Obstruction-flat asymptotically locally Euclidean metrics and asymptotics of the self-dual complex.

By

Antonio Gerardo Aché

A dissertation submitted in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

(Mathematics)

at the

UNIVERSITY OF WISCONSIN – MADISON

2012

Date of final oral examination: May 07, 2012

The dissertation is approved by the following members of the Final Oral Committee:

Jeff A. Viaclovsky, Professor, Mathematics

Sean T. Paul, Associate Professor, Mathematics

Sigurd Angenent, Professor, Mathematics

Mikhail Feldman, Professor, Mathematics

Alexander Nagel, Professor, Mathematics

Abstract

The first part of this thesis is based on the results in [AVis]. We show that any asymptotically locally Euclidean (ALE) metric which is obstruction-flat or extended obstruction-flat must be ALE of a certain optimal order. Moreover, our proof applies to very general elliptic systems and in any dimension $n \geq 3$. The proof is based on the technique of Cheeger-Tian for Ricci-flat metrics. We also apply this method to obtain a singularity removal theorem for (extended) obstruction-flat metrics with isolated C^0-orbifold singular points.

The second part of this thesis is based on [AVnt], where we restrict our attention to Riemannian 4-manifolds and analyze the indicial roots of the self-dual deformation complex on a cylinder $(\mathbb{R} \times Y^3, dt^2 + g_Y)$, where Y^3 is a space of constant curvature. An application is the optimal decay rate of solutions on a self-dual manifold with cylindrical ends having cross-section Y^3. We also resolve a conjecture of Kovalev-Singer in the case where Y^3 is a hyperbolic rational homology 3-sphere, and show that there are infinitely many examples for which the conjecture is true, and infinitely many examples for which the conjecture is false. Applications to gluing theorems are also discussed.

Acknowledgements

This dissertation was written under the guidance of Professor Jeff Viaclovsky to whom I am greatly indebted for dedicating to me all the time and effort that any graduate student could wish for. I would also like to thank him for supporting me with his NSF grant for several semesters, for introducing me to the field of conformal geometry and for giving me the opportunity to travel and meet wonderful mathematicians who work at many different institutions. I want to express my gratitude to Fedor Nazarov, Sigurd Angenent, Mikhail Feldman, Alexander Kiselev, Jeffrey Streets, Richard Kent, Alexander Nagel and Stephen Wainger for their time, interest and many helpful discussions that greatly contributed to this project. I am also indebted to the mathematics department at the University of Wisconsin-Madison for its generous support over the past years, specially during the summers of 2007, 2008, 2009, 2011 and the spring semester of 2010.

I want to thank my parents Gerardo and Maria Eugenia, my brothers David and José and my Aunt Isabel for their unconditional love and constant encouragement during the time that I spent at University of Wisconsin-Madison. Without this kind of support, I am sure that I would have given up on completing this dissertation a long time ago. I want to thank all my colleagues and fellow graduate students in the mathematics department for making my years in Madison very pleasant and for creating a friendly atmosphere in which to work and exchange ideas freely. In this regard, I want to give special thanks to Michael Lock, Lino Amorim, Kathleen Kiernan, Matthew Darnall, Garrett Alston, Michael Dabkowski, Rahbar Virk, James Knapp, Mustafa Kalafat, Guillermo Mantilla, Gabriel Pretel, Nathan Collins, Ali Godjali, Mathew Joseph, Dilip Raghavan, Soledad

Benguria, Kejia Wang, Ting-Ting Nan, Qin Li, Keith Dsouza, Zhenan Zhou, Yuanqi Wang, Junwu Tu, Yongqiang Zhao and Sun Song.

Finally, I want to thank all the people that I met at Universidad Simón Bolívar who continue to provide their honest advice, friendship and encouragement despite the distance that separates us. These people include Lázaro Recht, Pedro Berrizbeitia, Henryk Gzyl, Adolfo Quiroz, Alfredo Rios, Mercedes Rosas, Nestor Guillén and Héctor Chang.

Contents

Abstract i

Acknowledgements ii

1 Introduction 1

1.1 Background and Overview . . . 1

1.2 Statement of results in Part I . . . 4

1.2.1 The ambient obstruction tensor . . . 5

1.2.2 Extended obstruction tensors . . . 7

1.2.3 Singularity removal . . . 10

1.3 Statement of results in Part II . . . 12

1.3.1 Spherical cross-section . . . 15

1.3.2 Hyperbolic cross-section . . . 19

1.3.3 Flat cross-section . . . 22

1.3.4 Remarks and outline of Part II . . . 22

I Optimal decay at infinity and singularity removal theorems for Obstruction-flat metrics **24**

2 Linearized obstruction-flat equation **25**

2.1 Nonlinear terms in the obstruction-flat systems . . . 30

2.1.1 Scalar-flat condition and the modified equation . . . 34

3 The modified gauge condition **37**

3.1 Weighted Hölder and Sobolev Spaces 37

3.1.1 Divergence and the Lie derivative operator 39

3.1.2 A modified $\Box$ operator 43

4 Existence of divergence-free gauges **49**

4.1 Outline 49

4.2 The linearized equation in separated variables 50

4.2.1 Estimates for the linearized equation 54

4.3 Degenerate solutions of the linearized equations 59

4.4 Scaling and the nonlinear equation 62

4.5 Global divergence-free gauges 66

4.6 Turán's Lemma 70

4.6.1 An integral form of Turán's Lemma 71

4.6.2 Proof of Lemma 4.2.2 73

4.7 Optimal ALE order 77

4.7.1 Weighted Sobolev spaces and Δ^{k+1} 77

4.7.2 Optimal decay 79

4.8 Singularity removal theorems 81

II Asymptotics of the Self-Dual Deformation Complex **90**

5 Curvature decomposition in dimension 4 **91**

5.1 The anti self-dual part of the curvature tensor 91

5.1.1 The anti self-dual part of the Weyl tensor as a bilinear form . . . 92

6 The linearized equation and Dirac-type operators **99**

6.1 Linearization of W^- at a cylindrical metric 99

6.1.1 Conformal Killing operator and $\not{d}$ 101

6.1.2 The case of no radial components 103

6.1.3 The case of conformal variations 105

6.1.4 Completion of proof of Theorem 6.1.3 106

6.2 Some properties of $\not{d}$ 109

7 Analysis for the Self-Dual Deformation Complex **114**

7.1 The adjoint of $\mathcal{D}$ 114

7.2 Mixed solutions 125

7.3 Completion of proofs 132

7.4 The gluing problem 140

7.5 Appendix 143

7.5.1 The square of $\not{d}$ 143

Bibliography **148**

Chapter 1

Introduction

1.1 Background and Overview

In [FG85] Charles Fefferman and Robin Graham studied the problem of constructing a Poincaré metric associated to a given conformal structure. Let (M^n, g) be a Riemannian manifold of dimension $n \geq 3$ and consider the $n+1$-dimensional manifold $X^{n+1} = [0,1] \times M^n$. Given a 1-parameter family of metrics $\{g_r\}_{r\in[0,1]}$ on M one considers conformally compact metrics g_r^+ on X that can be written as

$$g_r^+ = r^{-2}\left(dr \otimes dr + g_r\right). \tag{1.1.1}$$

Constructing a Poincaré metric on X amounts to solving the following problem:

Problem 1.1.1 (Cauchy Problem)**.** *Find a 1-parameter family of metrics g_r with $r \in [0,1]$ defined on the base manifold M satisfying*

(a) $g_0 = g$,

(b) g_r *is even in* r,

(c) The conformally compact metric g_r^+ defined in (1.1.1) *is a formal power series solution (in r) of the Poincaré-Einstein equation*

$$Ric(g_r^+) = -n g_r^+. \tag{1.1.2}$$

Surprisingly, the solution to Problem 1.1.1 depends on the evenness of the dimension n. More precisely, in [FG85] it is shown that if n is odd, there exists a solution to the Cauchy problem which is unique up to the action of diffeomorphisms that fix the base manifold M. When n is even, however, one can in general only prescribe the family of metrics $\{g_r^+\}_r$ to satisfy

$$Ric(g_r^+) + g_r^+ = O(r^{n-2}), \text{ as } r \to 0, \tag{1.1.3}$$

therefore, in order to prove the existence of formal power series solutions of (1.1.2) a cancellation must occur, otherwise there is an *obstruction* to the existence of solutions to the Cauchy Problem 1.1.1. On the other hand, solutions of (1.1.3) always exist and are unique up to diffeomorphisms that fix the base manifold M and terms of order r^{n-2} as $r \to 0$. Given the unique solution of (1.1.3) associated to (M^n, g) (n even), the $(0,2)$-tensor

$$\mathcal{O}(g) = c_n \text{tf}\left(r^{2-n}\left(Ric(g^+) + ng^+\right)|_{TM}\right), \; c_n = \frac{2^{n-2}(n/2-1)!^2}{n-2},$$

where tf denotes"Trace-Free part" is called the *Ambient Obstruction Tensor.* The tensor $\mathcal{O}(g)$ is a natural tensor invariant of the metric g, i.e., in local coordinates, the components of $\mathcal{O}(g)$ are given by universal polynomials in g, g^{-1}, the components of the curvature tensor $Rm(g)$ and its derivatives. Moreover, $\mathcal{O}(g)$ is a conformal invariant of weight $2-n$, i.e., if $\varphi \in C^\infty(M)$ then $\mathcal{O}(\varphi^2 g) = \varphi^{2-n}\mathcal{O}(g)$ and $\mathcal{O}(g) = 0$ if g is locally conformally Einstein. In dimension 4, the obstruction tensor is equivalent to the *Bach tensor*, i.e., the gradient of the conformally invariant functional

$$\mathcal{W}(g) = \int_M |W|^2 dV_g,$$

where W is the *Weyl curvature tensor* associated to g. It has shown in [GH05] that in even dimensions higher than 4, the obstruction tensor $\mathcal{O}(g)$ has also a variational characterization, namely, $\mathcal{O}(g)$ is formally the gradient of the functional

$$\mathcal{Q}(g) = \int_M Q_g dV_g,$$

where Q_g is the *Q-curvature* of g, which is a scalar quantity defined on even dimensional manifolds introduced in [Bra95] (see also [Cha04]). In general, Q_g is not a pointwise conformal invariant like $|W|^2$, but the functional $\mathcal{Q}(g)$ restricted to a compact manifold M is a conformal invariant. In dimension 4, for example, Q_g is given by

$$Q_g = \frac{1}{3}\left(\Delta R + R^2 - 3|Ric|^2\right).$$

We say that a metric g is *obstruction-flat* if it satisfies $\mathcal{O}(g) = 0$ and we have already mentioned above that conformally Einstein metrics are obstruction-flat, however, it is not clear in general what geometric conditions guarantee that obstruction-flat metrics are conformally Einstein. In dimension 4, an important class of examples of obstruction-flat (or Bach-flat) metrics is the class of *self-dual* and *anti self-dual metrics* (see Sections 1.3 and 5.1 for definitions). By means of a gluing construction it was shown in [Flo91] the existence of self-dual metrics on $(\mathbb{CP}^2)^{\# m}$, i.e., connected sums of n copies of $\mathbb{CP}^2$ with the natural orientation. However, for $m \geq 4$ the spaces $(\mathbb{CP}^2)^{\# m}$ *do not* admit Einstein metrics (see for example [Bes08, Chapter 6]).

In the we study the problem of the optimal decay at infinity for obstruction-flat metrics which are Asymptotically Locally Euclidean (ALE). We also obtain singularity removal theorems for obstruction-flat metrics with constant scalar curvature which have isolated orbifold singularities. For the second part of this thesis, we specialize to dimension four and present a complete analysis of the self-dual deformation complex at a cylindrical metric and discuss applications to the analytic gluing problem. In addition, for the case of hyperbolic cross-sections we resolve a conjecture of Kovalev and Singer stated in [KS01] (see Section 1.3 and the references therein).

1.2 Statement of results in Part I

We first recall the definition of an ALE metric.

Definition 1.2.1. A complete Riemannian manifold (M, g) is called *asymptotically locally Euclidean* or *ALE* of order τ if it has finitely many ends, and for each end there exists a finite subgroup $\Gamma \subset SO(n)$ acting freely on $\mathbf{R}^n \setminus B(0,R)$ and a diffeomorphism $\Psi : M \setminus K \to (\mathbf{R}^n \setminus B(0,R))/\Gamma$ where K is a subset of M containing all other ends, and such that under this identification,

$$(\Psi_* g)_{ij} = \delta_{ij} + O(r^{-\tau}), \tag{1.2.1}$$

$$\partial^{|k|}(\Psi_* g)_{ij} = O(r^{-\tau-k}), \tag{1.2.2}$$

for any partial derivative of order k, as $r \to \infty$, where r is the distance to some fixed basepoint. We say that (M, g) is ALE of order 0 if we can find a coordinate system as above with $(\Psi_* g)_{ij} = \delta_{ij} + o(1)$, and $\partial^{|k|}(\Psi_* g)_{ij} = o(r^{-k})$ for any $k \geq 1$ as $r \to \infty$.

ALE spaces are ubiquitous in modern geometric analysis, and we do not attempt to give a complete list of references here. A crucial result in the Ricci-flat case was obtained by Cheeger-Tian: if (M^n, g) is Ricci-flat ALE of order 0, there exists a change of coordinates at infinity so that (M^n, g) is ALE of order n, where n is the dimension [CT94]. This generalized and improved earlier work of Bando-Kasue-Nakajima [BKN89], who employed improved Kato inequalities together with a Moser iteration argument.

The Cheeger-Tian method has the advantage of finding the *optimal* order of curvature decay, without relying on Kato inequalities.

Another interesting class of metrics is that of Bach-flat scalar-flat ALE metrics in dimension 4, or more generally any metric satisfying a system of the form

$$\Delta Ric = Rm * Ric, \tag{1.2.3}$$

where the right hand side is shorthand for a contraction of the full curvature tensor with the Ricci tensor. In the case of anti-self-dual scalar-flat metrics, or scalar-flat metrics with harmonic curvature, it was proved in [TV05a] that such spaces are ALE of order τ for any $\tau < 2$, using the technique of Kato inequalities. Subsequently, this was generalized to Bach-flat metrics and metrics with harmonic curvature in dimension 4 in [Str10a], using the Cheeger-Tian technique. In this thesis, we will simplify and generalize the Streets argument so that it also works for higher order systems and yields the optimal ALE order. A simplification from [Str10a] is that we do not need to perform the entire radial separation of variables on symmetric tensors to obtain the optimal decay rate. Rather, we show this optimal decay can be obtained directly in Euclidean coordinates without running into very complicated formulas in radial coordinates, see Proposition 2.0.2.

1.2.1 The ambient obstruction tensor

We now continue the discussion in Section 1.1 and present a more detailed list of properties of the ambient obstruction tensor introduced by Charles Fefferman and Robin Graham. Let (M^n, g) be an n-dimensional Riemannian manifold, where $n > 2$. Recall that the curvature tensor admits the decomposition

$$Rm = W + A_g \owedge g, \tag{1.2.4}$$

where W is the Weyl tensor, $\owedge$ is the Kulkarni-Nomizu product, A_g is the *Schouten tensor* defined as

$$A_g = \frac{1}{n-2}\left(Ric - \frac{R}{2(n-1)}g\right), \tag{1.2.5}$$

where R denotes the scalar curvature. Define the *n-dimensional Bach Tensor* by (see [CF08, GH05])

$$B_{ij} = \Delta A_{ij} - \nabla^k \nabla_i A_{kj} + A^{kl} W_{ikjl}, \tag{1.2.6}$$

where Δ denotes the rough Laplacian (our convention is to use the analyst's Laplacian).

If the dimension n is even, then the *ambient obstruction tensor* introduced in [FG85, FG12], and denoted by $\mathcal{O}$ is a symmetric $(0,2)$-tensor that has the following properties:

1. $\mathcal{O}(g)$ is trace-free.

2. If $n = 4$, $\mathcal{O}_{ij}(g)$ equals $B_{ij}(g)$ where $B_{ij}(g)$ is the Bach tensor of g.

3. If g is conformal to an Einstein metric then $\mathcal{O}(g) = 0$.

4. $\mathcal{O}(g)$ has an expansion of the form

$$\mathcal{O}_{ij} = \Delta^{\frac{n}{2}-2} B_{ij} + l.o.t., \tag{1.2.7}$$

   where $l.o.t.$ denotes quadratic and higher order terms in curvature involving fewer derivatives.

5. $\mathcal{O}(g)$ is variational, in fact $\mathcal{O}(g)$ is the gradient of the functional

$$\mathcal{F}(g) = \int_M Q_g dV_g,$$

where Q_g is the *Q-curvature* of g. In particular, $\mathcal{O}(g)$ is divergence-free.

1.2.2 Extended obstruction tensors

If (M^n, g) is even-dimensional, there is also a family of symmetric $(0,2)$-tensors called *extended obstruction tensors* introduced in [Gra09] and denoted by $\Omega^{(k)}(g)$ where $1 \leq k \leq \frac{n}{2} - 2$ which have the following properties:

1. $\Omega^{(k)}(g)$ is trace-free.

2. When the dimension n is seen as a formal parameter, $\Omega^{(k)}(g)$ has a pole at $n = 2(k+1)$, and its residue at $n = 2(k+1)$ is a multiple of the obstruction tensor in that dimension, for example,

$$\Omega^{(1)} = \frac{1}{4-n} B_{ij},$$

and when $n = 4$, B_{ij} equals the obstruction tensor.

3. If (M, g) is locally conformally flat then $\Omega^{(k)}(g) = 0$.

4. $\Omega^{(k)}(g)$ has an expansion of the form

$$\Omega^{(k)}_{ij} = \frac{1}{(4-n)(6-n)\dots(2k-n)} \Delta^{k-1} B_{ij} + l.o.t., \tag{1.2.8}$$

where $l.o.t.$ denotes quadratic and higher order terms in curvature involving fewer derivatives.

To simplify notation, we define $\Omega^{(k)}(g) = \mathcal{O}(g)$ for $k = \frac{n}{2} - 1$.

The main theorem in this thesis gives the optimal decay rate for obstruction-flat or extended obstruction-flat scalar-flat ALE metrics:

Theorem 1.2.2. *Let (M^n, g) be even-dimensional, scalar-flat, and $\Omega^{(k)}$-flat for some k with $1 \leq k \leq \frac{n}{2} - 1$. If (M^n, g) is ALE of order zero, then there exists a change of coordinates at infinity so that g is ALE of order $n - 2k$.*

The method of Cheeger-Tian is to show that after a suitable change of coordinates, g may be written as $g = g_0 + h$, where g_0 is Euclidean, and h is divergence-free. One then considers the linearization of the (extended) obstruction tensor at the flat metric. In the divergence-free gauge, this becomes becomes a power of the Laplacian (the trace is controlled using the scalar-flat condition). An analysis of the decay rates of solutions of the gauged linearized equation, together with an estimate on the nonlinear terms in the equation, then yields Theorem 1.2.2.

The main technical complication is that the assumption of ALE of order 0 does not directly yield a divergence-free gauge. As in [CT94], we obtain initially a modified divergence-free gauge $\delta_t h = 0$ (see Section 4). In this gauge, we must rule out certain solutions of the linearized equation which we call *degenerate solutions* (see Definition 4.2.1). Once these degenerate solutions are ruled out, we are able to find a change of coordinates so that (M, g) is ALE of order $\beta > 0$. This step requires a technique of Leon Simon called the *Three Annulus Lemma*, which was also employed by Cheeger-Tian [Sim85, CT94]. We generalize this technique so that it applies to higher-order equations. For this step, we show that Turán's Lemma implies the necessary estimates, which we prove in Section 4.6. Once this step is complete, it is relatively easy to find a divergence-free gauge using standard Fredholm Theory, and then to prove the optimal decay order. This work is carried out in Sections 2–4.7.

Remark 1.2.3. In the case of the obstruction tensor, which is conformally invariant, one may obtain many examples through the following construction. Let (M^n, g) be an even-dimensional compact Einstein manifold with positive scalar curvature, and let G_x denote the Green's functions of the conformal Laplacian at a point x. The metric $\hat{g} = G_x^p g$, where $p = \frac{4}{n-2}$, is asymptotically flat and scalar-flat [LP87]. Since Einstein spaces are obstruction-flat [GH05, Theorem 2.1], $\hat{g}$ is also obstruction-flat and asymptotically flat of order at least 2. If (M^n, g) is instead locally conformally flat with positive scalar curvature, the same construction yields an $\Omega^{(1)}$-flat asymptotically flat space of order at least $n-2$.

Our method applies to much more general systems than just the obstruction tensors, and works in any dimension $n \geq 3$. Given two tensor fields A, B, the notation $A * B$ will mean a linear combination of contractions of $A \otimes B$ yielding a symmetric 2-tensor.

Theorem 1.2.4. *Let $k = 1$ if $n = 3$, or $1 \leq k \leq \frac{n}{2} - 1$ if $n \geq 4$. Assume that (M, g) is scalar-flat, ALE of order 0, and satisfies*

$$\Delta_g^k Ric = \sum_{j=2}^{k+1} \sum_{\alpha_1 + \ldots + \alpha_j = 2(k+1) - 2j} \nabla_g^{\alpha_1} Rm * \ldots * \nabla_g^{\alpha_j} Rm. \tag{1.2.9}$$

Then (M, g) is ALE of order $n - 2k$.

For $k = 1$, this is simply

$$\Delta Ric = Rm * Rm. \tag{1.2.10}$$

We emphasize that this is more general than (1.2.3), since the right hand side is allowed to be quadratic in the full curvature tensor. This is satisfied in particular by scalar-flat Kähler metrics and metrics with harmonic curvature in any dimension, and also anti-self-dual metrics in dimension 4. These special cases were previously considered in [Che09] using improved Kato inequalites and a Moser iteration technique. We emphasize that our argument yields the optimal decay rate without requiring any improved Kato inequalities, and therefore applies to the more general system (1.2.9). The optimal decay for scalar-flat anti-self-dual ALE metrics was previously considered in [CLW08, Proposition 13]. The case of extremal Kähler ALE metrics was considered in [CW11]. As mentioned above, the cases of Bach-flat metrics and metrics with harmonic curvature in dimension 4 were considered in [Str10a]. However, we note that (1.2.10) is more general than (1.2.3).

Remark 1.2.5. We do not need such a strong requirement on the decay of partial derivatives of arbitrarily high order in (1.3.16) in Definition 1.3.10, but have assumed this here in the introduction for simplicity of stating the result. We only need to assume this up to a finite number of partial derivatives, see Remark 4.5.1.

1.2.3 Singularity removal

The methods used to prove the above results can also be applied to analyze isolated singularities. Similar results were proved in [BKN89, Che09, CW11, CLW08, Str10a, Tia90, TV05b]. We next recall the definition of a C^0-orbifold point.

Definition 1.2.6. Let g be a metric defined on $B_\rho(0) \setminus \{0\}$, where $B_\rho(0)$ is a metric ball in a flat cone. We say that the origin is a C^0-orbifold point if there exists a coordinate system around the origin such that

$$g_{ij} = \delta_{ij} + o(1), \tag{1.2.11}$$

$$\partial^l g_{ij} = o(r^{-|l|}), \tag{1.2.12}$$

for any multi-index l with $|l| \geq 1$ as $r \to 0$. We say that the origin is a smooth orbifold point, if after lifting to the universal cover of $B_\rho(0) \setminus \{0\}$, the metric extends to a smooth metric over the origin, after diffeomorphism.

Remark 1.2.7. As in the ALE case, we will not need to assume (1.2.12) for partial derivatives of arbitrarily high order, only up to a certain finite number of derivatives, see Remark 4.8.2.

Applying the Cheeger-Tian technique directly to the singularity, we obtain the following.

Theorem 1.2.8. *Let $B_\rho(0)$ be as above and even-dimensional, and let g be (extended) obstruction-flat in $B_\rho(0)\setminus\{0\}$ with constant scalar curvature. If the origin is a C^0-orbifold point for g, then the metric extends to a smooth orbifold metric in $B_\rho(0)$.*

As in the ALE case, this theorem also applies to much more general higher-order systems:

Theorem 1.2.9. *Let $k = 1$ if $n = 3$, or $1 \leq k \leq \frac{n}{2} - 1$ if $n \geq 4$. Assume that $(B_\rho(0) \setminus \{0\}, g)$ has constant scalar curvature and satisfies*

$$\Delta_g^k Ric = \sum_{j=2}^{k+1} \sum_{\alpha_1+\ldots+\alpha_j=2(k+1)-2j} \nabla_g^{\alpha_1} Rm * \ldots * \nabla_g^{\alpha_j} Rm. \tag{1.2.13}$$

If the origin is a C^0-orbifold point for g, then the metric extends to a smooth orbifold metric in $B_\rho(0)$.

Theorem 1.2.9 will be proved in Section 4.8, the proof of which uses the same method as that of Theorem 1.2.2, with a few minor modifications.

1.3 Statement of results in Part II

Let (M^4, g) be a four-dimensional Riemannian manifold and let Rm denote the Riemannian curvature tensor of g. Recall that Rm admits an orthogonal decomposition of the form

$$Rm = W + \frac{1}{2}E \owedge g + \frac{1}{24}Rg \owedge g, \tag{1.3.1}$$

where W is the Weyl tensor, E is the traceless Ricci tensor of g, R is the scalar curvature, and $\owedge$ is the Kullkarni-Nomizu product. If (M^4, g) is oriented, there is a further decomposition of (1.3.1). The Hodge-$*$ operator associated to g acting on 2-forms is a mapping $* : \Lambda^2 \mapsto \Lambda^2$ satisfying $*^2 = Id$, and Λ^2 admits a decomposition of the form

$$\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-, \tag{1.3.2}$$

where $\Lambda^2_\pm$ are the ± 1 eigenspaces of $*|_{\Lambda^2}$. Sections of Λ^2_+ and Λ^2_- are called self-dual and anti-self-dual 2-forms, respectively. The curvature tensor can be viewed as an operator $\mathcal{R} : \Lambda^2 \to \Lambda^2$, and we let $\mathcal{W}$ and $\mathcal{E}$ denote the operators associated to the Weyl and traceless Ricci tensors, respectively. With respect to the decomposition (1.3.2), the full curvature operator decomposes as

$$\mathcal{R} = \left(\begin{array}{c|c} \mathcal{W}^{+} + \frac{R}{12} I & \frac{1}{2}\mathcal{E}\pi_{-} \\ \hline \frac{1}{2}\mathcal{E}\pi_{+} & \mathcal{W}^{-} + \frac{R}{12} I \end{array} \right), \tag{1.3.3}$$

where $\pi_{\pm}$ is the projection onto $\Lambda^2_{\pm}$, and the self-dual and anti-self-dual Weyl tensors are defined by $\mathcal{W}^{\pm} = \pi_{\pm}\mathcal{W}\pi_{\pm}$.

Definition 1.3.1. Let (M^4, g) be an oriented four-manifold. Then g is called *self-dual* if $\mathcal{W}^{-} = 0$, and g is called *anti-self-dual* if $\mathcal{W}^{+} = 0$. In either case g is said to be *half-conformally-flat.*

By reversing orientation, a self-dual metric becomes an anti-self-dual metric, so without loss of generality, we will only consider self-dual metrics.

Since Poon's example of a 1-parameter family of self dual metrics on $\mathbb{CP}^2\#\mathbb{CP}^2$ in 1988 [Poo86], there has been an explosion of examples of self-dual metrics on various four-manifolds. We do not attempt to give a complete history here, but only mention a few results closely related to our results in this thesis. In 1989, Donaldson and Friedman developed a twistor space gluing procedure which invoked many non-trivial results in algebraic geometry [DF89]. In 1991, Floer produced examples on $n\#\mathbb{CP}^2$ by an analytic gluing procedure [Flo91]. Then in 2001, Kovalev and Singer generalized Floer's analytic gluing result to cover the case of gluing orbifold self-dual metrics [KS01]. We will describe the relation of our work with these prior works in more detail below, but first will state our main results.

Since the SD condition is conformally invariant, we are free to conformally change an end to obtain different types of asymptotics. For simplifying computations, the most useful type of geometry is that of cylindrical ends:

Definition 1.3.2. Let (Y^3, g_Y) be a compact 3-manifold with constant curvature. A complete Riemannian manifold (M^4, g) is called *asymptotically cylindrical* or *AC* with cross-section Y of order τ if there exists a diffeomorphism $\psi : M \setminus K \to \mathbb{R}_+ \times Y$ where K is a subset of M containing all other ends, satisfying

$$(\psi_* g)_{ij} = (g_C)_{ij} + O(e^{-\tau t}), \tag{1.3.4}$$

$$\partial^{|k|}(\psi_* g)_{ij} = O(e^{-\tau t}), \tag{1.3.5}$$

for any partial derivative of order k, as $t \to \infty$, where $g_C = dt^2 + g_Y$ is the product cylindrical metric.

Self-dual metrics have a rich obstruction theory. If (M, g) is a self-dual four-manifold, the deformation complex is given by

$$\Gamma(T^*M) \xrightarrow{\mathcal{K}_g} \Gamma(S^2_0(T^*M)) \xrightarrow{\mathcal{D}} \Gamma(S^2_0(\Lambda^2_-)), \tag{1.3.6}$$

where $\mathcal{K}_g$ is the conformal Killing operator defined by

$$(\mathcal{K}_g(\omega))_{ij} = \nabla_i \omega_j + \nabla_j \omega_i - \frac{1}{2}(\delta\omega) g, \tag{1.3.7}$$

with $\delta\omega = \nabla^i \omega_i$, and $\mathcal{D} = (\mathcal{W}^-)'_g$ is the linearized anti-self-dual Weyl curvature operator. This complex is then wrapped-up into a single operator

$$F : \Gamma(S^2_0(T^*M)) \longrightarrow \Gamma(S^2_0(\Lambda^2_-)) \oplus \Gamma(T^*M), \tag{1.3.8}$$

defined by

$$F(h) = (\mathcal{D}h, 2\delta h), \tag{1.3.9}$$

and $(\delta h)_j = \nabla^i h_{ij}$. This operator is mixed order elliptic of order $(2, (0, 1))$ in the sense of Douglis-Nirenberg [DN55], with formal L^2-adjoint

$$F^* : \Gamma(S^2_0(\Lambda^2_-)) \oplus \Gamma(T^*M) \longrightarrow \Gamma(S^2_0(T^*M)), \tag{1.3.10}$$

given by

$$F^*(Z, \omega) = \mathcal{D}^* Z - \mathcal{K}_g \omega. \tag{1.3.11}$$

Definition 1.3.3. The *indicial roots* of F on the cylinder $(\mathbb{R}\times Y^3, g_C)$, are those complex numbers λ for which there is a solution h of $F(h) = 0$ such that the components of h have the form $e^{\lambda t} p(y, t)$ where p is a polynomial in t with coefficients in $C^\infty(Y)$. The indicial roots of F^* are defined analogously for pairs (Z, ω). [1]

We will first determine the indicial roots of F^*. The indicial roots of F can then be obtained by using an index theorem, as we will show below. One could equivalently first analyze the indicial roots of F, however, for purposes of computation it turns out to be somewhat easier to completely analyze the cokernel (although the computations are in principle equivalent).

1.3.1 Spherical cross-section

Our first result deals with cross-section Y having constant positive curvature. In Theorem 7.3.1 below, we determine *all* indicial roots of F^*, but for simplicity we only state the following here in the introduction:

[1] *Our definition of indicial roots differs from that in [LM85] by a factor of* $\sqrt{-1}$.

Theorem 1.3.4. *Let M be $\mathbb{R} \times S^3/\Gamma$ with product metric $g = dt^2 + g_{S^3/\Gamma}$, where $g_{S^3/\Gamma}$ is a metric of constant curvature* 1*. Let $\mathcal{I}^*$ denote the set of indicial roots of F^*. If $\beta \in \mathcal{I}^*$ satisfies $|Re(\beta)| < 2$ then $\beta = 0$ or $\beta = \pm 1$. In these cases, the corresponding solutions are of the form $(0, \omega)$, where ω is dual to a conformal Killing field (that is, $\mathcal{K}_g \omega = 0$). Consequently,*

- *Case (0): $0 \in \mathcal{I}^*$, and the corresponding solutions are given by $(0, dt)$, or $(0, \omega_0)$ for ω_0 dual to a Killing field on S^3/Γ.*

- *Case (1): $\pm 1 \in \mathcal{I}^*$ if and only if Γ is trivial. In this case, the corresponding solutions are given by $(0, \omega)$, where ω is given by $e^{\pm t}(\phi dt \mp d\phi)$ where ϕ is a lowest nontrivial eigenfunction of Δ_{S^3} with eigenvalue* 3*.*

Remark 1.3.5. The indicial roots $\beta \in \mathcal{I}^*$ satisfying $|Re(\beta)| \geq 2$ fall into two classes. The indicial roots in the first class are integers and the corresponding solutions are of the form $(Z, 0)$; these are Cases (2) and (3) in Theorem 7.3.1. The indicial roots in the other class have non-zero imaginary part, and the corresponding solutions are of the form (Z, ω) with Z nontrivial and $\mathcal{K}_g(\omega) \neq 0$; these are Cases (4) and (5) in Theorem 7.3.1.

We can also completely characterize the indicial roots of the forward operator F. This follows from the above determination of the cokernel indicial roots, together with the index theorem of Lockhart and McOwen; it turns out that these are the same. We will describe all kernel elements explicitly below in Theorem 7.3.3, but for purposes of brevity in the introduction we only state here the following theorem which generalizes a well-known result of Floer [Flo91]. In order to state the theorem, we define the symmetric product of 1-forms ω_1 and ω_2 by

$$\omega_1 \odot \omega_2 = \omega_1 \otimes \omega_2 + \omega_2 \otimes \omega_1. \tag{1.3.12}$$

Theorem 1.3.6. *Let M be $\mathbb{R} \times S^3/\Gamma$ with product metric $g = dt^2 + g_{S^3/\Gamma}$, where $g_{S^3/\Gamma}$ is a metric of constant curvature 1, and let $\mathcal{I}$ denote the set of indicial roots of F. Then $\mathcal{I} = \mathcal{I}^*$. For $\beta = 0 \in \mathcal{I}$, the corresponding solutions of $F(h) = 0$ are given by*

$$span\{3dt \otimes dt - g_{S^3}, dt \odot \omega_0\}, \tag{1.3.13}$$

where ω_0 is a dual to a Killing field on S^3/Γ.

Next, $\beta = \pm 1 \in \mathcal{I}$ if and only if Γ is trivial. In this case, the corresponding kernel elements are given by

$$h_\phi = p(t)\phi(3dt \otimes dt - g_{S^3}) + q(t)(dt \odot d\phi), \tag{1.3.14}$$

where $p(t) = C_3 e^t - C_4 e^{-t}$ and $q(t) = C_3 e^t + C_4 e^{-t}$, for some constants C_3 and C_4, and ϕ is a lowest nonconstant eigenfunction of Δ_{S^3}.

Morever, solutions in (1.3.13) and (1.3.14) are in the image of the conformal Killing operator. All other indicial roots $\beta \in \mathcal{I}$ satisfy $|Re(\beta)| \geq 2$.

Remark 1.3.7. As in Remark 1.3.5, the indicial roots $\beta \in \mathcal{I}$ satisfying $|Re(\beta)| \geq 2$ fall into two classes. The indicial roots in the first class are integers and the corresponding solutions are not in the image of the conformal Killing operator; these are Cases (2) and (3) in Theorem 7.3.3. The indicial roots in the other class have non-zero imaginary part, and the corresponding solutions *are* in the image of the conformal Killing operator; these are Cases (4) and (5) in Theorem 7.3.3.

A corollary is the optimal result:

Corollary 1.3.8. *Let* (M^4, g) *be the the cylinder* $\mathbb{R} \times Y^3$*, where* $Y^3 = S^3/\Gamma$ *with* $\Gamma \subset SO(4)$ *a finite subgroup acting freely on* S^3 *with product metric* $g = dt^2 + g_Y$*, where* g_Y *is a metric of constant curvature* 1*.*

(a) Let (Z, ω) *be a solution of* $\mathcal{D}^* Z = \mathcal{K}_g \omega$*. If* $Z = o(e^{2|t|})$ *and* $\omega = o(e^{2|t|})$ *as* $|t| \to \infty$ *then* $Z = 0$*, and* ω *is dual to a conformal Killing field.*

(b) Let h *be a solution of* $\mathcal{D}h = 0$ *and* $\delta h = 0$*. If* $h = o(e^{2|t|})$ *as* $|t| \to \infty$ *then* h *can be written as a linear combination of elements in* (1.3.13) *and* (1.3.14)*.*

Standard analysis in weighted spaces then implies the following corollary for AC manifolds with spherical cross-section:

Corollary 1.3.9. *Let* (M^4, g) *be self-dual and asymptotically cylindrical with cross-section* $(Y^3 = S^3/\Gamma, g_Y)$ *with* $\Gamma \subset SO(4)$ *a finite subgroup acting freely on* S^3*, where* g_Y *is a metric of constant curvature* 1*.*

(a) Let (Z, ω) *be a solution of* $\mathcal{D}^* Z = \mathcal{K}_g \omega$*. If* $Z = o(e^{2t})$ *and* $\omega = o(e^{2t})$ *then* ω *is dual to a conformal Killing field, and* $Z = O(e^{-2t})$ *as* $t \to \infty$*.*

(b) Let h *be a solution of* $\mathcal{D}h = 0$ *and* $\delta h = 0$*. If* $h = o(e^{2|t|})$ *as* $|t| \to \infty$ *then* h *has an asymptotic expansion with leading term as in* (1.3.13) *or* (1.3.14)*.*

In Section 8, we apply Corollary 1.3.9 to fix a gap in the proof of a key step in the main gluing result in [KS01].

To state the next result, we require the following definition.

Definition 1.3.10. A complete Riemannian manifold (M^4, g) is called *asymptotically locally Euclidean* or *ALE* of order τ if it has finitely many ends, and for each end there exists a finite subgroup $\Gamma \subset SO(4)$ acting freely on S^3 and a diffeomorphism $\psi : M \setminus K \to (\mathbf{R}^4 \setminus B(0,R))/\Gamma$ where K is a subset of M containing all other ends, and such that under this identification,

$$(\psi_* g)_{ij} = \delta_{ij} + O(r^{-\tau}), \tag{1.3.15}$$

$$\partial^{|k|}(\psi_* g)_{ij} = O(r^{-\tau-k}), \tag{1.3.16}$$

for any partial derivative of order k, as $r \to \infty$, where r is the distance to some fixed basepoint.

It is known that any self-dual ALE metric is ALE of order 2, after a possible change of coordinates at infinity. ALE of any order $\tau < 2$ was first shown by [TV05a], while ALE of order exactly 2 was shown in [Str10b], see also [Che09, AVis]. This order is optimal, so without loss of generality we will assume that all ALE spaces are ALE of order 2.

We also have the following optimal decay result for self-dual ALE spaces:

Theorem 1.3.11. *Let (M, g) be self-dual and asymptotically locally Euclidean.*

(a) Any solution of $\mathcal{D}^* Z = \mathcal{K}_g \omega$ satisfying $Z = o(1)$ and $\omega = o(r^{-1})$ must satisfy $\omega = 0$ and $Z = O(r^{-4})$ as $r \to \infty$.

(b) Any solution of $\mathcal{D} h = 0$ and $\delta h = 0$ satisfying $h = o(1)$ must satisfy $Z = O(r^{-2})$ as $r \to \infty$.

1.3.2 Hyperbolic cross-section

We first define

$$H^1_C(Y) = \{B \in S^2_0(T^*Y) \mid d^\nabla B = 0, tr(B) = 0\}, \tag{1.3.17}$$

where $(d^{\nabla}B)_{klj}$ is given by $(d^{\nabla}B)_{klj} = \nabla_k B_{lj} - \nabla_l B_{kj}$, to be the vector space of traceless Codazzi tensor fields. For the case of hyperbolic cross-section, we have the following:

Theorem 1.3.12. *Let (M^4, g) be the the cylinder $\mathbb{R} \times Y^3$, where (Y^3, g_Y) is compact and hyperbolic with constant curvature -1, with product metric $g = dt^2 + g_Y$, and let $\mathcal{I}^*$ denote the set of indicial roots of F^*. Then there exists an $\epsilon > 0$ such that if $\beta \in \mathcal{I}^*$ with $|Re(\beta)| < \epsilon$ then $\beta \in \{0, \pm i\}$. The corresponding kernel of F^* has dimension*

$$1 + b_1(Y) + 2\dim(H^1_C(Y)). \tag{1.3.18}$$

The corresponding kernel of F has the same dimension and is spanned by

$$\{3dt \otimes dt - g_Y, dt \odot \omega, cos(t) \cdot B, \sin(t) \cdot B\}, \tag{1.3.19}$$

where ω is any harmonic 1-form ω, and B is any traceless Codazzi tensor on Y^3.

Remark 1.3.13. The element $(0, dt)$ is in the cokernel, which accounts for the 1 in (1.3.18). The other cokernel elements in case $b_1(Y) \neq 0$ arise from non-trivial harmonic 1-forms, and those in case $H^1_C(Y) \neq \{0\}$ of course arise from non-trivial traceless Codazzi tensor fields. These elements are written down explicitly in Section 7.1, see Propositions 7.1.5(b) and 7.1.9(b). We only note here that the nontrivial solutions in this case satisfy $Z = O(1)$ as $|t| \to \infty$ or are periodic in t.

We define[2]

$$H^2_+(\mathbb{R} \times Y^3) = \{Z \in S^2_0(\Lambda^2_-) \mid \mathcal{D}^* Z = 0 \text{ and } Z = O(e^{\epsilon|t|}) \text{ as } |t| \to \infty \text{ for every } \epsilon > 0\}.$$

[2]Note that this definition is more general than the definition in [KS01, Section 4.2.1] in that we allow solutions which have polynomial growth in t.

In [KS01, Conjecture 4.11], it was conjectured that $H^2_+(\mathbb{R}\times Y^3) = \{0\}$ for any hyperbolic rational homology 3-sphere. Theorem 1.3.12 shows that this is true if and only if Y^3 does not admit any non-trivial traceless Codazzi tensor field. Using this, and some examples of certain hyperbolic 3-manifolds of [Kap94, DeB06], we obtain infinitely many examples for which the conjecture is true, and infinitely many examples for which the conjecture is false:

Theorem 1.3.14. *Let (Y^3, g_Y) be a hyperbolic rational homology 3-sphere, with g_Y of constant curvature -1, and $M = \mathbb{R} \times Y^3$ with the product metric $g = dt^2 + g_Y$. Then $H^2_+(\mathbb{R}\times Y^3) = \{0\}$ if and only if Y^3 admits no non-trivial traceless Codazzi tensor fields. Furthermore, there are infintely many hyperbolic rational homology 3-spheres satisfying $H^2_+(\mathbb{R} \times Y^3) = \{0\}$, and infinitely many satisfying $H^2_+(\mathbb{R} \times Y^3) \neq \{0\}$.*

We also have the following application to AC manifolds with hyperbolic cross-section:

Corollary 1.3.15. *Let (M^4, g) be self-dual and asymptotically cylindrical with cross-section (Y^3, g_Y) a hyperbolic rational homology 3-sphere with g_Y of constant curvature -1, satisfying $H^1_C(Y) = \{0\}$.*

(a) Let (Z, ω) be a solution of $\mathcal{D}^*Z = \mathcal{K}_g\omega$. Then there exists a constant $\epsilon > 0$, such that if (Z, ω) solves $\mathcal{D}^*Z = \mathcal{K}_g\omega$ and satisfies $Z = o(e^{\epsilon|t|})$ and $\omega = o(e^{\epsilon|t|})$ as $t \to \infty$ then ω is dual to a conformal Killing field and $Z = o(e^{-\epsilon|t|})$ as $t \to \infty$.

(b) Let h be a solution of $\mathcal{D}h = 0$ and $\delta h = 0$. Then there exists a constant $\epsilon > 0$, such that if $h = o(e^{\epsilon|t|})$ as $|t| \to \infty$, then h admits an expansion

$$h = c \cdot (dt \otimes dt - 3g_Y) + O(e^{-\epsilon|t|}) \tag{1.3.20}$$

for some constant c as $|t| \to \infty$.

1.3.3 Flat cross-section

Finally, in the case that (Y^3, g) is a flat torus, we have the following:

Theorem 1.3.16. *Let (M^4, g) be the the cylinder $\mathbb{R} \times Y^3$, where (Y^3, g_Y) is compact and flat, with product metric $g = dt^2 + g_Y$, and let $\mathcal{I}$ denote the set of indicial roots of F. Then there exists an $\epsilon > 0$ such that if $\beta \in \mathcal{I}$ with $|Re(\beta)| < \epsilon$ then $\beta = 0$. The corresponding kernel of F has dimension 14 and is spanned by*

$$\{3dt \otimes dt - g_Y, dt \odot \omega, B, tB\}, \tag{1.3.21}$$

where ω is any parallel 1-form and B is any parallel traceless symmetric 2-tensor on Y^3.

The corresponding cokernel of F has dimension 14 and is spanned by

$$\{(0, dt), (0, \omega_0), (Z, 0), (tZ, 0)\}, \tag{1.3.22}$$

where ω_0 is a parallel 1-form, and Z is any parallel section of $S^2_0(\Lambda^2_-)$.

Remark 1.3.17. One can easily use our computations to explicitly determine *all* indicial roots in the case of flat cross-section $Y^3 = T^3$. However, in the interest of brevity this is omitted.

1.3.4 Remarks and outline of Part II

We next give a brief outline of the second part of this thesis. Sections 5.1 and 6.1 will be concerned with the derivation of the linearized anti-self-dual Weyl tensor in separated variables. In these sections, there is overlap with computations in Floer's paper [Flo91]. However, the main formula given in Floer for $(\mathcal{W}^-)'$ at a cylindrical metric is incorrect [Flo91, Proposition 5.1] (in addition to mistakes in the coefficients, Floer's formula omits crucial terms involving the trace component h_{00}). The correct formula (which moreover holds for any cross-section Y^3 with constant curvature) is given in Theorem 6.1.3. Section 6.2 contains required formulas for a Dirac-type operator, as well as some necessary eigenvalue computations. Section 7.1 contains the core analysis of the kernel of $\mathcal{D}^*$. The analysis in Section 7.2 is necessary to determine the possibilities for the 1-form ω appearing in the adjoint equation. The proofs of all the main theorems are then completed in Section 7.3. In Section 7.4, we discuss the application of our results to gluing theorems.

Finally, the Appendix contains the derivation of a crucial formula relating the square of the Dirac operator to the linearized Einstein equation on the cross-section. In the case of spherical cross-section, Floer writes down such a formula [Flo91, Lemma 5.1], but which has errors in the coefficients. The correct formula (which moreover holds for general cross-section Y) is given in Corollary 7.5.1.

Part I

Optimal decay at infinity and singularity removal theorems for Obstruction-flat metrics

Chapter 2

Linearized obstruction-flat equation

We begin with some notation: in the following δ will denote the divergence operator, which can act either on a symmetric 2-tensor h, or on a 1-form ω. In the former case $\delta h = \nabla^i h_{ij}$ and in the latter case $\delta\omega = \nabla^i \omega_i$. The L^2-adjoint of δ will be denoted by δ^*, which is $-(1/2)\mathcal{L}$, where $\mathcal{L}$ is the Lie derivative operator, defined by $(\mathcal{L}\omega)_{ij} = \nabla_i\omega_j + \nabla_j\omega_i$. The trace of a symmetric 2-tensor h will be denoted by $tr(h)$.

We now analyze the linearizations of (1.2.7) and (1.2.8). Note that from the dependence of the lower order terms in both equations on the curvature tensor it follows that the linearized equations $\left(\Omega^{(k)}\right)'_{g_0}(h) = 0$ for $1 \leq k \leq \frac{n}{2} - 1$ at a flat metric g_0 are equivalent to $\Delta^{k-1} B'_{g_0}(h) = 0$. With this observation we have the following, which holds in any dimension $n \geq 3$:

Proposition 2.0.1. *At a flat metric g_0 we have*

$$\begin{aligned} \Delta^{k-1}\left(B'_{g_0}(h)\right) = \Delta^{k-1}\Bigg(&-\frac{1}{2(n-2)}\Delta^2 h - \frac{1}{2(n-1)(n-2)}\nabla^2\Delta\mathrm{tr}(h) \\ &-\frac{1}{2(n-1)}\nabla^2\delta\delta h - \frac{1}{(n-2)}\Delta\delta^*\delta h + \frac{1}{2(n-1)(n-2)}\left(\Delta^2\mathrm{tr}(h) - \Delta\delta\delta h\right) g_0\Bigg). \end{aligned} \tag{2.0.1}$$

Proof. The Bach tensor can be written as $B_{ij} = \Delta A_{ij} - \nabla^k\nabla_i A_{kj} + A^{kl}W_{ikjl}$, and at a flat metric $\left(A^{kl}W_{ikjl}\right)'_{g_0}(h) = 0$ for any $h \in S^2\left(T^*\mathbb{R}^n\right)$, so then

$$B'_{ij}(h) = \Delta A'_{ij}(h) - \nabla^k\nabla_i A'_{jk}(h).$$

By the Bianchi identity and using again that g_0 is flat we have

$$\begin{aligned}-\nabla^k\nabla_i(A'_{g_0})_{jk}(h) = -\nabla_i\nabla^k(A_{g_0})'_{kj}(h) &= -\frac{1}{2(n-1)}\nabla_i\nabla_j R'_{g_0}(h) \\ &= \frac{1}{2(n-1)}\nabla_i\nabla_j\left(\Delta\mathrm{tr}(h) - \delta(\delta h)\right).\end{aligned} \tag{2.0.2}$$

On the other hand

$$\begin{aligned}\Delta A'_{g_0}(h) &= \frac{1}{(n-2)}\Delta\left(Ric'_{g_0}(h) - \frac{R'_{g_0}}{2(n-1)}(h)g_0\right) \\ &= \frac{1}{(n-2)}\Delta\left(-\frac{1}{2}\Delta h - \frac{1}{2}\nabla^2\mathrm{tr}(h) - \delta^*\delta h + \frac{1}{2(n-1)}\left[\Delta\mathrm{tr}(h) - \delta(\delta h)\right]g_0\right).\end{aligned} \tag{2.0.3}$$

Adding (2.0.2) and (2.0.3) together and taking Δ^{k-1} we obtain (2.0.1).

□

The linearized equations are *not* strictly elliptic due to the diffeomorphism invariance of the (extended) obstruction-flat equations. Note, however, that if h is divergence free and if the trace of h is harmonic then from (2.0.1) the linearized equations reduce to

$$\Delta^{k+1}h = 0, \tag{2.0.4}$$

$$\Delta\mathrm{tr}(h) = 0, \tag{2.0.5}$$

$$\delta h = 0, \tag{2.0.6}$$

and (2.0.4) is an elliptic equation on h. We point out that we will *not* be able to prescribe that h be divergence-free at first, so we will follow the approach in [CT94] and introduce a modified divergence operator in Section 2.1.1. The proof of Theorem 1.2.2 relies on the following crucial proposition.

Proposition 2.0.2. *Let $n > 2$, and $h \in S^2(T^*\mathbb{R}^n)$ be a solution on $\mathbb{R}^n \backslash B_\rho(0)$ for some*

$\rho > 0$ of the system

$$\Delta^{k+1} h = 0, \tag{2.0.7}$$

$$\delta h = 0, \tag{2.0.8}$$

for $1 \leq k \leq \frac{n}{2} - 1$ (or for $k = 1$ when $n = 3$) satisfying $h = O(|x|^{1-\epsilon})$ with $\epsilon > 0$. Then, h can be written as

$$h = h_c + O\left(|x|^{-n+2k}\right), \tag{2.0.9}$$

where h_c is constant. The result also holds for $k = 0$ if in addition we assume that $\mathrm{tr}(h) = 0$.

Proof. Consider first the case $1 \leq k \leq \frac{n}{2} - 2$. Since the components of h satisfy the scalar equation $\Delta^{k+1} f = 0$, using a classical expansion, we may write a solution h of (2.0.7)-(2.0.8) as

$$h(x) = h_c + \sum_{l=0}^{\infty} h_l(x), \tag{2.0.10}$$

where h_c is constant and each h_l is a homogeneous solution of (2.0.7)-(2.0.8) of degree $2(k+1) - n - l$. If h_l is one of such solutions, we can write

$$h_l(x) = \sum_{j=0}^{k} \left(\sum_{\{s:2(k-j)+s=l\}} |x|^{2(j+1)-n-s} h_{s,j}(x) \right), \tag{2.0.11}$$

where the components of $h_{s,j}(x)$ in (2.0.11) are spherical harmonics of degree s. In order to prove the claim for $1 \leq k \leq \frac{n}{2} - 2$ it suffices to show that $h_0 = h_1 \equiv 0$. For $l = 0$, we have

$$(h_0)_{ij}(x) = |x|^{2(k+1)-n} c_{ij},$$

where c_{ij} is constant. From (2.0.8) we obtain

$$\sum_{i=1}^{n} c_{ij} x_i = 0,$$

for each j and this clearly implies that h_0 is identically zero. For h_1 we have

$$(h_1)_{ij}(x) = |x|^{2(k+1)-n} u_{ij}\left(\frac{x}{|x|^2}\right),$$

where $u_{ij}(x)$ is a homogeneous polynomial of degree 1 that we will write as

$$u_{ij}(x) = \sum_{l=1}^{n} A_{ijl} x_l. \tag{2.0.12}$$

From (2.0.8) we have for every j

$$0 = \sum_{i=1}^{n} \partial_i \left(|x|^{2(k+1)-n} u_{ij}(x/|x|^2)\right) = -\sum_{i=1}^{n} (n-2k) x_i |x|^{2(k-1)-n} u_{ij}(x) + |x|^{2k-n} \sum_{i=1}^{n} \partial_i u_{ij}(x),$$

which becomes

$$(n-2k) \sum_{i=1}^{n} \sum_{l=1}^{n} A_{ijl} x_i x_l = |x|^2 \sum_{i=1}^{n} A_{iji}. \tag{2.0.13}$$

For fixed j and every $x \in \mathbb{R}^n$, with $x \neq 0$. If in (2.0.13) we let x be the vector with coordinates $x_i = \delta_{ip}$ for fixed p, one obtains the identity

$$A_{pjp} = \frac{1}{n-2k} \sum_{l=1}^{n} A_{ljl} \text{ for fixed } j. \tag{2.0.14}$$

An obvious consequence of (2.0.14) is that A_{pjp} is independent of p for fixed j, in particular, for every p and j

$$A_{pjp} = \frac{n}{n-2k} A_{pjp}, \tag{2.0.15}$$

and then $A_{pjp} = 0$ since $k \neq 0$. For the components of the form A_{ljm} with $l \neq m$, the coefficient of $x_l x_m$ in the left-hand side of (2.0.13) is $(n-2k)(A_{ljm} + A_{mjl})$ while in the right-hand side there are no off-diagonal terms, so we conclude that

$$A_{ljm} = -A_{mjl} \text{ for } l \neq m, \tag{2.0.16}$$

If l, j, m are all different we obtain from the symmetry of A_{ljm} in l, j and from (2.0.16) the identity

$$A_{ljm} = -A_{mjl} = -A_{jml} = A_{lmj} = A_{mlj} = -A_{jlm} = -A_{ljm},$$

therefore, in this case $A_{ljm} = 0$. For the components of the form A_{llj} and A_{jll} when $l \neq j$, it is easy to see that $A_{llj} = -A_{ljl}$ and $A_{jll} = A_{ljl}$ and as we saw above this implies that both components are zero, so we conclude that all polynomials u_{ij} are identically zero.

For the case $k = \frac{n}{2} - 1$ there is only one difference with the argument above: $h_0(x)$ is logarithmic, i.e., a solution of the form $h_{ij}(x) = \log(|x|)c_{ij}$ with c_{ij} constant, however the condition $\delta h = 0$ implies that $\sum_{j=1}^{n} c_{ij} x_j = 0$ for every i and hence $c_{ij} = 0$ for all i, j so this solution in fact does *not* occur.

For the case $k = 1$ and $n = 3$ we write h as

$$h(x) = \sum_{l=0}^{\infty} h_l(x), \tag{2.0.17}$$

where $h_l(x)$ is a homogeneous solution of degree $-l$ of $\Delta^2 h(x) = 0$ on $\mathbb{R}^n \backslash \{0\}$. In this case, the solution $h_0(x)$ has the form $h_0(x) = h_C + h_{0,1}(x)$ where the components of h_C are constant and the components of $h_{0,1}$ are spherical harmonics of degree 1. The solutions $h_l(x)$ with $l \geq 1$ have the form $h_l(x) = |x|^{-l}\left(h_{l,l-1}(x) + h_{l,l+1}(x)\right)$ where the components of $h_{l,l\pm1}(x)$ are spherical harmonics of degree $l \pm 1$. We only have to prove that if $\delta h_{0,1}(x) = 0$ on $\mathbb{R}^n \backslash \{0\}$ then $h_{0,1}(x) \equiv 0$. For that purpose write the components of $h_{0,1}(x)$ as $(h_{0,1})_{ij}(x) = u_{ij}\left(\frac{x}{|x|}\right)$ where $u_{ij}(x)$ are linear functions given by (2.0.12). The condition $\delta h_{0,1}(x) = 0$ for all $x \neq 0$ becomes

$$\sum_{i=1}^{3} \sum_{l=1}^{3} A_{ijl} x_i x_l = |x|^2 \sum_{i=1}^{3} A_{iji},$$

and we can argue as in the case $1 \leq k \leq \frac{n}{2} - 1$ to conclude that $A_{ijl} = 0$ for all $i, j, l = 1, 2, 3$.

The above proof can be extended to the case of $k = 0$ provided the trace vanishes. However, we omit the proof, and instead refer the reader to the proof given in [CT94, page 538], which is an alternative argument using Obata's Theorem. □

2.1 Nonlinear terms in the obstruction-flat systems

In this section we derive an expression for the error terms in the linearization of the (extended) obstruction tensors, i.e., the difference

$$\Omega^{(k)}(g_0 + h) - \Omega^{(k)}(g_0) - \left(\Omega^{(k)}\right)'_{g_0}(h), \tag{2.1.1}$$

where g_0 is a flat metric in $\mathbb{R}^n$. Given two tensor fields A, B by $A * B$ we mean a linear combination of contractions of $A \otimes B$ using the metric g_0, and for a positive integer j, $A^{-j} * B$ means contractions of j copies of the inverse of A with B.

Proposition 2.1.1. *Let g_0 be a flat metric on $\mathbb{R}^n$ and let $h \in S^2(T^*\mathbb{R}^n)$ be such that $g_0 + h$ is another Riemannian metric on $\mathbb{R}^n$. For the (extended) obstruction tensors $\Omega^{(k)}$*

with $1 \le k \le \frac{n}{2} - 1$*, we have*

$$\begin{aligned}\Omega^{(k)}(g_0+h) = {} & \left(\Omega^{(k)}\right)'_{g_0}(h) - g_0^{-1} * h * (g_0+h)^{-1} * \nabla^{2(k+1)} h \\ & + \sum_{j=2}^{\mathcal{I}_{\|}} (g_0+h)^{-j} * \left(\sum_{\alpha_1+\ldots+\alpha_j = 2(k+1)} \nabla_{g_0}^{\alpha_1} h * \ldots * \nabla_{g_0}^{\alpha_j} h \right),\end{aligned} \tag{2.1.2}$$

for some integer $\mathcal{I}_k \ge 2(k+1)$*. For the scalar curvature we have,*

$$\begin{aligned}R(g_0+h) = {} & R'_{g_0}(h) + (g_0+h)^{-1} * h * \nabla_{g_0}^2 h + (g_0+h)^{-2} * (\nabla_{g_0} h * \nabla_{g_0} h) \\ & + (g_0+h)^{-3} (\nabla_{g_0} * \nabla_{g_0} h * h).\end{aligned} \tag{2.1.3}$$

Proof. For any tensor T, we have

$$\nabla_{g_0+h} T = \nabla_{g_0} T + (g_0+h)^{-1} * \nabla_{g_0} h * T. \tag{2.1.4}$$

From this it follows that

$$Rm(g_0+h) = Rm(g_0) + (g_0+h)^{-1} * \nabla_{g_0}^2 h + (g_0+h)^{-2} * \nabla_{g_0} h * \nabla_{g_0} h. \tag{2.1.5}$$

Here we see Rm as a $(1,3)$-curvature tensor. It follows that the $Ric(g_0+h)$ has an expansion similar to (2.1.6). For the scalar curvature, on the other hand, we have

$$\begin{aligned} R(g_0+h) &= (g_0+h)^{-1} * (Ric(g_0+h)) \\ &= (g_0+h)^{-1} * \left((g_0+h)^{-1} * \nabla_{g_0}^2 h + (g_0+h)^{-2} * \nabla_{g_0} h * \nabla_{g_0} h\right) \\ &= \left((g_0+h)^{-2} * \nabla_{g_0}^2 h + (g_0+h)^{-3} * \nabla_{g_0} h * \nabla_{g_0} h\right).\end{aligned} \tag{2.1.6}$$

Using the identity

$$(g_0+h)^{-1} - g_0^{-1} = -g_0^{-1} * h * (g_0+h)^{-1}, \tag{2.1.7}$$

we obtain

$$\begin{aligned}R(g_0+h) = {} & (g_0+h)^{-1} * \nabla_{g_0}^2 h + (g_0+h)^{-2} \left(h * \nabla_{g_0}^2 h + \nabla_{g_0} h * \nabla_{g_0} h\right) \\ & + (g_0+h)^{-3} * h * \nabla_{g_0} h * \nabla_{g_0} h,\end{aligned}$$

another application of (2.1.7) yields

$$R(g_0+h) = \nabla^2_{g_0} h + (g_0+h)^{-1} * \nabla^2_{g_0} h + (g_0+h)^{-2} \left(*h * \nabla^2_{g_0} h + \nabla_{g_0} h * \nabla_{g_0} h \right) \\ +(g_0+h)^{-3} * h * \nabla_{g_0} h * \nabla_{g_0} h, \tag{2.1.8}$$

and since the only term in (2.1.8) that contributes to the linearization of R at g_0 is $\nabla^2_{g_0} h$ equation (2.1.3) follows. In order to find a similar expansion for the Bach tensor $B(g_0+h)$ we note that for any metric g, $B(g)$ can be written schematically as

$$B(g) = \Delta_g Ric(g) + \nabla^2_g R(g) + \left(\Delta_g R(g)\right) g + Ric(g) * Rm(g), \tag{2.1.9}$$

where the term $Ric(g) * Rm(g)$ has in local coordinates the form $R^{pq} R_{ipjq}$ and the indices p, q are raised using the metric g. If for example we consider the term $\Delta_g Ric(g)$ at $g = g_0 + h$ in (2.1.9), we first take one covariant derivative of $Ric(g)$ and obtain Taking a covariant derivative, and assuming that g_0 is flat, we have

$$\nabla_{g_0+h} Ric(g_0+h) = (g_0+h)^{-1} * \left(\nabla^3_{g_0} h\right) + (g_0+h)^{-2} * \nabla^2_{g_0} h * \nabla_{g_0} h \\ + (g_0+h)^{-3} * \nabla_{g_0} h * \nabla_{g_0} h * \nabla_{g_0} h, \tag{2.1.10}$$

Taking another covariant derivative and one contraction with respect to g_0+h, we obtain

$$\begin{aligned} \Delta_{g_0+h} Ric(g_0+h) &= (g_0+h)^{-2} * \left(\nabla^4_{g_0} h\right) \\ &+ (g_0+h)^{-3} * \left(\nabla^2_{g_0} h * \nabla^2_{g_0} h + \nabla^3_{g_0} h * \nabla_{g_0} h\right) \\ &+ (g_0+h)^{-4} * \nabla^2_{g_0} h * \nabla_{g_0} h * \nabla_{g_0} h \\ &+ (g_0+h)^{-5} \left(\nabla_{g_0} h * \nabla_{g_0} h * \nabla_{g_0} h * \nabla_{g_0} h\right). \end{aligned} \tag{2.1.11}$$

Using (2.1.7) once more we may write (2.1.11) as

$$\Delta_{g_0+h} Ric(g_0+h) = \nabla^4_{g_0} h + \sum_{j=1}^{5} \sum_{\alpha_1+\ldots+\alpha_j=4} (g_0+h)^{-j} * \nabla^{\alpha_1}_{g_0} h * \ldots * \nabla^{\alpha_j}_{g_0} h, \tag{2.1.12}$$

and using that g_0 is flat we conclude that

$$\Delta_{g_0+h} Ric(g_0+h) = \Delta_{g_0} Ric'_{g_0}(h) + \sum_{j=1}^{5} \sum_{\alpha_1+\ldots+\alpha_j=4} (g_0+h)^{-j} * \nabla_{g_0}^{\alpha_1} h * \ldots * \nabla_{g_0}^{\alpha_j} h. \tag{2.1.13}$$

From similar computations for the terms in $\nabla_g^2 R(g)$, $(\Delta_g R(g))\, g$ and $Ric(g) * g$ in (2.1.9) we conclude that

$$B(g_0+h) = B'_{g_0}(h) + \sum_{j=1}^{\mathcal{I}_1} \sum_{\alpha_1+\ldots+\alpha_j=4} (g_0+h)^{-j} * \nabla_{g_0}^{\alpha_1} h * \ldots * \nabla_{g_0}^{\alpha_j} h, \tag{2.1.14}$$

where $\mathcal{I}_1$ is some integer with $\mathcal{I}_1 \geq 7$. For any $m \geq 1$ it follows that

$$\begin{aligned} \Delta_{g_0+h}^{m-1} B(g+h) &= \Delta^{m-1} B'_{g_0}(h) \\ &\quad + \sum_{j=1}^{\mathcal{I}_m} (g_0+h)^{-j} * \left(\sum_{\alpha_1+\ldots+\alpha_j=m+4} \nabla_{g_0}^{\alpha_1} h * \ldots * \nabla_{g_0}^{\alpha_j} h \right). \end{aligned} \tag{2.1.15}$$

where $\mathcal{I}_m$ is some integer with $\mathcal{I}_m \geq m+7$. Next, using scaling arguments it is clear that the terms $l.o.t.$ in (1.2.7) and (1.2.8) have the form

$$\sum_{j=2}^{k+1} \left(\sum_{\alpha_1+\ldots+\alpha_j=2(k+1)-2j} \nabla^{\alpha_1} Rm * \ldots * \nabla^{\alpha_j} Rm \right), \tag{2.1.16}$$

see for example the proof of Theorem 2.1 in [GH05].

Equation (2.1.2) follows from combining (1.2.7) or (1.2.8) in Section 1.2.2 with (2.1.15) and with the identity Finally, equation (2.1.3) follows directly from (2.1.6) and (2.1.7). □

Next, defining

$$c_{n,k} = \begin{cases} \frac{1}{(4-n)(6-n)\ldots(2k-n)} & \text{if } 1 \leq k \leq \frac{n}{2}-2 \\ 1 & \text{if } k = \frac{n}{2}-1, \end{cases} \tag{2.1.17}$$

from Proposition 2.1.1, we may write

$$\Omega^{(k)}(g_0+h)-\Omega^{(k)}(g_0)=c_{n,k}\Delta^{k-1}B'_{g_0}(h)+F^{(k)}(h,g_0), \tag{2.1.18}$$

where $F^{(k)}(h,g_0)$ is the remainder in (2.1.2). For the scalar curvature we will write

$$R(g_0+h)-R(g_0)=R'_{g_0}(h)+F'(h,g_0), \tag{2.1.19}$$

where $F'(h,g_0)$ is the error term in (2.1.3). From now on, we will use ∇ to denote ∇_{g_0}, therefore all operators Δ, δ, tr are taken with respect to g_0.

2.1.1 Scalar-flat condition and the modified equation

In order to address the difficulty of not initially being able to prescribe h to be divergence-free, we follow [CT94] and introduce a *modified divergence operator* given by

$$\delta_t h=\delta h-t i_{r^{-1}\frac{\partial}{\partial r}}h, \tag{2.1.20}$$

and we will show in Section 4 that we can find a gauge where $\delta_t h=0$. A difference with the approach in [CT94] is that the obstruction-flat systems are *not* elliptic even if we are able to prescribe $\delta_t h=0$ because $\Delta^{k-1}B'_{g_0}(h)$ is traceless regardless of the gauge condition. Note that if $\delta_t h=0$ we obtain

$$\begin{aligned}\Delta^{k-1}B'_{g_0}(h)=\Delta^{k-1}\Bigg(&-\frac{1}{2(n-2)}\Delta^2 h-\frac{1}{2(n-1)(n-2)}\nabla^2\Delta\mathrm{tr}(h)\\&-\frac{t}{2(n-1)}\nabla^2\delta i_{r^{-1}\frac{\partial}{\partial r}}h-\frac{t}{(n-2)}\Delta\delta^* i_{r^{-1}\frac{\partial}{\partial r}}h\\&+\frac{1}{2(n-2)(n-1)}\left(\Delta^2\mathrm{tr}(h)-t\Delta\delta\left(i_{r^{-1}\frac{\partial}{\partial r}}h\right)\right)g_0\Bigg).\end{aligned} \tag{2.1.21}$$

At this point we use the scalar-flat condition on g_0+h. Assuming again that $\delta_t h=0$, the linearization of the scalar curvature at g_0 becomes

$$R'_{g_0}(h)=-\Delta\mathrm{tr}(h)+\delta\delta h=-\Delta\mathrm{tr}(h)+t\delta i_{r^{-1}\frac{\partial}{\partial r}}h, \tag{2.1.22}$$

so from the scalar-flat equation we have

$$\Delta \mathrm{tr}(h) = t\delta i_{r^{-1}\frac{\partial}{\partial r}} h + F'(h, g_0), \tag{2.1.23}$$

where $F'(h, g_0)$ is the remainder in (2.1.19). Inserting (2.1.23) into (2.1.21) we obtain

$$\begin{aligned} \Delta^{k-1} B'_{g_0}(h) = \Delta^{k-1} \Bigg(-\frac{1}{2(n-2)} \Delta^2 h - \frac{t}{2(n-2)} \nabla^2 \delta i_{r^{-1}\frac{\partial}{\partial r}} h \\ - \frac{t}{(n-2)} \Delta \delta^* i_{r^{-1}\frac{\partial}{\partial r}} h \Bigg) + \mathcal{E}^{(k)}(h, g_0), \end{aligned} \tag{2.1.24}$$

where $\mathcal{E}^{(k)}(h, g_0)$ is given by

$$\mathcal{E}^{(k)}(h, g_0) = \frac{1}{2(n-1)(n-2)} \Big(-\Delta^{k-1} \nabla^2 F'(h, g_0) + \Delta^k F'(h, g_0) g_0 \Big). \tag{2.1.25}$$

We now define a linear operator $\mathcal{P}_t^{(k)}$ by

$$\mathcal{P}_t^{(k)}(h) = \frac{c_{n,k}}{n-2} \Delta^{k-1} \left(-\frac{1}{2} \Delta^2 h - \frac{t}{2} \nabla^2 \delta i_{r^{-1}\frac{\partial}{\partial r}} h - t \Delta \delta^* i_{r^{-1}\frac{\partial}{\partial r}} h \right). \tag{2.1.26}$$

Clearly, the operator $\mathcal{P}_t^{(k)}$ is strictly elliptic. From (2.1.2) and (2.1.24), if $\delta_t h = 0$ and $g_0 + h$ is scalar-flat, the (extended) obstruction-flat system may be written as

$$0 = \mathcal{P}_t^{(k)} h + \mathcal{R}^{(k)}(h, g_0) \text{ for } 1 \leq k \leq \frac{n}{2} - 1, \tag{2.1.27}$$

where

$$\mathcal{R}^{(k)}(h, g_0) = c_{n,k} \mathcal{E}^{(k)}(h, g_0) + F^{(k)}(h, g_0).$$

Writing $\mathcal{E}^{(k)}(h, g_0)$ schematically as $\nabla^{2k} F'(h, g_0)$, we easily see using the proof of Proposition 2.1.1 that $\mathcal{R}^{(k)}(h, g_0)$ has the same form as the remainder in (2.1.2)

$$\begin{aligned} \mathcal{R}^{(k)}(h, g_0) = (g_0 + h)^{-1} * h * \nabla_{g_0}^{2(k+1)} h \\ + \sum_{j=2}^{2(k+1)} (g_0 + h)^{-j} * \left(\sum_{\alpha_1 + \ldots + \alpha_j = 2(k+1)} \nabla_{g_0}^{\alpha_1} h * \ldots * \nabla_{g_0}^{\alpha_j} h \right), \end{aligned} \tag{2.1.28}$$

so that none of the terms in $\mathcal{R}^{(k)}(h, g_0)$ contribute to the linearization. This shows that (2.1.27) defines a family of elliptic equations on $\mathbb{R}^n \backslash \{0\}$. This same argument applies equally to the system (1.2.9). We have proved

Corollary 2.1.2. *Let $1 \leq k \leq \frac{n}{2} - 1$. If $g = g_0 + h$ is scalar-flat and $\Omega^{(k)}$-flat with $\delta_t h = 0$, then h satisfies*

$$\mathcal{P}_t^{(k)} h + \mathcal{R}^{(k)}(h, g_0) = 0, \tag{2.1.29}$$

where $\mathcal{P}_t^{(k)}$ is the linear operator given by (2.1.26) and $\mathcal{R}^{(k)}(h, g_0)$ is the error term given by (2.1.28). The operator $\mathcal{P}_t^{(k)}$ is strictly elliptic.

Let $k = 1$ if $n = 3$, or $1 \leq k \leq \frac{n}{2} - 1$ if $n \geq 4$. If $g = g_0 + h$ is scalar-flat and solves (1.2.9) with $\delta_t h = 0$, then h also satisfies (2.1.29) with a remainder term of the same form.

Chapter 3

The modified gauge condition

3.1 Weighted Hölder and Sobolev Spaces

In this section we introduce weighted spaces that will be useful in the analysis needed to construct divergence-free gauges. We start by reviewing some of the notation in [CT94]. If we write $\mathbb{R}^n$ as $\mathbb{R}^n = C(S^{n-1})$, we define maps $\psi_a : C(S^{n-1}) \to C(S^{n-1})$ for $a > 0$ given by $\psi_a(r,x) = (ar,x)$. The weighted Hölder norms are defined as follows: if $u > 0$, let A_u be the natural scaling on tensors of type (p,q), i.e.

$$A_u = \underbrace{(\psi_{u^{-1}})_* \otimes \cdots \otimes (\psi_{u^{-1}})_*}_{p} \otimes \underbrace{\psi_u^* \otimes \cdots \otimes \psi_u^*}_{q}. \tag{3.1.1}$$

Given a tensor T of type (p,q) we have

$$|T_{(u,x)}|_{m,\alpha;0} = |u^{p-q} A_u T_{(1,x)}|_{m,\alpha;0}, \tag{3.1.2}$$

where $|\cdot|_{m,\alpha;0}$ is the $C^{m,\alpha}$-norm with respect to the flat metric g at the point $(1,x)$. We can now define Hölder norms by

$$|T|_{m,\alpha;0} = \sup_{(u,x)} |T_{(u,x)}|_{m,\alpha;0}, \tag{3.1.3}$$

and also weighted Hölder norms given by

$$|T|_{m,\alpha;l} = |r^{-l} T|_{m,\alpha;0}, \tag{3.1.4}$$

for any $l \in \mathbb{R}$. We say that a tensor T of type (p,q) is in $\mathcal{T}^{p,q}_{m,\alpha;l}$ if $|T|_{m,\alpha;l} < \infty$. We use $A_{c,d}(0)$ to denote the annulus $A_{c,d}(0) = \{(r,x)|c < r < d\}$. For a tensor h of type (u,v) and $\delta \in \mathbb{R}$ we define a weighted L^p norm by

$$\|h\|_{L'^{p,u,v}_{\delta}} = \left(\int_{\mathbb{R}^n} |h|^p |x|^{-\delta p - n} dx\right)^{\frac{1}{p}}, \tag{3.1.5}$$

where $|\cdot|$ is the usual pointwise norm on tensors of type (u,v). For any nonnegative integer m, we also define weighted Sobolev norms by

$$\|h\|_{W'^{m,p,u,v}_{\delta}} = \sum_{j=0}^{m} \|\nabla^j h\|_{L'^{p,u,v}_{\delta-j}}, \tag{3.1.6}$$

and then $W'^{m,p,u,v}_{\delta}$ is the space

$$W'^{m,p,u,v}_{\delta} = \left\{h : \|h\|_{W'^{m,p,u,v}_{\delta}} < \infty\right\}. \tag{3.1.7}$$

For further properties of these weighted Sobolev spaces see [Bar86].

For notational convenience, if h is a symmetric $(0,2)$-tensor we will use $|||h|||_{a,b}$ to denote the norm $\|h\|_{L_0^{2,0,2}(A_{a,b}(0))}$, i.e., the L^2 norm of h with weight 0 on the annulus $A_{a,b}(0)$. Another way to construct the norm $|||\cdot|||$ is as follows: consider the weighted inner product on the slices (r, S^{n-1}) given by

$$\langle\langle h_1, h_2 \rangle\rangle = r^{-(n-1)} \int_{(r,S^{n-1})} \langle h_1, h_2 \rangle dV_{g_{S^{n-1}(r)}}, \tag{3.1.8}$$

where $\langle \cdot, \cdot, \rangle$ is the usual pointwise inner product. It follows that

$$|||h|||^2_{a,b} = \int_a^b r^{-1} \|h\|^2 dr, \tag{3.1.9}$$

where $\|\cdot\|$ is the norm defined by the inner product $\langle\langle \cdot, \cdot \rangle\rangle$ in (3.1.8). The norms $|||\cdot|||$ are scale invariant in the sense that if h is a $(0,2)$ tensor and if we let $q = a^{-2}\psi_a^* h$ then

$$|||h|||_{a,aL} = |||q|||_{1,L}. \tag{3.1.10}$$

Finally, we say that a (p,q)-tensor T is *radially parallel* if

$$\nabla_{\frac{\partial}{\partial r}} T = 0. \tag{3.1.11}$$

3.1.1 Divergence and the Lie derivative operator

We now consider the operator $\Box : \Lambda^1(\mathbb{R}^n) \to \Lambda^1(\mathbb{R}^n)$ defined by $\Box\xi = \delta L_\xi g_0$, where L_ξ is the Lie derivative operator. This operator is formally self-adjoint and elliptic. From now on we use $\tilde{\Delta}_H$, $\tilde{\mathrm{tr}}$,$\tilde{\delta}$ and $\tilde{d}$ to denote the Hodge laplacian, trace, divergence and exterior differentiation in the cross section metric $g_{S^{n-1}}$ respectively. Following [CT94, Section 2], if we write ξ in polar coordinates as

$$\xi = f dr + \omega, \tag{3.1.12}$$

we have

$$L_\xi g_0 = 2f' dr \otimes dr + \left(\omega' - 2r^{-1}\omega + \tilde{d}f\right) \boxtimes dr + L_\omega g_{S^{n-1}} + 2rf g_{S^{n-1}}, \tag{3.1.13}$$

here we use primes to denote differentiation respect to r. Also, given a 1-form α we denote by $\alpha \boxtimes dr$ the symmetric product $\alpha \otimes dr + dr \otimes \alpha$. If now h is a symmetric $(0,2)$-tensor written in polar coordinates as

$$h = h_{00} dr \otimes dr + \alpha \boxtimes dr + B, \tag{3.1.14}$$

where B is a symmetric $(0,2)$ tensor whose radial components are zero, the divergence of h is given by

$$\begin{aligned} \delta h = &\left(h_{00}' + (n-1)r^{-1}h_{00} + r^{-2}\tilde{\delta}\alpha \boxtimes dr - r^{-3}\tilde{\mathrm{tr}}(B)\right) dr \\ &+ \alpha' + (n-1)r^{-1}\alpha + r^{-2}\tilde{\delta}B. \end{aligned} \tag{3.1.15}$$

Combining (3.1.13) and (3.1.15), the operator $\Box$ takes the form

$$\Box\xi = \left(2f'' + 2(n-1)r^{-1}f' + r^{-2}\left(-\tilde{\Delta}_H f - 2(n-1)f\right) + r^{-2}\tilde{\delta}\omega' - 4r^{-3}\tilde{\delta}\omega\right) dr$$
$$+ \omega'' + (n-3)r^{-1}\omega' + r^{-2}\left(-\tilde{\Delta}_H\omega + \tilde{d}\tilde{\delta}\omega\right) + \tilde{d}f' + r^{-1}(n+1)\tilde{d}f. \tag{3.1.16}$$

As pointed out in [CT94, Section 2], any 1-form defined on $\mathbb{R}^n\backslash\{0\}$ can be written as an infinite sum of forms of two types

1. Type I: $p(r)\psi$ where $\psi \in \Lambda^1(T^*S^{n-1})$, $\tilde{\delta}\psi = 0$ and $\tilde{\Delta}_H\psi = \mu\psi$,

2. Type II: $r^{-1}l(r)\phi dr + u(r)r\tilde{d}\phi$ where $\phi \in \Lambda^0(T^*S^{n-1})$ and $\tilde{\Delta}_H\phi = \nu\phi$.

Moreover, the operator $\Box$ preserves these two types of forms. If ξ is a 1-form of type I or II, the equation $\Box\xi = 0$ reduces to a second order linear system of ordinary differential equations of at most two equations. In order to see what these systems look like we consider the change of variable $r = e^s$ and use $p(s)$ to denote $p(e^s)$ for forms of type I and we use $l(s)$, $u(s)$ to denote $l(e^s)$ and $u(e^s)$ respectively for forms of type II. With this notation we have for example

$$p' = e^{-s}\partial_s p,\ p'' = e^{-2s}\left(\partial_s^2 p - \partial_s p\right). \tag{3.1.17}$$

On forms of type I, $\Box\xi$ is given by

$$\Box\xi = e^{-2s}\left(\partial_s^2 p + (n-4)\partial_s p - \mu p\right)\psi, \tag{3.1.18}$$

while on forms of type II, $\Box\xi$ takes the form

$$\begin{aligned}\Box\xi = e^{-2s}&\left(2\partial_s^2 l + 2(n-4)\partial_s l - 4(n-2+\frac{\nu}{4})l - \nu\partial_s u + 4\nu u\right)\phi ds \\ &+ e^{-2s}\left(\partial_s^2 u + (n-4)\partial_s u - 2\nu u + \partial_s l + nl\right)\tilde{d}\phi.\end{aligned} \tag{3.1.19}$$

The solutions of (3.1.18) and (3.1.19) are given by the following

$$\alpha = \frac{4-n}{2},\ \theta = \sqrt{\alpha^2+\mu},\ a^{\pm} = \alpha \pm \theta. \tag{3.1.20}$$

All solutions of $\Box\xi = 0$ of type I are given by

$$\xi = r^{a^{\pm}}\psi. \tag{3.1.21}$$

In this case, since $r\psi$ is radially parallel, we see that the order of growth of ξ is $a^{\pm}-1$. For solutions ξ of type II, we set

$$\beta = \frac{2-n}{2},\ \zeta = \sqrt{\beta^2+\nu},\ b^{\pm} = \beta \pm \zeta, \tag{3.1.22}$$

and then ξ is either of the form

$$r^{b^{\pm}}\tilde{d}\phi + b^{\pm}r^{b^{\pm}-1}\phi dr, \tag{3.1.23}$$

or of the form

$$2r^{b^{\pm}+2}\tilde{d}\phi + b^{\mp}r^{b^{\pm}+1}\phi dr. \tag{3.1.24}$$

See [CT94, Section 2] for more details. The above computations motivate the following definition

Definition 3.1.1. The set E of all numbers $a^{\pm}-1$, $b^{\pm}-1$ or $b^{\pm}+1$ with $a^{\pm}$, $b^{\pm}$ defined by (3.1.20), (3.1.22), is called the set of *exceptional values for* $\Box$. If $\gamma \in \mathbb{R}\backslash E$ then γ is said to be *nonexceptional*.

Remark 3.1.2. All elements in E are integers, in fact, computing the eigenvalues of $\tilde{\Delta}_H$ on 1-forms in S^{n-1} as in [Fol89, Theorem C], one can prove that all numbers in (3.1.21) have the form

$$a_j^{\pm} = -\frac{(n-4)}{2} \pm \frac{1}{2}(n-2+2j) \ \text{ for } j = 1, 2, \ldots, \tag{3.1.25}$$

and all numbers in (3.1.22) have the form

$$b_j^{\pm} = -\frac{(n-2)}{2} \pm \frac{1}{2}(n-2+2j) \ \text{ for } j = 0, 1, 2, \ldots. \tag{3.1.26}$$

An important property of $\Box$ is

Proposition 3.1.3. *On $\mathbb{R}^n \setminus \{0\}$, if γ is nonexceptional then $\Box : W'^{2,p,0,1}_{\gamma} \to W'^{0,p,0,1}_{\gamma-2}$ is an isomorphism with bounded inverse.*

Proof. Compare [Bar86, Theorem 1.7]. $\Box$

Finally, note that from (3.1.25) and (3.1.26) it follows that 1 is an exceptional value for $\Box$, which means that there are elements in the kernel of $\Box$ with linear growth, i.e., forms ξ with $\Box\xi = 0$ satisfying $\xi = r\eta$ where η is radially parallel.

All 1-forms of type I in the kernel of $\Box$ which have linear growth have the form

$$\xi = r^2\psi, \tag{3.1.27}$$

where ψ is dual to a Killing field in S^{n-1}. For forms of type II, all solutions of $\Box\xi = 0$ that have linear growth correspond to the eigenvalues $\nu = 0, 2n$ of $\tilde{\Delta}_H$ on functions, moreover, in that case the solution corresponding to $\nu = 0$ is

$$\xi = rdr, \tag{3.1.28}$$

and the solution corresponding to $\nu = 2n$ is

$$\xi = 2r\phi\tilde{d}r + r^2\tilde{d}\phi. \tag{3.1.29}$$

Note that the forms in (3.1.27) and (3.1.29) have linear growth because $r\psi$ and $r\tilde{d}\phi$ are radially parallel.

3.1.2 A modified $\Box$ operator

Given $t \neq 0$, let $\Box_t$ be the modified operator

$$\Box_t \xi \equiv \delta_t L_\xi g_0 = \Box \xi - t i_{r^{-1}\frac{\partial}{\partial r}} L_\xi g_0. \tag{3.1.30}$$

In order to compute $\Box_t$ for ξ as in (3.1.12) we start by noting that

$$i_{r^{-1}\frac{\partial}{\partial r}} L_\xi g_0 = 2r^{-1} f' dr + r^{-1}\left(\omega' - 2r^{-1}\omega + \tilde{d} f\right).$$

The modified operator $\Box_t$ is then computed to be

$$\begin{aligned}\Box_t \xi = &\left(2f'' + 2(n-1-t)r^{-1}f' + r^{-2}\left(-\tilde{\Delta}_H f - 2(n-1)f\right)\right) dr \\ &+ r^{-2}\left(\tilde{\delta}\omega' - 4r^{-3}\tilde{\delta}\omega\right) dr + \omega'' + (n-3-t)r^{-1}\omega' \\ &+ r^{-2}\left(-\tilde{\Delta}_H \omega + 2t\omega + \tilde{d}\tilde{\delta}\omega\right) + \tilde{d}f' + r^{-1}(n+1-t)\tilde{d}f.\end{aligned}$$

Using again the change of variable $r = e^s$ and the notation in Section 3.1.1, we see that $\Box_t \xi$ for ξ a 1-form of type I is given by

$$\Box_t \xi = e^{-2s}\left(\partial_s^2 p + (n-4-t)\partial_s p - (\mu - 2t)p\right)\psi. \tag{3.1.31}$$

On forms of type II, $\Box_t \xi$ is given by

$$\begin{aligned}\Box_t \xi = &e^{-2s}\left(2\partial_s^2 l + 2(n-4-t)\partial_s l - 4(n-2-\frac{t}{2}+\frac{\nu}{4})l - \nu\partial_s u + 4\nu u\right)\phi ds \\ &+ e^{-2s}\left(\partial_s^2 u + (n-4-t)\partial_s u - 2(\nu - t)u + \partial_s l + (n-t)l\right)\tilde{d}\phi.\end{aligned} \tag{3.1.32}$$

We conclude that in these cases, the system

$$\Box_t \xi = 0, \tag{3.1.33}$$

reduces again to a constant coefficient system of ordinary differential equations. From (3.1.31), we easily see that the possible growth and decay rates of solutions (3.1.33) on

1-forms of type I are given by

$$-\frac{1}{2}(n-4-t) \pm \frac{1}{2}\sqrt{(n-4-t)^2+4(\mu-2t)}. \tag{3.1.34}$$

For forms of type II, we note that we can simplify (3.1.32) by setting

$$\begin{aligned} l(s) &= e^{-\frac{(n-4-t)}{2}s} m(s), \\ u(s) &= e^{-\frac{(n-4-t)}{4}s} w(s), \end{aligned}$$

so we obtain a system of equations involving only m and w, namely

$$\begin{aligned} \partial_s^2 m - \left\{(n-4-t)^2/4 + 2(n-2-t/2+\nu/4)\right\} m - \frac{\nu}{2}\partial_s w + \frac{\nu}{4}(n+4-t)w = 0, \\ \partial_s^2 w - \left\{\frac{(n-4-t)^2}{4} + 2(\nu-t)\right\} w + \partial_s m + \frac{(n+4-t)}{2} m = 0. \end{aligned}$$

Let

$$\begin{aligned} \beta(\nu,t) =& \frac{(n-4-t)^2}{4} + 2(n-2-t/2+\nu/4), \\ \gamma(\nu,t) =& -\frac{\nu}{4}(n+4-t), \\ \sigma(\nu,t) =& \frac{(n-4-t)^2}{4} + 2(\nu-t), \\ \tau(t) =& -\frac{(n+4-t)}{2}. \end{aligned}$$

In what follows we will omit to write the dependence of $\beta, \gamma, \sigma, \tau$ on ν and t. If in addition we let

$$\begin{aligned} m_1 &= m, \\ m_2 &= \partial_s m_1, \\ w_1 &= w, \\ w_2 &= \partial_s w_1, \end{aligned}$$

then the vector

$$X(s) = \begin{pmatrix} m_1(s) \\ m_2(s) \\ w_1(s) \\ w_2(s) \end{pmatrix},$$

satisfies

$$\partial_s X(s) = AX(s),$$

where A is the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \beta & 0 & \gamma & \frac{\nu}{2} \\ 0 & 0 & 0 & 1 \\ \tau & -1 & \sigma & 0 \end{pmatrix}. \tag{3.1.35}$$

The characteristic polynomial $p(\lambda)$ of the matrix A in (3.1.35) is given by

$$p(\lambda) = \lambda^4 - (\sigma + \beta - \frac{\nu}{2})\lambda^2 + (\gamma - \frac{\tau\nu}{2})\lambda + \beta\sigma - \gamma\tau, \tag{3.1.36}$$

and note that

$$\gamma - \frac{\tau\nu}{2} = -\frac{\nu}{4}(n + 4 - t) + \frac{(n + 4 - t)}{2}\frac{\nu}{2} = 0.$$

The roots of the characteristic polynomial p in (3.1.36) are then given by

$$\lambda^{\pm}(\nu, t) = \pm\sqrt{\frac{\beta + \sigma - \nu/2}{2} \pm \frac{1}{2}\sqrt{(\beta + \sigma - \nu/2)^2 + 4\gamma\tau - 4\beta\sigma}}. \tag{3.1.37}$$

We can simplify the expression in (3.1.37) by fixing t and studying the dependence of the functions $\frac{\beta+\sigma-\nu/2}{2}$ and $(\beta + \sigma - \nu/2)^2 + 4\gamma\tau - 4\beta\sigma$ on the parameter ν. More precisely, if we let

$$\theta_1(\nu) = \frac{\beta + \sigma - \nu/2}{2},$$

and

$$\theta_2(\nu) = (\beta + \sigma - \nu/2)^2 + 4\gamma\tau - 4\beta\sigma,$$

then

$$\frac{\partial}{\partial \nu}\theta_1(\theta) = 2,$$
$$\frac{\partial}{\partial \nu}\theta_2(\theta) = 16 - 8t,$$

and after evaluating $\theta_1(\nu)$ and $\theta_2(\nu)$ at $\nu = 0$, we obtain

$$\lambda^{\pm}(\nu, t) = \sqrt{\frac{1}{4}(n-4-t)^2 + (n-2) - \frac{3}{2}t + \nu \pm \frac{1}{2}\sqrt{(2(n-2)+t)^2 + (16-8t)\nu}}. \tag{3.1.38}$$

From (3.1.38), we conclude that the possible growth and decay rates of solutions of (3.1.33) on forms of type II are

$$\begin{cases} -\frac{(n-2-t)}{2} \pm \lambda^{\pm}(\nu, t) & \text{for } \nu > 0, \\ -\frac{(n-2-t)}{2} \pm \frac{1}{2}\sqrt{(n-t)^2 + 4t} & \text{for } \nu = 0. \end{cases} \tag{3.1.39}$$

As in the discussion at the end of Subsection 3.1.1, we are interested in solutions of (3.1.33) which are *essentially linear*, i.e., solutions ξ that satisfy for every $0 < \gamma < 1$,

$$\xi = O(r^{1+\gamma}) \text{ as } r \to \infty, \ \xi = O(r^{1-\gamma}) \text{ as } r \to 0. \tag{3.1.40}$$

For these solutions we have

Proposition 3.1.4. *There exists $t_0 > 0$ such that for $0 < |t| < t_0$ there is a number $\gamma_0(t)$ with $0 < \gamma_0(t) < 1$ such that if ξ is a 1-form in the kernel of $\Box_t$ satisfying (3.1.40) for some $0 < \gamma < \gamma_0(t)$ then ξ is dual to a Killing vector field in $\mathbb{R}^n$.*

Proof. Since $\Box_t \xi = \delta_t L_\xi g_0$, it follows that a 1-form which is dual to a Killing field in $\mathbb{R}^n$ is in the kernel of $\Box_t$ *regardless* of the gauge condition. Next, since the general solution may be written as an infinite sum of 1-forms of Type I and Type II, it suffices to prove the proposition for 1-forms of either type. When $t = 0$ and as pointed out in Section 3.1.1, all solutions of (3.1.33) in separated variables which satisfy (3.1.40) correspond to the eigenvalues $\mu = 2(n-2)$ on forms of type I and $\nu = 0, 2n$ on forms of type II. For $t \neq 0$ small, the growth rates are small perturbations of the rates $a_j^\pm - 1$ and $b_j^\pm - 1$ with $a_j^\pm$, $b_j^\pm$ given by (3.1.25) and (3.1.26), moreover the rates corresponding to eigenvalues $\mu > 2n$ or $\nu > 2n$ have real parts bounded away from 1 and hence, for these solutions the proposition follows. It only remains to consider the kernel corresponding to the eigenvalues $\mu = 2(n-2)$ and $\nu = 0, 2n$. From [Fol89, Theorem C], all eigenvalues μ corresponding to 1-forms of type I are

$$\mu = (j+1)(j+n-3) \text{ for } j = 1, 2, \dots, \tag{3.1.41}$$

and then from (3.1.34) all solutions of $\Box_t \xi = 0$ with ξ a 1-form of type I can be written as

$$\xi = r^{c_j^\pm(t)-1} \left(r\psi_j \right), \tag{3.1.42}$$

where

$$c_j^\pm(t) = \frac{-(n-4-t) \pm \sqrt{(n-2+2j)^2 - 2nt + t^2}}{2}, \tag{3.1.43}$$

and ψ_j is a 1-form with eigenvalue $(j+1)(j+n-3)$. It follows that if $t \neq 0$ is sufficiently small, all solutions given by (3.1.43) are such that $Re\left(c_j^{\pm}(t)) - 1\right)$ is bounded away from 1 except for $c_1^{+}(t)-1$ which equals 1 for *any* t. In this case $r^2\psi_1$ is dual to a Killing field in $\mathbb{R}^n$. For those 1-forms of type II corresponding to the eigenvalues $\nu = 0, 2n$, one can show in a similar way using (3.1.39) that the growth rates are strictly bounded away from 1 for $t \neq 0$ sufficiently small. $\square$

Chapter 4

Existence of divergence-free gauges

4.1 Outline

In order to construct divergence-free gauges, we use the approach in [CT94] which consists in using the ALE of order 0 condition to initially prove that we can fix a gauge such that the modified divergence-free condition $\delta_t h = 0$ for $t \neq 0$ is satisfied. One is then interested in the δ_t-free kernel of the modified operator $\mathcal{P}_t^{(k)}$. This kernel can be classified into three types:

1. Growth solutions, i.e., solutions that are $O(r^{\beta'})$ on $\mathbb{R}^n \backslash \{0\}$ for some $\beta' > 0$,
2. Decay solutions, i.e., solutions that are $O(r^{-\beta'})$ on $\mathbb{R}^n \backslash \{0\}$ for some $\beta' > 0$,
3. "Degenerate" solutions, i.e., solutions that are $O(r^{\gamma})$ as $r \to \infty$ and $O(r^{-\gamma})$ as $r \to 0$ for some $\gamma > 0$ sufficiently small (depending upon t).

The main step is to prove that solutions with the behavior described in (3) do *not* occur (see Proposition 4.3.1). We can then prove a growth estimate for solutions of the linear elliptic equation $\mathcal{P}_t^{(k)} h = 0$ (see Lemma 4.2.3) which we call the *Three Annulus Lemma*. The next step is to use scaling properties of the nonlinear system (2.1.29), and elliptic estimates to prove a nonlinear version of the Three Annulus Lemma (Lemma 4.4.4), so that the behavior of solutions of (2.1.29) can be modeled after the behavior of solutions of the linearized equation. Consequently, we can use the nonlinear Three Annulus Lemma and the ALE of order zero condition to rule out the behavior described in (1) and (3) for solutions of the nonlinear equation. It follows that the only possible behavior at infinity for solutions of the nonlinear equation in the δ_t-gauge is that of decay solutions in (2), which yields a gauge where the metric g is ALE of positive order (see Corollary 4.5.4). With this improvement, one we can easily construct a global divergence-free gauge, see Proposition 4.5.5.

The Three Annulus Lemma was introduced in [Sim85] and used in [CT94] for the Ricci-flat case. Even though our statement of the Three Annulus Lemma is very similar to that of [CT94], we base our proof on a result called Turán's Lemma that we discuss in Section 4.6. Our case is complicated by the fact that higher powers of log may enter into the asymptotic expansions since the system is of higher order.

4.2 The linearized equation in separated variables

We consider solutions on $\mathbb{R}^n \backslash \{0\}$ of the system

$$\mathcal{P}_t^{(k)} h = 0, \tag{4.2.1}$$

for $1 \leq k \leq \frac{n}{2} - 1$ if $n \geq 4$, or for $k = 1$ if $n = 3$, with $\mathcal{P}_t^{(k)}$ given by (2.1.26). With notation as in [CT94], we write the general solution of $\mathcal{P}_t^{(k)} h = 0$ as an infinite sum of the form

$$h = \sum_{j=0}^{\infty} \left(l_j \phi_j dr \otimes dr + k_j \tau_j \boxtimes dr + f_j B_j + p_j \phi_j r^2 \underline{\tilde{g}} \right), \tag{4.2.2}$$

where

1. The functions l_j,k_j, f_j and p_j are radial, and the $(0,2)$ tensors $\phi_j dr \otimes dr$, $\tau_j \boxtimes dr$, B_j, and $\phi_j r^2 \underline{\tilde{g}}$ are radially parallel.

2. The components $\phi_j \in \Lambda^0(T^*S^{n-1})$, $\tau_j \in \Lambda^1(T^*S^{n-1})$, $B_j \in S_0^2(T^*S^{n-1})$ are eigentensors of the rough laplacian. Here $S_0^2(T^*S^{n-1})$ is the traceless component of $S^2(T^*S^{n-1})$.

3. The set

$$\{l_j\phi_j dr \otimes dr + k_j\tau_j \boxtimes dr + f_j B_j + p_j\phi_j r^2 \underline{\tilde{g}}\}_j,$$

is orthogonal in the inner product $\langle\langle \cdot, \cdot \rangle\rangle$ in (3.1.8).

It is clear that $\mathcal{P}_t^{(k)}$ preserves the expansion (4.2.2) in the sense that

$$\begin{aligned}
&\mathcal{P}_t^{(k)}\left(l_j\phi_j dr \otimes dr + k_j\tau_j \boxtimes dr + f_j B_j + p_j\phi_j \underline{\tilde{g}}\right) \\
&= F_j^{(1)}\phi_j dr \otimes dr + F_j^{(2)}\tau_j \boxtimes dr + F_j^{(3)} B_j + F_j^{(4)}\phi_j r^2 \underline{\tilde{g}},
\end{aligned}$$

where $F_j^{(c)} = F_j^{(c)}(l_j, k_j, f_j, p_j)$ for $c = 1, 2, 3, 4$. It follows that if $\mathcal{P}_t^{(k)} h = 0$ then for each j we must have

$$\mathcal{P}_t^{(k)}\left(l_j\phi_j dr \otimes dr + k_j\tau_j \boxtimes dr + f_j B_j + p_j\phi_j r^2 \underline{\tilde{g}}\right) = 0, \tag{4.2.3}$$

and

$$F_j^{(c)}(l_j, k_j, f_j, p_j) \equiv 0, \text{ for } c = 1, 2, 3, 4. \tag{4.2.4}$$

The system (4.2.4) is a linear system of ordinary differential equations which is homogeneous of order $2(k+1)$. Using the change of variable $r = e^s$, (4.2.4) can be written as a constant coefficient linear system of ordinary differential equations, in particular, for every $j = 0, 1, \ldots,$ the system (4.2.4) reduces to a first order constant coefficient linear system of the form

$$\dot{X} = M_j X, \tag{4.2.5}$$

where M_j is a matrix of order $8(k+1) \times 8(k+1)$. Let Φ_j is the characteristic polynomial of the matrix A_j and suppose that we factor Φ_j as

$$\Phi_j(z) = \prod_{a=1}^{m_j} (z - \zeta_{j,a})^{n_{j,a}}, \tag{4.2.6}$$

with

$$\sum_{a=1}^{m_j} n_{j,a} = 8(k+1). \tag{4.2.7}$$

Note that in general, the roots $\zeta_{j,a}$ depend on t and k, however, for simplicity we omit this dependence in the notation above. Each of the functions $l_j,\ k_j,\ f_j,\ p_j$ may be expressed as a linear combination of functions of the form

$$(\log(r))^b r^{\zeta_{j,a}} \text{ with } b = 0, \ldots, n_{j,a} - 1. \tag{4.2.8}$$

Fix j and let $T_j^{(c)}$ with $c = 1, 2, 3, 4,$ be the $(0, 2)$ tensors

$$T_j^{(1)} = \phi_j dr \otimes dr,\ T_j^{(2)} = \tau_j \boxtimes dr,\ T_j^{(3)} = B_j,\ T_j^{(4)} = \phi_j r^2 \underline{\tilde{g}}, \tag{4.2.9}$$

so that we can write

$$l_j \phi_j dr \otimes dr + k_j \tau_j \boxtimes dr + f_j B_j + p_j \phi_j r^2 \underline{\tilde{g}} = \sum_{c=1}^{4} q_{j,c} T_j^{(c)}, \tag{4.2.10}$$

where

$$q_{j,1} = l_j,\ q_{j,2} = k_j,\ q_{j,3} = f_j \text{ and } q_{j,4} = p_j. \tag{4.2.11}$$

It follows that if h satisfies $\mathcal{P}_t^{(k)}h = 0$ on $\mathbb{R}^n\backslash\{0\}$ then h can be expanded as an infinite sum of the form

$$h = \sum_{j=0}^{\infty}\sum_{c=1}^{4} q_{j,c}T_j^{(c)}, \tag{4.2.12}$$

where

1. The $(0,2)$-tensors $T_j^{(c)}$ are radially parallel.
2. The set $\{T_j^{(c)}\}_{j,c}$ is orthogonal with respect to the norm $\langle\langle\cdot,\cdot\rangle\rangle$.
3. The radial functions $q_{j,c}$ are linear combinations of functions of the form (4.2.8).

Let us write the radial function $q_{j,c}(r)$ as

$$q_{j,c}(r) = \sum_{a=1}^{m_j}\sum_{b=0}^{n_{j,a}-1} d_{a,b,c}(\log(r))^b r^{\zeta_{j,a}}, \tag{4.2.13}$$

where $d_{a,b,c}$ are complex numbers. From (4.2.13) we introduce the following sets

$$A_j^+ = \{1 \leq a \leq m_j : Re(\zeta_{j,a}) > 0\}, \tag{4.2.14}$$

$$A_j^- = \{1 \leq a \leq m_j : Re(\zeta_{j,a}) < 0\}, \tag{4.2.15}$$

$$A_j^0 = \{1 \leq a \leq m_j : Re(\zeta_{j,a}) = 0\}. \tag{4.2.16}$$

We will use $q_{j,c}^{\pm}$ to denote

$$q_{j,c}^{\pm}(r) = \sum_{a\in A_j^{\pm}}\sum_{b=0}^{n_{j,a}-1} d_{a,b,c}(\log(r))^b r^{\zeta_{j,a}}, \tag{4.2.17}$$

and $q_{j,c}^0$ will be used to denote

$$q_{j,c}^0(r) = \sum_{a\in A_j^0}\sum_{b=0}^{n_a-1} d_{a,b,c}(\log(r))^b r^{\zeta_{j,a}}. \tag{4.2.18}$$

With this we have a decomposition of the form

$$h = h^+ + h^- + h^0, \tag{4.2.19}$$

where

$$h^\pm = \sum_{j=0}^{\infty}\sum_{c=1}^{4} q_{j,c}^\pm T_j^{(c)}, \text{ and } h^0 = \sum_{j=0}^{\infty}\sum_{c=1}^{4} q_{j,c}^0 T_j^{(c)}. \tag{4.2.20}$$

Definition 4.2.1. A solution h of (4.2.8) is a *degenerate solution of* (4.2.1) if $h = h^0$.

It will also be important for us to consider for any nonnegative integer j the number

$$\beta_j = \min\{|Re(\zeta_{j,a})| : a \in A_j^\pm\}. \tag{4.2.21}$$

4.2.1 Estimates for the linearized equation

Consider a solution of (4.2.1) on an annulus, i.e., a solution of the problem

$$\mathcal{P}_t^{(k)} h = 0 \text{ on } A_{a,b}(0), \tag{4.2.22}$$

where $0 < a < b$. Note that since $A_{a,b}(0)$ and $\mathbb{R}^n \backslash \{0\}$ have the same cross section, if h is a solution of (4.2.22) then we can repeat the analysis in Subsection 4.2 to decompose h as $h = h^+ + h^- + h^0$. For solutions of (4.2.22), however, the infinite sum (4.2.12) may *not* be defined outside of $A_{a,b}(0)$. By definition of the norm $|||\cdot|||$, if we expand a solution of h of (4.2.22) satisfying $|||h|||_{a,b} < \infty$ as in (4.2.12) we see that

$$|||h|||_{a,b}^2 = \sum_{j=0}^{\infty}\sum_{c=1}^{4} \lambda_j^{(c)} \int_a^b |q_{j,c}(r)|^2 r^{-1} dr, \tag{4.2.23}$$

where

$$\lambda_j^{(c)} = \langle\langle T_j^{(c)}, T_j^{(c)} \rangle\rangle. \tag{4.2.24}$$

We consider again the numbers β_j in (4.2.21) and we define

$$\beta = \inf_{j=0,1,\dots} \{\beta_j\}. \tag{4.2.25}$$

The number β is well-defined and positive for t sufficiently small, since the equation (4.2.1) is a perturbation of $\Delta^{k+1}h = 0$, which has indicial roots contained in $\mathbb{Z} \subset \mathbb{C}$ (compare (2.0.10)-(2.0.11)).

We have the following property for solutions of (4.2.22):

Lemma 4.2.2. *For t sufficiently small, let $0 < \beta' < \frac{1}{2}\beta$. Let h be a solution of* (4.2.22) *on an annulus of the form $A_{a,L^2a}(0)$ where $a > 0$ and $L > 1$, and consider the decomposition*

$$h = h^+ + h^- + h_0 \text{ on } A_{a,L^2a}(0). \tag{4.2.26}$$

Then there exists $L_0 = L_0(\beta, \beta') > 1$ such that if $L > L_0$, then

$$|||h^+|||_{La,L^2a} \geq L^{\beta'} |||h^+|||_{a,La}, \tag{4.2.27}$$

and

$$|||h^-|||_{La,L^2a} \leq L^{-\beta'} |||h^-|||_{a,La}. \tag{4.2.28}$$

Proof. By the scale invariance of the norms $|||\cdot|||$, it suffices to prove the lemma for $a = 1$. The proof is completed in Section 4.6 using Turán's Lemma. □

We note that we are only able to prove (4.2.27) and (4.2.28) for $0 < \beta' < \frac{1}{2}\beta$ and not for $0 < \beta' < \beta$ as in [CT94]. However, the estimates (4.2.27) and (4.2.28) are sufficient for our purpose. Next, we use this to prove

Lemma 4.2.3 (Three Annulus Lemma). *Let $a > 0$, $L > 1$ and suppose that h is a solution of* (4.2.1) *in $A_{La,L^3a}(0)$ for t sufficiently small. Suppose in addition that in the decomposition* (4.2.26)*, $h_0 \equiv 0$. For any $0 < \beta' < \frac{1}{2}\beta$, there exists $L_0 = L_0(\beta, \beta')$ with $L_0 > 1$ such that for $L > L_0$, if*

$$|||h|||_{aL,aL^2} \geq L^{\beta'} |||h|||_{a,aL}, \tag{4.2.29}$$

then

$$|||h|||_{aL^2,aL^3} \geq L^{\beta'} |||h|||_{aL,aL^2}, \tag{4.2.30}$$

and if

$$|||h|||_{aL^2,aL^3} \leq L^{-\beta'} |||h|||_{aL,aL^2}, \tag{4.2.31}$$

then

$$|||h|||_{aL,aL^2} \leq L^{-\beta'} |||h|||_{a,La}. \tag{4.2.32}$$

Moreover, at least one of (4.2.30), (4.2.32) *holds (whether or not at least one of* (4.2.29), (4.2.32) *holds).*

Proof. By scaling properties of the norms $||| \cdot |||$, it suffices to prove the lemma for the case $a = 1$. Suppose that (4.2.29) holds. From the decomposition $h = h^+ + h^-$ and the Cauchy-Schwarz inequality we clearly have

$$|||h|||^2_{L,L^2} \leq 2\left(|||h^+|||^2_{L,L^2} + |||h^-|||^2_{L,L^2}\right), \tag{4.2.33}$$

and then

$$2\left(|||h^+|||^2_{L,L^2} + |||h^-|||^2_{L,L^2}\right) \geq L^{2\beta'}\left(|||h^+|||^2_{1,L} + |||h^-|||^2_{1,L} + 2\langle\langle\langle h^+, h^-\rangle\rangle\rangle_{1,L}\right). \tag{4.2.34}$$

Here $\langle\langle\langle\cdot,\cdot\rangle\rangle\rangle$ is the inner product associated to the norm $|||\cdot|||$. From Lemma 4.2.2, for L large enough (depending on β and β') we can estimate $\langle\langle\langle h^+, h^-\rangle\rangle\rangle_{1,L}$ in (4.2.34) as

$$\langle\langle\langle h^+, h^-\rangle\rangle\rangle_{1,L} \geq -|||h^+|||_{1,L}|||h^-|||_{1,L} \geq -L^{-\beta'}|||h^+|||_{L,L^2}|||h^-|||_{1,L}, \tag{4.2.35}$$

and then for a fixed $0 < \epsilon < \frac{1}{2}$, there exists a positive constant $c(\epsilon)$ such that (4.2.34) and (4.2.35) imply

$$\begin{aligned} L^{2\beta'}&\left(|||h^+|||_{1,L}^2 + |||h^-|||_{1,L}^2 - 2c(\epsilon)L^{-2\beta'}|||h^+|||_{L,L^2}^2 - 2\epsilon|||h^-|||_{1,L}^2\right) \\ &\leq 2\left(|||h^+|||_{L,L^2}^2 + |||h^-|||_{L,L^2}^2\right), \end{aligned} \tag{4.2.36}$$

and hence

$$2(1+c(\epsilon))\left(|||h^+|||_{L,L^2}^2 + |||h^-|||_{L,L^2}^2\right) \geq (1-2\epsilon)L^{2\beta'}\left(|||h^+|||_{1,L}^2 + |||h^-|||_{1,L}^2\right). \tag{4.2.37}$$

We have shown that for fixed $0 < \epsilon < \frac{1}{2}$ there exists a positive constant $q(\epsilon)$ such that

$$\left(|||h^+|||_{L,L^2}^2 + |||h^-|||_{L,L^2}^2\right) \geq q(\epsilon)L^{2\beta'}\left(|||h^+|||_{1,L}^2 + |||h^-|||_{1,L}^2\right). \tag{4.2.38}$$

Combining Lemma 4.2.2 with (4.2.38) we obtain

$$\left(|||h^+|||_{L,L^2}^2 + |||h^-|||_{L,L^2}^2\right) \geq q(\epsilon)L^{2\beta'}|||h^-|||_{1,L}^2 \geq q(\epsilon)L^{4\beta'}|||h^-|||_{L,L^2}^2, \tag{4.2.39}$$

and therefore

$$|||h^+|||_{L,L^2}^2 \geq \left(q(\epsilon)L^{4\beta'} - 1\right)|||h^-|||_{L,L^2}^2. \tag{4.2.40}$$

On the other hand, for fixed $0 < \epsilon < \frac{1}{2}$ we choose $c(\epsilon)$ as before so that

$$|||h|||_{L^2,L^3}^2 \geq (1-2\epsilon)|||h^+|||_{L^2,L^3}^2 - 2c(\epsilon)|||h^-|||_{L^2,L^3}^2, \tag{4.2.41}$$

and by virtue of Lemma 4.2.2, for any β'' with $\beta' < \beta'' < \frac{1}{2}\beta$, there exists $L_0 = L_0(\beta, \beta'') > 0$ such that if $L > L_0$ then

$$|||h|||^2_{L^2,L^3} \geq (1-2\epsilon)L^{2\beta''}|||h^+|||^2_{L,L^2} - 2c(\epsilon)L^{-2\beta''}|||h^-|||_{L,L^2}. \tag{4.2.42}$$

If L is large enough so that $L^{-2\beta''} < \frac{1}{2}$ we have from (4.2.40)

$$\begin{aligned} L^{2\beta''}|||h^+|||^2_{L,L^2} &\geq L^{2\beta''}\left(\frac{1}{2} + L^{-2\beta''}\right)|||h^+|||^2_{L,L^2} \geq \frac{1}{2}L^{2\beta''}|||h^+|||^2_{L,L^2} + |||h^+|||^2_{L,L^2} \\ &\geq \frac{1}{2}L^{2\beta''}|||h^+|||^2_{L,L^2} + \left(q(\epsilon)L^{4\beta'} - 1\right)|||h^-|||^2_{L,L^2}. \end{aligned} \tag{4.2.43}$$

Finally, from (4.2.42) and (4.2.43) it follows that

$$|||h|||^2_{L^2,L^3} \geq v(\epsilon, L, \beta', \beta'')L^{2\beta'}|||h^+|||^2_{L,L^2} + w(\epsilon, L, \beta', \beta'')L^{2\beta'}|||h^-|||^2_{L,L^2}, \tag{4.2.44}$$

where

$$v(\epsilon, L, \beta', \beta'') = \frac{1}{2}(1-2\epsilon)L^{2(\beta''-\beta')}, \tag{4.2.45}$$

and

$$w(\epsilon, L, \beta', \beta'') = (1-2\epsilon)(q(\epsilon)L^{2\beta'} - L^{-2\beta'}) - 2c(\epsilon)L^{-2(\beta'+\beta'')}. \tag{4.2.46}$$

It is clear that we can choose L large enough so that $v(\epsilon, L, \beta', \beta''), w(\epsilon, L, \beta', \beta'') \geq 2$ and then

$$|||h|||^2_{L^2,L^3} \geq 2L^{2\beta'}\left(|||h^+|||^2_{L,L^2} + |||h^-|||^2_{L,L^2}\right) \geq L^{2\beta'}|||h|||^2_{L,L^2}, \tag{4.2.47}$$

as needed. The proof for the case (4.2.31) is analogous. For the rest of the proposition, note that by the Cauchy-Schwarz inequality we must have either $|||h^+|||_{L,L^2} \geq \frac{1}{2}|||h|||_{L,L^2}$ or $|||h^-|||_{L,L^2} \geq \frac{1}{2}|||h|||_{L,L^2}$. If $|||h^+|||_{L,L^2} \geq \frac{1}{2}|||h|||_{L,L^2}$, then for fixed $0 < \epsilon < 1$ there exists $c_0(\epsilon) > 1$ such that

$$(1-c_0(\epsilon))|||h^+|||^2_{L,L^2} + (1-\epsilon)|||h^-|||^2_{L,L^2} \leq |||h|||^2_{L,L^2}, \tag{4.2.48}$$

and since $|||h|||^2_{L,L^2} \leq 4|||h^+|||^2_{L,L^2}$ we conclude that for some positive constant $c_1(\epsilon)$ we have

$$|||h^+|||^2_{L,L^2} \geq c_1(\epsilon)|||h^-|||^2_{L,L^2}. \tag{4.2.49}$$

On the other hand, if we fix $0 < \epsilon < 1$ there exists a constant $c_2(\epsilon) > 0$ such that

$$|||h|||^2_{L^2,L^3} \geq (1-\epsilon)|||h^+|||^2_{L^2,L^3} - c_2(\epsilon)|||h^-|||^2_{L^2,L^3}, \tag{4.2.50}$$

and from Lemma 4.2.2, for any $\beta' < \beta'' < \frac{1}{2}\beta$ there exists $L_0 = L_0(\beta, \beta'')$ such that if $L > L_0$ then

$$|||h|||^2_{L^2,L^3} \geq (1-\epsilon)L^{2\beta''}|||h^+|||^2_{L,L^2} - c_2(\epsilon)L^{-2\beta''}|||h^-|||^2_{L,L^2}, \tag{4.2.51}$$

and from (4.2.49) we have

$$|||h|||^2_{L^2,L^3} \geq C(\epsilon, L, \beta', \beta'')L^{2\beta'}|||h^+|||^2_{L,L^2}, \tag{4.2.52}$$

where

$$C(\epsilon, L, \beta', \beta'') = (1-\epsilon)L^{2(\beta''-\beta')} - \frac{c_2(\epsilon)}{c_1(\epsilon)}L^{-2(\beta'+\beta'')}. \tag{4.2.53}$$

If we choose L large enough so that $C(\epsilon, L, \beta', \beta'') \geq 4$ we obtain

$$|||h|||^2_{L^2,L^3} \geq 4L^{2\beta'}|||h^+|||^2_{L,L^2} \geq L^{2\beta'}|||h|||^2_{L,L^2}, \tag{4.2.54}$$

as needed. In a similar way we can show that if $|||h^-|||^2_{L,L^2} \geq \frac{1}{2}|||h|||^2_{L,L^2}$, then we have the inequality $|||h|||^2_{L,L^2} \leq L^{-2\beta'}|||h|||^2_{1,L}$, which completes the proof. □

4.3 Degenerate solutions of the linearized equations

We now turn our attention to degenerate solutions of (4.2.1). If $t = 0$, then constants are non-trivial degenerate solutions, which are also divergence-free. However, these are not

δ_t-free for $t \neq 0$. The main result of this section is that there are in fact *no* degenerate solutions of (4.2.1) for all t nonzero and sufficiently small:

Proposition 4.3.1. *There exists $t_0 > 0$ such that if $0 < |t| < t_0$ there are no degenerate solutions of* (4.2.1) *subject to $\delta_t h = 0$ on any annulus $A_{c,d}(0)$. In particular, for $t \neq 0$ sufficiently small, Lemma* 4.2.3 *holds.*

Proof. We only need to consider the case that h is a finite sum in (4.2.12). In this case, h extends to a solution of $\mathcal{P}_t^{(k)} h = 0$ on $\mathbb{R}^n \backslash \{0\}$ subject to $\delta_t h = 0$. Let $\rho > 0$, let $t_0, t, \gamma_0(t)$ be as in Proposition 3.1.4 and let φ be a C^∞ function such that $\varphi|_{B_\rho(0)} \equiv 0$ and $\varphi|_{\mathbb{R}^n \backslash B_{2\rho}(0)} \equiv 1$. Choose $p > n$ and a number $0 < \gamma < \gamma_0(t) < 1$, then $\varphi \delta h \in {W'}^{0,p,0,1}_{\gamma-1}$ and $(1-\varphi)\delta h \in W^{0,p,0,1}_{-\gamma-1}$. Since γ is nonexceptional it follows from Proposition 3.1.3 that there exists $X_1 \in {W'}^{2,p,0,1}_{\gamma+1}$ such that $\Box X_1 = \varphi \delta h$ and $X_2 \in {W'}^{2,p,0,1}_{-\gamma+1}$ such that $\Box X_2 = (1-\varphi)\delta h$, therefore

$$\Box \left(X_1 + X_2 \right) = \delta h, \tag{4.3.1}$$

moreover, there exists h_0 such that $\delta h_0 \equiv 0$, $\varphi h_0 \in {W'}^{1,p,0,2}_{\gamma}$, $(1-\varphi) h_0 \in {W'}^{1,p,0,2}_{-\gamma}$, and

$$h = L_{(X_1+X_2)} g_0 + h_0 \text{ on } \mathbb{R}^n \backslash \{0\}. \tag{4.3.2}$$

Note that from the weighted Sobolev inequality (see [Bar86, Theorem 1.2]), we must have

$$h_0 = O(r^{-\gamma}) \text{ as } r \to 0, \quad \text{and } h_0 = O(r^{\gamma}) \text{ as } r \to \infty. \tag{4.3.3}$$

Let $X = X_1 + X_2$, from $\mathcal{P}_t^{(k)} h = 0$ and $\delta_t h = 0$, h satisfies

$$\Delta^{k-1} \left(\frac{1}{2} \Delta^2 h + \frac{1}{2} \nabla^2 \delta \delta h + \Delta \delta^* \delta h \right) = 0 \tag{4.3.4}$$

on $\mathbb{R}^n\backslash\{0\}$. From diffeomorphism invariance, any Lie derivative $L_X g_0$ is in the kernel of the linearized operator, and similarly, $R'_{g_0}(L_X g_0) = 0$. It follows that $L_X g_0$ satisfies (4.3.4) and so does h_0 on $\mathbb{R}^n\backslash\{0\}$. From $\delta h_0 = 0$, h_0 satisfies

$$\Delta^{k+1} h_0 = 0, \tag{4.3.5}$$

so we can expand h_0 in terms of homogeneous solutions of (4.3.5) on $\mathbb{R}^n\backslash\{0\}$ and by (4.3.3) and the proof of Proposition 2.0.2 we conclude that $h_0 = \log(r) \cdot C + C'$ where C, C' are matrices whose components are constant, but since $\delta h_0 \equiv 0$ it follows from Proposition 2.0.2 that $C = 0$ and h_0 is constant in $\mathbb{R}^n$, in particular h_0 is a Lie derivative. We can now write $h = L_Y g_0$ where Y is a solution of $\Box_t Y = \delta_t L_Y g_0 \equiv 0$ and clearly Y is essentially linear in the sense of (3.1.40) for some γ with $0 < \gamma < \gamma_0(t)$. But from Proposition 3.1.4, we know that for $t \neq 0$ sufficiently small, if any such solution Y is non-zero then it must be dual to a Killing field which shows that $h \equiv 0$ as needed. In the case $n = 3$, all solutions $\Delta^2 h_0 = 0$ satisfying (4.3.3) are of the form $h_0 = C + h_1$ where the components of C are constant and the components of h_1 are spherical harmonics of order 1, however, if $\delta h_0 \equiv 0$ then $h_1 \equiv 0$ as seen in the proof of Proposition 2.0.2. $\square$

Remark 4.3.2. The argument in [Str10a, Corollary 3.7] is incomplete since only radially parallel solutions are ruled out there. One must moreover rule out degenerate solutions (those with oscillatory behavior and possibly times a power of log) which are *not* radially parallel.

4.4 Scaling and the nonlinear equation

In this subsection we prove the nonlinear version of the Three Annulus Lemma. We assume that (M^n, g) is ALE of order 0, as in Definition 1.3.10. In the following we use the ALE coordinate system to transfer the problem to $\left(\mathbb{R}^n \backslash B_\rho(0)\right)/\Gamma$. We have the following elliptic Schauder estimate for solutions of (2.1.29):

Lemma 4.4.1. *Let* $0 < a < d$ *and let* $h \in \mathcal{T}^{0,2}_{m,\alpha;0}(A_{a,d})$ *be a solution of* (2.1.29)*. There exists* $\chi > 0$ *such that if* $\|h\|_{\mathcal{T}^{0,2}_{m,\alpha;0}(A_{a,d}(0))} < \chi$*, then for every* b, c *with* $a < b < c < d$ *one has*

$$\|h\|_{\mathcal{T}^{0,2}_{m,\alpha;0}(A_{b,c}(0))} \leq C|||h|||_{a,d}, \tag{4.4.1}$$

with $C = C(\lambda_1, \lambda_2, n, m, \alpha, b-a, d-c, t)$*, where* $0 < \lambda_1 \leq \lambda_2$ *are ellipticity constants of* (2.1.29).

Proof. The result follows from standard interior elliptic regularity estimates, see for example [Èĭd69, Chapter II]. Note the leading order term is a power of the Laplacian, but lower order coefficients are negative powers of r. However, the Schauder estimate depends only on an appropriate weighted norm of the coefficients, which in this case is bounded, as one can easily verify. □

The following scaling lemma will be used to reduce the nonlinear problem to the linear case.

Lemma 4.4.2. *Let* $\{h_i\}$ *be a sequence of solutions of* (2.1.29) *satisfying*

$$\|h_i\|_{\mathcal{T}^{0,2}_{m,\alpha;0}(A_{a,L^3 a}(0))} < \chi_i, \tag{4.4.2}$$

where $\{\chi_i\}$ *is a sequence of positive numbers such that* $\chi_i \to 0$*. Suppose in addition that for some positive constant* C *we have*

$$|||h_i|||_{a,La} + |||h_i|||_{L^2a,L^3a} \leq C|||h_i|||_{La,L^2a}. \tag{4.4.3}$$

Let $q_i = |||h_i|||^{-1}_{La,L^2a} h_i$*, then on any annulus* $A_{c,d}(0)$ *with* $a < c < d < L^3 a$*, there exists a subsequence* q_{i_j} *that converges in* $\mathcal{T}^{0,2}_{m,\alpha';0}(A_{c,d}(0))$ *with* $\alpha' < \alpha$ *to* $\tilde{q}$ *satisfying* (4.2.1)*.*

Proof. The sequence $\{q_i\}$ satisfies

$$|||q_i|||_{a,La} + |||q_i|||_{La^2,L^3a} \leq C, \tag{4.4.4}$$

in particular, if c, d as in the statement, we have from Lemma 4.4.1 the inequality

$$\|q_i\|_{\mathcal{T}^{0,2}_{m,\alpha;0}(A_{c,d}(0))} \leq C', \tag{4.4.5}$$

for some positive constant C'. By the Arzela-Ascoli theorem there exists a subsequence $\{q_{i_j}\}$ that converges in $\mathcal{T}^{0,2}_{m,\alpha';0}(A_{c,d}(0))$ with $\alpha' < \alpha$ to $\tilde{q}$. It only remains to prove that $\tilde{q}$ solves (4.2.1). For that purpose we write (2.1.29) as

$$\mathcal{P}^{(k)}_t(c_i q_i) + \mathcal{R}^{(k)}(c_i q_i, g_0) = 0, \tag{4.4.6}$$

where $c_i = |||h_i|||_{La,L^2a}$. Note that $c_i \to 0$ as $i \to \infty$. From the estimate (4.4.5) and equation (2.1.28) it follows that

$$\mathcal{R}^{(k)}(c_i q_i, g_0) = O(c_i^2) \text{ as } c_i \to 0, \tag{4.4.7}$$

where the bound in the right-hand side of (4.4.7) is with respect to the Hölder norm in $\mathcal{T}^{0,2}_{m-2(k+1),\alpha';0}(A_{c,d}(0))$, so (4.4.6) takes the form

$$c_i \mathcal{P}^{(k)}_t(q_i) = O(c_i^2) \text{ as } c_i \to 0. \tag{4.4.8}$$

Passing to the subsequence $\{q_{i_j}\}$ we conclude that the limit $\tilde{q}$ satisfies $\mathcal{P}^{(k)}_t \tilde{q} = 0$. □

Remark 4.4.3. Using scaling properties of the norms $|||\cdot|||_{c,d}$ and $\|\cdot\|_{\mathcal{T}^{(0,2)}_{m,\alpha;0}\left(A_{c,d}(0)\right)}$ and the ALE of order 0 conditions, we can construct a nontrivial sequence $\{q_i\}_i$ of solutions of (2.1.29) on the annulus $A_{a,aL^3}(0)$ satisfying $\|q_i\|_{\mathcal{T}^{(0,2)}_{m,\alpha;0}(A_{L,L^3}(0))} \to 0$ as $i \to \infty$. More precisely, if h is defined on $A_{a,\infty}(0)$, then from the ALE of order 0 condition, we see that for any sequence of positive numbers b_i with $b_i \to \infty$ we have

$$\lim_{i\to\infty} \|h\|_{\mathcal{T}^{(0,2)}_{m,\alpha;0}\left(A_{b_ia,b_iaL^3}(0)\right)} = 0,$$

from (3.1.2) and (3.1.10) we see that $q_i = b_i^{-2}\psi_i^* h$, satisfies

$$|||q_i|||_{a,aL^3} = |||h|||_{b_ia,b_iaL^3},$$

$$\lim_{i\to\infty} \|q_i\|_{\mathcal{T}^{(0,2)}_{m,\alpha;0}\left(A_{a,aL^3}(0)\right)} = 0,$$

as needed.

Using Lemma 4.2.3 and Proposition 4.3.1 we have the following nonlinear version of the Three Annulus Lemma:

Lemma 4.4.4. *Let $\rho, t > 0$ and let h be a solution of (2.1.29) on $A_{\rho,\infty}(0)$ with $\delta_t h = 0$. Let $\beta' > 0$ and $L_0 > 1$ be as in Lemma 4.2.3 and let $L, a > 0$ be such that $L_0 a > \rho$ and $L > L_0$. There exist $\chi = \chi(n, \lambda, \Lambda) > 0$ so that if $|h|_{\mathcal{T}^{0,2}_{m,\alpha;0}(A_{\rho,\infty}(0))} < \chi$, then if*

$$|||h|||_{La,L^2a} \geq L^{\beta'}|||h|||_{a,La}, \tag{4.4.9}$$

then

$$|||h|||_{L^2a,L^3a} \geq L^{\beta'}|||h|||_{La,L^2a}, \tag{4.4.10}$$

and if

$$|||h|||_{L^2a,L^3a} \leq L^{-\beta'}|||h|||_{La,L^2a}, \tag{4.4.11}$$

then

$$|||h|||_{La,L^2a} \leq L^{-\beta'}|||h|||_{a,La}. \tag{4.4.12}$$

Moreover, there exists $t_0 > 0$ such that if $0 < |t| < t_0$ then at least one of the inequalities (4.4.10),(4.4.12) *must hold.*

Proof. For simplicity we will only prove the last assertion, i.e., at least one of (4.4.10) and (4.4.12) must hold (the other two assertions are proved analogously). Suppose this is not the case, then, there exists a sequence of positive numbers χ_i with $\lim_{i\to\infty}\chi_i = 0$ and solutions h_i of (2.1.29) with $|h_i|_{\mathcal{T}^{0,2}_{m,\alpha;0}(A_{\rho,\infty}(0))} < \chi_i$ such that neither (4.4.10) nor (4.4.11) hold for h_i (such a sequence can be obtained from Remark 4.4.3, for example). In that case we have,

$$|||h_i|||_{a,La} + |||h_i|||_{L^2a,L^3a} \leq 2L^{\beta'}|||h_i|||_{La,L^2a}.$$

Choose a sequence of positive numbers $\{\epsilon_l\}_l$ with $0 < \epsilon_l < 1$ such that $\epsilon_l \searrow 0$. If we rescale h_i by $\tilde{h}_i = |||h_i|||^{-1}_{La,L^2a}h_i$, we can use Lemma 4.2.3 to obtain a subsequence $\tilde{h}_{i,1}$ with $|||\tilde{h}_{i,1}|||_{a,L^3a} \leq C$ for some positive number C, that converges in $\mathcal{T}^{0,2}_{m,\alpha;0}(A_{a+\epsilon_1,L^3a-\epsilon_1}(0))$ to a solution q_1 of (4.2.1) satisfying $\delta_t q_1 = 0$. From the sequence $\{\tilde{h}_{i,1}\}$ we can obtain a subsequence $\{\tilde{h}_{i,2}\}_i$ that converges in $\mathcal{T}^{0,2}_{m,\alpha;0}(A_{a+\epsilon_2,L^3a-\epsilon_2}(0))$ to a solution q_2 of (4.2.1) satisfying $\delta_t q_2 = 0$. By repeating this construction we can produce for every $l > 1$ a subsequence $\{\tilde{h}_{i,l}\}_i$ of $\{\tilde{h}_i\}_i$ satisfying

1. $|||\tilde{h}_{i,l}|||_{a,L^3a} \leq C$,
2. $\{\tilde{h}_{i,l}\}_i$ is a subsequence of $\{\tilde{h}_{i,l-1}\}_i$,
3. $\{\tilde{h}_{i,l}\}_i$ converges in $\mathcal{T}_{m,\alpha;0}(A_{a+\epsilon_l,L^3a-\epsilon_l}(0))$,

4. The limit q_l of $\{\tilde{h}_{i,l}\}_i$ as $i \to \infty$ is a solution of (4.2.1) that satisfies $\delta_t q_l = 0$,

5. The restriction q_l to $A_{a+\epsilon_{l-1}, L^3 a - \epsilon_{l-1}}(0)$ is q_{l-1}.

It follows that from the diagonal sequence $\{h_l^*\} = \{\tilde{h}_{l,l}\}$, we can obtain a solution $\tilde{h}$ of (4.2.1) on $\mathcal{T}^{0,2}_{m,\alpha;0}(A_{a,L^3a}(0))$ satisfying $\delta_t \tilde{h} = 0$ and $|||\tilde{h}|||_{a,L^3a} < \infty$. This solution does not satisfy (4.4.10) or (4.4.12) and this contradicts Proposition 4.3.1 if $t > 0$ is sufficiently small.

□

4.5 Global divergence-free gauges

From (1.3.15) and (1.3.16) it follows that $\Psi_* g - g_0$ satisfies the estimate

$$| \left(\Psi_* g - g_0 \right)_{(r,x)} |_{m,\alpha;0} = o(1) \text{ as } r \to \infty. \tag{4.5.1}$$

for all $m \geq 0$.

Remark 4.5.1. We do not need to assume decay on derivatives of arbitrary order in (1.3.16). Our proof only requires that the estimate (1.3.16) be satisfied only for all multi-indices l such that $1 \leq |l| \leq 2(k+1)+1$. Then (4.5.1) holds for all $m \leq 2(k+1)$.

With the decay in (4.5.1) we can prove the existence of δ_t-free gauges on certain annuli by means of the implicit function theorem. The following result is contained in [CT94, Theorem 3.1] and does *not* depend on the metric g being $\Omega^{(k)}$-flat:

Proposition 4.5.2. *Let t_0 be as in Proposition 3.1.4 and let t be such that $0 < |t| < t_0$. There exists $\chi = \chi(t, m)$ such that if $(C(N^{n-1}), g_0)$ is a Ricci-flat cone and $\tilde{g}$ is a metric* *on* $A_{c,\infty}(0) \subset C(N^{n-1})$ *such that*

$$\|\tilde{g} - g_0\|_{\mathcal{T}^{0,2}_{m,\alpha;0}(A_{c,\infty}(0))} < \chi, \tag{4.5.2}$$

then there exists a diffeomorphism $\phi = \phi(\tilde{g})$, $\phi : A_{c,\infty}(0) \to A_{c,\infty}(0)$ *such that*

$$\phi^* \tilde{g} \in \mathcal{T}^{0,2}_{m,\alpha;0}(A_{c,\infty}(0)), \tag{4.5.3}$$

and

$$\delta_t(\phi^* \tilde{g} - g_0) = 0. \tag{4.5.4}$$

Moreover, if

$$\|\delta_t(\tilde{g})\|_{\mathcal{T}^{0,1}_{m-1,\alpha;-1}(A_{c,\infty}(p))} < \epsilon, \tag{4.5.5}$$

then

$$\|\phi^* \tilde{g} - \tilde{g}\|_{\mathcal{T}^{0,2}_{m,\alpha;0}(A_{c,\infty}(p))} < \delta(\epsilon), \tag{4.5.6}$$

where $\delta(\epsilon) \searrow 0$ *as* $\epsilon \to 0$.

In our application of this Lemma, we will simply let $\tilde{g} = \Psi_* g$, and g_0 the flat metric on the Euclidean cone (recall Definition 1.3.10). We will then write $h = \phi^* \tilde{g} - g_0$. From Proposition 4.5.2 and using that g is ALE of order 0 we have

Lemma 4.5.3. *Let* h *and* $A_{c,\infty}(0)$ *be as above, and let* $L_0 > 0$ *be as in Lemma 4.2.3. For any* $L > L_0$ *and any* $a > c$ *we have* $\lim_{i\to\infty} |||h|||_{L^i a, L^{i+1} a} = 0$, *moreover, we have the inequality*

$$|||h|||_{L^i a, L^{i+1} a} \leq L^{-\beta'(i-i_0)} |||h|||_{L^{i_0} a, L^{i_0+1} a} \text{ for all } i \geq i_0. \tag{4.5.7}$$

Proof. By Lemma 4.4.4, for any $i > i_0$ we have either

$$|||h|||_{L^{i+1},L^{i+2}} \geq L^{\beta'} |||h|||_{L^i,L^{i+1}}, \tag{4.5.8}$$

or

$$|||h|||_{L^i,L^{i+1}} \leq L^{-\beta'} |||h|||_{L^{i-1},L^i}, \tag{4.5.9}$$

Suppose that we have (4.5.8), then we conclude again from Lemma 4.4.4 that we must have for any integer $s \geq 1$ the inequality

$$|||h|||_{L^{i+s},L^{i+s+1}} \geq L^{\beta' s} |||h|||_{L^i,L^{i+1}}. \tag{4.5.10}$$

Since g is ALE of order 0, we can use (4.5.5) in Proposition 4.5.2 to conclude that for $\epsilon > 0$ sufficiently small and $s > 1$ large we must have

$$\|\Psi_* g - g_0\|_{\mathcal{T}^{0,2}_{m,\alpha;0}(A_{L^{i+s},L^{i+s+1}}(0))} < \epsilon, \tag{4.5.11}$$

$$\|\phi^* \Psi_* g - \Psi_* g\|_{\mathcal{T}^{0,2}_{m,\alpha;0}(A_{L^{i+s},L^{i+s+1}(0)})} < \frac{1}{2}, \tag{4.5.12}$$

and therefore

$$\|h\|_{\mathcal{T}^{0,2}_{m,\alpha;0}(A_{L^{i+s},L^{i+s+1}}(0))} < 1. \tag{4.5.13}$$

In particular, we must have

$$|||h|||_{L^{i+s},L^{i+s+1}} \leq c_n \log(L), \tag{4.5.14}$$

where c_n is a dimensional constant. It is clear that (4.5.10) contradicts (4.5.14) for s large. It follows that (4.5.9) holds and then, by Lemma 4.4.4 we must have (4.5.7). □

From Lemma 4.5.3 we have the following improvement in the ALE order of g:

Corollary 4.5.4. *If g is ALE of order 0, scalar flat and (extended) obstruction-flat or satisfies* (1.2.9)*, then there exists an annulus of the form $A_{c',\infty}(0)$ and a diffeomorphism $\phi : A_{c',\infty}(0) \to A_{c',\infty}(0)$ such that $|\phi^*\Psi_* g - g_0|_{m,\alpha;0} = O(r^{-\beta'})$ as $r \to \infty$ and therefore, g is ALE of order β'.*

Proof. From Lemma 4.4.1, Proposition 4.5.2, and Lemma 4.5.3, there exists a constant $C > 0$ such that for any $(r, x) \in A_{c',\infty}(0)$ with r sufficiently large we have

$$|(\phi^*\Psi_* g - g_0)_{(r,x)}|_{m,\alpha;0} \leq C r^{-\beta'},$$

so the claim follows. □

From [CT94, Sections 2 and 3] we have

Proposition 4.5.5. *Suppose g is a metric defined on $\mathbb{R}^n \backslash B_\rho(0)$ and satisfies*

$$g - g_0 = O(r^{-\tau}) \text{ as } r \to \infty, \tag{4.5.15}$$

then there exists a diffeomorphism $\phi : \mathbb{R}^n \backslash B_\rho(0) \to \mathbb{R}^n \backslash B_\rho(0)$ such that $h = \phi^* g - g_0$ satisfies $h \in \mathcal{T}^{0,2}_{m,\alpha;-\tau}(\mathbb{R}^n \backslash B_\rho(0))$ and $\delta_{g_0} h = 0$.

We summarize the results of this section in the following corollary

Corollary 4.5.6. *If g is ALE of order 0, scalar flat and (extended) obstruction-flat or satisfies* (1.2.9)*, then there exists a diffeomorphism $\Phi : M \backslash K \to (\mathbb{R}^n \backslash B_\rho(0))/\Gamma$ for some $\rho > 0$ and a compact set $K \subset M$ such that $h = \Phi_* g - g_0$ satisfies $h \in \mathcal{T}^{0,2}_{m,\alpha;-\beta'}$ and $\delta_{g_0} h = 0$.*

4.6 Turán's Lemma

We close this chapter by stating a classical result proved by Pál Turán which is sometimes called *Turán's Lemma* and showing how one can apply it to prove Lemma 4.2.2. Given complex numbers $z_1, \ldots, z_d$ and a nonnegative integer l we use S_l denote the sum

$$c_1, \ldots, c_d \in \mathbb{C}, \quad S_l = \sum_{j=1}^{d} c_j z_j^l. \tag{4.6.1}$$

The following is the version of Turán's Lemma that we will use throughout this section:

Lemma 4.6.1. *Let $z_1, \ldots, z_d$ where $d > 1$ be complex numbers with $|z_j| \geq 1$ for $j = 1, \ldots, d$ and let m be an integer with $m \geq 1$. Then*

$$|S_0|^2 \leq C \max\{|S_{m+1}|^2, \ldots, |S_{m+d}|^2\}, \tag{4.6.2}$$

where the positive constant $C = C(m, d)$ can be estimated as

$$C \leq A(d) \left(\frac{m+d}{d}\right)^{2(d-1)}, \tag{4.6.3}$$

for some constant $A(d)$ depending only on d.

A way of proving Lemma 4.6.1 is by means of the following elementary lemma whose proof we omit (see also [Naz93, Section 1.1]).

Lemma 4.6.2. *Let $m \geq 1$ be an integer and let $p(z)$ be the polynomial*

$$u(z) = \prod_{j=1}^{d} \left(1 - \frac{z}{z_j}\right) \sigma_m \left(\prod_{j=1}^{d} \left(1 - \frac{z}{z_j}\right)^{-1} \right), \tag{4.6.4}$$

where for a C^∞ function f(z) at the origin, $\sigma_m(f)(z)$ denotes the Taylor polynomial of degree m of $f(z)$ around 0. Then $u(z)$ is a polynomial of degree $m + d$ that has the form

$$u(z) = 1 + a_1 z^{m+1} + \ldots + a_d z^{m+d}, \tag{4.6.5}$$

and vanishes at the points $z_1 \dots, z_d$. Moreover, for $j = 1, \dots, d$ we have $|a_j| \le |b_{m+j}|$ where b_{m+j} is the coefficient of the z^{m+j} in the polynomial

$$v(z) = (1+z)^d \sigma_m \left(\frac{1}{(1-z)^d} \right), \tag{4.6.6}$$

In particular, we have

$$\max\{|a_1|, \dots, |a_d|\} \le d2^d \binom{m+d-1}{d-1}. \tag{4.6.7}$$

4.6.1 An integral form of Turán's Lemma

Let h be a positive number and consider the arithmetic progression $\{t_l = lh\}_l$ where l are nonnegative integers. Consider also complex numbers $\zeta_1, \dots, \zeta_d$ such that $Re(\zeta_j) \ge 0$ for $j = 1, \dots, d$. If we let $p(t)$ denote

$$p(t) = \sum_{j=1}^{d} c_j e^{\zeta_j t}, \tag{4.6.8}$$

then

$$p(lh) = \sum_{j=1}^{d} c_j z_j^l(h), \tag{4.6.9}$$

where $z_j(h) = e^{\zeta_j h}$. Since $|z_j(h)| \ge 1$ we have from Lemma 4.6.1 an inequality of the form

$$|p(0)|^2 \le C(m,d) \left(\max_{l=m+1,\dots,m+d} \left\{ |p(lh)|^2 \right\} \right) \le C(m,d) \left(\sum_{l=m+1}^{m+d} |p(lh)|^2 \right), \tag{4.6.10}$$

for any integer $m \ge 1$ where $C(m,d) \le A(d) \left(\frac{m+d}{d}\right)^{2(d-1)}$.

Lemma 4.6.3. *Let $\zeta_1, \dots, \zeta_d$ be complex numbers satisfying $Re(\zeta_j) \ge 0$ for $j = 1, \dots, d$, and let $p(t) = \sum_{j=1}^{d} c_j e^{\zeta_j t}$. Then for any positive numbers $0 < a < b$*

$$|p(0)|^2 \le A(d) \left(\frac{b}{b-a} \right)^{2(d-1)} \frac{(b+a)}{(b-a)^2} \int_a^b |p(t)|^2 dt. \tag{4.6.11}$$

Proof. Let $c = \frac{a+b}{2}$, set $h_0 = \frac{b-c}{d} = \frac{b-a}{2d}$ and let m be the integer part of $\frac{c}{h_0}$ (i.e. the only integer m such that $m \leq \frac{c}{h_0} < m+1$). Note that in this case $m \geq 1$. For any $h > 0$ by (4.6.10) we have

$$|p(0)|^2 \leq A(d) \left(\frac{m+d}{d}\right)^{2(d-1)} \sum_{l=m+1}^{m+d} \left(|p(lh)|^2\right). \tag{4.6.12}$$

By our choice of m and h_0 we have

$$(m+d)h_0 \leq c + dh_0 = b, \tag{4.6.13}$$

so then $a \leq lh_0 \leq b$ for $l = m+1, \ldots, m+d$. Let $\eta = \frac{2a}{a+b}$, i.e. $c\eta = a$. Note that $0 < \eta < 1$. By taking the average of (4.6.12) respect to h on the interval $[\eta h_0, h_0]$ we have

$$|p(0)|^2 \leq A(d) \left(\frac{m+d}{d}\right)^{2(d-1)} \frac{1}{h_0(1-\eta)} \int_{\eta h_0}^{h_0} \left(\sum_{l=m+1}^{m+d} |p(lh)|^2\right) dh, \tag{4.6.14}$$

It is easy to see that if $f : \mathbb{R} \mapsto \mathbb{R}$ is a continuous nonnegative function and and T_0, T_1 are positive numbers with $T_0 < T_1$, then

$$\int_{T_0}^{T_1} \left(\sum_{l=m+1}^{m+d} f(lh)\right) dh \leq \frac{d}{2} \int_{(m+1)T_0}^{(m+d)T_1} f(h)dh. \tag{4.6.15}$$

Therefore, we have

$$|p(0)|^2 \leq A(d)\frac{d}{2} \left(\frac{m+d}{d}\right)^{2(d-1)} \frac{1}{h_0(1-\eta)} \int_a^b |p(t)|^2 dt, \tag{4.6.16}$$

where we have used that $(m+1)\eta h_0 > c\eta = a$. We also have $(1-\eta)h_0 = \frac{(b-a)^2}{2d(b+a)}$. On the other hand

$$\left(\frac{m+d}{d}\right)^{2(d-1)} = \left(\frac{(m+d)h_0}{dh_0}\right)^{2(d-1)} \leq 2^{2(d-1)} \left(\frac{b}{b-a}\right)^{2(d-1)}, \tag{4.6.17}$$

so the result follows. □

Corollary 4.6.4. *If $p(t)$ is as before then for any $R > 0$ we have*

$$\|p\|^2_{L^\infty[0,R]} \leq \frac{A(d)}{R} \int_{\frac{3R}{2}}^{2R} |p(t)|^2 dt, \tag{4.6.18}$$

and also

$$\int_0^R |p(t)|^2 dt \leq A(d) \int_{3\frac{R}{2}}^{2R} |p(t)|^2 dt. \tag{4.6.19}$$

Proof. Let $t_0 \in [0, R]$ be such that $\|p\|_{L^\infty[0,R]} = |p(t_0)|$, and consider $p_{t_0}(\tau)$ given by

$$p_{t_0}(\tau) = \sum_{j=1}^{d} c_j e^{\zeta_j t_0} e^{\zeta_j \tau}. \tag{4.6.20}$$

Clearly $p_{t_0}(t - t_0) = p(t)$ and $|p_{t_0}(0)| = \|p\|_{L^\infty[0,R]}$. By Lemma 4.6.3 we have

$$|p_{t_0}(0)|^2 \leq 4 \cdot 2^{d-1} A(d) \left(\frac{2R - t_0}{R}\right)^{d-1} \left(\frac{\frac{7}{2}R - 2t_0}{R^2}\right) \int_{\frac{3R}{2} - t_0}^{2R - t_0} |p_{t_0}(\tau)|^2 d\tau. \tag{4.6.21}$$

Using a change of variables in (4.6.21) we obtain

$$\|p\|^2_{L^\infty[0,R]} \leq 4 \cdot 4^{d-1} A(d) \left(\frac{7}{2R}\right) \int_{\frac{3}{2}R}^{2R} |p(t)|^2 dt, \tag{4.6.22}$$

which proves (4.6.18). Once we have shown (4.6.18) we can prove (4.6.19) by writing

$$\frac{1}{R} \int_0^R |p(t)|^2 dt \leq \|p\|^2_{L^\infty[0,R]} \leq \frac{A(d)}{R} \int_{\frac{3R}{2}}^{2R} |p(t)|^2 dt. \tag{4.6.23}$$

□

4.6.2 Proof of Lemma 4.2.2

By making the change of variables $r = e^t$, it it is clear from (4.2.23) that it suffices to show the following lemma

Lemma 4.6.5. *Let $p(t)$ be a sum of the form*

$$p(t) = \sum_{j=1}^{d} \sum_{s=0}^{n_j} c_{j,s} t^s e^{\zeta_j t}, \tag{4.6.24}$$

where $n_j \geq 0$ are integers, and $c_{j,s} \in \mathbb{C}$ are fixed. Let $M = \sum_{j=1}^{d} n_j$. Then

1. *If $\lambda = \min\{Re(\zeta_1), \ldots, Re(\zeta_d)\} > 0$ then for every positive integer l,*

$$e^{\lambda R} \int_{(l-1)R}^{lR} |p(t)|^2 dt \leq A(M+d) \int_{lR}^{(l+1)R} |p(t)|^2 dt. \tag{4.6.25}$$

2. *If $\lambda = \min\{-Re(\zeta_1), \ldots, -Re(\zeta_d)\} > 0$, then for any positive integer l,*

$$\int_{lR}^{(l+1)R} |p(t)|^2 dt \leq C(M+d) e^{-\lambda R} \int_{(l-1)R}^{lR} |p(t)|^2 dt. \tag{4.6.26}$$

Proof. We first consider functions $p : \mathbb{R} \mapsto \mathbb{C}$ that have the form

$$p(t) = \sum_{j=1}^{d} c_j e^{\zeta_j t}, \tag{4.6.27}$$

with $c_j \in \mathbb{C}$, i.e., we do not consider the case of roots ζ_j with multiplicities. Suppose that all numbers ζ_j, $j = 1, \ldots, d$ have positive real part. Let us first prove (4.6.25) for $l = 1$. Consider the function $\tilde{p}(t)$ given by

$$\tilde{p}(t) = \sum_{j=1}^{d} c_j e^{(\zeta_j - \lambda)t}. \tag{4.6.28}$$

Since $Re(\zeta_j - \lambda) \geq 0$ for all $j = 1, \ldots, d$, we have from Corollary 4.6.4

$$\int_0^R |\tilde{p}(t)|^2 dt \leq A(d) \int_{\frac{3R}{2}}^{2R} |\tilde{p}(t)|^2 dt, \tag{4.6.29}$$

multiplying both sides of (4.6.29) by $e^{3\lambda R}$ we have

$$\begin{aligned} e^{\lambda R} \int_0^R e^{2\lambda t} |\tilde{p}(t)|^2 dt &\leq e^{3\lambda R} \int_0^R |\tilde{p}(t)|^2 dt \leq A(d) e^{3\lambda R} \int_{\frac{3R}{2}}^{2R} |\tilde{p}(t)|^2 dt \\ &\leq A(d) \int_{\frac{3R}{2}}^{2R} e^{2\lambda t} |\tilde{p}(t)|^2 dt \leq A(d) \int_R^{2R} e^{2\lambda t} |\tilde{p}(t)|^2 dt, \end{aligned} \tag{4.6.30}$$

and $e^{2\lambda t}|\tilde{p}(t)|^2 = |p(t)|^2$. For the case $l > 1$, we write any $t \in [(l-1)R, (l+1)R]$ as $t = (l-1)R + \tau$ where $\tau \in [0, 2R]$ and then we write $p(t)$ as

$$p(t) = q_l(\tau) = \sum_{j=1}^{d} c_{j,l} e^{\zeta_j \tau}, \tag{4.6.31}$$

where $c_{j,l} = c_j e^{(l-1)R}$. Applying the above argument to $q_l(\tau)$ then (4.6.25) follows after a change of variables. If now all numbers $Re(\lambda_j)$ are negative for $j = 1 \ldots, d$, it suffices to prove (4.6.26) for $l = 1$ since as before, the general case $l \geq 1$ follows after a change of variables. Let $\tilde{p}(t) = \sum_{j=1}^{d} c_j e^{(\zeta_j+\lambda)t}$ and write $t \in [0, 2R]$ as $t = 2R - \tau$ where $\tau \in [0, 2R]$, then $c_j e^{(\zeta_j+\lambda)t} = c_j e^{(\zeta_j+\lambda)2R} e^{-(\zeta_j+\lambda)\tau}$ and $Re(-(\zeta_j + \lambda)) \geq 0$, so by (4.6.25), if we let $q(\tau) = \sum_{j=1}^{d} c_j e^{(\zeta_j+\lambda)2R} e^{-(\zeta_j+\lambda)\tau}$ we obtain

$$\int_0^R |q(\tau)|^2 d\tau \leq A(d) \int_{\frac{3R}{2}}^{2R} |q(\tau)|^2 d\tau. \tag{4.6.32}$$

On the other hand,

$$\int_0^R |q(\tau)|^2 d\tau = \int_R^{2R} |\tilde{p}(t)|^2 dt, \text{ and } \int_{\frac{3R}{2}}^{2R} |q(\tau)|^2 d\tau = \int_0^{\frac{R}{2}} |\tilde{p}(t)|^2 dt, \tag{4.6.33}$$

so from (4.6.32), we have

$$\begin{aligned} e^{\lambda R} \int_R^{2R} e^{-2\lambda t} |\tilde{p}(t)|^2 dt \leq e^{-\lambda R} \int_R^{2R} |\tilde{p}(t)|^2 dt \leq A(d) e^{-\lambda R} \int_0^{\frac{R}{2}} |\tilde{p}(t)|^2 dt \\ \leq A(d) \int_0^{\frac{R}{2}} e^{-2\lambda t} |\tilde{p}(t)|^2 dt \leq A(d) \int_0^R e^{-2\lambda t} |\tilde{p}(t)|^2 dt, \end{aligned} \tag{4.6.34}$$

and $e^{-2\lambda t}|\tilde{p}(t)|^2 = |p(t)|^2$.

For the general case involving multiplicity, we will only prove the statement for the case $\min\{Re(\zeta_1), \ldots, Re(\zeta_d)\} > 0$ since the other case will follow as in the above argument. We consider first the case corresponding to $n_1 = 1$ and $n_2 = \ldots = n_d = 0$. Let $\epsilon > 0$ and let $p_\epsilon(t)$ be given by

$$p_\epsilon(t) = c_{1,0} e^{\zeta_1 t} + \frac{c_{1,1}}{\epsilon} \left(e^{(\zeta_1+\epsilon)t} - e^{\zeta_1 t} \right) + \sum_{j=2}^{d} c_{j,0} e^{\zeta_j t}, \tag{4.6.35}$$

then by (4.6.25) for the case with no multiplicity, we have

$$e^{\lambda R} \int_{(l-1)R}^{lR} |p_\epsilon(t)|^2 dt \leq A(d+1) \int_{lR}^{(l+1)R} |p_\epsilon(t)|^2 dt, \tag{4.6.36}$$

where λ is defined as before, and since $A(d+1)$ does not depend on ϵ we can take the limit of (4.6.36) as ϵ tends to zero and obtain (4.6.25). For the higher multiplicity case, (4.6.25) can be proved using induction. □

4.7 Optimal ALE order

In this Section, we complete the proof of Theorems 1.2.2 and 1.2.4.

4.7.1 Weighted Sobolev spaces and Δ^{k+1}

In this section we state some properties of the weighted Sobolev spaces introduced in Section 4.3 that will be useful to improve the decay estimate for the metric g derived in Section 4.5. Throughout this section we will work only with $(0,2)$ tensors so when we write ${W'}^{m,p}_{\delta}$ we actually mean the space ${W'}^{m,p,0,2}_{\delta}$. We start by defining the set of exceptional values for Δ^{k+1}.

Definition 4.7.1. A number $\delta \in \mathbb{R}$ is said to be *exceptional* for Δ^{k+1} if δ is in the set

$$E = \begin{cases} \{j \in \mathbb{Z} : j \neq -1, -2, \dots, 2(k+1) - (n-1)\} & \text{if } n > 2(k+1) \\ \mathbb{Z} & \text{if } n = 2(k+1). \end{cases} \tag{4.7.1}$$

We say that δ is *nonexceptional* if $\delta \in \mathbb{R} \backslash E$.

Remark 4.7.2. The exceptional values for Δ^{k+1} correspond to the growth rates of solutions of $\Delta^{k+1} h = 0$ on the complement of a ball, however, when $n = 2(k+1)$, as observed in the proof of Proposition 2.0.2, there are solutions of $\Delta^{k+1} h = 0$ on $\mathbb{R}^n \backslash \{0\}$ that are $O(\log(r))$ as $r \to \infty$.

Lemma 4.7.3. *If δ is nonexceptional, the map $\Delta^{k+1} : {W'}^{2(k+1),p}_{\delta} \to {W'}^{0,p}_{\delta-2(k+1)}$ is an isomorphism.*

Proof. See [Bar86, Theorem 1.7]. □

Lemma 4.7.4. *Suppose that* h *is defined on* $\mathbb{R}^n \backslash B_\rho(0)$ *and satisfies* $h = O(r^\delta)$ *as* $r \to \infty$ *and assume that* $\Delta^{k+1} h = O(r^{\delta' - 2(k+1)})$ *as* $r \to \infty$ *with* $\delta' < \delta$*. Then, for any* $\tau > 0$ *such that* $\delta' + \tau$ *is nonexceptional there exists* $h' \in {W'}^{2(k+1),p}_{\delta'+\tau}$ *and a ball* $B_{\rho'}(0)$ *such that*

$$\Delta^{k+1}(h - h') = 0 \quad \text{on } \mathbb{R}^n \backslash B_{\rho'}(0), \tag{4.7.2}$$

Furthermore, if $2(k+1) < n$*, there exists an exceptional value* $j \leq \max\{\delta, \delta' + \tau\}$ *such that*

$$h - h' = p_j + O(r^{j-1}) \text{ as } r \to \infty, \tag{4.7.3}$$

where p_j *is homogeneous of degree* j *and satisfies* $\Delta^{(k+1)}(p_j) = 0$ *on* $\mathbb{R}^n \backslash \{0\}$*. If* $n = 2(k+1)$*, we may also have*

$$h - h' = A \cdot \log(r) + O(1) \text{ as } r \to \infty, \tag{4.7.4}$$

where the components of A *are constant.*

Proof. Let φ be a function in $C^\infty(\mathbb{R}^n)$ such that $\varphi \equiv 0$ on $B_\rho(0)$ and $\varphi \equiv 1$ on $\mathbb{R}^n \backslash B_{2\rho}(0)$, then $\Delta^{k+1}(\varphi h) \in {W'}^{0,p}_{\delta' - 2(k+1) + \tau}$ for every $\tau > 0$. If we choose τ in such a way that $\delta' + \tau$ is nonexceptional, by Lemma 4.7.3 there exists $h' \in {W'}^{2(k+1)}_{\delta'+\tau}$ such that

$$\Delta^{(k+1)}(h\varphi - h') = 0, \tag{4.7.5}$$

and on $\mathbb{R}^n \backslash B_{2\rho}(0)$ we have (4.7.2). The expansion (4.7.3) follows from the expansion at infinity of solutions of $\Delta^{k+1} \tilde{h} = 0$ on $\mathbb{R}^n \backslash B_1(0)$ (compare with the proof of Proposition 4.3.1). □

4.7.2 Optimal decay

Suppose that (M^n, g) is ALE of order 0, scalar-flat, and either $\Omega^{(k)}$-flat or satisfies (1.2.9). Corollary 4.5.6 showed that g is ALE of order β' for some $\beta' > 0$. In the next proposition we obtain the optimal value for β' as stated in Theorems 1.2.2 and 1.2.4.

Proposition 4.7.5. *Let h be as in Corollary 4.5.6. If g is $\Omega^{(k)}$-flat or satisfies (1.2.9), then $h \in \mathcal{T}^{0,2}_{m,\alpha;2k-n}$ and g is ALE of order $n-2k$.*

Proof. Let us treat first the case $(k+1) = \frac{n}{2}$. Since $\delta h = 0$, h satisfies (2.1.29) with $t = 0$, i.e.

$$\frac{c_{n,\frac{n}{2}-1}}{2(n-2)} \Delta^{\frac{n}{2}} h = \mathcal{R}^{(\frac{n}{2}-1)}(h, g_0), \tag{4.7.6}$$

where $\mathcal{R}^{(\frac{n}{2}-1)}(h, g_0)$ is given by equation (2.1.28). Since $h = O(r^{-\beta'})$ as $r \to \infty$, the slowest decaying terms in (2.1.28) are those of the form $\nabla^{\alpha_1} h * \nabla^{\alpha_2} h$ with $\alpha_1 + \alpha_2 = n$, so from (4.7.6) we obtain $\Delta^{\frac{n}{2}} h = O(r^{-2\beta'-n})$, and by Lemma (4.7.4), for any $\tau > 0$ such that $-2\beta' + \tau$ is nonexceptional there exists $h_1 \in W'^{n,p}_{-2\beta'+\tau}$ such that $\Delta^{\frac{n}{2}}(h - h_1) = 0$ on the complement of some ball. From the weighted Sobolev inequality, if we take $p > n$ then $h_1 = O(r^{-2\beta'+\tau})$ as $r \to \infty$ and clearly we can assume that $-2\beta' + \tau < -\beta'$, so that both h, h_1 have pointwise decay at infinity but h_1 has a better decay at infinity than h. By Lemma 4.7.4, and since -1 is the least negative exceptional value for $\Delta^{\frac{n}{2}}$, the difference $h - h_1$ has an expansion at infinity of the form

$$h - h_1 = F_1 + O(r^{-2}), \tag{4.7.7}$$

where F_1 is a homogeneous solution of degree -1 of $\Delta^{\frac{n}{2}} h = 0$ on $\mathbb{R}^n \backslash \{0\}$. From the proof of Proposition 2.0.2, any such F_1 has the form

$$(F_1)_{ij}(x) = u_{ij}\left(\frac{x}{|x|^2}\right), \tag{4.7.8}$$

where u_{ij} are linear functions. We now claim that on the complement of some ball, h satisfies

$$h = F_1 + O(r^{-1-\epsilon}), \tag{4.7.9}$$

for some $\epsilon > 0$. If $-2\beta' < -1$ we can choose $\tau > 0$ sufficiently small so that h satisfies (4.7.9). If not, we have $h = O(r^{-2\beta'+\tau})$ with $-2\beta' + \tau < -\beta'$ and again from (4.7.6) it follows that $\Delta^{\frac{n}{2}} h = O(r^{-4\beta'+2\tau})$ and we can argue as above to obtain $h = O(r^{-\min\{1,4\beta'-2\tau'\}})$ for some $\tau' > 0$ such that $-4\beta' + \tau' < -2\beta' + \tau$. It is clear that we can use induction to obtain (4.7.9). Note that if δF_1 is not identically zero then $\delta F_1 = O(r^{-2})$ as $r \to \infty$ but we do *not* have $\delta F_1 = O(r^{-2-\epsilon})$ as $r \to \infty$ for any $\epsilon > 0$, however, by (4.7.9) one has

$$\delta h = \delta F_1 + O(r^{-2-\epsilon}) \text{ as } r \to \infty, \tag{4.7.10}$$

therefore, from $\delta h \equiv 0$ it follows that $\delta F_1 \equiv 0$, and by Proposition 2.0.2, $F_1 \equiv 0$ which shows that $h = O(r^{-\gamma})$ with $\gamma > 1$. By (4.7.6) one obtains $\Delta^{\frac{n}{2}} h = O(r^{-2\gamma-n})$ and repeating the argument used to obtain (4.7.9) we can show that on the complement of some ball, h has an expansion of the form

$$h = F_2 + O(r^{-2-\epsilon}) \text{ as } r \to \infty, \tag{4.7.11}$$

where F_2 is homogeneous of degree -2 and satisfies $\Delta^{\frac{n}{2}} F_2 \equiv 0$ on $\mathbb{R}^n \backslash \{0\}$ and ϵ is some positive number. This proves that $h = O(r^{-2})$ as $r \to \infty$ as needed. For the case $2(k+1) < n$, the only difference with the previous proof is that we have to consider homogeneous solutions of $\Delta^{k+1} h \equiv 0$ on $\mathbb{R}^n \backslash \{0\}$ that decay like $r^{2(k+1)-n}$ and like r^{2k+1-n}, but as shown in the proof of Proposition 2.0.2, these solutions are *not* divergence-free unless they are identically zero. $\square$

4.8 Singularity removal theorems

In this section we present the proofs of Theorems 1.2.8 and 1.2.9.

Lemma 4.8.1. *Let $g = g_0 + h$ be a metric defined on $B_\rho(0)\backslash\{0\}$ with constant scalar curvature, and assume that g is either $\Omega^{(k)}$-flat or satisfies* (1.2.13)*. Suppose in addition that $\delta_t h = 0$ on $B_\rho(0)\backslash\{0\}$. Then, on $B_\rho(0)\backslash\{0\}$, h satisfies the equation*

$$\mathcal{P}_t^{(k)} h + \mathcal{R}^{(k)}(h, g_0) = 0, \tag{4.8.1}$$

where $\mathcal{P}_t^{(k)}$ and $\mathcal{R}^{(k)}(h, g_0)$ have the same expressions as in (2.1.26) *and* (2.1.28) *respectively. The operator $\mathcal{P}_t^{(k)}$ is elliptic.*

Proof. Suppose that $R(g_0 + h) = c$ where c is a constant. If we also have $\delta_t h = 0$ on $B_\rho(0)\backslash\{0\}$, then from $R(g_0 + h) - R(g_0) = c$ we conclude that h satisfies the equation

$$\Delta \mathrm{tr}(h) = -c + t\delta i_{r^{-1}\frac{\partial}{\partial r}} h + F'(h, g_0), \tag{4.8.2}$$

on $B_\rho(0)\backslash\{0\}$. We now write the equation $\Omega^{(k)}(g_0 + h) - \Omega^{(k)}(g_0) = 0$ as

$$0 = (\Omega^{(k)})'_{g_0}(h) + F^{(k)}(h, g_0), \tag{4.8.3}$$

with $F^{(k)}(h, g_0)$ as in (2.1.2). Using (2.0.1), we see that if we insert (4.8.2) into (4.8.3) then h satisfies (4.8.1) on $B_\rho(0)\backslash\{0\}$. The rest of the claim follows easily, and the same argument works for (1.2.13). □

Recalling the C^0-orbifold condition as defined in Definition 1.2.6, we now assume that there exists a coordinate system around the origin such that

$$g_{ij} = \delta_{ij} + o(1), \tag{4.8.4}$$

$$\partial^l g_{ij} = o(r^{-|l|}), \tag{4.8.5}$$

for any multi-index l with $|l| \geq 1$ as $r \to 0$.

Remark 4.8.2. As in the ALE case (compare Remark 4.5.1), we do not need an assumption on derivatives of arbitrary order. If l in (4.8.5) only satisfies $|l| \leq 2(k+1)+1$, then we have $|\,(g-g_0)_{(r,x)}\,|_{m,\alpha;0} = o(1)$ as $r \to 0$ for any $m \leq 2(k+1)$, which is sufficient for our proof.

Next, we state the existence of a divergence-free gauge.

Lemma 4.8.3. *Suppose that g defined on $B_\rho(0)$ has constant scalar curvature and is (extended) obstruction-flat or satisfies* (1.2.13)*. Suppose also that the origin is a C^0-orbifold point for g. Then for some $\rho' < \rho$ there exists a diffeomorphism $\phi : B_{\rho'}(0)\backslash\{0\} \to B_{\rho'}(0)\backslash\{0\}$ such that $\delta_{g_0}\phi_* g = 0$ on $B_{\rho'}(0)\backslash\{0\}$. Moreover, there exists $\sigma > 0$ such that $|\phi_* g - g_0| = O(r^\sigma)$ as $r \to 0$ and $\partial^l \phi_* g = O(r^{\sigma-|l|})$ for any multi-index l with $|l| \geq 1$.*

Proof. This follows from a straightforward modification of the proof of Corollary 4.5.6.

□

We will also need the following

Lemma 4.8.4. *If the components of $h \in S^2(T^*\mathbb{R}^n)$ are linear functions then $h = L_X g_0$ for some quadratic vector field.*

Proof. Let S^2_1 be the subspace of $S^2(T^*\mathbb{R}^n)$ consisting of all elements whose components are linear functions. If $h \in S^2_1$, we can write the components of h as

$$h_{ij}(x) = \sum_{l=1}^{n} A_{ijl} x_l, \tag{4.8.6}$$

where A_{ijk} is symmetric in i, j and therefore $\dim(S^2_1) = \frac{n^2(n-1)}{2}$. On the other hand, if we let Γ^1_2 be the space of all vector fields whose components are functions which are homogeneous of degree 2, then any $X \in \Gamma_2^1$ can be written as $X = X_i \frac{\partial}{\partial x^i}$ where

$$X_i(\xi) = \sum_{l,m} a_{ilm} \xi_l \xi_m, \tag{4.8.7}$$

with a_{ilm} symmetric in l, m, and therefore $\dim(\Gamma_2^1) = \frac{n^2(n-1)}{2} = \dim(S_1^2)$. Since there are no quadratic Killing vector fields, the map $\mathcal{L} : \Gamma_2^1 \to S_1^2$ defined by $\mathcal{L}(X) = L_X g_0$ is an isomorphism. □

Lemma 4.8.5. *Let X a vector field that is homogeneous of degree 2 and let K_X be the diffeomorphism generated by taking the flow of X to time 1 (which exists for r sufficiently small). If g_0 is the Euclidean metric we have*

$$K_X^* g_0 - L_X g_0 - g_0 = O(r^2) \text{ as } r \to 0. \tag{4.8.8}$$

Proof. Let ϕ_t be the flow of X, then we have for any $t > 0$

$$\phi_t^* g_0 = g_0 + t L_X g_0 + E(t), \tag{4.8.9}$$

where $E(t)$ is an error term that can be estimated as

$$|E(t)| \leq \frac{t^2}{2} \sup_{s \in [0,t]} \left(\left| \frac{\partial^2}{\partial s^2} \phi_s \right| \right), \tag{4.8.10}$$

here $|\cdot|$ is the usual pointwise norm on $S^2(T^*\mathbb{R}^n)$. In particular

$$K_X^* g_0 = g_0 + L_X g_0 + E(1). \tag{4.8.11}$$

Since X is homogeneous of degree 2, we have for any $p \in \mathbb{R}^n$

$$\frac{\partial}{\partial t} |\phi_t(p)|^2 = 2 \langle \frac{\partial}{\partial t} \phi_t(p), \phi_t(p) \rangle \leq C |\phi_t(p)|^3, \tag{4.8.12}$$

for some constant $C > 0$ that only depends on X. Letting r denote the distance of p to the origin, it follows from (4.8.12) that for $0 < r < \frac{1}{C}$ and $0 \leq t \leq 1$ we have the inequality

$$|\phi_t(p)| \leq \frac{2r}{2 - Crt}, \tag{4.8.13}$$

and then $|\phi_t(p)| = O(r)$ as $r \to 0$. A similar argument shows that for all first order partial derivatives one has

$$|\partial_l \phi_t| = O(1) \text{ as } r \to 0. \tag{4.8.14}$$

Since g_0 is the Euclidean metric, we have

$$(\phi_t^* g_0)_{ij} = \sum_{k,j} \partial_k(\phi_t)_i \partial_l(\phi_t)_j,$$

and using the chain rule we can write schematically

$$\frac{\partial^2}{\partial t^2}\phi_t^* g_0 = (\partial^2 X)(\phi_t) * X(\phi_t) * \partial\phi_t * \partial\phi_t + (\partial X)(\phi_t) * (\partial X)(\phi_t) * \partial\phi_t * \partial\phi_t, \tag{4.8.15}$$

and by (4.8.14),(4.8.15) we conclude that for r sufficiently small

$$\left| \frac{\partial^2}{\partial t^2}\phi_t^* g_0 \right| \leq C' r^2, \tag{4.8.16}$$

for C' depending only on X. By (4.8.10),(4.8.11) and (4.8.16) the result follows. □

Lemma 4.8.6. *Let g a metric defined on $B_\rho(0)$ with a C^0-orbifold point at the origin. Suppose that g has constant scalar curvature and is (extended) obstruction-flat or satisfies* (1.2.13) *on $B_\rho(0)\backslash\{0\}$. Then there exists a change of coordinates $\tilde{\phi}$, defined in some small neighborhood around the origin, such that $\tilde{\phi}_* g$ satisfies*

$$g = g_0 + O(|x|^2), \tag{4.8.17}$$

$$\partial^l g = O(|x|^{2-|l|}), \tag{4.8.18}$$

for any multi-index l with $|l| \geq 1$ as $|x| \to 0$.

Proof. Let ϕ be the gauge given by Lemma 4.8.3 and if we take $h = \phi_* g - g_0$ we obtain

$$\frac{c_{n,k}}{2(n-2)}\Delta^{k+1}h = \mathcal{R}^{(k)}(h, g_0). \tag{4.8.19}$$

Since $h = O(r^\sigma)$ as $r \to 0$ we have $\Delta^{k+1}h = O(r^{2\sigma-2(k+1)})$ as $r \to 0$ and as in the proof of Proposition 4.7.5, for $p > n$ and for $\tau > 0$ such that $2\sigma - \tau$ is nonexceptional and positive, there exists $h' \in W'^{n,p}_{2\sigma-\tau}$ such that $\Delta^{k+1}(h - h') = 0$ on $B_\rho(0)\backslash\{0\}$. Since both h, h' are $o(1)$ as $r \to 0$ we conclude that

$$h - h' = G_1 + O(r^2) \text{ as } r \to 0, \tag{4.8.20}$$

and the components of G_1 are linear functions. As in Proposition 4.7.5 we can use induction to show that h satisfies

$$h = G_1 + O(r^{1+\epsilon}) \text{ as } r \to 0, \tag{4.8.21}$$

for some $\epsilon > 0$. The strategy for proving (4.8.17), (4.8.18) is slightly different to that used to prove Proposition 4.7.5, but is still based on an argument used in [CT94]. From $\delta h = 0$ on $B_{\rho'}(0)\backslash\{0\}$, it follows that $\delta G_1 \equiv 0$ and by Lemma 4.8.4, $G_1 = L_X g_0$ for some vector field X such that $X(p)$ is homogeneous of degree 2 in p. Assume that ρ' is sufficiently small so that K_X, the diffeomorphism obtained by taking the flow of X to time 1, is defined. By Lemma 4.8.5

$$K_X^* g_0 - g_0 - L_X g_0 = O(r^2) \text{ as } r \to 0, \tag{4.8.22}$$

and from

$$(\phi_* g - K_X^* g_0) + (K_X^* g_0 - g_0 - L_X g_0) = O(r^{\min\{2,1+\epsilon\}}) \text{ as } r \to 0, \tag{4.8.23}$$

we conclude that

$$K_{-X}^{*}\phi_{*}g - g_0 = O(r^{\min\{1+\epsilon,2\}}) \text{ as } r \to 0. \tag{4.8.24}$$

As in Corollary 4.5.6, we can find a diffeomorphism ϕ' defined on a smaller ball such that

$$h' = \phi'_{*}K_{-X}^{*}\phi_{*}g - g_0, \tag{4.8.25}$$

is divergence-free and $h' = O(r^{\min\{1+\epsilon,2\}})$ as $r \to 0$. With this new h' we argue again as in the proof of Proposition 4.7.5 to obtain (4.8.17) and (4.8.18) as needed. □

Lemma 4.8.7. *In the coordinate system constructed in Lemma* 4.8.6 *we have*

$$\nabla^l Rm = \partial^l Rm + O(r^{1-|l|}) \text{ as } r \to 0, \tag{4.8.26}$$

for any multi-index l with $|l| \geq 1$ and

$$\Delta_g^m Ric(g) = \Delta_{g_0}^m Ric(g) + O(r^{-2(m-1)}) \text{ as } r \to 0. \tag{4.8.27}$$

with $m \geq 1$.

Proof. To show (4.8.26), we consider first the case $|l| = 1$ and we write ∇Rm schematically as

$$\nabla Rm = \partial Rm + \Gamma * Rm, \tag{4.8.28}$$

and note that the terms $\Gamma * Rm$ are $O(r)$ as $r \to 0$. The general case follows easily by induction. For (4.8.27) we start also with the case $m = 1$ and write

$$\begin{aligned} \Delta_g Ric &= g^{-1} * \nabla\nabla Ric = g^{-1} * \nabla\left(\partial Ric + \Gamma * Ric\right) \\ &= g^{-1} * \left(\partial^2 Ric + \partial\Gamma * Ric + \Gamma * \partial Ric + \Gamma * \Gamma * Ric\right) \\ &= g^{-1} * \partial^2 Ric + \partial\Gamma * Ric + \Gamma * \partial Ric + \Gamma * \Gamma * Ric, \end{aligned} \tag{4.8.29}$$

and the term $g^{-1} * \partial^2 Ric$ has the form $g^{kl}\partial_k\partial_l Ric_{ij}$ which we can also write as

$$\Delta_{g_0} Ric_{ij} + \left(g^{kl} - \delta^{kl}\right)\partial_{kl} Ric_{ij}. \tag{4.8.30}$$

The terms $\left(g^{kl} - \delta^{kl}\right)\partial_{kl} Ric_{ij}$, $\partial\Gamma * Ric$, and $\Gamma * \partial Ric$ in (4.8.29) are $O(1)$ as $r \to 0$ and the terms $\Gamma * \Gamma * Ric$ are $O(r^2)$ as $r \to 0$ as needed. The other cases follow also by induction. □

Proof of Theorems 1.2.8 and 1.2.9. By Lemma 4.8.6, we can find a change of coordinates ϕ around the origin such that $\phi_* g$ satisfies (4.8.17) and (4.8.18). Recall that the obstruction-flat systems have the form

$$\Delta_g^{k-1} B_{ij} = \sum_{j=2}^{k+1} \sum_{\alpha_1+\ldots+\alpha_j = 2(k+1)-2j} \nabla_g^{\alpha_1} Rm * \ldots * \nabla_g^{\alpha_j} Rm. \tag{4.8.31}$$

From the expression (1.2.6) we can write the Bach tensor as

$$B_{ij} = \Delta A_{ij} - \nabla_i \nabla^k A_{kj} + Rm * Rm, \tag{4.8.32}$$

and using that g has constant scalar curvature together with the Bianchi identity we can rewrite (4.8.31) as

$$\Delta_g^k Ric = \sum_{j=2}^{k+1} \sum_{\alpha_1+\ldots+\alpha_j = 2(k+1)-2j} \nabla_g^{\alpha_1} Rm * \ldots * \nabla_g^{\alpha_j} Rm, \tag{4.8.33}$$

which is exactly the same form as (1.2.13). From Lemma 4.8.7, (4.8.33) becomes

$$\Delta_{g_0}^k Ric = T, \tag{4.8.34}$$

where $T = O(r^{-2(k-1)})$ as $r \to 0$. Note that $T \in L^p$ near the origin for $p = \infty$ if $k = 1$ and for any $1 \leq p < \frac{n}{2(k-1)}$ if $k > 1$. On the other hand, since Ric is bounded near the origin, $Ric \in L^p$ for any such p. It follows that $Ric(g)$ is a weak solution of $\Delta_{g_0}^k Ric = T$

on $B_\rho(0)\backslash\{0\}$ and it is easy to prove that in that case $Ric(g)$ extends to a weak solution of (4.8.34) on $B_\rho(0)$. We conclude that $Ric \in W^{2k,p}$. Choose

$$p = \begin{cases} \infty & \text{if } k = 1 \\ (1-\epsilon)\frac{n}{2(k-1)} & \text{with } 0 < \epsilon < \frac{1}{2k-1} \text{ if } k > 1. \end{cases} \tag{4.8.35}$$

Observe also that $0 < 2k - 1 - \frac{n}{p} \leq 1$ so by the Sobolev inequality, for any α such that $0 < \alpha < 2k - 1 - \frac{n}{p}$ we have

$$\|\nabla Ric\|_{C^\alpha(B_\rho(0))} \leq C\|\nabla Ric\|_{W^{2k-1,p}(B_\rho(0))}, \tag{4.8.36}$$

with $C = C(n,p,k,\rho)$ and then $Ric \in C^{1,\alpha}(B_\rho(0))$. Note that from the estimates (4.8.17) and (4.8.18) we have $g \in C^{1,\alpha}$, which is sufficient for the existence of harmonic coordinates at the origin [DK81, Lemma 1.2.]. In this harmonic coordinate system the metric g is also $C^{1,\alpha}$ and solves (4.8.33). Moreover, by (4.8.36) and [DK81, Corollary 1.4], $Ric \in C^{1,\alpha}$ near the origin.

We then have that g is a solution of (4.8.34) and is also a solution of an equation of the form

$$\frac{1}{2}g^{ij}\partial^2_{ij}g_{kl} + Q_{kl}(\partial g, g) = -Ric_{kl}(g), \tag{4.8.37}$$

where $Q(\partial g, g)$ is an expression that is quadratic in ∂g, polynomial in g and has $\sqrt{|g|}$ in its denominator. Letting p and α be as above, we know that g and $Ric(g)$ are $C^{1,\alpha}$ at the origin, in particular they both are in $W^{1,p}$. Using elliptic regularity in (4.8.37) we conclude that $g \in W^{3,p}$ and the Sobolev inequality (compare (4.8.36)) implies that $g \in C^{2,\alpha}$. Furthermore, since $Ric \in C^{1,\alpha}$ it also follows that $g \in C^{3,\alpha}$ (see [DK81, Theorem 4.5]). With this regularity in g we can write (4.8.33) as (4.8.34) in harmonic coordinates, i.e., we can write (4.8.33) as an equation of the form $\Delta^k_{g_0} Ric = T'$ with $T' \in L^p$.

Next, we claim that $\Delta_{g_0}^k Ric \in W^{1,p}$. To see this, take one derivative of (4.8.33). Since $g \in C^{3,\alpha}$, one sees that all of the terms on the right hand side are $O(r^{-2(k-1)})$ as $r \to 0$, which is in L^p for p as in (4.8.35). This shows that $\Delta_g^k Ric \in W^{1,p}$. Replacing $\Delta_g^k Ric$ with $\Delta_{g_0}^k Ric$ will introduce terms as in Lemma 4.8.7, but using the fact that Ric is now in $C^{1,\alpha}$, we see that these terms are also in $W^{1,p}$, and consequently $\Delta_{g_0}^k Ric \in W^{1,p}$. Elliptic regularity then implies that $Ric \in W^{2k+1,p}$. By the Sobolev inequality, $Ric \in C^{2,\alpha}$ and then (4.8.37) implies $g \in C^{4,\alpha}$. It is clear that we can bootstrap the above argument to prove that g is smooth at the origin. $\square$

Part II

Asymptotics of the Self-Dual Deformation Complex

Chapter 5

Curvature decomposition in dimension 4

5.1 The anti self-dual part of the curvature tensor

Let (M^4, g) be an orientable 4-manifold. As mentioned in the introduction, according to the decomposition

$$Rm = W^+ + W^- + \frac{1}{2}E \owedge g + \frac{1}{24}R_g g \owedge g, \tag{5.1.1}$$

we have the associated curvature operators are written as

$$\mathcal{R} = \mathcal{W}^+ + \mathcal{W}^- + \mathcal{E} + \frac{1}{24}\mathcal{S}, \tag{5.1.2}$$

where

$$\left(\mathcal{W}^{\pm} + \frac{1}{24}\mathcal{S}\right) : \Lambda^2_{\pm}\left(T^*M\right) \mapsto \Lambda^2_{\pm}\left(T^*M\right), \tag{5.1.3}$$

and

$$\mathcal{E} : \Lambda^2_{\pm}\left(T^*M\right) \mapsto \Lambda^2_{\mp}\left(T^*M\right). \tag{5.1.4}$$

Some basic properties of the tensors $W^{\pm}$ and of the curvature operators $\mathcal{W}^{\pm}$ are the following

1. Viewed as a $(1,3)$ tensor, for any C^2 function f, $W^{\pm}(e^{-2f}g) = W^{\pm}(g)$.

2. Letting $\mathcal{C} : S^2\left(\Lambda^2\left(T^*M\right)\right) \mapsto S^2\left(T^*M\right)$ be the *Ricci Contraction Map* defined by $\left(\mathcal{C}U\right)_{ab} = g^{cd}U_{acbd}$, then

$$\mathcal{C}W^{\pm} = 0. \tag{5.1.5}$$

3. Both $\mathcal{W}^+$ and $\mathcal{W}^-$ are traceless.

We note that our convention is that if P_{ijkl} is a tensor satisfying $P_{ijkl} = -P_{jikl} = -P_{ijlk} = P_{klij}$, then the associated operator $\mathcal{P} : \Lambda^2 \to \Lambda^2$ is given by

$$\left(\mathcal{P}\omega\right)_{ij} = \frac{1}{2}\sum_{k,l} P_{ijkl}\omega_{kl}. \tag{5.1.6}$$

5.1.1 The anti self-dual part of the Weyl tensor as a bilinear form

Consider a warped product metric on $M = \mathbb{R} \times Y^3$ of the form

$$g = dt^2 + g_Y, \tag{5.1.7}$$

Where g_Y is a smooth metric on Y, possibly depending on t. Our ultimate goal is a formula for the linearized anti-self-dual Weyl curvature $\mathcal{D}$, which maps from

$$\mathcal{D} : S_0^2(T^*M) \to S_0^2(\Lambda_-^2). \tag{5.1.8}$$

Using the decomposition $T^*M = \{dt\} \oplus T^*Y$, we have

$$S^2(T^*M) = S^2(dt) \oplus (dt \odot T^*Y) \oplus S^2(T^*Y), \tag{5.1.9}$$

which we will write as

$$\tilde{h} = h_{00} dt \otimes dt + (\alpha \otimes dt + dt \otimes \alpha) + h. \tag{5.1.10}$$

Next, we have

$$\Lambda^2(dt \oplus T^*Y) = (\Lambda^1(dt) \otimes \Lambda^1(T^*Y)) \oplus \Lambda^2(T^*Y). \tag{5.1.11}$$

Given an orientation, we then have

$$\Lambda^2(dt \oplus T^*Y) = (\Lambda^1(dt) \otimes \Lambda^1(T^*Y)) \oplus \Lambda^1(T^*Y). \tag{5.1.12}$$

Under this decomposition, the self-dual forms correspond to

$$dt \wedge \alpha + \tilde{*}\alpha, \tag{5.1.13}$$

while the anti-self-dual forms correspond to

$$dt \wedge \alpha - \tilde{*}\alpha, \tag{5.1.14}$$

where $\tilde{*}$ is the Hodge-$*$ operator on Y. Consequently, we have the identification

$$S^2_0(\Lambda^2_-) = S^2_0(T^*Y), \tag{5.1.15}$$

and we can therefore view $\mathcal{D}$ as a mapping

$$\mathcal{D} : S^2_0(T^*M) \to S^2_0(T^*Y). \tag{5.1.16}$$

In order to proceed, we must first write down $\mathcal{W}^-$, considered as an element of $S^2_0(T^*Y)$.

Proposition 5.1.1.

$$(\mathcal{W}^-)_{ij} = \Phi_{ij} - \Psi_{ij} + \Omega_{ij}, \tag{5.1.17}$$

where

$$\Phi_{ij} = \mathrm{tf}(R_{0i0j}(g)), \tag{5.1.18}$$

$$\Psi_{ij} = \mathrm{tf}\left(\mathrm{Sym}_{ij}\left(\sum_{k,l}\epsilon_{ikl}R_{0jkl}(g)\right)\right), \tag{5.1.19}$$

$$\Omega_{ij} = \frac{1}{4}\mathrm{tf}\left(\sum_{k,l}\sum_{p,q}\epsilon_{ikl}\epsilon_{jpq}R_{klpq}(g)\right), \tag{5.1.20}$$

where the symbols ϵ_{ijk} *are the components of the volume element defined by*

$$e_i \wedge e_j \wedge e_k = \epsilon_{ijk}e_1 \wedge e_2 \wedge e_3, \tag{5.1.21}$$

Sym *in* (5.1.19) *denotes the* Symmetrization Operator *given by*

$$\mathrm{Sym}_{ij}(F) = \frac{1}{2}\left(F_{ij} + F_{ji}\right), \tag{5.1.22}$$

and for $h \in S^2\left(T^*Y\right)$*,* $\mathrm{tf}(h)$ *denotes the traceless component of* h *with respect to the metric* g_Y*.*

Proof. Given g as in (5.1.7) and considering the decomposition (5.1.2), we conclude from (5.1.2),(5.1.3) and (5.1.4) that for any $\omega, \omega' \in \Lambda^2_-(M)$

$$\left\langle\left(\mathcal{W}^- + \frac{1}{24}\mathcal{S}\right)\omega, \omega'\right\rangle = \langle\mathcal{R}\omega, \omega'\rangle.$$

In order to compute $\langle\mathcal{R}\omega, \omega'\rangle$ we use the isomorphism between $\Omega^1(Y) \oplus \Omega^1(Y)$ and $\Lambda^2(\mathbb{R} \times Y)$ given as follows: any 2-form ω in $\Lambda^2(\mathbb{R} \times Y)$ can be written uniquely as

$$\omega = dt \wedge \pi^*(\xi) + \pi^*(\tilde{*}\eta), \tag{5.1.23}$$

where π is the projection map $\pi : \mathbb{R} \times Y \mapsto Y$, ξ and η are 1-forms in $\Omega^1(Y)$ and $\tilde{*}$ is the Hodge-$*$ operator with respect to the metric g_Y. Given a local orthonormal oriented basis $\{e_1, e_2, e_3\}$ of $\Gamma(TY)$, the operator $\tilde{*} : \Omega^1(Y) \mapsto \Omega^2(Y)$ takes the form

$$\tilde{*}(\zeta)_{ij} = \sum_{k=1}^{3} \epsilon_{ijk}\zeta_k. \tag{5.1.24}$$

Using i, k, l to denote indices in $\{1, 2, 3\}$ we see that the 1-forms ξ, η in (5.1.23) can be written in coordinates as

$$\xi_k = \omega_{0k}, \;\; \eta_i = \frac{1}{2}\sum_{j,k} \epsilon_{ijk}\omega_{jk} = \sum_{j<k} \epsilon_{ijk}\omega_{jk},$$

Moreover, in these coordinates, the Hodge-$*$ operator can be computed as

$$*\omega = dt \wedge \pi^*(\eta) + \pi^*(\tilde{*}\xi), \tag{5.1.25}$$

therefore, $\omega \in \Lambda^2_{\pm}(T^*Y)$ if and only if $\xi = \pm\eta$. Given 2-forms $\omega, \omega' \in \Lambda^2(M)$, let us write $\omega = dt \wedge \pi^*(\xi) + \pi^*(\tilde{*}\eta)$ and $\omega' = dt \wedge \xi' + \pi^*(\tilde{*}\eta')$, then we can express $\langle \mathcal{R}\omega, \omega' \rangle$ as

$$\langle \mathcal{R}\omega, \omega' \rangle = \alpha_{ij}\xi_i\xi'_j + \beta_{ij}\xi_i\eta'_j + \beta_{ji}\eta_i\xi'_j + \gamma_{ij}\eta_i\eta'_j \tag{5.1.26}$$

where clearly

$$\alpha_{ij} = R_{0i0j}(g), \;\; \beta_{ij} = \frac{1}{2}\sum_{k,l} \epsilon_{jkl}R_{0ikl}(g), \;\; \gamma_{ij} = \frac{1}{4}\sum_{k,l}\sum_{p,q} \epsilon_{ikl}\epsilon_{jpq}R_{klmn}(g). \tag{5.1.27}$$

If now $*\omega = -\omega$ and $*\omega' = -\omega'$, from (5.1.26) and (5.1.25) we obtain

$$\langle \mathcal{R}\omega, \omega' \rangle = (\alpha_{ij} - (\beta_{ij} + \beta_{ji}) + \gamma_{ij})\,\xi_i\xi'_j. \tag{5.1.28}$$

This shows that with the isomorphism defined by (5.1.23), we can identify the map $\left(\mathcal{W}^- + \frac{1}{24}\mathcal{S}\right) : \Lambda^2_-(T^*M) \mapsto \Lambda^2_-(T^*M)$ with a bilinear form in $S^2(T^*Y)$ such that in the local orthonormal basis $\{e_1, e_2, e_3\}$ has components

$$\alpha_{ij} - (\beta_{ij} + \beta_{ji}) + \gamma_{ij}. \tag{5.1.29}$$

with $\alpha_{ij}, \beta_{ij}, \gamma_{ij}$ given by (5.1.27). Since the scalar curvature operator $\mathcal{S}$ contributes a pure-trace term, we are done. $\square$

We next give a more detailed description of some of the terms appearing in (5.1.17) and for that purpose we will make use of the following notation:

- All letters $i, j, k, l, \ldots$ will denote non-zero indices.
- Given $H \in S^2\left(\Lambda^2\left(M\right)\right)$, by $c_Y H$ we will mean the map defined as

$$\left(c_Y H\right)_{jk} = g_Y^{il} H_{ijlk},$$

- By $\mathrm{tr}_Y(c_Y H)$ we will mean $g_Y^{ij}(c_Y H)_{ij}$,
- An upper dot will denote a t-derivative.
- We will use $\dot{g}_Y * \dot{g}_Y$ to denote linear combinations of contractions of $\dot{g}_Y \otimes \dot{g}_Y$ using the metric g_Y.

Proposition 5.1.2. *We have the identity*

$$\frac{1}{4} \sum_{k,l,u,v} \epsilon_{ikl} \epsilon_{juv} H_{kluv} = -\left(c_Y H - \frac{1}{2} \mathrm{tr}_Y\left(c_Y H\right) g_Y\right)_{ij}.$$

Proof. Suppose $i = j$, and let p, q with $p < q$ be indices such that $\{1, 2, 3\} = \{i, p, q\}$

$$\frac{1}{4} \sum_{k,l} \sum_{u,v} \epsilon_{ikl} \epsilon_{juv} H_{kluv} = \frac{1}{4} \sum_{k,l} \sum_{u,v} \epsilon_{ikl} \epsilon_{iuv} H_{klmn} = H_{pqpq}.$$

Note that the trace of $c_Y H$ is given by

$$\frac{1}{2} \mathrm{tr}_Y\left(c_Y H\right) = H_{pqpq} + H_{ipip} + H_{iqiq}.$$

Therefore

$$\begin{aligned} H_{pqpq} &= \frac{1}{2}\mathrm{tr}_Y(c_Y H) - (H_{ipi} + H_{iqiq}) = \frac{1}{2}\mathrm{tr}_Y(c_Y H) - (c_Y H)_{ii} \\ &= \frac{1}{2}\mathrm{tr}_Y(c_Y H)\delta_{ii} - (c_Y H)_{ii} = -\left(c_Y H - \frac{\mathrm{tr}_Y(c_Y H)}{2} g_Y\right)_{ii}. \end{aligned}$$

If now $i \neq j$ and p is such that $\{1,2,3\} = \{i,j,p\}$, we have

$$\begin{aligned} &\frac{1}{4}\sum_{k,l}\sum_{u,v} \epsilon_{ikl}\epsilon_{juv}H_{kluv} = \epsilon_{ijp}\epsilon_{jip}H_{jpjp} \\ &= -H_{jpip} = -\left(c_Y H\right)_{ij} = -\left(c_Y H - \frac{\mathrm{tr}_Y\left(c_Y H\right)}{2} g_Y\right)_{ij}, \end{aligned}$$

and the claim follows. □

We will also need to compute the Christoffel symbols and components of the curvature tensor of g in terms of the metric g_Y:

Proposition 5.1.3. *The Christoffel symbols of the metric $g = dt^2 + g_Y$ are given by*

$$\begin{aligned} &\Gamma^k_{i0}(g) = \frac{1}{2} g_Y^{kl} \left(\dot{g}_Y\right)_{il}, \ \Gamma^0_{ij}(g) = -\frac{1}{2}\left(\dot{g}_Y\right)_{ij}, \ \Gamma^k_{ij}(g) = \Gamma^k_{ij}(g_Y), \\ &\Gamma^k_{00}(g) = \Gamma^0_{0i}(g) = 0. \end{aligned}$$

For the components of the curvature tensor we have

$$\begin{aligned} R^0_{0ij} &= -\frac{1}{2}\left(\ddot{g}_Y\right)_{ij} + (\dot{g}_Y * \dot{g}_Y)_{ij}, \\ R^k_{ijl}(g) &= R^k_{ijl}(g_Y) + (\dot{g}_Y * \dot{g}_Y)^k_{ijl}. \end{aligned}$$

In particular, if g_Y is independent of t, then

$$\Gamma^\gamma_{\alpha\beta}(g) = \begin{cases} 0 & \text{if any } \alpha, \beta, \gamma \text{ equals } 0 \\ \Gamma^\gamma_{\alpha\beta}(g_Y) & \text{otherwise} \end{cases}, \tag{5.1.30}$$

and consequently

$$R^{\nu}_{\alpha\beta\mu} = \begin{cases} 0 & \textit{if any } \alpha, \beta, \mu, \nu \textit{ equals } 0 \\ R^{\nu}_{\alpha\beta\mu}(g_Y) & \textit{otherwise} \end{cases} . \tag{5.1.31}$$

Proof. The proof follows from a straightforward computation. □

We can now write out a more convenient expression for Ω_{ij} in (5.1.17)

Proposition 5.1.4. *The term Ω_{ij} in* (5.1.17) *has the form*

$$\Omega_{ij} = \left(-E(g_Y) + \dot{g}_Y * \dot{g}_Y\right)_{ij}, \tag{5.1.32}$$

where $E(g_Y)$ is the traceless Ricci tensor of g_Y.

Proof. Recall that Ω_{ij} is given by

$$\Omega_{ij} = \mathrm{tf}\left(\frac{1}{4}\sum_{k,l,m,n} \epsilon_{ikl}\epsilon_{jmn} R_{klmn}(g)\right), \tag{5.1.33}$$

and from Proposition 5.1.2 we must have

$$\Omega_{ij} = -\mathrm{tf}\left(c_Y Rm(g) - \frac{1}{2}\mathrm{tr}_Y(c_Y Rm(g)) g_Y\right). \tag{5.1.34}$$

With the expressions obtained for the components of Rm in Proposition 5.1.3 we have

$$-\left(c_Y Rm(g)\right)_{ij} = \left(-Ric(g_Y) + \dot{g}_Y * \dot{g}_Y\right)_{ij},$$

and then

$$\mathrm{tr}_Y\left(c_Y Rm(g)\right) = R_{g_Y} + \dot{g}_Y * \dot{g}_Y,$$

which implies (5.1.32). □

Chapter 6

The linearized equation and Dirac-type operators

6.1 Linearization of W^- at a cylindrical metric

Consider the cylindrical metric

$$g = dt^2 + g_Y, \tag{6.1.1}$$

defined on $M = \mathbb{R} \times Y$, where g_Y is a fixed metric of constant curvature $\kappa = +1, 0,$ or -1. We note that g is locally conformally flat, and therefore is self-dual. We are interested in studying the linearization of W^- at g. Given $\tilde{h} \in S^2(M)$, we consider a path of metrics $g(\epsilon)$ with $\epsilon \in (-\delta, \delta)$ for some $\delta > 0$ satisfying $g(0) = g$ and $g'(0) = \tilde{h}$. The linearization of W^- at g in the direction of $\tilde{h}$ is the map

$$\left(W^-\right)'_g(\tilde{h}) = \frac{\partial}{\partial \epsilon} W^-(g_\epsilon)|_{\epsilon=0}.$$

We next define a Dirac-type operator:

Definition 6.1.1. Let $\{e_1, e_2, e_3\}$ be a local orthonormal basis of $\Gamma(TY)$. Then, for any $h \in S^2(T^*Y)$ the operator $\not{d}h$ is given in these coordinates by

$$(\not{d}h)_{ij} = \mathrm{Sym}_{ij}\left(\sum_{k,l} \epsilon_{ikl} d^{\nabla} h_{klj}\right), \tag{6.1.2}$$

where $(d^\nabla h)_{klj}$ is given by $(d^\nabla h)_{klj} = \nabla_k h_{lj} - \nabla_l h_{kj}$.

We also recall the conformal Killing operator:

Definition 6.1.2. For an n-dimensional manifold (M^n, g), the *conformal Killing operator* with respect to the metric g is the map $\mathcal{K}_g : \Lambda^1(T^*M) \to S^2_0(T^*M)$, given by

$$\mathcal{K}_g(\tilde{\omega}) = \mathcal{L}_g(\tilde{\omega}) - \frac{2}{n}(\delta\tilde{\omega})g,$$

where $\mathcal{L}_g$ is the Lie derivative operator.

In cylindrical coordinates, a tensor $\tilde{h} \in S^2(M)$ can be decomposed as

$$\tilde{h} = h_{00} dt \otimes dt + \alpha \odot dt + h,$$

where $h_{00} \in \Lambda^0(M)$, $\alpha \in \Lambda^1(T^*Y)$ and $h \in S^2(T^*Y)$, so we will use the notation $\tilde{h} = \{h_{00}, \alpha, h\}$. The main result of this section is the following

Theorem 6.1.3. *For the cylindrical metric given by $g = dt^2 + g_Y$, the linearization $(W^-)'_g(\tilde{h})$ with $tr_g(\tilde{h}) = 0$, is given by*

$$\left(W^-\right)'(h_{00}, \alpha, h) = \frac{1}{2}\mathcal{K}_{g_Y}\left(-\frac{1}{2}dh_{00} + \dot{\alpha} - *d\alpha\right) - \frac{1}{2}\mathrm{tf}(\ddot{h}) + \frac{1}{2}\not{d}\dot{h} - E'(h), \tag{6.1.3}$$

where $E'(h)$ is the linearization of the traceless Ricci tensor at g_Y. Equivalently, after computing $E'(h)$ explicitly, $(W^-)'_g(\tilde{h})$ is given by

$$\begin{aligned}\left(W^-\right)'(h_{00}, \alpha, h) = \frac{1}{2}\mathcal{K}_{g_Y}&\left(-\frac{1}{2}dh_{00} - \delta_Y h + \dot{\alpha} - *d\alpha + \frac{1}{2}d\mathrm{tr}_Y(h)\right)\\ &-\frac{1}{2}\mathrm{tf}(\ddot{h}) - \kappa \cdot \mathrm{tf}(h) + \frac{1}{2}\not{d}\dot{h} + \frac{1}{2}\Delta_{g_Y}\mathrm{tf}(h).\end{aligned} \tag{6.1.4}$$

where Δ_{g_Y} is the rough laplacian on $S^2(T^*Y)$, and $(\delta_Y h)_j = \nabla^i_Y h_{ij}$ is the divergence.

The remainder of the section will be concerned with the proof of Theorem 6.1.3.

6.1.1 Conformal Killing operator and $\not{d}$

The operator $\not{d}$ enjoys the following properties:

Proposition 6.1.4. *For the operator $\not{d}$, we have*

$$\not{d} : S^2(T^*Y) \to S_0^2(T^*Y), \tag{6.1.5}$$

$$\not{d}(ug_Y) = 0 \text{ for any } u \in C^2(Y), \tag{6.1.6}$$

$$\not{d} : S_0^2(T^*Y) \to S_0^2(T^*Y) \text{ is formally self-adjoint.} \tag{6.1.7}$$

Proof. For the first property, in an orthonormal basis

$$\begin{aligned}\mathrm{tr}_Y\left(\not{d}h\right) &= \sum_{i,j}\sum_{k,l}\delta_{ij}\mathrm{Sym}_{ij}\left(\epsilon_{ikl}d^{\nabla}h_{klj}\right) = \sum_{i=1}^{3}\sum_{k,l\neq i}\epsilon_{ikl}\left(\nabla_k h_{li} - \nabla_l h_{ki}\right) \\ &= \sum_{i=1}^{3}\sum_{k,l\neq i}\epsilon_{ikl}\nabla_k h_{li} - \sum_{i=1}^{3}\sum_{k,l\neq i}\epsilon_{ilk}\nabla_l h_{ki} = 0.\end{aligned}$$

For (6.1.6), in an orthonormal basis and using that g_Y is parallel we have

$$\begin{aligned}\not{d}\left(ug_Y\right)_{ij} &= \sum_{k,l}\mathrm{Sym}_{ij}\left(\epsilon_{ikl}d^{\nabla}\left(ug_Y\right)_{klj}\right) \\ &= \mathrm{Sym}_{ij}\left(\sum_{k,l}\left(\epsilon_{ikl}(\nabla_k u)(g_Y)_{lj} - \epsilon_{ikl}(\nabla_l u)(g_Y)_{kj}\right)\right) \\ &= \mathrm{Sym}_{ij}\left(\sum_{k=1}^{3}\epsilon_{ikj}\nabla_k u - \sum_{l=1}^{3}\epsilon_{ijl}\nabla_l u\right) = -2\mathrm{Sym}_{ij}\left(\sum_{k=1}^{3}\epsilon_{ijk}\nabla_k u\right),\end{aligned}$$

and since $\epsilon_{ijk}\nabla_k u$ is skew-symmetric in i, j, it follows that $\not{d}\left(ug_Y\right) = 0$.

Finally, let h, h' be elements in $S^2(T^*Y)$, then in an orthonormal basis we have

$$\begin{aligned}\int_Y \langle \not{d}h, h'\rangle dV_{g_Y} &= \frac{1}{2}\sum_{i,j}\sum_{k,l}\int_Y \epsilon_{ikl}\left(\nabla_k h_{lj} - \nabla_l h_{kj}\right)h'_{ij}dV_{g_Y} \\ &= \sum_{i,j}\sum_{k,l}\int_Y \epsilon_{ikl}\nabla_k h_{lj}h'_{ij}dV_g = -\sum_{i,j}\sum_{k,l}\int_Y \epsilon_{ikl}h_{lj}\nabla_k h'_{ij}dV_{g_Y} \\ &= \sum_{i,j}\sum_{k,l}\int_Y \epsilon_{lki}\nabla_k h'_{ij}h_{lj}dV_{g_Y} = \int_Y \langle h, \not{d}h'\rangle dV_{g_Y}.\end{aligned}$$

□

For the operators $\not{d}, \mathcal{K}_g$ and $\mathcal{D}$ we have

Proposition 6.1.5. *The operators $\not{d}$ and $\mathcal{K}_g$ satisfy the following identities*

$$\not{d}\mathcal{L}_{g_Y}(\omega) = \mathcal{K}_{g_Y}(\tilde{*}d\omega), \tag{6.1.8}$$

$$\mathcal{D}\left(\mathcal{K}_{g_Y}(\tilde{\omega})\right) = 0 \text{ for any } \tilde{\omega} \in \Lambda^1(T^*M). \tag{6.1.9}$$

Proof. Identity (6.1.8) is a consequence of the following computation: let $h = \mathcal{L}_{g_Y}(\omega)$ then

$$\begin{aligned}(\not{d}h)_{ij} &= \mathrm{Sym}_{ij}\left(\sum_{k,l}\epsilon_{ikl}\left(\nabla_k h_{lj} - \nabla_l h_{kj}\right)\right)\\ &= \mathrm{Sym}_{ij}\left(\sum_{k,l}\epsilon_{ikl}\left(\nabla_k\nabla_l\omega_j + \nabla_k\nabla_j\omega_l - \nabla_l\nabla_k\omega_j - \nabla_l\nabla_j\omega_k\right)\right).\end{aligned} \tag{6.1.10}$$

Commuting covariant derivatives in (6.1.10) we obtain

$$(\not{d}h)_{ij} = \mathrm{Sym}_{ij}\left(\sum_{k,l}\epsilon_{ikl}\left(\nabla_j\nabla_k\omega_l - \nabla_j\nabla_l\omega_k - R^p_{klj}\omega_p - R^p_{kjl}\omega_p + R^p_{ljk}\omega_p\right)\right). \tag{6.1.11}$$

Note that $-R^p_{klj} - R^p_{kjl} + R^p_{ljk} = -2R^p_{klj}$ by the algebraic Bianchi identity, so (6.1.11) becomes

$$(\not{d}h)_{ij} = \mathrm{Sym}_{ij}\left(\sum_{k,l}\epsilon_{ikl}\left(\nabla_j\nabla_k\omega_l - \nabla_j\nabla_l\omega_k\right) - 2\sum_{k,l}\epsilon_{ikl}R^p_{klj}\omega_p\right). \tag{6.1.12}$$

Since g_Y has constant sectional curvature equal to κ

$$\begin{aligned}&-2\sum_{k,l}\epsilon_{ikl}R^p_{klj}\omega_p = -2\kappa\sum_{k,l}\epsilon_{ikl}\left(\delta^p_k\delta_{lj} - \delta^p_l\delta_{kj}\right)\omega_p\\ &= -2\kappa\left(\sum_{k,l}\epsilon_{ikl}\omega_k\delta_{lj} - \sum_{k,l}\epsilon_{ikl}\omega_l\delta_{kj}\right) = -2\kappa\left(\sum_k\left(\epsilon_{ikj} - \epsilon_{ijk}\right)\omega_k\right).\end{aligned}$$

Since $\epsilon_{ikj} - \epsilon_{ijk}$ is skew-symmetric in i, j we obtain

$$\begin{aligned}
(\not{d}h)_{ij} &= \mathrm{Sym}_{ij}\left(\sum_{k,l} \epsilon_{ikl}\left(\nabla_j \nabla_k \omega_l - \nabla_j \nabla_l \omega_k\right)\right) \\
&= \mathrm{Sym}_{ij}\left(\nabla_j\left(\sum_{k,l} \epsilon_{ikl}\left(\nabla_k \omega_l - \nabla_l \omega_k\right)\right)\right) \\
&= 2\mathrm{Sym}_{ij}\left(\nabla_j\left(\tilde{*}d\omega\right)_i\right) = \nabla_j\left(\tilde{*}d\omega\right)_i + \nabla_i(\tilde{*}d\omega)_j \\
&= \left(\mathcal{L}_{g_Y}(\tilde{*}d\omega)\right)_{ij}.
\end{aligned}$$

Since $\not{d}h$ is traceless, we actually obtain $\not{d}\mathcal{L}_{g_Y}(\omega) = \mathcal{K}_{g_Y}(\tilde{*}d\omega)$ as needed. For proving (6.1.9), we note that by diffeomorphism invariance of W^- and since g is locally conformally flat we have $\mathcal{D}(\mathcal{L}_g(\tilde{\omega})) = 0$ for any 1-form $\tilde{\omega} \in \Lambda^1(T^*M)$. By the conformal invariance of W^-, we have $\mathcal{D}(fg) = 0$ for any $f \in C^\infty(M)$, therefore the composition of $\mathcal{D}$ and $\mathcal{K}_g$ is zero. □

6.1.2 The case of no radial components

We first compute $(W^-)'(\tilde{h})$ assuming that $\tilde{h}$ has no radial components, i.e. $\tilde{h}$ has the form $\tilde{h} = \{0, 0, h\}$.

Proposition 6.1.6. *The linearization of W^- at $g = dt^2 + g_Y$ in the direction $\tilde{h} = \{0, 0, h\}$ is*

$$\left(W^-\right)'(\tilde{h}) = -\frac{1}{2}\mathrm{tf}(\ddot{h}) + \frac{1}{2}\left(\not{d}\dot{h}\right) - E'_{g_Y}(h). \tag{6.1.13}$$

Proof. We start by linearizing the component Ω_{ij} in (5.1.17). Note that

$$\frac{\partial}{\partial \epsilon}\left(\dot{g}_Y(\epsilon) * \dot{g}_Y(\epsilon)\right)|_{\epsilon=0} = 0, \tag{6.1.14}$$

for any variation which is purely spherical, that is, a variation which only deforms the cross-section metric on Y. From Proposition 5.1.4 and (6.1.14) it is clear that for $\tilde{h} = \{0, 0, h\}$ we have

$$\Omega'_{ij}(\tilde{h}) = -E'(h). \tag{6.1.15}$$

For the term Φ_{ij} in (5.1.17), we consider a purely spherical deformation g_ϵ of g in the direction of h so that from (5.1.18) we have

$$\Phi_{ij}(g_\epsilon) = \mathrm{tf}_{g_Y(\epsilon)}\left(R_{0i0j}(dt^2 + g_Y(\epsilon))\right),$$

and from Proposition 5.1.3

$$R_{0i0j}(g_\epsilon) = \left(-\frac{1}{2}\ddot{g_Y}(\epsilon) + \dot{g}_Y(\epsilon) * \dot{g}_Y(\epsilon)\right)_{ij}, \tag{6.1.16}$$

then

$$\left(\Phi'_g\right)_{ij}(\tilde{h}) = \frac{\partial}{\partial \epsilon}\left(-\mathrm{tf}_{g_Y}(\ddot{g}_Y(\epsilon)) + \dot{g}_Y(\epsilon) * \dot{g}_Y(\epsilon)\right)|_{\epsilon=0} = -\frac{1}{2}\mathrm{tf}_{g_Y}\ddot{h}. \tag{6.1.17}$$

Finally, for the components Ψ_{ij} we recall that we can express $\Psi_{ij}(dt^2 + g_Y)$ as

$$\Psi_{ij} = \mathrm{Sym}_{ij}\left(\sum_{k,l}\epsilon_{jkl}(g_Y)R_{0ikl}(dt^2 + g_Y)\right).$$

Note that taking the tracefree part is not necessary, see Proposition 6.1.4. Before linearizing $\epsilon_{jkl}(g_Y)R_{0ikl}(dt^2 + g_Y)$, we note that if we evaluate Ψ_{ij} along a purely spherical deformation g_ϵ of g in the direction of h, the symbol ϵ_{jkl} may depend on $g_Y(\epsilon)$ and so we must write

$$\Psi_{ij}(g_\epsilon) = \mathrm{Sym}_{ij}\left(\sum_{k,l}\epsilon_{jkl}(g_Y(\epsilon))R_{0ikl}(g_\epsilon)\right),$$

however, since $R_{0ijk}(dt^2+g_Y)=0$ for all choices of i,j,k as seen in (5.1.31), we conclude that the linearization of Ψ_{ij} in the direction $\tilde{h}=\{0,0,h\}$ is

$$\mathrm{Sym}_{ij}\left(\sum_{k,l}\epsilon_{jkl}\left(R_g'\right)_{0ikl}(\tilde{h})\right).$$

Linearizing Rm at g in the direction of $\tilde{h}$, and using Proposition 5.1.3, we obtain

$$\left(R_g'\right)_{0ikl}(\tilde{h})=\frac{1}{2}\left(\nabla_0\nabla_l\tilde{h}_{ik}-\nabla_0\nabla_k\tilde{h}_{il}-\nabla_i\nabla_l\tilde{h}_{0k}+\nabla_i\nabla_k\tilde{h}_{0l}\right). \tag{6.1.18}$$

It is easy to see that

$$\nabla_0\nabla_k\tilde{h}_{il}=\nabla_k\dot{h}_{il},\ \text{and}\ \nabla_i\nabla_k\tilde{h}_{0l}=\nabla_i\nabla_l h_{0j}=0,$$

so we have proved

$$\begin{aligned}\Psi_{ij}'(\tilde{h})&=\mathrm{Sym}_{ij}\left(\sum_{k,l}\epsilon_{jkl}\left(R_g'\right)_{0ikl}(\tilde{h})\right)\\&=-\frac{1}{2}\mathrm{Sym}_{ij}\left(\sum_{k,l}\epsilon_{jkl}\left(\nabla_k\dot{h}_{il}-\nabla_l\dot{h}_{ik}\right)\right)=-\frac{1}{2}(\not{d}\dot{h})_{ij}.\end{aligned} \tag{6.1.19}$$

The proposition follows from combining (6.1.15), (6.1.17) and (6.1.19). □

6.1.3 The case of conformal variations

Using conformal invariance, we next extend the formula in Proposition 6.1.6 to tensors of the form $\{h_{00},0,h\}$.

Proposition 6.1.7. *The linearization of W^- at g in the direction $\tilde{h}=\{h_{00},0,h\}$ is*

$$\mathcal{D}(h_{00}dt\otimes dt+h)=\mathcal{D}(h)-\frac{1}{2}\left(\nabla_Y^2 h_{00}-\frac{1}{3}(\Delta_{g_Y}h_{00})g_Y\right).$$

Proof. Since the cylinder is locally conformally flat, for any C^2 function v we have

$$\mathcal{D}\left(v\left(dt^2+g_Y\right)\right)=0,$$

therefore

$$\begin{aligned}\mathcal{D}\left(h_{00}dt\otimes dt+h\right)&=\mathcal{D}\left(h_{00}\left(dt^2+g_Y\right)-h_{00}g_Y+h\right)\\&=\mathcal{D}(h-h_{00}g_Y)=\mathcal{D}(h)-\mathcal{D}(h_{00}g_Y).\end{aligned}$$

Since $h_{00}g_Y$ is a scalar tensor we have by Corollary 6.1.6 and (6.1.6)

$$\mathcal{D}(h_{00}g_Y)=-E'_{g_Y}\left(h_{00}g_Y\right)=-E'_{g_Y}(h_{00}g_Y). \qquad (6.1.20)$$

Next, consider a path $\{g_s\}$ of metric on Y given by $g_s=e^{su}g_Y$, then $g_0=g_Y$ and $\partial_s g_s|_{s=0}=ug_Y$. Since g_Y is Einstein, a standard formula for conformal changes gives

$$E(g_s)=-\frac{s}{2}\left(\nabla^2_{g_Y}u-\frac{1}{3}(\Delta_{g_Y}u)g_Y\right)+\frac{s^2}{4}\left(du\otimes du-\frac{1}{3}|\nabla_{g_Y}u|^2g_Y\right).$$

Differentiating at $s=0$, we obtain

$$E'_{g_Y}(ug_Y)=-\frac{1}{2}\left(\nabla^2_{g_Y}u-\frac{1}{3}(\Delta_{g_Y}u)g_Y\right),$$

and the proposition follows. □

6.1.4 Completion of proof of Theorem 6.1.3

Consider now a variation $\tilde{h}$ of the form $\tilde{h}=\{0,\alpha,0\}$.

Proposition 6.1.8. *The linearization of W^- at g in the direction $\{0,\alpha,0\}$ is given by*

$$\mathcal{D}(\{0,\alpha,0\})=\frac{1}{2}\mathcal{K}_{g_Y}\left(\dot{\alpha}-\tilde{*}d\alpha\right). \qquad (6.1.21)$$

Proof. Choose ω so that $\dot{\omega} = \alpha$. In this case the conformal Killing operator equals

$$\mathcal{K}_g(\omega) = \left\{ -\frac{1}{2}\delta_Y \omega, \alpha, \mathcal{K}_{g_Y}(\omega) + \left(\frac{1}{6}\delta_Y \omega\right) g_Y \right\}.$$

We write

$$\begin{aligned} \mathcal{D}(\{0,\alpha,0\}) &= \mathcal{D}(\{0,\alpha,0\} - \mathcal{K}_g(\omega)) \\ &= \mathcal{D}\left(\{0,\alpha,0\} - \left\{ -\frac{1}{2}\delta_Y \omega, \alpha, \mathcal{K}_{g_Y}(\omega) + \left(\frac{1}{6}\delta_Y \omega\right) g_Y \right\}\right) \\ &= \mathcal{D}\left(\frac{1}{2}\delta_Y \omega, 0, -\mathcal{K}_{g_Y}(\omega) - \left(\frac{1}{6}\delta_Y \omega\right) g_Y\right). \end{aligned}$$

Recall that for any C^2 function u we have $\mathcal{D}(udt^2) = \mathcal{D}(-ug_Y)$, using (6.1.20) we obtain

$$\mathcal{D}(\{0,\alpha,0\}) = \mathcal{D}\left(-\mathcal{K}_{g_Y}(\omega) - \left(\frac{1}{6}\delta_Y \omega\right) g_Y - \left(\frac{1}{2}\delta_Y \omega\right) g_Y\right) = \mathcal{D}\left(-\mathcal{L}_{g_Y}(\omega)\right) \quad (6.1.22)$$

From Corollary 6.1.6 and from (6.1.8) and (6.1.22) we obtain

$$\mathcal{D}(\{0,\alpha,0\}) = \mathcal{K}_{g_Y}\left(\frac{1}{2}\ddot{\omega}\right) - \frac{1}{2}\not{d}\mathcal{L}_{g_Y}(\dot{\omega}), \quad (6.1.23)$$

and since $\dot{\omega} = \alpha$, we have

$$\ddot{\omega} = \dot{\alpha}, \quad (6.1.24)$$

$$\not{d}\mathcal{L}_{g_Y}(\dot{\omega}) = \mathcal{K}_{g_Y}(\tilde{*}d\dot{\omega}) = \mathcal{K}_{g_Y}(\tilde{*}d\alpha), \quad (6.1.25)$$

so from (6.1.23), (6.1.24) and (6.1.25) we obtain (6.1.21). □

With (6.1.21) we are ready to prove Theorem 6.1.3.

Proof of Theorem 6.1.3. Combining Corollary 6.1.6, Proposition 6.1.7 and (6.1.21) we obtain (6.1.3). In order to prove (6.1.4) we linearize E at g_Y in the direction of h

$$E'(h) = \left(Ric(g) - \frac{1}{3}R_g g\right)'_{g_Y}(h) = Ric'_{g_Y}(h) - \frac{1}{3}R'_{g_Y}(h) g_Y - \frac{1}{3}R_{g_Y} h.$$

The linearization of Ric is

$$\left(Ric'_{g_Y}(h)\right)_{ij} = -\frac{1}{2}\Delta_L h_{ij} - \frac{1}{2}\nabla^2_{ij}\mathrm{tr}(h) + \frac{1}{2}\left(\nabla_i\delta_j h + \nabla_j\delta_i h\right), \tag{6.1.26}$$

where $\Delta_L h$ is the *Lichnerowicz* Laplacian given by

$$\Delta_L h_{ij} = \Delta_{g_Y} h_{ij} + 2R_{iljp}h^{lp} - R^p_i h_{jp} - R^p_j h_{ip}. \tag{6.1.27}$$

Since g_Y has constant sectional curvature κ, Δ_L can be computed as

$$\begin{aligned}\Delta_L h_{ij} &= \Delta_{g_Y} h_{ij} + 2\kappa\left(\left(g_Y\right)_{ij}\left(g_Y\right)_{lp} - \left(g_Y\right)_{ip}\left(g_Y\right)_{lj}\right)h^{lp} - 2\kappa\delta^p_i h_{jp} - 2\kappa\delta^p_j h_{ip} \\ &= \Delta_{g_Y} h_{ij} + 2\kappa\mathrm{tr}_Y(h)\left(g_Y\right)_{ij} - 2\kappa h_{ij} - 4\kappa h_{ij} \\ &= \Delta_{g_Y} h_{ij} - 6\kappa\mathrm{tf}(h)_{ij}. \end{aligned} \tag{6.1.28}$$

On the other hand, the linearization of R_g is

$$R'(h) = -\Delta_{g_Y}\mathrm{tr}(h) + \delta_Y\delta_Y h - \langle Ric(g_Y), h\rangle_{g_Y}. \tag{6.1.29}$$

and

$$\frac{1}{3}(R_{g_Y}h - \langle Ric(g_Y), h\rangle_{g_Y} g_Y) = 2\kappa\mathrm{tf}(h). \tag{6.1.30}$$

Combining (6.1.26) and (6.1.29), we conclude that $E'_{g_Y}(h)$ is given by

$$\begin{aligned}E'_{g_Y}(h) = &-\frac{1}{2}\left(\Delta_L h + \nabla^2\mathrm{tr}(h) - \frac{2}{3}\Delta_{g_Y}\mathrm{tr}_Y(h)g_Y\right) \\ &+ \frac{1}{2}\left(\mathcal{L}_{g_Y}(\delta_Y h) - \frac{1}{3}(\delta_Y\delta_Y h)g_Y\right) - \frac{1}{3}(R_{g_Y}h - \langle Ric(g_Y), h\rangle_{g_Y}),\end{aligned} \tag{6.1.31}$$

and using (6.1.28) and (6.1.30), we finally obtain

$$E'_{g_Y}(h) = -\frac{1}{2}\left(\Delta_{g_Y}\mathrm{tf}(h) + \overset{\circ}{\nabla}{}^2\,\mathrm{tr}_Y(h)\right) + \frac{1}{2}\mathcal{K}_{g_Y}(\delta_Y h) + \kappa\cdot\mathrm{tf}(h), \tag{6.1.32}$$

where $\overset{\circ}{\nabla}{}^2$ denotes the traceless Hessian operator. From (6.1.3) and (6.1.32), (6.1.4) follows easily. □

6.2 Some properties of $\not{d}$

In this section we derive several useful identities for the operator $\not{d}$ introduced in Section 6.1 apart from those proved in Subsection 6.1.1. First, we have a crucial formula for the square of $\not{d}$:

Proposition 6.2.1. *The operator $\not{d}^2 : S^2(T^*Y) \mapsto S_0^2(T^*Y)$ is given by*

$$\not{d}^2 h = -4\Delta_{g_Y} \operatorname{tf}(h) - 2 \overset{\circ}{\nabla}{}^2 \operatorname{tr}_Y(h) + 3\mathcal{K}_{g_Y}(\delta_Y h) + 12\kappa \cdot \operatorname{tf}(h). \tag{6.2.1}$$

Proof. The proof is moved to Appendix 7.5.1. □

Next, we have

Proposition 6.2.2. *For any $h \in S^2(T^*Y)$ we have $\delta_Y(\not{d}h) = \tilde{*} d\delta_Y h$.*

Proof. In a local orthonormal basis we have

$$\begin{aligned}
(\delta_Y(\not{d}h))_i &= \sum_{j=1}^3 \nabla_j (\not{d}h)_{ij} \\
&= \sum_{j=1}^3 \sum_{k,l} \epsilon_{ikl} \nabla_j \nabla_k h_{lj} + \sum_{j=1}^3 \sum_{p,q} \epsilon_{jpq} \nabla_j \nabla_p h_{qi}.
\end{aligned} \tag{6.2.2}$$

Commuting covariant derivatives we have

$$\begin{aligned}
&\sum_{j=1}^3 \sum_{k,l} \epsilon_{ikl} \nabla_j \nabla_k h_{lj} = \sum_{j=1}^3 \sum_{k,l} \epsilon_{ikl} \left(\nabla_k \nabla_j h_{lj} - R^s_{jkl} h_{sj} - R^s_{jkj} h_{ls} \right) \\
&= \sum_{j=1}^3 \sum_{k,l} \epsilon_{ikl} \left(\nabla_k \nabla_j h_{lj} + \kappa\big(- h_{jj} g_{kl} + h_{kj} g_{jl} - h_{lj} g_{kj} + h_{lk} g_{jj} \big) \right) \\
&= (\tilde{*} d\delta_Y h)_i + \kappa \sum_{k,l} \left(\sum_{j=1}^3 \{ -h_{jj}\epsilon_{ikl} g_{kl} + \epsilon_{ikl} h_{kl} - \epsilon_{ikl} h_{lk} + 3\epsilon_{ikl} h_{lk} \} \right).
\end{aligned}$$

Since all terms in the sum consist of a term skew-symmetric in k and l times a term symmetric in k and l, the sum is zero, so we obtain

$$\sum_{j=1}^{3}\sum_{k,l}\epsilon_{ikl}\nabla_j\nabla_k h_{lj} = \left(\tilde{*}d\delta_Y h\right)_i . \tag{6.2.3}$$

We also have

$$\begin{aligned}\sum_{j=1}^{3}\sum_{p,q}\epsilon_{jpq}\nabla_j\nabla_p h_{qi} &= \sum_{j=1}^{3}\sum_{p,q}\epsilon_{jpq}\left(\nabla_p\nabla_j h_{qi} - R^s_{jpq}h_{si} - R^s_{jpi}h_{qs}\right)\\ &= \sum_{j=1}^{3}\sum_{p,q}\epsilon_{jpq}\left(\nabla_p\nabla_j h_{qi} + \kappa\big(-h_{ij}g_{pq} + h_{pi}g_{qj} + h_{qj}g_{pi} - h_{pq}g_{ji}\big)\right),\end{aligned}$$

and clearly the last 4 terms sum to zero. So we have

$$\sum_{j=1}^{3}\sum_{p,q}\epsilon_{jpq}\nabla_j\nabla_p h_{qi} = \sum_{j=1}^{3}\sum_{p,q}\epsilon_{jpq}\nabla_p\nabla_j h_{qi}$$

By reindexing j and p on the right hand side, we obtain

$$\sum_{j=1}^{3}\sum_{p,q}\epsilon_{jpq}\nabla_j\nabla_p h_{qi} = \sum_{j=1}^{3}\sum_{p,q}\epsilon_{pjq}\nabla_j\nabla_p h_{qi} = -\sum_{j=1}^{3}\sum_{p,q}\epsilon_{jpq}\nabla_j\nabla_p h_{qi},$$

so this sum vanishes. Combining this with (6.2.2) and (6.2.3), the proposition then follows. □

Corollary 6.2.3. *For any* $h \in S^2(T^*Y)$ *we have*

$$\not{d}\Delta_{g_Y} h = \Delta_{g_Y}\not{d}h. \tag{6.2.4}$$

Proof. From (6.2.1) we have

$$\not{d}^3 h = -4\not{d}\Delta_{g_Y}\mathrm{tf}h - 2\not{d}\overset{\circ}{\nabla}{}^2\,\mathrm{tr}_Y(h) + 3\not{d}\mathcal{K}_{g_Y}(\delta_Y h) + 12\kappa\cdot\not{d}(\mathrm{tf}(h)),$$

and clearly

$$-4\not{d}\Delta_{g_Y}\mathrm{tf}h = -4\not{d}\Delta_{g_Y}h, \tag{6.2.5}$$

$$\not{d}\,\overset{\circ}{\nabla}{}^2\,\mathrm{tr}_Y(h) = \frac{1}{2}\not{d}\mathcal{K}_{g_Y}(d\mathrm{tr}_{g_Y}h) = 0, \tag{6.2.6}$$

$$3\not{d}\mathcal{K}_{g_Y}(\delta_Y h) = 3\mathcal{K}_{g_Y}(\tilde{*}d\delta_Y h) = 3\mathcal{K}_{g_Y}(\delta_Y\not{d}h), \tag{6.2.7}$$

$$\not{d}(\mathrm{tf}(h)) = \not{d}h. \tag{6.2.8}$$

On the other hand we have

$$\begin{aligned}\not{d}^3h = \not{d}^2\not{d}h &= -4\Delta_{g_Y}\not{d}h - 2\,\overset{\circ}{\nabla}{}^2\,\mathrm{tr}_Y(\not{d}h) + 3\mathcal{K}_{g_Y}(\delta_Y\not{d}h) + 12\kappa\cdot\not{d}h \\ &= -4\Delta_{g_Y}\not{d}h + 3\mathcal{K}_{g_Y}(\delta_Y\not{d}h) + 12\kappa\cdot\not{d}h,\end{aligned}$$

and this proves the claim. □

For the next lemma we will use Δ_H to denote the Hodge-Laplacian on $\Lambda^1(T^*Y)$ which is given by

$$\begin{aligned}\Delta_H\omega &= -d\delta_Y\omega - \delta_Y d\omega \\ &= d\tilde{*}d\tilde{*}\omega - \tilde{*}d\tilde{*}d\omega,\end{aligned}$$

which is related to the rough Laplacian on 1-forms by the Weitzenböck formula

$$\Delta_{g_Y} = -\Delta_H + 2\kappa. \tag{6.2.9}$$

Lemma 6.2.4. *The operator $\mathcal{K}_{g_Y}(\omega)$ satisfies*

$$\begin{aligned}\delta_Y\mathcal{K}_{g_Y}(\omega) &= \Delta_{g_Y}\omega + \frac{1}{3}\cdot d\delta_Y\omega + 2\kappa\omega \\ &= -\Delta_H\omega + \frac{1}{3}\cdot d\delta_Y\omega + 4\kappa\omega,\end{aligned}$$

and also

$$\begin{aligned}\Delta_{g_Y}\mathcal{K}_{g_Y}(\omega) &= \mathcal{K}_{g_Y}\left(\left(\Delta_{g_Y}+4\kappa\right)\omega\right)\\ &= \mathcal{K}_{g_Y}\left(\left(-\Delta_H+6\kappa\right)\omega\right).\end{aligned}$$

Proof. Both identities follow from straightforward computations that can be found for example in [Str10a, Appendix]. □

Corollary 6.2.5. *If h is a divergence-free eigentensor of Δ_{g_Y} with eigenvalue $-\lambda$ in $S_0^2(T^*Y)$ which satisfies $\Delta_{g_Y}(h) = -\lambda \cdot h$ then $\not{d}h = \pm 2\sqrt{\lambda+3\kappa}\cdot h$ and if h has the form $h = \mathcal{K}_{g_Y}(\omega)$ with $\Delta_H\omega = \nu\omega$ and $\delta_Y\omega = 0$ then $\not{d}h = \pm\sqrt{\nu}\cdot h$. Both signs occur if (Y^3, g_Y) admits an orientation-reversing isometry (which is always true for S^3 or $\kappa = 0$).*

Proof. Note that in either of the above cases we must have $\not{d}^2 h = c^2 \cdot h$ for some constant c. To see this, if $\Delta_{g_Y} h = -\lambda h$ and $\delta_Y h = 0$ with $\lambda > 0$ then by Proposition 6.2.1 we have

$$\not{d}^2 h = -4\cdot\Delta_{g_Y} h + 12\kappa\cdot h = (4\lambda + 12\kappa)\cdot h. \tag{6.2.10}$$

In this case $c^2 = (4\lambda + 12\kappa)$. For the second case, from Lemma 6.2.4 we have the identities

$$\Delta_{g_Y} h = (6\kappa - \nu)\cdot\mathcal{K}_{g_Y}(\omega) = (6\kappa-\nu)\cdot h, \tag{6.2.11}$$

and

$$\delta_{g_Y}\mathcal{K}_{g_Y}(\omega) = (4\kappa - \nu)\cdot\mathcal{K}_{g_Y}(\omega) \tag{6.2.12}$$

$$= (4\kappa-\nu)\cdot h. \tag{6.2.13}$$

Since we also have

$$\mathrm{tr}_Y(h) = 0, \tag{6.2.14}$$

we easily obtain from Proposition 6.2.1

$$\not{d}^2 h = \nu \cdot h, \tag{6.2.15}$$

so in this case $c^2 = \nu$.

In both cases, observe that if we fix an eigenvalue υ of Δ_{g_Y} on $S_0^2(T^*Y)$ then the eigenspace $E_\upsilon = \{h \in S_0^2(T^*Y) : \Delta_{g_Y} h = -\upsilon \cdot h\}$ is not necessarily $SO(3)$-irreducible, and decomposes into $E_\upsilon = A_c^+ \oplus A_c^-$, where $A_c^\pm = \{h \in S_0^2(T^*Y) : \not{d}h = \pm c \cdot h\}$ and are $SO(3)$-invariant. To see this, given any eigentensor, writing the equation $\not{d}^2 h = c^2 \cdot h$ as

$$(\not{d} + c \cdot I)\,(\not{d} - c \cdot I)\,h = 0, \tag{6.2.16}$$

we conclude that either $+c$ or $-c$ occurs as an eigenvalue of $\not{d}$. If Y^3 admits an orientation-reversing isometry, then since the operator $\not{d}$ changes sign under reversal of orientation, pulling an eigentensor back along an orientation-reversing isometry shows that both A_c^+ and A_c^- are nontrivial and of the same dimension. □

Corollary 6.2.6. *If ω is an eigenform of the Hodge Laplacian on 1-forms with eigenvalue ν satisfying $\delta_Y \omega = 0$ then $\tilde{*} d\omega = \pm\sqrt{\nu} \cdot \omega$. Both signs occur if (Y^3, g_Y) admits an orientation-reversing isometry (which is always true for S^3 or $\kappa = 0$).*

Proof. Obviously, since $\delta_Y \omega = 0$, then

$$(\tilde{*} d)^2 \omega = \tilde{*} d \tilde{*} d = -\delta_Y d\omega = \Delta_H \omega = \nu \cdot \omega. \tag{6.2.17}$$

Using a similar argument as in Corollary 6.2.5, we conclude that $\tilde{*} d\omega = \pm\sqrt{\nu} \cdot \omega$, with both signs occurring on S^3. □

Chapter 7

Analysis for the Self-Dual Deformation Complex

7.1 The adjoint of $\mathcal{D}$

The adjoint operator will map from

$$\mathcal{D}^* : S_0^2(\Lambda_-^2) \to S_0^2(T^*M), \tag{7.1.1}$$

and using the decompositions in Subsection 5.1.1 we will think of this as

$$\mathcal{D}^* : S_0^2(T^*Y) \to S^2(\nu) \oplus (\nu \odot T^*(Y)) \oplus S^2(T^*Y). \tag{7.1.2}$$

Proposition 7.1.1. *The adjoint operator is given by*

$$\mathcal{D}^* Z = \Big\{ -\frac{1}{2}\delta_Y^2 Z, \frac{1}{2}\delta_Y \dot{Z} + \frac{1}{2}\delta_H \tilde{*} \delta_Y Z, -\frac{1}{2}\ddot{Z} - \kappa Z - \frac{1}{2}\not{d}\dot{Z} + \frac{1}{2}\Delta_{g_Y} Z \\ - \frac{1}{2}\mathcal{L}_{g_Y}(\delta_Y Z) + \frac{1}{2}(\delta_Y^2 Z) g_Y \Big\}. \tag{7.1.3}$$

Where δ_H is the Hodge divergence on forms given by $\delta_H = d^*$.

Proof. Let $Z \in S_0^2(T^*Y)$, from the decomposition (5.1.9) we can see $S_0^2(T^*Y)$ as embedded in $S^2(T^*M)$, so taking $\langle , \rangle$ to be the inner product induced by the cylindrical metric g on $S^2(T^*M)$ we observe that since Z is traceless with respect to g_Y we have for any

$\tilde{h} = h_{00} dt \otimes dt + dt \odot \alpha + h,$

$$\langle \mathcal{D}\tilde{h}, Z\rangle = \left\langle \frac{1}{2}\mathcal{L}_{g_Y}\left(-\frac{1}{2}dh_{00} - \delta_Y h + \dot{\alpha} - \tilde{*}d\alpha + \frac{1}{2}d\mathrm{tr}_Y(h)\right), Z\right\rangle$$
$$+ \left\langle -\frac{1}{2}\ddot{h} - \kappa h + \frac{1}{2}\not{d}\dot{h}, Z\right\rangle + \left\langle \frac{1}{2}\Delta_{g_Y} h, Z\right\rangle. \tag{7.1.4}$$

Formal integration by parts then yields

$$\int_0^\infty \int_Y \langle \mathcal{D}\tilde{h}, Z\rangle dt dV_{g_Y} = -\frac{1}{2}\int_0^\infty \left(\left(h_{00}\cdot \delta_Y^2 Z\right) + \langle \alpha, \delta_Y \dot{Z} + \delta_H \tilde{*}\delta_Y Z\rangle\right) dt dV_{g_Y}$$
$$+ \int_0^\infty \int_Y \left\langle h, \frac{1}{2}\left(\delta_Y^2 Z\right) g_Y\right\rangle dt dV_{g_Y}$$
$$- \int_0^\infty \int_Y \left(\left\langle h, \frac{1}{2}\mathcal{L}_{g_Y}(\delta_Y Z) - \frac{1}{2}\ddot{Z} - \kappa Z\right\rangle + \frac{1}{2}\langle \not{d}\dot{h}, Z\rangle\right) dt dV_{g_Y}.$$

Note that by the inner product

$$\langle \alpha, \delta_Y \dot{Z} + \delta_H \tilde{*}\delta_Y Z\rangle,$$

we mean the usual inner product on 1-forms, however, using the decomposition in (7.1.2), we identify a 1-form ξ with the tensor

$$\{0, \xi, 0\} = \xi \otimes dt + dt \otimes \xi,$$

so we obtain

$$\langle \alpha, \delta_Y \dot{Z} + \delta_H \tilde{*}\delta_Y Z\rangle = \frac{1}{2}\langle \{0, \alpha, 0\}, \{0, \delta_Y \dot{Z} + \delta_H \tilde{*}\delta_Y Z, 0\}\rangle.$$

Finally, the proposition follows using that $\not{d}$ is formally self-adjoint. □

Proposition 7.1.2. *We have the decompositions*

$$S_0^2(T^*Y) = Ker(\delta_Y) \oplus Im(\mathcal{K}_{g_Y}), \tag{7.1.5}$$

and

$$\Lambda^1(T^*Y) = Im(d) \oplus Ker(d^*). \tag{7.1.6}$$

Proof. Since δ_Y is the formal adjoint of $-\frac{1}{2}\mathcal{K}_{g_Y}$, (7.1.5) follows from standard Fredholm theory. The Hodge decomposition theorem says that

$$\Lambda^1(T^*Y) = \mathcal{H}^1(T^*Y) \oplus \left(d\Lambda^0(T^*Y)\right) \oplus \left(d^*\Lambda^2(T^*Y)\right), \tag{7.1.7}$$

where $\mathcal{H}^1(T^*Y)$ is the space of harmonic 1-forms in $\Lambda^1(T^*Y)$, and (7.1.6) follows easily from this since $\mathcal{H}^1(T^*Y)$ and $d^*\Lambda^2(T^*Y)$ are both contained in $Ker(d^*)$. □

Using this decomposition we obtain:

Corollary 7.1.3. *Any time dependent $Z \in S^2_0(T^*Y)$ can be written uniquely as an infinite linear combination of elements of three types, namely*

1. *Elements of type* I*:*

$$f(t) \cdot \mathcal{K}_{g_Y}(d\phi), \tag{7.1.8}$$

where ϕ is an eigenfunction of Δ_H on $\Lambda^0\left(T^*Y\right)$,

2. *Elements of type* II*:*

$$f(t) \cdot \mathcal{K}_{g_Y}(\omega), \tag{7.1.9}$$

where ω is an eigenform of Δ_H on $\Lambda^1\left(T^*Y\right)$ satisfying $\delta_Y\omega = 0$,

3. *Elements of type* III*:*

$$f(t) \cdot B, \tag{7.1.10}$$

where B is an eigentensor of Δ_{g_Y} on $S^2_0(T^*Y)$ satisfying $\delta_Y B = 0$.

In all of the three above cases $f(t)$ denotes a real-valued function.

From Propositions 6.2.2, 6.2.4, and 6.2.5, we observe that the image of $\mathcal{D}^*$ on an element of type I has the form

$$\mathcal{D}^*\left(f(t)\cdot\mathcal{K}_{g_Y}(d\phi)\right) = a_1(t)\cdot\phi dt\otimes dt + a_2(t)\cdot d\phi\odot dt$$
$$+a_3(t)\cdot\mathcal{K}_{g_Y}(d\phi) + a_4(t)\cdot\phi g_Y, \tag{7.1.11}$$

where each coefficient a_i depends on f and the eigenvalue of Δ_H corresponding to ϕ. On elements of type II the image of $\mathcal{D}^*$ is

$$\mathcal{D}^*\left(f(t)\cdot\mathcal{K}_{g_Y}(\omega)\right) = b_1(t)\cdot\omega\odot dt + b_2(t)\cdot\mathcal{K}_{g_Y}(\omega), \tag{7.1.12}$$

where each b_i depends on f and the eigenvalue of Δ_H on divergence-free 1-forms corresponding to η. Finally, on elements of type III we have

$$\mathcal{D}^*\left(f(t)\cdot B\right) = \tilde{f}(t)\cdot B, \tag{7.1.13}$$

where $\tilde{f}$ is determined by f and the eigenvalue of Δ_{g_Y} corresponding to B. In a similar way to Corollary 7.1.3 one can prove that all elements in $S^2_0(T^*M)$ can be written uniquely as an infinite sum of elements as in the right hand sides of (7.1.11), (7.1.12) and (7.1.13), so it follows that in order to find the general solution of $\mathcal{D}^*Z = 0$ it suffices to consider solutions Z of types I, II and III separately. For example, if Z has the form (7.1.8) then writing $\mathcal{D}^*Z$ as in (7.1.11) one sees that in order to obtain $\mathcal{D}^*Z = 0$ one must solve for f in (7.1.8) so that in (7.1.12) one has $a_1 = a_2 = a_3 = a_4 = 0$ and in general this amounts to solving an ordinary differential equation on f. We start by considering solutions of type III. For that purpose we use the following:

Lemma 7.1.4. *If λ is an eigenvalue of $-\Delta_{g_Y}$ on divergence-free sections of $S^2_0(T^*Y)$, then* *(a) if* $\kappa = 1$, $\lambda \geq 6$,

(b) if $\kappa = -1$, $\lambda \geq 3$ *with equality achieved only for nontrivial Codazzi tensors* $h \in S^2_0(T^*Y)$, *that is,* $d^\nabla h = 0$.

(c) if $\kappa = 0$, $\lambda \geq 0$ *with equality for parallel sections in* $S^2_0(T^*Y)$.

Proof. These are due to Koiso, we only give a brief argument [Koi78]. For (a), the inequality

$$\int_{S^3} |\nabla_i h_{jk} + \nabla_j h_{ki} + \nabla_k h_{ij}|^2 dV \geq 0, \tag{7.1.14}$$

easily implies that $\lambda \geq 6$. For (b), the inequality

$$\int_Y |\nabla_i h_{jk} - \nabla_j h_{ik}|^2 dV \geq 0, \tag{7.1.15}$$

implies that $\lambda \geq 3$, with equality exactly for Codazzi tensors. Finally, the $\kappa = 0$ case is trivial. □

The classification of type III solutions is given by the following.

Proposition 7.1.5. *Let* $0 \leq \lambda_1 < \lambda_2 < \ldots$, *be the eigenvalues of* $-\Delta_{g_Y}$ *on divergence-free tensors in* $S^2_0(T^*Y)$ *and let* $\beta_j = \sqrt{\lambda_j + 3\kappa}$. *For each eigenvalue* λ_j *there exist trace-free and divergence-free eigentensors* $B^\pm_j$ *and* $C^\pm_j$ *satisfying*

$$\not{d} B^\pm_j = \pm\beta_j B^\pm_j, \quad \not{d} C^\pm_j = \pm\beta_j C^\pm_j, \tag{7.1.16}$$

such that the general solution of $\mathcal{D}^* Z = 0$ *with* Z *satisfying*

$$\delta_Y Z = 0, \tag{7.1.17}$$

$$\mathrm{tr}_Y Z = 0, \tag{7.1.18}$$

can be written in in the following way:

(a) If $\kappa = 1$ *then*

$$Z = \sum_{j=1}^{\infty} \left(e^{(\beta_j+1)t} B_j^+ + e^{(\beta_j-1)t} C_j^+ + e^{(-\beta_j+1)t} B_j^- + e^{(-\beta_j-1)t} C_j^- \right).$$

Letting $\alpha_j^{\pm} = \beta_j \pm 1$, *we have* $0 < |\alpha_1^{\pm}| < |\alpha_2^{\pm}| < \dots$, *and* $|\alpha_1^{\pm}| = 2$.

(b) If $\kappa = -1$

$$Z = \sum_{j=1}^{\infty} \left\{ e^{\beta_j t} \left(B_j^+ \cos(t) + C_j^+ \sin(t) \right) + e^{-\beta_j t} \left(B_j^- \cos(t) + C_j^- \sin(t) \right) \right\},$$

with $\beta_1 = 0$ *and where* $B_1^{\pm}$ *and* $C_1^{\pm}$ *are trace-free Codazzi tensors.*

(c) If $\kappa = 0$,

$$Z = B_1 + tC_1 + \sum_{j=2}^{\infty} \left(e^{\beta_j t} B_j^+ + t e^{\beta_j t} C_j^+ + e^{-\beta_j t} B_j^- + t e^{-\beta_j t} C_j^- \right),$$

where B_1 *and* C_1 *are parallel sections of* $S_0^2(T^*Y)$.

Proof. Let Z be a solution of $\mathcal{D}^* Z = 0$ of type III, that is, $Z = fB$ with B satisfying (7.1.17), (7.1.18) and $\Delta_{g_Y} B = -\lambda \cdot B$, then f and B satisfy the equation

$$0 = \mathcal{D}^*(fB) = \left\{ 0, 0, -\frac{1}{2}\ddot{f} B - \frac{1}{2}\dot{f} \not{d} B - \left(\kappa + \frac{\lambda}{2} \right) fB \right\},$$

from (6.2.5) we have

$$\not{d}\dot{Z} = \not{d}\left(\dot{f} \cdot B \right) = \pm \dot{f} \left(2\sqrt{\lambda + 3\kappa} \cdot B \right). \tag{7.1.19}$$

It follows that f is a solution of the ordinary differential equation

$$-\frac{1}{2}\ddot{f} \pm \sqrt{\lambda + 3\kappa} \cdot \dot{f} - \left(\kappa + \frac{\lambda}{2} \right) f = 0. \tag{7.1.20}$$

Letting $\beta = \sqrt{\lambda + 3\kappa}$, then the characteristic roots of (7.1.20) are

$$\pm\beta \pm \sqrt{\kappa}.$$

The expansions follow from considering the different solutions obtained for κ in the set $\{1, -1, 0\}$, and Lemma 7.1.4. □

We now turn to solutions of $\mathcal{D}^*(Z) = 0$ with Z of types I and II. We will need to introduce the operator

$$\Box_{\mathcal{K}} : \Lambda^1(T^*Y) \mapsto \Lambda^1(T^*Y),$$

given by

$$\Box_{\mathcal{K}}\eta = \delta_Y \mathcal{K}_{g_Y}(\eta).$$

We have the following

Lemma 7.1.6. *Let $\eta \in \Lambda^1(T^*Y)$, then*

$$\Box_{\mathcal{K}}\eta = (\delta_Y d + \frac{4}{3} d\delta_Y + 4\kappa)\eta. \tag{7.1.21}$$

Also, if η is an eigenform of Δ_H on 1-forms then

$$\Box_{\mathcal{K}}\eta = c \cdot \eta,$$

where $c = (-\frac{4}{3}\mu + 4\kappa)$ if $\eta = d\phi$ and $\Delta_H \phi = \mu\phi$ or $c = (4\kappa - \nu)$ if $\Delta_H \eta = \nu\eta$ and $\delta_Y \eta = 0$. Moreover, in either case the constant c is nonzero unless $Z = 0$.

Proof. The expression (7.1.21) for $\mathcal{K}_{g_Y}$ is a direct consequence of Lemma 6.2.4. Suppose now that $\Delta_H \eta = \lambda \cdot \eta$. If $\eta = d\phi$ with $\Delta_H \phi = \nu\phi$, then observe that

$$\begin{aligned} \Box_{\mathcal{K}}\eta &= \left(\delta d + \frac{4}{3} d\delta + 4\kappa\right)\eta \\ &= \left(\frac{4}{3} d\delta + 4\kappa\right)\eta = \left(-\frac{4}{3}\mu + 4\kappa\right)\eta. \end{aligned}$$

If $\delta_Y \eta = 0$, then

$$\begin{aligned}\Box_{\mathcal{K}}\eta &= \left(\delta_Y d + \frac{4}{3} d\delta_Y + 4\kappa\right)\eta \\ &= (\delta_Y d + 4\kappa)\,\eta = -\Delta_H \eta + 4\kappa\eta = (-\nu + 4\kappa)\,\eta.\end{aligned}$$

Finally, in order to show that $c = 0$ does not occur we note that when $\kappa = 1$, there are eigenforms of Δ_H corresponding to the eigenvalue $\mu = 3$ on closed forms and to $\nu = 4$ on co-closed forms and in these cases $c = 0$. However, for any of these eigenvalues, the corresponding eigenforms are conformally Killing. In the hyperbolic case $\kappa = -1$, the constant c is strictly negative for either closed or co-closed eigenforms of the Hodge Laplacian. In the flat case $\kappa = 0$, the constant c equals zero only for parallel forms, but in this case $Z = 0$. $\Box$

Next, assume that Z is a non-trivial solution of $\mathcal{D}^* Z = 0$ with Z of type I or II and $c \neq 0$, where c is the constant in Lemma 7.1.6. The first component of (7.1.3) yields

$$0 = \delta_Y(\delta_Y Z) = f(t) \cdot \delta_Y \Box_{\mathcal{K}} \eta = f(t) \cdot \delta_Y (c\eta). \tag{7.1.22}$$

Since Z is non-trivial and $c \neq 0$, we conclude that $\delta_Y \omega = 0$ and hence, solutions of type I do *not* occur. Furthermore, we can prove

Proposition 7.1.7. *We have*

$$\dot{f}\tilde{*} d\omega = -\nu f \omega. \tag{7.1.23}$$

Proof. The second component of (7.1.3) yields

$$0 = \dot{f} \cdot \delta_Y \mathcal{K}_{g_Y}(\omega) - f \delta_Y \tilde{*} \delta_Y \mathcal{K}_{g_Y}(\omega), \tag{7.1.24}$$

which by Lemma 6.2.4 is equivalent to

$$\dot{f}(-\nu + 4\kappa)\omega + f\delta_H \tilde{*}(-\nu + 4\kappa)\omega = 0 \tag{7.1.25}$$

and writing δ_H on 2-forms as $\tilde{*}d\tilde{*}$ we obtain

$$\dot{f}\omega = -f\tilde{*}d\omega, \tag{7.1.26}$$

and after taking $\tilde{*}d$ on both sides we conclude that

$$\dot{f}\tilde{*}d\omega = -f\tilde{*}d\tilde{*}d\omega = -fd^*d\omega = -\nu f\omega, \tag{7.1.27}$$

as needed. □

Proposition 7.1.8. *Let $Z = f(t) \cdot \mathcal{K}_{g_Y}\omega$ satisfy $\mathcal{D}^*Z = 0$, where ω satisfies $\delta_Y\omega = 0$ and*

$$\Delta_H\omega = \nu \cdot \omega. \tag{7.1.28}$$

Then

$$f(t) = e^{\alpha t}, \tag{7.1.29}$$

with

$$\alpha = \pm(\sqrt{\nu}), \tag{7.1.30}$$

or

$$f(t) = c_0, \tag{7.1.31}$$

for a constant c_0, if ω is a non-trivial harmonic 1-form.

Proof. From (7.1.23) we have

$$-\frac{1}{2}\not{d}\dot{Z} = -\frac{1}{2}\dot{f}\not{d}\mathcal{K}_{g_Y}(\omega) = -\frac{1}{2}\dot{f}\mathcal{K}_{g_Y}(\tilde{*}d\omega)$$
$$= \frac{\nu}{2}f\mathcal{K}_{g_Y}(\omega), \tag{7.1.32}$$

we also have from Lemma 6.2.4

$$\delta_Y Z = f(t)\cdot\delta_Y\mathcal{K}_{g_Y}(\omega)$$
$$= f(t)\cdot(-\nu+4\kappa)\cdot\omega, \tag{7.1.33}$$
$$\Delta_{g_Y} Z = f(t)\cdot\Delta_{g_Y}\mathcal{K}_{g_Y}(\omega)$$
$$= f(t)\cdot(6\kappa-\nu)\cdot\mathcal{K}_{g_Y}(\omega), \tag{7.1.34}$$
$$\delta_Y^2 Z = f(t)\cdot\delta_Y^2\mathcal{K}_{g_Y}(\omega)$$
$$= f(t)\cdot(-\nu+4\kappa)\cdot\delta_Y\omega = 0. \tag{7.1.35}$$

The equation on the purely spherical component of $\mathcal{D}^*(Z)$ is

$$0 = -\frac{1}{2}\ddot{Z} - \kappa Z - \frac{1}{2}\not{d}\dot{Z} + \frac{1}{2}\Delta_{g_Y}Z - \frac{1}{2}\mathcal{L}_{g_Y}(\delta_Y Z) + \frac{1}{2}(\delta_Y^2 Z)g_Y, \tag{7.1.36}$$

which by (7.1.33), (7.1.34),(7.1.35) and (7.1.32) simplifies to

$$0 = \left(-\frac{1}{2}\ddot{f} - \kappa f + \frac{1}{2}(6\kappa-\nu)f - \frac{1}{2}(4\kappa-\nu)\right)\mathcal{K}_{g_Y}(\omega) - \frac{1}{2}\dot{f}\mathcal{K}_{g_Y}(\tilde{*}d\omega)$$
$$= -\frac{1}{2}\ddot{f}\mathcal{K}_{g_Y}(\omega) + \frac{1}{2}\nu f\mathcal{K}_{g_Y}(\omega), \tag{7.1.37}$$

which we write as

$$\left(\ddot{f}-\nu\right)\mathcal{K}_{g_Y}(\omega) = 0. \tag{7.1.38}$$

and for 1-forms ω that are *not* dual to Killing fields we obtain solutions

$$f(t) = e^{\pm\sqrt{\nu}t}. \tag{7.1.39}$$

The solutions with f as in (7.1.31) correspond to $\nu = 0$ which is the case of a harmonic 1-form. In this case, the tensor $Z = t \cdot \mathcal{K}_{g_Y}(\omega)$ is ruled out by (7.1.26) above, and only the solution $Z = c_0 \cdot \mathcal{K}_{g_Y}(\omega)$ occurs. □

Let $0 = \nu_0 < \nu_1 < \ldots,$ be all the eigenvalues of Δ_H on co-closed forms in $\Lambda^1(T^*Y)$. Note that there are non-trivial eigenforms corresponding to ν_0 if and only if $b_1(T^*Y) \neq 0$ since such eigenforms are harmonic 1-forms. In particular, for $\kappa = 1$ there are no nontrivial 1-forms in $\Lambda^1(T^*Y)$. We close this section with the following.

Proposition 7.1.9. *Let $Z \in S^2_0(T^*Y)$ be a solution of $\mathcal{D}^*Z = 0$. Then Z can be written as*

$$Z = \mathcal{K}_{g_Y}(\omega) + Z_0,$$

where Z_0 is divergence-free and has an expansion as in Proposition 7.1.5. Also for each eigenvalue ν_j, $j = 0, 1, \ldots$, there are eigenforms $\omega_j^{\pm}$ such that $\mathcal{K}_{g_Y}(\omega)$ can be written uniquely as an infinite sum of the following form

(a) If $\kappa = 1$ then

$$\mathcal{K}_{g_Y}(\omega) = \sum_{j=2}^{\infty} (e^{\sqrt{\nu_j}t}\mathcal{K}_{g_Y}(\omega_j^+) + e^{-\sqrt{\nu_j}t}\mathcal{K}_{g_Y}(\omega_j^-)).$$

where $c_j^{\pm}$ are constants. In the case $Y = S^3$, $\nu_j = (j+1)^2$.

(b) If $\kappa = -1$ then

$$\mathcal{K}_{g_Y}(\omega) = \mathcal{K}_{g_Y}(\omega_0) + \sum_{j=1}^{\infty} \left(e^{\sqrt{\nu_j}t}\mathcal{K}_{g_Y}(\omega_j^+) + e^{-\sqrt{\nu_j}t}\mathcal{K}_{g_Y}(\omega_j^-)\right),$$

where ω_0 is a harmonic 1-form.

(c) If $\kappa = 0$ then

$$\mathcal{K}_{g_Y}(\omega) = \sum_{j=1}^{\infty} \left(e^{\sqrt{\nu_j} t} \mathcal{K}_{g_Y}(\omega_j^+) + e^{-\sqrt{\nu_j} t} \mathcal{K}_{g_Y}(\omega_j^-) \right).$$

Proof. From Proposition 7.1.8, we can write the 1-form ω as an infinite sum of the form

$$\omega = \omega_0 + \sum_{j=1}^{\infty} \left(e^{\sqrt{\nu_j} t} \omega_j^+ + c_j^- e^{-\sqrt{\nu_j} t} \omega_j^- \right). \tag{7.1.40}$$

In case $\kappa = 1$, there are no harmonic 1-forms, and all eigenforms corresponding to $\nu_1 = 4$ are dual to Killing fields, so the sum starts at $j = 2$ in this case. The form of the eigenvalues ν_j in the case $\kappa = 1$ follows from [Fol89]. The $\kappa = -1$ case follows directly from (7.1.40). In the case $\kappa = 0$, any harmonic 1-form is parallel. □

7.2 Mixed solutions

Returning to the full system

$$\mathcal{D}^* Z = \mathcal{K}_g(\tilde{\omega}), \tag{7.2.1}$$

we note that since $\mathcal{D}^* Z$ is divergence-free, the 1-form $\tilde{\omega}$ automatically satisfies the equation

$$\delta_g \mathcal{K}_g(\tilde{\omega}) \equiv \Box_{\mathcal{K},g}(\tilde{\omega}) = 0, \tag{7.2.2}$$

so we next analyze solutions of (7.2.2) at a cylindrical metric $g = dt^2 + g_Y$. The conformal Killing operator on a 1-form $\tilde{\omega} = f dt + \omega$ is

$$\mathcal{K}_g(\tilde{\omega}) = \left(\frac{3}{2} \dot{f} - \frac{1}{2} \delta_Y \omega \right) dt \otimes dt + (\dot{\omega} + df) \odot dt + \mathcal{L}_{g_Y}(\omega) - \frac{1}{2} \left(\dot{f} + \delta_Y \omega \right) g_Y. \tag{7.2.3}$$

The divergence of a traceless symmetric 2-tensor

$$\tilde{h} = h_{00} dt \otimes dt + (\alpha \otimes dt + dt \otimes \alpha) + h \tag{7.2.4}$$

is given by

$$\delta \tilde{h} = (\dot{h}_{00} + \delta_{S^3}\alpha) dt + \dot{\alpha} + \delta_{S^3} h. \tag{7.2.5}$$

Combining (7.2.3) and (7.2.5), we obtain

$$\Box_{\mathcal{K},g}\tilde{\omega} = \left(\frac{3}{2}\ddot{f} + \frac{1}{2}\delta_Y \dot{\omega} - \Delta_H f\right) dt + \ddot{\omega} + \delta_Y \mathcal{L}_{g_Y}(\omega) - \frac{1}{2} d\delta_Y \omega.$$

Commuting covariant derivatives as in Lemma 6.2.4, we have

$$\delta_Y \mathcal{L}_{g_Y}(\omega) = -\Delta_H \omega + d\delta_Y \omega + 4\kappa\omega,$$

so $\Box_{\mathcal{K},g}\tilde{\omega}$ takes the form

$$\Box_{\mathcal{K},g}(\tilde{\omega}) = \left(\frac{3}{2}\ddot{f} + \frac{1}{2}\delta_Y \dot{\omega} - \Delta_H f\right) dt + \ddot{\omega} - \Delta_H \omega + \frac{1}{2} d\delta_Y \omega + 4\kappa\omega + \frac{1}{2} d\dot{f}.$$

Any 1-form $\tilde{\omega} \in \Lambda^1(T^*M)$ can be written as an infinite sum of 1-forms of two types, namely

(i) Forms of type (a)

$$c(t)\phi dt + k(t) d\phi, \tag{7.2.6}$$

where ϕ is an eigenfunction of the Hodge laplacian $\tilde{\Delta}_H$ on $\Lambda^0(T^*Y)$ with eigenvalue μ and $c = c(t)$, $k = k(t)$ are functions of t,

(ii) Forms of type (b)

$$m(t)\eta, \tag{7.2.7}$$

where η is an eigenform of the Hodge Laplacian Δ_H on $\Lambda^1(T^*Y)$ satisfying

$$\delta_Y \eta = 0,$$

and $m = m(t)$.

Let us start by solving $\Box_{\mathcal{K},g}\tilde{\omega} = 0$ assuming that $\tilde{\omega}$ is of type (a). In this case, from (7.2.6) we conclude that the functions l and m satisfy the following system of ordinary differential equations

$$\begin{aligned} \ddot{l} &= \frac{2}{3}\mu l + \frac{\mu}{3}\dot{m}, \\ \ddot{m} &= -\frac{1}{2}\dot{l} + \left(\frac{3}{2}\mu - 4\kappa\right) m. \end{aligned} \tag{7.2.8}$$

If we let $l_1 = l$, $l_2 = \dot{l}_1$, $m_1 = m$ and $m_2 = \dot{m}_1$, the system (7.2.8) is equivalent to the first order linear system

$$\dot{X} = AX,$$

where X and A are given by

$$X = \begin{pmatrix} l_1 \\ l_2 \\ m_1 \\ m_2 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{2}{3}\mu & 0 & 0 & \frac{\mu}{3} \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{1}{2} & \left(\frac{3}{2}\mu - 4\kappa\right) & 0 \end{pmatrix}.$$

The characteristic roots of the matrix A are $\pm\alpha^{\pm}(\mu)$ where $\alpha^{\pm}(\mu)$ is given by

$$\alpha^{\pm} = \alpha^{\pm}(\mu) = \sqrt{\mu - 2\kappa \pm 2\sqrt{\kappa^2 - \frac{\mu}{3}\kappa}}, \tag{7.2.9}$$

We now consider solutions of $\Box_{\mathcal{K}}\tilde{\omega} = 0$ with $\tilde{\omega}$ of type (b). If $\tilde{\omega}$ is as in (7.2.7), the system $\Box_{\mathcal{K}}\tilde{\omega} = 0$ takes the form

$$\ddot{m} - \nu m + 4\kappa m = 0, \tag{7.2.10}$$

and the characteristic roots of this equation are

$$\pm\sqrt{\nu - 4\kappa}.$$

Let $0 = \mu_0 < \mu_1 < \dots$ be all the eigenvalues of Δ_H on $\Lambda^0(T^*Y)$ and let ν_j for $j = 0, 1, \dots,$ denote all the eigenvalues of Δ_H on co-closed forms in $\Lambda^1(T^*Y)$. In particular if $\kappa = 1$ and $\Gamma = \{e\}$, $\mu_j = j(j+2)$. We have

Proposition 7.2.1. *Let $(Z, \tilde{\omega})$ be a solution of* (7.2.1). *Then up to addition of 1-forms which are dual to conformal Killing fields, the 1-form $\tilde{\omega}$ can be written as follows.*

(a) If $\kappa = 1$ and $\Gamma = \{e\}$, $\tilde{\omega}$ is an infinite sum of the form

$$\sum_{j=2}^{\infty} e^{\pm\beta_j t}\left(\left\{\phi_{1j}^{\pm}\cos(\gamma_j t) + \phi_{2j}^{\pm}\sin(\gamma_j t)\right\}dt + \left\{c_{1j}^{\pm}\cos(\gamma_j t)d\phi_{1j}^{\pm} + c_{2j}^{\pm}\sin(\gamma_j t)d\phi_{2j}^{\pm}\right\}\right)$$
$$+\sum_{j=2}^{\infty}\left(e^{\delta_j t}\omega_j^{+} + e^{-\delta_j t}\omega_j^{-}\right),$$

where $\phi_{1j}^{\pm}$ and $\phi_{2j}^{\pm}$ are eigenfunctions of Δ_H corresponding to $\mu_j = j(j+2)$, and the coefficients $c_{1j}^{\pm}$ and $c_{2j}^{\pm}$ are constants, for $j \geq 2$. The rates β_j satisfy $\sqrt{6} < \beta_j$ for $j \geq 2$ and are given by $\beta_j = Re(\alpha_j^{+})$, $\gamma_j = Im(\alpha_j^{+})$, where

$$\alpha_j^{\pm} = \sqrt{j(j+2) - 2 \pm \frac{2}{3}\sqrt{-1}\cdot\sqrt{3(j-1)(j+3)}}.$$

Also, $\omega_j^{\pm}$ are eigenforms corresponding to the eigenvalues $\nu_j = (j+1)^2$ of Δ_H on co-closed forms, and $\delta_j = \sqrt{\nu_j - 4}$.

If $\Gamma \neq \{e\}$ then $\pm\alpha_j^{\pm}$ or $\pm\delta_j$ will occur as indicial roots if and only if the corresponding eigenfunction or eigenform descends to the quotient S^3/Γ, respectively.

(b) If $\kappa = -1$, then

$$\tilde{\omega} = \sum_{j=1}^{\infty} \left(\left\{ e^{\pm\sigma_j^+ t} \phi_{1j}^{\pm} + e^{\pm\sigma_j^- t} \phi_{2j}^{\pm} \right\} dt + c_{1j}^{\pm} e^{\pm\sigma_j^+ t} d\phi_{1j}^{\pm} + c_{2j}^{\pm} e^{\pm\sigma_j^- t} d\phi_{2j}^{\pm} \right)$$
$$+ \sum_{j=0}^{\infty} \left(e^{\tau_j t} \omega_j^+ + e^{-\tau_j t} \omega_j^- \right),$$

where $\omega_0^{\pm}$ are harmonic 1-forms in $\Lambda^1(T^*Y)$. The numbers $\sigma_j^{\pm}$ for $j \geq 1$ are real and are given by

$$\sigma_j^{\pm} = \sqrt{\mu_j + 2 \pm 2\sqrt{1 + \frac{\mu_j}{3}}}, \tag{7.2.11}$$

where μ_j are the eigenvalues with respect to the hyperbolic metric. The numbers τ_j are also real and are given by $\tau_j = \sqrt{\nu_j + 4}$, where ν_j are the eigenvalues with respect to the hyperbolic metric. The coefficients $c_{1j}^{\pm}$ and $c_{2j}^{\pm}$ for $j \geq 1$ are constants.

(c) If $\kappa = 0$,

$$\tilde{\omega} = \sum_{j=1}^{\infty} e^{\pm t\sqrt{\mu_j}} \left(\left\{ \phi_{1j}^{\pm} + t\phi_{2j}^{\pm} \right\} dt + \left\{ c_{1j}^{\pm} d\phi_{1j}^{\pm} + t c_{2j}^{\pm} d\phi_{2j}^{\pm} \right\} \right)$$
$$+ \sum_{j=1}^{\infty} \left(e^{t\sqrt{\nu_j}} \omega_j^+ + e^{-t\sqrt{\nu_j}} \omega_j^- \right),$$

where the notation is as above, but with eigenvalues μ_j and ν_j corresponding to the metric on T^3.

Proof. For the case $\kappa = 1$, the only real roots in (7.2.9) correspond to the eigenvalues $\mu = 0, 3$ of Δ_H on $\Lambda^0(T^*Y)$, however, we see in either case that for the solution $\tilde{\omega}$ of (7.2.2) obtained, $\mathcal{K}_g(\tilde{\omega})$ is not in the image of $\mathcal{D}^*$. To clarify this observation, we note that for $\mu = 0$, $\mathcal{K}_g(\tilde{\omega})$ has the form

$$l(t) dt \otimes dt + k(t) g_Y, \tag{7.2.12}$$

i.e., l and k only depend on t and for $\mu = 3$, $\mathcal{K}_g(\tilde{\omega})$ has the form

$$c_1(t)\phi dt \otimes dt + c_2(t)d\phi \odot dt + c_3(t)\phi g_Y, \tag{7.2.13}$$

where ϕ is a spherical harmonic of order 1 (and hence $d\phi$ is conformally Killing with respect to the metric g_Y). From (7.1.11), (7.1.12) and (7.1.13), we conclude that elements of the form (7.2.12) or (7.2.13) in the image of $\mathcal{D}^*$ can only arise from evaluating $\mathcal{D}^*$ at an element of the form $f(t) \cdot \mathcal{K}_{g_Y}(d\psi)$, but in either case $\mathcal{K}_{g_Y}(d\psi) = 0$ and therefore $\tilde{\omega}$ must be conformally Killing with respect to the metric g. Similarly, for forms of type (b) in the case $\kappa = 1$, we see that the solution of (7.2.2) obtained for the least positive eigenvalue of Δ_H on co-closed forms in $\Lambda^1(T^*Y)$ (which is $\nu = 4$), is dual to a Killing field in Y, and in this case $\mathcal{K}_g(\tilde{\omega})$ is not in the image of $\mathcal{D}^*$ either. It is also easy to see from (7.2.9) that for $\mu > 3$ all the rates $\alpha^{\pm}(\mu)$ satisfy $|Re(\alpha^{\pm}(\mu))| > \sqrt{6}$. The rest of the Proposition follows from straightforward computations.

□

We are now ready to describe the general solution of (7.2.1). If $(Z, \tilde{\omega})$ is a solution of (7.2.1), then we can write Z as

$$Z = Z_0 + \tilde{Z},$$

where Z_0 satisfies $\mathcal{D}^* Z_0 = 0$ and $\tilde{Z}$ is a non-zero solution of (7.2.1). We now prove that this solution $\tilde{Z}$ indeed exists.

Proposition 7.2.2. *Let $\tilde{\omega} \in \Lambda^1(T^*M)$ be a solution of (7.2.2) of type (a) or type (b) which is not conformally Killing. There exists a nonzero $\tilde{Z} \in S^2_0(T^*M)$ such that $\mathcal{D}^*\tilde{Z} = \mathcal{K}_g\tilde{\omega}$.*

Proof. For the proof, we consider a solution of (7.2.1) with $\tilde{\omega}$ of type (a), that is,

$$\tilde{\omega} = l\phi dt + md\phi,$$

where ϕ is an eigenfunction of Δ_H on $\Lambda^0(T^*Y)$ with eigenvalue μ and l, m are solutions of (7.2.8). On the other hand, for the element of type I, $f(t) \cdot \mathcal{K}_{g_Y}(d\phi)$, the operator $\mathcal{D}^*$ can be computed explicitly following the results in Section 7.1 as

$$\mathcal{D}^*\left(f\mathcal{K}_{g_Y}(d\phi)\right) = \Big\{\mu(2\kappa - \frac{2}{3}\mu)f\phi, \dot{f}(2\kappa - \frac{2}{3}\mu)d\phi,$$
$$-\frac{1}{2}\left(\ddot{f} - \frac{\mu}{3}f\right)\mathcal{L}_{g_Y}(d\phi) - \frac{\mu}{3}\left(\ddot{f} - (\mu - 2\kappa)f\right)\phi g_Y\Big\}.$$

Suppose that $d\phi$ is not conformally Killing with respect to g_Y, then $\mathcal{L}_{g_Y}(d\phi)$ and ϕg_Y are linearly independent. If we write $Z = f\mathcal{K}_{g_Y}(d\phi)$, then we can solve for f such that $\mathcal{D}^* Z = \mathcal{K}_g(\tilde{\omega})$ by considering the system

$$\begin{aligned} \mu(2\kappa - \frac{2}{3}\mu)f &= \frac{3}{2}\dot{l} + \frac{\mu}{2}m, \\ (2\kappa - \frac{2}{3}\mu)\dot{f} &= l + \dot{m}, \\ -\frac{1}{2}\left(\ddot{f} - \frac{\mu}{3}f\right) &= m, \\ -\frac{\mu}{3}\left(\ddot{f} - (\mu - 2\kappa)f\right) &= -\frac{1}{2}(\dot{l} - \mu m). \end{aligned} \tag{7.2.14}$$

Note that from the condition that $d\phi$ is not conformally Killing we see that in the cases $\kappa = \pm 1$ we have $\mu(2\kappa - \frac{2}{3}\mu) \neq 0$ and then, from (7.2.8) we see that if we set

$$f = \frac{1}{\mu(2\kappa - \frac{2}{3}\mu)}\left(\frac{3}{2}\dot{l} + \frac{\mu}{2}m\right),$$

then f is a a nontrivial solution of the system (7.2.14) unless $\tilde{\omega}$ is conformally Killing with respect to the metric $g = dt^2 + g_Y$ and hence $Z = f\mathcal{K}_{g_Y}(d\phi)$ is a nontrivial solution of (7.2.1). The case $\kappa = 0$ is similar.

If now $\tilde{\omega}$ is of type (b), we can write $\tilde{\omega} = m\eta$ where $\eta \in \Lambda^1(T^*Y)$ is a co-closed eigenform of Δ_H with eigenvalue ν. Let us also consider an element of type II written as $Z = f\mathcal{K}_{g_Y}(\eta)$ where f is a function. Assuming that η is not conformally Killing, we have

$$\mathcal{D}^* Z = \left\{ 0, \frac{1}{2}(4\kappa - \nu)\left(\dot{f} \pm \sqrt{\nu} f\right)\eta, \frac{1}{2}\left(-\ddot{f} \mp \sqrt{\nu}\dot{f}\right)\mathcal{L}_{g_Y}(\eta) \right\},$$

where the sign of $\pm\sqrt{\nu}$ arises from Corollary 6.2.6. In order to solve $\mathcal{D}^* Z = \mathcal{K}_g(\tilde{\omega})$, we consider $\tilde{\omega} = m\eta$, where m solves (7.2.10) so (7.2.1) reduces to the system

$$\begin{aligned} \frac{1}{2}(4\kappa - \nu)\left(\dot{f} \pm \sqrt{\nu} f\right) &= \dot{m}, \\ -\frac{1}{2}\left(\ddot{f} \pm \sqrt{\nu}\dot{f}\right) &= m. \end{aligned} \tag{7.2.15}$$

and again since η is not conformally Killing with respect to g_Y it follows that $4\kappa - \nu$ is non-zero and if we find f satisfying

$$\dot{f} \pm \sqrt{\nu} f = \frac{2\dot{m}}{4\kappa - \nu},$$

then f is a solution of (7.2.15) and $f\mathcal{K}_{g_Y}(\eta)$ is a solution of (7.2.1) with $\tilde{\omega} = m\eta$. We can choose f to be

$$f(t) = e^{\pm\sqrt{\nu}t} f_0 + \frac{2e^{\pm\sqrt{\nu}t}}{4\kappa - \nu}\int_0^t \dot{m}(s) e^{\mp\sqrt{\nu}s} ds,$$

where f_0 is a constant. It is clear that we can choose the constant f_0 so that f is a solution of (7.2.10). The case $\kappa = 0$ is similar. □

7.3 Completion of proofs

We first state the following which determines all indicial roots of F^* in the spherical case:

Theorem 7.3.1. *Let M be $\mathbb{R} \times S^3/\Gamma$ with product metric $g = dt^2 + g_{S^3/\Gamma}$, where $g_{S^3/\Gamma}$ is a metric of constant curvature 1. Let $\mathcal{I}^*$ denote the set of indicial roots of F^*.*

- *Case (0): $0 \in \mathcal{I}^*$.*
- *Case (1): If $\Gamma = \{e\}$ then $j = \pm 1 \in \mathcal{I}^*$. If Γ is non-trivial, then $j = \pm 1 \notin \mathcal{I}^*$.*

All solutions in Case (0) and Case (1) are of the form $(0, \omega)$, where ω is dual to a conformal Killing field (that is, $\mathcal{K}_g \omega = 0$).

- *Case (2): If B is a nontrivial eigentensor of $\Delta_{S^3/\Gamma}$ on divergence-free symmetric 2-tensors, with eigenvalue $j^2 + 2j - 2$ with $j \geq 2$, then $\{\pm j, \pm(j+2)\} \in \mathcal{I}^*$.*
- *Case (3): If ω is an eigenform of $\Delta_{S^3/\Gamma}$ on divergence-free 1-forms with eigenvalue $(j+1)^2$, with $j \geq 2$, then $\pm(j+1) \in \mathcal{I}^*$.*

All solutions in Case (2) and Case (3) are of the form $(Z, 0)$.

- *Case (4) If u is an eigenfunction of $\Delta_{S^3/\Gamma}$ with eigenvalue $j(j+2), j \geq 2$ then*

$$\pm\alpha_j^{\pm} = \pm\sqrt{j(j+2) - 2 \pm \frac{2}{9}\sqrt{-1} \cdot \sqrt{(j+3)(j-1)}} \in \mathcal{I}^* \tag{7.3.1}$$

- *Case (5) If ω is an eigenform of $\Delta_{S^3/\Gamma}$ on divergence-free 1-forms with eigenvalue $(j+1)^2$, with $j \geq 2$, then*

$$\pm\delta_j = \pm\sqrt{(j+1)^2 - 4} \in \mathcal{I}^*. \tag{7.3.2}$$

All solutions in Case (4) and Case (5) are of the form (Z, ω) with both Z and ω nontrivial and $\mathcal{K}_g \omega \neq 0$.

Remark 7.3.2. If $\Gamma = \{e\}$, then all of the above indicial roots do in fact occur. For nontrivial Γ, exactly which roots occur depends on which eigentensors descend from S^3 to the quotient S^3/Γ.

Proof of Theorem 7.3.1. This follows from combining Propositions 7.1.9 and 7.2.2 for the case $\kappa = 1$. □

Proof of Theorem 1.3.4. This follows immediately from Theorem 7.3.1, since Cases (2) and (3) obviously have real part larger than 2, and it is easy to see that $|Re(\alpha_j^{\pm})| > \sqrt{6}$ and $|Re(\delta_j)| \geq \sqrt{5}$ for all $j \geq 2$. The determination of the conformal Killing fields follows easily from Section 7.2. □

Next we state the following Theorem, which immediately implies Theorem 1.3.6.

Theorem 7.3.3. *Let M be $\mathbb{R} \times S^3/\Gamma$ with product metric $dt^2 + g_{S^3/\Gamma}$, where $g_{S^3/\Gamma}$ is a metric of constant curvature 1. Let $\mathcal{I}$ denote the set of indicial roots of F.*

- *Case (0): $0 \in \mathcal{I}$. The corresponding kernel is*

$$span\{3dt \otimes dt - g_{S^3}, dt \odot \omega_0\} \tag{7.3.3}$$

  where ω_0 is dual to a Killing field on S^3/Γ.

- *Case (1): If $\Gamma = \{e\}$ then $j = \pm 1 \in \mathcal{I}$. If Γ is non-trivial, then $j = \pm 1 \notin \mathcal{I}$. The corresponding kernel elements are given by*

$$h_\phi = p(t)\phi(3dt \otimes dt - g_{S^3}) + q(t)(dt \odot d\phi), \tag{7.3.4}$$

  where $p(t) = C_3e^t - C_4e^{-t}$ and $q(t) = C_3e^t + C_4e^{-t}$, for some constants C_3 and C_4, and ϕ is a lowest nonconstant eigenfunction of $\Delta_{S^3/\Gamma}$. In particular, if Γ is nontrivial, then $j = \pm 1$ are not indicial roots.

All solutions in Case (0) and Case (1) are in the image of the conformal Killing operator. More precisely,

$$3dt \otimes dt - g_{S^3} = \mathcal{K}_g(2tdt) \ \textit{and} \ dt \odot \omega_0 = \mathcal{K}_g(t\omega_0), \tag{7.3.5}$$

and

$$h_\phi = \mathcal{K}_g\Big\{\frac{1}{2}\left(C_3(t+3)e^t - C_4(t-3)e^{-t}\right)\phi dt + \frac{1}{2}\left(-C_3te^t - C_4te^{-t}\right)d\phi\Big\}. \tag{7.3.6}$$

- *Case (2): If B is a nontrivial eigentensor of $\Delta_{S^3/\Gamma}$ on divergence-free symmetric 2-tensors, with eigenvalue j^2+2j-2 with $j \geq 2$, then $\{\pm j, \pm(j+2)\} \in \mathcal{I}$.*

- *Case (3): If ω is an eigenform of $\Delta_{S^3/\Gamma}$ on divergence-free 1-forms with eigenvalue $(j+1)^2$, with $j \geq 2$, then $\pm(j+1) \in \mathcal{I}$.*

The kernel elements in Case (2) are of the form $h = f(t)B$, and in Case (3) are of the form $h = f_0(t)\cdot\omega \odot dt + f_1(t)\cdot\mathcal{K}_{S^3}(\omega)$. Neither of these are in the image of the conformal Killing operator $\mathcal{K}_g$ of the cylinder.

- *Case (4) If u is an eigenfunction of $\Delta_{S^3/\Gamma}$ with eigenvalue $j(j+2), j \geq 2$ then $\pm\alpha_j^\pm \in \mathcal{I}$, where $\alpha_j^\pm$ were defined in (7.3.1).*

- *Case (5) If ω is an eigenform of $\Delta_{S^3/\Gamma}$ on divergence-free 1-forms with eigenvalue $(j+1)^2$, with $j \geq 2$, then $\pm\delta_j \in \mathcal{I}$, where δ_j were defined in (7.3.2).*

All solutions in Case (4) and Case (5) are in the image of the conformal Killing operator of the cylinder. More precisely, they are exactly those solutions of $\square_{\mathcal{K},g}\omega = 0$ which are not conformally Killing.

Remark 7.3.4. As before, if $\Gamma = \{e\}$, then all of the above indicial roots do in fact occur. For nontrivial Γ, exactly which roots occur depends on which eigentensors descend to the quotient.

Proof of Theorem 7.3.3. From the index theorem of Lockhart-McOwen, it follows that the real parts of indicial roots of F are the same as those of F^* and the dimensions of the space of solutions of the form $e^{\lambda t}p(y,t)$ where p is a polynomial in t with coefficients in $C^\infty(Y)$ are the same for all indicial roots with the same real part. We consider Cases (0) – (5) in order.

For Case (0), the corresponding kernel of F^* is of the form $(0, \omega)$, where ω is dual to a bounded conformal Killing field on the cylinder. By direct calculation, elements in (7.3.3) form the corresponding space of kernel elements.

For Case (1), the corresponding kernel of F^* is of the form $(0, \omega)$, where ω is dual to a conformal Killing field which grows like e^t on one end. For S^3, this is an 8-dimensional space, while if Γ is nontrivial, this space is empty. Again, by direct calculation, elements in (7.3.4) form the corresponding 8-dimensional space of kernel elements in the case of the sphere. The formulas in (7.3.5) and (7.3.6) can also easily be verified by direct calculation, which we omit.

For Case (2), we consider solutions of the form

$$f(t) \cdot B,$$

where f is a function, and $B \in S^2_0(T^*Y)$ is an eigentensor of Δ_{g_Y} with eigenvalue $-\lambda$ satisfying $\mathrm{tr}_Y(B) = 0$ and $\delta_Y B = 0$. In this case the equation $F = 0$ takes the form

$$-\frac{1}{2}\ddot{f} \cdot B - f \cdot B - \frac{\lambda}{2} f \cdot B + \dot{f} \cdot \not{d} B = 0.$$

Case (2) follows as in the proof of Proposition 7.1.5, and the index theorem.

For Case (3), we consider solutions of the form

$$\tilde{h} = f_0(t) \cdot \omega \odot dt + f_1(t) \cdot \mathcal{K}_{g_Y}(\omega),$$

which are *not* in the image of $\mathcal{K}_g$, where $\omega \in \Lambda^1(T^*Y)$ is an eigenform of Δ_H with eigenvalue $\nu > 0$ satisfying $\delta_Y \omega = 0$. Case (3) then follows as in Proposition 7.1.9, and the index theorem.

For Cases (4) and (5), we consider $\tilde{h}$ of the form

$$\tilde{h} = \mathcal{K}_g(\tilde{\omega}),$$

where $\tilde{\omega} \in \Lambda^1(T^*M)$. The equation $\delta \tilde{h} = 0$ says that ω is a solution of $\Box_{\mathcal{K},g}\tilde{\omega} = 0$, and the solutions of this equations were completely classified in Section 7.2 into those of types (a) and (b). Cases (4) and (5) then follow from Proposition 7.2.2 and the index theorem. □

Proof of Corollaries 1.3.8 and 1.3.9. Corollary 1.3.8 follows immediately from Theorem 1.3.4. Corollary 1.3.9 then follows using a standard argument that solutions of elliptic equations in weighted spaces admit asymptotic expansions with leading terms solutions on the cylinder corresponding to indicial roots [LM85]. □

Proof of Theorem 1.3.11. Applying the divergence operator to the equation $\mathcal{D}^* Z = \mathcal{K}_g \omega$, we see that ω satisfies $\Box_{\mathcal{K}} \omega = 0$. An integration by parts shows that $\mathcal{K}_g \omega = 0$, which implies that $\omega = 0$ since there are no nontrivial decaying conformal Killing fields. We next convert (M, g) into a manifold with a cylindrical end, using the conformal factor u^{-2} which is smooth and positive and equal to r^{-2} outside of some compact set, and let $\hat{g} = r^{-2} g$. From conformal invariance of $\mathcal{D}^*$, we have that $\mathcal{D}^*_{\hat{g}} Z = 0$. Using Corollary

1.3.9, we conclude that $|Z|_{\hat{g}} = O(e^{-2t})$, where $t = \log(u)$ as $t \to \infty$. This implies that $|Z|_g = O(r^{-4})$ as $r \to \infty$.

Next, if h solves $\mathcal{D}h = 0$ and $\delta h = 0$, then $B'(h) = \mathcal{D}^*\mathcal{D}h = 0$, where B' is is the linearized Bach tensor [Ito95]. Since B' is asymptotic to Δ^2 as $r \to \infty$, [AVis, Proposition 2.2], implies that there is no $O(r^{-1})$ term in the asymptotic expansion of h and therefore $h = O(r^{-2})$ as $r \to \infty$. □

Proof of Theorem 1.3.12. The cokernel statements follow from combining Propositions 7.1.9 and 7.2.2 for the case $\kappa = -1$. The kernel statements follow from an analysis similar to the one outlined in the proof of Theorem 7.3.3, using the index theorem. For the indicial roots $\{0, \pm i\}$, the corresponding kernel of F^* is of dimension $1 + b_1(Y) + 2\dim(H^1_C(Y))$. From Theorem 6.1.3, we see that $3dt \otimes dt - g_Y$ is in the kernel of F. For a harmonic 1-form ω, from Theorem 6.1.3 we also see that $\omega \odot dt$ is also in the kernel of F. For a traceless Codazzi tensor B on Y^3, from Theorem 6.1.3 it follows that

$$(c_1 \cos(t) + c_2 \sin(t))B, \tag{7.3.7}$$

is in the kernel of F for any constants c_1 and c_2. By counting dimensions and using the index theorem, this accounts for all kernel elements of F corresponding to the indicial roots in $\{0, \pm i\}$.

□

Proof of Theorem 1.3.14. A compact hyperbolic 3-manifold (Y, g_Y) corresponds to a discrete cocompact subgroup $\Gamma \subset O_\circ(3,1)$ without torsion. The space of locally conformally flat deformations of Y is given by $H^1(\Gamma, \mathfrak{g})$, where $\mathfrak{g}$ is the lie algebra to $O(4,1)$ viewed as a Γ–module under the adjoint representation. If Y^3 is a hyperbolic rational homology 3-sphere, then by assumption $b^1(Y) = 0$, so Theorem 1.3.12 implies that $H^2_+(\mathbb{R} \times Y^3) = \{0\}$

if and only if $H^1_C(Y) = \{0\}$.

In [Kap94] it was shown that infinitely many (p, q)-surgeries on a hyperbolic 2-bridge knot satisfy $H^1(\Gamma, \mathfrak{g}) = \{0\}$ (a 2-bridge knot is any knot that my be embedded in $\mathbb{R}^3$ with only 2 local maxima, and the figure 8 knot is an example of a hyperbolic 2-bridge knot). These have $p \geq 2$ and are therefore rational homology 3-spheres, and all but finitely many are hyperbolic by Thurston's hyperbolic Dehn surgery Theorem (see, for example, [HK05] or [PP00]). By [Laf83, Lemma 6], there is an injection $H^1_C(Y) \hookrightarrow H^1(\Gamma, \mathfrak{g})$, so these examples are therefore an infinite family of hyperbolic rational homology 3-spheres satisfying $H^2_+(\mathbb{R} \times Y^3) = \{0\}$.

Next, it was shown by DeBlois that there are infinitely many hyperbolic rational homology 3-spheres containing closed embedded totally geodesic surfaces [DeB06] (these examples are n-fold cyclic branched covers of S^3 branched along a certain 2-component link). By [Laf83, Theorem 2], such a surface yields a non-trivial traceless Codazzi tensor field on Y. Thus by Theorem 1.3.12, the examples of DeBlois are an infinite family of examples of hyperbolic rational homology 3-spheres satisfying $H^2_+(\mathbb{R} \times Y^3) \neq \{0\}$.

□

Proof of Corollary 1.3.15. This follows from Theorem 1.3.12, again using a standard argument that solutions of elliptic equations in weighted spaces admit asymptotic expansions with leading terms solutions on the cylinder corresponding to indicial roots [LM85]. □

Proof of Theorem 1.3.16. The cokernel statements follow from combining Propositions 7.1.9 and 7.2.2 for the case $\kappa = 0$. The kernel statements follow from an analysis similar to the one outlined in the proof of Theorem 7.3.3, using the index theorem. □

7.4 The gluing problem

We will next describe the setup to the gluing theorem of Kovalev-Singer. A brief statement of the theorem is as follows.

Theorem 7.4.1 (Floer, Kovalev-Singer, Lebrun-Singer, Donaldson-Friedman [Flo91, KS01, LS94, DF89])**.** *Let $(X_1, [g_1])$ and $(X_2, [g_2])$ be self-dual conformal structures on compact 4-manifolds X_i satisfying $H^2_c(X_i, [g_i]) = 0$ for $i = 1, 2$. Then the connect sum $X_1 \# X_2$ admits self-dual conformal structures.*

Donaldson-Friedman proved this using twistor theory, using methods from the deformation theory of singular complex 3-folds. The proofs of Floer, Kovalev-Singer and Lebrun-Singer are analytic, and thus generalize more easily to the setting of orbifolds. Consequently, the gluing can be performed at isolated orbifold points $p_i \in X_i, i = 1, 2$, provided they are compatible. This means that there is an orientation-reversing intertwining map between the actions of the respective orbifold groups $\Gamma_i \subset SO(4)$ at the gluing points.

We will next outline the idea of the analytic proof. Let $r_i(x) = d(p_i, x)$ in sufficiently small neighborhoods of p_i, and extend to smooth positive functions on each X_i. Consider the conformal cylindrification of X_i, which is $\tilde{X}_i = X_i \setminus \{p_i\}$ with metric $\tilde{g}_i = r_i^{-2} g_i$, and let $t_i = -\log r_i$. These metrics are then “glued” together with a cylindrical region in between using cutoff functions, we refer the reader to [KS01, Section 2.3] for the exact formulas. We only need to remark here that the main argument of [KS01] is to reduce the gluing problem to the study of the deformation complex on the component cylindrified spaces. A weight $\delta > 0$ and a weight function are chosen so that in the limit, the weight function is $e^{\delta t}$ in the middle cylindrical region, and $e^{\delta t_1}$ on $\tilde{X}_1$ and $e^{-\delta t_2}$ on

$\tilde{X}_2$. One next considers the operators

$$F_1 : e^{\delta t_1} C^{k,\alpha}(A_1) \xrightarrow{\mathcal{D} \oplus 2\delta_{\tilde{g}_1}} e^{\delta t_1} C^{k-2,\alpha}(B_1) \oplus e^{\delta t_1} C^{k-1,\alpha}(C_1), \tag{7.4.1}$$

$$F_0 : e^{\delta t} C^{k,\alpha}(A_0) \xrightarrow{\mathcal{D} \oplus 2\delta_{\tilde{g}_0}} e^{\delta t} C^{k-2,\alpha}(B_0) \oplus e^{\delta t} C^{k-1,\alpha}(C_0), \tag{7.4.2}$$

$$F_2 : e^{-\delta t_2} C^{k,\alpha}(A_2) \xrightarrow{\mathcal{D} \oplus 2\delta_{\tilde{g}_2}} e^{-\delta t_2} C^{k-2,\alpha}(B_2) \oplus e^{-\delta t_2} C^{k-1,\alpha}(C_2), \tag{7.4.3}$$

where $A_i = T^*\tilde{X}_i, B_i = S^2_0(T^*\tilde{X}_i)$, and $C_i = S^2_0(\Lambda^2_-)(T^*\tilde{X}_i)$. The adjoints of these operators are maps

$$F_1^* : e^{-\delta t_1} C^{k,\alpha}(B_1) \oplus e^{-\delta t_1} C^{k-1,\alpha}(C_1) \to e^{-\delta t_1} C^{k-2,\alpha}(A_1) \tag{7.4.4}$$

$$F_0^* : e^{-\delta t} C^{k,\alpha}(B_0) \oplus e^{-\delta t} C^{k-1,\alpha}(C_0) \to e^{-\delta t} C^{k-2,\alpha}(A_0) \tag{7.4.5}$$

$$F_2^* : e^{\delta t_2} C^{k,\alpha}(B_2) \oplus e^{\delta t_2} C^{k-1,\alpha}(C_2) \to e^{\delta t_2} C^{k-2,\alpha}(A_2), \tag{7.4.6}$$

given by

$$F_i^*(Z, \omega) = \mathcal{D}^* Z - \mathcal{K}_{\tilde{g}_i} \omega. \tag{7.4.7}$$

Note the duals of the Hölder spaces are not Hölder spaces, but we are only interested in the kernel and cokernel, which will be smooth by elliptic regularity, so this slight abuse of notation does not matter.

On the middle cylindrical region, Corollary 1.3.8 shows that F_0 is an isomorphism for $0 < \delta < 2$. On $\tilde{X}_1$, we consider solutions of $F_1^*(Z, \omega) = 0$ with both $Z = O(e^{-\delta t_1})$ and $\omega = O(e^{-\delta t_1})$ as $t_1 \to \infty$. Corollary 1.3.9 implies that ω is a conformal Killing field, and $Z = O(e^{-2t_1})$ as $t_1 \to \infty$. The conformal transformation formula $\mathcal{D}^*_{\hat{g}} Z = r_1^2 \mathcal{D}^*_g Z$ shows that Z is a solution of $\mathcal{D}^*_g Z = 0$ on $X_2 \backslash \{p_2\}$ satisfying $Z = O(1)$ as $r_1 \to 0$. The operator $\mathcal{D}\mathcal{D}^*$ is an elliptic operator with leading term Δ^2 ([AVis, Str10b]). Consequently, the singularity is removable, so Z extends to a smooth solution of $\mathcal{D}^* Z = 0$ on X_1, and Z then vanishes by the assumption that $H^2_c(X_1, [g_1]) = 0$.

On $\tilde{X}_2$, we consider solutions of $F_2^*(Z,\omega) = 0$ with both $Z = O(e^{\delta t_2})$ and $\omega = O(e^{\delta t_2})$ as $t_2 \to \infty$. Vanishing of Z again follows from Corollary 1.3.9 and the assumption that $H_c^2(X_2, [g_2]) = 0$.

Remark 7.4.2. The argument given on [KS01, page 1259-1260] to handle the case of $\tilde{X}_2$ is incorrect, because there was a mistake in the order of growth given there. Namely, the growth rate given on the bottom on page 1258 for $H^{2,\pm}$ should be $|\Psi|_0 = O(r^{-2\pm\delta})$, and not $|\Psi|_0 = O(r^{-2\mp\delta})$ as written there and then applied incorrectly in the subsequent argument. Indeed, on X_2 the weight function is $e^{-\delta t}$, while the argument given there to remove the singularity (quoting Biquard's Theorem from [Biq91]) requires $\delta > 0$. The above argument fixes this gap.

The remainder of the proof then proceeds as in [KS01].

Remark 7.4.3. We note that there can be asymptotic cokernel arising from conformal Killing fields on the factors. Namely, on $\tilde{X}_1$ there are conformal Killing fields in the cokernel satisfying $\omega = O(e^{-\delta t_1})$ as $t_1 \to \infty$. The conformal transformation formula $\mathcal{K}_{\hat{g}}(\omega) = r_1^{-2}\mathcal{K}_g(r_1^2\omega)$ shows that $r_1^2\omega$ is a conformal Killing field on X_1 satisfying $|r_1^2\omega|_{g_1} = O(r_1^{1+\delta})$ as $r_1 \to 0$. Thus the asymptotic cokernel contains conformal Killing fields on X_1 which vanish at p_1 and whose first derivatives vanish at p. Similarly, on X_2 there are conformal Killing fields in the cokernel satisfying $\omega = O(e^{\delta t_2})$ as $t_2 \to \infty$. Then $r_2^2\omega$ is a conformal Killing field on X_2 satisfying $|r^2\omega|_{g_2} = O(r_2^{1-\delta})$ as $r_2 \to 0$. Thus the asymptotic cokernel also contains conformal Killing fields on X_2 which vanish at p_2. However, the existence of this cokernel does not affect finding a self-dual metric, since we only need to find a zero of the first component of F (and do not necessarily need to find a zero of the divergence map).

7.5 Appendix

7.5.1 The square of $\not{d}$

In this appendix, we give the proof of Proposition 6.2.1.

Proof of Proposition 6.2.1. In a local orthonormal basis we have

$$\not{d}^2 h_{ij} = \sum_{a,b} \epsilon_{iab} \nabla_a (\not{d} h)_{bj} + \sum_{c,d} \epsilon_{jcd} \nabla_c (\not{d} h)_{di}.$$

Expanding the right hand side, we obtain

$$\begin{aligned}
\not{d}^2 h_{ij} &= \sum_{a,b} \sum_{k,l} \epsilon_{iab} \epsilon_{bkl} \nabla_a \nabla_k h_{lj} + \sum_{a,b} \sum_{m,n} \epsilon_{iab} \epsilon_{jmn} \nabla_a \nabla_m h_{nb} \\
&+ \sum_{c,d} \sum_{p,q} \epsilon_{jcd} \epsilon_{dpq} \nabla_c \nabla_p h_{qi} + \sum_{c,d} \sum_{u,v} \epsilon_{jcd} \epsilon_{iuv} \nabla_c \nabla_u h_{vd} \\
&= I + II + III + IV.
\end{aligned}$$

Note that $I+III$ is twice the symmetrization of I and $II+IV$ is twice the symmetrization of II, so it will suffice to compute I and II. A straightforward computation shows that if we let a, b be indices such that $\{i, a, b\} = \{1, 2, 3\}$ then

$$I = \nabla_a \nabla_i h_{aj} - \nabla_a \nabla_a h_{ij} + \nabla_b \nabla_i h_{bj} - \nabla_b \nabla_b h_{ij}.$$

Commuting covariant derivatives we have

$$\begin{aligned}
\nabla_a \nabla_i h_{aj} &= \nabla_i \nabla_a h_{aj} - R^p_{aia} h_{pj} - R^p_{aij} h_{ap} \\
&= \nabla_i \nabla_a h_{aj} - \kappa \left(\delta^p_a (g_Y)_{ia} - \delta^p_i (g_Y)_{aa} \right) h_{pj} \\
&- \kappa \left(\delta^p_a (g_Y)_{ij} - \delta^p_i (g_Y)_{aj} \right) h_{ap} \\
&= \nabla_i \nabla_a h_{aj} + \kappa \big(h_{ij} - h_{aa} (g_Y)_{ij} - (g_Y)_{aj} h_{ai} \big).
\end{aligned}$$

Similarly,

$$\nabla_b \nabla_i h_{bj} = \nabla_i \nabla_b h_{bj} + \kappa\big(h_{ij} - h_{bb}(g_Y)_{ij} - (g_Y)_{bj} h_{bi}\big).$$

It follows that

$$\begin{aligned} \nabla_a \nabla_i h_{aj} + \nabla_b \nabla_i h_{bj} &= \nabla_i \left(\delta_Y\right)_j - \nabla_i \nabla_i h_{ij} \\ &\quad + 2\kappa h_{ij} + \kappa\left(h_{aa} + h_{bb} + h_{ii}\right)(g_Y)_{ij} \\ &\quad - \kappa\left(h_{ii}(g_Y)_{ij} + h_{ai}(g_Y)_{aj} + h_{bi}(g_Y)_{bj}\right) \\ &= \nabla_i \left(\delta_Y\right)_j - \nabla_i \nabla_i h_{ij} + 3\kappa \mathrm{tf}(h), \end{aligned}$$

and clearly

$$I = \nabla_i \left(\delta_Y h\right)_j - \Delta h_{ij} + 3\kappa \mathrm{tf}(h).$$

We conclude that

$$I + III = \mathcal{L}_{g_Y}(\delta_Y h) - 2\Delta h + 6\kappa \mathrm{tf}(h).$$

For II we consider two cases. If $i \neq j$ we let l be a an index such that $\{i, j, l\}$ such that $\{i, j, l\} = \{1, 2, 3\}$, then it is easy to see that II equals

$$II = -\nabla_j \nabla_i h_{ll} + \nabla_j \nabla_l h_{il} + \nabla_l \nabla_i h_{lj} - \nabla_l \nabla_l h_{ij} \tag{7.5.1}$$

$$= A_1 + A_2 + A_3 + A_4. \tag{7.5.2}$$

For the terms in (7.5.1) we have

$$\begin{aligned} A_1 &= -\nabla_j \nabla_i h_{ll} = -\nabla_j \nabla_i \mathrm{tr}_Y(h) + \nabla_j \nabla_i h_{ii} + \nabla_j \nabla_i h_{jj}, \\ A_2 &= \nabla_j \nabla_l h_{il} = \nabla_j (\delta_Y h)_i - \nabla_j \nabla_i h_{ii} - \nabla_j \nabla_j h_{ij}, \end{aligned}$$

For A_3 we commute covariant derivatives

$$
\begin{aligned}
A_3 = \nabla_l \nabla_i h_{lj} &= \nabla_i \nabla_l h_{lj} - R^p_{lil} h_{pj} - R^p_{lij} h_{lp} \\
&= \nabla_i \nabla_l h_{lj} - \kappa \left(\delta^p_l g_{il} - \delta^p_i g_{ll} \right) h_{pj} - \kappa \left(\delta^p_l g_{ij} - \delta^p_i g_{lj} \right) h_{lp} \\
&= \nabla_i \nabla_l h_{lj} + \kappa h_{ij} \\
&= \nabla_i \left(\delta_Y h \right)_j - \nabla_i \nabla_i h_{ij} - \nabla_i \nabla_j h_{ij} + \kappa h_{ij},
\end{aligned}
$$

and finally

$$
A_4 = -\nabla_l \nabla_l h_{ij} = -\Delta_{g_Y} h_{ij} + \nabla_i \nabla_i h_{ij} + \nabla_j \nabla_j h_{ij}.
$$

We then have

$$
A_1 + A_2 = -\nabla_j \nabla_i \mathrm{tr}_Y(h) + \nabla_j (\delta_Y h)_i + \nabla_j \nabla_i h_{jj} - \nabla_j \nabla_j h_{ij},
$$

and

$$
\begin{aligned}
A_3 + A_4 = \nabla_i \left(\delta_Y h \right)_j &- \nabla_i \nabla_i h_{ij} - \nabla_i \nabla_j h_{ij} + \kappa h_{ij} \\
&- \Delta_{g_Y} h_{ij} + \nabla_i \nabla_i h_{ij} + \nabla_j \nabla_j h_{ij} \\
&= \nabla_i \left(\delta_Y h \right)_j + \kappa h_{ij} - \Delta_{g_Y} h_{ij} \\
&- \nabla_i \nabla_j h_{jj} + \nabla_j \nabla_j h_{ij},
\end{aligned}
$$

so then

$$
\begin{aligned}
II &= A_1 + A_2 + A_3 + A_4 \\
&= -\nabla_j \nabla_i \mathrm{tr}_Y(h) - \Delta_{g_Y} h_{ij} \\
&+ \mathcal{L}_{g_Y}(\delta_Y h) + \kappa h_{ij} + \nabla_j \nabla_i h_{jj} - \nabla_i \nabla_j h_{jj}.
\end{aligned}
$$

Commuting covariant derivatives we have

$$\begin{aligned}\nabla_j \nabla_i h_{jj} - \nabla_i \nabla_j h_{jj} &= -R^p_{jij} h_{pj} - R^p_{jij} h_{jp} = -2R^p_{jij} h_{pj} \\ &= -2\kappa \left(\delta^p_j g_{ij} - \delta^p_i g_{jj}\right) h_{jp} \\ &= 2\kappa h_{ij},\end{aligned}$$

so we have shown

$$II = -\Delta h_{ij} - \nabla_i \nabla_j \mathrm{tr}_Y(h) + \mathcal{L}_{g_Y}(\delta_Y h) + 3\kappa h_{ii},$$

so then $II + IV$ is given by

$$II + IV = -2\Delta_{g_Y} h - 2\nabla_i \nabla_j \mathrm{tr}_Y(h) + 2\mathcal{L}_{g_Y}(\delta_Y h) + 6\kappa h_{ij}. \tag{7.5.3}$$

If now $i = j$ we let a, b be indices such that $\{i, a, b\} = \{1, 2, 3\}$ so that we have

$$II = \nabla_a \nabla_a h_{bb} - \nabla_a \nabla_b h_{ab} - \nabla_b \nabla_a h_{ab} + \nabla_b \nabla_b h_{aa},$$

which simplifies to

$$\begin{aligned}II = \Delta_{g_Y} \mathrm{tr}_Y(h) - \nabla_i \nabla_i \mathrm{tr}_Y(h) - (\delta_Y \delta_Y h) + \nabla_i \left(\delta h\right)_i - \Delta_{g_Y} h_{ii} \\ + \nabla_i \nabla_i h_{ii} + \nabla_a \nabla_i h_{ai} + \nabla_b \nabla_i h_{ib}.\end{aligned}$$

Commuting covariant derivatives we obtain

$$\begin{aligned}II = \Delta_{g_Y} \mathrm{tr}_Y(h) - \nabla_i \nabla_i \mathrm{tr}_Y(h) - (\delta_Y \delta_Y h) - \Delta_{g_Y} h_{ii} \\ + \nabla_i (\delta_Y h)_i - R^p_{aia} h_{pi} - R^p_{aii} h_{ap} - R^p_{bii} h_{pb} - R^p_{bib} h_{ip},\end{aligned}$$

and it is easy to see from this expression that

$$II = \Delta_{g_Y} \mathrm{tr}_Y(h) - \nabla_i \nabla_i \mathrm{tr}_Y(h) - (\delta_Y \delta_Y h) - \Delta_{g_Y} h_{ii} + 2\nabla_i \left(\delta_Y h\right)_i + 3\kappa \mathrm{tf}(h), \tag{7.5.4}$$

so then

$$II + IV = 2\Delta_{g_Y}\mathrm{tr}_Y(h) - 2\nabla_i\nabla_i\mathrm{tr}_Y(h) - 2\left(\delta_Y\delta_Y h\right)$$
$$- 2\Delta_{g_Y}h_{ii} + 4\nabla_i\left(\delta_Y h\right)_i + 6\kappa\mathrm{tf}(h).$$

Now, since we are using a local orthonormal basis to compute $\not{d}^2 h$, it is clear that in either case $i = j$ or $i \neq j$ the results in (7.5.3) and (7.5.4) are equivalent to

$$II + IV = -2\Delta_{g_Y}\mathrm{tf}(h) - 2\overset{\circ}{\nabla}{}^2 h + 2\mathcal{L}_{g_Y}(\delta_Y h) - 2\left(\delta_Y\delta_Y\right) g_Y$$
$$+ \frac{2}{3}\left(\Delta_{g_Y}\mathrm{tr}_Y(h)\right) g_Y + 6\kappa\mathrm{tf}(h).$$

Expressing $I + III$ as

$$I + III = -2\Delta_{g_Y}\mathrm{tf}(h) - \frac{2}{3}\Delta_{g_Y}\mathrm{tr}_Y(h)g_Y + \mathcal{L}_{g_Y}(\delta_Y h) + 6\kappa\mathrm{tf}(h),$$

we conclude that

$$\not{d}^2 h = -4\Delta_{g_Y}\mathrm{tf}(h) - 2\overset{\circ}{\nabla}{}^2\,\mathrm{tr}_Y(h) + 3\mathcal{K}_{g_Y}(\delta_Y h) + 12\kappa\mathrm{tf}(h),$$

as needed. □

The following corollary should be compared with [Flo91, Lemma 5.1]:

Corollary 7.5.1. *For any $h \in S^2(T^*Y)$ we have*

$$E_g'(h) = \frac{1}{8}\not{d}^2 h - \frac{1}{4}\overset{\circ}{\nabla}{}^2\,\mathrm{tr}_Y(h) + \frac{1}{8}\mathcal{K}_{g_Y}(\delta_Y h) - \frac{\kappa}{2}\mathrm{tf}(h). \tag{7.5.5}$$

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