Weighted Discriminants and Mass Formulas for Number Fields

By

Silas Johnson

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The dissertation is approved by the following members of the Final Oral Committee:

Professor N. Boston, Professor, Mathematics

Professor J. Ellenberg, Professor, Mathematics

Professor D. Erman, Assistant Professor, Mathematics

Professor L. Maxim, Associate Professor, Mathematics

Professor M. M. Wood, Assistant Professor, Mathematics

Abstract

This dissertation studies *alternate discriminants*, a class of invariants for number fields based on the standard discriminant, and their mass formulas. We study alternate discriminants both in their combinatorial properties and in their relationships to field-counting heuristics of Malle and Bhargava.

Chapter 1 is an exposition of existing results and conjectures on counting number fields by standard discriminant.

In Chapter 2, we define *weighted discriminants*, the objects of primary interest in the remainder of this work, as well as other relevant ideas. We then prove our main theorem, which restricts the number of counting functions of a certain type for any given group that can have mass formulas.

In Chapter 3, we extend the techniques and machinery of Chatper 2 to analyze questions about class groups of number fields. Where possible, we compare our predicted asymptotics for class groups to known results and to conjectures of Cohen-Lenstra and Cohen-Martinet.

In Chapter 4, we calculate mass formulas, or prove that none exist, for number fields with several specific Galois groups.

Finally, Chapter 5 discusses some further questions that we leave open for future work.

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Contents

Chapter 1

Introduction

An old question in number theory asks about the distribution of number fields when counted by discriminant:

Question 1.1. *How many number fields K are there with $|\operatorname{Disc} K| < X$, and how does this number grow as $X \to \infty$?*

Normally, one restricts K to have a particular degree, and often a particular Galois group and/or signature. Not much is known about this question except when the degree of K is small.

We are interested in what happens in Question 1.1 if Disc is replaced by some *alternate discriminant*. The study of these alternate discrimnants forms the main body of this work.

1.1 Notation

Let K be a number field. The absolute discriminant of K will be denoted by $\operatorname{Disc} K$ or $\operatorname{Disc}(K/\mathbb{Q})$. The Galois group of the Galois closure of K will be denoted $\operatorname{Gal}(K/\mathbb{Q})$, although in most cases we will not use this notation unless $K/\mathbb{Q}$ is already a Galois extension. If L is an extension of K, we will use $\operatorname{Disc}(L/K)$ and $N_{L/K}$ to represent the relative discriminant and norm, respectively. If $H \subseteq \operatorname{Gal}(K/\mathbb{Q})$, we use K_H to denote the fixed field of H inside K.

$G_{\mathbb{Q}}$ will represent the absolute Galois group of $\mathbb{Q}$. If p is a prime, $\mathbb{Q}_p$ will be the p-adic completion of $\mathbb{Q}$. $G_{\mathbb{Q}_p}$ will be the absolute Galois group of $\mathbb{Q}_p$, and $I_{\mathbb{Q}_p}$ will be its inertial subgroup. For a number field K, $\mathfrak{p}$ will denote one of the primes of K above p. If $K_{\mathfrak{p}}/\mathbb{Q}_p$ is an extension of local fields, then $I_{p,i}$ will denote the ith ramification group, using lower numbering, with $I_{p,0}$ being the inertia group and $I_{p,1}$ the wild inertia group.

Γ will always denote a finite group. Unless stated otherwise, (H, H') will always denote a pair where H is a nontrivial subgroup of Γ, and H' is a maximal subgroup of H. If $a, b \in \Gamma$, $a \sim b$ means that a and b are conjugate in Γ.

We will also use the standard notation that two functions $f(X)$ and $g(X)$ are *asymptotic* to one another, denoted $f(X) \sim g(X)$, if $\lim\limits_{X\to\infty} \dfrac{f(X)}{g(X)} = 1$, and that $f(X) = o(g(X))$ if $\lim\limits_{X\to\infty} \dfrac{f(X)}{g(X)} = 0$. It will be clear from context whether $\sim$ refers to conjugate elements of a group or asymptotic functions.

1.2 The Malle-Bhargava Heuristic

The following heuristic gives a prediction for the asymptotics of number fields with a fixed Galois group. It is a refinement by Bhargava [2] of a conjecture originally posed by Malle [19].

Let Γ be any finite group. Denote by $N(\Gamma, \mathrm{Disc}, X)$ the number of Galois extensions $K/\mathbb{Q}$ with $\mathrm{Gal}(K/\mathbb{Q}) = \Gamma$ and $|\operatorname{Disc}(K/\mathbb{Q})| < X$. Also, let ρ be the regular representation of Γ. For any prime p and any map $\phi : G_{\mathbb{Q}_p} \to \Gamma$, let $c(\phi)$ be the Artin conductor of $\rho \circ \phi$. Following Kedlaya's formulation in [15], define the *total mass* at p as:

$$M(\mathbb{Q}_p, \rho, \Gamma) = \frac{1}{|\Gamma|} \sum_{\phi} \frac{1}{p^{c(\phi)}} \tag{1.1}$$

where the sum ranges over all continuous homomorphisms $\phi : G_{\mathbb{Q}_p} \to \Gamma$.

Remark. Bhargava's formulation of the total mass lacks the factor of $\frac{1}{|\Gamma|}$, and instead of Galois representations, counts étale Γ-extensions of $\mathbb{Q}_p$, weighted by the inverse of the order of their automorphism groups. This fits into the broader philosophy that "objects in nature appear with frequency inversely proportional to the size of their automorphism groups", the objects in this case being local completions of number fields.

Bhargava forms an Euler product from the total mass at each prime:

$$M(\rho, \Gamma, s) = C(\Gamma) \cdot C_\infty \prod_p \left(\frac{1}{|\Gamma|} \sum_{\phi} \frac{1}{p^{c(\phi)s}} \right) \tag{1.2}$$

Here $C(\Gamma)$ is an unknown constant that should depend only on Γ, and C_∞ is another constant that depends on maps $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) \to \Gamma$. Alternatively, C_∞ can be included in the Euler product, representing the factor at the infinite place of $\mathbb{Q}$.

This Euler product is equivalent to a Dirichlet series $\sum_{n=1}^{\infty} m_n n^{-s}$. Bhargava's heuristic is then:

Heuristic 1.2. *For some constant $C(\Gamma)$, we expect that*

$$\sum_{n=1}^{X} m_n \sim N(\Gamma, \mathrm{Disc}, X) \tag{1.3}$$

as $X \to \infty$.

This heuristic derives from the assumption that the local completions at each prime of a number field behave independently of each other. For each prime p, the p-part of the Euler product $M(\rho, \Gamma, s)$ tracks the possible completions at p.

To count non-Galois fields by their discriminant, we can replace ρ by a different permutation representation of Γ. For example, counting S_3 sextic fields by discriminant uses the Artin conductor of the 6-dimensional regular representation of S_3. On the other hand, counting non-cyclic cubic fields by discriminant uses the standard 3-dimensional representation of S_3. In principle, we could even replace ρ by any other representation of Γ, and there is no obvious reason to suspect that this makes Heuristic 1.2 less likely to hold.

1.3 Summary of Previous Results

Field counting is a relatively well-understood problem for fields of small degree. In this section, we present the main results known for degree at most 6. In Chapter 3, we will give a more detailed discussion of how these results compare to the predictions of Heuristic 1.2.

Throughout this section, if Γ is a group that has only one conjugacy class of index-n subgroups, then for K a Galois Γ-number field, Disc_n will denote the absolute discriminant of the unique isomorphism class of degree-n subfields of K.

1.3.1 Quadratic fields

Given a squarefree odd positive integer d, there is one quadratic field of discriminant $\pm d$ and one of discriminant $\pm 4d$. These are $\mathbb{Q}(\sqrt{\pm d})$ depending on whether d is 1 or 3 mod 4. There are also two fields, $\mathbb{Q}(\sqrt{\pm 2d})$, of discriminant $\pm 8d$. Since the number of odd squarefree integers less than X is asymptotic to $\frac{4}{\pi^2}X$, we have

$$N(C_2, \mathrm{Disc}, X) \sim \frac{4}{\pi^2}X + \frac{4}{\pi^2}\frac{X}{4} + 2 \cdot \frac{4}{\pi^2}\frac{X}{8} = \frac{6}{\pi^2}X = \frac{1}{\zeta(2)}X \tag{1.4}$$

1.3.2 Cubic fields

A famous result of Davenport and Heilbronn [11] counts cubic fields by discriminant:

$$N(S_3, \mathrm{Disc}_3, X) \sim \frac{1}{3\zeta(3)}X \tag{1.5}$$

Furthermore, their results imply that asymptotically, $\frac{1}{4}$ of non-cyclic cubic fields are totally real, and the other $\frac{3}{4}$ are complex.

For cyclic cubic fields, Cohn obtained the following result [10, 24]:

$$N(C_3, \mathrm{Disc}, X) \sim \left(\frac{11\sqrt{3}}{36\pi} \prod_{p \equiv 1 \bmod 6} \frac{(p+2)(p-1)}{p(p+1)} \right) X \tag{1.6}$$

1.3.3 Quartic fields

Bhargava [1] has shown that:

$$N(S_4, \mathrm{Disc}_4, X) \sim \left(\frac{5}{24} \prod_p (1 + p^{-2} - p^{-3} - p^{-4}) \right) X \tag{1.7}$$

He also proves that $\frac{1}{10}$ of S_4 quartic fields are totally real, and $\frac{3}{10}$ are totally complex.

Cohen, Diaz y Diaz, and Olivier [9] have also shown that the number of quartic fields K with $\mathrm{Gal}(K/\mathbb{Q}) \simeq D_4$ and $|\mathrm{Disc}(K/\mathbb{Q})| < X$ is asymptotic to cX, with $c \approx 0.052326$.

The other possible Galois groups for a quartic number field are C_4, $V_4 = C_2 \times C_2$, and A_4. If Γ is any of these groups, then $N(\Gamma, \mathrm{Disc}_4, X) = o(X)$ [1].

1.3.4 Quintic fields

Bhargava's results on quintic fields [3] show that

$$N(S_5, \mathrm{Disc}_5, X) \sim \left(\frac{13}{120}\prod_p (1 + p^{-2} - p^{-4} - p^{-5})\right) X \tag{1.8}$$

and that if Γ is any other transitive subgroup of S_5, then

$$N(\gamma, \mathrm{Disc}_5, X) = o(X) \tag{1.9}$$

Bhargava also shows that of quintic fields with discriminant less than X, the proportion having 1, 3, and 5 real embeddings is $\frac{15}{26}$, $\frac{10}{26}$, and $\frac{1}{26}$, respectively.

1.3.5 Sextic fields with Galois group S_3

Very little is known about counting fields of degree larger than 5, but a result of Bhargava and Wood [4] deals with counting degree-6 S_3 fields by discriminant. They show:

$$N(S_3, \mathrm{Disc}, X) \sim \left(\frac{2}{9}\left(\frac{4}{3} + \frac{1}{3^{5/3}} + \frac{2}{3^{7/3}}\right)\prod_p (1 + p^{-1} + p^{-4/3})(1 - p^{-1})\right) X \tag{1.10}$$

1.4 Philosophy and Goals

Some of the asymptotics in Section 1.3 contain very simple constant factors, and others do not. This simplicity is at least partially related to a *universal mass formula*, an equation relating the sets of possible local completions of the fields in question at each prime p.

Our main goal is to investigate the existence of these mass formulas. Instead of varying the group Γ and looking for mass formulas when groups are counted by discriminant, however, we will fix Γ and find mass formulas for different *counting functions*. These counting functions are objects that could be used in place of the c in Heuristic 1.2.

First, we will need to define what a reasonable coutning function is. In Section 2.2, we discuss the need for some restrictions on the counting function, and we propose *weighted discriminant counting functions* as a reasonable set to consider. This definition is based on a counting function for D_4 described by Wood in [26].

Much of the rest of this work is devoted to studying the number of weighted discriminant counting functions that can have universal mass formulas for a fixed finite group Γ. In Chapter 2, we prove a general theorem, valid when $|\Gamma|$ is a prime power, and along the way study several other properties of weighted discriminant counting functions. In Chapter 4, we look at a number of specific groups or families of groups, and find all weighted discriminant counting functions with universal mass formulas for each.

We leave open the question of how the presence of a universal mass formula affects the validity of Heuristic 1.2. This is a difficult question, but it is a significant motivating factor behind our line of study. A theorem along the lines of "Heuristic 1.2 is valid whenever there is a universal mass formula" is almost certainly unrealistic. Not only would such a result solve many open problems in field and class group counting at once, but there exist cases (as in (3.3.1), for example) where Heuristic 1.2 gives the wrong main term even though there is a universal mass formula. Nonetheless, we hope that in the future, understanding universal mass formulas may play a role in developing a "meta-heuristic" that gives some predictions as to the validity of Heuristic 1.2 for different groups and counting functions.

Chapter 2

Mass Formulas For ℓ-Groups

In this chapter, we begin with a discussion of counting functions and mass formulas. The primary objects of study are *weighted discriminant counting functions*, a generalization of the discriminant of a number field. Our main result is:

Theorem 2.1. *Let Γ be any finite ℓ-group, for a prime ℓ. There are only finitely many natural weighted discriminant counting functions for Γ which have a universal mass formula.*

2.1 Counting Functions and Mass Formulas

Let Γ be a finite group.

Let $S_{\mathbb{Q}_p,\Gamma}$ be the set of continuous homomorphisms $G_{\mathbb{Q}_p} \to \Gamma$, where $G_{\mathbb{Q}_p}$ denotes the absolute Galois group of $\mathbb{Q}_p$. We define a *counting function* for Γ to be any mapping

$$c : \bigcup_p S_{\mathbb{Q}_p,\Gamma} \to \mathbb{R}$$

satisfying the following conditions:

- $c(\phi) = c(\gamma\phi\gamma^{-1})$ for any $\gamma \in \Gamma$
- $c(\phi) = 0$ if ϕ is unramified

Furthermore, a counting function is called *proper* if it satisfies the following condition: Let p, p' be any two primes not dividing $|\Gamma|$, and let $I_{\mathbb{Q}_p}$ and $I_{\mathbb{Q}_{p'}}$ be the absolute inertia groups of $\mathbb{Q}_p$ and $\mathbb{Q}_{p'}$. If $\phi : G_{\mathbb{Q}_p} \to \Gamma$ and $\phi' : G_{\mathbb{Q}_{p'}} \to \Gamma$ with $\phi(I_{\mathbb{Q}_p}) = \phi'(I_{\mathbb{Q}_{p'}})$, then $c(\phi) = c(\phi')$. That is, for tame primes, c depends only on the image of the absolute inertia group.

We follow Wood's notation in [26] here, except that we allow c to take values in $\mathbb{R}$. If c takes only values in $\mathbb{Z}_{\geq 0}$, we call it *natural*. All counting functions we will consider in this chapter are assumed to be natural, except in section 2.4.

Also as in [26], we define the *total mass at* p of a counting function c to be

$$M(\mathbb{Q}_p, \Gamma, c) = \frac{1}{|\Gamma|} \sum_{\phi \in S_{\mathbb{Q}_p, \Gamma}} \frac{1}{p^{c(\phi)}}$$

Note that this sum is finite, so the right-hand side is well-defined. Kedlaya [15] and Wood [26] omit the factor of $\frac{1}{|\Gamma|}$, but we divide it out for simplicity. We use the following proposition to show that all the coefficients of the Laurent polynomial $M(\mathbb{Q}_p, \Gamma, c)$ are still integers after dividing by $|\Gamma|$:

Proposition 2.2. *Let $I_{\mathbb{Q}_p}$ be the inertia subgroup of $G_{\mathbb{Q}_p}$. Given any continuous homomorphism $\phi : I_{\mathbb{Q}_p} \to \Gamma$, there are either 0 or $|\Gamma|$ extensions $\widetilde{\phi} : G_{\mathbb{Q}_p} \to \Gamma$ such that $\widetilde{\phi} \mid_{I_{\mathbb{Q}_p}} = \gamma\phi\gamma^{-1}$ for some $\gamma \in \Gamma$.*

Note that if c is a proper counting function and $\widetilde{\phi}$ and $\widetilde{\phi}'$ are two different extensions of maps conjugate to ϕ, then $c(\widetilde{\phi}) = c(\widetilde{\phi}')$. Thus given Proposition 2.2, the total mass, with the factor of $\frac{1}{|\Gamma|}$, actually counts the conjugacy classes of maps from $I_{\mathbb{Q}_p} \to \Gamma$ that have such extensions, rather than the raw number of maps $G_{\mathbb{Q}_p} \to \Gamma$. The total mass is still a Laurent polynomial with integer coefficients, as desired, since there are an integer number of such conjugacy classes.

Remark. This proposition does not actually say anything in particular about $G_{\mathbb{Q}_p}$ and $I_{\mathbb{Q}_p}$; it applies generally to any semidirect product.

2.1.1 Proof of Proposition 2.2

A standard fact about local fields (see, for example, [25]) is that there is a surjective map $\mu : G_{\mathbb{Q}_p} \to G_{\mathbb{F}_p} \simeq \hat{\mathbb{Z}}$, with kernel $I_{\mathbb{Q}_p}$. $\hat{\mathbb{Z}}$ is free as a topological group; let 1 denote a generator, and let $f \in G_{\mathbb{Q}_p}$ be any preimage of 1. By the universal property of free objects in the category of topological groups, there is a unique map $i : \hat{\mathbb{Z}} \to G_{\mathbb{Q}_p}$ with $i(1) = \gamma$. Then $\mu \circ i$ is a map from $\hat{\mathbb{Z}}$ to itself that takes 1 to 1; by the universal property again, such a map is unique. The identity is such a map, so $\mu \circ i$ must be the identity. Thus the map μ splits, so $G_{\mathbb{Q}_p} \simeq I_{\mathbb{Q}_p} \rtimes \hat{\mathbb{Z}}$.

Now take any map $\phi : I_{\mathbb{Q}_p} \to \Gamma$, and let f be the generator of $\hat{\mathbb{Z}}$ inside $G_{\mathbb{Q}_p}$. An extension $\widetilde{\phi}$ is uniquely specified by a map $\phi' = \gamma\phi\gamma^{-1}$ and a choice of $\phi'(f)$ that is consistent with the action of f on $I_{\mathbb{Q}_p}$.

Let H be the image of ϕ in Γ, and for each $h = \phi(x)$, let $a(h) = \phi(fxf^{-1})$. Then $a : H \to \Gamma$ is injective, and we must choose $\phi(f)$ so that $\phi(f) \cdot h \cdot \phi(f)^{-1} = a(h)$ for each $h \in H$. The following lemma shows how many ways we can do this:

Lemma 2.3. *The set*

$$C_a = \{\gamma \in \Gamma : \gamma h \gamma^{-1} = a(h) \text{ for all } h \in H\}$$

is either empty or a coset of the centralizer $C(H)$.

Proof. Assume C_a is nonempty, and $\gamma \in C_a$. We will show that $C_a = \gamma C(H)$. If $\delta \in C_a$, then

$$\gamma^{-1}\delta h \delta^{-1}\gamma = \gamma^{-1}a(h)\gamma = \gamma^{-1}(\gamma h \gamma^{-1})\gamma = h$$

which shows that $\gamma^{-1}\delta \in C(H)$, so $\delta \in \gamma C(H)$, and thus $C_a \subseteq \gamma C(H)$. Conversely, let $c \in C(H)$. Then

$$\gamma c h c^{-1} \gamma^{-1} = \gamma h \gamma^{-1} = a(h)$$

so $\gamma c \in C_a$. This implies that $\gamma C(H) \subseteq C_a$, so $\gamma C(H) = C_a$ unless C_a is empty. $\square$

Note that the number of ways to choose $\phi(f)$ is equal to $|C_a|$ as defined in Lemma 2.3. Furthermore, $|C_a|$ is invariant under conjugation of ϕ by elements of Γ. Thus the number of ways to choose an extension of ϕ is the number of conjugates of ϕ times $|C_a|$.

If C_a is empty, then there are no extensions of ϕ. Otherwise, $|C_a| = |C(H)|$, and the number of distinct conjugates of ϕ is $\frac{|\Gamma|}{|C(H)|}$, so the total number of extensions is $|\Gamma|$, as desired.

2.1.2 Mass Formulas

Definition 2.4. A *character Laurent polynomial* is a sum

$$f(x) = \sum_{i=k_1}^{k_2} \sigma_i(x) x^{-i}$$

defined for integers x, where each σ_i is a $\mathbb{Z}$-linear combination of Dirichlet characters modulo divisors of $|\Gamma|$. Note that i may take negative values if $k_1 < 0$.

We use the convention that if χ is a character with modulus n and $(x, n) > 1$, then $\chi(x) = 0$, and we assume that each character has its smallest possible modulus. That is, we exclude, for example, the character with $\chi(x) = 1$ when $5 \nmid x$, and $\chi(x) = 0$ when $5|x$; instead, we use $\chi(x) = 1$ for all x. This is necessary for Theorem 2.16 to hold in the case where Γ is not an ℓ-group, but it does not affect the value of any $\chi(x)$ where $(x, |\Gamma|) = 1$.

Definition 2.5. If f is a character Laurent polynomial and S is a set of primes, we say that the pair (c, Γ) has f as an S-mass formula (or a mass formula for S) if for all primes $p \in S$,

$$M(\mathbb{Q}_p, \Gamma, c) = f(p)$$

We generally say that f is an S-mass formula for c, or that c has an S-mass formula, since the reference to Γ is implicit in the counting function c. If S is the set of all primes, then we call f a *universal mass formula*. If S is the set of all primes not dividing $|\Gamma|$, then we call f a *tame mass formula*.

Masses and mass formulas can also be defined over a base field other than $\mathbb{Q}$ by replacing the fields $\mathbb{Q}_p$ by all nonarchimedean completions of the base field, and replacing p elsewhere by the residue characteristic. We will currently consider only $\mathbb{Q}$ as a base field, and consider more general base fields in future work.

Remark. If f is a mass formula in which only the trivial Dirichlet character appears (i.e. f is a Laurent polynomial with integer coefficients), then we call f a *pure mass formula*. This corresponds to the definition of "mass formula" used by Kedlaya and Wood, except that we allow powers of p other than negative integers to appear in f, accounting for non-natural counting functions. Our definition of a mass formula, with characters allowed to appear in the coefficients, gives the more elegant result on tame mass formulas in Theorem 2.16.

Example. Let $\Gamma = C_2$. Each surjective $\phi \in \bigcup_p S_{\mathbb{Q}_p, \Gamma}$ corresponds to a distinct quadratic extension of $\mathbb{Q}_p$. Define a counting function c so that $c(\phi)$ is the discriminant exponent (the power of p appearing in the discriminant) of this extension. This counting function is proper, and it has a universal pure mass formula, as we can verify by computing masses explicitly using [13]. If $p \neq 2$, there are two ramified quadratic extensions of $\mathbb{Q}_p$, each with discriminant exponent 1. In addition, there is one unramified quadratic extension, and one non-surjective map $G_{\mathbb{Q}_p} \to C_2$ (the trivial map), so the mass at p is $1+p^{-1}$. For $p = 2$, there are still two unramified maps $G_{\mathbb{Q}_p} \to C_2$, but now there are two quadratic extensions of $\mathbb{Q}_2$ with discriminant exponent 2, and four quadratic extensions with discrimiant exponent 3. The mass at 2 is thus $1+2^{-2}+2\cdot 2^{-3} = 1+2^{-1}$. Since this agrees numerically with the mass at all other primes, the mass formula $f(p) = 1 + p^{-1}$ is universal.

The following two results are due to Kedlaya [15, Corollaries 5.4-5.5]. We restate them here in a form that takes into account our modified definitions in this section:

Proposition 2.6. *Let a be an integer not divisible by $|\Gamma|$. Then for any proper counting function c, (Γ, c) has a pure S-mass formula, where S is the set of all primes congruent to a modulo $|\Gamma|$.*

Proposition 2.7. *Let c be any proper counting function. Then (Γ, c) has a pure tame mass formula if and only if Γ has a rational character table.*

Kedlaya only allows a subset of the counting functions we allow here, so we will show that Proposition 2.6 extends to the counting functions we are considering. We omit this for Proposition 2.7, since we will study the tame mass formula in more depth in section 2.5.

Proof. Consider the quotient $G_{\mathbb{Q}_p}/G_{1,\mathbb{Q}_p}$, for $p \nmid |\Gamma|$, where the latter group is the absolute wild inertia group. This quotient is a semidirect product of the absolute tame inertia group $G_{0,\mathbb{Q}_p}/G_{1,\mathbb{Q}_p}$ with $\hat{\mathbb{Z}}$. Let the topological generators of $G_{0,\mathbb{Q}_p}/G_{1,\mathbb{Q}_p}$ and $\hat{\mathbb{Z}}$ be s

and t, respectively. Then a continuous homomorphism $\phi : G_{\mathbb{Q}_p}/G_{1,\mathbb{Q}_p} \to \Gamma$ is described entirely by $\phi(s)$ and $\phi(t)$, where these choices must be compatible with the relation $tst^{-1} = s^p$. Furthermore, if c is a proper counting function, then $c(\phi)$ is determined only by the choice of $\phi(s)$.

Now suppose q is another prime with $q = p + a \cdot |\Gamma|$, where $a \in \mathbb{Z}$. Then for any $\sigma \in \Gamma$, $\sigma^q = \sigma^p \cdot \sigma^{a \cdot |\Gamma|} = \sigma^p$. This shows that the number of pairs (σ, τ) with $\sigma, \tau \in \Gamma$ and $\tau\sigma\tau^{-1} = \sigma^p$ is the same as the number with $\tau\sigma\tau^{-1} = \sigma^q$, and thus there is a one-to-one correspondence between $S_{\mathbb{Q}_p,\Gamma}$ and $S_{\mathbb{Q}_q,\Gamma}$, which preserves the value of any proper counting function c.

From this, it follows that the total masses of c at p and q are the same Laurent polynomial in p and q, and thus c has a pure mass formula for all primes congruent to p modulo $|\Gamma|$. □

Remark. In the following sections, we will discuss global maps $\phi : G_{\mathbb{Q}} \to \Gamma$. We call such a map a Γ-extension of $\mathbb{Q}$.

A Γ-extension of $\mathbb{Q}$ is also equivalent to the data of a Galois extension $K/\mathbb{Q}$ together with a choice of isomorphism $\mathrm{Gal}(K/\mathbb{Q}) \xrightarrow{\sim} \Gamma$. We will sometimes refer to these extensions in terms of the map ϕ, and sometimes in terms of the field K, taking the isomorphism $\mathrm{Gal}(K/\mathbb{Q}) \to \Gamma$ to be implicit.

Recall that if H is a subgroup of Γ, then K_H denotes the fixed field of H in K.

2.2 Weighted discriminants

We use the term *alternate discriminant* to refer to any "reasonable" rational-valued function on the set of Γ-extensions of $\phi : G_{\mathbb{Q}} \to \Gamma$. A "reasonable" function, broadly speaking, should be one determined locally at each prime p by the restriction of ϕ to $G_{\mathbb{Q}_p}$.

If we require alternate discriminants to be determined locally in this way, then an alternate discriminant is equivalent to an integer-valued counting function, and a positive-integer-valued alternate discriminant is equivalent to a natural counting function. If an alternate discriminant D is defined locally, then we can construct a counting function corresponding to it. If ϕ_p is the restriction of some Γ-extension to $G_{\mathbb{Q}_p}$, then we define $c(\phi_p)$ to be the power of p appearing in $D(\phi)$. Conversely, given a counting function c for Γ, we can build an alternate discriminant corresponding to c: Let $\phi : G_{\mathbb{Q}} \to \Gamma$ be a Γ-extension. Then if ϕ_p is the restriction of ϕ to $G_{\mathbb{Q}_p}$, and

$$D_c(\phi) = \prod_p p^{c(\phi_p)}$$

From the perspective of searching for universal mass formulas, this broad class of invariants is not very interesting, even if we require counting functions to be proper. As we will see in Theorem 2.16, any proper counting function c is guaranteed to have a tame mass formula. Then, since the condition of properness imposes no restrictions on how the counting function can behave at primes dividing $|\Gamma|$, we can assign values to c in such a way that it forces the tame mass formula to be universal.

Thus, we seek a natural way to define counting functions (or alternate discriminants) globally, and prohibit entirely contrived behavior at the wild prime. To that end, in this paper we consider *weighted discriminants*, a class of alternate discriminants defined as follows:

Definition 2.8. A *weight function* for Γ is a function $w : \{(H, H')\} \to \mathbb{Z}$, where the domain of w consists of ordered pairs (H, H') where $H \subset \Gamma$ and H' is a maximal subgroup of H.

The *weighted discriminant* given by a weight function w is

$$D_w(K) = \prod_{(H,H')} N_{K_H/\mathbb{Q}}(\mathrm{Disc}(K_{H'}/K_H))^{w(H,H')}$$

where Disc is the standard relative discriminant and N is the norm.

Since $N_{K_H/\mathbb{Q}}(\mathrm{Disc}(K_{H'}/K_H))^{w(H,H')}$ can be determined locally from the ramification groups of $K/\mathbb{Q}$, D_w is an alternate discriminant and can also be defined in terms of a counting function c_w. We call a counting function of this form a *weighted discriminant counting function.*

If $w(H, H') \in Z_{\geq 0}$ for each (H, H'), we call w *positive integral.* If w is positive integral, then its counting function c_w is natural, but the converse need not hold. See Section 4.11 for an example of a non-integer-valued weight function whose counting function is nonetheless natural.

Remark. Changing the isomorphism $\mathrm{Gal}(K/\mathbb{Q}) \to \Gamma$ by an outer automorphism of Γ may change the value of $D_w(K)$, but an inner automorphism will not.

Remark. It is possible for two different weight functions to give the same counting function. For example, let $\Gamma = C_2 \times C_2$, and let H_1, H_2, and H_3 be its order-2 subgroups, with 1 denoting the trivial subgroup. If we let w be the weight function with $w(\Gamma, H_1) = 1$ and all other weights equal to 0, and w' be the weight function with $w(H_2, 1) = 2$ and all other weights zero, then $c_w = c'_w$.

2.3 An Explicit Formula for c_w

In this section, we give an explicit formula for $c_w(\phi)$ in terms of the weight function w and the ramification groups of the map ϕ, which we will use in the proof of Theorem 2.1.

Let $\phi : G_{\mathbb{Q}} \to \Gamma$ be a map, and let $K/\mathbb{Q}$ be the corresponding field extension. If $\mathfrak{p}$ is a prime of K above p, we denote by $I_{\mathfrak{p},i}$ the ith ramification group in lower numbering at $\mathfrak{p}$, for the extension $K/\mathbb{Q}$. As in [25], $i = 0$ and $i = -1$ correspond to the inertia and decomposition groups, respectively. Throughout this section, Disc denotes the standard discriminant ideal, and $\mathcal{D}$ the different ideal.

Let H' be a maximal subgroup of $H \subseteq \Gamma$. Using the fact that the discriminant of a field extension is the norm of the different ideal, and that

$$\operatorname{Disc} K/K_H = N_{K_{H'}/K_H}(\operatorname{Disc} K/H') \cdot (\operatorname{Disc} K_{H'}/K_H)^{|H'|}$$

we first obtain

$$\operatorname{Disc} K_{H'}/K_H = \left(\frac{N_{K/K_H}\mathcal{D}(K/K_H)}{N_{K/K_H}\mathcal{D}(K/K_{H'})} \right)^{\frac{1}{|H'|}}$$

Norming down to $\mathbb{Q}$ gives:

$$N_{K_H/\mathbb{Q}}(\operatorname{Disc}(K_{H'}/K_H)) = N_{K/\mathbb{Q}} \left(\frac{\mathcal{D}(K/K_H)}{\mathcal{D}(K/K_{H'})} \right)^{\frac{1}{|H'|}}$$

Now we take the valuation at p of both sides, and use the fact that if $\mathfrak{p}$ is a prime above p and $K/\mathbb{Q}$ is Galois, then $N_{K/\mathbb{Q}}(\mathfrak{p}) = p^{f_{K/\mathbb{Q}}(p)}$, where f denotes the degree of the residue field extension.

$$v_p(N_{K_H/\mathbb{Q}}(\operatorname{Disc}(K_{H'}/K_H))) = \frac{f_{K/\mathbb{Q}}(p)}{|H'|} \sum_{\mathfrak{p}|p} (v_{\mathfrak{p}}(\mathcal{D}(K/H)) - v_{\mathfrak{p}}(\mathcal{D}(K/H')))$$

Using the formula in [25] for the different in terms of the ramification groups of an extension, and that $f_{K/\mathbb{Q}}(p) = \frac{|I_{p,-1}|}{|I_{p,0}|}$, the right side becomes

$$\frac{|I_{p,-1}|}{|I_{p,0}| \cdot |H'|} \sum_{\mathfrak{p}|p} \left[\sum_{i \geq 0} (|I_{\mathfrak{p},i} \cap H| - |I_{\mathfrak{p},i} \cap H'|) \right]$$

Now choose any prime $\mathfrak{p}$ above p. The ramification groups of the other primes above p are conjugates of $I_{\mathfrak{p},i}$. There are $|\Gamma|/|I_{\mathfrak{p},-1}|$ of these, so we can rewrite the previous line as

$$\frac{|I_{\mathfrak{p},-1}|}{|I_{\mathfrak{p},0}| \cdot |H'|} \cdot \frac{1}{|I_{\mathfrak{p},-1}|} \sum_{\gamma \in \Gamma} \left[\sum_{i \geq 0} \left(|\gamma I_{\mathfrak{p},i} \gamma^{-1} \cap H| - |\gamma I_{\mathfrak{p},i} \gamma^{-1} \cap H'| \right) \right]$$

If ϕ is a map $G_{\mathbb{Q}} \to \Gamma$ with ramification groups $I_{\mathfrak{p},i}$, and ϕ_p is its restriction to $G_{\mathbb{Q}_p}$, then we set

$$c_{H,H'}(\phi_p) := \frac{1}{|I_{\mathfrak{p},0}| \cdot |H'|} \cdot \sum_{\gamma \in \Gamma} \left[\sum_{i \geq 0} \left(|\gamma I_{\mathfrak{p},i} \gamma^{-1} \cap H| - |\gamma I_{\mathfrak{p},i} \gamma^{-1} \cap H'| \right) \right]$$

Note that this expression does not depend on the choice of $\mathfrak{p}$, since we sum over all conjugates of $I_{\mathfrak{p},i}$.

Now if w is any weight function with corresponding weighted discriminant D_w, define the counting function

$$c_w(\phi_p) = \sum_{(H,H')} c_{H,H'}(\phi_p) \cdot w(H, H')$$

Let $\phi : G_{\mathbb{Q}} \to \Gamma$, with ϕ_p the restriction of ϕ to $G_{\mathbb{Q}_p}$. Since

$$c_w(\phi_p) = \sum_{(H,H')} v_p(N_{K_H/\mathbb{Q}}(\mathrm{Disc}(K_{H'}/K_H))) \cdot w(H, H')$$

we have

$$\begin{aligned} D_{c_w}(K) &= \prod_p p^{c_w(\phi_p)} \\ &= \prod_p \prod_{(H,H')} p^{v_p(N_{K_H/\mathbb{Q}}(\mathrm{Disc}(K_{H'}/K_H)))\cdot w(H,H')} \\ &= \prod_{(H,H')} N_{K_H/\mathbb{Q}}(\mathrm{Disc}(K_{H'}/K_H))^{w(H,H')} \\ &= D_w(K) \end{aligned}$$

Thus if $c_{H,H'}$ and c_w are defined as above, then c_w is the counting function corresponding to the weighted discriminant D_w.

Finally, if $p \nmid |\Gamma|$, then c_w depends only on the inertia groups $I_{\mathfrak{p},0}$, and in particular not on the decomposition group. This implies:

Corollary 2.9. *Given any weight function w, the corresponding counting function c_w is proper.*

2.4 The Overall Weight of a Subgroup

We can now use the explicit formula in section 2.3 to show that a weighted discriminant counting function actually depends on far fewer parameters than the original definition would suggest.

Definition 2.10. Let w be a weight function for Γ, and let $I \subseteq \Gamma$ be any subgroup of Γ. The *overall weight* of I is the quantity

$$\bar{w}(I) = \sum_{(H,H')} \left(\frac{w(H,H')}{|I| \cdot |H'|} \sum_{\gamma \in \Gamma} \left(|\gamma I \gamma^{-1} \cap H| - |\gamma I \gamma^{-1} \cap H'| \right) \right) \tag{2.1}$$

where the first sum, as usual, ranges over pairs where $H \subseteq \Gamma$ and H' is a maximal subgroup of H.

This quantity is crafted so that $\bar{w}(I)$ depends only on the weight function w. Note also that if I and I' are conjugate in Γ, then $\bar{w}(I) = \bar{w}(I')$, so we can speak of the overall weight of a conjugacy class of subgroups.

More importantly, if $\phi_p : G_{\mathbb{Q}_p} \to \Gamma$ with ramification groups $I_{p,i}$, then using the explicit formula in section 2.3,

$$c_w(\phi_p) = \sum_{i \geq 0} \frac{|I_{p,i}|}{|I_{p,0}|} \bar{w}(I_{p,i})$$

This shows that the counting function attached to w depends only on the overall weights. That is, if w_1 and w_2 are weight functions and $\bar{w}_1(I) = \bar{w}_2(I)$ for every subgroup $I \subseteq \Gamma$, then $c_{w_1} = c_{w_2}$ (and thus $D_{w_1} = D_{w_2}$).

In fact, we can go even farther than this:

Proposition 2.11. *If w is a weight function for Γ, then c_w and D_w are completely determined by the overall weights of conjugacy classes of cyclic subgroups of Γ.*

Proof. Let I be any subgroup of Γ. I is the union of its cyclic subgroups, so by the principle of inclusion-exclusion,

$$|I| = \sum_{n \geq 1} \left((-1)^{n+1} \sum_{C_1, \ldots, C_n \subseteq I} |C_1 \cap \ldots \cap C_n| \right)$$

Using this, we can express the overall weight of I as

$$\bar{w}(I) = \sum_{(H,H')} \left(\frac{w(H,H')}{|I| \cdot |H'|} \sum_{\gamma \in \Gamma} \sum_{n \geq 1} \sum_{C_1, \ldots, C_n \subseteq I} (-1)^{n+1} |\gamma(C_1 \cap \ldots \cap C_n)\gamma^{-1} \cap (H \setminus H')| \right)$$

where $C_1, \ldots, C_n$ are cyclic subgroups of I. Then we can rearrange the sums to obtain

$$\bar{w}(I) = \sum_{n \geq 1} \sum_{C_1, \ldots, C_n \subseteq I} \sum_{(H,H')} \left(\frac{w(H,H')}{|I| \cdot |H'|} \sum_{\gamma \in \Gamma} (-1)^{n+1} |\gamma(C_1 \cap \ldots \cap C_n)\gamma^{-1} \cap (H \setminus H')| \right)$$

or

$$\bar{w}(I) = \sum_{n\geq 1} \sum_{C_1,\ldots,C_n\subseteq I} \left((-1)^{n+1} \frac{|C_1 \cap \ldots C_n|}{|I|} \cdot \bar{w}\left(C_1 \cap \ldots C_n\right) \right) \tag{2.2}$$

Since $C_1 \cap \ldots C_n$ is itself cyclic, this expresses $\bar{w}(I)$ in terms of the overall weights of cyclic groups, as desired. (Note that the overall weight of the trivial group must always be 0, from equation 2.1.) □

A converse of Proposition 2.11 also holds:

Proposition 2.12. *For any choice of one real number for each conjugacy class of nontrivial cyclic subgroups of* Γ*, there is a weight function* w *such that for any nontrivial cyclic subgroup* $C \subseteq \Gamma$*,* $\bar{w}(C)$ *is the real number assigned to the conjugacy class of* C*.*

The proof of Proposition 2.12 is long and technical, so we postpone it to the end of this chapter.

These two results together imply:

Corollary 2.13. *The set of (not necessarily natural) weighted discriminant counting functions for a finite group* Γ *can be viewed as a vector space over* $\mathbb{R}$*, with basis vectors corresponding to the conjugacy classes of cyclic subgroups of* Γ*.*

This suggests that we should think of a weighted discriminant counting function as being determined by the choice of overall weights $\bar{w}(C)$, rather than the weights $w(H, H')$. We can even determine if c_w is natural in part by looking at the overall weights:

Proposition 2.14. *If* c_w *is natural, then all the overall weights of cyclic subgroups given by* w *are nonnegative integers.*

Proof. First, suppose that c_w is natural. Let C be any cyclic subgroup of Γ. Recall from the proof of Proposition 2.7 that if ℓ is a prime not dividing $|\Gamma|$, then maps $G_{\mathbb{Q}_\ell} \to \Gamma$ are determined by a pair (s,t) in Γ with $tst^{-1} = s^\ell$. If $\ell \equiv 1 \bmod |\Gamma|$, then this requirement is trivially satisfied. By letting s be a generator of C and $t = 1$, we can construct a tamely ramified map $\phi : G_{\mathbb{Q}_\ell} \to \Gamma$ whose image of inertia is C. Then $c_w(\phi) = \bar{w}(C)$, so $\bar{w}(C)$ must be a nonnegative integer. $\square$

Looking at counting functions through the lens of overall weights makes one more fact clear, which will be useful in the proof of Theorem 2.1:

Corollary 2.15. *Let $\phi_p : G_{\mathbb{Q}_p} \to \Gamma$ and $\phi'_\ell : G_{\mathbb{Q}_\ell} \to \Gamma$, with $p \neq \ell$, and let $I_{\ell,i}(\phi'_\ell)$ and $I_{p,i}(\phi_p)$ denote the ramification groups of ϕ'_ℓ and ϕ_p.*

If all of the following hold:

- Γ *is an ℓ-group*
- c *is a natural weighted discriminant counting function for* Γ
- $I_{\ell,0}(\phi'_\ell) = I_{p,0}(\phi_p)$

then $c_w(\phi'_\ell) \geq 2c_w(\phi_p)$.

Proof. Since ϕ_p is tamely ramified, the inertia group $I_{p,0}(\phi_p)$ is cyclic. Since Γ is an ℓ-group, we have $I_{\ell,0}(\phi'_\ell) = I_{\ell,1}(\phi'_\ell) = I_{p,0}(\phi_p)$. Furthermore, all of the higher ramification groups $I_{\ell,i}(\phi'_\ell)$ must be cyclic as well, so their overall weights are nonnegative, by

Proposition 2.14. Thus

$$\begin{aligned} c_w(\phi'_\ell) &= \sum_{i\geq 0} \frac{|I_{\ell,i}(\phi'_\ell)|}{|I_{\ell,0}(\phi'_\ell)|}\bar{w}(I_{\ell,i}(\phi'_\ell)) \\ &\geq \sum_{i=0,1} \frac{|I_{\ell,i}(\phi'_\ell)|}{|I_{\ell,0}(\phi'_\ell)|}\bar{w}(I_{\ell,i}(\phi'_\ell)) \\ &= 2\bar{w}\left(I_{\ell,0}(\phi'_\ell)\right) \\ &= 2\bar{w}\left(I_{p,0}(\phi_p)\right) \\ &= 2c_w(\phi_p) \end{aligned}$$

□

Remark. The overall weight of a subgroup parallels Malle's *index* in [19]. In particular, if w is the weight function corresponding to the standard discriminant, then the overall weight of each subgroup is equal to the index of its generator, viewing Γ via its regular representation.

2.5 Tame Mass Formulas and Their Coefficients

In this section, we prove a more general form of Proposition 2.7 for non-pure mass formulas:

Theorem 2.16. *Any proper natural counting function c has exactly one tame mass formula. The tame mass formula is of the form*

$$f(x) = \sum_C \sigma_C(x)x^{-i_C}$$

where the sum ranges over conjugacy classes of cyclic subgroups $C \subseteq \Gamma$. Each "coefficient" σ_C is a sum of distinct Dirichlet characters modulo divisors of $|\Gamma|$, one of which is the trivial character.

Furthermore, if c is a weighted discriminant counting function with weight function w, then $i_C = \bar{w}(C)$.

Remark. Proposition 2.7, in this context, implies that the mass formula given by Theorem 2.16 is pure if and only if Γ has a rational character table.

We first require a simple result about sums of characters:

Proposition 2.17. *Let A be an abelian group, and B a subgroup of A. Let σ be the sum of all irreducible characters of A that are trivial on B. Then*

$$\sigma(a) = \begin{cases} 0 & \text{if } a \notin B \\ [A : B] & \text{if } a \in B \end{cases}$$

Proof. Since A is abelian, a character of A is a map $A \to \mathbb{C}$. A character trivial on B descends to a character of A/B, and this gives a bijection between irreducible characters of A that are trivial on B and irreducible characters of A/B. By orthogonality relations for characters, the sum of all irreducible characters of A/B is 0 for nonidentity elements, and $|A/B|$ for the identity; this proves Proposition 2.17. □

We now prove Theorem 2.16. We use the notation $g_1 \sim g_2$ to mean that g_1 and g_2 are conjugate as elements of Γ, and $[g]$ to denote the conjugacy class of g.

Proof. Let a be an integer relatively prime to $|\Gamma|$. Since c is proper, there exists a pure mass formula f_a for all primes congruent to a modulo $|\Gamma|$, by Proposition 2.6. This is unique, since if there were another such pure mass formula f'_a, then f_a and f'_a would be two different Laurent polynomials which agree at infinitely many values, which is impossible.

For convenience, we will assume from here on that c is a weighted discriminant counting function with weight function w; if it is not, the argument is the same but with the overall weights $\bar{w}(C)$ replaced by unknown integers.

From the argument in the proof of Proposition 2.6, we can see that

$$f_a(x) = \sum_C n_C x^{-\bar{w}(C)}$$

where the sum runs over conjugacy classes of cyclic subgroups of Γ, and n_C is the number of maps $G_{\mathbb{Q}_p} \to \Gamma$ with $p \equiv a \bmod |\Gamma|$ whose inertia group is conjugate to C.

Now let f be a character Laurent polynomial, and assume f is a tame mass formula for c. If $p \equiv a \bmod |\Gamma|$, then we must have $f(p) = f_a(p)$, so f is of the form

$$f(x) = \sum_C \left(\sum_{\chi_j} b_{C,j} \chi_j(x) \right) x^{-\bar{w}(C)}$$

where the inner sum runs over all Dirichlet characters modulo divisors of $|\Gamma|$. If p is sufficiently large compared to all the $b_{C,j}$ and n_C, then for every C, we must have

$$\sum_{C' : \bar{w}(C') = \bar{w}(C)} \sum_{\chi_j} b_{C',j} \chi_j(p) = \sum_{C' : \bar{w}(C') = \bar{w}(C)} n_{C'} \tag{2.3}$$

Since the χ_j are periodic, this must in fact hold for all $p \equiv a \bmod |\Gamma|$. That is, the value of each "coefficient" σ in f on each $a \in (\mathbb{Z}/|\Gamma|\mathbb{Z})^*$ is determined by the corresponding coefficient of f_a.

Each coefficient of f is a function on the conjugacy classes (i.e. the elements) of $(\mathbb{Z}/|\Gamma|\mathbb{Z})^*$, and irreducible characters of $(\mathbb{Z}/|\Gamma|\mathbb{Z})^*$ are a basis for class functions on $(\mathbb{Z}/|\Gamma|\mathbb{Z})^*$ as a $\mathbb{C}$-vector space. There is thus a unique $\mathbb{C}$-linear combination of irreducible characters of $(\mathbb{Z}/|\Gamma|\mathbb{Z})^*$ for each coefficient of f that makes that coefficient agree with the corresponding coefficient of each of the f_a.

Each irreducible character of $(\mathbb{Z}/|\Gamma|\mathbb{Z})^*$ is equal to a unique Dirichlet character with modulus a divisor of $|\Gamma|$ that is as small as possible. This shows that there is a unique function f of the form

$$f(x) = \sum_C \sigma_C(x) x^{-\bar{w}(C)}$$

such that $f(p)$ agrees with the mass of c at p for all $p \nmid |\Gamma|$, where each σ_C is a complex linear combination of Dirichlet characters modulo divisors of $|\Gamma|$. That is, if a tame mass formula exists, then it is unique.

To complete the proof of Theorem 2.16, we will now show that that each coefficient σ_C in f is a sum of distinct characters, with the trivial character appearing in the sum.

Let p be a prime not dividing $|\Gamma|$, and let f_p be the pure mass formula for the set of primes congruent to p modulo $|\Gamma|$, as discussed above. As described above, f_p also has a term corresponding to each conjugacy class of cyclic subgroups of Γ.

Let γ be an element of Γ. The coefficient $\sigma_{\langle\gamma\rangle}$ is $\frac{1}{|\Gamma|}$ times the number of maps $G_{\mathbb{Q}_p} \to \Gamma$ with inertia group conjugate to $\langle\gamma\rangle$. Each such map is specified by an ordered pair $(s,t) \in \Gamma^2$, where $\langle t\rangle$ is conjugate to $\langle\gamma\rangle$, and $sts^{-1} = t^p$. (In the language of number fields, t is the generator of inertia, and s is the Frobenius element.)

If $\gamma^p \notin [\gamma]$, then there are no such pairs. Otherwise, the number of choices for t is the number of elements of Γ generating a subgroup conjugate to $\langle\gamma\rangle$, and the number of choices for s is equal to the number of elements of $\mathrm{Cent}(\gamma)$, the centralizer of x in Γ.

In the latter case, let n be the order of γ in Γ. If a and b are coprime to n, and $\gamma \sim \gamma^a$ and $\gamma \sim \gamma^b$, then we have $g_1\gamma g_1^{-1} = \gamma^a$, and $g_2\gamma g_2^{-1} = \gamma^b$, and

$$g_2 g_1 \gamma g_1^{-1} g_2^{-1} = g_2 \gamma^a g_2^{-1} = \gamma^{ab}$$

Thus $[\gamma] \cap \langle\gamma\rangle$ is naturally in bijection with a subgroup $S \subseteq (\mathbb{Z}/n\mathbb{Z})^*$, via $\gamma^k \mapsto k$.

We can now calculate $\sigma_{\langle\gamma\rangle}(p)$. For each element of $[\gamma]$, we have one choice for t, but we also need to count elements of Γ not in $[\gamma]$ but generating a subgroup conjugate to $\langle\gamma\rangle$. Overall, then, a choice of t is described by a choice of an element of $[\gamma]$ and a coset of S in $(\mathbb{Z}/n\mathbb{Z})^*$. The number of choices for s, as above, is $|\operatorname{Cent}(\gamma)|$. The coefficient is then

$$\frac{1}{|\Gamma|} \cdot |[\gamma]| \cdot \frac{\phi(n)}{|S|} \cdot |\operatorname{Cent}(\gamma)| = \frac{\phi(n)}{|S|} = [(\mathbb{Z}/n\mathbb{Z})^* : S]$$

since $|[\gamma]| \cdot |\operatorname{Cent}(\gamma)| = |\Gamma|$.

Now, if $\gamma \sim \gamma^p$, then $p \in S$ when p is taken as an element of $(\mathbb{Z}/n\mathbb{Z})^*$. Thus $\sigma_{\langle\gamma\rangle}(p)$ should be 0 if $p \notin S$ and $[(\mathbb{Z}/n\mathbb{Z})^* : S]$ if $p \in S$.

Let $\sigma_{n,S}$ be the sum of all irreducible characters of $(\mathbb{Z}/n\mathbb{Z})^*$ that are trivial on S. By Proposition 2.17, $\sigma_{\langle\gamma\rangle}(p) = \sigma_{n,S}(p)$. Thus $\sigma_{n,S} = \sigma_{\langle\gamma\rangle}$, the "coefficient" of f corresponding to the conjugacy class of $\langle\gamma\rangle$. Finally, $\sigma_{n,S}$ is a sum of distinct Dirichlet characters including the trivial character, as desired.

$\square$

Remark. When Γ is an ℓ-group, as in the proof of Theorem 2.1, ℓ divides the modulus of every nontrivial Dirichlet character in the coefficients of f. Theorem 2.16 then implies that $f(\ell)$ is a polynomial with one term for each conjugacy class of cyclic subgroups of Γ and all coefficients equal to 1.

2.6 Proof of Theorem 2.1

We now are equipped to prove our main theorem, Theorem 2.1. Let Γ be an ℓ-group, and c a natural weighted discriminant counting function for Γ, with weight function w, and corresponding weighted discriminant D_w. Assume that c has a universal mass formula f. Our method of proof will be to put an upper bound on each of the overall weights given by w. Since c is natural, this fact, combined with Propositions 2.11 and 2.14, will show that there are only finitely many choices for the overall weights and thus for c.

2.6.1 Preliminaries

If f is universal, it must be exactly the unique tame mass formula described in Theorem 2.16. Let $[C_1], \ldots, [C_s]$ be the conjugacy classes of cyclic subgroups of Γ. By Theorem 2.16, f is of the form

$$f(p) = \sum_{[C_j]} \sigma_{C_j}(p) p^{-\bar{w}(C_j)}$$

where each σ_{C_j} is a sum of Dirichlet characters containing the trivial character exactly once. Since each nontrivial character vanishes at ℓ, we have

$$f(\ell) = \sum_{[C_j]} \ell^{-\bar{w}(C_j)}$$

The mass formula f is universal if and only if this quantity is equal to the total mass of c at ℓ.

Note that $f(\ell)$ need only be numerically equal to the total mass; the two quantities will never be abstractly the same poylnomial in ℓ. For example, if we take $\Gamma = C_2$, and D_w to be the standard discriminant, then the tame mass formula is

$$f(p) = 1 + p^{-1}$$

At $\ell = 2$, there are two quadratic extensions of $\mathbb{Q}_2$ of discriminant 4 and four extensions of discriminant 8 [13], so the total mass is

$$1 + \ell^{-2} + 2\ell^{-3}$$

However, since

$$1 + \cdot 2^{-1} = 1 + \cdot 2^{-2} + 2 \cdot 2^{-3} = \frac{3}{2}$$

the mass formula $1 + p^{-1}$ is universal.

2.6.2 The Determining Equation

For each weight function w, we have a tame mass formula f_w. f_w is universal if and only if $f_w(\ell)$, which we computed in (2.6.1), is equal to the total mass at ℓ. Since w is completely determined by the overall weights of cyclic subgroups (by Proposition 2.11), a weighted discriminant counting function with a universal mass formula is equivalent to a choice of $(\bar{w}(C_1), \ldots, \bar{w}(C_s))$ satisfying the "determining equation":

$$\sum_{[C_j]} \ell^{-\bar{w}(C_j)} = \sum_{\phi : G_{\mathbb{Q}_\ell} \to \Gamma} \ell^{-c(\phi)} \tag{2.4}$$

Note that by the arguments in section 2.4, each exponent on the right side is a linear combination (with rational coefficients) of the overall weights of cyclic subgroups.

For any C_j, using cyclotomic extensions of $\mathbb{Q}_\ell$, we can construct a totally ramified map $\phi : G_{\mathbb{Q}_\ell} \to \Gamma$ with image C_j, using the following lemma:

Lemma 2.18. *There is a totally ramified cyclic extension of $\mathbb{Q}_\ell$ of degree ℓ^k for any k. Equivalently, there is a surjective totally ramified map $G_{\mathbb{Q}_\ell} \to C_{\ell^k}$ for any k.*

Proof. We can construct such an extension easily from cyclotomic extensions of $\mathbb{Q}_\ell$. Adjoining a primitive ℓ^mth root of unity gives a totally ramified extension with Galois group $(\mathbb{Z}/\ell^m\mathbb{Z})^*$ [25]. By choosing a large enough m and taking a subfield corresponding to the appropriate quotient of $(\mathbb{Z}/\ell^m\mathbb{Z})^*$, we obtain the desired extension. $\square$

By Corollary 2.15, $c(\phi) \geq 2\bar{w}(C_j)$. On the right side of equation (2.4), we can split off all the maps obtained from Lemma 2.18 to get

$$\sum_{[C_j]} \ell^{-\bar{w}(C_j)} = \sum_{[C_j]} \ell^{-b_j \bar{w}(C_j)} + \sum_{\text{other } \phi} \ell^{-c(\phi)} \tag{2.5}$$

where each b_j is at least 2.

We will now put upper bounds on the overall weights $\bar{w}(C_j)$ by studying the ℓ-adic valuation of equation (2.5).

2.6.3 The Upper Bound

Let M be the largest of the $\bar{w}(C_j)$, and let t be the number of terms on the right side of equation (2.5).

Each term on the right is a power of ℓ, and one of them has ℓ-adic valuation less than or equal to $-2M$. Since each power of ℓ added to this term can increase the valuation by at most 1, the largest possible valuation of the right side is then $-2M + t - 1$.

Meanwhile, the valuation of the left side is greater than or equal to $-M$, since no term has a valuation smaller than this. Thus for f to be universal, we must have

$$-M \leq -2M + t - 1$$

This implies that $M \leq t - 1$. Thus there is an upper bound on the overall weights of cyclic subgroups of Γ, which completes the proof of Theorem 2.1.

2.7 Proof of Proposition 2.12

We now complete the proof of Proposition 2.12.

Recall that the overall weight of a subgroup is

$$\bar{w}(I) = \sum_{(H,H')} \left(\frac{w(H,H')}{|I| \cdot |H'|} \sum_{\gamma \in \Gamma} \left(|\gamma I \gamma^{-1} \cap H| - |\gamma I \gamma^{-1} \cap H'| \right) \right)$$

For notational convenience, let $w'(H,H') = \frac{w(H,H')}{|H'|}$, and $\bar{w}'(I) = |I| \cdot \bar{w}(I)$. Also let

$$u(I,(H,H')) = \sum_{\gamma \in \Gamma} \left| \gamma I \gamma^{-1} \cap (H \setminus H') \right|$$

Then equation (2.1) simplifies to

$$\bar{w}'(I) = \sum_{(H,H')} w'(H,H') u(I,(H,H'))$$

Let $C_1, \ldots, C_n$ be the conjugacy classes of cyclic subgroups of Γ, and $P_1, \ldots, P_m$ be all the pairs (H,H') such that H' is maximal in H and $H \subseteq \Gamma$. Proposition 2.12 states that given an ordered n-tuple of real numbers $(b_1, \ldots, b_n)$, we can find $w'(P_j)$ for each j such that $\sum_j w'(P_j) u(C_i, P_j) = b_i$ for each i. In other words, the rank of the matrix $[u(C_i, P_j)]$ is n.

Since we clearly have $m \geq n$, this is equivalent to the statement that the rows of this matrix are linearly independent. That is, if there exists a real number $a_{[C]}$ for each conjugacy class of cyclic subgroups C of Γ, with

$$\sum_{[C]} a_{[C]} u(C, P_j) = 0 \tag{2.6}$$

for all j, then $a_{[C]} = 0$ for all C. This statement is what we will prove in the remainder of this section.

2.7.1 Converting subsets to elements

For C a cyclic subgroup of Γ, the number of conjugates of C is $\frac{|\Gamma|}{|N_\Gamma(C)|}$, where $N_\Gamma(C)$ is the normalizer of C in Γ. Thus we have

$$\sum_{[C]} a_{[C]} \sum_{\gamma} \left|\gamma C \gamma^{-1} \cap (H \setminus H')\right| = \sum_{C} a_{[C]} \cdot |N_\Gamma(C)| \cdot |C \cap (H \setminus H')| \tag{2.7}$$

where the sum on the right side ranges over all cyclic subgroups of Γ. The right side is also equal to

$$\sum_{C} a_{[C]} \cdot |N_\Gamma(C)| \sum_{x \in C} |\{x\} \cap (H \setminus H')|$$

which, when we reverse the order of summation, becomes

$$\sum_{x \in \Gamma} \sum_{C \supseteq \langle x \rangle} a_{[C]} \cdot |N_\Gamma(C)| \cdot |\{x\} \cap (H \setminus H')|$$

Now if we set

$$a'_{[\langle x \rangle]} = \sum_{C \supseteq \langle x \rangle} a_{[C]} \cdot |N_\Gamma(C)| \tag{2.8}$$

for each $x \neq 1_\Gamma$, then this can be written as

$$\sum_{x \in \Gamma} a'_{[\langle x \rangle]} \, |\{x\} \cap (H \setminus H')| = \sum_{x \in H \setminus H'} a'_{[\langle x \rangle]}$$

We have now converted equation (2.6) to

$$\sum_{x \in H \setminus H'} a'_{[\langle x \rangle]} = 0 \tag{2.9}$$

for all pairs (H, H').

2.7.2 Finishing the proof

To make use of equation 2.9, we need the following lemma:

Lemma 2.19. *If $a'_{[\langle x\rangle]} = 0$ for every $x \neq 1_\Gamma$, then $a_{[C]} = 0$ for every nontrivial cyclic $C \subseteq \Gamma$.*

Proof. We induct on the number of cyclic subgroups of Γ containing C. If the only such subgroup is C itself, then from equation (2.8), we have $a_{[C]} = \frac{1}{|N_\Gamma(C)|} a'_{[C]} = 0$. Otherwise,

$$a'_{[C]} = |N_\Gamma(C)| a_{[C]} + \sum_{\substack{C' \supset C \\ C' \neq C}} a_{[C']} \cdot |N_\Gamma(C')|$$

Since each C' in the right sum is contained in fewer cyclic subgroups than C, by induction, $a_{[C']} = 0$. Thus we also have $a_{[C]} = 0$, as desired. $\square$

Now it suffices to prove that if

$$\sum_{x \in H \setminus H'} a'_{[\langle x\rangle]} = 0$$

for each pair (H, H'), then $a'_{[\langle x\rangle]} = 0$ for every $x \neq 1_\Gamma$. To do this, suppose that the order of $x \in \Gamma$ is $\prod p_k^{r_k}$ where the p_k are distinct primes. Let $\sigma(x) = \sum r_k$. We will induct on $\sigma(x)$.

If $\sigma(x) = 1$, then let $H = \langle x\rangle$ and $H' = 1$. Every element of $\langle x\rangle$ is a generator of $\langle x\rangle$, so we have

$$\sum_{x \in H \setminus H'} a'_{[\langle x\rangle]} = (|\langle x\rangle| - 1)\, a'_{[\langle x\rangle]} = 0$$

and thus $a'_{[\langle x\rangle]} = 0$, as desired.

Otherwise, let $H = \langle x\rangle$ and H' be any maximal subgroup of H. If $h \in \langle x\rangle$, then either h generates $\langle x\rangle$, or the order of h is a proper divisor of the order of x, from which $\sigma(h) < \sigma(x)$. Thus either $a'_{[\langle h\rangle]} = a'_{[\langle x\rangle]}$ or $a'_{[\langle h\rangle]} = 0$ (by induction). Then

$$\sum_{x \in H \setminus H'} a'_{[\langle x\rangle]} = B a'_{[\langle x\rangle]} = 0$$

where B is the number of generators of $\langle x \rangle$.

Thus $a'_{[\langle x \rangle]} = 0$ for all $x \in \Gamma$. This completes the proof of Proposition 2.12.

Chapter 3

Counting Class Groups with Alternate Discriminants

In this chapter, we turn our attention to class groups of number fields, using the machinery of Chapter 2. We make use of one of the basic theorems from class field theory:

Theorem 3.1. *Let K be any number field, and K^{nr} its maximal unramified abelian extension. Then* $\mathrm{Gal}(K^{nr}/K)$ *is naturally isomorphic to the class group $Cl(K)$.*

This implies:

Corollary 3.2. *Let K be any number field and A any abelian group. The number of extensions L/K that are unramified at every place, with* $\mathrm{Gal}(L/K) = A$*, is equal to*

$$\#\{\text{surjections } Cl(K) \to A\} \cdot \frac{1}{|\operatorname{Aut}(A)|}$$

We will take advantage of this by tuning Heuristic 1.2 to count such extensions L/K, rather than counting surjections $Cl(K) \to A$ directly.

3.1 Extended Counting Functions; Infinite Weights

We first introduce some notation that allows us to fit questions involving restricted ramification into the framework of alternate discriminants and counting functions.

Definition 3.3. An *extended counting function* is a counting function that takes values in $\mathbb{R} \cup \infty$ instead of $\mathbb{R}$.

An extended counting function is natural if its values are in $\mathbb{Z}_{\geq 0} \cup \{\infty\}$. If c is an extended counting function, the alternate discriminant D_c will also take values in $\mathbb{R} \cup \{\infty\}$. D_c may be equal to a weighted discriminant D_w, where w is an *extended weight function* that takes values in $\mathbb{R} \cup \{\infty\}$. For the purpose of computing the total mass at a prime p, we use the convention $\frac{1}{p^\infty} = 0$.

If c_w is an extended weighted discriminant counting function for a group Γ, and c_w takes the value ∞ at least once, then by Proposition 2.11, there is some cyclic subgroup $C \subseteq \Gamma$ whose overall weight is $\bar{w}(C) = \infty$. This has two main effects:

1. Globally, when counting number fields by the weighted discriminant D_w, any field K for which any ramification group at any prime is equal to C has $D_w(K) = \infty$. This excludes K from the count.

2. Locally, when computing the total mass at p, if $\phi : G_{\mathbb{Q}_p} \to \Gamma$ has any ramification group equal to C, then $c_w(\phi) = \infty$. This means $p^{-c_w(\phi)} = 0$, so ϕ does not count toward the total mass.

This allows us to remove from consideration all fields, local and global, with certain types of ramification, and ask the same kinds of questions as in Chapters 1 and 2.

3.2 Cohen-Lenstra for Quadratic Fields

Let $\Gamma = A \rtimes C_2$, where A is abelian and C_2 acts on A by multiplication by -1. Let $\phi : G_{\mathbb{Q}_p} \to \Gamma$ with ramification groups $I_0, I_1, I_2, \ldots$, where I_0 is the inertia group and I_1

is the wild inertia group. Define the *Cohen-Lenstra counting function* for Γ by:

$$c_{CL}(\phi) = \begin{cases} \sum_{n=0}^{\infty}(|I_n|-1) & \text{if } I_0 \cap A = \{1\} \\ \infty & \text{otherwise} \end{cases}$$

Note that the only subgroups of Γ having trivial intersection with A are cyclic of order 2, generated by an element of $\Gamma \setminus A$, so the sum in the first case is equal to the number of ramification groups I_n that are nontrivial.

In terms of overall weights, c_{CL} is given by $\bar{w}(C) = \infty$ if $C \subseteq A$, and $\bar{w}(C) = 1$ otherwise (i.e. if $C \cap A$ is trivial).

c_{CL} is also a natural extended weighted discriminant counting function. It is associated to an extended weight function w_{CL} defined by:

$$w_{CL}(H, H') = \begin{cases} 1 & \text{if } H = \Gamma \text{ and } H' = A \\ \infty & \text{if } H \subseteq A \\ 0 & \text{otherwise} \end{cases}$$

This illustrates the utility of the weighted discriminant $D_{w_{CL}}$. If L is a Γ-extension of $\mathbb{Q}$, let L_A be the quadratic subfield fixed by A. Then $D_{w_{CL}}(L) = \mathrm{Disc}(L_A/\mathbb{Q})$ if L/L_A is unramified, and $D_{w_{CL}}(L) = \infty$ otherwise.

The following proposition allows us to make a better interpretation of $D_{w_{CL}}$:

Proposition 3.4. *Let K be a quadratic extension of $\mathbb{Q}$ (or any other number field having class number 1). Let L be an unramified A-extension of K, where A is an abelian group. Then $L/\mathbb{Q}$ is Galois, and* $\mathrm{Gal}(L/\mathbb{Q}) = A \rtimes C_2$, *where C_2 acts on A by multiplication by -1.*

Proof. We follow the outline of a proof given in [28], using facts about class field theory found in [22]. Suppose that $L/\mathbb{Q}$ is not Galois; let L^g be the Galois closure of L over $\mathbb{Q}$. Let $f(x)$ be a generating polynomial (of degree $2|A|$) for L, so L^g is the splitting field of f. Since $K/\mathbb{Q}$ is Galois, f must split into two irreducible polynomials f_1, f_2 of degree $|A|$ in K, with L the splitting field of f_1. The nontrivial automorphism of K takes f_1 to f_2, so the splitting field of f_2 must also be everywhere unramified over K. Then L^g is the compositum of these two unramified abelian extensions of K, so it must also be unramified and abelian over K.

Let $G = \mathrm{Gal}(L^g/K)$. We claim first that $\Gamma = \mathrm{Gal}(L^g/\mathbb{Q})$ is a semidirect product $G \rtimes C_2$. We have a map $\Gamma \to C_2$ with kernel G. Since L^g/K is unramified at every place, no inertia group at any prime can intersect G nontrivially; this also means the inertia group at each prime must be trivial or have order 2, since $[\Gamma : G] = 2$. Some prime of L^g must be ramified in $L^g/\mathbb{Q}$, so choose p to be any such prime, and let I_p be its inertia group. Then the map $\Gamma \to C_2$ is injective on I_p, and $I_p \cap G$ is trivial. Thus the inclusion $I_p \hookrightarrow \Gamma$ splits Γ as a semidirect product, as desired.

Now let $H = \mathrm{Gal}(L^g/L)$, and denote by σ a generator of $\mathrm{Gal}(K/\mathbb{Q})$. Since L^g/K is unramified, $\mathrm{Gal}(K/\mathbb{Q})$ acts on G in a way that agrees with its action on the class group $Cl(K)$. $K/\mathbb{Q}$ is quadratic, so every ideal of $\mathbb{Q}$ is either split, inert, or ramified in K. σ fixes all the ideals of K that are either inert or ramified. If an ideal (p) of $\mathbb{Q}$ splits into p_1p_2 in K, then $\sigma(p_1) = p_2$. Since $[p_1][p_2] = [(p)] = 1$ in $Cl(K)$, σ acts on $Cl(K)$, and thus on G, as multiplication by -1.

Every element of $\mathrm{Gal}(L^g/\mathbb{Q})$ is either g or $g\sigma$ for some $g \in G$. Let $h \in H$. Then

$ghg^{-1} = h$ since G is abelian. Furthermore,

$$g\sigma h(g\sigma)^{-1} = g\sigma hg\sigma = g(\sigma h\sigma)(\sigma g\sigma) = gh^{-1}g^{-1} \in H$$

This shows that H is normal in $\mathrm{Gal}(L^g/\mathbb{Q})$, and thus $L/\mathbb{Q}$ is in fact Galois, as desired. This also shows that σ acts on A by multiplication by -1. $\square$

By Proposition 3.4, counting Γ-extensions of $\mathbb{Q}$ by $D_{w_{CL}}$ actually counts pairs (K, L), where K is a quadratic field and L is an unramified A-extension of K. Furthermore, these pairs are counted by $\mathrm{Disc}(K/\mathbb{Q})$. Combining this with Corollary 3.2, we obtain:

Proposition 3.5. *Let $N_{CL}(X)$ be the number of Γ-number fields L with $D_{w_{CL}}(K) < X$, where $\Gamma = A \rtimes C_2$ as above. Let $N_{surj}(X)$ be the total number of surjections to A from class groups of quadratic fields K with $|\,\mathrm{Disc}(K)| < X$. Then*

$$N_{CL}(X) = N_{surj}(X) \cdot \frac{1}{|\,\mathrm{Aut}(A)|}$$

Remark. Nothing in the proof of Proposition 3.4 requires that the ground field be $\mathbb{Q}$, only that its ring of integers be a principal ideal domain. The same will be true later of Proposition 3.7.

3.2.1 A Mass Formula

We now prove an interesting fact about the counting function c_{CL} defined in Section 3.2:

Theorem 3.6. *The Cohen-Lenstra counting function, c_{CL}, has a universal mass formula.*

To do this, we will need to list all maps $\phi : G_{\mathbb{Q}_p} \to \Gamma$ whose inertia group has trivial intersection with A. We can exclude the others, since if the inertia group of ϕ intersects A nontrivially, then $c_{CL}(\phi) = \infty$, so ϕ does not contribute to the total mass at p.

Taking $\Gamma = A \rtimes C_2$, let σ be a generator of C_2. Then $\Gamma = A \cup \sigma A$, and $\sigma a \sigma = a^{-1}$ for $a \in A$. If $a_1, a_2 \in A$, then $\sigma a_1 \sigma a_1 = a_1^{-1} a_1 = 1$, and $\sigma a_1 \sigma a_2 = a_1^{-1} a_2 \in A$. Thus any subgroup of Γ containing more than one element of $\Gamma \setminus A$ must contain a nontrivial element of A. This shows that the only subgroups of Γ having trivial intersection with A are isomorphic to C_2, and there are $|A|$ of them, one generated by each element of $\Gamma \setminus A$.

Thus if $\phi : G_{\mathbb{Q}_p} \to \Gamma$ with $c_{CL}(\phi) \neq \infty$, then ϕ factors through C_2. Furthermore, if ϕ' is the corresponding map to C_2, then from the definition of c_{CL}, we have

$$c_{CL}(\phi) = c_{\text{Disc}}(\phi')$$

where c_{Disc} is the counting function for C_2 corresponding to the standard discriminant. Furthermore, the maps from $G_{\mathbb{Q}_p}$ to Γ (excluding those for which $c_{CL} = \infty$) and to C_2 are the same, except there are $|A|$ times as many maps to Γ, since ϕ is ϕ' composed with any of the $|A|$ maps $C_2 \to \Gamma$ whose image has trivial intersection with A. Since Γ is $|A|$ times the size of C_2, this factor cancels out in the total mass. Then for any prime p, we have

$$M(\mathbb{Q}_p, \Gamma, c_{CL}) = M(\mathbb{Q}_p, C_2, c_{\text{Disc}}) = 1 + p^{-1}$$

Since c_{Disc} has a universal mass formula (see Section 4.2), so does c_{CL}, as desired.

3.3 Malle-Bhargava and Cohen-Lenstra Heuristics for Quadratic Fields

In the setup of Section 3.2, the mass at each prime for (Γ, c_{CL}) is the same as the mass for (C_2, c_{Disc}). Heuristic 1.2 predicts that for any abelian group A, the number of surjections from the class groups of quadratic fields of discriminant $\leq X$ to A will approach a constant multiple of the number of fields of discriminant $\leq X$. If the value of $C(\Gamma)$ is correct, this parallels the "moment form" of the Cohen-Lenstra heuristics [6, 27] that states that the average number of surjections from the class group to A should be 1 for any A.

This is reasonable if we restrict to imaginary quadratic fields (which can be formulated as a change in the value of C_∞) and groups with $|A|$ odd. For real quadratic fields, Cohen and Lenstra predict a different distribution of class groups, so this may not match the actual count of number fields. When $|A|$ is even, the class group heuristics are known to be false by genus theory, and we will see in (3.3.1), as an example, that Heuristic 1.2 fails badly for $A = C_2$.

Remark. We think it is not out of the question that the value of $C(\Gamma)$ may change when we introduce restrictions on ramification at an infinite set of primes, but we have not yet seen an example where this happens.

3.3.1 $A = C_2$ and genus theory for imaginary quadratic fields

Gauss's formulation of genus theory shows that if K is a quadratic field with discriminant $D < 0$, and D has k prime factors, then the 2-part of the class group of K has rank $(k-1)$ [5]. The number of surjections from the class group to C_2 is then $2^{k-1} - 1$. Take $D = -an$, where n is squarefree, $a = 1$ if $n \equiv 3 \bmod 4$, and $a = 4$ otherwise. Then k is the number of prime factors of n if $n \equiv 2, 3 \bmod 4$, and one more than this if $n \equiv 1 \bmod 4$. Note that in the first case, $2^{k-1} = \frac{1}{2}d(n)$, where $d(n)$ is the number of divisors of n. In the second case, $2^{k-1} = d(n)$.

Let $N_2(X)$ be the total number of surjections to C_2 from the class groups of imaginary quadratic fields with $|D| < X$. Then we have

$$N_2(X) = \sum_{\substack{n<X \\ \text{squarefree} \\ n\equiv 3 \bmod 4}} \left(\frac{1}{2}d(n) - 1\right) + \sum_{\substack{n<\frac{1}{4}X \\ \text{squarefree} \\ n\equiv 2 \bmod 4}} \left(\frac{1}{2}d(n) - 1\right) + \sum_{\substack{n<\frac{1}{4}X \\ \text{squarefree} \\ n\equiv 1 \bmod 4}} (d(n) - 1) \quad (3.1)$$

We can rearrange the three sums into:

$$N_2(X) = \sum_{\substack{n<\frac{1}{4}X \\ \text{squarefree}}} \left(\frac{1}{2}d(n) - 1\right) + \sum_{\substack{n<\frac{1}{4}X \\ \text{squarefree} \\ n\equiv 1 \bmod 4}} \frac{1}{2}d(n) + \sum_{\substack{\frac{1}{4}X\le n<X \\ \text{squarefree} \\ n\equiv 3 \bmod 4}} \left(\frac{1}{2}d(n) - 1\right) \quad (3.2)$$

In general, let $m_n = d(n)$ if n is squarefree, and 0 otherwise. Then $\sum m_n n^{-s}$ is equivalent to the Euler product

$$f(s) = \prod_p \left(1 + 2p^{-s}\right)$$

Then

$$\frac{f(s)}{\zeta(s)^2} = \prod_p \left(1 - 3p^{-2s} + 2p^{-3s}\right)$$

which converges absolutely for $\mathrm{Re}(s) > \frac{1}{2}$. Thus by [23, Exercise 4.4.17],

$$\sum_{n<X} m_n \sim \prod_p \left(1 - 3p^{-2} + 2p^{-3}\right) \cdot X \log X \quad (3.3)$$

This shows that the first term in equation (3.2) for $N_2(X)$ is at least $BX \log X$ for some constant B, and thus even the main term does not agree with Heuristic 1.2, which predicts BX.

3.3.2 $A = C_3$ and results of Davenport-Heilbronn

With $A = C_3$, we have $\Gamma = S_3$. Again using techniques in [23], we can translate Heuristic 1.2 into a prediction that the number of surjections from class groups of quadratic fields of discriminant $< X$ to C_3 is asymptotic to

$$2C(S_3) \cdot C_\infty \cdot \frac{1}{\zeta(2)} X$$

A corollary to the work of Davenport-Heilbronn on cubic fields is that the actual number of surjections is, on average, 1 per imaginary quadratic field and $\frac{1}{3}$ per real quadratic field. It is an elementary result [9] that the number of quadratic fields of discriminant $< X$ is asymptotic to $\frac{1}{\zeta(2)} X$, of which half are real and half are imaginary. Thus we should have $C(S_3) \cdot C_\infty = \frac{1}{3}$.

In Heuristic 1.2, as formulated in [26], the constant C_∞ is equal to

$$\frac{1}{|\Gamma|} \sum_{\phi: \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \to \Gamma} 1$$

where we can think of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ as $G_{\mathbb{Q}_v}$ with v the infinite prime of $\mathbb{Q}$.

In the current setup, all four maps from $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) \to \Gamma$ are allowed (i.e. none cause L/K to be ramified at an archimedean place), so we get $C_\infty = \frac{2}{3}$ and $C(S_3) = \frac{1}{2}$. If we instead count S_3 cubic [9, 11] or sextic [4] fields by discriminant, we also get $C(S_3) = \frac{1}{2}$. Thus in this case, at least, $C(S_3)$ appears to be the same regardless of the counting function used.

3.4 Class Groups of Cyclic Cubic Fields

It is difficult to extend the results in Section 3.2 to general class groups of number fields. However, a result similar to Proposition 3.4 is possible for unramified quadratic extensions of cyclic cubic fields (i.e. surjections from the class group to C_2):

Proposition 3.7. *Let $K/\mathbb{Q}$ be a Galois cubic field, L/K be an unramified quadratic extension, and L^g be the Galois closure of $K/\mathbb{Q}$. Then* $\mathrm{Gal}(L^g/\mathbb{Q}) = A_4$*, the alternating group on four elements.*

Proof. Let $G = \mathrm{Gal}(L^g/K)$, and $\mathrm{Gal}(K/\mathbb{Q}) = \langle\sigma\rangle$.

Let f be the generating polynomial for $L/\mathbb{Q}$, so f is degree 6. Since $K/\mathbb{Q}$ is Galois, f must split into three irreducible quadratic polynomials f_1, f_2, f_3 in K, and σ permutes these three. As in Proposition 3.4, the fields generated by f_1, f_2, f_3 are all unramified over K, and L^g is their compositum, so L^g/K is unramified. This also implies that G is isomorphic to either C_2, C_2^2, or C_2^3, since L^g is the compositum of three quadratic extensions of K. Since $|\mathrm{Gal}(L^g/K)|$ is coprime to $|\mathrm{Gal}(K/\mathbb{Q})|$, $\mathrm{Gal}(L^g/\mathbb{Q})$ is a semidirect product $A \rtimes C_3$.

The action of σ on $Cl(K)$ satisfies $x + \sigma(x) + \sigma^2(x) = 0$, since any prime of $\mathbb{Q}$ is either inert, totally ramified, or split completely (with σ permuting the primes above it) in K. As before, this must agree with the action of σ on G.

If $G = C_2$, then σ must act trivially on it, which does not agree with the action on $Cl(L)$. If $G = C_2^3$, then we can choose generators i_1, i_2, i_3 for G such that the fields generated by f_1, f_2, and f_3 are the fixed fields of $\langle i_2, i_3\rangle$, $\langle i_1, i_3\rangle$, and $\langle i_1, i_2\rangle$, respectively. Since $\sigma(f_n) = f_{n+1}$, we must have $\sigma(i_n) = i_{n+1}$. Then

$$i_1 + \sigma(i_1) + \sigma^2(i_1) = i_1 + i_2 + i_3 \neq 0$$

which is impossible.

Thus $G = C_2^2$. The action of σ on G cannot be trivial, so it must permute the three nonidentity elements. Finally, $C_2^2 \rtimes C_3$ with this action is isomorphic to A_4. $\square$

3.4.1 The counting function

As above, let L^g be a degree-12 A_4-extension of $\mathbb{Q}$, with K its (unique) cubic subfield, and let L be one of the three degree-6 subfields, fixed by a subgroup $H \subseteq A_4$. We wish to count fields L^g for which L/K is unramified, ordered by $\mathrm{Disc}(K/\mathbb{Q})$.

Define an extended weight function w with $w(A_4, V_4) = 1$, $w(V_4, H) = \infty$, and $w = 0$ otherwise. Then the weighted discriminant D_w counts fields in the correct way. This weight function has overall weights $\bar{w}(C_2) = \infty$ and $\bar{w}(C_3) = 1$.

Using the determining equations in Section 4.13, we can see that the corresponding counting function does not have a universal mass formula in this case.

3.4.2 Predictions from Malle-Bhargava

Referring again to Section 4.13, the mass at primes $p \neq 3$ is $1 + (1 + \chi(p))p^{-1}$, where χ is the nontrivial Dirichlet character modulo 3. The mass at 3 is $1 + 2 \cdot 3^{-2}$. These are the same as the masses for counting C_3 fields by discriminant, so as in Section 3.2, Heuristic 1.2 will predict, on average, a constant number of surjections from the class group to C_2 over all cyclic cubic fields.

This agrees with the prediction by Cohen and Martinet in [7] of 2^{-2} surjections per field, at least in the order of the main term. Whether the exact value of the main term agrees depends on the value of $C(A_4)$ in Heuristic 1.2.

3.5 Class Groups of Non-Cyclic Cubic Fields

We now consider the same questions as in Section 3.4, but for non-cyclic cubic fields.

Let K be a non-cyclic cubic number field, and let L be an unramified quadratic extension of K. Let L^g and K^g be the Galois closures of L and K, respectively, and let f be a generating polynomial for L. A proof is given in [12] that $\Gamma = \mathrm{Gal}(L^g/\mathbb{Q}) \simeq S_4$.

We now have a tower of fields $L^g/L/K/\mathbb{Q}$, with $K/\mathbb{Q}$ noncyclic cubic, L/K unramified everywhere, and $\mathrm{Gal}(L^g/\mathbb{Q}) = S_4$. In addition, we know that $\mathrm{Gal}(L^g/K) = D_4$, the Sylow 2-subgroup of S_4. We must also have $\mathrm{Gal}(L^g/K^g) = V_4$ (the subgroup containing all products of two 2-cycles), since it must be a normal subgroup of S_4 of order 4. Since $K^g \neq L$, this implies that $\mathrm{Gal}(L^g/L)$ is either V_4 (a non-normal copy) or C_4.

If $\mathrm{Gal}(L^g/L) = C_4$, then the inertia group at any prime cannot intersect any conjugate of $D_4 \setminus C_4$, and thus it cannot contain any 2-cycles. The only remaining option for the inertia group is C_3, and then all the inertia groups will be contained in A_4. This means the fixed field of A_4 will be unramified everywhere, which is impossible. Thus $\mathrm{Gal}(L^g/L) = V_4$.

3.5.1 The counting function

We can now construct a counting function that counts 2-parts of class groups of non-cyclic cubic fields.

Let w be a weight function with $w(S_4, D_4) = 1$, $w(D_4, V_4) = \infty$ for the non-normal copies of V_4, and $w = 0$ otherwise. Then the weighted discriminant D_w counts fields in the appropriate way. Using the notation of Section 4.14, the corresponding counting funtion c_w has overall weights $w_{2a} = 1$, $w_{2b} = \infty$, $w_3 = 2$, and $w_4 = \infty$ The determining equations in Section 4.14 reduce to

$$3^{-w_3} = 2 \cdot 3^{-2w_3} + 2 \cdot 3^{-(w_{2a}+w_3)} + 3 \cdot 3^{-(w_{2a}+2w_3)} \tag{3.4}$$

$$2^{-w_{2a}} = 2^{-2w_{2a}} + 2 \cdot 2^{-3w_{2a}} \tag{3.5}$$

These values satisfy both determining equations, so the Cohen-Lenstra counting function for this situation does have a universal mass formula.

3.5.2 Predictions from Malle-Bhargava

Once again, we refer to Chapter 4 for total masses. The mass at any prime $p \neq 2, 3$ is $1 + p^{-1} + p^{-2}$. The mass at 3 is $1 + 3^{-1} + 2 \cdot 3^{-3} + 2 \cdot 3^{-4} + 3 \cdot 3^{-5}$, and the mass at 2 is $1 + 2 \cdot 2^{-2} + 2 \cdot 2^{-3}$. These are the same as the masses for counting S_3-cubic fields by discriminant, so Heuristic 1.2 predicts a constant number of surjections to C_2 per non-cyclic cubic field, depending on the value of $C(S_4)$.

Cohen and Martinet [7] predict that this number should be $\frac{1}{2}$ for complex fields and $\frac{1}{4}$ for totally real fields, and Bhargava [1] confirms that this prediction is correct. In either case, the Euler product in Heuristic 1.2 is

$$f(s) = C(S_4) \cdot C_\infty \cdot g(s) \cdot \prod_p \left(1 + p^{-s} + p^{-2s}\right) \tag{3.6}$$

where

$$g(s) = \frac{1 + 2 \cdot 2^{-2s} + 2 \cdot 2^{-3s}}{1 + 2^{-s} + 2^{-2s}} \cdot \frac{1 + 3^{-s} + 2 \cdot 3^{-3s} + 2 \cdot 3^{-4s} + 3 \cdot 3^{-5s}}{1 + 3^{-s} + 3^{-2s}}$$

Let $\sum m_n n^{-s}$ be the corresponding Dirichlet series. Since

$$f(s) = C(S_4) \cdot g(s) \cdot \frac{\zeta(s)}{\zeta(3s)}$$

we have, as in (3.3.1),

$$\sum_{n<X} m_n \sim C(S_4) \cdot C_\infty g(1) \cdot \frac{1}{\zeta(3)} \cdot X = \frac{C(S_4)C_\infty}{\zeta(3)} X \tag{3.7}$$

Let $C_\infty^{\mathbb{R}}$ and $C_\infty^{\mathbb{C}}$ be the constants that arise when restricting to totally real and complex fields, respectively. From Davenport and Heilbronn [11], the number of non-cyclic cubic fields of discriminant $< X$ is asymptotic to $\frac{1}{12\zeta(3)}X$ for totally real fields, and $\frac{1}{4\zeta(3)}X$ for complex fields. We should thus have $C(S_4) \cdot C_\infty^{\mathbb{R}} = \frac{1}{12} \cdot \frac{1}{4} = \frac{1}{48}$ and $C(S_4) \cdot C_\infty^{\mathbb{C}} = \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{8}$, in order for the prediction of Heuristic 1.2 to give the correct asymptotic.

In fact, in this setup, inertia groups are only allowed to contain 2-cycles and 3-cycles, so of the 10 maps $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) \to \Gamma$, only 7 are allowed, and 1 of these 7 results in K being totally real. Thus $C_\infty^{\mathbb{R}} = \frac{1}{24}$ and $C_\infty^{\mathbb{C}} = \frac{1}{4}$. Both of these give $C(S_4) = \frac{1}{2}$, which is the same value needed to get the correct asymptotic for counting quartic fields, given Bhargava's results in [1].

3.6 A Conjecture on Mass Formulas

In each case we have studied in this chapter, we defined a counting function that would count surjections from the class group of a number field K to an abelian group A. We found, in all three cases, that this counting function had a universal mass formula if and only if the counting function corresponding to the standard discriminant of K had a universal mass formula. This suggests a broad conjecture, but first we must define the Cohen-Lenstra counting function in general.

Definition 3.8. Let G be a finite group, $H \subseteq G$ a subgroup containing no normal subgroup of G, and A any finite abelian group.

Let K be a number field with Galois closure K^g. Let G and H be the Galois groups of $K^g/\mathbb{Q}$ and K^g/K, respectively. Let L be an everywhere unramified A-extension of K, and let L^g be the Galois closure of L.

Suppose that under these circumstances, there is only one possible group Γ that can appear as $\mathrm{Gal}(L^g/\mathbb{Q})$. Then let $\widetilde{H}$ and $\widetilde{A}$ be the subgroups of Γ fixing K and L, respectively.

Define the *Cohen-Lenstra counting function* $c^{CL}_{G,H,A}$ to be the extended weighted discriminant counting function for Γ such that the corresponding alternate discriminant is ∞ if the fixed field of $\widetilde{A}$ is ramified at any place over the fixed field of $\widetilde{H}$, and otherwise is the discriminant of the fixed field of $\widetilde{H}$.

We omit the proof that this counting function can be defined by a weighted discriminant, but it is easy to see that assigning appropriate weights to a chain of maximal subgroups from Γ down to $\widetilde{H}$, and infinite weights where appropriate, will accomplish this.

Now let $c^{\mathrm{Disc}}_{G,H}$ be the counting function for G that corresponds to the standard discriminant of the fixed field of H. Then, as a first attempt, the conjecture suggested by the results of this chapter is:

Conjecture 3.9. *The total masses at each prime for $c^{CL}_{G,H,A}$ and $c^{\mathrm{Disc}}_{G,H}$ are the same.*

A weaker version of this conjecture, which would be a trivial corollary to Conjecture 3.9, is:

Conjecture 3.10. *$c^{CL}_{G,H,A}$ has a universal mass formula if and only if $c^{\mathrm{Disc}}_{G,H}$ does.*

Remark. Conjectures 3.9 and 3.10 only apply in situations where Γ is uniquely determined from G, H, and A, like the ones in this chapter. Extending them to cases where there is more than one possible Γ would require a significant expansion of our idea of a counting function. We would instead need an object that counts maps to several different groups simultaneously, along with a sensible notion of the total mass of such an object. It is not at all obvious how to construct this object in any elegant way.

Chapter 4

Mass Formulas For Finite Groups

In this chapter, we will consider several examples of groups Γ for which it is possible to list all natural weighted discriminant counting functions with universal mass formulas. In some cases, when Γ is an ℓ-group, we will use the bounds on overall weights given in the proof of Theorem 2.1. In other cases, and where Γ is not an ℓ-group, we will use more *ad hoc* methods. The calculation of the determining equations is done using the Jones-Roberts database of local fields [13] and accompanying paper [14].

In particular, we will show that:

Theorem 4.1. *If p and q are distinct odd primes, then*

- *The groups C_p, C_4, C_8, $C_4 \times C_2$, C_9, A_4, C_{2p}, D_p (for $p \geq 5$), $C_p \times C_q$, have no natural weighted discriminant counting function with a universal mass formula.*
- *The groups C_2, S_3, S_4, have exactly one such counting function, which corresponds to either the standard discriminant or the standard discriminant of some subfield.*
- *The groups $C_2 \times C_2$, Q_8, $(C_2)^3$, D_4 have exactly one such counting function (up to symmetry in some cases), which is different from the standard discriminant. For all of these groups except Q_8, this counting function is the same as the one guaranteed to exist by [26, Theorem 1.1].*

Throughout this chapter, p and q will denote distinct odd primes unless noted otherwise.

In each of the following sections, we study a hypothetical counting function with a universal mass formula. We find the determining equation in terms of the overall weights of this counting function; the nonnegative integer solutions to the determining equation correspond to the counting functions (if any) with a universal mass formula. When a group Γ has exactly one conjugacy class of cyclic subgroups of order n, w_n will denote the overall weight of this class.

4.1 Mass Formulas for D_4

We begin with the case of $\Gamma = D_4$, as it provides a good illustration of the techniques involved in computing overall weights and the determining equation. For most of the other groups we consider in this chapter, these computations are simpler, so we only summarize them. For D_4, however, we will lay out the calculation in full detail.

D_4 has four conjugacy classes of cyclic subgroups: two non-central copies of C_2, the center (also isomorphic to C_2), and one isomorphic to C_4. We denote the overall weights of these by w_{2a}, w_{2b}, w_{2c}, and w_4, respectively.

4.1.1 Overall Weights

First, we must compute the overall weights of the three noncyclic subgroups of D_4 (itself and two nonconjugate copies of V_4) in terms of w_{2a}, w_{2b}, w_{2c}, and w_4, using Proposition 2.11.

Let I be the copy of V_4 generated by the subgroups whose overall weights are w_{2a}

and w_{2c}. The $n = 1$ term in equation (2.2) contributes $\frac{1}{2}w_{2a} + \frac{1}{2}w_{2a} + \frac{1}{2}w_{2c}$ to $\bar{w}(I)$, since I contains two conjugate copies of C_2 whose overall weight is each w_{2a}. All other terms are zero, since the intersection of any two cyclic subgroups of I is trivial. Thus the overall weight of I is

$$\bar{w}(V_4) = w_{2a} + \frac{1}{2}w_{2c}$$

By the same argument, the overall weight of the other copy of V_4 is $\frac{1}{2}w_{2b} + \frac{1}{2}w_{2c}$.

Now let $I = D_4$. The $n = 1$ term in equation (2.2) contributes $\frac{2}{4}w_{2a} + \frac{2}{4}w_{2b} + \frac{1}{4}w_{2c} + \frac{1}{2}w_4$. The $n = 2$ term contributes $-\frac{1}{4}w_{2c}$, and all other terms are zero. Thus the overall weight of I is

$$\bar{w}(D_4) = \frac{1}{2}w_{2a} + \frac{1}{2}w_{2b} + \frac{1}{2}w_4$$

4.1.2 Extensions of $\mathbb{Q}_2$ and the Determining Equation

Now we use [13] to find all maps from $G_{\mathbb{Q}_2} \to D_4$. First, we list all field extensions of $\mathbb{Q}_2$ whose Galois group is a subgroup of D_4, with their (lower-numbered) ramification filtrations. The filtrations of the fields listed below begin with the inertia group.

- 2 C_2-extensions with filtration C_2, C_2
- 4 C_2-extensions with filtration C_2, C_2, C_2
- 1 V_4-extension with filtration C_2, C_2
- 1 C_4-extension with filtration C_2, C_2
- 2 V_4-extensions with filtration C_2, C_2, C_2
- 2 C_4-extensions with filtration C_2, C_2, C_2

- 4 V_4-extensions with filtration V_4, V_4, C_2, C_2
- 8 C_4-extensions with filtration C_4, C_4, C_4, C_2, C_2
- 2 D_4-extensions with filtration V_4, V_4
- 2 D_4-extensions with filtration V_4, V_4, C_2, C_2
- 2 D_4-extensions with filtration C_4, C_4, C_4, C_2, C_2
- 8 D_4-extensions with filtration $D_4, D_4, V_4, V_4, C_2, C_2$
- 4 D_4-extensions with filtration $D_4, D_4, C_4, C_4, C_2, C_2, C_2, C_2$

In each case where the Galois group is D_4, the C_2's in the ramification filtration are the center of the group.

Each of these extensions corresponds to several maps $G_{\mathbb{Q}_2} \to D_4$. The number of maps per extension is the number of injections from the Galois group of the extension into D_4. Note that several extensions have the same ramification filtration; since our counting functions do not depend on the Galois group, we will sort the corresponding maps together when listing maps $G_{\mathbb{Q}_2} \to D_4$.

Now considering just the maps to D_4, we have:

- 24 with ramification filtration C_2, C_2
- 48 with filtration C_2, C_2, C_2
- 48 with filtration V_4, V_4, C_2, C_2

The maps above are evenly divided in terms of which conjugacy class of C_2 the ramification groups land in. We also have:

- 16 maps with filtration V_4, V_4
- 16 with filtration V_4, V_4, C_2, C_2
- 32 with filtration C_4, C_4, C_4, C_2, C_2
- 32 with filtration $D_4, D_4, C_4, C_4, C_2, C_2, C_2, C_2$
- 64 with filtration $D_4, D_4, V_4, V_4, C_2, C_2$

In the second list, all copies of C_2 are required to be the center of D_4. In both lists, any V_4's in the ramification filtration occur equally often as each copy of V_4.

Theorem 2.16 allows us to calculate one side of the determining equation, and the above list gives the other side. The determining equation is

$$\begin{aligned} & 2^{-w_{2a}} + 2^{-w_{2b}} + 2^{-w_{2c}} + 2^{-w_4} && (4.1) \\ & \quad = 2^{-2w_{2a}} + 2^{-2w_{2b}} + 2^{-2w_{2c}} + 2 \cdot 2^{-3w_{2a}} + 2 \cdot 2^{-3w_{2b}} + 2 \cdot 2^{-3w_{2c}} \\ & \quad + 2 \cdot 2^{-(3w_{2a}+w_{2c})} + 2 \cdot 2^{-(3w_{2b}+w_{2c})} + 2 \cdot 2^{-(2w_{2a}+2w_{2c})} + 2 \cdot 2^{-(2w_{2b}+2w_{2c})} \\ & \quad + 2^{-(2w_{2a}+w_{2c})} + 2^{-(2w_{2b}+w_{2c})} + 4 \cdot 2^{-(3w_4+w_{2c})} \\ & \quad + 4 \cdot 2^{-(w_{2a}+w_{2b}+2w_4+w_{2c})} + 4 \cdot 2^{-(2w_{2a}+w_{2b}+w_{2c}+w_4)} + 4 \cdot 2^{-(w_{2a}+2w_{2b}+w_{2c}+w_4)} && (4.2) \end{aligned}$$

4.1.3 Results

Following the proof of Theorem 2.1, let m be the largest of w_{2a}, w_{2b}, w_{2c}, and w_4. The valuation of the left side of the determining equation is at least $-m$. The right side is a sum of 16 powers of 2, and one of them has valuation at most $-3m$, so the right side cannot have valuation larger than $-3m+15$. Thus $-m \leq -3m+15$, from which $m \leq 7$.

An exhaustive search (carried out using Sage) shows that the only positive integer solution to the defining equation is $w_{2a} = w_{2b} = 1$ and $w_{2c} = w_4 = 2$. The corresponding counting function is the same as the one given by Wood in [26], which comes from the wreath product structure of $D_4 \simeq C_2 \wr C_2$.

4.2 Mass Formulas for C_2

The determining equation for C_2 is

$$2^{-w_2} = 2^{-2w_2} + 2 \cdot 2^{-3w_2} \tag{4.3}$$

Let $x = 2^{-w_2}$; then this becomes $x = x^2 + 2x^3$, which has solutions $x = 0$, $x = -1$, and $x = \frac{1}{2}$. Thus the only nonnegative integer solution for w_2 is $w_2 = 1$, which corresponds to D_w being the standard discriminant for C_2.

4.3 Mass Formulas for C_p

The determining equation for C_p is

$$p^{-w_p} = (p-1)p^{-2w_p} \tag{4.4}$$

This simplifies to $1 = (p-1)p^{-w_p}$, which has no solutions because $p - 1$ cannot be a power of p.

4.4 Mass Formulas for C_4

The determining equation is

$$2^{-w_2} + 2^{-w_4} = 2^{-2w_2} + 2 \cdot 2^{-3w_2} + 4 \cdot 2^{-(w_2+3w_4)} \tag{4.5}$$

In the language of the proof of Theorem 2.1, we have $t = 7$, so w_2 and w_4 are at most 6, but we can do slightly better. In fact, if $w_2 \geq 2$ and $w_2 \geq w_4$, then the 2-adic valuation of the left side is greater than or equal to $-w_2$, and the right side is less than or equal to $-3w_2+3$, which is less than $-w_2$. If $w_4 \geq 3$ and $w_4 \geq w_2$, then the valuation of the left side is greater than or equal to $-w_4$, and the right side is less than or equal to $-3w_4 + 4$, which is less than $-w_4$.

This leaves only seven pairs (w_2, w_4) to check, and none of them satisfy the determining equation. It is interesting to note, however, that $w_2 = \frac{1}{2}$ and $w_4 = 0$ satisfies the determining equation, even though the resulting counting function is not natural.

4.5 Mass Formulas for $V_4 = C_2 \times C_2$

Let x, y, and z be the overall weights of the three cyclic subgroups of V_4. The determining equation is

$$\begin{aligned} 2^{-x} + 2^{-y} + 2^{-z} = {} & 2^{-2x} + 2^{-2y} + 2^{-2z} + 2 \cdot 2^{-3x} + 2 \cdot 2^{-3y} + 2 \cdot 2^{-3z} \\ & + 2 \cdot 2^{-(2x+y+z)} + 2 \cdot 2^{-(x+2y+z)} + 2 \cdot 2^{-(x+y+2z)} \end{aligned} \tag{4.6}$$

Without loss of generality, assume $x \leq y \leq z$, so the 2-adic valuation of the left side is at least $-z$. The 2-adic valuation of the right side is at most $-3z + 9$, and if $z \geq 5$, this is impossible.

A brute-force search finds that the only solution is $(x, y, z) = (1, 1, 2)$ and its permutations. The alternate discriminant given by these overall weights is different from the standard discriminant, which has $(x, y, z) = (2, 2, 2)$.

This counting function is the same, however, as the one given by Kedlaya [15] (and restated in [26]) for a direct product of groups. Specifically, let π_x and π_y be the projection maps from V_4 onto the subgroups with overall weights x and y, and let c_2 be the counting function for C_2 with a universal mass formula (given in Section 4.2). Let c_4 be the counting function described above for V_4. Then if $\phi : G_{\mathbb{Q}_p} \to V_4$ for any prime p, we have

$$c_4(\phi) = c_2(\pi_x \circ \phi) + c_2(\pi_y \circ \phi)$$

4.6 Mass Formulas for C_6

We now come to our first non-ℓ-group.

Remark. The argument in Section 4.16 for $\Gamma = C_{2p}$ applies to C_6 as well; we introduce this case separately to illustrate explicitly how the problem of finding mass formulas changes when Γ is not an ℓ-group.

When Γ is not an ℓ-group, there is one determining equation for each prime dividing $|\Gamma|$, and not every overall weight must appear in every determining equation. In fact, the overall weights appearing on the left side of the determining equation for ℓ include at least those of cyclic subgroups of order a power of ℓ, and exclude those of order not divisible by ℓ. This is because if $\ell \nmid |C|$, then any map $G_{\mathbb{Q}_\ell} \to \Gamma$ with inertia group C is tamely ramified, and the tame mass formula automatically gives the correct number of such maps. This causes the $\ell^{-\bar{w}(C)}$ term to drop out of the left side of the determining equation.

It is also possible that $\ell^{-\bar{w}(C)}$ will not appear on the left side of the determining equation for ℓ even if ℓ divides $|C|$, and in fact, this happens for C_6. To account for this, we will need to explicitly write down the tame mass formula in some cases where Γ is not an ℓ-group.

Let χ be the nontrivial multiplicative character modulo 3, so $\chi(x)$ is 1 if $x \equiv 1 \bmod 3$, -1 if $x \equiv 2 \bmod 3$, and 0 otherwise. Then the tame mass formula is

$$f(p) = 1 + p^{-w_2} + (1 + \chi(p))p^{-w_3} + (1 + \chi(p))p^{-w_6} \tag{4.7}$$

and the determining equations are

$$2^{-w_2} = 2^{-2w_2} + 2 \cdot 2^{-3w_2} \tag{4.8}$$

$$3^{-w_3} + 3^{-w_6} = 2 \cdot 3^{-2w_3} + 2 \cdot 3^{-(w_3+w_6)} \tag{4.9}$$

Note that 2^{-w_6} does not appear in the first equation. This is because the existence of a totally ramified C_6-extension of $\mathbb{Q}_p$ depends only on the residue of p mod 3, not mod 6. Since $2 \nmid 3$, these extensions in a sense "behave like tame extensions" of $\mathbb{Q}_2$.

As in Section 4.2, the only solution to the first determining equation is $w_2 = 1$. Letting $x = 3^{-w_3}$ and $y = 3^{-w_6}$, the second equation becomes $x + y = 2x^2 + 2xy$; solving for y gives either $x = \frac{1}{2}$ or $y = -x$. Neither of these is possible, so there is no solution.

4.7 Mass Formulas for S_3

The determining equations are

$$2^{-w_2} = 2^{-2w_2} + 2 \cdot 2^{-3w_2} \tag{4.10}$$

$$3^{-w_3} = 2 \cdot 3^{-2w_3} + 2 \cdot 3^{-(w_2+w_3)} + 3 \cdot 3^{-(w_2+2w_3)} \tag{4.11}$$

From section 4.2, we already know that w_2 must equal 1. Then setting $x = 3^{-w_3}$, the second determining equation becomes

$$\frac{1}{3} + x = \frac{1}{3} + 2x^2 + \frac{2}{3}x + x^2$$

This gives $x = \frac{1}{9}$ and $w_3 = 2$, which is the standard discriminant of an S_3 cubic field.

4.8 Mass Formulas for C_8

The determining equation is

$$2^{-w_2} + 2^{-w_4} + 2^{-w_8} = 2^{-2w_2} + 2 \cdot 2^{-3w_2} + 4 \cdot 2^{-(w_2+3w_4)} + 8 \cdot 2^{-(w_2+w_4+3w_8)} \tag{4.12}$$

A slightly refined use of the bound from Theorem 2.1 shows that none of the overall weights can be greater than 3, and a search of the 64 possible triples (w_2, w_4, w_8) yields no solution.

4.9 Mass Formulas for $C_4 \times C_2$

$C_4 \times C_2$ has 5 cyclic subgroups: two isomorphic to C_4, one isomorphic to C_2 contained in both C_4's, and two isomorphic to C_2 not contained in either C_4. Let a, b, c, d, and

e be the overall weights of these subgroups, in that order. The determining equation is then

$$
\begin{aligned}
&2^{-a} + 2^{-b} + 2^{-c} + 2^{-d} + 2^{-e} && (4.13)\\
&= 2^{-2c} + 2 \cdot 2^{-3c} + 2^{-2d} + 2 \cdot 2^{-3d} + 2^{-2e} + 2 \cdot 2^{-3e} \\
&+ 3 \cdot 2^{-(c+2d+e)} + 3 \cdot 2^{-(c+d+2e)} \\
&+ 4 \cdot 2^{-(3a+c)} + 4 \cdot 2^{-(3b+c)} \\
&+ 4 \cdot 2^{-\left(2a+b+\frac{1}{2}c+\frac{1}{2}d+\frac{1}{2}e\right)} + 4 \cdot 2^{-\left(a+2b+\frac{1}{2}c+\frac{1}{2}d+\frac{1}{2}e\right)} && (4.14)
\end{aligned}
$$

Let m be the largest of a, b, c, d, and e. The left side has valuation at least $-m$. The right side can be written as a sum of 14 powers of 2, one of which has valuation less than or equal to $-3m+2$, so its valuation is at most $-3m+15$. Thus $-m \leq -3m+15$, from which $m \leq 7$.

Searching, again using Sage, gives no solutions to the defining equation.

4.9.1 Counterexamples

This group provides several examples of unexpected behavior in the overall weights of subgroups. First, the overall weight of the full group $C_4 \times C_2$ is

$$\frac{1}{2}a + \frac{1}{2}b - \frac{1}{4}c + \frac{1}{4}d + \frac{1}{4}e$$

This shows that even if the overall weights of cyclic subgroups are all nonnegative integers, the overall weights of other subgroups need not be integers, or even nonnegative. Second, if w is a weight function with these overall weights, then there are maps

$\phi : G_{\mathbb{Q}_2} \to C_4 \times C_2$ (with discriminant 2^{24}, listed in [13]) with

$$c_w(\phi) = 2a + b + \frac{1}{2}c + \frac{1}{2}d + \frac{1}{2}e$$

Thus even if the overall weights of cyclic subgroups are all integers, the counting function c_w is not guaranteed to be natural.

4.10 Mass Formulas for $(C_2)^3$

$(C_2)^3$ has 7 cyclic subgroups; let w_γ be the overall weight of the subgroup $\langle\gamma\rangle$ for $1 \neq \gamma \in (C_2)^3$. The determining equation is then

$$\sum_{\gamma} 2^{-w_\gamma} = \sum_{\gamma} \left(2^{-2w_\gamma} + 2 \cdot 2^{-3w_\gamma}\right) + 2 \sum_{\{\gamma_1,\gamma_2\}} 2^{-(w_{\gamma_1} + w_{\gamma_2} + 2w_{\gamma_1\gamma_2})} \tag{4.15}$$

where the sum on the right is over unordered pairs $\{\gamma_1, \gamma_2\}$.

Let m be the maximum value of any w_γ, so the valuation of the left side is greater than or equal to $-m$. The right side is a sum of 35 powers of 2, one of which is 2^{-3m+1}, so its valuation is at most $-3m + 35$. This implies $m \leq 17$.

An exhaustive search (using Sage) shows that the only solution to the determining equation is when the 7 overall weights of cyclic subgroups are a permutation of $(1, 1, 1, 2, 2, 2, 3)$. This does not generate the standard discriminant, which would have overall weights $(4, 4, 4, 4, 4, 4, 4)$. As in Section 4.5, it is Kedlaya's direct product counting function.

4.11 Mass Formulas for Q_8

The quaternion group Q_8 has one subgroup isomorphic to C_2 (its center); let w_2 be its overall weight. It has three nonconjugate subgroups isomorphic to C_4; let a, b, and c be their overall weights. The determining equation is

$$\begin{aligned} 2^{-w_2} + 2^{-a} + 2^{-b} + 2^{-c} = 2^{-2w_2} + 2 \cdot 2^{-3w_2} + 4 \cdot 2^{-(w_2+3a)} + 4 \cdot 2^{-(w_2+3b)} \\ + 4 \cdot 2^{-(w_2+3c)} + 4 \cdot 2^{-(2a+b+c)} + 4 \cdot 2^{-(a+2b+c)} + 4 \cdot 2^{-(a+b+2c)} \end{aligned} \tag{4.16}$$

Let m be the largest of w_2, a, b, and c. Let M be the largest of the exponents that appears on the right side of this equation. Then we have $M \geq 3m$. The 2-adic valuation of the left side is at least $-m$, and the valuation of the right hand side is at most $-M + 9$. Therefore, we must have $-m \leq -M + 9 \leq -3m + 9$, and hence $m \leq 4$ (since m is integer-valued).

Another exhaustive Sage search shows that $w_2 = a = b = c = 1$ is the only integer solution to the determining equation, so this gives the only weighted discriminant counting function for Q_8 with a universal mass formula.

Remark. There is no integer-valued weight function that produces this counting function. One non-integer valued weight function that produces it, taking the standard presentation of Q_8 in terms of generators $\{i, j, k\}$, is $w(\langle i \rangle, \pm 1) = w(\langle j \rangle, \pm 1) = w(\langle k \rangle, \pm 1) = \frac{1}{4}$, $w(\pm 1, 1) = 1$, and all other weights equal to 0. Nonetheless, this counting function is natural. The existence of counting functions like this one is the reason why we consider natural counting functions, as opposed to positive-integer-valued weight functions.

4.12 Mass Formulas for C_9

The determining equation for C_9 is

$$3^{-w_3} + 3^{-w_9} = 2 \cdot 3^{-2w_3} + 6 \cdot 3^{-(w_3+2w_9)} \tag{4.17}$$

Let m be the larger of w_3 and w_9, so the left side has 3-adic valuation at least $-m$. The right side has 3-adic valuation at most $-2m+3$, so $m \leq 2$. Checking all nine possibilities for (w_2, w_3) yields no integer solution.

4.13 Mass Formulas for A_4

A_4 has two conjugacy classes of cyclic subgroups: one isomorphic to C_2 and one to C_3, with overall weights w_2 and w_3, respectively. The intersection of any two cyclic subgroups is trivial, so equation (2.2) easily gives the overall weights of the other two subgroups of A_4 as $\bar{w}(C_2 \times C_2) = \frac{3}{2}w_2$ and $\bar{w}(A_4) = \frac{1}{2}w_2 + w_3$.

If χ is the nontrivial Dirichlet character modulo 3, then the tame mass formula is

$$f(p) = 1 + p^{-w_2} + (1 + \chi(p))p^{-w_3} \tag{4.18}$$

and the determining equations are:

$$2^{-w_2} = 2^{-2w_2} + 4 \cdot 2^{-3w_2} + 2 \cdot 2^{-4w_2} \tag{4.19}$$

$$3^{-w_3} = 2 \cdot 3^{-2w_3} \tag{4.20}$$

Neither of these equations have integer solutions for w_2 or w_3.

4.14 Mass Formulas for S_4

S_4 has two different conjugacy classes of subgroups isomorphic to C_2. Let w_{2a} be the overall weight of the class of 2-cycles, and let w_{2b} be the overall weight of the class of products of 2-cycles. Also, let V_{4a} denote a non-normal subgroup of S_4 isomorphic to $C_2 \times C_2$ and V_{4b} denote a normal subgroup isomorphic to $C_2 \times C_2$ (i.e. the subgroup consisting of all products of 2-cycles).

Using Proposition 2.11, we can determine the overall weights of the other conjugacy classes of subgroups of S_4:

$$\begin{aligned}
\bar{w}(V_{4a}) &= w_{2a} + \frac{1}{2}w_{2b} \\
\bar{w}(V_{4b}) &= \frac{3}{2}w_{2b} \\
\bar{w}(D_4) &= \frac{1}{2}w_{2b} + \frac{1}{2}w_{2a} + \frac{1}{2}w_4 \\
\bar{w}(A_4) &= \frac{1}{2}w_{2b} + w_3 \\
\bar{w}(S_3) &= w_{2a} + w_3 \\
\bar{w}(S_4) &= \frac{1}{2}w_{2a} + \frac{1}{2}w_3 + \frac{1}{2}w_4
\end{aligned}$$

Since S_4 has a rational character table, the tame mass formula is $1 + p^{-w_{2a}} + p^{-w_{2b}} + p^{-w_3} + p^{-w_4}$. Using the information in [13], the determining equation at 3 is:

$$3^{-w_3} = 2 \cdot 3^{-2w_3} + 2 \cdot 3^{-(w_{2a}+w_3)} + 3 \cdot 3^{-(w_{2a}+2w_3)} \tag{4.21}$$

Letting $s = 3^{-w_3}$ and $t = 3^{-w_{2a}}$ and simplifying, this becomes

$$1 = 2s + 2t + 3st$$

Multiplying both sides by 3 and using a factoring trick gives:

$$(3s+2)(3t+2) = 7$$

Both factors on the left are rational numbers with denominator a power of 3, so one of them must in fact be an integer. If it is the first factor, then $s = 1$ or $s = \frac{1}{3}$ ($w_3 = 0, 1$). In the former case, $3t + 2 = \frac{7}{5}$, which is impossible. In the latter case, $3t + 2 = \frac{7}{3}$, so $t = \frac{1}{9}$ (and $w_{2a} = 2$). Thus the only solutions to this equation are $w_3 = 1$ and $w_{2a} = 2$, or $w_{2a} = 1$ and $w_3 = 2$.

Using the information in [13], the determining equation at 2 is:

$$\begin{aligned} 2^{-w_{2a}} + 2^{-w_{2b}} + 2^{-w_4} = 2^{-2w_{2a}} &+ 2^{-2w_{2b}} + 2 \cdot 2^{-3w_{2a}} + 2 \cdot 2^{-3w_{2b}} \\ &+ 2^{-(2w_{2a}+w_{2b})} + 2 \cdot 2^{-3w_{2b}} + 2 \cdot 2^{-(3w_{2a}+w_{2b})} + 2 \cdot 2^{-(2w_{2a}+2w_{2b})} \\ &+ 2 \cdot 2^{-4w_{2b}} + 4 \cdot 2^{-(3w_4+w_{2b})} + 2^{-(w_{2b}+w_3)} + 2 \cdot 2^{-(3w_{2b}+w_3)} \\ &+ 4 \cdot 2^{-(2w_{2a}+2w_{2b}+w_4)} + 4 \cdot 2^{-(w_{2a}+3w_{2b}+w_4)} + 4 \cdot 2^{-(w_{2a}+2w_{2b}+2w_4)} \end{aligned} \tag{4.22}$$

In this equation, the first line of the right side comes from extensions of $\mathbb{Q}_2$ with inertia group C_2. The second and third lines contain terms for extensions with inertia group $C_2 \times C_2$, C_4, and A_4, and the last line contains terms for inertia group D_4.

Following the proof of 2.1, let m be the largest of w_{2a}, w_{2b}, and w_4. The left side of the determining equation has valuation $\geq -m$. The right side is a sum of 15 powers of 2, one of which has valuation $\leq -3m + 2$. If this equation is satisfied, we then have $-m \leq -3m + 2 + 14$, so $m \leq 8$.

An exhaustive search, aided by the restrictions on w_3 and w_{2a} from the other determining equation, gives

$$(w_{2a}, w_{2b}, w_3, w_4) = (1, 2, 2, 3)$$

as the only solution, which corresponds to the standard discriminant of an S_4 quartic number field.

4.15 Mass Formulas for D_p ($p \geq 5$)

D_p has two conjugacy classes of subgroups: one isomorphic to C_p and one to C_2. The calculations in [14, Proposition 2.3.1] give p totally ramified degree-p extensions of $\mathbb{Q}_p$ with Galois group C_p. There are also 3 degree-p extensions whose Galois closure has Galois group D_p; one of these has inertia group C_p, and the others have inertia group D_p. This information is enough to compute the total mass at p and thus the determining equations:

$$2^{-w_2} = 2^{-2w_2} + 2 \cdot 2^{-3w_2} \tag{4.23}$$

$$p^{-w_p} = (p-1)p^{-2w_p} + (p-1)p^{-(w_2+w_p)} \tag{4.24}$$

The first equation, as in Section 4.2, implies that $w_2 = 1$. Then, letting $x = p^{-w_p}$, the second equation simplifies to $x = (p-1)x^2 + \frac{p-1}{p}x$. The only solutions to this are $x = 0$ and $x = \frac{1}{p(p-1)}$, neither of which gives a positive integer value of w_p.

4.16 Mass Formulas for C_{2p}

Since $C_{2p} \simeq C_2 \times C_p$, let α and δ be the projection maps from C_{2p} onto C_2 and C_p. Then for any $\phi : G_{\mathbb{Q}_\ell} \to C_{2p}$, we have $\phi = (\alpha \circ \phi) \times (\delta \circ \phi)$. We can use this to identify all such ϕ.

First, we determine the tame mass formula. If $\ell \equiv 1 \bmod p$, there are p ramified C_p-extensions of $\mathbb{Q}_\ell$, and thus $p(p-1)$ ramified maps from $G_{\mathbb{Q}_\ell} \to C_p$. If $\ell \not\equiv 1 \bmod p$, there are only the p unramified maps to C_p. Either way, there are 4 maps $G_{\mathbb{Q}_\ell} \to C_2$, two of which are ramified. Putting these together, we get that the mass at $\ell \neq 2, p$ is

$$\begin{cases} 1 + \ell^{-w_2} + (p-1)\ell^{-w_p} + (p-1)\ell^{-w_{2p}} & \text{if } \ell \equiv 1 \bmod p \\ 1 + \ell^{-w_2} & \text{otherwise} \end{cases}$$

Thus if $\chi_1, \ldots, \chi_{p-2}$ are the nontrivial Dirichlet characters modulo p, the tame mass formula is

$$1 + \ell^{-w_2} + \left(1 + \sum_{n=1}^{p-2} \chi_n(\ell)\right)\left(\ell^{-w_p} + \ell^{-w_{2p}}\right) \tag{4.25}$$

For $\ell = 2$, there are no nontrivial maps $G_{\mathbb{Q}_2} \to C_p$, since $2 \not\equiv 1 \bmod p$. The maps $G_{\mathbb{Q}_2} \to C_2$ are the same as the ones accounted for in Section 4.2, so the determining equation for 2 is

$$2^{-w_2} = 2^{-2w_2} + 2 \cdot 2^{-3w_2} \tag{4.26}$$

which has the solution $w_2 = 1$.

For $\ell = p$, there are $p(p-1)$ ramified maps $G_{\mathbb{Q}_p} \to C_p$, and there are still 2 ramified and 2 unramified maps to C_2. This gives $2p(p-1)$ maps to C_{2p} with inertia group C_p, and $2p(p-1)$ totally ramified maps. The former have wild inertia also equal to C_p, and all higher ramification groups are trivial. To compute the ramification groups of the totally ramified maps, we will use a trick involving discriminants of the corresponding extensions of $\mathbb{Q}_p$.

Let ϕ be a totally ramified surjective map from $G_{\mathbb{Q}_p} \to C_p$. ϕ corresponds to a local field extension $K/\mathbb{Q}_p$, which has a degree-p subfield L. All such $L/\mathbb{Q}_p$ have discriminant p^{2p-2}. Then K/L is a tamely ramified quadratic extension, so the norm to $\mathbb{Q}_p$ of

$\mathrm{Disc}(K/L)$ is p. Thus

$$\mathrm{Disc}(K/\mathbb{Q}_p) = \mathrm{Disc}(L/\mathbb{Q}_p)^2 \cdot N_{L/\mathbb{Q}_p}\,\mathrm{Disc}(K/L) = p^{4p-3}$$

The inertia group of $K/\mathbb{Q}_p$ is $I_0 = C_{2p}$, the wild inertia group is $I_1 = C_p$, and we must have $I_1 = \ldots = I_n = C_p$ and $I_{n+1} = 1$ for some n. Then by the formula given in [25] for the discriminant,

$$4p - 3 = \sum_{j=0}^{\infty} (|I_j| - 1) = 2p - 1 + n(p-1)$$

from which $n = 2$. Thus

$$c(\phi) = w_{2p} + w_p$$

We can now compute the determining equation for p:

$$p^{-w_p} + p^{-w_{2p}} = (p-1)p^{-2w_p} + (p-1)p^{-(w_{2p}+w_p)} \tag{4.27}$$

Let $x = p^{-w_p}$ and $y = p^{-w_{2p}}$. Then the determining equation becomes

$$x + y = (p-1)x^2 + (p-1)xy$$

or, rearranging:

$$x(1-(p-1)x) = y((p-1)x - 1)$$

Since $x \neq \frac{1}{p-1}$, this implies $x = -y$, which is impossible as x and y are both positive.

Thus there is no solution to the determining equations, and no universal mass formula.

4.17 Mass Formulas for C_{pq}

Assume that $p < q$. As in Section 4.16, any map to C_{pq} is the direct product of maps to C_p and C_q. We can use this to find the tame mass formula and the total masses at p

and q, and thus the determining equations.

For $\ell \neq p$, there are p totally ramified C_p-extensions of $\mathbb{Q}_\ell$, and thus $p(p-1)$ totally ramified maps from $G_{\mathbb{Q}_\ell} \to C_p$, if $\ell \equiv 1 \bmod p$, and none otherwise. This implies that there are $pq(p-1)(q-1)$ totally ramified maps from $G_{\mathbb{Q}_\ell} \to C_{pq}$ if $\ell \equiv 1 \bmod pq$, and none otherwise. Also, there are $pq(p-1)$ maps from $G_{\mathbb{Q}_\ell} \to C_{pq}$ with inertia group C_p and $pq(q-1)$ with inertia group C_q if ℓ is 1 modulo p and q, respectively.

Let $\chi_1, \dots, \chi_{p-2}$ be the nontrivial Dirichlet characters modulo p, and $\psi_1, \dots, \psi_{q-2}$ be the nontrivial Dirichlet characters modulo q. Then the tame mass formula is:

$$1+\left(1+\sum_i \chi_i(\ell)\right)\ell^{-w_p}+\left(1+\sum_j \psi_j(\ell)\right)\ell^{-w_q}+\left(1+\sum_i \chi_i(\ell)\right)\left(1+\sum_j \psi_j(\ell)\right)\ell^{-w_{pq}} \tag{4.28}$$

Since $p < q$, $p \neq 1 \bmod q$, so the only C_q-extension of $\mathbb{Q}_p$ is the unramified one, which gives q maps $G_{\mathbb{Q}_p} \to C_q$. There are p totally ramified C_p extensions of $\mathbb{Q}_p$, which give $p(p-1)$ ramified maps $G_{\mathbb{Q}_p} \to C_p$. Combining these, we get $pq(p-1)$ ramified maps $G_{\mathbb{Q}_p} \to C_{pq}$ (none of which are totally ramified), so the determining equation for p is

$$p^{-w_p} = (p-1)p^{-2w_p} \tag{4.29}$$

This has no solution, as in Section 4.3, so there is no universal mass formula. For completeness, we compute the determining equation for q anyway.

If $q \neq 1 \bmod p$, then the determining equation for q is the same as that for p (with p changed to q). If $q \equiv 1 \bmod p$, then there are p totally ramified C_p-extensions of $\mathbb{Q}_q$, and thus $p(p-1)$ totally ramified maps $G_{\mathbb{Q}_q} \to C_p$. Combining these with the totally ramified maps to C_q, we get $pq(p-1)(q-1)$ totally ramified maps $G_{\mathbb{Q}_q} \to C_{pq}$.

We use the same method as in Section 4.16 to compute the ramification groups of these maps. Such a map corresponds to a totally ramified extension $K/\mathbb{Q}_q$, which has a degree-q subfield L. $L/\mathbb{Q}_q$ is a totally ramified C_q-extension of $\mathbb{Q}_q$, which must have (standard) discriminant q^{2q-2}. K/L is degree p and tamely ramified, so the norm (to $\mathbb{Q}_q$) of its discriminant is q^{p-1}. Thus

$$\mathrm{Disc}(K/\mathbb{Q}_q) = (\mathrm{Disc}(L/\mathbb{Q}_q))^p \cdot N_{L/\mathbb{Q}_q} \mathrm{Disc}\, K/L = q^{2pq-2p+p-1} = q^{2pq-p-1}$$

If I_n denotes the nth ramification group of $K/\mathbb{Q}_q$, then by the formula given for the discriminant in [25], we have

$$2pq - p - 1 = \sum_{n=0}^{\infty} (|I_n| - 1) = pq - 1 + (q-1) \cdot n_{\max}$$

where $n_{\max}$ is the largest n for which I_n is nontrivial. Clearly, then, $n_{\max} = p$, so $I_0 = pq$, $I_n = C_q$ for $1 \le n \le p$, and I_n is trivial for $n > p$.

Taking into account the maps with inertia group C_q, we can then obtain the determining equation:

$$q^{-w_q} + (p-1)q^{-w_{pq}} = (q-1)q^{-2w_q} + (p-1)(q-1)q^{-(w_{pq}+w_q)} \tag{4.30}$$

Remark. If we allow non-integer weights, setting $w_p = \log_p(p-1)$ and $w_q = \log_q(q-1)$ gives a universal mass formula, regardless of whether or not $q \equiv 1 \bmod p$. Even more remarkably, this does not depend at all on the value of w_{pq}! A similar non-natural counting function also arises in Section 4.16 for $\Gamma = C_{2p}$.

4.18 Mass Formulas for $C_p \times C_p$

Choose γ_0, γ_1 such that $C_p \times C_p = \langle \gamma_0, \gamma_1 \rangle$, and let α and δ be the projection maps from $C_p \times C_p$ onto the subgroups generated γ_0 and γ_1, respectively.

$C_p \times C_p$ has $p+1$ subgroups isomorphic to C_p; let $a_0, a_1, \ldots, a_p$ be their overall weights.

As in Section 4.16, any map $\phi : G_{\mathbb{Q}_\ell} \to C_p \times C_p$ is uniquely specified by $\alpha \circ \phi$ and $\delta \circ \phi$. From [14], there are p ramified C_p-extensions of C_p, which means there are $p(p-1)$ ramified maps $G_{\mathbb{Q}_p} \to C_p$ and p unramified maps. Putting these together, we get p^2 unramified maps to $C_p \times C_p$, and $p^2(p-1)^2 + 2p^2(p-1) = p^2(p-1)(p+1)$ ramified maps.

Of these ramified maps, none of them can be totally ramified. A totally ramified map would correspond to a totally ramified $C_p \times C_p$-extension of $\mathbb{Q}_p$, which would have $p+1$ distinct subfields, all of which would be totally ramified C_p-extensions of $\mathbb{Q}_p$. However, there are only p such extensions, so this is impossible.

Each ramified map thus has inertia group C_p, and by symmetry, the inertia group will land in each of the $p+1$ subgroups of $C_p \times C_p$ equally often. Furthermore, this implies (either by choosing a different basis for $C_p \times C_p$, or by the discriminant trick of Section 4.16) that each map has only inertia and wild inertia groups, and no higher ramification groups.

From this information, we can compute the determining equation:

$$\sum_{n=0}^{p} p^{-a_n} = (p-1)\sum_{n=0}^{p} p^{-2a_n} \tag{4.31}$$

We could use Theorem 2.1 to put bounds on a_n, but the symmetry of the determining equation allows a nicer method to show it has no integer solutions. Let $x_n = p^{-a_n}$, and rearrange the determining equation to get:

$$\sum_{n=0}^{p} x_n(1-(p-1)x_n) = 0 \tag{4.32}$$

If $a_n \geq 1$, then $x_n(1-(p-1)x_n) \geq 0$, so we must have $a_n = 0$ for some n.

Assume, without loss of generality, that $a_0 = 0$. Then we have

$$\sum_{n=1}^{p} x_n(1 - (p-1)x_n) = p - 2 \tag{4.33}$$

If $a_n = 0$, then $x_n(1 - (p-1)x_n) < 0$. Otherwise, $x_n \leq \frac{1}{p}$ and $1 - (p-1)x_n < 1$, so $x_n(1 - (p-1)x_n) < \frac{1}{p}$. Thus

$$\sum_{n=1}^{p} x_n(1 - (p-1)x_n) < sum_{n=1}^{p} \frac{1}{p} = 1 \leq p - 2 \tag{4.34}$$

Thus there are no solutions to the determining equation.

Chapter 5

Further Questions

We conclude, in this chapter, with a discussion of some other questions that arise from this work.

5.1 Existence of Universal Mass Formulas

Theorem 2.1 gives a bound on the number of weighted discriminant counting functions for any given ℓ-group that can have universal mass formulas. The first natural folllow-up question is whether or not Theorem 2.1 extends to all finite groups. As we saw in Chapter 4, removing the condition that Γ be an ℓ-group complicates the problem significantly, because we now have more than one determining equation, and not every overall weight appears in every determining equation. Nonetheless, the results from Chapter 4 suggest that similar techniques may be viable for extending Theorem 2.1 to non-ℓ-groups.

We could also ask what happens over a number field $K \neq \mathbb{Q}$. In this setup, $\mathbb{Q}_p$ in the definition of the total mass would be replaced by local completions of K, with the size of the residue field replacing p. Extending Theorem 2.1 to this case should be less difficult; most of the proof remains valid, although certain parts (such as Lemma 2.18) will require modification.

Based on Chapter 4, it also appears that universal mass formulas may be even more rare than the finiteness given by Theorem 2.1. In fact, we have not yet found any group for which more than one weighted discriminant counting function has a universal mass formula. We cannot see any reason why no such group should exist, but we would also expect one to be very difficult to find.

Furthermore, all of the universal mass formulas we have found are for groups with rational character tables, and the only ones we have found for the standard discriminant have been when Γ is a symmetric group. It would be interesting to know if either of these holds true more generally.

In contrast to Theorem 2.1, Wood's work [26, Theorem 1.1] guarantees the existence of at least one counting function with a universal mass formula for an infinite class of Γ (the symmetric groups, and those built up from them using repeated cross products and wreath products). Every group for which we have found a universal mass formula is in this class, except for the quaternion group Q_8. This raises the question of whether all groups with universal mass formulas can be classified in some way. We suspect they cannot, but only because we know of no particular property of Q_8 that causes it to have a universal mass formula.

5.2 Counting Functions from Artin Conductors

As we alluded to in Section 1.2, we can define a counting function for Γ given any linear representation of Γ, using the Artin conductor.

Specifically, let ρ be any representation of Γ and f be the Artin conductor. Then for a map $\phi : G_{\mathbb{Q}_p} \to \Gamma$, define the counting function c_ρ by

$$c_\rho(\phi) = f(\rho \circ \phi) \tag{5.1}$$

It is not even necessary that ρ be a representation; ρ can in fact be any formal sum

$$\rho = \bigoplus_i a_i \rho_i$$

where the a_i are real numbers and each ρ_i is an irreducible representation of Γ. Then we define

$$f(\rho \circ \phi) = \sum_i a_i f(\rho_i \circ \phi)$$

Taking this viewpoint, we can express the spaces of Artin conductor counting functions and weighted discriminant counting functions (requring neither to be natural) as $\mathbb{R}$-vector spaces A and W, respectively. The first question to ask is whether or not $A = W$; that is, if every Artin conductor counting function is also a weighted discriminant counting function, and vice versa. If not, then how do their dimensions compare, and what is the nature and dimension of $A \cap W$?

We saw in Sections 2.4 and 2.7 that $\dim W$ is equal to the number of conjugacy classes of cyclic subgroups of Γ. Clearly, $\dim A$ is at most the number of conjugacy classes of elements of Γ (which is the same as the number of conjugacy classes of cyclic subgroups if and only if Γ has a rational character table). It is possible, however, that $\dim A$ could be smaller than this.

Lastly, if $A = W$, then Theorem 2.1 applies to Artin conductor counting functions as well. If not, this would be a natural direction in which to extend Theorem 2.1.

5.3 Field and Class Group Counting

Our repeated references to Heuristic 1.2 raise another question: when does this heuristic actually give the correct count of number fields? That is, under what circumstances does a $C(\Gamma)$ exist such that equation (1.3) holds? Most importantly for the present work, does choosing a counting function with a universal mass formula have any influence on the validity of Heuristic 1.2? Wood speculates in [26], based on computational data, that there is no simple rational value for $C(D_4)$ when D_4 fields are counted by the weighted discriminant with a universal mass formula. Aside from this, however, we know of no proven results on counting fields by anything but the standard discriminant, so it is difficult to make any guesses.

It would also be interesting to see, in cases where $C(\Gamma)$ does exist, whether or not its value changes when we change the counting function or impose ramification restrictions. The results of Chapter 3 suggest that the value may not change with the counting function, even when we use an extended counting function. Again, due to the dearth of known results, we cannot say whether we expect this to hold in general.

Also in Chapter 3, we discussed the more local question of how masses for the "Cohen-Lenstra counting function" compare to masses for the standard discriminant counting function. We left open a proof of Conjectures 3.9 and 3.10. We also leave open the question of how to modify these conjectures to accommodate the case where the Galois group Γ is unknown or not uniquely determined.

Bibliography

[1] M. Bhargava, *The density of discriminants of quartic rings and fields*, Ann. of Math., 162 (2005), pp. 1031–1063.

[2] ——, *Mass formulae for extensions of local fields, and conjectures on the density of number field discriminants*, Int. Math. Res. Not., (2007).

[3] ——, *The density of discriminants of quintic rings and fields*, Ann. of Math., 172 (2010), pp. 1559–1591.

[4] M. Bhargava and M. M. Wood, *The density of discriminants of S_3-sextic number fields*, Proc. Amer. Math. Soc., 136 (2008), pp. 1581–1587.

[5] D. A. Buell, *Class groups of quadratic fields*, Math. of Computation, 30 (1976), pp. 610–623.

[6] H. Cohen and H. W. Lenstra, *Heuristics on class groups of number fields*, in Number Theory, Lecture Notes in Mathematics 1068, Noordwijkerhout, 1983, Springer.

[7] H. Cohen and J. Martinet, *Class groups of number fields: Numerical heuristics*.

[8] ——, *Étude heuristique des groupes de classes des corps de nombres*, Journal Für Die Reine Und Angewandte Mathematik, (1990), pp. 39–76.

[9] H. Cohen, F. D. y Diaz, and M. Olivier, *Counting discriminants of number fields*, J. Thor. Nombres Bordeaux, 18 (2006), pp. 573–593.

[10] H. Cohn, *The density of abelian cubic fields*, Proc. Amer. Math. Soc., 5 (1954), pp. 476–477.

[11] H. Davenport and H. Heilbronn, *On the density of discriminants of cubic fields II*, Proc. Roy. Soc. London Ser. A, (1971), pp. 405–420.

[12] H. Heilbronn, *On the 2-classgroup of cubic fields*, in Studies in Pure Mathematics (Presented to Richard Rado), Academic Press, London, 1971, pp. 117–119.

[13] J. W. Jones and D. P. Roberts, *Database of local fields.* Online database. http://math.la.asu.edu/ jj/localfields/.

[14] ——, *A database of local fields.* Accompanying paper to database, http://arxiv.org/abs/math/0309309.

[15] K. S. Kedlaya, *Mass formulas for local galois representations*, Int. Math. Res. Not., (2007). With appendix by Daniel Gulotta.

[16] M. Krasner, *Nombre des extensions d'un degré donné d'un corps p-adique*, in Les Tendances Géométriques en Algèbre et Thèorie des Nombres, Paris, 1966.

[17] S. Mäki, *On the density of abelian number fields*, Suomalainen Tiedeakatemia, Helsinki, 1985.

[18] ——, *The conductor density of abelian number fields*, J. London Math. Soc., 47 (1993), pp. 18–30.

[19] G. Malle, *On the distribution of galois group, ii*, Experiment. Math., 13.

[20] ——, *On the distribution of class groups of number fields*, Experiment. Math., 19 (2010), pp. 465–474.

[21] J. S. Milne, *Algebraic number theory.* Lecture notes. http://www.jmilne.org/math/CourseNotes/ANT210.pdf.

[22] ——, *Class field theory.* Lecture notes. http://www.jmilne.org/math/CourseNotes/CFT310.pdf.

[23] M. R. Murty, *Problems in Analytic Number Theory*, Springer, New York, 2008.

[24] D. P. Roberts, *Density of cubic field discriminants*, Math. of Computation, 70 (2000), pp. 1699–1705.

[25] J.-P. Serre, *Local Fields*, Springer-Verlag, New York, 1979. M. J. Greenberg, trans.

[26] M. M. Wood, *Mass formulas for local galois representations to wreath products and cross products*, Algebra and Number Theory, 2 (2008), pp. 391–405.

[27] ——, *The distribution of sandpile groups of random graphs.* http://arxiv.org/abs/1402.5149, 2014.

[28] ——, *Algebraic curves and varieties over finite fields.* Course lectures, Fall 2011.

[29] M. M. Wood and T. Yasuda, *Mass formulas for local galois representations and quotient singularities I: A comparison of counting functions*, Int. Math. Res. Not., (to appear). Preprint: http://arxiv.org/abs/1309.2879.

[30] D. J. Wright, *Distribution of discriminants of abelian extensions*, Proc. London Math. Soc, 58 (1989), pp. 17–50.