Decoupling inequalities for quadratic forms

by

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A dissertation submitted in partial fulfillment
of the requirements for the degree of

Doctor of Philosophy
Mathematics

at the
University of Wisconsin-Madison
2022

Date of Final Oral Exam: 4/19/2022
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Abstract

In this dissertation, we consider decoupling inequalities. The study of this topic originated from classical problems in number thoery, for example, Waring's problem, Wely sum estimate, and estimates of the Riemann zeta function.

We prove sharp $l^q L^p$ decoupling inequalities for $p, q \in [2, \infty)$ and arbitrary tuples of quadratic forms. estimates. The proof of our main result is based on scale-dependent Brascamp–Lieb inequalities.

Acknowledgements

I would like to express my deep gratitude to my advisor, Shaoming Guo. I was very fortunate to be his first Ph.D. student. He always suggested interesting research problems, shared his ideas, and made me motivated. He was not only an advisor but also a good friend. We used to go for a drive, and I sincerely enjoyed a trip to Devil's Lake with him.

I would like to thank Andreas Seeger for taking care of me during my first semester. I would like to thank Betsy Stovall, Brian Street, and Ruixiang Zhang for teaching enlightening harmonic analysis courses. I would like to thank Botong Wang for being committee members. I would like to thank my collaborators including Shengwen Gan, Joris Roos, Hong Wang, Shukun Wu, Po-Lam Yung, Pavel Zorin-Kranich, and Ruixiang Zhang. It was a great pleasure collaborating with outstanding mathematicians. Finally, I would like to thank my family for their love and support.

Contents

Chapter 1

Introduction

In this thesis, we study decoupling inequalities for quadratic forms. The study of this topic originated from classical problems in number theory, for example, Waring's problem, Wely sum estimate, and estimates of the Riemann zeta function. To motivate our main theorem (Theorem 1.3.3), we begin by describing Waring's problem.

1.1 Waring's problem

The classical Waring's problem traces back to Lagrange's four-square theorem. In 1770, Lagrange proved that every positive integer can be represented as the sum of four integer squares. In other words, for every positive integer n, there exist some integers a_1, a_2, a_3, a_4 such that

$$n = a_1^2 + a_2^2 + a_3^2 + a_4^2. \tag{1.1.1}$$

A generalization of Lagrange's theorem to higher degrees is well understood. For every positive integer $k \geq 2$, denote by $g(k)$ the smallest integer s (possibly infinity) of kth powers of positive integers needed to represent all positive integers. If $g(k)$ is finite, then according to the definition, for every positive integer n, there exist some integers $a_1, \ldots, a_{g(k)}$ such that

$$n = a_1^k + a_2^k + \cdots + a_{g(k)}^k. \tag{1.1.2}$$

The classical Waring's problem is to find the number $g(k)$. A satisfactory answer to this problem is already known a long time ago. It is proved by Hilbert 1909 that $g(k)$ is finite for every k. The formula of $g(k)$ is also known by many mathematicians, for example, Dickson 1936 and Niven 1944. We do not state the formula here.

Modern Waring's problems concern only sufficiently large integers. For $k \geq 2$, we denote by $G(k)$ the smallest number s such that every "sufficiently large" integer is the sum of at most s kth powers of positive integers. If $G(k)$ is finite, then for every sufficiently large integer n, there exist some integers $a_1, \ldots, a_{G(k)}$ such that

$$n = a_1^k + a_2^k + \cdots + a_{G(k)}^k. \tag{1.1.3}$$

A modern Waring's problem is to find the number $G(k)$. It is conjectured that $G(k) = k+1$. For large k, the best upper bound of $G(k)$ is proved by Trevor D. Wooley 1995;

$$G(k) \leq k(\log k + \log\log k + 2 + O(\log\log k/\log k)). \tag{1.1.4}$$

There is still some gap between the upper bound and the conjectured number, but asymptotically it already gives an almost satisfactory answer to the modern Waring's problem.

Let us move on to a stronger modern Waring's problem. Denote $R_{s,k}(n)$ the number of representations of the positive integer n as sum of s kth powers. For s sufficiently large, a precise asymptotic of $R_{s,k}(n)$ is already known (see the equation (2.1) of Jean Bourgain 2016). The asymptotic formula is involved, so we do not state it here. Denote $\tilde{G}(k)$ the smallest integer s for which the asymptotic holds true. It is known that

$$G(k) \leq \tilde{G}(k). \tag{1.1.5}$$

As in the case of $G(k)$, it is conjectured that $\tilde{G}(k) = k + 1$. A stronger modern Waring's problem is to find the number $\tilde{G}(k)$. The best upper bounds of $\tilde{G}(k)$ are obtained by Vaughan, Bourgain-Demeter-Guth-Wooley, and Bourgain. Let us first state the theorem of Bourgain-Demeter-Guth-Wooley.

Theorem 1.1.1 (Jean Bourgain, Demeter, and Guth 2016; Trevor D. Wooley 2012)**.** *For $k \geq 3$,*

$$\tilde{G}(k) \leq k^2 + 1 - \max_{\substack{1 \leq j \leq k-1 \\ 2^j \leq k^2}} \left\lceil \frac{kj - 2^j}{k + 1 - j} \right\rceil, \tag{1.1.6}$$

where $\lceil x \rceil$ is the smallest integer no smaller than x.

The bound (1.1.6) is derived by Trevor D. Wooley 2012 under the assumption of the main conjecture in Vinogradov's mean value theorem. The contribution of Jean Bourgain, Demeter, and Guth 2016 is a verification of the conjecture for high degrees. We explain the conjecture in the next section. We note that for large k, the bound (1.1.6) is improved by Jean Bourgain 2016, where the main conjecture in Vinogradov's mean value theorem is still used as a main ingredient. For the case $k = 3$, the best upper bound of $\tilde{G}(k)$ is obtained by Vaughan 1986. Lastly, we refer to Vaughan and T. D. Wooley 2002 and Trevor D. Wooley 2014 for historical backgrounds on Waring's problem.

1.2 Translation-dilation invariant systems

Let us introduce the main conjecture in Vinogradov's mean value theorem. Let $s \geq 1$ and $n \geq 2$ be integers. Consider the system of Diophantine equations

$$\begin{aligned} x_1 + \cdots + x_s &= x_{s+1} + \cdots + x_{2s} \\ x_1^2 + \cdots + x_s^2 &= x_{s+1}^2 + \cdots + x_{2s}^2 \\ \vdots \quad &= \quad \vdots \\ x_1^n + \cdots + x_s^n &= x_{s+1}^n + \cdots + x_{2s}^n. \end{aligned} \tag{1.2.1}$$

This system is called Vinogradov system. Denote by $J_{s,n}(N)$ the number of integral solutions $(x_1, \ldots, x_{2s}) \in [-N, N]^{2s}$ satisfying Vinogradov system. The main conjecture in Vinogradov's mean value theorem states that for every $\epsilon > 0$, $s \geq 1$, $n, N \geq 2$,

$$J_{s,n}(N) \leq C_\epsilon N^\epsilon (N^s + N^{2s - \frac{n(n+1)}{2}}). \tag{1.2.2}$$

This upper bound is sharp up to the loss of N^ϵ. As mentioned in the previous section, this conjecture is verified independently by Jean Bourgain, Demeter, and Guth 2016 and Trevor D. Wooley 2016; Trevor D. Wooley 2019.

Theorem 1.2.1 (Jean Bourgain, Demeter, and Guth 2016; Trevor D. Wooley 2016; Trevor D. Wooley 2019)**.** (1.2.2) *holds true.*

The proofs of Bourgain-Demeter-Guth and Wooley are different. From now on, we focus only on the proof of Bourgain-Demeter-Guth. They proved a slightly stronger inequality, called a decoupling inequality for the moment curve. The decoupling inequality implies the main conjecture in Vinogradov's mean value theorem. Let us postpone the introduction of decoupling inequalities to the next section, and explain a generalization of Vinogradov system.

Vinogradov system is naturally extended to translation-dilation invariant systems. Let us give the definition of the systems. Let $s, d \geq 1$ and $n \geq 2$. Consider

$$\begin{aligned} F_1(\mathbf{x}_1) + \cdots + F_1(\mathbf{x}_s) &= F_1(\mathbf{x}_{s+1}) + \cdots + F_1(\mathbf{x}_{2s}) \\ F_2(\mathbf{x}_1) + \cdots + F_2(\mathbf{x}_s) &= F_2(\mathbf{x}_{s+1}) + \cdots + F_2(\mathbf{x}_{2s}) \\ \vdots \qquad &= \qquad \vdots \\ F_n(\mathbf{x}_1) + \cdots + F_n(\mathbf{x}_s) &= F_n(\mathbf{x}_{s+1}) + \cdots + F_n(\mathbf{x}_{2s}), \end{aligned} \tag{1.2.3}$$

where $F_i(\mathbf{x}) \in \mathbb{Z}[x_1, \ldots, x_d]$ and $\mathbf{x}_j = (x_{j1}, \ldots, x_{jd}) \in \mathbb{Z}^d$ for every i and j.

Definition 1.2.2. Consider the system $\mathbf{F} = (F_1, \ldots, F_n)$ given by (1.2.3). We say that the system is translation-dilation invariant if

1. the polynomials $F_1, \ldots, F_n$ are each homogeneous of positive degree, and
2. there exist polynomials

   $$c_{jl}(\boldsymbol{\xi}) \in \mathbb{Z}[\xi_1, \ldots, \xi_d] \ \ (1 \leq j \leq n \ \text{ and } \ 0 \leq l \leq j), \tag{1.2.4}$$

   with $c_{jj} = 1$ for $1 \leq j \leq n$, having the property that whenever $\xi \in \mathbb{Z}^d$, then

   $$F_j(\mathbf{x} + \boldsymbol{\xi}) = c_{j0}(\boldsymbol{\xi}) + \sum_{l=1}^{j} c_{jl}(\boldsymbol{\xi}) F_l(\mathbf{x}) \ \ (1 \leq j \leq n). \tag{1.2.5}$$

Given a translation-dilation invariant system $\mathbf{F}$, denote by $J_{\mathbf{F},s,n}(N)$ the number of integral solutions of the system with $1 \leq x_{ik} \leq N$ for every i, k. It is asked by Parsell, Prendiville, and Trevor D. Wooley 2013 to find the lower bound and upper bound of $J_{\mathbf{F},s,n}(N)$. Even though there has been significant progress, for example, by Parsell, Prendiville, and Trevor D. Wooley 2013; Jean Bourgain and Demeter 2015; Jean Bourgain, Demeter, and Guth 2016; Shaoming Guo and Ruixiang Zhang 2019; Shaoming Guo and Zorin-Kranich 2020b, this problem is still widely open. Our main theorem (Theorem 1.3.3) concerns the quadratic case of translation-dilation invariant systems in the sense that all the degrees of F_i are less or equal to two. We discuss more in the next section.

1.3 Decoupling inequalities

In this section, we introduce decoupling inequalities and state our main theorem.

By row operations, any translation-dilation invariant system can be rewritten as

$$\mathbf{F}_m := \big(\xi_1, \dots, \xi_d, P_1(\xi_1, \dots, \xi_d), \dots, P_{n-d}(\xi_1, \dots, \xi_d)\big). \tag{1.3.1}$$

Let us call this a translation-dilation invariant manifold. For simplicity, we sometimes call this a TDI manifold. The degree of a TDI manifold, denoted by $\deg(\mathbf{F}_m)$, is defined by the highest degree of P_i. For each TDI manifold and set $\square \subset [0,1]^d$, we define an extension operator associated with the system by

$$E_\square^{\mathbf{F}_m} f(x) := \int_\square f(\xi) e\big(x' \cdot \xi + x'' \cdot P(\xi)\big)\, d\xi, \tag{1.3.2}$$

where $x = (x', x'') \in \mathbb{R}^d \times \mathbb{R}^{n-d}$, $P = (P_1, \dots, P_{n-d})$, and $e(t) = e^{2\pi i t}$.

Let us introduce decoupling inequalities. For some technical reasons, we first define some weight functions; for each ball $B(c_B, r_B) \subset \mathbb{R}^n$ with center c_B and radius r_B, define an associated weight

$$w_B(\cdot) := \Big(1 + \frac{|\cdot - c_B|}{r_B}\Big)^{-10n}. \tag{1.3.3}$$

Definition 1.3.1. Let $D_{p,q}(\mathbf{F}_m, \delta)$ be the smallest constant D such that

$$\big\|E_{[0,1]^d}^{\mathbf{F}_m} f\big\|_{L^p(w_B)} \le D\Big(\sum_{\substack{\square \subset [0,1]^d \\ l(\square) = \delta}} \big\|E_\square^{\mathbf{F}_m} f\big\|_{L^p(w_B)}^q\Big)^{1/q} \tag{1.3.4}$$

holds for every measurable function f and every ball $B \subset \mathbb{R}^n$ of radius $\delta^{-\deg(\mathbf{F}_m)}$. This inequality is called a $l^q L^p$ decoupling inequality for the TDI manifold $\mathbf{F}_m$.

The decoupling problem for the TDI manifold $\mathbf{F}_m$ is to find the decoupling constants $D_{p,q}(\mathbf{F}_m, \delta)$. A major motivation of finding the decoupling constants is that a decoupling inequality for $\mathbf{F}_m$ gives a upper bound of $J_{\mathbf{F}_m,s,n}(N)$. In particular, a decoupling inequality for a moment curve $(t, t^2, \dots, t^n)$ gives sharp solution counting of Vinogradov system.

Theorem 1.3.2 (Jean Bourgain, Demeter, and Guth 2016)**.** *Let $n \ge 2$ and $\mathbf{F}_V = (t, t^2, \dots, t^n)$. Then for every $\epsilon > 0$ and $0 < \delta < 1$*

$$D_{p,2}(\mathbf{F}_V, \delta) \le C_\epsilon \delta^{-\epsilon} \max\big(1, \delta^{-(\frac{1}{2} - \frac{n(n+1)}{2p})}\big). \tag{1.3.5}$$

Moreover, this inequality implies Theorem 1.2.1.

Before stating our main theorem, let us first review related results in the literature. Decoupling theory originated from the work of T. Wolff 2000 and was further developed by Łaba and Thomas Wolff 2002, Łaba and Pramanik 2006, Garrigós and Seeger 2009, Garrigós and Seeger 2010, and Jean. Bourgain 2013. A breakthrough came with the resolution of the ℓ^2-decoupling conjecture for paraboloids by Jean Bourgain and Demeter 2015. Subsequently, Jean Bourgain, Demeter, and Guth 2016 resolved the Main Conjecture in

Vinogradov's mean value theorem using decoupling theory. We also refer to Jean Bourgain and Demeter 2016b, Jean Bourgain, Demeter, and Shaoming Guo 2017, Shaoming Guo and Ruixiang Zhang 2019, and Shaoming Guo and Zorin-Kranich 2020b for extensions of Jean Bourgain, Demeter, and Guth 2016 to higher dimensions.

In this thesis, we study sharp decoupling inequalities for quadratic d-surfaces in $\mathbb{R}^{d+n}$ with $d, n \geq 1$. The cases $n = 1, d \geq 1$, that is, quadratic hypersurfaces, were the objects studied in Jean Bourgain and Demeter 2015 and Jean Bourgain and Demeter 2017b. Since these works, there have been a number of other works studying sharp decoupling inequalities for quadratic d-surfaces in $\mathbb{R}^{d+n}$ with $n \geq 2$, that is, manifolds of co-dimension greater than one. Bourgain's improvement on the Lindelöf hypothesis (Jean. Bourgain 2017) relies on a decoupling inequality in the case $d = n = 2$, which was later generalized and extended to a more general family of manifolds with dimension and codimension 2 in Jean Bourgain and Demeter 2016a. Further sharp decoupling inequalities for (non-degenerate) quadratic d-surfaces of co-dimension 2 were proven, for $2 \leq d \leq 4$, in Demeter, Shaoming Guo, and Shi 2019 and Shaoming Guo and Zorin-Kranich 2020a. More recently, in Shaoming Guo, Changkeun Oh, Roos, Yung, and Zorin-Kranich 2019, the classification of sharp decoupling inequalities for quadratic 3-surfaces in $\mathbb{R}^5$ was completed, and sharp decoupling inequalities were proved in the "degenerate" cases, which were not covered by previously mentioned works. The approach to the "degenerate" cases in Shaoming Guo, Changkeun Oh, Roos, Yung, and Zorin-Kranich 2019 stands out from the previously mentioned works, in that it relies on small cap decouplings for the parabola and the 2-surface $(\xi_1, \xi_2, \xi_1^2, \xi_1\xi_2)$ (we refer also to Demeter, Guth, and Wang 2020 for further discussion of small cap decouplings). For manifolds of co-dimension $n > 2$, the only result for quadratic forms that are not monomials prior to the current article was by Changkeun Oh 2018, who proved sharp decoupling inequalities for non-degenerate 3-surfaces in $\mathbb{R}^6$. In this thesis, we provide a unified approach that takes care of all the above examples, and indeed all quadratic d-surfaces in $\mathbb{R}^{d+n}$ for arbitrary combinations of d and n.

Theorem 1.3.3 (Shaoming. Guo, Changkeun. Oh, Ruixiang. Zhang, and Zorin-Kranich 2021)**.** *For all $p, q \geq 2$, the sharp decoupling theorem holds true for every translation-dilation invariant manifold of degree 2.*

We refer to Theorem 3.1.1 for the complete statement of the theorem.

Beyond decoupling theory, problems associated with quadratic d-surfaces ($d \geq 2$) of co-dimension bigger than one have also attracted much attention, in particular in Fourier restriction theory and related areas. We refer to Christ 1985, Christ 1982, Mockenhaupt 1996, Bak and S. Lee 2004, Bak, Jungjin Lee, and S. Lee 2017, Juyoung Lee and S. Lee 2022, Shaoming Guo and Changkeun Oh 2020 for the restriction problems associated with manifolds of co-dimension two and higher, Jean Bourgain 1991, Rogers 2006, R. Oberlin 2007 for the planar variant of the Kakeya problem, and D. M. Oberlin 2004 for sharp $L^p \to L^q$ improving estimates for a quadratic 3-surface in $\mathbb{R}^5$. Recently, Gressman 2019a, Gressman 2019b and Gressman 2021 has made significant progress in proving sharp L^p-improving estimates for Radon transforms of intermediate dimensions. Perhaps more interestingly, he connected this problem with Brascamp-Lieb inequalities and geometric invariant theory.

One major difficulty in the development of the above mentioned problems in the setting of co-dimension bigger than one is the lack of a good notion of "curvature". This is in strong contrast with the case of co-dimension one, where we have Gaussian curvatures and the notion of rotational curvatures, introduced by Phong and Stein 1986a and Phong and Stein 1986b.

1.4 Organization of the thesis

The rest of the dissertation is organized as follows. In Chapter 2, we prove sharp decoupling inequalities for all degenerate surfaces of codimension two in $\mathbb{R}^5$ given by two quadratic forms in three variables. Together with previous work by Demeter, Guo, and Shi in the non-degenerate case, this provides a classification of decoupling inequalities for pairs of quadratic forms in three variables. In Chapter 3, we prove sharp $l^q L^p$ decoupling inequalities for $p, q \in [2, \infty)$ and arbitrary tuples of quadratic forms. Connections to prior results on decoupling inequalities for quadratic forms are also explained. We also include some applications of our results to exponential sum estimates and to Fourier restriction estimates.

Chapter 2

Decoupling inequalities for two quadratic forms in $\mathbb{R}^5$

2.1 Introduction

We begin by recalling the definition of decoupling constants. Let $d, n \geq 1$ be integers. For real quadratic forms $Q_1, \ldots, Q_n$ in d variables, consider the surface

$$\mathcal{S}_0 = \{(\boldsymbol{t}, Q_1(\boldsymbol{t}), \ldots, Q_n(\boldsymbol{t})) \,|\, \boldsymbol{t} \in [0,1]^d\}. \tag{2.1.1}$$

For a dyadic cube $\square \subset [0,1]^d$ with side length δ, we will use $f_\square$ to denote a function with

$$\operatorname{supp}(\widehat{f_\square}) \subset \{(\boldsymbol{t}, Q_1(\boldsymbol{t}) + \delta^{(1)}, \ldots, Q_n(\boldsymbol{t}) + \delta^{(n)}) \,|\, \boldsymbol{t} \in \square, |\delta^{(1)}|, \ldots, |\delta^{(n)}| \leq \delta^2\}. \tag{2.1.2}$$

For $2 \leq p < \infty$ and a dyadic number $\delta \in (0,1)$, the *decoupling constant* $\mathcal{D}_{\mathcal{S}_0}(\delta, p)$ is the smallest constant A such that the inequality

$$\Big\| \sum_{\square \in \mathcal{P}(\delta)} f_\square \Big\|_{L^p(\mathbb{R}^{d+n})} \leq A \Big(\sum_{\square \in \mathcal{P}(\delta)} \big\| f_\square \big\|^p_{L^p(\mathbb{R}^{d+n})} \Big)^{1/p}, \tag{2.1.3}$$

where $\mathcal{P}(\delta)$ is the partition of $[0,1]^d$ into dyadic cubes with side length δ, holds for every choice of functions $f_\square$ satisfying (2.1.2); replacing the $\ell^p L^p$ norm on the right hand side of (2.1.3) by $\ell^\infty L^\infty$ gives the definition when $p = \infty$.

In this chapter we are interested in the case $d = 3$, $n = 2$. We will also use existing results for smaller values of d and n, which necessitates defining (2.1.3) in more generality. We will denote by (P, Q) a pair of real quadratic forms in three variables and by $\mathcal{S}$ the surface

$$\mathcal{S} := \{(r, s, t, P(r,s,t), Q(r,s,t)) \,|\, (r,s,t) \in [0,1]^3\}. \tag{2.1.4}$$

Our goal is to prove, for every $2 \leq p < \infty$, an essentially sharp bound on $\mathcal{D}_{P,Q}(\delta, p) := \mathcal{D}_{\mathcal{S}}(\delta, p)$ as $\delta \to 0$. In order to formulate our results in a concise way, we introduce the *sharp decoupling exponent*

$$\gamma_{P,Q}(p) := \inf\{\gamma \geq 0 \,|\, \mathcal{D}_{P,Q}(\delta, p) \lesssim \delta^{-\gamma}\}. \tag{2.1.5}$$

In other words, $\gamma_{P,Q}(p)$ is the smallest exponent γ such that, for every $\epsilon > 0$, we have

$$\mathcal{D}_{P,Q}(\delta, p) \lesssim_\epsilon \delta^{-\gamma-\epsilon} \text{ for every } \delta \in (0,1). \tag{2.1.6}$$

2.1.1 Previous results

The case of linearly dependent P and Q is equivalent to the case $n = 1$ of (2.1.1). In this case, sharp decoupling inequalities were proved by Bourgain and Demeter Jean Bourgain and Demeter 2017b, Theorem 1.1. Henceforth, we assume that P and Q are linearly independent.

Moreover, we assume that there is no linear change of variables in (r, s, t) such that P and Q both omit one of the variables, as otherwise we can reduce to the case $d = n = 2$ that was considered in Jean Bourgain and Demeter 2016a.

We say that the pair (P, Q) is *non-degenerate* if both of the following conditions hold:

$$\det(\nabla P, \nabla Q, u) \not\equiv 0 \text{ for all } u \in \mathbb{R}^3 \setminus \{0\}, \tag{2.1.7}$$

$$P \text{ and } Q \text{ do not share a common linear real factor.} \tag{2.1.8}$$

Examples show that, for any pair (P, Q) of quadratic polynomials in 3 variables, the sharp decoupling exponent satisfies

$$\gamma_{P,Q}(p) \geq \begin{cases} 3(\frac{1}{2} - \frac{1}{p}) & \text{if } 2 \leq p \leq 14/3, \\ 3 - \frac{10}{p} & \text{if } 14/3 \leq p \leq \infty. \end{cases} \tag{2.1.9}$$

In the non-degenerate case, Demeter, Shi, and the first author Demeter, Shaoming Guo, and Shi 2019 (see also Shaoming Guo and Zorin-Kranich 2020a for a simplified proof) proved that, in fact,

$$\gamma_{P,Q}(p) = \begin{cases} 3(\frac{1}{2} - \frac{1}{p}) & \text{if } 2 \leq p \leq 14/3, \\ 3 - \frac{10}{p} & \text{if } 14/3 \leq p \leq \infty. \end{cases} \tag{2.1.10}$$

Therefore, it is a natural question to find the minimal requirements for P and Q such that the decoupling inequality (2.1.6) holds with the smallest possible sharp decoupling exponent (2.1.10).

2.1.2 Classification of pairs (P, Q)

We say that two pairs of quadratic forms (P, Q) and (P', Q') are equivalent if there exist two invertible real matrices $L_1 \in M_{3\times 3}$ and $L_2 \in M_{2\times 2}$ such that

$$(P'(r, s, t), Q'(r, s, t)) = L_2 \cdot (P(L_1 \cdot (r, s, t)), Q(L_1 \cdot (r, s, t))). \tag{2.1.11}$$

This defines an equivalence relation, which we denote by

$$(P, Q) \equiv (P', Q'). \tag{2.1.12}$$

By changing coordinates, it is easy to see that

$$\mathcal{D}_{P,Q}(\delta, p) \approx \mathcal{D}_{P',Q'}(\delta, p), \tag{2.1.13}$$

with an implicit constant depending only on L_1 and L_2 in (2.1.11), and in particular $\gamma_{P,Q}(p) = \gamma_{P',Q'}(p)$. The following result describes all possible sharp decoupling exponents for two quadratic forms in three variables that do not omit any variable.

Theorem 2.1.1. *Let (P, Q) be a pair of linearly independent quadratic forms. Assume that there is no linear change of variables in (r, s, t) after which P and Q both omit one of the variables. Then exactly one of the following alternatives holds.*

1. *(P, Q) is non-degenerate, that is, both (2.1.7) and (2.1.8) hold. In this case, the sharp decoupling exponent is given by (2.1.10).*

2. *(2.1.7) holds, but (2.1.8) fails. In this case, $(P, Q) \equiv (rs, rt)$, and*

$$\gamma_{P,Q}(p) = \begin{cases} 2 - \frac{4}{p} & \text{if } 2 \le p \le 6, \\ 3 - \frac{10}{p} & \text{if } 6 \le p \le \infty. \end{cases} \tag{2.1.14}$$

3. *(2.1.7) fails, but (2.1.8) holds. In this case, either $(P, Q) \equiv (r^2, s^2 \pm t^2)$, or $(P, Q) \equiv (r^2, s^2 + rt)$. In both subcases,*

$$\gamma_{P,Q}(p) = \begin{cases} 3(\frac{1}{2} - \frac{1}{p}) & \text{if } 2 \le p \le 4, \\ \frac{5}{2} - \frac{7}{p} & \text{if } 4 \le p \le 6, \\ 3 - \frac{10}{p} & \text{if } 6 \le p \le \infty. \end{cases} \tag{2.1.15}$$

Theorem 2.1.1 combines several results. Our main result is the bound $\le$ in (2.1.15) in the case $(P, Q) \equiv (r^2, s^2 + rt)$, which we repeat in Theorem 2.1.2 and discuss in more detail below.

The classification of pairs of quadratic forms is the content of Proposition 2.2.1. The upper bound $\le$ in (2.1.14) in the case $(P, Q) \equiv (rs, rt)$ is the content of Proposition 2.3.1. The upper bound $\le$ in (2.1.15) in the case $(P, Q) \equiv (r^2, s^2 \pm t^2)$ follows directly from the corresponding inequalities for the parabola (r, r^2), see Jean Bourgain and Demeter 2015, and the surfaces $(s, t, s^2 \pm t^2)$, see Jean Bourgain and Demeter 2017b, Theorem 1.1. Finally, examples that show the lower bounds $\ge$ in (2.1.10), (2.1.14), and (2.1.15) are discussed in Section 2.6.

For a pair of linearly independent quadratic forms (P, Q) in three variables that omit at least one variable (possibly after a linear change of variables), the sharp decoupling exponent is also given by (2.1.14). The upper bound follows from flat decoupling and the decoupling inequality for two quadratic forms in two variables that was proved in Jean Bourgain and Demeter 2016a, similarly to the proof of Proposition 2.3.1. The lower bound follows from Proposition 2.6.1 with $d' = n' = 2$ when $2 \le p \le 6$ (and from (2.1.9) when $6 \le p \le \infty$).

2.1.3 The main decoupling inequality

Let us state the main new part of Theorem 2.1.1 more explicitly.

Theorem 2.1.2. *Let $\mathcal{S}$ be the surface given by $(r, s, t, r^2, s^2 + rt)$. Then, for every $\epsilon > 0$, we have*

$$\mathcal{D}_{\mathcal{S}}(\delta, p) \lesssim_{\epsilon,p} \begin{cases} \delta^{-3(\frac{1}{2} - \frac{1}{p}) - \epsilon} & \text{if } 2 \le p \le 4, \\ \delta^{-(\frac{5}{2} - \frac{7}{p}) - \epsilon} & \text{if } 4 \le p \le 6, \\ \delta^{-(3 - \frac{10}{p}) - \epsilon} & \text{if } 6 \le p \le \infty. \end{cases} \tag{2.1.16}$$

It is well-known that, for an integer $s \geq 1$, the study of the decoupling constant $\mathcal{D}_{\mathcal{S}}(\delta, p)$ with $p = 2s$ is closely related to the problem of counting integer solutions to the Diophantine system

$$\begin{aligned} x_1 + \cdots + x_s &= x_{s+1} + \cdots + x_{2s}, \\ y_1 + \cdots + y_s &= y_{s+1} + \cdots + y_{2s}, \\ z_1 + \cdots + z_s &= z_{s+1} + \cdots + z_{2s}, \\ P(x_1, y_1, z_1) + \cdots + P(x_s, y_s, z_s) &= P(x_{s+1}, y_{s+1}, z_{s+1}) + \cdots + P(x_{2s}, y_{2s}, z_{2s}), \\ Q(x_1, y_1, z_1) + \cdots + Q(x_s, y_s, z_s) &= Q(x_{s+1}, y_{s+1}, z_{s+1}) + \cdots + Q(x_{2s}, y_{2s}, z_{2s}). \end{aligned} \tag{2.1.17}$$

Indeed, let $J_{\mathcal{S},s}(N)$ denote the number of integral solutions of (2.1.17), where all variables x_i, y_i, z_i with $1 \leq i \leq 2s$ take values in $\{0, 1, \ldots, N\}$. Then, by the argument in Jean Bourgain, Demeter, and Guth 2016, Corollary 4.2, we have

$$J_{\mathcal{S},s}(N) \lesssim N^3 \mathcal{D}_{\mathcal{S}}(N^{-1}, 2s)^{2s}. \tag{2.1.18}$$

Theorem 2.1.1 implies sharp estimates on $J_{\mathcal{S},s}(N)$ for every N and every $s \geq 1$. For instance, if we take $P = r^2$ and $Q = s^2 + rt$, then Theorem 2.1.2 implies that

$$J_{\mathcal{S},s}(N) \lesssim_{s,\epsilon} N^{3s+\epsilon} + N^{5s-4+\epsilon} + N^{6s-7+\epsilon}, \tag{2.1.19}$$

for every $\epsilon > 0$. In particular, when $s = 2$ (which corresponds to $p = 4$), we have that $J_{\mathcal{S},2}(N) \lesssim_\epsilon N^{6+\epsilon}$. Notice that if we set $x_i = x_{i+2}, y_i = y_{i+2}, z_i = z_{i+2}$ for every $i = 1, 2$, then we obtain a trivial lower bound $J_{\mathcal{S},2}(N) \geq N^6$. In this sense, the number of integral solutions to the system (2.1.17) still shows diagonal behavior when $s = 2$.

In Section 2.4, we will present a simple direct proof of the bound $J_{\mathcal{S},2}(N) \lesssim_\epsilon N^{6+\epsilon}$ that relies on elementary counting methods, rather than decoupling inequalities. Such a bound on $J_{\mathcal{S},s}$ usually cannot be used to derive a sharp decoupling inequality, that is, a sharp bound on $\mathcal{D}_{\mathcal{S}}(\delta, p)$. Nevertheless, some features of the counting argument in Section 2.4 remain visible in our proof of Theorem 2.1.2.

It is a bit surprising that the decoupling theory for the surface in Theorem 2.1.2 admits three different regimes. This is not reflected by the lower bounds for $J_{\mathcal{S},s}$ obtained by Parsell, Prendiville, and Trevor D. Wooley 2013, since there is no even integer in the interval $(4, 6) \subset \mathbb{R}$. For this reason, we discuss lower bounds directly for decoupling inequalities in Section 2.6.

In Theorem 2.1.1, we see that there are several different regimes for sharp decoupling exponents, and in case 3 of that theorem we see that equal decoupling exponents can arise in different ways.

2.2 Classification of pairs of quadratic forms in 3 variables

In this section we prove the classification part of Theorem 2.1.1.

Proposition 2.2.1. *Let (P, Q) be a degenerate pair of linearly independent quadratic forms. Moreover, assume that there is no linear change of variables in (r, s, t) after which P and Q both omit one of the variables. Then exactly one of the following alternatives holds.*

1. $(P, Q) \equiv (rs, rt)$,
2. $(P, Q) \equiv (r^2, s^2 \pm t^2)$, *or*
3. $(P, Q) \equiv (r^2, s^2 + rt)$.

The key step of proving Proposition 2.2.1 is the following result.

Lemma 2.2.2. *For two general quadratic forms in* 3 *variables* P *and* Q*, Condition* (2.1.7) *is equivalent to*

$$\text{no non-trivial linear combination of } P, Q \text{ is a complete square.} \tag{2.2.1}$$

Proof of Proposition 2.2.1 assuming Lemma 2.2.2. The hypothesis that (P, Q) is a degenerate pair means that at least one of the conditions (2.1.8), (2.1.7) fails.

Assume that (2.1.8) fails, that is, that the two quadratic forms P and Q share a common real linear factor. Without loss of generality, we may assume that the common factor is r. Then, up to a linear change of variables, there are two cases, $(P, Q) = (r^2, rs)$ or $(P, Q) = (rs, rt)$. Here we used the assumption that P and Q are linearly independent. The former case is not admissible, as the t variable is omitted.

Suppose now that (2.1.8) holds and (2.1.7) fails. By Lemma 2.2.2, (2.1.7) fails if and only if some non-trivial linear combination of P, Q is a complete square. Hence, after a change of variables as in (2.1.11), we may assume $P(r, s, t) = r^2$.

Now consider $Q(0, s, t)$, which is a quadratic form of two variables. First of all, we know that it cannot have rank zero, as otherwise $Q(r, s, t)$ will share a common factor with $P(r, s, t) = r^2$. Therefore, $Q(0, s, t)$ can have rank either one or two. Making a change of variables in s and t, we may assume that $Q(0, s, t)$ equals either $s^2 \pm t^2$ (if it has rank 2) or s^2 (if it has rank 1). In the rank 2 case, we have

$$Q(r, s, t) = s^2 \pm t^2 + c_1 r^2 + c_2 rs + c_3 rt. \tag{2.2.2}$$

Here c_1, c_2, c_3 are real numbers. We now add multiples of $P(r, s, t) = r^2$ to $Q(r, s, t)$ and complete squares. This process removes all the mixed terms in (2.2.2) and hence $(P, Q) \equiv (r^2, s^2 \pm t^2)$.

The case of $Q(0, s, t)$ having rank one is similar, but one of the mixed terms cannot be removed, hence $(P, Q) \equiv (r^2, s^2 + crt)$. The coefficient c does not vanish, since otherwise (P, Q) would omit the variable t. Rescaling the variable t, we may assume $c = 1$. □

Proof of Lemma 2.2.2. If some non-trivial linear combination of P, Q is of the form $(u_1 r + u_2 s + u_3 t)^2$, then Condition (2.1.7) fails for that u.

Conversely, suppose that Condition (2.1.7) fails for some $u \neq 0$. Without loss of generality, we may assume $u = (0, 0, 1)$. Write $P = a_{rr} r^2 + a_{ss} s^2 + a_{tt} t^2 + a_{rs} rs + a_{rt} rt + a_{st} st$

and $Q = b_{rr}r^2 + b_{ss}s^2 + b_{tt}t^2 + b_{rs}rs + b_{rt}rt + b_{st}st$. Then

$$\begin{aligned} 0 \equiv \det(\nabla P, \nabla Q, u) &= \det \begin{pmatrix} \partial_r P & \partial_r Q \\ \partial_s P & \partial_s Q \end{pmatrix} \\ &= \det \begin{pmatrix} 2ra_{rr} + sa_{rs} + ta_{rt} & 2rb_{rr} + sb_{rs} + tb_{rt} \\ 2sa_{ss} + ra_{rs} + ta_{st} & 2sb_{ss} + rb_{rs} + tb_{st} \end{pmatrix} \\ &= 2r^2(a_{rr}b_{rs} - a_{rs}b_{rr}) + 2s^2(a_{rs}b_{ss} - a_{ss}b_{rs}) + t^2(a_{rt}b_{st} - a_{st}b_{rt}) \\ &+ rs(4a_{rr}b_{ss} + a_{rs}b_{rs} - 4a_{ss}b_{rr} - a_{rs}b_{rs}) \\ &+ rt(2a_{rr}b_{st} + a_{rt}b_{rs} - a_{rs}b_{rt} - 2a_{st}b_{rr}) \\ &+ st(a_{rs}b_{st} + 2a_{rt}b_{ss} - 2a_{ss}b_{rt} - a_{st}b_{rs}). \end{aligned} \tag{2.2.3}$$

Since all coefficients must vanish, we obtain

$$\begin{aligned} 0 &= a_{rr}b_{rs} - a_{rs}b_{rr} = a_{rs}b_{ss} - a_{ss}b_{rs} = a_{rt}b_{st} - a_{st}b_{rt} = a_{rr}b_{ss} - a_{ss}b_{rr} \\ &= 2a_{rr}b_{st} + a_{rt}b_{rs} - a_{rs}b_{rt} - 2a_{st}b_{rr} = a_{rs}b_{st} + 2a_{rt}b_{ss} - 2a_{ss}b_{rt} - a_{st}b_{rs}. \end{aligned} \tag{2.2.4}$$

Replacing (P, Q) by suitable linear combinations, we may assume without loss of generality $a_{rs} = 0$. We distinguish several cases.

Case 1: $b_{rs} = 0$. Then the equations simplify to

$$0 = a_{rt}b_{st} - a_{st}b_{rt} = a_{rr}b_{ss} - a_{ss}b_{rr} = a_{rr}b_{st} - a_{st}b_{rr} = a_{rt}b_{ss} - a_{ss}b_{rt}. \tag{2.2.5}$$

This shows that $(a_{rr}, a_{ss}, a_{rt}, a_{st})$ and $(b_{rr}, b_{ss}, b_{rt}, b_{st})$ lie in the same one-dimensional subspace of $\mathbb{R}^4$. Hence, subtracting a suitable multiple of Q from P, we may assume $(a_{rr}, a_{ss}, a_{rt}, a_{st}) = 0$. But then $P = a_{tt}t^2$, and we are done.

Case 2: $b_{rs} \neq 0$. Then from the first two equations in (2.2.4) we obtain $a_{rr} = a_{ss} = 0$, and the remaining equations simplify to

$$0 = a_{rt}b_{st} - a_{st}b_{rt} = a_{rt}b_{rs} - 2a_{st}b_{rr} = 2a_{rt}b_{ss} - a_{st}b_{rs}. \tag{2.2.6}$$

Case 2.1: if $a_{rt} = a_{st} = 0$, then $P = a_{tt}t^2$, and we are done.
Case 2.2: if exactly one of a_{rt}, a_{st} vanishes, then suppose without loss of generality $a_{rt} = 0$ and $a_{st} \neq 0$. Then from the last equation $b_{rs} = 0$, contradiction.
Case 2.3: both $a_{rt} \neq 0$ and $a_{st} \neq 0$. Then, multiplying the last two equations, we obtain

$$2a_{st}b_{rr} \cdot 2a_{rt}b_{ss} = a_{rt}b_{rs} \cdot a_{st}b_{rs}. \tag{2.2.7}$$

Since all a's don't vanish, this gives $4b_{rr}b_{ss} = b_{rs}^2$. Hence $b_{rr}r^2 + b_{ss}s^2 + b_{rs}rs$ is a complete square. Making a change of variables only in (r, s), we may assume $b_{rs} = 0$. Notice that we retain the relation $a_{rs} = 0$ after this change of variables, since we have $a_{rr} = a_{ss} = 0$ in Case 2. Hence we are back in Case 1. $\square$

2.3 Sharp decouplings for the surface (r, s, t, rs, rt)

In this section, we will prove the upper bound in (2.1.14).

Proposition 2.3.1. *Let $\mathcal{S}$ be the surface given by (r,s,t,rs,rt). Then, for every $\epsilon > 0$, we have*

$$\mathcal{D}_{\mathcal{S}}(\delta,p) \lesssim_{\epsilon,p} \begin{cases} \delta^{-(2-\frac{4}{p})-\epsilon} & \text{if } 2 \le p \le 6, \\ \delta^{-(3-\frac{10}{p})-\epsilon} & \text{if } 6 \le p \le \infty. \end{cases} \tag{2.3.1}$$

Proof. By interpolation with orthogonality at $p = 2$ and a trivial estimate at $p = \infty$, it suffices to prove the case $p = 6$. First, we notice that

$$(rs, rt) \equiv (rs, r^2 + rt). \tag{2.3.2}$$

Denote

$$\mathcal{S}' = \{(r,s,t,rs,r^2+rt) : (r,s,t) \in [0,1]^3\}. \tag{2.3.3}$$

Our goal is to prove that

$$\Big\| \sum_{\square \in \mathcal{P}(\delta)} f_\square \Big\|_6 \lesssim_\epsilon \delta^{-4(\frac{1}{2}-\frac{1}{6})-\epsilon} \Big(\sum_{\square} \|f_\square\|_6^6 \Big)^{1/6}, \tag{2.3.4}$$

where

$$\operatorname{supp}(\widehat{f_\square}) \subset \{(r,s,t,rs+\delta', r^2+rt+\delta'') \,|\, (r,s,t) \in \square, |\delta'|, |\delta''| \le \delta^2\}. \tag{2.3.5}$$

For an integer $0 \le j \le \delta^{-1}$, let

$$S_j = [0,1] \times [0,1] \times [j\delta, (j+1)\delta]. \tag{2.3.6}$$

By flat decoupling, we obtain

$$\Big\| \sum_{\square \in \mathcal{P}(\delta)} f_\square \Big\|_6 \lesssim \delta^{-2(\frac{1}{2}-\frac{1}{6})} \Big(\sum_j \| \sum_{\square \subset S_j} f_\square \|_6^6 \Big)^{1/6}. \tag{2.3.7}$$

It remains to prove

$$\Big\| \sum_{\square \subset S_j} f_\square \Big\|_6 \lesssim_\epsilon \delta^{-2(\frac{1}{2}-\frac{1}{6})-\epsilon} \Big(\sum_{\square} \|f_\square\|_6^6 \Big)^{1/6}, \tag{2.3.8}$$

uniformly in j. To this end, we use the decoupling inequality for the surface

$$\mathcal{S}'' := \{(r,s,rs,r^2) : (r,s) \in [0,1]^2\}, \tag{2.3.9}$$

which was proved in Jean Bourgain and Demeter 2016a (see also Shaoming Guo and Zorin-Kranich 2020a, where this result is discussed from a more general perspective). In our notation, Jean Bourgain and Demeter 2016a, Theorem 1.2 implies that, for every $\epsilon > 0$, we have

$$\mathcal{D}_{\mathcal{S}''}(\delta, 6) \lesssim_\epsilon \delta^{-2(\frac{1}{2}-\frac{1}{6})-\epsilon}. \tag{2.3.10}$$

Using Shaoming Guo and Zorin-Kranich 2020a, Theorem 2.2 with $\mathcal{H} = \{(r,s,t) \,|\, t = 0\}$, we can now deduce (2.3.8) with $j = 0$. The estimates (2.3.8) for other values of j can be obtained from the case $j = 0$ using the affine transformation

$$(r,s,t,\xi,\eta) \mapsto (r,s,t+t_0,\xi,\eta+rt_0)$$

in the frequency space, where $t_0 = j\delta$. $\square$

2.4 A counting argument

Consider the Diophantine system (2.1.17) with $P(x, y, z) = x^2$ and $Q(x, y, z) = y^2 + xz$. In this section, we will give a direct proof of the estimate

$$J_{\mathcal{S},2}(N) \lesssim_\epsilon N^{6+\epsilon}, \tag{2.4.1}$$

for every $\epsilon > 0$, without invoking decoupling theory. Recall that this corresponds to the case $p = 4$ in Theorem 2.1.2, which is the most interesting case there. As mentioned in the introduction, the argument used in the following proof will shed some light on how to prove the related decoupling inequality in Theorem 2.1.2.

In the current situation, the system of equations (2.1.17) becomes

$$\begin{aligned} x_1 + x_2 &= x_3 + x_4, \\ y_1 + y_2 &= y_3 + y_4, \\ z_1 + z_2 &= z_3 + z_4, \\ x_1^2 + x_2^2 &= x_3^2 + x_4^2, \\ y_1^2 + x_1 z_1 + y_2^2 + x_2 z_2 &= y_3^2 + x_3 z_3 + y_4^2 + x_4 z_4. \end{aligned} \tag{2.4.2}$$

The first and fourth equations in (2.4.2) imply that $\{x_1, x_2\}$ is a permutation of $\{x_3, x_4\}$. Without loss of generality let us assume that $x_1 = x_3$ and $x_2 = x_4$. Also keeping in mind that $z_1 - z_3 = z_4 - z_2$, the last equation in (2.4.2) can then be written as

$$y_1^2 - y_3^2 + (x_1 - x_2)(z_1 - z_3) = y_4^2 - y_2^2. \tag{2.4.3}$$

We now distinguish two cases: $x_1 = x_2$ and $x_1 \neq x_2$.

Case 1: $x_1 = x_2$. Then we have $x_1 = x_2 = x_3 = x_4$ and this is the only constraint on the x_i variables. Similarly, the only constraint on the y_i variables now becomes

$$\begin{aligned} y_1 + y_2 &= y_3 + y_4, \\ y_1^2 - y_3^2 &= y_4^2 - y_2^2. \end{aligned} \tag{2.4.4}$$

Finally, the only constraint on the z_i variables is the linear equation $z_1 + z_2 = z_3 + z_4$. To summarize, this case leads to a contribution of $N \cdot N^2 \cdot N^3 = N^6$ to $J_{\mathcal{S},2}(N)$.

Case 2: $x_1 \neq x_2$. In this case we have two free choices among the x_i variables. Suppose that x_1, x_2, x_3, x_4 have been fixed.

Case 2.1: $z_1 = z_3$. Then also $z_2 = z_4$, so we may choose two of the z_i variables freely. Suppose that z_1, z_2, z_3, z_4 have been fixed. The remaining constraints are now

$$\begin{aligned} y_1 + y_2 &= y_3 + y_4, \\ y_1^2 - y_3^2 &= y_4^2 - y_2^2, \end{aligned} \tag{2.4.5}$$

which gives two free choices of y_i variables. Summarizing, this case yields a contribution of $\approx N^6$ to $J_{\mathcal{S},2}(N)$.

Case 2.2: $z_1 \neq z_3$. This is the critical case. First note that there are $\approx N^3$ valid choices of the z_i variables. Assume that z_1, z_2, z_3, z_4 have been fixed. It remains to analyze the constraints on the remaining variables y_1, y_2, y_3, y_4, which can be written as

$$\begin{aligned} y_1 - y_3 &= y_4 - y_2, \\ y_1^2 - y_3^2 + C &= y_4^2 - y_2^2, \end{aligned} \tag{2.4.6}$$

where $C = (x_1 - x_2)(z_1 - z_3) \neq 0$. We will now make critical use of the fact that all involved quantities are integers. Observe that necessarily $y_1 \neq y_3$. Next, the first equation implies that $y_4^2 - y_2^2$ is divisible by $y_1 - y_3$. Since also $y_1^2 - y_3^2$ is divisible by $y_1 - y_3$, the second equation implies that $y_1 - y_3$ must divide C. Since C is $\lesssim N^2$, we have $d(C) \lesssim_\epsilon N^\epsilon$ for all $\epsilon > 0$, where $d(C)$ denotes the number of divisors of C. Let D be one of these divisors and suppose that $y_1 - y_3 = D$. We then have the constraints

$$\begin{aligned} y_1 - y_3 &= D, \\ y_4 - y_2 &= D, \\ y_1 + y_3 + \tfrac{C}{D} &= y_4 + y_2. \end{aligned} \tag{2.4.7}$$

For each fixed D, there are $\lesssim N$ valid choices of y_1, y_2, y_3, y_4. Summarizing, this case gives a contribution of $\lesssim_\epsilon N^{6+\epsilon}$ to $J_{\mathcal{S},2}(N)$, for all $\epsilon > 0$.

2.5 Sharp decouplings for the surface $(r, s, t, r^2, s^2 + rt)$

In this section, we will prove Theorem 2.1.2. By interpolation with orthogonality at $p = 2$ and a trivial estimate at $p = \infty$, it suffices to prove the cases $p = 4$ and $p = 6$. For $p = 6$ we can use the same argument as in Proposition 2.3.1, using flat decoupling in the t variable and the decoupling inequality of Jean Bourgain and Demeter 2016a for the surface (r, s, r^2, s^2), lifted to 5 dimensions using Shaoming Guo and Zorin-Kranich 2020a, Theorem 2.2.

It remains to consider $p = 4$. Let us give the proof.

Notation 2.5.1. For a dyadic number $\delta \in (0, 1)$ and a dyadic box $\alpha \subset [0, 1]^3$ with side lengths at least δ, we use $\mathcal{P}(\alpha, \delta)$ to denote the partition of α into dyadic cubes of side length δ. For three real numbers k_1, k_2, k_3, we use $\mathcal{P}^{(k_1,k_2,k_3)}(\alpha, \sigma)$ to denote a partition of α into rectangular boxes of dimension $\sigma^{k_1} \times \sigma^{k_2} \times \sigma^{k_3}$. We also write $\mathcal{P}(\delta) = \mathcal{P}([0, 1]^3, \delta)$ and $\mathcal{P}^{(k_1,k_2,k_3)}(\sigma) = \mathcal{P}^{(k_1,k_2,k_3)}([0, 1]^3, \sigma)$ for brevity.

Theorem 2.1.2 will be proved by iterating the following two propositions, which decouple in different coordinates.

Proposition 2.5.2. *Let $\alpha_0 \in \mathcal{P}^{(1,0,1)}(\sigma)$. For each $\alpha \in \mathcal{P}^{(1,2,1)}(\alpha_0, \sigma)$, let g_α be a function with*

$$\operatorname{supp}(\widehat{g_\alpha}) \subseteq \{(r, s, t, r^2 + \sigma', s^2 + rt + \sigma'') \mid (r, s, t) \in \alpha, |\sigma'|, |\sigma''| \leq \sigma^2\}. \tag{2.5.1}$$

Then, for every $\epsilon' > 0$, we have

$$\Big\| \sum_{\alpha \in \mathcal{P}^{(1,2,1)}(\alpha_0,\sigma)} g_\alpha \Big\|_4 \lesssim_{\epsilon'} \sigma^{-2(\frac{1}{2}-\frac{1}{4})-\epsilon'} \Big(\sum_{\alpha \in \mathcal{P}^{(1,2,1)}(\alpha_0,\sigma)} \|g_\alpha\|_4^4 \Big)^{1/4}, \tag{2.5.2}$$

uniformly in α_0 and g_α.

Proposition 2.5.3. *Let $\alpha_0 \in \mathcal{P}^{(0,1,0)}(\sigma)$. For each $\alpha \in \mathcal{P}^{(2,1,2)}(\alpha_0, \sigma)$, let g_α be a function with*

$$\operatorname{supp}(\widehat{g_\alpha}) \subseteq \{(r, s, t, r^2 + \sigma', s^2 + rt + \sigma'') \mid (r, s, t) \in \alpha, |\sigma'| \leq \sigma^4, |\sigma''| \leq \sigma^2\}. \tag{2.5.3}$$

Then, for every $\epsilon' > 0$, we have

$$\Big\| \sum_{\alpha \in \mathcal{P}^{(2,1,2)}(\alpha_0,\sigma)} g_\alpha \Big\|_4 \lesssim_{\epsilon'} \sigma^{-4(\frac{1}{2}-\frac{1}{4})-\epsilon'} \Big(\sum_{\alpha \in \mathcal{P}^{(2,1,2)}(\alpha_0,\sigma)} \|g_\alpha\|_4^4 \Big)^{1/4}, \tag{2.5.4}$$

uniformly in α_0 and g_α.

The Fourier support restrictions in (2.5.1) and (2.5.3) arise naturally in the proofs, but it would be sufficient to prove the above results under the more restrictive conditions $|\sigma'|, |\sigma''| \le \sigma^4$.

The proofs of Propositions 2.5.2 and 2.5.3, as well as Theorem 2.1.2, rely on translation-dilation invariance, which we now explain. Let

$$\alpha_0 = [0,\sigma^{k_1}] \times [0,\sigma^{k_2}] \times [0,\sigma^{k_3}], \quad \alpha = (r_0,s_0,t_0) + \tau\alpha_0.$$

Then, the affine map

$$\begin{aligned} L_\alpha(\xi) := (&\tau\xi_1 + r_0, \tau\xi_2 + s_0, \tau\xi_3 + t_0, \\ &\tau^2\xi_4 + \tau 2r_0\xi_1, \tau^2\xi_5 + \tau(2s_0\xi_2 + t_0\xi_1 + r_0\xi_3) + s_0^2 + r_0t_0) \end{aligned} \tag{2.5.5}$$

maps frequency parallelepipeds such as in (2.1.2), (2.5.1), and (2.5.3) to other frequency parallelepipeds of a similar form. Thus, in order to decouple on α, we will first pull the Fourier transforms of the relevant functions back by L_α, and decouple on $[0,\sigma^{k_1}]\times[0,\sigma^{k_2}]\times[0,\sigma^{k_3}]$ instead, where

$$(k_1,k_2,k_3) \in \{(0,0,0),(0,1,0),(1,0,1)\}.$$

Proof of Theorem 2.1.2 with $p = 4$ assuming Propositions 2.5.2 and 2.5.3. For a dyadic box $\alpha \subseteq [0,1]^3$, we write

$$f_\alpha := \sum_{\square \in \mathcal{P}(\alpha,\delta)} f_\square. \tag{2.5.6}$$

Set $\sigma = \delta^\epsilon$. By flat decoupling, we obtain

$$\Big\| \sum_{\alpha_0 \in \mathcal{P}^{(1,0,1)}(\sigma)} f_{\alpha_0} \Big\|_4 \lesssim \sigma^{-2(1-\frac{2}{4})} \Big(\sum_{\alpha_0 \in \mathcal{P}^{(1,0,1)}(\sigma)} \|f_{\alpha_0}\|_4^4 \Big)^{1/4}. \tag{2.5.7}$$

We iterate the following two estimates. Let $k \in \{0,1,\dots\}$ with $\delta \le \sigma^{2k+3}$.

Given $\alpha \in \mathcal{P}^{(2k+1,2k,2k+1)}(\sigma)$, by a rescaled version of Proposition 2.5.2, we obtain

$$\begin{aligned} \|f_\alpha\|_4 &= \Big\| \sum_{\alpha' \in \mathcal{P}^{(2k+1,2k+2,2k+1)}(\alpha,\sigma)} f_{\alpha'} \Big\|_4 \\ &\lesssim_\epsilon \sigma^{-2(\frac{1}{2}-\frac{1}{4})-\epsilon} \Big(\sum_{\alpha' \in \mathcal{P}^{(2k+1,2k+2,2k+1)}(\alpha,\sigma)} \|f_{\alpha'}\|_4^4 \Big)^{1/4}. \end{aligned} \tag{2.5.8}$$

Given $\alpha \in \mathcal{P}^{(2k+1,2k+2,2k+1)}(\sigma)$, by a rescaled version of Proposition 2.5.3, we obtain

$$\begin{aligned} \|f_\alpha\|_4 &= \Big\| \sum_{\alpha' \in \mathcal{P}^{(2k+3,2k+2,2k+3)}(\alpha,\sigma)} f_{\alpha'} \Big\|_4 \\ &\lesssim_\epsilon \sigma^{-4(\frac{1}{2}-\frac{1}{4})-\epsilon} \Big(\sum_{\alpha' \in \mathcal{P}^{(2k+3,2k+2,2k+3)}(\alpha,\sigma)} \|f_{\alpha'}\|_4^4 \Big)^{1/4}. \end{aligned} \tag{2.5.9}$$

Let K be the largest integer such that $\delta \leq \sigma^{2K+1}$. Using (2.5.7) and applying the estimates (2.5.8) and (2.5.9) for $k = 0, \ldots, K-1$, we obtain

$$\Big\| \sum_{\Box \in \mathcal{P}(\delta)} f_\Box \Big\|_4 \lesssim_{K,\epsilon} \sigma^{-(6K+4)(\frac{1}{2}-\frac{1}{4})-2K\epsilon} \Big(\sum_{\alpha \in \mathcal{P}^{(2K+1,2K,2K+1)}(\sigma)} \|f_\alpha\|_4^4 \Big)^{1/4}. \tag{2.5.10}$$

For every $\alpha \in \mathcal{P}^{(2K+1,2K,2K+1)}(\sigma)$, we have $|\mathcal{P}(\alpha,\delta)| \leq \sigma^{-7}$. Hence, by flat decoupling, we obtain

$$\|f_\alpha\|_4 \lesssim \sigma^{-7\cdot(1-\frac{2}{4})} \Big(\sum_{\Box \in \mathcal{P}(\alpha,\delta)} \|f_\Box\|_4^4 \Big)^{1/4}. \tag{2.5.11}$$

Combining the last two estimates, we obtain

$$\Big\| \sum_{\Box \in \mathcal{P}(\delta)} f_\Box \Big\|_4 \lesssim_{K,\epsilon} \sigma^{-(6K+18)(\frac{1}{2}-\frac{1}{4})-2K\epsilon} \Big(\sum_{\Box \in \mathcal{P}(\delta)} \|f_\Box\|_4^4 \Big)^{1/4}. \tag{2.5.12}$$

Since $\sigma^{-2K} \leq \delta^{-1}$, this concludes the proof. $\Box$

It remains to prove Propositions 2.5.2 and 2.5.3, which will be our objective in the next two subsections. The key ingredients are a decoupling inequality on "small balls" for the parabola $\{(s,s^2) \,|\, s \in [0,1]\}$ and a decoupling inequality on "thin slabs" for the surface $\{(r,t,r^2,rt) \,|\, (r,t) \in [0,1]^2\}$. The smallness and thinness of these balls and slabs are what allowed us to decouple certain frequency variables down to scale σ^2 in Propositions 2.5.2 and 2.5.3 when the other frequency variables are limited to an interval of length σ. This is crucial in letting us make progress, as we decouple in alternate coordinates in the above proof of Theorem 2.1.2.

2.5.1 Decoupling on small balls and proof of Proposition 2.5.2

We will need the following "small ball" decoupling inequality. The term "small ball" refers to the fact that it can be localized to spatial scale δ^{-1}, whereas the usual decoupling inequality (2.1.3) can only be localized to the larger spatial scale δ^{-2}. As a side note, optimal decoupling inequalities for the parabola at spatial scales between δ^{-1} and δ^{-2} were recently established in Demeter, Guth, and Wang 2020. In that paper, "small ball" decoupling inequalities are referred to as "small cap" inequalities. They mean the same thing: One features the spatial side of the problem, while the other features the frequency side. Here, we prefer the name "small ball", because in Proposition 2.5.3 we need a decoupling inequality similar in spirit that also features the spatial side of the problem.

Theorem 2.5.4 (cf. Shaoming Guo, Li, and Yung 2021, Lemma 4.2)**.** *Let $\delta \in (0,1)$ be a dyadic number, and for each $\theta \in \mathcal{P}([0,1],\delta)$ let f_θ be a tempered distribution on $\mathbb{R}^2$ with*

$$\operatorname{supp} \widehat{f_\theta} \subseteq \{(s, s^2+\delta') \,|\, s \in \theta, |\delta'| \leq \delta\}. \tag{2.5.13}$$

Then, for every $\epsilon > 0$, we have

$$\Big\| \sum_{\theta \in \mathcal{P}([0,1],\delta)} f_\theta \Big\|_{L^4(\mathbb{R}^2)} \lesssim_\epsilon \delta^{-(\frac{1}{2}-\frac{1}{4})-\epsilon} \Big(\sum_{\theta \in \mathcal{P}([0,1],\delta)} \|f_\theta\|_{L^4(\mathbb{R}^2)}^4 \Big)^{1/4}. \tag{2.5.14}$$

In Shaoming Guo, Li, and Yung 2021, Lemma 4.2, a version of Theorem 2.5.4 is stated for the extension operator. Although it is possible to deduce Theorem 2.5.4 from Shaoming Guo, Li, and Yung 2021, Lemma 4.2 by the argument in Jean Bourgain and Demeter 2017a, Section 5, it is preferable to observe that the proof continues to work at the level of generality in Theorem 2.5.4.

Proof of Proposition 2.5.2. By affine scaling, we may assume that $\alpha_0 = [0,\sigma]\times[0,1]\times[0,\sigma]$. The inequality (2.5.2) will follow from the fiberwise inequality

$$\Big\| \sum_{\alpha\in\mathcal{P}^{(1,2,1)}(\alpha_0,\sigma)} g_\alpha(x_1,\cdot,x_3,x_4,\cdot)\Big\|_{L^4(\mathbb{R}^2)} \lesssim_{\epsilon'} \sigma^{-2(\frac{1}{2}-\frac{1}{4})-\epsilon'}\Big(\sum_\alpha \|g_\alpha(x_1,\cdot,x_3,x_4,\cdot)\|^4_{L^4(\mathbb{R}^2)}\Big)^{1/4} \tag{2.5.15}$$

with a constant independent of x_1, x_3, x_4. The inequality (2.5.15) holds by Theorem 2.5.4, since for every choice of x_1, x_3, x_4 the Fourier support of

$$g_\alpha(x_1,\cdot,x_3,x_4,\cdot) \tag{2.5.16}$$

is contained in an $O(\sigma^2)$-neighborhood of a σ^2-arc of the unit parabola in $\mathbb{R}^2$. Indeed, the Fourier support is contained in the projection of the right hand side of (2.5.1) to $\mathbb{R}^2$ by omitting the first, third and fourth coordinate. Since $|rt| \le \sigma^2$ when $(r,s,t)\in\alpha_0$, the projection is contained inside $\{(s, s^2+\sigma''') \,|\, |\sigma'''| \le 2\sigma^2\}$, as claimed. □

2.5.2 Decoupling on thin slabs and proof of Proposition 2.5.3

In view of the proof of Proposition 2.5.2, it would be natural to use a small ball decoupling for the 2-dimensional surface (r,t,r^2,rt) in $\mathbb{R}^4$. If we had such a small ball decoupling, then we could hope to have the estimate (2.5.4) under the assumption that

$$\operatorname{supp}(\widehat{g_\alpha}) \subseteq \{(r,s,t,r^2+\sigma', s^2+rt+\sigma'') : (r,s,t)\in\alpha, |\sigma'|\le\sigma^2, |\sigma''|\le\sigma^2\}. \tag{2.5.17}$$

One would then be able to finish the proof of Theorem 2.1.2 using the same bootstrapping argument. Unfortunately, although such a small cap decoupling holds for the more "elliptic" surface (r,t,r^2,t^2), it fails for its "hyperbolic" variant (r,t,r^2,rt).

For a dyadic rectangle $R\subset[0,1]^2$, we let $\mathcal{P}(R,\delta)$ be the partition of R into squares of side length δ. We denote by $\mathcal{V}(\delta)$ the smallest constant such that, for every collection of functions g_α indexed by $\alpha\in\mathcal{P}([0,1]^2,\delta)$ with

$$\operatorname{supp}(\widehat{g_\alpha}) \subset \{(r,t,r^2+\delta', rt+\delta'') : (r,t)\in\alpha,\ |\delta'|\le\delta^2,\ |\delta''|\le\delta\}, \tag{2.5.18}$$

the following inequality holds:

$$\Big\|\sum_{\alpha\in\mathcal{P}([0,1]^2,\delta)} g_\alpha\Big\|_{L^4(\mathbb{R}^4)} \le \mathcal{V}(\delta)\Big(\sum_{\alpha\in\mathcal{P}([0,1]^2,\delta)} \|g_\alpha\|^4_{L^4(\mathbb{R}^4)}\Big)^{\frac{1}{4}}. \tag{2.5.19}$$

Theorem 2.5.5. *For every $\epsilon>0$ and every dyadic $\delta\in(0,1)$, we have*

$$\mathcal{V}(\delta) \lesssim_\epsilon \delta^{-1/2-\epsilon}. \tag{2.5.20}$$

Proof of Proposition 2.5.3 assuming Theorem 2.5.5. By affine scaling, we may assume $\alpha_0 = [0,1] \times [0,\sigma] \times [0,1]$. By Fubini's theorem, it suffices to show the fiberwise inequality

$$\Big\| \sum_{\alpha \in \mathcal{P}^{(2,1,2)}(\alpha_0,\sigma)} g_\alpha(\cdot, x_2, \cdot, \cdot, \cdot) \Big\|_{L^4(\mathbb{R}^4)} \lesssim_{\epsilon'} \sigma^{-1-\epsilon'} \Big(\sum_{\alpha} \| g_\alpha(\cdot, x_2, \cdot, \cdot, \cdot) \|_{L^4(\mathbb{R}^4)}^4 \Big)^{1/4}, \quad (2.5.21)$$

uniformly in x_2. This follows from Theorem 2.5.5 with $\delta = \sigma^2$, because for each fixed x_2 the Fourier support of

$$g_\alpha(\cdot, x_2, \cdot, \cdot, \cdot) \quad (2.5.22)$$

is contained in the projection of the Fourier support of g_α modulo the second coordinate, and this projection satisfies an inclusion of the form (2.5.18) because $s^2 \leq \sigma^2$ when $s \in [0,\sigma]$. □

2.5.3 Proof of decoupling on thin slabs

Theorem 2.5.5 will follow from

Proposition 2.5.6. *For each $\epsilon > 0$, there exists $K > 0$ such that for any $\delta \in (0,1)$*

$$\mathcal{V}(\delta) \leq K^{1/2+\epsilon} \mathcal{V}(K\delta) + C_K \delta^{-1/2}, \quad (2.5.23)$$

where C_K is a constant depending only on K.

Proof of Theorem 2.5.5. By iterating the result in Proposition 2.5.6, $\frac{(\epsilon-1)\log\delta}{\log K}$-many times (assuming without loss of generality that this is a positive integer), we obtain

$$\mathcal{V}(\delta) \leq \delta^{(\epsilon-1)\frac{1}{2}-\epsilon} \mathcal{V}(\delta^\epsilon) + \widetilde{C}_K (\log \delta^{-1}) \delta^{-1/2}. \quad (2.5.24)$$

It remains to note that

$$\mathcal{V}(\delta^\epsilon) \lesssim \delta^{-2\epsilon} \quad (2.5.25)$$

by the triangle inequality and Hölder's inequality. □

It remains to prove Proposition 2.5.6. We will apply a bilinear method, together with a Bourgain–Guth type argument Jean Bourgain and Guth 2011. We need the following bilinear estimate.

Lemma 2.5.7. *Let $K^{-1} > \delta > 0$. Let $j_1, j_2 \in \mathbb{Z}$ with $|j_1 - j_2| \geq 2$. Let $R_1 = [j_1 K^{-1}, (j_1 + 1)K^{-1}] \times [0,1] \subset [0,1]^2$ and $R_2 = [j_2 K^{-1}, (j_2+1)K^{-1}] \times [0,1] \subset [0,1]^2$. Then*

$$\Big\| \Big| \Big(\sum_{\beta \in \mathcal{P}(R_1,\delta)} g_\beta \Big) \Big(\sum_{\beta' \in \mathcal{P}(R_2,\delta)} g_{\beta'} \Big) \Big|^{\frac{1}{2}} \Big\|_{L^4} \leq C_K \delta^{-1/2} \Big(\sum_{\beta \in \mathcal{P}(\delta)} \| g_\beta \|_{L^4}^4 \Big)^{\frac{1}{4}}. \quad (2.5.26)$$

Lemma 2.5.7 would in fact still work under the more relaxed Fourier support assumption $|\delta'| \leq \delta$ in (2.5.18).

Proof of Lemma 2.5.7. By Plancherel's theorem,

$$\left\| \left| \Big(\sum_{\beta \in \mathcal{P}(R_1,\delta)} g_\beta \Big) \Big(\sum_{\beta' \in \mathcal{P}(R_2,\delta)} g_{\beta'} \Big) \right|^{\frac{1}{2}} \right\|_{L^4} = \left\| \sum_{\beta \in \mathcal{P}(R_1,\delta)} \sum_{\beta' \in \mathcal{P}(R_2,\delta)} \widehat{g_\beta} * \widehat{g_{\beta'}} \right\|_{L^2}^{\frac{1}{2}}. \tag{2.5.27}$$

We claim that the collection

$$\{\mathrm{supp}(\widehat{g_\beta}) + \mathrm{supp}(\widehat{g_{\beta'}})\}_{\beta \in \mathcal{P}(R_1,\delta), \beta' \in \mathcal{P}(R_2,\delta)} \tag{2.5.28}$$

has overlap bounded by a constant depending on K: if $\beta_1, \beta_3 \in \mathcal{P}(R_1,\delta)$ and $\beta_2, \beta_4 \in \mathcal{P}(R_2,\delta)$, and $(r_i,t_i) \in \beta_i$ for $i = 1,2,3,4$ are such that

$$(r_1, t_1, r_1^2, r_1t_1) + (r_2, t_2, r_2^2, r_2t_2) = (r_3, t_3, r_3^2, r_3t_3) + (r_4, t_4, r_4^2, r_4t_4) + O(\delta), \tag{2.5.29}$$

then the distances between β_i and β_{i+2} are $O(K^2\delta)$ for $i = 1,2$.

The geometric reason for this is that the pieces of the surface (r,t,r^2,rt) with (r,t) restricted to R_1 and R_2, respectively, are transverse. Indeed, if $(r_1,t_1) \in R_1$ and $(r_2,t_2) \in R_2$, then for bases of tangent spaces at these points we have

$$\det \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 2r_1 & 0 & 2r_2 & 0 \\ t_1 & r_1 & t_2 & r_2 \end{pmatrix} = 2(r_1 - r_2)^2 \geq 2K^{-2}. \tag{2.5.30}$$

However, it is formally easier to verify the bounded overlap property of the collection (2.5.28) algebraically. Suppose that (2.5.29) holds. Looking at the third component of (2.5.29), we obtain

$$\begin{aligned} O(\delta) &= r_1^2 - r_3^2 + r_2^2 - r_4^2 = (r_1 - r_3)(r_1 + r_3) + (r_2 - r_4)(r_2 + r_4) \\ &= (r_4 - r_2)(r_1 + r_3) + (r_2 - r_4)(r_2 + r_4) + O(\delta) \\ &= (r_2 - r_4)(r_2 + r_4 - r_1 - r_3) + O(\delta). \end{aligned} \tag{2.5.31}$$

Since $|r_2 + r_4 - r_1 - r_3| \gtrsim 1/K$, it follows that $r_2 - r_4 = O(K\delta)$. Similarly, $r_1 - r_3 = O(K\delta)$. Looking at the fourth component of (2.5.29), we obtain

$$\begin{aligned} O(\delta) &= r_1t_1 - r_3t_3 + r_2t_2 - r_4t_4 = r_1(t_1 - t_3) + r_2(t_2 - t_4) + O(K\delta) \\ &= r_1(t_4 - t_2) + r_2(t_2 - t_4) + O(K\delta) \\ &= (r_2 - r_1)(t_2 - t_4) + O(K\delta). \end{aligned} \tag{2.5.32}$$

Since $|r_2 - r_1| \gtrsim 1/K$, it follows that $t_2 - t_4 = O(K^2\delta)$. Similarly, $t_1 - t_3 = O(K^2\delta)$. This shows that the collection (2.5.28) has bounded overlap. Therefore,

$$(2.5.27) \leq C_K \Big(\sum_{\beta \in \mathcal{P}(R_1,\delta)} \sum_{\beta' \in \mathcal{P}(R_2,\delta)} \|\widehat{g_\beta} * \widehat{g_{\beta'}}\|_{L^2}^2 \Big)^{\frac{1}{4}}. \tag{2.5.33}$$

By Plancherel's theorem and Hölder's inequality, this implies

$$
\begin{aligned}
(2.5.27) &\le C_K \Big(\sum_{\beta \in \mathcal{P}(R_1,\delta)} \sum_{\beta' \in \mathcal{P}(R_2,\delta)} \|g_\beta g_{\beta'}\|_2^2 \Big)^{\frac14} \\
&\le C_K \Big(\sum_{\beta \in \mathcal{P}(R_1,\delta)} \|g_\beta\|_4^2 \Big)^{\frac14} \Big(\sum_{\beta' \in \mathcal{P}(R_2,\delta)} \|g_{\beta'}\|_4^2 \Big)^{\frac14} \qquad (2.5.34) \\
&\le C_K \delta^{-2(\frac12 - \frac14)} \Big(\sum_{\beta \in \mathcal{P}(R_1,\delta)} \|g_\beta\|_{L^4}^4 \Big)^{\frac18} \Big(\sum_{\beta' \in \mathcal{P}(R_2,\delta)} \|g_{\beta'}\|_4^4 \Big)^{\frac18}.
\end{aligned}
$$

This completes the proof of Lemma 2.5.7. □

Proof of Proposition 2.5.6. Let $K > 0$ be a large number that is to be determined. For each $j \in \mathbb{Z}$ with $0 \le j \le K-1$, we define the strip $R_j = [jK^{-1}, (j+1)K^{-1}] \times [0,1]$. We define

$$
G_j := \sum_{\alpha \in \mathcal{P}(R_j,\delta)} g_\alpha. \qquad (2.5.35)
$$

For each $x \in \mathbb{R}^4$, we define the significant part

$$
\mathcal{C}(x) := \Big\{ j' \in \{0,\dots,K-1\} : \Big| \sum_{j=0}^{K-1} G_j(x) \Big| < 10K |G_{j'}(x)| \Big\}. \qquad (2.5.36)
$$

Note that $\mathcal{C}(x) \ne \emptyset$ unless $\sum_j G_j(x) = 0$. By considering two possible cases $|\mathcal{C}(x)| \ge 3$ and $|\mathcal{C}(x)| \le 2$, we obtain

$$
\Big| \sum_{\alpha \in \mathcal{P}([0,1]^2,\delta)} g_\alpha(x) \Big| \le 10 \max_j |G_j(x)| + 10K \max_{j,j' : |j-j'| \ge 2} |G_j(x) G_{j'}(x)|^{\frac12}; \qquad (2.5.37)
$$

indeed, for $j' \notin \mathcal{C}(x)$, we have $|G_{j'}(x)| \le \frac{1}{10K} \Big| \sum_{j=0}^{K-1} G_j(x) \Big|$, so

$$
\Big| \sum_{j' \notin \mathcal{C}(x)} G_{j'}(x) \Big| \le \tfrac{1}{10} \Big| \sum_{j' \notin \mathcal{C}(x)} G_{j'}(x) \Big| + \tfrac{1}{10} \sum_{j' \in \mathcal{C}(x)} |G_{j'}(x)|, \qquad (2.5.38)
$$

which implies, if $|\mathcal{C}(x)| \le 2$, that $\Big| \sum_{j' \notin \mathcal{C}(x)} G_{j'}(x) \Big| \le \frac29 \max_j |G_j(x)|$ and hence

$$
\Big| \sum_{j'=0}^{K-1} G_{j'}(x) \Big| \le \big(2 + \tfrac29\big) \max_j |G_j(x)|. \qquad (2.5.39)
$$

Raising both sides of (2.5.37) to the fourth power and integrating in x, we obtain

$$
\| \sum_{\alpha \in \mathcal{P}([0,1]^2,\delta)} g_\alpha \|_4^4 \lesssim \sum_{j=1}^{K} \|G_j\|_4^4 + K^4 \sum_{j,j' : |j-j'| \ge 2} \| |G_j G_{j'}|^{\frac12} \|_4^4. \qquad (2.5.40)
$$

By Lemma 2.5.7, the second term is bounded by

$$C_K^4\delta^{-2}\Big(\sum_{\alpha\in\mathcal{P}([0,1]^2,\delta)}\|g_\alpha\|_4^4\Big). \tag{2.5.41}$$

In order to conclude the proof, it suffices to show that, for each $j=0,\dots,K-1$, we have

$$\|G_j\|_4 \le CK^{1/2}\mathcal{V}(K\delta)\Big(\sum_{\alpha\in\mathcal{P}(R_j,\delta)}\|g_\alpha\|_4^4\Big)^{\frac14} \tag{2.5.42}$$

and take K large enough so that $C\le K^{\epsilon}$.

By an affine transformation, we may assume without loss of generality that $j=0$. We define the scalings:

$$L:(\xi_1,\xi_2,\xi_3,\xi_4)\mapsto(\tfrac{\xi_1}{K},\xi_2,\tfrac{\xi_3}{K^2},\tfrac{\xi_4}{K}),$$
$$L':(\xi_1,\xi_2)\mapsto(\tfrac{\xi_1}{K},\xi_2).$$

Note that L' scales $[0,1]^2$ to the strip R_0. The map L is chosen so that L' is the restriction of L to the first two coordinates, and so that L leaves the surface (r,t,r^2,rt) invariant.

For each $\beta\in\mathcal{P}([0,1]^2,K\delta)$, we define a function H_β by

$$\widehat{H_\beta}(\xi_1,\xi_2,\xi_3,\xi_4):=\sum_{\alpha\in\mathcal{P}(L'(\beta),\delta)}\widehat{g_\alpha}(L(\xi_1,\xi_2,\xi_3,\xi_4)). \tag{2.5.43}$$

Then,

$$\operatorname{supp}(\widehat{H_\beta})\subseteq\{(r,t,r^2+\delta',rt+\delta''):(r,t)\in\beta,\ |\delta'|\le(K\delta)^2,\ |\delta''|\le K\delta\}. \tag{2.5.44}$$

Thus, by the definition of the constant $\mathcal{V}(K\delta)$,

$$\Big\|\sum_{\beta\in\mathcal{P}([0,1]^2,K\delta)}H_\beta\Big\|_4\le\mathcal{V}(K\delta)\Big(\sum_{\beta\in\mathcal{P}([0,1]^2,K\delta)}\|H_\beta\|_4^4\Big)^{\frac14}. \tag{2.5.45}$$

Changing back to the original variables and applying flat decoupling, we obtain

$$\begin{aligned}\|G_j\|_4&\lesssim\mathcal{V}(K\delta)\Big(\sum_{\beta\in\mathcal{P}([0,1]^2,K\delta)}\Big\|\sum_{\alpha\in\mathcal{P}(L'(\beta),\delta)}g_\alpha\Big\|_4^4\Big)^{\frac14}\\&\lesssim K^{1/2}\mathcal{V}(K\delta)\Big(\sum_{\alpha\in\mathcal{P}(R_j,\delta)}\|g_\alpha\|_4^4\Big)^{\frac14},\end{aligned} \tag{2.5.46}$$

where we used that $|\mathcal{P}(L'(\beta),\delta)|=K$ in the last step. This finishes the proof of (2.5.42). $\square$

Remark 2.5.8. We have already seen, at the beginning of Section 2.5.2, that one could not replace the condition $|\delta'|\le\delta^2$ on the right hand side of (2.5.18) by $|\delta'|\le\delta$ and still hope to prove Theorem 2.5.5. It may be helpful to see what goes wrong in the proof of Theorem 2.5.5 if we had the condition $|\delta'|\le\delta$ instead. In that case, when we rescale, on the right hand side of (2.5.44), we would only get $|\delta'|\le K^2\delta$, which would not allow us to close the induction on scale argument. This signifies the advantage to decouple on thin slabs as in Theorem 2.5.5.

2.6 Lower bounds

In this section, we prove lower bounds for decoupling constants defined in (2.1.3). In the case when p is an even integer, such bounds were proved for the related problem of bounding multidimensional Vinogradov mean values in Parsell, Prendiville, and Trevor D. Wooley 2013, Theorem 3.1. However, the construction given there does not detect the optimality of the bound (2.1.16) for $p \in (4, 6)$.

Proposition 2.6.1. *Let $Q_1(\mathbf{t}), \dots, Q_n(\mathbf{t})$ be quadratic forms in d variables. Suppose that $Q_1, \dots, Q_{n'}$ do not depend on $t_{d'+1}, \dots, t_d$ for some partitions $n = n' + n''$ and $d = d' + d''$. Let $\mathcal{K}'' := d'' + 2n''$.*

Let $\mathcal{S}$ be the surface defined in (2.1.1) *and let $\mathcal{D}_{\mathcal{S}}(\delta, p)$ be the associated decoupling constant, defined in* (2.1.3). *Then, for $2 \le p < \infty$, we have*

$$\mathcal{D}_{\mathcal{S}}(\delta, p) \gtrsim \delta^{-d'(1/2-1/p)} \cdot \delta^{-d''(1-1/p)+\mathcal{K}''/p}. \tag{2.6.1}$$

We postpone the proof of Proposition 2.6.1 till the end of this section and indicate how it recovers the lower bounds in Shaoming Guo and Zorin-Kranich 2020a, Section 1.5.

Corollary 2.6.2. *Let $Q_1, \dots, Q_n$ be quadratic forms in d variables and $\mathcal{K} := d + 2n$. Let $\mathcal{S}$ be the surface defined in* (2.1.1) *and let $\mathcal{D}_{\mathcal{S}}(\delta, p)$ be the associated decoupling constant, defined in* (2.1.3). *Then, for $2 \le p < \infty$, we have*

$$\mathcal{D}_{\mathcal{S}}(\delta, p) \gtrsim \max(\delta^{-d(1/2-1/p)}, \delta^{-d(1-1/p)+\mathcal{K}/p}). \tag{2.6.2}$$

Proof of Corollary 2.6.2 assuming Proposition 2.6.1. The hypothesis of Proposition 2.6.1 clearly holds for arbitrary quadratic forms $Q_1, \dots, Q_n$ with either $d' = n' = 0$ or $d'' = n'' = 0$. □

By considering functions f_θ of tensor product form, we also obtain the following lower bound.

Lemma 2.6.3. *In the situation of Corollary 2.6.2, if $V \le \mathbb{R}^d$ is a linear subspace, $\tilde{Q}_j := Q_j|_V$, $j = 1, \dots, n$, are restrictions of Q_j's to V, and $\tilde{\mathcal{S}}$ is the surface*

$$\{(\mathbf{t}, \tilde{Q}_1(\mathbf{t}), \dots, \tilde{Q}_n(\mathbf{t})) : |\mathbf{t}| \le 1\}, \tag{2.6.3}$$

then

$$\mathcal{D}_{\mathcal{S}}(\delta, p) \gtrsim \mathcal{D}_{\tilde{\mathcal{S}}}(\delta, p). \tag{2.6.4}$$

Now we can justify the lower bounds on the sharp decoupling exponents in Theorem 2.1.1.

For surfaces (2.1.4), we have $d = 3$ and $\mathcal{K} = 7$. Hence, (2.6.2) above implies the lower bounds

$$\mathcal{D}_{\mathcal{S}}(\delta, p) \gtrsim \max(\delta^{-3(1/2-1/p)}, \delta^{-3+10/p}). \tag{2.6.5}$$

This shows the lower bounds on sharp decoupling exponents in (2.1.9).

In case 2 of Theorem 2.1.1, assume without loss of generality that $(P, Q) = (rs, rt)$. Then we apply (2.6.4) for the subspace given by $r = 0$. On this subspace, we apply (2.6.2) with $\tilde{d} = 2$ and $\tilde{\mathcal{K}} = 2$. This gives the additional lower bound

$$\mathcal{D}_{\mathcal{S}}(\delta, p) \gtrsim \delta^{-2(1-1/p)+2/p} = \delta^{-2+4/p}. \tag{2.6.6}$$

In case 3 of Theorem 2.1.1, assume without loss of generality that $P = r^2$. Applying (2.6.1) with $d' = 1$ and $n' = 1$, we obtain

$$\mathcal{D}_{\mathcal{S}}(\delta, p) \gtrsim \delta^{-1\cdot(1/2-1/p)}\delta^{-2\cdot(1-1/p)+4/p} = \delta^{-5/2+7/p}. \tag{2.6.7}$$

This shows the middle lower bound in (2.1.15).

Proof of Proposition 2.6.1. Write points in $\mathbb{R}^{d+n}$ as $(x', x'', y', y'') \in \mathbb{R}^{d'+d''+n'+n''}$. For $\theta \in \mathcal{P}(\delta)$, write $\theta = \theta' \times \theta''$. Choose functions f_θ of the form $f_\theta = g_{\theta'}(x', y')h_\theta(x'', y'')$ with the following properties

1. $\widehat{g_{\theta'}}$ and $\widehat{h_\theta}$ are positive smooth functions,
2. $\int \widehat{g_{\theta'}} = \int \widehat{h_\theta} = 1$,
3. $\widehat{g_{\theta'}}$ is supported in a ball of radius $\approx \delta^2$,
4. $\widehat{f_\theta}$ is supported in and adapted to a rectangular box of dimensions
$$\underbrace{\delta^2/10 \times \cdots \times \delta^2/10}_{d' \text{ times}} \times \underbrace{\delta/10 \times \cdots \times \delta/10}_{d'' \text{ times}} \times \underbrace{\delta^2/10 \times \cdots \times \delta^2/10}_{n \text{ times}} \tag{2.6.8}$$
inside the set (2.1.2).

Note that $g_{\theta'}$ depends only on the projection of θ onto $\mathbb{R}^{d'+n'}$, whereas h_θ has to depend on θ because of the geometry of the set (2.1.2).

On one hand, $\|f_\theta\|_p \sim \delta^{-(2d'+d''+2n)/p}$, and by definition we have

$$\Big\| \sum_{\theta\in\mathcal{P}(\delta)} f_\theta \Big\|_p \le \mathcal{D}_{\mathcal{S}}(\delta, p)\Big(\sum_{\theta\in\mathcal{P}(\delta)} \|f_\theta\|_p^p \Big)^{1/p} \sim \mathcal{D}_{\mathcal{S}}(\delta, p)\delta^{-d/p}\delta^{-(2d'+d''+2n)/p}. \tag{2.6.9}$$

On the other hand,

$$\begin{aligned} \Big\| \sum_{\theta\in\mathcal{P}(\delta)} f_\theta \Big\|_p &\gtrsim \inf_{\substack{x''\in\mathbb{R}^{d''}, y''\in\mathbb{R}^{n''},\\ |x''|,|y''|\le 1/100}} \Big\| \sum_{\theta\in\mathcal{P}(\delta)} f_\theta \Big\|_{L^p(\mathbb{R}^{d'}\times\{x''\}\times\mathbb{R}^{n'}\times\{y''\})} \\ &= \inf_{\substack{x''\in\mathbb{R}^{d''}, y''\in\mathbb{R}^{n''},\\ |x''|,|y''|\le 1/100}} \Big\| \sum_{\theta'} c_{\theta',x'',y''} g_{\theta'} \Big\|_{L^p(\mathbb{R}^{d'}\times\mathbb{R}^{n'})} \end{aligned} \tag{2.6.10}$$

where $c_{\theta',x'',y''} := \sum_{\theta''} h_\theta(x'', y'')$ is independent of x', y' and satisfies

$$|c_{\theta',x'',y''}| \sim \delta^{-d''} \tag{2.6.11}$$

uniformly in θ' and $|x''|, |y''| \le 1/100$. This is because there is almost no cancellation in the sum over θ''.

Let $\phi = \eta(\delta^2 \cdot)$, where η is a fixed positive Schwartz function on $\mathbb{R}^{d'} \times \mathbb{R}^{n'}$ with $\operatorname{supp} \widehat{\eta} \subset B(0, 1/10)$. Then, by Hölder's inequality,

$$\begin{aligned} \Big\| \sum_{\theta'} c_{\theta',x'',y''} g_{\theta'} \Big\|_{L^p(\mathbb{R}^{d'}\times\mathbb{R}^{n'})} &\ge \|\phi\|_{1/(1/2-1/p)}^{-1} \Big\| \phi \sum_{\theta'} c_{\theta',x'',y''} g_{\theta'} \Big\|_{L^2(\mathbb{R}^{d'}\times\mathbb{R}^{n'})} \\ &\sim \delta^{2\cdot(d'+n')(1/2-1/p)} \Big\| \sum_{\theta'} c_{\theta',x'',y''} \phi g_{\theta'} \Big\|_{L^2(\mathbb{R}^{d'}\times\mathbb{R}^{n'})}. \end{aligned} \tag{2.6.12}$$

Since the Fourier supports of $\phi g_{\theta'}$ are disjoint for different (θ')'s, we obtain

$$\begin{aligned}\Big\|\sum_{\theta'} c_{\theta',x'',y''}\phi g_{\theta'}\Big\|_{L^2(\mathbb{R}^{d'}\times\mathbb{R}^{n'})} &= \Big(\sum_{\theta'}|c_{\theta',x'',y''}|^2\big\|\phi g_{\theta'}\big\|^2_{L^2(\mathbb{R}^{d'}\times\mathbb{R}^{n'})}\Big)^{1/2} \\ &\sim \delta^{-d'/2}\cdot\delta^{-d''}\cdot\delta^{-2\cdot(d'+n')/2},\end{aligned} \tag{2.6.13}$$

uniformly for $|x'|,|y'|\leq 1/100$. Combining the above estimates, we obtain

$$\mathcal{D}_{\mathcal{S}}(\delta,p)\delta^{-d/p}\delta^{-(2d'+d''+2n)/p}\gtrsim \delta^{2\cdot(d'+n')(1/2-1/p)}\cdot\delta^{-d'/2}\cdot\delta^{-d''}\cdot\delta^{-2\cdot(d'+n')/2}. \tag{2.6.14}$$

This implies the claim (2.6.1). $\square$

Chapter 3

Decoupling inequalities for quadratic forms

3.1 Introduction

Let $n, d \geq 1$. We denote by $\mathbf{Q}(\xi) = (Q_1(\xi), \ldots, Q_n(\xi))$ an n-tuple of real quadratic forms in d variables. The graph of such a tuple, $S_{\mathbf{Q}} = \{(\xi, \mathbf{Q}(\xi)) \in [0,1]^d \times \mathbb{R}^n\}$, is a d-dimensional submanifold of $\mathbb{R}^{d+n}$. We often write a spatial vector in $\mathbb{R}^{d+n}$ as (x, y) with $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ and $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$. Similarly we often write a frequency vector in $\mathbb{R}^{d+n}$ as (ξ, η) with $\xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d$ and $\eta = (\eta_1, \ldots, \eta_n) \in \mathbb{R}^n$.

Let $\square \subset [0,1]^d$. Define the Fourier extension operator

$$E_{\square}^{\mathbf{Q}} g(x, y) := \int_{\square} g(\xi) e^{i(x \cdot \xi + y \cdot \mathbf{Q}(\xi))} d\xi, \tag{3.1.1}$$

with $x \in \mathbb{R}^d, y \in \mathbb{R}^n$. For $q, p \geq 2$ and dyadic $\delta \in (0,1)$, let $D_{q,p}(\mathbf{Q}, \delta)$ be the smallest constant D such that

$$\big\|E_{[0,1]^d}^{\mathbf{Q}} g\big\|_{L^p(w_B)} \leq D \Big(\sum_{\substack{\square \subset [0,1]^d \\ l(\square) = \delta}} \big\|E_{\square}^{\mathbf{Q}} g\big\|_{L^p(w_B)}^q \Big)^{1/q} \tag{3.1.2}$$

holds for every measurable function g and every ball $B \subset \mathbb{R}^{d+n}$ of radius δ^{-2}, where w_B is a smooth version of the indicator function of B (see (3.1.28) in the subsection of notation) and the sum on the right hand side runs through all dyadic cubes of side length δ. In this chapter, we determine an optimal asymptotic behavior of $D_{q,p}(\mathbf{Q}, \delta)$ as δ tends to zero, for every choice of $q, p \geq 2$ and $\mathbf{Q}$.

Let us introduce more definitions. We will formulate a slightly more general (and essentially equivalent) version of (3.1.2). This version uses functions with Fourier supports in small neighborhoods of $S_{\mathbf{Q}}$, instead of Fourier extension operators, and lends itself more readily to induction on dimension d.

It is convenient to define the Fourier supports in terms of symmetries of $S_{\mathbf{Q}}$. The group $\mathbf{A}$ generated by translations and scalings of $\mathbb{R}^d$ consists of affine maps of the form $A(\xi) = \delta\xi + a$ with $a \in \mathbb{R}^d$ and $\delta \in (0, \infty)$ (in particular, $\mathbf{A} \cong \mathbb{R}^d \rtimes (0, \infty)$). This group acts on $\mathbb{R}^{d+n}$ by affine transformations

$$\mathcal{A}(A)(\xi,\eta) = (\delta\xi + a, \delta^2\eta + \delta\nabla\mathbf{Q}(a)\cdot\xi + \mathbf{Q}(a)). \tag{3.1.3}$$

This $\mathbf{A}$-action leaves $S_{\mathbf{Q}}$ invariant. For a cube $\square \subset \mathbb{R}^d$, let $A_\square \in \mathbf{A}$ be the map such that $A_\square([0,1]^d) = \square$, and denote the corresponding affine transformation on $\mathbb{R}^{d+n}$ by $\mathcal{A}_\square := \mathcal{A}(A_\square)$. We define the associated uncertainty region by

$$\mathcal{U}_\square = \mathcal{U}_\square(\mathbf{Q}) := \mathcal{A}_\square\Big([-2,2]^d \times \prod_{j=1}^{n} 4d(\|\mathrm{Hess}\, Q_j\| + 1)[-1,1]\Big). \tag{3.1.4}$$

The main feature of the definition (3.1.4) is that the uncertainty region $\mathcal{U}_\square$ contains the convex hull of the graph of $\mathbf{Q}$ on $\square$ and is not much larger than this convex hull. Another convenient property is that

$$2\square \subseteq 2\square' \implies \mathcal{U}_\square \subseteq \mathcal{U}_{\square'}. \tag{3.1.5}$$

We will denote by $f_\square$ an arbitrary function with $\mathrm{supp}\,\widehat{f_\square} \subseteq \mathcal{U}_\square$.

Let $q, p \geq 2$. Let $\delta < 1$ be a dyadic number. Let $\mathcal{P}(\delta)$ be the partition of $[0,1]^d$ into dyadic cubes of side length δ. Let $\mathcal{D}_{q,p}(\mathbf{Q},\delta)$ be the smallest constant D such that

$$\|\sum_{\square\in\mathcal{P}(\delta)} f_\square\|_{L^p(\mathbb{R}^{d+n})} \leq D\Big(\sum_{\square\in\mathcal{P}(\delta)} \|f_\square\|^q_{L^p(\mathbb{R}^{d+n})}\Big)^{1/q} \tag{3.1.6}$$

holds for every $f_\square$ with $\mathrm{supp}\,\widehat{f_\square} \subseteq \mathcal{U}_\square$. If $p = q$, we often write

$$\mathcal{D}_p(\mathbf{Q},\delta) := \mathcal{D}_{q,p}(\mathbf{Q},\delta). \tag{3.1.7}$$

Let $\Gamma_{q,p}(\mathbf{Q})$ be the smallest constant Γ such that, for every $\epsilon > 0$, we have

$$\mathcal{D}_{q,p}(\mathbf{Q},\delta) \leq C_{p,q,\mathbf{Q},\epsilon}\delta^{-\Gamma-\epsilon}, \text{ for every dyadic } \delta < 1, \tag{3.1.8}$$

where $C_{p,q,\mathbf{Q},\epsilon}$ is a constant that is allowed to depend on $p, q, \mathbf{Q}$ and ϵ. If $p = q$, we often write

$$\Gamma_p(\mathbf{Q}) := \Gamma_{p,p}(\mathbf{Q}). \tag{3.1.9}$$

For a tuple $\widetilde{\mathbf{Q}} = (\widetilde{Q}_1(\xi), \dots, \widetilde{Q}_{\tilde{n}}(\xi))$ of quadratic forms with $\xi \in \mathbb{R}^d$, denote

$$\mathrm{NV}(\widetilde{\mathbf{Q}}) := |\{1 \leq d' \leq d : \partial_{\xi_{d'}}\widetilde{Q}_{\tilde{n}'} \not\equiv 0 \text{ for some } 1 \leq \tilde{n}' \leq \tilde{n}\}|. \tag{3.1.10}$$

Here for a function F, we use $F \not\equiv 0$ to mean that it does not vanish constantly, and $\mathrm{NV}(\widetilde{\mathbf{Q}})$ refers to “the number of variables that $\widetilde{\mathbf{Q}}$ depends on”. For instance, for $\widetilde{\mathbf{Q}} = ((\xi_1+\xi_3)^2, (\xi_1+\xi_3+\xi_4)^2)$, we have that $\mathrm{NV}(\widetilde{\mathbf{Q}}) = 3$.

For $0 \leq n' \leq n$ and $0 \leq d' \leq d$, define

$$\mathfrak{d}_{d',n'}(\mathbf{Q}) := \inf_{\substack{M\in\mathbb{R}^{d\times d}\\ \mathrm{rank}(M)=d'}} \inf_{\substack{M'\in\mathbb{R}^{n\times n}\\ \mathrm{rank}(M')=n'}} \mathrm{NV}(M'\cdot(\mathbf{Q}\circ M)), \tag{3.1.11}$$

where $\mathbf{Q}\circ M$ is the composition of $\mathbf{Q}$ with M. We abbreviate $\mathfrak{d}_{n'}(\mathbf{Q}) := \mathfrak{d}_{d,n'}(\mathbf{Q})$.

As these quantities will be crucial throughout the entire paper, it may make sense to look at them from a few angles. Let $H \subset \mathbb{R}^d$ be a linear subspace of codimension m. Let Rot_H be a rotation[1] on $\mathbb{R}^d$ that maps $\{\xi \in \mathbb{R}^d : \xi_{d-m+1} = \cdots = \xi_d = 0\}$ to H. We define

$$Q|_H(\xi') := Q((\xi', \mathbf{0}) \cdot (\mathrm{Rot}_H)^T), \text{ with } \xi' \in \mathbb{R}^{d-m} \text{ and } \mathbf{0} = (0, \dots, 0) \in \mathbb{R}^m. \tag{3.1.12}$$

Here $(\mathrm{Rot}_H)^T$ refers to the transpose of Rot_H. Similarly, for $\mathbf{Q} = (Q_1, \dots, Q_n)$, we denote

$$\mathbf{Q}|_H := (Q_1|_H, \dots, Q_n|_H). \tag{3.1.13}$$

With the above notation, we can also write

$$\mathfrak{d}_{d',n'}(\mathbf{Q}) = \inf_H \inf_{M \in \mathrm{GL}_{d'}} \inf_{\substack{M' \in \mathbb{R}^{n' \times n} \\ \mathrm{rank}(M') = n'}} \mathrm{NV}(M' \cdot (\mathbf{Q}|_H \circ M)), \tag{3.1.14}$$

where H runs through all linear sub-spaces in $\mathbb{R}^d$ of dimension d'. In other words, this is the minimal number of variables that n' many of the forms in $\mathbf{Q}$ depend on, after restricting them to sub-spaces of dimension d', up to linear changes of variables in their definition domain $\mathbb{R}^{d'}$ and their value domain $\mathbb{R}^n$. From (3.1.14), one can also see that

$$\mathfrak{d}_{d',n'}(\mathbf{Q}) = \inf_{H \text{ of dim } d'} \mathfrak{d}_{d',n'}(\mathbf{Q}|_H). \tag{3.1.15}$$

For example, if we take $d = 3, n = 3$ and $\mathbf{Q} = ((\xi_1 + \xi_2 + \xi_3)^2, (\xi_1 + \xi_2)^2, (\xi_1 + \xi_2)^2)$, then $\mathfrak{d}_0(\mathbf{Q}) = \mathfrak{d}_1(\mathbf{Q}) = 0$, $\mathfrak{d}_2(\mathbf{Q}) = 1$ and $\mathfrak{d}_3(\mathbf{Q}) = 2$.

Theorem 3.1.1. *Let $d \geq 1$ and $n \geq 1$. Let $\mathbf{Q} = (Q_1, \dots, Q_n)$ be a collection of quadratic forms in d variables. Let $2 \leq q \leq p < \infty$. Then the $\ell^q L^p$ decoupling exponent for the d-surface $S_{\mathbf{Q}} = \{(\xi, \mathbf{Q}(\xi)) : \xi \in [0,1]^d\}$ equals*

$$\Gamma_{q,p}(\mathbf{Q}) = \max_{0 \leq d' \leq d} \max_{0 \leq n' \leq n} \Big(d'\big(1 - \frac{1}{p} - \frac{1}{q}\big) - \mathfrak{d}_{d',n'}(\mathbf{Q})\big(\frac{1}{2} - \frac{1}{p}\big) - \frac{2(n-n')}{p} \Big). \tag{3.1.16}$$

Moreover, for $2 \leq p < q \leq \infty$, we have

$$\Gamma_{q,p}(\mathbf{Q}) = \Gamma_{p,p}(\mathbf{Q}) + d(1/p - 1/q). \tag{3.1.17}$$

The expression (3.1.16) simplifies in the case $q = p$ in the following way.

Corollary 3.1.2. *In the situation of Theorem 3.1.1, we have*

$$\Gamma_p(\mathbf{Q}) = \max_{d/2 < d' \leq d} \max_{0 \leq n' \leq n} \Big((2d' - \mathfrak{d}_{d',n'}(\mathbf{Q}))\big(\frac{1}{2} - \frac{1}{p}\big) - \frac{2(n-n')}{p} \Big). \tag{3.1.18}$$

for every $p \geq 2$.

Proof. For every $d' \leq d/2$ and $0 \leq n' \leq n$, we have $2d' - \mathfrak{d}_{d',n'}(\mathbf{Q}) \leq d \leq 2d - \mathfrak{d}_{d,n'}(\mathbf{Q})$. Hence, the (d', n') term in (3.1.16) is not larger than the (d, n') term. □

[1] There are infinitely many such rotations: We pick an arbitrary one.

Taking $d' = d$ and $n' \in \{0, n\}$ in (3.1.18), we see that, for every tuple $\mathbf{Q} = (Q_1, \dots, Q_n)$ of quadratic forms depending on d variables, it always holds that

$$\Gamma_p(\mathbf{Q}) \geq \max\Big(d(\frac{1}{2} - \frac{1}{p}), 2d(\frac{1}{2} - \frac{1}{p}) - \frac{2n}{p}\Big) \text{ for every } p \geq 2. \tag{3.1.19}$$

Similarly,

$$\Gamma_{2,p}(\mathbf{Q}) \geq \max\Big(0, d(\frac{1}{2} - \frac{1}{p}) - \frac{2n}{p}\Big) \text{ for every } p \geq 2. \tag{3.1.20}$$

We say that $\mathbf{Q} = (Q_1, \dots, Q_n)$ is *strongly non-degenerate* if

$$\mathfrak{d}_{d-m,n'}(\mathbf{Q}) \geq n'd/n - m, \tag{3.1.21}$$

for every n' and every m with $0 \leq m \leq d$.

Corollary 3.1.3 (Best possible $\ell^2 L^p$ decoupling)**.** *We have*

$$\Gamma_{2,p}(\mathbf{Q}) = \max\Big(0, d(\frac{1}{2} - \frac{1}{p}) - \frac{2n}{p}\Big) \textit{ for every } 2 \leq p < \infty \tag{3.1.22}$$

if and only if $\mathbf{Q}$ *is strongly non-degenerate.*

We say that $\mathbf{Q} = (Q_1, \dots, Q_n)$ is *non-degenerate* if

$$\mathfrak{d}_{d-m,n'}(\mathbf{Q}) \geq n'd/n - 2m, \tag{3.1.23}$$

for every n' and every m with $0 \leq m < d/2$.

Corollary 3.1.4 (Best possible $\ell^p L^p$ decoupling)**.** *We have*

$$\Gamma_{p,p}(\mathbf{Q}) = \max\Big(d(\frac{1}{2} - \frac{1}{p}), 2d(\frac{1}{2} - \frac{1}{p}) - \frac{2n}{p}\Big) \textit{ for every } 2 \leq p < \infty \tag{3.1.24}$$

if and only if $\mathbf{Q}$ *is non-degenerate.*

In view of (3.1.20) and (3.1.19), Corollary 3.1.3 and Corollary 3.1.4 characterize tuples of quadratic forms that possess "best possible" $\ell^2 L^p$ decoupling constants and $\ell^p L^p$ decoupling constants, respectively.

We say that $\mathbf{Q} = (Q_1, \dots, Q_n)$ is *weakly non-degenerate* if

$$\mathfrak{d}_{d-m,n}(\mathbf{Q}) \geq d - 2m, \tag{3.1.25}$$

for every $0 \leq m < d/2$.

Corollary 3.1.5. *A tuple* $\mathbf{Q} = (Q_1, \dots, Q_n)$ *of quadratic forms is weakly non-degenerate if and only if there exists some* $p_c > 2$ *such that*

$$\Gamma_p(\mathbf{Q}) = d(\frac{1}{2} - \frac{1}{p}), \quad 2 \leq p \leq p_c. \tag{3.1.26}$$

If $\mathbf{Q}$ *is weakly non-degenerate, then the largest possible* p_c *is given by*

$$2 + \min\bigg(\frac{4(n-n')}{d - \big(\mathfrak{d}_{d-m,n'}(\mathbf{Q}) + 2m\big)}\bigg), \tag{3.1.27}$$

where the minimum on the right hand side is taken over all n' *and* m *satisfying* $n' \leq n-1, m < d/2$ *and* $d > \mathfrak{d}_{d-m,n'}(\mathbf{Q}) + 2m$*.*

One reason that we are interested in the exponent p_c in Corollary 3.1.5 is that, when applying our main results to exponential sum estimates (see Corollary 3.2.1 below), the exponent p_c is the largest for which we can still expect square root cancellation; see right below Corollary 3.2.1 for what we mean by square root cancellation.

Organization of Chapter 3

In Section 3.2 we state a few applications of our main theorem. In Section 3.3, we compute the decoupling exponent provided by the main theorem more explicitly for several examples of tuples of quadratic forms $\mathbf{Q}$, including some of those tuples $\mathbf{Q}$ for which sharp decoupling inequalities have been previously established in the literature, and a few tuples $\mathbf{Q}$ (in particular, arbitrary pairs of forms and tuples of simultaneously diagonalizable forms) for which our results are new.

The upper bounds of $\Gamma_{q,p}(\mathbf{Q})$ in Theorem 3.1.1 with $q \leq p$ are proven in Section 3.4 and Section 3.5. The lower bounds of $\Gamma_{q,p}(\mathbf{Q})$ in Theorem 3.1.1 with $q \leq p$ are proven in Section 3.6. In Section 3.7, we show that the optimal decoupling inequalities for $q > p$ follow from the case $q = p$ of Theorem 3.1.1.

In Section 3.8, we provide the proofs of Corollary 3.1.3, Corollary 3.1.4 and Corollary 3.1.5. In Section 3.9, we prove the Fourier restriction estimate in Corollary 3.2.3.

Notation

For two positive constants A_1, A_2 and a set $\mathcal{I}$ of parameters, we use $A_1 \lesssim_{\mathcal{I}} A_2$ to mean that there exists $C > 0$ depending on the parameters in $\mathcal{I}$ such that $A_1 \leq C A_2$. Typically, $\mathcal{I}$ will be taken to be $\{\mathbf{Q}, d, n, p, q, \epsilon\}$ where $\epsilon > 0$ is a small number. Similarly, we define $A_1 \gtrsim A_2$. Moreover, $A_1 \sim A_2$ means $A_1 \lesssim A_2$ and $A_1 \gtrsim A_2$.

Let $\delta \in (0,1)$ be a dyadic number. We denote by $\mathcal{P}(Q, \delta)$ the dyadic cubes of side length δ in Q for every dyadic cube $Q \subset [0,1]^d$. Let $\mathcal{P}(\delta)$ be the partition of $[0,1]^d$ into dyadic cubes of side length δ. Let $\square$ be a cube with side length $l(\square)$, we use $c \cdot \square$ to denote the cube of side length $c \cdot l(\square)$ and of the same center as $\square$.

For two linear spaces V, V', we use $V' \leq V$ to mean that V' is a linear subspace of V. For a sequence of real numbers $\{A_j\}_{j=1}^M$, we abbreviate $\prod\!\!\!\prod A_j := \big(\prod_{j=1}^M |A_j|\big)^{1/M}$. For $E > 0$ and a ball $B = B(c_B, r_B) \subset \mathbb{R}^{d+n}$ with center c_B and radius r_B, define an associated weight

$$w_{B,E}(\cdot) := \Big(1 + \frac{|\cdot - c_B|}{r_B}\Big)^{-E}. \tag{3.1.28}$$

The power E is large number depending on d, n, e.g., $E = 10(d+n)$, and will be omitted from the notation $w_{B,E}$. All implicit constants in the paper are allowed to depend on E. Also, we define averaged integrals:

$$\|f\|_{\text{Ł}^p(B)} := \Big(\frac{1}{|B|}\int_B |f|^p\Big)^{1/p} \text{ and } \|f\|_{\text{Ł}^p(w_B)} := \Big(\frac{1}{|B|}\int |f|^p w_B\Big)^{1/p}.$$

For a dyadic box $\square \subset [0,1]^d$, a function $f_\square$ is always implicitly assumed to satisfy supp $\widehat{f_\square} \subset \mathcal{U}_\square$, unless otherwise stated.

We would like to make the convention that all vectors are column vectors, unless they are variables of functions or otherwise stated. Below are a few more conventions we make on notation: We will use dyadic cubes of side lengths δ, δ^b with $b < 1$ and $1/K_j$ with $1 \leq j \leq d$. One can always keep in mind that $\log_{K_j} \log_{K_j}(1/\delta) \geq K_j$. We will always use $\square$ to denote a dyadic cube of the smallest scale δ, J to denote a dyadic cube of an intermediate scale δ^b, and W or W_j to denote a dyadic cube of a large scale $1/K_j$. We will introduce certain multi-linear estimates during the proof, and the degree of the multi-linearity will always be called M.

3.2 Applications

3.2.1 Exponential sum estimates.

Let $\mathbf{Q} = (Q_1, \ldots, Q_n)$ be a collection of quadratic forms of integral coefficients defined on $\mathbb{R}^d$. Let $\mathbf{w} = (w_1, \ldots, w_d) \in \mathbb{N}^d$.

Corollary 3.2.1. *For every $d, n \geq 1$, every $p \geq 2, \epsilon > 0$, there exists $C_{\mathbf{Q},\epsilon,p}$ such that*

$$\Big\| \sum_{1 \leq d' \leq d} \sum_{0 \leq w_{d'} \leq W} e^{2\pi i (\mathbf{w} \cdot x + \mathbf{Q}(\mathbf{w}) \cdot y)} \Big\|_{L^p([0,1]^d \times [0,1]^n)} \leq C_{\mathbf{Q},\epsilon,p} W^{\Gamma_p(\mathbf{Q}) + \frac{d}{p} + \epsilon} \tag{3.2.1}$$

for every integer W.

If $\Gamma_p(\mathbf{Q}) = d(1/2 - 1/p)$, then the above corollary says that

$$\begin{aligned} &\Big\| \sum_{1 \leq d' \leq d} \sum_{0 \leq w_{d'} \leq W} e^{2\pi i (\mathbf{w} \cdot x + \mathbf{Q}(\mathbf{w}) \cdot y)} \Big\|_{L^p([0,1]^d \times [0,1]^n)} \\ &\leq C_{\mathbf{Q},\epsilon,p} W^{\epsilon} \Big\| \sum_{1 \leq d' \leq d} \sum_{0 \leq w_{d'} \leq W} e^{2\pi i (\mathbf{w} \cdot x + \mathbf{Q}(\mathbf{w}) \cdot y)} \Big\|_{L^2([0,1]^d \times [0,1]^n)}, \end{aligned} \tag{3.2.2}$$

by which we mean square root cancellation holds for the exponential sum at such p.

The derivation of exponential sum estimates of the form in the above corollary from decoupling inequalities has been standard, see for instance Section 2 Jean Bourgain and Demeter 2015 and Section 4 Jean Bourgain, Demeter, and Guth 2016. We will not repeat the argument here.

Let $s \geq 1$ be an integer. Consider the system of Diophantine equations

$$\begin{aligned} &\mathbf{w}_1 + \cdots + \mathbf{w}_s = \mathbf{w}_{s+1} + \cdots + \mathbf{w}_{2s}, \\ &\mathbf{Q}(\mathbf{w}_1) + \cdots + \mathbf{Q}(\mathbf{w}_s) = \mathbf{Q}(\mathbf{w}_{s+1}) + \cdots + \mathbf{Q}(\mathbf{w}_{2s}). \end{aligned} \tag{3.2.3}$$

For a large integer W, let $\mathcal{J}_{\mathbf{Q}}(W)$ be the number of solutions to (3.2.3) where $0 \leq w_{d'} \leq W$ for every d'. As a immediate corollary of (3.2.1), we obtain

Corollary 3.2.2. *For every $d, n \geq 1$, integer $s \geq 1$, and every $\epsilon > 0$, there exists $C_{\mathbf{Q},\epsilon,s}$ such that*

$$\mathcal{J}_{\mathbf{Q}}(W) \leq C_{\mathbf{Q},\epsilon,s} W^{2s\Gamma_{2s}(\mathbf{Q}) + d + \epsilon}, \tag{3.2.4}$$

for every W.

3.2.2 Fourier restriction estimates.

Let $\mathbf{Q} = (Q_1, \dots, Q_n)$ be a collection of quadratic forms defined on $\mathbb{R}^d$. We say that $\mathbf{Q}$ is *linearly independent* if $Q_1, \dots, Q_n$ are linearly independent. We are interested in the Fourier restriction problems: Find an optimal range of p such that

$$\|E^{\mathbf{Q}}_{[0,1]^d} g\|_{L^p(\mathbb{R}^{d+n})} \lesssim_{d,n,p,\mathbf{Q}} \|g\|_p \tag{3.2.5}$$

holds true for every function g. By a simple change of variables, one can see that the restriction estimate (3.2.5) cannot hold true for any $p < \infty$ if $\mathbf{Q}$ is linearly dependent. As an application of Corollary 3.1.2, we prove some restriction estimates for every linearly independent $\mathbf{Q}$ for some range of p.

Corollary 3.2.3. *Let $\mathbf{Q} = (Q_1, \dots, Q_n)$ be a collection of linearly independent quadratic forms defined on $\mathbb{R}^d$. Then*

$$\|E^{\mathbf{Q}}_{[0,1]^d} g\|_{L^p(\mathbb{R}^{d+n})} \lesssim_{d,n,p,\mathbf{Q}} \|g\|_p \tag{3.2.6}$$

for every

$$p > p_{\mathbf{Q}} := 2 + \max_{m \geq 1} \max_{n' \leq n} \frac{4n'}{2m + \mathfrak{d}_{d-m,n'}(\mathbf{Q})}. \tag{3.2.7}$$

The proof of this corollary will be presented in Section 3.9. One significance of this corollary is that the range (3.2.7) is sharp for Parsell–Vinogradov systems. Let us be more precise. Let $d \geq 2$. Denote $\xi^\alpha := \xi_1^{\alpha_1} \dots \xi_d^{\alpha_d}$ for $\xi = (\xi_1, \dots, \xi_d)$ and a multi-index $\alpha = (\alpha_1, \dots, \alpha_d)$. For $\mathbf{Q} := (\xi^\alpha)_{|\alpha|=2}$, we have $n = d(d+1)/2$, and it has been shown by Christ Christ 1982 and Mockenhaupt Mockenhaupt 1996 that (3.2.6) holds if and only if

$$p > 2 + \frac{4n}{d+1} = 2d + 2. \tag{3.2.8}$$

Let us also mention that, for this tuple $\mathbf{Q}$, the full range of $L^q \to L^p$ estimates generalizing (3.2.6) has been obtained in Bak and S. Lee 2004; D. M. Oberlin 2005. The next claim shows that the range (3.2.7) coincides with (3.2.8).

Claim 3.2.4. *Let $\mathbf{Q} := (\xi^\alpha)_{|\alpha|=2}$. Then*

$$\max_{m \geq 1} \max_{n' \leq n} \frac{4n'}{2m + \mathfrak{d}_{d-m,n'}(\mathbf{Q})} = \frac{4n}{2 + \mathfrak{d}_{d-1,n}(\mathbf{Q})} = \frac{4n}{d+1} = 2d. \tag{3.2.9}$$

In other words, the max *is attained at $m = 1, n' = n$.*

Proof of Claim 3.2.4. By definition, $\mathfrak{d}_{d-1,n}(\mathbf{Q}) = d - 1$. Hence it suffices to show the leftmost expression in (3.2.9) is equal to $2d$.

Fix $m \geq 1$ and $n' \leq n$. Denote $l := \mathfrak{d}_{d-m,n'}(\mathbf{Q})$. Notice that by the definition of $\mathfrak{d}_{d-m,n'}(\mathbf{Q})$, we see that $l + m \leq d$. Our goal is to show that $\frac{4n'}{2m+l} \leq 2d$. We claim that

$$n' \leq \binom{l+1}{2} + \binom{m+1}{2} + m(d-m). \tag{3.2.10}$$

Indeed, by definition (3.1.14), there exist a linear subspace $H \subset \mathbb{R}^d$ of dimension $d-m$ and a linear subspace $\mathcal{Q}$ of the span of $\mathbf{Q}$ of dimension n' such that the restrictions of the forms from $\mathcal{Q}$ to H depend only on l variables. Since the system $\mathbf{Q} = (\xi^\alpha)_{|\alpha|=2}$ is a basis for the space of all quadratic forms in d variables, the above statement does not depend on H and the l variables inside H, so we may assume $H = \{\xi : \xi_{d-m+1} = \cdots = \xi_d = 0\}$ and the l variables are $\xi_1, \ldots, \xi_l$. In this case, $\mathcal{Q}$ is contained in the space of all quadratic forms that depend either only on $\xi_1, \ldots, \xi_l$, or on at least one of the variables $\xi_{d-m+1}, \ldots, \xi_d$. The right-hand side of (3.2.10) is precisely the dimension of the latter space, which concludes the proof of (3.2.10).

Given (3.2.10), it remains to show

$$4\left(\frac{l(l+1)}{2} + \frac{m(m+1)}{2} + m(d-m)\right) \leq 2d(2m+l), \tag{3.2.11}$$

which is equivalent to

$$2(l+m)(l-m+1) \leq 2dl. \tag{3.2.12}$$

This holds because $l + m \leq d$ and $l - m + 1 \leq l$. □

3.3 Examples: Old and new

3.3.1 Example: Hypersurfaces with nonvanishing Gaussian curvatures

We take $n = 1$. Let Q be a quadratic form depending on d variables. Without loss of generality we assume that $\mathfrak{d}_1(Q) = d$. Via a change of coordinate, we can write $Q(\xi)$ as $\xi_1^2 \pm \xi_2^2 \pm \cdots \pm \xi_d^2$. This is the (hyperbolic) paraboloid case. It is easy to see $\mathfrak{d}_0(Q) = 0, \mathfrak{d}_1(Q) = d$.

Lemma 3.3.1. *Let $\widetilde{Q} : \mathbb{R}^d \to \mathbb{R}$ be a quadratic form. Let $M \in M_{d\times d}$ with rank d'. Then*

$$\mathfrak{d}_1(\widetilde{Q}(\cdot M)) \geq \mathfrak{d}_1(\widetilde{Q}(\cdot)) - 2(d-d'). \tag{3.3.1}$$

Proof of Lemma 3.3.1. A lemma of this form was already proved and used by Bourgain and Demeter, see Lemma 2.6 in Jean Bourgain and Demeter 2017b. We use $\mathrm{Hess}(\widetilde{Q})$ to denote the Hessian of the quadratic form $\widetilde{Q}$. What we need to prove is, for every $M \in M_{d\times d}$ with rank d', it holds that

$$\mathrm{rank}(M\mathrm{Hess}(\widetilde{Q})M^T) \geq \mathrm{rank}(\mathrm{Hess}(\widetilde{Q})) - 2(d-d'). \tag{3.3.2}$$

This follows immediately form Sylvester's rank inequality:

$$\mathrm{rank}(AB) \geq \mathrm{rank}(A) + \mathrm{rank}(B) - n, \tag{3.3.3}$$

for two arbitrary matrices $A, B \in M_{n\times n}$. □

By Lemma 3.3.1, we know that $\mathfrak{d}_1(Q|_H) \geq d - 2m$ for every linear subspace of codimension m, which means Q is non-degenerate. Therefore we can apply Corollary 3.1.4 and obtain

$$\Gamma_p(Q) = \max\Big(d - \frac{2d+2}{p}, d(\frac{1}{2} - \frac{1}{p})\Big). \tag{3.3.4}$$

This recovers the $\ell^p L^p$ decoupling results of Bourgain and Demeter in Jean Bourgain and Demeter 2015 and Jean Bourgain and Demeter 2017b. Moreover, if we take $Q(\xi) = \xi_1^2 + \cdots + \xi_d^2$, then it is elementary to see that $\mathfrak{d}_1(Q|_H) \geq d - m$ for every linear subspace of co-dimension m, which means Q is strongly non-degenerate. Therefore we can apply Corollary 3.1.3 and obtain

$$\Gamma_{2,p}(Q) = \max\Big(0, \frac{d}{2} - \frac{d+2}{p}\Big). \tag{3.3.5}$$

This recovers the $\ell^2 L^p$ decoupling results of Bourgain and Demeter in Jean Bourgain and Demeter 2015.

3.3.2 Example: Co-dimension two manifolds in $\mathbb{R}^4$

Take $d = n = 2$. Let $Q_1(\xi) = A_1\xi_1^2 + 2A_2\xi_1\xi_2 + A_3\xi_2^2$ and $Q_2(\xi) = B_1\xi_1^2 + 2B_2\xi_1\xi_2 + B_3\xi_2^2$. Under the assumption that

$$\operatorname{rank}\begin{bmatrix} A_1, & A_2, & A_3 \\ B_1, & B_2, & B_3 \end{bmatrix} = 2, \tag{3.3.6}$$

Bourgain and Demeter Jean Bourgain and Demeter 2016a proved that

$$\Gamma_p(\mathbf{Q}) = \max\Big(2(\frac{1}{2} - \frac{1}{p}), 2(1 - \frac{4}{p})\Big), \tag{3.3.7}$$

with $\mathbf{Q} = (Q_1, Q_2)$. This decoupling inequality is particularly interesting as it is one key ingredient in Bourgain's improvement on the Lindelöf Hypothesis in Jean. Bourgain 2017.

Let us see how Theorem 3.1.1 recovers this result. We take $d = n = 2$ and notice that $\mathfrak{d}_2(\mathbf{Q}) = 2$ (indeed, if $\mathfrak{d}_2(\mathbf{Q}) \leq 1$, then Q_1, Q_2 would be linearly dependent, since the space of quadratic forms in one variable is one-dimensional, contradicting (3.3.6)). Moreover, it is straightforward to see that $\mathfrak{d}_1(\mathbf{Q}) > 0$ as the assumption (3.3.6) says that Q_1 and Q_2 are linearly independent. Therefore, $\mathbf{Q}$ is non-degenerate in the sense of (3.1.23). We can apply Corollary 3.1.4 and recover the result of Bourgain and Demeter Jean Bourgain and Demeter 2016a.

3.3.3 Example: Degenerate three-dimensional submanifolds of $\mathbb{R}^5$

Take $d = 3$, $n = 2$ and $\mathbf{Q} = (\xi_1^2, \xi_2^2 + \xi_1\xi_3)$. Note that $\mathfrak{d}_0(\mathbf{Q}) = 0$, $\mathfrak{d}_1(\mathbf{Q}) = 1$, $\mathfrak{d}_2(\mathbf{Q}) = 3$, and therefore $\mathbf{Q}$ fails to satisfy the non-degeneracy condition (3.1.23). On the other hand, one can also compute, for instance via (3.1.14), that $\mathfrak{d}_{2,2}(\mathbf{Q}) = 1, \mathfrak{d}_{2,1}(\mathbf{Q}) = 0$ and $\mathfrak{d}_{d',n'}(\mathbf{Q}) = 0$ whenever $d' \leq 1$. We apply Theorem 3.1.1 and obtain that

$$\Gamma_p(\mathbf{Q}) = \max\Big(3(\frac{1}{2} - \frac{1}{p}), \frac{5}{2} - \frac{7}{p}, 3 - \frac{10}{p}\Big), \tag{3.3.8}$$

after some elementary computation. This recovers the main result of Guo, Oh, Roos, Yung and Zorin-Kranich Shaoming Guo, Changkeun Oh, Roos, Yung, and Zorin-Kranich 2019, via an entirely different approach: The proof in Shaoming Guo, Changkeun Oh, Roos, Yung, and Zorin-Kranich 2019 relies on bilinear Fourier restriction estimates, small cap decoupling inequalities for the parabola and the manifold $(\xi_1, \xi_2, \xi_1^2, \xi_1\xi_2)$ and a more sophisticated induction argument; while the proof in the current paper relies on more sophisticated Brascamp-Lieb inequalities and multi-linear Fourier restriction estimates.

3.3.4 Simultaneously diagonalizable forms

Corollary 3.3.2. *Let $\mathbf{Q} = (Q_1, \ldots, Q_n)$ be a collection of quadratic forms without mixed terms. Then*

$$\Gamma_p(\mathbf{Q}) = \max_{0 \le n' \le n} \Big(d\big(\frac{1}{2} - \frac{1}{p}\big) + \big(\frac{1}{2} - \frac{1}{p}\big)(d - \mathfrak{d}_{n'}(\mathbf{Q})) - \frac{2(n-n')}{p} \Big), \tag{3.3.9}$$

for every $p \ge 2$.

Proof of Corollary 3.3.2. We first apply Corollary 3.1.2 and obtain

$$\Gamma_p(\mathbf{Q}) = \max_{d/2 \le d' \le d} \max_{0 \le n' \le n} \Big((2d' - \mathfrak{d}_{d',n'}(\mathbf{Q}))\big(\frac{1}{2} - \frac{1}{p}\big) - \frac{2(n-n')}{p} \Big). \tag{3.3.10}$$

In order to obtain (3.3.9), it suffices to prove that

$$\max_{d/2 \le d' \le d} (2d' - \mathfrak{d}_{d',n'}(\mathbf{Q})) = 2d - \mathfrak{d}_{n'}(\mathbf{Q}), \tag{3.3.11}$$

for every n'. By the equivalent definition of $\mathfrak{d}_{d',n'}(\mathbf{Q})$ as in (3.1.14), this is equivalent to proving

$$\min_{0 \le m \le d/2} \inf_{\substack{H \text{ of} \\ Setcodim\ m}} \big(\mathfrak{d}_{n'}(\mathbf{Q}|_H) + 2m \big) = \mathfrak{d}_{n'}(\mathbf{Q}), \tag{3.3.12}$$

for every n', which is the same as saying

$$\mathfrak{d}_{n'}(\mathbf{Q}) - 2m \le \mathfrak{d}_{n'}(\mathbf{Q}|_H) \tag{3.3.13}$$

for every $1 \le n' \le n$ and every plane H of codimension m with $1 \le m \le d/2$.

We argue by contradiction and assume that

$$\mathfrak{d}_{n'}(\mathbf{Q}|_H) \le \mathfrak{d}_{n'}(\mathbf{Q}) - 2m - 1, \tag{3.3.14}$$

for some n' and some linear subspace H of codimension m. By the definition (3.1.11), we can find $M_{d-m} \in GL_{d-m}(\mathbb{R})$ and $M' \in M_{n \times n'}$ of rank n' such that

$$\mathrm{NV}(\bar{\mathbf{P}}) = \mathfrak{d}_{n'}(\mathbf{Q}|_H), \tag{3.3.15}$$

where for $\xi' \in \mathbb{R}^{d-m}$ we define

$$\begin{aligned} \bar{\mathbf{P}}(\xi') &:= (Q_1|_H(\xi' \cdot M_{d-m}), \ldots, Q_n|_H(\xi' \cdot M_{d-m})) \cdot M' \\ &= (Q_1((\xi' \cdot M_{d-m}, \mathbf{0}) \cdot \mathrm{Rot}_H), \ldots, Q_n((\xi' \cdot M_{d-m}, \mathbf{0}) \cdot \mathrm{Rot}_H)) \cdot M'. \end{aligned} \tag{3.3.16}$$

Here $\mathbf{0} = (0, \ldots, 0) \in \mathbb{R}^m$ and Rot_H is a rotation matrix acting on $\mathbb{R}^d$. Let $M_d \in GL_d(\mathbb{R})$ be a matrix such that

$$(\xi' \cdot M_{d-m}, \mathbf{0}) = (\xi', \mathbf{0}) \cdot M_d, \text{ for every } \xi' \in \mathbb{R}^{d-m}. \tag{3.3.17}$$

With this notation, we can write

$$\begin{aligned} \bar{\mathbf{P}}(\xi') &= (Q_1((\xi', \mathbf{0}) \cdot M_d \cdot \mathrm{Rot}_H), \ldots, Q_n((\xi', \mathbf{0}) \cdot M_d \cdot \mathrm{Rot}_H)) \cdot M' \\ &=: (\bar{P}_1(\xi'), \ldots, \bar{P}_{n'}(\xi')). \end{aligned} \tag{3.3.18}$$

Recall (3.3.15). It implies that

$$\mathrm{NV}(\lambda_1 \bar{P}_1 + \cdots + \lambda_{n'} \bar{P}_{n'}) \leq \mathfrak{d}_{n'}(\mathbf{Q}|_H) \leq \mathfrak{d}_{n'}(\mathbf{Q}) - 2m - 1, \tag{3.3.19}$$

for all choices of $\lambda_1, \ldots, \lambda_{n'} \in \mathbb{R}$. Now if we denote

$$\bar{\mathbf{Q}}(\xi) := (\bar{Q}_1(\xi), \ldots, \bar{Q}_{n'}(\xi)) := (Q_1(\xi), \ldots, Q_n(\xi)) \cdot M', \tag{3.3.20}$$

then from the definition of $\mathfrak{d}_{n'}(\mathbf{Q})$ and the fact that $Q_1, \ldots, Q_n$ are diagonal quadratic forms, we can find some $\lambda_1, \ldots, \lambda_{n'}$ such that

$$\mathfrak{d}_1(\lambda_1 \bar{Q}_1 + \cdots + \lambda_{n'} \bar{Q}_{n'}) = \mathrm{NV}(\lambda_1 \bar{Q}_1 + \cdots + \lambda_{n'} \bar{Q}_{n'}) \geq \mathfrak{d}_{n'}(\mathbf{Q}). \tag{3.3.21}$$

Recall the definition of $\bar{\mathbf{P}}$ in (3.3.16) and the relation in (3.3.17) and (3.3.18). Lemma 3.3.1 then says that

$$\mathrm{NV}(\lambda_1 \bar{P}_1 + \cdots + \lambda_{n'} \bar{P}_{n'}) \geq \mathfrak{d}_{n'}(\mathbf{Q}) - 2m, \tag{3.3.22}$$

which is a contradiction to (3.3.19). $\square$

Corollary 3.3.3. *For $1 \leq n' \leq n$, define*

$$Q_{n'}(\xi) := \sum_{1 \leq d' \leq d} a_{n',d'} \xi_{d'}^2. \tag{3.3.23}$$

Then for every $p \geq 2$, with $\mathbf{Q} = (Q_1, \ldots, Q_n)$,

$$\Gamma_p(\mathbf{Q}) = \max\Big(d(\frac{1}{2} - \frac{1}{p}), 2d(\frac{1}{2} - \frac{1}{p}) - \frac{2n}{p}\Big) \tag{3.3.24}$$

if and only if, for every $1 \leq n' \leq n$, every $n \times (\lfloor d - \frac{n'd}{n} \rfloor + 1)$ submatrix of

$$\begin{bmatrix} a_{1,1}, & a_{1,2}, & \ldots, & a_{1,d} \\ \cdots & & & \\ a_{n,1}, & a_{n,2}, & \ldots, & a_{n,d} \end{bmatrix} \tag{3.3.25}$$

has rank at least $n - n' + 1$. Here for $A \in \mathbb{R}$, $\lfloor A \rfloor$ refers to the largest integer $\leq A$.[2]

When $n = 2$, a condition of the form (3.3.25) already appeared in Heath-Brown and Pierce Heath-Brown and Pierce 2017. Let $\mathbf{Q} = (Q_1, Q_2)$ be a pair of quadratic forms with integer coefficients. Heath-Brown and Pierce Heath-Brown and Pierce 2017 studied the problem of representing a pair of integers (n_1, n_2) by the pair of (Q_1, Q_2) for general Q_1 and Q_2. If Q_1 and Q_2 are assumed to be simultaneously diagonalizable, say of the form (3.3.23), then the condition in Heath-Brown and Pierce 2017 becomes that every 2×2 minor of

$$\begin{bmatrix} a_{1,1}, & a_{1,2}, & \ldots, & a_{1,d} \\ a_{n,1}, & a_{n,2}, & \ldots, & a_{n,d} \end{bmatrix} \tag{3.3.26}$$

has rank 2, see Condition 3 there.

[2]This notation is used only in Corollary 3.3.3 and its proof.

Proof of Corollary 3.3.3. Let us show the "only if" part by contradiction. Suppose that, for some $1 \le n' \le n$, some $n \times (\lfloor d - \frac{n'd}{n} \rfloor + 1)$ submatrix of (3.3.25) has rank $n - n'$ or less. Then

$$\mathfrak{d}_{n'}(\mathbf{Q}) \le d - (\lfloor d - \frac{n'd}{n} \rfloor + 1) < \frac{n'd}{n}. \tag{3.3.27}$$

Therefore, $\mathbf{Q}$ is not non-degenerate in the sense of 3.1.23, and (3.3.24) cannot hold true by Corollary 3.1.4.

Let us show the other direction of the equivalence. First of all, notice that the two terms on the right hand side of (3.3.24) match at $p = p_{n,d} := 2 + 4n/d$. By Corollary 3.3.2, it suffices to show that

$$d\big(\frac{1}{2} - \frac{1}{p}\big) + \big(\frac{1}{2} - \frac{1}{p}\big)(d - \mathfrak{d}_{n'}(\mathbf{Q})) - \frac{2(n-n')}{p} \le 2d\big(\frac{1}{2} - \frac{1}{p}\big) - \frac{2n}{p} \tag{3.3.28}$$

for every $1 \le n' \le n$ and every $p \ge p_{n,d}$. By rearranging the terms, what we need to show becomes

$$\mathfrak{d}_{n'}(\mathbf{Q}) \ge n'd/n \tag{3.3.29}$$

for every $1 \le n' \le n$. We argue by contradiction and assume that

$$\mathfrak{d}_{n'}(\mathbf{Q}) < n'd/n \tag{3.3.30}$$

for some $1 \le n' \le n$. By definition, there exist $M \in \mathbb{R}^{d \times d}$ of rank d and $M' \in \mathbb{R}^{n \times n}$ of rank n' such that

$$\mathfrak{d}_{n'}(\mathbf{Q}) = \mathrm{NV}(M' \cdot (\mathbf{Q} \circ M)). \tag{3.3.31}$$

Since the assumption (3.3.25) is invariant under the row operations, we may assume that M' is a diagonal matrix with diagonal entries $1, \dots, 1, 0, \dots, 0$. By the inequality (3.3.30), we have

$$\dim \bigcap_{i=1}^{n'} \bigcap_{\xi \in \mathbb{R}^d} \ker \nabla Q_i(\xi) > d - \frac{n'd}{n}. \tag{3.3.32}$$

It remains to observe that

$$\ker \nabla Q_i(\xi) = \{\eta \in \mathbb{R}^d : \sum_{j=1}^{d} \xi_j \eta_j a_{i,j} = 0\} \quad \text{and}$$

$$\bigcap_{\xi \in \mathbb{R}^d} \ker \nabla Q_i(\xi) = \{\eta \in \mathbb{R}^d : \eta_j a_{i,j} = 0 \text{ for all } j = 1, \dots, d\},$$

so that (3.3.32) implies that an $n' \times (\lfloor d - \frac{n'd}{n} \rfloor + 1)$ submatrix of (3.3.25) vanishes. $\square$

3.3.5 Decoupling theory for two quadratic forms

Corollary 3.3.4. *Let $\mathbf{Q} = (Q_1, Q_2)$ be two linearly independent quadratic forms defined on $\mathbb{R}^d$ satisfying $\mathfrak{d}_2(\mathbf{Q}) = d$.*

(1) Let $1 \leq k < d/2$. Then $\mathbf{Q}$ *satisfies* $\mathfrak{d}_1(\mathbf{Q}) = k$ *and the weakly non-degenerate condition if and only*

$$\Gamma_p(\mathbf{Q}) = \max\Big(d(\frac{1}{2}-\frac{1}{p}), (2d-k)(\frac{1}{2}-\frac{1}{p})-\frac{2}{p}, 2d(\frac{1}{2}-\frac{1}{p})-\frac{4}{p}\Big), \tag{3.3.33}$$

for every $p \geq 2$.

(2) $\mathbf{Q}$ *is non-degenerate if and only if it is weakly non-degenerate and satisfies* $\mathfrak{d}_1(\mathbf{Q}) \geq d/2$.

Proof of Corollary 3.3.4. Let us start with proving the first part of the corollary. We denote the right hand side of (3.3.33) by $\Gamma'_p(\mathbf{Q})$. By Corollary 3.1.2, $\Gamma_p(\mathbf{Q})$ is given by

$$\max_{d/2\leq d'\leq d}\max\Big((2d'-\mathfrak{d}_{d',2}(\mathbf{Q}))(\frac{1}{2}-\frac{1}{p}), (2d'-\mathfrak{d}_{d',1}(\mathbf{Q}))(\frac{1}{2}-\frac{1}{p})-\frac{2}{p}, 2d(\frac{1}{2}-\frac{1}{p})-\frac{4}{p}\Big). \tag{3.3.34}$$

Let us first show that (3.3.33) holds, that is, $\Gamma_p(\mathbf{Q}) = \Gamma'_p(\mathbf{Q})$ for every $p \geq 2$, if and only if

$$\begin{aligned} &\max_{d'}(2d'-\mathfrak{d}_{d',1}(\mathbf{Q})) = 2d-k \\ &\max_{d'}(2d'-\mathfrak{d}_{d',2}(\mathbf{Q})) = d. \end{aligned} \tag{3.3.35}$$

To show that (3.3.35) implies (3.3.33), we apply (3.3.34), move the $\max_{d/2\leq d'\leq d}$ inside the second max and obtain (3.3.33). To show the other direction of the equivalence, the constraint $k < d/2$ will come into play. Notice that under this assumption,

$$\Gamma'_p(\mathbf{Q}) = \begin{cases} d(\frac{1}{2}-\frac{1}{p}) & \text{if } p \leq 2+\frac{4}{d-k}, \\ (2d-k)(\frac{1}{2}-\frac{1}{p})-\frac{2}{p} & \text{if } 2+\frac{4}{d-k} \leq p \leq 2+\frac{4}{k}, \\ 2d(\frac{1}{2}-\frac{1}{p})-\frac{4}{p} & \text{if } p \geq 2+\frac{4}{k}. \end{cases} \tag{3.3.36}$$

Note that we are now under the assumption that $\Gamma_p(\mathbf{Q}) = \Gamma'_p(\mathbf{Q})$ for every $p \geq 2$. When p is slightly larger than 2, we have

$$\Gamma_p(\mathbf{Q}) = \max_{d/2\leq d'\leq d}(2d'-\mathfrak{d}_{d',2}(\mathbf{Q}))(\frac{1}{2}-\frac{1}{p}), \tag{3.3.37}$$

as the contributions from the other two terms in (3.3.34) are already negative. This implies

$$\max_{d/2\leq d'\leq d}(2d'-\mathfrak{d}_{d',2}(\mathbf{Q})) = d. \tag{3.3.38}$$

We use (3.3.38) to further simplify $\Gamma_p(\mathbf{Q})$ to

$$\max\Big(d(\frac{1}{2}-\frac{1}{p}), \max_{d/2\leq d'\leq d}(2d'-\mathfrak{d}_{d',1}(\mathbf{Q}))(\frac{1}{2}-\frac{1}{p})-\frac{2}{p}, 2d(\frac{1}{2}-\frac{1}{p})-\frac{4}{p}\Big). \tag{3.3.39}$$

By comparing $\Gamma_p(\mathbf{Q})$ with $\Gamma'_p(\mathbf{Q})$ for $2+\frac{4}{d-k} \leq p \leq 2+\frac{4}{k}$, we see that

$$\max_{d/2\leq d'\leq d}(2d'-\mathfrak{d}_{d',1}) = 2d-k. \tag{3.3.40}$$

This finishes the proof that (3.3.33) is equivalent to (3.3.35).

It remains to show that (3.3.35) is equivalent to that $\mathbf{Q}$ is weakly non-degenerate and satisfies $\mathfrak{d}_1(\mathbf{Q}) = k$. Since the second equation in (3.3.35) is already equivalent to the weakly non-degenerate condition, what we need to prove becomes $\mathfrak{d}_{d,1}(\mathbf{Q}) = \mathfrak{d}_1(\mathbf{Q}) = k$ if and only if

$$\max_{d'}(2d' - \mathfrak{d}_{d',1}(\mathbf{Q})) = 2d - k, \tag{3.3.41}$$

which follows immediately from

$$\max_{d'}(2d' - \mathfrak{d}_{d',1}(\mathbf{Q})) = 2d - \mathfrak{d}_{d,1}(\mathbf{Q}). \tag{3.3.42}$$

To prove (3.3.42), it suffices to prove

Claim 3.3.5.

$$\mathfrak{d}_{d,1}(\mathbf{Q}) - \mathfrak{d}_{d',1}(\mathbf{Q}) \leq 2(d - d') \tag{3.3.43}$$

for every $d/2 \leq d' \leq d$.

The proof of Claim 3.3.5 will be presented in the end of this subsection. So far we have finished the proof of the first part of the corollary.

Let us turn to the second part and show that $\mathbf{Q}$ is non-degenerate if and only if it is weakly non-degenerate and satisfies $\mathfrak{d}_1(\mathbf{Q}) \geq d/2$. By definition, we need to show that $\mathfrak{d}_1(\mathbf{Q}) \geq d/2$ if and only if

$$\mathfrak{d}_{d-m,1}(\mathbf{Q}) \geq d/2 - 2m \tag{3.3.44}$$

for every $0 \leq m \leq d/2$. By taking $m = 0$, we see that (3.3.44) implies $\mathfrak{d}_1(\mathbf{Q}) \geq d/2$. The other direction immediately follows from Claim 3.3.5. This finishes the second part of the corollary. □

Proof of Claim 3.3.5. We take $M_0 \in \mathbb{R}^{d\times d}$ of rank d' and $M_0' \in \mathbb{R}^{2\times 2}$ of rank one such that

$$\mathfrak{d}_{d',1}(\mathbf{Q}) = \inf_{\substack{M\in\mathbb{R}^{d\times d}\\ \mathrm{rank}(M)=d'}} \inf_{\substack{M'\in\mathbb{R}^{n\times n}\\ \mathrm{rank}(M')=1}} \mathrm{NV}(M'\cdot(\mathbf{Q}\circ M)) = \mathrm{NV}(M_0'\cdot(\mathbf{Q}\circ M_0)). \tag{3.3.45}$$

Therefore there exist $\lambda_1, \lambda_2 \in \mathbb{R}$ such that

$$\mathfrak{d}_{d',1}(\mathbf{Q}) = \mathrm{NV}(\widetilde{Q}(\cdot M_0)) = \mathfrak{d}_{d,1}(\widetilde{Q}(\cdot M_0)) \tag{3.3.46}$$

where $\widetilde{Q} = \lambda_1 Q_1 + \lambda_2 Q_2$ and $\mathbf{Q} = (Q_1, Q_2)$. We now apply Lemma 3.3.1 and obtain

$$\begin{aligned}\mathfrak{d}_{d',1}(\mathbf{Q}) - \mathfrak{d}_{d,1}(\mathbf{Q}) &= \mathfrak{d}_{d,1}(\widetilde{Q}(\cdot M_0)) - \mathfrak{d}_{d,1}(\mathbf{Q})\\ &\geq \mathfrak{d}_{d,1}(\widetilde{Q}(\cdot M_0)) - \mathfrak{d}_{d,1}(\widetilde{Q})\\ &\geq -2(d-d').\end{aligned} \tag{3.3.47}$$

This completes the proof of Claim 3.3.5. □

3.4 Transversality

3.4.1 Brascamp–Lieb inequalities

A central tool in most existing proofs of decoupling inequalities are the Brascamp–Lieb inequalities for products of functions in $\mathbb{R}^m$ which are constant along some linear subspaces. Scale-invariant inequalities of this kind have been characterized in Bennett, Carbery, Christ, and Tao 2008. A novelty of our approach is that we for the first time take full advantage of scale-dependent versions of Brascamp–Lieb inequalities. First inequalities of this kind were proved in Bennett, Carbery, Christ, and Tao 2008; Bennett, Carbery, Christ, and Tao 2010, and a unified description taking into account both minimal and maximal scales was obtained in Maldague 2021. We will only use the results of Maldague 2021 in a special symmetric case when all functions f_j below play similar roles. This special case is captured in the following definition.

Definition 3.4.1. Let $m, m' \in \mathbb{N}$. Let $(V_j)_{j=1}^M$ be a tuple of linear subspaces $V_j \subseteq \mathbb{R}^m$ of dimension m'. For a linear subspace $V \subseteq \mathbb{R}^m$, let $\pi_V : \mathbb{R}^m \to V$ denote the orthogonal projection onto V. For $0 \le \alpha \le M$ and $R \ge 1$, we denote by $\mathrm{BL}((V_j)_{j=1}^M, \alpha, R, \mathbb{R}^m)$ (for *Brascamp–Lieb constant*) the smallest constant such that the inequality

$$\int_{[-R,R]^m} \prod_{j=1}^{M} f_j(\pi_{V_j}(x))^\alpha \mathrm{d}x \le \mathrm{BL}((V_j)_{j=1}^M, \alpha, R, \mathbb{R}^m) \prod_{j=1}^{M} \Big(\int_{V_j} f_j(x_j) \mathrm{d}x_j \Big)^\alpha \tag{3.4.1}$$

holds for any functions $f_j : V_j \to [0, \infty)$ that are constant at scale 1, in the sense that V_j can be partitioned into cubes with unit side length on each of which f_j is constant. If the dimension m of the total space $\mathbb{R}^m$ is clear from the context, $\mathrm{BL}((V_j)_{j=1}^M, \alpha, R, \mathbb{R}^m)$ is often abbreviated to $\mathrm{BL}((V_j)_{j=1}^M, \alpha, R)$.

We also need a Kakeya variant of Brascamp–Lieb inequalities, in which each function $f_j \circ \pi_{V_j}$ is replaced by a sum of functions of the form $f_{j,l} \circ \pi_{V_{j,l}}$, where $V_{j,l}$ are different subspaces. The first almost optimal inequality of this kind was the multilinear Kakeya inequality, proved in Bennett, Carbery, and Tao 2006, which generalizes the Loomis–Whitney inequality. A simplified induction on scales proof was later given by Guth Guth 2015. An endpoint version of the multilinear Kakeya inequality was proved by Guth Guth 2010 using the polynomial method. Endpoint Kakeya type extensions of Brascamp–Lieb inequalities were further developed in Carbery and Valdimarsson 2013; Ruixiang Zhang 2018; Zorin-Kranich 2020. It will be convenient to use the following formulation, although a non-endpoint result such as Maldague 2021, Theorem 2 would also suffice for the purpose of proving decoupling inequalities with the optimal range of exponents.

Theorem 3.4.2 (Kakeya–Brascamp–Lieb, Zorin-Kranich 2020)**.** *Fix integers $m' \le m$. Let $\mathcal{V}_j$, $1 \le j \le M$, be families of linear subspaces of $\mathbb{R}^m$ of dimension m'. Let $1 \le \alpha \le M$ and $R \ge 1$. Assume that*

$$A := \sup_{V_1 \in \mathcal{V}_1, \dots, V_M \in \mathcal{V}_M} \mathrm{BL}((V_j)_{j=1}^M, \alpha, R) < \infty. \tag{3.4.2}$$

Then, for any non-negative integrable functions $f_{j,V_j} : V_j \to \mathbb{R}$ constant at scale 1, we have

$$\int_{B(0,R)} \prod_{j=1}^{M} \Big(\sum_{V_j \in \mathcal{V}_j} f_{j,V_j}(\pi_{V_j}(x)) \Big)^\alpha \mathrm{d}x \le C^\alpha A \prod_{j=1}^{M} \Big(\sum_{V_j \in \mathcal{V}_j} \int_{V_j} f_{j,V_j}(x) \mathrm{d}x \Big)^\alpha, \tag{3.4.3}$$

where the constant C depends only on the dimension m.

The uniform bound (3.4.2) is clearly necessary for (3.4.3) to hold. In the scale invariant case, such uniform bounds for Brascamp-Lieb constants were obtained in Bennett, Bez, Flock, and S. Lee 2018; Bennett, Bez, Cowling, and Flock 2017. We need the following corresponding result in the scale-dependent case.

Theorem 3.4.3 (Maldague 2021, Theorem 3)**.** *In the situation of Definition 3.4.1, fix a tuple $(V_j)_{j=1}^M$ and an exponent $1 \leq \alpha \leq M$. Let*

$$\kappa := \sup_{V \leq \mathbb{R}^m} \Big(\dim V - \frac{\alpha}{M} \sum_{j=1}^{M} \dim \pi_{V_j} V \Big), \tag{3.4.4}$$

where the supremum is taken over all linear subspaces of $\mathbb{R}^m$.

Then there exists a constant $C_0 < \infty$ and a neighborhood of the tuple $(V_j)_{j=1}^M$ in the M-th power of the Grassmanian manifold of all linear subspaces of dimension m' of $\mathbb{R}^m$ such that, for any tuple $(\tilde{V}_j)_{j=1}^M$ in this neighborhood and any $R \geq 1$, we have

$$\mathrm{BL}((\tilde{V}_j)_{j=1}^M, \alpha, R) \leq C_0 R^{\kappa}. \tag{3.4.5}$$

3.4.2 Transversality for quadratic forms

Let $\mathbf{Q} = (Q_1, \ldots, Q_n)$ be a sequence of quadratic forms defined on $\mathbb{R}^d$. The subspaces in the subsequent application of Kakeya–Brascamp–Lieb inequalities will be the tangent spaces to the manifold $S_{\mathbf{Q}}$:

$$V_\xi = V_\xi(\mathbf{Q}) := \mathrm{lin}\{(e_j, \partial_j \mathbf{Q}(\xi)), j = 1, \ldots, d\}, \quad \xi \in \mathbb{R}^d. \tag{3.4.6}$$

Here e_j is the j-th coordinate vector and lin refers to linear span. Transversality of pieces of this manifold will be measured by the exponent κ defined in (3.4.4) evaluated at tangent spaces somewhere at the respective pieces: The smaller the exponent, the more transverse are the pieces. It is an observation going back to Jean Bourgain and Demeter 2016b (for scale-invariant Brascamp–Lieb inequalities) that the most transverse situations arise when the pieces are not concentrated near a low degree subvariety in the following sense.

Definition 3.4.4. A subset $\mathcal{W} \subseteq \mathcal{P}(1/K)$ will be called *θ-uniform* if, for every non-zero polynomial P in d variables with real coefficients of degree $\leq d$, we have

$$|\{W \in \mathcal{W} : 2W \cap Z_P \neq \emptyset\}| \leq \theta |\mathcal{W}|.$$

Here Z_P refers to the zero set of P. When using the notation $\mathcal{W} = \{W_1, \ldots, W_M\} = \{W_j\}_{j=1}^M$ for θ-uniform sets, we always mean that the W_j's are pairwise distinct.

Lemma 3.4.5. *Let $\theta \in [0,1]$, $\alpha \geq 1$, and $K \in 2^{\mathbb{N}}$. Then there exists $C_{\theta,K,\alpha} < \infty$ such that, for every θ-uniform set $\mathcal{W} = \{W_1, \ldots, W_M\} \subseteq \mathcal{P}(1/K)$ with $\alpha \leq M$ and every $R \geq 1$, we have*

$$\sup_{\xi_j \in W_j} \mathrm{BL}((V_{\xi_j})_{j=1}^M, \alpha, R, \mathbb{R}^{d+n}) \leq C_{\theta,K,\alpha} R^{\kappa_{\mathbf{Q}}(\alpha \cdot (1-\theta))},$$

where

$$\kappa_{\mathbf{Q}}(\alpha) := \sup_{V \leq \mathbb{R}^{d+n}} \Big(\dim V - \alpha \sup_{\xi \in \mathbb{R}^d} \dim \pi_{V_\xi} V \Big). \tag{3.4.7}$$

In the remaining part, if $\mathbf{Q}$ is clear from the context, we often abbreviate $\kappa_{\mathbf{Q}}(\alpha)$ to $\kappa(\alpha)$.

Proof of Lemma 3.4.5. Since there are only finitely many θ-uniform sets $\mathcal{W}$, and, for any fixed θ-uniform set $\mathcal{W} = \{W_1, \ldots, W_M\}$, the set $\prod_{j=1}^M \overline{W_j}$ is compact, by Theorem 3.4.3, it suffices to show that, for any $\xi_j \in \overline{W_j}$, and every subspace $V \leq \mathbb{R}^{d+n}$, we have

$$\dim V - \frac{\alpha}{M} \sum_{j=1}^{M} \dim \pi_{V_{\xi_j}} V \leq \dim V - \alpha(1-\theta) \sup_{\xi \in \mathbb{R}^d} \dim \pi_{V_\xi} V.$$

This is equivalent to

$$\frac{1}{M} \sum_{j=1}^{M} \dim \pi_{V_{\xi_j}} V \geq (1-\theta) \sup_{\xi \in \mathbb{R}^d} \dim \pi_{V_\xi} V.$$

If $v_1, \ldots, v_m$ is a basis of V, then

$$\dim \pi_{V_\xi} V = \operatorname{rank} \begin{pmatrix} e_1 & \partial_1 \mathbf{Q}(\xi) \\ \vdots & \vdots \\ e_d & \partial_d \mathbf{Q}(\xi) \end{pmatrix} \cdot \begin{pmatrix} v_1 & \ldots & v_m \end{pmatrix}, \tag{3.4.8}$$

where on the right hand side we have the product of two matrices. Each minor determinant of this matrix is a polynomial of degree at most d. Consider the largest minor (of size $d' \times d'$, say) whose determinant is a non-vanishing polynomial; call this polynomial P (if $d' = 0$, then $P = 1$). Then

$$d' = \sup_{\xi \in \mathbb{R}^d} \dim \pi_{V_\xi} V.$$

By Definition 3.4.4, we have $P(\xi_j) \neq 0$ for at least $(1-\theta)M$ many j's. Therefore,

$$\frac{1}{M} \sum_{j=1}^{M} \dim \pi_{V_{\xi_j}} V \geq \frac{1}{M} \sum_{j : P(\xi_j) \neq 0} \dim \pi_{V_{\xi_j}} V \geq \frac{1}{M} \sum_{j : P(\xi_j) \neq 0} d' \geq (1-\theta) d'. \qquad \square$$

This finishes the proof of the lemma.

From the proof of Lemma 3.4.5, we see that the sup in $\sup_{\xi \in \mathbb{R}^d} \dim \pi_{V_\xi} V$ is attained at almost every point, with respect to the d-dimensional Lebesgue measure. Therefore, we introduce the following notation

$$\dim \pi V := \sup_{\xi \in \mathbb{R}^d} \dim \pi_{V_\xi} V. \tag{3.4.9}$$

Next, we will find a more explicit description of the exponent (3.4.7) in terms of the quadratic forms $\mathbf{Q}$. The following result relates the terms in (3.4.7) to the quantities introduced in (3.1.11).

Lemma 3.4.6. *Let $\mathbf{Q}$ be an n-tuple of quadratic forms in d variables. For a linear subspace $V \subseteq \mathbb{R}^{d+n}$, let*

$$d' := \dim \pi V, \quad n' := \dim V - \dim \pi V.$$

Then

$$n' \leq n \text{ and } \mathfrak{d}_{n'}(\mathbf{Q}) \leq d'.$$

Lemma 3.4.6 relies on the following algebraic result.

Lemma 3.4.7. *Let $\mathbb{F} = \mathbb{R}(\xi_1, \ldots, \xi_d)$ be the field of rational functions in d variables. Let $A = (\sum_k a_{i,j,k}\xi_k)_{i,j}$ be a $(N_1 \times N_2)$-matrix whose entries are linear maps with real coefficients. Suppose that* $\operatorname{rank}_{\mathbb{F}} A = r$. *Then there exist real invertible matrices B, B' such that*

$$BAB' = \begin{pmatrix} * & * \\ * & 0 \end{pmatrix}, \tag{3.4.10}$$

where the zero block has size $(N_1 - r) \times (N_2 - r)$.

Standard linear algebra shows that there exist invertible matrices B, B' with entries in $\mathbb{F}$ such that (3.4.7) holds. The point of this Lemma 3.4.7 is that we can find B, B' with real entries. Note that Lemma 3.4.7 may fail if entries of A are not assumed to be linear forms. We will include a proof of Lemma 3.4.7 below and we also note that after finishing the first version of the paper, Zipei Nie Nie 2021 pointed out to us that Lemma 3.4.7 is in fact known in the literature and follows from Flanders 1962, Lemma 1. We thank him for this comment.

Proof of Lemma 3.4.6 assuming Lemma 3.4.7. The claim $n' \leq n$ follows from the fact that the tangent spaces V_ξ have codimension n.

After linear changes of variables in $\mathbb{R}^d$ and $\mathbb{R}^n$, we may assume that V is spanned by linearly independent vectors of the form

$$(e_1, v_1), \ldots, (e_s, v_s), (0, \tilde{e}_1), \ldots, (0, \tilde{e}_j),$$

where e_i are unit coordinate vectors in $\mathbb{R}^d$, $\tilde{e}_i$ are unit coordinate vectors in $\mathbb{R}^n$, v_i are vectors in $\mathbb{R}^n$ and $s \leq \min(d, \dim V)$. Note also that $\dim V = s + j$ and that $j \leq n$. As in (3.4.8), we have

$$\dim \pi_{V_\xi} V = \operatorname{rank}_{\mathbb{R}} \begin{pmatrix} e_1 & \partial_1 \mathbf{Q}(\xi) \\ \vdots & \vdots \\ e_d & \partial_d \mathbf{Q}(\xi) \end{pmatrix} \cdot \begin{pmatrix} e_1^T & \ldots & e_s^T & 0 & \ldots & 0 \\ v_1^T & \ldots & v_s^T & \tilde{e}_1^T & \ldots & \tilde{e}_j^T \end{pmatrix}.$$

Since all entries of the product matrix on the right-hand side are polynomials in ξ, we have

$$d' = \sup_\xi \dim \pi_{V_\xi} V = \operatorname{rank}_{\mathbb{F}} \begin{pmatrix} e_1 & \partial_1 \mathbf{Q}(\xi) \\ \vdots & \vdots \\ e_d & \partial_d \mathbf{Q}(\xi) \end{pmatrix} \cdot \begin{pmatrix} e_1^T & \ldots & e_s^T & 0 & \ldots & 0 \\ v_1^T & \ldots & v_s^T & \tilde{e}_1^T & \ldots & \tilde{e}_j^T \end{pmatrix},$$

where $\mathbb{F}$ is the field of rational functions in d variables. This is because the rank equals the size of the largest minor with non-vanishing determinant, and the determinant of any minor, viewed as an element of $\mathbb{F}$, vanishes if and only if its value vanishes for every ξ. The latter matrix can be written in the block form

$$\begin{pmatrix} I + L_1 & L_3 \\ L_2 & B \end{pmatrix}, \tag{3.4.11}$$

where I is the $s \times s$ identity matrix, L_1, L_2, L_3 are matrices whose entries are linear combinations of monomials of degree 1, and

$$B = \begin{pmatrix} \partial_{s+1} Q_1(\xi) & \dots & \partial_{s+1} Q_j(\xi) \\ \vdots & & \vdots \\ \partial_d Q_1(\xi) & \dots & \partial_d Q_j(\xi) \end{pmatrix}.$$

Let $r := \operatorname{rank}_{\mathbb{F}} B$. Any $r \times r$-minor determinant P of the matrix B is a homogeneous polynomial of degree r, and P coincides with the lowest degree homogeneous part of the corresponding $(r+s) \times (r+s)$-minor determinant of (3.4.11), obtained by adjoining the first s rows and columns. Therefore,

$$d' \geq s + r.$$

Let us continue to prove $\mathfrak{d}_{n'}(\mathbf{Q}) \leq d'$. Recall that $n' = s + j - d' \leq j - r$. By the definition in (3.1.11), it suffices to find linear changes of variables in $\mathbb{R}^d$ and $\mathbb{R}^n$, after which $Q_{r+1}, \dots, Q_j$ no longer depend on variables $\xi_{s+r+1}, \dots, \xi_d$. Notice that row and column operations on B with coefficients in $\mathbb{R}$ correspond to linear changes of variables in $\mathbb{R}^d$ and $\mathbb{R}^n$, respectively. By Lemma 3.4.7, by row and column operations with coefficients in $\mathbb{R}$, B can be brought in a form in which it has a $(j-r) \times (d-s-r)$-block of zeroes. This means that, after a change of variables, $Q_{r+1}, \dots, Q_j$ do not depend on variables $\xi_{s+r+1}, \dots, \xi_d$. □

Proof of Lemma 3.4.7. Let k_1 be the largest index such that ξ_{k_1} appears in A. Swapping rows and columns, we may assume $a_{1,1,k_1} \neq 0$. Using elementary row and column operations, we may further assume that $a_{1,1,k_1} = 1$, $a_{1,j,k_1} = 0$, and $a_{i,1,k_1} = 0$ for all $j \neq 0$ and $i \neq 0$. Thus, we may assume

$$A = \begin{pmatrix} \xi_{k_1} + * & * \\ * & A' \end{pmatrix},$$

where ξ_{k_1} does not appear in entries $*$ and A' is an $(N_1 - 1) \times (N_2 - 1)$-matrix. If $A' \neq 0$, we repeat the same procedure in A', and so on. If this process stops after at most r iterations, then we are done. Otherwise, we have brought the upper left corner of A into the form

$$\begin{pmatrix} \xi_{k_1} + * & * & \dots & * \\ * & \xi_{k_2} + * & \dots & * \\ \vdots & & \ddots & \vdots \\ * & \dots & * & \xi_{k_{r+1}} + * \end{pmatrix}, \tag{3.4.12}$$

where $a_{i,j,k} = 0$ if $i \neq j$ and $k \geq k_{\min(i,j)}$. The determinant of this matrix is a polynomial whose leading term in the lexicographic ordering is $\xi_{k_1} \cdots \xi_{k_{r+1}}$:

$$\det(3.4.12) = \xi_{k_1} \cdots \xi_{k_{r+1}} + \text{lower order terms}.$$

This can be seen by induction on the size of this matrix. Indeed, if $k_1 = \dots = k_l > k_{l+1}$, then ξ_{k_1} appears in this matrix only in the first l diagonal entries, so

$$\det(3.4.12) = \xi_{k_1}^l \cdot \det \begin{pmatrix} \xi_{k_{l+1}} + * & * & \dots & * \\ * & \xi_{k_{l+2}} + * & \dots & * \\ \vdots & & \ddots & \vdots \\ * & \dots & * & \xi_{k_{r+1}} + * \end{pmatrix} + \text{lower order terms}.$$

In particular, the matrix (3.4.12) is invertible (over $\mathbb{F}$), so that $\operatorname{rank}_{\mathbb{F}} A \geq r+1$, a contradiction. $\square$

Corollary 3.4.8. *For any $\alpha \geq 1$, the exponent defined in* (3.4.7) *satisfies*

$$\kappa(\alpha) \leq \sup_{0\leq n'\leq n} \Big(n' + (1-\alpha)\mathfrak{d}_{n'}(\mathbf{Q}) \Big). \tag{3.4.13}$$

Proof of Corollary 3.4.8. Let $V \subseteq \mathbb{R}^{d+n}$ be a linear subspace. With the notation from Lemma 3.4.6, we obtain

$$\dim V - \alpha \dim \pi V = (d' + n') - \alpha d' = n' + (1-\alpha) d' \leq n' + (1-\alpha)\mathfrak{d}_{n'}(\mathbf{Q}).$$

The conclusion follows after taking the supremum over all subspaces V. $\square$

3.4.3 Ball inflation

A so-called ball inflation inequality, based on scale invariant Kakeya–Brascamp–Lieb inequalities, was first introduced in Jean Bourgain, Demeter, and Guth 2016, Theorem 6.6. Here, we formulate a version of this inequality based on scale-dependent Kakeya-Brascamp-Lieb inequalities. Recall that $\mathcal{U}_J$ was defined in (3.1.4).

Proposition 3.4.9 (Ball inflation)**.** *Let $K \in 2^{\mathbb{N}}$ be a dyadic integer and $0 < \rho \leq 1/K$. Let $\{W_j\}_{j=1}^M \subset \mathcal{P}(1/K)$ be a θ-uniform set of cubes. Then, for any $1 \leq t \leq p < \infty$, any functions f_J with $\operatorname{supp} \widehat{f_J} \subset \mathcal{U}_J$ and any $x_0 \in \mathbb{R}^{d+n}$, we have*

$$\begin{aligned} &\Big\| \prod_{j=1}^{M} \Big(\sum_{J\in\mathcal{P}(W_j,\rho)} \|f_J\|_{Ł^t(w_{B(x,1/\rho)})}^t \Big)^{1/t} \Big\|_{Ł^p_{x\in B(x_0,\rho^{-2})}} \\ &\leq C_{\theta,K,p,t}\, \rho^{-(\frac{d}{t} - \frac{d+n}{p} + \frac{\kappa((1-\theta)p/t)}{p})} \prod_{j=1}^{M} \Big(\sum_{J\in\mathcal{P}(W_j,\rho)} \|f_J\|_{Ł^t(w_{B(x_0,\rho^{-2})})}^t \Big)^{1/t}. \end{aligned} \tag{3.4.14}$$

Proof of Proposition 3.4.9. Let $R := \rho^{-1}$. Without loss of generality, we set $x_0 = 0$. Let $\Omega := B(0, R^2)$. Let $\alpha := p/t$. The p-th power of the left-hand side of (3.4.14) equals

$$\fint_{x\in\Omega} \prod_{j=1}^{M} \Big(\sum_{J\in\mathcal{P}(W_j,\rho)} \|f_J\|_{Ł^t(w_{B(x,1/\rho)})}^t \Big)^{\alpha}, \tag{3.4.15}$$

where $\fint_\Omega := |\Omega|^{-1}\int_\Omega$ denotes the average integral. For each cube $J \in \mathcal{P}(W_j,\rho)$ with center ξ_J, we cover Ω with a family $\mathcal{T}_J$ of disjoint tiles T_J, which are rectangular boxes with n long sides of length $2\rho^{-2}$ centered at 0 pointing in the directions $V_{\xi_J}^{\perp}$ and d short sides of length ρ^{-1} pointing in complementary directions (the length of the long sides equals the diameter of $B(x_0,\rho^{-2})$, so that we only need one layer of tiles in the directions $V_{\xi_J}^{\perp}$). We can choose these tiles so that they are contained in $C_0\Omega$ with $C_0 \lesssim 1$. We let $T_J(x)$ be the tile containing x, and for $x \in \cup_{T_J\in\mathcal{T}_J} T_J$ we define

$$F_J(x) := \sup_{y\in T_J(x)} \|f_J\|_{Ł^t(w_{B(y,1/\rho)})}.$$

Then
$$\fint_{x\in\Omega}\prod_{j=1}^{M}\Big(\sum_{J\in\mathcal{P}(W_j,\rho)}\|f_J\|_{\not L^t(w_{B(x,1/\rho)})}^t\Big)^\alpha \le \fint_\Omega \prod_{j=1}^{M}\big(\sum_{J\in\mathcal{P}(W_j,\rho)}|F_J|^t\big)^\alpha.$$

Since the function F_J is constant on each tile $T_J \in \mathcal{T}_J$, we can write its restriction to Ω in the form $\tilde{F}_J \circ \pi_J$, where π_J is the orthogonal projection onto V_{ξ_J}. To apply Theorem 3.4.2, we apply the change of variables $x \to Rx$ such that the resulting functions $F_J(Rx)$ are constant at the unit scale:

$$\fint_\Omega \prod_{j=1}^{M}\big(\sum_{J\in\mathcal{P}(W_j,\rho)}|F_J|^t\big)^\alpha = R^{-(d+n)}\int_{B(0,R)}\prod_{j=1}^{M}\big(\sum_{J\in\mathcal{P}(W_j,\rho)}F_J(R\cdot)^t\big)^\alpha. \tag{3.4.16}$$

By Theorem 3.4.2 and Lemma 3.4.5 with $R=\rho^{-1}$, we bound the last expression by

$$\begin{aligned}&R^{-(d+n)}C_{\theta,K,\alpha}R^{\kappa(\alpha(1-\theta))}\prod_{j=1}^{M}\Big(\sum_{J\in\mathcal{P}(W_j,\rho)}\int_{B(0,C_0R)\subset\mathbb{R}^d}(\tilde{F}_J(R\cdot))^t\Big)^\alpha\\ &\lesssim R^{-(d+n)}C_{\theta,K,\alpha}R^{\kappa(\alpha(1-\theta))}R^{\alpha d}\prod_{j=1}^{M}\Big(\sum_{J\in\mathcal{P}(W_j,\rho)}\fint_{B(0,C_0R^2)\subset\mathbb{R}^{d+n}}|F_J|^t\Big)^\alpha.\end{aligned} \tag{3.4.17}$$

The conclusion will now follow from the bound

$$\|F_J\|_{\not L^t(C_0\Omega)} \lesssim \|f_J\|_{\not L^t(w_\Omega)}, \tag{3.4.18}$$

which is a standard application of the uncertainty principle, see e.g. Shaoming Guo and Zorin-Kranich 2020b, (3.13). □

The ball inflation inequality in Proposition 3.4.9 is sufficient for proving ℓ^pL^p decoupling. For the proof of ℓ^qL^p decoupling with $q<p$ we need a slightly more general statement.

Corollary 3.4.10 (Ball inflation, ℓ^qL^t version)**.** *In the situation of Proposition 3.4.9, for any $1\le\tilde{q}\le t$, we have*

$$\begin{aligned}&\Big\|\prod_{j=1}^{M}\Big(\sum_{J\in\mathcal{P}(W_j,\rho)}\|f_J\|_{\not L^t(w_{B(x,1/\rho)})}^{\tilde{q}}\Big)^{1/\tilde{q}}\Big\|_{\not L^p_{x\in B(x_0,\rho^{-2})}}\\ &\le C_{\theta,K,p,t,\tilde{q}}(|\log\rho|+2)^{K^d}\rho^{-(\frac{d}{t}-\frac{d+n}{p}+\frac{\kappa((1-\theta)p/t)}{p})}\prod_{j=1}^{M}\Big(\sum_{J\in\mathcal{P}(W_j,\rho)}\|f_J\|_{\not L^t(w_{B(x_0,\rho^{-2})})}^{\tilde{q}}\Big)^{1/\tilde{q}}.\end{aligned} \tag{3.4.19}$$

Proof of Corollary 3.4.10. This follows from Proposition 3.4.9 by a dyadic pigeonholing argument in the proof of Jean Bourgain, Demeter, and Guth 2016, Theorem 6.6. For the sake of completeness, we still include the proof here. We follow the presentation in Shaoming Guo and Zorin-Kranich 2020b, Appendix A.

For each $1 \le j \le M$, partition

$$\mathcal{P}(W_j, \rho) = \mathcal{J}_{j,\infty} \cup \bigcup_{\iota=0}^{\lfloor d \log_2(1/\rho) \rfloor} \mathcal{J}_{j,\iota}, \tag{3.4.20}$$

where for $0 \le \iota \le \log(1/\rho)$

$$\begin{aligned} \mathcal{J}_{j,\iota} &:= \Big\{ J \in \mathcal{P}(W_j, \rho) : 2^{-\iota-1} < \frac{\|f_J\|_{\L^t(w_B)}}{\max_{J' \in \mathcal{P}(W_j,\rho)} \|f_{J'}\|_{\L^t(w_B)}} \le 2^{-\iota} \Big\}, \\ \mathcal{J}_{j,\infty} &:= \Big\{ J \in \mathcal{P}(W_j, \rho) : \|f_J\|_{\L^t(w_B)} \le 2^{-\lfloor d \log_2(1/\rho) \rfloor} \max_{J' \in \mathcal{P}(W_j,\rho)} \|f_{J'}\|_{\L^t(w_B)} \Big\}. \end{aligned} \tag{3.4.21}$$

Since $M \le K^d$, the claim (3.4.19) follows by the triangle inequality from

$$\begin{aligned} &\Big\| \prod_{j=1}^{M} \Big(\sum_{J \in \mathcal{J}_{j,\iota_j}} \|f_J\|_{\L^t(w_{B(x,1/\rho)})}^{\tilde{q}} \Big)^{1/\tilde{q}} \Big\|_{\L^p_{x \in \Omega}} \\ &\le C_{\theta,K,p,t,\tilde{q}} \rho^{-(\frac{d}{t} - \frac{d+n}{p} + \frac{\kappa((1-\theta)p/t)}{p})} \prod_{j=1}^{M} \Big(\sum_{J \in \mathcal{P}(W_j,\rho)} \|f_J\|_{\L^t(w_\Omega)}^{\tilde{q}} \Big)^{1/\tilde{q}}, \end{aligned} \tag{3.4.22}$$

which we will show for every choice of $\iota_1, \dots, \iota_M \in \{0, \dots, \lfloor d \log_2(1/\rho) \rfloor\} \cup \{\infty\}$.

Since $\tilde{q} \le t$, by Hölder's inequality, the left hand side of (3.4.22) is bounded by

$$\Big(\prod |\mathcal{J}_{j,\iota_j}|^{\frac{1}{\tilde{q}} - \frac{1}{t}} \Big) \Big\| \prod_{j=1}^{M} \Big(\sum_{J \in \mathcal{J}_{j,\iota_j}} \|f_J\|_{\L^t(w_{B(x,1/\rho)})}^{t} \Big)^{1/t} \Big\|_{\L^p_{x \in B}}. \tag{3.4.23}$$

By Proposition 3.4.9, the last display is bounded by

$$C_{\theta,K,p,t,\tilde{q}} \rho^{-(\frac{d}{t} - \frac{d+n}{p} + \frac{\kappa((1-\theta)p/t)}{p})} \Big(\prod |\mathcal{J}_{j,\iota_j}|^{\frac{1}{\tilde{q}} - \frac{1}{t}} \Big) \prod_{j=1}^{M} \Big(\sum_{J \in \mathcal{J}_{j,\iota_j}} \|f_J\|_{\L^t(w_B)}^{t} \Big)^{1/t}. \tag{3.4.24}$$

It remains to observe that, for every ι, we have

$$|\mathcal{J}_{j,\iota}|^{\frac{1}{\tilde{q}} - \frac{1}{t}} \Big(\sum_{J \in \mathcal{J}_{j,\iota}} \|f_J\|_{\L^t(w_B)}^{t} \Big)^{1/t} \lesssim \Big(\sum_{J \in \mathcal{P}(W_j,\rho)} \|f_J\|_{\L^t(w_B)}^{\tilde{q}} \Big)^{1/\tilde{q}}. \tag{3.4.25}$$

If $\iota \ne \infty$, this follows as the summands on the left hand side are comparable. For $\iota = \infty$, we have

$$\begin{aligned} &|\mathcal{J}_{j,\infty}|^{\frac{1}{\tilde{q}} - \frac{1}{t}} \Big(\sum_{J \in \mathcal{J}_{j,\infty}} \|f_J\|_{\L^t(w_B)}^{t} \Big)^{1/t} \\ &\le |\mathcal{J}_{j,\infty}|^{\frac{1}{\tilde{q}}} \max_{J \in \mathcal{J}_{j,\infty}} \|f_J\|_{\L^t(w_B)} \\ &\le \rho^{-d} 2^{-\lfloor d \log_2(1/\rho) \rfloor} \max_{J' \in \mathcal{P}(W_j,\rho)} \|f_{J'}\|_{\L^t(w_B)} \\ &\lesssim \Big(\sum_{J \in \mathcal{P}(W_j,\rho)} \|f_J\|_{\L^t(w_B)}^{\tilde{q}} \Big)^{1/\tilde{q}}. \end{aligned}$$

□

3.5 Induction on scales

The upper bounds of $\Gamma_{q,p}(\mathbf{Q})$ in Theorem 3.1.1 will be proved by induction on the dimension d. The main inductive step is contained in the following result, whose proof will occupy the whole Section 3.5. One can apply Theorem 3.5.1 repeatedly and then obtain the upper bounds in part (3.1.16) of Theorem 3.1.1.

Theorem 3.5.1. *Let $d \geq 1$ and $n \geq 1$. Let $\mathbf{Q} = (Q_1, \dots, Q_n)$ be a collection of quadratic forms in d variables. Let $2 \leq q \leq p < \infty$ and*

$$\Lambda := \sup_H \Gamma_{q,p}(\mathbf{Q}|_H), \tag{3.5.1}$$

where the sup is taken over all hyperplanes $H \subset \mathbb{R}^d$ that pass through the origin. Then

$$\Gamma_{q,p}(\mathbf{Q}) \leq \max\Big(\Lambda, \max_{0 \leq n' \leq n} \Big[d\big(1 - \frac{1}{p} - \frac{1}{q}\big) - \mathfrak{d}_{n'}(\mathbf{Q})\big(\frac{1}{2} - \frac{1}{p}\big) - \frac{2(n-n')}{p}\Big]\Big). \tag{3.5.2}$$

In the proof of Theorem 3.5.1, we may assume that

$$\Gamma := \Gamma_{q,p}(\mathbf{Q}) > \Lambda, \tag{3.5.3}$$

since otherwise (3.5.2) already holds. The assumption (3.5.3) is convenient, because it means that the multilinear terms in Proposition 3.5.6 below dominate the lower-dimensional terms. On a technical level, it allows us to define the quantities (3.5.32) that are central to the bootstrapping argument.

3.5.1 Stability of decoupling constants and lower dimensional contributions

In order to make use of the quantity (3.5.1), we need to show that we have a bound for the decoupling constants $\mathcal{D}_{q,p}(\mathbf{Q}|_H, \delta)$ that is uniform in the hyperplanes H. More generally, it turns out that decoupling constants can be bounded locally uniformly in the coefficients of the quadratic forms $\mathbf{Q}$. Although it is possible to obtain such uniform bounds by keeping track of the dependence on $\mathbf{Q}$ in all our proofs, we use this opportunity to record a compactness argument for decoupling constants whose validity is not restricted to quadratic forms.

Theorem 3.5.2. *For every $2 \leq q \leq p < \infty$, $\epsilon > 0$, and real quadratic forms $Q_1, \dots, Q_n$ in d variables, there exist $C_{\mathbf{Q},\epsilon,q,p} < \infty$ and a neighborhood $\mathcal{Q}$ of $(Q_1, \dots, Q_n)$ such that, for every $(\tilde{Q}_1, \dots, \tilde{Q}_n) \in \mathcal{Q}$ and every $\delta \in (0,1)$, we have*

$$\mathcal{D}_{q,p}((\tilde{Q}_1, \dots, \tilde{Q}_n), \delta) \leq C_{\mathbf{Q},\epsilon,q,p} \delta^{-\Gamma_{q,p}(\mathbf{Q}) - \epsilon},$$

where $\Gamma_{q,p}(\mathbf{Q})$ is given by (3.1.8).

Lemma 3.5.3 (Affine rescaling)**.** *Let $2 \leq q \leq p < \infty$. For any dyadic numbers $0 < \delta \leq \sigma \leq 1$ and every $J \in \mathcal{P}(\sigma)$,*

$$\|\sum_{\square \in \mathcal{P}(J,\delta)} f_\square\|_{L^p} \leq \mathcal{D}_{q,p}(\mathbf{Q}, \delta/\sigma)\Big(\sum_{\square \in \mathcal{P}(J,\delta)} \|f_\square\|_{L^p}^q\Big)^{1/q}. \tag{3.5.4}$$

Such a lemma has also been standard in the decoupling literature, see for instance Jean Bourgain and Demeter 2015, Section 4 or Shaoming Guo and Zorin-Kranich 2020a, Lemma 1.23.

Proof of Theorem 3.5.2. Let $\sigma = \sigma(\mathbf{Q}, \epsilon, q, p)$ be a small number, which will be determined later. We may assume that $\delta \le \sigma/4$. We consider a tuple $(\tilde{Q}_1, \dots, \tilde{Q}_n)$ such that

$$\sup_i \|\mathrm{Hess}(\tilde{Q}_i - Q_i)\| < \sigma^2/(10d+10). \tag{3.5.5}$$

Then, for every $J \in \mathcal{P}(\sigma)$ and $\Box \in \mathcal{P}(J, \delta)$, we have

$$\mathcal{U}_\Box(\tilde{\mathbf{Q}}) \subseteq \mathcal{U}_J(\mathbf{Q}). \tag{3.5.6}$$

Take a collection of functions $f_\Box$ with $\mathrm{supp}\, \hat{f_\Box} \subset \mathcal{U}_\Box(\tilde{\mathbf{Q}})$ for each $\Box \in \mathcal{P}(\delta)$. Using (3.5.6) and the definition of $\Gamma_{q,p}(\mathbf{Q})$, we obtain

$$\begin{aligned} \|\sum_{\Box \in \mathcal{P}(\delta)} f_\Box\|_{L^p} &\le C_{\mathbf{Q},\epsilon} \sigma^{-\Gamma_{q,p}(\mathbf{Q}) - \epsilon/2} (\sum_{J \in \mathcal{P}(\sigma)} \|f_J\|_{L^p}^q)^{1/q} \\ &\le C_{\mathbf{Q},\epsilon} \sigma^{-\Gamma_{q,p}(\mathbf{Q}) - \epsilon/2} \mathcal{D}_{q,p}(\tilde{\mathbf{Q}}, \delta/\sigma) (\sum_{\Box \in \mathcal{P}(\delta)} \|f_\Box\|_{L^p}^q)^{1/q}. \end{aligned} \tag{3.5.7}$$

The last inequality follows from the affine rescaling (Lemma 3.5.3). Hence, we obtain

$$\mathcal{D}_{q,p}(\tilde{\mathbf{Q}}, \delta) \le C_{\mathbf{Q},\epsilon} \sigma^{-\Gamma_{q,p}(\mathbf{Q}) - \epsilon/2} \mathcal{D}_{q,p}(\tilde{\mathbf{Q}}, \delta/\sigma). \tag{3.5.8}$$

We iterate this inequality $\log_{\sigma^{-1}}(\delta^{-1})$-times and obtain

$$\mathcal{D}_{q,p}(\tilde{\mathbf{Q}}, \delta) \le C_{\mathbf{Q},\epsilon} (\delta^{-1})^{\log_{\sigma^{-1}} C_{\mathbf{Q},\epsilon}} \delta^{-\Gamma_{q,p}(\mathbf{Q}) - \epsilon/2}. \tag{3.5.9}$$

It suffices to take σ small enough so that $\log_{\sigma^{-1}} C_{\mathbf{Q},\epsilon} \le \epsilon/2$. $\Box$

Corollary 3.5.4. *Let $2 \le q \le p < \infty$. For each $\epsilon > 0$, there exists $C_{\mathbf{Q},\epsilon,q,p} < \infty$ such that, for every linear subspace $H \subset \mathbb{R}^d$ of co-dimension one, we have*

$$\mathcal{D}_{q,p}(\mathbf{Q}|_H, \delta) \le C_{\mathbf{Q},\epsilon,q,p} \delta^{-\Lambda - \epsilon}, \tag{3.5.10}$$

where Λ was defined in (3.5.1).

Proof. Recall from (3.1.12) that the $\mathbf{Q}|_H$ are parametrized by the orthogonal group $O(d)$. Since $\mathbf{Q}|_H$ depends continuously on the rotation used to define it, the group $O(d)$ is compact, and by Theorem 3.5.2, we obtain the claim. $\Box$

To prepare for the broad-narrow analysis of Bourgain and Guth Jean Bourgain and Guth 2011 in the following section, we need the following lemma that takes care of the case when frequency cubes are clustered near sub-varieties of low degrees.

Lemma 3.5.5 (Shaoming Guo and Zorin-Kranich 2020a, Corollary 2.18)**.** *For every $d \geq 1$, $D > 1$ and $\epsilon > 0$, there exists $c = c(D, \epsilon) > 0$ such that the following holds. For every sufficiently large K, there exist*

$$K^c \leq K_1 \leq K_2 \leq \cdots \leq K_D \leq \sqrt{K} \tag{3.5.11}$$

such that for every non-zero polynomial P in d variables of degree at most D, there exist collections of pairwise disjoint cubes $\mathcal{W}_j \subset \mathcal{P}(1/K_j)$, $j = 1, 2, \ldots, D$, such that

$$\mathcal{N}_{1/K}(Z_P) \cap [0,1]^d \subset \bigcup_{j=1}^{D} \bigcup_{W \in \mathcal{W}_j} W \tag{3.5.12}$$

and

$$\Big\| \sum_{W \in \mathcal{W}_j} f_W \Big\| \lesssim_{D, \mathbf{Q}, \epsilon, q, p} K_j^{\Lambda+\epsilon} \Big(\sum_{W \in \mathcal{W}_j} \| f_W \|_p^q \Big)^{1/q}. \tag{3.5.13}$$

Here $\mathcal{N}_{1/K}(Z_P)$ denotes the $1/K$ neighborhood of the zero set of P.

This lemma was stated in Shaoming Guo and Zorin-Kranich 2020a only for $p = q$ and with Fourier support condition that is slightly different from (3.1.4). The same proof works also for $q \leq p$ and with Fourier support condition (3.1.4) without any change, and we will therefore not repeat it here. The main hypothesis of Shaoming Guo and Zorin-Kranich 2020a, Corollary 2.18 is Shaoming Guo and Zorin-Kranich 2020a, Hypothesis 2.4, which is exactly what we verified in Corollary 3.5.4.

3.5.2 Multilinear decoupling

For a positive integer K, a transversality parameter $\theta > 0$, and $0 < \delta < K^{-1}$, the *multilinear decoupling constant*

$$\mathrm{MulDec}(\delta, \theta, K) = \mathrm{MulDec}(\mathbf{Q}, \delta, \theta, K) \tag{3.5.14}$$

is the smallest constant such that the inequality

$$\begin{aligned} &\Big(\int_{\mathbb{R}^{d+n}} \big(\prod_{j=1}^{M} \| f_{W_j} \|_{\not L^p(B(x,K))} \big)^p \mathrm{d}x \Big)^{1/p} \\ &\qquad \leq \mathrm{MulDec}_{q,p}(\mathbf{Q}, \delta, \theta, K) \prod_{j=1}^{M} \Big(\sum_{\square \in \mathcal{P}(W_j, \delta)} \| f_\square \|_{L^p(\mathbb{R}^{d+n})}^q \Big)^{\frac{1}{q}} \end{aligned} \tag{3.5.15}$$

holds for every choice of functions $f_\square$ and every θ-uniform set $\{W_1, \ldots, W_M\} \subseteq \mathcal{P}(K^{-1})$ with $1 \leq M \leq K^d$.

We use a version of the Bourgain–Guth reduction of linear to multilinear estimates Jean Bourgain and Guth 2011. Estimates of a similar form already appeared in works of Bourgain and Demeter, see Jean Bourgain and Demeter 2017b and Jean Bourgain and Demeter 2016b for instance. The version below is a minor variant of Shaoming Guo and Zorin-Kranich 2020a, Proposition 2.33. This is the place where the uniform bound in Theorem 3.5.2 is used.

Proposition 3.5.6. *Let $2 \le q \le p < \infty$. Let Λ be given by* (3.5.1)*. Then, for each $\epsilon > 0$ and $\theta > 0$, there exists K such that*

$$\mathcal{D}_{q,p}(\mathbf{Q},\delta) \lesssim_{\epsilon,\theta} \delta^{-\Lambda-\epsilon} + \delta^{-\epsilon} \max_{\delta \le \delta' \le 1; \delta' \, dyadic} \Big[\big(\frac{\delta'}{\delta}\big)^{\Lambda} \mathrm{MulDec}_{q,p}(\mathbf{Q},\delta',\theta,K) \Big]. \tag{3.5.16}$$

Proof of Proposition 3.5.6. Let $\{f_\square\}_{\square \in \mathcal{P}(\delta)}$ be a collection of functions with supp $\hat{f}_\square \subset \mathcal{U}_\square$. In the proof, for each dyadic cube J with $l(J) \ge \delta$, we denote

$$f_J := \sum_{\square \in \mathcal{P}(J,\delta)} f_\square. \tag{3.5.17}$$

Let K be a large constant that is to be determined. For each ball $B' \subset \mathbb{R}^{d+n}$ of radius K, we initialize

$$\mathcal{S}_0(B') := \{W \in \mathcal{P}(1/K) \big| \; \|f_W\|_{L^p(B')} \ge K^{-d} \max_{W' \in \mathcal{P}(1/K)} \|f_{W'}\|_{L^p(B')}\}. \tag{3.5.18}$$

We repeat the following algorithm. Let $\iota \ge 0$. If $\mathcal{S}_\iota(B') = \emptyset$ or $\mathcal{S}_\iota(B')$ is θ-uniform, then we set

$$\mathcal{T}(B') := \mathcal{S}_\iota(B') \tag{3.5.19}$$

and terminate. Otherwise, there exists a sub-variety Z of degree at most d such that

$$|\{W \in \mathcal{S}_\iota(B') | 2W \cap Z \neq \emptyset\}| \ge \theta |\mathcal{S}_\iota(B')|. \tag{3.5.20}$$

Fix any such variety Z. Note that $2W \cap Z \neq \emptyset \implies W \subseteq \mathcal{N}_{2^d/K}(Z)$. For $j \in \{1, \dots, d\}$, let $\mathcal{W}_{\iota,j}(B') := \mathcal{W}_j$ be as in Lemma 3.5.5 with K replaced by $K/2^d$. Repeat this algorithm with

$$\mathcal{S}_{\iota+1}(B') := \mathcal{S}_\iota(B') \setminus \bigcup_{j=1}^{d} \bigcup_{W \in \mathcal{W}_{\iota,j}(B')} \mathcal{P}(W, 1/K). \tag{3.5.21}$$

This algorithm terminates after $O(\log K)$ steps, with an implicit constant depending on θ, as in each step we remove at least the set on the left-hand side of (3.5.20), which constitutes a fixed proportion θ of $\mathcal{S}_\iota(B')$.

To process the cubes in $\mathcal{W}_{\iota,j}$ and to avoid multiple counting, we define

$$\widetilde{\mathcal{W}}_{\iota,j} := \Big(\mathcal{W}_{\iota,j} \setminus \bigcup_{0 \le \iota' < \iota} \mathcal{W}_{\iota',j}\Big) \setminus \bigcup_{1 \le j' < j} \bigcup_{\iota'} \bigcup_{W \in \mathcal{W}_{\iota',j'}} \mathcal{P}(W, 1/K_j). \tag{3.5.22}$$

So far we see that every cube in (3.5.18) can be covered by exactly one cube in

$$\Big(\bigcup_{\iota} \bigcup_{j} \widetilde{\mathcal{W}}_{\iota,j}\Big) \bigcup \mathcal{T}(B'). \tag{3.5.23}$$

Therefore, by the triangle inequality we obtain

$$\begin{aligned} \| \sum_{\square \subset [0,1]^d} f_\square \|_{L^p(B')} \le & \Big(\sum_{W \in \mathcal{P}(1/K)} \|f_W\|_{L^p(B')}^q \Big)^{1/q} \\ & + \sum_{\iota \lesssim \log K} \sum_{j=1}^{d} \Big\| \sum_{W \in \widetilde{\mathcal{W}}_{\iota,j}} f_W \Big\|_{L^p(B')} + \sum_{W \in \mathcal{T}(B')} \|f_W\|_{L^p(B')}. \end{aligned} \tag{3.5.24}$$

On the right hand side of (3.5.24), the first term is used to take care of the cubes that are not counted in (3.5.18). Next, we will see how to handle all these three terms. The second term on the right hand side will be processed via a standard localization argument (see for instance Shaoming Guo and Zorin-Kranich 2020a, Remark 1.24) and Lemma 3.5.5. It is bounded by

$$C_{\mathbf{Q},\epsilon,p,q} \log K \sum_{j=1}^{d} K_j^{\Lambda+\epsilon} \Big(\sum_{W \in \mathcal{P}(1/K_j)} \|f_W\|_{L^p(w_{B'})}^q \Big)^{1/q}. \tag{3.5.25}$$

Recall in Lemma 3.5.5 that $K^c \leq K_j \leq \sqrt{K}$ for some $c = c(d,\epsilon)$ and every j. This allows us to absorb $\log K$ by K_j^ϵ, which is the only place where the lower bound K^c in (3.5.11) is used. To bound the last term, we use the definition of $\mathcal{T}(B')$ and obtain

$$K^d \max_{W \in \mathcal{T}(B')} \|f_W\|_{L^p(B')} \leq K^{2d} \Big(\sum_{\substack{\{W_1,\dots,W_M\} \subseteq \mathcal{P}(1/K) \\ \theta-\text{uniform}}} \prod_{j=1}^{M} \|f_{W_j}\|_{L^p(B')}^p \Big)^{1/p}. \tag{3.5.26}$$

The above estimate seems rather crude, but we can allow any K-dependent constant in the estimate for this term. We plug (3.5.25) and (3.5.26) in (3.5.24), integrate over the centers of balls B', and obtain

$$\begin{aligned} \| \sum_{\square \in \mathcal{P}(\delta)} f_\square \|_{L^p(\mathbb{R}^{d+n})} &\leq C_{\mathbf{Q},\epsilon,q,p} \sum_{j=0}^{d} K_j^{\Lambda+2\epsilon} \Big(\sum_{W \in \mathcal{P}(1/K_j)} \|f_W\|_{L^p(\mathbb{R}^{d+n})}^q \Big)^{1/q} \\ &+ K^{2d} \sum_{\substack{\{W_1,\dots,W_M\} \subseteq \mathcal{P}(1/K) \\ \theta-\text{uniform}}} \Big(\sum_{B' \subset \mathbb{R}^{d+n}} \prod_{j=1}^{M} \|f_{W_j}\|_{L^p(B')}^p \Big)^{1/p}. \end{aligned} \tag{3.5.27}$$

Here we let $K_0 := K$. The terms under the sum in the former term have the same form as that on the left hand side, and therefore are ready for an iteration argument. In other words, we will apply (rescaled versions of) (3.5.27) to each term $\|f_W\|_{L^p(\mathbb{R}^{d+n})}$ under the sum in the former term. By the definition of the multi-linear decoupling constant, the latter term can be controlled by

$$K^{2d} 2^{K^d} \mathrm{MulDec}_{q,p}(\mathbf{Q}, \delta, \theta, K) \Big(\sum_{\square \in \mathcal{P}(\delta)} \|f_\square\|_{L^p(\mathbb{R}^{d+n})}^q \Big)^{1/q}, \tag{3.5.28}$$

where we used that there are only 2^{K^d} subsets of $\mathcal{P}(1/K)$, and hence at most that many θ-uniform subsets. We plug (3.5.28) in (3.5.27). Now it is standard argument to iterate (3.5.27) and obtain the desired estimate in the proposition. We leave out the details and refer to Jean Bourgain and Demeter 2016b, Section 8 or Jean Bourgain and Demeter 2016b, Proposition 8.4. $\square$

Recall that we have assumed (3.5.3). For most of Section 3.5, we fix some $0 < \epsilon < \Gamma - \Lambda$, a transversality parameter $\theta > 0$, and a corresponding K as in Proposition 3.5.6.

The mutlilinear decoupling constant will be estimated by the same procedure as in Jean Bourgain and Demeter 2015. For a detailed exposition of this argument we refer to

Jean Bourgain and Demeter 2017a, Theorem 10.16 or Shaoming Guo and Zorin-Kranich 2020a, Section 2.6. We use a compressed version of this argument, in which each step is expressed as an inequality between the quantities (3.5.32) below. This version of the Bourgain–Demeter argument was originally motivated by decoupling for higher degree polynomials, see Shaoming Guo and Zorin-Kranich 2020b.

For a θ-uniform set $\{W_j\}_{j=1}^M \subset \mathcal{P}(1/K)$ and a choice of functions $f_\Box$, $\Box \in \mathcal{P}(\delta)$, we write

$$\tilde{A}_2(b) := \Big\| \prod_{j=1}^{M} \Big(\sum_{J \in \mathcal{P}(W_j, \delta^b)} \|f_J\|^2_{\not{L}^2(w_{B(x,\delta^{-2b})})} \Big)^{1/2} \Big\|_{L^p_{x\in\mathbb{R}^{d+n}}},$$

$$\tilde{A}_t(b) := \Big\| \prod_{j=1}^{M} \Big(\sum_{J \in \mathcal{P}(W_j, \delta^b)} \|f_J\|^{\tilde{q}}_{\not{L}^t(w_{B(x,\delta^{-b})})} \Big)^{1/\tilde{q}} \Big\|_{L^p_{x\in\mathbb{R}^{d+n}}},$$

$$\tilde{A}_p(b) := \Big\| \prod_{j=1}^{M} \Big(\sum_{J \in \mathcal{P}(W_j, \delta^b)} \|f_J\|^{q}_{\not{L}^p(w_{B(x,\delta^{-2b})})} \Big)^{1/q} \Big\|_{L^p_{x\in\mathbb{R}^{d+n}}},$$

where $0 < b \le 1$ and

$$\frac{1}{t} = \frac{1/2}{p} + \frac{1/2}{2}, \quad \frac{1}{\tilde{q}} = \frac{1/2}{q} + \frac{1/2}{2}. \tag{3.5.29}$$

Note that $2 \le \tilde{q} \le t \le p$. For

$$0 < b < 1 \text{ and } * = 2, t, p, \tag{3.5.30}$$

let $a_*(b)$ be the infimum over all exponents a such that, for every θ-uniform set $\{W_j\}_{j=1}^M \subseteq \mathcal{P}(1/K)$, every $\delta < 1/K$, and every choice of functions $f_\Box$, $\Box \in \mathcal{P}(\delta)$, we have

$$\tilde{A}_*(b) \lesssim_{a,\theta,K} \delta^{-a} \prod_{j=1}^{M} \Big(\sum_{\Box \in \mathcal{P}(W_j,\delta)} \|f_\Box\|^q_{L^p(\mathbb{R}^{d+n})} \Big)^{1/q} \tag{3.5.31}$$

with the implicit constant independent of the choice of the tuples (W_j) and $(f_\Box)$, and in particular independent of b as we will send $b \to 0$. It follows from Hölder's inequality that this $a_*(b) < \infty$. Recall that $\Gamma := \Gamma_{q,p}(\mathbf{Q})$. As in Shaoming Guo and Zorin-Kranich 2020b, Section 3.6, we define

$$a_* := \liminf_{b\to 0} \frac{\Gamma - a_*(b)}{b}, \quad * \in \{2, t, p\}. \tag{3.5.32}$$

The next lemma says that a_* is non-trivial.

Lemma 3.5.7. *Under the above notation, it holds that*

$$a_* < \infty, \tag{3.5.33}$$

for $* = 2, t, p$.

Proof of Lemma 3.5.7. By Hölder's inequality and Bernstein's inequality, the left-hand side of (3.5.15) is bounded by

$$\delta^{-Cb}\tilde{A}_*(b) \tag{3.5.34}$$

for any $* \in \{2, t, p\}$ and any $0 < b < 1$ with some constant C depending on $*$. Therefore, we obtain that

$$\mathrm{MulDec}_{q,p}(\mathbf{Q}, \delta, \theta, K) \lesssim_{\epsilon,\theta,K} \delta^{-\left(Cb + a_*(b) + \epsilon\right)}, \tag{3.5.35}$$

for every $\epsilon > 0$ and $1 > b > 0$. This, together with Proposition 3.5.6 and the assumption (3.5.3), implies that

$$\Gamma \leq Cb + a_*(b). \tag{3.5.36}$$

This finishes the proof of the lemma. $\square$

3.5.3 Using linear decoupling

By Hölder's inequality, we obtain

$$\tilde{A}_p(b) \leq \prod_{j=1}^{M} \Big\| \Big(\sum_{J \in \mathcal{P}(W_j, \delta^b)} \|f_J\|^q_{\text{Ł}^p(w_{B(x,\delta^{-2b})})} \Big)^{1/q} \Big\|_{L^p_{x \in \mathbb{R}^{d+n}}}. \tag{3.5.37}$$

By Minkowski's inequality, this is further bounded by

$$\begin{aligned} &\prod_{j=1}^{M} \Big(\sum_{J \in \mathcal{P}(W_j, \delta^b)} \Big\| \|f_J\|_{\text{Ł}^p(w_{B(x,\delta^{-2b})})} \Big\|^q_{L^p_{x \in \mathbb{R}^{d+n}}} \Big)^{1/q} \\ &\lesssim \prod_{j=1}^{M} \Big(\sum_{J \in \mathcal{P}(W_j, \delta^b)} \|f_J\|^q_{L^p(\mathbb{R}^{d+n})} \Big)^{1/q}. \end{aligned} \tag{3.5.38}$$

By the definition of the decoupling exponent and affine scaling (Lemma 3.5.3), this is

$$\lesssim_\epsilon \delta^{-(\Gamma+\epsilon)(1-b)} \prod_{j=1}^{M} \Big(\sum_{\square \in \mathcal{P}(W_j, \delta)} \|f_\square\|^q_{L^p} \Big)^{1/q}. \tag{3.5.39}$$

Hence

$$a_p(b) \leq (\Gamma + \epsilon)(1-b),$$

for every $\epsilon > 0$, which means $a_p(b) \leq \Gamma(1-b)$. It follows that

$$a_p \geq \Gamma. \tag{3.5.40}$$

3.5.4 Using L^2 orthogonality

By L^2 orthogonality, see e.g. Shaoming Guo and Zorin-Kranich 2020b, Appenidx B for details, we have

$$\begin{aligned} \tilde{A}_2(b) &= \Big\| \prod_{j=1}^{M} \Big(\sum_{J \in \mathcal{P}(W_j, \delta^b)} \|f_J\|^2_{\text{Ł}^2(w_{B(x,\delta^{-2b})})} \Big)^{1/2} \Big\|_{L^p_{x \in \mathbb{R}^{d+n}}} \\ &\lesssim \Big\| \prod_{j=1}^{M} \Big(\sum_{J \in \mathcal{P}(W_j, \delta^{2b})} \|f_J\|^2_{\text{Ł}^2(w_{B(x,\delta^{-2b})})} \Big)^{1/2} \Big\|_{L^p_{x \in \mathbb{R}^{d+n}}}. \end{aligned} \tag{3.5.41}$$

We further apply Hölder's inequality and obtain

$$\lesssim \delta^{-d\cdot 2b(1/2-1/\tilde{q})}\Big\|\prod_{j=1}^{M}\Big(\sum_{J\in\mathcal{P}(W_j,\delta^{2b})}\|f_J\|^{\tilde{q}}_{\bar{L}^t(w_{B(x,\delta^{-2b})})}\Big)^{1/\tilde{q}}\Big\|_{L^p_{x\in\mathbb{R}^{d+n}}} \tag{3.5.42}$$

Note that the last expression is exactly $\delta^{-db(1-2/\tilde{q})}\tilde{A}_t(2b)$. Hence

$$a_2(b) \le db(1-2/\tilde{q}) + a_t(2b).$$

It follows that

$$a_2 \ge -d(1-2/\tilde{q}) + 2a_t. \tag{3.5.43}$$

3.5.5 Ball inflation

Using Corollary 3.4.10 with $\rho = \delta^b$ and taking L^p norms in x_0 on both sides of (3.4.19), we obtain

$$\begin{aligned}\tilde{A}_t(b) &= \Big\|\prod_{j=1}^{M}\Big(\sum_{J\in\mathcal{P}(W_j,\delta^{b})}\|f_J\|^{\tilde{q}}_{\bar{L}^t(w_{B(x,\delta^{-b})})}\Big)^{1/\tilde{q}}\Big\|_{L^p_{x\in\mathbb{R}^{d+n}}} \\ &\lesssim_\epsilon \delta^{-b(\gamma+\epsilon)}\Big\|\prod_{j=1}^{M}\Big(\sum_{J\in\mathcal{P}(W_j,\delta^{b})}\|f_J\|^{\tilde{q}}_{\bar{L}^t(w_{B(x,\delta^{-2b})})}\Big)^{1/\tilde{q}}\Big\|_{L^p_{x\in\mathbb{R}^{d+n}}},\end{aligned} \tag{3.5.44}$$

for every $\epsilon > 0$, where

$$\begin{aligned}\gamma &:= \frac{d}{t} - \frac{d+n}{p} + \frac{\kappa((1-\theta)p/t)}{p} \\ &\le \frac{d}{t} - \frac{d+n}{p} + \frac{1}{p}\sup_{0\le n'\le n}\big(n' + (1-\frac{p}{t}(1-\theta))\mathfrak{d}_{n'}(\mathbf{Q})\big),\end{aligned} \tag{3.5.45}$$

and the log factors in Corollary 3.4.10 have been absorbed by $\delta^{-b\epsilon}$. In the last step we used Corollary 3.4.8. In the end, we apply Hölder's inequality to the last term in (3.5.44) and obtain

$$\tilde{A}_t(b) \lesssim \delta^{-b(\gamma+\epsilon)}\tilde{A}_2(b)^{1/2}\tilde{A}_p(b)^{1/2}. \tag{3.5.46}$$

It follows that

$$a_t(b) \le b\gamma + a_p(b)/2 + a_2(b)/2.$$

Substituting this inequality into the definition (3.5.32), we obtain

$$a_t \ge -\gamma + a_p/2 + a_2/2. \tag{3.5.47}$$

3.5.6 Proof of Theorem 3.5.1

Inequalities (3.5.29), (3.5.40), (3.5.43), (3.5.47) imply

$$\Gamma \le a_p \le 2\gamma - a_2 + 2a_t \le 2\gamma + d(1-2/\tilde{q}) = 2\gamma + d(1/2 - 1/q).$$

Inserting the definitions of the respective terms into this inequality, we obtain

$$\Gamma_{q,p}(\mathbf{Q}) \le 2\Big(\frac{d}{t} - \frac{d+n}{p} + \frac{1}{p}\sup_{0\le n'\le n}\big(n' + (1-\frac{p}{t}(1-\theta))\mathfrak{d}_{n'}(\mathbf{Q})\big)\Big) + d\big(\frac{1}{2}-\frac{1}{q}\big).$$

Both sides of this inequality depend continuously on θ, and we consider its limit when $\theta \to 0$. This gives

$$\Gamma_{q,p}(\mathbf{Q}) \le 2\Big(\frac{d}{t} - \frac{d+n}{p} + \sup_{0\le n'\le n}\big(\frac{n'}{p} + (\frac{1}{p}-\frac{1}{t})\mathfrak{d}_{n'}(\mathbf{Q})\big)\Big) + d\big(\frac{1}{2}-\frac{1}{q}\big).$$

Substituting the ansatz (3.5.29) for t, we obtain

$$\begin{aligned}\Gamma_{q,p}(\mathbf{Q}) &\le d(\frac{1}{p}+\frac{1}{2}) - 2\frac{d+n}{p} + \sup_{0\le n'\le n}\big(\frac{2n'}{p} + (\frac{2}{p}-\frac{1}{p}-\frac{1}{2})\mathfrak{d}_{n'}(\mathbf{Q})\big) + d\big(\frac{1}{2}-\frac{1}{q}\big)\\ &= \sup_{0\le n'\le n}\big((\frac{1}{2}-\frac{1}{p})(d-\mathfrak{d}_{n'}) - \frac{2(n-n')}{p}\big) + d\big(\frac{1}{2}-\frac{1}{q}\big).\end{aligned}$$

This finishes the proof of Theorem 3.5.1.

3.6 Lower bounds in Theorem 3.1.1

In this section, we show the lower bounds for $\ell^q L^p$ decoupling constants in Theorem 3.1.1 for $q \le p$. We will prove that

$$\Gamma_{q,p}(\mathbf{Q}) \ge \max\Big(\sup_{H}\Gamma_{q,p}(\mathbf{Q}|_H), \max_{0\le n'\le n}\Big(d\big(1-\frac{1}{p}-\frac{1}{q}\big) - \mathfrak{d}_{n'}(\mathbf{Q})\big(\frac{1}{2}-\frac{1}{p}\big) - \frac{2(n-n')}{p}\Big)\Big), \tag{3.6.1}$$

where H is a hyperplane passing through the origin, for every $p \ge 2, q \ge 2$. Note that here we do not necessarily require $q \le p$. One can apply the above inequality repeatedly and then obtain the lower bounds in Theorem 3.1.1 for $q \le p$.

First of all, we relate the decoupling exponent with the decoupling exponents on subspaces. Here, the distinction between the cases $q \le p$ and $q > p$ becomes apparent.

Lemma 3.6.1. *Let $\mathbf{Q}$ be an n-tuple of quadratic forms in d variables and $H \subseteq \mathbb{R}^d$ a linear subspace of dimension d'. Then, for any $2 \le q \le p < \infty$, we have*

$$\Gamma_{q,p}(\mathbf{Q}) \ge \Gamma_{q,p}(\mathbf{Q}|_H), \tag{3.6.2}$$

and, for any $2 \le p < q \le \infty$, we have

$$\Gamma_{q,p}(\mathbf{Q}) \ge \Gamma_{q,p}(\mathbf{Q}|_H) + (d-d')\big(\frac{1}{p}-\frac{1}{q}\big). \tag{3.6.3}$$

Proof of Lemma 3.6.1. For notational convenience, assume that $\mathbb{R}^d = H \times \mathbb{R}^{d''}$ with $d'' = d - d'$. The bound (3.6.2) will follow from

$$\mathcal{D}_{q,p}(\mathbf{Q}, C\delta) \gtrsim \mathcal{D}_{q,p}(\mathbf{Q}|_H, \delta), \tag{3.6.4}$$

for some absolute constant C. To see this, let $\{\tilde{f}_{\Box'} : \Box' \in \mathcal{P}([0,1]^{d'}, \delta)\}$, be a tuple of functions on $\mathbb{R}^{d'+n}$ that nearly extremizes the inequality (3.1.6) for $\mathbf{Q}|_H$. Fix a bump function ϕ such that $\operatorname{supp}\widehat{\phi} \subseteq B(0,\delta^2) \subset \mathbb{R}^{d''}$ and, for

$$\Box = \Box' \times \Box'' \in \mathcal{P}([0,1]^d, \delta) = \mathcal{P}([0,1]^{d'}, \delta) \times \mathcal{P}([0,1]^{d''}, \delta),$$

consider the functions

$$f_\Box = f_{\Box'\times\Box''} = \begin{cases} \tilde{f}_{\Box'} \otimes \phi, & \Box'' = \Box''_0 := [0,\delta]^{d''}, \\ 0, & \Box'' \neq \Box''_0. \end{cases}$$

Then $\operatorname{supp}\widehat{f_\Box} \subseteq C\mathcal{U}_\Box$ and

$$\|\sum_{\Box} f_\Box\|_p = \|\phi\|_p \|\sum_{\Box'} f_{\Box'}\|_p, \quad \|f_{\Box'\times\Box''}\|_p = \mathbf{1}_{\Box''=\Box''_0}\|\phi\|_p\|f_{\Box'}\|_p,$$

which implies (3.6.4). Here $\mathbf{1}$ denotes an indicator function, that takes the value 1 if the statement in the subscript is true, and 0 otherwise.

To see (3.6.3), we define $f_{\Box'\times\Box''_0}$, as above. For other $\Box'' \in \mathcal{P}([0,1]^{d''}, \delta)$, let $a'' \in \mathbb{R}^{d''}$ be the center of $\Box''$ and define

$$f_{\Box'\times\Box''} := A_{\Box''} f_{\Box'\times\Box''_0}(\cdot + c_{\Box''}),$$

where $c_{\Box''} \in \mathbb{R}^{d+n}$ are very large vectors and the linear operators $A_{\Box''}$ are given by affine transformations in the Fourier space:

$$\widehat{A_{\Box''} f}(\xi,\eta) := \widehat{f}(\xi - (0,a''), \eta + Q(0,a'') - \nabla\mathbf{Q}(0,a'')\cdot\xi).$$

If $c_{\Box''}$ are sufficiently far apart, then functions $f_{\Box'\times\Box''}$ and $f_{\tilde{\Box}'\times\tilde{\Box}''}$ are almost disjointly supported for $\Box'' \neq \tilde{\Box}''$, so that

$$\begin{aligned}\|\sum_{\Box} f_\Box\|_p &\sim \big(\sum_{\Box''}\|\sum_{\Box'} f_{\Box'\times\Box''}\|_p^p\big)^{1/p} \\ &= \|\phi\|_p\big(\sum_{\Box''}\|\sum_{\Box'} \tilde{f}_{\Box'}\|_p^p\big)^{1/p} = \|\phi\|_p\delta^{-d''/p}\|\sum_{\Box'} \tilde{f}_{\Box'}\|_p\end{aligned}$$

and

$$\begin{aligned}\big(\sum_{\Box}\|f_\Box\|_p^q\big)^{1/q} &= \big(\sum_{\Box''}\sum_{\Box'}\|f_{\Box'\times\Box''}\|_p^q\big)^{1/q} \\ &= \|\phi\|_p\big(\sum_{\Box''}\sum_{\Box'}\|\tilde{f}_{\Box'}\|_p^q\big)^{1/q} = \|\phi\|_p\delta^{-d''/q}\big(\sum_{\Box'}\|\tilde{f}_{\Box'}\|_p^q\big)^{1/q}.\end{aligned}$$

This implies

$$\mathcal{D}_{q,p}(\mathbf{Q}, C\delta) \gtrsim \delta^{-d''(1/p-1/q)}\mathcal{D}_{q,p}(\mathbf{Q}|_H, \delta),$$

and therefore (3.6.3). $\Box$

To show the lower bound in (3.6.1), it remains to prove

Proposition 3.6.2. *Let $\mathbf{Q}$ be an n-tuple of quadratic forms in d variables. For $0 \le n' \le n$ and $2 \le q, p \le \infty$, we have*

$$\Gamma_{q,p}(\mathbf{Q}) \ge d\big(1 - \frac{1}{p} - \frac{1}{q}\big) - \mathfrak{d}_{n'}(\mathbf{Q})\big(\frac{1}{2} - \frac{1}{p}\big) - \frac{2(n-n')}{p}. \tag{3.6.5}$$

Proof of Proposition 3.6.2. Let $d' = \mathfrak{d}_{n'}(\mathbf{Q})$. After linear changes of variables, we may assume that $Q_1, \dots, Q_{n'}$ depend only on $\xi_1, \dots, \xi_{d'}$. Write frequency points in $\mathbb{R}^{d+n}$ as

$$(\xi', \xi'', \eta', \eta'') \in \mathbb{R}^{d'+d''+n'+n''}, \tag{3.6.6}$$

with

$$\begin{aligned} \xi' &= (\xi_1, \dots, \xi_{d'}),\ \xi'' = (\xi_{d'+1}, \dots, \xi_d), \\ \eta' &= (\eta_1, \dots, \eta_{n'}),\ \eta'' = (\eta_{n'+1}, \dots, \eta_n), \end{aligned} \tag{3.6.7}$$

and $d' + d'' = d, n' + n'' = n$. Similarly, we write spatial points in $\mathbb{R}^{d+n}$ as

$$(x', x'', y', y'') \in \mathbb{R}^{d'+d''+n'+n''}. \tag{3.6.8}$$

For a dyadic cube $\Box \in \mathcal{P}(\delta)$, write $\Box = \Box' \times \Box''$ with $\Box' \subset \mathbb{R}^{d'}$ and $\Box'' \subset \mathbb{R}^{d''}$. Choose functions $f_\Box$ of the form

$$f_\Box(x', x'', y', y'') = g_{\Box'}(x', y') h_\Box(x'', y'') \tag{3.6.9}$$

with the following properties[3]

1. $\widehat{g_{\Box'}}$ and $\widehat{h_\Box}$ are positive smooth functions satisfying
$$\int \widehat{g_{\Box'}} = \int \widehat{h_\Box} = 1, \tag{3.6.10}$$

2. $\widehat{g_{\Box'}}$ is supported on a ball of radius $\approx \delta^2$ contained in
$$\{(\xi', \eta') : \xi' \in \delta \cdot \Box', |\eta_1 - Q_1(\xi')| \le \delta^2, \dots, |\eta_{n'} - Q_{n'}(\xi')| \le \delta^2\}, \tag{3.6.11}$$
where $\delta \cdot \Box'$ is the box of the same center as $\Box'$ and side length δ times that of $\Box'$.

3. $\widehat{h_\Box}$ is supported on a rectangular box of dimensions comparable to
$$\underbrace{\delta^1 \times \cdots \times \delta^1}_{d'' \text{ times}} \times \underbrace{\delta^2 \times \cdots \times \delta^2}_{n'' \text{ times}} \tag{3.6.12}$$
contained in
$$\bigcup_{\xi' \in \delta \cdot \Box'} \{(\xi'', \eta'') : \xi'' \in \Box'', |\eta_{\tilde{n}'} - Q_{\tilde{n}'}(\xi', \xi'')| \le \delta^2, n' < \tilde{n}' \le n\}. \tag{3.6.13}$$

[3]Here and below we use $\Box$ instead of $\Box''$ in $h_\Box$ as $Q_{n'+1}, \dots, Q_n$ still depend on ξ'.

On one hand, by the uncertainty principle,

$$\|f_\square\|_p \sim \delta^{-(2d'+d''+2n)/p}, \tag{3.6.14}$$

and by definition we have

$$\begin{aligned}\Big\|\sum_{\square\in\mathcal{P}(\delta)} f_\square\Big\|_p &\le \mathcal{D}_{q,p}(\mathbf{Q},\delta)\Big(\sum_{\square\in\mathcal{P}(\delta)}\|f_\square\|^q\Big)^{1/q}\\ &\sim \mathcal{D}_{q,p}(\mathbf{Q},\delta)\delta^{-d/q}\delta^{-(2d'+d''+2n)/p}.\end{aligned} \tag{3.6.15}$$

On the other hand, with

$$U=\{(x'',y'')\in\mathbb{R}^{d''}\times\mathbb{R}^{n''}:|x''|,|y''|\le 10^{-d-n}/(\sup_j\|\mathrm{Hess}\,Q_j\|+1)\}, \tag{3.6.16}$$

we have

$$\begin{aligned}\Big\|\sum_{\square\in\mathcal{P}(\delta)} f_\square\Big\|_p &\gtrsim \inf_{(x'',y'')\in U}\Big\|\sum_{\square\in\mathcal{P}(\delta)} f_\square\Big\|_{L^p(\mathbb{R}^{d'}\times\{x''\}\times\mathbb{R}^{n'}\times\{y''\})}\\ &= \inf_{(x'',y'')\in U}\Big\|\sum_{\square'} c_{\square',x'',y''}g_{\square'}\Big\|_{L^p(\mathbb{R}^{d'}\times\mathbb{R}^{n'})}\end{aligned} \tag{3.6.17}$$

where

$$c_{\square',x'',y''}:=\sum_{\square''}h_{\square'\times\square''}(x'',y'')=\sum_{\square''}h_\square(x'',y'') \tag{3.6.18}$$

satisfies

$$|c_{\square',x'',y''}|\sim\delta^{-d''} \tag{3.6.19}$$

uniformly in $\square'$ and $(x'',y'')\in U$. This is because $h_\square(0,0)=1$ and

$$|h_\square(x'',y'')-h_\square(0,0)|\le\int|e(x''\cdot\xi''+y''\cdot\eta'')-1||\widehat{h_\square}(\xi'',\eta'')|\mathrm{d}\xi''\mathrm{d}\eta''\le\frac12\int|\widehat{h_\square}(\xi'',\eta'')|\mathrm{d}\xi''\mathrm{d}\eta''=\frac12,$$

so that all summands in (3.6.18) are close to 1.

Let $\phi_\delta(\cdot)=\phi(\delta^2\cdot)$, where ϕ is a fixed positive Schwartz function on $\mathbb{R}^{d'}\times\mathbb{R}^{n'}$ with $\operatorname{supp}\widehat{\phi}\subset B(0,1/10)$. Then, by Hölder's inequality,

$$\begin{aligned}&\Big\|\sum_{\square'} c_{\square',x'',y''}g_{\square'}\Big\|_{L^p(\mathbb{R}^{d'}\times\mathbb{R}^{n'})}\\ &\ge\|\phi_\delta\|^{-1}_{1/(1/2-1/p)}\Big\|\phi_\delta\sum_{\square'} c_{\square',x'',y''}g_{\square'}\Big\|_{L^2(\mathbb{R}^{d'}\times\mathbb{R}^{n'})}\\ &\sim\delta^{2\cdot(d'+n')(1/2-1/p)}\Big\|\sum_{\square'} c_{\square',x'',y''}\phi_\delta\cdot g_{\square'}\Big\|_{L^2(\mathbb{R}^{d'}\times\mathbb{R}^{n'})}.\end{aligned} \tag{3.6.20}$$

Since the Fourier supports of $\phi_\delta\cdot g_{\square'}$ are disjoint for different $(\square')$'s for sufficiently small δ, we obtain

$$\begin{aligned}\Big\|\sum_{\square'} c_{\square',x'',y''}\phi_\delta\cdot g_{\square'}\Big\|_{L^2(\mathbb{R}^{d'}\times\mathbb{R}^{n'})}&=\Big(\sum_{\square'}|c_{\square',x'',y''}|^2\big\|\phi_\delta\cdot g_{\square'}\big\|^2_{L^2(\mathbb{R}^{d'}\times\mathbb{R}^{n'})}\Big)^{1/2}\\ &\sim\delta^{-d'/2}\cdot\delta^{-d''}\cdot\delta^{-2\cdot(d'+n')/2},\end{aligned} \tag{3.6.21}$$

uniformly in $(x'', y'') \in U$. Combining the above estimates, we obtain

$$\begin{aligned} &\mathcal{D}_{q,p}(\mathbf{Q}, \delta)\delta^{-d/q}\delta^{-(2d'+d''+2n)/p} \\ &\gtrsim \delta^{2\cdot(d'+n')(1/2-1/p)} \cdot \delta^{-d'/2} \cdot \delta^{-d''} \cdot \delta^{-2\cdot(d'+n')/2}. \end{aligned} \tag{3.6.22}$$

This implies

$$\Gamma_{q,p}(\mathbf{Q}) \geq d(1 - 1/q - 1/p) - d'(1/2 - 1/p) - 2(n - n')/p,$$

as desired. □

3.7 Sharp $\ell^q L^p$ decoupling inequalities with $q > p$

Proof of Theorem 3.1.1 with $q > p$. The upper bound $\leq$ follows from the Hölder inequality between ℓ^p and ℓ^q sums in the definitions of $\Gamma_{q,p}$ and $\Gamma_{p,p}$.

Let us prove the lower bound. Recall from Corollary 3.1.2 that

$$\Gamma_{p,p}(\mathbf{Q}) = \max_{d/2 \leq d' \leq d} \max_{0 \leq n' \leq n} \Big((2d' - \mathfrak{d}_{d',n'}(\mathbf{Q}))\big(\frac{1}{2} - \frac{1}{p}\big) - \frac{2(n - n')}{p} \Big). \tag{3.7.1}$$

We will show that

$$\Gamma_{q,p}(\mathbf{Q}) \geq \max_{d' \leq d} \max_{0 \leq n' \leq n} \Big((2d' - \mathfrak{d}_{d',n'}(\mathbf{Q}))\big(\frac{1}{2} - \frac{1}{p}\big) - \frac{2(n - n')}{p} \Big) + d(1/p - 1/q) \tag{3.7.2}$$

via an induction on d. The base case $d = 1$ is easy, as quadratic forms depending on one variable ξ_1 are all multiples of ξ_1^2. Let us assume we have proven (3.7.2) for $d = d_0$, that is, we have established (3.7.2) for all $\mathbf{Q}$ depending on d_0 variables. We aim to prove it for $d = d_0 + 1$, that is, for $\mathbf{Q}$ depending on $d_0 + 1$ variables. First of all, we apply Proposition 3.6.2 and obtain

$$\Gamma_{q,p}(\mathbf{Q}) \geq \max_{0 \leq n' \leq n} \Big((2(d_0+1) - \mathfrak{d}_{d_0+1,n'}(\mathbf{Q}))\big(\frac{1}{2} - \frac{1}{p}\big) - \frac{2(n - n')}{p} \Big) + (d_0+1)\big(\frac{1}{p} - \frac{1}{q}\big), \tag{3.7.3}$$

which is the right hand side of (3.7.2) with $d' = d_0 + 1$. It remains to prove that

$$\Gamma_{q,p}(\mathbf{Q}) \geq \max_{d' \leq d_0} \max_{0 \leq n' \leq n} \Big((2d' - \mathfrak{d}_{d',n'}(\mathbf{Q}))\big(\frac{1}{2} - \frac{1}{p}\big) - \frac{2(n - n')}{p} \Big) + (d_0 + 1)\big(\frac{1}{p} - \frac{1}{q}\big). \tag{3.7.4}$$

Let $H \subset \mathbb{R}^{d_0+1}$ be a linear subspace of dimension d_0. By Lemma 3.6.1, we obtain

$$\Gamma_{q,p}(\mathbf{Q}) \geq \Gamma_{q,p}(\mathbf{Q}|_H) + \frac{1}{p} - \frac{1}{q}. \tag{3.7.5}$$

Now we apply our induction hypothesis to $\Gamma_{q,p}(\mathbf{Q}|_H)$ as $\mathbf{Q}|_H$ depend on d_0 variables, and obtain

$$\Gamma_{q,p}(\mathbf{Q}) \geq \max_{d' \leq d_0} \max_{n' \leq n} \Big((2d' - \mathfrak{d}_{d',n'}(\mathbf{Q}|_H))\big(\frac{1}{2} - \frac{1}{p}\big) - \frac{2(n - n')}{p} \Big) + (d_0 + 1)\big(\frac{1}{p} - \frac{1}{q}\big). \tag{3.7.6}$$

In order to prove (3.7.4), we first take the sup over H in (3.7.6) and realize that it suffices to prove

$$\inf_H \mathfrak{d}_{d',n'}(\mathbf{Q}|_H) \leq \mathfrak{d}_{d',n'}(\mathbf{Q}), \tag{3.7.7}$$

for every H of dimension d_0 and every $d' \leq d_0$. This follows from the definition of $\mathfrak{d}_{d',n'}$. □

The following example shows that Proposition 3.6.2 does not by itself always give the correct lower bound for $\Gamma_{q,p}$ when $q > p$. Let us take the extreme case $q = \infty$.

Example 3.7.1. Let $d = 4$, $n = 2$, and

$$\mathbf{Q} = (\xi_1^2 + \xi_2\xi_4, \xi_3\xi_4)$$

We have

$$\mathfrak{d}_{4,2} = 4, \quad \mathfrak{d}_{4,1} = 2, \quad \mathfrak{d}_{3,2} = 1,$$

and all other $\mathfrak{d}_{d',n'}$ are 0. Let $p = 2 + 4n/d = 4$. Then direct computation shows that $\Gamma_{\infty,p} = 9/4$. However Proposition 3.6.2 only shows $\Gamma_{\infty,p} \geq 2$.

3.8 Proofs of Corollaries 3.1.3–3.1.5

3.8.1 Proof of Corollary 3.1.3

We apply Theorem 3.1.1 with $q = 2$ to the tuple of quadratic forms $\mathbf{Q}$, and by (3.1.20), we know that (3.1.22) holds true if and only if

$$\begin{aligned} &\max_{0 \leq d' \leq d} \max_{0 \leq n' \leq n} \Big((d' - \mathfrak{d}_{d',n'}(\mathbf{Q}))(\frac{1}{2} - \frac{1}{p}) - \frac{2(n-n')}{p} \Big) \\ &\leq \max \Big(0, d(\frac{1}{2} - \frac{1}{p}) - \frac{2n}{p} \Big) \end{aligned} \tag{3.8.1}$$

for every $p \geq 2$. Both sides of (3.8.1) are finite maxima of affine linear functions in $1/p$. The two arguments of the max on the right hand side coincide at $p_0 := 2 + 4n/d$. Hence, (3.8.1) holds for every $p \in [2, \infty]$ if and only if it holds for all $p \in \{2, p_0, \infty\}$.

For $p = 2$, we have $LHS(3.8.1) = 0 = RHS(3.8.1)$. For $p = \infty$, we have

$$LHS(3.8.1) = \max_{0 \leq d' \leq d} \max_{0 \leq n' \leq n} (d' - \mathfrak{d}_{d',n'}(\mathbf{Q}))/2 = d/2,$$

where the maximum is attained at $d' = d$ and $n' = 0$, and therefore (3.8.1) holds with equality at $p = \infty$. For $p = p_0$, (3.8.1) is equivalent to

$$\max_{0 \leq d' \leq d} \max_{0 \leq n' \leq n} \Big((d' - \mathfrak{d}_{d',n'}(\mathbf{Q}))\frac{2n}{dp_0} - \frac{2(n-n')}{p_0} \Big) \leq 0. \tag{3.8.2}$$

A direct calculation shows that (3.8.2) is equivalent to the strong non-degeneracy condition (3.1.21).

3.8.2 Proof of Corollary 3.1.4

The proof is basically the same as that for Corollary 3.1.3. We apply Corollary 3.1.2 to the tuple of quadratic forms $\mathbf{Q}$, and by (3.1.19), we know that (3.1.24) holds true if and only if

$$\begin{aligned} &\max_{d/2 < d' \leq d} \max_{0 \leq n' \leq n} \Big((2d' - \mathfrak{d}_{d',n'}(\mathbf{Q}))(\frac{1}{2} - \frac{1}{p}) - \frac{2(n-n')}{p} \Big) \\ &\leq \max \Big(d(\frac{1}{2} - \frac{1}{p}), 2d(\frac{1}{2} - \frac{1}{p}) - \frac{2n}{p} \Big) \end{aligned} \tag{3.8.3}$$

for every $p \geq 2$. The two numbers on the right hand side coincide at $p_0 = 2 + 4n/d$. As in the proof of Corollary 3.1.3, (3.8.3) holds for every $p \in [2, \infty]$ if and only if it holds for all $p \in \{2, p_0, \infty\}$. For $p \in \{2, \infty\}$, the condition (3.8.3) again always holds with equality. Hence, (3.8.3) holds for every $p \in [2, \infty]$ if and only if it holds at $p = p_0$, which is further equivalent to

$$\max_{d/2<d'\leq d} \max_{0\leq n'\leq n} \Big((2d' - \mathfrak{d}_{d',n'}(\mathbf{Q}))\frac{2n}{dp_0} - \frac{2(n-n')}{p_0}\Big) \leq \frac{2n}{p_0}. \tag{3.8.4}$$

A direct calculation shows (3.8.4) is equivalent to the non-degeneracy condition (3.1.23).

3.8.3 Proof of Corollary 3.1.5

Recall from Corollary 3.1.2 that

$$\Gamma_p(\mathbf{Q}) = \max_{d/2<d'\leq d} \max_{0\leq n'\leq n} \gamma_{d',n'}(1/p), \quad \gamma_{d',n'}(1/p) = (2d' - \mathfrak{d}_{d',n'}(\mathbf{Q}))\big(\frac{1}{2} - \frac{1}{p}\big) - \frac{2(n-n')}{p}.$$

The functions $\gamma_{d',n'}$ are affine. For $n' < n$ and arbitrary d', we have $\gamma_{d',n'}(1/2) < 0$. Moreover, for arbitrary d', we have $\gamma_{d',n}(1/2) = 0$. For every $p \in (2, \infty)$, the condition (3.1.25) is equivalent to

$$\forall d' \in (d/2, d] \quad \gamma_{d',n}(1/p) \leq d(1/2 - 1/p).$$

In particular, if (3.1.25) fails, then (3.1.26) fails for any $p_c > 2$.

Suppose now that the condition (3.1.25) is satisfied. Then, in particular, $\mathfrak{d}_{d,n}(\mathbf{Q}) = d$, and it follows that Then, Corollary 3.1.2 implies

$$\Gamma_p(\mathbf{Q}) = \max\Big(d\big(\frac{1}{2} - \frac{1}{p}\big), \max_{d/2<d'\leq d} \max_{0\leq n'<n} \gamma_{d',n'}(1/p)\Big).$$

Since the latter double maximum is a piecewise affine function of $1/p$ and is strictly negative for $p = 2$, we see that there exists $p_c > 2$ satisfying (3.1.26). The largest possible p_c is the minimum of solutions $p \in (2, \infty)$ of the equations

$$d\big(\frac{1}{2} - \frac{1}{p}\big) = \gamma_{d',n'}(1/p) \tag{3.8.5}$$

for $d/2 < d' = d - m \leq d$ and $0 \leq n' \leq n - 1$. These solutions are given by the formula

$$p(d', n') = 2 + \frac{4n - 4n'}{2d' - d - \mathfrak{d}_{d',n'}(\mathbf{Q})}. \tag{3.8.6}$$

This shows (3.1.27), since the minimum in (3.1.27) is restricted in such a way as to be taken over numbers in $(2, \infty)$.

We note also that, for $n' = 0$ and $m = 0$, we have $\mathfrak{d}_{d,0}(\mathbf{Q}) = 0$, which shows that the minimum in (3.1.27) is taken over a non-empty set, and is at most $2 + 4n/d$.

3.9 Fourier restriction: proof of Corollary 3.2.3

In this section we prove Corollary 3.2.3. The proof is standard, and it relies on an epsilon removal lemma of Tao Tao 1999, the broad-narrow analysis of Bourgain and Guth Jean Bourgain and Guth 2011 and the decoupling inequalities established in the current paper. The use of decoupling inequalities in this context is also standard, see for instance Guth Guth 2018. As $\mathbf{Q}$ will be fixed throughout the proof, we will leave out the dependence of the extension function $E^{\mathbf{Q}}g$ on $\mathbf{Q}$ and simply write Eg.

Let us begin with the epsilon removal lemma. In order to prove (3.2.6), it suffices to prove that for every $\epsilon > 0$, there exists $C_{d,n,p,\mathbf{Q},\epsilon} = C_\epsilon$ such that

$$\|E_{[0,1]^d} g\|_{L^p(B)} \le C_\epsilon \delta^{-\epsilon} \|g\|_p, \tag{3.9.1}$$

for every $\delta \le 1, \epsilon > 0$, $p > p_{\mathbf{Q}}$ and every ball $B \subset \mathbb{R}^{d+n}$ of radius δ^{-2}. Here and below, we will leave out the dependence of our implicit constants on d, n, p and $\mathbf{Q}$. Such a reduction first appeared in Tao 1999, see also Jean Bourgain and Guth 2011; Kim 2017. For a version of epsilon removal lemmas for manifolds of co-dimension bigger than one, we refer to Section 4 in Shaoming Guo and Changkeun Oh 2020.

In order to prove (3.9.1), we will apply the broad-narrow analysis and the decoupling inequalities in the current paper, together with an induction argument on δ. Let us assume that we have proven (3.9.1) with δ' in place of δ for every $1 \ge \delta' \ge 2\delta$. Under this induction hypothesis, we will prove (3.9.1). Let us begin with one corollary of Proposition 3.4.9.

Corollary 3.9.1 (Multilinear restriction estimate)**.** *Let $K \in 2^{\mathbb{N}}$ be a dyadic integer and $0 < \delta \le 1/K$. Let $\theta > 0$ and $\{W_j\}_{j=1}^M \subseteq \mathcal{P}(1/K)$ be a θ-uniform set of cubes. Let $B \subset \mathbb{R}^{d+n}$ be a ball of radius δ^{-2}. Then, for each $2 \le p < \infty$ and $\epsilon' > 0$, we have*

$$\Big\| \prod_{j=1}^{M} |E_{W_j} g| \Big\|_{L^p(B)} \le C_{\theta,K,\epsilon'} \delta^{-\gamma(p,\theta,\mathbf{Q})-\epsilon'} \prod_{j=1}^{M} \|g\|_{L^2(W_j)}, \tag{3.9.2}$$

where

$$\gamma(p,\theta,\mathbf{Q}) := \sup_{0 \le n' \le n} \Big(\frac{2n'}{p} + \big(\frac{2}{p} - (1-\theta)\big)\mathfrak{d}_{n'}(\mathbf{Q}) \Big). \tag{3.9.3}$$

Proof of Corollary 3.9.1. The proof is essentially via the argument of passing from multi-linear Kakeya estimates to multi-linear restriction estimates as in Bennett, Carbery and Tao Bennett, Carbery, and Tao 2006. Let us first show that

$$\Big\| \prod_{j=1}^{M} |E_{W_j} g| \Big\|_{Ł^p(B)} \le C_{\theta,K,\epsilon'} \delta^{-2(\frac{d}{2} - \frac{d+n}{p} + \frac{\kappa((1-\theta)p/2)}{p}) - \epsilon'} \prod_{j=1}^{M} \Big(\sum_{\square \in \mathcal{P}(W_j,\delta)} \|E_\square g\|^2_{Ł^2(w_B)} \Big)^{1/2}, \tag{3.9.4}$$

for every $\epsilon' > 0$. We take a Schwartz function ψ such that ψ is positive on the ball of radius one centered at the origin, and Fourier transform of ψ has a compact support. Let us define the function $\psi_B(x) := \psi(\delta^2 x)$. By Hölder's inequality and L^2-orthogonality (see for instance Shaoming Guo and Zorin-Kranich 2020b, Appenidx B), we see that

$$\begin{aligned}\Big\|\prod_{j=1}^{M}|E_{W_j}g|\Big\|_{\text{Ł}^p(B)} &\lesssim \Big\|\prod_{j=1}^{M}|\psi_B E_{W_j}g|\Big\|_{\text{Ł}^p(B)}\\ &\lesssim \delta^{-\epsilon'(d+n)/2}\Big\|\prod_{j=1}^{M}\|\psi_B E_{W_j}g\|_{\text{Ł}^2(B(x,\delta^{-\epsilon'}))}\Big\|_{\text{Ł}^p_{x\in B}} \\ &\lesssim \delta^{-\epsilon'(d+n)/2}\Big\|\prod_{j=1}^{M}\Big(\sum_{J\in\mathcal{P}(W_j,\delta^{\epsilon'})}\|\psi_B E_J g\|^2_{\text{Ł}^2(w_{B(x,\delta^{-\epsilon'})})}\Big)^{1/2}\Big\|_{\text{Ł}^p_{x\in B}}.\end{aligned} \tag{3.9.5}$$

We apply (3.4.14) and L^2-orthogonality, and bound the above term by

$$\begin{aligned}&\delta^{-\epsilon'(d+n)/2}\delta^{-\epsilon'(\frac{d}{2}-\frac{d+n}{p}+\frac{\kappa((1-\theta)p/2)}{p})}\Big\|\prod_{j=1}^{M}\Big(\sum_{J\in\mathcal{P}(W_j,\delta^{\epsilon'})}\|\psi_B E_J g\|^2_{\text{Ł}^2(w_{B(x,\delta^{-2\epsilon'})})}\Big)^{1/2}\Big\|_{\text{Ł}^p_{x\in B}}\\ &\lesssim \delta^{-\epsilon'(d+n)/2}\delta^{-\epsilon'(\frac{d}{2}-\frac{d+n}{p}+\frac{\kappa((1-\theta)p/2)}{p})}\Big\|\prod_{j=1}^{M}\Big(\sum_{J\in\mathcal{P}(W_j,\delta^{2\epsilon'})}\|\psi_B E_J g\|^2_{\text{Ł}^2(w_{B(x,\delta^{-2\epsilon'})})}\Big)^{1/2}\Big\|_{\text{Ł}^p_{x\in B}}.\end{aligned} \tag{3.9.6}$$

We repeat this process and obtain

$$\delta^{-\epsilon'(d+n)/2}\delta^{-2(\frac{d}{2}-\frac{d+n}{p}+\frac{\kappa((1-\theta)p/2)}{p})}\Big\|\prod_{j=1}^{M}\Big(\sum_{J\in\mathcal{P}(W_j,\delta)}\|\psi_B E_J g\|^2_{\text{Ł}^2(w_{B(x,\delta^{-2})})}\Big)^{1/2}\Big\|_{\text{Ł}^p_{x\in B}}. \tag{3.9.7}$$

We rename $\epsilon'(d+n)/2$ by ϵ', and the above term is bounded by the right hand side of (3.9.4).

By Plancherel theorem, we see that

$$\|E_{\square}g\|_{\text{Ł}^2(w_B)} \lesssim \delta^d\|g\|_{L^2(\square)}. \tag{3.9.8}$$

Therefore, by (3.9.4), we obtain that

$$\Big\|\prod_{j=1}^{M}|E_{W_j}g|\Big\|_{L^p(B)} \le C_{\theta,K,\epsilon'}\delta^{-2\kappa((1-\theta)p/2)/p-\epsilon'}\prod_{j=1}^{M}\|g\|_{L^2(W_j)}. \tag{3.9.9}$$

It suffices to apply Corollary 3.4.8 to bound κ. $\square$

We let θ be a small number, which will be determined later. Its choice depends only on how close p is to $p_{\mathbf{Q}}$. Therefore the dependence of the forthcoming constants on θ will also be compressed. Readers can take $\theta = 0$ for convenience. We define p_c to be the smallest number such that $\gamma(p_c, \theta, \mathbf{Q}) = 0$. More explicitly,

$$p_c = \max_{1\le n'\le n}\Big(2 + \frac{2n' + 2\theta\mathfrak{d}_{n'}(\mathbf{Q})}{(1-\theta)\mathfrak{d}_{n'}(\mathbf{Q})}\Big). \tag{3.9.10}$$

To prove (3.9.1), we run the broad-narrow analysis of Bourgain and Guth Jean Bourgain and Guth 2011, in a way that is almost the same as in the proof of Proposition 3.5.6. We repeat the proof there until before the step (3.5.27), with $q = p$ and $f_\Box$ replaced by $\psi_B E_\Box g$ for every dyadic box $\Box$ of side length δ. Next, instead of summing over all balls B' of radius K in $\mathbb{R}^{d+n}$, we sum over $B' \subset B$, a ball of radius δ^{-2}, and obtain

$$\begin{aligned}\|\sum_{\Box\subset[0,1]^d}\psi_B E_\Box g\|_{L^p(B)} &\le C_{\epsilon'}\sum_{j=0}^{d}K_j^{\Lambda+2\epsilon'}\Big(\sum_{W\in\mathcal{P}(1/K_j)}\|\psi_B E_W g\|^p_{L^p(w_B)}\Big)^{1/p}\\ &+ K^d\sum_{1\le M\le K^d}\sum_{\substack{W_1,\dots,W_M\in\mathcal{P}(1/K)\\ \theta-\text{uniform}}}\Big(\sum_{B'\subset B}\prod_{j=1}^{M}\|\psi_B E_{W_j} g\|^p_{L^p(B')}\Big)^{1/p},\end{aligned} \tag{3.9.11}$$

where

$$\Lambda := \sup_H \Gamma_p(\mathbf{Q}|_H), \tag{3.9.12}$$

and the sup is taken over all hyperplanes $H \subset \mathbb{R}^d$ that pass through the origin. Regarding the second term on the right hand side of (3.9.11), we notice that each term $|E_{W_j}g|$ is essentially constant on B' and therefore we can apply Corollary 3.9.1 and bound it by $C_{\epsilon',K}\delta^{-\epsilon'}\|g\|_2$, whenever $p > p_c$. So far we have obtained

$$\|\psi_B E_{[0,1]^d} g\|_{L^p(B)} \le C_{\epsilon'}\sum_{j=0}^{d}K_j^{\Lambda+2\epsilon'}\Big(\sum_{W\in\mathcal{P}(1/K_j)}\|\psi_B E_W g\|^p_{L^p(w_B)}\Big)^{1/p} + C_{\epsilon',K}\delta^{-\epsilon'}\|g\|_2, \tag{3.9.13}$$

for every $\epsilon' > 0$ and $p > p_c$. After arriving at this form, we are ready to apply an inductive argument as the terms on the right hand side of (3.9.13) are of the same form as that on the left hand side, with just different scales. To be precise, we will apply our induction hypothesis to each $\|\psi_B E_W g\|_{L^p(w_B)}$. All these terms can be handled in exactly the same way. Without loss of generality, we take $W = [0, 1/K_j]^d$. Recall that

$$E_W g(x, y) = \int_W g(\xi)e(\xi\cdot x + \mathbf{Q}(\xi)\cdot y)d\xi, \tag{3.9.14}$$

where $x \in \mathbb{R}^d, y \in \mathbb{R}^n$. We apply the change of variables $\xi \mapsto \xi/K_j$, the induction hypothesis and obtain

$$\|\psi_B E_W g\|_{L^p(w_B)} \le CC_\epsilon\delta^{-\epsilon}K_j^{-d}K_j^{\frac{d+2n}{p}}K_j^{\frac{d}{p}}\|g\|_{L^p(W)}, \tag{3.9.15}$$

where C is some new large constant that is allowed to depend on d, n, p and $\mathbf{Q}$. This, together with (3.9.13), implies that

$$\|E_{[0,1]^d} g\|_{L^p(B)} \le CC_{\epsilon'}C_\epsilon\delta^{-\epsilon}\sum_{j=0}^{d}K_j^{\Lambda-d+\frac{2d+2n}{p}+2\epsilon'}\|g\|_p + C_{\epsilon',K}\delta^{-\epsilon'}\|g\|_p, \tag{3.9.16}$$

for every $\epsilon' > 0$. Recall from (3.5.11) that there exists a small number $c = c_{\epsilon'}$ such that

$$K^c \le K_1 \le K_2 \le \cdots \le K_d \le \sqrt{K}. \tag{3.9.17}$$

From (3.9.16) we see that if p is such that

$$\Lambda - d + (2d + 2n)/p < 0, \tag{3.9.18}$$

then we can pick ϵ' small enough and K sufficiently large, depending on ϵ', such that

$$CC_{\epsilon'} K^{\Lambda - d + (2d+2n)/p + 2\epsilon'} \leq 1/(2(d+1)). \tag{3.9.19}$$

After fixing ϵ' and K, we see that in order to control the second term in (3.9.16), we just need to set the constant C_ϵ from (3.9.1) to be $2C_{\epsilon',K}$ and then we can close the induction step.

Notice that there were two constraints on p, including $p > p_c$ and (3.9.18). Recall the definition of Λ in (3.9.12). One can apply Theorem 3.1.1 and see that

$$\Lambda = \max_{d' \leq d-1} \max_{0 \leq n' \leq n} \Big[(2d' - \mathfrak{d}_{d',n'}(\mathbf{Q}))\big(\frac{1}{2} - \frac{1}{p}\big) - \frac{2(n-n')}{p} \Big]. \tag{3.9.20}$$

Elementary computation shows that

$$p > \max\big(p_c, 2 + \max_{m \geq 1} \max_{n' \leq n} \frac{4n'}{2m + \mathfrak{d}_{d-m,n'}(\mathbf{Q})}\big) = \max(p_c, p_{\mathbf{Q}}). \tag{3.9.21}$$

As p_c is a continuous function depending on θ, to see that we have the range $p > p_{\mathbf{Q}}$, it suffices to show that

$$\max_{1 \leq n' \leq n} \left(2 + \frac{2n'}{\mathfrak{d}_{n'}(\mathbf{Q})}\right) \leq 2 + \max_{m \geq 1} \max_{n' \leq n} \frac{4n'}{2m + \mathfrak{d}_{d-m,n'}(\mathbf{Q})}. \tag{3.9.22}$$

This inequality follows from

$$2\mathfrak{d}_{d,n'}(\mathbf{Q}) \geq 2 + \mathfrak{d}_{d-1,n'}(\mathbf{Q}), \tag{3.9.23}$$

for every $n' \geq 1$, which holds true because $\mathfrak{d}_{d,n'}(\mathbf{Q}) > \mathfrak{d}_{d-1,n'}(\mathbf{Q})$ as long as $\mathfrak{d}_{d,n'}(\mathbf{Q}) > 0$. Recall that we assumed $\mathbf{Q}$ is linearly independent, and therefore we indeed have that $\mathfrak{d}_{d,n'}(\mathbf{Q}) > 0$ for every $n' \geq 1$. This verifies the range $p > p_{\mathbf{Q}}$ and thus finishes the proof of the corollary.

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