Additions and scaling limits of invariant random matrix ensembles

by

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Abstract

Random matrices are known as important universal objects in modern probability theory, and have lots of connections with other areas in mathematics, statistics, physics and the other subjects. Among the large class of random matrices, the invariant matrix ensembles, whose distribution is symmetric under certain action, are of great importance and well-studied. In this thesis, I present two new problems related to the scaling limits and additions of invariant matrix ensembles. They deal with different objects, but both rely on the analysis of some explicit expressions given by the corresponding ensembles, which come from the symmetry they have.

The first direction is about the additions of rectangular matrices for general $\beta > 0$. In Xu 2023, we define the operation by introducing type BC Bessel functions, some symmetric function known in special function literature, in the probabilistic context. Then we prove two global limiting results of the empirical measure of such additions in low and high temperature regime, by heavily using the rich properties of type BC Bessel functions.

In the second direction, in Gorin and Xu 2024+ my co-author and I study the edge limit of uniformly random sorting network. It is one of the central objects in combinatorics and integrable probability. We prove local result that as $n \to \infty$, after proper rescling, the spacing on a fixed row k of the RSN converges in distribution to some limiting distribution. We further identify this limit as the corner process of the anti-symmetric GUE, and therefore estabilish a direct connection between sorting networks and invariant random matrices.

Declaration

I declare that this thesis has been composed solely by myself and that it has not been submitted, in whole or in part, in any previous application for a degree. Except where states otherwise by reference or acknowledgment, the work presented is entirely my own.

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Chapter 1

Introduction

In this thesis, we present a problem about the scaling limits of eigenvalues of certain β-ensembles in Chapter 2. In particular, it considers the additions of rectangular β-matrices respectively. Chapter 3 is about another problem, that studies the scaling limit of uniformly random sorting network. We identify its edge limit with an object originated from anti-symmetric Gaussian unitary ensemble, which is also a class of invariant square random matrix. Appendix A contains some technical details and backgrounds related to 2. The following sections in Chapter 1 give a brief overview of related backgrounds and the main results.

1.1 Invariant random matrices

Random matrices are important objects in modern probability theory. On one hand, the scaling limits of the eigenvalues of certain large random matrices appear universally in many other stochastic systems, and on the other hand, there are lots of connections between random matrices and other areas in mathematics, statistics, physics and the other subjects. Among the large class of random matrices, the invariant matrix ensembles, whose distribution is symmetric under certain action, are known to have explicit expressions of distribution, and therefore well-studied by the random matrix community.

According to the symmetry, there are more than one types of invariant matrix ensembles, and in this thesis we consider the following two types. The first type are $N \times N$ square real/complex/real quaternion matrices, invariant under conjugation of orthogonal/unitary/sympletic matrice, and there are two most classical examples. For simplicity consider real matrices. Let X be a $N \times N$ matrix, Y a be $N \times M$ $(N \leq M)$ matrix, whose entries are i.i.d $\mathscr{N}(0,1)$ random variables. Then $\frac{X+X^T}{2}$ is called *Gaussian Orthogonal ensemble*, YY^T is called *Laguerre Orthogonal ensemble*, and by definition one can see that they are invariant under conjugation of orthogonal group $O(N)$.

The second type are $M \times N$ real/complex/real quaternion rectangular matrices, invariant under the left and right multiplication of the corresponding Lie group elements according to their size and base (skew) field of their entries. One classical example is the matrix Y defined as above. A usual setting of great interest in random matrix theory is to study the asymptotic behavior of the (random) eigenvalues of certain matrix ensemble, when the size of the matrices (dimension) tends to infinity. While there are also interesting problems about the random eigenvectors, in this thesis we focus on the former. We refer the readers to Forrester 2010 for a detailed survey of classical results on invariant matrix ensembles.

1.2 β-ensembles

In literature, the β-ensemble is often referred to the probability density function on N-tuples of real numbers $x_1 \geq x_2 \cdots \geq x_N$

$$\frac{1}{Z_{n,\beta}} \prod_{1 \leq i < j \leq N} (x_i - x_j)^\beta \prod_{i=1}^{N} \exp\left[-V(x_i)\right], \tag{1.2.1}$$

where $Z_{n,\beta}$ is a normalizing constant. When $\beta = 1, 2, 4$, $V(x) = \frac{\beta}{2} x^2$, this gives the eigenvalue density function of Gaussian Orthogonal/Unitary/Sympletic ensemble, and when $\beta = 1, 2, 4$, $V(x) = \left[\frac{\beta}{2}(a+1) - 1\right] \log(x) - \frac{\beta}{2} x$ this gives Laguerre Orthogonal/Unitary/Sympletic ensemble. The case $\beta = 1, 2, 4$ correspond to real/complex/quaternionic matrices. For general $\beta > 0$ the density function still gives a legitimate probability measure of N particles on $\mathbb{R}$, that is called log-gas system in statistical mechanics, but this in general no longer correspond to concrete random matrices.

For GβE and LβE (i.e, when $\beta > 0$ and $V(x)$ is of one of the two forms above), it's worthy to mention that Dumitriu and Edelman 2002 give a concrete tridiagonal matrix realization of the density function. Such construction is later extended by Krishnapur, Rider, and Virag 2016 to a larger class of potentials $V(x)$. The tridiagonal structure is known as a main tool to study β-ensembles. However, in this text we generalize certain matrix ensembles from $\beta = 1, 2, 4$ to $\beta > 0$ in an alternate integrable approach, in terms of multivariate symmetric functions. It turns out that for Gaussian and Laguerre ensembles, such generalization fits well with the density function above.

1.3 Global limit of rectangular additions

Compared to additions of self-adjoint matrices, additions of rectangular matrices are relatively less studied. The classical result in this direction is given by Benaych-Georges 2009, which considers the additions of two independent $M \times N$ matrices, whose singular vectors are uniformly distributed. It proves a different global limiting result, that the empirical measure

$$\mu_M := \frac{1}{2M} \sum_{i=1}^{M} (\delta_{c_i} + \delta_{-c_i}) \tag{1.3.1}$$

of the addition converges weakly in probability to some limiting measure, characterized as the *rectangular free convolution* in free probability. Here c_i's are the singular values of the matrix $C = A + B$.

In Xu 2023, weconsidered the same type of addition as in Benaych-Georges 2009, more precisely, let $C = A + B$ of independent $M \times N$ rectangular matrices A and B with uniformly distributed singular vectors, i.e,

$$A \overset{d}{=} UAV, \ B \overset{d}{=} UBV, \tag{1.3.2}$$

where $U \in U(N), V \in U(M)$ are uniformly random (distributed according to the Haar measure) unitary matrices. It's worthy to note that the rectuangular matrices with i.i.d Gaussian entries satisfy this invariance, so the results of rectangular additions generalizes those of Laguerre ensembles.

Such addition can be thought as a binary operation of random vectors $(\vec{a}, \vec{b}) \mapsto \vec{c}$, where the input and output are singular values of the matrix A, B and C. we generalize the addition from $\beta = 1, 2, 4$ to $\beta > 0$, when concrete matrix structure no longer exists, using the theory of symmetric functions called *type BC Bessel function* in the literature.

After defining the β-addition, we study the behavior of singular values of C in low and high temperature regime, and obtained several limiting results. In the low temperature regime (N, M are fixed and $\beta \to \infty$) the random singular values crystallize at certain deterministic positions. In high temperature regime, we proved a Law of Large Numbers on the empirical measure of C in the flavor as described at the beginning of this section, which at the same time defines a new member in the family of convolutions that we called q-γ convolution. This extends the connection of random matrix addition and free probability done by Voiculescu 1991 (Hermitian, fixed temperature) , Benaych-Georges 2009 (rectangular, fixed temperature) and Benaych-Georges, Cuenca, and Gorin 2022 (Hermitian, high temperature) to rectangular, high temperature setting. See the following for the main result.

Theorem 1.3.1 (High temperature global limit, Xu 2023). *Take $\{\vec{a}_N\}_{N=1}^{\infty}$, $\{\vec{b}_N\}_{N=1}^{\infty}$ to be two independent sequences of random singular values of (virtual) $N \times M$ matrices, and let $\vec{c}_N = \vec{a}_N \boxplus_{N,M}^{\beta} \vec{b}_N$.*

Suppose for some deterministic $\{m_{2k}(A)\}_{k\geq 1}, \{m_{2k}(B)\}_{k\geq 1}$, as $N \to \infty$,

$$\frac{1}{N}\sum_{i=1}^{N} a_{N,i}^{2k} \longrightarrow m_{2k}(A), \quad \frac{1}{N}\sum_{i=1}^{N} b_{N,i}^{2k} \longrightarrow m_{2k}(B),$$

then as $N, M \to \infty$, $\frac{M}{N} \to q \geq 1$, $\frac{N\beta}{2} \to \gamma > 0$,

$$\frac{1}{N}\sum_{i=1}^{N} c_{N,i}^{2k} \longrightarrow m_{2k}(C),$$

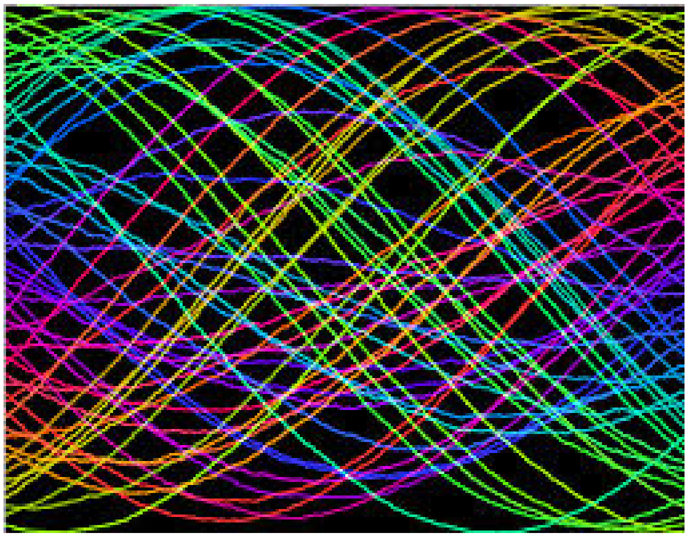

Figure 1.1: A simulation of uniformly random 2000-element sorting network (selected trajectories shown). Picture by A. Holroyd.

for some deterministic $\{m_{2k}(C)\}_{k\geq 1}$*. We write*

$$\{m_{2k}(C)\}_{k\geq 1} = \{m_{2k}(A)\}_{k\geq 1} \boxplus_{q,\gamma} \{m_{2k}(B)\}_{k\geq 1},$$

the q-γ *convolution. This is a new member in the family of convolutions.*

Moreover, we characterize this convolution in terms of its *cumulants*, and give explicit combinatorial moment-cumulant relation that allows calculating q-γ convolution by hands/computer simulation.

1.4 Edge limit of random sorting networks

Random sorting network (abbreviated as RSN) is a stochastic system with strong combinatorial flavor, introduced in O. Angel, A. Holroyd, et al. 2007. Consider the permutation group of n elements $\mathfrak{S}_n$. We use the one-row notation for the elements of $\mathfrak{S}_n$ representing them as sequences $(a_1\ a_2\ \dots\ a_n)$. We let τ_k be the adjacent swap $(k, k+1)$, $1 \leq k \leq n-1$, so that

$$(a_1\ a_2\ \dots\ a_n) \cdot \tau_k = (a_1\ a_2\ a_{k-1}\ a_{k+1}\ a_k\ a_{k+2}\ \dots\ a_n).$$

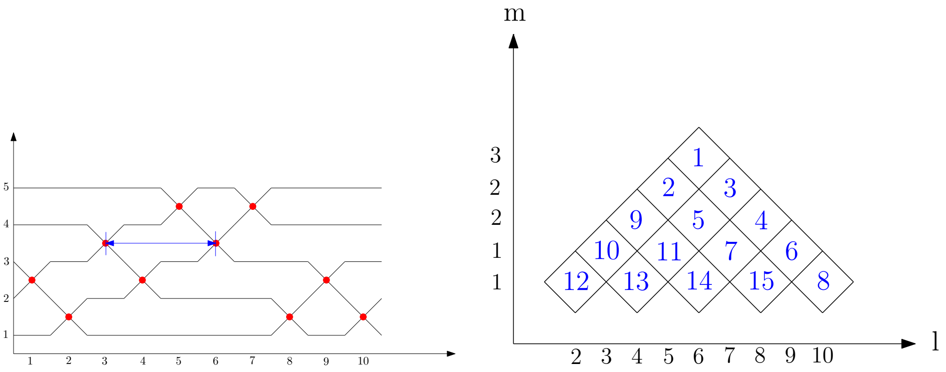

Figure 1.2: A sorting network $(s_1 \dots s_N)$, $N = \binom{n}{2}$ can be represented as a diagram of n wires, with wires at heights k and $k+1$ being swapped at time i whenever $s_i = k$. The left figure shows the wiring diagram of the sorting network $(2,1,3,2,4,3,4,1,2,1)$ with $n = 5$. The blue double arrow shows a spacing in row 3, which is a time interval between two adjacent swaps τ_3 and it has length 3 in our example. The right figure gives an example of a size 5 standard Young tableaux with a coordinate system: the numbers in each row and column increase without repetitions.

A sorting network of size n is a shortest sequence of swaps, whose product equals the reverse permutation $(n\ n-1\ \dots\ 2\ 1)$. One can show that each sorting network has $N = \binom{n}{2}$ swaps. The object was extensively studied in combinatorics, see e.g Stanley 1984. Sorting networks are "integrable", because of a classical combinatorial bijection between a sorting network of size n and a *standard Young tableaux* (abbreviated by SYT) of size n given by P. Edelman and C. Greene 1987. See the right side of Figure 1.2 for an example of SYT.

Denote the set of all such shortest sequences by Ω_n, a random sorting network of size n is a uniform chosen element in Ω_n. It is beneficial to think about the sorting network by its *wiring diagram*, see Figure 1.2 and the explanations there. Then the probabilistic results can somehow be thought as studying the "limit shape" of the wiring diagram, when $n \to \infty$, and therefore can be separated into two groups: global and local. The global side has already many established results, among which probably the most celebrated one is conjectured in O. Angel, A. Holroyd, et al. 2007 and proved in D. Dauvergne 2022, D. Dauvergne and B. Virag 2020, individual trajectories in the wiring diagram become (random) sine curves.

There are fewer results on the local side, and our work belongs to it. More precisely, in

Gorin and Xu 2024+, we study the asymptotic distribution of the *spacing* in row k, which is a distance between two occurrences of the same swap τ_k in the sorting network, cf. the left side of Figure 1.2. Compared to the previous work V. Gorin and M. Rahman 2019, which studies the spacing on row $k = \lfloor \alpha n \rfloor$ ($\alpha \in (0,1)$ is any fixed constant), so that k lies in the "bulk" of the wiring diagram, in our setting $k \in \mathbb{Z}_{\geq 1}$ is a fixed and independent of n, which gives the behavior of the wiring diagram on the edge.

Our first result considers the so-called *first swapping time* of a RSN on a certain row k. We prove that after proper rescaling, this converges in distribution to certain random variable, that is identified with the smallest positive eigenvalue of a $2k \times 2k$ anti-symmetric GUE, studied in P. J. Forrester and E. Nordenstam 2009. This gives one more direct bridge between sorting networks and random matrices. As the second contribution, we connect the two definitions of spacings with first swapping time and give their quantitative relations. This allows us to deduce the asymptotic distributions of spacing expressed in terms of the Fredholm determinants.

Chapter 2

Rectangular additions in low and high temperatures

2.1 Overview

Addition is one of the most natural operations on matrices. For deterministic matrices, there was a classical question posed by Weyl Weyl 1912 in 1912, which considers eigenvalues $c_1 \leq ... \leq c_N$ of $C = A + B$, where A, B are two arbitrary self-adjoint $N \times N$ matrices with fixed real eigenvalues $a_1 \leq ... \leq a_N$, $b_1 \leq ... \leq b_N$, and try to describe all the possible values of $c_1 \leq ... \leq c_N$. Solved by the end of XX^{th} centery due to combined efforts by Horn, Klyochko, Kuntson-Tao, and others, see e.g Horn 1962, Klyachko 1998, Knutson and Tao 2001. In random matrices, people usually assume the summands A and B are random, independent and share some certain symmetries, and the study of this type of questions have significant connections with free probability theory.

A well-known classical result connecting random matrix addition and free probability is stated by Volculescu in Voiculescu 1991, which considers the addition of two independent real/complex/real quaternionic self-adjoint matrices, and relates its asymptotic behavior as the size of the matrix grows with the notion of *free convolution.* There's also another classical result of similar flavor in rectangular setting. Take $\{A_M\}_{M=1}^{\infty}$, $\{B_M\}_{M=1}^{\infty}$ to be two independent sequences of $M \times N$ ($M \leq N$) matrices with real/complex/real quaternionic entries, that are uniformly chosen from the set of rectangular matrices with given singular values $a_{M,1} \geq ... \geq a_{M,M} \geq 0$ and $b_{M,1} \geq ... \geq b_{M,M} \geq 0$, and let $C_M = A_M + B_M$ with random singular values $c_{M,1} \geq ... \geq c_{M,M} \geq 0$.

Definition 2.1.1. *For a $M \times N$ ($M \leq N$) matrix A with singular values $a_1, ..., a_M \geq 0$, define its (symmetric) empirical measure to be*

$$\mu_A := \frac{1}{2M} \sum_{i=1}^{M} (\delta_{a_i} + \delta_{-a_i}).$$

Theorem 2.1.2. *Benaych-Georges 2009, Proposition 2.1 Define $\{A_M\}_{M=1}^{\infty}$, $\{B_M\}_{M=1}^{\infty}$ as above. Assume that $M, N \to \infty$ in a way that $N(M)/M \to q$ for some constant $q \geq 1$, and there exists deterministic probability measures μ_A, μ_B on $\mathbb{R}$, such that*

$$\lim_{M\to\infty} \mu_{A_M} = \mu_A, \quad \lim_{M\to\infty} \mu_{B_M} = \mu_B.$$

Then the random empirical measure of C, $\mu_{C_M} = \frac{1}{2M}\sum_{i=1}^{M}(\delta_{c_{M,i}} + \delta_{-c_{M,i}})$, converges weakly in probability to some deterministic probability measure μ_C on $\mathbb{R}$.

$\mu_C = \mu_A \boxplus_q \mu_B$ is called the rectangular free convolution *of μ_A and μ_B.*

The rectangular free convolution is a deterministic binary operation of measures on $\mathbb{R}$, that itself does not rely on random matrix structure, and it was well-studied in free probability theory from different aspects. In particular, for each measure μ with finite moments, there exists a collection of *rectangular free cumulants* $\{c_l^q\}_{l=1}^{\infty}$ (see Benaych-Georges 2009, Section 3.1) that are polynomials of moments with explicit expressions, and these quantities linearize free convolution, i.e, $c_l^q(\mu_A \boxplus \mu_B) = c_l^q(\mu_A) + c_l^q(\mu_B)$ for all l's. It turns out that the existence of such cumulants is a common feature of the various version of convolutions in free probability theory, and each convolution is characterized by its corresponding cumulants.

On the other side, there has been lots of papers studying additions of β-ensembles of random matrix theory that generalize the above theory in different parameter regimes. The parameter $\beta > 0$ is interpreted in physics as inverse temperature, and the cases $\beta = 1, 2, 4$ correspond to matrices with real/complex/real quaternionic entries. There are two classes of matrix ensembles, the $N \times N$ self-adjoint matrix and the $M \times N$ rectangular matrices, and the most classical examples are Gaussian ensembles and Laguerre ensembles, respectively. For the first class, Gorin and A. W. Marcus 2020 studies the limit behavior of eigenvalues of $C = A + B$ when N is fixed and $\beta \to \infty$, Mergny and Potters 2022 first considers the limiting behavior of additions in high temperature regime in physic literature, and shortly after Benaych-Georges, Cuenca, and Gorin 2022 proves a Law of Large Numbers similar to Theorem 2.1.2 when $N \to \infty, \theta \to 0, N\theta \to \gamma > 0$, and A. Marcus, Spielman, and Srivastava 2019,Arizmendi and Perales 2018,Gorin and A. W. Marcus 2020 extend the theory of convolution and cumulants to finite matrix additions for $\beta > 0$. However, extending Theorem 2.1.2 to general $\beta > 0$ remained open. The second class is relatively less understood. Benaych-Georges 2009,Benaych-Georges 2011 study the so-called rectangular free convolution for $\beta = 1, 2$, $N, M \to \infty$ and $N/M \to q \geq 1$, and Gribinski and A. Marcus 2022,Gribinski 2022 study the finite free convolution and cumulants for rectangular matrix additions for $\beta = 1, 2$.

The matrix ensemble considered in this text belongs to the second class, and we study the limiting behavior of singular values of $C = A + B$ in both low and high temperature regimes, more precisely, when M, N are fixed, $\theta \to \infty$, and when $M, N \to \infty, \theta \to 0, M\theta \to \gamma > 0, N\theta \to q\gamma$ for some $q \geq 1$. Note that even defining the operation $C = A + B$ for $\beta \neq 1, 2, 4$ is non-trivial, and this is one of our tasks.

Our approach is based on distributions of rectangular matrices in a version of characteristic function. The symmetry of self-adjoint/rectangular matrices with fixed eigenvalues/singular values are given by invariance under actions of classical Lie group $O(N)/U(N)/Sp(N)$, and when $\beta = 1, 2, 4$, the matrix characteristic functions have matrix integral representations with representation-theoretic background. Such functions have natural analytic continuation to all $\beta > 0$, and can be identified as eigenfunctions of certain differential operators. See the following papers Benaych-Georges, Cuenca, and Gorin 2022, Gorin and

A. W. Marcus 2020, V.Gorin and Shkolnikov 2018 for application of such idea in random matrices, and also Bufetov and Gorin 2018, Bufetov and Gorin 2019, Huang 2021 for the study of more general N-particle system using symmetric characteristic functions of similar flavor. While the above works deal with self-adjoint matrices, or more generally N-particle systems that are corresponding to root system of type A with a single root multiplicity $\theta = \beta/2$, rectangular matrices are corresponding to root system of type BC, that has two distinct root multiplicities parameterized by β in a more involved way. For more connections of type BC Lie theoretic object with probability, see e.g Kornyik, Voit, and J. Woerner 2021, Voit 2022, Voit and J. H. C. Woerner 2021.

In this text, the randomness of a M-tuple of nonnegative real numbers (which should be thought as singular values of some $M \times N$ random matrices) is characterized by a multivariate symmetric function, known as type BC Bessel function in special functions literature. It is also a special case of the symmetric Dunkl kernel, that generalizes the usual Fourier kernel to nontrivial root multiplicities, see Anker 2017 for a review. Motivated by the asymptotic behavior in high temperature regime, we apply and further develop a philosophy, that the limit of partial derivatives of logarithm of our characteristic function at 0 give a collection of cumulants, and the existence of such cumulants is equivalent to the existence of limiting moments, which implies that the empirical measure of the random M-tuples satisfy a Law of Large Numbers. These new q-γ cumulants are designed to linearize the rectangular addition in the regime $M\theta \to \gamma, N\theta \to q\gamma$, while the operation itself in this limit regime is called q-γ convolution. Similar to classical and free cumulants, they have nice combinatorial relation with moments. Finally, we point out that there's a surprising identification of q-γ theory with the rectangular finite free probability theory, which was developed in A. Marcus, Spielman, and Srivastava 2019,Gribinski and A. Marcus 2022,Gribinski 2022 while studying finite rectangular matrix additions.

2.1.1 Rectangular matrix addition

Throughout the text we always take $\beta = 2\theta > 0$, and $\beta = 1, 2, 4$ ($\theta = \frac{1}{2}, 1, 2$) correspond to the (skew) field real, complex and real quaterion (whose real dimension are given by β). For $M \leq N$, given two $M \times N$ independent random matrices A and B, we study the randomness of the sum

$$C = A + B.$$

Inspired by the classical theory of summing independent random variables $X + Y$, namely, that the charateristic function satiesfies

$$\phi_{X+Y}(t) = \phi_X(t) \cdot \phi_Y(t),$$

where $t \in \mathbb{R}$ is the parameter variable, we have

Proposition 2.1.3. *For $\theta = \frac{1}{2}, 1, 2$, let A and B be $M \times N$ rectangular independent random matrices, let Z be $N \times M$ an arbitrary deterministic matrix with real/complex/real quaternionic entries, and let $C = A + B$. We have*

$$\mathbb{E}\Big[\exp\Big(Re(Tr(CZ))\Big)\Big] = \mathbb{E}\Big[\exp\Big(Re(Tr(AZ))\Big)\Big] \cdot \mathbb{E}\Big[\exp\Big(Re(Tr(BZ))\Big)\Big]. \tag{2.1.1}$$

Proof. $Re(Tr(CZ)) = Re(Tr(AZ)) + Re(Tr(BZ))$, and since A, B are independent, the expectation of the exponential function factors. $\square$

Let us now rewrite (2.1.1) in terms of singular values of A, B and C, and for simplicity first take $\theta = 1$, i.e, we deal with complex matrices. In this text, we are considering the summands A and B that have distribution invariant under left and right unitary actions, i.e,

$$A \overset{d}{=} UAV, \ B \overset{d}{=} UBV, \tag{2.1.2}$$

where $U \in U(M), V \in U(N)$ are arbitrary unitary matrices. One example is the real/complex/real quaternionic $M \times N$ Wishart matrix, with i.i.d mean 0 Gaussian entries.

Note that if A, B satisfy (2.1.2), so is C. For simplicity, in the following discussions we usually stick to A. By singular value decomposition, it's useful to write A as $U\Lambda V$, where

$$\Lambda = \begin{bmatrix} a_1 & & & & 0 & ... & 0 \\ & a_2 & & & 0 & ... & 0 \\ & & ... & & & & \\ & & & ... & & & \\ & & & a_M & 0 & ... & 0 \end{bmatrix}_{M\times N}, \tag{2.1.3}$$

$\vec{a} = (a_1, ..., a_M) \in \mathbb{R}_{\geq 0}$, and $U \in U(M), V \in U(N)$ are random elements under Haar measures on the corresponding unitary groups. For now, assume that the singular values of A are deterministic. One can consider rectangular matrices A with real/real quaternionic entries and define invariant distribution in exactly the same way, while replacing the Haar distributed U, V by elements in orthogonal/unitary sympletic groups $O(M)/Sp(M)$, $O(N)/Sp(N)$. Similarly for B.

The eigenvectors of AA^*, A^*A, BB^*, B^*B are distributed uniformly, and so are eigenvectors of CC^*, C^*C. Hence the nontrivial randomness of C is about its singular values. Also, because of the singular value decomposition in (2.1.2), we can replace the parameter matrix Z by the form in (2.1.5), where $z_1, ..., z_M$ are its singular values. Therefore, we can rewrite the matrix Fourier transform of A in (2.1.1) as a function $\mathbb{B}(\vec{a}; z_1, ..., z_M; \theta, N)$, such that

$$\mathbb{B}(\vec{a}; z_1, z_2, ..., z_M; \theta, N) = \int dU \int dV \exp\left(\frac{1}{2}Tr(U\Lambda V Z + Z^* V^* \Lambda^* U^*)\right), \tag{2.1.4}$$

where Λ is defined in the same way as in (2.1.3), and

$$Z = \begin{bmatrix} z_1 & & & & \\ & z_2 & & & \\ & & ... & & \\ & & & ... & \\ & & & & z_M \\ 0 & & ... & & 0 \\ & & ... & & \\ 0 & & ... & & 0 \end{bmatrix}_{N\times M}, \tag{2.1.5}$$

θ takes the value $\frac{1}{2}, 1, 2$, $(z_1, ..., z_M) \in \mathbb{R}_M$, and $U \in O(M)/U(M)/Sp(M)$, $V \in O(N)/U(N)/Sp(N)$ are integrated under Haar measures respectively.

The function $\mathbb{B}(\vec{a}; z_1, ..., z_M; \theta, N)$ is known as multivariate type BC Bessel function in a general theoretical framework of special functions. We briefly review this framework in Appendix A, and give more information about the type BC Bessel function in Section 2.2, 2.3 based on the theory.

We note that because of the symmetry of Haar measure, $\mathbb{B}(\vec{a}; z_1, ..., z_M; \theta, N)$ is symmetric in both $(a_1, ..., a_M)$ and $(z_1, ..., z_M)$, and without loss of generality we can always take $a_1 \geq a_2 \geq ... \geq a_M$. Same for matrix B and C. Then Proposition 2.1.3 is rewritten as the following result.

Proposition 2.1.4. *For* $\theta = \frac{1}{2}, 1, 2$, *fix* $a_1 \geq ... \geq a_M \geq 0$, $b_1 \geq ... \geq b_M \geq 0$, *let* $A_{M\times N}$ *and* $B_{M\times N}$ *be real/complex/real quaternionic rectangular matrices with deterministic singular values* $\{a_i\}_{i=1}^M, \{b_i\}_{i=1}^M$ *and invariant distribution, as in (2.1.2),* $C = A + B$ *with singular values* $\vec{c} = (c_1 \geq ... \geq c_M \geq 0)$, *then*

$$\mathbb{E}\mathbb{B}(\vec{c}; z_1, ..., z_M; \theta, N) = \mathbb{B}(\vec{a}; z_1, ..., z_M; \theta, N) \cdot \mathbb{B}(\vec{b}; z_1, ..., z_M; \theta, N), \quad (z_1, ..., z_M) \in \mathbb{R}^M. \tag{2.1.6}$$

For general $\beta > 0$, there is no skew field of real dimension β and therefore no concrete β-

rectangular matrices. But motivated by Proposition 2.1.4, we first identify an invariant $M \times N$ matrix with uniform "singular vectors" and deterministic singular values $a_1, ..., a_M$, with the M-tuples $\vec{a}$. Moreover, it's known (see e.g Forrester 2010, Section 13.4.3, Ghaderipoor and Tellambura 2008) that multivariate Bessel functions admit a natural exterpolation from $\theta = \frac{1}{2}, 1, 2$ to arbitrary real $\theta > 0$. We still denote this function as $\mathbb{B}(\vec{a}; z_1, ..., z_M; \theta, N)$ for all $\theta > 0$. Hence, the $M \times N$ matrix addition (of independent summands) is extended to $\theta > 0$, by generalizing Proposition 2.1.4.

Definition 2.1.5. *Fix $\theta > 0$, $M \leq N$, $\vec{a} = (a_1 \geq ... \geq a_M \geq 0)$, $\vec{b} = (b_1 \geq ... \geq b_M \geq 0)$, let $\vec{c}$ be a symmetric random vector in $\mathbb{R}^M_{\geq 0}$, such that*

$$\mathbb{E}\Big[\mathbb{B}(\vec{c}; z_1, ..., z_M; \theta, N)\Big] = \mathbb{B}(\vec{a}; z_1, ..., z_M; \theta, N) \cdot \mathbb{B}(\vec{b}; z_1, ..., z_M; \theta, N), \quad (z_1, ..., z_M) \in \mathbb{R}^M. \tag{2.1.7}$$

We write

$$\vec{c} = \vec{a} \boxplus^{\theta}_{M,N} \vec{b}.$$

From the probablistic point of view, $\vec{c}$ is identified as the singular values of the "virtual" random $M \times N$ matrix $C = A \boxplus^{\theta}_{M,N} B$, and $\mathbb{B}(\vec{c}; z_1, ..., z_M; \theta, N)$ serves as the characterstic function of C. From the analytic point of view, the operation in (2.1.7) has been studied previously, in the context of Dunkl kernel and Dunkl translation, see Anker 2017, Section 3.6. The expectation symbol on the left of (2.1.7) holds in the sense that, there exists a unique generalized function[1] $\mathfrak{m}$ on $\mathbb{R}^M_{\geq 0}$ depending on $\vec{a}$ and $\vec{b}$, such that for any $(z_1, ..., z_M) \in \mathbb{R}^M$, when testing on $\mathbb{B}(\cdot; z_1, ..., z_M; \theta, N)$ we get the right side of (2.1.7), and in particular when taking $z_1 = ... = z_M = 0$ we have $\mathfrak{m}(1) = 1$. Note that $\mathfrak{m}$ is symmetric in the sense that, for any proper test function f and any permutation σ,

$$\langle \mathfrak{m}, f(c_1, ..., c_M) \rangle = \langle \mathfrak{m}, f(c_{\sigma(1)}, ..., c_{\sigma(M)}) \rangle,$$

where $\langle \mathfrak{m}, f \rangle$ denotes the value of the functional $\mathfrak{m}$ testing on the f. Moreover, by Anker

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[1] Throughout this text, we use the term "generalized function" instead of "distribution" to denote a linear functional on smooth functions, in order to avoid confusion with the probability distribution.

2017, Lemma 3.23 $\mathfrak{m}$ is compactly supported.

The rectangular addition $\vec{a} \boxplus^{\theta}_{M,N} \vec{b}$ can also be naturally generalized to independent random M-tuples $\vec{a}, \vec{b}$, by first taking the conditional event that $\vec{a}, \vec{b}$ taking some fixed value, then applying Definition 2.1.5. Formally, for random M-tuples $\vec{a}$ we replace the type BC Bessel function by

$$G_{N;\theta}(z_1, ..., z_M) := \mathbb{E}\Big[\mathbb{B}(\vec{a}, z_1, ..., z_M; N, \theta)\Big], \tag{2.1.8}$$

the type BC Bessel generating function of $\vec{a}$, and we assume the randomness of $\vec{a}$ to be reasonable, in the sense that the right side of (2.1.8) is finite and well-behaved as an analytic function of $(z_1, ..., z_M) \in \mathbb{R}_M$. See Section 2.2.5 for more details.

2.1.2 Low and high temperature behavior

By viewing $\vec{c} = \vec{a} \boxplus^{\theta}_{M,N} \vec{b}$ as the random M-tuples of singular values of some $M \times N$ virtual rectangular matrix with invariant distribution, it's then natural to study the behavior of $\vec{c}$ from a random matrix point of view. The distribution of $\vec{c}$ depends on summands $\vec{a}, \vec{b}$ and parameters $M \leq N, \theta > 0$. In various regimes of parameters, one can propose the following questions:

1. For fixed M, N, θ, is the distribution of $\vec{c}$ given by a probability measure on $\mathbb{R}^M_{\geq 0}$, and how do $\vec{a}, \vec{b}$ explicitly determine this measure?

2. What's the "low temperature" behavior of $\vec{c}$, i.e, when taking M, N to be fixed, and $\theta \to \infty$?

3. What's the "fixed temperature" behavior of $\vec{c}$, i.e, when taking θ to be fixed, and $M, N \to \infty$?

4. What's the "high temperature" behavior of $\vec{c}$, i.e, when taking $\theta \to 0$, and $M, N \to \infty$, growing in potentially different speed?

This text answers question 2 and 4. For question 1, it is well-believed (but still open) that the generalized function under $\vec{c}$ is indeed a probability measure, see e.g Anker 2017, Section 3.5, and this is related to the positivity conjecture of Littlewood-Richardson coefficients in the theory of symmetric functions, see Stanley 1989, Conjecture 8.3, Roesler 2003a or Gorin and A. W. Marcus 2020, Conjecture 2.1. We do not rely on this conjecture and instead analyze moments of the distribution of $\vec{c}$, which can be defined no matter the positivity conjecture holds or not. See Proposition 2.2.26 for the precise statement.

In low temperature regime, we observe that the random M-tuples are becoming "frozen" at some deterministic positions. More precisely we have the following statement. Let $1 \leq M \leq N$, z be a formal variable, $(a_1, ..., a_M), (b_1, ..., b_M) \in \mathbb{R}^M_{\geq 0}$, we define a polynomial $P_{M,N}(z)$ by

$$P_{M,N}(z) = \sum_{l=0}^{M} (-1)^l \Bigg(\sum_{i\geq 0, j\geq 0, i+j=l} \frac{(M-i)!(M-j)!}{M!(M-l)!} \frac{(N-i)!(N-j)!}{N!(N-l)!} e_i(a_1^2, ..., a_M^2) e_j(b_1^2, ..., b_M^2) \Bigg) z^{M-l}. \tag{2.1.9}$$

Theorem 2.1.6. *Fix $M \leq N$, given $\vec{a}$ and $\vec{b}$, let $\vec{c} = \vec{a} \boxplus^{\theta}_{M,N} \vec{b}$. Then as $\theta \to \infty$, the distribution of $\vec{c^2} = (c_1^2, ..., c_M^2)$ converges on polynomial test functions to δ-measures on roots of $P_{M,N}(z)$.*

Remark 2.1.7. *The polynomial $P_{M,N}(z)$ has previously appeared in Gribinski and A. Marcus 2022, and it's shown in Gribinski and A. Marcus 2022, Theorem 2.3 that all roots of $P_{M,N}(z)$ are real and nonnegative, given that $a_1^2, ..., a_M^2, b_1^2, ..., b_M^2$ are all real and nonnegative, using the theory of stable polynomials.*

In the fixed temperature regime, it was shown in Benaych-Georges 2009 (see Theorem 2.1.2) that for $\theta = \frac{1}{2}, 1$, as $M, N \to \infty$ in a way that $\frac{N}{M} \to q$, we get the rectangular free convolution. We believe the same result holds for any fixed $\theta > 0$.

In high temperature regime, when taking $\theta \to 0$, $N \to \infty$ and let M, the number of singular values to be fixed, the type BC multivariate Bessel function becomes a simple symmetric combination of exponents:

$$\mathbb{B}(\vec{a}, N\theta z_1, ..., N\theta z_M; \theta, N) \longrightarrow \frac{1}{M!} \sum_{\sigma \in S_M} \prod_{i=1}^{M} e^{a_i^2 z_{\sigma(i)}^2}. \tag{2.1.10}$$

See Appendix D for more details. Such limit expression has a clear probabilistic interpretation. Given deterministic M-tuples $\vec{a}$ and $\vec{b}$ as before, let $\vec{c} = (c_1, ..., c_M \geq 0)$ be obtained by choosing uniformly random an element σ in S_M, and taking

$$(c_1^2, ..., c_M^2) = (a_1^2 + b_{\sigma(1)}^2, ..., a_M^2 + b_{\sigma(M)}^2).$$

Moreover, when taking $M \to \infty$ after and assume the empirical measure

$$\frac{1}{M} \sum_{i=1}^{M} \delta_{x_i^2} \ (x_i = a_{i,M} \text{ or } b_{i,M})$$

of $\{\vec{a}_M\}_{M=1}^{\infty}$, $\{\vec{b}_M\}_{M=1}^{\infty}$ converge to some probability measure μ_a, μ_b on $\mathbb{R}_{\geq 0}$ weakly, then so is $\{\vec{c}_M\}_{M=1}^{\infty}$, and the empirical measure

$$\mu_c = \mu_a * \mu_b,$$

where $*$ denotes the usual convolution of measures on $\mathbb{R}$.

We see two different limiting behavior of $\vec{a} \boxplus_{M,N}^{\theta} \vec{b}$ as $M \to \infty$. For $\theta = 0$ we get the usual convolution, and for $\theta > 0$ we get the rectangular free convolution. This motivates us to look at the intermediate regime between the above two settings, such that we take $M \to \infty, N \to \infty, \theta \to 0, M\theta \to \gamma > 0$, $N\theta \to q\gamma$ for some $q \geq 1$. We are interested in the sequence of (random) virtual singular values $\{\vec{c}_M = (c_{M,1} \geq ... \geq c_{M,M} \geq 0)\}_{M=1}^{\infty}$, and we study the limiting behavior of the symmetric empirical measures of $\vec{c}_M$.

Remark 2.1.8. *The same intermediate regime was considered in S. N. M. R. Allez J. B. and Vivo 2013, where the authors study the limiting behavior of Laguerre ensemble, and proved that its empirical measure converges in this regime to some deterministic probability measure $\mu_{q,\gamma}$, which interpolates between Marcenko-Pastur law (when $\gamma \to \infty$) and some version of Gamma distribution (when $\gamma \to 0$). As an application of the theory we develop in this text, we reobtain this result in Section 2.5.5. There are also similar results for Gaussian ensemble, see J. B. R. Allez and Guionnet 2012.*

Definition 2.1.9. *Let* $\{\vec{a}_M\}_{M=1}^{\infty}$ *be a sequence of random M-tuples such that* $\vec{a}_M = (a_{M,1} \geq ... \geq a_{M,M} \geq 0)$. *Denote*

$$p_k^M = \frac{1}{2M} \sum_{i=1}^{M} \left[a_{M,i}^k + (-a_{M,i})^k \right].$$

We say $\{\vec{a}_M\}$ *converges in moments, if there exist deterministic nonnegative real numbers* $\{m_k\}_{k=1}^{\infty}$ *such that for any s=1,2,... and any* $k_1, ..., k_s \in \mathbb{Z}_{\geq 1}$, *we have*

$$\lim_{M\to\infty} \mathbb{E} \prod_{i=1}^{s} p_{k_i}^M = \prod_{i=1}^{s} m_{k_i}. \tag{2.1.11}$$

We write

$$\vec{a}_M \xrightarrow[M\to\infty]{m} \{m_k\}_{k=1}^{\infty}. \tag{2.1.12}$$

Remark 2.1.10. *Definition 2.1.9 is intuitively stating that the empirical measure of* $(a_{M,1}, ..., a_{M,M})$ *is converging weakly to some deterministic probability measure with moments* $\{m_k\}_{k=1}^{\infty}$, *as long as the moment problem of* $\{m_k\}_{k=1}^{\infty}$ *has a unique solution. By definition* $p_k^M = 0$ *for all odd k's, and therefore one can immediately see that* $m_k = 0$ *for all odd k's. The reason why we use symmetric empirical measure is that there's no canonical choice of the sign of singular values.*

Remark 2.1.11. *The convergence is well-posed as long as the randomness of* $\vec{a}_M$*'s are given by compactly supported generalized function, where the expectation* $\mathbb{E}$ *is testing the generalized function by the polynomial function* p_k^M *of* $\vec{a}_M$.

We prove a law of large numbers of the symmetric empirical measure of $\vec{c}_M$, which is interpreted as the empirical measure of the $M \times N$ matrix C with singular values $c_{M,1}, ..., c_{M,M}$. We assume the the distribution of each $\vec{a}_M$, $\vec{b}_M$ is given by some real valued compactly supported generalized function or exponentially decaying measure. For the precise meaning of the latter notion and more details of this technicality, see Section 2.2.5.

Theorem 2.1.12. *Fix* $\gamma > 0, q \geq 1$. *For* $M = 1, 2, ...$, *let* $N(M) \geq M$, $\theta(M) > 0$ *be two* *sequences satisfying* $N \to \infty$, $\theta \to 0$, $M\theta \to \gamma$, $N\theta \to q\gamma$ *as* $M \to \infty$. *Suppose for two sequences of random tuples* $\{\vec{a}_M\}_{M=1}^{\infty}, \{\vec{b}_M\}_{M=1}^{\infty}$,

$$\vec{a}_M \xrightarrow[M\to\infty]{m} \{m_k^a\}_{k=1}^{\infty}, \quad \vec{b}_M \xrightarrow[M\to\infty]{m} \{m_k^b\}_{k=1}^{\infty}.$$

Then

$$\vec{a}_M \boxplus_{M,N}^{\theta} \vec{b}_M \xrightarrow[M\to\infty]{m} \{m_k^c\}_{k=1}^{\infty},$$

where $\{m_k^c\}_{k=1}^{\infty}$ *is a sequence of deterministic nonnegative real numbers.*

We say $\{m_k^c\}_{k=1}^{\infty}$ *is the q-*γ *convolution of* $\{m_k^a\}_{k=1}^{\infty}$ *and* $\{m_k^b\}_{k=1}^{\infty}$, *written as*

$$\{m_k^c\}_{k=1}^{\infty} = \{m_k^a\}_{k=1}^{\infty} \boxplus_{q,\gamma} \{m_k^b\}_{k=1}^{\infty}.$$

We provide more properties of the q-γ convolution in the following two theorems.

Theorem 2.1.13. *There exists a invertible map* $\mathrm{T}_{m\to k}^{q,\gamma} : \mathbb{R}^{\infty} \to \mathbb{R}^{\infty}$, *that corresponds each* $\{m_{2k}\}_{k=1}^{\infty}$ *with a collection of* q-γ cumulants $\{k_l\}_{l=1}^{\infty}$, *i.e,* $\{k_l\}_{l=1}^{\infty} = \mathrm{T}_{m\to k}^{q,\gamma}(\{m_{2k}\}_{k=1}^{\infty})$. *The q-*$\gamma$ *cumulants linearizes q-*γ *convolution: for* $l = 1, 2, ...$,

$$k_l(\{m_{2k}^c\}_{k=1}^{\infty}) = k_l(\{m_{2k}^a\}_{k=1}^{\infty}) + k_l(\{m_{2k}^b\}_{k=1}^{\infty}).$$

$k_l = 0$ *for all odd* l*'s. Also* m_k^a, m_k^b, m_k^c *are 0 for all odd* k*'s.*

Treating each k_l *as a variable of degree* l, *then each* m_{2k} *is a homogeneous polynomial in* k_l*'s of degree* $2k$, *whose coefficients are polynomials of* q, γ *with explicit combinatorial description. Conversely, treating* m_{2k} *as a variable of degree* $2k$, *each even q-*γ *cumulant* k_{2l} *is a homogeneous polynomial in* m_{2k}*'s of degree* $2l$.

Theorem 2.1.14. *When* $q \to 0, q\gamma \to \infty$, *the q-*$\gamma$ *convolution of* $\{m_k^a\}_{k=1}^{\infty}$ *and* $\{m_k^b\}_{k=1}^{\infty}$ *turns into usual convolution of the two corresponding independent random variables, and the q-*γ *cumulants of* $\{m_{2k}^a\}_{k=1}^{\infty}$, $\{m_{2k}^b\}_{k=1}^{\infty}$ *turn into the usual cumulants after proper rescaling. Similarly, when* q *is fixed,* $\gamma \to \infty$, *the q-*γ *convolution turns into rectangular free convolution, and the q-*γ *cumulants turn into rectangular free cumulants after proper rescaling.*

Theorem 2.1.12 is proved in Section 2.4. Theorem 2.1.13 summarizes results in Section 2.5.1 and 2.5.2, such that the combinatorial moment-cumulant formula is given in Theorem 2.5.5, and the relation between moment generating function and cumulant generating function is given in Theorem 2.5.8. Theorem 2.1.14 summarizes the connections of q-γ convolution to classical and free convolution, which are given in Theorem 2.5.12 and Theorem 2.5.15 respectively. We also provide a limit transition of our q-γ convolution to the γ-convolution defined in Benaych-Georges, Cuenca, and Gorin 2022 in Theorem 2.5.10, which is related to the asymptotic behavior of self-adjoint matrix additions in high temperature regime.

2.1.3 Duality between low and high temperatures

It is observed that in β-random matrix theory, there's a duality between parameters β and $\frac{4}{\beta}$ (or θ with $\frac{1}{\theta}$). For example in Desrosiers 2009, the author gives an equality of average products of characteristic polynomials of Gaussian/Chiral β-ensembles at β and $\frac{4}{\beta}$. Similarly in Forrester 2022, it's shown that for Gaussian/Laguerre/Jocobi β-ensembles, the one-point or higher-point functions that describe the linear statistic of eigenvalues at low and high temperature can be identified with each other. The phenomena is not yet fully understood, and one analog exists in the theory of symmetric polynomials, where there is an automorphism that sends Jack polynomial to its dual by taking the transpose of its labelling Young diagram and invert the parameter θ at the same time, see Stanley 1989, Section 3 or Macdonald 1995, (10.17) for the precise statement. Since in this text we are considering low and high temperature regimes at the same time, the duality is indicating some connection between the two regimes. When M, N are fixed, $\theta \to \infty$, $\vec{c} = \vec{a} \boxplus^{\theta}_{M,N} \vec{b}$ concentrate at roots of $P_{M,N}(z)$, which are identified as the rectangular finite free convolution of $\vec{a}$ and $\vec{b}$ defined in A. Marcus, Spielman, and Srivastava 2019, Gribinski and A. Marcus 2022. When $M, N \to \infty$, $\theta \to 0$, $M\theta \to \gamma$, $N\theta \to q\gamma$, $\vec{c}$ converges in moments to the q-γ convolution of $\vec{a}$, $\vec{b}$. We find that the (M, N)-rectangular finite free convolution and the q-γ convolution match with each other under certain identification of parameters. More precisely, Gribinski 2022 introduces a degree M polynomial as the so-called rectangular R-transform, that linearizes rectangular finite free convolution. We treat the coefficients of rectangular R-transform as the rectangular finite cumulants, and show that if identifying M in rectangular finite convolution with $-\gamma$ in q-γ convolution, the moment-cumulant relation of rectangular finite convolution and q-γ convolution match perfectly. In addition, since both M and γ are positive, these two operations are analytic continuation of each other, and they together extend the moment-cumulant relation to $\gamma \in \mathbb{R}_{\geq 0} \bigcup \mathbb{Z}_{\leq -1}$. See Section 2.6 for more details.

We also note that similar identification of low and high temperature regimes holds also for self-adjoint matrix addtions. Benaych-Georges, Cuenca, and Gorin 2022 studies addition of two $N \times N$ self-adjoint matrices in the high temperature regime $N \to \infty, \theta \to 0, N\theta \to \gamma > 0$, and introduces the so-called γ-convolution and γ-cumulants. On the other hand, Arizmendi and Perales 2018 introduces a family of $d \times d$ free cumulants from finite self-adjoint matrix additions in low temperature, and the authors of these two papers discovered that their moment-cumulant relations can also match by identifying d with $-\gamma$. We believe that such matching appearing in both self-adjoint and rectangular matrix additions should not be just coincidence.

2.1.4 Techniques and difficulties

Unlike many other classes of β-ensembles, we don't have a density function of our object $\vec{c} = \vec{a} \boxplus_{M,N}^{\theta} \vec{b}$, and because of the openness of the positivity conjecture we can't even guarantee that such density exists. Therefore, the proof of main results in low and high temperature regime, Theorem 2.1.6 and Theorem 2.1.12, both rely heavily on moment calculations. We characterize its distribution using the type BC Bessel generating function $G_{N;\theta}(z_1, ..., z_M; \vec{c})$, which is a new object in random matrix literature, and apply two different approaches in the low and high temperature regime respectively, to extract the moment information of $\vec{c}$.

In order to apply such approaches, it's necessary to figure out the correct notion of

Bessel function $\mathbb{B}(\vec{c}; z_1, ..., z_M; \theta, N)$ for rectangular matrices. On one hand, we start from the case $\theta = \frac{1}{2}, 1, 2$ and define $\mathbb{B}(\vec{c}; z_1, ..., z_M; \theta, N)$ as the matrix integral in (2.1.4), based on the probabilistic intuition of rectangular random matrices. On the other hand, for arbitrary $\theta > 0$, we define our type BC Bessel function to be a symmetric Dunkl kernel, that is known as the joint eigenfunction of the corresponding type BC Dunkl operators, with eigenvalues given by the symmetric moments of $\vec{c}$. While there are infinite versions of Dunkl kernels, we choose the root multiplicities $m_{\pm e_i}$, $m_{\pm e_i \pm e_j}$ in a unique way that

1. For $\theta = \frac{1}{2}, 1, 2$, it coincides with (2.1.4).
2. For general $\theta > 0$, it has nice explicit power series expansion that naturally extrapolates from $\theta = \frac{1}{2}, 1, 2$.

We find such root multiplicities and verify the analytic and combinatorial properties of $\mathbb{B}(\cdot; z_1, ..., z_M; \theta, N)$ in Section 2.2, by applying the general theory of special functions and symmetric spaces under random matrix motivations.

In low temperature regime, we use the explicit expansion of Bessel generating function to calculate the limiting distribution of $\vec{c}$. And in high temperature regime, we study the asymptotic behavior of the action of Dunkl operators on $G_{N;\theta}(z_1, ..., z_M; \vec{c})$, which extracts moment information. More precisely, in Theorem 2.4.8 we build an equivalence of the following two conditions of a sequence of random M-tuples $\vec{c}_M = (c_{M,1}, ..., c_{M,M}) \in \mathbb{R}^M_{\geq 0}$, $M = 1, 2, ...$, in the regime $M, N \to \infty, \theta \to 0, M\theta \to \gamma, N\theta \to q\gamma$:

1. $\{\vec{c}_M\}_{M=1}^{\infty}$ converges in moments as in Definition 2.1.9.
2. The l^{th} order partial derivative in z_1 of $\ln\Big(G_{N;\theta}(z_1, ..., z_M; \vec{c})\Big)$ at 0 converges to some real number for all $l = 1, 2, ...$, and the partial derivatives in more than one variables among $z_1, ..., z_M$ at 0 all converge to 0.

The nontrivial limit of l^{th} order derivative in condition 2 gives the q-γ cumulant k_l of $\{\vec{c}_M\}_{M=1}^{\infty}$, up to some constant. Note that this equivalence itself is independent of the addition operation, and can be applied to a single sequence of (virtual) rectangular matrices. See Section 2.5.5 as an example.

Remark 2.1.15. *There exists an analog of Theorem 2.4.8 in fixed temperature regime (for*

$\beta = 1, 2$), given in Benaych-Georges 2011 using a different approach that relies on the concrete matrix structure.

Compared to the previous studies of rectangular additions, which mostly deal with real/complex matrices, our text defines and considers the general β-additions that do not rely on concrete matrix structure. Compared to the study of self-adjoint additions, there are some extra technicality that arise in this text. Firstly, there are two parameters M, N of the matrix size, and we allow M and N to grow in different speed. More importantly, because of the more involved root multiplicities, the type BC Bessel generating functions, type BC Dunkl operators have more complicated expressions, and this makes the combinatoric in the asymptotic analysis more complicated as well. Because of the above two issues, and because of the fact that rectangular matrices are relatively less studied in literatures, it takes more efforts for us to properly define the rectangular version of empirical measures, moments, cumulants etc, and figure out the limit regime that nontrivial behavior and connection to known objects occur. The readers will also see a more complicated moment-cumulant relation of our q-γ convolution, that can degenerate to the usual, free, rectangular free and γ-convolutions which are operations that characterize several other random matrix additions.

2.1.5 Further Studies

We point out several possible directions for further studies in rectangular matrix addition. In the regime $\theta \to \infty$ and M, N fixed, we believe that the fluctuation of $\vec{c} = \vec{a} \boxplus_{M,N}^{\theta} \vec{b}$ around roots of $P_{M,N}(z)$ will converge in distribution to some Gaussian vector, since the similar limiting behavior holds for Laguerre β ensemble. We are only able to prove this for the single row matrix, i.e, when $M = 1$, and the general case remains as an open problem.

This text does not consider the fixed temperature regime, where $M, N \to \infty$, $N/M \to q \geq 1$ and θ is fixed. As mentioned previously, we believe that for general $\theta > 0$, we will get rectangular free convolution of the empirical measure in the limit.

In the regime $M, N \to \infty, \theta \to 0, M\theta \to \gamma, N\theta \to q\gamma$, we prove a Law of Large

Numbers of the empirical measure of rectangular matrix addition, and it might be of interest to go further, and prove a Central Limit Theorem of it under proper assumption of the summands: for a class of well-behaved test function ϕ, testing the empirical measure with ϕ always gives a Gaussian random variable in the limit. We refer to Guera and Memin 2023 for the result in this flavor on a collection of β-ensembles.

We also note that we only consider the global behavior of the limiting empirical measure in this text. However, our approach of using Dunkl operators to extract moment information, might be applicable in the study of the bulk or edge limit of certain matrix ensemble, including but not limited to rectangular matrix addition.

The chapter is organized as follows. In Section 2.2 we introduce type BC Bessel function and Bessel generating function, which play the role of characteristic function for rectangular matrices. In Section 2.3 we study the low temperature behavior. In Section 2.4 we prove the main theorem in high temperature regime, and introduce the q-γ cumulants in an analytic way. Then we study the moment-cumulant relation of q-γ convolution in more details, provide an explicit combinatorial description, and point out its connection with the classical free probability theory in Section 2.5. Finally in Section 2.6, we check the quantitative connection between low and high temperature regimes.

2.2 Bessel functions and Dunkl operators

2.2.1 Symmetric polynomials

Symmetric polynomials are common objects appearing in combinatoric, representation theory, and random matrices. This section recalls basic definitions of several objects in this subject, that we will use in this text. For a detailed introduction of classical results of symmetric polynomials, see e.g Macdonald 1995.

Definition 2.2.1. *A* partition λ *is a M-tuple of nonnegative integers* $(\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_M \geq 0)$. *We identity* $(\lambda_1, ..., \lambda_M)$ *with* $(\lambda_1, ..., \lambda_M, 0, ..., 0)$, *and denote the* length *of* λ *by* $l(\lambda) \in \mathbb{Z}_{\geq 1}$, *which is the number of strictly positive* λ_i*'s. We say a partition is even, if*

$\lambda_1, ..., \lambda_{l(\lambda)}$ *are all even.*

Let $|\lambda| = \sum_{i=1}^{l(\lambda)} \lambda_i$*. For two partitions* λ, μ *such that* $|\lambda| = |\mu|$*, there's a lexicographical order between them, that is,* $\lambda > \mu$ *if and only if for some* $j \in \mathbb{Z}_{\geq 1}$*,*

$$\lambda_1 = \mu_1, ..., \lambda_{j-1} = \mu_{j-1} \text{ and } \lambda_j > \mu_j.$$

The combinatoric expressions of symmetric polynomials are often given by sums over partitions, for which we introduce the following notions.

Definition 2.2.2. *A* Young diagram *is graphical representation of a partition. Given a partition* λ*, view it as a collection of* $|\lambda|$ *boxes, that there are* λ_i *boxes in the* i^{th} *row. In this text we do not distinguish a partition and its corresponding Young diagram. Let* $s = (i, j) \in \lambda$ *be the coordinate of the box on the* j^{th} *column and the* i^{th} *row in* λ*. Moreover, let* $\lambda_j^{'}$ *be the number of boxes on the* j^{th} *column of* λ*, and*

$$a(s) = a(i,j) = \lambda_i - j, \quad l(s) = l(i,j) = \lambda_j^{'} - i, \quad \lambda^{'} = (\lambda_1^{'}, ..., \lambda_{\lambda_1}^{'}).$$

Definition 2.2.3. *For* $M \in \mathbb{Z}_{\geq 1}$*, a symmetric polynomial* $g(z_1, ..., z_M)$ *is a multivariate polynomial of variables* $z_1, ..., z_M$ *with complex coefficients, such that for any* $\sigma \in S_M$*, the symmetric group of* M *elements, we have*

$$f(z_1, ..., z_M) = f(z_{\sigma(1)}, ..., z_{\sigma(M)}).$$

We denote the space of all symmetric polynomials in M *variables by* Λ_M*, which has the structure of an (complex) algebra.*

We introduce several classical symmetric polynomials as elements in Λ_M.

Definition 2.2.4. *The monomial symmetric polynomial* m_λ*'s are a collection of elements in* λ *indexed by partition* λ*, such that for* $l(\lambda) \leq M$*,*

$$m_\lambda(\vec{z}) = \sum_{(k_1,...,k_N)} \sum_{1 \leq i_1 < i_2 < ... < i_k \leq M} z_{i_1}^{k_1} z_{i_2}^{k_2} ... z_{i_k}^{k_N},$$

where $(k_1, ..., k_N)$ *go over all rearrangements of* $\lambda_1 \geq ... \geq \lambda_N$ *without repetitions. We also take* $m_\lambda(\vec{z}) = 0$ *for* $l(\lambda) > M$.

The elementary symmetric polynomials $\{e_k\}_{i=1}^M$ *are*

$$e_k(\vec{z}) = \sum_{1 \leq i_1 < i_2 < ... < i_k \leq M} z_{i_1} z_{i_2} ... z_{i_N}.$$

By definition $e_k = m_{1^k}$, *where* 1^k *denotes the partition* $(1, 1, ..., 1)$ *of length* k.

The power sums $\{p_k\}_{k=1}^\infty$ *are*

$$p_k(\vec{z}) = z_1^k + z_2^k + ... + z_M^k.$$

By definition $e_k = m_{(k)}$, *where* (k) *denotes the length 1 partition* λ *such that* $\lambda_1 = k$.

Remark 2.2.5. *It's clear from definition that* $\{m_\lambda\}$ *form a linear basis of* Λ_M. *Another important fact is that,* $\{e_k\}_{k=1}^M$ *and* $\{p_k\}_{k=1}^\infty$ *are two sets of algebraic generators of* Λ_M, *see Macdonald 1995, Chapter 1.*

The Jack polynomials play a central role in this text. Let $\vec{z}$ denote $(z_1, ..., z_M)$ for some $M \geq 1$, fix $\theta > 0$, and let X be a formal auxiliary variable, $\partial_i, i = 1, 2, ..., M$, be the partial derivative operator in z_i; $V(\vec{z}) = \prod_{1 \leq i < j \leq M}(z_i - z_j)$ be the Vandermonde determinant.

Definition 2.2.6. *Macdonald 1995, Chapter VI Let* $D_M(X; \theta)$ *be a differential operator of the form*

$$D_M(X; \theta) = V(\vec{z})^{-1} \det \left[z_i^{M-j} \left(z_i \frac{\partial}{\partial z_i} + (M - j)\theta + X \right) \right]_{1 \leq i,j \leq M}. \tag{2.2.1}$$

$D_M(X; \theta)$ *is a generating function (with variable* X*) of linear differential operators* $D_M^1, ..., D_M^M$ *acting on* Λ_M, *such that*

$$D_M(X; \theta) = \sum_{r=0}^M D_M^r X^{M-r}.$$

The Jack polynomials in M-variables are a collection of elements $P_\lambda(\vec{z})$ *in* Λ_M, *indexed by partitions* λ *such that* $l(\lambda) \leq M$. *Each* $P_\lambda(\vec{z})$ *is uniquely determined by the following* *two properties:*

$$P_\lambda(\vec{z}) = m_\lambda(\vec{z}) + \sum_{\mu<\lambda} u^\lambda_\mu(\theta) m_\mu(\vec{z}), \tag{2.2.2}$$

where $u^\lambda_\mu(\theta) \in \mathbb{R}$ *are parameterized by* θ*, and*

$$D_M(X;\theta)P_\lambda(\vec{z}) = c^\lambda_\lambda(\theta)P_\lambda(\vec{z}), \tag{2.2.3}$$

where $c^\lambda_\lambda(\theta) = \prod_{i=1}^M (X + \theta^{-1}\lambda_i + M - i)$.

Proposition 2.2.7. *Macdonald 1995, chapter VI For* $M \geq l(\mu)$*,* $u^\lambda_\mu(\theta)$ *in (2.2.2) is independent of* M*.*

Because of last proposition, we write $P_\lambda(\cdot;\theta) = m_\lambda(\cdot) + \sum_{\mu<\lambda} u^\lambda_\mu(\theta)m_\mu(\cdot)$, where $\cdot$ denotes $(z_1, ..., z_M)$ for arbitrary $M \geq l(\lambda)$, which does not affect the combinatorial expansion in m_μ's. We also introduce another version of Jack polynomial.

Definition 2.2.8. *The dual of Jack polynomial* $Q_\lambda(\cdot;\theta)$ *as*

$$Q_\lambda(\cdot;\theta) = b_\lambda(\theta) \cdot P_\lambda(\cdot;\theta),$$

where $b_\lambda(\theta) = \prod_{s\in\lambda} \frac{a(s)+\theta l(s)+\theta}{a(s)+\theta l(s)+1}$.

Remark 2.2.9. *It's a nontrivial fact that Jack polynomials satisfying the two defining properties exist. The differential operator* $D_M(X;\theta)$ *was discovered by Sekiguchi in Sekiguchi 1976.*

Given two Jack polynomials $P_v(\cdot;\theta)$ and $P_\mu(\cdot;\theta)$, their project $P_v(\cdot;\theta) \cdot P_\mu(\cdot;\theta)$ is again a symmetric polynomial, and hence can be written as a unique linear combination of Jack polynomials. Namely we have the following equality, where $C^{v,\mu}_\lambda(\theta)$ is the coefficient of $P_\lambda(\cdot;\theta)$ in the expansion:

$$P_v(\cdot;\theta)P_\mu(\cdot;\theta) = \sum_\lambda C^{v,\mu}_\lambda(\theta)P_\lambda(\cdot;\theta). \tag{2.2.4}$$

We note that $C^{v,\mu}_\lambda(\theta)$ is also independent of M because of Proposition 2.2.7.

2.2.2 Type BC Bessel functions

For positive integers $M \leq N$, take a M-tuples of nonnegative real numbers $\vec{a} = (a_1 \geq a_2 \geq ... \geq a_M)$ as the given data. The idea of type BC Bessel function $\mathbb{B}(\vec{a}, z_1, ..., z_M); \theta)$ is a version of multivariate symmetric Fourier kernel, with certain nontrivial root multiplicities given by parameter $\theta > 0$. In the special functions literature this is a special case of the so-called symmetric Dunkl kernel, see Section 2.2.3.

Definition 2.2.10. *For* $\theta = \frac{1}{2}, 1, 2$, $M \leq N$, *the type BC multivariate Bessel functions are defined with parameter* θ, *and M-tuples of ordered real labels* $a = (a_1 \geq a_2 \geq \cdots a_M)$, *that*

$$\mathbb{B}(\vec{a}; z_1, z_2, ..., z_M; \theta, N) = \int dU \int dV \ \exp\left(\frac{1}{2} Tr(U\Lambda VZ + Z^* V^* \Lambda^* U^*)\right),$$

where

$$\Lambda = \begin{bmatrix} a_1 & & & & 0 & ... & 0 \\ & a_2 & & & 0 & ... & 0 \\ & & ... & & & & \\ & & & ... & & & \\ & & & a_M & 0 & ... & 0 \end{bmatrix}_{M \times N}, \tag{2.2.5}$$

$$Z = \begin{bmatrix} z_1 & & & & \\ & z_2 & & & \\ & & ... & & \\ & & & ... & \\ & & & & z_M \\ 0 & & ... & & 0 \\ & & ... & & \\ 0 & & ... & & 0 \end{bmatrix}_{N \times M}, \tag{2.2.6}$$

$U \in O(M)/U(M)/Sp(M)$, $V \in O(N)/U(N)/Sp(N)$ *are integrated under Haar measures.*

Definition 2.2.10 provides an explicit connection with rectangular matrices, where the integral is of the form as a "Fourier transform"/characteristic function of $A = U\Lambda V$. However, since there is no (skew) field with real dimension β for general $\beta > 0$, one need to define the Bessel functions in an alternate way that does not rely on explicit matrix structure. For this purpose, we introduce the notion of type BC Jacobi polynomial, which is indeed the multivariate Jacobi polynomial in Appendix A with a specified root multiplicity function parametrized by $\theta > 0$, and was studied in Okounkov and Olshanski 2006.

For $M \in \mathbb{Z}_{\geq 1}$, let W denote the BC_M Weyl group $W = S_M \ltimes \mathbb{Z}_2^M$, which acts on functions of $\vec{x} = (x_1, ..., x_M)$. The S_M part permutes $x_1, ..., x_M$, and the $\mathbb{Z}_2^M$ part acts by $f(\vec{x}) \mapsto f(x_1^{\pm}, ..., x_M^{\pm})$.

Definition 2.2.11. *Okounkov and Olshanski 2006 Take three parameters $\theta > 0, a, b > -1$. The type BC Jacobi polynomials are a collection of functions $J_\lambda(\vec{x}; \theta, a, b)$ on the M-dimensional torus*

$$\mathbb{T} = \{(x_1, ..., x_M) \subset \mathbb{C}^M, |x_1| = ... = |x_M| = 1\},$$

indexed by partition λ. And J_λ's are determined by the following:

(1). $J_\lambda(\vec{x}; \theta, a, b) = x_1^{\lambda_1} \cdots x_M^{\lambda_M} + ...$, where the dots stand for lower monomials in the lexicographic order as in Definition 2.2.1, and J_λ is W-invariant,

(2). J_λ's are mutually orthogonal in $L^2(\mathbb{T}, w)$, with scalar product given by

$$\langle f, g\rangle := \int_{\mathbb{T}} f(\vec{x})\bar{g}(\vec{x})w(\vec{x}) \cdot Haar(d\vec{x}),$$

where

$$w(\vec{x}) = \prod_{1\leq i<j\leq M} |x_i - x_j|^{2\theta}|1 - x_i x_j|^{2\theta} \prod_{1\leq i\leq M} |1 - x_i|^{2a+1}|1 + x_i|^{2b+1}.$$

Let $\Phi_\lambda(x_1, ..., x_M; \theta, a, b)$ be the normalized type BC multivariate Jacobi polynomials where $\Phi_\lambda(0, ..., 0; \theta, a, b) = 1$.

Remark 2.2.12. *By taking $x_i = e^{2z_i i}$ for i=1,2,...,M, each J_λ is identified with the Jacobi* *polynomial* $\mathfrak{J}_{\tilde{\lambda}}$ *in Definition A.1.12, where* $\tilde{\lambda_k} = 2\tilde{\lambda_k}$ *for* $k = 1, 2, ..., l(\lambda)$.

We define the type BC Bessel function as a limit of type BC Jacobi polynomial, then present a more concrete power series expression of it in terms of Jack polynomials, using the limit transition.

Definition 2.2.13. *Take* $\theta > 0$, $M \le N$, $\lambda = \lfloor \epsilon^{-1}(a_1, ..., a_M) \rfloor$, $a = \theta(N - M + 1) - 1$, $b = \theta - 1$, ρ *be a fixed vector defined as in (A.1.1), the type BC multivariate Bessel function labeled by* $\vec{a} = (a_1 \ge a_2 \ge ... \ge a_M)$ *is an multivariate analytic function in both* $\vec{a}$ *and* $(z_1, ..., z_M)$, *defined by*

$$\mathbb{B}(i\vec{a}, z_1, ..., z_M; \theta, N) := \lim_{\epsilon \to 0} \Phi_{\lfloor \frac{\vec{a}}{2\epsilon} - \frac{\rho}{2} \rfloor}(e^{2\epsilon z_1 i}, ..., e^{2\epsilon z_M i}; a, b, \theta).$$

Remark 2.2.14. *Because of Remark 2.2.12, each* $\mathbb{B}(\vec{a}, z_1, ..., z_M; \theta, N)$ *is identified with* $f_{\vec{a}}$ *in Definition A.1.10. Moreover, by specifying* a, b *in this way we take the root multiplicities* $m_{\pm e_i} = 2\theta(N - M), m_{\pm 2e_i} = 2\theta - 1, m_{\pm e_i \pm e_j} = 2\theta$, *which are parameterized by a single variable* $\theta > 0$.

Definition 2.2.15. *For a partition* μ, $t \in \mathbb{R}$, $\theta > 0$, *let*

$$H(\mu) = \prod_{s \in \mu} [a(s) + 1 + \theta l(s)], \tag{2.2.7}$$

$$H'(\mu) = \prod_{s \in \mu} [a(s) + \theta + \theta l(s)], \tag{2.2.8}$$

and

$$(t)_\mu = \prod_{s \in \mu} [(t + j - 1 - \theta(i - 1)]. \tag{2.2.9}$$

Proposition 2.2.16. *The limit in Definition 2.2.13 exists, and*

$$
\begin{aligned}
&\mathbb{B}(\vec{a}, z_1, ..., z_M; \theta, N)\\
=&\sum_{\mu}\prod_{i=1}^{M}\frac{\Gamma(\theta N-\theta(i-1))}{\Gamma(\theta N-\theta(i-1)+\mu_i)}\frac{1}{H(\mu)}2^{-2|\mu|}\frac{P_\mu(a_1^2,\cdots,a_M^2;\theta)P_\mu(z_1^2,\cdots,z_M^2;\theta)}{P_\mu(1^M;\theta)}\\
=&\sum_{\mu}\prod_{i=1}^{M}\frac{\Gamma(\theta N-\theta(i-1))}{\Gamma(\theta N-\theta(i-1)+\mu_i)}\frac{\Gamma(\theta M-\theta(i-1))}{\Gamma(\theta M-\theta(i-1)+\mu_i)}\frac{H^{'}(\mu)}{H(\mu)}2^{-2|\mu|}P_\mu(a_1^2,\cdots,a_M^2;\theta)P_\mu(z_1^2,\cdots,z_M^2;\theta),
\end{aligned}
\tag{2.2.10}
$$

where μ is summed over all partitions of length at most M.

Proof. The existence of the limit is guaranteed by Proposition A.1.11, Theorem A.1.14, Remark 2.2.12 and 2.2.14. More precisely, by Roesler 2003b, Theorem 2.32, the multivariate Bessel function is meromorphic on m, the root multiplicity function, and the pole set K^{sing} of m is explicitly given in C. F. Dunkl and Jeu 1994. One can check that for all $\theta > 0$, $m \notin K^{sing}$. Hence we can do an analytically continuation of (2.2.10) from nonnegative root multiplicities to all $\theta > 0$.

We do a concrete calculation for the explicit expression on the right of (2.2.10). By Okounkov and Olshanski 2006, Proposition 2.3,

$$
\begin{aligned}
&\Phi_{\lfloor\frac{\vec{a}}{2\epsilon}-\frac{\rho}{2}\rfloor}(e^{2\epsilon z_1 i}, ..., e^{2\epsilon z_M i}; \theta, a, b)\\
=&\sum_{\mu\le\lfloor\frac{\vec{a}}{2\epsilon}-\frac{\rho}{2}\rfloor}\frac{I_\mu(\lfloor\frac{\vec{a}}{2\epsilon}-\frac{\rho}{2}\rfloor;\theta;\sigma+M)P_\mu(2cos(2\epsilon z_j)-2;\theta)}{C(M,\mu;\theta;a,b)},
\end{aligned}
$$

where $I_\mu(x_1, ..., x_M; \theta, h)$ is defined in Okounkov and Olshanski 2006, Proposition 2.2, $\sigma = (a + b + 1)/2$, and

$$C(M, \mu; \theta, a, b) = I_\mu(\mu; \theta, \sigma + \theta M)J_\mu(1^M; \theta, a, b).$$

By comparing Okounkov and Olshanski 2006, (2.3) and (2.4), we see that as an inhomogeneous polynomial of $x_1, ..., x_M$, $I_\mu(x_1, ..., x_M; \theta; h)$ has highest degree term $P_\mu(x_1^2, ..., x_M^2; \theta)$.

Therefore asymptotically

$$I_\mu(\lfloor \frac{\vec{a}}{2\epsilon} - \frac{\rho}{2} \rfloor; \theta; \sigma + M) \approx P_\mu(a_1^2, ..., a_M^2; \theta) 2^{-2|\mu|} \epsilon^{-2|\mu|}.$$

On the other hand, when ϵ is small,

$$P_\mu(2cos(2\epsilon z_j) - 2; \theta) \approx P_\mu(-(2\epsilon z_j)^2; \theta) = P_\mu(z_j^2; \theta)(-4)^{|\mu|}\epsilon^{2|\mu|},$$

so it remains to match the coefficients. This follows by (see Macdonald 1995, (10, 20))

$$P_\mu(1^M; \theta) = \frac{(M\theta)_\mu}{H'(\mu)}, \tag{2.2.11}$$

and (see Okounkov and Olshanski 2006, Remark 2.5)

$$\begin{aligned} &C(M, \mu; \theta, \theta(N - M + 1) - 1, \theta - 1) \\ =&4^\mu \cdot \frac{H(\mu)}{H'(\mu)} \prod_{i=1}^{M} \frac{\Gamma(\mu_i + (M - i + 1)\theta)}{\Gamma((M - i + 1)\theta)} \frac{\Gamma(\mu_i + (N - i + 1)\theta)}{\Gamma((N - i + 1)\theta)}. \end{aligned} \qquad \square$$

The following example gives a connection of $\mathbb{B}(\cdot, z_1, ..., z_M; \theta, N)$ with the usual single variable Bessel function.

Example 2.2.17. *When* $M = 1$*,*

$$\mathbb{B}(a, iz; \theta, N) = \Gamma(N\theta) \cdot (\frac{az}{2})^{-(N\theta-1)} B_{N\theta-1}(az),$$

where B_α *is the Bessel function of the first kind.*

Definition 2.2.13 generalizes the notion of the type BC Bessel function to any $\theta > 0$. In particular, when $\theta = \frac{1}{2}, 1, 2$, Definition 2.2.13 provides an explicit power series expansion of the matrix integral in Definition 2.2.10. There are more than one way to show the equivalence of these two expressions, and the one we present below relies on the representation theory lying behind the concrete objects.

Theorem 2.2.18. *For $\theta = \frac{1}{2}, 1, 2,$*

$$\int dU \int dV \ exp(i \cdot Tr(U\Lambda V Z + Z^* V^* \Lambda^* U^*))$$
$$= \lim_{\epsilon \to 0} \Phi_{\lfloor \frac{\vec{a}}{\epsilon} - \frac{\rho}{2} \rfloor}(e^{2\epsilon z_1 i}, ..., e^{2\epsilon z_M i}; a, b, \theta)$$

where the matrix integral on the left is defined in the same way as in Definition 2.2.10, only differs by a constant $2i$ in the exponent.

Proof. $\Phi_{\lfloor \frac{\vec{a}}{\epsilon} - \frac{\rho}{2} \rfloor}(e^{2z_1 i}, ..., e^{2z_M i}; a, b, \theta)$ is identified with spherical function of $O(M+N)/O(M) \times O(N), U(M+N)/U(M) \times U(N), Sp(M+N)/Sp(M) \times Sp(N)$ respectively according to Theorem A.1.16, and the root multiplicity list in Appendix B. After limit transition in Proposition A.1.11 or Remark A.1.17, it suffices to identify the matrix integral with the corresponding Euclidean spherical function, which we again refer to Helgason 2001, Helgason 2012. ☐

Remark 2.2.19. *The expression of matrix integral in Definition 2.2.10 as power series in Definition 2.2.13 is not new, and could be found in Forrester 2010, Section 13.4.3 with a different proof. See the Appendix C for more information and yet another short proof of this result.*

2.2.3 Type BC Dunkl operators

As a special class of differential operators, Dunkl operators were introduced in C. Dunkl 1989, and can be thought as a generalization of the usual partial derivatives on multivariate analytic functions, that take Fourier kernels as eigenfunction. We briefly review the basic general theory of Dunkl opertors in Appendix A, and in this section, we specify to a special class of rational Dunkl operators under root system of type BC, which is parametrized by a single variable $\theta > 0$ and plays a central role in Section 2.3. For the convenience of readers, we redefine this operator in a more concrete and straightforward way.

Definition 2.2.20. *For $N \geq M \geq 2, \theta > 0$, let D_i be a differential operator acting on* *analytic functions on* $\mathbb{C}^M$ *with variables* $z_1, .., z_M$*, that*

$$D_i = \partial_i + \Big[\theta(N-M+1) - \frac{1}{2}\Big]\frac{1-\sigma_i}{z_i} + \theta\sum_{j\neq i}\Big[\frac{1-\sigma_{ij}}{z_i - z_j} + \frac{1-\tau_{ij}}{z_i+z_j}\Big], \tag{2.2.12}$$

where σ_i *interchanges* z_i *and* $-z_i$*,* σ_{ij} *interchanges* z_i *and* z_j*, and* τ_{ij} *interchanges* z_i *and* $-z_j$*.*

Remark 2.2.21. D_i*'s are special cases of the rational Dunkl operator in Definition A.1.9, such that the reflections* s_α *for* $\alpha \in R$ *are specified as following:*

$$\sigma_i = s_{e_i}, \quad \sigma_{ij} = s_{e_i - e_j}, \quad \tau_{ij} = s_{e_i+e_j}.$$

Moreover, the root multiplicity function is given by $m_{\pm e_i} = 2\theta(N-M), m_{\pm 2e_i} = 2\theta - 1, m_{\pm e_i \pm e_j} = 2\theta$*, the same as type BC Bessel function in Section 2.2.2.*

Proposition 2.2.22. *C. Dunkl 1989 The Dunkl operators of same root multiplicities commute, i.e,*

$$D_i D_j = D_j D_i$$

for any $1 \leq i, j \leq M$*.*

The following result provides connection of type BC multivariate Bessel functions and Dunkl operators, namely, the former are eigenfunctions of the latter.

Definition 2.2.23. *Fix* $M \geq 1$*. For* $k = 1, 2, ...,$ *denote*

$$\mathrm{P}_k = D_1^k + ... + D_M^k.$$

Theorem 2.2.24. *Given* $\vec{a} = (a_1 \geq ... \geq a_M)$ *for each* $k = 1, 2, ...,$

$$\mathrm{P}_{2k}\mathbb{B}(\vec{a}, z_1, .., z_M; \theta) = \left(\sum_{i=1}^{M}(a_i)^{2k}\right) \cdot \mathbb{B}(\vec{a}, z_1, ..., z_M; \theta). \tag{2.2.13}$$

Proof. This is a special case of Definition A.1.10. $\square$

Remark 2.2.25. *From Proposition 2.2.16, one can see that* $\mathbb{B}(\vec{a}, z_1, ..., z_M; \theta)$ *is symmetric under actions of Weyl group of root system* BC_M*, namely, invariant by interchanging* z_i *with* z_j *and replacing* z_i *by* $-z_i$*. Similarly, it's necessary to take symmetric power sum of* $D_i's$ *with even power, which satisfies the same symmetry.*

2.2.4 Matrix addition and moments

For $\vec{c} = \vec{a} \boxplus^{\theta}_{M,N} \vec{b}$, we assume in this section that $\vec{a}, \vec{b}$ are deterministic, and recall from Definition 2.1.5 that the distribution $\mathfrak{m}$ of $\vec{c}$ is given by testing on type BC Bessel function. Note that polynomials are bounded and smooth on compact sets, and therefore are legitimate test functions of $\mathfrak{m}$. Moreover, by Proposition 2.1.4 Bessel function is analytic and symmetric on $\mathbb{C}^M$, so we can view it as a generating function of symmetric polynomials of M variables $c_1, ..., c_M$. More precisely, by expanding Bessel functions on both sides of (2.1.7) using (2.2.10), we have the following:

Proposition 2.2.26. *For each partition* λ *with* $l(\lambda) \leq M$*, let* $\vec{c} = \vec{a} \boxplus^{\theta}_{M,N} \vec{b}$*, then*

$$\mathbb{E}P_\lambda(c_1^2, ..., c_M^2; \theta) = \sum_{|v|+|\mu|=|\lambda|} \frac{H(\lambda)}{H(v)H(\mu)} \frac{\prod_{i=1}^M \Gamma(\theta(N-i+1))\Gamma(\theta(N-i+1)+\lambda_i)}{\prod_{i=1}^M \Gamma(\theta(N-i+1)+v_i)\Gamma(\theta(N-i+1)+\mu_i)}$$
$$\frac{P_\lambda(1^M;\theta)}{P_v(1^M;\theta)P_\mu(1^M;\theta)} C_\lambda^{v,\mu}(\theta) P_v(a_1^2, ..., a_M^2; \theta) P_\mu(b_1^2, ..., b_M^2; \theta) \tag{2.2.14}$$

where v, μ *are two partitions of length at most* M*.*

Proposition 2.2.26 provides explicit data of the distribution of random singular values $\vec{a} \boxplus^{\theta}_{M,N} \vec{b}$ in terms of moments. It is believed, but not yet proved that $\mathfrak{m}$ (with such moments) is indeed a (symmetric) positive probability measure on $\mathbb{R}^M$. For $\beta = 1, 2, 4$ this holds automatically because the probability measure is constructed explicitly by the matrix structure, (and Roesler 2007, Corollary 4.8 provides an explicit expression of this measure for $N \geq 2M$), while for general $\beta > 0$, the randomness of $\vec{a} \boxplus^{\theta}_{M,N} \vec{b}$ holds and is studied in this text in the weaker sense given by (2.2.14).

2.2.5 Type BC Bessel generating functions

For $\vec{a} = (a_1 \geq ... \geq a_M \geq 0)$, we assume that $\vec{a}$ is random, and its distribution is given by a symmetric generalized function $\mathfrak{m}$, testing on smooth functions and in particular polynomials on $\mathbb{R}^M_{\geq 0}$.

Definition 2.2.27. *Fix $M \leq N, \theta > 0$. Given a compactly supported symmetric generalized function $\mathfrak{m}$ on $\mathbb{R}^M_{\geq 0}$ defined as above, let the Bessel generating function of $\mathfrak{m}$ be a function of $z_1, ..., z_M$ given by*

$$G_{N,\theta}(z_1, ..., z_M; \mathfrak{m}) := \langle \mathfrak{m}, \mathbb{B}(\vec{a}, z_1, ..., z_M; N, \theta)\rangle, \tag{2.2.15}$$

where the bracket denotes testing $\mathfrak{m}$ by $\mathbb{B}(\vec{a}, z_1, ..., z_M; \theta, N)$, in which $\vec{a}$ are the variables and $z_1, ..., z_M$ are parameters.

We also define the Bessel generating function for a class of fast decaying probability measures, for potential applications of our theory (see e.g Section 2.5.5). As preparation, we state a uniform upper bound of multivariate Bessel functions.

Proposition 2.2.28. *For any $\theta > 0$, $M \leq N$, $\vec{a} = (a_1 \geq ... \geq a_M) \in \mathbb{R}^M_{\geq 0}$, $z = (z_1, ..., z_M) \in \mathbb{R}^M$, we have*

$$0 \leq \mathbb{B}(\vec{a}, z_1, ..., z_M; \theta, N) \leq \left[1 + \frac{1}{\theta}\left(\frac{a_1|z|}{2}\right)^2 e^{\frac{a_1|z|}{2}}\right]^M, \tag{2.2.16}$$

and for any $k_1, ..., k_s \in \mathbb{Z}_{\geq 1}$,

$$\left|\left(\prod_{i=1}^{s} P_{2k_i}\right)\mathbb{B}(\vec{a}, z_1, ..., z_M; \theta, N)\right| \leq \prod_{i=1}^{s}\left(\sum_{j=1}^{M} a_i^{2k_i}\right)\left[1 + \frac{1}{\theta}\left(\frac{a_1|z|}{2}\right)^2 e^{\frac{a_1|z|}{2}}\right]^M. \tag{2.2.17}$$

Proof. From Proposition 2.1.4, it's clear that $\mathbb{B}(\vec{a}, z_1, ..., z_M; \theta, N) \geq 0$, and since $P_\mu(1^M; \theta) =$

$\frac{(M\theta)_\mu}{H'(\mu)}$,

$$\begin{aligned}
&\mathbb{B}(\vec{a}, z_1, ..., z_M; \theta, N)\\
=&\sum_\mu \prod_{i=1}^M \frac{\Gamma(\theta N - \theta(i-1))}{\Gamma(\theta N - \theta(i-1)+\mu_i)} \frac{(M\theta)_\mu}{H(\mu)H'(\mu)} 2^{-2|\mu|} \frac{P_\mu(a_1^2, \cdots, a_M^2; \theta) P_\mu(z_1^2, \cdots, z_M^2; \theta)}{P_\mu(1^M; \theta)^2}\\
\leq&\sum_\mu \prod_{i=1}^M \frac{\Gamma(\theta N - \theta(i-1))}{\Gamma(\theta N - \theta(i-1)+\mu_i)} \frac{(M\theta)_\mu}{H(\mu)H'(\mu)} 2^{-2|\mu|} a_1^{2|\mu|} z_1^{2|\mu|}\\
\leq&\sum_\mu \prod_{i=1}^M \left[\frac{\Gamma(\theta N - \theta(i-1))}{\Gamma(\theta N - \theta(i-1)+\mu_i)} \frac{\Gamma(\theta M - \theta(i-1)+\mu_i)}{\Gamma(\theta M - \theta(i-1))}\right]\\
&\qquad\qquad\cdot \frac{1}{\prod_{i=1}^M \mu_i!} \frac{1}{\prod_{i=1}^M \prod_{j=0}^{\mu_i-1}(\theta+j)} 2^{-2|\mu|} a_1^{2|\mu|} z_1^{2|\mu|}\\
\leq&\sum_{\mu_1 \geq \ldots \geq \mu_M \geq 0} \frac{1}{\prod_{i=1}^M \mu_i!} \frac{1}{\prod_{i=1}^M \prod_{j=0}^{\mu_i-1}(\theta+j)} \left(\frac{a_1 z_1}{2}\right)^{2|\mu|}\\
\leq&\prod_{i=1}^M \left(\sum_{\mu_i=0}^\infty \frac{1}{\mu_i! \prod_{j=0}^{\mu_i-1}(\theta+j)} \left(\frac{a_1 z_1}{2}\right)^{2\mu_i}\right) \leq \prod_{i=1}^M \left(1 + \sum_{\mu_i=1}^\infty \frac{1}{\theta} \frac{1}{[(\mu_i-1)!]^2} \left(\frac{a_1|z|}{2}\right)^{2\mu_i}\right)\\
\leq&\prod_{i=1}^M \left(1 + \frac{1}{\theta}\left(\frac{a_1|z|}{2}\right)^2 e^{\frac{a_1|z|}{2}}\right) = \left[1 + \frac{1}{\theta}\left(\frac{a_1|z|}{2}\right)^2 e^{\frac{a_1|z|}{2}}\right]^M.
\end{aligned} \tag{2.2.18}$$

This verifies (2.2.16). (2.2.17) follows from (2.2.16) and Theorem 2.2.24. □

Definition 2.2.29. *We say a measure* $\mathfrak{m}$ *on M-tuples* $a_1 \geq ... \geq a_M \geq 0$ *is* exponentially decaying *with exponent* $R > 0$, *if*

$$\int e^{MRa_1} \mu(da_1, ..., da_M) < \infty.$$

By Proposition 2.2.28 and Definition 2.2.29, the Bessel generating function of $\mathfrak{m}$, where $\mathfrak{m}$ is a compactly supported generalized function or exponentially decaying measure, is well-defined on a domain near 0. Moreover, we will take $\mathfrak{m}$ to be of total mass 1, which means $\langle \mathfrak{m}, 1 \rangle = 1$, where 1 is the constant function 1. So we have

$$G_{N,\theta}(0, ..., 0; \mathfrak{m}) = 1.$$

Now we generalize the addition to random vectors $\vec{a}$ and $\vec{b}$ following Definition 2.1.5.

Definition 2.2.30. *Given $\theta > 0$, $M \leq N$, let $\vec{a} = (a_1 \geq ... \geq a_M \geq 0)$, $\vec{b} = (b_1 \geq ... \geq b_M \geq 0)$ be two random M-tuples whose distribution are given by generalized functions $\mathfrak{m}_a$ and $\mathfrak{m}_b$ on $\mathbb{R}^M_{\geq 0}$. Let $\vec{c}$ be a symmetric random vector in $\mathbb{R}^M_{\geq 0}$ whose distribution is given by generalized function $\mathfrak{m}_c$, such that*

$$G_{N,\theta}(z_1, ..., z_M; \mathfrak{m}_c) = G_{N,\theta}(z_1, ..., z_M; \mathfrak{m}_a) \cdot G_{N,\theta}(z_1, ..., z_M; \mathfrak{m}_b). \tag{2.2.19}$$

We write

$$\vec{c} = \vec{a} \boxplus^{\theta}_{M,N} \vec{b}.$$

Since $\mathbb{B}(z_1, ..., z_M; \theta, N)$ is behaving nice enough in the analytic sense, one can interchange the differentiation over $z_1, ..., z_M$ and the pairing with $\mathfrak{m}$, and therefore Theorem 2.2.24 generalizes to the following.

Theorem 2.2.31. *Let $\mathfrak{m}$ be a symmetric compactly supported generalized function on $\mathbb{R}^M$, or a exponential decaying measure as in Definition 2.2.29 with exponent R. Let $k_1, ..., k_s \in \mathbb{Z}_{\geq 1}$. Then $G_{N,\theta}(z_1, .., z_M; \mathfrak{m})$ is analytic as a function of $(z_1, ..., z_M)$ (in the domain $\{z \in \mathbb{R}^M : |z| < R\}$ in the second case). Moreover,*

$$\left(\prod_{i=1}^{s} \mathrm{P}_{2k_i}\right) G_{N,\theta}(z_1, .., z_M; \mathfrak{m})\Big|_{z_1=...z_M=0} = \Big\langle \mathfrak{m}, \prod_{i=1}^{s} \Big(\sum_{j=1}^{M} (a_j)^{2k_i} \Big) \Big\rangle. \tag{2.2.20}$$

The above properties also hold for

$$G_{N,\theta}(z_1, ..., z_M; \mathfrak{m}_c) = G_{N,\theta}(z_1, ..., z_M; \mathfrak{m}_a) \cdot G_{N,\theta}(z_1, ..., z_M; \mathfrak{m}_b),$$

where $\mathfrak{m}_a$, $\mathfrak{m}_b$ are of the above two types.

Proof. This follows from dominated convergence theorem, where the uniform upper bounds of $\mathbb{B}(\cdot, z_1, ..., z_M; \theta, N)$ and its derivatives are given by Proposition 2.2.28. □

2.3 Concentration in low temperature

In this section, we fix the size of matrices M,N and the input as deterministic input $\vec{a}$, $\vec{b}$, and study the behavior of $\vec{c} = \vec{a} \boxplus^{\theta}_{M,N} \vec{b}$ as $\theta \to \infty$. According to the statistical physic interpretation, when $\theta \to \infty$ the temperature is going down to 0, and hence the random vector $\vec{c}$ will freeze at some deterministic M-tuples.

2.3.1 Finite Law of Large Numbers

Before taking the limit, we consider the expected characteristic polynomial of CC^* for each $\theta < \infty$. It turns out that the expression does not really depend on θ. The following lemma will be used later in the proof. Let $C^{v,\mu}_{\lambda}(\theta)$ be the coefficient defined in 2.2.4.

Lemma 2.3.1. *When* $\lambda = 1^l$, $C^{v,\mu}_{\lambda}(\theta) \neq 0$ *only when* $v = 1^i$, $\mu = 1^j$, *and* $i + j = l$. *Moreover,*

$$C^{1^i,1^j}_{1^l}(\theta) = \frac{\prod_{m=1}^{l}(\frac{l\theta-m\theta+1}{l\theta-m\theta+1})}{\prod_{m=1}^{i}(\frac{i\theta-m\theta+\theta}{i\theta-m\theta+1})\prod_{m=1}^{j}(\frac{j\theta-m\theta+\theta}{j\theta-m\theta+1})}. \tag{2.3.1}$$

Proof. This is studied in Gorin and A. W. Marcus 2020, and for the convenience of the readers we reproduce the proof. Applying the automorphism ω_θ of the algebra of symmetric functions (see Macdonald 1995, Chapter VI, Section 10, which acts on Jack polynomials in the following way:

$$\omega_\theta(P_\lambda(\cdot;\theta)) = Q_{\lambda'}(\cdot;\theta^{-1}), \tag{2.3.2}$$

(2.2.4) becomes

$$Q_{(i,0,\ldots)}(\cdot;\theta^{-1}) \cdot Q_{(j,0,\ldots)}(\cdot;\theta^{-1}) = \sum_{\mu} C^{1^i,1^j}_{\mu}(\theta) \cdot Q_{\mu'}(\cdot;\theta^{-1}). \tag{2.3.3}$$

Recall that $Q_\lambda(\cdot;\theta) = b_\lambda(\theta)P_\lambda(\cdot;\theta) = \frac{H(\lambda)}{H'(\lambda)}P_\lambda(\cdot;\theta)$. By comparing the coefficient of the leading monomial z_1^l, we have

$$C^{1^i,1^j}_{1^l}(\theta) = \frac{b_{1^i}(\theta^{-1})b_{1^j}(\theta^{-1})}{b_{1^l}(\theta^{-1})}, \tag{2.3.4}$$

and $C_{1^l}^{v,\mu} = 0$ if v or μ has more than one column. $\square$

Theorem 2.3.2. *Fix $M \leq N$, given $\vec{a}$ and $\vec{b}$, let $\vec{c} = \vec{a} \boxplus_{M,N}^{\theta} \vec{b}$. Take z as a formal variable, and let*

$$P_{M,N}^{\theta}(z) = \mathbb{E}\prod_{i=1}^{M}(z - c_i^2). \tag{2.3.5}$$

Then the explicit expression of $P_{M,N}^{\theta}(z)$ is θ-independent, and

$$P_{M,N}^{\theta}(z) = P_{M,N}(z)$$

for all $\theta > 0$, where $P_{M,N}(z)$ is defined in (2.1.9).

Proof. Rewrite the product on the right side of (2.3.5) as

$$\prod_{i=1}^{M}(z - c_i^2) = \sum_{l=0}^{M}(-1)^l e_l(c_1^2, ..., c_M^2) z^{M-l},$$

it turns out that $P_{M,N}^{\theta}(z)$ is given by the moments of $\{c_i^2\}_{i=1}^{M}$ only in terms of elementary symmetric polynomials.

Taking the partition $\lambda = (1^j, 0^{M-j})$, $P_\lambda(x;\theta) = e_j(x)$ for any $\theta > 0$. We use Proposition 2.2.26 and it remains to specify the coefficients. From Lemma 2.3.1 we get

$$\frac{H(1^l)}{H(1^i)H(1^j)} C_{1^l}^{1^i,1^j}(\theta) = \frac{H'(1^l)}{H'(1^i)H'(1^j)} = \frac{l!}{i!j!}.$$

Moreover, direct calculation yields

$$\frac{e_l(1^M)}{e_i(1^M)e_j(1^M)} = \frac{i!(M-i)!j!(M-j)!}{M!l!(M-l)!},$$

and when $\lambda = 1^l, v = 1^i, \mu = 1^j$,

$$\frac{\prod_{i=1}^{M}\Gamma(\theta(N-i+1))\Gamma(\theta(N-i+1)+\lambda_i)}{\prod_{i=1}^{M}\Gamma(\theta(N-i+1)+v_i)\Gamma(\theta(N-i+1)+\mu_i)} = \frac{(N-i)!}{N!}\frac{(N-j)!}{(N-l)!}.$$

Combine all these together finishes the proof. $\square$

We highlight the connection of our result with the so-called finite free probability, which was initiated in recent years by Marcus, Spielman and Srivastava and studies convolution of polynomials. Given two polynomials $p(z)=\sum_{i=0}^{M} z^{M-i}a_i, q(z)=\sum_{i=0}^{M} z^{M-i}b_i$ with degree at most M, A. Marcus, Spielman, and Srivastava 2019 defines the rectangular additive convolution for two $M\times M$ matrices, and Gribinski and A. Marcus 2022 generalizes it to arbitrary rectangular matrices, such that the $(M,N)^{th}$ rectangular additive convolution of $p(z)$ and $q(z)$ is defined as

$$p(z)\boxplus\boxplus_M^N q(z)=\sum_{l=0}^{M} z^{M-l}(-1)^l\left(\frac{(M-i)!(M-j)!}{M!(M-l)!}\frac{(N-i)!(N-j)!}{N!(N-l)!}\right)a_ib_j.$$

In Gribinski and A. Marcus 2022, it is shown that taking $p(z)=\chi_z(AA^*)$, $q(z)=\chi_z(BB^*)$, where A and B are two $M\times N$ real/complex matrices, taking $\chi_z(\cdot)$ to be the characteristic polynomial $\det(zI-\cdot)$, and let $U_{M\times M}, V_{N\times N}$ are independent Haar orthogonal/unitary, then $p(z)\boxplus\boxplus_M^N q(z)=\mathbb{E}\chi_z((A+UBV)(A+UBV)^*)$. Theorem 2.3.2 generalizes this operation from $\beta=1,2$ to arbitrary $\beta>0$, with a different approach not relying on the concrete matrix structure. In particular it shows that the rectangular additive convolution is $\beta-$independent.

Our next result is the law of large number of $\vec{c}=\vec{a}\boxplus_{M,N}^{\theta}\vec{b}$ in the regime $\theta\to\infty$. As preparation we state a combinatorial result. Given partitions v,μ,λ such that $v_1,\mu_1\le\lambda_1$, $l(v),\ l(\mu),\ l(\lambda)\le M$, let $\{k_l\}$ be an index set that $l=1,2,...,\lambda_k-\lambda_{k+1}, k=1,2,...,M$, and $\{i_{k_l}\},\{j_{k_l}\}$ be two collections of nonnegative integers. We do not distinguish $\{i_{k_l}\}_{l=1}^{\lambda_k-\lambda_{k+1}}$ with $\{i_{k_{\sigma(l)}}\}_{l=1}^{\lambda_k-\lambda_{k+1}}$, where $\sigma\in S_{\lambda_k-\lambda_{k+1}}$ is an arbitrary permutation, and same for $\{j_{k_l}\}_{l=1}^{\lambda_k-\lambda_{k+1}}$.

Proposition 2.3.3. *Let $C_\lambda^{v,\mu}$ be the coefficient of $m_\lambda(\cdot)$ in the expansion*

$$m_v(\cdot)\cdot m_\mu(\cdot)=\sum_\lambda C_\lambda^{v,\mu}m_\lambda(\cdot).$$

Then

$$C_{\lambda'}^{v',\mu'} = \#\ \textit{ways to choose}\ \{i_{k_l}\}, \{j_{k_l}\}\ \textit{such that for}\ m = 1, 2, ..., M,$$

$$\begin{cases} v_m = \sum_{k=1}^{M} \sum_{l=1}^{\lambda_k - \lambda_{k+1}} I_{i_{k_l} \geq m}; \\ \mu_m = \sum_{k=1}^{M} \sum_{l=1}^{\lambda_k - \lambda_{k+1}} I_{j_{k_l} \geq m}. \end{cases} \tag{2.3.6}$$

Proof. We choose $\{i_{k_l}\}, \{j_{k_l}\}$ in an explicit way. By definition of $C_{\lambda}^{v,\mu}$, we are combining column v'_{l_1} with column μ'_{l_2} to get a column λ'_{l_3}, where l_1, l_2, l_3 are chosen among $1, 2, ..., \lambda_1$, and v'_{l_1}, μ'_{l_2} might be of length 0. Inspired by this, let $\{i_{k_l}\}_{l=1}^{\lambda_k - \lambda_{k+1}}$ be the length of (distinct) columns of v, $\{j_{k_l}\}_{l=1}^{\lambda_k - \lambda_{k+1}}$ be the length of (distinct) columns of μ, which are chosen to contribute to $\lambda'_{\lambda_{k+1}+1}, ..., \lambda'_{\lambda_k}$.

We immediately see that the above way to choose $\{i_{k_l}\}, \{j_{k_l}\}$ satisfy (2.3.6), whose total number is equal to $C_{\lambda'}^{v',\mu'}$. It remains to check each way of choosing $\{i_{k_l}\}, \{j_{k_l}\}$ can be interpreted in this way. Given a sequence of nonnegative integers $\{i_{k_l}\}, \{j_{k_l}\}$ satisfying (2.3.6), we have for $m = 1, 2, ..., M$,

$$\begin{cases} v_m - v_{m+1} = \sum_{k=1}^{M} \sum_{l=1}^{\lambda_k - \lambda_{k+1}} I_{i_{k_l} = m}; \\ \mu_m - \mu_{m+1} = \sum_{k=1}^{M} \sum_{l=1}^{\lambda_k - \lambda_{k+1}} I_{j_{k_l} = m}. \end{cases} \tag{2.3.7}$$

Then one can split $\{i_{k_l}\}_{l=1}^{\lambda_k - \lambda_{k+1}} (k = 1, 2, ..., M)$ into disjoint groups, such that the number of elements in group m is exactly $v_m - v_{m+1}$, which is equal to the number of length m columns in v. Vice versa for $\{j_{k_l}\}_{l=1}^{\lambda_k - \lambda_{k+1}} (k = 1, 2, ..., M)$. □

Proof of Theorem 2.1.6: The weak convergence to a delta function on polynomial test functions is equivalent to the statement that, given any arbitrary collection of polynomials

$f_1, ..., f_n$ of M variables, we have

$$\lim_{\theta\to\infty} \mathbb{E}\prod_{i=1}^{n} f_i(\vec{c^2}) = \lim_{\theta\to\infty} \prod_{i=1}^{n} \Big[\mathbb{E} f_i(\vec{c^2})\Big]. \tag{2.3.8}$$

Since $\vec{c}$ is symmetric in distribution, if suffices to consider symmetric polynomials in Λ_M, which can be generated (in the sense of algebra) by elementary symmetric functions $e_1, ..., e_M$. Since (2.3.8) is multilinear in $f_i's$, we reduce to showing for any positive integers $k_1, ..., k_M$,

$$\lim_{\theta\to\infty} \mathbb{E}\prod_{i=1}^{M} e_i(\vec{c^2})^{k_i} \overset{?}{=} \lim_{\theta\to\theta} \prod_{i=1}^{M} \Big[\mathbb{E} e_i(\vec{c^2}\Big]^{k_i}. \tag{2.3.9}$$

Once we show this, the deterministic limit of $\vec{c}$ will be a M-tuples $\vec{\lambda}$, such that $\mathbb{E}\Big[e_i(\vec{c})\Big] = e_i(\vec{\lambda})$ for all $i = 1, 2, ..., M$. Then Theorem 2.3.2 identifies $\vec{\lambda}$ with roots of $P^{\theta}_{M,N}(z)$.

We connect the left side of (2.3.9) with Jack polynomials, using the following result (Stanley 1989, Proposition 7.6):

$$\lim_{\theta\to\infty} P_\lambda(z_1, ..., z_M; \theta) = \prod_{i=1}^{M} [e_i(z_1, ..., z_M)]^{\lambda_i - \lambda_{i+1}}, \tag{2.3.10}$$

for any partition λ. Then, let $\lambda_i = k_i + ... + k_M$, the left side of (2.3.9) becomes the limit of $\mathbb{E}P_\lambda(\vec{c^2}; \theta)$, since each product of $e_i(\vec{c^2})'s$ has bounded expectation for $\theta > 0$ due to the fact that $\vec{c}$ is bounded supported. Again by Proposition 2.2.26,

$$\begin{aligned}\mathbb{E}P_\lambda(c_1^2, ..., c_M^2; \theta) = \sum_{|v|+|\mu|=|\lambda|} &\frac{H(\lambda)}{H(v)H(\mu)} \frac{\prod_{i=1}^{M} \Gamma(\theta(N-i+1))\Gamma(\theta(N-i+1)+\lambda_i)}{\prod_{i=1}^{M} \Gamma(\theta(N-i+1)+v_i)\Gamma(\theta(N-i+1)+\mu_i)} \\ &\frac{P_\lambda(1^M; \theta)}{P_v(1^M; \theta)P_\mu(1^M; \theta)} C_\lambda^{v,\mu}(\theta) P_v(a_1^2, ..., a_M^2; \theta) P_\mu(b_1^2, ..., b_M^2; \theta).\end{aligned} \tag{2.3.11}$$

Taking $\theta \to \infty$,

$$\frac{\prod_{i=1}^{M} \Gamma(\theta(N-i+1))\Gamma(\theta(N-i+1)+\lambda_i)}{\prod_{i=1}^{M} \Gamma(\theta(N-i+1)+v_i)\Gamma(\theta(N-i+1)+\mu_i)} \longrightarrow \prod_{m=1}^{M} (N-m+1)^{\lambda_m - v_m - \mu_m},$$

and since by definition

$$P_v(\cdot;\theta)P_\mu(\cdot;\theta) = \sum_\lambda C_\lambda^{v,\mu}(\theta) P_\lambda(\cdot;\theta),$$

applying ω_θ on both sides (c.f. the proof of Lemma 2.3.1), and use the fact that (see Stanley 1989, Proposition 7.6)

$$\lim_{\theta\to 0} P_\lambda(z_1,...,z_M;\theta) = m_\lambda(z_1,...,z_M), \tag{2.3.12}$$

we have

$$\begin{aligned} C_\lambda^{v,\mu}(\theta)\frac{H(\lambda)}{H(v)H(\mu)} &\longrightarrow C_{\lambda'}^{v',\mu'} \cdot \frac{\lim_{\theta\to\infty} H'(\lambda)}{\lim_{\theta\to\infty} H'(v)\cdot \lim_{\theta\to\infty} H'(\mu)} \\ &= C_{\lambda'}^{v',\mu'} \cdot \frac{\prod_{s\in\lambda}(l(s)+1)}{\prod_{s\in v}(l(s)+1)\cdot\prod_{s\in\mu}(l(s)+1)}. \end{aligned} \tag{2.3.13}$$

Moreover, applying (2.3.10) again on $\frac{P_\lambda(1^M;\theta)}{P_v(1^M;\theta)P_\mu(1^M;\theta)}$, $P_v(a_1^2,...,a_M^2;\theta)P_\mu(b_1^2,...,b_M^2;\theta)$, the right side of (2.3.11) goes to

$$\begin{aligned} &\sum_{|v|+|\mu|=|\lambda|} C_{\lambda'}^{v',\mu'} \frac{\prod_{s\in\lambda}(l(s)+1)}{\prod_{s\in v}(l(s)+1)\cdot\prod_{s\in\mu}(l(s)+1)} \frac{\prod_{i=1}^M \binom{M}{i}^{\lambda_i-\lambda_{i+1}}}{\prod_{i=1}^M \binom{M}{i}^{v_i-v_{i+1}} \prod_{i=1}^M \binom{M}{i}^{\mu_i-\mu_{i+1}}} \\ &\qquad \prod_{m=1}^M (N-m+1)^{\lambda_i-v_i-\mu_i} \prod_{i=1}^M \left[e_i(a_1^2,...,a_M^2)\right]^{v_i-v_{i+1}} \prod_{i=1}^M \left[e_i(b_1^2,...,b_M^2)\right]^{\mu_i-\mu_{i+1}}. \end{aligned} \tag{2.3.14}$$

On the other hand, by Theorem 2.3.2, the right side of (2.3.9) is equal to

$$\begin{aligned} &\prod_{k=1}^M \left[\mathbb{E}\left[e_i(\vec{c^2})\right]\right]^{\lambda_k-\lambda_{k+1}} \\ =&\prod_{k=1}^M \left[\sum_{i+j=k} \frac{(M-i)!(M-j)!}{M!(M-k)!}\frac{(N-i)!(N-j)!}{N!(N-k)!} e_i(a_1^2,...,a_M^2)e_j(b_1^2,...,e_M^2)\right]^{\lambda_k-\lambda_{k+1}}. \end{aligned} \tag{2.3.15}$$

It remains to check that (2.3.14) is equal to (2.3.15).

We open the bracket in (2.3.15), and identify each term in the sum with a unique collection of nonnegative integer valued indices $\{k_l\}_{l=1}^{\lambda_k-\lambda_{k+1}} (k = 1, 2, ..., M)$, such that $i_{k_l} + j_{k_l} = k$ for each l. Moreover, such term is a multiple of

$$\prod_{i=1}^{M} \left[e_i(a_1^2, ..., a_M^2)\right]^{v_i - v_{i+1}} \prod_{i=1}^{M} \left[e_i(b_1^2, ..., b_M^2)\right]^{\mu_i - \mu_{i+1}},$$

where for $i = 1, 2, ..., M$,

$$\begin{aligned} v_i - v_{i+1} &= \sum_{k=1}^{M} \sum_{l=1}^{\lambda_k-\lambda_{k+1}} I_{i_{k_l}=m}, \\ \mu_i - \mu_{i+1} &= \sum_{k=1}^{M} \sum_{l=1}^{\lambda_k-\lambda_{k+1}} I_{j_{k_l}=m}, \\ \lambda_i - \lambda_{i+1} &= \sum_{k=1}^{M} \sum_{l=1}^{\lambda_k-\lambda_{k+1}} I_{k=m}, \end{aligned} \tag{2.3.16}$$

which matches (2.3.7). Hence it remains to match the coefficients, i.e, to show that

$$\begin{aligned} &\sum_{i_{k_l}+j_{k_l}=k} \prod_{k=1}^{M} \prod_{l=1}^{\lambda_k-\lambda_{k+1}} \left[\frac{(M-i_{k_l})!(M-j_{k_l})!}{M!(M-k)!} \frac{(N-i_{k_l})!(N-j_{k_l})!}{N!(N-k)!}\right] \\ &\overset{?}{=} C_{\lambda'}^{v',\mu'} \frac{\prod_{s\in\lambda}[l(s)+1]}{\prod_{s\in v}[l(s)+1] \cdot \prod_{s\in\mu}[l(s)+1]} \\ &\quad \cdot \frac{\prod_{i=1}^{M} \binom{M}{i}^{\lambda_i-\lambda_{i+1}}}{\prod_{i=1}^{M} \binom{M}{i}^{v_i-v_{i+1}} \prod_{i=1}^{M} \binom{M}{i}^{\mu_i-\mu_{i+1}}} \prod_{m=1}^{M} (N-m+1)^{\lambda_m-v_m-\mu_m}. \end{aligned} \tag{2.3.17}$$

We first rewrite the left side:

$$\frac{(M-i_{k_l})!(M-j_{k_l})!}{(M-k)!M!} = \prod_{m=1}^{M} (M-m+1)^{\sum_{k=1}^{M}(I_{m\le k} - I_{m\le i_{k_l}} - I_{m\le j_{k_l}})}, \tag{2.3.18}$$

hence

$$\begin{aligned}
&\prod_{k=1}^{M} \prod_{l=1}^{\lambda_k-\lambda_{k+1}} \left[\frac{(M-i_{k_l})!(M-j_{k_l})!}{(M-k)!M!} \frac{(N-i_{k_l})!(N-j_{k_l})!}{(N-k)!N!} \right] \\
&= \prod_{m=1}^{M} (M-m+1)^{\sum_{k=1}^{M} \sum_{l=1}^{\lambda_k-\lambda_{k+1}} \left[I_{m\le k} - I_{m\le i_{k_l}} - I_{m\le j_{k_l}} \right]} \\
&\cdot \prod_{m=1}^{M} (N-m+1)^{\sum_{k=1}^{M} \sum_{l=1}^{\lambda_k-\lambda_{k+1}} \left[I_{m\le k} - I_{m\le i_{k_l}} - I_{m\le j_{k_l}} \right]} .
\end{aligned} \tag{2.3.19}$$

On the right side,

$$\prod_{m=1}^{M} (N-m+1)^{\lambda_m - v_m - \mu_m} = \prod_{m=1}^{M} (N-m+1)^{\sum_{k=1}^{M} \sum_{l=1}^{\lambda_k-\lambda_{k+1}} (I_{m\le k} - I_{m\le i_{k_l}} - I_{m\le j_{k_l}})} . \tag{2.3.20}$$

And

$$\binom{M}{i} = \prod_{m=1}^{M} (M-m+1)^{1 - I_{m\ge i+1} - I_{m\ge M-i+1}}, \tag{2.3.21}$$

hence

$$\begin{aligned}
&\frac{\prod_{i=1}^{M} \binom{M}{i}^{\lambda_i-\lambda_{i+1}}}{\prod_{i=1}^{M} \binom{M}{i}^{v_i-v_{i+1}} \prod_{i=1}^{M} \binom{M}{i}^{\mu_i-\mu_{i+1}}} \\
&= \prod_{i=1}^{M} \binom{M}{i}^{\sum_{k=1}^{M} \sum_{l=1}^{\lambda_k-\lambda_{k+1}} \left[I_{k=i} - I_{i_{k_l}=i} - I_{j_{k_l}=i} \right]} \\
&= \prod_{m=1}^{M} (M-m+1)^{\left(\sum_{i=1}^{M} \left[1 - I_{m\ge i+1} - I_{m\ge M-i+1} \right] \right) \left(\sum_{k=1}^{M} \sum_{l=1}^{\lambda_k-\lambda_{k+1}} \left[I_{k=i} - I_{i_{k_l}=i} - I_{j_{k_l}=i} \right] \right)} \\
&= \prod_{m=1}^{M} (M-m+1)^{\sum_{k=1}^{M} \sum_{l=1}^{\lambda_k-\lambda_{k+1}} \left[I_{m\le k} - I_{m\le i_{k_l}} - I_{m\le j_{k_l}} \right] + \left[I_{m\ge M-i_{k_l}+1} + I_{m\ge M-j_{k_l}+1} - I_{m\ge M-k+1} \right]} .
\end{aligned} \tag{2.3.22}$$

Finally,

$$\begin{aligned}
&\prod_{m=1}^{M}(M-m+1)^{\sum_{k=1}^{M}\sum_{l=1}^{\lambda_k-\lambda_{k+1}}\left[-\left(I_{m\geq M-i_{k_l}+1}+I_{m\geq M-j_{k_l}+1}-I_{m\geq M-k+1}\right)\right]}\\
=&\prod_{m=1}^{M}(M-m+1)^{\sum_{k=1}^{M}\sum_{l=1}^{\lambda_k-\lambda_{k+1}}\left[-\left(I_{i_{k_l}\geq M-m+1}-I_{j_{k_l}\geq M-m+1}+I_{k\geq M-m+1}\right)\right]}\\
=&\prod_{m=1}^{M} m^{\sum_{k=1}^{M}\sum_{l=1}^{\lambda_k-\lambda_{k+1}}\left[I_{m\leq k}-I_{m\leq i_{k_l}}-I_{m\leq j_{k_l}}\right]}\\
=&\frac{\prod_{j=1}^{\lambda_1}\lambda_j'!}{\prod_{j=1}^{\lambda_1}v_j'!\prod_{j=1}^{\lambda_1}\mu_j'!}=\frac{\prod_{s\in\lambda}[l(s)+1]}{\prod_{s\in v}[l(s)+1]\prod_{s\in\mu}[l(s)+1]}.
\end{aligned} \tag{2.3.23}$$

(2.3.17) follows from (2.3.20), (2.3.22), (2.3.23) and Proposition 2.3.3. ☐

2.3.2 Gaussian Fluctuation for $1 \times N$ matrix

Take $M = 1$, so that A and B are two $1 \times N$ matrices with singular values $a_1, b_1 \geq 0$, and let $c_1 = a_1 \boxplus_{1,N}^{\theta} b_1$. When taking $\theta \to \infty$, Theorem 2.1.6 shows

$$c_1^2 \longrightarrow \lambda_1^2$$

in moments, where

$$\mathbb{E}c_1^2 = \mathbb{E}e_1(c^2) = a_1^2 + b_1^2 = \lambda_1^2. \tag{2.3.24}$$

Based on this result, we consider further the fluctuation of c_1 around λ_1 in $\theta \to \infty$ regime, which turns out to be a Gaussian random variable under proper rescaling.

Theorem 2.3.4. *For $a_1, b_1 \geq 0$, let $\lambda_1^2 = a_1^2 + b_1^2$, and $c_1 = a_1 \boxplus_{1,N}^{\theta} b_1$. As $\theta \to \infty$, we have:*

$$\sqrt{\theta}(c_1^2 - \lambda_1^2) \xrightarrow{d} Z, \tag{2.3.25}$$

where $Z \sim \mathscr{N}(0, \frac{2}{N}a_1^2 b_1^2)$.

Remark 2.3.5. *We expect the Gaussian fluctuation behavior of $\vec{c} = \vec{a} \boxplus_{M,N}^{\theta} \vec{b}$ when $\theta \to \infty$, for general $M > 1$, and we leave the generalization of Theorem 2.3.4 as an open problem.*

Proof. We first show that the convergence holds in the sense of moments. By Proposition 2.2.26, for all $l = 1, 2, ...$

$$\mathbb{E}c_1^{2l} = \sum_{k_1+k_2=l} \frac{l!}{k_1!k_2!} \frac{\Gamma(\theta N)\Gamma(\theta N + l)}{\Gamma(\theta N + k_1)\Gamma(\theta N + k_2)} a_1^{2k_1} b_1^{2k_2}, \tag{2.3.26}$$

since in (2.2.14) $\lambda = (l, 0, ...)$, $v = (k_1, 0, ...)$, $\mu = (k_2, 0, ...)$ and $C_\lambda^{v,\mu}(\theta) \equiv 1$.

Then for $m \in \mathbb{Z}_{\geq 0}$,

$$\begin{aligned}
&\mathbb{E}(c_1^2 - \lambda_1^2)^m) \\
=&\sum_{k_1+k_2+k=m} \frac{(-1)^k m!}{(k_1+k_2)!k!} \frac{(k_1+k_2)!}{k_1!k_2!} \frac{\Gamma(\theta N)\Gamma(\theta N + k_1 + k_2)}{\Gamma(\theta N + k_1)\Gamma(\theta N + k_2)} a_1^{2k_1} b_1^{2k_2} (a_1^2 + b_1^2)^k \\
=&\sum_{k_1+k_2+k_3+k_4=m} (-1)^k \frac{m!}{k_1!k_2!k_3!k_4!} \frac{\Gamma(\theta N)\Gamma(\theta N + k_1 + k_2)}{\Gamma(\theta N + k_1)\Gamma(\theta N + k_2)} a_1^{2k_1+2k_3} b_1^{2k_2+2k_4}.
\end{aligned}$$

This implies for fixed $l_1 + l_2 = m \geq 0$, coefficient of monomial $a_1^{2l_1} b_1^{2l_2}$ in $\mathbb{E}[\sqrt{\theta}(c_1^2 - \lambda_1^2)]^m$ is

$$\sqrt{\theta}^{l_1+l_2} m! \sum_{k_3=0}^{l_1} \sum_{k_4=0}^{l_2} \frac{(-1)^{k_3}}{(l_1-k_3)!k_3!} \frac{(-1)^{k_4}}{(l_2-k_4)!k_4!} \frac{\Gamma(\theta N)\Gamma(\theta N + l_1 + l_2 - k_3 - k_4)}{\Gamma(\theta N + l_1 - k_3)\Gamma(\theta N + l_2 - k_4)} \tag{2.3.27}$$

$$=\sqrt{\theta}^{l_1+l_2} m! \sum_{k_3=0}^{l_1} \sum_{k_4=0}^{l_2} \frac{(-1)^{l_1-k_3}}{(l_1-k_3)!k_3!} \frac{(-1)^{l_2-k_4}}{(l_2-k_4)!k_4!} \frac{\Gamma(\theta N)\Gamma(\theta N + +k_3 + k_4)}{\Gamma(\theta N + k_3)\Gamma(\theta N + k_4)}. \tag{2.3.28}$$

It remains to match the above expression with moments of Z. We use the following lemma, whose prove is postponed.

Lemma 2.3.6. *For any $l = 1, 2, ...,$ with z as a formal variable,*

(a).

$$\sum_{p=0}^{l} \frac{(-1)^{(l-p)}}{(l-p)!p!} (z+p)(z+p+1)...(z+p+q-1) = 0 \tag{2.3.29}$$

if $q = 0, 1, 2, ..., l-1$.

(b).

$$\sum_{p=0}^{l} \frac{(-1)^{(l-p)}}{(l-p)!p!} (z+p)(z+p+1)...(z+p+q-1) = 1 \tag{2.3.30}$$

if $q = l$.

Without loss of generality, assume that $l_1 \geq l_2$ in (2.3.28), and we rewrite (2.3.28) as

$$\sqrt{\theta}^{l_1+l_2} m! \sum_{k_4=0}^{l_2} \frac{(-1)^{l_2-k_4}}{(l_2-k_4)!k_4!} \frac{\Gamma(\theta N)}{\Gamma(\theta N+k_4)} \left[\sum_{k_3=0}^{l_1} \frac{(-1)^{l_1-k_3}}{(l_1-k_3)!k_3!} (\theta N+k_3)(\theta N+k_3+1)...(\theta N+k_3+k_4-1) \right].$$

By Lemma 2.3.6, the sum in the bracket is nonzero only when $k_4 = l_1$, which implies $l_1 = l_2$, $m = l_1 + l_2 = 2l_1$, and (2.3.28) becomes

$$\sqrt{\theta}^{2l_1} \frac{(2l_1)!}{l_1!} \frac{1}{\theta N(\theta N+1)...(\theta N+l_1-1)}.$$

Therefore, the odd moments of $\sqrt{\theta}(c_1^2 - \lambda_1^2)$ are all zero, and the $2k^{th}$ moment of $\sqrt{\theta}(c_1^2 - \lambda_1^2)$ is equal to

$$\sqrt{\theta}^{2k} \frac{(2k)!}{k!} \frac{1}{\theta N(\theta N+1)...(\theta N+k-1)} a_1^{2k} b_1^{2k},$$

which converges to

$$\frac{(2k)!}{k!} \frac{1}{N^k} a_1^{2k} b_1^{2k} = (2k-1)!!(\frac{2}{N})^k a_1^{2k} b_1^{2k}$$

as $\theta \to \infty$. This coincides with the moments of $Z \sim \mathscr{N}(0, \frac{2}{N} a_1^2 b_1^2)$. By Example 2.2.17 and Marserevic and Pogany 2019, Theorem 2, which states that products of two usual Bessel functions can be written as a convex combination of Bessel functions, we have that for each $\theta > 0$, c_1^2 is supported by a legitimate probability measure μ_θ. The convergence of second moment when $\theta \to \infty$ implies that $\{\mu_\theta\}_{\theta > 0}$ are tight, hence (2.3.25) follows from the moment convergence. □

Proof of Lemma 2.3.6. (a). Expand the polynomial of z. For each coefficient of $z^0, z^1, z^2, ..., z^q$, it can be written as a polynomial of p with degree at most q with integer coefficients, and hence an integral linear combination of $p, p(p-1), p(p-1)(p-2), ..., p(p-1)\cdots(p-q+1)$, so the left side of (2.3.29) becomes an integral linear combination of binomial sums of

$$\sum_{p=q}^{l} \frac{(-1)^{l-p}}{(l-p)!(p-q)!} = (1 + (-1))^{l-q},$$

which equals to 0 when $q \leq l-1$.

(b). Following the same idea as (a), the only nonvanishing term is the coefficient of z^0, which equals to

$$\sum_{p=0}^{l} \frac{(-1)^{l-p}}{(l-p)!p!} p(p-1)\cdots(p-l+1) = \sum_{p=l}^{l} \frac{(-1)^{l-p}}{(l-p)!(p-l)!} = 1. \quad \square$$

2.4 Law of large number in high temperature

In this section, fix two parameters $\gamma > 0, q \geq 1$. we explore the behavior of empirical measures of a $M \times N$ random matrix C, in the regime that taking $M, N \to \infty$, $\theta \to 0$, $M\theta \to \gamma$, $N\theta \to q\gamma$. To simplify the notation, sometimes we only write $M\theta \to \infty, N\theta \to q\gamma$ to denote the same regime.

2.4.1 Main results

Consider M-tuples of real numbers $\vec{c} = (c_1 \geq ... \geq c_M \geq 0)$, which should be thought as singular values of some (virtual) rectangular matrix. Suppose that there is a sequence of random M-tuples $\{\vec{c}_M\}_{M=1}^{\infty}$ where $\vec{c}_M = (c_{M,1} \geq ... \geq c_{M,M} \geq 0)$, and the distribution of $\vec{c}_M$ is given in the sense as in Theorem 2.2.31. Denote its empirical measure by $\mu_M = \frac{1}{M}\sum_{i=1}^{M}(\delta_{c_{M,i}} + \delta_{-c_{M,i}})$.

We set up a condition in terms of moments, that under some mild technical assumption, is equivalent to the weak convergence, in probability, of the random empirical measures $\{\mu_M\}$ to some limiting probability measure μ when $M \to \infty$. The moments of μ are all finite and given by m_k's.

Definition 2.4.1. *(LLN) Let $\{\vec{c}_M\}_{M=1}^{\infty}$ be a sequence of random M-tuples defined as above. For $k = 1, 2, ...,$ denote*

$$p_k^M = \frac{1}{M}\sum_{i=1}^{2M}[c_{M,i}^k + (-c_{M,i})^k].$$

We say $\{\vec{c}_M\}$ satisfies a law of large numbers*, if there exists deterministic real numbers*

$\{m_k\}_{k=1}^{\infty}$ *such that for any s=1,2,... and any* $k_1, ..., k_s \in \mathbb{Z}_{\geq 1}$, *we have*

$$\lim_{M\to\infty} \mathbb{E}\prod_{i=1}^{s} p_{k_i}^{M} = \prod_{i=1}^{s} m_{k_i}. \tag{2.4.1}$$

Denote the Bessel generating function of $\vec{c}_M$ by

$$G_{M,N;\theta}(z_1, ..., z_M) := G_{N,\theta}(z_1, ..., z_M; \mathfrak{m}_{\vec{c}_M}).$$

Recall from the Section 2.2.5 that $G_{M,N;\theta}(0, ..., 0) = 1$, and $G_{M,N;\theta}(z_1, ..., z_M)$ is analytic on a domain near $(0, ..., 0)$. Under these conditions, $\ln(G_{M,N;\theta}(z_1, ..., z_M))$ is analytic near $(0, ..., 0)$, and $\ln(G_{M,N;\theta}(0, ..., 0)) = 0$.

Next, we introduce a condition of the partial derivatives of $\ln(G_{M,N;\theta}(z_1, ..., z_M))$ at 0, as $M \to \infty$.

Definition 2.4.2. *(q-γ-LLN-appropriateness) Given the sequence* $\{\vec{c}_M\}_{M=1}^{\infty}$, *if for a sequence of real numbers* $\{k_l\}_{l=1}^{\infty}$, *the following limits hold:*

(a). $$\lim_{M\theta\to\gamma, N\theta\to q\gamma} \frac{\partial^l}{\partial z_i^l} ln(G_{M,N;\theta})\Big|_{z_1=,...,z_M=0} = (l-1)! \cdot k_l, \text{ for all } l, i \in \mathbb{Z}_{\geq 1}.$$

(b). $$\lim_{M\theta\to\gamma, N\theta\to q\gamma} \frac{\partial}{\partial z_{i_1}} ... \frac{\partial}{\partial z_{i_r}} ln(G_{M,N;\theta})\Big|_{z_1,..,z_M=0} = 0, \text{ for all } r \geq 2, \text{ and } i_1, ..., i_r \in \mathbb{Z}_{\geq 1}$$

such that the set $\{i_1, ..., i_r\}$*is of cardinality at least two.*

We say $\{k_l\}_{l=1}^{\infty}$ *are the limiting q-γ cumulants of* $\{\vec{c}_M\}$.

Remark 2.4.3. *By Proposition 2.2.26,* k_l *are always 0 for all odd l's.*

Remark 2.4.4. *Writing*

$$g^{M,N,\theta}(z) = \frac{\partial}{\partial z} ln(G_{M,N;\theta}(z, 0, ..., 0)) = \sum_{l=1}^{\infty} k_l^{M,N,\theta} z^{l-1},$$

we have

$$k_l^{M,N,\theta} \longrightarrow k_l$$

as $M\theta \to \gamma, N\theta \to q\gamma$.

Our main theorem connects Definition 2.4.1, 2.4.2 and gives a quantitative relation between moments and q-γ cumulants of the limiting empirical measure of $\{\mu_M\}_{M=1}^{\infty}$, which is stated using generating function. Consider $\mathbb{R}[[z]]$, the space of all formal power series of variable z with real coefficients.

Definition 2.4.5. *Let* $a(z)$ *be an element in* $\mathbb{R}[[z]]$. *We define four linear operators acting on* $\mathbb{R}[[z]]$ *to itself, such that for any* $n = 0, 1, 2, ...$

$$(1).\ \partial(z^n) := n \cdot z^{n-1}$$

$$(2).\ d(z^n) := \begin{cases} 0 & n = 0; \\ z^{n-1} & n \geq 1, \end{cases}$$

$$(3).\ d^{'}(z^n) := \begin{cases} 0 & n \ \textit{is}\ \textit{even}; \\ 2z^{n-1} & n\ \textit{is}\ \textit{odd}, \end{cases}$$

$$(4).\ *_a (z^n) := a(z) \cdot z^n.$$

Definition 2.4.6. *Let* $\mathrm{T}^{\mathrm{q},\gamma}_{\mathrm{k}\to\mathrm{m}} : \mathbb{R}^\infty \to \mathbb{R}^\infty$ *be an operation sending a countable sequence* $\{k_l\}_{l=1}^{\infty}$ *to another sequence* $\{m_{2k}\}_{k=1}^{\infty}$, *such that for each* $k = 1, 2, ...$

$$m_{2k} = [z^0]\left(\partial + 2\gamma d + ((q-1)\gamma - \frac{1}{2})d^{'} + *_g\right)^{2k-1} g(z), \tag{2.4.2}$$

where $[z^0]$ *takes the constant term of the formal power series in* $\mathbb{R}[[z]]$, *and*

$$g(z) = \sum_{l=1}^{\infty} k_l z^{l-1}.$$

Remark 2.4.7. *Note by a simple induction on* $k = 1, 2, ...$ *that (2.4.2) implies each* m_{2k} *is given by*

a positive constant time k_{2k} + *a polynomial of* $k_2, k_4, ..., k_{2k-2}$.

Hence, $\mathrm{T}^{\mathrm{q},\gamma}_{\mathrm{k}\to\mathrm{m}}$ *is an invertible map, such that given a sequence of real numbers* $\{m_{2k}\}_{k=1}^{\infty}$*, there exists a unique real sequence* $\{k_l\}_{l=1}^{\infty}$ *with* $k_l = 0$ *for all odd* l*'s, and* $\mathrm{T}^{q,\gamma}_{k\to m}(\{k_l\}_{l=1}^{\infty}) = \{m_{2k}\}_{k=1}^{\infty}$*. More precisely,* $\{m_{2j}\}_{j=1}^{k}$ *are corresponding to* $\{k_l\}_{l=1}^{2k}$*. We denote the inverse map by*

$$\{k_l\} = \mathrm{T}^{\mathrm{q},\gamma}_{\mathrm{m}\to\mathrm{k}}(\{m_{2k}\}).$$

In Section 2.5 we provide various points of views on the maps $\mathrm{T}^{q,\gamma}_{k\to m}$ *and* $\mathrm{T}^{q,\gamma}_{m\to k}$*.*

We are ready to present the main result now.

Theorem 2.4.8. *(Convergence of empirical measure in high temperature) The sequence of random M-tuples* $\{\vec{c}_M\}_{M=1}^{\infty}$ *satisfies LLN, if and only if it is q-*γ*-LLN-appropriate.*

If this occurs, we have

$$\{m_{2k}\}_{k=1}^{\infty} = \mathrm{T}^{q,\gamma}_{k\to m}(\{k_l\}_{l=1}^{\infty}), \tag{2.4.3}$$

where $\{k_l\}_{l=1}^{\infty}$ *are the q-*γ *cumulants corresponding to* $\{m_{2k}\}_{k=1}^{\infty}$*.*

2.4.2 Asymptotic expression under Dunkl actions

The proof of Theorem 2.4.8 is relying on the actions of Dunkl operator introduced in Section 2.2.3 on Bessel generating functions. Before proceeding to the proof, we first study the explicit expression of this action in detail.

Consider a symmetric function $F(z_1, .., z_M)$ which is analytic on a complex domain near 0. Then the Talor expansion of F of k^{th} order is

$$F(z_1, ..., z_M) = \sum_{\lambda:|\lambda|\le k,\ l(\lambda)\le M} c_F^{\lambda} \cdot m_{\lambda}(\vec{z}) + O(||z||^{k+1}), \tag{2.4.4}$$

where $m_\lambda(\vec{z})$ is the monomial symmetric polynomial indexed by λ. If we further assume F to be a symmetric function in $z_1^2, ..., z_M^2$, then

$$c_F^{\lambda} \text{ is nonzero only if } \lambda \text{ is even.} \tag{2.4.5}$$

Fix $M \geq 1$. Recall we denote

$$P_k = D_1^k + ... + D_M^k,$$

where D_i is defined in Section 2.2.3.

The following theorem is a technical result on the explicit expansion of $\exp(F(z_1, ..., z_M))$ under the action of P_k's, and it serves as a stepping stone to the proof of Theorem 2.4.8.

Theorem 2.4.9. *Fix $k = 2, 4, ...$ and a even partition λ and $|\lambda| = 2k$. Let $F(z_1, ..., z_M)$ be a symmetric function on $\mathbb{R}^M$ satisfying (2.4.5), analytic on a domain near $(0, ..., 0)$ and $F(0, ..., 0) = 0$. Then*

$$\begin{aligned} M^{-l(\lambda)} \left[\prod_{i=1}^{l(\lambda)} P_{\lambda_i}\right] \exp(F(z_1, ..., z_M)\Big|_{z_1 = ... z_M = 0} &= b_\lambda^\lambda \cdot c_F^\lambda + \sum_{\mu : |\mu| = k, l(\mu) > l(\lambda)} b_\mu^\lambda \cdot c_F^\mu \\ &+ L(c_F^{(i)}, 1 \leq i \leq 2k-1) + R_1(c_F^v, |v| < 2k) + M^{-1} R_2(c_F^v, |v| \leq 2k), \end{aligned} \tag{2.4.6}$$

where b_μ^λ are coefficients that are uniformly bounded in the limit regime $M \to \infty, N \to \infty, \theta \to 0, M\theta \to \gamma, N\theta \to q\gamma$, and the notation (i) denotes the partition $(i, 0, ..., 0)$. In particular,

$$\begin{aligned} \lim_{M\theta \to, N\theta \to q\gamma} b_\lambda^\lambda = \prod_{i=1}^{l(\lambda)} \Big[&\lambda_i(\lambda_i - 2 + 2q\gamma)(\lambda_i - 2 + 2q\gamma)(\lambda_i - 4 - 2q\gamma)(\lambda_i - 4 + 2\gamma) \\ &...(2 + 2q\gamma)(2 + 2\gamma)2q\gamma\Big], \end{aligned} \tag{2.4.7}$$

and

$$\begin{aligned} L(c_F^{(i)}, 1 \leq i \leq 2k-1) = &\prod_{i=1}^{l(\lambda)} \left([z^0](\partial + 2\gamma d + ((q-1)\gamma - \frac{1}{2})d' + *_g)^{\lambda_i - 1} g(z) \right) \\ &-2k(2k - 2 + 2q\gamma)(2k - 2 + 2\gamma)(2k - 4 - 2q\gamma)(2k - 4 + 2\gamma) \\ &...(2 + 2q\gamma)(2 + 2\gamma)2q\gamma \cdot c_F^{(k)} \cdot \mathrm{I}_{l(\lambda) = 1}. \end{aligned} \tag{2.4.8}$$

The operator $\partial, d, d^{'}$ and $*_g$ are defined in the same way as in Definition 2.4.6, and

$$g(z) := \sum_{i=1}^{\infty} nc_F^{(n)} z^{n-1}.$$

Here, L, R_1 and R_2 are all polynomials of c_F^v's, whose corresponding variables are given in the parenthesis, and the coefficient of each monomial is uniformly bounded in the limit regime. Moreover, each monomial in R_1 contains at least one C_F^v where $l(v) \geq 2$. If we assign c_F^v with degree $|v|$, each summand on the right of (2.4.6) is homogeneous of degree k.

We postpone the proof of Theorem 2.4.9 to next section, and using its result, we are able to prove Theorem 2.4.8.

Proof of Theorem 2.4.8. We first assume the sequence $\{\vec{c}_M\}_{M=1}^{\infty}$ is q-γ-appropriate, with limiting q-γ-cumulants $\{k_l\}_{l=1}^{\infty}$. We need to show $\{\vec{c}_M\}_{M=1}^{\infty}$ is satisfying LLN with moments $\{m_{2k}\}_{k=1}^{\infty} = \mathrm{T}_{k\to m}^{q,\gamma}(\{k_l\}_{l=1}^{\infty})$.

Denote the type-BC Bessel generating function of $\vec{c}_M$ by $G_{M,N;\theta}(z_1, ..., z_M)$. By Theorem 2.2.31, the left side of (2.4.1) before taking the limit is given by

$$M^{-s}\left(\prod_{i=1}^{s} P_{2k_i}\right) G_{M,N;\theta}(z_1, ..., z_M)\Big|_{z_1=...=z_M=0}. \tag{2.4.9}$$

For each $M = 1, 2, ...$, without loss of generality assume $k_1 \geq k_2 \geq ... \geq k_s$ and identify $(2k_1, ..., 2k_s)$ with a partition λ. Also since $G_{M,N;\theta}$ is analytic on a domain near 0 and $G_{M,N\theta}(0, ..., 0) = 1$, there is a function $F_{M,N;\theta}(z_1, ..., z_M)$ analytic near 0 and $\exp(F_{M,N;\theta}(z_1, ..., z_M)) = G_{M,N;\theta}(z_1, ..., z_M)$, $F_{M,N;\theta}(0, ..., 0) = 0$. We write $F_{M,N;\theta}$ in terms of its k^{th} order Talor polynomial

$$F_{M,N;\theta}(z_1, ..., z_M) = \sum_{\mu:|\mu|\leq k,\ l(\mu)\leq M} c_{F_{M,N;\theta}}^{\lambda} \cdot m_\mu(\vec{z}) + O(||z^{k+1}||).$$

After the above identifications (2.4.9) satisfies the condition of Theorem 2.4.9. Then we turn it into the expression on the right of (2.4.6), and take the limit $M\theta \to \infty$,$N\theta \to q\gamma$.

By q-γ-appropriateness,

$$\lim_{M\theta\to\infty,\ N\theta\to q\gamma} c^{(n)}_{F_{M,N;\theta}} = \frac{k_n}{n}, \quad \lim_{M\theta\to\infty,\ N\theta\to q\gamma} c^{(n)}_{F_{M,N;\theta}} = 0, \text{ if } l(\mu) > 1.$$

Hence $\sum_{\mu:|\mu|=k, l(\mu)>l(\lambda)} b^{\lambda}_{\mu} \cdot c^{v}_{F_{M,N;\theta}}$ turns to 0, since each summand contains some term converging to 0, and

$$b^{\lambda}_{\lambda} \cdot c^{\lambda}_{F_{M,N;\theta}} \longrightarrow \begin{cases} 0 & \text{if } s > 1 \\ (2k_1 - 2 + 2q\gamma)(2k_1 - 2 + 2\gamma)(2k_1 - 4 - 2q\gamma)(2k_1 - 4 + 2\gamma) & \\ ...(2 + 2q\gamma)(2 + 2\gamma)2q\gamma \cdot c^{(2k_1)}_{F} k_{2k_1} & \text{if } s = 1. \end{cases}$$

The polynomial L converges to

$$\prod_{i=1}^{s} \left([z^0](\partial + 2\gamma d + ((q-1)\gamma - \frac{1}{2})d' + *_g)^{2k_i - 1} g(z) \right)$$
$$-(2k_1 - 2 + 2q\gamma)(2k_1 - 2 + 2\gamma)(2k_1 - 4 - 2q\gamma)(2k_1 - 4 + 2\gamma)...(2 + 2q\gamma)(2 + 2\gamma)2q\gamma \cdot k_{2k_1} \cdot \mathrm{I}_{s=1},$$

where $g(z) = \sum_{n=1} k_n z^{n-1}$, since $\sum_{n=1}^{\infty} n c^{(n)}_{F_{M,N;\theta}} z^{n-1}$ converges coefficient-wise to $g(z)$.

The polynomial R_1 converges to 0, also because each summand contains some factor $c^{v}_{F_{M,N;\theta}}$ with $l(v) > 1$, that vanishes in the limit regime. The polynomial $M^{-1}R_2$ vanishes as well in the limit since all its coefficients converge to 0. Combining all the results above gives

$$\lim_{M\to\infty} \mathbb{E} \prod_{i=1}^{s} p^{M}_{2k_i} = \prod_{i=1}^{s} \left([z^0](\partial + 2\gamma d + ((q-1)\gamma - \frac{1}{2})d' + *_g)^{2k_i - 1} g(z) \right),$$

which is equal to $\prod_{i=1}^{s} m_{2k_i}$ that $\{m_{2k}\}_{k=1}^{\infty} = \mathrm{T}^{q,\gamma}_{k\to m}(\{k_l\}_{l=1}^{\infty})$. Hence the LLN condition of $\{\vec{c}_M\}_{M=1}^{\infty}$ is proved.

Now we go in the opposite direction, that assuming $\{\vec{c}_M\}_{M=1}^{\infty}$ satisfies LLN for some

$\{m_k\}_{k=1}^{\infty}$, i.e, for all even partition λ,

$$M^{-l(\lambda)}\left[\prod_{i=1}^{l(\lambda)} P_{\lambda_i}\right] \exp\Big(F_{M,N;\theta}(z_1,...,z_M)\Big)\Big|_{z_1=...=z_M=0} \xrightarrow[M\theta\to\infty]{N\theta\to q\gamma} \prod_{i=1}^{l(\lambda)} m_{\lambda_i},$$

where $F_{M,N;\theta}(z_1,...,z_M)=\sum_{\mu:l(\mu)\leq M} c^{v}_{F_{M,N;\theta}}\cdot m_\mu(\vec{z})$ is an analytic function near 0 satisfying $\exp(F_{M,N;\theta})=G_{M,N;\theta}$. We need to show:

$$c^{\lambda}_{F_{M,N;\theta}} \longrightarrow \begin{cases} 0 & l(\lambda)>1 \text{ or } \lambda \text{ is not even} \\ \dfrac{k_{2k}}{2k} & \lambda=(2k) \end{cases} \tag{2.4.10}$$

in the limit regime $M\theta\to\gamma, N\theta\to q\gamma$, where $\{k_l\}_{l=1}^{\infty}=\mathrm{T}^{\mathrm{q},\gamma}_{\mathrm{m}\to\mathrm{k}}(\{m_{2k}\}_{k=1}^{\infty})$.

Note that we only need to consider the case $|\lambda|$ is even, and $c^{v}_{F_{M,N;\theta}}=0$ for all v not even, since the type BC Bessel function of each $\vec{c}_M$ is a symmetric function in $z_1^2,...,z_M^2$, by Definition 2.2.27 and Proposition 2.2.26. We proceed by induction on $|\lambda|$.

For $|\lambda|=0$ there's nothing to show. Suppose the result holds for all $|\lambda|\leq 2k-2$, we now consider the partition that $|\lambda|=2k$. By Theorem 2.4.9, for each M,N,θ, we have a (finite) system of linear equations of

$$\{c^{\mu}_{F_{M,N;\theta}}\}_{v:\ |v|=2k,\ l(v)>l(\lambda),\ v\text{ is even}}.$$

$$\begin{aligned} b^{\lambda}_{\lambda}\cdot c^{\lambda}_{F_{M,N;\theta}}+&\sum_{v:\ |v|=2k,\ l(v)>l(\lambda),\ \mu\text{ is even}} b^{\lambda}_{v}\cdot c^{v}_{F_{M,N;\theta}} \\ &=M^{-l(\lambda)}\left[\prod_{i=1}^{l(\lambda)} P_{\lambda_i}\right]\exp\Big(F_{M,N;\theta}(z_1,...,z_M)\Big)\Big|_{z_1=...z_M=0}-L(c^{(i)}_{F_{M,N;\theta}},1\leq i\leq 2k-1) \\ &\qquad -R_1(c^{v}_{F_{M,N;\theta}},|v|<2k)-M^{-1}R_2(c^{v}_{F_{M,N;\theta}},|v|\leq 2k). \end{aligned} \tag{2.4.11}$$

We observe that if we write it in the matrix form in the lexicographical order of v's introduced in Section 2.2.1, the above system is upper triangular, and again by Theorem 2.4.9, its diagonal entries b^{μ}_{μ}'s all converge to some nonzero constant in the limit regime, and the off-diagonal entries are uniformly bounded. Hence the matrix is invertible asymptotically, and its inverse has uniformly bounded entries.

Claim: If $\lambda \neq (2k)$, the right side of (2.4.11) converges to 0 in the limit regime.

Proof of the claim: $R_1 \to 0$ by induction hypothesis (recall that each of its term involves some partition v with $l(v) \geq 2$), and $R_2 \to 0$ since the coefficients all vanish in the limit.

By the LLN condition,

$$M^{-l(\lambda)} \left[\prod_{i=1}^{l(\lambda)} P_{\lambda_i}\right] \exp(F(z_1, ..., z_M)\Big|_{z_1=...z_M=0} \longrightarrow \prod_{i=1}^{l(\lambda)} m_{\lambda_i},$$

and by Theorem 2.4.9 and Definition 2.4.6, when $\lambda \neq (2k)$, each $\lambda_i < 2k$ and

$$L(c^{(i)}_{F_{M,N;\theta}}, 1 \leq i \leq 2k-1) = \prod_{i=1}^{l(\lambda)} m^{M,N;\theta}_{\lambda_i},$$

where $\{m^{M,N;\theta}_k\}_{k=1}^{\infty} = \mathrm{T}^{q,\gamma}_{k\to m}(\{l \cdot c^{(l)}_{F_{M,N;\theta}}\}_{l=1}^{\infty})$. By induction hypothesis $\{l \cdot c^{(l)}_{F_{M,N;\theta}}\}_{l=1}^{\infty} \longrightarrow \{k_l\}_{l=1}^{\infty}$ pointwisely for $l < 2k$, and hence $m^{M,N;\theta}_j \to m_j$ pointwisely for $j < k$, and

$$L(c^{(i)}_{F_{M,N;\theta}}, 1 \leq i \leq 2k-1) \longrightarrow \prod_{i=1}^{l(\lambda)} m_{\lambda_i}$$

as well. □

Because of this claim, we conclude that when $M\theta \to \gamma, N\theta \to q\gamma$, the solutions of the linear system converge to the zero vector, in particular,

$$c^{\lambda}_{F_{M,N;\theta}} \longrightarrow 0 \text{ for all } |\lambda| = 2k, \lambda \neq (2k). \tag{2.4.12}$$

It remains to consider $\lambda = (2k)$. This time we write down a single identity

$$\begin{aligned}b_{(2k)}^{(2k)} \cdot c_{F_{M,N;\theta}}^{(2k)} + &\sum_{v:\ |v|=2k,\ l(v)>1,\ \mu \text{ is even}} b_v^{(2k)} \cdot c_{F_{M,N;\theta}}^{v} \\ &= M^{-1} P_{2k} \left[\exp(F_{M,N;\theta}(z_1, ..., z_M))\right]\Big|_{z_1=...z_M=0} - L(c_{F_{M,N;\theta}}^{(i)}, 1 \le i \le 2k-1) \\ &\quad - R_1(c_{F_{M,N;\theta}}^{v}, |v| < 2k) - M^{-1} R_2(c_{F_{M,N;\theta}}^{v}, |v| \le 2k).\end{aligned} \tag{2.4.13}$$

We have that

$$\text{LHS} = b_{(2k)}^{(2k)} \cdot c_{F_{M,N;\theta}}^{(2k)} + o(1),$$

where $g_{M,N;\theta}(z) = \sum_{n=1}^{\infty} n c_{F_{M,N;\theta}}^{(n)} z^{n-1}$, because of Theorem 2.4.9 and (2.4.12). And

$$\text{RHS} = m_{2k} - [z^0]\Big(\partial + 2\gamma d + ((q-1)\gamma - \frac{1}{2})d' + *_{g_{M,N;\theta}}\Big)^{2k-1} g_{M,N;\theta}(z) + b_{(2k)}^{(2k)} \cdot c_{F_{M,N;\theta}}^{(2k)} + o(1)$$

because of Theorem 2.4.9 and the LLN assumption. Hence, when $M\theta \to \gamma, N\theta \to q\gamma$,

$$[z^0]\Big(\partial + 2\gamma d + ((q-1)\gamma - \frac{1}{2})d' + *_{g_{M,N;\theta}}\Big)^{2k-1} g_{M,N;\theta}(z) \longrightarrow m_{2k}.$$

By Definition 2.4.6, the invertibility of $\mathrm{T}_{k\to m}^{q,\gamma}$ and the induction hypothesis, this is equivalent to

$$(2k) \cdot c_{F_{M,N;\theta}}^{(2k)} \longrightarrow k_{2k}$$

in the limit regime, that k_{2k} is in the image of $\mathrm{T}_{m\to k}^{q,\gamma}(\{m_{2j}\}_{j=1}^{\infty})$. This finishes the induction step and therefore the proof. □

2.4.3 Proof of Theorem 2.4.9

We start by reducing $F(z_1, ..., z_M)$ from a (locally) analytic function to its $2k^{th}$ Talor polynomial.

Lemma 2.4.10. *For $F(z_1, ..., z_M)$ of the form (2.4.4), denote $F'(z_1, ..., z_M) = \sum_{\lambda:|\lambda|\le 2k,\ l(\lambda)\le M} c_F^{\lambda}\cdot$*

$m_\lambda(\vec{z})$. Then for a partition λ with $|\lambda| = 2k$, we have

$$\left[\prod_{i=1}^{l(\lambda)} P_{\lambda_i}\right] \exp(F^{'}(z_1, ..., z_M))\Big|_{z_1=...=z_M=0} = \left[\prod_{i=1}^{l(\lambda)} P_{\lambda_i}\right] \exp(F(z_1, ..., z_M))\Big|_{z_1,...,z_M=0}.$$

Proof. Since F is analytic near 0, write $\exp(F(z_1, ..., z_M))$ and $\exp(F^{'}(z_1, ..., z_M))$ as symmetric power series. Their difference $R(\vec{z})$ is a power series of order $O(||z||^{k+1})$. Since $\prod_{i=1}^{l(\lambda)} P_{\lambda_i}$ is a homogeneous polynomial of D_i's and each D_i reduces the total power of a monomial by 1,

$$\left[\prod_{i=1}^{l(\lambda)} P_{\lambda_i}\right] R(\vec{z}) = 0. \quad \square$$

By Lemma 2.4.10, in the remaining of this section we take

$$F(z_1, ..., z_M) = \sum_{\lambda:|\lambda|\leq 2k,\ l(\lambda)\leq M, \lambda \text{ even}} c_F^\lambda \cdot m_\lambda(\vec{z}).$$

$\prod_{i=1}^{l(\lambda)} P_{\lambda_i}$ is a sum of products of D_i's ($i = 1, 2, ..., M$). For each product of the form $D_1^{n_1}...D_M^{n_M}$ acting on $\exp(F(z_1, ..., z_M))$, by (2.2.22) the order does not matter. Recall

$$D_i = \partial_i + \left[\theta(N - M + 1) - \frac{1}{2}\right] \frac{1 - \sigma_i}{z_i} + \theta \sum_{j\neq i} \left[\frac{1 - \sigma_{ij}}{z_i - z_j} + \frac{1 - \tau_{ij}}{z_i + z_j}\right].$$

Observe that $D_i[\exp(F(z_1, ..., z_M))]$ is of the form $H(z_1, ..., z_M) \exp(F(z_1, ..., z_M))$, where $H(z_1, ..., z_M)$ is a polynomial of $z_1, ..., z_M$, and for any i, $D_i[H(z_1, ..., z_M) \exp(F(z_1, ..., z_M))]$ is still $\exp(F(z_1, ..., z_M))$ multiplied by a polynomial. More precisely,

$$\begin{aligned}&\partial_i[H(z_1, ..., z_M) \exp(F(z_1, ..., z_M))] \\ &\quad = \Big(\partial_i H(z_1, ..., z_M) + H(z_1, ..., z_M)\partial_i F(z_1, ..., z_M)\Big) \cdot \exp(F(z_1, ..., z_M)),\end{aligned} \tag{2.4.14}$$

$$\begin{aligned}&\frac{1 - \sigma_i}{z_i}[H(z_1, ..., z_M) \exp(F(z_1, ..., z_M))] \\ &\quad = \Big(\frac{1 - \sigma_i}{z_i} H(z_1, ..., z_M)\Big) \cdot \exp(F(z_1, ..., z_M)),\end{aligned} \tag{2.4.15}$$

$$\begin{aligned}\frac{1-\sigma_{ij}}{z_i-z_j}&[H(z_1,...,z_M)\exp(F(z_1,...,z_M))]\\&=\Big(\frac{1-\sigma_{ij}}{z_i-z_j}H(z_1,...,z_M)\Big)\cdot\exp(F(z_1,...,z_M)),\end{aligned}\tag{2.4.16}$$

$$\begin{aligned}\frac{1-\tau_{ij}}{z_i+z_j}&[H(z_1,...,z_M)\exp(F(z_1,...,z_M))]\\&=\Big(\frac{1-\tau_{ij}}{z_i+z_j}H(z_1,...,z_M)\Big)\cdot\exp(F(z_1,...,z_M)).\end{aligned}\tag{2.4.17}$$

We see that $\prod_{i=1}^{l(\lambda)}[D_1^{n_1}...D_M^{n_M}]\exp(F(z_1,...,z_M))\Big|_{z_1,...,z_M=0}$ is obtained by acting a polynomial of ∂_i, $\frac{1-\sigma_i}{z_i}$, $\frac{1-\sigma_{ij}}{z_i-z_j}$, $\frac{1-\tau_{ij}}{z_i+z_j}$ on $F(z_1,...,z_M)$, then take the constant term. Then we have the following basic observation.

Proposition 2.4.11. *For any M-tuples of nonnegative integers $n_1,...,n_M$,*

$$\Big[D_1^{n_1}\cdots D_M^{n_M}\Big]\exp(F(z_1,...,z_M))\Big|_{z_1,...,z_M=0}\tag{2.4.18}$$

is a homogeneous polynomial in c_F^v's of degree $\sum_{i=1}^M n_i$, if taking c_F^v to be of degree $|v|$. Moreover, the coefficients of this polynomial are all uniformly bounded in the limit regime $M\theta\to\gamma$, $N\theta\to q\gamma$.

Proof. Each of ∂_i, $\frac{1-\sigma_i}{z_i}$, $\frac{1-\sigma_{ij}}{z_i-z_j}$, $\frac{1-\tau_{ij}}{z_i+z_j}$ reduces the degree of a monomial by 1, the constant term of $\Big[D_1^{n_1}...D_M^{n_M}\Big]\exp(F(z_1,...,z_M))$ is then obtained from some monomials of $z_1,...,z_M$ of degree $\sum_{i=1}^M n_i$. Since c_F^v is the coefficient of $m_v(\vec{z})$ which is of degree $|v|$, by assigning c_F^v with degree $|v|$ one can pass the degree of the original monomials to their resulting constant terms.

Each D_i is a sum of 2M single operators ∂_i, $\frac{1-\sigma_i}{z_i}$, $\frac{1-\sigma_{ij}}{z_i-z_j}$ and $\frac{1-\tau_{ij}}{z_i+z_j}$, in which $2M-2$ terms involves a factor θ, and hence $D_1^{n_1}...D_M^{n_M}$ is a sum of $2M^{\sum_{i=1}^M n_i}$ products of single operators. The constant term of each of these products acting on $\exp(F(z_1,...,z_M))$ is changing with M,N,θ as a muliple of $\theta^{\sum_{i=1}^M n_i-\#\partial_i\text{'s in the product}}$, and the number of such products is of order $O(M^{\sum_{i=1}^M n_i-\#\partial_i\text{'s in the product}})$. Hence as $M\theta\to\gamma$ the coefficient is uniformly bounded. ☐

Proposition 2.4.12. *For a partition λ, we have*

$$\begin{aligned} &M^{-l(\lambda)}\left[\prod_{i=1}^{l(\lambda)} P_{\lambda_i}\right]\exp(F(z_1,...,z_M))\Big|_{z_1,...,z_M=0} \\ &\qquad = \left[\prod_{i=1}^{l(\lambda)}(D_i)^{\lambda_i}\right]\exp(F(z_1,...,z_M))\Big|_{z_1,...,z_M=0} + O(\frac{1}{M}), \end{aligned} \tag{2.4.19}$$

in the limit regime $M\theta \to \gamma, N\theta \to q\gamma$, where $O(\frac{1}{M})$ is a homogeneous polynomial of c_F^v's (taking c_F^v to be of degree $|v|$) whose coefficients are of order $O(\frac{1}{M})$.

Proof. Each P_{λ_i} is a sum of M terms $D_j^{\lambda_i} (j = 1, 2, ..., M)$, hence $\prod_{i=1}^{l(\lambda)} P_{\lambda_i}$ is a sum of $M^{l(\lambda)}$ such terms, in which $O(M^{l(\lambda)-1})$ terms have not all distinct indices j's. By Proposition 2.4.11 each of these terms has uniformly bounded coefficient, hence they together contribute $O(\frac{1}{M})$. As for the remaining terms with all distinct indices, by symmetry of $F(z_1, ..., z_M)$, their action are all the same as $\prod_{i=1}^{l(\lambda)}(D_i)^{\lambda_i}$. □

After all the reductions above, it remains to study

$$\left[\prod_{i=1}^{l(\lambda)}(D_i)^{\lambda_i}\right]\exp(F(z_1,...,z_M))\Big|_{z_1,...,z_M=0}$$

for an arbitrary partition λ, whose expression should match the right side of (2.4.6). The expression on the right side of (2.4.6) can be splitted into three parts: the linear polynomials of c_F^v's, the terms involving only c_F^v's where v are length 1 partitions, and all the other remaining terms. In the next two Propositions, we deal with the first two cases separately. Before that we present several lemmas that will be used in the proof. Consider the action of $\prod_{i=1}^{l(\lambda)} D_i^{\lambda_i}$ on $m_\mu(\vec{z})$. Each D_i is a combination of $\partial_i + [\theta(N - M + 1) - \frac{1}{2}]\frac{1-\sigma_i}{z_i}$ and $\theta\left[\frac{1-\sigma_{ij}}{z_i-z_j} + \frac{1-\tau_{ij}}{z_i+z_j}\right]$ with $M - 1$ choices of $j \neq i$, hence $\prod_{i=1}^{l(\lambda)} D_i^{\lambda_i}$ will lead to a big sum, whose summands are products of these two terms.

Lemma 2.4.13. *For arbitrary partitions λ and μ, the constant term of $\prod_{i=1}^{l(\lambda)} D_i^{\lambda_i} m_\mu(\vec{z})$ has a generic part, which is contributed by the summands of $\prod_{i=1}^{l(\lambda)} D_i^{\lambda_i}$ in which all the* *indices j are distinct, and all bigger than $l(\lambda)$. The remaining part is of order $O(\frac{1}{M})$ in the limit regime $M\theta \to \gamma, N\theta \to q\gamma$.*

Proof. For $k = 0, 1, ..., |\lambda|$, the number of summands in the remaining part (which means their exists a pair of indices that coincides) with k components of $\theta\left[\frac{1-\sigma_{ij}}{z_i-z_j} + \frac{1-\tau_{ij}}{z_i+z_j}\right]$ is of order $O(M^{k-1})$, and the power of θ in these summands is k. Since $M\theta \to \gamma > 0$, the remaining part is a finite sum of order $O(M^{k-1}\theta^k)$, which is $O(\frac{1}{M})$. □

Because of this, we only consider the limit of the generic part of the expression. For simplicity we write $l = l(\lambda)$.

Lemma 2.4.14. *The generic part of constant term of $\prod_{i=1}^{l} D_i^{\lambda_i} m_\mu(\vec{z})$ is given by*

$$\left[D_l^{\lambda_l-1}\partial_l\right]\cdots\left[D_2^{\lambda_2-1}\partial_2\right]\cdot\left[D_1^{\lambda_1-1}\partial_1\right]m_\mu(\vec{z}). \tag{2.4.20}$$

Proof. For $m = 1$, because of the symmetry, $m_\mu(\vec{z})$ is invariant under the action of σ_i, σ_{ij} and τ_{ij} and hence $D_i m_\mu(\vec{z})$ is equal to $\partial_i m_\mu(\vec{z})$.

For $m = 2, 3, ..., l$, after acting $\left[D_{m-1}^{\lambda_{m-1}-1}\partial_m\right]\cdots\left[D_1^{\lambda_1-1}\partial_1\right]$ on $m_\mu(\vec{z})$, since the generic part has distinct indices j's bigger than $l(\lambda)$, we get some polynomial $H(z_1, ..., z_M)$, where the operators act on variables $z_1, ..., z_{m-1}$ and z_j's ($j > l$). Hence $H(z_1, ..., z_M)$ is still symmetric as function of z_m^2 and $z_{j'}^2$ for another different j' in the first D_m, invariant again under the action of σ_i, σ_{ij} and τ_{ij}. We conclude that

$$D_m\left[D_{m-1}^{\lambda_{m-1}-1}\partial_{m-1}\right]\cdots\left[D_1^{\lambda_1-1}\partial_1\right]m_\mu(\vec{z}) = \partial_m\left[D_{m-1}^{\lambda_{m-1}-1}\partial_{m-1}\right]\cdots\left[D_1^{\lambda_1-1}\partial_1\right]m_\mu(\vec{z}).$$

□

Remark 2.4.15. *One can replace $m_\mu(\vec{z})$ by $F(z_1, ..., z_M)$ or $\exp(F(z_1, ..., z_M))$ in last lemma, since these functions satisfy the same symmetry.*

The next lemma considers the concrete action of D_i on a polynomial of $z_1, ..., z_l$.

Lemma 2.4.16. *For an arbitrary l-tuple $(n_1, ..., n_l) \in \mathbb{Z}^l_{\geq 0}$ and arbitrary $i = 1, 2, ..., l$, we have that for the generic part of D_i,*

$$D_i[z_1^{n_1}...z_l^{n_l}] = \Big(\partial_i + \left[\theta(N-M+1) - \frac{1}{2}\right] d_i^{'} + 2\theta(M-1)d_i\Big)[z_1^{n_1}\cdots z_l^{n_l}] + \sum_{j\neq i}(z_j p_1^j + z_j p_2^j), \tag{2.4.21}$$

where p_1^j, p_2^j are some polynomials of $z_1, ..., z_l$ depending on $(n_1, ..., n_l)$, and d_i, $d_i^{'}$ are linear operators on polynomials of $z_1, ..., z_M$ such that

$$d_i(z_i^n) = \begin{cases} 0 & n = 0; \\ 2z_i^{n-1} & n > 0, \end{cases} \qquad d_i^{'}(z_i^n) = \begin{cases} 0 & n \text{ is even;} \\ 2z_i^{n-1} & n \text{ is odd.} \end{cases} \tag{2.4.22}$$

Note that the the action depends whether the power of z_i is odd or even.

Proof. This follows directly from definition. More precisely, for $j > l$,

$$\theta\frac{1-\sigma_{ij}}{z_i - z_j}[z_1^{n_1}\cdots z_l^{n_l}] = d_i[z_1^{n_1}\cdots z_l^{n_l}] + z_j p_1^j,$$

$$\theta\frac{1-\tau_{ij}}{z_i + z_j}[z_1^{n_1}\cdots z_l^{n_l}] = d_i[z_1^{n_1}\cdots z_l^{n_l}] + z_j p_2^j,$$

and

$$\theta\frac{1-\sigma_i}{z_i}[z_1^{n_1}\cdots z_l^{n_l}] = d_i^{'}[z_1^{n_1}\cdots z_l^{n_l}].$$

□

Proposition 2.4.17. *For even partition λ with $|\lambda| = 2k$, we have*

$$\left[\prod_{i=1}^{l(\lambda)}(D_i)^{\lambda_i}\right] \exp(F(z_1, ..., z_M))\Big|_{z_1,...,z_M=0} = b_\lambda^\lambda \cdot c_F^\lambda + \sum_{\mu:|\mu|=2k,\ l(\mu)>l(\lambda)} b_\mu^\lambda \cdot c_F^\mu + R + O(\frac{1}{M}).$$

In particular,

$$\lim_{M\theta\to\gamma, N\theta\to q\gamma} b_\lambda^\lambda = \prod_{i=1}^{l(\lambda)}\Big[\lambda_i(\lambda_i-2+2q\gamma)(\lambda_i-2+2q\gamma)(\lambda_i-4-2q\gamma)(\lambda_i-4+2\gamma)\cdots(2+2q\gamma)(2+2\gamma)2q\gamma\Big].$$

The summand R is a polynomial of c_F^v that $|v| < 2k$. And $O(\frac{1}{M})$ denotes a linear polynomial of c_F^v such that $|v| = 2k$, and the coefficients are of order $O(\frac{1}{M})$ in the limit regime of this section.

Proof. Since the expression on the right is homogeneous of degree 2k, all the nonlinear terms are collected as R, and it suffices to consider the linear terms, which is

$$\sum_{\mu:|\mu|=2k,\ \mu \text{ is even}} b_\mu^\lambda \cdot c_F^\mu.$$

We classify all the even partition μ with $|\mu| = 2k$ in terms of their length. When $l(\mu) > l(\lambda)$, there's nothing to show. When $l(\mu) \leq l(\lambda)$, we want to show when $\mu \neq \lambda$, b_μ^λ is of order $O(\frac{1}{M})$.

Writing $\exp(F(z_1, ..., z_M))$ as power series of $F(z_1, ..., z_m)$. Since each term in D_i reduces the total power of a monomial by 1, and $F(z_1, ..., z_M) = \sum_{\mu:|\mu|\leq 2k} c_F^\mu \cdot m_\mu(\vec{z})$ where each $m_\mu(\vec{z})$ is homogeneous of degree $|\mu|$, we see that b_μ^λ is obtained from the action of $\prod_{i=1}^{l(\lambda)} D_i^{\lambda_i}$ on the single symmetric monomial $m_\mu(\vec{z})$.

Again let $l = l(\lambda)$. By Lemma 2.4.13 and 2.4.14, we first consider the generic part of $\prod_{i=1}^{l(\lambda)} D_i^{\lambda_i} m_\mu(\vec{z})$. When $l(\mu) < l$, each monomial of $m_\mu(\vec{z})$ is missing some variable among $z_1, ..., z_M$, say z_m. Then when acting the ∂_m in (2.4.20), we get 0 since $\left[D_{m-1}^{\lambda_{m-1}-1}\partial_{m-1}\right]...\left[D_1^{\lambda_1-1}\partial_1\right]$ does not produce any power of z_m to $m_\mu(\vec{z})$. And the remaining part of the action gives $O(\frac{1}{M})$.

What remains is to consider the case $|\mu| = |\lambda|$, and we calculate the limit of b_λ^λ. Again b_λ^λ is obtained from the action of $\prod_{i=1}^{l(\lambda)} D_i$ on $m_\lambda(\vec{z})$. By Lemma 2.4.13, 2.4.14, we consider only the generic part of the Dunkl product, and act it on the monomials of $m_\mu(\vec{z})$ separately. The monomials with variables other than $z_1, ..., z_l$ are missing some variables, say z_m $(1 \leq m \leq l)$. Then (2.4.20) again tells that these monomials only contribute $O(\frac{1}{M})$.

Now we consider monomials formed by $z_1, ..., z_l$. For an arbitrary l-tuple $(n_1, ..., n_l)$

and arbitrary $i = 1, 2, ..., l$, by Lemma 2.4.16 we have

$$\begin{aligned} D_i[z_1^{n_1}...z_l^{n_l}] =& \Big(\partial_i + [\theta(N-M+1) - \frac{1}{2}][1-(-1)^{n_i}]d_i + \theta(M-1)d_i + \theta(M-1)d_i\Big)[z_1^{n_1}\cdots z_l^{n_l}] \\ &+ \sum_{j\neq i}(z_j p_1^j + z_j p_2^j), \end{aligned} \tag{2.4.23}$$

One can see from the above expression that, after a single action of D_i, $z_1^{n_1}..., z_l^{n_l}$ splits into two parts. The z_i-power of the first part decreases by 1. The second part has a common factor z_j, and its z_i-powers decreases as well while the powers of other variables are unchanged. For the action of $\prod_{i=1}^l D_i^{\lambda_i}$, we repeat the above action by another $|\lambda| - 1$ times. Since all indices j are distinct, the second part has no chance to become a constant. Hence we only apply the first part each time we apply one more single D_i, and $\prod_{i=1}^l D_i^{\lambda_i}$ results in reducing power of z_i by λ_i. In the monomials of $m_\mu(\vec{z})'s$ where $|\mu| = |\lambda|$, only $z_1^{\lambda_1}, ..., z_l^{\lambda_l}$ survives as a nonzero constant. More precisely (we use $\approx$ to omit the $O(\frac{1}{M})$ part),

$$\begin{aligned} b_\lambda^\lambda \approx& [z^0]\prod_{i=1}^l D_i^{\lambda_i} m_\lambda(\vec{z}) \approx [z^0]\prod_{i=1}^l D_i^{\lambda_i} z_1^{\lambda_1}....z_M^{\lambda_M} \\ \approx& [z^0]\Big[\partial_l + \Big(2(M-1)\theta + 2(N-M+1)\theta - 1\Big)d_l\Big]^{\frac{\lambda_l}{2}}\Big[\partial_l + 2(M-1)\theta d_l\Big]^{\frac{\lambda_l}{2}-1}\partial_l \\ & \cdots\Big[\partial_1 + \Big(2(M-1)\theta + 2(N-M+1)\theta - 1\Big)d_1\Big]^{\frac{\lambda_1}{2}}\Big[\partial_1 + 2(M-1)\theta d_1\Big]^{\frac{\lambda_1}{2}-1}\partial_1\Big[z_1^{\lambda_1}\cdots z_l^{\lambda_l}\Big] \\ =& \Big[\partial_l + (2N\theta - 1)d_l\Big]^{\frac{\lambda_l}{2}}\Big[\partial_l + 2(M-1)\theta d_l\Big]^{\frac{\lambda_l}{2}-1}\partial_l \\ & \cdots\Big[\partial_1 + (2N\theta - 1)d_1\Big]^{\frac{\lambda_1}{2}}\Big[\partial_1 + 2(M-1)\theta d_1\Big]^{\frac{\lambda_1}{2}-1}\partial_1\Big[z_1^{\lambda_1}\cdots z_l^{\lambda_l}\Big] \\ \xrightarrow[M\theta\to\gamma]{N\theta\to q\gamma}& \Big[\partial_l + (2q\gamma - 1)d_l\Big]^{\frac{\lambda_l}{2}}\Big[\partial_l + 2\gamma d_l\Big]^{\frac{\lambda_l}{2}-1}\partial_l \\ & \cdots\Big[\partial_1 + (2q\gamma - 1)d_1\Big]^{\frac{\lambda_1}{2}}]\Big[\partial_1 + 2\gamma d_1\Big]^{\frac{\lambda_1}{2}-1}\partial_1\Big[z_1^{\lambda_1}\cdots z_l^{\lambda_l}\Big] \\ =& \prod_{i=1}^l(\lambda_i - 1 + 2q\gamma - 1)(\lambda_i - 2 + 2\gamma)(\lambda_i - 3 + 2q\gamma - 1)\cdots(2 + 2\gamma)(1 + 2q\gamma - 1) \\ =& \prod_{i=1}^l \lambda_i(\lambda_i - 2 + 2q\gamma)(\lambda_i - 2 + 2\gamma)(\lambda_i - 4 + 2q\gamma)(\lambda_i - 4 + 2\gamma)\cdots(2 + 2q\gamma)(2 + 2\gamma)2q\gamma. \end{aligned}$$

The above argument also implies for $|\mu| = |\lambda|$, $\prod_{i=1}^{l(\lambda)} D_i^{\lambda_i} m_\mu(\vec{z})$ is $O(\frac{1}{M})$, and so is b_μ^λ. $\square$

The next proposition deals with the terms involving only length 1 partitions, and identify them with L in (2.4.6).

Proposition 2.4.18. *For even partition λ with $|\lambda| = 2k$, we have*

$$\left[\prod_{i=1}^{l(\lambda)} (D_i)^{\lambda_i}\right] \exp(F(z_1, ..., z_M))\Big|_{z_1=...z_M=0}$$
$$= \prod_{i=1}^{l(\lambda)} \left([z^0](\partial + 2\gamma d + ((q-1)\gamma - \frac{1}{2})d^{'} + *_g)^{\lambda_i - 1} g(z) \right) + R + O(\frac{1}{M}), \tag{2.4.24}$$

where $g(z) = \sum_{n=1}^{\infty} n c_F^{(n)} z^{n-1}$, and ∂, d, $d^{'}$ and $*_g$ are defined in Definition 2.4.6. Moreover, R is a homogeneous polynomial of c_F^v's that $|v| \leq 2k$, and each monomial contains at least one c_F^v that $l(v) > 1$, and $O(\frac{1}{M})$ is a homogeneous polynomial of c_F^v's whose coefficients are of order $O(\frac{1}{M})$ in the limit regime $M\theta \to \gamma, N\theta \to q\gamma$.

Proof. Again by Lemma 2.4.13, 2.4.14, we only take the generic part of the action of Dunkl operators, namely, all indices j's involved are distinct and bigger than $l(\lambda)$, and the remaining part becomes $O(\frac{1}{M})$ in (2.4.24).

Moreover, we only consider the polynomials involving only $c_F^{(n)}$'s ($n = 2, 4, ..., 2k$), which are corresponding to $m_{(n)}(\vec{z})$, and all other terms are collected in R and $O(\frac{1}{M})$. Hence, we only look at the action

$$\left[\prod_{i=1}^{l(\lambda)} (D_i)^{\lambda_i}\right] \exp(\sum_{n=2}^{2k} c_F^{(n)} m_{(n)}(\vec{z}))\Big|_{z_1,...,z_M=0}$$
$$= \left[\prod_{i=1}^{l(\lambda)} (D_i)^{\lambda_i}\right] \prod_{t=1}^{M} \exp(\sum_{n=2}^{2k} c_F^{(n)} (z_t^n))\Big|_{z_1,...,z_M=0}. \tag{2.4.25}$$

Claim:

$$\begin{aligned}\left[\prod_{i=1}^{l(\lambda)}(D_i)^{\lambda_i}\right]&\prod_{t=1}^{M}\exp(\sum_{n=2}^{2k}c_F^{(n)}(z_j^n))\Big|_{z_1=\ldots z_M=0}\\ &=\prod_{i=1}^{l(\lambda)}\left((D_i)^{\lambda_i-1}\partial_i\Big[\prod_{t=1}^{M}\exp(\sum_{n=2}^{2k}c_F^{(n)}(z_t^n)\Big]\Big|_{z_i=0}\right).\end{aligned}\tag{2.4.26}$$

Proof of the Claim: Since the indices j's of D_i's are distinct and bigger than $l(\lambda)$, for $i_1 \neq i_2$, $D_{i_1}^{\lambda_{i_1}}$ and $D_{i_2}^{\lambda_{i_2}}$ are acting on two groups of disjoint variables. Hence the action of each $(D_i)^{\lambda_i}$ factors. Moreover, the first D_i acts as ∂_i for the same reason as in the proof of Lemma 2.4.14. □

Without loss of generality consider $i = 1$.

$$\partial_1\Big[\prod_{t=1}^{M}\exp\Big(\sum_{n=2}^{2k}c_F^{(n)}(z_t^n)\Big)\Big]=g(z_1)\prod_{t=1}^{M}\exp\Big(\sum_{n=2}^{2k}c_F^{(n)}(z_t^n)\Big).$$

By (2.4.14)-(2.4.17), it suffices to consider the explicit action of D_i on

$$H(z_1)\prod_{t=1}^{M}\exp\left(\sum_{n=2}^{2k}c_F^{(n)}(z_t^n)\right),$$

where $H(z_1)$ is a polynomial of z_1, and

$$D_1\Big[H(z_1)\prod_{t=1}^{M}\exp\left(\sum_{n=2}^{2k}c_F^{(n)}(z_t^n)\right)\Big]=\Big[D_1H(z_1)+g(z_1)\Big]\prod_{t=1}^{M}\exp\left(\sum_{n=2}^{2k}c_F^{(n)}(z_t^n)\right).$$

Hence we have for $i = 1, 2, ..., l(\lambda)$,

$$D_i^{\lambda_i-1}\partial_i\left[\prod_{t=1}^{M}\Big(\exp(\sum_{n=2}^{2k}c_F^{(n)}(z_t^n)\Big)\right]\Big|_{z_1=\ldots=z_M=0}=(D_i+*_g)^{\lambda_i-1}g(z_i)\Big|_{z_i=0}.$$

Again by Lemma 2.4.13, 2.4.14, 2.4.16, up to $O(\frac{1}{M})$ error,

$$(D_i + *_g)^{\lambda_i - 1} g(z_i)\Big|_{z_i=0} \approx \left(\partial_i + 2(M-1)\theta d_i + \left[\theta(N-M+1) - \frac{1}{2}\right] d_i^{'}\right)^{\lambda_i - 1} g(z_i)\Big|_{z_i=0}$$
$$\xrightarrow[M\theta\to\gamma]{N\theta\to q\gamma} \left(\partial_i + 2\gamma d_i + \left[(q-1)\gamma - \frac{1}{2}\right] d_i^{'} + *_g\right)^{\lambda_i - 1} g(z_i)\Big|_{z_i=0}, \tag{2.4.27}$$

and plugging this back to (2.4.26) gives

$$\left[\prod_{i=1}^{l(\lambda)} (D_i)^{\lambda_i}\right] \prod_{t=1}^{M} \exp\Big(\sum_{n=1}^{2k} c_F^{(n)}(z_j^n)\Big)\Big|_{z_1=\ldots z_M=0}$$
$$= \prod_{i=1}^{l(\lambda)} \left([z^0]\Big(\partial + 2\gamma d + \left[(q-1)\gamma - \frac{1}{2}\right] d' + *_g\Big)^{\lambda_i - 1} g(z)\right) + O(\frac{1}{M}). \tag{2.4.28}$$

Proposition 2.4.18 then follows. □

Combining all the results above in this section, we arrive at the expansion (2.4.6) representing action of Dunkl operators on $\exp(F(z_1, ..., z_M))$.

Proof of Theorem 2.4.9: By Proposition 2.4.11 and 2.4.12 the left side of (2.4.6) is a homogeneous polynomial of c_F^v's of degree $2k$ with uniformly bounded coefficients in the limit regime. The right side of (2.4.6) is a combination of Proposition 2.4.17 (which gives $b_\lambda^\lambda \cdot c_F^\lambda + \sum_{\mu:|\mu|=k,\ l(\mu)>l(\lambda)} b_\mu^\lambda \cdot c_F^v$) and (2.4.18) (which gives polynomial L), and note that the only possible overlap of linear terms and terms involving only length 1 partitions is when λ itself is $(2k)$, which gives the term subtracted in (2.4.8). □

2.4.4 q-γ convolution.

After stating the equivalence in Theorem 2.4.8, Theorem 2.1.12 follows as a direct consequence.

Proof of Theorem 2.1.12. For each $M \le N, \theta > 0$, let $G_{M,N,\theta}^a$, $G_{M,N,\theta}^b$, $G_{M,N,\theta}^c$ denote the type BC Bessel generating function of $\vec{a}_M, \vec{b}_M$ and $\vec{c}_M = \vec{a}_M \boxplus^{\theta}_{M,N} \vec{b}_M$. Then

$$G^c_{M,N,\theta}(z_1, ..., z_M) = G^a_{M,N,\theta}(z_1, ..., z_M) \cdot G^b_{M,N,\theta}(z_1, ..., z_M),$$

and hence partial derivatives of $\ln(G^c_{M,N,\theta})$ are equal to the sum of the ones of $\ln(G^b_{M,N,\theta})$ and $\ln(G^a_{M,N,\theta})$. By assumption of the theorem, $\{\vec{a}_M\}$ and $\{\vec{b}_M\}$ satisfy LLN condition, then by Theorem 2.4.8 they are q-γ-LLN appropriate. Hence by Definition 2.4.2 $\{\vec{c}_M\}$ is also q-γ-LLN appropriate. By Theorem 2.4.8 again $\{\vec{c}_M\}$ satisfies LLN. $\square$

2.5 q-γ cumulants and moments

Fix $q \geq 1, \gamma > 0$, in this section we continue with the limit regime $M, N \to \infty$, $\theta \to 0$, $M\theta \to \gamma$, $N\theta \to q\gamma$. Definition 2.4.6 introduces a map $\mathrm{T}^{q,\gamma}_{k\to m}$ in terms of operators, that sends the real sequence $\{k_l\}_{l=1}^{\infty}$ to another real sequence $\{m_k\}_{k=1}^{\infty}$. We keep the interpretation from Theorem 2.4.8, that is, we call $\{k_l\}_{l=1}^{\infty}$ the q-γ cumulants and take $k_l = 0$ for all odd l's. In this section we give a more combinatorical descriptions of $T^{q,\gamma}_{k\to m}$. After that, we also provide an explicit relation of $T^{q,\gamma}_{m\to k}$ in terms of generating functions, and by taking q, γ to some extreme values, we set the connections of our q-γ-cumulants to the usual cumulants and (rectangular) free cumulants in free probability theory, and also to the γ-cumulants defined in Benaych-Georges, Cuenca, and Gorin 2022, that arises in the high temperature regime of self-adjoint matrix additions.

2.5.1 From q-γ cumulants to moments

We start by introducing some basic notions of set partitions, which are necessary for the statement of the main theorem.

For $k \in \mathbb{Z}_{\geq 1}$, a *set partition* π of $[k]$ is a way to write $[k] := \{1, 2, ..., k\}$ as disjoint union of sets $B_1, ..., B_n$ for some m. We write $\pi = B_1 \sqcup B_2 \sqcup ... \sqcup B_m$, and denote the space of all set partitions of $[k]$ by $P(k)$. Given a set partition π, for each B_i let $\min(B_i)$ and $\max(B_i)$ denote the minimal and maximal number in the subset B_i of $[k]$, and for simplicity, we label $B_1, ..., B_n$ by $\min(B_i)$ in increasing order.

In this text, we are in particular interested in the non-crossing partitions.

Definition 2.5.1. *Fix $k \in \mathbb{Z}_{\geq 1}$, a set partition $\pi = B_1 \sqcup ... \sqcup B_n$ of $[k]$ is non-crossing if for any $l = 2, ..., m$, and any $j = 1, 2, ..., l-1$, the elements in B_j are either bigger than $\max(B_l)$ or smaller than $\min(B_l)$. See Figure 2.1. Denote the set of all non-crossing partition of $[k]$ by $NC(k)$.*

Each set partition can be realized visually as a collection of blocks $B_1, ..., B_n$ with k legs in total, and the block B_i has $|B_i|$ legs, which is the number of elements in B_i. See Figure 2.1. From this point of view, π is non-crossing if and only if the legs of one block does not cross any other blocks.

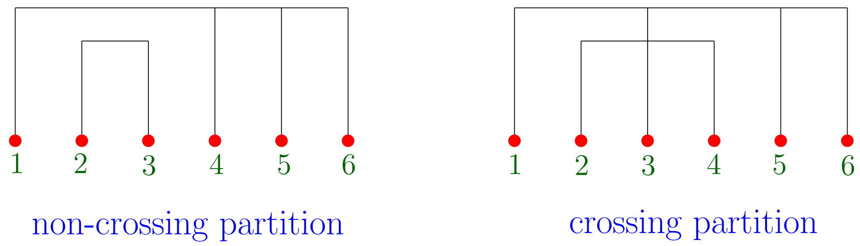

Figure 2.1: The graph on the left represents a noncrossing partition π of [6], where $B_1 = \{1, 4, 6\}$, $B_2 = \{2, 3\}$, $B_3 = \{5\}$, and the graph on the right represents a crossing partition $\pi^{'}$ of [6], where $B_1 = \{1, 3, 6\}$, $B_2 = \{2, 4\}$, $B_3 = \{5\}$.

Next we define a quantity associated with the non-crossing set partition π.

Definition 2.5.2. *Given $\pi = B_1 \sqcup ... \sqcup B_m \in NC(k)$, for $i = 1, 2, ..., n$, let $P_i = \#$ of elements in $B_1, ..., B_i$ bigger than $\min(B_i)$, and $Q_i = \#$ of elements in $B_1, ..., B_i$ bigger than $\max(B_i)$(note that $P_i - Q_i = |B_i| - 1$ by definition). Let $C_1, C_2, ...$ be the countable sequence of constants*

$$2q\gamma,\ 2\gamma+2,\ 2q\gamma+2,\ 2\gamma+4,\ 2q\gamma+4,\ 2\gamma+6,\ 2q\gamma+6\ ...$$

respectively. Then we define

$$W(\pi)=\prod_{i=1}^{m}\Big[C_{Q_i+1}C_{Q_i+2}\cdots C_{P_i}\Big]. \tag{2.5.1}$$

Example 2.5.3. *In Figure 2.2, π is a non-crossing partition of $[14]$, such that $B_1 = \{1,7,8\}$, $B_2 = \{2,3,6\}$, $B_3 = \{4,5\}$, $B_4 = \{9,10,13,14\}$, and $B_5 = \{11,12\}$. Moreover, $P_1 = 2$, $Q_1 = 0$, $P_2 = 4$, $Q_2 = 2$, $P_3 = 4$, $Q_3 = 3$, $P_4 = 3$, $Q_4 = 0$, $P_5 = 1$, $Q_5 = 0$, so $W(\pi) = C_1 \cdot C_3C_4 \cdot C_4 \cdot C_1C_2C_3 \cdot C_1 = C_1^3C_2C_3^2C_4^2 = (2q\gamma)^3(2\gamma+2)(2q\gamma+2)^2(2\gamma+4)^2$.*

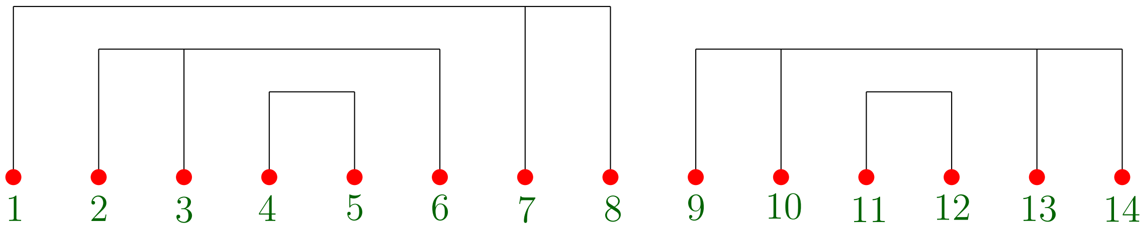

Figure 2.2: The graphical representation of the non -crossing partition in Example 2.5.3.

We also introduce a notion of *even partition* that will be used later.

Definition 2.5.4. *We say π is even if $|B_1|,\dots,|B_n|$ are all even, and denote the collection of all non-crossing even set partition of $[2k]$ by $\mathfrak{NC}(2k)$, for some $k \in \mathbb{Z}_{\geq 1}$.*

The following main theorem of this section gives the combinatorial expression of moments as polynomials of q-γ cumulants, whose coefficients are given by $W(\pi)$.

Theorem 2.5.5. *(q-γ cumulants to moments formula) Let $\{k_l\}_{l=1}^{\infty}$, $\{m_k\}_{k=1}^{\infty}$ be two real sequences such that $k_l = 0$ for all odd l's, and $\{m_{2k}\}_{k=1}^{\infty} = \mathrm{T}^{q,\gamma}_{k\to m}(\{k_l\}_{l=1}^{\infty})$. Then for any $k = 1,2,\dots$,*

$$m_{2k-1}=0,\quad m_{2k}=\sum_{\pi\in\mathfrak{NC}(2k)}W(\pi)\prod_{B\in\pi}k_{|B_i|}. \tag{2.5.2}$$

Remark 2.5.6. *It is well known (see e.g Nica and Speicher 2006, Proposition 9.8) that non-crossing partitions are in bijection with the so-called Lukasiewicz paths, i.e, lattice paths whose steps take value in $\mathbb{Z}_{\geq -1}$ with starting and ending point at height 0. Therefore, the cumulant expression in (2.5.2) for m_{2k} can be rewritten as a weighted sum over*

Lukasiewicz paths of length 2k. Such interpretation is used in the subsequent work Keating and Xu 2024+ and C. Cuenca and Moll n.d., which studies random partitions under certain Jack measures. In particular, the moment expression in C. Cuenca and Moll n.d., Theorem 3.7 is similar to (2.5.2) and Benaych-Georges, Cuenca, and Gorin 2022, Theorem 3.10. But notice that in C. Cuenca and Moll n.d. Lukasiewicz paths are not allowed to have horizontal steps at height 0.

Example 2.5.7. *By manipulating (2.5.2), we have the explicit expression of the first few q-γ cumulants in terms of moments:*

$$\begin{aligned} m_2 =& 2q\gamma k_2, \\ m_4 =& 2q\gamma(2\gamma+2)(2q\gamma+2)k_4 + [(2q\gamma)^2 + 2q\gamma(2\gamma+2)]k_2^2, \\ m_6 =& \Big[2q\gamma(2\gamma+2)(2q\gamma+2)(2\gamma+4)(2q\gamma+4)\Big]k_6 \\ &+ \Big[2q\gamma(2\gamma+2)(2q\gamma+2)(3\times 2q\gamma + 2\gamma + 2 + 2q\gamma + 2 + 2\gamma + 4 + 2q\gamma + 4)\Big]k_4 k_2 \\ &+ \Big[(2q\gamma)^3 + 2(2q\gamma)^2(2\gamma+2) + 2q\gamma(2\gamma+2)(2q\gamma+2)\Big]k_2^3. \end{aligned} \tag{2.5.3}$$

Proof of Theorem 2.5.5. Recall from Definition 2.4.6 that

$$m_{2k} = [z^0]\left(\partial + 2\gamma d + \left[(q-1)\gamma - \frac{1}{2}\right]d^{'} + *_g\right)^{2k-1} g(z)$$

where $g(z) = \sum_{l=1}^{\infty} k_l z^{l-1}$, i.e, we act the operator $D := \partial + 2\gamma d + ((q-1)\gamma - \frac{1}{2})d^{'} + *_a$ on $g(z)$ by $2k-1$ times, then take the constant term of the resulting expression. Here $a(z) = \sum_{l=1}^{\infty} a_l z^{l-1}$ such that $a_l = k_l$, and the resulting polynomial $D^{p-1}g(z)$, before the acting D by the p^{th} time, contains only odd powers of z when p is odd, and contains only even power of z when p is even.

Because of this, we claim that D^{2k-1} acting on $g(z)$ is equivalent to the alternate product of two operators

$$D^{'} \circ (D^{''} \circ D^{'})^{k-1}. \tag{2.5.4}$$

More precisely, $D^{''} \circ D^{'}$ has the following explicit effect:

$$... \longrightarrow \sum_{l=1}^{\infty} b_l z^{l-1} \xrightarrow{D^{''}} \sum_{l=1}^{\infty} c_l z^{l-1} \xrightarrow{D^{'}} \sum_{l=1}^{\infty} d_l z^{l-1} \longrightarrow ...,$$

where $D^{'}$ acts on the polynomial $b(z) = \sum_{l=1} b_l z^{l-1}$ which contains only odd powers, with output $c(z) = \sum_{l=1} b_l z^{l-1}$ such that

$$c_l = \begin{cases} 0 & \text{l is even;} \\ (2q\gamma + l - 1)b_{l+1} + \sum_{j=1}^{l} a_j b_{l+1-j} & \text{l is odd,} \end{cases} \tag{2.5.5}$$

and then $D^{''}$ acts on $c(z)$ which contains only odd powers, with output $d(z) = \sum_{l=1}^{\infty} d_l z^{l-1}$ such that

$$d_l = \begin{cases} 0 & \text{l is odd;} \\ (2\gamma + l)c_{l+1} + \sum_{j=1}^{l} a_j c_{l+1-j} & \text{l is even.} \end{cases} \tag{2.5.6}$$

We verify the expression (2.5.2) inductively on k, by visualizing the action of $D^{'}$ or $D' \circ D^{''}$ under the graphical representation of set partitions.

When $k = 1$,

$$\begin{aligned} m_2 =& [z^0]\Big(D^{'}(g(z)\Big) \\ =& (2q\gamma)k_2 + \sum_{j=1}^{1} a_1 k_{1+1-1} = (2q\gamma)k_2 + a_1 k_1 \end{aligned} \tag{2.5.7}$$

by (2.5.5).

Correspondingly, we realize this expression by the following concrete operations:

(0). Start with an empty set partition, corresponding to a blank graph with no leg.

(1). Draw a single leg, and label it with monomial k_1.

(2). Add one more leg on the right of the first leg. Now we have two options: connect this leg with the first leg to form a block of size 2, and label the new block by $C_1 = 2q\gamma k_2$ or keep them to be separate as two blocks with single leg, and label the second block by

a_1.

After step 2 the first configuration corresponds to monomial k_2 with weight $2q\gamma$, and the second one corresponds to monomial $k_1 \cdot a_1$ with weight 1 (which is the product of the labels of the two blocks). Adding them together gives m_2. To match (2.5.2), note that $W(\{1,2\}) = C_1 = 2q\gamma$, and $k_1 = a_1 = 0$ so the second term vanishes.

To obtain m_{2k}, suppose $m_2, ..., m_{2k-2}$ obtained from step 0, 1, 2,..., $2k-3$, $2k-2$ all match (2.5.2). We give the operations in step $2k-1$, $2k$:

($2k-1$). Given a configuration $\pi = B_1 \sqcup ... \sqcup B_n$ of $2k-2$ legs in total, with a corresponding weight $\tilde{W}(\pi)$, insert one more leg right after the first leg to B_1, so that we get a new configuration $\pi^{'} = B_1^{'} \sqcup ... \sqcup B_n^{'}$, where $B_1^{'} = \{1\} \cup \{j+1 : j \in B_1\}$, and $B_i^{'} = \{j+1 : j \in B_i\}$.

Then one have $|B_1^{'}|$ options: either keep everything unchanged and let $B_i^{''} = B_i^{'}$ for $i = 1, 2, ..., n$,

$$\tilde{W}(\pi^{''}) = \begin{cases} \tilde{W}(\pi) \cdot (2\gamma + |B_1^{''}| - 1) & \text{if } |B_1^{''}| \text{ is odd;} \\ \tilde{W}(\pi) \cdot (2q\gamma + |B_1^{''}| - 2) & \text{if } |B_1^{''}| \text{ is even,} \end{cases}$$

or split $B_1^{'}$ into two non-crossing blocks $B_1^{''} \sqcup B_2^{''}$, where $\min(B_1^{''}) = 1$, $\min(B_2^{''}) = 2$, and let $B_i^{''} = B_{i-1}^{'}$ for $i = 3, 4, ..., n+1$, $\tilde{W}(\pi^{''}) = \tilde{W}(\pi)$.

($2k$). Repeat the operations in step $2k-1$ one more time, and still denote the output configuration with 2k legs by $\pi^{''} = B_1^{''} \sqcup ... \sqcup B_{n^{''}}^{''}$($n^{''}$ denotes the number of blocks) with weight $W^{''}(\pi^{''})$.

Then delete all configurations with at least one odd block. For each remaining configuration $\pi^{''}$, assign it with monomial

$$W^{''}(\pi^{''}) k_{|B_1^{''}|} \cdot a_{|B_2^{''}|} ... \, a_{|B_{n^{''}}^{''}|}. \tag{2.5.8}$$

We claim that the above steps represent the action of (2.5.4). Indeed, both step 0-2 and $D^{'}$ generate $(2q\gamma)k_2 + a_1 k_1$, and for $l \geq 3$, step l is corresponding to the $(l-1)^{th}$ term (from left to right) in the product. More precisely, the expression of d_l (c_l resp.) in

(2.5.6) ((2.5.5) resp.) is recording the $l+1$ options one can choose on a configuration whose first block is of size l, that choosing to enlarge the first block by 1 gives one extra factor $(2\gamma+l)$ $(2q\gamma+l-1$ resp.), and splitting d_l ((c_l resp.) into a_j and c_{l+1-j} corresponds to splitting the first block into two new blocks of size $l+1-j$ and j respectively. Therefore, acting (2.5.4) on $g(z)$ and take the constant term is equivalent to a chain of compositions of (2.5.5) and (2.5.6). Compared to (2.5.8), the output is also a large sum of monomials of k_l's (coefficients of $g(z)$) and a_l's (coefficients of $a(z)$), that each non-vanishing monomial is corresponding to a unique non-crossing even partition π (recall $k_l = a_l = 0$ for odd l's), and the unique k_l it has is giving the size of the first block of π. To see that each non-crossing even set partition can be realized in this way, we do induction on k and assume this holds for the set partitions of size up to $2k-2$. For $\pi = B_1 \sqcup ... \sqcup B_n \in \mathfrak{NC}(2k)$, just combine B_1 with B_2 and B_3(both might be $\varnothing$) as a single block, then remove two legs from this new block. What we get is an element $\tilde{\pi}$ in $\mathfrak{NC}(2k-2)$, which can be realized by induction hypothesis, and one can construct $\bar{\pi}$ from $\tilde{\pi}$ using the step $2k-1$ and $2k$ above.

Since $a_l = k_l$ for all $l = 1, 2, ...$, to match m_k with the right side of (2.5.2), it remains to match the coefficient of each monomials. Given $\pi \in \mathfrak{NC}(2k-1)$ ($\mathfrak{NC}(2k-1)$ resp.), define its degeneration $\tilde{\pi} = \tilde{B}_1 \sqcup ... \sqcup \tilde{B}_n \in \mathfrak{NC}(2k-2)$ ($\mathfrak{NC}(2k-2)$ resp.) by taking $\tilde{B}_1$ as the combination of the first two blocks removing 1 leg, or simply removing 1 leg from B_1, similarly as in last paragraph. Then compared to $W(\tilde{\pi})$, we only replace $C_{|\tilde{Q}_1|+1}...C_{|\tilde{P}|_1}$ by $C_{|Q_2|+1}...C_{|P_2|} \cdot C_{|Q_1|+1}...C_{|P_1|}$ in $W(\pi)$ when we choose to split the first block, and one can check that $W(\pi) = W(\tilde{\pi})$ after the change, and when we choose to enlarge the first block by 1, $W(\pi)$ has one more factor $C_{|P_1|} = C_{|\tilde{B}_1}|$ than $W(\tilde{\pi})$.

On the other hand, for $l = 1, 2, ..., 2k$, as pointed out in step $2k-1$ and $2k$, since $a_l = k_l = 0$ for l odd, in order to get a non-vanishing term after $2k$ steps, one can only enlarge $|\tilde{B}_1|$ by 1 in even steps, when $|\tilde{B}_1|$ is odd, then multiply the new factor $2q\gamma + |\tilde{B}_1| - 1$ to the monomial, and enlarge $|\tilde{B}_1|$ by 1 in odd steps, when $|\tilde{B}_1|$ is even, then multiply the new factor $2\gamma + |\tilde{B}_1|$. In both cases this factor matches $C_{|\tilde{B}_1}|$. This finishes the proof.

□

2.5.2 From moments to q-γ-cumulants

Recall that $T^{q,\gamma}_{k\to m}$ is invertible, and for each $l = 1, 2, ...$, k_{2l} is a polynomial of $m_2, m_4, ..., m_{2l}$ with leading term as a multiple of m_{2l}. For example, by reversing (2.5.3), we have

$$\begin{aligned}
k_2 =& \frac{1}{2q\gamma} m_2,\\
k_4 =& \frac{1}{2q\gamma(2\gamma+2)(2q\gamma+2)}\Big[m_4 - (1+\frac{\gamma+1}{q\gamma})m_2^2\Big],\\
k_6 =& \frac{1}{2q\gamma(2\gamma+2)(2q\gamma+2)(2\gamma+4)(2q\gamma+4)}\\
&\cdot\Bigg(m_6 - \Big[(3\times 2q\gamma + 2\gamma + 2 + 2q\gamma + 2 + 2\gamma + 4 + 2q\gamma + 4)\cdot\frac{1}{2q\gamma}\Big]\Big[m_4 - (1+\frac{\gamma+1}{q\gamma})m_2^2\Big]m_2\\
&- \Big[1 + \frac{\gamma+1}{q\gamma} + \frac{(\gamma+1)(q\gamma+1)}{(q\gamma)^2}\Big]m_2^3\Bigg).
\end{aligned} \tag{2.5.9}$$

For the more general cases, we express the generating function of q-γ cumulants by the generating function of moments.

Theorem 2.5.8. *Let $\{m_{2k}\}_{k=1}^{\infty}$, $\{k_l\}_{l=1}^{\infty}$ be two real sequences such that $\{k_l\}_{l=1}^{\infty} = \mathrm{T}^{q,\gamma}_{m\to k}(\{m_{2k}\}_{k=1}^{\infty})$. Then $k_l = 0$ for all odd l's, and*

$$\begin{cases}
\exp\Big[\gamma\sum_{k=1}^{\infty}\frac{m_{2k}}{k}y^{2k}\Big] = \sum_{n=0}^{\infty} c_n\cdot y^{2n},\\
\exp\Big[\sum_{l=1}^{\infty}\frac{k_{2l}}{2l}y^{2l}\Big] = \sum_{n=0}^{\infty}\frac{c_n}{(q\gamma)_n(\gamma)_n}2^{-2n}y^{2n}
\end{cases} \tag{2.5.10}$$

for some auxiliary sequence $\{c_n\}_{n=0}^{\infty}$. Here we use the Pochhammmer symbol notation

$$(x)_n := \begin{cases} x(x+1)\cdot(x+n-1), & \text{if } n\in\mathbb{Z}_{\geq 1},\\ 1, & \text{if } n = 0.\end{cases}$$

Alternatively one has the more compact expression

$$\exp\Big[\sum_{l=1}^{\infty}\frac{k_{2l}}{2l}y^{2l}\Big] = [z^0]\Bigg\{\sum_{n=0}^{\infty}\frac{(yz)^{2n}}{(q\gamma)_n(\gamma)_n}2^{-2n}\cdot\exp\Big[\gamma\sum_{k=1}^{\infty}\frac{m_{2k}}{k}z^{-2k}\Big]\Bigg\}. \tag{2.5.11}$$

Before giving the proof, we first present two technical results that will be used.

Lemma 2.5.9. *(a). The following Talor series expansion holds:*

$$\sum_{k=0}^{\infty} Q_{(k)}(a_1^2, ..., a_M^2; \theta) y^{2k} = \prod_{i=1}^{M} (1 - a_i^2 y^2)^{-\theta}. \tag{2.5.12}$$

(b). For $\theta > 0$, $y \in \mathbb{C}$ and $\vec{a} = (a_1 \geq ... \geq a_M \geq 0)$,

$$\mathbb{B}(\vec{a}, y, 0^{M-1}; \theta) = \sum_{k=0}^{\infty} \frac{1}{(N\theta)_k (M\theta)_k} 2^{-2k} Q_{(k)}(a_1^2, ..., a_M^2; \theta) y^{2k}, \tag{2.5.13}$$

where $Q_{(k)}(a_{i,M}^2; \theta)$ is defined in Definition 2.2.8, and (k) denotes the partition $(k, 0, ..., 0) \in \Lambda_M$. Moreover, the power series converges uniformly in a domain near 0.

Proof. (a) is a well known result that can be found in Macdonald 1995, p378 and p380. (b) follows from Proposition 2.2.16 and (2.2.3), after specifying all but one variables to 0. □

Proof of Theorem 2.5.8. First we note that (2.5.10) and (2.5.11) are equivalent by comparing the coefficients for y^{2n} for each $n = 0, 1, 2, ...$, and we will prove (2.5.10).

For now, we assume that there exists a probability measure μ supported on $[a, b] \subset \mathbb{R}_{\geq 0}$, such that for $k = 1, 2, ...$

$$m_{2k} = \int x^k d\mu.$$

We take a sequence of deterministic M-tuples $\{\vec{a}_M\}_{M=1}^{\infty}$ such that $\vec{a}_M = (a_{1,M}, ..., a_{M,M}) \in [-\sqrt{b}, \sqrt{b}]^M$, and define $\mu_M = \frac{1}{M} \sum_{i=1}^{M} \delta_{a_{i,M}^2}$. We choose $\{\vec{a}_M\}$ in a way that $\mu_M \to \mu$ weakly as $M \to \infty$. This implies that the moments of μ_M also converge pointwisely to the corresponding moments of μ, i.e,

$$\frac{1}{M} \sum_{i=1}^{M} a_{i,M}^2 \longrightarrow m_{2k}.$$

In other words, $\{\vec{a}_M\}_{M=1}^{\infty}$ satisfies the LLN condition, and by Theorem 2.4.8 $\{\vec{a}_M\}_{M=1}^{\infty}$ is q-γ-LLN-appropriate.

By Lemma 2.5.9 (a),

$$\begin{aligned}
&\sum_{k=0}^{\infty} Q_{(k)}(a_{i,M}^2;\theta)y^{2k}=\prod_{i=1}^{M}(1-a_{i,M}^2y^2)^{-\theta}\\
=&\exp\Big[-\theta\sum_{i=1}^{M}\ln\Big(1-a_{i,M}^2y^2\Big)\Big]=\exp\Big[\theta M\sum_{k=1}^{\infty}\frac{y^{2k}}{k}\frac{1}{M}\sum_{i=1}^{M}(a_{i,M})^{2k}\Big]
\end{aligned} \tag{2.5.14}$$

as a formal power series. Taking $M\to\infty,\theta\to 0, M\theta\to\gamma$, the above equality becomes

$$\sum_{k=0}^{\infty} c_k\cdot y^{2k}=\exp\Big[\gamma\sum_{k=1}^{\infty}\frac{m_k}{k}y^{2k}\Big], \tag{2.5.15}$$

where c_k is the pointwise limit of $Q_{(k)}(a_{i,M}^2;\theta)$ in the above limit regime. This defines $\{c_k\}_{k=1}^{\infty}$ in terms of $\{m_{2k}\}_{k=1}^{\infty}$.

On the other hand, since μ_M is deterministic, its type BC Bessel generating function is equal to its Bessel function. And we have for $l=1,2,...$

$$(\frac{\partial}{\partial y})^{2l}\ln[\mathbb{B}(\vec{a}_M,y,0^{M-1};\theta)]\Big|_{y=0}\xrightarrow[M\theta\to\gamma]{N\theta\to q\gamma}(2l-1)!\cdot k_{2l}. \tag{2.5.16}$$

By Lemma 2.5.9 (b), the above equation is equivalent to

$$(\frac{\partial}{\partial y})^{2l}\ln\Big[\sum_{k=0}^{\infty}\frac{1}{(N\theta)_k(M\theta)_k}2^{-2k}Q_{(k)}(a_{i,M}^2;\theta)y^{2k}\Big]\Big|_{y=0}\xrightarrow[M\theta\to\gamma]{N\theta\to q\gamma}(2l-1)!\cdot k_{2l}. \tag{2.5.17}$$

Also, since type BC Bessel function is analytic over $z_1,...,z_M$, so is its logarithm near 0, and by Talor expanding $\ln\Big[\sum_{k=0}^{\infty}\frac{1}{(N\theta)_k(M\theta)_k}2^{-2k}Q_k(a_{i,M}^2;\theta)y^{2k}\Big]$ we see that each k_{2l} is a polynomial of finitely many terms $\frac{1}{(N\theta)_k(M\theta)_k}2^{-2k}Q_k(a_{i,M}^2;\theta)$, each of which converges to $\frac{1}{(q\gamma)_k(\gamma)_k}2^{-2k}c_k$.

We claim that as $M\theta\to\gamma, N\theta\to q\gamma$, $\sum_{k=0}^{\infty}\frac{1}{(N\theta)_j(M\theta)_j}2^{-2k}Q_{(k)}(a_{i,M}^2;\theta)y^{2k}$ converges uniformly on a domain near 0. Indeed, the pointwise convergence of coefficient of y is already given above, and to obtain a tail bound of the power series, first note that we have assumed $a_{i,M}$'s are uniformly bounded, then by writing each $Q_{(k)}(a_{1,M}^2,...,a_{M,M}^2;\theta)$ as a contour integral of the right side of (2.5.12) on the circle $\{z:|z|=r\}$ for some r small enough, we see that $Q_{(k)}(a_{1,M}^2, ..., a_{M,M}^2; \theta)$ are uniformly bounded by $C \cdot r^{-2k}$ for some constant C and r. By (2.5.14), (2.5.15), the limit is

$$\sum_{k=0}^{\infty} \frac{c_k}{(q\gamma)_k(\gamma)_k} y^{2k},$$

but since the uniform convergence of analytic functions implies convergence of derivatives, it's also equal to $\exp\left[\sum_{l=1}^{\infty} \frac{k_{2l}}{2l} y^{2l}\right]$ by (2.5.17). Hence these two functions are equal.

It remains to generalize to the case where $m_2, m_4, ...$ are arbitrary real sequence. For each $l = 1, 2, ...,$ k_{2l} is a polynomial $h_l(m_2, m_4, ..., m_{2l})$ of degree at most l, where the expression of h_l is given by (2.5.11), while on the other hand, for each k_{2l}, (2.5.2) gives another polynomial of degree at most l, such that $k_{2l} = h_l^{'}(m_2, m_4, ..., m_{2l})$. What we need to show is that $h_l = h_l^{'}$ for all l's.

Fix $l \geq 1$. We have already shown that $h_l(m_2, m_4, ..., m_{2l}) = h_l^{'}(m_2, m_4, ..., m_{2l})$, when $m_2, m_4, ..., m_{2l}$ are the first l moments of some compactly supported probability measure μ on $\mathbb{R}_{\geq 0}$. Clearly there exists more than l such choices of $m_2, m_4, ..., m_{2l}$, and therefore by fundamental theorem of algebra these two polynomials coincide. The same argument holds for arbitrary $l \geq 1$. □

2.5.3 Connections to self-adjoint additions

Let A,B be two independent $N \times N$ matrices, uniformly chosen from the sets of self-adjoint matrices with deterministic eigenvalues $a_1 \geq ... \geq a_N$ and $b_1 \geq ... \geq b_N$ respectively. The study of eigenvalues of $C = A + B$ dates back to Voiculescu 1991, which considers the empirical measure of C in fixed temperature regime. In high temperature regime, it was first considered by Mergny and Potters 2022 which introduced the notion of "high temperature convolution", and studied later in several texts including Benaych-Georges, Cuenca, and Gorin 2022, Mergny 2022 and Mergny n.d. In particular, it was proved rigorously that when $N \to \infty$, $\theta \to 0$ and $N\theta \to \gamma$, assuming the empirical measure of A, B converge to some deterministic probability measure μ_A, μ_B on $\mathbb{R}$, then the empirical measure of C converges to some deterministic probability measure μ_C, which is named as the γ-convolution of μ_A and μ_B.

There is a collection of quantities $\{k_l^\gamma\}_{l=1}^\infty$ introduced in Mergny and Potters 2022 and Benaych-Georges, Cuenca, and Gorin 2022 independently by different approaches, such that

$$k_l^\gamma(\mu_C) = k_l^\gamma(\mu_A) + k_l^\gamma(\mu_B)$$

for each $l \geq 1$. In the following we refer to Benaych-Georges, Cuenca, and Gorin 2022. We write $\{m_k^{'}\}_{k=1}^\infty = \mathrm{T}_{k\to m}^\gamma(\{k_l^\gamma\}_{l=1}^\infty)$, where $m_k^{'} \in \mathbb{R}$ denotes the k^{th} moment of the limiting empirical measure, and $\mathrm{T}_{k\to m}^\gamma$ is a map that gives moment-cumulant relation of γ-convolution. While in this text we are considering addition of a different type of matrices, we find a limit transition in high temperature regime from rectangular addition to self-adjoint addition, which is stated in terms of cumulants.

Theorem 2.5.10. *Given a real sequence $\{k_l\}_{l=1}^\infty$ such that $k_l = 0$ for all odd l's, let $\{m_{2k}\}_{k=1}^\infty = \mathrm{T}_{k\to m}^{q,\gamma}(\{k_l\}_{l=1}^\infty)$, $m_k^{'} = \frac{m_{2k}}{(q\gamma)^k}$, $k_l^{'} = 2^{2l-1}k_{2l}$ for $l = 1, 2, \ldots$. Then*

$$\lim_{q\to\infty}\{m_k^{'}\}_{k=1}^\infty = \mathrm{T}_{k\to m}^\gamma(\{k_l^{'}\})_{l=1}^\infty. \tag{2.5.18}$$

Proof. This follows from a straightforward limit transition of (2.5.10) under the assigned rescaling, and the moment-cumulant relation of γ-convolution in Benaych-Georges, Cuenca, and Gorin 2022, Theorem 3.11. □

Moreover, we point out that the combinatorial moment-cumulant formula of γ-convolution, given in Benaych-Georges, Cuenca, and Gorin 2022, Theorem 3.10 can be expressed in an alternate way similar to our Theorem 2.5.5.

Proposition 2.5.11. *Let $\{m_k^{'}\}_{k=1}^\infty = \mathrm{T}_{k\to m}^\gamma(\{k_l^\gamma\}_{l=1}^\infty)$. Then for each $k = 1, 2, \ldots$,*

$$m_k^{'} = \sum_{\pi\in NC(k)} W(\pi)\prod_{B\in\pi} k_{|B_i|}^\gamma, \tag{2.5.19}$$

where $W(\pi)$ is defined in the same way as in Definition 2.5.2, after replacing the values of C_1, C_2... to be $\gamma+1$, $\gamma+2$...

Proof. By Benaych-Georges, Cuenca, and Gorin 2022, Definition 3.7 and Theorem 3.8,

$$m_k = [z^0](\partial + \gamma d + *_g)^{k-1} g(z),\ k = 1, 2, ...$$

The statement then follows from the similar argument as in the proof of Theorem 2.5.5. $\square$

2.5.4 Connections to the classical convolutions

Recall from previous sections that, limit of $\boxplus^{\theta}_{M,N}$ gives the q-γ convolution $\boxplus_{q,\gamma}$ of two (virtual) probability measures on $\mathbb{R}$, which is linearized by q-γ cumulants. We show in this section that, under certain limit transition of the parameters q, γ, $\boxplus_{q,\gamma}$ converge to the usual convolution, classical free convolution, and rectangular free convolution respectively.

We first provide the connection of q-γ cumulants to usual cumulants. For this we recall the combinatorial classical moment-cumulant formula: for $k = 1, 2, ...$

$$m_k = \sum_{\pi' \in P(k)} \prod_{i=1}^{n} k'_{|B_i|}$$

where $\{k'_l\}_{l=1}^{\infty}$ stands for the usual cumulants, and $P(k)$ is the set of all set partitions of $[k]$. We denote the map that sends $\{m_k\}_{k=1}^{\infty}$ to $\{k'_l\}_{l=1}^{\infty}$ by $\mathrm{T}^{0}_{m \to k}$.

Theorem 2.5.12. *Given a real sequence* $\{m_{2k}\}_{k=1}^{\infty}$, *let* $\{k_l\}_{l=1}^{\infty} = \mathrm{T}^{q,\gamma}_{m\to k}(\{m_{2k}\}_{k=1}^{\infty})$, $k'_l = (q\gamma)^l 2^{2l-1}(l-1)! k_{2l}$, *then*

$$\lim_{\gamma \to 0,\ q\gamma \to \infty} \{k'_l\}_{l=1}^{\infty} = \mathrm{T}^{0}_{m\to k}(\{m_{2k}\}_{k=1}^{\infty}). \tag{2.5.20}$$

Proof. By Theorem 2.5.5, after rescaled by $(q\gamma)^k$, the coefficient $W(\pi)$ does not vanish asymptotically only if for each $i = 1, 2, ..., m$, $2l_i - 1 := P_i - Q_i$, there are l_i terms in $C_{Q_i+1} \cdots C_{P_i}$ that contain $q\gamma$. Hence each Q_i must be even.

Recall that $NC(k)$ denote the space of all (not neccesary even) non-crossing partitions. We say a non-crossing even partition π of $[2k]$ is *equivalent to* $\pi' \in NC(k)$, if there exists some $\pi' \in NC(k) = B'_1 \sqcup ... \sqcup B'_m$, such that by replacing all element $j \in B_i$ by $\{2j-1, 2j\}$, we get the set B_i, for any $i = 1, 2, ..., m$.

Claim: For $\pi = B_1 \sqcup ... \sqcup B_m \in \mathfrak{NC}(2k)$, each Q_i is even if and only if π is equivalent to some $\tilde{\pi} \in NC(k)$.

Proof of the claim. The "if" part is clear. For the "only if" part, just notice that when π is even, non-crossing and each Q_i is even, $max(B_i)$ turn out to be all even. The statement then follows by going over all the legs in the graphical representation of π from right to left. □

Set $\tilde{C_i} = i$, then after taking the limit,

$$\frac{C_{Q_i+1} \cdots C_{P_i}}{(q\gamma)^{l_i}} \xrightarrow[q\gamma\to\infty]{\gamma\to 0} 2^{2l_i-1}\tilde{C}_{Q_i+1} \cdots \tilde{C}_{P_i}.$$

In other words,

$$\begin{aligned} m_{2k} &= \sum_{\pi=B_1\sqcup...\sqcup B_m\in\mathfrak{NC}(2k)} W(\pi) \prod_{B\in\pi} k_{|B_i|} \\ &\xrightarrow[q\gamma\to\infty]{\gamma\to 0} \sum_{\tilde{\pi}=\tilde{B}_1\sqcup...\sqcup\tilde{B}_m\in NC(k)} \prod_{i=1}^{m} \left[(Q_i+1)\cdots(P_i)\cdot k'_{|\tilde{B}_i|}\right] \\ &= \lim_{\gamma\to 0} \sum_{\tilde{\pi}=\tilde{B}_1\sqcup...\sqcup\tilde{B}_m\in NC(k)} \prod_{i=1}^{m} \left[(\gamma+Q_i+1)\cdots(\gamma+P_i)\cdot k'_{|\tilde{B}_i|}\right] \\ &= \lim_{\gamma\to 0} \mathrm{T}^{\gamma}_{k\to m}(\{k'_l\}_{l=1}^{\infty})_k = \mathrm{T}^{0}_{k\to m}(\{k'_l\}_{l=1}^{\infty})_k \\ &= \sum_{\pi'=B'_1\sqcup...\sqcup B'_m\in P(k)} \prod_{i=1}^{m} k'_{|B'_i|}. \end{aligned} \tag{2.5.21}$$

The two equalities in the second to last row hold by Proposition 2.5.11 and Benaych-Georges, Cuenca, and Gorin 2022, Theorem 8.2 respectively. Then (2.5.20) follows from acting $\mathrm{T}^0_{m\to k}$ on both sides. □

Corollary 2.5.13. *For two real sequences* $\{m^a_{2k}\}_{k=1}^{\infty}$, $\{m^b_{2k}\}_{k=1}^{\infty}$, *set* $m^a_{2k-1} = m^b_{2k-1} = 0$

for $k = 1, 2, ...,$ and define

$$\{m_k^c\}_{k=1}^{\infty} := \lim_{\gamma\to 0,\ q\gamma\to\infty} \Big[\{m_k^a\}_{k=1}^{\infty} \boxplus_{q,\gamma} \{m_k^b\}_{k=1}^{\infty}\Big]. \tag{2.5.22}$$

Then $m_{2k-1}^c = 0$ for $k = 1, 2, ...,$ and the usuall cumulants of $\{m_{2k}^c\}_{k=1}^{\infty}$'s are given by the sum of the corresponding usual cumulants of $\{m_{2k}^a\}_{k=1}^{\infty}$ and $\{m_{2k}^b\}_{k=1}^{\infty}$, i.e,

$$\mathrm{T}_{m\to k}^{0}\Big(\{m_{2k}^c\}_{k=1}^{\infty}\Big) = \mathrm{T}_{m\to k}^{0}\Big(\{m_{2k}^a\}_{k=1}^{\infty}\Big) + \mathrm{T}_{m\to k}^{0}\Big(\{m_{2k}^b\}_{k=1}^{\infty}\Big). \tag{2.5.23}$$

Remark 2.5.14. *Suppose μ_a, μ_b are two probability measures on $\mathbb{R}_{\geq 0}$ that for $k = 1, 2, ...$*

$$m_{2k}^a = \int x^k d\mu_a,\ m_{2k}^b = \int x^k d\mu_b,$$

then $\{m_{2k}^c\}_{k=1}^{\infty}$ are the moments of the usual convolution of μ_a and μ_b.

Next, we consider $\boxplus_{q,\gamma}$ and match its asymptotic behavior with the classical and rectangular free convolution. Before that we recall the definitions of their corresponding cumulants. For $k \in \mathbb{Z}_{\geq 1}$, let $\mathrm{T}_{r\to m}^{'}$ denote the map sending the real sequence $\{r_l\}_{l=1}^{\infty}$ of classical free cumulants to the sequence $\{m_k\}_{k=1}^{\infty}$ of moments. Then there is a moment-cumulant formula:

$$m_k = \sum_{\pi = B_1 \sqcup ... \sqcup B_m \in NC(k)} \prod_{B\in\pi} r_{|B_i|} \tag{2.5.24}$$

for $\{m_k\}_{k=1}^{\infty} = \mathrm{T}_{r\to m}^{'}(\{r_l\}_{l=1}^{\infty})$. See e.g Novak 2014 for a reference.

Similarly, as defined in Benaych-Georges 2009, Section 3.1, rectangular free cumulants are a real sequence $\{c_l^q\}_{l=1}^{\infty}$ parametrized by $q \geq 1$, such that for $l = 1, 2, ..., c_{2l-1}^q = 0$, and c_{2l}^q are related with moments $\{m_k\}_{k=1}^{\infty}$ by the following identities:

$$m_{2k} = \sum_{\pi\in\mathfrak{NC}(2k)} q^{-e(\pi)} \prod_{B\in\pi} c_{|B_i|}, \tag{2.5.25}$$

where $e(\pi)$ = # of block B_i's with even $\min(B_i)$, and $m_{2k-1} = 0$ for $k = 1, 2, ...$ Denote the map sending even moments to rectangular free cumulants by $\mathrm{T}_{m\to k}^{\infty}$, i.e,

$\mathrm{T}^{\infty}_{m\to k}(\{m_{2k}\}_{k=1}^{\infty}) = \{c_l\}_{l=1}^{\infty}$.

Theorem 2.5.15. *Given a real sequence* $\{m_{2k}\}_{k=1}^{\infty}$, $q \geq 1$, *let*

$$\{k_l\}_{l=1}^{\infty} = \mathrm{T}^{q,\gamma}_{m\to k}(\{m_{2k}\}_{k=1}^{\infty}), \quad r_l^{\gamma} = (2\gamma)^{l-1} q^{\frac{l}{2}} \cdot k_l.$$

Then we have the following.

(a).

$$\lim_{\gamma\to\infty} \{r_l^{\gamma}\}_{l=1}^{\infty} = \mathrm{T}^{\infty}_{m\to k}(\{m_{2k}\}_{k=1}^{\infty}).$$

(b).

$$\lim_{\gamma\to\infty} \{r_l^{\gamma}\}_{l=1}^{\infty} = \mathrm{T}^{'}_{m\to r}(\{m_{2k}\}_{k=1}^{\infty})$$

when $q = 1$.

Remark 2.5.16. *(b) is a special case of (a) when* $q = 1$. *Such connection of rectangular free convolution and classical free convolution was first pointed out in Benaych-Georges 2009, Remark 2.2.*

Proof. It suffices to prove (a). By Theorem 2.5.5,

$$m_{2k} = \sum_{\pi\in\mathfrak{NC}(2k)} W(\pi) \prod_{B\in\pi} k_{|B_i|},$$

where $C_i (i = 1, 2, ...)$ are $2q\gamma, 2\gamma + 2, 2q\gamma + 2, 2\gamma + 4, 2q\gamma + 4...$ Hence by taking $\gamma \to \infty$, the right side above becomes

$$\sum_{\pi\in\mathfrak{NC}(2k)} q^{-c(\pi)} \prod_{B\in\pi} c_{|B_i|}, \tag{2.5.26}$$

where $c(\pi) := \#$ of block B_i's such that Q_i is odd. Since

$$\begin{aligned} Q_i \text{ is odd } &\iff \text{ there are odd elements of } B_1, ..., B_{i-1} \text{ bigger than } \max(B_i) \\ &\iff \text{ there are odd elements of } B_1, ..., B_{i-1} \text{ smaller than } \min(B_i) \iff \min(B_i) \text{ is even,} \end{aligned}$$

$c(\pi) = e(\pi)$, and (2.5.26) is equal to the right side of (2.5.25). $\square$

Recall also that similar to q-γ convolution, free convolution and rectangular free convolution are both binary operation of two probability measures linearized by free cumulants. Therefore Theorem 2.5.15 implies the following.

Corollary 2.5.17. *Given $q \geq 1$, for two real sequences $\{m_{2k}^a\}_{k=1}^{\infty}$, $\{m_{2k}^b\}_{k=1}^{\infty}$, set $m_{2k-1}^a = m_{2k-1}^b = 0$ for $k = 1, 2, ...$, and define*

$$\{m_k^c\}_{k=1}^{\infty} := \lim_{\gamma \to \infty} \Big[\{m_k^a\}_{k=1}^{\infty} \boxplus_{q,\gamma} \{m_k^b\}_{k=1}^{\infty}\Big]. \tag{2.5.27}$$

Then the free cumulants of $\{m_k^c\}$'s are given by the sum of the corresponding rectangular free cumulants of $\{m_k^a\}_{k=1}^{\infty}$ and $\{m_k^b\}_{k=1}^{\infty}$, i.e,

$$\mathrm{T}_{m \to k}^{\infty}\Big(\{m_k^c\}_{k=1}^{\infty}\Big) = \mathrm{T}_{m \to k}^{\infty}\Big(\{m_k^a\}_{k=1}^{\infty}\Big) + \mathrm{T}_{m \to k}^{\infty}\Big(\{m_k^b\}_{k=1}^{\infty}\Big). \tag{2.5.28}$$

Remark 2.5.18. *Suppose μ_a, μ_b are two symmetric probability measures on $\mathbb{R}$ that for $k = 1, 2, ...$*

$$m_k^a = \int x^k d\mu_a,\ m_k^a = \int x^k d\mu_a,$$

then $\{m_k^c\}_{k=1}^{\infty}$ are the moments of the rectangular free convolution of μ_a and μ_b.

Remark 2.5.19. *Similar results hold for classical free convolution when $q = 1$.*

2.5.5 Law of large numbers of Laguerre β ensembles

For $M \leq N$ and $\theta = \frac{1}{2}, 1, 2$, a $M \times N$ Wishart matrix X is a rectangular random matrix, whose entries are real/complex/real quaternionic i.i.d Gaussian random variables $\mathscr{N}(0,1)/\mathscr{N}(0,1) + i\mathscr{N}(0,1)/\mathscr{N}(0,1) + i\mathscr{N}(0,1) + j\mathscr{N}(0,1) + k\mathscr{N}(0,1)$. One can check directly that X satisfies the same invariant property given in Section 2.2.4, with M random singular values $\vec{x}_M = (x_{1,M} \geq ... \geq x_{M,M} \geq 0)$.

The density of $\vec{x}_M$ is (see e.g Forrester 2010, Chapter 3)

$$f(\vec{x}_M; M, N, \theta) = \frac{1}{Z_{M,N,\theta}} \prod_{i=1}^{M} \left[x_{M,i}^{2\theta(N-M+1)-1} \exp(-\frac{1}{2} x_{M,i}^2) \right] \prod_{1 \le j \le k \le M} (x_{M,j}^2 - x_{M,k}^2)^{2\theta}, \tag{2.5.29}$$

where $Z_{N,M,\theta}$ is the normalizing constant. While for general $\theta > 0$ there's again no skew field of real dimension 2θ, $f(\vec{x}_M; M, N, \theta)$ continues to make sense, and is defined as the so-called Laguerre β ensemble.

Remark 2.5.20. *It's easy to check that $f(\vec{x}_M; M, N, \theta)$ is an exponential decaying measure defined in Definition 2.2.29, and therefore by Theorem 2.2.31, its type BC Bessel generating function is defined and well-behaved under the action of type BC Dunkl operators.*

Proposition 2.5.21. *Let $G^L_{M,N,\theta}(z_1, ..., z_M)$ denote the type BC Bessel generating function*

$$\int_{x_{M,1} \ge \cdots \ge x_{M,M} \ge 0} \mathbb{B}(\vec{x}_M, z_1, ..., z_M; \theta, N) f(\vec{x}_M; M, N, \theta) dx_{M,1} \cdots dx_{M,M},$$

then

$$G^L_{M,N,\theta}(z_1, ..., z_M) = \exp\left[\frac{1}{2}(z_1^2 + ... + z_M^2)\right].$$

Proof. For $\theta = \frac{1}{2}, 1, 2$, one can use Definition 2.2.10 and check this by hand. For general $\theta > 0$, this is a special case of Roesler 2003b, Proposition 2.37.(2), such that in that identity y is set to be 0, and our $\mathbb{B}(\vec{a}, z_1, ..., z_M; \theta, N)$ is a symmetric version of $E_k(x, z)$, see Roesler 2003b, Definition 2.35 □

For each M, N, θ, Denote the random empirical measure of $f(\vec{x}_M; M, N, \theta)$ by $\mu_{M,N,\theta} := \frac{1}{M} \sum_{i=1}^{M} \delta_{x_{M,i}^2}$.

Theorem 2.5.22. *As $M \to \infty, N \to \infty, \theta \to 0, M\theta \to \gamma, N\theta \to q\gamma$,*

$$\mu_{M,N,\theta} \longrightarrow \mu_{q,\gamma}$$

weakly in moments, where $\mu_{q,\gamma}$ is a probability measure on $\mathbb{R}_{\geq 0}$, which is uniquely determined by its moments:

$$m_k^{'} = \int_{\mathbb{R}\geq 0} x^k d\mu_{q,\gamma} = \sum_{\pi} \prod_{i=1}^{k} C_{P_i}, \text{ for } k = 1, 2, ... \tag{2.5.30}$$

where C_l's, P_i's are defined in the same way as in Section 2.5.1, and π goes over all non-crossing perfect matching of $[2k]$.

Remark 2.5.23. *The weak convergence of $\mu_{M,N,\theta}$ to $\mu_{q,\gamma}$ in probability was already proved in S. N. M. R. Allez J. B. and Vivo 2013, in which the authors give an explicit density of $\mu_{q,\gamma}$ in terms of Whitakker function.*

Proof. By taking logarithm and partial derivatives of $G^L_{M,N,\theta}$ we have that $\{\vec{x}_M\}$ is *q*-γ-LLN appropriate with q-γ cumulant $k_2 = 1$, $k_l = 0$ for $l \neq 2$. By Theorem 2.5.5, only the set partitions that are formed by blocks of size two survive, and in this case $P_i = Q_i + 1$. (2.5.30) is then specified from (2.5.2).

Since the existence of probability measure $\mu_{q,\gamma}$ is known by S. N. M. R. Allez J. B. and Vivo 2013, it remains to show that the moments in (2.5.30) does correspond to a unique probability measure. This is the so-called Stieltjes moment problem, since the (potential) corresponding measure lies on $[0, \infty)$, see e.g Akhiezer 2020. We need to check

$$\sum_{k=1}^{\infty} (m_k^{'})^{-\frac{1}{2k}} = \infty.$$

Again by (2.5.30), m_k is a sum of $\prod_{i=1}^{k} C_{P_i}$'s. Among these summands the biggest term corresponds to $P_i = i$ for $i = 1, 2, ..., k$, and

$$\prod_{i=1}^{k} C_{P_i} \leq C(C+2)(C+4)\cdots(C+2k-2) = 2^k \frac{\Gamma(\frac{C}{2}+k)}{\Gamma(\frac{C}{2})},$$

where $C := max\{2q\gamma - 2, 2\gamma\}$. The number of non-crossing perfect matching is $Cat(k) = \frac{1}{k+1}\binom{2k}{k}$, the k^{th} Catalan number. Multiplying these two gives an upper bound of m_k. By

Stirling approximation, it turns out that

$$(m_k^{'})^{\frac{1}{2k}} \leq C_1 \cdot \sqrt{k}$$

for some positive constant C_1. Hence the series diverges. □

The limiting measure $\mu_{q,\gamma}$ is an q-γ analog of the Gaussian and semicircle law, in the sense that their only nonvanishing (q-γ/classical/free) cumulant is $k_2 = 1$.

Moreover, the connections to the usual and free convolution in Section 2.5.4 continues to hold in this special case. Indeed, one can show from (2.5.30) that

$$\frac{m_k^{'}}{(q\gamma)^k} \xrightarrow[q\gamma\to\infty]{\gamma\to 0} \sum_{\pi} \prod_{i=1}^{k} (2),$$

where π goes over all set partitions of $[k]$ into k blocks (which is indeed a single one), since any other non-crossing perfect matching has coefficient with $C_2 = 2\gamma + 2$, and therefore after rescaled by $(q\gamma)^k$ this term vanishes in the limit. The sum on the right is equal to 2^k, which means $m_k^{'} = 2^k$, and

$$\mu_{q,\gamma} \longrightarrow \delta_2$$

weakly when $q\gamma \to \infty, \gamma \to 0$.

On the other hand, by taking $q = 1$, $\gamma \to \infty$, (2.5.30) becomes

$$\frac{m_k^{'}}{(2\gamma)^k} \longrightarrow \# \text{ of non-crossing perfect matchings of } [2k] = Cat(k). \tag{2.5.31}$$

$Cat(k)$ is exactly the $2k^{th}$ moment of the semicircle law.

2.6 Duality between convolutions in high and low temperature

After studying the behavior of rectangular matrix additions in both high and low temperatures, we present a quantitative connection between these two regimes.

Recall that in low temperature regime, given two deterministic M-tuples $\vec{a}$, $\vec{b}$, the limit of $\vec{c} = \vec{a} \boxplus^{\theta}_{M,N} \vec{b}$ is a deterministic M-tuples $\vec{\lambda}$, where $\vec{\lambda}$ is the (M,N)-rectangular finite convolution of $\vec{a}$ and $\vec{b}$. For a M-tuples $\vec{a} = (a_1, ..., a_M)$, let $r_i = a_i^2$ for $i = 1, 2, ..., M$. Let $m_k^{'} = \frac{1}{M}(r_1^k + ... + r_d^k)$ be the finite version of moments, for $k = 1, 2, ..., M$. Then the (M,N)-rectangular finite convolution of $\vec{a}$ and $\vec{b}$ can be thought as a deterministic binary operation of $\{m_k^{'}(\vec{a})\}_{k=1}^{M}$ and $\{m_k^{'}(\vec{b})\}_{k=1}^{M}$. Similarly, we view the q-γ convolution of $\{m_k^a\}\}_{k=1}^{\infty}$ and $\{m_k^b\}\}_{k=1}^{\infty}$ as a deterministic binary operation of $\{m_k^a\}\}_{k=1}^{M}$ and $\{m_k^b\}\}_{k=1}^{M}$.

Theorem 2.6.1. *By identifying M with $-\gamma$, $\frac{M}{N}$ with q, m_{2k} with $m_k^{'}(-N)^k$ for $k = 1, 2, ..., M$, the (M,N)-rectangular convolution matches the q-γ convolution as binary operation of first M nontrivial moments.*

The theorem is claiming that under the above identification, the moment-cumulant formula of these two convolutions are the same, and therefore we need to introduce a version of cumulants for the rectangular finite convolution. For this we refer to Gribinski 2022, which considers sum of two invariant $M \times N$ $(M = N\lambda,\ \lambda \in [0,1])$ rectangular matrices as in Section 2.3, and defines the rectangular finite R-transform as the analog of the R-transform in (classical) free probability theory, in the sense that it linearizes the finite rectangular addition.

Definition 2.6.2. *Gribinski 2022, Definition 3.7 $R^{M,\lambda}_{S_{p_A}}(z)$ is the unique polynomial of degree M verifying*

$$R^{M,\lambda}_{S_{p_A}}(z) \equiv \frac{-1}{M} z \frac{d}{dz} \ln\left(\mathbb{E}e^{-T^{(N,M)}_{S_{p_A}} zNM}\right) \quad mod\ [z^{M+1}], \tag{2.6.1}$$

where $T^{(N,M)}_{S_{p_A}}$ is a random variable. By Gribinski 2022, p13, for $i = 1, 2, ..., M$,

$$\mathbb{E}(T^{(N,M)}_{S_{p_A}})^i = \frac{i!(m-i)!}{m!} \frac{(d-i)!}{d!} a_i, \tag{2.6.2}$$

where $a_i = e_i(\vec{r})$.

Inspired by the fact that (classical) R-transform is the generating function of free cumulants, We define the rectangular finite cumulants, such that

$$R^{d,\lambda}_{S_{p_A}}(z) = \sum_{l=1}^{d} k_l^{m,d} z^l. \tag{2.6.3}$$

$k_1^{m,d}, ..., k_d^{m,d}$ uniquely determine $r_1, ..., r_d$.

Proof of Theorem 2.6.1. We prove that under the following identification of parameters

$$\begin{aligned} k_l^{N,M} &\longleftrightarrow \frac{k_{2l}}{2}\gamma^{l-1} \text{ for } l = 1, 2, ..., M \\ M &\longleftrightarrow -\gamma \\ \frac{N}{M} &\longleftrightarrow q \\ N^n a_n &\longleftrightarrow c_n \\ m_{2k} &\longleftrightarrow m_k^{'} \cdot (-N)^k, \end{aligned} \tag{2.6.4}$$

the moment-cumulant relation in rectangular finite convolution and (2.5.10) match exactly.

We match the second formula of (2.5.10) and (2.6.1), which play the role of cumulant generating function in their own setting. Let $y^2 = (-M)z$, then the first formula of (2.5.10) becomes

$$\begin{aligned} &\exp\Big(\sum_{l=1}^{\infty} \frac{k_{2l}}{2l}\gamma^l z^l\Big) = \sum_{k=0}^{\infty} \frac{c_k}{(q\gamma)_k(\gamma)_k}(-M)^k z^k \\ \Longrightarrow &\sum_{l=1}^{\infty} k_{2l}\gamma^{l-1} z^l = -\frac{1}{d} z \frac{d}{dz} \ln\Big(\sum_{k=0}^{\infty} \frac{c_k}{(q\gamma)_k(\gamma)_k}(-M)^k z^k\Big). \end{aligned} \tag{2.6.5}$$

It remains to match the right side of (2.6.5) and (2.6.1), i.e, matching

$$\mathbb{E}e^{-T^{(N,M)}_{S_{p_A}} zNM} \qquad \text{with} \qquad \sum_{k=0}^{\infty} \frac{c_k}{(q\gamma)_k(\gamma)_k}(-M)^k z^k$$

for $k = 1, 2, ..., M$. This follows by Talor expanding $e^{-T^{(N,M)}_{S_{p_A}} zNM}$, (2.6.2) and (2.6.4).

Then we identify the first formula of (2.5.10) with the moment generating function in rectangular finite convolution. In the latter setting, recall that $a_n = e_n(\vec{r})$ for $n =$

$1, 2, ..., M$, and $m_k' = \frac{1}{M} p_k(\vec{r})$ for $k = 1, 2, ...$ Moreover, take $r_i = 0$ for all $i > M$, and identify a_n with $e_n(\vec{r})$ formally for $n > M$ (both have value 0) as well. Then on the rectangular finite addition side,

$$\begin{aligned}&\sum_{n=0}^{\infty} N^n a_n y^{2n} = \sum_{n=0}^{\infty} e_n(\vec{r})(Ny^2)^n \\ = &\prod_{n=1}^{\infty}(1 + r_n N y^2) = \exp\Big(-\sum_{k=1}^{\infty} \frac{p_k(\vec{r})(-N)^k y^{2k}}{k}\Big) = \exp\Big(-M \sum_{k=1}^{\infty} \frac{m_k'(-N)^k y^{2k}}{k}\Big).\end{aligned} \tag{2.6.6}$$

This matches the first formula of (2.5.10) under the identification of parameters. □

Remark 2.6.3. *After identifying the $k_1^{m,d}, ..., k_d^{m,d}$ with the first d even q-γ cumulants, one can define $k_l^{m,d}$ for $l \geq d+1$ for rectangular finite convolution, by the moment-cumulant relation of q-γ convolution under the same parameter identification in (2.6.4).*

Remark 2.6.4. *Note that in (2.6.4), both M and γ are positive, hence there's no choice of parameters that the finite rectangular cumulants coincide with q-γ cumulants. Instead, one can combine the domain of these two groups of parameters, and treat the result as an extension of the moment-cumulant relation to, say, $\gamma \in \mathbb{R}_{>0} \bigcup \mathbb{Z}_{\leq -1}$.*

Chapter 3

Random Sorting Networks: edge limit

3.1 Overview

3.1.1 Motivation

The main object in this article is the uniformly random sorting network which we abbreviate as RSN. Let us start by giving basic definitions. Consider the permutation group of n elements $\mathfrak{S}_n$. We use the one-row notation for the elements of $\mathfrak{S}_n$ representing them as sequences $(a_1\ a_2\ ...\ a_n)$. We let τ_k be the adjacent swap $(k, k+1)$, $1 \le k \le n-1$, so that

$$(a_1\ a_2\ \dots\ a_n) \cdot \tau_k = (a_1\ a_2\ a_{k-1}\ a_{k+1}\ a_k\ a_{k+2}\ \dots\ a_n).$$

A sorting network of size n is a shortest sequence of swaps, whose product equals the reverse permutation $(n\ n-1\ ...\ 2\ 1)$. By counting inversions in permutations, one shows that the length N of such shortest sequence is $N = \binom{n}{2}$. Let Ω_n denote the set of all sorting networks of size n. Thus, elements of Ω_n are sequences $(s_1, \dots, s_N)$ such that

$$\tau_{s_1} \tau_{s_2} \cdots \tau_{s_N} = (n\ n-1\ ...\ 2\ 1).$$

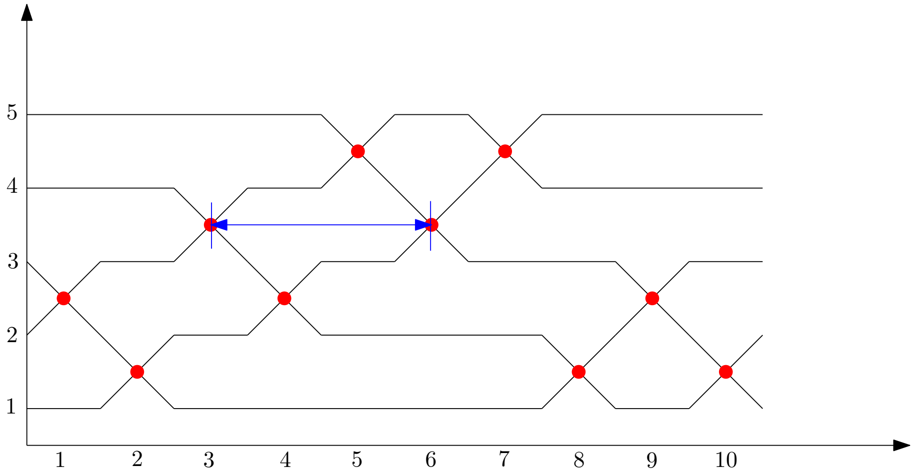

Figure 3.1: A sorting network $(s_1 \dots s_N)$, $N = \binom{n}{2}$ can be represented as a diagram of n wires, with wires at heights k and $k+1$ being swapped at time i whenever $s_i = k$. The figure shows the wiring diagram of the sorting network $(2, 1, 3, 2, 4, 3, 4, 1, 2, 1)$ with $n = 5$. The blue double arrow shows a spacing in row 3, which is a time interval between two adjacent swaps τ_3 and it has length 3 in our example.

Sorting networks can be drawn as wiring diagrams, see Figure 3.1. Whenever $s_t = k$, we say that swap $(k, k+1)$ occurs at time t; in the wiring diagram this corresponds to two wires intersecting at the point $(t, k + \frac{1}{2})$.

The study of sorting networks is a fruitful topic, with the first mathematical results going back to Stanley 1984 where Stanley computed the number of elements in Ω_n:

$$|\Omega_n| = \frac{\binom{n}{2}!}{\prod_{j=1}^{n-1}(2n-1-2j)^j}.$$

A recent point of interest is the study of uniformly random sorting networks initiated by Angel, Holroyd, Romik, and Virag in O. Angel, A. Holroyd, et al. 2007. The probabilistic results can be split into two groups: global and local. On the global side, O. Angel, A. Holroyd, et al. 2007 proved that the density of swaps approximates the semi-circle law[1] as $n \to \infty$; O. Angel, A. Holroyd, et al. 2007 predicted and D. Dauvergne and B. Virag 2020; D. Dauvergne 2022 proved (among other results) that individual trajectories in the

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[1] There is no known direct connection to the Wigner semi-circle law of the random matrix theory and the match seems coincidental.

wiring diagram become (random) sine curves; see also M. Kotowski 2016; M. Rahmana, B. Virag, and M. Vizer 2019 for related results.

Our paper belongs to the second group of results, O. Angel, V. Gorin, and A. E. Holroyd 2012; A. Rozinov 2016; O. Angel, D. Dauvergne, et al. 2019; V. Gorin and M. Rahman 2019, which studies local limits. The most basic local characteristic of the sorting network is *spacing* in row k, which is a distance between two occurrences of the same swap $s_i = k$ in the sorting network, cf. Figure 3.1. A. Rozinov 2016 and V. Gorin and M. Rahman 2019 discovered a link between the asymptotic laws of spacings as $n \to \infty$ and random matrix theory. Namely, A. Rozinov 2016 matched the asymptotic law of the spacing in row 1 with the distance between two eigenvalues of 2×2 matrix of Gaussian Orthogonal Ensemble (GOE) of real symmetric random matrices. V. Gorin and M. Rahman 2019 dealt with spacings in row αn with $0 < \alpha < 1$ and showed that after proper rescaling they converge to the *Gaudin–Mehta law*, which is the universal asymptotic law for spacings between eigenvalues of real symmetric matrices of very large sizes.

Comparing the results of A. Rozinov 2016 and V. Gorin and M. Rahman 2019, we see that the former links the spacings in the extreme row (i.e. at the end-point of the edge) to 2×2 real symmetric matrices, while the latter links the spacings in the bulk rows to the real symmetric matrices of infinite sizes. The observed gap in the asymptotics between 2×2 and $\infty \times \infty$ matrices motivated the central question of our paper: we would like to understand, whether distributions related to $k \times k$ random matrices can be also found in the asymptotic laws for random sorting networks.

The answer presented in this text is both Yes and No. On one side, we are so far unable to find any connections of sorting networks to eigenvalues of real symmetric $k \times k$ matrices (which would be the most direct interpolation between A. Rozinov 2016 and V. Gorin and M. Rahman 2019). On the other hand, by slightly adjusting our point view, we find that the law of the asymptotic spacing in row k can be expressed in terms of the eigenvalues of $2k \times 2k$ random *anti-symmetric* Gaussian matrices — they are known in the literature as aGUE, since their analysis reveals connections to the tools used for the Gaussian Unitary

Ensemble[2], see M. L. Mehta 2004, Chapter 13 and P. J. Forrester and E. Nordenstam 2009.

3.1.2 Main Results

Here is the precise definition of the random matrix object appearing in our asymptotic results.

Definition 3.1.1. *Let Y be an $\ell \times \ell$ matrix whose entries are i.i.d. standard Gaussian random variables $N(0,1)$. Then*

$$M_\ell = \frac{\mathbf{i}}{2}\left(Y - Y^T\right)$$

is called $\ell \times \ell$ anti-symmetric GUE or aGUE for being short.

Note that the eigenvalues of M_ℓ are real and come in pairs: λ is an eigenvalue if and only if so is $-\lambda$. When ℓ is odd, M_ℓ is necessary degenerate, i.e. it has a zero eigenvalue; for even ℓ, zero is not an eigenvalue almost surely.

We also would like to deal with eigenvalues of all M_ℓ together.

Definition 3.1.2. *Let M be an infinite aGUE, i.e, an $\infty \times \infty$ matrix whose l^{th} corner is a $l \times l$ aGUE. The aGUE–corners process is a point process $\mathcal{X}$ on $\mathbb{Z}_{\geq 0} \times \mathbb{R}_{\geq 0}$: we put a particle at (l, u) whenever u is one of the $\lfloor \frac{l}{2} \rfloor$ positive eigenvalues of the top-left $l \times l$ corner of M.*

Remark 3.1.3. *aGUE is studied in P. J. Forrester and E. Nordenstam 2009, and it turns out that its corners process $\mathcal{X}$ has a determinantal structure. We give the correlation kernel of $\mathcal{X}$ in Theorem 3.2.9.*

Definition 3.1.4. *$\mathbf{T}_{FS}(k)$ is the smallest positive eigenvalue of a $2k \times 2k$ aGUE.*

The determinantal structure for the distribution of eigenvalues of M_{2k} (see M. L. Mehta 2004, Chapter 13 and P. J. Forrester and E. Nordenstam 2009) leads to an expression for

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[2] However, up to multiplication by $\mathbf{i}$, matrix elements of aGUE are real, rather than complex.

the distribution of $\mathbf{T}_{FS}(k)$ as a series, which can be identified with a Fredholm determinant:

$$\mathbb{P}\left[\mathbf{T}_{\mathrm{FS}}(k) > t\right] = 1 + \sum_{l=1}^{\infty} \frac{(-1)^l}{l!} \int_0^t \cdots \int_0^t \det[K^{(k)}(u_i, u_j)]\, \mathrm{d}u_1 \cdots \mathrm{d}u_l,$$

where $K^{(k)}$ is a correlation kernel, expressed through the Hermite polynomials $H_i(x)$ as

$$K^{(k)}(u_1, u_2) = e^{-\frac{1}{2}u_1^2 - \frac{1}{2}u_2^2} \sum_{\ell=0}^{k-1} \frac{H_{2\ell}(u_1) H_{2\ell}(u_2)}{\sqrt{\pi}(2\ell)! 2^{2\ell-1}}. \tag{3.1.1}$$

Here we use the "physicist's normalization", so that $H_i(x)$, $i = 0, 1, \ldots,$ is a polynomial of degree i with leading coefficient 2^i and such that

$$\int_{\mathbb{R}} H_i(x) H_j(x) e^{-x^2} dx = \delta_{ij} j! 2^j \sqrt{\pi}. \tag{3.1.2}$$

We can now state our first theorem.

Theorem 3.1.5 (First swapping law). *Fix $k \in \mathbb{Z}_{\geq 1}$, and let $\mathbf{T}_{FS,n}(k)$ denote the first occurrence time of the swap $(k, k{+}1)$ in a uniformly random sorting network $(s_1, \ldots, s_N) \in \Omega_n$, $N = \binom{n}{2}$:*

$$\mathbf{T}_{FS,n}(k) = \min\{t \geq 1 : \, s_t = k\}.$$

Then we have the following convergence in law:

$$\lim_{n\to\infty} \frac{\mathbf{T}_{FS,n}(k)}{n^{\frac{3}{2}}} = \frac{1}{2} \mathbf{T}_{FS}(k). \tag{3.1.3}$$

In order to connect the first swaps to the spacings, we are going to use translation invariance of the random sorting networks (cf. O. Angel, A. Holroyd, et al. 2007, Theorem 1, (i)), which is based on the following combinatorial fact:

Proposition 3.1.6. *Let $N = \binom{n}{2}$. Then $(s_1, ..., s_N) \in \Omega_n$ if and only if $(s_2, ..., s_N, n{-}s_1) \in \Omega_n$.*

Applying Proposition 3.1.6 one readily proves that for all $1 \le r \le r+t \le N$,

$$\mathbb{P}\left[\mathbf{T}_{\mathrm{FS}}(k) > t\right] = \mathbb{P}\left[\text{there are no swaps } (k,k+1) \text{ at times } r+1, r+2, \dots, r+t\right],$$

which can be used to identify the distribution of a spacing in row k. Before doing so, we need to specify the definition of a spacing. In fact, there are two natural definitions, which lead to distinct random variables. For the definition it is convenient to extend a sorting network to a $2N$–periodic $\mathbb{Z}$–indexed sequence:

Definition 3.1.7. *Given $(s_1 \dots s_N) \in \Omega_n$, we extend it to a sequence $(s_t)_{t\in\mathbb{Z}}$ by requiring $s_{t+N} = n - s_t$ for all $t \in \mathbb{Z}$. We call $(s_t)_{t\in\mathbb{Z}}$ the* periodic extension *of $(s_1 \dots s_N) \in \Omega_n$.*

For instance $(1\,2\,1) \in \Omega_3$ is extended to the infinite $\mathbb{Z}$–indexed sequence $(\dots 2\,1\,2\,1\,2\,\dots)$ with 1s at odd positions and 2s at even positions. The reason for this particular definition is contained in the following straightforward corollary of Proposition 3.1.6:

Corollary 3.1.8. *If $(s_1, \dots, s_N)$ is a uniformly random element of Ω_n, then its periodic extension $(s_t)_{t\in\mathbb{Z}}$ is translation-invariant: for each $a \in \mathbb{Z}$ we have a distributional identity*

$$(s_{t+a})_{t\in\mathbb{Z}} \overset{d}{=} (s_t)_{t\in\mathbb{Z}}.$$

Definition 3.1.9 (First definition of the spacing)**.** *Fix $n = 1, 2, \dots,$ $k \in \{1, 2, ..., n-1\}$ and $a \in \mathbb{Z}$. Let $(s_t)_{t\in\mathbb{Z}}$ be the periodic extension of a uniformly random sorting network in Ω_n. Set*

$$X = \max\{t \le a : s_t = k\}, \qquad Y = \min\{t > a : s_t = k\}.$$

We define the spacing on the k^{th} row of a sorting network of size n as $\mathrm{Sp}_{k,n} := Y - X$.

Remark 3.1.10. *While the definition depends on the choice of a, Corollary 3.1.8 implies that the distribution of $\mathrm{Sp}_{k,n}$ is the same for all $a \in \mathbb{Z}$. Hence, we omit a from the notation.*

The $n \to \infty$ limit of $\mathrm{Sp}_{k,n}$ turns out to be connected to $\mathbf{T}_{\mathrm{FS}}(k)$ of Definition 3.1.4.

Theorem 3.1.11. *Fix $k \in \mathbb{Z}_{\geq 1}$. We have the following convergence in distribution:*

$$\lim_{n\to\infty} \frac{\mathrm{Sp}_{k,n}}{n^{\frac{3}{2}}} = \frac{1}{2} Z_k,$$

where Z_k is a positive random variable of density $g_k(x)$, $x > 0$, given by

$$g_k(x) = x \, \frac{\partial^2}{\partial x^2} \big(\mathbb{P} \left[\mathbf{T}_{\mathrm{FS}}(k) > x \right] \big). \tag{3.1.4}$$

Here is an alternative definition of the spacing.

Definition 3.1.12 (Second definition of the spacing)**.** *Fix $n = 1, 2, \dots,$ $k \in \{1, 2, ..., n-1\}$ and $a \in \mathbb{Z}$. Let $(s_t)_{t\in\mathbb{Z}}$ be the periodic extension of a uniformly random sorting network in Ω_n. Set*

$$Y = \min\{t > a : s_t = k\}.$$

We define the spacing $\widehat{\mathrm{Sp}}_{k,n}$ on the k^{th} row of a sorting network of size n as a random variable whose law is the distribution of $Y - a$ conditional on the event $\{s_a = k\}$.

Remark 3.1.13. *As in Definition 3.1.9, the choice of a does not affect the distribution of $\widehat{\mathrm{Sp}}_{k,n}$.*

Both definitions of the spacing have their own merits. Definition 3.1.12 is the one preferred in theoretical physics and random matrix literature. However, a disadvantage of Definition 3.1.12 is that it is hard to sample or observe $\widehat{\mathrm{Sp}}_{k,n}$, as it fails to be a random variable on the state space of all sorting networks; on the other hand, $\mathrm{Sp}_{k,n}$ is a function of a sorting network. The $n \to \infty$ limit of $\widehat{\mathrm{Sp}}_{k,n}$ is still connected to $\mathbf{T}_{\mathrm{FS}}(k)$ of Definition 3.1.4, but in a slightly different way.

Theorem 3.1.14. *Fix $k \in \mathbb{Z}_{\geq 1}$. We have the following convergence in distribution:*

$$\lim_{n\to\infty} \frac{\widehat{\mathrm{Sp}}_{k,n}}{n^{\frac{3}{2}}} = \frac{1}{2} \widehat{Z}_k,$$

where $\widehat{Z}_k$ is a positive random variable of density $\widehat{g}_k(x)$, $x > 0$, given by

$$\widehat{g}_k(x) = \frac{\sqrt{\pi}}{2}\frac{(2k-2)!!}{(2k-1)!!}\frac{\partial^2}{\partial x^2}\big(\mathbb{P}\left[\mathbf{T}_{\mathrm{FS}}(k) > x\right]\big). \tag{3.1.5}$$

Remark 3.1.15. *In $k = 1$ case $\mathbf{T}_{\mathrm{FS}}(k)$ becomes the absolute value of a Gaussian random variable, $|N(0, \frac{1}{2})|$. Hence, $\widehat{Z}_1$ has density $2xe^{-x^2}$, $x > 0$, which matches the result of A. Rozinov 2016.*

In addition, we present an alternative expression for $\widehat{Z}_k$ by identifying it with a marginal of a certain two-dimensional determinantal point process; we refer to A. Borodin 2011 and references therein for general discussion of the determinantal point processes.

Definition 3.1.16. *Let $\mathcal{X}_k^{'}$ be a determinantal point process on $\mathbb{Z}_{\geq 2} \times \mathbb{R}_{\geq 0}$, such that with respect to the product of the counting measure on the first coordinate and the Lebesgue measure on the second coordinate, the correlation kernel is*

$$\begin{aligned} K_k^{'}(x_1, u_1; x_2, u_2) = \mathbf{1}_{\{u_2<u_1,\, x_2<x_1\}}\, &\frac{(u_2-u_1)^{x_1-x_2-1}}{(x_1-x_2-1)!} \\ &+ \frac{1}{(2\pi\mathbf{i})^2} \oint\limits_{C_z[0,\infty)} dz \oint\limits_{C_w[0,x_1)} dw\, \frac{\Gamma(-w)}{\Gamma(z+1)} \frac{\Gamma(z+x_2+1)}{\Gamma(x_1-w)} \frac{\Gamma(-\frac{z+x_2}{2})}{\Gamma(\frac{-x_1+w+1}{2})} \\ &\times \frac{z+x_2-2k}{z+x_2-2k+1} \frac{x_1-w-2k}{x_1-w-2k-1} \frac{u_1^w u_2^z}{w+z+x_2-x_1+1}. \end{aligned}$$

Here $u_1, u_2 \in \mathbb{R}_{>0}$ and $x_1, x_2 \in \mathbb{Z}_{\geq 2}$; when u_1 or u_2 is equal to 0, the kernel is defined as the limit as u_1 or u_2 tends to 0. $C_z[0,\infty), C_w[0, x_1)$ are counterclockwise–oriented contours which enclose the integers in $[0,\infty)$ and $[0, x_1)$, respectively, and are arranged so that C_z and $x_1 - x_2 - 1 - C_w$ are disjoint, as in Figure 3.5.

Remark 3.1.17. *For $m \geq 0$, there are m particles in the $2m^{th}$ and $2m+1^{th}$ levels of $\mathcal{X}_k^{'}$ (i.e. with first coordinates $2m$ and $2m+1$, respectively), except that on the $2k^{th}$ level there are only $k-1$ particles.*

If we denote the j^{th} particle(from top to bottom) on the l^{th} level by x_j^l and set $x_j^l = 0$

for $j >$ # particles in the l^{th} level, then the particles of $\mathcal{X}_k'$ interlace, i.e,

$$x_{j+1}^{l+1} \leq x_j^l \leq x_j^{l+1}.$$

While we do not prove this, we expect that compared to $\mathcal{X}$, $\mathcal{X}_k'$ can be thought as the corner process of aGUE conditioned on the event that the smallest positive eigenvalues of the $2k^{th}$ corner is 0, see Remark 3.3.2.

Theorem 3.1.18. *Let $\widehat{Z}_k$ be as in Theorem 3.1.14 and let Z_l' denote the coordinate of the closest to the origin particle on the l^{th} level in the point process $\mathcal{X}_k'$.*

- *If $k = 1$, then the law of $\widehat{Z}_1$ coincides with that of Z_3'.*
- *If $k \geq 2$, then the law of $\widehat{Z}_k$ coincides with that of $\min\{Z_{2k-1}', Z_{2k+1}'\}$.*

3.1.3 Methods

Here is a sketch of our proof of Theorem 3.1.5:

- The Edelman-Greene bijection P. Edelman and C. Greene 1987 maps the problem to the study of the asymptotic distribution of the entries of uniformly random standard Young tableau of staircase shape, see Section 3.4.2.
- We replace standard Young tableau by a continuous version, which we call Poissonized Young tableau. In Section 3.3.2 we show that the error of this replacement is negligible in our limit regime.
- We use the formula of V. Gorin and M. Rahman 2019 for the correlation kernel of the determinantal point process, describing the entries of a Poissonized Young tableau.
- Asymptotic analysis of this formula leads in Theorems 3.3.5 and 3.3.6 to the limiting process described in terms of double contour integral formulas for its correlation kernel.

- Expanding the integrals in residues, performing resummations, and using the Gibbs (conditional uniformity) properties of the point processes under consideration, we reveal in Section 3.3.4 a match to the distribution of eigenvalues of aGUE matrices.

We remark that (in contrast to the results in the bulk, i.e., for k of order αn, $0 < \alpha < 1$, as in V. Gorin and M. Rahman 2019; O. Angel, D. Dauvergne, et al. 2019) we do not claim any results for the joint asymptotic distribution of several swaps; Theorem 3.1.5 only deals with one-dimensional marginal distribution. A technical reason is that the Edelman-Green bijection does not have any straightforward continuous version acting on the eigenvalues of aGUE of finite sizes: it seems that one needs to deal simultaneously with aGUE of all sizes, which is not a particularly transparent operation. In future, it would be interesting to find a way to overcome this difficulty.

Theorems 3.1.11 and 3.1.14 are proven in Section 3.4 as corollaries of Theorem 3.1.5. The central idea is to develop discrete versions of (3.1.4), (3.1.5) valid for random sorting networks of finite sizes. In fact, these discrete statements are valid for any periodic point processes and, thus, can be of independent interest, see Proposition 3.4.2.

Finally, Theorem 3.1.18 is proven in Section 3.4 by repeating the arguments of Theorem 3.1.5 for the sorting networks conditional on the event $\{s_1 = k\}$. This requires asymptotic analysis of entries of standard Young tableau of a slightly different shape, which is still possible by our methods. Thus, we arrive at the identities of Theorem 3.1.18 in an indirect way, by showing that the two sides of the identity are $n \to \infty$ limits of the same discrete distributions.

3.2 Preliminaries

One of the key tools in studying sorting networks is the Edelman-Green bijection (see Section 3.4.2 for the details), which maps them to Standard Young Tableaux (SYT). In this and the next section we develop asymptotic results for SYT, which will be ultimately used in Section 3.4 to prove the theorems about the random sorting networks announced in the introduction.

In addition, in the last subsection of this section we recall the properties of the eigenvalues of aGUE, which will be useful later on.

3.2.1 Young diagrams and Young tableaux

A partition is a sequence of non-negative integers $\lambda_1 \geq \lambda_2 \geq ... \geq 0$ such that $|\lambda| := \sum_{i=1}^{\infty} \lambda_i < \infty$. The length of λ, denoted by $l(\lambda)$, is the number of strictly positive $\lambda_i s$.

We identify a partition λ with a collection of $N = |\lambda|$ boxes, such that there are λ_i boxes in row i. We use coordinate (i, j) to denote the j-th box at row i. In other words, the coordinates of boxes are $\{(i, j) \mid 1 \leq j \leq \lambda_i,\ i = 1, 2, \ldots\}$. Such a combinatorial object is named a *Young diagram*, still denoted as λ. In particular, we say the Young diagram is of staircase shape and write $\lambda = \Delta_n$ for some $n \in \mathbb{Z}_{\geq 2}$, if $\lambda_i = n - i$ for $i = 1, 2, \ldots, n$. We also use the diagram $\lambda = \Delta_n \setminus (n - k, k)$ for $1 \leq k \leq n - 1$, which has $\lambda_i = n - i$ for $i \neq n - k$ and $\lambda_{n-k} = k - 1$.

A *standard Young tableau* (SYT) of shape λ is an insertion of numbers 1,2,...,$|\lambda|$ into boxes of λ without repetitions, such that the numbers in each row and column strictly increase from left to right and from top to bottom. The numbers within a SYT are its *entries*. Fixing the Young diagram λ, we can get a uniformly random SYT of shape λ, denoted as $\mathbf{T}_\lambda$ by considering a uniform measure on the space of all SYT of shape λ. See Figure 3.2 for an example with $\lambda = \Delta_6$.

A *Poissonized Young tableau* (PYT) of shape λ is an insertion of $|\lambda|$ real numbers in $[0, 1]$ into boxes of λ, such that the numbers strictly increase along each row and column. We use PYT(λ) to denote the space of all PYT of shape λ. Note that a given PYT of shape λ can be transformed canonically to a SYT of the same shape, by replacing the k-th smallest entry with k. In the opposite direction, we can get a uniformly random PYT of shape λ, denoted as $\mathbb{T}_\lambda$ by the following steps: first generate a uniformly random SYT of shape λ, then independently sample $|\lambda|$ i.i.d. random variables uniform in [0,1], and replace the entry k with the k-th smallest random variable.

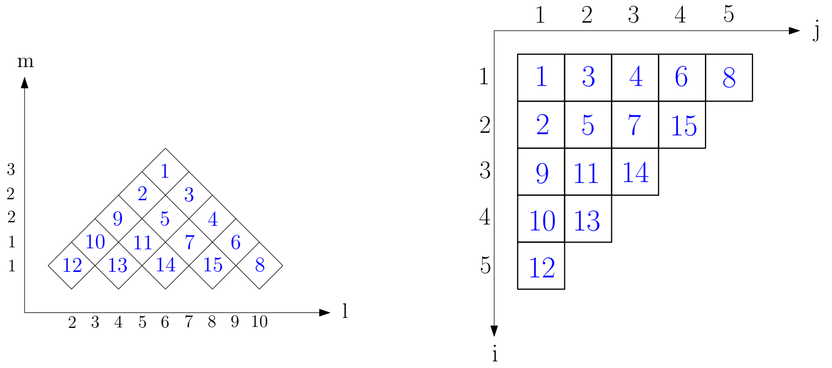

Figure 3.2: An example of staircase shaped standard Young tableau, with $n = 6$ in two different coordinate systems: the right one is defined in Section 3.2.1 and left one is defined in Section 3.2.2.

3.2.2 Rotation of Young Tableaux

Let T_{Δ_n} denote a standard Young tableau of shape Δ_n. We make a change of coordinates $(i, j) \to (l, m)$ for the entries in T_{Δ_n} by letting

$$l = n - i + j, \qquad m = \begin{cases} \frac{n-i-j}{2} + 1, & n - i - j \text{ is even,} \\ \frac{n-i-j+1}{2}, & n - i - j \text{ is odd.} \end{cases}$$

We let $T_{\Delta_n}(l, m)$ denote the entry of T_{Δ_n} on the i^{th} row and the j^{th} column, where i and j are reconstructed from l and m by inverting the above formulas. In more detail, we have

$$i = \begin{cases} n - \frac{l}{2} - m + 1, & l \text{ is even,} \\ n - \frac{l}{2} - m + \frac{1}{2}, & l \text{ is odd;} \end{cases} \qquad j = \begin{cases} \frac{l}{2} - m + 1, & l \text{ is even,} \\ \frac{l}{2} - m + \frac{1}{2}, & l \text{ is odd.} \end{cases}$$

The allowed values for (i, j) are: $i \geq 1$, $j \geq 1$, $i + j \leq n$. The allowed values for (l, m) are $l = 2, 3, 4, \ldots, 2n - 2$ and $m = 1, 2, \ldots, \lfloor \frac{\min(l, 2n-l)}{2} \rfloor$. The transformation $(i, j) \to (l, m)$ is essentially a clockwise rotation by 45 degrees, as can be seen in Figure 3.2.

Similarly, for standard Young tableau $T_{\Delta_n \setminus (n-k,k)}$ we make the same change of coordinates as above and deal with $T_{\Delta_n \setminus (n-k,k)}(l, m)$. Formally, the entry at $(l, m) = (2k, 1)$ does not exist, because it corresponds to the removed box at $(n - k, k)$, but we can also think about this entry as being $N = \binom{n}{2}$; in this way $T_{\Delta_n \setminus (n-k,k)}$ becomes T_{Δ_n} with the additional restriction $T_{\Delta_n}(2k, 1) = N$.

For Poissonized Young tableau (PYT) $\mathbb{T}_\lambda$ of shapes $\lambda = \Delta_n$ or $\lambda = \Delta_n \setminus (n-k, k)$, we define the change of coordinate in exactly the same way and use (m, l) coordinate system.

3.2.3 Point processes associated with PYT

Let $\mathbb{T}_\lambda$ be a PYT of shape λ. As in Section 3.2.2, we will focus on the case $\lambda = \Delta_n$ or $\lambda = \Delta_n \setminus (n-k, k)$, for which we induce a point process on $\mathbb{Z}_{\geq 0} \times \mathbb{R}_{\geq 0}$ from its entries.

Definition 3.2.1. *The projection of a* PYT $\mathbb{T}$ *with shape* Δ_n *or* $\Delta_n \setminus (n-k, k)$ *is a point configuration on* $\mathbb{Z}_{\geq 0} \times [0, 1]$, *such that there's a particle at* (l, u), *if for some* $l \in \mathbb{Z}_{\geq 2}$ *and* $m \in \mathbb{Z}_{\geq 1}$,

$$u = 1 - \mathbb{T}(l, m).$$

By definition, $u = 1 - \mathbb{T}(l, m)$ is the m^{th} lowest particle on level $\{l\} \times \mathbb{R}_{\geq 0}$ (except for level $l = 2k$ in $\mathbb{T}_{\Delta_n \setminus (n-k,k)}$, where $\mathbb{T}(2k, 1)$ is missing), and the particles on neighboring levels interlace by the setting of PYT, see Figure 3.3.

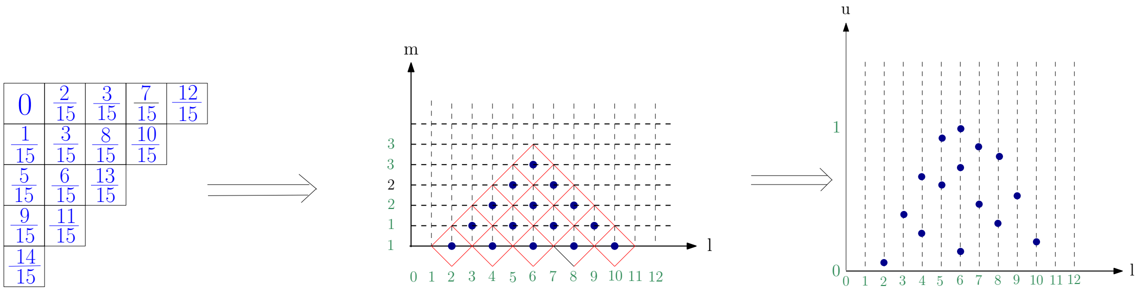

Figure 3.3: Producing a point configuration from a PYT T of shape Δ_6 with entries given in the figure as in Section 3.2.2 and 3.2.3: first rotate Δ_6 clockwise by 45 degrees, then project its entries into a configuration of interlacing particles.

If we take $\mathbb{T}_\lambda$ to be uniformly random, then the projection of $\mathbb{T}_\lambda$ becomes a point process on $\mathbb{Z}_{\geq 0} \times [0, 1]$. Note that this random point configuration is simple almost surely. We also would like to rescale the u–coordinate.

Definition 3.2.2. *The point process* $\mathcal{X}_n'$ *on* $\mathbb{Z}_{\geq 0} \times \mathbb{R}_{\geq 0}$ *is obtained by taking a uniformly random* PYT $\mathbb{T}_{\Delta_n}$ *, projecting its entries on* $\mathbb{Z}_{\geq 0} \times [0,1]$*, and then rescaling the second coordinate of each particle by the map* $(l,u) \mapsto (l, n^{\frac{1}{2}} \cdot u)$*. Similarly,* $\mathcal{X}_{k,n}'$ *is obtained from uniformly random PYT* $\mathbb{T}_{\Delta_n \setminus (n-k,k)}$ *by the same procedure.*

We study the asymptotic behavior of $\mathcal{X}_n'$ and $\mathcal{X}_{k,n}'$ in Section 3.3.3. Our analysis uses the technique of determinantal point processes with correlation kernels given by double contour integrals, which we present next.

3.2.4 Determinantal Point Process

Here's a brief review of some standard definitions, for a thorough introduction see A. Borodin 2011; D. Daley and D. Vere-Jones 2003. Let S be a locally compact Polish space, say $\mathbb{Z}$ or $\mathbb{R}^n$. We consider the space of discrete subsets in S, which consists of countable subsets X of S without accumulation points, and identify each such point configuration with the measure $\sum_{x \in X} \delta_x$. The topology of this space is given as the weak topology of Radon measures on S. We further use Borel $\sigma-$algebra corresponding to this topology.

A point process is a random discrete subset of S, and we assume that it is simple a.s, i.e, all the points have distinct positions. A *determinantal point process* $\mathcal{X}$ on S is a point process with correlation kernel $K : S \times S \to \mathbb{R}$, and a reference measure μ on S, such that for any $f : S^k \to \mathbb{R}$ with compact support,

$$\mathbb{E}\left[\sum_{\substack{x_1,\ldots,x_k \in \mathcal{X} \\ x_1,\ldots,x_k \text{ are distinct}}} f(x_1,\ldots,x_k) \right] = \int\limits_{S^k} \det\left[K(x_i,x_j)\right] f(x_1,\ldots,x_k)\, \mu^{\otimes k}(dx_1,\ldots,dx_k). \tag{3.2.1}$$

Expectations of the form given by the l.h.s. of (3.2.1) determine the law of $\mathcal{X}$ under mild conditions on K, see A. Lenard 1973. This will always be the case in this text as the correlation kernels we consider will be continuous.

The determinantal point processes appearing in this paper live in space $S = \mathbb{Z} \times [0,1]$ or $S = \mathbb{Z} \times \mathbb{R}_{\geq 0}$, with reference measure being the product of counting measure and Lebesgue measure, denoted by $\#_{\mathbb{Z}} \times \mathcal{L}([0,1])$ or $\#_{\mathbb{Z}} \times \mathcal{L}(\mathbb{R}_{\geq 0})$ resp. We use the following lemma, whose proof we omit.

Lemma 3.2.3. *Let* Y_n *be a determinantal point process on* $\mathbb{Z}\times\mathbb{R}_{>0}$ *with reference measure* $\#_{\mathbb{Z}}\times\mathcal{L}(\mathbb{R}_{>0})$ *and correlation kernel* K_n*. Then the sequence* Y_n *converges weakly as* $n\to\infty$ *to a determinantal point process* $\mathcal{X}$ *with correlation kernel* K *on* $\mathbb{Z}\times\mathbb{R}_{\geq 0}$ *(with reference measure* $\#_{\mathbb{Z}}\times\mathbb{R}_{\geq 0}$*) and with almost surely no particles at* $(i,0)$*,* $i\in\mathbb{Z}$*, if*

$K_n\to K$ *uniformly on compact subsets of* $\mathbb{Z}\times\mathbb{R}_{\geq 0}$*.*

3.2.5 Uniformly Random Projection

When PYT $\mathbb{T}_{\Delta_n}$ and $\mathbb{T}_{\Delta_n\setminus(n-k,k)}$ are uniformly random, their projections are point processes on $\mathbb{Z}_{\geq 0}\times[0,1]$, whose correlation functions were computed in V. Gorin and M. Rahman 2019. We restate the result there for our case in the following theorem, which is a stepping stone of the proofs of our main results.

Theorem 3.2.4. *The processes* $\mathcal{X}_n'$ *and* $\mathcal{X}_{k,n}'$ *of Definition 3.2.2 are determinantal point process on* $\{2,3,\ldots,2n-2\}\times\mathbb{R}_{>0}$ *with correlation kernel* $K_\lambda(x_1,u_1;x_2,u_2)$ *given for* $x_1,x_2\in\{2,3,\ldots,2n-2\}$ *and* $u_1,u_2\in\mathbb{R}_{>0}$*, by*

$$K_\lambda(x_1,u_1;x_2;u_2)=\mathbf{1}_{\{u_2<u_1,x_2<x_1\}}\frac{(u_2-u_1)^{x_1-x_2-1}n^{-\frac{1}{2}(x_1-x_2)}}{(x_1-x_2-1)!}\tag{3.2.2}$$
$$+\frac{1}{(2\pi i)^2}\oint_{C_z[0,\lambda_1+n-x_2)}dz\oint_{C_w[0,x_1)}dw\frac{\Gamma(-w)}{\Gamma(z+1)}\frac{G_\lambda(z+x_2-n)}{G_\lambda(x_1-n-1-w)}\cdot\frac{u_2^z\,u_1^w\,n^{-\frac{1}{2}(z+w+1)}}{w+z+x_2-x_1+1},$$

where

$$G_\lambda(u)=\Gamma(u+1)\prod_{i=1}^{\infty}\frac{u+i}{u-\lambda_i+i}=\frac{\Gamma(u+1+n)}{\prod\limits_{i=1}^{n}(u-\lambda_i+i)},$$

with $\lambda=\Delta_n$ *for* $\mathcal{X}_n'$ *and* $\lambda=\Delta_n\setminus(n-k,k)$ *for* $\mathcal{X}_{k,n}'$*. The contours* $C_z[0,\lambda_1+n-x_2)$ *and* $C_w[0,x_1)$ *are as shown in Figure 3.4. Both are counter-clockwise, encloses only the integers in the respective half open intervals* $[0,\lambda_1+n-x_2)$ *and* $[0,x_1)$*, and are arranged so that* C_z *and* $x_1-x_2-1-C_w$ *are disjoint, as in Figure 3.4.*

Remark 3.2.5. *The* Γ*–functions in double-contour integrals are used for the convenience of notations, but the integral is, in fact, a rational function of* z *and* w *multiplied by* $u_2^z\, u_1^w\, n^{-\frac{1}{2}(z+w+1)}$. *Indeed, we have*

$$\frac{\Gamma(-w)}{\Gamma(z+1)}\frac{G_\lambda(z+x_2-n)}{G_\lambda(x_1-n-1-w)}=\frac{(z+1)(z+2)\cdots(z+x_2)}{\prod\limits_{i=1}^{n}(z+x_2-n-\lambda_i+i)}\cdot\frac{\prod\limits_{i=1}^{n}(-w+x_1-n-1-\lambda_i+i)}{(-w)(1-w)\cdots(x_1-1-w)}. \tag{3.2.3}$$

Proof of Theorem 3.2.4. This is a corollary of V. Gorin and M. Rahman 2019, Theorem 1.5. In the notation of V. Gorin and M. Rahman 2019, we choose:

$$t_1=1-\frac{u_1}{n^{\frac{1}{2}}};\ t_2=1-\frac{u_2}{n^{\frac{1}{2}}};\ x_1^{[GR]}=x_1-n;\ x_2^{[GR]}=x_2-n;\qquad \lambda=\Delta_n \text{ or } \Delta_n\setminus(n-k,k).$$

Since our match of parameters involves rescaling of the real spatial coordinate, we also need to use the following Lemma 3.2.6, whose proof we omit. □

Lemma 3.2.6. *Let* Y_n *be a determinantal point process on* $\mathbb{Z}\times(0,1)$ *with reference measure* $\#_{\mathbb{Z}}\times\mathcal{L}((0,1))$ *and correlation kernel* K_n*. For* $c_n\in\mathbb{Z}$*,* $\alpha\in\mathbb{R}$*, we define the rescaled and shifted point process*

$$Y_n^{'}=\{(x-c_n,n^\alpha u)\mid(x,u)\in Y_n\}.$$

Then, the correlation kernel of $Y_n^{'}$ *with reference measure* $\#_{\mathbb{Z}}\times\mathcal{L}(\mathbb{R}_{>0})$ *is*

$$K_n^{'}(x_1,u_1;\,x_2,u_2)=n^{-\alpha}K_n\left(x_1+c_n,\frac{u_1}{n^\alpha};\,x_2+c_n,\frac{u_2}{n^\alpha}\right).$$

Remark 3.2.7. *We can extend* $\mathcal{X}_n^{'}$ *and* $\mathcal{X}_{k,n}^{'}$ *to point processes on* $\{2,3,\ldots,2n-2\}\times\mathbb{R}_{\geq 0}$*, in such a way that almost surely there is no particle at* $(x,0)$ *for any* $x\in\{2,3,\ldots,2n-2\}$*. Moreover, the kernel* K_λ *can be extended to the same space based on its regular behavior at* u_1 *or* $u_2=0$*.*

In order to see that the double contour integral in the definition of the kernel is well-behaved as $u_1\to 0$ *or* $u_2\to 0$*, we expand the integral as a sum of residues at the poles inside* *the integration contours. We start from the case* $2 \leq x_1 \leq x_2$ *with contours of the top panel of Figure 3.4. In this case the denominator* $w + z + x_2 - x_1 + 1$ *is never singular inside the integration contours, the* w*–residues are at simple poles at finitely many non-negative integers and* z*–residues are also at simple poles at finitely many non-negative integers. The residue at* $w = m_1$, $z = m_2$ *for* $m_1, m_2 \in \mathbb{Z}_{\geq 0}$ *has the form* $c(m_1, m_2) \cdot u_1^{m_1} u_2^{m_2}$, *where* $c(m_1, m_2)$ *does not depend on* u_1, u_2. *Hence, the residue is continuous as* $u_1, u_2 \to 0$ *and so is the double integral.*

Proceeding to the case $2 \leq x_2 < x_1$ *with contours of the bottom panel of Figure 3.4, the additional feature is the potential singularity of the denominator* $w + z + x_2 - x_1 + 1$. *Let us first compute the* w*–integral as a sum of the residues, getting a sum of the terms of the form* $u_1^{m_1} \times$ *(one-dimensional* z*-integral) with non-negative values for* m_1. *Up to the factors not depending on* u_1 *or* u_2, *the corresponding* z*–integral has the form*

$$\oint_{C_z[0,\lambda_1+n-x_2)} \frac{\Gamma(z + x_2 + 1)}{\Gamma(z + 1)} \cdot \frac{1}{\prod_{i=1}^{n} (z + x_2 - \lambda_i + i - n)} \cdot \frac{u_2^z n^{-\frac{1}{2}z}}{z + m_1 + x_2 - x_1 + 1} dz.$$

The simple poles of the last integral lead to u_2*–dependence of the form* $u_2^{m_2}$, $m_2 \geq 0$, *which is again continuous at* $u_2 = 0$. *It would be more complicated, if the integral had a double pole: the residue at such a pole would have involved the* z*–derivative of* u_2^z, *which leads to the appearance of* $\ln(u_2)$ *factor that is singular at* $u_2 = 0$. *However, we claim that there is no double pole. Indeed, using* (3.2.3), $m_1 \notin \{x_1 - n - 1 - \lambda_i + i\}_{i=1}^n$. *Therefore, for the pole of* $\frac{1}{z+m_1+x_2-x_1+1}$, *we have* $(-m_1 - x_2 + x_1 - 1) \notin \{-x_2 + \lambda_i - i + n\}_{i=1}^n$. *Hence, this pole is outside the set of poles of* $\frac{1}{\prod_{i=1}^n (z+x_2-\lambda_i+i-n)}$.

3.2.6 Anti-symmetric GUE

Let us recall a statement from P. J. Forrester and E. Nordenstam 2009, Theorems 2.9 and 3.3:

Proposition 3.2.8. *For* M *in Definition 3.1.2 and* $l = 1, 2, \ldots$ *we have:*

(a) The l^{th} *corner of* M *has* l *real eigenvalues.*

(b) The eigenvalues come in pairs, i.e, λ *is an eigenvalue of the* l^{th} *corner if and only if* $-\lambda$ *is an eigenvalue of the* l^{th} *corner. When* l *is odd,* 0 *is necessarily an eigenvalue of the* l^{th} *corner.*

(c) The eigenvalues of the l^{th} *corner interlace with the eigenvalues of the* $(l+1)^{st}$ *corner a.s, which means that*

$$\lambda_{j+1}^{l+1} \leq \lambda_j^l \leq \lambda_j^{l+1}, \tag{3.2.4}$$

where $j = 1, 2, \ldots, l$ *and* λ_j^l *denotes the* j^{th} *largest eigenvalue of the* l^{th} *corner.*

(d) Conditionally on the positive eigenvalues of the l^{th} *corner, the positive eigenvalues of the* i^{th} *corners,* $i = 2, ..., l-1$*, are jointly distributed uniformly on the polytope determined by the interlacing inequalities* (3.2.4).

(e) The joint density of positive eigenvalues $\left(\lambda_1^l \geq \cdots \geq \lambda_{\lfloor \frac{l}{2} \rfloor}^l\right)$ *of the* l^{th} *corner is*

$$p(\lambda_1^l, ..., \lambda_{\lfloor \frac{l}{2} \rfloor}^l) = \begin{cases} C_l \displaystyle\prod_{1 \leq i < j \leq \frac{l}{2}} \Big((\lambda_i^l)^2 - (\lambda_j^l)^2 \Big) \prod_{j=1}^{\frac{l}{2}} e^{-(\lambda_j^l)^2}, & l \ \textit{is even}; \\ \\ C_l \displaystyle\prod_{1 \leq i < j \leq \frac{l-1}{2}} \Big((\lambda_i^l)^2 - (\lambda_j^l)^2 \Big) \prod_{j=1}^{\frac{l-1}{2}} \Big[\lambda_j^l \, e^{-(\lambda_j^l)^2} \Big], & l \ \textit{is odd}, \end{cases}$$

where C_l *is an explicitly known normalization constant.*

Note that the above five properties uniquely characterize the joint distribution of $\{\lambda_j^k\}_{1 \leq j \leq k < \infty}$. An alternative characterization is through the correlation functions of the corresponding point process $\mathcal{X}$:

Theorem 3.2.9 (P. J. Forrester and E. Nordenstam 2009, Proposition 4.3)**.** *The aGUE corners process* $\mathcal{X}$ *of Definition 3.1.2 is a determinantal point process with correlation kernel* $K(x, u; t, v)$ *(with respect to the reference measure* $\#_{\mathbb{Z}} \otimes \mathcal{L}(\mathbb{R}_{\geq 0})$*), such that for*

$x \geq y$,

$$K(x,u;y,v) = e^{-u^2} \sum_{l=1}^{[y/2]} \frac{H_{x-2l}(u) H_{y-2l}(v)}{N_{y-2l}}, \tag{3.2.5}$$

and for $x < y$,

$$K(x,u;y,v) = -e^{-u^2} \sum_{l=-\infty}^{0} \frac{H_{x-2l}(u) H_{y-2l}(v)}{N_{y-2l}}. \tag{3.2.6}$$

Here $H_j(u)$ is the j^{th} "physicist's Hermite polynomial" with leading coefficient 2^j, so that

$$\int_{\mathbb{R}} H_i(u) H_j(u) e^{-u^2} du = 2\delta_{ij} N_j, \qquad N_j = j! 2^{j-1} \sqrt{\pi}.$$

Remark 3.2.10. *The $x = y$ case of (3.2.5) differs from the kernel of (3.1.1) by the factor $e^{-\frac{1}{2}u^2 + \frac{1}{2}v^2}$, which cancels out when we compute determinants.*

3.3 Local Limit of Standard Staircase Shaped Tableaux

The aim of this section is to investigate the scaling limit near the bottom-left corner for the uniformly random standard Young tableaux of shapes Δ_n and $\Delta_n \setminus (n-k,k)$ as $n \to \infty$.

3.3.1 Statement of the result

We use the (l,m) coordinate system for SYT, as in Section 3.2.2. We recall that $\mathcal{X}$ is the point process corresponding to the corners of aGUE as in Definition 3.1.2. Let $\mathcal{X}(l,m)$ denote the m^{th} particle of $\mathcal{X}$ in the level l, sorted in the increasing order; in other words, $\mathcal{X}(l,m)$ is the m^{th} positive eigenvalue of $l \times l$ corner of the infinite aGUE matrix M (with $m = 1$ corresponding to the smallest positive and $m = \lfloor \frac{l}{2} \rfloor$ to the largest positive eigenvalue).

Further, we set $\mathcal{X}_k'$ to be the point process of Definition 3.1.16 with an extra particle inserted at $(2k, 0)$. We let $\mathcal{X}_k'(l,m)$ denote the m^{th} particle of $\mathcal{X}_k'$ in the level $x = l$, sorted in the increasing order.

Theorem 3.3.1. *For each $n \in \mathbb{Z}_{\geq 2}$, let $T_{\Delta_n}(l,m)$ and $T_{\Delta_n \setminus (n-k,k)}(l,m)$ be uniformly random standard Young tableaux of shapes Δ_n and $\Delta_n \setminus (n-k,k)$, respectively, in the* *coordinate system of Section 3.2.2. Recall* $N = \binom{n}{2}$*. Then for each fixed* k*, as* $n \to \infty$ *we have*

$$\left\{ n^{\frac{1}{2}} \left(1 - \frac{T_{\Delta_n}(l,m)}{N} \right) \right\}_{l \geq 2,\ 1 \leq m \leq \lfloor \frac{l}{2} \rfloor} \longrightarrow \left\{ \mathcal{X}(l,m) \right\}_{l \geq 2,\ 1 \leq m \leq \lfloor \frac{l}{2} \rfloor} \qquad \textit{and} \tag{3.3.1}$$

$$\left\{ n^{\frac{1}{2}} \left(1 - \frac{T_{\Delta_n \setminus (n-k,k)}(l,m)}{N} \right) \right\}_{l \geq 2,\ 1 \leq m \leq \lfloor \frac{l}{2} \rfloor} \longrightarrow \left\{ \mathcal{X}_k^{'}(l,m) \right\}_{l \geq 2,\ 1 \leq m \leq \lfloor \frac{l}{2} \rfloor}, \tag{3.3.2}$$

in the sense of convergence of finite-dimensional distributions.

Remark 3.3.2. *The SYT* $T_{\Delta_n \setminus (n-k,k)}$ *can be thought as* T_{Δ_n} *conditioned on its largest entry being at* $(n-k,k)$*. Hence, Theorem 3.3.1 suggest a conjecture:* $\mathcal{X}_k^{'}$ *should have the same law as* $\mathcal{X}$ *conditioned on having a particle at* $(2k,0)$*, i.e. on* $\mathcal{X}(2k,1) = 0$*. Note that additional efforts are needed to prove this, because the topology of the convergence in Theorem 3.3.1 is not strong enough to make conclusions about the conditional distributions.*

The rest of this section is devoted to the proof of Theorem 3.3.1. We first couple the two types of SYT with the uniformly random PYT of the same shape, and induce two point processes $\mathcal{X}_n^{'}, \mathcal{X}_{k,n}^{'}$ on $\mathbb{Z}_{\geq 0} \times \mathbb{R}_{\geq 0}$ respectively from these two PYT, as in Definition 3.2.2. Then we prove that $\mathcal{X}_n^{'} \to \mathcal{X}^{'}$ and $\mathcal{X}_{k,n}^{'} \to \mathcal{X}_k^{'}$ as $n \to \infty$, where $\mathcal{X}^{'}$ and $\mathcal{X}_k^{'}$ are determinantal point processes on $\mathbb{Z}_{\geq 0} \times \mathbb{R}_{\geq 0}$ with explicit correlation kernels given by double contour integrals. Next, we identify $\mathcal{X}^{'}$ with $\mathcal{X}$, the corners process of aGUE of Definition 3.1.2.

3.3.2 Coupling with PYT

The first step of the proof is to introduce a coupling between SYT and PYT of the same shape and to show that under this coupling the difference between random SYT and PYT is negligible in the asymptotic regime of our interest.

Let T_{Δ_n} and $\mathbb{T}_{\Delta_n}$ be uniformly random SYT and PYT, respectively, of shape Δ_n. We couple these two tableaux in the following way: for $N = \binom{n}{2}$, let $(P_1 < P_2 < \cdots < P_N)$ be a uniformly random point on the simplex $\{(x_1, \cdots, x_N) \mid 0 < x_1 < x_2 < \cdots < x_N < 1\}$.

Given a uniformly random SYT T_{Δ_n}, we replace entry k in T_{Δ_n} by P_k for $k = 1, 2, \cdots, N$; it is straightforward to check that the result is a uniformly random PYT $\mathbb{T}_{\Delta_n}$. In the opposite direction, T_{Δ_n} is reconstructed by $\mathbb{T}_{\Delta_n}$ by replacing the kth largest entry in $\mathbb{T}_{\Delta_n}$ by k for each $k = 1, 2, \ldots, N$.

Lemma 3.3.3. *Fix any pair of integers* (i, j) *with* $i \geq 2$ *and* $1 \leq j \leq \lfloor \frac{i}{2} \rfloor$. *Using the coupling described above, we have for each* $\delta > 0$

$$n^{1-\delta} \left| \mathbb{T}_{\Delta_n}(i,j) - \frac{T_{\Delta_n}(i,j)}{N} \right| \to 0$$

in probability, as $n \to \infty$.

Proof. Let $T_{\Delta_n}(i,j) = \xi \in \{1, 2, \ldots, N\}$, where ξ is a random variable. Then, by the construction of the coupling, $\mathbb{T}_{\Delta_n}(i,j) = P_\xi$. By Chebyshev's inequality,

$$\begin{aligned} &\mathbb{P}\left[n^{1-\delta} \left| P_\xi - \frac{\xi}{N} \right| > \epsilon \right] \\ &\leq \left(\frac{n^{1-\delta}}{\epsilon} \right)^2 \mathbb{E}\left(P_\xi - \frac{\xi}{N} \right)^2 \\ &= \frac{n^{2-2\delta}}{\epsilon^2} \sum_{j=1}^{N} \mathbb{E}\left(P_\xi - \frac{\xi}{N} \right)^2 \Big| \xi = j \cdot \mathbb{P}\left[\xi = j \right] \end{aligned} \tag{3.3.3}$$

For a fixed j, the random variable P_j is distributed according to Beta distribution with parameters j and $N + 1 - j$, which has mean $\frac{j}{N+1}$ and variance (see, e.g., N. Johnson, S. Kotz, and N. Balakrishnan 1995, Chapter 25).

$$\mathbb{E}\left(P_j - \frac{j}{N+1} \right)^2 = \frac{j(N+1-j)}{(N+1)^2(N+2)} \leq \frac{\left(\frac{N+1}{2}\right)^2}{(N+1)^2(N+2)} = \frac{1}{4(N+2)}. \tag{3.3.4}$$

Thus,

$$\mathbb{E}\left(P_j - \frac{j}{N} \right)^2 = \mathbb{E}\left(P_j - \frac{j}{N+1} \right)^2 + \left(\frac{j}{N(N+1)} \right)^2 \leq \frac{1}{4(N+2)} + \frac{1}{(N+1)^2} \leq \frac{1}{N}. \tag{3.3.5}$$

Since $N = \frac{n(n-1)}{2}$, (3.3.3) and (3.3.5) imply that as $n \to \infty$

$$\mathbb{P}\left[n^{1-\delta}\left|P_\xi - \frac{\xi}{N}\right| > \epsilon\right] \leq \frac{n^{2-2\delta}}{\epsilon^2 N} \to 0. \quad \square$$

In a similar way, we couple uniformly random SYT $T_{\Delta_n \setminus (n-k,k)}$ and PYT $\mathbb{T}_{\Delta_n \setminus (n-k,k)}$ of shape $\Delta_n \setminus (n-k,k)$: we let $(P_1 < P_2 < \cdots < P_{N-1})$ be a uniformly random point on the simplex $\{(x_1, \cdots, x_{N-1}) \mid 0 < x_1 < x_2 < \cdots < x_{N-1} < 1\}$. Given a uniformly random SYT $T_{\Delta_n \setminus (n-k,k)}$, we replace entry k in $T_{\Delta_n \setminus (n-k,k)}$ by P_k for $k = 1, 2, \cdots, N-1$; it is straightforward to check that the result is a uniformly random PYT $\mathbb{T}_{\Delta_n \setminus (n-k,k)}$. Repeating the proof of Lemma 3.3.3 we arrive at:

Lemma 3.3.4. *Fix any pair of integers (i, j) with $i \geq 2$ and $1 \leq j \leq \lfloor \frac{i}{2} \rfloor$. Using the coupling described above, we have for each $\delta > 0$*

$$n^{1-\delta}\left|\mathbb{T}_{\Delta_n \setminus (n-k,k)}(i,j) - \frac{T_{\Delta_n \setminus (n-k,k)}(i,j)}{N}\right| \to 0$$

in probability, as $n \to \infty$.

3.3.3 Limiting Processes

In this section we analyze asymptotic behavior of the point processes associated to random PYT $\mathbb{T}_{\Delta_n}$ and $\mathbb{T}_{\Delta_n \setminus (n-k,k)}$ by the procedure of Section 3.2.3.

Theorem 3.3.5. *The point process $\mathcal{X}_n^{'}$ (corresponding to random PYT $\mathbb{T}_{\Delta_n}$ via Definition 3.2.2) converges weakly as $n \to \infty$ to a point process $\mathcal{X}^{'}$ on $\mathbb{Z}_{\geq 2} \times \mathbb{R}_{\geq 0}$. $\mathcal{X}^{'}$ is a determinantal point process (with respect to reference measure $\#_{\mathbb{Z}_{\geq 2}} \otimes \mathcal{L}(\mathbb{R}_{\geq 0})$) with correlation kernel*

$$\begin{aligned} K^{'}(x_1, u_1; x_2, u_2) &= I_{\{u_2 < u_1, x_2 < x_1\}} \frac{(u_2 - u_1)^{x_1 - x_2 - 1}}{(x_1 - x_2 - 1)!} \\ &\quad + \frac{1}{(2\pi i)^2} \oint_{C_z[0,\infty)} dz \oint_{C_w[0,x_1)} dw \frac{\Gamma(-w)}{\Gamma(z+1)} \frac{\Gamma(z + x_2 + 1)}{\Gamma(x_1 - w)} \frac{\Gamma(-\frac{z+x_2}{2})}{\Gamma(\frac{-x_1+w+1}{2})} \frac{u_1^w u_2^z}{w + z + x_2 - x_1 + 1}. \end{aligned}$$

Here $u_1, u_2 \in \mathbb{R}_{>0}$ and $x_1, x_2 \in \mathbb{Z}_{\geq 2}$; when u_1 or u_2 is equal to 0, the kernel is defined as *the limit as u_1 or u_2 tends to 0. $C_z[0,\infty), C_w[0,x_1)$ are counterclockwise–oriented contours which enclose the integers in $[0,\infty)$ and $[0,x_1)$, respectively, and are arranged so that C_z and $x_1-x_2-1-C_w$ are disjoint, as in Figure 3.5.*

Theorem 3.3.6. *The point process $\mathcal{X}'_{k,n}$ (corresponding to random PYT $\mathbb{T}_{\Delta_n\setminus(n-k,k)}$ via Definition 3.2.2) converges weakly to a point process $\mathcal{X}'_k$ on $\mathbb{Z}_{\geq 2}\times\mathbb{R}_{\geq 0}$. $\mathcal{X}'_k$ is a determinantal point process (with respect to reference measure $\#_{\mathbb{Z}_{\geq 2}}\otimes\mathcal{L}(\mathbb{R}_{\geq 0})$) with correlation kernel given by*

$$\begin{aligned}K'_k(x_1,u_1;x_2,u_2) &= I_{\{u_2<u_1,x_2<x_1\}}\frac{(u_2-u_1)^{x_1-x_2-1}}{(x_1-x_2-1)!}\\ &+\frac{1}{(2\pi i)^2}\oint_{C_z[0,\infty)}dz\oint_{C_w[0,x_1)}dw\frac{\Gamma(-w)}{\Gamma(z+1)}\frac{\Gamma(z+x_2+1)}{\Gamma(x_1-w)}\frac{\Gamma(-\frac{z+x_2}{2})}{\Gamma(\frac{-x_1+w+1}{2})}\\ &\times\frac{z+x_2-2k}{z+x_2-2k+1}\cdot\frac{x_1-w-2k}{x_1-w-2k-1}\cdot\frac{u_1^w u_2^z}{w+z+x_2-x_1+1}.\end{aligned}$$

Here $u_1,u_2\in\mathbb{R}_{>0}$ and $x_1,x_2\in\mathbb{Z}_{\geq 0}$; the value of K'_k when u_1 or u_2 equals 0 is to be understood as the limit as u_1 or u_2 tends to 0. The contours $C_z[0,\infty)$ and $C_w[0,x_1)$ are the same as in Theorem 3.3.5.

In the proof we use a known asymptotic property of the Gamma-function:

$$\frac{\Gamma(y+m)}{(m-1)!}=m^y(1+O(m^{-1})),\qquad m\to+\infty, \tag{3.3.6}$$

where $O(m^{-1})$ term is uniform as long as y is uniformly bounded.

Proof of Theorem 3.3.5. We show the convergence in distribution by verifying conditions of Lemma 3.2.3, namely, we show that the correlation kernel $K'_n(x_1,u_1;x_2,u_2)n^{\frac{x_1-x_2}{2}}$ of $\mathcal{X}'_n$, where K'_n is given by (3.2.2), converges to K' uniformly on compact subsets of $x_1,x_2\in\mathbb{Z}_{\geq 2}$ and $u_1,u_2\in\mathbb{R}_{\geq 0}$. Note that the just introduced multiplication by $n^{\frac{x_1-x_2}{2}}$ does not change the value of the correlations functions $\det[K'_n(x_i,u_i;x_j,u_j)]$. Let us fix arbitrary $x_1,x_2\in\mathbb{Z}_{\geq 2}$, strictly positive u_1, u_2, and analyze the asymptotic behavior of the double contour integral in (3.2.2).

The first step is to deform the contours of integration from the ones of Figure 3.4 to the ones of Figure 3.5. Using (3.2.5), the z–dependent factors of the integrand are

$$\frac{(z+1)(z+2)\cdots(z+x_2)}{\prod\limits_{i=1}^{n}(z+x_2-2n+2i)}\cdot\left(\frac{u_2}{\sqrt{n}}\right)^{z}\cdot\frac{1}{w+z+x_2-x_1+1}. \tag{3.3.7}$$

The form of (3.3.7) implies that there are no z–poles of the integrand to the right from the contour of Figure 3.4; hence, the value of the integral does not change in the deformation. For each fixed u_2 and large enough n, on the new contours of Figure 3.5, the expression (3.3.7) rapidly decays as $\Re z \to +\infty$ (uniformly in n). This observation allows us to control the part of the integral corresponding to large $\Re z$, and it remains to study the $n \to \infty$ asymptotics of the integrand for finite z and w.

We use (3.3.6) to compute the asymptotic behavior of $\prod_{i=1}^{n}$ in (3.3.7):

$$\begin{aligned}
&\prod_{i=1}^{n}(z+x_2-2n+2i) = (-1)^n\prod_{i=1}^{n}(-z-x_2+2n-2i)\\
&= (-1)^n 2^n \prod_{i=1}^{n}\left(-\frac{z-x_2}{2}+n-i\right) = (-1)^n 2^n \frac{\Gamma(-\frac{z+x_2}{2}+n)}{\Gamma(-\frac{z+x_2}{2})}\\
&= (-1)^n 2^n (n-1)!\frac{n^{-\frac{z+x_2}{2}}}{\Gamma(-\frac{z+x_2}{2})}\left(1+O(n^{-1})\right).
\end{aligned}$$

Similarly,

$$\prod_{i=1}^{n}(x_1-w-2n+2i-1) = (-1)^n 2^n (n-1)!\frac{n^{\frac{-x_1+w+1}{2}}}{\Gamma(\frac{-x_1+w+1}{2})}\left(1+O(n^{-1})\right).$$

Thus, for $\lambda = \Delta_n$, we have

$$\frac{G_\lambda(z+x_2-n)}{G_\lambda(x_1-n-1-w)} = \frac{\Gamma(z+x_2+1)}{\Gamma(x_1-w)}\frac{\Gamma(-\frac{z+x_2}{2})}{\Gamma(\frac{-x_1+w+1}{2})}n^{\frac{-x_1+w+1}{2}}n^{\frac{z+x_2}{2}}\left(1+O(n^{-1})\right).$$

Hence, the integrand in the double contour integral part of $K_n'(x_1,u_1;x_2,u_2)n^{\frac{x_1-x_2}{2}}$, converges as $n \to \infty$ to

$$\frac{\Gamma(-w)}{\Gamma(z+1)} \frac{\Gamma(z+x_2+1)}{\Gamma(x_1-w)} \frac{\Gamma(-\frac{z+x_2}{2})}{\Gamma(\frac{-x_1+w+1}{2})} \frac{u_1^w u_2^z}{w+z+x_2-x_1+1}.$$

Combining with the straightforward asymptotics of the indicator term, we conclude that for all strictly positive u_1 and u_2,

$$\lim_{n\to\infty} K_n^{'}(x_1,u_1;x_2,u_2) n^{\frac{x_1-x_2}{2}} = K_n^{'}(x_1,u_1;x_2,u_2).$$

Clearly, the convergence is uniform over u_1, u_2 in compact subsets of $(0,+\infty)$. It remains to show that there is no explosion for u_1 or u_2 near 0, which is done in exactly the same way as we did in Remark 3.2.7, i.e. by interpreting the integral as a sum of residues at simple $z-$ and $w-$poles at non-negative integers. □

Proof of Theorem 3.3.6. The proof is the same as for Theorem 3.3.5, with only some slight difference in calculation. This time, for $\lambda = \Delta_n \setminus (n-k,k)$ we have

$$G_\lambda(z+x_2-n) = \frac{\Gamma(z+x_2+1)}{\prod_{i=1}^{n}(z+x_2-2n+2i)} \cdot \frac{z+x_2-2k}{z+x_2-2k+1},$$

$$G_\lambda(x_1-n-1-w) = \frac{\Gamma(x_1-w)}{\prod_{i=1}^{n}(x_1-w-2n+2i-1)} \cdot \frac{x_1-w-2k-1}{x_1-w-2k}.$$

The additional fractions $\frac{z+x_2-2k}{z+x_2-2k+1}$ and $\frac{x_1-w-2k-1}{x_1-w-2k}$ (as compared to $\lambda = \Delta_n$ case) propagate to the final answer without changes. □

3.3.4 Connection to Anti-symmetric GUE

In this section we prove that the distribution of the limiting process of Theorem 3.3.5 coincides with the corners process of aGUE.

Proposition 3.3.7. *$K^{'}(2k,u_1;2k,u_2)$ of Theorem 3.3.5 for $u_1,u_2\in\mathbb{R}_{\geq 0}$, $k\in\mathbb{Z}_{>0}$, is equal to*

$$2^{k+1}\sum_{i=0}^{\infty}\sum_{j=0}^{k-1}(-1)^{i+k}\frac{u_1^{2j}}{(2k-2j-1)!(2j)!\,\Gamma(j+\frac{1}{2}-k)}\cdot\frac{u_2^{2i}\cdot\prod_{a=1}^{k}(2i+2a-1)}{i!}\cdot\frac{1}{2j+2i+1}. \tag{3.3.8}$$

Remark 3.3.8. *When $k=1$, we simplify $K^{'}(2,u_1;2,u_2)=\frac{2}{\sqrt{\pi}}e^{-u_2^2}$. There's only one particle on level $\{2\}\times\mathbb{R}_{\geq 0}$ which corresponds to the limit of the entry of the SYT at lower corner, with coordinate $(n-1,1)$.*

When $k=2$, we simplify $K^{'}(4,u_1;4,u_2)=\frac{1}{\sqrt{\pi}}[(1-2u_1^2)(1-2u_2^2)+2]e^{-u_2^2}$. There are two particles on level $\{4\}\times\mathbb{R}_{\geq 0}$ which correspond to the two entries of the SYT with coordinates $(n-3,1)$ and $(n-2,2)$.

Proof of Proposition 3.3.7. $K^{'}(2k,u_1;2k,u_2)$ of Theorem 3.3.5 is given by

$$\frac{1}{(2\pi\mathbf{i})^2}\oint_{C_z[0,\infty)}dz\oint_{C_w[0,2k)}dw\frac{\Gamma(-w)}{\Gamma(z+1)}\frac{\Gamma(z+2k+1)}{\Gamma(2k-w)}\frac{\Gamma(-\frac{z+2k}{2})}{\Gamma(\frac{-2k+w+1}{2})}\frac{(u_1)^w(u_2)^z}{w+z+1},$$

The w–integral is evaluated as the sum of residues at simple poles at even integers $w=0,2,4\ldots,2k-2$ and we have

$$\begin{aligned}\mathrm{Res}_{w=2j}&\left[\frac{\Gamma(-w)}{\Gamma(2k-w)\Gamma(\frac{-2k+w+1}{2})}\cdot\frac{(u_1)^w}{w+z+1}\right]\\&=\mathrm{Res}_{w=2j}\left[\frac{1}{\Gamma(\frac{-2k+w+1}{2})(-w)(-w+1)\cdots(-w+2k-1)}\cdot\frac{(u_1)^w}{w+z+1}\right]\\&\qquad=-\left[\frac{1}{\Gamma(\frac{-2k+2j+1}{2})\,(2k-1-2j)!(2j)!}\cdot\frac{(u_1)^{2j}}{2j+z+1}\right],\end{aligned}$$

which matches the j–dependent factors in (3.3.7). The remaining z–integral for the j–th term becomes

$$\frac{1}{(2\pi\mathbf{i})}\oint_{C_z[0,\infty)}\frac{\Gamma(z+2k+1)\Gamma(-\frac{z+2k}{2})}{\Gamma(z+1)}\frac{(u_2)^z}{2j+z+1}dz,$$

The last integral is evaluated as the sum of the residues at $z = 0, 2, 4, \ldots$. Using the fact that the residue of the Gamma function at a simple pole at $(-n)$, $n = 1, 2, \ldots$, is $\frac{(-1)^n}{n!}$, we get

$$\begin{aligned} \mathrm{Res}_{z=2i}\left[\frac{\Gamma(z+2k+1)\Gamma(-\frac{z+2k}{2})}{\Gamma(z+1)}\frac{(u_2)^z}{2j+z+1}\right] &= -2\frac{(-1)^{i+k}}{(i+k)!}\cdot\frac{(2i+2k)!}{(2i)!}\cdot\frac{(u_2)^{2i}}{2j+2i+1} \\ &= -2^{k+1}(-1)^{i+k}\prod_{a=1}^{k}(2i+2a-1)\cdot\frac{(u_2)^{2i}}{2j+2i+1}, \end{aligned}$$

which matches the i–dependent factors in (3.3.7). ☐

Proposition 3.3.9. *$K^{'}(2k, u_1; 2k, u_2)$ of Theorem 3.3.5 for $u_1, u_2 \in \mathbb{R}_{\geq 0}$, $k \in \mathbb{Z}_{>0}$, is also equal to*

$$e^{-(u_2)^2}\sum_{l=0}^{k-1}\frac{H_{2l}(u_1)H_{2l}(u_2)}{2^{2l-1}(2l)!\sqrt{\pi}}, \tag{3.3.9}$$

where $H_i(u)$ are Hermite polynomials, as in (3.1.1) *and Theorem 3.2.9.*

Proof. Our task is to show that (3.3.8) and (3.3.9) match. The proof is induction in k. For $k = 1$, this is a content of Remark 3.3.8. Subtracting (3.3.8) at $k+1$ and at k, we get

$$\begin{aligned} 2^{k+1}\sum_{i=0}^{\infty}\sum_{j=0}^{k}(-1)^{i+k}&\frac{u_1^{2j}}{(2k-2j+1)!(2j)!\,\Gamma(j+\frac{1}{2}-k)}\cdot\frac{u_2^{2i}\cdot\prod_{a=1}^{k}(2i+2a-1)}{i!}\cdot\frac{1}{2j+2i+1} \\ &\times[-2(j-k-1/2)(2i+2k+1)-(2k-2j+1)(2k-2j)] \\ = 2^{k+1}\sum_{j=0}^{k}&\frac{u_1^{2j}}{(2k-2j)!(2j)!\,\Gamma(j+\frac{1}{2}-k)}\sum_{i=0}^{\infty}(-1)^{i+k}\frac{u_2^{2i}\cdot\prod_{a=1}^{k}(2i+2a-1)}{i!} \end{aligned} \tag{3.3.10}$$

On the other hand, subtracting (3.3.9) at $k+1$ and at k, we get

$$e^{-(u_2)^2}\frac{H_{2k}(u_1)H_{2k}(u_2)}{2^{2k-1}(2k)!\sqrt{\pi}}, \tag{3.3.11}$$

The explicit expression for the Hermite polynomials Szego 1939, Chapter 5.5 is

$$H_{2k}(u_1) = \sum_{j=0}^{k}(-1)^{k-j}2^{2j}\frac{(2k)!}{(k-j)!(2j)!}u_1^{2j}$$

In order to match with u_1–dependent part of (3.3.10) we write

$$\Gamma(j+\tfrac{1}{2}-k) = \frac{\Gamma(\frac{1}{2})}{(\frac{1}{2}-1)(\frac{1}{2}-2)\cdots(\frac{1}{2}+j-k)} = \frac{(-1)^{k-j}2^{k-j}\sqrt{\pi}}{1\cdot 3\cdot 5\cdots(2k-2j-1)} = \frac{(-1)^{k-j}2^{2k-2j}\sqrt{\pi}(k-j)!}{(2k-2j)!}$$

Hence, (3.3.10) gets transformed into

$$\begin{aligned}2^{k+1}\sum_{j=0}^{k}\frac{u_1^{2j}}{(2j)!(-1)^{k-j}2^{2k-2j}\sqrt{\pi}(k-j)!}&\sum_{i=0}^{\infty}(-1)^{i+k}\frac{u_2^{2i}\cdot\prod_{a=1}^{k}(2i+2a-1)}{i!}\\ &= \frac{2^{1-k}}{(2k)!\sqrt{\pi}}H_{2k}(u_1)\sum_{i=0}^{\infty}(-1)^{i+k}\frac{u_2^{2i}\cdot\prod_{a=1}^{k}(2i+2a-1)}{i!}\end{aligned}$$

Comparing with (3.3.11), it remains to show that:

$$e^{-(u_2)^2}\frac{H_{2k}(u_2)}{2^k} \overset{?}{=} \sum_{i=0}^{\infty}(-1)^{i+k}\frac{u_2^{2i}\cdot\prod_{a=1}^{k}(2i+2a-1)}{i!}.$$

Or, equivalently,

$$H_{2k}(u_2) \overset{?}{=} e^{(u_2)^2}2^k\sum_{i=0}^{\infty}(-1)^{i+k}\frac{u_2^{2i}\cdot\prod_{a=1}^{k}(2i+2a-1)}{i!}. \tag{3.3.12}$$

The identity (3.3.12) is a corollary of the Rodriguez formula for Hermite polynomials Szego 1939, Chapter 5.5

$$H_n(u) = e^{u^2}(-1)^n(\partial_u)^n e^{-u^2}$$

Indeed, we have

$$\begin{aligned}e^{u^2}(-1)^{2k}(\partial_u)^{2k}e^{-u^2} &= e^{u^2}(\partial_u)^{2k}\left[\sum_{i=0}^{\infty}\frac{(-u^2)^i}{i!}\right] = e^{u^2}(\partial_u)^{2k}\left[\sum_{i=0}^{\infty}(-1)^{i+k}\frac{u^{2i+2k}}{(i+k)!}\right]\\ &= e^{u^2}\left[\sum_{i=0}^{\infty}(-1)^{i+k}\frac{u^{2i}(2i+1)(2i+2)\cdots(2i+2k)}{(i+k)!}\right]\\ &= e^{u^2}\left[\sum_{i=0}^{\infty}(-1)^{i+k}\frac{u^{2i}2^k(2i+1)(2i+3)\cdots(2i+2k-1)}{i!}\right]. \quad \square\end{aligned} \tag{3.3.13}$$

Theorem 3.3.10. *The point process $\mathcal{X}'$ in Theorem 3.3.5 has the same distribution as the* *point process $\mathcal{X}$ — the corner process of aGUE of Definition 3.1.2.*

Proof. **Step 1.** For each $k = 1, 2, \ldots$, each of the processes $\mathcal{X}'$ and $\mathcal{X}$ has k particles on level $2k$ (i.e. with the first coordinate $2k$). Let us show the marginal distributions describing the joint law of these k particles are the same. By Proposition 3.3.9, for $\mathcal{X}'$ these particles form a determinantal point process with kernel (3.3.9). By Theorem 3.2.9, for $\mathcal{X}$ these particles also form a determinantal point process with kernel given by plugging $x = y = 2k$ into (3.2.5). The kernels match and, hence, so do the point processes.

Step 2. Let us now fix k and compare the joint distribution of particles in $\mathcal{X}'$ and in $\mathcal{X}$ on levels $l = 1, 2, \ldots, 2k$. By part (d) in Proposition 3.2.8, the conditional law of the particles on levels $l = 1, 2, \ldots, 2k-1$ given k particles on level k is uniform (subject to interlacing conditions) for $\mathcal{X}$. On the other hand, $\mathcal{X}'$ possesses the same conditional uniformity, because it was obtained by a scaling limit from uniformly random PYT $\mathbb{T}_{\Delta_n}$ and conditional uniformity is preserved in limit transitions. Combining with Step 1, we conclude that the joint distributions of particles on levels $l = 1, 2, \ldots, 2k$ are the same for $\mathcal{X}'$ and $\mathcal{X}$. Since k is arbitrary, we are done. □

3.3.5 Proof of Theorem 3.3.1

Theorems 3.3.5 and 3.3.6 show that the point process $\mathcal{X}'_n$, $\mathcal{X}'_{k,n}$ induced from $n^{\frac{1}{2}}(1 - \mathbb{T}(l, m))$, the rescaled entries of PYT of shapes Δ_n and $\Delta_n \setminus (n-k, k)$, converge weakly as $n \to \infty$ to $\mathcal{X}'$ and $\mathcal{X}'_k$, respectively. In particular, treating the entries of the PYT of these two shapes as infinite random vectors, they converges in the sense of finite dimensional distribution to the positions of the corresponding particles in $\mathcal{X}'_n$ and $\mathcal{X}'_{k,n}$, respectively.

On the other hand, by Lemma 3.3.3, there's a coupling of PYT $\mathbb{T}$ and SYT T that, for any (l, m),

$$n^{\frac{1}{2}}\left[\left(1 - \tfrac{T(l,m)}{N}\right) - \left(1 - \mathbb{T}(l, m)\right)\right]$$

converges to 0 in probability. Combining the above two results, we obtain the $n \to \infty$

convergence for finite dimensional distributions of

$$\left\{n^{\frac{1}{2}}\left(1-\frac{T(l,m)}{N}\right)\right\}_{l\geq 2,\ 1\leq m\leq \lfloor \frac{l}{2}\rfloor}$$

to the limits given by particles of $\mathcal{X}'$ and $\mathcal{X}'_k$. It remains to use Theorem 3.3.10 to match $\mathcal{X}'$ with $\mathcal{X}$. □

3.4 Asymptotics for spacings

We start this section by studying general rotationally-invariant point processes on a circle, giving two different definitions of a spacing between particles in such processes, and proving that their distribution functions satisfy relations, which are discrete versions of (3.1.4) and (3.1.5). When specialized to the random sorting networks setting, these spacings are precisely the ones discussed in the introduction. After that we recall the Edelman-Greene bijection between standard Young tableaux and sorting networks. In the last subsection we combine all the ingredients to finish the proofs of Theorems 3.1.5, 3.1.11, 3.1.14, and 3.1.18.

3.4.1 Generalities on spacings

Consider a random point process $\mathscr{P}$ on a discrete circle of length $K \in \mathbb{Z}_{\geq 2}$. We index the possible positions of the particles by 1, 2, ..., K in the clockwise order and refer to the midpoint between 1 and K as the origin, see Figure 3.6. Throughout this section we assume that the distribution of the point process is invariant under rotations of the circle; we also silently assume that almost surely $\mathscr{P}$ has at least one particle.

We use $\mathscr{P}$ to define several random variables with positive integer values. We define the waiting time $W \in \{1, 2, ..., K\}$ to be the smallest value of $\ell \geq 1$ such that there is a particle at position ℓ. We let the spacing $\mathrm{Sp}_1 \in \{1, 2, ..., K\}$ to be the distance between two adjacent particles on the circle: one to the left from the origin and one to the right from the origin. By distance we mean here one plus the number of the holes between the particles (counted along the arc including the origin), so that if there are particles at positions 1 and K, then the distance is 1.

Next, we define the conditional spacing $\mathrm{Sp}_2 \in \{1, 2, ..., K\}$ to be a random variable whose distribution is that of W conditional on having a particle at position K. Note that W and Sp_1 are random variables (functions) on the probability space of $\mathscr{P}$, but Sp_2 should be defined on a different probability space.

Example 3.4.1. *Figure 3.6 gives two possible configurations of a point process on the discrete circle of length $K = 8$. On the first configuration, $\mathrm{Sp}_1 = 4$ and $W = 1$. On the second configuration, $\mathrm{Sp}_1 = W = 3$. Since there is a particle at K, we can also think of Sp_2 being equal to 3 for the second configuration.*

Let $f_1(\cdot)$, $f_2(\cdot)$, and $g(\cdot)$ be the probability mass functions of Sp_1, Sp_2, and W, respectively:

$$f_1(\ell) = \mathbb{P}\left[\mathrm{Sp}_1 = \ell\right], \qquad f_2(\ell) = \mathbb{P}\left[\mathrm{Sp}_2 = \ell\right], \qquad g(\ell) = \mathbb{P}\left[W = \ell\right], \qquad l = 1, 2, \ldots.$$

We also define ρ to the the probability that there is a particle at position K (in other words, this is the first correlation function or density of the point process $\mathscr{P}$).

Proposition 3.4.2. *For any rotationally invariant point process $\mathscr{P}$, we have*

$$-\Delta g(\ell) \cdot \ell = f_1(\ell), \qquad \ell = 1, 2, \ldots, \tag{3.4.1}$$

$$-\Delta g(\ell) = \rho \cdot f_2(\ell), \qquad \ell = 1, 2, \ldots, \tag{3.4.2}$$

where Δ is the forward difference operator: $\Delta g(\ell) := g(\ell + 1) - g(\ell)$.

Proof. We claim that

$$-\Delta g(\ell) = \mathbb{P}\left[\text{there are particles at positions } K \text{ and } \ell, \text{ but not at } 1, 2, \ldots, \ell - 1\right]. \tag{3.4.3}$$

Indeed, using rotational invariance of the law of $\mathscr{P}$, we have

$$\begin{aligned} g(\ell) &= \mathbb{P}\left[\text{there is a particle at position } \ell, \text{ but not at } 1,2,\ldots,\ell-1\right], \\ g(\ell+1) &= \mathbb{P}\left[\text{there is a particle at position } \ell, \text{ but not at } K,1,2,\ldots,\ell-1\right]. \end{aligned}$$

In order to prove (3.4.2), it is now sufficient to rewrite the probability in the right-hand side of (3.4.3) as the product of probability of having a particle at K (which is ρ) and conditional probability.

In order to prove (3.4.1), note that by definition

$$f_1(\ell) = \sum_{a=0}^{\ell-1} \mathbb{P}\left[\text{there are particles at } K-a \text{ and } \ell-a, \text{ but not at } K-a+1,\ldots,\ell-a-1\right]. \tag{3.4.4}$$

By rotational invariance all terms in the right-hand side of (3.4.4) are equal. There are ℓ of them and each one is computed by (3.4.3), leading to (3.4.1). □

Remark 3.4.3. *A continuous version of Proposition 3.4.2 for translationally invariant point processes on the real line* $\mathbb{R}$ *says that the following is true (under technical regularity assumptions on the point process, which we do not spell out, and which are needed to guarantee the existence of all the densities below): Suppose that* Sp_1, Sp_2 *and* W *are spacing (distance between closest particles to the left and to the right from the origin), conditional spacing, and waiting time for an arrival of a particle, and let* $f_1(x)$, $f_2(x)$, $g(x)$, $x \geq 0$ *be probability densities of these random variables. Then we have*

$$\begin{aligned} -x \cdot \partial_x g(x) &= f_1(x), \\ -\partial_x g(x) &= \rho \cdot f_2(x). \end{aligned}$$

where ρ *is equal to the first correlation function for the process.*

3.4.2 Edelman-Greene bijection

The relation of the random Young tableaux (which we were studying in Sections 3.2 and 3.3) with the random sorting networks relies on the Edelman–Green bijection P. Edelman and C. Greene 1987, which we now present.

The bijection takes a standard Young tableau T of shape Δ_n as an input and outputs a sorting network. This correspondence maps the uniform measure on standard Young tableaux of shape Δ_n to the uniform measure on sorting networks of size n.

The bijection proceeds through the following algorithm, in which we use the standard (i, j) coordinate system for Young tableaux, as in Section 3.2.1:

1. Given a standard Young tableau T, find the box $(n-\ell, \ell)$ which contains the largest entry of T. Necessarily, this entry is $N = \frac{n(n-1)}{2}$ and $1 \leq \ell \leq n-1$.

2. Set the first swap s_1 of the corresponding sorting network to be ℓ.

3. Define the sliding path in the following way: the first box of the path is $(n-\ell, \ell)$. Compare the entries at $(n-\ell-1, \ell)$ and at $(n-\ell, \ell-1)$, and take the box with the larger entry as the second box (if only one of the boxes is inside the Young diagram, then take that one). Repeat this procedure (each time decreasing by 1 either the first of the second coordinate of the box and moving in the direction of the larger entry), until you arrive at the box $(1, 1)$, which is the last box of the sliding path.

4. Slide the entries on the sliding path in such a way that the maximal entry N is removed and all the remaining entries on the path are moved to the neighboring boxes on the path. No entry remains at $(1, 1)$ after the sliding. Then we increase all entries by 1, and fill the box $(1, 1)$ with new entry 1.

5. After transformations of steps 1-4, T becomes a new standard Young tableau of shape Δ_n. Repeat steps 1-4 additional $(N-1)$ times to get swaps s_2,s_3,...,s_N of the sorting network.

Using the Edelman-Green bijection, the random variable $T_{FS,n}(k)$ of Theorem 3.1.5 — the first time swap k occurs — gets recast in terms of the uniformly random Young tableau.

Lemma 3.4.4. *Let $T_{\Delta_n}(l,m)$ be a uniformly random standard Young tableau of shape Δ_n in the rotated coordinate system of Section 3.2.2. The following distributional identity holds:*

$$T_{FS,n}(k) \overset{d}{=} N + 1 - T_{\Delta_n}(2k,1) \tag{3.4.5}$$

Proof. In each iteration of the Edelman–Greene algorithm the entry $(n-k,k)$ (in the standard (i,j) coordinate system for Young tableaux, as in Section 3.2.1) grows by 1 until it becomes N, at which point the swap s_k is added to the sorting network for the first time. The entry at $(n-k,k)$ in the (i,j) coordinate system is the same as the entry $(2k,1)$ in the rotated (l,m) coordinate system, leading to (3.4.5). □

We can also compute the distribution of the conditional spacing $\widehat{\mathrm{Sp}}_{k,n}$ of Definition 3.1.12.

Lemma 3.4.5. *Let $T_{\Delta_n \setminus (n-k,k)}(l,m)$ be a uniformly random standard Young tableau of shape $\Delta_n \setminus (n-k,k)$ in the rotated coordinate system of Section 3.2.2. The following distributional identity holds:*

$$\widehat{\mathrm{Sp}}_{k,n} \overset{d}{=} \begin{cases} N - \max\big(T_{\Delta_n\setminus(n-k,k)}(2k-1,1), T_{\Delta_n\setminus(n-k,k)}(2k+1,1)\big), & \text{if } 2 \le k \le n-2; \\ N - T_{\Delta_n\setminus(n-k,k)}(3,1), & \text{if } k = 1; \\ N - T_{\Delta_n\setminus(n-k,k)}(2n-3,1), & \text{if } k = n-1. \end{cases} \tag{3.4.6}$$

Proof. In order to obtain the law of $\widehat{\mathrm{Sp}}_{k,n}$, we need to condition on the first swap being k; through the Edelman-Green bijection this is the same as conditioning on the largest entry N in the tableau to be in the box $(n-k,k)$ in the (i,j) coordinate system. Note that if we condition a uniformly random standard Young tableau of shape Δ_n on the position of largest entry $T(n-k,k)=N$, then the rest is a uniformly random standard Young tableu of shape $\Delta_n \setminus (n-k,k)$.

Once we do conditioning, the value of $1+\widehat{\mathrm{Sp}}_{k,n}$ becomes the number of the iterations of the Edelman-Greene algorithm when the largest entry of the tableau is at $(n-k,k)$ for the second time. After the first step of the algorithm, the entry at $(n-k,k)$ is

$$1+\max\bigl(T_{\Delta_n\setminus(n-k,k)}(2k-1,1), T_{\Delta_n\setminus(n-k,k)}(2k+1,1)\bigr)$$

and at each further step the entry grows by 1, until it reaches N. Hence, we need $N - \max\bigl(T_{\Delta_n\setminus(n-k,k)}(2k-1,1), T_{\Delta_n\setminus(n-k,k)}(2k+1,1)\bigr)$ additional iterations, matching the first case in (3.4.6). The second and the third cases correspond to the situations when $(n-k,k)$ has only one neighboring box in the tableau. □

3.4.3 Proofs of the main theorems

Proof of Theorem 3.1.5. By Lemma 3.4.4, we need to find the $n \to \infty$ asymptotics of

$$\mathbf{T}_{FS,n}(k) \overset{d}{=} N+1-T_{\Delta_n}(2k,1) = \frac{N}{n^{1/2}} \cdot n^{1/2}\left(\frac{1}{N}+1-\frac{T_{\Delta_n}(2k,1)}{N}\right).$$

Recalling $N \sim \frac{1}{2}n^2$ and using (3.3.1) in Theorem 3.3.1, we get (3.1.3) with the right-hand side being $\frac{1}{2}$ times the law of the smallest eigenvalue in $2k \times 2k$ aGUE. The law of the latter is given by the Fredholm determinant through Theorem 3.2.9 for the case $s=t=2k$ and general properties of the determinantal point processes (cf.A. Borodin 2011). □

Remark 3.4.6. *When $k=1$, there is only one particle at level $x=2$ and $\mathbf{T}_{FS}(1)$ equals in distribution to the coordinate of this particle. The density of this random variable is given by $K(2,u;2,u)=\frac{2}{\sqrt{\pi}}e^{-u^2}$, $u \geq 0$, as in Remark 3.3.8. The $k=1$ result was first obtained in A. Rozinov 2016, Theorem 1.7.*

Proof of Theorem 3.1.18. This is a combination of Lemma 3.4.5 with (3.3.2) in Theorem 3.3.1. □

Remark 3.4.7. *When $k = 1$, Z_3' is the coordinate of the unique particle of $\mathcal{X}_1'$ on level $x = 3$. The density of this random variable is given by $K_1'(3, u; 3, u) = 2ue^{-u^2}$, $u \geq 0$. Although without giving a formal definition, A. Rozinov 2016 used our second definition of spacing implicitly, and obtained this $k = 1$ result in A. Rozinov 2016, Theorem 1.7.*

Proof of Theorem 3.1.11. Let $\Lambda_- = a - X \in \mathbb{Z}_{\geq 0}$ and $\Lambda_+ = Y - a \in \mathbb{Z}_{>0}$. We have $\Lambda_-, \Lambda_+ \geq 0$, and $\Lambda_- + \Lambda_+ = \mathrm{Sp}_{k,n}$. Note that $\Lambda_+ \overset{d}{=} \mathbf{T}_{FS,n}(k)$.

By translation invariance of Corollary 3.1.8,

$$\mathbb{P}\left[\Lambda_- \geq r,\, \Lambda_+ \geq q\right] = \mathbb{P}\left[\Lambda_- \geq r',\, \Lambda_+ \geq q'\right],$$

if $r, q, r', q' \geq$ and $r + q = r' + q'$. Hence, for any $u, v \geq 0$ by Theorem 3.1.5

$$\begin{aligned}\mathbb{P}\left[2\frac{\Lambda_-}{n^{\frac{3}{2}}} \geq u,\, 2\frac{\Lambda_+}{n^{\frac{3}{2}}} \geq v\right] =& \mathbb{P}\left[\Lambda_- \geq u\frac{n^{\frac{3}{2}}}{2},\, \Lambda_+ \geq v\frac{n^{\frac{3}{2}}}{2}\right] = \mathbb{P}\left[\Lambda_- \geq 0,\, \Lambda_+ \geq u\frac{n^{\frac{3}{2}}}{2} + v\frac{n^{\frac{3}{2}}}{2}\right]\\ =& \mathbb{P}\left[2\frac{\Lambda_+}{n^{\frac{3}{2}}} \geq \frac{u\frac{n^{\frac{3}{2}}}{2} + v\frac{n^{\frac{3}{2}}}{2}}{\frac{n^{\frac{3}{2}}}{2}}\right] \xrightarrow{n\to\infty} W(u+v),\end{aligned}$$

where $W(\cdot)$ is the distribution function of $\mathbf{T}_{FS}(k)$. Smoothness of $W(\cdot)$ and the above computation show that $\left(2\frac{\Lambda_-}{n^{\frac{3}{2}}}, 2\frac{\Lambda_+}{n^{\frac{3}{2}}}\right)$ converges in distribution as $n \to \infty$ to a random vector (Z_1, Z_2), whose probability density we denote $f(x, y)$, $x, y \geq 0$. We also let $F(a, b) := \mathbb{P}\left[Z_1 > a, Z_2 > b\right]$; we already know that $F(a, b) = 1 - W(a + b)$, $a, b \geq 0$. Denoting F_1 and F_2 the partial derivatives of F in the first and second coordinates, respectively, we have $F_1(a, b) = F_2(a, b) = -W'(a+b)$. Recall that $Z_1 + Z_2$ is the scaled limit of $\mathrm{Sp}_{k,n}$ that we are interested in. We have:

$$
\begin{aligned}
\mathbb{P}\left[Z_1+Z_2>b\right] &= \int_0^b \int_{b-y}^{\infty} f(x,y)dxdy + \int_b^{\infty}\int_0^{\infty} f(x,y)dxdy \\
&= \int_0^b \left[-F_2(b-y,y)\right]dy + \int_0^{\infty}\left[-F_1(x,b)\right]dx \\
&= \int_0^b W'(b)dy + \int_0^{\infty} W'(x+b)dx \\
&= bW'(b) + \int_b^{\infty} W'(x)dx
\end{aligned}
$$

Differentiating the last identity in b we get the desired (3.1.4). □

Proof of Theorem 3.1.14. We fix k and deal with periodic extension of the sorting network of Definition 3.1.7. Given a random sorting network $(s_t)_{t\in\mathbb{Z}}$, we define a point process $\mathscr{P}$ on discrete circle of length $2N$ by declaring that a spot i is occupied by a particle if and only if $s_i = k$; in other words, this is the point process of times of swaps s_k. By Corollary 3.1.8 the process $\mathscr{P}$ is rotationally invariant and, therefore, the results of Section 3.4.1 apply. Since we would like to find the distribution of the conditional spacing, we rely on (3.4.2) — the desired quantity is $f_2(\ell)$ in that formula, which is being connected to $g(\ell)$ and ρ. The asymptotics of $g(\ell)$ is computed in Theorem 3.1.5. In order to find ρ, we notice that by the Edelman–Greene bijection (cf. O. Angel, A. Holroyd, et al. 2007, Proposition 9), it is computed as

$$\rho = \frac{\#\{\text{Standard Young tableaux of shape } \Delta_n \setminus (n-k,k)\}}{\#\{\text{Standard Young tableaux of shape } \Delta_n\}}.$$

Both numerator and denominator are explicitly known (they can be computed by the hook length formula J. S. Frame, G. B. Robinson, and R. M. Thrall. 1954). Applying the Stirling's formula, we get as $n \to \infty$:

$$
\begin{aligned}
\frac{1}{\rho} =& \binom{n}{2} \frac{\prod_{1\le j<k}(2k-2j)}{\prod_{1\le j<k}(2k-1-2j)} \frac{\prod_{k<i\le n}(2i-2k)}{\prod_{k<i\le n}(2i-2k-1)} \\
=& \binom{n}{2} \frac{(2n-2-2k)!!}{(2n-1-2k)!!} \frac{(2k-2)!!}{(2k-1)!!} \sim \frac{\sqrt{\pi}}{4} n^{\frac{3}{2}} \frac{(2k-2)!!}{(2k-1)!!}.
\end{aligned}
\tag{3.4.7}
$$

Now take $0 \leq a < b$ and sum (3.4.2) over $\lfloor a\frac{n^{\frac{3}{2}}}{2}\rfloor \leq \ell \leq \lfloor b\frac{n^{\frac{3}{2}}}{2}\rfloor$. We get

$$\sum_{\lfloor a\frac{n^{\frac{3}{2}}}{2}\rfloor}^{\lfloor b\frac{n^{\frac{3}{2}}}{2}\rfloor} f_2(\ell) = \frac{1}{\rho}\left(g\left(\lfloor a\frac{n^{\frac{3}{2}}}{2}\rfloor\right) - g\left(\lfloor b\frac{n^{\frac{3}{2}}}{2}\rfloor\right)\right). \tag{3.4.8}$$

We send $n \to \infty$ using Theorem 3.1.5, (3.4.7) and the following lemma, whose proof we leave as an exercise for the reader (the monotonicity condition is implied by (3.4.2)).

Lemma 3.4.8. *Let* $\xi_1, \xi_2, \ldots$ *be* $\mathbb{Z}_{\geq 0}$*–valued random variables, such that for each* n *the probabilities* $\mathbb{P}\left[\xi_n = \ell\right]$ *depend on* $\ell = 0, 1, 2\ldots$ *in a monotone way. Suppose that we have a distributional limit* $\lim\limits_{n\to\infty} \frac{2\xi_n}{n^{3/2}} \overset{d}{=} \xi$*, where* ξ *is an absolutely continuous random variable with density* $h(x)$*. Then for each* $a \geq 0$

$$\lim_{n\to\infty} \frac{n^{3/2}}{2}\mathbb{P}\left[\xi_n = \lfloor a\frac{n^{\frac{3}{2}}}{2}\rfloor\right] = h(a). \tag{3.4.9}$$

Hence, in terms of $\widehat{\mathrm{Sp}}_{k,n}$ of Theorem 3.1.14, as $n \to \infty$ (3.4.8) implies

$$\lim_{n\to\infty} \mathbb{P}\left[a \leq 2\frac{\widehat{\mathrm{Sp}}_{k,n}}{n^{\frac{3}{2}}} \leq b\right] = \frac{\sqrt{\pi}}{2}\frac{(2k-2)!!}{(2k-1)!!}\left[\frac{\partial}{\partial b}\big(\mathbb{P}\left[\mathbf{T}_{\mathrm{FS}}(k) > b\right]\big) - \frac{\partial}{\partial a}\big(\mathbb{P}\left[\mathbf{T}_{\mathrm{FS}}(k) > a\right]\big)\right]. \tag{3.4.10}$$

Thus, $2\frac{\widehat{\mathrm{Sp}}_{k,n}}{n^{\frac{3}{2}}}$ converges in distribution to a random variable of density given by (3.1.5). □

Remark 3.4.9. *We expect that an alternative proof of Theorem 3.1.11 can be obtained by following the lines of the just presented argument, but using* (3.4.1) *instead of* (3.4.2)*.*

Remark 3.4.10. *Theorem 3.1.14 and Theorem 3.1.18 give two different expressions for the distribution of the same random variable* $\widehat{Z}_k$*. It would be interesting to find a direct (i.e. avoiding taking the limit from the sorting networks) proof that these expressions coincide.*

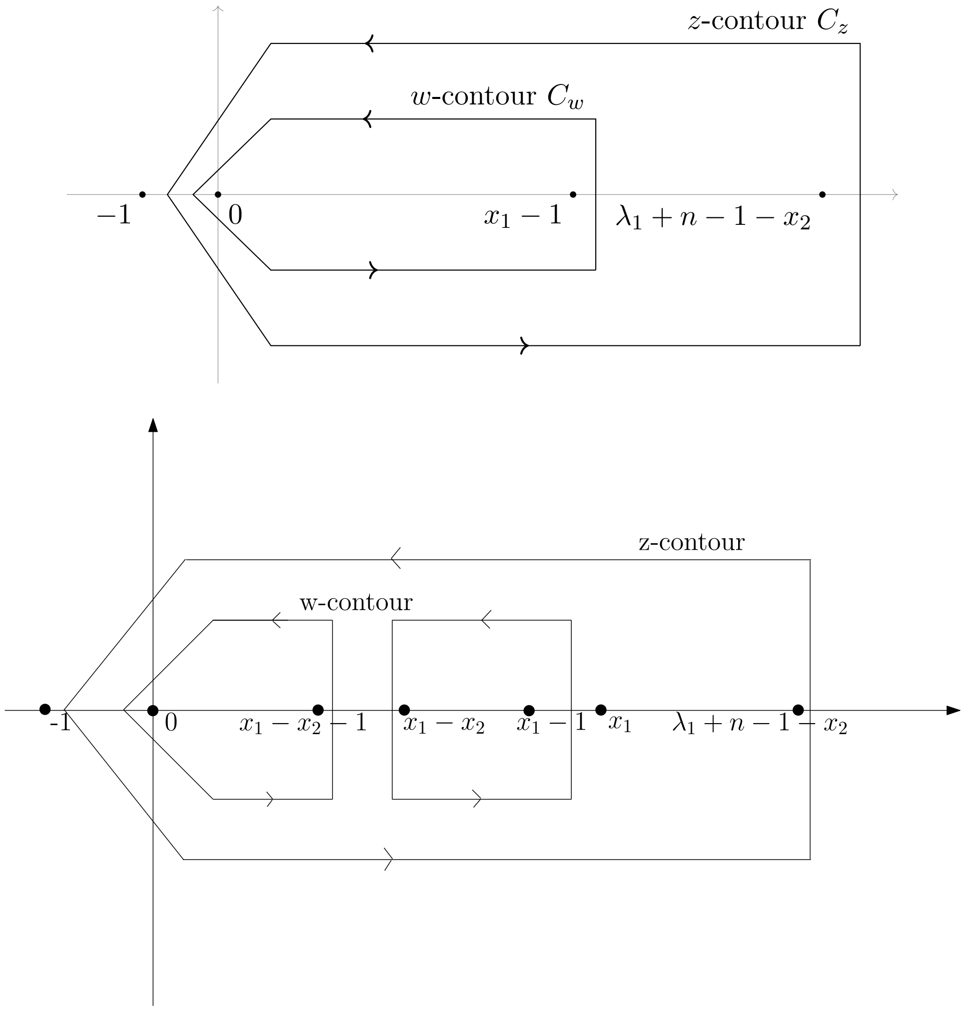

Figure 3.4: The contours in the statement of Theorem 3.2.4, correspond to the case $2 \leq x_1 \leq x_2$ (top) and $2 \leq x_2 < x_1$ (bottom). In the bottom picture, w-contour splits into two components, in order to guarantee that C_z and $x_1 - x_2 - 1 - C_w$ are disjoint

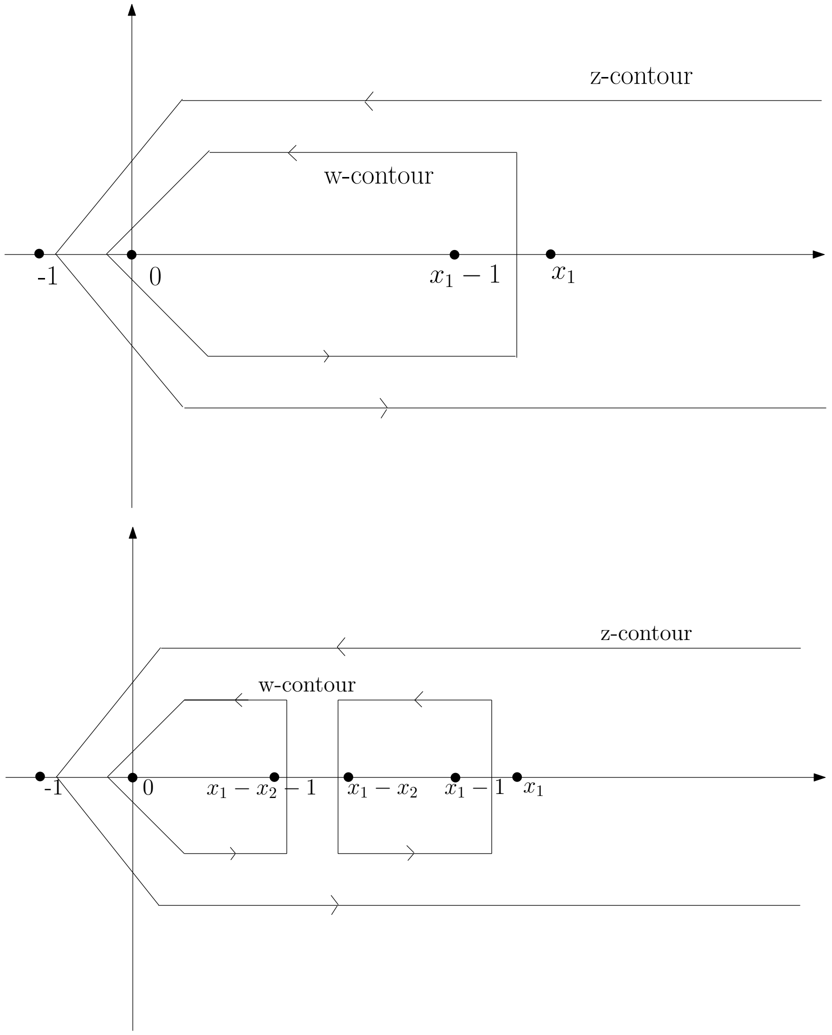

Figure 3.5: The contours in the statements of Theorem 3.3.5 and 3.3.6, for the case $1 \leq x_1 \leq x_2$ and $1 \leq x_2 < x_1$ respectively.

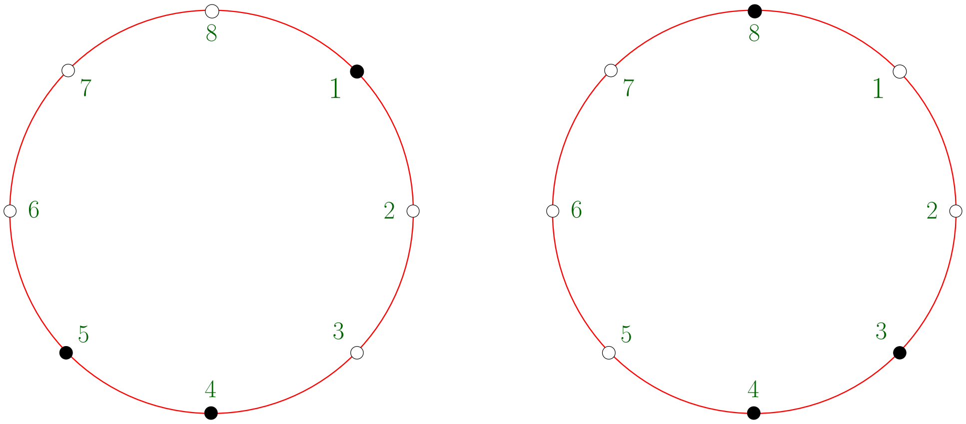

Figure 3.6: Two configurations of point process on discrete circle of length $K = 8$ described in Example 3.4.1. Particles are shown in black.

Appendix A

Supplementary materials: Symmetric functions and Harish-Chandra integrals

A.1 Dunkl operators and hypergeometric functions

In this appendix, we give a brief review of the basic settings of multivariate hypergeometric functions defined by abstract root systems, the differential operators acting on them, their connection to symmetric spaces and their spherical functions, and the limit transition to multivariate Bessel functions. The purpose is to provide a theoretical background to the particular objects appearing and used in this text, and explain the connections between them. In Section 2.2 to 2.5, we specify from general theory to the special case and provide more concrete formulas, that we operate with in Section 3-6.

A large part of our presentation is a simplification of Remling and Roesler 2015 Section 2 and 3, which gives a brief and clear review of the theory with more explanations of the concepts. For more detailed exposition of Dunkl theory, see Roesler 2003b and Anker 2017.

For any $M \geq 1$, consider the Euclidean space $\mathbb{R}^M$ with the standard scalar product $\langle x, y\rangle = \sum_{i=1}^{M} x_i y_i$. For $\alpha \in \mathbb{R}^M \backslash \{0\}$, denote the reflection of point x about the hyperplane

$\langle\alpha\rangle^{\perp}$ by σ_α, such that

$$\sigma_\alpha(x) = x - 2\frac{\langle\alpha, x\rangle}{\langle\alpha,\alpha\rangle^2}\alpha.$$

Clearly each σ_α is an element in the orthogonal group $O(M)$.

Definition A.1.1. *A root system R is a finite set of vectors in $\mathbb{R}^M \setminus \{0\}$, such that $\sigma_\alpha(R) = R$ for all $\alpha \in R$. We say R is irreducible if it cannot be decomposed into two disjoint subsets whose elements are mutually orthogonal. R is crystallographic, if for any $\alpha, \beta \in R$ we have*

$$\frac{2\langle\alpha,\beta\rangle}{\langle\beta,\beta\rangle} \in \mathbb{Z}.$$

$\{\sigma_\alpha\}_{\alpha\in R}$ generate a subgroup of $O(M)$, which is called the Weyl group of root system R and record the symmetry that R has.

Each root system can be written as a disjoint union $R = R_+ \bigcup(-R_+)$, such that R_+ and $-R_+$ are separated by some hyperplane through the origin. We call R_+ the positive part of R.

Remark A.1.2. *The choice of R_+ is not unique, but all the choices are identical under a linear transformation.*

Remark A.1.3. *In this text, we do not require R to be reduced, i.e, $R\bigcap\mathbb{R}\alpha = \pm\alpha$, for all $\alpha \in R$.*

Example A.1.4. *In practise, people care mostly about the classical root systems of type A-D. The following two crystallographic root systems appear in random matrices: the root system of A_M, $M = 1, 2, ...$, that is*

$$R = \{e_i - e_j, 1 \le i < j \le M\},$$

and the root system BC_M, $M = 1, 2, ...$, that is

$$R = \{\pm e_i, 1 \le i \le M, \pm 2e_i, 1 \le i \le M, \pm e_i \pm e_j, 1 \le i < j \le M\},$$

where e_i*,* $i = 1, 2, ..., M$ *denotes the* i^{th} *standard basis in* $\mathbb{R}^M$*.*

After recalling the notion of root system, we are now able give the definition of hypergeometric and Bessel functions, which were introduced in a series of works by Heckman and Opdam, see Heckman and Schlichtkrull 1995,Opdam 1995,Opdam 2000 for details. Fix $M \in \mathbb{Z}_{\geq 1}$, take $\mathfrak{A}$ to be a M-dimensional Euclidean space, let $\mathfrak{A}_{\mathbb{C}}$ be the complexification of $\mathfrak{A}$, which is isomorphic to $\mathbb{C}^M$, and let R be a crystallographic root system on $\mathfrak{A}$ with Weyl group W, R^+ be the positive part of R, and P^+ be the set of dominant weights associated with R^+, i.e,

$$P^+ := \Big\{\lambda \in \mathfrak{A} : \frac{\langle \lambda, \alpha\rangle}{\langle \alpha, \alpha\rangle} \in \mathbb{Z}^+ \text{ for all } \alpha \in R^+\Big\}.$$

Remark A.1.5. *View a partition* $(\lambda_1, ..., \lambda_M)$ *as a vector with nonnegative integer entries. For root system of type A,* P^+ *can be identified with the set of all partitions of length at most* M*, and for root system of type BC,* P^+ *can be identified with the set of all even partitions of length at most* M*.*

For $\mu, \lambda \in P^+$*, we write* $\mu \leq \lambda$ *if* $\mu_i \leq \lambda_i$ *for all* $i = 1, 2, ..., M$*.*

A root multiplicity function $m_\alpha : \alpha \in R$ on R is a W-invariant map which assigns to each root in R a real number. For $\alpha \in R$, let

$$\rho = \rho(m) := \frac{1}{2} \sum_{\alpha \in R^+} m_\alpha \alpha, \tag{A.1.1}$$

and α_i, ρ_i be the i^{th} component of α, ρ respectively.

Let s_α to be the reflaction operator about $\{\alpha\}^\perp$ such that $s_\alpha f(x) := f(\sigma_\alpha(x))$. The following differential operator acts on any smooth function on $\mathfrak{A}$.

Definition A.1.6. *For* $i = 1, 2, ..., M$*, given a root multiplicity function* m_α*, the* trigonometric Dunkl operator *associated with* R *and* m *is*

$$T_i = \partial_i + \sum_{\alpha \in R^+} m_\alpha \frac{\alpha_i}{1 - e^{-2\langle \alpha, \cdot \rangle}}(1 - s_\alpha) - \rho_i \tag{A.1.2}$$

Let $S(\mathfrak{A}_{\mathbb{C}})$ denote the space of complex polynomials p in M variables, such that when identifying the i^{th} variable with the i^{th} standard basis e_i in $\mathfrak{A}$, $p \in S(\mathfrak{A}_{\mathbb{C}})$ is invariant under action of W.

Definition A.1.7. *For* $\lambda \in \mathfrak{A}_{\mathbb{C}}$*, the* hypergeometric function associated with R *is an analytic W-invariant function* $F_\lambda(x; m)$ *on* $\mathfrak{A}$*, such that for each* $p \in S(\mathfrak{A}_{\mathbb{C}})$*,*

$$p(T)F_\lambda = p(\lambda)F_\lambda. \tag{A.1.3}$$

We set $F_\lambda(0; m) = 1$*.*

Theorem A.1.8. *Heckman and Schlichtkrull 1995 There exists an open set of root multiplicity functions* M_{reg}*, which contains all nonnegative m's, such that if* $m \in M_{reg}$*, for each* $\lambda \in \mathfrak{A}_{\mathbb{C}}$ *there exists a unique function* $F_\lambda(z; m)$ *satisfying Definition A.1.7. Moreover,* $F : \mathfrak{A}_{\mathbb{C}} \times M_{reg} \times \mathfrak{A} \to \mathbb{C}$ *is analytic.*

Similar to hypergeometric functions, the multivariate Bessel functions are also defined as W-invariant eigenfunctions of some differential operators, which are called rational Dunkl operators.

Definition A.1.9. *For i=1,2,...,M, given a root multiplicity function* m_α*, the* rational Dunkl operator *associated with R and m is*

$$D_i = \partial_i + \sum_{\alpha \in R^+} m_\alpha \frac{\alpha_i}{2\langle \alpha, \cdot \rangle}(1 - s_\alpha) \tag{A.1.4}$$

Definition A.1.10. *For* $\lambda \in \mathfrak{A}_{\mathbb{C}}$*,* $m \geq 0$*, the* Bessel function associated with R *is an analytic W-invariant function* $f_\lambda(x; m)$ *on* $\mathfrak{A}$*, such that for each* $p \in S(\mathfrak{A}_{\mathbb{C}})$*,*

$$p(D)f_\lambda = p(\lambda)f_\lambda. \tag{A.1.5}$$

Note that both F_λ, f_λ are invariant in both λ and z. Indeed, Bessel functions can be obtained from hypergeometric functions by a limit transition. This is simply because under the same limit transition, the trigonometric Dunkl operator T_i converges to the corresponding rational Dunkl operator D_i.

Proposition A.1.11. *Saïd and Ørsted 2005a, Anker 2017, Section 4.4 For* $\lambda \in \mathfrak{A}_{\mathbb{C}}$*,* $m \geq 0$*,*

$$f_\lambda(z; m) = \lim_{\epsilon \to 0} F_{\epsilon^{-1}\lambda}(\epsilon z; m). \tag{A.1.6}$$

From now consider only $\lambda \in P^+$. Let $M_\lambda := \sum_{\mu \in W.\lambda} e^{i\langle \mu, \cdot \rangle}$ be the trigonometric symmetric monomial on $\mathbb{T}$ indexed by λ, where $\mathbb{T}$ is a torus obtained as the quotient space of $\mathfrak{A}$, on which $e^{i\langle \mu, \cdot \rangle}$ is periodic.

For simplicity take m to be a nonnegative root multiplicity function (which is in M_{reg} by Theorem A.1.8). Let

$$w_m(z) := \prod_{\alpha \in R^+} \left| e^{i\langle \alpha, z \rangle} - e^{-i\langle \alpha, z \rangle} \right|^{m_\alpha}. \tag{A.1.7}$$

Definition A.1.12. *The* Jacobi polynomials *(Heckman-Opdam polynomials) associated with* R *and* $m \geq 0$ *are a collection of functions* $\mathfrak{J}_\lambda$ *on* T *indexed by* $\lambda \in P^+$*, where*

$$\mathfrak{J}_\lambda(\cdot; m) = \sum_{\mu \in P^+,\ \mu \leq \lambda} c_{\lambda\mu}(m) M_\mu,$$

and coefficients $c_{\lambda\mu}(m)$*'s are uniquely determined by*

(1). $c_{\lambda\lambda}(m) = 1$

(2). $\mathfrak{J}_\lambda$*'s are mutually orthogonal in* $L^2(\mathbb{T}; w_m)$*.*

Note that $\mathfrak{J}_\lambda$'s form an orthogonal basis of $L^2(\mathbb{T}; w_m)^W$, the subspace of W-invariant elements in $L^2(\mathbb{T}; w_m)$.

Example A.1.13. *When taking* R *to be the type A root system, the Jacobi polynomials of type A are Jack polynomial* $P_\lambda(\vec{x}; \theta) \in \Lambda_M$*'s, where* x_i *is identified with* e^{iz_i}*.*

The following important result identifies each hypergeometric function with a Jacobi polynomial, when the corrsponding weight λ is positive integer-valued.

Theorem A.1.14. *Heckman and Schlichtkrull 1995, Section 4.4 For all* $\lambda \in P^+$*,* $m \geq 0$*, the function* $F_{\lambda+\rho}(x; m)$ *extends holomorphically to* $\mathfrak{A}_{\mathbb{C}}$*, and*

$$F_{\lambda+\rho}(iz; m) = c(\lambda + \rho, m)\mathfrak{J}_\lambda(z; m), \tag{A.1.8}$$

where $c(\lambda + \rho, m)$ *is a constant depending on* λ *and* m*.*

Remark A.1.15. *For some multiplicity function* m *that is not nonnegative, as long as the* L^2 *kernel* $w_m(x)$ *is integrable on* T*,* $\mathfrak{J}_\lambda$*'s are still well-defined and Theorem A.1.14 holds for such* m*. See Section 2.2.2.*

When the root multiplicity function m takes some special values and $\lambda \in P^+$, the Jacobi polynomial $\mathfrak{J}_\lambda$'s can be identified with the spherical functions on one of the classical symmetric spaces.

The theory of symmetric spaces are classical, and the standard references are Helgason 2001, Helgason 2012, which discuss the classification problem, representation theory and analytic properties of symmetric spaces. In a word, a Riemannian symmetric space is a certain quotient of classical Lie groups G/K or U/K, where G is noncompact, U is compact and K is a compact subgroup. G/K and U/K are of the so-called noncompact type and compact type, respectively, and there's a duality between one noncompact symmetric space and one compact symmetric space.

Let G/K and U/K be the dual of each other, and let g, u, l denote the Lie algebra of G, U, K respectively, and let $\mathfrak{A}$ denote the maximal abelian subspace in g/l, then by duality $i\mathfrak{A}$ is the maximal abelian subspace in u/l. The restricted root system of G/K (or U/K) is set to be on $\mathfrak{A}$ (or $i\mathfrak{A}$), and they share the same root multiplicities. In Appendix B we list the (restricted) root multiplicity functions of several compact symmetric spaces that are connected to random matrices.

For a noncompact symmetric space G/K, its spherical function is defined as a nonzero K-biinvariant function $\phi_\lambda : G \to \mathbb{C}$, which is an eigenfunction of any so-called invariant differential operator on G/K, indexed by $\lambda \in \mathfrak{A}_{\mathbb{C}}$. For a compact symmetric space U/K

with restricted root system R, its spherical function $\psi : U \to \mathbb{C}$ are indexed by the highest weights of unitary irreducible K-spherical representations (the representation with a 1-dimensional invariant subspace V_λ^K) π_λ of U , where the space of all highest weight is identified as $P^+(R)$, and each spherical function is given by

$$\psi_\lambda(u) = \langle \pi_\lambda(u)e_\lambda, e_\lambda \rangle,$$

where e_λ is the unique unit vector in V_λ^K.

For a spherical function ϕ of G/K, by Cartan decomposition and the K-biinvariance ϕ is determined by the value on $\mathfrak{A}$, and same for ψ of U/K. Moreover we have the following result.

Theorem A.1.16. *For $x \in \mathfrak{A}$, any $\lambda \in P^+$,*

$$\psi_\lambda(\exp(ix)) = \phi_{\lambda+\rho}(\exp(ix)) = F_{\lambda+\rho}(ix; m) = c(\lambda+\rho, m)\mathfrak{J}_\lambda(x; m). \tag{A.1.9}$$

The first equality is given in Helgason 2012, and the second equality is Heckman and Schlichtkrull 1995, Theorem 5.2.2. Because of this identification, we no longer distinguish spherical function (indexed by $\lambda \in P^+$ of noncompact and compact symmetric spaces, and treat them simply as an analytic function on M dimensional Euclidean space $\mathfrak{A}$.

Remark A.1.17. *The Heckman-Opdam Laplacian is an analog of the usual Laplace operator on $\mathbb{R}^M$, given by $p_2(T) = \sum_{i=1}^M T_i^2$. In the case of Theorem A.1.16,*

$$p_2(T) = \Delta + \langle \rho, \rho \rangle,$$

where Δ is the Laplace-Beltrami operator on G/K (or U/K). Moreover, the limit transition in Proposition A.1.11 specifies to a contraction of a Riemannian symmetric space to a corresponding Euclidean symmetric space, and convergence of the spherical function of the former to the latter. See Saïd and Ørsted 2005b, Theorem 3.4.

The spherical functions of Riemannian and Euclidean symmetric spaces can both be *written as the so-called Harish-Chandra integrals, see Helgason 2012, Chapter IV or Saïd and Ørsted 2005b, Section 2 for their explicit forms. We give calculations of the Harish-Chandra integral corresponding to several Euclidean symmetric spaces in A.3.*

A.2 Root multiplicities

Let U be a classical compact Lie group, and K be a Lie subgroup of U. We list several examples that U/K is a compact Riemannian symmetric space of rank M, and give its root multiplicities. For a complete list of classifications of irreducible Riemannian symmetric spaces, see Helgason 2001, Chapter X.

Let $O(M), U(M), Sp(M)$ denote the $M \times M$ orthogonal/unitary/compact sympletic group. Let $M \leq N$, $1 \leq i \neq j \leq M$, $\{e_i\}_{i=1}^{M}$ be the standard basis of $\mathfrak{A} = \mathbb{R}^M$.

Compact symmetric space	$m_{e_i-e_j}$	$m_{\pm e_i \pm e_j}$	$m_{\pm e_i}$	$m_{\pm 2e_i}$
$U(M+1)/O(M+1)$	1	0	0	0
$U(M+1)$	2	0	0	0
$U(2M+2)/Sp(M+1)$	4	0	0	0
$O(N+M)/O(N) \times O(M)$	0	1	$N-M$	0
$U(N+M)/U(N) \times U(M)$	0	2	$2(N-M)$	1
$Sp(N+M)/Sp(N) \times Sp(M)$	0	4	$4(N-M)$	3

Now let m be a root multiplicity function such that $m_{\pm e_i \pm e_j} = 2\theta$, $m_{\pm e_i} = 2\theta(N-M)$, $m_{\pm 2e_i} = 2\theta - 1$, the $\theta = \frac{1}{2}, 1, 2$ cases correspond to the last three rows in the above table. On the other hand, $m_{e_i-e_j} = 2\theta$ corresponds to the first three rows when $\theta = \frac{1}{2}, 1, 2$, and the Bessel functions with these root multiplicities are of type A, and were studied in Benaych-Georges, Cuenca, and Gorin 2022.

A.3 Power series expression of Harish-Chandra integrals

In this appendix we consider matrix integrals which appear as spherical functions of certain Euclidean type symmetric spaces, whose root systems are of type A or type BC. These so-called Harish-Chandra integral originated in the representation theory gain independent interests from physics and special functions, and were also studied intensively.

One goal of people is to calculate out an explicit expression of these integrals. There were a large amount of literature studying this, but while different people care about Harish-Chandra integral of different forms, the explicit expressions they provide and the level of explicitness also differ a lot. In the remaining pages, we try to give a relatively systematic summary of explicit expressions of Harish-Chandra integrals related to several classical symmetric spaces, highlight their connections with some classical random matrix ensembles, and we give the expressions in terms of the symmetric polynomials. While the results and proof ingredient here are well known by experts, parts of them might not been formally published and might be helpful to readers.

Fix $N \in \mathbb{Z}_{>0}$. Let U denote the compact orthogonal/unitary/unitary sympletic Lie group $O(N)/U(N)/Sp(N)$, and dU be the corresponding Haar measure while the parameter $\theta = \frac{1}{2}, 1, 2$.

Proposition A.3.1. *Forrester 2010, Proposition 13.4.1*

For $N \geq 2$, $\theta = \frac{1}{2}, 1, 2$, *let* $A = diag(a_1, ..., a_N), Z = diag(z_1, ..., z_N)$,

$$\int \exp(Tr(ZUAU^{-1})dU = \sum_{\mu} \frac{1}{H(\mu)} \frac{P_\mu(a_1, ..., a_N; \theta) P_\mu(z_1, ..., z_N; \theta)}{P_\mu(1^N; \theta)}. \tag{A.3.1}$$

For $\theta = \frac{1}{2}, 1, 2$, (A.3.1) corresponds by a limit transition to spherical function of compact symmetric spaces $U(N)/O(N), U(N), U(2N)/Sp(N)$ of type A_{N-1}, where θ is the (restricted) root multiplicity $k_{e_i - e_j} = \frac{1}{2} m_{e_i - e_j}$ $(1 \leq i \neq j \leq N)$. See Okounkov and Olshanski 1997, Section 4. In probablistic context, (A.3.1) is introduced as "characteristic function" of $N \times N$ real/complex/real quaternionic self-adjoint random matrices whose distribution is invariant under unitary conjugations. The typical examples are GOE/GUE/GSE. See Gorin and A. W. Marcus 2020, Benaych-Georges, Cuenca, and Gorin 2022 for more details.

Olshanski and Vershik 1996, Section 4 gives a proof of Proposition A.3.1 for the case $\theta = 1$. Inspired by their approach, we provide another proof of Theorem 2.2.18 following the same line.

Fix $M \leq N$, $\theta = \frac{1}{2}, 1, 2$, define

$$\Lambda = \begin{bmatrix} a_1 & & & & 0 & \dots & 0 \\ & a_2 & & & 0 & \dots & 0 \\ & & \dots & & & & \\ & & & \dots & & & \\ & & & a_M & 0 & \dots & 0 \end{bmatrix}_{M \times N},$$

$$Z = \begin{bmatrix} z_1 & & & & \\ & z_2 & & & \\ & & \dots & & \\ & & & \dots & \\ & & & & z_M \\ 0 & & \dots & & 0 \\ & & \dots & & \\ 0 & & \dots & & 0 \end{bmatrix}_{N \times M},$$

$U \in O(M)/U(M)/Sp(M)$, $V \in O(N)/U(N)/Sp(N)$ are integrated under Haar measures.

Lemma A.3.2. *Macdonald 1995, Chapter I, (7.8) For $m \in \mathbb{Z}_{\geq 1}$, expand p_1^m in terms of Schur polynomials, i.e,*

$$p_1^m = \sum_{|\lambda|=m} C_m^\lambda S_\lambda. \tag{A.3.2}$$

Then

$$C_m^\lambda = \frac{m!}{\prod_{s \in \lambda}[a(s) + l(s) + 1]}. \tag{A.3.3}$$

Remark A.3.3. *C_m^λ can be interpreted in view of both representation theory of symmetric group and combinatoric: $C_m^\lambda = \chi_\lambda(1^m) = dim_{S_m}(\lambda)$, the character of S_m at identity, and it's equal to the number of standard Young tableaux of shape λ.*

Proposition A.3.4. *Forrester 2010, Proposition 13.4.1 For* $\theta = \frac{1}{2}, 1, 2$,

$$\begin{aligned} &\int dU \int dV \ \exp(Tr(U\Lambda VZ + Z^*V^*\Lambda^*U^*)) \\ =&\sum_{\mu}\prod_{j=1}^{M} \frac{\Gamma(\theta N - \theta(j-1))}{\Gamma(\theta N - \theta(j-1) + \mu_j)} \frac{1}{H(\mu)} \frac{P_\mu(a_1^2, \cdots, a_M^2; \theta) P_\mu(z_1^2, \cdots, z_M^2; \theta)}{P_\mu(1^M; \theta)}, \end{aligned} \tag{A.3.4}$$

where $H(\mu)$ *is defined in (2.2.7).*

Remark A.3.5. *Forrester 2010, Chapter 13 provides a different self-contained proof of Proposition A.3.1 and A.3.4.*

Proof. Throughout this proof, for a symmetric polynomial $f(x_1, ..., x_M)$ and a $M \times M$ matrix X, let $f(X)$ be the value of f evaluated at eigenvalues of X.

For $\theta = 1$ this integral and its various generalizations were well-studied in physic literature, e.g, in Ghaderipoor and Tellambura 2008, Schlittgen and Wettig 2003,Guhr and Wettig 1996. Ghaderipoor and Tellambura 2008 gives the same power series expansion as the right side, which degenerates to

$$\sum_{\mu} \frac{\prod_{i=1}^{M}(N-i)!}{\prod_{i=1}^{M}(N-i+\mu_i)!} \frac{\prod_{i=1}^{M}(M-i)!}{\prod_{i=1}^{M}(M-i+\mu_i)!} S_\mu(a_1^2, ..., a_M^2) S_\mu(b_1^2, ..., b_M^2),$$

and its proof relies on the well known fact that Schur polynomials $s_\mu(x_1, ..., x_N)$ are the characters of $U(N)$ and $SL_N(\mathbb{R})$.

For $\theta = \frac{1}{2}$,

$$\exp(Tr(U\Lambda VZ + Z^*V^*\Lambda^*U^*) = \exp(Tr(2U\Lambda VZ) = \sum_{l(\mu)\leq M} \frac{C^{\mu}_{|\mu|}}{|\mu|!} S_\mu(2U\Lambda VZ).$$

By Macdonald 1995, Chapter VII, Section 3, (2.23) and Macdonald 1995, Chapter VI, (10.22),

$$\int S_\mu(U\Lambda VZ) dU = \begin{cases} 0 & \mu \text{ is not even;} \\ \Omega_\mu(\Lambda VZ) = \dfrac{P_\lambda(Z^TV^T\Lambda^T\Lambda VZ; \frac{1}{2})}{P_\mu(1^M; \frac{1}{2})} & \mu \text{ is even, } \mu = 2\lambda. \end{cases} \tag{A.3.5}$$

$$\begin{aligned}\int P_\lambda(Z^TV^T\Lambda^T\Lambda VZ;\frac{1}{2})dV &= \int P_\lambda(ZZ^TV^T\Lambda^T\Lambda V;\frac{1}{2})dV\\ =\frac{P_\lambda(ZZ^T;\frac{1}{2})P_\lambda(\Lambda^T\Lambda;\frac{1}{2})}{P_\lambda(1^N;\frac{1}{2})} &= \frac{P_\lambda(Z^TZ)P_\lambda(\Lambda\Lambda^T)}{P_\lambda(1^N;\frac{1}{2})},\end{aligned} \tag{A.3.6}$$

where the second equality holds by Macdonald 1995, p. VII.4.2. Then

$$\begin{aligned}&\int\int \exp(Tr(U\Lambda VZ + Z^*V^*\Lambda^*U^*)dUdv\\ =&\sum_\lambda \frac{C^{2\lambda}_{2|\lambda|}}{|\lambda|!}\frac{4^{|\lambda|}}{P_\lambda(1^M;\frac{1}{2})P_\lambda(1^N;\frac{1}{2})}P_\lambda(Z^TZ;\frac{1}{2})P_\lambda(\Lambda\Lambda^T;\frac{1}{2})\\ =&\sum_\lambda \frac{C^{2\lambda}_{2|\lambda|}}{|\lambda|!}\frac{4^{|\lambda|}}{P_\lambda(1^N;\frac{1}{2})}\frac{1}{P_\lambda(1^M;\frac{1}{2})}P_\lambda(a_1^2,...,a_M^2;\frac{1}{2})P_\lambda(z_1^2,...,z_M^2;\frac{1}{2})\\ =&\sum_\lambda \frac{4^{|\lambda|}}{\prod_{s\in 2\lambda}[a(s)+l(s)+1]}\frac{\prod_{s\in\lambda}[a(s)+\frac{1}{2}l(s)+\frac{1}{2}]}{\prod_{s\in\lambda}[\frac{N}{2}+j-1-\frac{1}{2}(i-1)]}\frac{1}{P_\lambda(1^M;\frac{1}{2})}\\ &\cdot P_\lambda(a_1^2,...,a_M^2;\frac{1}{2})P_\lambda(z_1^2,...,z_M^2;\frac{1}{2})\\ =&\sum_\lambda\prod_{i=1}^M \frac{\Gamma(\frac{N}{2}-\frac{1}{2}(i-1))}{\Gamma(\frac{1}{2}N-\frac{1}{2}(i-1)+\lambda_i)}\frac{1}{\prod_{s\in\lambda}[a(s)+1+\frac{1}{2}l(s)]}\frac{P_\lambda(a_1^2,\cdots,a_M^2;\frac{1}{2})P_\lambda(z_1^2,\cdots,z_M^2;\frac{1}{2})}{P_\lambda(1^M;\frac{1}{2})}\end{aligned}$$

where the third equality follows from Lemma A.3.2 and (2.2.11).

For $\theta = 2$, let η denote the map embedding real quaternion into $M_{2\times 2}(\mathbb{C})$, such that for $x = a + bi + cj + dk$, $a, b, c, d \in \mathbb{R}$,

$$\eta(x) = \begin{bmatrix} a+bi & c+di \\ -c+di & a-bi \end{bmatrix}. \tag{A.3.7}$$

Similarly for η embeds $GL_M(Sp)$ into $GL_{2M}(\mathbb{C})$ by

$$\eta(X) = [\eta(X_{ij})]_{1\le i,j\le M}. \tag{A.3.8}$$

$$\exp(Tr(U\Lambda VZ + Z^*V^*\Lambda^*U^*) = \exp(Tr(\eta(U\Lambda VZ)) = \sum_{l(\mu)\le M}\frac{C^\mu_{|\mu|}}{|\mu|!}S_\mu(\eta(U\Lambda VZ)).$$

By Macdonald 1995, Chapter VII, (6.13), (6.14), (6.20), Exercise 2.7 and Chapter VI, (10.22),

$$\int S_\mu(\eta(U\Lambda VZ))dU = \begin{cases} 0 & \mu' \text{ is not even;} \\ \Omega_\mu(\Lambda VZ) = \dfrac{P_\lambda(Z^*V^*\Lambda^*\Lambda VZ;2)}{P_\mu(1^M;2)} & \mu' \text{ is even, } \mu = \lambda\cup\lambda, \end{cases} \tag{A.3.9}$$

where $*$ denotes the conjugate transpose of quaternionic matrices.

$$\begin{aligned} &\int P_\lambda(Z^*V^*\Lambda^*\Lambda VZ;2)dV = \int P_\lambda(ZZ^*V^*\Lambda^*\Lambda V;2)dV \\ =&\frac{P_\lambda(ZZ^*;2)P_\lambda(\Lambda^*\Lambda;2)}{P_\lambda(1^N;2)} = \frac{P_\lambda(Z^*Z;2)P_\lambda(\Lambda\Lambda^*;2)}{P_\lambda(1^N;2)}, \end{aligned} \tag{A.3.10}$$

where the second equality holds by Macdonald 1995, Chapter VII, Exercise 6.4. Then

$$\begin{aligned} &\int\int \exp(Tr(U\Lambda VZ + Z^*V^*\Lambda^*U^*)dUdV \\ =&\sum_\lambda \frac{C^{\lambda\cup\lambda}_{|\lambda\cup\lambda|}}{|\lambda\cup\lambda|!}\frac{1}{P_\lambda(1^M;2)P_\lambda(1^N;2)}P_\lambda(Z^*Z;2)P_\lambda(\Lambda\Lambda^*;2) \\ =&\sum_\lambda \frac{C^{\lambda\cup\lambda}_{|\lambda\cup\lambda|}}{|\lambda\cup\lambda|!}\frac{1}{P_\lambda(1^N;2)}\frac{1}{P_\lambda(1^M;2)}P_\lambda(a_1^2,...,a_M^2;2)P_\lambda(z_1^2,...,z_M^2;2) \\ =&\sum_\lambda \frac{1}{\prod_{s\in\lambda\cup\lambda}[a(s)+l(s)+1]}\frac{\prod_{s\in\lambda}[a(s)+2l(s)+2]}{\prod_{s\in\lambda}[2N+j-1-2(i-1)]}\frac{1}{P_\lambda(1^M;2)} \\ &\cdot P_\lambda(a_1^2,...,a_M^2;2)P_\lambda(z_1^2,...,z_M^2;2) \\ =&\sum_\lambda\prod_{i=1}^M \frac{\Gamma(2N-2(i-1))}{\Gamma(2N-2(i-1)+\lambda_i)}\frac{1}{\prod_{s\in\lambda}[a(s)+1+2l(s)]}\frac{P_\lambda(a_1^2,\cdots,a_M^2;2)P_\lambda(z_1^2,\cdots,z_M^2;2)}{P_\lambda(1^M;2)}, \end{aligned}$$

where the third equality again follows from Lemma A.3.2 and (2.2.11) □

A.4 Limit transition of type BC Bessel functions

In this section we provide a limit transition of $\mathbb{B}(\vec{a}, z_1, ..., z_M; \theta)$ to a simple symmetric combination of exponents. This transition implies that in the $\theta = 0$ regime, the rectangular addition $\vec{a} \boxplus^\theta_{M,N} \vec{b}$ becomes the usual convolution of the empirical measures $\frac{1}{M}\sum_{i=1}^M \delta_{a_i^2}$

and $\frac{1}{M}\sum_{i=1}^{M}\delta_{b_i^2}$.

Proposition A.4.1. *Given $\vec{a}=(a_1\geq a_2\geq ...\geq a_M)$, take M to be fixed, $N\to\infty,\theta\to 0, N\theta\to\infty$, then*

$$\mathbb{B}(\vec{a},N\theta z_1,...,N\theta z_M;\theta,N)\longrightarrow\frac{1}{M!}\sum_{\sigma\in S_M}\prod_{i=1}^{M}e^{a_i^2z_{\sigma(i)}^2}. \tag{A.4.1}$$

Proof. This follows from a straightforward calculation. Indeed, by Proposition 2.2.16,

$$\mathbb{B}(\vec{a},N\theta z_1,...,N\theta z_M;\theta,N)=\sum_{\mu\in\text{YD}}\prod_{j=1}^{l(\mu)}\frac{(N\theta)^{\mu_j}}{[\theta(N-j+1)]\cdots[\theta(N-j+1)+\mu_j-1]}$$
$$\cdot\frac{\prod_{s\in\mu}\Big[a(s)+\theta l(s)+\theta\Big]}{\prod_{s\in\mu}\Big[M\theta+(j-1)-\theta(i-1)\Big]}\cdot\frac{1}{\prod_{s\in\mu}\Big[a(s)+1+\theta l(s)\Big]}P_\mu(a_1^2,...,a_M^2;\theta)P_\mu(z_1^2,...,z_M^2;\theta). \tag{A.4.2}$$

When taking the limit in the above way,

$$\prod_{j=1}^{l(\mu)}\frac{(N\theta)^{\mu_j}}{[\theta(N-j+1)]\cdots[\theta(N-j+1)+\mu_j-1]}\longrightarrow 1,$$

and

$$\frac{1}{\prod_{s\in\mu}\Big[a(s)+1+\theta l(s)\Big]}\longrightarrow\prod_{j=1}^{l(\mu)}\frac{1}{\mu_j!}.$$

Also note that $a(s)+\theta l(s)+\theta$ does not go to 0 only if $a(s)=0$, and $M\theta+(j-1)-\theta(i-1)$ does not go to 0 only if $j=1$. These terms contribute

$$\prod_{i\geq 1}k_i!\cdot\frac{\Big(M-l(\mu)\Big)!}{M!}=\prod_{i\geq 0}k_i!\cdot\frac{1}{M!},$$

where k_i denotes the number of rows in μ of length i. And the remaining part of

$$\frac{\prod_{s\in\mu}\Big[a(s)+\theta l(s)+\theta\Big]}{\prod_{s\in\mu}\Big[M\theta+(j-1)-\theta(i-1)\Big]}$$

converges to 1. Together with (2.3.12), we have the limit is equal to

$$\sum_{\mu} \frac{\prod_{i\geq 0} k_i!}{M!} \frac{1}{\prod_{j=1}^{l(\mu)} \mu_j!} m_\mu(a_1^2, ..., a_M^2) m_\mu(z_1^2, ..., z_M^2)$$

which is the Talor expansion of

$$\frac{1}{M!} \sum_{\sigma \in S_M} \prod_{i=1}^{M} e^{a_i^2 z_{\sigma(i)}^2}.$$

□

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