

ALGEBRAIC INTERSECTION SPACES

By

Christian Geske

A DISSERTATION SUBMITTED IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

(MATHEMATICS)

at the

UNIVERSITY OF WISCONSIN – MADISON

2019

Date of final oral examination: April 16, 2019

The dissertation is approved by the following members of the Final Oral Committee:

L. Maxim, Professor, Mathematics (Advisor)

B. Wang, Assistant Professor, Mathematics

A. Căldăraru, Professor, Mathematics

J. Rodriguez, Visiting Assistant Professor, Mathematics

Abstract

We define a variant of intersection space theory that applies to many compact complex and real analytic spaces X , including all complex projective varieties; this is a significant extension to a theory which has so far only been shown to apply to a particular subclass of spaces with smooth singular sets. We verify existence of these so-called *algebraic* intersection spaces and show that they are the (reduced) chain complexes of known topological intersection spaces in the case that both exist. We next analyze “local duality obstructions”, which we can choose to vanish, and verify that algebraic intersection spaces satisfy duality in the absence of these obstructions. We conclude by defining an untwisted algebraic intersection space pairing, whose signature is equal to the Novikov signature of the complement in X of a tubular neighborhood of the singular set.

Acknowledgements

I thank, above all, my advisor Laurentiu Maxim for the wholehearted support he has shown me throughout my graduate studies. His topological awareness for what “should be true” has propelled me down the fruitful avenues of research that culminated in multiple papers and, ultimately, this thesis. His never-flagging optimism and impressive mental store of relevant references have, in periods of doubt, repeatedly reignited my mathematical willpower and inspired new approaches to a problem. Further, his genuine interest in my short-term and long-term success has bestowed me with opportunities of which graduate students dream.

I also thank my committee members: Botong Wang, Andrei Căldăraru, and Jose Rodriguez. During my many seminar talks, Botong always displayed an unparalleled attention that he would direct into questions striking at the heart of the subject matter. These little inquiries could bring me to reevaluate my perspective on topics with which I had thought I was familiar. And, as a young graduate student, my exposure to Andrei’s precise, never ambiguous, and well-structured expository style influenced my own developing style.

I am grateful to the singularity theory postdocs Alexandra Kjuchukova and Manuel González Villa, who contributed to the notion that mathematical research is built on collaboration. In particular, I thank Alexandra for her ceaseless encouragement and for the hours of discussion we shared as I broached problems in knot theory. I thank Tommy Wong, Suky Su, and Yongqiang Liu, Laurentiu’s former students, for professional advice and for setting a model for postgraduate existence. And to my “academic sister” Eva

Elduque, I express my appreciation for often being my primary mathematical motivator, be it while we struggled through Alexandru Dimca's *Sheaves in Topology*, collaborated on a paper, or lamented the difficulties of graduate student life.

I would also like to thank Markus Banagl and Jörg Schürmann for valuable input and interest in my work, and Julius L. Shaneson for his collaboration.

Lastly, I would be remiss not to include my family, who instilled in me scientific curiosity and the idea that you never put less than maximal effort into the things you care about, and my partner, Sally Wu, who has been an incredible source of stability and encouragement since the moment we converged. She taught me that collaboration is not only essential to research, but to living too.

The author also gratefully acknowledges the support provided by the NSF-RTG grant #1502553 at the University of Wisconsin-Madison.

Contents

Abstract	i
Acknowledgements	ii
1 Introduction	1
2 Background on Intersection Spaces	7
2.1 Pseudomanifolds	7
2.1.1 Intersection Homology and Cohomology	8
2.1.2 Duality	10
2.1.3 Witt Spaces and Signature	11
2.2 Intersection Spaces in the Isolated Singularities Case	12
2.2.1 Moore Approximations	13
2.2.2 Intersection Space	13
2.2.3 Relation to Smooth Deformations of Hypersurfaces	15
2.3 Intersection Spaces in the Depth One Case	17
2.3.1 Fiberwise Truncations	17
2.3.2 Intersection Space	19
2.4 Motivation for Algebraic Intersection Spaces	22
3 Preliminary Results	23
3.1 Triangulated Categories	23
3.2 Tubular Neighborhoods of Singular Sets	25

3.3	Intersection Homology and Cohomology	27
3.4	Linear Algebra	30
4	Local Construction	33
4.1	Denotations and Assumptions	33
4.2	Duality and the Image of the Boundary	33
4.3	Local Intersection Approximation	36
4.4	Local Duality Isomorphism	40
5	Global Construction	48
5.1	Denotations and Assumptions	48
5.2	Intersection Space	48
6	A Worked out Example	53
6.1	Denotations and Assumptions	53
6.2	Setting Up the Example	54
6.3	The Intersection Space	59
6.4	Comparison with the AF intersection space.	61
7	Intersection Space Pairing	66
7.1	Denotations and Assumptions	66
7.2	Families of Sections	68
7.3	Signature	75
	Bibliography	80

Chapter 1

Introduction

Singular compact complex varieties typically lack the Poincaré duality enjoyed by their nonsingular counterparts. One approach to rectifying this disparity can be found in Goresky-MacPherson's (middle-perversity) intersection homology and cohomology [14], [15]. These theories endow both singular and nonsingular compact complex varieties with an intrinsic duality, extending Poincaré duality. Moreover, intersection homology and cohomology exhibit stability under small resolutions. However, they suffer from instability under smooth deformations, and intersection cohomology fails in general to have a cup product structure.

A natural question is: does a duality-satisfying (co)homology-type theory exist that behaves well under smooth deformations?

This question has been broached and, for hypersurfaces with isolated singularities, answered partially in the affirmative in [8] and [9], in which an alternate theory is utilized: the intersection space theory introduced in [3]. A generalization of this approach to hypersurfaces with more complicated singular sets has not yet been discovered, because intersection space theory has not been defined for the vast majority of singular spaces.

We briefly discuss the history and current limitations of intersection space theory. Given a perversity $\bar{p}$ (in the sense of intersection homology theory) and a real

n -dimensional stratified pseudomanifold X with isolated singularities, Banagl in [3] defined CW-complexes $I^{\bar{p}}X$, the **perversity $\bar{p}$ intersection spaces**. The cohomology of an intersection space $I^{\bar{p}}X$, unlike the intersection cohomology of X , is always equipped with a cup product structure. Moreover, if X is closed and oriented, there exist duality isomorphisms:

$$\tilde{H}^*(I^{\bar{p}}X; \mathbb{Q}) \cong \tilde{H}_{n-*}(I^{\bar{q}}X; \mathbb{Q})$$

where $\bar{q}$ is the complementary perversity to $\bar{p}$. Banagl continued in [3] to prove the uniqueness of the Betti numbers of $I^{\bar{p}}X$ (dependent only on X and its stratification), even though the intersection space itself need not be unique (even up to homotopy).

The essential component to the construction of intersection spaces in the isolated singularities case is the selection of **Moore approximations** for the links of the singular points. Banagl describes the existence of such Moore approximations in the case that the connected components of the links are simply-connected CW-complexes, though this is far from a necessary condition for existence.

Beyond isolated singularities, Banagl in [3] also defined intersection spaces for depth one stratified pseudomanifolds whose singular strata have trivial link bundles. This idea was generalized by Banagl and Chriestenson in [7], where intersection spaces were defined for depth one Thom-Mather stratified pseudomanifolds. In contrast to the isolated singularities case, there exist clear restrictions to the existence of such intersection spaces, dependent on the structure of the link bundles of the singular strata (as indicated in the following paragraph).

In this extended depth one setting, the primary ingredient for intersection space

construction is the selection of *fiberwise truncations* for the link bundles of the singular strata. There exist bundles which fail to have fiberwise truncations, and therefore stratified spaces to which Banagl and Chriestenson's process cannot associate intersection spaces. Even when a fiberwise truncation exists, there can arise *local duality obstructions* that preclude the intersection space from satisfying duality.

Passing beyond depth one, Agustín-Fernández identify in [1] a variant/extension of intersection space theory that applies to arbitrary depth spaces whose strata all have trivial link bundles (though it applies in some other cases too). They do not prove that their topological intersection spaces satisfy duality nor do they identify local duality obstructions, however they abstract their construction to a sheaf level by defining *intersection space constructible complexes*, from which they establish a generic duality for generic choices of intersection space constructible complexes (which may or may not be realized by topological intersection spaces).

Special cases where the stratification of the singular space is more elaborate have been studied, for example in [5], but no all-encompassing picture has been painted. Despite the limited collection of spaces for which it is defined, intersection space theory has had applications in multiple fields: fiber bundle theory [4], algebraic geometry and smooth deformations [8] and [9], perverse sheaves [6], and theoretical physics [3, Chapter 3].

This document devises a variant/extension of intersection space theory applicable in particular to all complex projective varieties, and so enables future endeavors in these fields.

More specifically, this document selects as its target the collection of compact orientable Whitney stratified pseudomanifolds X that are subvarieties of a real/complex

analytic manifold, and from them systematically extracts chain complex alternatives to topological intersection spaces, which we equip with the moniker *perversity $\bar{p}$ algebraic intersection spaces*. Though deprived of a topology, these algebraic intersection spaces carry homology, which we require to be an extension to, not replacement of, the already existing intersection space homology. On the topological side, we define *perversity $\bar{p}$ topological intersection spaces*, which may or may not exist, and whose chain complexes yield algebraic intersection spaces. The study of what conditions allow an algebraic intersection space to be made topological could be the source of interesting future research.

The introduction is followed in Chapter 2 with background on intersection spaces. This chapter is intended to familiarize the reader with the theory and to motivate the later algebraic intersection space construction.

Chapter 3 collects general theorems and lemmas that will enable, or in some cases merely streamline, the arguments to be made toward the main results of the paper. Except perhaps for Section 3.2 on tubular neighborhoods of singular sets, detailed reading of this section should be left until the corresponding theorem or lemma is referred to in a proof from the final four chapters, which form the core of the paper.

Suppose for the remainder of the introduction that k is a field and that X is a compact orientable n -dimensional Whitney stratified pseudomanifold that is a subvariety of a real/complex analytic manifold (for example X is a complex projective variety). $T \subset X$ denotes a tubular neighborhood of the singular set.

Just as with intersection spaces, algebraic intersection spaces are built in parts: first locally, then globally. Chapter 4 describes the local construction, which rests on a *$\bar{p}$ algebraic intersection approximation for T with coefficients in k* . It further

proves these approximations always exist.

Proposition 1.0.1 (4.3.3). *$\bar{p}$ denotes a perversity function. A $\bar{p}$ algebraic intersection approximation $(A_\bullet, f_\bullet)$ for T with coefficients in k always exists.*

The concluding Section 4.4 describes **local duality obstructions**, an unfortunate feature of certain algebraic intersection approximations that prevents a global duality. On the bright side, there always *exist* local intersection approximations for which there are no local duality obstructions (here proved for even-dimensional Witt spaces):

Theorem 1.0.2 (4.4.4). *Suppose that X is a Witt space of even dimension. Then there exists a Witt algebraic intersection approximation $(A_\bullet, f_\bullet)$ for T with coefficients in k for which all the local duality obstructions vanish.*

Chapter 5 takes the local construction and converts it into a global algebraic intersection space $I_{f_\bullet}X$ associated to X . In the case that the local duality obstructions vanish, duality isomorphisms are constructed between the homology and cohomology of complementary perversity algebraic intersection spaces:

Theorem 1.0.3 (5.2.1). *$\bar{p}$ and $\bar{q}$ denote complementary perversity functions. Assume the local duality obstructions vanish for algebraic intersection approximations $(A_\bullet^{\bar{p}}, f_\bullet^{\bar{p}})$, $(A_\bullet^{\bar{q}}, f_\bullet^{\bar{q}})$ for T with coefficients in k . Then there exist non-canonical duality isomorphisms:*

$$D : H_*(I_{f_\bullet^{\bar{p}}}X) \xrightarrow{\cong} H^{n-*}(I_{f_\bullet^{\bar{q}}}X).$$

Chapter 6 explicitly constructs a topological intersection space for a depth two pseudomanifold. It is compared against the [1] construction, from which it is shown to differ on the level of homology.

When X is a Witt space, Chapter 7 extracts the signature of an ***intersection space pairing*** on the middle-dimensional homology of the algebraic intersection space (again in the case that the local duality obstructions vanish), which turns out to be equal to the Novikov signature of X minus an open tubular neighborhood of the singular set:

Theorem 1.0.4 (7.3.1, 7.3.2). *Suppose $k = \mathbb{Q}$ and that X is a closed, oriented Witt space of dimension $n = 2m$, m even. If $IX_\bullet$ is an algebraic intersection space obtained from a Witt algebraic approximation whose local duality obstructions vanish, then there is a symmetric intersection space pairing:*

$$(-, -)_{IX} : H_m(IX_\bullet) \times H_m(IX_\bullet) \rightarrow \mathbb{Q}.$$

whose signature equals the Novikov signature of $(X - T^\circ, \partial T)$.

Chapter 2

Background on Intersection Spaces

2.1 Pseudomanifolds

All intersection spaces are associated to topological spaces called ***topological stratified pseudomanifolds***, which may be defined recursively as follows (see [2, Section 4.1] and [13, Chapter 2] for further discussion):

- A 0-dimensional topological stratified pseudomanifold is a 0-dimensional manifold (i.e. a countable set of points with the discrete topology).
- An n -dimensional topological stratified pseudomanifold is a Hausdorff topological space X with a filtration by closed subspaces

$$X = X_n \supset X_{n-1} = X_{n-2} \supset X_{n-3} \supset \cdots \supset X_1 \supset X_0 \supset X_{-1} = \emptyset$$

such that:

1. Every $X_{n-k} - X_{n-k-1}$ is a (possibly empty) topological manifold of dimension $n - k$ without boundary, called a ***stratum*** of X .
2. $X - X_{n-2}$ is dense in X .
3. For all $x \in X_{n-k} - X_{n-k-1}$, there exists an open neighborhood U of x in X , a compact topological stratified pseudomanifold L of dimension $k - 1$, and

a stratum preserving homeomorphism $U \cong \mathbb{R}^{n-k} \times {}^\circ cL$ (${}^\circ cL$ denotes the open cone on L), i.e. a homeomorphism mapping $U \cap X_{n-\ell}$ to $\mathbb{R}^{n-k} \times {}^\circ cL_{k-1-\ell}$ for all $\ell \leq k$ (for $\ell > k$ we require $U \cap X_{n-\ell} = \emptyset$).

This notion can be extended to **stratified pseudomanifolds with boundary** $(X, \partial X)$, which (for our purposes) are defined in the same way as n -dimensional stratified pseudomanifolds, except the dense stratum $X_n - X_{n-2}$ is allowed to have boundary ∂X , as long as this boundary has a collar neighborhood in X that does not intersect X_{n-2} . A compact stratified pseudomanifold without boundary is said to be **closed**.

2.1.1 Intersection Homology and Cohomology

For stratified pseudomanifolds, intersection (co)homology is often a more suitable alternative to usual (co)homology. There are various choices of intersection (co)homologies, determined by the selection of a perversity function.

A **perversity function** is a function $\bar{p} : \mathbb{N}_{\geq 2} \rightarrow \mathbb{N}$ such that $\bar{p}(2) = 0$ and $\bar{p}(k) \leq \bar{p}(k+1) \leq \bar{p}(k) + 1$ for all k . Among these are the **lower middle perversity** $\bar{m}(k) = \lfloor \frac{k-2}{2} \rfloor$ and **upper middle perversity** $\bar{n}(k) = \lfloor \frac{k-1}{2} \rfloor$. Two perversity functions $\bar{p}$ and $\bar{q}$ are said to be **complementary** if $\bar{p}(k) + \bar{q}(k) = k - 2$. Observe that the lower and upper middle perversity functions are complementary.

Suppose X is a stratified pseudomanifold (with or without boundary) and $\bar{p}$ is a perversity function. Let $C_\bullet(X)$ denote the usual singular chain complex on the topological space X . A singular i -simplex $\sigma : \Delta^i \rightarrow X$ is called **$\bar{p}$ -allowable** if $\sigma^{-1}(X_{n-k})$ belongs to the $i - k + \bar{p}(k)$ -skeleton of Δ^i for all $k \geq 2$. A singular chain $\xi \in C_\bullet(X)$ is **$\bar{p}$ -allowable**

if both ξ and $\partial\xi$ can be written as an integral combination of $\bar{p}$ -allowable singular simplices. The collection of $\bar{p}$ -allowable chains determines a subcomplex $IC_{\bullet}^{\bar{p}}(X)$ of $C_{\bullet}(X)$, called the $\bar{p}$ -intersection chain complex.

The homology of $IC_{\bullet}^{\bar{p}}(X)$ is denoted by $IH_{*}^{\bar{p}}(X)$ and called the *$\bar{p}$ -intersection homology of X* . Applying $\text{Hom}(-, \mathbb{Z})$ to $IC_{\bullet}^{\bar{p}}(X)$ yields the *$\bar{p}$ -intersection cochain complex $IC_{\bar{p}}^{\bullet}(X)$* and the corresponding *$\bar{p}$ -intersection cohomology of X* , denoted by $IH_{\bar{p}}^{*}(X)$. If A is an abelian group, one can instead begin from $C_{\bullet}(X) \otimes A$ when defining $\bar{p}$ -allowable chains to obtain *$\bar{p}$ -intersection homology and cohomology with coefficients in A* , $IH_{*}^{\bar{p}}(X; A)$ and $IH_{\bar{p}}^{*}(X; A)$.

If X has boundary ∂X , then $C_{\bullet}(\partial X) \subset IC_{\bullet}^{\bar{p}}(X)$, and from the resulting quotient complex one obtains the *relative $\bar{p}$ -intersection homology and cohomology*, $IH_{*}^{\bar{p}}(X, \partial X)$ and $IH_{\bar{p}}^{*}(X, \partial X)$. One can also consider the corresponding relative intersection (co)homologies with coefficients in an abelian group A .

Like usual (co)homology, intersection (co)homology is a topological invariant, i.e. does not depend on the particular stratification of the pseudomanifold (see for example [13, Section 5.5]). However, it is no longer a homotopy invariant but instead is a *stratified homotopy invariant* (see [13, Definition 4.1.9]). Furthermore, intersection cohomology does not generally have a cup product structure.

A basic example demonstrating that intersection homology is not a homotopy invariant is the *cone formula* (see [13, Theorem 4.2.1]). If L is an $(n-1)$ -dimensional closed manifold, then the (closed) cone cL is an n -dimensional pseudomanifold with boundary

for which:

$$IH_*^{\bar{p}}(cL) \cong \begin{cases} 0 & \text{if } * \geq n - 1 - \bar{p}(n) \\ H_*(L) & \text{if } * < n - 1 - \bar{p}(n) \end{cases}$$

The isomorphism in low degrees is induced by the inclusion $L \hookrightarrow cL$.

2.1.2 Duality

Suppose X is an n -dimensional stratified pseudomanifold (with or without boundary). Then X is said to be **orientable** if the dense stratum $X - X_{n-2}$ is orientable. An **orientation** of X is an orientation of this dense stratum. For more discussion on orientability of pseudomanifolds, see [13, Section 8.1].

Remarkably, using intersection (co)homology, Poincaré duality can be recovered for pseudomanifolds when the coefficient group is a field. Duality was first identified in [14], however the version we state here is closer to [13, Theorem 8.2.4].

Theorem (Poincaré Duality [14], [13]). *Let k denote a field and $(\bar{p}, \bar{q})$ denote a pair of complementary perversity functions. Suppose X is an n -dimensional closed, oriented, stratified pseudomanifold. Then there exist duality isomorphisms:*

$$D : IH_{\bar{p}}^*(X; k) \xrightarrow{\cong} IH_{n-*}^{\bar{q}}(X; k)$$

Likewise, when X has boundary we have Lefschetz duality:

Theorem (Lefschetz Duality [13]). *Let k denote a field and $(\bar{p}, \bar{q})$ denote a pair of complementary perversity functions. Suppose X is an n -dimensional compact oriented*

stratified pseudomanifold with boundary ∂X . Then there exist duality isomorphisms:

$$D : IH_{\bar{p}}^*(X, \partial X; k) \xrightarrow{\cong} IH_{n-*}^{\bar{q}}(X; k)$$

$$D : IH_{\bar{p}}^*(X; k) \xrightarrow{\cong} IH_{n-*}^{\bar{q}}(X, \partial X; k)$$

2.1.3 Witt Spaces and Signature

Closed, oriented manifolds X of dimension $2n$, n even, have a well-defined signature associated to the nondegenerate symmetric ***intersection pairing***:

$$H_n(X; \mathbb{Q}) \otimes H_n(X; \mathbb{Q}) \rightarrow \mathbb{Q}$$

induced by Poincaré duality (recall that the signature of a symmetric pairing is the difference between the number of positive and negative eigenvalues in a matrix representation of the pairing). The same cannot be said if we replace *manifold* by *pseudomanifold*, owing to the presence of different perversity functions in the pseudomanifold Poincaré duality statement. If $\bar{p}$ and $\bar{q}$ are complementary, then we only get a nondegenerate bilinear pairing:

$$IH_n^{\bar{p}}(X; \mathbb{Q}) \otimes IH_n^{\bar{q}}(X; \mathbb{Q}) \rightarrow \mathbb{Q}$$

However, there is a particular class of pseudomanifolds X , called ***Witt spaces*** (see for example [13, Chapter 9]), for which there are natural isomorphisms $IH_*^{\bar{m}}(X; \mathbb{Q}) \cong IH_*^{\bar{n}}(X; \mathbb{Q})$ where $\bar{m}$ and $\bar{n}$ denote the lower-middle and upper-middle perversity functions. If X is Witt, we often omit the superscript $\bar{m}$ or $\bar{n}$ (which of these we select doesn't matter) and simply write $IH_*(X; \mathbb{Q})$.

The simplest examples of spaces that are Witt are those that can be equipped with pseudomanifold stratifications whose nonempty strata all have *even* codimension.

Equidimensional closed complex subvarieties of complex analytic manifolds can be equipped with such stratifications, and therefore are all Witt.

If X is a closed, oriented Witt space of dimension $2n$, n even, then we obtain a symmetric pairing (see [13, Section 9.3]):

$$IH_n(X; \mathbb{Q}) \otimes IH_n(X; \mathbb{Q}) \rightarrow \mathbb{Q}$$

whose signature is called the **(Witt) signature of X** . If X is in fact a manifold, this recovers the usual signature of X .

If M is a compact *manifold* with boundary ∂M of dimension $2n$, n even, the **Novikov signature of $(M, \partial M)$** is obtained from a symmetric pairing:

$$\text{im } j_n \otimes \text{im } j_n \rightarrow \mathbb{Q}$$

where $j_n : H_n(M; \mathbb{Q}) \rightarrow H_n(M, \partial M; \mathbb{Q})$ is induced by the inclusion of pairs $(M, \emptyset) \hookrightarrow (M, \partial M)$. The Novikov signature can be related to Witt signature in the following way. Let $\widehat{M}$ denote the space $M \cup_{\partial M} c(\partial M)$ obtained by coning off the boundary ∂M . Then the Novikov signature of $(M, \partial M)$ equals the Witt signature of $\widehat{M}$ (see for example [19]).

2.2 Intersection Spaces in the Isolated Singularities Case

The essential ingredient to an intersection space in the isolated singularities case is the selection of Moore approximations of links of singularities, so we begin there.

2.2.1 Moore Approximations

Suppose L is a topological space. Then a **degree k Moore approximation for L** is pair $(L_{<k}, f_{<k})$ where $L_{<k}$ is a topological space and $f_{<k} : L_{<k} \rightarrow L$ is a continuous map satisfying:

- $H_i(L_{<k}) = 0$ for $i \geq k$
- $f_{<k*} : H_i(L_{<k}) \rightarrow H_i(L)$ is an isomorphism for $i < k$.

In [3], Banagl discusses the existence of Moore approximations for simply-connected CW-complexes L , though simple-connectivity is far from necessary for existence.

Moore approximations can be related to intersection homology in the following way. Suppose $\bar{p}$ is a perversity function, L is an $(n - 1)$ -dimensional manifold, and $f^{\bar{p}} : L_{<n-1-\bar{p}(n)} \rightarrow L$ is a degree $n - 1 - \bar{p}(n)$ Moore approximation. By the cone formula described in Subsection 2.1.1, it follows that the composition:

$$L_{<n-1-\bar{p}(n)} \xrightarrow{f^{\bar{p}}} L \hookrightarrow cL$$

induces an isomorphism $H_*(L) \xrightarrow{\simeq} IH_*^{\bar{p}}(cL)$.

2.2.2 Intersection Space

Suppose X is an n -dimensional ($n \geq 2$) compact oriented stratified pseudomanifold with **isolated singularities**, i.e. the pseudomanifold stratification of X has the form:

$$X = X_n \supset X_0 \supset X_{-1} = \emptyset.$$

By definition, and because X is compact, X_0 is a finite set of points $\{x_1, \dots, x_s\}$. The $x_1, \dots, x_s$ have disjoint conic neighborhoods $cL_1, \dots, cL_s$ where each L_i is an $(n-1)$ -dimensional compact manifold. Let $T \subset X$ denote the disjoint union of the cL_i . Observe that ∂T is the disjoint union of the L_i .

Let $\bar{p}$ denote a perversity function. To every degree $n-1-\bar{p}(n)$ Moore approximation:

$$f^{\bar{p}} : \partial T_{<n-1-\bar{p}(n)} \rightarrow \partial T$$

in [3] Banagl associates a **$\bar{p}$ -intersection space for X** :

$$I_{f^{\bar{p}}}X := \text{cone} \left(\partial T_{<n-1-\bar{p}(n)} \xrightarrow{f^{\bar{p}}} \partial T \hookrightarrow X - T^\circ \right),$$

the mapping cone on the composition of the Moore approximation with the inclusion into $X - T^\circ$, where T° denotes the interior of T .

Banagl proceeds to establish a duality associated to reduced Betti numbers of intersection spaces:

Theorem ([3] Theorem 2.12). *$\bar{p}$ and $\bar{q}$ denote complementary perversity functions. Suppose X is an n -dimensional closed, oriented, stratified pseudomanifold with isolated singularities. Suppose $I_{f^{\bar{p}}}X$ is a $\bar{p}$ -intersection space for X and $I_{f^{\bar{q}}}X$ is a $\bar{q}$ -intersection space. Then there exist non-canonical duality isomorphisms:*

$$D : \tilde{H}_*(I_{f^{\bar{p}}}X; \mathbb{Q}) \xrightarrow{\sim} \tilde{H}^{n-*}(I_{f^{\bar{q}}}X; \mathbb{Q})$$

A fundamental observation used to to prove this duality (essentially made at the end of Subsection 2.2.1) is that the composition:

$$H_* \left(\partial T_{<n-1-\bar{p}(n)} \right) \xrightarrow{f_*^{\bar{p}}} H_*(\partial T) \rightarrow IH_*^{\bar{p}}(T)$$

is an isomorphism. Banagl moreover proves that the rational (co)homology of an intersection space depends only on the perversity $\bar{p}$ and not on the particular choice of Moore approximation $f^{\bar{p}}$.

If X happens to be Witt, then $\bar{m}$ -intersection spaces are the same as $\bar{n}$ -intersection spaces. In this Witt case, we more simply denote an $\bar{m}$, $\bar{n}$ intersection space by $I_f X$.

2.2.3 Relation to Smooth Deformations of Hypersurfaces

Suppose $f : \mathbb{C}^{n+1} \rightarrow \mathbb{C}$ is a holomorphic function. By a **deformation of f** we mean a holomorphic function:

$$F : \mathbb{C}^{n+1} \times \Delta \rightarrow \mathbb{C}$$

where $\Delta \subset \mathbb{C}$ is a small open disk centered at the origin, and we require $f(-) = F(-, 0)$. For $\lambda \in \Delta$ we let $f_\lambda(-)$ denote $F(-, \lambda)$ so that $f = f_0$. Let $V(-)$ denote the zero set of the function $(-)$. The restriction of the projection $\mathbb{C}^{n+1} \times \Delta \rightarrow \Delta$ to $V(F) \rightarrow \Delta$ is called a **deformation of $V(f)$** . Observe that $V(f_\lambda) = V(f) \times \{\lambda\} \subset V(F)$.

Suppose that $f : \mathbb{C}^{n+2} \rightarrow \mathbb{C}$ is a homogenous polynomial. Then we define a **homogeneous deformation of f** to be a deformation of f such that all the f_λ are homogeneous polynomials. In this case, the $V(f_\lambda)$ determine projective hypersurfaces in $\mathbb{C}P^{n+1}$ and $V(F)$ naturally sits inside of $\mathbb{C}P^{n+1} \times \Delta$. The restriction of the projection $\mathbb{C}P^{n+1} \times \Delta \rightarrow \Delta$ to $V(F) \rightarrow \Delta$ is called a **deformation of (projective) $V(f)$** . Observe that if $U \subset \mathbb{C}P^{n+1}$ is a standard affine open chart, then the restriction $(U \times \Delta) \cap V(F) \rightarrow \Delta$ is a deformation of the affine hypersurface $V(f) \cap U$.

Suppose $X \subset \mathbb{C}P^{n+1}$ is a projective hypersurface and $\pi : \mathcal{X} \rightarrow \Delta$ is a deformation of X . Let $\mathcal{X}_\lambda$ denote the fiber over $\lambda \in \Delta$. Note that $\mathcal{X}_0 = X$. We say that π is a **smooth**

deformation of X if the $\mathcal{X}_\lambda$ are smooth projective varieties for all $\lambda \neq 0$ (in which case $\mathcal{X} - \mathcal{X}_0$ is also smooth) and moreover the restriction $\mathcal{X} - \mathcal{X}_0 \xrightarrow{\pi} \Delta - \{0\}$ is a smooth submersion. In particular, by Ehresmann's fibration theorem, the $\mathcal{X}_\lambda$ for $\lambda \neq 0$ are all diffeomorphic - we call these $\mathcal{X}_\lambda$ **smoothings of** X .

Suppose $f : \mathbb{C}^{n+1} \rightarrow \mathbb{C}$ is a holomorphic function, and let $p \in V(f) \subset \mathbb{C}^{n+1}$. For $\epsilon > 0$ let S_ϵ denote a sphere of radius ϵ centered at p (which has real dimension $2n + 1$) and let K_ϵ denote $S_\epsilon \cap V(f)$. Then (for example see [12, Section 3.1], though the notion was introduced by Milnor in [18]) if $\epsilon > 0$ is sufficiently small, the map:

$$f/|f| : S_\epsilon \setminus K_\epsilon \rightarrow S^1$$

is a smooth fiber bundle over the unit circle called the **Milnor fibration of** f **at** p and whose fiber F is called the **Milnor fiber of** f **at** p (the fibration is invariant of the choice of sufficiently small $\epsilon > 0$). As with all fiber bundles over the circle, the fiber F has an associated **monodromy homeomorphism** $h : F \rightarrow F$ that is well-defined up to isotopy.

Suppose $V(f) \subset \mathbb{C}P^{n+1}$ is a projective hypersurface and $p \in V(f) \subset \mathbb{C}P^{n+1}$. The point p belongs to some standard affine chart $U \subset \mathbb{C}P^{n+1}$ and f determines an algebraic function $f_U : U \rightarrow \mathbb{C}$ such that $p \in V(f_U)$. By the **Milnor fiber of** p **in** $V(f)$ we mean the Milnor fiber of f_U at p .

We can now state the following theorem about smooth deformations and intersection spaces (here we assume lower-middle or upper-middle perversity because projective complex varieties are Witt):

Theorem ([8] Theorem 4.1). *Suppose $X \subset \mathbb{C}P^{n+1}$ is a projective hypersurface with a single (isolated) singularity p . Suppose $I_f X$ is an intersection space for X . If $\mathcal{X}_\lambda$ is a*

smoothing of X , then:

(i) $H_*(I_f X; \mathbb{Q}) \cong H_*(\mathcal{X}_\lambda; \mathbb{Q})$ for $*$ $\neq n, 2n$.

(ii) $H_n(I_f X; \mathbb{Q}) \cong H_n(\mathcal{X}_\lambda; \mathbb{Q})$ if and only if $h_* : H_n(F; \mathbb{Q}) \rightarrow H_n(F; \mathbb{Q})$ is the identity, where F is the Milnor fiber of p in X and h is the associated monodromy.

If $h_* \neq \text{id}$, there is still a relationship between $H_n(I_f X; \mathbb{Q})$ and $H_n(\mathcal{X}_\lambda; \mathbb{Q})$: their dimensions differ by the rank of the linear map $h_* - \text{id}$ (see [8, Theorem 4.5]). Examples of isolated hypersurface singularities for which $h_* = \text{id}$ include all nodal singularities for n odd.

2.3 Intersection Spaces in the Depth One Case

The essential ingredient to an intersection space in the depth one case is the selection of fiberwise truncations for the boundaries of tubular neighborhoods of the singular sets, so we begin there.

2.3.1 Fiberwise Truncations

Suppose $\pi : E \rightarrow B$ is a topological fiber bundle with fiber L and topological structure group G , defined for example in [11, Definition 4.2]. In particular, G acts on the left on L . If G acts on the left on another topological space F , then (E, π) determines an **associated bundle with fiber F** equipped with the same base space [11, Section 4.4]. Intuitively, the associated bundle is obtained by replacing fiber L with F , but preserving the nature of the transition functions (which are determined by elements of

G). A G -equivariant continuous map $F \rightarrow L$ determines a morphism of bundles from the associated bundle with fiber F into (E, π) .

Suppose now that $f_{<k} : L_{<k} \rightarrow L$ is a **degree k G -equivariant Moore approximation**, i.e. a Moore approximation in which $L_{<k}$ admits a left G -action such that $f_{<k}$ is G -equivariant. The associated bundle with fiber $L_{<k}$ is denoted by $(\text{ft}_{<k}E, \pi_{<k})$ and the morphism of bundles induced by $f_{<k}$ is denoted by $F_{<k} : \text{ft}_{<k}E \rightarrow E$. The data $(\text{ft}_{<k}E, \pi_{<k}, F_{<k})$ is called a **degree k fiberwise truncation of (E, π)** , originally defined by Banagl-Chriestenson in [7, Definition 6.1].

The (closed) cone cL is naturally equipped with a left G -action. The associated bundle with fiber cL is denoted by $(DE, D\pi)$ and called the **associated cone bundle**. There are natural inclusions $E \hookrightarrow DE$ and $B \hookrightarrow DE$. If (E, π) is a fiber bundle of manifolds for which the fiber L is a closed manifold of strictly positive dimension, then DE is naturally a stratified pseudomanifold with boundary E and stratification:

$$DE = X_n \supset X_{n-c} = B \supset X_{-1} = \emptyset$$

where $n = \dim E + 1$ and $c = \dim L + 1$.

Fiberwise truncations relate to intersection homology in the following way:

Theorem ([7] Proposition 6.5). $\bar{p}$ denotes a perversity function. Suppose $\pi : E \rightarrow B$ is a fiber bundle of closed manifolds with closed manifold fiber L of strictly positive dimension. Set $n = \dim E + 1$ and $c = \dim L + 1$. Let DE denote the associated cone bundle. If $f^{\bar{p}} : \text{ft}_{<c-1-\bar{p}(c)}E \rightarrow E$ is a degree $c - 1 - \bar{p}(c)$ fiberwise truncation, then the composition:

$$H_*(\text{ft}_{<c-1-\bar{p}(c)}E) \xrightarrow{f^{\bar{p}}_*} H_*(E) \rightarrow IH_*^{\bar{p}}(DE)$$

is an isomorphism.

As a side note, this theorem presents an obstruction to what bundles admit fiberwise truncations of a given degree: for the presented composition to be an isomorphism, in particular $H_*(E) \rightarrow IH_*^{\bar{p}}(DE)$ needs to be surjective, which is not true in general.

2.3.2 Intersection Space

Suppose X is an n -dimensional stratified pseudomanifold (with or without boundary).

We say that X has **depth one** if its stratification has the form:

$$X = X_n \supset X_{n-c} \supset X_{-1} = \emptyset.$$

where X_{n-c} is a nonempty manifold of codimension $c \geq 2$, which we will typically denote by Σ and call the **singular set of X** .

For example, a stratified pseudomanifold X with (a nonempty set of) isolated singularities is **depth one**, with $\Sigma = X_0$. If DE is the cone bundle associated to a fiber bundle $\pi : E \rightarrow B$ of manifolds whose fiber is a closed manifold of strictly positive dimension, then DE has the structure of a depth one stratified pseudomanifold with $\Sigma = B$.

In order to use fiberwise truncations, we need a bundle structure. If X has depth one, then every point in Σ has a neighborhood of the form $\mathbb{R}^{n-c} \times cL$ (where L is some closed manifold of dimension $c-1$) that naturally projects to $\mathbb{R}^{n-c} \times \{0\} \subset \Sigma$. However, in the general class of depth one stratified pseudomanifolds, we may not be able to glue these projections into a global fiber bundle with base Σ .

To obtain such a bundle structure, we restrict ourselves to the class of depth one **Thom-Mather stratified pseudomanifolds** (see [17] for details). Thom-Mather

manifolds are in particular smoothly stratified, i.e. their strata are smooth manifolds.

Specifically, suppose X is a depth one Thom-Mather stratified pseudomanifold whose singular set Σ is closed and connected. Then Σ admits a ***closed tubular neighborhood*** T (really a family of tubular neighborhoods) equipped with a retraction $\pi : T \rightarrow \Sigma$ such that:

- ∂T is a smoothly embedded closed submanifold of $X - \Sigma$.
- the restriction $\pi|_{\partial T} : \partial T \rightarrow \Sigma$ is a smooth fiber bundle of manifolds with some closed manifold fiber L of strictly positive dimension.
- (T, π) is isomorphic to the cone bundle associated to $(\partial T, \pi|_{\partial T})$.

See [7, Proposition 8.2] for details.

With notation and assumptions as above, let $\bar{p}$ denote a perversity function. Let $n = \dim X$ and c be the codimension of Σ . To every degree $c - 1 - \bar{p}(c)$ fiberwise truncation:

$$f^{\bar{p}} : \partial T_{<c-1-\bar{p}(c)} \rightarrow \partial T$$

in [7] Banagl associates a ***$\bar{p}$ -intersection space for X*** :

$$I_{f^{\bar{p}}} X := \text{cone} \left(\partial T_{<c-1-\bar{p}(c)} \xrightarrow{f^{\bar{p}}} \partial T \hookrightarrow X - T^\circ \right),$$

the mapping cone on the composition of the fiberwise truncation with the inclusion into $X - T^\circ$, where T° denotes the interior of T . This is a generalization of the isolated singularities intersection case (they agree when both constructions apply).

Unlike in the case of isolated singularities, in the depth one case there can arise ***local duality obstructions*** associated to fiberwise truncations that, if they do not ***vanish***,

can preclude intersection spaces associated to closed, oriented pseudomanifolds from satisfying duality. We detail a more general version of these obstructions in Section 4.4. If the duality obstructions do vanish, then we have a duality:

Theorem ([7] Theorem 9.5). *$\bar{p}$ and $\bar{q}$ denote complementary perversity functions. Suppose X is an n -dimensional closed, oriented, depth one, Thom-Mather stratified pseudo-manifold with connected singular set. Suppose $I_{f\bar{p}}X$ is a $\bar{p}$ -intersection space for X and $I_{f\bar{q}}X$ is a $\bar{q}$ -intersection space. If the local duality obstructions associated to $f\bar{p}$ and $f\bar{q}$ vanish, then there exist non-canonical duality isomorphisms:*

$$D : \tilde{H}_*(I_{f\bar{p}}X; \mathbb{Q}) \xrightarrow{\sim} \tilde{H}^{n-*}(I_{f\bar{q}}X; \mathbb{Q})$$

If X happens to be Witt, then $\bar{m}$ -intersection spaces are the same as $\bar{n}$ -intersection spaces (recall we are referring to the lower middle and upper middle perversity functions). In this Witt case, we more simply denote an $\bar{m}$, $\bar{n}$ intersection space by I_fX . Moreover, we can associate a signature to I_fX :

Theorem ([7]). *Suppose X is a closed, oriented, depth one, Thom-Mather stratified Witt space of dimension $2n$, n even, with connected singular set. Suppose I_fX is an intersection space for X and that the local duality obstructions associated to f vanish. Then (associated to an appropriate choice of duality isomorphism) there is a symmetric bilinear pairing:*

$$H_n(I_fX; \mathbb{Q}) \otimes H_n(I_fX; \mathbb{Q}) \rightarrow \mathbb{Q}$$

whose signature equals the Novikov signature of the manifold with boundary $(X - T^\circ, \partial T)$, where T is a tubular neighborhood of the singular set.

2.4 Motivation for Algebraic Intersection Spaces

The following is an informal discussion motivating the *algebraic intersection spaces* to be defined in this document.

In the constructions we've described of intersection spaces $I_{f^{\bar{p}}}X$ associated to a space X , an essential ingredient was the selection of topological spaces $A^{\bar{p}}$ and maps $f^{\bar{p}} : A^{\bar{p}} \rightarrow \partial T$ (where ∂T is the boundary of a tubular neighborhood of the singular set of X) such that the composition:

$$H_*(A^{\bar{p}}) \xrightarrow{f^{\bar{p}}_*} H_*(\partial T) \rightarrow IH_*^{\bar{p}}(T)$$

is an isomorphism. In the isolated singularities case, the map $f^{\bar{p}}$ was an appropriate Moore approximation, and in the depth one case, an appropriate fiberwise truncation. A weakness to the theory is that it requires $H_*(\partial T) \rightarrow IH_*^{\bar{p}}(T)$ be surjective, which is not generally true (and less often the more intricate the singular set becomes). This seems to obstruct an entirely general intersection space construction.

We circumvent this obstruction by working with the *image* of this map, rather than $IH_*^{\bar{p}}(T)$ itself. At the same time, we abstract the relevant notions to a *chain complex* level (though being sure to define topological analogs along the way). By doing so, we retain the Betti numbers of the intersection space, which may or may not have physical relevance (as suggested by the relationship to smooth deformations in the case of isolated singularities). There are certain features of intersection spaces that we expect to reappear in our *algebraic intersection spaces*. Namely, the existence of local duality obstructions, the presence of global duality when these obstructions vanish, and, for Witt spaces, a symmetric pairing whose signature is related to the Novikov signature of $(X - T^\circ, \partial T)$. We will see these features arise as we progress through the paper.

Chapter 3

Preliminary Results

3.1 Triangulated Categories

The set of tools consisting of the language and results of triangulated categories will streamline a number of the arguments made throughout this paper. We draw from [21] throughout this section.

Lemma 3.1.1. *Suppose R is a commutative, unital ring and:*

$$A_{\bullet} \xrightarrow{f_{\bullet}} B_{\bullet} \xrightarrow{g_{\bullet}} C_{\bullet} \xrightarrow{-1}$$

is an exact triangle of chain complexes of R -modules. For $i \in \mathbb{Z}$ set:

$$\begin{aligned} Z_i &= \operatorname{im} \left[H_i(A_{\bullet}) \xrightarrow{f_*} H_i(B_{\bullet}) \right] \\ Y_i &= \operatorname{coker} \left[H_i(B_{\bullet}) \xrightarrow{g_*} H_i(C_{\bullet}) \right]. \end{aligned}$$

Interpret $H_{\bullet}(A_{\bullet})$, $Z_{\bullet}$, and $Y_{\bullet}$ as chain complexes with zero differential. Then there is an exact triangle:

$$H_{\bullet}(A_{\bullet}) \rightarrow Z_{\bullet} \rightarrow Y_{\bullet} \xrightarrow{-1}$$

whose maps are those induced by the maps of the long exact sequence in homology associated to the given exact triangle.

Proof. The given exact triangle induces a long exact sequence in homology, which provides us the natural identification:

$$Y_i = \ker \left[H_{i-1}(A_\bullet) \xrightarrow{f_*} H_{i-1}(B_\bullet) \right].$$

So there is a short exact sequence of chain complexes:

$$0 \rightarrow Y_{\bullet+1} \rightarrow H_\bullet(A_\bullet) \rightarrow Z_\bullet \rightarrow 0.$$

By [21] Example 10.4.9, this induces an exact triangle:

$$Y_{\bullet+1} \rightarrow H_\bullet(A_\bullet) \rightarrow Z_\bullet \xrightarrow{-1}.$$

By the second axiom of triangulated categories, stated in [21], this new triangle induces by translation another exact triangle:

$$H_\bullet(A_\bullet) \rightarrow Z_\bullet \rightarrow Y_\bullet \xrightarrow{-1}.$$

□

A similar statement for cochain complexes is given below.

Lemma 3.1.2. *Suppose R is a commutative, unital ring and:*

$$C^\bullet \xrightarrow{f_\bullet} B^\bullet \xrightarrow{g_\bullet} A^\bullet \xrightarrow{+1}$$

is an exact triangle of cochain complexes of R -modules. For $i \in \mathbb{Z}$ set:

$$Z^i = \text{coim} \left[H^i(B^\bullet) \xrightarrow{g_*} H^i(A^\bullet) \right]$$

$$Y^i = \ker \left[H^i(C^\bullet) \xrightarrow{f_*} H^i(B^\bullet) \right].$$

Interpret $H^\bullet(A^\bullet)$, $Z^\bullet$, and $Y^\bullet$ as cochain complexes with zero differential. Then there is an exact triangle:

$$Y^\bullet \rightarrow Z^\bullet \rightarrow H^\bullet(A^\bullet) \xrightarrow{+1}$$

whose maps are those induced by the maps of the long exact sequence in cohomology associated to the given exact triangle.

Proof. The first isomorphism theorem gives a natural identification:

$$Z^i = \operatorname{im} \left[H^i(B^\bullet) \xrightarrow{g_*} H^i(A^\bullet) \right].$$

The long exact sequence in cohomology of the given exact triangle provides further identifications:

$$\begin{aligned} Z^i &= \ker \left[H^i(A^\bullet) \rightarrow H^{i+1}(C^\bullet) \right] \\ Y^i &= \operatorname{im} \left[H^{i-1}(A^\bullet) \rightarrow H^i(C^\bullet) \right]. \end{aligned}$$

So there is a short exact sequence of cochain complexes:

$$0 \rightarrow Z^\bullet \rightarrow H^\bullet(A^\bullet) \rightarrow Y^{\bullet+1} \rightarrow 0.$$

Proceed with the same argument given in the proof of Lemma 3.1.1 to obtain the desired exact triangle. $\square$

3.2 Tubular Neighborhoods of Singular Sets

Throughout this section, let $\mathcal{C}$ denote the category of real subanalytic sets (we refer to [20] for a description of this category and its properties). Let X denote a compact subvariety of a real analytic manifold and let $\Sigma \subset X$ denote its singular set.

Lemma 3.2.1. *There exists a $\mathcal{C}$ -map $f : X \rightarrow \mathbb{R}_{\geq 0}$ such that $f^{-1}0 = \Sigma$.*

Proof. Let M be the real analytic manifold containing X . [20] D.19 provides a $\mathcal{C}$ -map $M \rightarrow \mathbb{R}$ with zero set Σ . Square this map and restrict it to X to obtain the desired $\mathcal{C}$ -map. This argument does not require that X be compact. $\square$

Suppose f is a map satisfying the conditions of Lemma 3.2.1. Because X is compact, f is proper and therefore can be smoothly Whitney stratified into subanalytic sets (see [20] 1.19 and the following remark). Let $\mathcal{S}$ denote such a stratification of X and $\mathcal{S}'$ the stratification of $\mathbb{R}_{\geq 0}$. Because 0-dimensional subanalytic sets are discrete (see [20] 1.15) there exists a minimal $\epsilon_0 > 0$ for which $\{\epsilon_0\} \in \mathcal{S}'$.

A triple $\xi = (f, \mathcal{S}, \epsilon_0)$ as in the previous paragraph is called **global tubular data** for the singular set of X . Given such data and $0 < \epsilon < \epsilon_0$ we let $T = T(\epsilon)$ denote $f^{-1}[0, \epsilon]$ and call it a **(closed) tubular neighborhood** of Σ in X associated to ξ . T , its boundary ∂T , and its interior T° will always be equipped with Whitney stratifications induced by $\mathcal{S}$ (see [16] I.1.3.1).

Lemma 3.2.2. *Let ξ be tubular data. Then:*

- (i) *the stratified homeomorphism type of $T(\epsilon)$ does not depend on ϵ .*
- (ii) *the inclusion $\partial T(\epsilon) \hookrightarrow T(\epsilon) - \Sigma$ is a codimension preserving stratified homotopy equivalence.*
- (iii) *for $0 < \epsilon' < \epsilon < \epsilon_0$ the inclusions $T(\epsilon')^\circ \hookrightarrow T(\epsilon)^\circ$ and $T(\epsilon')^\circ - \Sigma \hookrightarrow T(\epsilon)^\circ - \Sigma$ are stratified homotopy equivalences.*

Proof. By Thom's isotopy lemma [12] Theorem 1.3.5 and contractibility of the range, the restriction $f^{-1}(0, \epsilon_0) \rightarrow (0, \epsilon_0)$ is a trivial stratified fiber bundle. We can use this

trivialization to construct a stratified homeomorphism $T(\epsilon') \cong T(\epsilon)$ for any $0 < \epsilon' < \epsilon < \epsilon_0$ and to show that the inclusions in question are codimension preserving stratified homotopy equivalences (see [13] Definition 2.9.10). $\square$

We say that tubular data ξ is *pseudomanifold compatible (pc)* if and only if $\mathcal{S} \in \xi$ induces on X the structure of a pseudomanifold. In other words: the top dimensional strata of $\mathcal{S}$ are dense and the remaining strata have codimension at least two.

Remark. Tubular data always exists. Pc tubular data ξ exists for example if X is complex and equidimensional. If ξ is pc, then T inherits the structure of a pseudomanifold with boundary, since ∂T admits a collar neighborhood (see [13] Definition 2.7.1) by Thom's isotopy lemma.

3.3 Intersection Homology and Cohomology

Let X denote a compact subvariety of a real analytic manifold. Assume X admits pc tubular data (e.g. X is complex and equidimensional). All tubular data in this section is assumed to be pc.

We implicitly draw from [13] throughout this section. For basic definitions see for example [13] Definition 3.1.4, Remark 3.1.5, and Definition 3.4.1.

We begin by proving that the intersection homologies/cohomologies of a tubular neighborhood T are independent of choices.

Lemma 3.3.1 (Invariance). *The intersection homologies and cohomologies of the tubular neighborhood T , its boundary ∂T , and the pair $(T, \partial T)$ do not depend on any choices.*

Proof. By [13] Corollary 4.1.11 codimension preserving stratified homotopy equivalences preserve intersection homologies and cohomologies. Moreover, the intersection homologies of a pseudomanifold with boundary are naturally isomorphic to those of the pseudomanifold minus the boundary. Together with Lemma 3.2.2 (ii) we then reduce to checking invariance for T° , $T^\circ - \Sigma$, and the pair they form. By Lemma 3.2.2 (iii) the choice of ϵ for fixed ξ doesn't affect these homologies and cohomologies.

Let ξ and $\tilde{\xi}$ be pc tubular data for X . Because X is compact, any tube for one data contains a smaller tube for the other data. In particular we can construct a sequence of tubes:

$$T_0 \subset \tilde{T} \subset T \subset \tilde{T}_0$$

corresponding in an alternating fashion to ξ and $\tilde{\xi}$. We may restrict these inclusions to the open tubes $(-)^\circ$ or to the open tubes minus the singular set $(-)^\circ - \Sigma$. Any composition of two of these restricted inclusions by Lemma 3.2.2 (iii) induces isomorphisms on intersection homologies and cohomologies. A simple argument shows that the same is true for the central inclusions, namely in the commutative diagram:

$$\begin{array}{ccc} T^\circ - \Sigma & \hookrightarrow & T^\circ \\ \uparrow & & \uparrow \\ \tilde{T}^\circ - \Sigma & \hookrightarrow & \tilde{T}^\circ \end{array}$$

the vertical maps induces isomorphisms on intersection homologies and cohomologies. From naturality of long exact sequences of pairs and the five lemma, we obtain also an isomorphism for intersection homologies and cohomologies of the pair. We have successfully compared the tubular data ξ and $\tilde{\xi}$. $\square$

Remark. Our definition of tubular neighborhood is actually unnecessarily restrictive. If $T(\epsilon)$ for $0 < \epsilon < \epsilon_0$ is any increasing family of closed neighborhoods of Σ such that:

$$(i) \bigcap_{\epsilon > 0} T(\epsilon) = \Sigma$$

(ii) the $T(\epsilon)$ are pseudomanifolds whose boundary $\partial T(\epsilon)$ is a submanifold of $X - \Sigma$

(iii) the $T(\epsilon)$ satisfy the conditions of Lemma 3.2.2

then such a $T(\epsilon)$ will do equally well, and will not affect the validity of Lemma 3.3.1.

We will use such a tubular neighborhood in Section 6.

The duality captured by any intersection space construction is inseparable from a Lefschetz duality described by the results of intersection homology and cohomology. To this end we will need orientability. If X as a pseudomanifold is oriented, then a tubular neighborhood T inherits an orientation from X .

Lemma 3.3.2. *Let $(\bar{p}, \bar{q})$ be complementary perversity functions, k a field, and T a tubular neighborhood of Σ . Then there is an exact triangle of (intersection) chain complexes:*

$$C_{\bullet}(\partial T; k) \rightarrow IC_{\bullet}^{\bar{p}}(T; k) \rightarrow IC_{\bullet}^{\bar{p}}(T, \partial T; k) \xrightarrow{-1}$$

and of (intersection) cochain complexes:

$$IC_q^{\bullet}(T, \partial T; k) \rightarrow IC_q^{\bullet}(T; k) \rightarrow C^{\bullet}(\partial T; k) \xrightarrow{+1}.$$

If X is oriented of dimension n , then there is a natural duality isomorphism between their (shifted) associated long exact sequences:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & H_i(\partial T; k) & \longrightarrow & IH_i^{\bar{p}}(T; k) & \longrightarrow & IH_i^{\bar{p}}(T, \partial T; k) \longrightarrow \cdots \\ & & \downarrow \mathfrak{z}|_D & & \downarrow \mathfrak{z}|_D & & \downarrow \mathfrak{z}|_D \\ \cdots & \longrightarrow & H^{n-i-1}(\partial T; k) & \longrightarrow & IH_q^{n-i}(T, \partial T; k) & \longrightarrow & IH_q^{n-i}(T; k) \longrightarrow \cdots \end{array}$$

Proof. Because ∂T is non-singular (recall the definition of tubular neighborhoods) there are quasi-isomorphisms $IC_{\bullet}^{\bar{p}}(\partial T; k) \simeq C_{\bullet}(\partial T; k)$ and $IC_q^{\bullet}(\partial T; k) \simeq C^{\bullet}(\partial T; k)$. We may

therefore replace the former with the latter when we only care about complexes up to quasi-isomorphism.

By the definition of the relative intersection chain complex given in [13] Definition 4.3.7, there is a short exact sequence of chain complexes:

$$0 \rightarrow IC_{\bullet}^{\bar{p}}(\partial T; k) \rightarrow IC_{\bullet}^{\bar{p}}(T; k) \rightarrow IC_{\bullet}^{\bar{p}}(T, \partial T; k) \rightarrow 0.$$

This (together with the first paragraph) produces the first distinguished triangle. The second is obtained analogously. The duality isomorphism between their shifted long exact sequences is described in the proof of [13] Corollary 8.3.10.

□

3.4 Linear Algebra

We will exclusively use field coefficients for the main results of the paper. To this end, we let k denote a field and establish a few lemmas.

Lemma 3.4.1. *Suppose $f : A \rightarrow B$ is a morphism of k -vector spaces with dual map $f^* : B^* \rightarrow A^*$. Then there are natural identifications:*

$$(\operatorname{coker} f)^* = \ker(f^*)$$

$$(\operatorname{im} f)^* = \operatorname{coim}(f^*).$$

Proof. We dualize the exact sequence:

$$A \xrightarrow{f} B \rightarrow \operatorname{coker} f \rightarrow 0$$

to obtain exact:

$$0 \rightarrow (\operatorname{coker} f)^* \rightarrow B^* \xrightarrow{f^*} A^*,$$

which proves the first identification by showing that $(\operatorname{coker} f)^*$ maps isomorphically onto $\ker(f^*)$. We prove the second identification by dualizing the exact sequence:

$$0 \rightarrow \operatorname{im} f \rightarrow B \rightarrow \operatorname{coker} f \rightarrow 0$$

to obtain exact:

$$0 \rightarrow (\operatorname{coker} f)^* \rightarrow B^* \rightarrow (\operatorname{im} f)^* \rightarrow 0$$

and utilizing the first identification $(\operatorname{coker} f)^* = \ker(f^*)$. $\square$

Lemma 3.4.2. *Suppose there is a commutative diagram of exact sequences of k -vector spaces:*

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & C_{i+1} & \xrightarrow{\partial_{i+1}} & A_i & \xrightarrow{g_i} & B_i & \xrightarrow{h_i} & C_i & \xrightarrow{\partial_i} & A_{i-1} & \longrightarrow & \cdots \\ & & \uparrow D''_{i+1} & \uparrow \mathbb{I} & \uparrow D'_i & \uparrow \mathbb{I} & & \uparrow D''_i & \uparrow \mathbb{I} & \uparrow D'_{i-1} & \uparrow \mathbb{I} & & \\ \cdots & \longrightarrow & F_{i+1} & \xrightarrow{\delta_{i+1}} & D_i & \xrightarrow{u_i} & E_i & \xrightarrow{v_i} & F_i & \xrightarrow{\delta_i} & D_{i-1} & \longrightarrow & \cdots \end{array}$$

Then for each pair $(r_\bullet, s_\bullet)$, where $r_\bullet : \operatorname{im} h_\bullet \rightarrow B_\bullet$ and $s_\bullet : \operatorname{im} v_\bullet \rightarrow E_\bullet$ are (families) of sections, there exists an induced isomorphism:

$$D_\bullet = D_\bullet^{(r_\bullet, s_\bullet)} : E_\bullet \xrightarrow{\cong} B_\bullet$$

whose description is found in the proof.

Proof. This is the content of [3, Lemma 2.46]. We recreate the argument here, because it is important in Chapter 7 to understand exactly how $D_\bullet^{(r_\bullet, s_\bullet)}$ relates to the choice of $(r_\bullet, s_\bullet)$.

Because the diagram commutes and the rows are exact, there is an induced diagram

of exact sequences:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \overbrace{\text{coker } \partial_{i+1}}^{\simeq \ker h_i} & \xrightarrow{\bar{g}_i} & B_i & \xrightarrow{h_i} & \overbrace{\ker \partial_i}^{\text{im } h_i} \longrightarrow 0 \\
& & \uparrow D'_i \wr & & & & \uparrow D''_i \wr \\
0 & \longrightarrow & \underbrace{\text{coker } \delta_{i+1}}_{\simeq \ker k_i} & \xrightarrow{\bar{u}_i} & E_i & \xrightarrow{v_i} & \underbrace{\ker \delta_i}_{\text{im } v_i} \longrightarrow 0
\end{array}$$

where we are abusing notation by allowing D'_i and D''_i to denote induced isomorphisms.

The sections r_i and s_i give splittings

$$B_i = \text{im } \bar{g}_i \oplus \text{im } r_i = \text{im } g_i \oplus \text{im } r_i$$

$$E_i = \text{im } \bar{u}_i \oplus \text{im } s_i = \text{im } u_i \oplus \text{im } s_i.$$

The isomorphism $D_i : E_i \rightarrow B_i$ induced by the splittings $(r_\bullet, s_\bullet)$ is described on components as follows.

$$D_i(u_i \alpha) = g_i D'_i(\alpha), \quad \alpha \in D_i$$

$$D_i(s_i \beta) = r_i D''_i(\beta), \quad \beta \in \text{im } v_i \subset F_i$$

□

Chapter 4

Local Construction

4.1 Denotations and Assumptions

Throughout Section 4 we let k denote a field and X a compact subvariety of a real analytic manifold with singular set Σ . Assume X admits pc tubular data and is oriented of dimension n (e.g. X is complex and equidimensional). Let T denote a pc tubular neighborhood of Σ .

If $\bar{p}$ is a perversity function, for $i \in \mathbb{Z}$ we define:

$$\begin{aligned} Z_i^{\bar{p}} &= \text{im} [H_i(\partial T; k) \rightarrow IH_i^{\bar{p}}(T; k)], \quad Z_{\bar{p}}^i = (Z_i^{\bar{p}})^* \\ Y_i^{\bar{p}} &= \text{coker} [IH_i^{\bar{p}}(T; k) \rightarrow IH_i^{\bar{p}}(T, \partial T; k)], \quad Y_{\bar{p}}^i = (Y_i^{\bar{p}})^*. \end{aligned}$$

We also write $Z_{\bullet}^{\bar{p}}$ and $Y_{\bullet}^{\bar{p}}$ (resp. $Z_{\bar{p}}^{\bullet}$ and $Y_{\bar{p}}^{\bullet}$) if we'd like to interpret these collections of vector spaces as chain (resp. cochain) complexes with zero differential.

4.2 Duality and the Image of the Boundary

Let $\bar{p}$ denote a perversity function. To begin, we'd like to overcome the obstruction discussed in Section 2.4. It will be essential to work with the *image* of $H_*(\partial T; k) \rightarrow IH_*^{\bar{p}}(T; k)$ as opposed to $IH_*^{\bar{p}}(T; k)$ itself, the latter being more in line with the original approach. We've already denoted this collection of vector spaces by $Z_*^{\bar{p}}$. The first step

in this transition is to understand the “Lefschetz dual” object to $Z_*^{\bar{p}}$, in the sense of Theorem 3.3.2.

Lemma 4.2.1. *For all $i \in \mathbb{Z}$ there are natural identifications:*

$$\begin{aligned} Z_{\bar{p}}^i &= \text{coim} [IH_{\bar{p}}^i(T; k) \rightarrow H^i(\partial T; k)] \\ Y_{\bar{p}}^i &= \ker [IH_{\bar{p}}^i(T, \partial T; k) \rightarrow IH_{\bar{p}}^i(T; k)] \end{aligned}$$

Proof. If we apply universal coefficients (see [13] Theorem 7.1.4 for the intersection cohomology version of universal coefficients) and the second identification of Lemma 3.4.1 to the map $H_i(\partial T; k) \rightarrow IH_i^{\bar{p}}(T; k)$ then we obtain the identification:

$$Z_{\bar{p}}^i = \text{coim} [IH_{\bar{p}}^i(T; k) \rightarrow H^i(\partial T; k)] .$$

If we next apply universal coefficients and the first identification of Lemma 3.4.1 to the map $IH_i^{\bar{p}}(T; k) \rightarrow IH_i^{\bar{p}}(T, \partial T; k)$ then we obtain the identification:

$$Y_{\bar{p}}^i = \ker [IH_{\bar{p}}^i(T, \partial T; k) \rightarrow IH_{\bar{p}}^i(T; k)] .$$

□

Lemma 4.2.2. *Suppose $(\bar{p}, \bar{q})$ are complementary perversity functions. There exists an exact triangle of chain complexes with zero differential:*

$$H_{\bullet}(\partial T; k) \rightarrow Z_{\bullet}^{\bar{p}} \rightarrow Y_{\bullet}^{\bar{p}} \xrightarrow{-1}$$

and an exact triangle of cochain complexes with zero differential:

$$Y_{\bar{q}}^{\bullet} \rightarrow Z_{\bar{q}}^{\bullet} \rightarrow H^{\bullet}(\partial T; k) \xrightarrow{+1} .$$

Moreover there is a natural duality isomorphism between their (shifted) long exact sequences:

$$\begin{array}{ccccccc}
 \cdots & \longrightarrow & H_i(\partial T; k) & \longrightarrow & Z_i^{\bar{p}} & \longrightarrow & Y_i^{\bar{p}} \longrightarrow \cdots \\
 & & \wr \downarrow D & & \wr \downarrow D & & \wr \downarrow D \\
 \cdots & \longrightarrow & H^{n-1-i}(\partial T; k) & \longrightarrow & Y_{\bar{q}}^{n-i} & \longrightarrow & Z_{\bar{q}}^{n-i} \longrightarrow \cdots
 \end{array}$$

Proof. Existence of the first exact triangle is a direct consequence of Lemma 3.1.1 and Theorem 3.3.2. Existence of the second exact triangle is a consequence of Lemma 3.1.2, Theorem 3.3.2, and Lemma 4.2.1.

For the isomorphism of long exact sequences, we first recall the diagram from Theorem 3.3.2:

$$\begin{array}{ccccccc}
 \cdots & \longrightarrow & H_i(\partial T; R) & \longrightarrow & IH_i^{\bar{p}}(T; R) & \longrightarrow & IH_i^{\bar{p}}(T, \partial T; R) \longrightarrow \cdots \\
 & & \wr \downarrow D & & \wr \downarrow D & & \wr \downarrow D \\
 \cdots & \longrightarrow & H^{n-i-1}(\partial T; R) & \longrightarrow & IH_{\bar{q}}^{n-i}(T, \partial T; R) & \longrightarrow & IH_{\bar{q}}^{n-i}(T; R) \longrightarrow \cdots
 \end{array} .$$

Because *this* diagram is an isomorphism of long exact sequences, appealing to the definition of $Z_i^{\bar{p}}$ we have:

$$\begin{aligned}
 D(Z_i^{\bar{p}}) &= \text{im} [H^{n-1-i}(\partial T; k) \rightarrow IH_{\bar{q}}^{n-i}(T, \partial T; k)] \\
 &= \ker [IH_{\bar{q}}^{n-i}(T, \partial T; k) \rightarrow IH_{\bar{q}}^{n-i}(T; k)] = Y_{\bar{q}}^{n-i}
 \end{aligned}$$

where in the last step we have used Lemma 4.2.1. This provides us the middle isomorphism of the desired diagram.

To construct the rightmost isomorphism of the desired diagram we first observe that:

$$\begin{aligned}
 D(\text{im} [IH_i^{\bar{p}}(T; k) \rightarrow IH_i^{\bar{p}}(T, \partial T; k)]) &= \text{im} [IH_{\bar{q}}^{n-i}(T, \partial T; k) \rightarrow IH_{\bar{q}}^{n-i}(T; k)] \\
 &= \ker [IH_{\bar{q}}^{n-i}(T; k) \rightarrow H^{n-i}(\partial T; k)] .
 \end{aligned}$$

Therefore D induces an isomorphism between $Y_i^{\bar{p}}$ and $Z_{\bar{q}}^{n-i}$ after we make the identification of Lemma 4.2.1.

That these isomorphisms fit into a *commutative* diagram follows from the fact that all our maps are induced from the *already existing* commutative diagram of long exact sequences from which we have been drawing. $\square$

4.3 Local Intersection Approximation

Let $\bar{p}$ denote a perversity function. The results of Section 4.2 are the source of several desired properties for our intersection space construction. Banagl constructs his intersection spaces by first selecting a “local approximation”, local in the sense that it takes as input only the tubular neighborhood of the singular set. We will do the same, but will also loosen some constraints by allowing approximations which are merely “algebraic”, not necessarily topological.

A $\bar{p}$ *algebraic intersection approximation for T with coefficients in k* is a pair $(A_{\bullet}, f_{\bullet})$ where $A_{\bullet}$ is a chain complex of k -vector spaces and:

$$f_{\bullet} : A_{\bullet} \rightarrow C_{\bullet}(\partial T; k)$$

is a chain map such that the composition:

$$H_{\bullet}(A_{\bullet}) \xrightarrow{f_*} H_{\bullet}(\partial T; k) \rightarrow Z_{\bullet}^{\bar{p}}$$

is an isomorphism. A $\bar{p}$ *topological intersection approximation for T with coefficients in k* is a pair (A, f) where A is a topological space and $f : A \rightarrow \partial T$ is a continuous map such that $(C_{\bullet}(A; k), f_{\#})$ is a local $\bar{p}$ algebraic intersection approximation.

Observe that the class of such approximations does not depend on the particular choice of tubular neighborhood (see Lemmas 3.2.2 and 3.3.1 and their proofs). These are extensions of Banagl's "approximations" as indicated by the following examples.

Example 4.3.1. Suppose $\Sigma = \{x\}$ is a single point. Then the tubular neighborhood of Σ is conic: $T = cL$ with cone point x where L is called the link of x . So $\partial T = L$ and the cone formula (see [13, Theorem 4.2.1]) implies:

$$IH_{\bullet}^{\bar{p}}(T; k) = H_{\bullet}^{<n-1-\bar{p}(n)}(L; k)$$

where $H_i^{<n-1-\bar{p}(n)}(L; k)$ agrees with $H_i(L; k)$ for $i < n-1-\bar{p}(n)$ and vanishes otherwise.

The map:

$$H_{\bullet}(L; k) \rightarrow H_{\bullet}^{<n-1-\bar{p}(n)}(L; k)$$

is surjective so that $Z_{\bullet}^{\bar{p}} = H_{\bullet}^{<n-1-\bar{p}(n)}(L; k)$. Consider a Moore approximation (defined in [3]):

$$f : L_{<n-1-\bar{p}(n)} \rightarrow L.$$

By its defining properties, the composition:

$$H_{\bullet}(L_{<n-1-\bar{p}(n)}; \mathbb{Q}) \xrightarrow{f_*} H_{\bullet}(L; \mathbb{Q}) \rightarrow H_{\bullet}^{<n-1-\bar{p}(n)}(L; \mathbb{Q})$$

is an isomorphism. Therefore $(L_{<n-1-\bar{p}(n)}, f)$ is a $\bar{p}$ topological intersection approximation for T with coefficients in $\mathbb{Q}$. //

Example 4.3.2. Suppose X has a Whitney stratification consisting of exactly two strata $\{X - \Sigma, \Sigma\}$ where Σ has codimension c . In particular, this means that Σ is smooth and connected. Suppose also that T is homeomorphic to the mapping cylinder of a fiber

bundle projection $\partial T \rightarrow \Sigma$ (e.g. T is a tubular neighborhood in the Thom-Mather sense). Suppose there exists a fiberwise truncation (defined in [7]):

$$f : ft_{<c-1-\bar{p}(c)}\partial T \rightarrow \partial T.$$

By [7] Proposition 6.5, the composition:

$$H_{\bullet}(ft_{<c-1-\bar{p}(c)}\partial T; \mathbb{Q}) \xrightarrow{f_*} H_{\bullet}(\partial T; \mathbb{Q}) \rightarrow IH_{\bullet}^{\bar{p}}(T; \mathbb{Q})$$

is an isomorphism. Therefore $(ft_{<c-1-\bar{p}(c)}\partial T, f)$ is a $\bar{p}$ topological intersection approximation for T with coefficients in $\mathbb{Q}$. //

Proposition 4.3.3 (Existence). *A $\bar{p}$ algebraic intersection approximation $(A_{\bullet}, f_{\bullet})$ for T with coefficients in k always exists.*

Proof. Pick a section s of the composition:

$$\ker[\partial_{\bullet} : C_{\bullet}(\partial T; k) \rightarrow C_{\bullet-1}(\partial T; k)] \rightarrow H_{\bullet}(\partial T; k) \rightarrow Z_{\bullet}^{\bar{p}}$$

where the first map is the quotient map from cycles to homology classes and the second is the obvious surjection. The composition:

$$Z_{\bullet}^{\bar{p}} \xrightarrow{s} \ker \partial_{\bullet} \hookrightarrow C_{\bullet}(\partial T; k)$$

is an algebraic intersection approximation. □

We next examine the “dual” object to an intersection approximation.

Lemma 4.3.4. *Suppose $(A_{\bullet}, f_{\bullet})$ is a $\bar{p}$ algebraic intersection approximation for T with coefficients in k . Then the composition (where the first map is the boundary map of Lemma 4.2.2):*

$$Y_{\bullet+1}^{\bar{p}} \rightarrow H_{\bullet}(\partial T; k) \rightarrow H_{\bullet}(cf_{\bullet})$$

is an isomorphism, where $cf_{\bullet}$ denotes the algebraic cone on $f_{\bullet}$.

Proof. We will describe three exact triangles, then use the octahedral axiom for triangulated categories (see [21]) to construct a fourth that will imply the theorem.

By definition of “algebraic cone”, we have an exact triangle:

$$A_{\bullet} \xrightarrow{f_*} C_{\bullet}(\partial T; k) \rightarrow cf_{\bullet} \xrightarrow{-1}.$$

A consequence of the definition of algebraic intersection approximation is that the map $f_* : H_{\bullet}(A_{\bullet}) \rightarrow H_{\bullet}(\partial T; k)$ is injective. Applying this to the long exact sequence from the aforementioned exact triangle shows that the map $H_{\bullet}(\partial T; k) \rightarrow H_{\bullet}(cf_{\bullet})$ is surjective and the boundary map $H_{\bullet}(cf_{\bullet}) \rightarrow H_{\bullet-1}(A_{\bullet})$ is the zero map. We next translate this exact triangle to obtain another (with the same long exact sequence in homology):

$$C_{\bullet}(\partial T; k) \rightarrow cf_{\bullet} \rightarrow A_{\bullet-1} \xrightarrow{-1}.$$

We apply Lemma 3.1.1 and our observations about the maps from this exact triangle to find an exact triangle:

$$H_{\bullet}(\partial T; k) \rightarrow H_{\bullet}(cf_{\bullet}) \rightarrow H_{\bullet-1}(A_{\bullet}) \xrightarrow{-1}$$

and translate it back to obtain another exact triangle:

$$H_{\bullet}(A_{\bullet}) \rightarrow H_{\bullet}(\partial T; k) \rightarrow H_{\bullet}(cf_{\bullet}) \xrightarrow{-1}.$$

By definition of an algebraic intersection approximation, there is an isomorphism of chain complexes (with zero differential) $H_{\bullet}(A_{\bullet}) \xrightarrow{\simeq} Z_{\bullet}^{\bar{p}}$. Consequently there is an exact triangle:

$$H_{\bullet}(A_{\bullet}) \rightarrow Z_{\bullet}^{\bar{p}} \rightarrow 0 \xrightarrow{-1}.$$

Lastly, by Lemma 4.2.2 there is an exact triangle:

$$H_{\bullet}(\partial T; k) \rightarrow Z_{\bullet}^{\bar{p}} \rightarrow Y_{\bullet}^{\bar{p}} \xrightarrow{-1}.$$

Altogether we have exact triangles:

$$H_{\bullet}(A_{\bullet}) \rightarrow H_{\bullet}(\partial T; k) \rightarrow H_{\bullet}(cf_{\bullet}) \xrightarrow{-1}$$

$$H_{\bullet}(\partial T; k) \rightarrow Z_{\bullet}^{\bar{p}} \rightarrow Y_{\bullet}^{\bar{p}} \xrightarrow{-1}$$

$$H_{\bullet}(A_{\bullet}) \rightarrow Z_{\bullet}^{\bar{p}} \rightarrow 0 \xrightarrow{-1}.$$

where the first map of the third exact triangle is the composition $H_{\bullet}(A_{\bullet}) \rightarrow H_{\bullet}(\partial T; k) \rightarrow Z_{\bullet}^{\bar{p}}$. This is the setting in which the octahedral axiom is applicable. The resulting exact triangle is:

$$H_{\bullet}(cf_{\bullet}) \rightarrow 0 \rightarrow Y_{\bullet}^{\bar{p}} \xrightarrow{-1}$$

and the boundary map (which must be an isomorphism) is the composition:

$$Y_{\bullet+1}^{\bar{p}} \rightarrow H_{\bullet}(\partial T; k) \rightarrow H_{\bullet}(cf_{\bullet}).$$

□

4.4 Local Duality Isomorphism

Let $(\bar{p}, \bar{q})$ denote complementary perversities. If our intersection spaces are to have a global duality, a local duality must first be understood. Suppose $(A_{\bullet}^{\bar{p}}, f_{\bullet}^{\bar{p}})$ and $(A_{\bullet}^{\bar{q}}, f_{\bullet}^{\bar{q}})$ are $\bar{p}$ and $\bar{q}$ algebraic intersection approximations for T with coefficients in k . We consider the diagram:

$$\begin{array}{ccccc} Z_{\bar{q}}^{n-r-1} & \longrightarrow & H^{n-r-1}(\partial T; k) & \longrightarrow & H^{n-r-1}(A_{\bullet}^{\bar{q}}) \\ D \uparrow & & D \uparrow & & \\ Y_{r+1}^{\bar{p}} & \longrightarrow & H_r(\partial T; k) & \longrightarrow & H_r(cf_{\bullet}^{\bar{p}}). \end{array}$$

The upper composition is by definition an isomorphism and the lower composition is by Lemma 4.3.4 also an isomorphism. So there exists a unique **local duality isomorphism**:

$$D : H_r(cf_{\bullet}^{\bar{p}}) \rightarrow H^{n-r-1}(A_{\bullet}^{\bar{q}})$$

that makes the *outer* box commute. This describes a **local intersection pairing**:

$$(-, -) : H_r^{\bar{p}}(cf_{\bullet}^{\bar{p}}) \times H_{n-r-1}(A_{\bullet}^{\bar{q}}) \rightarrow k, \quad (\alpha, \beta) = D(\alpha)(\beta).$$

We say the ***r*th local duality obstruction for $(A_{\bullet}^{\bar{p}}, f_{\bullet}^{\bar{p}})$, $(A_{\bullet}^{\bar{q}}, f_{\bullet}^{\bar{q}})$ vanishes** if and only if the *entire* diagram:

$$\begin{array}{ccccc} Z_{\bar{q}}^{n-r-1} & \longrightarrow & H^{n-r-1}(\partial T; k) & \longrightarrow & H^{n-r-1}(A_{\bullet}^{\bar{q}}) \\ D \uparrow & & D \uparrow & & D \uparrow \\ Y_{r+1}^{\bar{p}} & \longrightarrow & H_r(\partial T; k) & \longrightarrow & H_r(cf_{\bullet}^{\bar{p}}) \end{array}$$

commutes (this is not necessarily true, because the right box need not commute). The following theorem captures the physical notion that the local duality obstructions will vanish if $\text{im}(f_*)$ contains no pairs of “stably intersecting” cycles.

Proposition 4.4.1. *The ***r*th local duality obstruction for $(A_{\bullet}^{\bar{p}}, f_{\bullet}^{\bar{p}})$, $(A_{\bullet}^{\bar{q}}, f_{\bullet}^{\bar{q}})$ vanishes if and only if given any:***

$$\alpha \in \text{im } f_{\bullet}^{\bar{p}} \subset H_{\bullet}(\partial T; k), \beta \in \text{im } f_{\bullet}^{\bar{q}} \subset H_{\bullet}(\partial T; k) \text{ with } |\alpha| = r, \quad |\beta| = n - r - 1$$

we have the following vanishing of the intersection pairing on ∂T :

$$(\alpha, \beta) = 0.$$

This vanishing occurs for example if α and β are representable by disjoint cycles.

Proof. Fix $r \in \mathbb{Z}$ and consider the commutative diagram whose maps have been named:

$$\begin{array}{ccccc}
 Z_{\bar{q}}^{n-r-1} & \xrightarrow{l} & H^{n-r-1}(\partial T; k) & \xrightarrow{f_{\bar{q}}^*} & H^{n-r-1}(A_{\bullet}^{\bar{q}}) \\
 D \uparrow & & D \uparrow & & \\
 Y_{r+1}^{\bar{p}} & \xrightarrow{u} & H_r(\partial T; k) & \xrightarrow{v} & H_r(cf_{\bullet}^{\bar{p}}).
 \end{array}$$

The r th local duality obstruction vanishes iff:

$$f_{\bar{q}}^* D = (f_{\bar{q}}^* l) D (vu)^{-1} v$$

By commutativity we have equivalences:

$$\begin{aligned}
 [f_{\bar{q}}^* D = (f_{\bar{q}}^* l) D (vu)^{-1} v] &\iff [f_{\bar{q}}^* D = f_{\bar{q}}^* D u (vu)^{-1} v] \\
 &\iff [\forall \alpha \in H_r(\partial T; k), \ f_{\bar{q}}^* D(\alpha - u(vu)^{-1} v \alpha) = 0].
 \end{aligned}$$

Next observe that:

$$\{\alpha - u(vu)^{-1} v \alpha \mid \alpha \in H_r(\partial T; k)\} = \ker v = \text{im } [f_{\bar{q}}^* : H_r(A_{\bullet}^{\bar{p}}) \rightarrow H_r(\partial T; k)];$$

The second equality is a consequence of a long exact sequence. The “ $\subset$ ” part of the first equality can be directly verified. For the “ $\supset$ ” part of this equality, simply observe that if $\alpha \in \ker v$, then:

$$\alpha = \alpha - u(vu)^{-1} v \alpha.$$

Put together we have that the r th local duality obstruction vanishes iff:

$$D(\text{im } [f_{\bar{q}}^* : H_r(A_{\bullet}^{\bar{p}}) \rightarrow H_r(\partial T; k)]) \subset \ker [f_{\bar{q}}^* : H^{n-r-1}(\partial T; k) \rightarrow H^{n-r-1}(A_{\bullet}^{\bar{q}})].$$

This holds iff for all $\alpha \in \text{im } f_{\bar{q}}^*$ with $|\alpha| = r$ and $\beta = f_{\bar{q}}^*(\gamma) \in \text{im } f_{\bar{q}}^*$ with $|\beta| = n - r - 1$ we have:

$$0 = f_{\bar{q}}^* D(\alpha)(\gamma) = D(\alpha)(\beta) = (\alpha, \beta)$$

where we have used the fact that the duality isomorphism induces the intersection pairing on homology. $\square$

Example 4.4.2. Suppose $\Sigma = \{p\}$ consists of a single point with link L . Then:

$$IH_{\bullet}^{\bar{p}}(T; k) = H_{\bullet}^{<n-1-\bar{p}(n)}(L; k), \quad IH_{\bullet}^{\bar{q}}(T; k) = H_{\bullet}^{<n-1-\bar{q}(n)}(L; k)$$

Therefore, given any algebraic intersection approximations $(A_{\bullet}^{\bar{p}}, f_{\bullet}^{\bar{p}}), (A_{\bullet}^{\bar{q}}, f_{\bullet}^{\bar{q}})$ for T with coefficients in k , the subsets $\text{im } f_{\bullet}^{\bar{p}}$ and $\text{im } f_{\bullet}^{\bar{q}}$ of $H_{\bullet}(\partial T; k)$ contain classes of degree strictly less than $n - 1 - \bar{p}(n)$ and $n - 1 - \bar{q}(n)$ respectively. By definition of complementary perversities we have:

$$(n - 1 - \bar{p}(n)) - 1 + (n - 1 - \bar{q}(n)) - 1 = n - 2.$$

Since ∂T is an $(n-1)$ -dimensional manifold, no two of these classes can pair to a nonzero field element. So the local duality obstructions always vanish. //

Example 4.4.3. In this example we show that our local duality obstructions all vanish if and only if those of Banagl-Chriestenson [7] all vanish (when our local intersection approximations are fiberwise truncations). We assume $\mathbb{Q}$ -coefficients. Suppose X has a Whitney stratification consisting of exactly two strata $\{X - \Sigma, \Sigma\}$ where Σ has codimension c . Suppose also that T is homeomorphic to the mapping cylinder of a fiber bundle projection $\partial T \rightarrow \Sigma$, and that there exist fiberwise truncations:

$$f^{\bar{p}} : ft_{<c-1-\bar{p}(c)} \partial T \rightarrow \partial T.$$

$$f^{\bar{q}} : ft_{<c-1-\bar{q}(c)} \partial T \rightarrow \partial T.$$

By Example 4.3.2 these maps are shown to be topological local intersection approximations and:

$$Z_{\bullet}^{\bar{p}} = IH_{\bullet}^{\bar{p}}(T), \quad Z_{\bullet}^{\bar{q}} = IH_{\bullet}^{\bar{q}}(T)$$

from which we also conclude:

$$Y_{\bullet}^{\bar{p}} = IH_{\bullet}^{\bar{p}}(T, \partial T), \quad Y_{\bullet}^{\bar{q}} = IH_{\bullet}^{\bar{q}}(T, \partial T).$$

There are also isomorphisms:

$$H_{\bullet}(cf_{\#}^{\bar{p}}) \cong \tilde{H}_{\bullet}(cf^{\bar{p}}), \quad H_{\bullet}(cf_{\#}^{\bar{q}}) \cong \tilde{H}_{\bullet}(cf^{\bar{q}})$$

where $cf^{\bar{p}}$ and $cf^{\bar{q}}$ are topological mapping cones. Therefore by definition our r th local duality obstruction vanishes if and only if the entire diagram:

$$\begin{array}{ccccc} IH_{\bar{q}}^{n-r-1}(T) & \longrightarrow & H^{n-r-1}(\partial T) & \longrightarrow & H^{n-r-1}(ft_{<c-1-\bar{q}(c)}\partial T) \\ D \uparrow & & D \uparrow & & D \uparrow \\ IH_{r+1}^{\bar{p}}(T, \partial T) & \longrightarrow & H_r(\partial T) & \longrightarrow & \tilde{H}_r(cf^{\bar{p}}) \end{array}$$

commutes. Compare this to the diagram appearing in [7, Proposition 6.10]: up to labelling and the direction of duality isomorphisms, it is the same, since their $Q_{\geq c-1-\bar{p}(c)}$ is homotopy equivalent to our $cf^{\bar{p}}$ by [7, Equation (6.4)]. Banagl-Chriestenson prove that their local duality obstructions vanish if and only if the above diagram commutes for all r . In other words: the local duality obstructions of Banagl-Chriestenson all vanish if and only if our local duality obstructions all vanish (where both are associated to a fixed complementary pair of fiberwise truncations). //

Remark. If X is a Witt space (see [13] Definition 9.1.2 and Proposition 9.1.8) then so is T . In this case, by definition, the approximations for T for the lower $\bar{m}$ and upper $\bar{n}$ middle perversities would be indistinguishable. So for X Witt, an $\bar{m}$ approximation $(A_{\bullet}, f_{\bullet})$ is an $\bar{n}$ approximation - we call this a **Witt approximation for T with coefficients in k** - and we can talk about vanishing of duality obstructions for $(A_{\bullet}, f_{\bullet})$ alone. For simplicity, we state the following theorem for Witt approximations.

Theorem 4.4.4 (Existence). *Suppose X is a Witt space of even dimension $n = 2m$. Then there exists a Witt algebraic intersection approximation $(A_\bullet, f_\bullet)$ for T with coefficients in k for which all the local duality obstructions vanish.*

Proof. We suppress perversity superscripts and subscripts since they yield isomorphic objects below. Pick a Witt algebraic intersection approximation $(A_\bullet, f_\bullet)$ with *zero differential* as in Proposition 4.3.3. This allows us to assume $A_\bullet \subset H_\bullet(\partial T; k)$; since $A_\bullet = H_\bullet(A_\bullet)$ and the map $H_\bullet(A_\bullet) \rightarrow H_\bullet(\partial T; k)$ is injective by definition.

We will replace all the A_r for $r \geq m$ and leave unchanged all the A_r for $r < m$. We do this as follows. Fix $r < m$ and set $s = 2m - 1 - r \geq m$. Replace our given A_s with the subspace of $H_s(\partial T; k)$ on which $D(A_r) \subset H^s(\partial T; k)$ vanishes. We pick a map:

$$f_s : A_s \rightarrow C_s(\partial T; k)$$

by selecting a section of the quotient map $\{s\text{-cycles of } \partial T\} \rightarrow H_s(\partial T; k)$, and then using the composition:

$$A_s \hookrightarrow H_s(\partial T; k) \begin{matrix} \xrightarrow{\quad} \\ \leftarrow \end{matrix} \{s\text{-cycles}\} \hookrightarrow C_s(\partial T; k).$$

We must now check two things. First, that the composition:

$$A_s \hookrightarrow H_s(\partial T; k) \rightarrow Z_s$$

is an isomorphism. Second, that the r th local duality obstruction vanishes (observe that Proposition 4.4.1 implies we only have to check vanishing for $r < n$, because this leads to vanishing for all r). Because the differentials of our old and new $A_\bullet$ are zero, there is nothing else to worry about.

By Lemma 4.2.2 we have a commutative diagram of short exact sequences (where we have arbitrarily assigned names to some maps and considered injections as inclusions):

$$\begin{array}{ccccccc}
 0 & \longrightarrow & Z^s & \hookrightarrow & H^s(\partial T; k) & \xrightarrow{u} & Y^{s+1} \longrightarrow 0 \\
 & & \uparrow D|_{\mathbb{R}} & & \uparrow D|_{\mathbb{R}} & & \uparrow D|_{\mathbb{R}} \\
 0 & \longrightarrow & Y_{r+1} & \hookrightarrow & H_r(\partial T; k) & \xrightarrow{v} & Z_r \longrightarrow 0.
 \end{array}$$

By definition and from the diagram we have $\dim A_r = \dim Z_r = \dim Y^{s+1}$. By construction, the codimension of A_s in $H^s(\partial T; k)$ is equal to the dimension of A_r . Combining this with the diagram of short exact sequences we have:

$$\begin{aligned}
 \dim A_s &= \dim H^s(\partial T; k) - \dim Z_r = \dim H^s(\partial T; k) - \dim Y^{s+1} \\
 &= \dim Z^s = \dim Z_s.
 \end{aligned}$$

This is a start, for we have shown that the dimensions of A_s and Z_s coincide. We now need only show that A_s does has trivial intersection with the kernel of $H_s(\partial T; k) \rightarrow Z_s$ to conclude that A_s is suitable for an algebraic intersection approximation. Suppose towards a contradiction that it has *nontrivial* intersection with this kernel. From the short exact sequence:

$$0 \rightarrow Y_{s+1} \hookrightarrow H_s(\partial T; k) \rightarrow Z_s \rightarrow 0$$

it follows that $A_s \cap Y_{s+1}$ is nontrivial. Pick a function $g \in Y^{s+1}$ which does not vanish on $A_s \cap Y_{s+1}$. Our diagram identifies Y^{s+1} with Z_r , and $A_r \subset H_r(\partial T; k)$ maps isomorphically onto Z_r under v , so there exists $\alpha \in A_r$ with:

$$g = Dv(\alpha) = uD(\alpha)$$

The map u is none other than the restriction to Y_{s+1} . So the fact that $g = uD(\alpha)$ does not vanish on $A_s \cap Y_{s+1}$ implies $D(\alpha)$ does not vanish on A_s . This contradicts the

definition of A_s . Hence, in fact $A_s \cap Y_{s+1} = \langle 0 \rangle$ and the composition:

$$A_s \hookrightarrow H_s(\partial T; k) \rightarrow Z_s$$

is an isomorphism.

Next we verify vanishing of the r th local duality obstruction. We know that $D(A_r)$ vanishes on A_s . Therefore for all $\alpha \in A_r$ and $\beta \in A_s$ we have:

$$D(\alpha)(\beta) = (\alpha, \beta) = 0.$$

Now apply Proposition [4.4.1](#).

□

Chapter 5

Global Construction

5.1 Denotations and Assumptions

Throughout Chapter 5 we let k denote a field and X a compact subvariety of a real analytic manifold with singular set Σ . Assume X admits pc tubular data and is oriented of dimension n (e.g. X is complex and equidimensional). Let T denote a pc tubular neighborhood of Σ . Let $(\bar{p}, \bar{q})$ denote complementary perversity functions. Let $(A_{\bullet}^{\bar{p}}, f_{\bullet}^{\bar{p}})$ and $(A_{\bullet}^{\bar{q}}, f_{\bullet}^{\bar{q}})$, respectively $(A^{\bar{p}}, f^{\bar{p}})$ and $(A^{\bar{q}}, f^{\bar{q}})$, denote algebraic, respectively topological, intersection approximations for T with coefficients in k .

5.2 Intersection Space

We are now in a position to define a global space extending earlier definitions of intersection space. The **algebraic intersection space** $I_{f_{\bullet}^{\bar{p}}} X$ **associated to** $(A_{\bullet}^{\bar{p}}, f_{\bullet}^{\bar{p}})$ is the algebraic cone on the composition:

$$A_{\bullet}^{\bar{p}} \xrightarrow{f_{\bullet}^{\bar{p}}} C_{\bullet}(\partial T; k) \xrightarrow{\text{incl}_{\bullet}} C_{\bullet}(X - T^{\circ}; k).$$

The **topological intersection space** $I_{f^{\bar{p}}}X$ **associated to** $(A^{\bar{p}}, f^{\bar{p}})$ is the topological cone on the composition:

$$A^{\bar{p}} \xrightarrow{f^{\bar{p}}} \partial T \xrightarrow{\text{incl}} X - T^{\circ}.$$

We achieve a global duality assuming the local duality obstructions vanish.

Theorem 5.2.1. *Assume the local duality obstructions vanish for $(A_{\bullet}^{\bar{p}}, f_{\bullet}^{\bar{p}})$, $(A_{\bullet}^{\bar{q}}, f_{\bullet}^{\bar{q}})$. Then there exist non-canonical duality isomorphisms:*

$$D : H_*(I_{f_{\bullet}^{\bar{p}}}X) \xrightarrow{\cong} H^{n-*}(I_{f_{\bullet}^{\bar{q}}}X).$$

Proof. We temporarily omit the perversity superscripts and subscripts, as the following statements about distinguished triangles hold for both. We have a set of three distinguished triangles:

$$\begin{aligned} A_{\bullet} &\xrightarrow{f_{\bullet}} C_{\bullet}(\partial T; k) \rightarrow cf_{\bullet} \xrightarrow{-1} \\ C_{\bullet}(\partial T; k) &\xrightarrow{\text{incl}_{\bullet}} C_{\bullet}(X - T^{\circ}; k) \rightarrow C_{\bullet}(X - T^{\circ}, \partial T; k) \xrightarrow{-1} \\ A_{\bullet} &\xrightarrow{\text{incl}_{\bullet} \circ f_{\bullet}} C_{\bullet}(X - T^{\circ}; k) \rightarrow I_{f_{\bullet}}X \xrightarrow{-1}. \end{aligned}$$

The octahedral axiom implies the existence of a third distinguished triangle:

$$cf_{\bullet} \rightarrow I_{f_{\bullet}}X \rightarrow C_{\bullet}(X - T^{\circ}, \partial T; k) \xrightarrow{-1}.$$

The octahedral axiom moreover relates the maps in these four distinguished triangles; namely we have the following (every map below is a map from one of these distinguished triangles, and a shift by “ -1 ” in a subscript indicates we are considering a boundary

map):

$$[C_\bullet(X - T^\circ, \partial T; k) \rightarrow cf_{\bullet-1}] = [C_\bullet(X - T^\circ, \partial T; k) \rightarrow C_{\bullet-1}(\partial T; k) \rightarrow cf_{\bullet-1}] \quad (5.1)$$

$$[cf_\bullet \rightarrow A_{\bullet-1}] = [cf_\bullet \rightarrow I_{f_\bullet} X \rightarrow A_{\bullet-1}] \quad (5.2)$$

$$[C_\bullet(X - T^\circ; k) \rightarrow C_\bullet(X - T^\circ, \partial T; k)] = [C_\bullet(X - T^\circ; k) \rightarrow I_{f_\bullet} X \rightarrow C_\bullet(X - T^\circ, \partial T; k)] \quad (5.3)$$

$$[C_\bullet(\partial T; k) \rightarrow cf_\bullet \rightarrow I_{f_\bullet} X] = [C_\bullet(\partial T; k) \rightarrow C_\bullet(X - T^\circ; k) \rightarrow I_{f_\bullet} X] \quad (5.4)$$

$$[I_{f_\bullet} X \rightarrow C_\bullet(X - T^\circ, \partial T; k) \rightarrow C_{\bullet-1}(\partial T; k)] = [I_{f_\bullet} X \rightarrow A_{\bullet-1} \rightarrow C_{\bullet-1}(\partial T; k)] \quad (5.5)$$

We will only use the first of these in this proof, but the rest will be important later.

We now reintroduce perversity subscripts and superscripts. Consider two long exact sequences obtained from the aforementioned distinguished triangles:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & H^{n-r-1}(A_\bullet^{\bar{q}}) & \longrightarrow & H^{n-r}(I_{f_\bullet^{\bar{q}}} X) & \longrightarrow & H^{n-r}(X - T^\circ; k) \longrightarrow H^{n-r}(A_\bullet^{\bar{q}}) \longrightarrow \cdots \\ & & \uparrow D & & \uparrow D & & \uparrow D \\ \cdots & \longrightarrow & H_r(cf_\bullet^{\bar{p}}) & \longrightarrow & H_r(I_{f_\bullet^{\bar{p}}} X) & \longrightarrow & H_r(X - T^\circ, \partial T; k) \longrightarrow H_{r-1}(cf_\bullet^{\bar{p}}) \longrightarrow \cdots \end{array}$$

If we can prove that this diagram is commutative, then we can use Lemma 3.4.2 to construct (non-canonical) duality isomorphisms. The left (bigger) rectangle above commutes by exactness. We next use vanishing of duality obstructions to show that the right square also commutes. By our observation (1) about the boundary map of the lower long exact sequence, the square of interest can be decomposed:

$$\begin{array}{ccccc} H^{n-r}(X - T^\circ; k) & \longrightarrow & H^{n-r}(\partial T; k) & \longrightarrow & H^{n-r}(A_\bullet^{\bar{q}}) \\ \uparrow D & & \uparrow D & & \uparrow D \\ H_r(X - T^\circ, \partial T; k) & \longrightarrow & H_{r-1}(\partial T; k) & \longrightarrow & H_{r-1}(cf_\bullet^{\bar{p}}) \end{array}$$

The leftmost box in this decomposed diagram always commutes, and the rightmost box commutes owing to the vanishing of the $(r-1)$ th local duality obstruction.

We have successfully verified the hypotheses of Lemma 3.4.2. $\square$

There is an analogous statement for the topological intersection space.

Corollary 5.2.2. *Assume the local duality obstructions for $(A^{\bar{p}}, f^{\bar{p}})$, $(A^{\bar{q}}, f^{\bar{q}})$ vanish. Then there exist non-canonical duality isomorphisms:*

$$D : \tilde{H}_r(I_{f^{\bar{p}}}X; k) \xrightarrow{\simeq} \tilde{H}^{n-r}(I_{f^{\bar{q}}}X; k).$$

Proof. This is a consequence of the arguments from Theorem 5.2.1, since there for either perversity (omit the superscripts) there is an exact triangle:

$$C_{\bullet}(A; k) \xrightarrow{\text{incl}_{\bullet} \circ f_{\#}} C_{\bullet}(X - T^{\circ}; k) \rightarrow \tilde{C}_{\bullet}(I_f X) \xrightarrow{-1}.$$

associated to a topological mapping cone. □

Remark. While the cochain complex $C^{\bullet}(I_f X)$ of a *topological intersection space* is naturally a differential graded k -algebra under cup product, the dual complex $(I_{f_{\bullet}} X)^*$ of an *algebraic intersection space* does not seem to have a natural multiplicative structure. Therefore finding topological, as opposed to just algebraic, intersection spaces will prove to be an interesting task.

Example 5.2.3. We show that, when the local intersection approximation is a fiberwise truncation, our topological intersection space coincides with the Banagl-Chriestenson intersection space. With assumptions as in Example 4.3.2 we have a fiberwise truncation $f : ft_{<c-1-\bar{p}(c)}\partial T \rightarrow \partial T$. As in Example 4.3.2, this fiberwise truncation constitutes a topological intersection approximation for T . The associated topological intersection space is the cone on the composition:

$$ft_{<c-1-\bar{p}(c)}\partial T \rightarrow \partial T \rightarrow X - T^{\circ}.$$

This is precisely [7, Definition 9.2], the definition of the Banagl-Chriestenson intersection space. //

Remark. There is not an obvious general sheaf interpretation of algebraic intersection space cohomology. This is because the local intersection approximation takes as input the not entirely local map $C_{\bullet}(\partial T; k) \rightarrow IH_{\bullet}(T; k)$. This in contrast to the *AF intersection space pairs* of [1], but we will show in the following chapter that the AF intersection space is in general distinct from our algebraic intersection space: in an example, we will show that homologies of the two do not even coincide.

Chapter 6

A Worked out Example

6.1 Denotations and Assumptions

In this chapter, we will deal only with spaces with even-dimensional strata, so without further comment we use middle-perversity intersection homology. Let $X \subset \mathbb{C}P^2$ denote an irreducible degree three nodal hypersurface with exactly one singular point p . Let B denote a closed tubular neighborhood of p in X whose boundary is denoted by L . Let M denote $X - B^\circ$. Observe that:

- X is topologically a pinched torus.
- $B \cong cL$.
- $L \cong S^1 \sqcup S^1$.
- $M \cong S^1 \times D^1$.

Let $\overline{\mathcal{X}} \subset \mathbb{C}P^3$ denote the projective cone on X . Let ∞ denote $(0 : 0 : 0 : 1) \in \mathbb{C}P^3$. The vector bundle $\mathbb{C} \hookrightarrow \mathbb{C}P^3 - \{\infty\} \rightarrow \mathbb{C}P^2$ restricts to a vector bundle $\mathbb{C} \hookrightarrow \overline{\mathcal{X}} - \{\infty\} \rightarrow X$ which we denote by $(\mathcal{X}, \pi)$. We also let:

- ρ denote the restriction of vector bundle $\mathcal{X}$ over p .
- $\mathcal{B}$ denote the restriction of vector bundle $\mathcal{X}$ over B .

- $\mathcal{L}$ denote the restriction of vector bundle $\mathcal{X}$ over L .
- $\mathcal{M}$ denote the restriction of vector bundle $\mathcal{X}$ over M .
- $\bar{\rho}$ denote the line in $\mathbb{C}P^3$ connecting p with ∞ , i.e. the closure of ρ in $\overline{\mathcal{X}}$, also the singular set of $\overline{\mathcal{X}}$.
- $S(-)$ denote the sphere bundle associated to a vector bundle $(-)$.
- $D(-)$ denote the disk bundle associated to a vector bundle $(-)$.

6.2 Setting Up the Example

We will explicitly construct a topological intersection space for the projective cone $\overline{\mathcal{X}}$ on $X \subset \mathbb{C}P^2$. Moreover we will show that the corresponding local duality obstructions vanish. This example is of interest, because it is depth two with pseudomanifold stratification $\overline{\mathcal{X}} \supset \bar{\rho} \supset \{\infty\}$, so the topological methods of [7] do not apply. We will also use this example to distinguish our construction from the construction of [1].

To rigorously carry out this construction we need to analyze $\overline{\mathcal{X}}$ in detail. Topologically it is the Thom space of the vector bundle $(\mathcal{X}, \pi)$ as described for example in [10, Page 18], therefore is the homotopy pushout of the following diagram involving disk and circle bundles:

$$\overline{\mathcal{X}} = \text{hp} (D\mathcal{X} \leftarrow S\mathcal{X} \rightarrow cS\mathcal{X}).$$

where $cS\mathcal{X}$ is the cone on $S\mathcal{X}$. The singular set $\bar{\rho}$ is the homotopy pushout:

$$\bar{\rho} = \text{hp} (D\rho \leftarrow S\rho \rightarrow cS\rho)$$

We then define:

$$T = \text{hp} (D\mathcal{B} \leftarrow S\mathcal{B} \rightarrow cS\mathcal{X}).$$

Claim 1. T is a closed tubular neighborhood of $\bar{\rho}$ in $\bar{\mathcal{X}}$ in the sense of the remark following Lemma 3.3.1. Moreover, the nonsingular boundary ∂T is the homotopy pushout:

$$\partial T = \text{hp} (D\mathcal{L} \leftarrow S\mathcal{L} \rightarrow S\mathcal{M})$$

and $X - T^\circ$ is the homotopy pushout:

$$X - T^\circ = \text{hp} (D\mathcal{M} \leftarrow S\mathcal{M} \rightarrow S\mathcal{M})$$

Proof. Consider the inclusions of diagrams:

$$\begin{array}{ccccc} D\mathcal{X} & \leftarrow & S\mathcal{X} & \rightarrow & cS\mathcal{X} \\ \uparrow & & \uparrow & & \uparrow \\ D\mathcal{B} & \leftarrow & S\mathcal{B} & \rightarrow & cS\mathcal{X} \\ \uparrow & & \uparrow & & \uparrow \\ D\rho & \leftarrow & S\rho & \rightarrow & cS\rho. \end{array}$$

Each vertical inclusion from the lower half of the diagram is the inclusion of a deformation retract, so that the inclusion of homotopy pushouts $\bar{\rho} \rightarrow T$ is also the inclusion of a deformation retract. This provides our family $T(\epsilon)$ as in the remark following Lemma 3.3.1. Therefore, once we check in the following paragraph that ∂T is a submanifold of $\bar{\mathcal{X}}$, we have our tubular neighborhood T .

By inspection, the boundary of ∂T in $\bar{\mathcal{X}}$ is the homotopy pushout:

$$\text{hp} (D\mathcal{L} \leftarrow S\mathcal{L} \rightarrow S\mathcal{M}).$$

Because $D\mathcal{L}$ and $S\mathcal{M}$ are manifolds with boundary $S\mathcal{L}$, it follows that the homotopy pushout ∂T is in fact a closed manifold. By another inspection the complement $\bar{\mathcal{X}} - T^\circ$

is the homotopy pushout:

$$\text{hp}(D\mathcal{M} \leftarrow S\mathcal{M} \rightarrow S\mathcal{M})$$

which again is a manifold with boundary ∂T . In particular, ∂T is a submanifold of $\overline{\mathcal{X}}$.

□

We will need to analyze various homologies and intersection homologies related to the tube in order to construct an intersection space. First we find:

Claim 2.

$$IH_*(S\mathcal{X}) = \begin{cases} \mathbb{Z} & \text{if } * = 0, 3 \\ \mathbb{Z}_3 & \text{if } * = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Since X is a pinched torus, it has normalization $\nu : S^2 \rightarrow X$ where S^2 is a two-sphere. Let $S\mathcal{Z}$ and $\hat{\nu}$ be such that the below is a pair of pullback diagrams:

$$\begin{array}{ccccc} S\mathcal{Z} & \xrightarrow{\hat{\nu}} & S\mathcal{X} & \longrightarrow & S^5 \\ \downarrow & & \downarrow & & \downarrow \\ S^2 & \xrightarrow{\nu} & X & \longrightarrow & \mathbb{C}P^2. \end{array}$$

Then $\hat{\nu}$ is a normalization and $S\mathcal{Z}$ is a principal circle bundle over S^2 . Each principal circle bundle corresponds to an element of $H^2(S^2; \mathbb{Z})$. Let us determine to which element $S\mathcal{Z}$ corresponds.

Since $S^5 \rightarrow \mathbb{C}P^2$ is the pullback of the universal circle-bundle $S^\infty \rightarrow \mathbb{C}P^\infty$ under the inclusion, by composing pullbacks it follows that $S\mathcal{Z} \rightarrow S^2$ is the pullback of the universal bundle under:

$$S^2 \xrightarrow{\nu} X \hookrightarrow \mathbb{C}P^2 \hookrightarrow \mathbb{C}P^\infty.$$

We analyze this map on second cohomology. Both ν and $\mathbb{C}P^2 \hookrightarrow \mathbb{C}P^\infty$ induce isomorphisms on second cohomology. Because X is degree 3 the map $\mathbb{Z} \cong H^2(\mathbb{C}P^2; \mathbb{Z}) \rightarrow H^2(X; \mathbb{Z}) \cong \mathbb{Z}$ is multiplication by ± 3 . Therefore the composition $H^2(\mathbb{C}P^\infty; \mathbb{Z}) \rightarrow H^2(S^2; \mathbb{Z})$ is multiplication by ± 3 . Hence $S\mathcal{Z}$ is the unique principal circle bundle corresponding to $\pm 3 \in H^2(S^2; \mathbb{Z})$. A standard argument then shows that:

$$H_*(S\mathcal{Z}) = \begin{cases} \mathbb{Z} & \text{if } * = 0, 3 \\ \mathbb{Z}_3 & \text{if } * = 1 \\ 0 & \text{otherwise.} \end{cases}$$

But $\hat{\nu} : S\mathcal{Z} \rightarrow S\mathcal{X}$ is a normalization, so as detailed in [10, I.1.6] induces an isomorphism $H_*(S\mathcal{Z}) \cong IH_*(S\mathcal{X})$. $\square$

Next we work on the rational intersection homology of tubular neighborhood T .

Claim 3.

$$IH_*(T; \mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } * = 0 \\ \mathbb{Q}^2 & \text{if } * = 2 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Given the description of T as the homotopy pushout of:

$$D\mathcal{B} \leftarrow S\mathcal{B} \rightarrow cS\mathcal{X}$$

and the fact that $D\mathcal{B}$ and $S\mathcal{B}$ are circle bundles over contractible B , we obtain a long exact sequence:

$$\cdots \rightarrow IH_i(B \times S^1; \mathbb{Q}) \rightarrow IH_i(B \times D^2; \mathbb{Q}) \oplus IH_i(cS\mathcal{X}; \mathbb{Q}) \rightarrow IH_i(T; \mathbb{Q}) \rightarrow \cdots$$

which, using the cone formula and Künneth for intersection homology (see [13, Theorem 4.2.1, Corollary 6.4.10]), becomes:

$$\begin{aligned} \cdots \rightarrow [H_*^{<1}(L; \mathbb{Q}) \otimes H_*(S^1; \mathbb{Q})]_i &\rightarrow [H_*^{<1}(L; \mathbb{Q}) \otimes H_*(D^2; \mathbb{Q})]_i \oplus IH_i^{<2}(S\mathcal{X}; \mathbb{Q}) \\ &\rightarrow IH_i(T; \mathbb{Q}) \rightarrow \cdots \end{aligned}$$

Recall that $L \cong S^1 \sqcup S^1$. For $i = 0$ the first map is given by:

$$H_0(L; \mathbb{Q}) \rightarrow H_0(L; \mathbb{Q}) \oplus IH_0(S\mathcal{X}; \mathbb{Q}).$$

which is obviously injective. Then for $i = 1$ we have exact:

$$H_0(L; \mathbb{Q}) \otimes H_1(S^1; \mathbb{Q}) \rightarrow IH_1(S\mathcal{X}; \mathbb{Q}) \rightarrow IH_1(T; \mathbb{Q}) \rightarrow 0.$$

But $IH_1(S\mathcal{X}; \mathbb{Q}) = 0 \implies IH_1(T; \mathbb{Q}) = 0$. For $i = 2$ we have:

$$0 \rightarrow IH_2(T; \mathbb{Q}) \rightarrow H_0(L; \mathbb{Q}) \otimes H_1(S^1; \mathbb{Q}) \rightarrow 0$$

which implies $IH_2(T; \mathbb{Q}) \cong \mathbb{Q}^2$. The remaining homology $\mathbb{Q}$ -vector spaces are trivially computed owing to the vanishing of many terms in the long exact sequence. $\square$

Next let's provide exact descriptions for $X - T^\circ$ and ∂T .

Claim 4. $X - T^\circ \cong S^1 \times D^3$ and $\partial T \cong S^1 \times S^2$.

Proof. By Claim 1 the space $X - T^\circ$ is the homotopy pushout:

$$\text{hp}(D\mathcal{M} \leftarrow S\mathcal{M} \rightarrow S\mathcal{M})$$

which (since $S\mathcal{M}$ has a collar neighborhood in $D\mathcal{M}$) is homeomorphic to $D\mathcal{M}$. But $H^2(M; \mathbb{Z}) = 0$ so the complex vector bundle $\mathcal{M}$ is trivial. Hence:

$$D\mathcal{M} \cong M \times D^2 \cong S^1 \times D^1 \times D^2 \cong S^1 \times D^3.$$

$\square$

6.3 The Intersection Space

We rely on the results of the preceding section to construct a topological intersection approximation. Let $S^2 \xrightarrow{f} S^1 \times S^2 \cong \partial T$ be the inclusion of a sphere such that f induces an isomorphism on second homology.

Claim 5. The pair (S^2, f) is a topological intersection approximation for T with coefficients in $\mathbb{Q}$.

Proof. We first must understand the map $H_\bullet(\partial T; \mathbb{Q}) \rightarrow IH_\bullet(T; \mathbb{Q})$ and its image $Z_\bullet$. For dimensional reasons, the description of the map is only unclear in degree two. In this case we have exact:

$$IH_3(T, \partial T; \mathbb{Q}) \rightarrow H_2(\partial T; \mathbb{Q}) \rightarrow IH_2(T; \mathbb{Q}).$$

Duality shows that:

$$IH_3(T, \partial T; \mathbb{Q}) \cong IH_1(T; \mathbb{Q})^* = 0$$

and consequently that the induced map $H_2(\partial T; \mathbb{Q}) \rightarrow IH_2(T; \mathbb{Q})$ is an injection. We explicitly specify:

$$Z_* = \begin{cases} \text{im } [H_0(\partial T; \mathbb{Q}) \hookrightarrow IH_0(T; \mathbb{Q})] & \text{if } * = 0 \\ \text{im } [H_2(\partial T; \mathbb{Q}) \hookrightarrow IH_2(T; \mathbb{Q})] & \text{if } * = 2 \\ 0 & \text{otherwise.} \end{cases}$$

The map $H_*(S^2; \mathbb{Q}) \xrightarrow{f_*} H_*(\partial T; \mathbb{Q})$ is an isomorphism for $* = 0, 2$ and $H_*(S^2; \mathbb{Q})$ vanishes otherwise. So the composition:

$$H_\bullet(S^2; \mathbb{Q}) \xrightarrow{f_*} H_\bullet(\partial T; \mathbb{Q}) \rightarrow Z_\bullet$$

is an isomorphism. □

We have a topological intersection approximation for the tube T , so are granted a topological intersection space $I_f \overline{\mathcal{X}}$. It is obtained by coning off an embedded S^2 in the boundary $\partial T \cong S^1 \times S^2$ of $\overline{\mathcal{X}} - T^\circ$. The long exact sequence associated to the inclusion $S^2 \hookrightarrow S^1 \times D^3 \cong \overline{\mathcal{X}} - T^\circ$ gives:

$$H_*(I_f \overline{\mathcal{X}}; \mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } * = 0, 1, 3 \\ 0 & \text{otherwise.} \end{cases}$$

Alternatively, check that $I_f \overline{\mathcal{X}} \simeq S^1 \vee S^3$. The dual Betti numbers of $I_f \overline{\mathcal{X}}$ seem to match up. In fact this is because:

Claim 6. The local duality obstructions vanish for the intersection approximation $S^2 \xrightarrow{f} \partial T$. Therefore, the intersection space $I_f \overline{\mathcal{X}}$ satisfies duality.

Proof. As usual we set:

$$Z_\bullet = \text{im } H_\bullet(\partial T; \mathbb{Q}) \rightarrow IH_\bullet(T; \mathbb{Q})$$

$$Y_\bullet = \text{coker } H_\bullet(T; \mathbb{Q}) \rightarrow IH_\bullet(T, \partial T; \mathbb{Q}).$$

Consider the diagram:

$$\begin{array}{ccccc} Z_{3-r}^* & \longrightarrow & H_{3-r}(\partial T; \mathbb{Q})^* & \xrightarrow{f^*} & H_{3-r}(S^2; \mathbb{Q})^* \\ D \uparrow & & D \uparrow & & D \uparrow \\ Y_{r+1} & \longrightarrow & H_r(\partial T; \mathbb{Q}) & \longrightarrow & H_r(cf_\#) \end{array}$$

which a priori need not commute. The left box always commutes, and the outer box commutes by construction. The local duality obstruction vanishes if and only if the right box also commutes.

Commutativity is obvious when $r \neq 1, 3$ because the upper-rightmost term vanishes. When $r = 1$ or $r = 3$ the map f^* is an isomorphism (for $r = 1$ see the proof of Claim 5),

from which it can be derived that all maps are isomorphisms, in which case the box again commutes (owing to the commutativity of the left box and the outer box). $\square$

Remark. With some effort, this example can be extended to the projective cone on *any* irreducible hypersurface in $\mathbb{C}P^2$ with isolated singularity. In this general case, the topological local intersection approximation will be composed of a wedge of spheres and circles. Again, the local duality obstructions will vanish.

6.4 Comparison with the AF intersection space.

We can compare with the method introduced in [1], and will show that their *AF intersection space pair* is distinct from our notion of algebraic intersection space even on the level of homology. Since the strata ρ and $\{\infty\}$ are contractible, [1, Theorem 3.30] implies that their construction applies. We avoid excruciating detail, choosing only to outline the construction of this AF intersection space pair $(I\overline{\mathcal{X}}_{AF}, \bar{\rho}_{AF})$.

[1] requires a decomposition of the tubular neighborhood T , which we provide in this paragraph. Keeping in mind the homotopy pushout descriptions:

$$\bar{\rho} = \text{hp}(D\rho \leftarrow S\rho \rightarrow cS\rho), \quad T = \text{hp}(D\mathcal{B} \leftarrow S\mathcal{B} \rightarrow cS\mathcal{X})$$

with cone point ∞ , we define:

$$\bar{\rho}_1 = \text{hp}(D\rho \leftarrow S\rho \rightarrow S\rho), \quad T_1 = \text{hp}(D\mathcal{B} \leftarrow S\mathcal{B} \rightarrow S\mathcal{B}), \quad E_1 = \text{hp}(D\mathcal{L} \leftarrow S\mathcal{L} \rightarrow S\mathcal{L})$$

$$\bar{\rho}_0 = cS\rho, \quad T_0 = cS\mathcal{X}, \quad E_0 = S\mathcal{X}.$$

Observe that:

- $T_1 \cap \bar{\rho} = \bar{\rho}_1$, $T_0 \cap \bar{\rho} = \bar{\rho}_0$, and $\bar{\rho}_0 \cap \bar{\rho}_1 = S\rho$.

- $E_1 = \partial T_1 \cap (X - T^\circ)$ and $E_0 = \partial T_0$.
- E_1 fibers trivially over $\bar{\rho}_1$ with fiber L .
- $T_1 = \text{cyl}(E_1 \rightarrow \bar{\rho}_1)$, the mapping cylinder of the bundle projection.
- the pair $(E_0, S\rho)$ fibers trivially over $\{\infty\}$ with fiber $(S\mathcal{X}, S\rho)$.
- $(T_0, \bar{\rho}_0) \cong \text{cyl}((E_0, S\rho) \rightarrow \{\infty\})$, the mapping cylinder pair of the pair of bundle projections. Since $\{\infty\}$ is a point set, this is actually a cone pair.
- $\partial T_0 \cap \partial T_1 = S\mathcal{L} = E_1|_{S\rho}$, the restricted fiber bundle over $S\rho$.
- $\partial T_0 \cap T_1 = S\mathcal{B} = \text{cyl}(E_1|_{S\rho} \rightarrow S\rho)$, the mapping cylinder of the bundle projection.
- $\partial T_0 = S\mathcal{M} \cup_{E_1|_{S\rho}} \text{cyl}(E_1|_{S\rho} \rightarrow S\rho)$

Keep these observations in mind when considering the construction detailed in the following paragraph.

The essence of the [1] construction for this example (up to homotopy, not word-for-word) is the following:

- (i) Select a fiberwise truncation $ft_{<1}E_1 \rightarrow E_1$ of the trivial bundle $E_1 \rightarrow \bar{\rho}_1$. Define

$$T_1^{AF} = \text{cyl}(ft_{<1}E_1 \rightarrow \bar{\rho}_1), \text{ the mapping cylinder of the bundle projection.}$$

- (ii) Define a Step 1 AF intersection space:

$$I\overline{\mathcal{X}}_{AF,1} = (X - T^\circ) \cup_{ft_{<1}E_1} T_1^{AF}$$

by gluing T_1^{AF} to $X - T^\circ$ via:

$$T_1^{AF} \hookleftarrow ft_{<1}E_1 \rightarrow E_1 \hookrightarrow X - T^\circ$$

Effectively, we have deleted T from X , then replaced T_1 with T_1^{AF} .

(iii) Define pair:

$$\partial T_0^{AF} = S\mathcal{M} \cup_{ft_{<1}E_1|_{S\rho}} \text{cyl}(ft_{<1}E_1|_{S\rho} \rightarrow S\rho) \subset I\overline{\mathcal{X}}_{AF,1}.$$

and interpret $(\partial T_0^{AF}, S\rho)$ as a pair of fiber bundles over the point set $\{\infty\}$.

(iv) Select a fiberwise truncation of pairs $(\partial T_0^{AF}, S\rho)_{<2} \rightarrow (\partial T_0^{AF}, S\rho)$ of the pair of fiber bundles $(\partial T_0^{AF}, S\rho) \rightarrow \{\infty\}$. Set $(T_0^{AF}, \bar{\rho}_0^{AF}) = \text{cyl}((\partial T_0^{AF}, S\rho)_{<2} \rightarrow \{\infty\})$, the mapping cylinder pair of the pair of bundle projections.

(v) Define the AF intersection space pair:

$$(I\overline{\mathcal{X}}_{AF}, \bar{\rho}_{AF}) = (I\overline{\mathcal{X}}_{AF,1}, \bar{\rho}_1) \cup_{(\partial T_0^{AF}, S\rho)_{<2}} (T_0^{AF}, \bar{\rho}_0^{AF})$$

by gluing $(T_0^{AF}, \bar{\rho}_0^{AF})$ to $(I\overline{\mathcal{X}}_{AF,1}, \bar{\rho}_1)$ via:

$$(T_0^{AF}, \bar{\rho}_0^{AF}) \hookleftarrow (\partial T_0^{AF}, S\rho)_{<2} \rightarrow (\partial T_0^{AF}, S\rho) \hookrightarrow (I\overline{\mathcal{X}}_{AF,1}, \bar{\rho}_1).$$

Effectively, we have replaced $(T_0, \bar{\rho}_0)$ with $(T_0^{AF}, \bar{\rho}_0^{AF})$.

The rational homology of the pair $(I\overline{\mathcal{X}}_{AF}, \bar{\rho}_{AF})$ has the potential to satisfy duality, and is what we will compare the rational homology of our algebraic intersection spaces against. Having outlined the construction, let's select fiberwise truncations and determine an explicit AF intersection space.

We include the following claims without proof, as they can be verified in a straightforward manner.

Claim 7. Let $L_{<1} = \{*\} \sqcup \{*\} \hookrightarrow L$ be the inclusion of two points into the two disjoint circles that make up L . Let $\mathcal{L}_{<1}$ denote the restriction of bundle $\mathcal{L} \rightarrow L$ above subspace

$L_{<1}$. The space $\mathcal{L}_{<1}$ like $\mathcal{L}$ can also be interpreted as a trivial bundle over ρ , but with fiber $L_{<1}$. Then:

$$ft_{<1}E_1 = \text{hp} (D\mathcal{L}_{<1} \leftarrow S\mathcal{L}_{<1} \rightarrow S\mathcal{L}_{<1}) \hookrightarrow E_1$$

is a fiberwise truncation of bundles over $\bar{\rho}_1$. It is an inclusion.

Claim 8. Define $B^{AF} = \text{cyl}(L_{<1} \rightarrow p)$. It is a subset of $\text{cyl}(L \rightarrow p) = B$. Let $\mathcal{B}^{AF}$ denote the restriction of $\mathcal{B} \rightarrow B$ above subspace B^{AF} . Then T_1^{AF} from (i) of the AF construction is:

$$T_1^{AF} = \text{hp} (D\mathcal{B}^{AF} \leftarrow S\mathcal{B}^{AF} \rightarrow S\mathcal{B}^{AF}).$$

It is a subset of T_1 .

Claim 9. Define $IX^{AF} = M \cup B^{AF}$. It is a subset of X that is homotopy equivalent to a wedge of two circles. Let $\mathcal{IX}^{AF}$ denote the restriction of $\mathcal{X} \rightarrow X$ above subspace IX^{AF} . It is a trivial bundle because IX^{AF} has vanishing second cohomology. Then the Step 1 AF intersection space $I\bar{\mathcal{X}}_{AF,1}$ is:

$$I\bar{\mathcal{X}}_{AF,1} = \text{hp} (D\mathcal{IX}^{AF} \leftarrow S\mathcal{IX}^{AF} \rightarrow S\mathcal{IX}^{AF})$$

and ∂T_0^{AF} from (iii) of the AF construction is:

$$\partial T_0^{AF} = S\mathcal{IX}^{AF}.$$

It is a subset of ∂T_0 .

Claim 10. The pair $(\partial T_0^{AF}, S\rho) = (S\mathcal{IX}^{AF}, S\rho)$, interpreted as a pair of bundles over $\{\infty\}$, has fiberwise truncation:

$$(S\mathcal{IX}^{AF}, S\rho)_{<2} = (IX^{AF}, p) \hookrightarrow (S\mathcal{IX}^{AF}, S\rho)$$

where the inclusion is any section of the trivial bundle pair $(S\mathcal{IX}^{AF}, S\rho) \rightarrow (IX^{AF}, p)$.

Because all our truncation are inclusions, the associated AF intersection space $I\overline{\mathcal{X}}_{AF}$ is a subset of $\overline{\mathcal{X}}$. We describe the pair $(I\overline{\mathcal{X}}_{AF}, \overline{\rho}_{AF})$:

Claim 11. Both $I\overline{\mathcal{X}}_{AF}$ and $\overline{\rho}_{AF}$ are contractible. Therefore $H_*(I\overline{\mathcal{X}}_{AF}, \overline{\rho}_{AF})$ vanishes identically.

Proof. By Claims 9 and 10 and the AF construction, one verifies that the AF intersection space pair is described by the following mapping cones:

$$\begin{aligned} I\overline{\mathcal{X}}_{AF} &\cong c(IX^{AF} \hookrightarrow D\mathcal{IX}^{AF}) \\ \overline{\rho}_{AF} &\cong c(p \hookrightarrow D\rho) \end{aligned}$$

where $(IX^{AF}, p) \hookrightarrow (D\mathcal{IX}^{AF}, D\rho)$ is the inclusion of a section of the trivial bundle pair $(S\mathcal{IX}^{AF}, S\rho) \rightarrow (IX^{AF}, p)$. But $IX^{AF} \subset D\mathcal{IX}^{AF}$ and $p \subset D\rho$ are deformation retracts. Therefore $I\overline{\mathcal{X}}_{AF}$ and $\overline{\rho}_{AF}$ are contractible. $\square$

We have shown that $H_*(I\overline{\mathcal{X}}_{AF}, \overline{\rho}_{AF})$ vanishes identically. On the other hand, suppose we are given *any* algebraic intersection approximation $(A_\bullet, f_\bullet)$ for T with coefficients in $\mathbb{Q}$, and associated algebraic intersection space $I_{f_\bullet}X$. Then we have the following exact sequence:

$$H_1(A_\bullet) \rightarrow H_1(\partial T; \mathbb{Q}) \rightarrow H_1(I_{f_\bullet}X)$$

But $H_1(A_\bullet) \cong Z_1 = 0$ (see Proof of Claim 5) and $H_1(\partial T; \mathbb{Q}) \cong \mathbb{Q}$ together imply $H_1(I_{f_\bullet}X) \neq 0$. In other words, our notion of intersection space is distinct from the AF notion. It seems difficult to compare them in general.

Chapter 7

Intersection Space Pairing

7.1 Denotations and Assumptions

Throughout Chapter 7 we let k denote a field and X a compact subvariety of a real analytic manifold with singular set Σ . Assume X admits pc tubular data and is oriented of *even* dimension $2n$ (e.g. X is complex and equidimensional). Let T denote a pc tubular neighborhood of Σ . Assume X is a *Witt space* and $(A_\bullet, f_\bullet)$ is a *Witt algebraic approximation* for T with coefficients in k *for which the local duality obstructions vanish*. Recall that an approximation for a Witt space is said to be Witt if it is either a lower $\bar{m}$ or upper $\bar{n}$ middle perversity approximation, and that distinguishing between the two is unnecessary as the constructed objects are naturally isomorphic. We thus omit any perversity subscripts and superscripts (assuming them to be either $\bar{m}$ or $\bar{n}$, distinction unnecessary).

We also use this section as a grand collection of *names and properties of maps*. We give names to the following natural maps, all of which sit inside exact sequences (see

preceding chapters to understand these sequences):

$$\begin{aligned}
H_{\bullet}(cf_{\bullet}) &\xrightarrow{u_{\bullet}} H_{\bullet}(I_{f_{\bullet}}X) \xrightarrow{v_{\bullet}} H_{\bullet}(X - T^{\circ}, \partial T; k) \\
H_{\bullet}(X - T^{\circ}; k) &\xrightarrow{h_{\bullet}} H_{\bullet}(I_{f_{\bullet}}X) \xrightarrow{g_{\bullet}} H_{\bullet-1}(A_{\bullet}) \\
H_{\bullet}(\partial T; k) &\xrightarrow{\iota_{\bullet}} H_{\bullet}(X - T^{\circ}; k) \xrightarrow{j_{\bullet}} H_{\bullet}(X - T^{\circ}, \partial T; k) \xrightarrow{\delta_{\bullet}} H_{\bullet-1}(\partial T; k) \\
H_{\bullet}(\partial T; k) &\xrightarrow{\ell_{\bullet}} H_{\bullet}(cf_{\bullet}) \xrightarrow{0} H_{\bullet-1}(A_{\bullet}) \xrightarrow{f_{\bullet-1}} H_{\bullet-1}(\partial T; k)
\end{aligned}$$

where maps that sit in the same row are sequential in a long exact sequence. Next we gather the relationships between these maps (all of which can be found in the proof of Theorem 5.2.1):

$$j_{\bullet} = h_{\bullet}v_{\bullet}$$

$$0 = g_{\bullet}u_{\bullet}$$

$$h_{\bullet}\iota_{\bullet} = u_{\bullet}\ell_{\bullet}$$

$$f_{\bullet-1}g_{\bullet} = \delta_{\bullet}v_{\bullet}$$

$$\ell_{\bullet} \text{ is surjective and } f_{\bullet} \text{ is injective.}$$

We consider it allowable to use these properties *without comment*. The dual of a map, $u_{\bullet}$ for example, will be denoted by $u^{\bullet}$. Lastly we name the duality isomorphisms:

$$D_{f_{\bullet}} : H_{\bullet}(cf_{\bullet}) \xrightarrow{\cong} H^{2n-1-\bullet}(A_{\bullet})$$

$$D_{\partial} : H_{\bullet}(\partial T; k) \xrightarrow{\cong} H^{2n-1-\bullet}(\partial T; k)$$

$$D_L : H_{\bullet}(X - T^{\circ}, \partial T; k) \xrightarrow{\cong} H^{2n-\bullet}(X - T^{\circ}; k)$$

$$D'_L : H_{\bullet}(X - T^{\circ}; k) \xrightarrow{\cong} H^{2n-\bullet}(X - T^{\circ}, \partial T; k)$$

where “ L ” indicates Lefschetz duality. For $\alpha \in H_{\bullet}(X - T^{\circ}; k)$ and $\beta \in H_{2n-\bullet}(X -$

$T^\circ, \partial T; k$) we have:

$$D'_L(\alpha)(\beta) = (-1)^{|\alpha||\beta|} D_L(\beta)(\alpha)$$

since these duality isomorphisms (or more specifically their inverses) can be understood in terms of cup products, which are anti-commutative. The duality isomorphisms are related to each other as follows (again these properties are allowable to use *without comment*):

$$D_{f_\bullet} \ell_\bullet = f^{2n-1-\bullet} D_\partial$$

$$D'_L \iota_\bullet = \delta^{2n-\bullet} D_\partial$$

$$D_L j_\bullet = j^{2n-\bullet} D'_L.$$

where the first is a direct consequence of the local duality obstructions vanishing, and the second two follow from commutativity of the duality isomorphism diagram relating the long exact sequence of the pair $(X - T^\circ, \partial T; k)$ in homology to the long exact sequence of the pair in cohomology.

7.2 Families of Sections

We would like our duality isomorphisms on the intersection space to have some geometric significance, and to give us a meaningful intersection space pairing. In this section, we describe how duality isomorphisms are selected.

Lemma 3.4.2 gives us insight into the particular nature of a duality isomorphism $D_{IX} : H_\bullet(I_{f_\bullet} X) \rightarrow H^{2n-\bullet}(I_{f_\bullet} X)$. Consider the commutative diagram of exact sequences

from the previous section:

$$\begin{array}{ccccccc}
 \cdots & \longrightarrow & H^{2n-i-1}(A_\bullet) & \xrightarrow{g^{2n-i}} & H^{2n-i}(I_{f_\bullet}X) & \xrightarrow{h^{2n-i}} & H^{2n-i}(X - T^\circ; k) \longrightarrow \cdots \\
 & & \uparrow D_{f_\bullet} & & & & \uparrow D_L \\
 \cdots & \longrightarrow & H_i(cf_\bullet) & \xrightarrow{u_i} & H_i(I_{f_\bullet}X) & \xrightarrow{v_i} & H_i(X - T^\circ, \partial T; k) \longrightarrow \cdots
 \end{array}$$

where $g^{2n-i} = (g_{2n-i})^*$ and $h^{2n-i} = (h_{2n-i})^*$ are dual maps to the maps on homology.

By Lemma 3.4.2, the intersection space duality isomorphism is constructed by selecting *families of sections*:

$$\begin{aligned}
 r^\bullet &: \text{im } h^\bullet \rightarrow H^\bullet(I_{f_\bullet}X) \\
 s_\bullet &: \text{im } v_\bullet \rightarrow H_\bullet(I_{f_\bullet}X)
 \end{aligned}$$

where we will utilize the identification $\text{im } h^\bullet = (\text{coim } h_\bullet)^*$ asserted by Lemma 3.4.1 to write $r^\bullet = (r_\bullet)^*$ where $r_\bullet : H_\bullet(I_{f_\bullet}X) \rightarrow \text{coim } h_\bullet$ is a retraction of $h_\bullet : \text{coim } h_\bullet \hookrightarrow H_\bullet(I_{f_\bullet}X)$. As in the proof of Lemma 3.4.2 (and with indices shifted for the family of sections on cohomology) we thus have duality isomorphism:

$$D_{IX} = D^{(r^\bullet, s_\bullet)} : H_\bullet(I_{f_\bullet}X) \xrightarrow{\sim} H^{2n-\bullet}(I_{f_\bullet}X)$$

which is entirely described by:

$$\begin{aligned}
 \alpha \in H_i(cf_\bullet) &\implies D_{IX}(u_i\alpha) = g^{2n-i}D_{f_\bullet}\alpha \\
 \beta \in \text{im } v_i &\implies D_{IX}(s_i\beta) = r^{2n-i}D_L\beta.
 \end{aligned}$$

This will allow us to describe the *intersection space pairing associated to* $(r^\bullet, s_\bullet)$, which is defined by:

$$(-, -)_{IX} = (-, -)^{(r^\bullet, s_\bullet)} : H_i(I_{f_\bullet}X) \times H_{2n-i}(I_{f_\bullet}X) \rightarrow k, \quad (\alpha, \beta)_{IX} = D_{IX}(\alpha)(\beta).$$

Lemma 7.2.1. *Suppose $(r^\bullet, s_\bullet)$ is a family of sections and $i \in \mathbb{Z}$. Consider the decompositions:*

$$H_i(I_{f_\bullet}X) = \text{im } u_i \oplus \text{im } s_i$$

$$H_{2n-i}(I_{f_\bullet}X) = \text{im } h_{2n-i} \oplus \ker r_{2n-i}.$$

Under the intersection space pairing for $(r^\bullet, s_\bullet)$ we have:

$$(\text{im } u_i, \text{im } h_{2n-i})_{IX} = 0, \quad (\text{im } s_i, \ker r_{2n-i})_{IX} = 0.$$

and:

$$\alpha \in H_i(\partial T; k), \quad \beta \in H_{2n-i}(I_{f_\bullet}X) \implies (u_i \ell_i \alpha, \beta)_{IX} = (-1)^i (v_{2n-i} \beta, \iota_i \alpha)_L$$

$$\gamma \in \text{im } v_i, \quad \delta \in H_{2n-i}(X - T^\circ; k) \implies (s_i \gamma, h_{2n-i} \delta)_{IX} = (\gamma, \delta)_L.$$

where $(-, -)_L$ is the intersection pairing associated to D_L , i.e. $(-, -)_L = D_L(-)(-)$.

Proof. Suppose $\alpha \in H_i(\partial T; k)$ and $\beta \in H_{2n-i}(I_{f_\bullet}X)$. Then:

$$(u_i \ell_i \alpha, \beta)_{IX} = D_{IX}(u_i \ell_i \alpha)(\beta) = (g^{2n-i} D_{f_\bullet} \ell_i \alpha)(\beta) = (D_{f_\bullet} \ell_i \alpha)(g_{2n-i} \beta).$$

Since exactness implies $\ker g_{2n-i} = \text{im } h_{2n-i}$, and $\text{im } u_i = \text{im } u_i \ell_i$, this proves that $(\text{im } u_i, \text{im } h_{2n-i})_{IX} = 0$. But let's go further with our computation of $(u_i \ell_i \alpha, \beta)_{IX}$; it is equal to:

$$\begin{aligned} (g^{2n-i} D_{f_\bullet} \ell_i \alpha)(\beta) &= (g^{2n-i} f^{2n-i-1} D_\partial \alpha)(\beta) \\ &= (v^{2n-i} \delta^{2n-i} D_\partial \alpha)(\beta) = D'_L(\iota_i \alpha)(v_{2n-i} \beta) \\ &= (-1)^{i(2n-i)} D_L(v_{2n-i} \beta)(\iota_i \alpha) = (-1)^i (v_{2n-i} \beta, \alpha)_L. \end{aligned}$$

Next suppose $\gamma \in \text{im } v_i$ and $\delta \in H_{2n-i}(X - T^\circ; k)$. Then:

$$\begin{aligned} (s_i \gamma, h_{2n-i} \delta)_{IX} &= D_{IX}(s_i \gamma)(h_{2n-i} \delta) = (r^{2n-i} D_L \gamma)(h_{2n-i} \delta) \\ &= (D_L \gamma)(r_{2n-i} h_{2n-i} \delta) \\ &= (D_L \gamma)(\delta) = (\gamma, \delta)_L. \end{aligned}$$

Note that it makes sense to evaluate $D_L \gamma$ on $\text{im } r_{2n-i} = \text{coim } h_{2n-i}$, since γ belonging to $\text{im } v_i$ implies $D_\partial \gamma$ belongs to:

$$\text{im } h^{2n-i} = (\text{coim } h_{2n-i})^* = \{\phi \in H^{2n-i}(X - T^\circ) : \phi(\ker h_{2n-i}) = 0\}$$

In the computations at the beginning of this paragraph, if we replace $h_{2n-i} \delta$ with an element of $\ker r_{2n-i}$, it easily follows that $(\text{im } s_i, \ker r_{2n-i})_{IX} = 0$. $\square$

Lemma 7.2.1 informs us how to carry out the intersection space pairing, but we will seek an even finer decomposition of the homology vector spaces. We say a family of sections $(r_\bullet, s_\bullet)$ is **untwisted** if and only if each of the following hold:

- (i) $\text{im } s_\bullet j_\bullet \subset \text{im } h_\bullet$ (observe that $j_\bullet = v_\bullet h_\bullet$ implies $\text{im } j_\bullet \subset \text{im } v_\bullet$, so $s_\bullet j_\bullet$ is in fact well-defined).
- (ii) $\ker r_\bullet \subset \text{im } s_\bullet$ and $(\ker r_\bullet, \text{im } s_{2n-\bullet} j_{2n-\bullet})_{IX} = 0$.

We will prove the existence of an untwisted family of sections, but first must establish a technical lemma.

Lemma 7.2.2. *Suppose $s_\bullet$ satisfies property (i) of untwisted, and $Q_\bullet$ is the maximal subspace of $\text{im } v_\bullet$ satisfying the vanishing:*

$$(Q_\bullet, h_{2n-\bullet}^{-1}(\text{im } s_{2n-\bullet} j_{2n-\bullet}))_L = 0.$$

Then $\text{im } v_\bullet = Q_\bullet \oplus \text{im } j_\bullet$.

Proof. Fix $i \in \mathbb{Z}$. First we check that Q_i has trivial intersection with $\text{im } j_i$. Suppose $j_i \alpha \in \text{im } j_i \cap Q_i$. Then:

$$\begin{aligned} 0 &= (j_i \alpha, h_{2n-i}^{-1}(\text{im } s_{2n-i} j_{2n-i}))_L = (D_L j_i \alpha) (h_{2n-i}^{-1}(\text{im } s_{2n-i} j_{2n-i})) \\ &= (j^{2n-i} D'_L \alpha) (h_{2n-i}^{-1}(\text{im } s_{2n-i} j_{2n-i})) . \end{aligned}$$

Since $j^{2n-i} = h^{2n-i} v^{2n-i}$ and $\text{im } s_{2n-i} j_{2n-i} \subset \text{im } h_{2n-i}$, this sequence of equalities continues into:

$$\begin{aligned} 0 &= (D'_L \alpha)(v_{2n-i} \text{im } s_{2n-i} j_{2n-i}) = (D'_L \alpha)(\text{im } j_{2n-i}) \implies j^{2n-i} D'_L \alpha = 0 \\ &\implies D_L j_i \alpha = 0 \\ &\implies j_i \alpha = 0. \end{aligned}$$

Having shown the intersection is trivial, we use dimension counting to complete the proof. The duality isomorphism of long exact sequences of the pair $(X - T^\circ, \partial T)$ implies $D_L(\text{im } j_i) = \text{im } j^{2n-i}$. Vanishing of the local duality obstruction implies $D_L(\text{im } v_i) = \text{im } h^{2n-i}$ (see the proof of Lemma 3.4.2); in particular this implies $(\text{im } v_i, \ker h_{2n-i})_L = 0$. By construction, and the fact that $(\text{im } v_i, \ker h_{2n-i})_L = 0$, the vector space Q_i has dimension greater than or equal to the difference:

$$\begin{aligned} &\dim_k \text{im } v_i - \dim_k h_{2n-i}^{-1}(\text{im } s_{2n-i} j_{2n-i}) + \dim_k \ker h_{2n-i} \\ &= \dim_k \text{im } v_i - \dim_k \text{im } j_{2n-i} \\ &= \dim_k \text{im } v_i - \dim_k \text{im } j_i \end{aligned}$$

Rearranged, this is:

$$\dim_k Q_i + \dim_k \text{im } j_i \geq \dim_k \text{im } v_i,$$

completing the proof. $\square$

Proposition 7.2.3. *There always exists an untwisted family of sections $(r^\bullet, s_\bullet)$.*

Proof. First let's check that there exist sections $s_\bullet : \text{im } v_\bullet \rightarrow H_\bullet(I_{f_\bullet}X)$ of $v_\bullet$ such that $\text{im } s_\bullet j_\bullet \subset \text{im } h_\bullet$. This is possible iff $v_\bullet$ maps $\text{im } h_\bullet$ onto $\text{im } j_\bullet$ (because then we can construct a restricted section $s_\bullet| : \text{im } j_\bullet \rightarrow \text{im } h_\bullet \subset H_\bullet(I_{f_\bullet}X)$ which by choice of a basis for $\text{im } v_\bullet$ can be extended to a full section $s_\bullet$). But $\text{im } j_\bullet = \text{im } v_\bullet h_\bullet$, so $v_\bullet$ indeed maps $\text{im } h_\bullet$ surjectively onto $\text{im } j_\bullet$.

Given these sections $s_\bullet$ satisfying (i), we next verify that there exist retractions $r_\bullet : H_\bullet(I_{f_\bullet}X) \rightarrow \text{coim } h_\bullet$ satisfying (ii). Let $Q_\bullet$ be as in Lemma 7.2.2. We will first show that $r_\bullet$ can be selected so that $\ker r_\bullet \subset s_\bullet Q_\bullet$; this is possible iff $s_\bullet Q_\bullet + \text{im } h_\bullet = H_\bullet(I_{f_\bullet}X)$. We already know from Lemma 7.2.1 that $\text{im } s_\bullet + \text{im } u_\bullet = H_\bullet(I_{f_\bullet}X)$. But $g_\bullet u_\bullet = 0$ implies $\text{im } u_\bullet \subset \ker g_\bullet = \text{im } h_\bullet$, where we have used exactness in the last step. So:

$$\text{im } s_\bullet + \text{im } h_\bullet = H_\bullet(I_{f_\bullet}X).$$

If we can show $\text{im } s_\bullet = s_\bullet Q_\bullet + \text{im } s_\bullet j_\bullet$ then we will be done with selecting our $r_\bullet$, since $\text{im } s_\bullet j_\bullet \subset \text{im } h_\bullet$ by property (i) of being untwisted. But this follows by applying $s_\bullet$ to the equality of Lemma 7.2.2.

It remains to verify that $\ker r_\bullet \subset s_\bullet Q_\bullet$ satisfies property (ii). This is clear, because Lemma 7.2.1 and the definition of $Q_\bullet$ imply:

$$(s_\bullet Q_\bullet, \text{im } s_{2n-\bullet} j_{2n-\bullet})_{IX} = (Q_\bullet, h_{2n-\bullet}^{-1}(\text{im } s_{2n-\bullet} j_{2n-\bullet}))_L = 0.$$

□

Lemma 7.2.4. *Suppose $(r^\bullet, s_\bullet)$ is an untwisted family of sections and $i \in \mathbb{Z}$. Then there*

exist further decompositions:

$$\operatorname{im} h_i = \operatorname{im} u_i \oplus \operatorname{im} s_i j_i$$

$$\operatorname{im} s_i = \ker r_i \oplus \operatorname{im} s_i j_i.$$

Proof. Let's begin with the decomposition of $\operatorname{im} h_i$. First we verify $\operatorname{im} u_i \subset \operatorname{im} h_i$: we have $g_i u_i = 0$, so $\operatorname{im} u_i \subset \ker g_i = \operatorname{im} h_i$ where we have used exactness in the last equality. Next, we know from Lemma 7.2.1 that $\operatorname{im} s_i j_i$ has trivial intersection with $\operatorname{im} u_i$. Finally, we count dimension (using rank and nullity of maps):

$$\begin{aligned} \operatorname{rk} j_i = \operatorname{rk} v_i h_i &\geq \operatorname{rk} h_i - \operatorname{nul} v_i \implies \operatorname{rk} h_i \leq \operatorname{nul} v_i + \operatorname{rk} j_i \\ &\implies \dim_k \operatorname{im} h_i \leq \dim_k \ker v_i + \dim_k \operatorname{im} j_i \\ &\implies \dim_k \operatorname{im} h_i \leq \dim_k \operatorname{im} u_i + \dim_k \operatorname{im} s_i j_i \end{aligned}$$

where in the last step we have used exactness of a long exact sequence and injectivity of s_i .

Now let's approach the decomposition of $\operatorname{im} s_i$. First we verify $\ker r_i \cap \operatorname{im} s_i j_i = \langle 0 \rangle$: because the family of sections is untwisted, we know $\operatorname{im} s_i j_i \subset \operatorname{im} h_i$, but Lemma 7.2.1 implies $\operatorname{im} h_i \cap \ker r_i = \langle 0 \rangle$. Finally for (2), we again count dimension:

$$\begin{aligned} \dim_k \operatorname{im} u_i + \dim_k \operatorname{im} s_i &= \dim_k H_i(I_{f\bullet} X) \\ &= \dim_k \operatorname{im} h_i + \dim_k \ker r_i \\ &= \dim_k \operatorname{im} u_i + \dim_k \operatorname{im} s_i j_i + \dim_k \ker r_i. \end{aligned}$$

where we have used Lemma 7.2.1 for the first two inequalities, and (1) of this Lemma for the last. We then obtain:

$$\dim_k \operatorname{im} s_i = \dim_k \operatorname{im} s_i j_i + \dim_k \ker r_i$$

as desired. $\square$

7.3 Signature

Suppose throughout this section that $(r^\bullet, s_\bullet)$ is a family of sections. We would like to prove that, in the case that n is even, an untwisted family of sections $(r^\bullet, s_\bullet)$ induces a symmetric pairing:

$$(-, -)_{IX} : H_n(I_{f_\bullet} X) \times H_n(I_{f_\bullet} X) \rightarrow k.$$

But first let's compute the signature when $k = \mathbb{Q}$ if we *assume* the induced pairing $(-, -)_{IX}$ is symmetric. We will do so by comparing to the already existing symmetric pairing on $\text{im } j_n$:

$$(-, -)_j : \text{im } j_n \times \text{im } j_n \rightarrow \mathbb{Q}, \quad (j_n \alpha, j_n \beta)_j = (j_n \alpha, \beta)_L = (j_n \beta, \alpha)_L;$$

this signature is called the **Novikov signature**, which is known (for example, [19]) to equal the signature of the pairing:

$$IH_n(\widehat{X - T^\circ}; \mathbb{Q}) \times IH_n(\widehat{X - T^\circ}; \mathbb{Q}) \rightarrow \mathbb{Q}$$

where $\widehat{X - T^\circ}$ is the space $(X - T^\circ) \cup_T \text{cone}(T)$.

Theorem 7.3.1. *Suppose n is even, $k = \mathbb{Q}$, and $(r^\bullet, s_\bullet)$ is an untwisted family of sections that induces a symmetric intersection space pairing:*

$$(-, -)_{IX} : H_n(I_{f_\bullet} X) \times H_n(I_{f_\bullet} X) \rightarrow \mathbb{Q}.$$

Then the signature of $(-, -)_{IX}$ is equal to the Novikov signature.

Proof. Remember that throughout this proof we are *assuming* $(-, -)_{IX}$ is symmetric. We will frequently use Lemmas 7.2.1 and 7.2.4 in this proof. Combined they give us the decompositions:

$$H_n(I_{f_\bullet}X) = \text{im } u_n \oplus \text{im } s_n j_n \oplus \ker r_n$$

$$\text{im } h_n = \text{im } u_n \oplus \text{im } s_n j_n$$

$$\text{im } s_n = \text{im } s_n j_n \oplus \ker r_n.$$

Let's observe how these components pair. By Lemma 7.2.1, the above decompositions, and symmetry of the intersection space pairing, we know that under the intersection space pairing:

$$\text{im } u_n \perp (\text{im } u_n \oplus \text{im } s_n j_n)$$

$$\text{im } s_n j_n \perp (\text{im } u_n \oplus \ker r_n)$$

$$\ker r_n \perp (\text{im } s_n j_n \oplus \ker r_n).$$

Therefore, in a basis that respects the direct sum decomposition $\text{im } u_n \oplus \text{im } s_n j_n \oplus \ker r_n$, the pairing $(-, -)_{IX}$ is represented by a symmetric block matrix of the form:

$$M = \begin{pmatrix} 0 & 0 & A \\ 0 & Y & 0 \\ A^T & 0 & 0 \end{pmatrix},$$

where Y is the symmetric matrix associated to the restricted pairing:

$$(-, -)_{IX} : \text{im } s_n j_n \times \text{im } s_n j_n \rightarrow \mathbb{Q}.$$

Let $p(t)$ and $q(t)$ be the respective characteristic polynomials for Y and AA^T . Linear algebra shows that the characteristic polynomial of the block matrix M is the product:

$$p(t)q(t^2).$$

Therefore the signature of M (the number of positive eigenvalues minus the number of negative eigenvalues) is equal to the signature of Y , i.e. the signature of the restricted pairing:

$$\mathrm{im} \, s_n j_n \times \mathrm{im} \, s_n j_n \rightarrow \mathbb{Q}.$$

It remains to prove that the signature of this restricted pairing is the same as the signature of $(-, -)_j$; this is established if we can prove that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{im} \, s_n j_n \times \mathrm{im} \, s_n j_n & \longrightarrow & k \\ s_n \times s_n \uparrow \wr & & \parallel \\ \mathrm{im} \, j_n \times \mathrm{im} \, j_n & \longrightarrow & k. \end{array}$$

Let $j_n \alpha$ and $j_n \beta$ in $\mathrm{im} \, j_n$ be given. Since $\mathrm{im} \, s_n j_n \subset \mathrm{im} \, h_n$ by untwistedness of the family of sections, there exists $\gamma \in H_n(X - T^\circ)$ such that $h_n \gamma = s_n j_n \beta$. By Lemma 7.2.1 we have:

$$(s_n j_n \alpha, s_n j_n \beta)_{IX} = (s_n j_n \alpha, h_n \gamma)_{IX} = (j_n \alpha, \gamma)_\partial = (j_n \gamma, \alpha)_\partial,$$

where symmetry of the j -pairing was used in the last step. Since $j_n = v_n h_n$ this further becomes:

$$(s_n j_n \alpha, s_n j_n \beta)_{IX} = (v_n h_n \gamma, \alpha)_\partial = (v_n s_n j_n \beta, \alpha) = (j_n \beta, \alpha)_\partial = (j_n \alpha, j_n \beta)_j.$$

This proves that the diagram in question commutes, and we are finished. $\square$

Next we verify that the pairing $(-, -)_{IX}$ induced by an untwisted family is indeed symmetric. If V and W are subspace of $H_n(I_{f_\bullet} X)$, we say that the pairing $(-, -)_{IX}$ is **symmetric on** (V, W) if and only if for all $\alpha \in V$ and $\beta \in W$ we have:

$$(\alpha, \beta)_{IX} = (\beta, \alpha)_{IX}.$$

Proposition 7.3.2. *Suppose n is even and $(r^\bullet, s_\bullet)$ is an untwisted family of sections. Then $(-, -)_{IX}$ is symmetric on $H_n(I_{f_\bullet}X)$.*

Proof. *Symmetry on $(\text{im } u_n, \text{im } u_n)$:* By Lemmas 7.2.1 and 7.2.4, we know the pairing $(\text{im } u_n, \text{im } u_n)_{IX} = 0$.

Symmetry on $(\text{im } u_n, \text{im } s_n j_n)$: By the same Lemmas, we know $(\text{im } u_n, \text{im } s_n j_n)_{IX} = 0$. We must then check $(\text{im } s_n j_n, \text{im } u_n)_{IX} = 0$. Let $s_n j_n \alpha$ and $u_n \beta$ be given. Since $\text{im } u_n \subset \text{im } h_n = \ker v_n$, we can write $u_n \beta = h_n \gamma$ and we observe:

$$j_n \gamma = v_n h_n \gamma = v_n u_n \beta = 0.$$

Using Lemma 7.2.1 and symmetry of the j -pairing we compute:

$$(s_n j_n \alpha, u_n \beta)_{IX} = (s_n j_n \alpha, h_n \gamma)_{IX} = (j_n \alpha, \gamma)_L = (j_n \gamma, \alpha)_L = 0.$$

Symmetry on $(\text{im } u_n, \ker r_n)$. Let $u_n \ell_n \alpha \in \text{im } u_n \ell_n = \text{im } u_n$ and $s_n \beta \in \ker r_n \subset \text{im } s_n$ be given. Since $\text{im } u_n \subset \text{im } h_n$ by Lemma 7.2.4, there exists γ such that $u_n \ell_n \alpha = h_n \gamma$. By Lemma 7.2.1:

$$(u_n \ell_n \alpha, s_n \beta)_{IX} = (v_n s_n \beta, \iota_n \alpha)_L = (\beta, \iota_n \alpha)_L$$

and:

$$(s_n \beta, u_n \ell_n \alpha)_{IX} = (s_n \beta, h_n \gamma)_{IX} = (\beta, \gamma)_L$$

Next note that $\beta \in \text{im } v_n$ implies $(\beta, -)_L \in \text{im } h^n$ vanishes on $\ker h_n$. So symmetry can be proven if $\iota_n \alpha - \gamma \in \ker h_n$. This follows from:

$$h_n \iota_n \alpha = u_n \ell_n \alpha = h_n \gamma.$$

Symmetry on $(\text{im } s_n j_n, \text{im } s_n j_n)$. Let $s_n j_n \alpha, s_n j_n \beta \in \text{im } s_n j_n$. The reasoning from the last part of the proof of Theorem 7.3.1 did not rely on symmetry of $(-, -)_{IX}$ and shows:

$$(s_n j_n \alpha, s_n j_n \beta)_{IX} = (j_n \alpha, j_n \beta)_j$$

which of course is symmetric.

Symmetry on $(\text{im } s_n j_n, \ker r_n)$. By Lemma 7.2.1 we have the vanishing of the pairing $(\text{im } s_n j_n, \ker r_n)_{IX} = 0$. By property (ii) of being untwisted, we have the reverse vanishing $(\ker r_n, \text{im } s_n j_n)_{IX}$. $\square$

Bibliography

- [1] M. AGUSTIN AND J. F. DE BOBADILLA, *Intersection space constructible complexes*, arXiv preprint arXiv:1804.06185, (2018).
- [2] M. BANAGL, *Topological invariants of stratified spaces*, Springer Science & Business Media, 2007.
- [3] —, *Intersection spaces, spatial homology truncation, and string theory*, Springer Science & Business Media, 2010.
- [4] —, *Isometric group actions and the cohomology of flat fiber bundles*, arXiv preprint arXiv:1105.0811, (2011).
- [5] —, *First cases of intersection spaces in stratification depth 2*, Journal of Singularities, 5 (2012), pp. 57–84.
- [6] M. BANAGL, N. BUDUR, L. MAXIM, ET AL., *Intersection spaces, perverse sheaves and type iib string theory*, Advances in Theoretical and Mathematical Physics, 18 (2014), pp. 363–399.
- [7] M. BANAGL AND B. CHRIESTENSON, *Intersection spaces, equivariant moore approximation and the signature*, Journal of Singularities, 16 (2016), pp. 147–179.
- [8] M. BANAGL AND L. MAXIM, *Deformation of singularities and the homology of intersection spaces*, Journal of Topology and Analysis, 4 (2012), pp. 413–448.

- [9] —, *Intersection spaces and hypersurface singularities*, Journal of Singularities, 5 (2012), pp. 48–56.
- [10] A. BOREL, *Intersection cohomology*, Springer Science & Business Media, 2009.
- [11] J. F. DAVIS AND P. KIRK, *Lecture notes in algebraic topology*, vol. 35, American Mathematical Soc., 2001.
- [12] A. DIMCA, *Singularities and topology of hypersurfaces*, Springer Science & Business Media, 2012.
- [13] G. FRIEDMAN, *Singular intersection homology*, Book in progress, (2014).
- [14] M. GORESKY AND R. MACPHERSON, *Intersection homology theory*, Topology, 19 (1980), pp. 135–162.
- [15] —, *Intersection homology ii*, Inventiones Mathematicae, 72 (1983), pp. 77–129.
- [16] —, *Stratified morse theory*, in Stratified Morse Theory, Springer, 1988, pp. 3–22.
- [17] J. MATHER, *Notes on topological stability*, Bulletin of the American Mathematical Society, 49 (2012), pp. 475–506.
- [18] J. MILNOR, *Singular Points of Complex Hypersurfaces.(AM-61)*, vol. 61, Princeton University Press, 2016.
- [19] P. H. SIEGEL, *Witt spaces: a geometric cycle theory for ko -homology at odd primes*, American Journal of Mathematics, 105 (1983), pp. 1067–1105.
- [20] L. VAN DEN DRIES, C. MILLER, ET AL., *Geometric categories and o-minimal structures*, Duke Math. J, 84 (1996), pp. 497–540.

- [21] C. A. WEIBEL, *An introduction to homological algebra*, Cambridge university press, 1995.