\frac{(n+1)!}{p^{(n+1)}(z_0)}\right)^{1/(n+1)} e^{irp(z_0)} r^{-\mu - 1/(n+1)} (1 + o(1)) \quad \text{as } r \to +\infty\,. \tag{3.14}$$

Proof. As with the analysis of Airy's integral, set $u(z) = p(z_0) - p(z)$. If p has a saddle of order n at z_0, write $p(z) = p(z_0) + \sum_{j=n+1}^{\infty} \frac{p^{(j)}(z_0)}{j!}(z - z_0)^j$, with $p^{(n+1)}(z_0) \neq 0$.

As with the analysis of Airy's integral, set $u(z) = p(z_0) - p(z) \geqslant 0$. Then we have:

$$u(z) = (z - z_0)^{n+1} \sum_{j=n+1}^{\infty} \frac{p^{(j)}(z_0)}{j!} (z - z_0)^{j-n-1}$$

$$u^{1/(n+1)} = (z - z_0) \left(\sum_{j=n+1}^{\infty} \frac{p^{(j)}(z_0)}{j!} (z - z_0)^{j-n-1} \right)^{1/(n+1)} .$$

Now, to first order, we have

$$z - z_0 \approx \lim_{z \to z_0} \left[\frac{z - z_0}{p(z) - p(z_0)} \right] u^{1/(n+1)}$$

$$= \lim_{z \to z_0} \left(\sum_{j=n+1}^{\infty} \frac{p^{(j)}(z_0)}{j!} (z - z_0)^{j-n-1} \right)^{-1/(n+1)} u^{1/(n+1)}$$

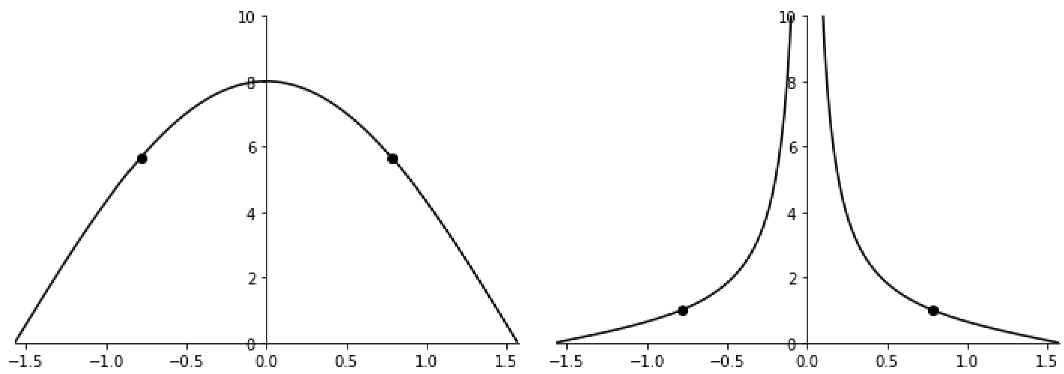
$$= \left(\frac{(n+1)!}{p^{(n+1)}(z_0)} \right)^{1/(n+1)} u^{1/(n+1)} .$$

Returning to (3.13), we can find our first-order approximation for J:

$$J \approx e^{rp(z_0)} \int_0^\infty (z(u) - z_0)^\mu h(z(u)) \frac{dz}{du} e^{-ru}\, du$$

$$= e^{rp(z_0)} \int_0^\infty (z(u) - z_0)^\mu h(z(u)) \left(\frac{(n+1)!}{p^{(n+1)}(z_0)} \right)^{1/(n+1)} \frac{u^{-n/(n+1)}}{n+1} e^{-ru}\, du\,.$$

Apply Watson's lemma to obtain (3.14). ☐



(a) (b)

Figure 3.1: Level curves of the imaginary part of example members of class $\mathcal{C}$, with saddle points indicated.

3.4 Geometry of the Steepest Descent Curves

The principal difficulty with applying the method of steepest descent is verifying the geometry of the steepest descent curves of $p(z)$ in (3.7). Our case is not unique, and we have been able to obtain only partial results in this direction.

Figures (3.2(a)) through (3.2(d)) contain images of the level curves for the imaginary part of the Green's function for different values of λ and the formula for $p(x)$ given in (3.16). Our computational results suggest that $p(z)$ is a member of the following class for λ with $\operatorname{Re}\lambda \neq 0$ and $\operatorname{Im}\lambda \neq 0$:

Definition 3.4.1. We say that a function $p : \mathbb{C} \to \mathbb{C}$ is in the class $\mathcal{C}$ if the following hold:

1. There is a saddle point of p, z_1, with $\operatorname{Re} z_1 \in (0, \pi/2)$ and $\operatorname{Im} z_1 > 0$.
2. There is a saddle point of p, z_2, with $\operatorname{Re} z_2 \in (-\pi/2, 0)$ and $\operatorname{Im} z_2 > 0$.
3. Level curves of $\operatorname{Im} p$ connect $\pi/2$ to z_1, $-\pi/2$ to z_2, and either z_1 to z_2 or both saddle points to $+i\infty$.

3.5 Asymptotics of the Greens function

We use the steepest descent method to compute the asymptotic behavior of the integral

$$I_\theta(N) = \frac{2}{\pi\lambda}\int_0^{\pi/2} \frac{\cos(N\theta t)}{\sqrt{1-4\lambda^{-2}\cos^2 t}} \left(\frac{2\lambda^{-1}\cos t}{1+\sqrt{1-4\lambda^{-2}\cos^2 t}}\right)^N dt \tag{3.15}$$

to first order. The integrand is even, so the integral is equal to

$$\begin{aligned}
&\frac{1}{2}\int_{-\pi/2}^{\pi/2} \frac{\cos(N\theta t)+i\sin(N\theta t)}{\sqrt{1-4\lambda^{-2}\cos^2 t}} \left(\frac{2\lambda^{-1}\cos t}{1+\sqrt{1-4\lambda^{-2}\cos^2 t}}\right)^N dt \\
&= \frac{1}{2}\int_{-\pi/2}^{\pi/2} \frac{e^{iN\theta t}}{\sqrt{1-4\lambda^{-2}\cos^2 t}} \left(\frac{2\lambda^{-1}\cos t}{1+\sqrt{1-4\lambda^{-2}\cos^2 t}}\right)^N dt \\
&= \frac{1}{2}\int_{-\pi/2}^{\pi/2} \frac{1}{\sqrt{1-\frac{4\cos^2 t}{\lambda^2}}} e^{iN\theta t + N\log\left(\frac{2\lambda^{-1}\cos t}{1+\sqrt{1-4\lambda^{-2}\cos^2 t}}\right)} dt\,.
\end{aligned}$$

Set

$$p(z) = i\theta z + \log\left(\frac{2\lambda^{-1}\cos z}{1+\sqrt{1-4\lambda^{-2}\cos^2 z}}\right), \tag{3.16}$$

taking principal value for the square root and logarithm when $z \in (-\pi/2, \pi/2)$ and extending analytically, away from singularities, so that the integrand in $I_\theta(N)$ becomes $\frac{e^{Np(t)}}{\sqrt{1-4\lambda^{-2}\cos^2 t}}$. Note that the conditions in Definition 3.4.1 that $\operatorname{Im} z_1 > 0$ and $\operatorname{Im} z_2 > 0$ are equivalent to the square root in (3.16) taking its principal value for, at any saddle point z,

$$-i\tan z = \theta\sqrt{1-4\lambda^{-2}\cos^2 z}\,.$$

If the principal branch of the square root has positive real part, then $\tan z$ must have positive imaginary part. But since, for real number x and y, $\tan(x+iy) = \frac{\sin 2x + i\sinh 2y}{\cos 2x + \cosh 2y}$, for $\operatorname{Im}\tan z$ to be positive, $\operatorname{Im} z$ must be positive.

The function $p(z)$ fails to be analytic when $\cos z = 0$ and when $1-4\lambda^{-2}\cos^2 z = 0$. Elsewhere, we have

$$p(z) = i\theta z + \log\left(\frac{2\lambda^{-1}\cos z}{1+\sqrt{1-4\lambda^{-2}\cos^2 z}}\right), \tag{3.17}$$

$$p'(z) = i\theta - \frac{\tan z}{\sqrt{1 - 4\lambda^{-2}\cos^2 z}}\,, \tag{3.18}$$

$$p''(z) = -\frac{\lambda^2 \sec^2 z - 4 - 4\sin^2 z}{\lambda^2 \left(1 - 4\lambda^{-2}\cos^2 z\right)^{3/2}}\,. \tag{3.19}$$

We set

$$\beta_\pm(\theta, \lambda) = \sqrt{-\frac{\lambda^2}{8\theta^2}\left(1 - \theta^2 \pm \sqrt{(1-\theta^2)^2 + 16\lambda^{-2}\theta^2}\right)}\,, \tag{3.20}$$

though we will suppress the dependence of $\beta_\pm(\theta, \lambda)$ on its arguments for much of the following exposition and simply write $\beta_\pm$.

We will need some facts regarding the location of $\beta_\pm$ in the complex plane. To that end, we refer to the quadrants of $\mathbb{C}\backslash(\mathbb{R} \cup i\mathbb{R})$ by the following names:

$$Q_1 = \{z \in \mathbb{C} : \operatorname{Re} z > 0 \text{ and } \operatorname{Im} z > 0\}\,,$$

$$Q_2 = \{z \in \mathbb{C} : \operatorname{Re} z < 0 \text{ and } \operatorname{Im} z > 0\}\,,$$

$$Q_3 = \{z \in \mathbb{C} : \operatorname{Re} z < 0 \text{ and } \operatorname{Im} z < 0\}\,,$$

$$Q_4 = \{z \in \mathbb{C} : \operatorname{Re} z > 0 \text{ and } \operatorname{Im} z < 0\}\,.$$

Lemma 3.5.1. *The quadrant containing $\beta_\pm$ depends on λ as follows:*

$$\operatorname{Re} \beta_\pm > 0\,, \tag{3.21}$$

$$-\operatorname{sgn} \operatorname{Im} \beta_\pm = \pm \operatorname{sgn} \operatorname{Im}(\lambda^2)\,. \tag{3.22}$$

Proof. (3.21) follows immediately from the choice of branch for the principal value of the square root.

Suppose that $\lambda \in Q_1$. Write $\lambda = re^{i\phi}$, so we have:

$$\beta_\pm^2 = -\frac{\lambda^2}{8\theta^2}\left(1 - \theta^2 \pm \sqrt{(1-\theta^2)^2 + 16\theta^2\lambda^{-2}}\right)$$

$$= -\frac{(1-\theta^2)r}{8\theta^2} e^{2i\phi} \left(1 \pm \sqrt{1 + \frac{16\theta^2}{(1-\theta^2)^2 r^2} e^{-2i\phi}} \right)$$

$$= -se^{2i\phi} \left(1 \pm \sqrt{1 + te^{-2i\phi}} \right)$$

$$= -se^{i\phi} \left(e^{i\phi} \pm \sqrt{e^{2i\phi} + t} \right)$$

for some $s, t > 0$. It is clear that $\operatorname{Im} \beta_+^2 < 0$.

For β_-^2, when $t = 0$, $-se^{i\phi} \left(e^{i\phi} - \sqrt{e^{2\phi i} + t} \right) = 0$. The partial derivative with respect to t is

$$-se^{i\phi} \frac{\partial}{\partial t} \left(e^{i\phi} - \sqrt{e^{2\phi i} + t} \right) = \frac{se^{i\phi}}{2} \left(e^{2i\phi} + t \right)^{-1/2} = \frac{s}{2} \left(1 + te^{-2i\phi} \right)^{-1/2} \in Q_1 \, .$$

For any choice of $t > 0$,

$$\operatorname{Im} \beta_-^2 = \operatorname{Im} \left(0 + \frac{s}{2} \int_0^t \left(e^{2i\phi} + u \right)^{-1/2} du \right) > 0$$

and so $\beta_- \in Q_1$.

Now,

$$\beta_\pm(-\lambda) = \beta_\pm(\lambda) \, ,$$

$$\beta_\pm(\overline{\lambda}) = \overline{\beta_\pm(\lambda)} \, ,$$

so we conclude that $\beta_+ \in Q_4$ when $\lambda \in Q_1 \cup Q_3$ and $\beta_+ \in Q_1$ when when $\lambda \in Q_2 \cup Q_4$; $\beta_- \in Q_1$ when $\lambda \in Q_1 \cup Q_3$ and $\beta_- \in Q_4$ when $\lambda \in Q_2 \cup Q_4$. This gives us the result. □

Lemma 3.5.2. *The imaginary part of* $\frac{\lambda}{2\beta_\pm}$ *depends on* λ *as follows:*

$$\operatorname{sgn} \operatorname{Im} \frac{\lambda}{2\beta_\pm} = \operatorname{sgn} \operatorname{Im} \lambda \, . \tag{3.23}$$

Proof. Suppose that $\lambda \in Q_1$. According to (3.22), $\beta_+ \in Q_4$. Thus, $\operatorname{Im} \frac{\lambda}{2\beta_+} > 0$.

Now consider $\lambda^{-1}\beta_-$. From (3.22), $\beta_- \in Q_1$, so we can put the λ^{-1} inside the square root without changing the sign, and we have

$$\lambda^{-1}\beta_{-} = \sqrt{-\frac{1}{8\theta^2}\left(1-\theta^2-\sqrt{(1-\theta^2)^2+16\theta^2\lambda^{-2}}\right)}\,. \tag{3.24}$$

Consider now just the expression under the radical, and write it as

$$-s\left(1-\sqrt{1+te^{-2i\phi}}\right) \tag{3.25}$$

for some $s, t > 0$ and $\phi \in (0, \pi/2)$. This is equal to 0 when $t = 0$ and has as partial derivative with respect to t,

$$-s\frac{\partial}{\partial t}\left(1-\sqrt{1+te^{-2i\phi}}\right) = \frac{s}{2}\left(1+te^{-2i\phi}\right)^{-1/2} e^{-2i\phi} = \frac{s}{2}(e^{2i\phi}+t)^{-1/2}e^{-i\phi}\,. \tag{3.26}$$

The expressions $(e^{2i\phi}+t)^{-1/2}$ and $e^{-i\phi}$ are both elements of Q_4, so their product has negative imaginary part. As in the previous lemma, this property propagates to (3.24) and so $\lambda^{-1}\beta_{-} \in Q_4$. Thus, $\operatorname{Im}\frac{\lambda}{2\beta_{-}} > 0$.

Now,

$$\frac{-\lambda}{2\beta_{\pm}(-\lambda)} = -\frac{\lambda}{2\beta_{\pm}(\lambda)}\,,$$
$$\frac{\overline{\lambda}}{2\beta_{\pm}(\overline{\lambda})} = \overline{\left(\frac{\lambda}{2\beta_{\pm}(\lambda)}\right)}\,,$$

so we conclude that $\operatorname{Im}\frac{\lambda}{2\beta_{\pm}} > 0$ when $\lambda \in Q_1 \cup Q_2$ and $\operatorname{Im}\frac{\lambda}{2\beta_{\pm}} < 0$ when $\lambda \in Q_3 \cup Q_4$. This gives us the result. □

Lemma 3.5.3. *Let λ be such that $\operatorname{Re}\lambda$ is not a multiple of 2 and $\operatorname{Im}\lambda \neq 0$. Then saddles of p are simple and occur at*

$$z_{\pm} = \pm(\operatorname{sgn}\operatorname{Im}(\lambda^2))\arccos\beta_{\pm} + 2\pi\,. \tag{3.27}$$

Proof. Saddles of p occur when $p'(z) = 0$. This implies that

$$i\theta = \frac{\tan z}{\sqrt{1 - 4\lambda^{-2}\cos^2 z}}\,, \tag{3.28}$$

the solution set to which is contained in the solution set of the following:

$$-\theta^2 = \frac{\tan^2 z}{1 - 4\lambda^{-2}\cos^2 z}\,,$$

$$-\theta^2 \cos^2 z \left(1 - 4\lambda^{-2}\cos^2 z\right) = 1 - \cos^2 z\,,$$

$$\frac{4\theta^2}{\lambda^2}\cos^4 z + (1 - \theta^2)\cos^2 z - 1 = 0\,, \tag{3.29}$$

$$\cos^2 z = -\frac{\lambda^2}{8\theta^2}\left(1 - \theta^2 \pm \sqrt{(1-\theta^2)^2 + 16\lambda^{-2}\theta^2}\right)\,, \tag{3.30}$$

$$\cos^2 z = \beta_\pm^2\,. \tag{3.31}$$

These are simple saddles unless $p''(z)$ is also equal to 0. This occurs when

$$\frac{\lambda^2}{\cos^2 z} - 4 - 4\sin^2 z = 0\,,$$

$$4\cos^4 z - 8\cos^2 z + \lambda^2 = 0\,,$$

$$\cos^2 z = 1 \pm \frac{\sqrt{4 - \lambda^2}}{2}\,.$$

Plugging this into (3.29), we see that a necessary condition for a higher order saddle is that

$$\frac{4\theta^2}{\lambda^2}\left(1 \pm \frac{\sqrt{4-\lambda^2}}{2}\right)^2 + (1-\theta^2)\left(1 \pm \frac{\sqrt{4-\lambda^2}}{2}\right) - 1 = 0\,,$$

$$(8\theta^2 + (1-\theta^2)\lambda^2)\sqrt{4-\lambda^2} + 4\theta^2(4-\lambda^2) = 0\,,$$

$$\lambda^2(4-\lambda^2)(16\theta^2 + (1-\theta^2)^2\lambda^2) = 0\,.$$

Recall that we defined S_0 to be the set of exceptional values in $\sigma(H)$ for which we do not expect an absorption principle to hold; in the case of $\mathbb{Z}^2$, $S_0 = \{-2, 0, 2\}$. From these calculations, we see that λ must be in S_0, or must be purely imaginary. We will be interested in λ approaching $S = \sigma(H)\backslash S_0$ from above and below in the complex plane, so we omit the analysis of the behavior at higher order saddles.

Choosing the principal branch of $\arccos z$ to be

$$\frac{\pi}{2} + i \log\left(\sqrt{1-z^2} + iz\right) , \tag{3.32}$$

all saddles are simple. These saddles are restricted to the set:

$$z = \pm_3 \arccos\left(\pm_2 \beta_{\pm_1}\right) + 2\pi n$$

for $n \in \mathbb{Z}$ and $\{\pm_1, \pm_2, \pm_3\} \subset \{+, -\}$. The choice of each plus-minus is determined by the criteria we identified in §3.4.

The first criterion from §3.4 is that the real part of any saddle must be in the interval $(-\pi/2, \pi/2)$. For our choice of principal branch of the inverse cosine (3.32), this is equivalent to $\operatorname{Im}\left(\sqrt{1-z^2} + iz\right) > 0$. If $z \in Q_3$, then z^2 has positive imaginary part and iz has negative imaginary part, hence $\operatorname{Im}\left(\sqrt{1-z^2} + iz\right) < 0$. If $z \in Q_4$, then z^2 has negative imaginary part and iz has positive imaginary part, hence $\operatorname{Im}\left(\sqrt{1-z^2} + iz\right) > 0$. We also have the identity

$$\begin{aligned} \arccos(-z) &= \frac{\pi}{2} + i\log\left(\sqrt{1-z^2} - iz\right) \\ &= \pi - \left(\frac{\pi}{2} + i\log\left(\sqrt{1-z^2} + iz\right)\right) \\ &= \pi - \arccos(z)\,. \end{aligned} \tag{3.33}$$

Taken together, these imply that $\operatorname{Re}\arccos(z) \in (-\pi/2, \pi/2)$ if and only if $\operatorname{Re} z > 0$. This is true for the principal branch of the square root, so we choose the plus sign for $\pm_2$. Note that the choice of sign for $\pm_3$ preserves $\operatorname{Re}\arccos(z) \in (-\pi/2, \pi/2)$.

The second criterion is that the imaginary part of any saddle must be greater than 0. For our choice of principal branch of the inverse cosine, we choose the plus sign for $\pm_3$ when $|\sqrt{1-z^2} + iz| > 1$ and the minus sign for $\pm_3$ when $|\sqrt{1-z^2} + iz| < 1$. Suppose that $z \in Q_3$. Then both $\sqrt{1-z^2}$ and iz are in Q_4. Note that, for z and w in the same quadrant,

$$|z+w|^2 > |z|^2 + |w|^2 .$$

We have

$$\begin{aligned} |\sqrt{1-z^2}+iz|^2 &> |\sqrt{1-z^2}|^2 + |iz|^2 \\ &= |1-z^2| + |z^2| \\ &\geqslant 1 . \end{aligned}$$

Similarly, for $z \in Q_4$, both $\sqrt{1-z^2}$ and iz are in Q_1, so $|\sqrt{1-z^2}+iz|^2 > 1$. Using again (3.33), we see that $\pm_3$ takes the minus sign if $\beta_{\pm_1} \in Q_1$ and the plus sign if $\beta_{\pm_1} \in Q_4$. Thus, from (3.22),

$$\pm_3 = -\operatorname{sgn}\operatorname{Im}\beta_{\pm_1} = \pm_1 \operatorname{sgn}\operatorname{Im}\lambda^2 .$$

This leaves us with the saddles of $p(z)$, which are those identified in (3.27). □

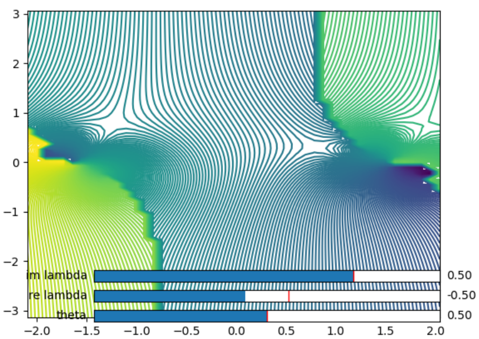
Lemma 3.5.4. *Let z be the location of a saddle as specified in (3.27). There is a $\delta \in \mathbb{Z}$ which makes the following identity true:*

$$\log\left(\frac{z^{-1}}{1+\sqrt{1-z^{-2}}}\right) = -(\operatorname{sgn}\operatorname{Im}\lambda) i \arccos z + 2\pi i \delta . \tag{3.34}$$

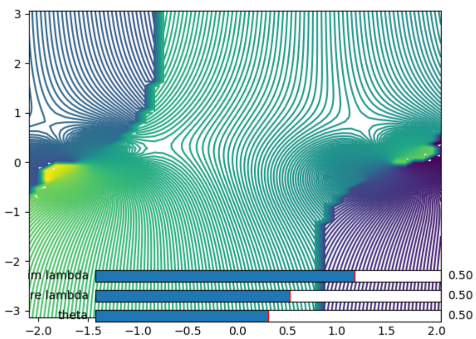
Proof. Note first that $z\sqrt{1-z^{-2}} = \pm i\sqrt{1-z^2}$. $\operatorname{Im} z\sqrt{1-z^2}$ is greater than 0 for $z \in Q_1 \cup Q_2$ and less than 0 for $z \in Q_3 \cup Q_4$. $\operatorname{Im} i\sqrt{1-z^2}$ is always greater than 0. We summarize this as $z\sqrt{1-z^{-2}} = (\operatorname{sgn}\operatorname{Im} z) i\sqrt{1-z^2}$.

$$\begin{aligned} \log\left(\frac{z^{-1}}{1+\sqrt{1-z^{-2}}}\right) &= \log\left(z - z\sqrt{1-z^{-2}}\right) \\ &= \log\left(z - (\operatorname{sgn}\operatorname{Im} z) i\sqrt{1-z^2}\right) \\ &= \log\left(-i(\operatorname{sgn}\operatorname{Im} z)\left((\operatorname{sgn}\operatorname{Im} z) iz + \sqrt{1-z^2}\right)\right) \\ &= -i(\operatorname{sgn}\operatorname{Im} z)\frac{\pi}{2} + i\left(\frac{\pi}{2} - \arccos((\operatorname{sgn}\operatorname{Im} z)z\right) + 2\pi i \delta . \end{aligned}$$

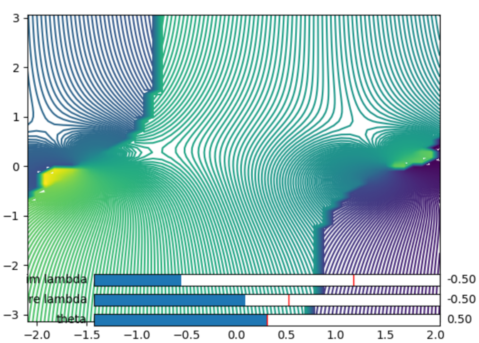
Figure 3.2



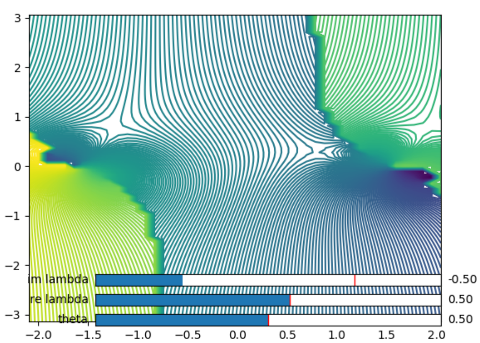
(a) Level curves of $\operatorname{Im} p$ for $\lambda = -.5 + .5i$



(b) Level curves of $\operatorname{Im} p$ for $\lambda = .5 + .5i$



(c) Level curves of $\operatorname{Im} p$ for $\lambda = -.5 - .5i$



(d) Level curves of $\operatorname{Im} p$ for $\lambda = .5 - .5i$

If $\operatorname{Im} z > 0$, this becomes

$$\log\left(\frac{z^{-1}}{1+\sqrt{1-z^{-2}}}\right) = -i\arccos z + 2\pi i\delta\,.$$

If $\operatorname{Im} z < 0$, this becomes

$$\begin{aligned}\log\left(\frac{z^{-1}}{1+\sqrt{1-z^{-2}}}\right) &= i\pi - i\arccos(-z) + 2\pi i\delta \\ &= i\arccos z + 2\pi i\delta\,.\end{aligned}$$

□

Theorem 3.5.5. *Take $\lambda \in \mathbb{C}$ with $|\operatorname{Re}\lambda| \in (0,2)$ and $\operatorname{Im}\lambda \neq 0$. Let $\theta \in [0,1]$ and $N \in \mathbb{N}$. Let $I_\theta(N)$ be as in (3.15) and suppose that p in (3.16) satisfies the conditions of Definition 3.4.1. The asymptotic behavior of $I_\theta(N)$ is described by*

$$I_\theta(N) = \frac{1}{2\lambda\pi\sqrt{1-4\lambda^{-2}\beta^2_{\operatorname{sgn}\operatorname{Re}\lambda}}}\left(\frac{2\pi}{p^{(2)}(z_{\operatorname{sgn}\operatorname{Re}\lambda})}\right)^{1/2}$$
$$\times \exp\left(-iN(\operatorname{sgn}\operatorname{Im}\lambda)\left((\operatorname{sgn}\operatorname{Re}\lambda)\theta\arccos\beta_- + \arccos\frac{\lambda}{2\beta_-}\right)\right)N^{-1/2}(1+o(1)) \quad (3.35)$$

as N goes to $+\infty$.

Proof. Deform the contour of integration from $[-\pi/2, \pi/2]$ to follow level curves of $\operatorname{Im} p$, defined in (3.16). By the hypothesis, these pass from the points $\pm\pi/2 + 0i$, through a pair of saddle points located at points given by (3.27), and then up to $+\infty i$.

Plug (3.27) and (3.19) into (3.14) to get

$$I_\theta(N) = \left(\frac{2}{4\lambda\pi\sqrt{1-4\lambda^{-2}\beta_+^2}}\left(\frac{2\pi}{p^{(2)}(z_+)}\right)^{1/2} e^{Np(z_+)}N^{-1/2}\right.$$
$$\left. + \frac{2}{4\lambda\pi\sqrt{1-4\lambda^{-2}\beta_-^2}}\left(\frac{2\pi}{p^{(2)}(z_-)}\right)^{1/2} e^{Np(z_-)}N^{-1/2}\right)(1+o(1))\,.$$

Note that the coefficients are doubled because we have two steepest descent curves emanating from each of z_+ and z_-. The term associated to z_- decays exponentially slower for all λ, so contributes the dominant term. The exponent is equal to:

$$p(z_-) = i\theta z_- + \log\left(\frac{2\lambda^{-1}\cos(z_-)}{1+\sqrt{1-4\lambda^{-2}\cos^2 z_-}}\right)$$
$$= -(\operatorname{sgn}\operatorname{Im}\lambda^2)i\theta\arccos\beta_- + \log\left(\frac{\left(\frac{\lambda}{2\beta_-}\right)^{-1}}{1+\sqrt{1-\left(\frac{\lambda}{2\beta_-}\right)^{-2}}}\right).$$

For $\lambda \in Q_j$, determine the sign of $\operatorname{Im}\frac{\lambda}{2\beta_-}$ from (3.23). By (3.34), the decay has the following form:

$$I_\theta(N) = \begin{cases} \frac{1}{2\lambda\pi\sqrt{1-4\lambda^{-2}\beta_+^2}} \left(\frac{2\pi}{p^{(2)}(z_+)}\right)^{1/2} e^{iN\left(-\theta \arccos \beta_- - \arccos \frac{\lambda}{2\beta_-}\right)} N^{-1/2} & \lambda \in Q_1 \\ \frac{1}{2\lambda\pi\sqrt{1-4\lambda^{-2}\beta_-^2}} \left(\frac{2\pi}{p^{(2)}(z_-)}\right)^{1/2} e^{iN\left(\theta \arccos \beta_- - \arccos \frac{\lambda}{2\beta_-}\right)} N^{-1/2} & \lambda \in Q_2 \\ \frac{1}{2\lambda\pi\sqrt{1-4\lambda^{-2}\beta_-^2}} \left(\frac{2\pi}{p^{(2)}(z_-)}\right)^{1/2} e^{iN\left(-\theta \arccos \beta_- + \arccos \frac{\lambda}{2\beta_-}\right)} N^{-1/2} & \lambda \in Q_3 \\ \frac{1}{2\lambda\pi\sqrt{1-4\lambda^{-2}\beta_+^2}} \left(\frac{2\pi}{p^{(2)}(z_+)}\right)^{1/2} e^{iN\left(\theta \arccos \beta_- + \arccos \frac{\lambda}{2\beta_-}\right)} N^{-1/2} & \lambda \in Q_4 \,. \end{cases}$$

□

3.6 Concordance with Literature

Recall that, in [48], Shaban and Vainberg derive radiation conditions for solutions to the Schrodinger equation. These are provided only for λ in the spectrum and are expressed in terms of the geometry of ϕ in the dual space rather than directly in terms of features of $\mathbb{Z}^d$.

In this section, we provide a brief reminder of their results specialized to the two-dimensional case and compare them to the results derived here in the previous theorem.

For $\lambda \notin [-d, d]$, the resolvent of the standard Laplacian is a bounded operator $R_\lambda : l^2(\mathbb{Z}^d) \to l^2(\mathbb{Z}^d)$ given by the formula

$$R_\lambda(f)(\xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{T}^d} \frac{\hat{f}(k) e^{ik\cdot\xi}}{\phi(k) - \lambda} \, dk \,,$$

where $\phi(k) = \sum_{j=1}^d \cos(k_j)$. The authors identify $\mathbb{T}^d = \mathbb{R}^d / 2\pi\mathbb{Z}^d$ with the cube $[-\pi, \pi]^d$ when $\lambda > 0$ and with the cube $[0, 2\pi]^d$ when $\lambda < 0$.

Let $\Gamma(\lambda) = \{k \in \mathbb{T}^d : \phi(k) = \lambda\}$. When $\lambda \in S$, $|\nabla\phi| \neq 0$; define an orientation on $\Gamma(\lambda)$ by choosing the normal vector $n = \nabla\phi = -(\sin k_1, \ldots, \sin k_d)$. Say that $\omega \in S^{d-1}$ is non-singular, if there is no point $k \in \Gamma(\lambda)$ with $\nabla\phi(k)$ pointing in the same direction as ω and with the curvature of $\Gamma(\lambda)$ at k equal to 0.

Shaban and Vainberg show that, for $d-2 < |\lambda| < d$, $\Gamma(\lambda)$ is strictly convex, so every direction $\omega \in S^{d-1}$ is non-singular and there is a unique point on $\Gamma(\lambda)$ whose normal is parallel to and points in the same direction as ω. For $d > 2$, there are pairs of $\lambda \in S$ and $\omega \in S^{d-1}$ for which ω is a singular direction; there are also pairs of $\lambda \in S$ and $\omega \in S^{d-1}$ for which there is more than one point in $\Gamma(\lambda)$ with normal parallel to and pointing in the same direction as ω. However, neither of those is true for the case $d = 2$, which concerns us here. The following result is true in this case but requires more nuance in higher dimensions.

Corollary 3.6.1. *Let $k(\omega, \lambda)$ be the unique point on $\Gamma(\lambda)$ whose normal is parallel to and points in the same direction as ω. Let $\mu(\omega, \lambda) = k(\omega, \lambda) \cdot \omega$, the projection of k on ω. $(\Delta - \lambda)\psi = f$ admits unique solutions $\psi_\pm$, each the pointwise limit of $R_\eta f$ over $\eta \in \mathbb{C}_\pm$ as $\eta \to \lambda \pm i0$. Asymptotic behavior of $\psi_\pm$ is given by*

$$\psi_\pm = \frac{e^{\pm i\mu(\omega,\lambda)|\xi|}}{|\xi|^{(d-1)/2}} a_\pm(\omega, \lambda) + O\left(|\xi|^{-(d+1)/2}\right) \tag{3.36}$$

as $|\xi| \to \infty$, the amplitude $a_\pm$ smooth in ω and λ.

This concludes our summary of the results from [48]. The following lemma expresses (3.36) in terms of θ. The remainder of the section is concerned with the equivalence of the exponents of each decay.

Define the following:

$$\begin{aligned} \beta_1 &= \frac{\lambda}{4\theta}\left((1+\theta)^2 - \sqrt{(1-\theta^2)^2 + \frac{16\theta^2}{\lambda^2}}\right), \\ \beta_2 &= \frac{-\lambda}{4\theta}\left((1-\theta)^2 - \sqrt{(1-\theta^2)^2 + \frac{16\theta^2}{\lambda^2}}\right). \end{aligned} \tag{3.37}$$

Lemma 3.6.2. *The exponent in (3.36) can be written*

$$\pm\, ik(\omega, \lambda) \cdot \xi = \mp\frac{i}{2}\left(\arccos\beta_1 + \arccos\beta_2 + \theta(\arccos\beta_1 - \arccos\beta_2)\right). \tag{3.38}$$

Proof. We rewrite the magnitude of the exponent:

$$\mu(\omega,\lambda)|\xi| = k(\omega,\lambda)\cdot\omega|\xi| = k(\omega,\lambda)\cdot\xi\,.$$

Recall that, for some point $(m,n)\in\mathbb{Z}^2$, $0\leqslant n\leqslant m$, pointing in the direction of ω, we write $\theta=\frac{m-n}{m+n}$. $(1+\theta,1-\theta)$ also points in the direction of ω. $k(\omega,\lambda)$ is the point $(x_1,x_2)\in\mathbb{T}^2$ such that:

$$\cos x_1+\cos x_2=\lambda\,, \tag{3.39}$$

$$-2\sin x_1 = s(1+\theta)\,, \tag{3.40}$$

$$-2\sin x_2 = s(1-\theta) \tag{3.41}$$

for some $s>0$. Note in particular that, for $\lambda>0$, we identify $\mathbb{T}$ with $[-\pi,\pi]^2$ and so $\{x_1,x_2\}\subset(-\pi,0)$. For $\lambda<0$, we identify $\mathbb{T}$ with $[0,2\pi]^2$, and so $\{x_1,x_2\}\subset(\pi,2\pi)$ We combine (3.40) and (3.41) to find

$$\begin{aligned}\frac{\sin x_1}{1+\theta}&=\frac{\sin x_2}{1-\theta}\,,\\ \cos^2 x_2&=\frac{4\theta}{(1+\theta)^2}+\frac{(1-\theta)^2}{(1+\theta)^2}\cos^2 x_1\,.\end{aligned}$$

Moving the $\cos x_1$ term in (3.39) to the right-hand side and squaring, this gives us

$$\begin{aligned}\cos^2 x_2 &= \lambda^2-2\lambda\cos x_1+\cos^2 x_1\,,\\ \frac{4\theta}{(1+\theta)^2}+\frac{(1-\theta)^2}{(1+\theta)^2}\cos^2 x_1 &= \lambda^2-2\lambda\cos x_1+\cos^2 x_1\,,\\ 0 &= 4\theta\cos^2 x_1-2(1+\theta)^2\lambda\cos x_1+(1+\theta)^2\lambda^2-4\theta\,,\\ \cos x_1 &= \frac{2(1+\theta)^2\lambda\pm\sqrt{4(1+\theta)^4\lambda^2-16\theta\left((1+\theta)^2\lambda^2-4\theta\right)}}{8\theta}\,,\\ \cos x_1 &= \frac{(1+\theta)^2\lambda\pm\lambda\sqrt{(1-\theta^2)^2+16\theta^2\lambda^{-2}}}{4\theta}\,.\end{aligned}$$

The equations (3.39)-(3.41) are invariant under exchange of (x_1, θ) with $(x_2, -\theta)$, so we find also

$$\cos x_2 = \frac{(1-\theta)^2\lambda \pm \lambda\sqrt{(1-\theta^2)^2 + 16\theta^2\lambda^{-2}}}{-4\theta}\,.$$

When $\lambda > 0$, $k \in [-\pi, \pi]^2$ and, in particular, $x_2 = 0$ when $\theta = 1$. This gives us

$$\begin{aligned} 1 &= \pm\frac{\lambda\sqrt{16\lambda^{-2}}}{-4}\,, \\ 1 &= \pm(-1) \end{aligned}$$

and so we choose the minus sign for $\cos x_2$ which, because of the invariance under exchange of (x_1, θ) and $(x_2, -\theta)$, implies that we take the minus sign for $\cos x_1$ as well.

When $\lambda < 0$, $k \in [0, 2\pi]^2$ and, in particular, $x_2 = \pi$ when $\theta = 1$. This gives us

$$\begin{aligned} -1 &= \pm\frac{\lambda\sqrt{16\theta^2\lambda^{-2}}}{-4}\,, \\ -1 &= \pm\frac{\lambda|\lambda|^{-1}4}{-4}\,, \\ -1 &= \pm 1 \end{aligned}$$

and so again we choose the minus sign for both $\cos x_1$ and $\cos x_2$.

Recall that we identify $\mathbb{T}^2$ with $[-\pi, \pi]$ if $\lambda > 0$ and with $[0, 2\pi]$ if $\lambda < 0$. The principal value of arccos maps $[-1, 1]$ to $[0, \pi]$, so we have the following values for x_1 and x_2:

$$x_1 = \begin{cases} -\arccos\beta_1 & \lambda > 0 \\ 2\pi - \arccos\beta_1 & \lambda < 0\,, \end{cases}$$

$$x_2 = \begin{cases} -\arccos\beta_2 & \lambda > 0 \\ 2\pi - \arccos\beta_2 & \lambda < 0\,. \end{cases}$$

Using these values of x_1 and x_2, we have an explicit expression for the exponent:

$$\begin{aligned}\pm ik(\omega,\lambda)\cdot\xi &= \pm i\frac{1+\theta}{2}x_1 \pm i\frac{1-\theta}{2}x_2 = \pm\frac{i}{2}\left(x_1+x_2+\theta(x_1-x_2)\right)\\ &= \mp\frac{i}{2}\left(\arccos\beta_1+\arccos\beta_2+\theta(\arccos\beta_1-\arccos\beta_2)\right)\,.\end{aligned}$$

after removing the $2\pi i$ from the exponent. □

Corollary 3.6.3. *The exponents in (3.35) and (3.38) are equal.*

Proof. We consider separately the parts that do and don't depend on θ and compute their cosines. Cosine obscures the sign of its argument, so this first part of the analysis is independent of the quadrant containing λ.

We require, first:

$$\begin{aligned}\cos(\arccos\beta_-) &= \cos\left(\frac{\arccos\beta_1-\arccos\beta_2}{2}\right)\,,\\ \beta_- &= \sqrt{\frac{1+\beta_1\beta_2+\sqrt{1-\beta_1^2}\sqrt{1-\beta_2^2}}{2}}\,,\\ 2\beta_-^2-1 &= \beta_1\beta_2+\sqrt{1-\beta_1^2}\sqrt{1-\beta_2^2}\,,\\ (2\beta_-^2-\beta_1\beta_2-1)^2 &= (1-\beta_1^2)(1-\beta_2^2)\,.\end{aligned} \tag{3.42}$$

Secondly, we require:

$$\begin{aligned}\cos\left(\arccos\frac{\lambda}{2\beta_-}\right) &= \cos\left(\frac{\arccos\beta_1+\arccos\beta_2}{2}\right)\,,\\ \frac{\lambda}{2\beta_-} &= \sqrt{\frac{1+\beta_1\beta_2-\sqrt{1-\beta_1^2}\sqrt{1-\beta_2^2}}{2}}\,,\\ \frac{\lambda^2}{2\beta_-^2}-1 &= \beta_1\beta_2-\sqrt{1-\beta_1^2}\sqrt{1-\beta_2^2}\,,\\ \left(\frac{\lambda^2}{2\beta_-^2}-\beta_1\beta_2-1\right)^2 &= (1-\beta_1^2)(1-\beta_2)^2\,.\end{aligned} \tag{3.43}$$

These are continuous in λ and θ, so (3.43) and (3.42) are sufficient if also we demonstrate the equality of exponents for a value of λ with $\operatorname{Re}\lambda > 0$ and one with $\operatorname{Re}\lambda < 0$. We choose $\theta = 1$ and $\lambda = \pm 1 + i0$.

In this case, we have:

$$\begin{aligned}
\beta_1 &= \lambda - \lambda\sqrt{\lambda^{-2}} = 0\,, \\
\beta_2 &= \lambda\sqrt{\lambda^{-2}} = \pm 1\,, \\
\beta_- &= \sqrt{\frac{\lambda^2}{2}\sqrt{\lambda^{-2}}} = \frac{1}{\sqrt{2}}\,, \\
\frac{\lambda}{2\beta_-} &= \frac{\lambda}{\sqrt{2\lambda^2\sqrt{\lambda^{-2}}}} = \frac{\pm 1}{\sqrt{2}}\,,
\end{aligned}$$

where the indicated choices of sign in the formulas above and below correspond to the sign of λ. It's easy to see that

$$\begin{aligned}
(\operatorname{sgn}\operatorname{Re}\lambda)\arccos\beta_- &= \pm\arccos\frac{1}{\sqrt{2}} = \frac{\pm\pi}{4} = \frac{\arccos(0) - \arccos(\pm 1)}{2} = \frac{\arccos\beta_1 - \arccos\beta_2}{2}\,, \\
\arccos\frac{\lambda}{2\beta_-} &= \arccos\frac{\pm 1}{\sqrt{2}} = \frac{\pi}{2} \pm \frac{\pi}{4} = \frac{\arccos(0) + \arccos(\pm 1)}{2} = \frac{\arccos\beta_1 + \arccos\beta_2}{2}\,.
\end{aligned}$$

Finally, we verify (3.42) and (3.43) with the following identities:

$$\beta_1\beta_2 = \frac{-\lambda^2}{8\theta^2}\left((1-\theta^2)^2 + \frac{8\theta^2}{\lambda^2} - (1+\theta^2)\sqrt{(1-\theta^2)^2 + \frac{16\theta^2}{\lambda^2}}\right)\,,$$

$$2\beta_-^2 - \beta_1\beta_2 - 1 = \frac{-\lambda^2}{8\theta^2}\left((1-\theta^4) - (1-\theta^2)\sqrt{(1-\theta^2)^2 + \frac{16\theta^2}{\lambda^2}}\right)\,,$$

$$\begin{aligned}
(2\beta_-^2 - \beta_1\beta_2 - 1)^2 = \frac{\lambda^4}{32\theta^4}\Bigg((1-\theta)^2(1+\theta)^2(1+\theta^4) + \frac{8\theta^2}{\lambda^2}(1-\theta^2)^2 \\
- (1+\theta^2)(1-\theta^2)^2\sqrt{(1-\theta^2)^2 + \frac{16\theta^2}{\lambda^2}}\Bigg)\,,
\end{aligned}$$

$$(1-\beta_1^2)(1-\beta_2^2) = \frac{\lambda^4}{32\theta^4}\Bigg((1-\theta)^2(1+\theta)^2(1+\theta^4) + \frac{8\theta^2}{\lambda^2}(1-\theta^2)^2 - (1+\theta^2)(1-\theta^2)^2\sqrt{(1-\theta^2)^2 + \frac{16\theta^2}{\lambda^2}}\Bigg),$$

and so (3.42) holds. Continuing,

$$\frac{\lambda^2}{2\beta_-^2} = \frac{\lambda^2}{4}\left((1-\theta^2) + \sqrt{(1-\theta^2)^2 + \frac{16\theta^2}{\lambda^2}}\right),$$

$$\frac{\lambda^2}{2\beta_-^2} - \beta_1\beta_2 - 1 = \frac{\lambda^2}{8\theta^2}\left((1-\theta^4) - (1-\theta^2)\sqrt{(1-\theta^2)^2 + \frac{16\theta^2}{\lambda^2}}\right),$$

$$\left(\frac{\lambda^2}{2\beta_-^2} - \beta_1\beta_2 - 1\right)^2 = \frac{\lambda^4}{32\theta^4}\Bigg((1-\theta)^2(1+\theta)^2(1+\theta^4) + \frac{8\theta^2}{\lambda^2}(1-\theta^2)^2 - (1+\theta^2)(1-\theta^2)^2\sqrt{(1-\theta^2)^2 + \frac{16\theta^2}{\lambda^2}}\Bigg),$$

and so (3.43) holds.

□

Chapter 4

Short-Range Perturbations of the Cayley Tree Laplacian

4.1 Introduction

This chapter is concerned with the Schrödinger equation on the Cayley tree. Let T_d, the degree d Cayley tree, be the infinite d-regular graph with no cycles. The analysis that we will conduct applies in the same form for any $d > 2$, so we will for simplicity restrict ourselves to $d = 3$.

Choose a point $0 \in T_3$. For $x, y \in T_3$, define P_{xy} to be the path that connects them. This path is unique since T_3 is a tree. Set $|x - y|$ to be the number of edges in P_{xy} and $|x| = |x - 0|$. For $x \in T_3$ and $E \subset T_3$, let $|x - E| = \min_{y \in E} |x - y|$. We define Δ by

$$(\Delta f)(x) = \sum_{y:|y-x|=1} f(y)\,.$$

We will consider the equation

$$H\psi = (\Delta + V - \lambda)\psi = f \tag{4.1}$$

for $V : T_3 \to \mathbb{R}$, f a complex-valued function with compact support on T_3, and $\lambda \in \mathbb{C}$.

Investigation into spectral properties of the Laplacian on the Cayley tree originated in connection to questions arising in algebra and combinatorics. The Cayley tree of degree $2p$ is the Cayley graph of the free group on p generators and so inform the study of that subject. The book [43] provides a survey of what was known about the Cayley tree by 1989. Interest in the Cayley tree as an object of study in relation to Schrodinger scattering arose in connection with progress on a Cayley-tree analog of the Anderson model.

The Anderson model, in which the standard Schrodinger Hamiltonian H_0 on $\mathbb{Z}^d$ is coupled with a random potential V, was introduced in [3] in 1958 in order to explain physically observed quantum mechanical effects of disorder: localization (ostensibly corresponding to purely singular spectrum) for relatively high $|\lambda|$ and extended states (ostensibly corresponding to absolutely continuous spectrum) for relatively low $|\lambda|$, with the two regimes separated by mobility edges.

Anderson localization from the mathematical perspective has been extensively studied, first for the one-dimensional case in [26] and [38] and eventually in the arbitrary dimension case in [25] and [2]. However, characterization of extended states and the region of absolute continuity in multiple dimensions remains elusive. In [34, 35], Klein demonstrates presence of absolutely continuous spectrum for randomly perturbed H on the Cayley tree, which inspired substantial interest in this line of inquiry in the hope that it would serve as a bridge to results in higher dimensional integer lattices.

The characterization of the unperturbed spectrum of Δ on T_3 has been known at least since [31]. The history of this field through 2010 is recorded in [50]. In [17], Denisov proves the following theorem about the preservation of the absolutely continuous part of the spectrum under perturbation:

Theorem 4.1.1 (Denisov)**.** *Let $V : T_d \to \mathbb{R}$ be bounded and obey*

$$\sum_{n=0}^{\infty} \frac{1}{(d-1)^n} \sum_{\alpha:|\alpha|=n} V^2(\alpha) < \infty\,, \tag{4.2}$$

where $|\alpha|$ is the distance from α to a designated origin. Then H has absolutely continuous spectrum of infinite multiplicity on $[-2\sqrt{d-1}, 2\sqrt{d-1}]$.

This theorem is sharp, as for any $p > 2$, there is a V satisfying (4.2) with $|V(\alpha)|^2$ replaced by $|V(\alpha)|^p$, but for which the absolutely continuous spectrum of H is empty. Examples of this type of potential are discussed in [33].

This result was extended by Denisov and Kiselev in [18]. Consider infinite paths $\omega = \{x_j\}_{j=0}^{\infty}$ starting at 0, with $|x_j| = j$. Put the infinite product measure ν on the ω for which $\nu(\{\omega : x_j = x\}$ is constant across all x with $|x| = j$. Finally, let

$$\|V_\omega\|^2 = \sum_{j=0}^{\infty} |V(x_j)|^2 \text{ for } x_j \in \omega\,.$$

With this nomenclature, we have the following theorem:

Theorem 4.1.2 (Denisov, Kiselev)**.** *Let $V : T_3 \to \mathbb{R}$ be bounded and suppose*

$$\nu(\omega : \|V\omega\| < \infty) > 0\,.$$

Then H has a.c. spectrum of infinite multiplicity on $[-2\sqrt{2}, 2\sqrt{2}]$.

In this chapter, we present some basic results about the spectral theory of the Cayley tree. We start by investigating the free Laplacian and finding fundamental solutions to the associated resolvent. Then we prove an absorption principle for compactly supported potentials, justify exponential decay of the Green's function for H perturbed by short-range potentials, and a factorization identity. Control over the decay of the perturbed Green's function is new; other results are subsumed by those provided in e.g. [17,18], but we approach the proofs from a perturbative perspective that is absent from the literature.

4.2 Notation

Let $x \neq 0$ and y be neighbors in T_3. Every element of T_3 other than 0 has a unique neighbor with smaller norm. If y is the unique such neighbor of x, with $|y| < |x|$, then write $y = x - 1$ and say that y is the proximal neighbor of x. Neighbors with larger norm are not unique; we will call them the distal neighbors of x. Let x and n be neighbors in T_3. Then we say that the set $\{y \in T_3 : |x - y| < |n - y|\}$ is the cone at x growing away from n and refer to it by $C_n(x)$. If $|n| = 1$, we refer to the cone centered at 0 growing away from n by C_n.

For $x \in T_3$ and $R \in \mathbb{N}$, let $B_R(x) = \{y \in T_3 : |x - y| \leqslant R\}$. If x is not specified, it is assumed to be 0. Similarly, let $S_R(x) = \{y \in T_3 : |x - y| = R\}$.

From [31], $\sigma(\Delta) = [-2\sqrt{2}, 2\sqrt{2}]$. Let $S_0 = \{-2\sqrt{2}, 2\sqrt{2}\}$, the exceptional points in the spectrum; let $S = \sigma(\Delta)\backslash S_0 = (-2\sqrt{2}, 2\sqrt{2})$. Throughout this paper, we will reserve the symbols ζ for elements of $\mathbb{C}\backslash S_0$, η for elements of $\mathbb{C}\backslash\sigma(\Delta)$, λ for elements of S, and $\lambda \pm i0$ for elements of S when they appear as limits of η from above and below, respectively.

Let R^0_λ denote $(\Delta - \lambda)^{-1}$ and R_λ denote $(\Delta + V - \lambda)^{-1}$. For $\lambda \in \mathbb{C}$, we will have $\mu_\lambda = \frac{\lambda \pm \sqrt{\lambda^2 - 8}}{4}$. For $\operatorname{Im}\eta \neq 0$, choose the root with the smaller modulus and let W_η be the class of functions $\psi : T_3 \to \mathbb{C}$ such that there is some R so that for all $x \in T_3\backslash B_R$, if y is a distal neighbor of x, we have $\psi(y) = \mu_\eta \psi(x)$. For $\lambda \in S$, let $\mu_{\lambda \pm i0}$ be the the limit of μ_η as $\eta \to \lambda$ from above or below, respectively. Let $W_{\lambda \pm i0}$ be the class of functions $\psi : T_3 \to \mathbb{C}$ such that there is some R so that for all $x \in T_3\backslash B_R$, if y is a distal neighbor of x, we have $\psi(y) = \mu_{\lambda \pm i0}\psi(x)$.

4.3 The Free Laplacian

Let $\eta \notin \sigma(\Delta)$. We examine the behavior of solutions ψ of

$$(\Delta - \eta)\psi = 0 \tag{4.3}$$

on cones C in T_3 with center 0. Note that the following analysis does not require that $(\Delta - \eta)\psi = 0$ at 0.

Lemma 4.3.1. *Let $C \subset T_3$ be a cone and label its center 0. For $\lambda \in S$, there is no nontrivial $\psi \in l^2(C)$ satisfying (4.3) on C. For $\eta \notin [-2\sqrt{2}, 2\sqrt{2}]$, there is a $\psi \in l^2(C)$ that satisfies (4.3) on $C\backslash\{0\}$. It is given by $\psi(x) = \mu^{|x|}\psi(0)$.*

Proof. Suppose that ψ satisfies (4.3). Let $S_n = \{x \in C : |x| = n\}$ for $n \in \mathbb{N}$. Counting adjacencies

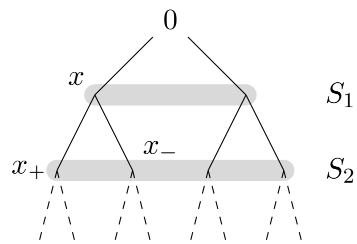


Figure 4.1: A cone, C

in S_{n-1} and S_{n+1} to elements of S_n, we see that, for $n \geqslant 1$,

$$\sum_{S_n}(\Delta - \eta)\psi = 2\sum_{S_{n-1}}\psi + \sum_{S_{n+1}}\psi - \eta\sum_{S_n}\psi = 0$$

and so

$$\sum_{S_{n+1}}\psi = \eta\sum_{S_n}\psi - 2\sum_{S_{n-1}}\psi\,.$$

We express this recurrence relation as

$$\begin{bmatrix} \sum_{S_{n+1}}\psi \\ \sum_{S_n}\psi \end{bmatrix} = \begin{bmatrix} \eta & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \sum_{S_n}\psi \\ \sum_{S_{n-1}}\psi \end{bmatrix},$$

which has eigenvalues equal to

$$2\mu_\pm = \frac{\eta \pm \sqrt{\eta^2 - 8}}{2}\,.$$

For $\eta \in [-2\sqrt{2}, 2\sqrt{2}]$, $|2\mu_\pm| = \sqrt{2}$. For $\eta \notin [-2\sqrt{2}, 2\sqrt{2}]$, one of the $2\mu_\pm$ has modulus less than $\sqrt{2}$ and one has modulus greater than $\sqrt{2}$. Let 2μ be the eigenvalue such that $|2\mu| < \sqrt{2}$ and $2\tilde{\mu}$ be the other one.

By Cauchy-Schwartz, we have

$$\|\psi\|_{l^1(S_n)} \leqslant \|1\|_{l^2(S_n)}^{1/2}\|\psi\|_{l^2(S_n)}^{1/2}$$

and so, for some constants c and $\tilde{c}$,

$$\|\psi\|_{l^2(S_n)}^{1/2} \geqslant \frac{\|\psi\|_{l^1(S_n)}}{|S_n|^{1/2}} \geqslant \frac{|\sum_{S_n}\psi|}{|S_n|^{1/2}} = \frac{|c(2\mu)^n + \tilde{c}(2\tilde{\mu})^n|}{2^{n/2}} \geqslant \frac{|\tilde{c}||(2\tilde{\mu})|^n - |c||(2\mu)|^n}{2^{n/2}}\,. \tag{4.4}$$

If $\eta \notin [-2\sqrt{2}, 2\sqrt{2}]$, (4.4) implies that $\psi \in l^2(C)$ only if $\tilde{c} = 0$, which is to say that $\sum_{S_n} \psi = c(2\mu)^n$ for $n \geqslant 1$. For $\eta \in [-2\sqrt{2}, 2\sqrt{2}]$, $|2\mu| = |2\tilde{\mu}| = \sqrt{2}$ and $\tilde{\mu} = \bar{\mu}$. Suppose $|c| < |\tilde{c}|$. Then, as for η off the spectrum, (4.4) implies that $\psi \in l^2(C)$ only if $|\tilde{c}| = 0$, which contradicts $|c| < |\tilde{c}|$. A similar argument shows that $|\tilde{c}|$ cannot be less than $|c|$. Suppose now that $|c| = |\tilde{c}|$. Rewrite the middle expression in (4.4) as

$$\begin{aligned} \frac{|c(2\mu)^n + \tilde{c}(2\tilde{\mu})^n|}{2^{n/2}} &= |c| \left| e^{in\theta + i\kappa} + e^{-in\theta + i\tilde{\kappa})} \right| \\ &= |c| \left| e^{in\theta + i(\kappa - \tilde{\kappa})/2} + e^{-in\theta - i(\kappa - \tilde{\kappa})/2} \right| \\ &= 2|c| |\cos(n\theta + (\kappa - \tilde{\kappa})/2)| \end{aligned} \tag{4.5}$$

for some θ, κ, and $\tilde{\kappa}$ in $\mathbb{R}$. The only way for (4.5) to converge to 0 with $\theta \in [0, \pi]$ is if $\theta = 0$, $\theta = \pi$ or $\theta = \pi/2$. If $\theta = 0$ or $\theta = \pi$, then $\mu = \tilde{\mu}$ and (4.4) implies that $\|\psi\|_{l^2(S_n)}^{1/2}$ is bounded below by $|c + \tilde{c}|$. If $\theta = \pi/2$, then $\mu = -\tilde{\mu}$ and (4.4) implies that $\limsup_{n\to\infty} \|\psi\|_{l^2(S_n)}^{1/2}$ is bounded below by $\max\{|c + \tilde{c}|, |c - \tilde{c}|\} \geqslant \max\{|c|, |\tilde{c}|\}$. In any case, $\|\psi\|_{l^2(S_n)}$ doesn't converge to 0 as n increases to ∞ unless $|c| = |\tilde{c}| = 0$, and so there is no nontrivial $\psi \in l^2(C)$ satisfying (4.3) when $\eta \in [-2\sqrt{2}, 2\sqrt{2}]$. Going forward, we only consider $\eta \notin [-2\sqrt{2}, 2\sqrt{2}]$.

Now we focus on the behavior of ψ around a point $x \in S_1$. Let x_+ and x_- be the distal neighbors of x as in figure 4.1. By repeating the preceding analysis with $x = 0$, we know that $\psi(x_+) + \psi(x_-) = 2\mu\psi(x)$, so:

$$\begin{aligned} 0 = (\Delta - \eta)\psi = \psi(0) + 2\mu\psi(x) - \eta\psi(x)\,, \\ \psi(x) = \psi(0)(\eta - 2\mu)^{-1} \end{aligned}$$

and we compute

$$(\eta - 2\mu)^{-1} = \frac{2}{\eta \mp \sqrt{\eta^2 - 8}} = \frac{\eta \pm \sqrt{\eta^2 - 8}}{4} = \mu\,.$$

By induction, we see that, for $x \in S_n$, $\psi(x) = \mu^n \psi(0)$. □

Lemma 4.3.2. *Let $\eta \notin \sigma(\Delta)$ and $f : T_3 \to \mathbb{R}$ with* $\operatorname{supp} f \subset B_R$ *for some R. Solutions to $(\Delta - \eta)^{-1}$*

in $l^2(T_3)$ are given by

$$((\Delta - \eta)^{-1} f)(x) = \sum_{y \in B_R} \frac{f(y)\mu^{|x-y|}}{3\mu - \eta}\,.$$

Proof. Let $f(x) = \delta_0(x)$, which is 1 when $x = 0$ and 0 otherwise. We look for a fundamental solution E_0 to

$$(\Delta - \eta)E_0 = \delta_0$$

in l^2. From Lemma 4.3.1, we know that any solution ψ must have $\psi(x) = \psi(0)\mu^{|x|}$ for $x \neq 0$, and that this will satisfy the equation for $x \neq 0$. At 0, we have

$$\left(\sum_{S_1} \psi\right) - \eta\psi(0) = \psi(0)(3\mu - \eta) = 1\,.$$

This gives us our fundamental solution: $E_0(x) = \frac{\mu^{|x|}}{3\mu - \eta}$.

For $f : T_3 \to \mathbb{R}$ with $f \in l^2(T_3)$, we can superpose the fundamental solutions to find a solution to

$$(\Delta - \eta)\psi = f\,.$$

We write $f(x) = \sum_{y \in T_3} f(y)\delta_y(x)$ and so

$$((\Delta - \eta)^{-1} f)(x) = \sum_{y \in T_3} f(y)((\Delta - \eta)^{-1}\delta_y)(x) = \sum_{y \in T_3} \frac{f(y)\mu^{|x-y|}}{3\mu - \eta}\,.$$

□

4.4 The Perturbed Laplacian

Lemma 4.4.1. *Let $\lambda \in S$ and let $V : T_3 \to \mathbb{R}$. The homogeneous equation*

$$(\Delta + V - \lambda)\psi = 0$$

has only trivial solutions in the classes $W_{\lambda \pm i0}$.

Proof. Suppose that $\psi \in W_{\lambda\pm i0}$ solves the homogeneous equation. There is some R such that, for $|x| > R$, $\psi(x) = \mu_{\lambda\pm i0}\psi(x-1)$. We have:

$$0 = (\Delta + V - \lambda)\psi \cdot \bar{\psi} - \psi \cdot (\Delta + V - \lambda)\bar{\psi} \tag{4.6}$$

$$= \Delta\psi \cdot \bar{\psi} - \psi \cdot \Delta\bar{\psi}\,. \tag{4.7}$$

For any $r > R$, we can sum over B_r:

$$0 = \sum_{B_r} \Delta\psi \cdot \bar{\psi} - \psi \cdot \Delta\bar{\psi} \tag{4.8}$$

$$= \sum_{x \in S_{r+1}} \psi(x)\bar{\psi}(x-1) - \psi(x-1)\bar{\psi}(x) \tag{4.9}$$

$$= \sum_{S_{r+1}} \bar{\mu}^{-1}_{\lambda\pm i0}\psi\bar{\psi} - \mu^{-1}_{\lambda\pm i0}\psi\bar{\psi} \tag{4.10}$$

$$= -2i\,\mathrm{Im}\,\mu^{-1}_{\lambda\pm i0} \sum_{S_{r+1}} |\psi|^2 \tag{4.11}$$

and since $\mathrm{Im}\,\mu_{\pm i0} \neq 0$ for $\lambda \in S$, we conclude that $\psi(x) = 0$ for $|x| > R$.

Now, suppose that $\psi|_{S_r} \equiv 0$ and $\psi|_{S_{r+1}} \equiv 0$. Then, for $x \in S_r$, $0 = (\Delta + V - \lambda)\psi(x) = (V - \lambda)\psi(x) + \sum_{|y-x|=1} \psi(y) = \psi(x-1)$. Since every element of S_{r-1} is adjacent to an element of S_r, $\psi|_{S_{r-1}} \equiv 0$. By induction on decreasing r, $\psi \equiv 0$. □

Theorem 4.4.2 (Absorption Principle)**.** *Let* $f : T_3 \to \mathbb{C}$ *and* $V : T_3 \to \mathbb{R}$ *have* $\mathrm{supp} f \subset B_R$ *and* $\mathrm{supp} V \subset B_R$ *for some* R. *For* $\lambda \in S$ *and* $\eta \in \mathbb{C}\backslash S$, *the pointwise limit* $\lim_{\eta\to\lambda\pm i0} R_\eta f$ *exists and is the unique function in* $W_{\lambda\pm i0}$ *that solves* $(\Delta + V - \lambda)\psi = f$.

Proof. The uniqueness in $W_{\lambda\pm i0}$ is a consequence of Lemma 4.4.1.

Suppose that $V = 0$. From Lemma 4.3.1, for $\mathrm{Im}\,\eta \neq 0$ and $f \in C_0(T_3)$, we have $(R^0_\eta f)(x) = \sum_{y \in B_R} \frac{f(y)\mu^{|x-y|}}{3\mu - \eta}$. The pointwise limits of $R^0_\eta f$ as $\eta \to \lambda \pm i0$ clearly exist, and the limit is a function in $W_{\lambda\pm i0}$. This limit solves $(\Delta-\lambda)\psi = f$, since

$$
\begin{aligned}
(\Delta-\lambda)\left(R^0_{\lambda\pm i0}f\right) &= (\Delta-\lambda)\left(\lim_{\eta\to\lambda\pm i0} R^0_\eta f\right) \\
&= \lim_{\eta\to\lambda\pm i0}\left\{(\Delta-\eta)\left(R^0_\eta f\right) + (\eta-\lambda)R^0_\eta f\right\} \\
&= f + \lim_{\eta\to\lambda\pm i0}(\eta-\lambda)R^0_\eta f \\
&= f
\end{aligned}
$$

and since $(R^0_\eta f)(x)$ is uniformly bounded for η in a neighborhood of λ. Furthermore, if $\psi \in W_{\lambda\pm i0}$ is such that $(\Delta-\lambda)\psi = f$, then

$$
R^0_{\lambda\pm i0}(\Delta-\lambda)\psi = \lim_{\eta\to\lambda\pm i0}\{R^0_\eta(\Delta-\eta)\psi + R^0_\eta(\eta-\lambda)\psi\} = \psi\,.
$$

This shows that $(\Delta-\lambda): W_{\lambda\pm i0} \to C_0$ and $R^0_{\lambda\pm i0}: C_0 \to W_{\lambda\pm i0}$ are inverse to each other, and that the absorption principle holds in the unperturbed case.

Suppose that $V \neq 0$ and $\operatorname{Im}\eta \neq 0$. Let $K = \operatorname{supp}V \cup \operatorname{supp}f$. Consider the operator $(I + VR^0_\eta): C(K) \to C(K)$. Suppose that $\phi \in C(K)$ and $(I+VR^0_\eta)\phi = f$. Let $\psi = R^0_\eta\phi \in W_\eta$. Then

$$
\begin{aligned}
(\Delta + V - \eta)\psi &= (\Delta-\eta)R^0_\eta\phi + VR^0_\eta\phi && (4.12)\\
&= (I + VR^0_\eta)\phi && (4.13)\\
&= f\,. && (4.14)
\end{aligned}
$$

Now suppose that $\psi \in W_\eta$ and $(\Delta+V-\eta)\psi = f$. Let $\phi = (\Delta-\eta)\psi$. For $x \notin K$, $[(\Delta-\eta)\psi](x) = f(x) - V(x)\psi(x) = 0$, so $\phi \in C(K)$. Then

$$
\begin{aligned}
(I + VR^0_\eta)\phi &= (\Delta-\eta)\psi + VR^0_\eta(\Delta-\eta)\psi && (4.15)\\
&= (\Delta+V-\eta)\psi && (4.16)\\
&= f\,. && (4.17)
\end{aligned}
$$

Since $(\Delta - \eta) : W_\eta \to C_0$ and $R^0_\eta : C_0 \to W_\eta$ are injective, they give a bijection between $\phi \in C(K)$ that solve $(I + VR^0_\eta)\phi = f$ and $\psi \in W_\eta$ that solve $(\Delta + V - \eta)\psi = f$. For $\lambda \in S$, similar reasoning shows that there is a bijection between $\phi \in C(K)$ that solve $(I + VR^0_{\lambda \pm i0})\phi = f$ and $\psi \in W_{\lambda \pm i0}$ that solve $(\Delta + V - \lambda)\psi = f$.

By the reasoning in Lemma 4.4.1, for $\lambda \in \mathbb{C}\backslash S_0$, $(\Delta + V - \lambda)\psi = 0$ has only trivial solutions in W_λ. By the bijective correspondence established in the preceding paragraphs, $(I + VR^0_\lambda)$ has trivial kernel in $C(K)$. As a consequence of the Fredholm alternative, $(I + VR^0_\lambda)$ is invertible on $C(K)$. Since $(I + VR^0_\eta) \to (I + VR^0_{\lambda \pm i0})$ in $L(C(K))$ as $\eta \to \lambda \pm i0$ and $C(K)$ is finite dimensional, $(I + VR^0_\eta)^{-1} \to (I + VR^0_{\lambda \pm i0})^{-1}$ as $\eta \to \lambda \pm i0$.

Let $\chi : C(K) \to C(T_3)$ be the inclusion map, so that $\chi(f(x)) = f(x)$ for $x \in K$ and $\chi(f(x)) = 0$ for $x \notin K$. Let $\chi^* : C(T_3) \to C(K)$ be adjoint to χ, restricting functions in $C(T_3)$ to K. For $\lambda \in \mathbb{C}\backslash S_0$, define $\mathcal{R}_\lambda = R^0_\lambda \chi (I + VR^0_\lambda)^{-1}\chi^*$. $R^0_\eta \to R^0_{\lambda \pm i0}$ and $\chi(I + VR^0_\eta)^{-1}\chi^* \to \chi(I + VR^0_{\lambda \pm i0})^{-1}\chi^*$ as $\eta \to \lambda \pm i0$, so $\mathcal{R}_\eta \to \mathcal{R}_{\lambda \pm i0}$ pointwise as $\eta \to \lambda \pm i0$.

$\chi(I + VR^0_\lambda)^{-1}\chi^* : C_0 \to C_0$ and $R^0_\lambda : C_0 \to W_\lambda$, so $\mathcal{R}_\lambda : C_0 \to W_\lambda$. We claim $\mathcal{R}_\lambda$ is the right inverse to $(\Delta + V - \lambda)$: $f \in C(K)$, so we have

$$(\Delta + V - \lambda)\mathcal{R}_\lambda f = (\Delta + V - \lambda)R^0_\lambda \chi(I + VR^0_\lambda)^{-1}\chi^* f \tag{4.18}$$

$$= [(\Delta - \lambda)R^0_\lambda + VR^0_\lambda]\chi(I + VR^0_\lambda)^{-1}\chi^* f \tag{4.19}$$

$$= (I + VR^0_\lambda)\chi(I + VR^0_\lambda)^{-1}\chi^* f \tag{4.20}$$

$$= f\,. \tag{4.21}$$

□

As mentioned at (1.31) in the introduction to this chapter, for V with sufficiently small L^∞ norm, the Born series $\sum_{j=0}^{\infty}(-1)^j (R^0_\eta V)^j R^0_\eta$ converges to R_η in the $L^2 \to L^2$ operator norm topology . In the case where there domain of R_η is restricted to functions supported in some ball in T_3, functions in the range of R_η can be bounded by functions with the asymptotic exponential decay exhibited by functions in the range of R^0_η.

In the following, we denote $(1+x^2)^{1/2}$ by the *Japanese bracket* $\langle x\rangle$.

Theorem 4.4.3. *Let $\eta \in \mathbb{C}^+$ and $\epsilon > 0$. There is a $C > 0$ such that, for any V with $|V(x)| < C\langle x\rangle^{-1-\epsilon}$ and any $f \in C_0(T_3)$,*

$$|R_\eta f(x)| < |g(x)|$$

for some $g \in W_\eta$.

Proof. From Lemma 4.3.2, we have

$$(R^0_\eta f)(x) = \sum_{y\in T_3} \frac{\mu^{|x-y|}}{3\mu-\eta} f(y) = \sum_{y\in B_R} \frac{\mu^{|x-y|}}{3\mu-\eta} f(y)\,,$$

for some R such that $\mathrm{supp} f \subset B_R$. Since this expression decays like $\mu^{|x|}$ outside of B_R in each direction, there is a C' depending on f and η so that

$$|(R^0_\eta f)(x)| < C'|\mu|^{|x|}\,. \tag{4.22}$$

As an inductive hypothesis, suppose that $|u| : T_3 \to \mathbb{C}$ satisfies $|u| \leqslant C''|\mu|^{|x|}$. Then we have

$$\begin{aligned} |(R^0_\eta Vu)(x)| &= \left|\sum_{y\in T_3} \frac{\mu^{|x-y|}}{3\mu-\eta} V(y)u(y)\right| \\ &\leqslant CC''|3\mu-\eta|^{-1} \sum_{y\in T_3} |\mu|^{|x-y|}|\mu|^{|y|}\langle y\rangle^{-1-\epsilon} \\ &= CC''|3\mu-\eta|^{-1}|\mu|^{|x|} \sum_{y\in T_3} |\mu|^{|x-y|+|y|-|x|}\langle y\rangle^{-1-\epsilon}\,. \end{aligned} \tag{4.23}$$

Consider the sum over $y \in T_3$. We will show that it is bounded by a constant depending on η and ϵ. Let $S_j(P_{0x}) = \{y \in T_3 : |y - P_{0x}| = j\} = \{y \in T_3 : |x-y| + |y| = |x| + 2j\}$. Write

$$\sum_{y\in T_3} |\mu|^{|x-y|+|y|-|x|}\langle y\rangle^{-1-\epsilon} = \sum_{j\in\mathbb{N}} \sum_{y\in S_j(P_{0x})} |\mu|^{2j}\langle y\rangle^{-1-\epsilon}\,.$$

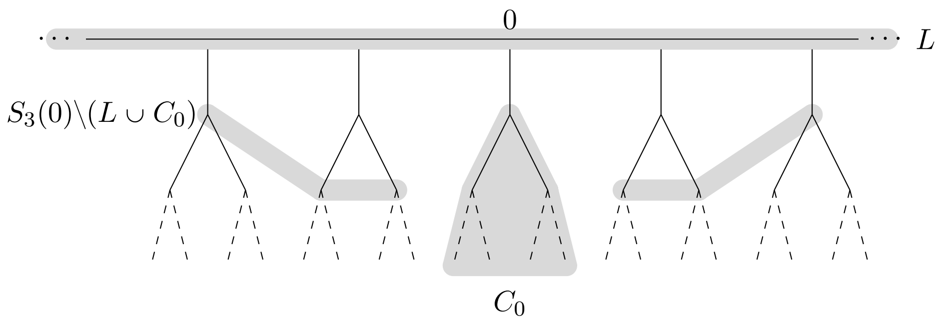


Figure 4.2: The decomposition of T_3

The magnitude $|\mu| < 2^{-1/2} < 1$, so

$$\sum_{j\in\mathbb{N}} \sum_{y\in S_j(P_{0x})} |\mu|^{2j}\langle y\rangle^{-1-\epsilon} \leqslant \sum_{j\in\mathbb{N}} \sum_{y\in S_j(P_{x'z'})} |\mu|^{2j}\langle y\rangle^{-1-\epsilon}$$

as long as $P_{0x} \subset P_{x'z'}$. We may consider any path in T_3 as a subset of a path of infinite length; suppose that L is such a line in T_3 containing P_{0x}, and let $S_j(L) = \{y \in T_3 : |y - L| = j\}$. It suffices to control

$$\sum_{j\in\mathbb{N}} \sum_{y\in S_j(L)} |\mu|^{2j}\langle y\rangle^{-1-\epsilon} .$$

For the sum over L, where $j = 0$, we have

$$\sum_{y\in S_0(L)} \langle y\rangle^{-1-\epsilon} \leqslant \sum_{y\in L} \langle y\rangle^{-1-\epsilon} \leqslant 3 + 2\epsilon^{-1} .$$

Let C_0 be the cone in $T_3\backslash L$ adjacent to 0. For the sum over C_0, we have

$$\begin{aligned}\sum_{j\in\mathbb{N}} \sum_{y\in C_0\cap S_j(L)} |\mu|^{2j}\langle y\rangle^{-1-\epsilon} &\leqslant \sum_{j\in\mathbb{N}} \sum_{y\in C_0\cap S_j(L)} |\mu|^{2j} \\ &\leqslant \sum_{0<j} 2^{j-1}|\mu|^{2j} \\ &= \frac{|\mu|^2}{1-2|\mu|^2} .\end{aligned}$$

Now excluding L and C_0, we split the remaining sum over spheres $S_k(0)$ of radius k centered at 0.

$$\begin{aligned}\sum_{j\in\mathbb{N}}\sum_{y\in S_j(0)\backslash(L\cup C_0)}|\mu|^{2j}\langle y\rangle^{-1-\epsilon} &\leqslant \sum_{2\leqslant k}2\langle k\rangle^{-1-\epsilon}\sum_{l=1}^{k-1}|\mu|^{2l}2^{l-1}\\ &\leqslant \sum_{2\leqslant k}\langle k\rangle^{-1-\epsilon}\frac{1-(2|\mu|^2)^k}{1-2|\mu|^2}\\ &\leqslant \frac{6+4\epsilon^{-1}}{1-2|\mu|^2}\,.\end{aligned}$$

Putting these together, we see that

$$\sum_{y\in T_3}|\mu|^{|x-y|+|y|-|x|}\langle y\rangle^{-1-\epsilon}\leqslant 10\frac{1+|\mu|^2+\epsilon^2+|\mu|^2\epsilon^2}{1-2|\mu|}\,.$$

Choose C so that (4.23) is less than $\frac{C''}{2}|\mu|^{|x|}$. Then, from (4.22) and by induction on powers of $R^0_\eta V$,

$$|(R_\eta f)(x)|=\left|\sum_{j=0}^{\infty}(-1)^j(R^0_\eta V)^j R^0_\eta f(x)\right|\leqslant 2C'|\mu|^{|x|}\,.$$

$2C'\mu^{|x|}$ is an element of W_η, so this completes the proof. □

Finally, we justify a factorization identity in this setting.

Theorem 4.4.4 (Factorization Identity)**.** *Suppose that* ψ *solves* $(\Delta-\eta+V)\psi=f$ *for some* V *and* f *in* $C_0(T_3)$*. Then, for* R *big enough that* $\mathrm{supp}f\cup\mathrm{supp}V\subset B_{R-1}$,

$$\mathrm{Im}\langle\mathcal{R}_{\eta+i0}f,f\rangle=i(\mu-\bar{\mu})\|\psi\|^2_{l^2(S_R)}\,.$$

Proof. Let $\eta=\tau+i\delta$, with $\tau\in S$ and $\delta>0$. There is some R such that $\mathrm{supp}f\subset B_R$ and, for all $|x|>R$, $\psi(x)=\mu\psi(x-1)$. We have

$$(\Delta-\eta+V)\psi=f\,.$$

Multiply by $\bar{\psi}$, sum over B_R and take the imaginary part to get:

$$\operatorname{Im}\sum_{B_R}\Delta\psi\bar{\psi} - \operatorname{Im}\eta\sum_{B_R}\psi\bar{\psi} + \operatorname{Im}\sum_{B_R}V\psi\bar{\psi} = \operatorname{Im}\sum_{B_R}f\bar{\psi}\,, \tag{4.24}$$

$$\frac{1}{2i}\sum_{B_R}\big(\Delta\psi\bar{\psi} - \psi\Delta\bar{\psi}\big) - \delta\sum_{B_R}\psi\bar{\psi} = \operatorname{Im}\sum_{B_R}f\bar{\psi}\,, \tag{4.25}$$

$$\frac{1}{2i}\sum_{x\in S_{R+1}}\big(\psi(x)\bar{\psi}(x-1) - \psi(x-1)\bar{\psi}(x)\big) - \delta\sum_{B_R}\psi\bar{\psi} = \operatorname{Im}\langle f, \mathcal{R}_\eta f\rangle\,. \tag{4.26}$$

Now, taking the limit as $\delta \to 0$:

$$\frac{1}{2i}\sum_{x\in S_{R+1}}\big(\mu\psi(x-1)\bar{\psi}(x-1) - \bar{\mu}\psi(x-1)\bar{\psi}(x-1)\big) = \operatorname{Im}\langle f, \mathcal{R}_{\eta+i0} f\rangle\,, \tag{4.27}$$

$$-i(\mu-\bar{\mu})\|\psi\|^2_{L^2(S_R)} = \operatorname{Im}\langle f, \mathcal{R}_{\eta+i0} f\rangle\,. \tag{4.28}$$

□

Appendix A

Appendix

A.1 Standard Results

In this Appendix, we collect results that are used in Chapter 2. Although some of them are standard, we provide their proofs for completeness.

Proof of Lemma 2.1.4. In Section 13 of [14], the following formula for the Green's function of operator $\mathcal{D}$ (i.e., the integral kernel of $R_z = (\mathcal{D} - z)^{-1}$) was obtained

$$G_z(x,y) = \begin{pmatrix} G_z^{11}(x,y) & G_z^{12}(x,y) \\ G_z^{21}(x,y) & G_z^{22}(x,y) \end{pmatrix} =$$

$$\begin{pmatrix} \displaystyle\int_{\mathbb{R}} \frac{\phi(x,k)\phi(y,k)}{k-z} d\sigma_d(k) & \displaystyle\int_{\mathbb{R}} \frac{\phi(x,k)\psi(y,k)}{k-z} d\sigma_d(k) \\ \displaystyle\int_{\mathbb{R}} \frac{\psi(x,k)\phi(y,k)}{k-z} d\sigma_d(k) & \displaystyle\int_{\mathbb{R}} \frac{\psi(x,k)\psi(y,k)}{k-z} d\sigma_d(k) \end{pmatrix} \tag{A.1}$$

and $\sigma_d = 2\sigma$. We now introduce an auxiliary parameter $\rho \in [1, \infty)$ to be chosen later as $\rho \sim 1 + \|A\|^2_{\mathrm{St}}$. Since $|P(2x,k)|^2 = \phi^2(x,k) + \psi^2(x,k)$ and $\sup_{k\in\mathbb{R}}(k^2+\rho^2)/(k^2+1) \lesssim \rho^2$, then

$$\sup_{x\geqslant 0} \int_{\mathbb{R}} \frac{|P(x,k)|^2}{k^2+1} d\sigma = \sup_{x\geqslant 0} \int_{\mathbb{R}} \frac{(k^2+\rho^2)|P(x,k)|^2}{(k^2+\rho^2)(k^2+1)} d\sigma \lesssim \rho \sup_{x\geqslant 0} \int_{\mathbb{R}} \frac{\rho|P(x,k)|^2}{k^2+\rho^2} d\sigma \,. \tag{A.2}$$

Hence, we only need to prove that

$$\sup_{x\geqslant 0} \operatorname{Im}(G_{i\rho}^{11}(x,x) + G_{i\rho}^{22}(x,x)) \lesssim 1 \,. \tag{A.3}$$

To control $G_{i\rho}(x,y)$, i.e., the integral kernel of the resolvent $R_{i\rho}$, we will use the standard perturbation series. If $R^0_{i\rho}$ denotes the resolvent of free Dirac operator, we write the second resolvent identity:

$$R_{i\rho} = R^0_{i\rho} - R_{i\rho} V R^0_{i\rho}, \quad V \stackrel{\text{def}}{=} \begin{pmatrix} -b & -a \\ -a & b \end{pmatrix}$$

and iterate it to get the series

$$R_{i\rho} = R^0_{i\rho} - R^0_{i\rho} V R^0_{i\rho} + R^0_{i\rho} V R^0_{i\rho} V R^0_{i\rho} + \ldots \,. \tag{A.4}$$

In the series (A.4), each term starting with the second one takes the form $(-1)^{j+1}(R^0_{i\rho}V)^j(R^0_{i\rho}VR^0_{i\rho})$ and $j = 0, 1, 2, \ldots$. If we denote its kernel by $k_j(x,y)$, then

$$G_{i\rho}(x,y) = G^0_{i\rho}(x,y) - k_0(x,y) + k_1(x,y) + \ldots \tag{A.5}$$

and $G^0_z(x,y)$ stands for the Green's function of free Dirac operator. Next, we will show convergence of this series for suitable choice of parameter ρ and will provide an estimate for it.

First, we claim that for every $j = 0, 1, \ldots$, we have

$$\|k_j(x,y)\| \leqslant C^{j+1} \frac{e^{-\rho|x-y|/2}\|A\|^{j+1}_{\mathrm{St}}}{\rho^{(j+1)/2}}\,, \tag{A.6}$$

where C is an absolute constant to be specified below. We will prove (A.6) by induction. To this end, we use formula (A.1) and residue calculus to obtain the bound

$$\|G^0_{i\rho}(x,y)\| \lesssim e^{-\rho|x-y|} + e^{-\rho(x+y)} \lesssim e^{-\rho|x-y|}\,.$$

Thus, for $k_0(x,y)$, we have

$$\|k_0(x,y)\| \lesssim \int_0^\infty e^{-\rho|x-\xi|}|\alpha(\xi)|e^{-\rho|y-\xi|}d\xi, \quad \alpha \stackrel{\text{def}}{=} |a| + |b|\,.$$

Continue $\alpha(\xi)$ to negative ξ by zero. We write

$$\|k_0(x,0)\| \lesssim \int_0^\infty e^{-\rho|x-\xi|}\alpha(\xi)e^{-\rho\xi}d\xi \leqslant e^{-\rho x}\int_0^x \alpha(\xi)d\xi + e^{\rho x}\int_x^\infty \alpha(\xi)e^{-2\rho\xi}d\xi\,. \tag{A.7}$$

Then, using Cauchy-Schwarz inequality, one has $\int_0^x \alpha(\xi)d\xi \lesssim (x+x^{1/2})\|A\|_{\mathrm{St}}$. By the change of variable,

$$e^{\rho x}\int_x^\infty \alpha(\xi)e^{-2\rho\xi}d\xi = e^{-\rho x}\int_0^\infty e^{-2\rho\eta}\alpha(x+\eta)d\eta\,.$$

We have

$$\int_0^\infty e^{-2\rho\eta}\alpha(x+\eta)d\eta = \int_0^1 e^{-2\rho\eta}\alpha(x+\eta)d\eta + \sum_{j=1}^\infty \int_j^{j+1} e^{-2\rho\eta}\alpha(x+\eta)d\eta \leqslant$$

$$\left(\int_0^1 e^{-4\rho\eta}d\eta\right)^{1/2}\left(\int_0^1 \alpha^2(x+\eta)d\eta\right)^{1/2} + \sum_{j=1}^\infty e^{-2\rho j}\left(\int_j^{j+1}\alpha^2(x+\eta)d\eta\right)^{1/2} \lesssim \frac{\|A\|_{\mathrm{St}}}{\rho^{1/2}}$$

by virtue of Cauchy-Schwarz inequality. Summing up, we get

$$\|k_0(x,0)\| \lesssim (x+x^{1/2}+\rho^{-1/2})e^{-\rho x}\|A\|_{\mathrm{St}} \lesssim \frac{e^{-\rho x/2}\|A\|_{\mathrm{St}}}{\rho^{1/2}}\,.$$

The Stummel condition is translation-invariant on the line which implies (A.6) for $j=0$:

$$\|k_0(x,y)\| \leqslant C\frac{e^{-\rho|x-y|/2}}{\rho^{1/2}}\|A\|_{\mathrm{St}}\,. \tag{A.8}$$

We can write $k_{j+1}(x,y) = \int_{\mathbb{R}^+} G^0_{i\rho}(x,\xi)V(\xi)k_j(\xi,y)d\xi$ and use the inductive assumption to conclude that

$$\|k_{j+1}(x,y)\| \leqslant C_1\int_{\mathbb{R}^+} e^{-\rho|x-\xi|}\alpha(\xi)\cdot\|k_j(\xi,y)\|d\xi \leqslant \frac{C_1C^{j+1}\|A\|^{j+1}_{\mathrm{St}}}{\rho^{(j+1)/2}}\int_{\mathbb{R}^+} e^{-\rho|x-\xi|}\alpha(\xi)e^{-\rho|\xi-y|/2}d\xi\,.$$

For $y=0$, we get

$$\int_{\mathbb{R}^+} e^{-\rho|x-\xi|}\alpha(\xi)e^{-\rho\xi/2}d\xi = e^{-\rho x/2}\cdot e^{-\rho x/2}\int_0^x e^{\rho\xi/2}\alpha(\xi)d\xi + e^{\rho x}\int_x^\infty \alpha(\xi)e^{-3\rho\xi/2}d\xi\,. \tag{A.9}$$

Then, we write

$$e^{-\rho x/2}\int_0^x e^{\rho\xi/2}\alpha(\xi)d\xi = \int_0^x e^{-\rho\eta/2}\alpha(x-\eta)d\eta \leqslant \int_0^1 e^{-\rho\eta/2}\alpha(x-\eta)d\eta + \sum_{j=1}^{\infty}\int_j^{j+1} e^{-\rho\eta/2}\alpha(x-\eta)d\eta \leqslant$$

$$\left(\int_0^1 e^{-\rho\eta}d\eta\right)^{1/2}\left(\int_0^1 \alpha^2(x-\eta)d\eta\right)^{1/2} + \sum_{j=1}^{\infty} e^{-\rho j/2}\left(\int_j^{j+1}\alpha^2(x-\eta)d\eta\right)^{1/2} \lesssim \frac{\|A\|_{\rm St}}{\rho^{1/2}}\,.$$

Estimating the second integral in (A.9) in a similar way, we have

$$\int_{\mathbb{R}^+} e^{-\rho|x-\xi|}\alpha(\xi)e^{-\rho\xi/2}d\xi \leqslant C_2\frac{e^{-\rho x/2}\|A\|_{\rm St}}{\rho^{1/2}}$$

and, using translation invariance of Stummel condition,

$$\int_{\mathbb{R}^+} e^{-\rho|x-\xi|}\alpha(\xi)e^{-\rho|\xi-y|/2}d\xi \leqslant C_2\frac{e^{-\rho|x-y|/2}\|A\|_{\rm St}}{\rho^{1/2}}\,.$$

Thus,

$$\|k_{j+1}(x,y)\| \leqslant \frac{C_1C_2C^{j+1}e^{-\rho|x-y|/2}\|A\|_{\rm St}^{j+2}}{\rho^{(j+2)/2}}\,.$$

Choosing C sufficiently large, e.g., larger than C_1C_2, we show (A.6) for $j+1$. This proves the claim. Now, (A.5) implies $\|G_{i\rho}(x,y)\| \lesssim e^{-\rho|x-y|/2}$ provided $\rho = 2C(1+\|A\|_{\rm St}^2)$. Thus, (A.2) finishes the proof. □

Lemma A.1.1. *Let $h \in L^2(\mathbb{R})$. Then,*

$$\lim_{t\to+\infty}\left\| e^{it\partial_{xx}^2}h - \frac{1}{1+i}\frac{e^{ix^2/(4t)}}{\sqrt{t}}\widehat{h}(x/(2t))\right\|_{L^2(\mathbb{R})} = 0, \tag{A.10}$$

and, taking inverse Fourier transform,

$$\lim_{t\to+\infty}\left\|\frac{1}{1+i}\left(\frac{e^{ix^2/(4t)}}{\sqrt{t}}\widehat{h}(x/(2t))\right)^{\vee} - e^{-it\xi^2}\,\check{h}(\xi)\right\|_{L^2(\mathbb{R})} = 0\,. \tag{A.11}$$

Suppose $\widehat{h} \in C_c^\infty(\mathbb{R})$*, then*

$$\sup_{t>1,\alpha,\beta\in\mathbb{R}} \left\| \int_{\alpha t}^{\beta t} \frac{e^{ix^2/(4t)}}{\sqrt{t}} \widehat{h}(x/(2t)) e^{ixk} dx \right\|_{L^\infty(\mathbb{R})} < C_{(h)} \,. \tag{A.12}$$

Proof. Formula (A.10) can be found in [57] (see formulas (4.10) and (4.12) there). Then, (A.11) is a direct corollary. Proof of (A.12) follows from a direct calculation:

$$\int_{\alpha t}^{\beta t} \frac{e^{ix^2/(4t)}}{\sqrt{t}} \widehat{h}(x/(2t)) e^{ixk} dx = \frac{e^{-itk^2}}{\sqrt{t}} \int_{\alpha t}^{\beta t} \exp\left(i \left(\frac{x}{2\sqrt{t}} + k\sqrt{t} \right)^2 \right) \widehat{h}(x/(2t)) dx$$

$$= 2e^{-itk^2} \int_{\sqrt{t}(0.5\alpha+k)}^{\sqrt{t}(0.5\beta+k)} \exp(i\xi^2) \widehat{h}(-k+\xi/\sqrt{t}) d\xi \,.$$

Now, consider an integral

$$\int_0^l \exp(i\xi^2) \widehat{h}(-k+\xi/\sqrt{t}) d\xi$$

for arbitrary $l \in \mathbb{R}, k \in \mathbb{R}, t \geqslant 1$ and let μ_0 be a bump function introduced in (2.34). We have

$$\int_0^l \exp(i\xi^2) \widehat{h}(-k+\xi/\sqrt{t}) d\xi = \int_0^l \exp(i\xi^2) \widehat{h}(-k+\xi/\sqrt{t}) \mu_0 d\xi + \int_0^l \exp(i\xi^2) \widehat{h}(-k+\xi/\sqrt{t})(1-\mu_0) d\xi \,.$$

The first integral is bounded uniformly in all parameters since $\widehat{h} \in C_c^\infty(\mathbb{R})$. For the second one, we can write

$$\int_0^l \exp(i\xi^2) \widehat{h}(-k+\xi/\sqrt{t})(1-\mu_0) d\xi = \int_0^l \Big(\exp(i\xi^2) \Big)' \frac{\widehat{h}(-k+\xi/\sqrt{t})(1-\mu_0)}{2i\xi} d\xi$$

$$= \exp(il^2) \frac{\widehat{h}(-k+l/\sqrt{t})(1-\mu_0(l))}{2il} - \int_0^l \exp(i\xi^2) \left(\frac{\widehat{h}(-k+\xi/\sqrt{t})(1-\mu_0(\xi))}{2i\xi} \right)' d\xi \,.$$

The first term is uniformly bounded because $1-\mu_0(0) = 0$. For the second one, we can show that each resulting integral is uniformly bounded, e.g.,

$$\left| \frac{1}{\sqrt{t}} \int_0^l \exp(i\xi^2) \frac{\widehat{h}'(-k+\xi/\sqrt{t})(1-\mu_0)}{2i\xi} d\xi \right| \lesssim \frac{1}{\sqrt{t}} \int_0^l |\widehat{h}'(-k+\xi/\sqrt{t})| d\xi \leqslant \|\widehat{h}'\|_1 \,,$$

$$\left| \int_0^l \exp(i\xi^2) \frac{\widehat{h}(-k+\xi/\sqrt{t})(1-\mu_0(\xi))}{\xi^2} d\xi \right| \leqslant \|\widehat{h}\|_\infty \int_0^l \frac{|1-\mu_0(\xi)|}{\xi^2} d\xi \lesssim \|\widehat{h}\|_\infty \,,$$

$$\left|\int_0^l \exp(i\xi^2)\frac{\widehat{h}(-k+\xi/\sqrt{t})\mu_0'(\xi)}{2i\xi}d\xi\right| \lesssim \|\widehat{h}\|_\infty$$

and (A.12) is proved. □

Lemma A.1.2. *Let $\epsilon \in (0,1), \nu > 0, a > 0$, and $|a\epsilon| \leqslant \nu$. We have*

$$\left|\int_0^a e^{iu^2} g(u\epsilon)du\right| \leqslant C_{(g,\nu)}\epsilon$$

provided that $g \in C^\infty(\mathbb{R})$ and $g(0) = 0$.

Proof. We have

$$\int_0^a (e^{iu^2})' g(u\epsilon)u^{-1}du = e^{ia^2}\epsilon\left(\frac{g(a\epsilon)}{a\epsilon}\right) - \epsilon g'(0) - \epsilon\int_0^a e^{iu^2}\left(\frac{g(u\epsilon)}{\epsilon u}\right)' du\,. \tag{A.13}$$

We can write $|g(\xi)| \leqslant C_{(g,\nu)}|\xi|$ for $\xi \in [-\nu,\nu]$ and the first term is controlled by $C_{(g,\nu)}\epsilon$ since $|a\epsilon| \leqslant \nu$. For the third one, we introduce $G(u) \overset{\text{def}}{=} (g(u)/u)' \in C^\infty(\mathbb{R})$ and write

$$G_1(u) \overset{\text{def}}{=} G(u) - G(0), \quad G(u) = G(0) + G_1(u)$$

so that

$$\int_0^a e^{iu^2} G(u\epsilon)du = G(0)\int_0^a e^{iu^2}du + \int_0^a e^{iu^2} G_1(\epsilon u)du\,.$$

The absolute value of the first term is bounded by $C_{(g)}$ uniformly in a. For the second one, we can iterate the argument since $G_1 \in C^\infty(\mathbb{R})$ and $G_1(0) = 0$. We get

$$\int_0^a e^{iu^2} G_1(\epsilon u)du = -0.5i\epsilon\int_0^a (e^{iu^2})'\frac{G_1(\epsilon u)}{\epsilon u}du$$
$$= -0.5i\epsilon\left(e^{ia^2}\frac{G_1(\epsilon a)}{\epsilon a} - G_1'(0) - \int_0^a e^{iu^2}\left(\frac{G_1(\epsilon u)}{\epsilon u}\right)' du\right)\,. \tag{A.14}$$

Writing a rough estimate

$$\left|\int_0^a e^{iu^2}\left(\frac{G_1(\epsilon u)}{\epsilon u}\right)' du\right| \leqslant C_{(g)}|a|$$

and substituting it into (A.14) gives

$$\left|\int_0^a e^{iu^2} G_1(\epsilon u)du\right| \leqslant C_{(g)}(\epsilon + \epsilon|a|) = C_{(g)}(\epsilon + \nu)\,.$$

We bring it to (A.13) to finish the proof of the Lemma. □

Consider H defined as

$$H(x) = \int_x^\infty e^{it^2} dt\,.$$

This integral can be related to the so-called erf–function whose properties are well-known. However, our purpose is to obtain a specific representation for H for $x \in [1, \infty)$ and we proceed directly as follows. We change variables and iteratively integrate by parts n times to get

$$H(x) = i\frac{e^{ix^2}}{2x} - \frac{i}{2}\int_{x^2}^\infty \frac{e^{iu}}{u^{3/2}}du = e^{ix^2}\left(\sum_{j=0}^{n-1} \frac{c_j}{x^{1+2j}} + c_n' e^{-ix^2}\int_{x^2}^\infty \frac{e^{iu}}{u^{n+1/2}}du\right) \stackrel{\text{def}}{=} e^{ix^2}(H_{1,n} + H_{2,n})$$

where $\{c_j\}$ and c_n' are some constants. Let μ_+ be the cutoff function that satisfies conditions: μ_+ is supported on $(1, \infty)$, $\mu_+(x) = 1$ for $x > 2$, $\mu_+ \in C^\infty(\mathbb{R})$. Define

$$H_{1,n}^{(m)} \stackrel{\text{def}}{=} H_{1,n}\mu_+, \quad H_{2,n}^{(m)} \stackrel{\text{def}}{=} H_{2,n}\mu_+\,.$$

Lemma A.1.3. *Let $n > 1$. We have $\widehat{H_{1,n}^{(m)}} \in L^1(\mathbb{R})$, $\widehat{H_{2,n}^{(m)}} \in L^1(\mathbb{R})$.*

Proof. Consider $H_{2,n}^{(m)}$ first. We have

$$|H_{2,n}^{(m)}| \leqslant C_n(1+|x|)^{-(2n+1)}, \quad |\partial_x H_{2,n}^{(m)}| \leqslant C_n(1+|x|)^{-(2n)}, \qquad |\partial_{xx}^2 H_{2,n}^{(m)}| \leqslant C_n(1+|x|)^{-(2n-1)}\,.$$

Therefore,

$$|\widehat{H_{2,n}^{(m)}}(\xi)| < C_n(1+|\xi|)^{-2}$$

and hence $\widehat{H_{2,n}^{(m)}} \in L^1(\mathbb{R})$. For $H_{1,n}^{(m)}$, consider the first term, $x^{-1}\mu_+$. Other terms can be handled similarly. We have $x^{-1}\mu_+ \in C^\infty(\mathbb{R}) \cap L^2(\mathbb{R})$ and all of its derivatives are in $L^2(\mathbb{R})$. Thus,

$\xi^j(\widehat{x^{-1}\mu_+}) \in L^2(\mathbb{R})$ for all $j \in \mathbb{Z}^+$. Therefore, $(\widehat{x^{-1}\mu_+})(\xi) \in L^1(|\xi| > 1)$. For $|\xi| < 1$, we can write an estimate

$$|\widehat{x^{-1}\mu_+}| < C|\log\xi|\,,$$

which can be verified directly:

$$\int_1^\infty \frac{\mu_+(x)}{x} e^{-i\xi x} dx = \int_2^\infty \frac{e^{-i\xi x}}{x} dx + O(1)\,.$$

For $\xi \in (0,1)$,

$$\int_2^\infty \frac{e^{-i\xi x}}{x} dx = \int_{2\xi}^\infty \frac{e^{-iu}}{u} du = \int_{2\xi}^1 \frac{e^{-iu}}{u} du + \int_1^\infty \frac{e^{-iu}}{u} du = O(|\log\xi| + 1)\,.$$

For $\xi \in (-1,0)$, the argument is analogous and we get the statement of the Lemma. $\square$

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