CM FIELDS, THE COLMEZ CONJECTURE, AND THETA LIFTING

By

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Abstract

The work in this thesis has two main focal points, The Colmez Conjecture and theta lifting for signature (1,1) quadratic spaces. These two topics both have important relations to the theory of complex multiplication of abelian varieties.

While trying to formulate a product formula for periods of abelian varieties, Colmez conjectured a relation between Faltings Heights of CM abelian varieties and logarithmic derivatives of associated L-functions. We start in the unitary case, where the CM field contains an imaginary quadratic field. By using recent work in the averaged version of the conjecture, and by studying the class functions that arise, we are able to reduce the unitary Colmez Conjecture to a simpler case. Furthermore, we use the Galois action to prove new cases of the Colmez Conjecture.

We then turn from unitary CM fields to imprimitive CM fields, which are CM fields which contain smaller CM subfields. We apply similar methods as in the unitary case to reduce the Colmez Conjecture to simpler CM types.

The final section of this thesis deals with theta lifting for signature (1,1) quadratic spaces. Kudla derived formulas for the integral of the theta lift of a weakly holomorphic modular form for a signature $(n, 2)$ quadratic space. Inspired by this work, we calculate the corresponding integral for a signature $(1, 1)$ quadratic space.

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Contents

Chapter 1

Complex Multiplication

1.1 Introduction to L-functions

The Riemann zeta function $\zeta(s)$ is the first example of an L-function to show up in number theory. Initially defined by the formula $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ for real part of s greater than 1. Riemann proved that this function has a meromorphic continuation to the complex plane with a simple pole at $s = 1$. The first immediate connection to number theory can be seen by the Euler product expansion of $\zeta(s)$, given by

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}.$$

This product expansion holds for real part of $s > 1$. As $s \in \mathbb{R}$ approaches 1 from the right, the divergence of the harmonic series implies that $\zeta(s)$ has a pole at $s = 1$. By the Euler product expansion, the pole at $s = 1$ implies the existence of infinitely many primes. Of course, this is not the first proof, nor the most straightforward proof, that there are infinitely many prime numbers, but the idea has been generalized by number theorists over the years. Values of L-functions at $s = 1$, and other special values, often encode a huge amount of arithmetic information.

The next simplest example of an L-function is defined as a Dirichlet series by

$$L(s, \chi_4) = 1 - 1/3^s + 1/5^s - 1/7^s + \cdots .$$

In contrast to $\zeta(s)$, this L-function doesn't have a pole at $s = 1$. In fact, by using the Taylor series for $\arctan(s)$, one can calculate that $L(1, \chi_4) = \frac{\pi}{4} = \frac{(2\pi)^1 \cdot 1}{4 \cdot \sqrt{|-4|}}$.

This L-function, and in particular this value $L(1, \chi_4)$, can be interpreted in terms of the arithmetic of the field $\mathbb{Q}(i)$. The relationship between this particular L-function and the field $\mathbb{Q}(i)$ is due to the fact that a prime $p \in \mathbb{Z}$ splits into a product of 2 Gaussian primes if and only if $p \equiv 1 \pmod 4$, which are exactly the primes such that the coefficient of $\frac{1}{p^s}$ is positive in $L(s, \chi_4)$.

The power of 2π tells us that up to complex conjugation, there is only 1 embedding of $\mathbb{Q}(i)$ into $\mathbb{C}$. The 1 in the numerator that is not an exponent tells us that the class number of $\mathbb{Q}(i)$ is 1, that is to say, the associated ring of integers $\mathbb{Z}[i]$ is a unique factorization domain. The 4 in the denominator tells us that the field $\mathbb{Q}(i)$ contains the 4^{th} roots of unity. The -4 inside the square root of the denominator is the discriminant of the extension $\mathbb{Q}(i)/\mathbb{Q}$, a fundamental invariant that describes the complexity of a number field.

Dirichlet's class number formula (Chapter 7 of [15] has a wonderful exposition of this topic) is a classical generalization of this result, and a lot of number theory has been developed to provide arithmetic interpretations of special values of L-functions. The setting where we will focus on is the theory of abelian varieties with complex multiplication.

1.2 Elliptic Curves

Complex multiplication is both a very classical and very modern part of number theory. In the first chapter of this thesis, we will describe some aspects of this beautiful story, with a particular focus on what is relevant to the Colmez Conjecture.

Consider the elliptic curve X given by the equation

$$y^2 = x^3 + x.$$

Being an abelian group, X has endomorphisms associated to every integer. For every $n \in \mathbb{Z}$, we denote by $[n]$ the multiplication by n map on X. For example, the map $[2]$ on X sends a point P to $[2]P := P + P$. And the map $[-2]$ on X sends P to $[-2]P := -P - P$, where $-P$ is the additive inverse of the point P. This defines an embedding $\mathbb{Z} \hookrightarrow \mathrm{End}(X)$.

For most elliptic curves (in a sense of "most" which can be made precise, but we won't do so here), these are all of the endomorphisms. However, for our particular choice of X, we also have the following endomorphism

$$X \xrightarrow{[i]} X$$

$$(x, y) \mapsto (-x, iy).$$

For a point $P = (x, y) \in X$, when we apply this map to P twice, we get

$$[i]^2(x, y) = [i]\Big([i](x, y)\Big) = [i](-x, iy) = (x, -y).$$

Since $(x, -y) = [-1](x, y)$ is the additive inverse of P, we have the relation that $[i]^2 = -1$. In fact, this gives an isomorphism $\mathbb{Z}[i] \cong X$ and we say that X has complex multiplication (or CM) by $\mathbb{Z}[i]$.

More so than just being the endomorphism ring, this implies a very strong relationship between the elliptic curve X and the arithmetic of the number field $\mathbb{Q}(i)$. Here we focus on abelian extensions of $\mathbb{Q}(i)$. We briefly follow the discussion in Chapter 6 of [22], which we strongly encourage the reader to read as an introduction to complex multiplication for our specific curve X.

For an integer n, let $X[n] \subseteq X(\overline{\mathbb{Q}})$ denote the set of all n torsion points on the curve X. That is, $P \in X[n]$ if and only if $[n]P = 0$.

Theorem 1.1. *Let $K_n := \mathbb{Q}(i)(X[n])$ denote the field generated over $\mathbb{Q}(i)$ by the coordinates of the points in $X[n]$. Then we have that,*

1. *K_n is an abelian extension of $\mathbb{Q}(i)$.*

2. *Moreover, if K is any abelian extension of $\mathbb{Q}(i)$, then there is some integer n with $K \subseteq K_n$.*

The above theorem generalizes to all imaginary quadratic fields. This is one of the ways in which complex multiplication provides an explicit version of Class Field Theory for these number fields. The interested reader should read Chapter 2 of [21] for a more advanced and complete discussion of the theory of complex multiplication of elliptic curves.

1.3 Abelian Varieties with Complex Multiplication

Class Field Theory was one of the crowning achievements of early 20th century number theory, and motivated by the successes of the 1 dimensional theory of the complex multiplication of elliptic curves, mathematicians sought to generalize this story to higher dimensions.

Imaginary quadratic fields are fundamental in the theory of complex multiplication of elliptic curves. A CM field plays the analogous role for the higher dimensional CM theory.

Definition 1.2. *A totally imaginary number field E is a CM field if E is a quadratic extension of a totally real number field F.*

Since $\mathbb{Q}$ is a totally real number field, any imaginary quadratic field is a CM field. For $n > 2$, let ζ_n denote a primitive n^{th} root of unity. Then $\mathbb{Q}(\zeta_n)$ is a CM Field with totally real subfield $\mathbb{Q}(\zeta_n + \zeta_n^{-1})$.

Alternatively, a CM field can also be defined as a number field with a well-defined non-trivial complex conjugation, in the following sense.

Lemma 1.3. *Let E be a number field of degree $2n$. The following are equivalent.*

1. *E is a CM field.*
2. *There is an non-trivial automorphism $\rho : E \to E$ such that for every embedding $\sigma : E \hookrightarrow \mathbb{C}$, we have that $\sigma(\rho(x)) = \overline{\sigma(x)}$ for $x \in E$ and $\overline{\sigma(x)}$ denotes the complex conjugate of $\sigma(x)$.*

Proof. First, suppose that E is a CM field. That is, E is a totally imaginary number field that is a degree 2 extension of a totally real number field F. Let ρ be the non-trivial element in $\mathrm{Gal}(E/F)$.

Let $\sigma : E \hookrightarrow \mathbb{C}$ be an embedding. We want to show that the two embeddings $\sigma \circ \rho$ and $\overline{\sigma}$ are equal. Since F is a totally real field, we have that ρ is the identity map on F and $\sigma(F) \subseteq \mathbb{R}$. Therefore, the two maps $\sigma \circ \rho$ and $\overline{\sigma}$ agree on F.

Two embeddings of E into $\mathbb{C}$ which agree on F can differ only by complex conjugation. So we must have that either $\sigma \circ \rho = \overline{\sigma}$ or $\sigma \circ \rho = \sigma$. Since ρ is not the identity map, we must have that $\sigma \circ \rho = \overline{\sigma}$.

On the other hand, suppose E is a number field with such a non-trivial automorphism E. Since complex conjugation has order 2, we have that ρ has order 2 as well. Since ρ is non-trivial, we must have that E is totally imaginary. Let F be the field fixed by ρ. This implies that for any embedding $\sigma : F \hookrightarrow \mathbb{C}$, we have that $\overline{\sigma} = \sigma$, and so $\sigma(F) \subseteq \mathbb{R}$, which means that F is a totally real field. $\square$

In the rest of this thesis, E will denote a CM field, $F \subseteq E$ will be the totally real number field with $[E : F] = 2$, and $\rho : E \to E$ will denote the complex conjugation, as in the above lemma.

Also very important in the higher dimensional CM theory is the notion of a CM type. CM types are trivial in the imaginary quadratic case (due to the Galois equivalence which we will discuss in Chapter 2 of this thesis) but they are needed in the higher dimensional theory. The embeddings of E into $\mathbb{C}$ come in conjugate pairs, and choosing one embedding from each of these pairs constitutes a CM type.

Definition 1.4. *Let E be a CM field with complex conjugation ρ. A subset Φ of* $\mathrm{Hom}(E, \mathbb{C})$ *is a CM type if the following two equalities hold,*

$$\Phi \cap \rho\Phi = \varnothing,$$

$$\Phi \cup \rho\Phi = \mathrm{Hom}(E, \mathbb{C}).$$

We also present the following equivalent definition of a CM type. This definition better serves to motivate some of Colmez's constructions to be defined later.

Definition 1.5. *Let E be a CM field with complex conjugation ρ. A function Φ :* $\mathrm{Hom}(E, \mathbb{C}) \to \{0, 1\}$ *is a CM type if $\Phi(\sigma) + \Phi(\rho\sigma) = 1$ for every $\sigma \in \mathrm{Hom}(E, \mathbb{C})$.*

To switch between the above two definitions of CM types, we identify a CM type $\Phi \subseteq \mathrm{Hom}(E, \mathbb{C})$ with its characteristic function $\Phi : \mathrm{Hom}(E, \mathbb{C}) \to \{0, 1\}$.

If E is a CM field and Φ is a CM type of E, we can not necessarily restrict Φ to CM subfields of E. However, Φ can be extended to larger CM fields. For M a CM field containing E, we can extend Φ to be a CM type of M via

$$\Phi^M = \{\sigma \in \mathrm{Hom}(M, \mathbb{C}) : \sigma|_E \in \Phi\}.$$

The Colmez Conjecture has an analytic side and a geometric side. The geometric side uses the (stable) Faltings height of an abelian variety, which we define here. In this thesis, we choose certain normalizations to the Faltings height to agree with [8],[27]. Let $2n = [E : \mathbb{Q}]$ and let Φ be a CM type of E. Let X_Φ be an abelian variety with CM by $(\mathcal{O}_E, \Phi)$. Here, $\mathcal{O}_E$ is the full ring of integers of E. This means that we have an embedding $\mathcal{O}_E \hookrightarrow \mathrm{End}(X_\Phi)$ and the action of $\mathcal{O}_E$ on the holomorphic differentials of X_Φ is described by Φ.

Since X_Φ has CM, we can find a number field L such that X_Φ is defined over L and has everywhere good reduction [14]. Let $\mathcal{X}_\Phi$ be the Neron model of X_Φ defined over the ring of integers $\mathcal{O}_L$. Let $\epsilon : \mathrm{Spec}(\mathcal{O}_L) \to \mathcal{X}_\Phi$ be the zero section of the Neron model. Let $\omega_{\mathcal{X}_\Phi/\mathcal{O}_L} = \epsilon^*(\Lambda^n \Omega_{\mathcal{X}_\Phi/\mathcal{O}_L})$ and take a non-zero $\alpha \in \omega_{\mathcal{X}_\Phi/\mathcal{O}_L}$. Then, the Faltings height of X_Φ is defined by

$$\mathrm{h}_{\mathrm{Fal}}(X_\Phi) := \frac{-1}{2[L : \mathbb{Q}]} \sum_{\sigma : L \hookrightarrow \mathbb{C}} \sum \log\left|\int_{X_\Phi^\sigma(\mathbb{C})} \alpha^\sigma \wedge \overline{\alpha^\sigma}\right| + \log|\omega_{\mathcal{X}_\Phi/\mathcal{O}_L}/\mathcal{O}_L\alpha|.$$

This is independent of L and α. Similarly, we define the Faltings height of a CM type Φ by

$$\mathrm{h}_{\mathrm{Fal}}(\Phi) = \frac{1}{[E : \mathbb{Q}]} \mathrm{h}_{\mathrm{Fal}}(X_\Phi).$$

Under these normalizations, Colmez showed [8] that $\mathrm{h}_{\mathrm{Fal}}(\Phi)$ does not change if we extend Φ to the extended CM type on a larger CM field containing E. This invariance under CM field extension will be very useful for us and we will use it often.

1.4 The Colmez Conjecture

In this section, we introduce the Colmez Conjecture [8] and state some applications and cases where the conjecture is known. The conjecture involves a good amount of background constructions, so we start with a very brief overview before delving into the specific details.

We start with a CM field E together with Φ, a CM type of E, and X_Φ an abelian variety with CM by $(\mathcal{O}_E, \Phi)$. Starting from Φ, we can obtain a class function A^0_Φ on $\mathrm{Gal}(E^c/\mathbb{Q})$. We can rewrite this class function in terms of the irreducible characters of the Galois group. From these irreducible characters, we create a meromorphic function $Z(s, A^0_\Phi)$ related to logarithmic derivatives of certain L-functions. The Colmez Conjecture is that a special value of this function is related to the Faltings height of X_Φ.

To go into more detail, we start with the Galois closure of E, which we will call E^c and is also a CM field. For a CM type Φ of E, we let Φ^c denote the extension of Φ to E^c. Now, we pick an identification of $\mathrm{Hom}(E^c, \mathbb{C})$ with $\mathrm{Gal}(E^c/\mathbb{Q})$. Of course, such an identification is only well defined up to conjugacy. However, we will eventually produce a class function A^0_Φ, and hence the choice of identification will not effect the final result.

Next, we have the reflex CM type $\widetilde{\Phi^c}$. The reflex type is most naturally defined via the function view of a CM type, via

$$\begin{aligned}\widetilde{\Phi^c} : \mathrm{Gal}(E^c/\mathbb{Q}) &\to \{0,1\} \\ g &\mapsto \Phi^c(g^{-1}).\end{aligned}$$

Then, we define a function $A_\Phi : \mathrm{Gal}(E^c/\mathbb{Q}) \to \mathbb{Q}$ by taking a normalized convolution of Φ^c and $\widetilde{\Phi^c}$ thusly

$$A_\Phi(g) := \frac{1}{\#\,\mathrm{Gal}(E^c/\mathbb{Q})} \sum_{\sigma \in \mathrm{Gal}(E^c/\mathbb{Q})} \Phi^c(\sigma)\widetilde{\Phi^c}(\sigma^{-1}g).$$

Then to obtain our class function A^0_Φ, we take the average of A_Φ among the conjugates in $\mathrm{Gal}(E^c/\mathbb{Q})$. More precisely,

$$A^0_\Phi(g) := \frac{1}{\#\,\mathrm{Gal}(E^c/\mathbb{Q})} \sum_{h \in \mathrm{Gal}(E^c/\mathbb{Q})} A_\Phi(hgh^{-1}).$$

As A^0_Φ is a class function, we can write A^0_Φ as

$$A^0_\Phi = \sum_\chi a_\chi \chi,$$

where χ ranges through the irreducible representations of $\mathrm{Gal}(E^c/\mathbb{Q})$ and $a_\chi \in \mathbb{C}$. For each character χ, define the function $Z(s,\chi)$ by

$$Z(s,\chi) := \frac{L'(s,\chi)}{L(s,\chi)} + \frac{1}{2}\log f_\chi,$$

where $L(s,\chi)$ is the Artin L-function of χ and f_χ is the Artin conductor of χ. Then, we linearly extend $Z(s,-)$ to A^0_Φ via

$$Z(s, A^0_\Phi) = \sum_\chi a_\chi Z(s,\chi).$$

We define $\mathbb{Q}^{\mathrm{cm}}$ as the compositum of all CM number fields. This field is infinite degree over $\mathbb{Q}$, and has a unique automorphism ρ acting as complex conjugation. Using the projection $\mathrm{Gal}(\mathbb{Q}^{\mathrm{cm}}/\mathbb{Q}) \twoheadrightarrow \mathrm{Gal}(E^c/\mathbb{Q})$, we can view A^0_Φ as a class function on $\mathrm{Gal}(\mathbb{Q}^{\mathrm{cm}}/\mathbb{Q})$.

In his paper [8], Colmez considers the space $\mathcal{CM}^0$, the $\mathbb{Q}$-vector space of class functions $f : \mathrm{Gal}(\mathbb{Q}^{\mathrm{cm}}) \to \mathbb{Q}$ such that the quantity $f(g) + f(\rho g)$ is independent of $g \in \mathrm{Gal}(\mathbb{Q}^{\mathrm{cm}}/\mathbb{Q})$. For any CM type Φ of E, the function A^0_Φ is an element of $\mathcal{CM}^0$.

Colmez defined a $\mathbb{Q}$-linear height function $\mathrm{ht} : \mathcal{CM}^0 \to \mathbb{R}$ such that if X_Φ is an abelian variety with CM by $(\mathcal{O}_E, \Phi)$, then

$$\mathrm{h}_{\mathrm{Fal}}(X_\Phi) = -\,\mathrm{ht}(A^0_\Phi).$$

Finally with all of the above set up, we can state the Colmez Conjecture from [8].

Conjecture 1.6 (Colmez)**.** *For any CM type* Φ*,* $\mathrm{ht}(A^0_\Phi) = Z(0, A^0_\Phi)$.

In the rest of this chapter, we discuss a bit of what is known about the Colmez Conjecture and how it can be used.

The Colmez Conjecture for imaginary quadratic fields is a reformulation of the classical Chowla-Selberg formula [7]. For abelian CM fields E, the conjecture was proven by combined work of Colmez [8], and Obus [16]. The first non-abelian case was due to Yang [24], [25], [26], who proved the Colmez Conjecture for certain classes of non-abelian quartic CM fields.

A major breakthrough recently came with the proof of the Average Colmez Conjecture, which was originally conjectured by Colmez in his 1993 paper and proven independently by two groups of mathematicians [1], [28] . Rather than focus on a single CM type, the average version of the conjecture makes a statement about all CM types of a CM field. For a CM field E, let $\Phi(E)$ denote the set of CM types of E.

Theorem 1.7 (Andreatta, Goren, Howard, Madapusi Pera and Yuan, Zhang)**.** *For a CM field E of degree $[E:\mathbb{Q}] = 2n$, we have*

$$\frac{1}{2^n}\sum_{\Phi\in\Phi(E)} \mathrm{h}_{Fal}(\Phi) = -\frac{1}{2^n}\sum_{\Phi\in\Phi(E)} Z(0, A^0_\Phi) = -\frac{1}{4n}Z(0,\chi_{E/F}) - \frac{1}{4}\log(2\pi).$$

The proof of the Average Colmez Conjecture spurred a lot of work in the Colmez Conjecture, in particular trying to leverage group theoretic constraints to prove further cases of Colmez. The Conjecture has been proven in the case where the Galois group is as large as possible for a CM field, a so-called Weyl CM field [2]. Along these lines, in [27], the authors study the case of unitary CM fields, which are CM fields containing imaginary quadratic field. In Chapter 2 of this thesis, we will discuss this work further.

Mathematicians have been able to work out some consequences from the known cases of the Colmez Conjecture. Many analytic techniques are known for bounding logarithmic derivatives of L-functions, and applications have come from using the Colmez Conjecture to translate these bounds into bounds on the Faltings height. Most notably was the work of Tsimerman in proving the Andre-Oort Conjecture for $\mathcal{A}_g$ [23], and there has also been work applying these bounds to Faltings heights of elliptic curves [9].

1.5 Outline of This Thesis

The new work for this thesis begins in the unitary setting for the Colmez Conjecture, which we discuss in Chapter 2 of this thesis. There, we further examine Yang and Yin's work [27]. The main result of that chapter is to show that the Colmez Conjecture for unitary CM fields reduces to certain special CM types. Furthermore, we can use these results and some group theoretic arguments to prove some new cases of the Colmez Conjecture.

In Chapter 3 of this thesis, we attempt to generalize the work from Chapter 2. A unitary CM field is a CM field which contains an imaginary quadratic field. In Chapter 3, we look at CM fields which contain CM subfields. Unfortunately, the complexity of the field theoretic set up does limit how far we can push the arguments in Chapter 2. However we do obtain relations amongst the different constructions of Colmez in this setting.

In Chapter 4, we derive a formula for a signature $(1, 1)$ theta lift of a weakly holomorphic modular form in terms of an Eisenstein series. This chapter of the thesis is separate from the previous chapters which focused on the Colmez Conjecture, and we provide an introduction to these ideas in that chapter.

Chapter 2

The Unitary Colmez Conjecture

2.1 Unitary CM Fields

In this chapter, we discuss unitary CM fields and what is known about the Colmez Conjecture in this setting.

Definition 2.1. *A unitary CM field E is a CM field that contains an imaginary quadratic subfield $k \subseteq E$.*

Let E be a unitary CM field. If F is the maximal totally real subfield of E and k is the imaginary quadratic subfield of E, then we have $E = kF$.

One very important feature of unitary CM fields is that we are able to stratify the set of CM Types by their signature. The signature of a CM type is related to the fact that an embedding $\sigma : E \hookrightarrow \mathbb{C}$ can be restricted to k to obtain an embedding $\sigma|_k : k \hookrightarrow \mathbb{C}$.

The discussion of signature will be greatly simplified by viewing $k \subset \mathbb{C}$. In this viewpoint, $\mathrm{Hom}(k, \mathbb{C})$ consists of two elements, the identity map, 1, and its complex conjugate ρ. The reader is correct to be skeptical of such a simplification. Chapter 3 of this thesis deals with the more general setting where a much more careful discussion of signature is needed.

Definition 2.2. *Let $E = kF$ be a unitary CM field of degree $2n$. A CM type $\Phi \subset$*

$\mathrm{Hom}(E,\mathbb{C})$ *has signature* $(n-r,r)$ *if exactly* $n-r$ *of the embeddings in* Φ *restrict to the identity map* $k \hookrightarrow \mathbb{C}$.

The study of unitary CM fields in connection to the Colmez Conjecture began with the work of Yang and Yin [27]. Building on the recent work on the average Colmez Conjecture, they proved the following theorems which will be useful in our discussion. Their first theorem says that the Colmez Conjecture holds if we average amongst CM types of a given fixed signature. Moreover, they are able to calculate a value for this average Faltings height.

Theorem 2.3 (Yang-Yin)**.** *Let* $E = kF$ *be a unitary CM field of degree* $2n$ *and denote by* $\Phi(E)_r$ *the set of all CM types of* E *of signature* $(n-r,r)$. *Let* $1 < r < n$, *let* ζ_k *be the zeta function for* k, *and let* $\chi_{k/\mathbb{Q}}$ *and* $\chi_{E/F}$ *be the quadratic characters associated to the quadratic extensions* $k/\mathbb{Q}$ *and* E/F *respectively. Then,*

$$\begin{aligned}\sum_{\Phi\in\Phi(E)_r} h_{Fal}(\Phi) &= \sum_{\Phi\in\Phi(E)_r} -Z(0, A^0_\Phi)\\ &= \frac{-1}{4}\binom{n}{r}Z(0,\zeta_k) + \binom{n-2}{r-1}Z(0,\chi_{k/\mathbb{Q}}) - \frac{1}{n}\binom{n-2}{r-1}Z(0,\chi_{E/F}).\end{aligned}$$

For the CM types not included in the above theorem, Yang and Yin are able to prove more, in fact they prove that the Colmez Conjecture holds for these CM types.

Theorem 2.4 (Yang-Yin)**.** *Let* $E = kF$ *be a unitary CM field of degree* $2n$

(a) If Φ *is a CM type of signature* $(n,0)$ *or* $(0,n)$, *then* $h_{Fal}(\Phi) = \frac{-1}{4}Z(0,\zeta_k)$.

(b) If Φ *is a CM type of signature* $(n-1,1)$ *or* $(1,n-1)$, *then*

$$h_{Fal}(\Phi) = \frac{-1}{4}Z(0,\zeta_k) + \frac{1}{n}Z(0,\chi_{k/\mathbb{Q}}) - \frac{1}{n^2}Z(0,\chi_{E/F}).$$

In fact, part (a) of the above theorem is a restatement of the classical Chowla-Selberg formula.

2.2 Signature $(n-2,2)$ CM Types

After Yang and Yin tackled signature $(n-1,1)$ CM types, one might wonder what we can say about signature $(n-2,2)$ CM types. In this end, we have the following result from [17].

Theorem 2.5 (Parenti)**.** *Let $E = kF$ be a unitary CM field of degree $2n$. Then, the Colmez Conjecture holds for E if and only if it holds for all CM types of signature $(n-2,2)$.*

In the remainder of this section, we discuss a proof of the above theorem.

Let's introduce some notation relevant to the theorem. Let $E = kF$ be a unitary CM field. Let F^c, E^c denote the Galois closures of F and E respectively. Then $E^c = kF^c$. Let $H := \mathrm{Gal}(F^c/F) \leq \mathrm{Gal}(F^c/\mathbb{Q}) =: G$ and let $h = \#H$. Then we can identify the embeddings of F into $\mathbb{C}$, which we will call $\{\sigma_1, \ldots, \sigma_n\}$, with coset representatives for $H\backslash G$.

An embedding $E \hookrightarrow \mathbb{C}$ is uniquely determined by a pair of embeddings $F \hookrightarrow \mathbb{C}$ and $k \hookrightarrow \mathbb{C}$. We denote by $\{1, \rho\}$ the two embeddings of k into $\mathbb{C}$, and for an embedding $\sigma : F \hookrightarrow \mathbb{C}$, we write $\rho^i\sigma$ for the embedding of E into $\mathbb{C}$ associated to the pair $\{\rho^i, \sigma\}$. If $i = 0$, we simply write σ for 1σ.

A CM type for E consists of a choice of one of the embeddings $k \hookrightarrow \mathbb{C}$ for each embedding $F \hookrightarrow \mathbb{C}$. Thus, we can parametrize CM types of E via subsets of $\{1, 2, \ldots, n\}$.

Given $S \subseteq \{1, 2, \dots, n\}$, the corresponding CM type of E is given by

$$\Phi_S = \{\rho^{j_i}\sigma_i : j_i = 1 \text{ if } i \in S, j_i = 0 \text{ if } i \notin S\}.$$

Thus Φ_S is a CM type of signature $(n-r, r)$ where $r = \#S$. We often write these CM types as sums,

$$\Phi_S = \sum_{i \in S} \rho\sigma_i + \sum_{i \notin S} \sigma_i = \operatorname{tr}_{E/k} + (\rho - 1)\sum_{i \in S} \sigma_i.$$

where $\operatorname{tr}_{E/k} = \sum_{i=1}^{n} \sigma_i$ consists of all of the embeddings $E \hookrightarrow \mathbb{C}$ which restrict to the identity on k.

The first step in the reduction to $(n-2, 2)$ signature CM types is the following computation of A^0_Φ.

Theorem 2.6. *Let $S \subseteq \{1, 2, \dots, n\}$ be a subset of size r. Then,*

$$A^0_{\Phi_S} = \frac{1}{2}\operatorname{tr}_{E^c/k} - \frac{r}{n}(1-\rho)\operatorname{tr}_{E^c/k} + \frac{r}{n^2}\chi_{\operatorname{Ind}_H^G}(\chi_0)$$
$$+ \frac{1}{hn^2}(1-\rho)\sum_{g \in G}\left(\sum_{i \neq j \in S} \sigma_i H \sigma_j^{-1}\right).$$

where χ_0 is the trivial character.

Proof. First, we need to extend Φ_S to Φ_S^c, the CM type on E^c. To do so, we need to determine which embeddings of E^c into $\mathbb{C}$, when restricted to E, are in Φ_S. Because E is a unitary CM field, we have an isomorphism $\operatorname{Gal}(E^c/\mathbb{Q}) \cong \operatorname{Gal}(E^c/k) \times \operatorname{Gal}(k/\mathbb{Q}) \cong G \times \mathbb{Z}/2$. Therefore, we have

$$\Phi_S^c = \operatorname{tr}_{E^c/k} + (\rho - 1)\sum_{i \in S} \sigma_i H.$$

Since we have written the CM type as elements of $\operatorname{Gal}(E^c/\mathbb{Q})$, finding the reflex type of Φ_S^c amounts to inverting every element thusly,

$$\widetilde{\Phi_S^c} = \operatorname{tr}_{E^c/k} + (\rho - 1)\sum_{j \in S} H\sigma_j^{-1}.$$

Taking the convolution of Φ_S^c and $\widetilde{\Phi_S^c}$ gives

$$\begin{aligned}
A_{\Phi_S} &= \frac{1}{[E^c:\mathbb{Q}]}\Phi_S^c\widetilde{\Phi_S^c} \\
&= \frac{1}{2hn}\left(\operatorname{tr}_{E^c/k} + (\rho-1)\sum_{i\in S}\sigma_i H\right)\left(\operatorname{tr}_{E^c/k} + (\rho-1)\sum_{j\in S} H\sigma_j^{-1}\right) \\
&= \frac{1}{2}\operatorname{tr}_{E^c/k} - \frac{r}{n}(1-\rho)\operatorname{tr}_{E^c/k} + \frac{1}{n}(1-\rho)\sum_{i,j\in S}\sigma_i H\sigma_j^{-1}.
\end{aligned}$$

Now we project A_{Φ_S} onto the space of class functions to obtain $A^0_{\Phi_S}$. Note that $\operatorname{tr}_{E^c/k} = \operatorname{Gal}(E^c/k)$ is a normal subgroup of $\operatorname{Gal}(E^c/\mathbb{Q})$ and so we have

$$A^0_{\Phi_S} = \frac{1}{2}\operatorname{tr}_{E^c/k} - \frac{r}{n}(1-\rho)\operatorname{tr}_{E^c/k} + \frac{1}{n}(1-\rho)\frac{1}{hn}\sum_{g\in G} g\left(\sum_{i,j\in S}\sigma_i H\sigma_j^{-1}\right)g^{-1}.$$

The main difficulty in the above equation is the final term. First, look at the diagonal terms where $i=j$,

$$\begin{aligned}
\frac{1}{hn}\sum_{g\in G} g\left(\sum_{i\in S}\sigma_i H\sigma_i^{-1}\right)g^{-1} &= \frac{1}{hn}\sum_{i\in S}\left(\sum_{g\in G} g\sigma_i H\sigma_i^{-1}g^{-1}\right) \\
&= \frac{1}{hn}\sum_{i\in S}\left(\sum_{g\in G} gHg^{-1}\right) \\
&= \frac{r}{hn}\sum_{g\in G} gHg^{-1}.
\end{aligned}$$

Combining the above equation and the following Lemma 2.7 will complete the proof. $\square$

Lemma 2.7. *Let $\chi_0 : H \to \mathbb{C}$ be the trivial character. As functions $G \to \mathbb{C}$, we have the relation*

$$\sum_{g\in G} gHg^{-1} = h\chi_{\operatorname{Ind}_H^G(\chi_0)}.$$

Proof. The induced representation $\operatorname{Ind}_H^G(\chi_0)$ is given by

$$\operatorname{Ind}_H^G(\chi_0) = \{f : G \to \mathbb{C} : f(xg) = f(g)\forall x \in H, g \in G\}.$$

where the action of G is by right translation. This space is exactly the space of functions $f : H\backslash G \to \mathbb{C}$. Thus, the representation $\mathrm{Ind}_H^G(\chi_0)$ is isomorphic to the representation coming from the action of G on $H\backslash G$ via $g \cdot H\sigma = H\sigma g^{-1}$.

The character of a permutation representation is the number of fixed points. That is, for $\sigma \in G$,

$$\begin{aligned}\chi_{\mathrm{Ind}_H^G(\chi_0)}(\sigma) &= \#\{i \in \{1, 2, \dots, n\} : \sigma \cdot H\sigma_i = H\sigma_i\} \\ &= \#\{i \in \{1, 2, \cdots, n\} : \sigma \in \sigma_i H \sigma_i^{-1}\}.\end{aligned}$$

On the other side of the equation, we have

$$\left(\sum_{g\in G} gHg^{-1}\right)(\sigma) = \#\{g \in G : \sigma \in gHg^{-1}\}.$$

For i such that $\sigma \in \sigma_i H \sigma_i^{-1}$, then every $g \in \sigma_i H$ satisfies $\sigma \in gHg^{-1}$, and so we have that $\sum_{g\in G} gHg^{-1} = h\chi_{\mathrm{Ind}_H^G(\chi_0)}$. □

For CM types of signature $(n, 0), (n - 1, 1)$, and $(n - 2, 2)$, Theorem 2.6 takes the form

$$\begin{aligned}A^0_{\Phi_{\{\varnothing\}}} &= \frac{1}{2}\operatorname{tr}_{E^c/k}, \\ A^0_{\Phi_{\{i\}}} &= \frac{1}{2}\operatorname{tr}_{E^c/k} - \frac{1}{n}(1-\rho)\operatorname{tr}_{E^c/k} + \frac{1}{n^2}(1-\rho)\chi_{\mathrm{Ind}_H^G(\chi_0)}, \\ A^0_{\Phi_{\{i,j\}}} &= \frac{1}{2}\operatorname{tr}_{E^c/k} - \frac{2}{n}(1-\rho)\operatorname{tr}_{E^c/k} + \frac{2}{n^2}(1-\rho)\chi_{\mathrm{Ind}_H^G(\chi_0)} \\ &\quad + \frac{1}{hn^2}(1-\rho)\sum_{g\in G}\Big(g\sigma_i H\sigma_j^{-1}g^{-1} + g\sigma_j H \sigma_i^{-1} g^{-1}\Big).\end{aligned}$$

Next we will rewrite A^0_Φ, for any CM type Φ, in terms of the above CM types.

Lemma 2.8. *For any subset $S \subseteq \{1, 2, \dots, n\}$ of size r, we have*

$$A^0_{\Phi_S} = \sum_{\{i,j\}\subseteq S} A^0_{\Phi_{\{i,j\}}} - (r-2)\sum_{i\in S} A^0_{\Phi_{\{i\}}} + \frac{(r-1)(r-2)}{2} A^0_{\Phi_{\{\varnothing\}}}.$$

Proof. This proposition is clear when $r = 0, 1$, where we interpret an empty sum as 0. For $r \geq 2$, we have

$$\begin{aligned}
\sum_{\{i,j\}\subseteq S} A^0_{\Phi_{\{i,j\}}} &= \sum_{\{i,j\}\subseteq S} \Bigg(\frac{1}{2} \operatorname{tr}_{E^c/k} - \frac{2}{n}(1-\rho) \operatorname{tr}_{E^c/k} + \frac{2}{n^2}(1-\rho)\chi_{\operatorname{Ind}_H^G(\chi_0)} \\
&\quad + \frac{1}{hn^2}(1-\rho) \sum_{g\in G} \Big(g\sigma_i H \sigma_j^{-1} g^{-1} + g\sigma_j H \sigma_i^{-1} g^{-1} \Big) \Bigg) \\
&= \frac{r(r-1)}{4} \operatorname{tr}_{E^c/k} - \frac{r(r-1)}{n}(1-\rho) \operatorname{tr}_{E^c/k} + \frac{r(r-1)}{n^2}(1-\rho)\chi_{\operatorname{Ind}_H^G(\chi_0)} \\
&\quad + \frac{1}{hn^2}(1-\rho) \sum_{g\in G} g \Big(\sum_{i\neq j\in S} \sigma_i H \sigma_j^{-1} \Big) g^{-1}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
A^0_{\Phi_S} - \sum_{\{i,j\}\subseteq S} A^0_{\Phi_{\{i,j\}}} &= \frac{-(r+1)(r-2)}{4} \operatorname{tr}_{E^c/k} + \frac{r(r-2)}{n}(1-\rho) \operatorname{tr}_{E^c/k} \\
&\quad - \frac{r(r-2)}{n^2}(1-\rho)\chi_{\operatorname{Ind}_H^G(\chi_0)} \\
&= -(r-2) \sum_{i\in S} A^0_{\Phi_{\{i\}}} + \frac{(r-1)(r-2)}{2} A^0_{\Phi_{\{\varnothing\}}}.
\end{aligned}$$

$\square$

With this result, we can finally prove our theorem.

Theorem 2.9. *Let $E = kF$ be a unitary CM field of degree $2n$. If the Colmez Conjecture holds for all CM types of E of signature $(n-2, 2)$, then the Colmez Conjecture holds for all CM types of E.*

Proof. Colmez defined a $\mathbb{Q}$-linear height function ht and conjectured that $\operatorname{ht}(A^0_\Phi) = Z(0, A^0_\Phi)$ for any CM type Φ of a CM field. We also note that the function $Z(0, \cdot)$ is $\mathbb{Q}$-linear.

In [27], Yang and Yin show that the Colmez Conjecture holds for all CM types of signature $(n, 0)$ and $(n-1, 1)$. Then, if the Colmez Conjecture also holds for any $\Phi_{i,j}$, and if $S \subset \{1, 2, \ldots n\}$ has size r, we have that

$$\begin{aligned}
\mathrm{ht}(A^0_{\Phi_S}) &= \mathrm{ht}\left(\sum_{\{i,j\}\subseteq S} A^0_{\Phi_{\{i,j\}}} - (\epsilon-2)\sum_{i\in S} A^0_{\Phi_{\{i\}}} + \frac{(\epsilon-1)(\epsilon-2)}{2} A^0_{\Phi_{\{\varnothing\}}}\right) \\
&= \sum_{\{i,j\}\subseteq S} \mathrm{ht}(A^0_{\Phi_{\{i,j\}}}) - (\epsilon-2)\sum_{i\in S} \mathrm{ht}(A^0_{\Phi_{\{i\}}}) + \frac{(\epsilon-1)(\epsilon-2)}{2}\,\mathrm{ht}(A^0_{\Phi_{\{\varnothing\}}}) \\
&= \sum_{\{i,j\}\subseteq S} Z(0, A^0_{\Phi_{\{i,j\}}}) - (\epsilon-2)\sum_{i\in S} Z(0, A^0_{\Phi_{\{i\}}}) + \frac{(\epsilon-1)(\epsilon-2)}{2} Z(0, A^0_{\Phi_{\{\varnothing\}}}) \\
&= Z\left(0, \sum_{\{i,j\}\subseteq S} A^0_{\Phi_{\{i,j\}}} - (\epsilon-2)\sum_{i\in S} A^0_{\Phi_{\{i\}}} + \frac{(\epsilon-1)(\epsilon-2)}{2} A^0_{\Phi_{\{\varnothing\}}}\right) \\
&= Z(0, A^0_{\Phi_S}).
\end{aligned}$$

$\square$

2.3 Galois Action

There is a Galois action on the set of CM types. An element $g \in \mathrm{Gal}(E^c/\mathbb{Q})$ acts on a CM type Φ by $g \cdot \Phi = \{g\sigma : \sigma \in \Phi\}$. If Φ_1 and Φ_2 are CM types that are equivalent under this action, then $A^0_{\Phi_1} = A^0_{\Phi_2}$.

As before, $E = kF$ is a unitary CM field where F^c is the Galois closure of F, $G = \mathrm{Gal}(F^c/\mathbb{Q})$ and $H = \mathrm{Gal}(F^c/F)$. Then $\mathrm{Gal}(E^c/\mathbb{Q}) \cong G \times \mathbb{Z}/2$. The $\mathbb{Z}/2$ component of $\mathrm{Gal}(E^c/\mathbb{Q})$ is complex conjugation, taking a CM type of signature $(n-r, r)$ to a corresponding CM type of signature $(r, n-r)$. The action of $\mathrm{Gal}(E^c/k) \cong G$ fixes the signature of a CM type.

The action of $\mathrm{Gal}(E^c/k) \cong G$ on the set of CM types of signature $(n-r, r)$ is isomorphic to the action of G on the subsets of G/H of size r.

One of the main results of [27] is that the Colmez Conjecture holds if we average amongst CM types of a given signature. That is, if $\Phi(E)_r$ denotes all CM types of E of signature $(n-r, r)$, then

$$\sum_{\Phi \in \Phi(E)_r} \mathrm{ht}(A_\Phi^0) = \sum_{\Phi \in \Phi(E)_r} Z(0, A_\Phi^0).$$

If all CM types of a given signature are equivalent, then this result immediately implies the Colmez Conjecture for those CM types. This is how Yang and Yin proved their result that the Colmez Conjecture holds for signature $(n, 0)$ and $(n-1, 1)$. In both of those cases, the Galois action forces all CM types of that signature to be equivalent. Combining this idea and our previous theorem gives the following theorem.

Theorem 2.10. *Let $E = kF$ be a unitary CM field. Let $H := \mathrm{Gal}(F^c/F) \leq \mathrm{Gal}(F^c/\mathbb{Q}) =: G$. If G acts doubly transitively on G/H, then the Colmez Conjecture is true for E.*

The first question to ask after such a result is whether or not this result actually tells us anything. That is to say, do there really exist such pairs (G, H)? Thankfully, the answer is yes.

In particular, $\mathrm{PSL}_n(\mathbb{F}_q)$ and $\mathrm{PGL}_n(\mathbb{F}_q)$ both act doubly transitively on $\mathbb{P}^{n-1}(\mathbb{F}_q)$, and so if we take H to be the stabilizer of a point in each of those groups. More precisely, we can take

$$H := \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \vdots & \ddots & & \\ 0 & a_{n2} & \dots & a_{nn} \end{bmatrix}.$$

When $n = 2$, H is the Borel subgroup of upper triangular matrices, which gives the main result of [18]. In that paper, we compute A^0_Φ explicitly, rather than using this signature $(n-2, 2)$ result.

For small values of n, q, the groups $\mathrm{PSL}_n(\mathbb{F}_q)$ and $\mathrm{PGL}_n(\mathbb{F}_q)$ can be realized as the Galois groups of totally real number fields over $\mathbb{Q}$. In particular, consulting the LMFDB [13] shows that for $q \leq 11$, the groups $\mathrm{PSL}_2(\mathbb{F}_q)$ and $\mathrm{PGL}_2(\mathbb{F}_q)$ appear as the Galois groups of totally real fields.

Furthermore, the fact that G is the Galois group of the Galois closure of F, and H is the subgroup that fixes the field F implies that the action of G on G/H induces an embedding $G \hookrightarrow \mathrm{Sym}(G/H)$ of G into the symmetric group on the set of cosets of G by H.

Thus, we may apply Theorem 2.10 to any G which is a doubly transitive subgroup of a symmetric group. As a corollary of the classification of finite simple groups, a classification of doubly transitive groups is known. There are infinite families of examples and sporadic examples. We list the other doubly transitive groups and refer the reader to [10], [4] for further details on these groups and their doubly transitive actions. As a caution to the reader, in our present discussion we change some of the notations from the above sources to match more modern usage.

The alternating and symmetric groups A_n and S_n are doubly transitive subgroups of S_n for any n. For these groups, we take H to be A_{n-1} and S_{n-1} respectively. The Colmez Conjecture was already discussed in this case by Yang and Yin [27].

Another family of examples is the symplectic groups $\mathrm{Sp}_{2m}(\mathbb{F}_2)$ for $m \geq 2$. For these groups, two choices of subgroups induce doubly transitive actions. This discussion involves quadratic forms, symmetric bilinear forms, and alternating bilinear forms in characteristic 2, so we will explain in detail. Let 0_m and 1_m denote the $m \times m$ zero and identity matrix respectively and define the matrix J by

$$J := \begin{bmatrix} 0_m & 1_m \\ -1_m & 0_m \end{bmatrix}.$$

If we denote x^T to be the transpose of a matrix x, we define the symplectic group as

$$\mathrm{Sp}_{2m}(\mathbb{F}_2) := \{x \in \mathrm{GL}_{2m}(\mathbb{F}_2) : x^T J x = J\}.$$

We can also view $\mathrm{Sp}_{2m}(\mathbb{F}_2)$ as the set of linear transformations which preserve the following alternating bilinear form ψ,

$$\psi : \mathbb{F}_2^{2m} \times \mathbb{F}_2^{2m} \to \mathbb{F}_2$$
$$\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right) \mapsto u_1 \cdot v_2 - u_2 \cdot v_1.$$

That is to say,

$$\mathrm{Sp}_{2m}(\mathbb{F}_2) := \{x \in \mathrm{GL}_{2m}(\mathbb{F}_2) : \psi(xu, xv) = \psi(u, v) \quad \forall u, v \in \mathbb{F}_2^{2m}\}.$$

However, ψ is also a symmetric form over $\mathbb{F}_2$ and the doubly transitive action we are interested in involves orthogonal groups of associated quadratic forms.

Definition 2.11. *If V is a vector space over $\mathbb{F}_2$, then a quadratic form on V is a function $q : V \to \mathbb{F}_2$ such that*

1. *$q(\lambda v) = \lambda^2 q(v)$ for all $\lambda \in \mathbb{F}_2, v \in V$;*

2. *The function $f(u, v) := q(u + v) - q(u) - q(v)$ is a symmetric bilinear form.*

Over $\mathbb{F}_2$, a quadratic form determines a symmetric bilinear form, but many quadratic forms can determine the same symmetric bilinear form. Up to isometry, there are two quadratic forms on $\mathbb{F}_2^{2m}$, given by

$$Q^+(v_1, \ldots, v_{2m}) = v_1 v_{m+1} + \cdots + v_m v_{2m},$$

$$Q^-(v_1, \ldots, v_{2m}) = v_1 v_{m+1} + \cdots + v_m v_{2m} + v_m + v_{2m}.$$

These quadratic forms can be distinguished by their Witt index, the largest dimension of an isotropic subspace.

We define two orthogonal groups, $\mathrm{O}_{2m}^+(\mathbb{F}_2)$ and $\mathrm{O}_{2m}^-(\mathbb{F}_2)$, as the isometry groups of these quadratic forms,

$$\mathrm{O}_{2m}^\pm(\mathbb{F}_2) = \{x \in \mathrm{GL}_{2m}(\mathbb{F}_2) : Q^\pm(xv) = Q^\pm(v) \quad \forall v \in \mathbb{F}_2^{2m}\}.$$

Both Q^+ and Q^- determine the same bilinear symmetric form, ψ, and thus we have that $\mathrm{O}_{2m}^+(\mathbb{F}_2) \subseteq \mathrm{Sp}_{2m}(\mathbb{F}_2)$ and $\mathrm{O}_{2m}^-(\mathbb{F}_2) \subseteq \mathrm{Sp}_{2m}(\mathbb{F}_2)$. Furthermore, $\mathrm{Sp}_{2m}(\mathbb{F}_2)$ acts doubly transitively on the cosets by either of the aforementioned subgroups.

Applying Theorem 2.10 to this situation shows that if F is a totally real number field whose Galois closure F^c has Galois group $\mathrm{Sp}_{2m}(\mathbb{F}_2)$ and F is the fixed field by either $\mathrm{O}_{2m}^+(\mathbb{F}_2)$ or $\mathrm{O}_{2m}^-(\mathbb{F}_2)$, then the Colmez Conjecture holds for $E := kF$.

Another class of examples is the unitary groups $\mathrm{PSU}_3(\mathbb{F}_q)$ and $\mathrm{PU}_3(\mathbb{F}_q)$ where q is a prime power. To describe these groups and the relevant subgroups, we will follow the convention of [10], however there are different choices possible for the bilinear form. Let

φ be the following bilinear form on $\mathbb{F}_{q^2}^3$,

$$\varphi : \mathbb{F}_{q^2}^3 \times \mathbb{F}_{q^2}^3 \to \mathbb{F}_{q^2}$$
$$\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \right) \mapsto u_1 v_3^q + u_2 v_2^q + u_3 v_1^q.$$

We define $\mathrm{U}_3(\mathbb{F}_q)$ to be the matrices which preserve this form,

$$\mathrm{U}_3(\mathbb{F}_q) := \{A \in \mathrm{GL}_3(\mathbb{F}_{q^2}) : \varphi(Au, Av) = \varphi(u, v) \quad \forall u, v \in \mathbb{F}_{q^2}^3\}.$$

The action of $\mathrm{U}_3(\mathbb{F}_q)$ on the 1-dimensional subspaces of $\mathbb{F}_{q^2}^3$ defines the projective group $\mathrm{PU}_3(\mathbb{F}_q)$ and taking those matrices of determinant 1 defines the group $\mathrm{PSU}_3(\mathbb{F}_q)$.

An isotropic vector $v \in \mathbb{F}_{q^2}^3$ is a vector such that $\varphi(v, v) = 0$. The groups $\mathrm{PU}_3(\mathbb{F}_q)$ and $\mathrm{PSU}_3(\mathbb{F}_q)$ both act doubly transitively on the set of 1-dimensional isotropic subspaces of $\mathbb{F}_{q^2}^3$, so therefore we may take H to be the stabilizer of any 1-dimensional isotropic subspace. A quick calculation shows that the stabilizer in $\mathrm{PSU}_3(\mathbb{F}_q)$ and $\mathrm{PU}_3(\mathbb{F}_q)$ of the subspace spanned by $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is exactly the upper triangular matrices of the respective group.

Applying Theorem 2.10 to this group action shows that the Colmez Conjecture holds for $E := kF$ where k is an imaginary quadratic field, F is a totally real number field such that $\mathrm{Gal}(F^c/\mathbb{Q}) \cong \mathrm{PSU}_3(\mathbb{F}_q)$ or $\mathrm{PU}_3(\mathbb{F}_q)$ and $\mathrm{Gal}(F^c/F)$ is the subgroup of upper triangular matrices.

There are a two more infinite families and a few more sporadic examples which we list here.

- The Suzuki groups $\mathrm{Sz}(q)$ for q an odd power of 2 is a doubly transitive subgroup of S_{q^2+1}.
- The Ree groups $\mathrm{R}(q)$ for q an odd power of 3 is a doubly transitive subgroup of S_{q^3+1}.
- The Mathieu groups $M_{11}, M_{12}, M_{22}, M_{23}, M_{24}$ are doubly transitive subgroups of $S_{11}, S_{12}, S_{22}, S_{23}$, and S_{24} respectively.
- The Mathieu group M_{11} is a doubly transitive subgroup of S_{12} and $\mathrm{PSL}_2(\mathbb{F}_{11})$ is a transitive subgroup of S_{11}.
- The alternating group A_7 is a transitive subgroup of S_8.
- The Higman-Sims group HS is a doubly transitive subgroup of S_{176}.
- The Conway group Co_3 is a doubly transitive subgroup of S_{276}.

The above list is a complete classification of doubly transitive subgroups of symmetric groups, which is a complete list of the fields where our previous work together with the Galois theoretic constraints imply the Colmez Conjecture.

Chapter 3

Imprimitive CM Fields

3.1 Imprimitive CM Fields

The work in this section is joint with Jiuya Wang.

In this section, we aim to generalize some of the results of the previous chapter. A unitary CM field is a CM field where the complex conjugation "comes from" an imaginary quadratic field. In this section, we will study CM fields whose complex conjugation "comes from" smaller CM fields.

Definition 3.1. *Let $K/\mathbb{Q}$ be a totally real field, and let L be a CM field with totally real subfield $F \subsetneq K$. Then LK is an L-imprimitive CM field.*

It is useful to look at a field diagram in this case. And to set up notation we will use in the rest of this chapter, let $[K : F] = n$ and $[F : \mathbb{Q}] = f$.

$$\begin{tikzcd}
 & LK \arrow[dl, dash] \arrow[dr, dash] & \\
K \arrow[dr, dash, "n"'] & & L \arrow[dl, dash, "2"'] \\
 & F \arrow[dr, dash, "f"] & \\
 & & \mathbb{Q}
\end{tikzcd}$$

Taking $L = k$ to be an imaginary quadratic field, then a k-imprimitive CM field is exactly the same as a unitary CM field. And the reason we exclude F being equal to K is that if $F = K$, then the imprimitivity condition says nothing. This notion of imprimitive lets us interpolate between a unitary CM field and a general CM field.

One ultimate hope of these ideas would be a sort of inductive procedure for the Colmez Conjecture. Perhaps it is too optimistic to hope to deduce the Colmez Conjecture for LK from the Colmez Conjecture for L, but maybe something can be said in that case.

3.2 Signatures of CM Types in Imprimitive CM Fields

To emulate some of our results from the unitary case, we will define a notion of signature for an imprimitive CM field. In the unitary case, the signature of a CM type depended on a distinguished embedding of the imaginary quadratic field into $\mathbb{C}$. For a general CM field L, there is no such embedding, and hence, the signature of an imprimitive CM field is slightly more complicated than in the unitary case.

Let Φ_0 be a CM type of L. This choice determines a labelling of the embeddings of L into $\mathbb{C}$. That is to say, we label the embeddings of L into $\mathbb{C}$ by $\tau_1, \rho\tau_1, \ldots, \tau_f, \rho\tau_f$, where $\Phi_0 = \{\tau_1, \ldots, \tau_f\}$ and ρ denotes complex conjugation.

This choice of Φ_0 also determines a labelling of $\mathrm{Hom}(LK, \mathbb{C})$. Label the elements of $\mathrm{Hom}(LK, \mathbb{C})$ by $\{\tau_{1,1}, \ldots, \tau_{1,n}, \ldots, \tau_{f,1}, \ldots, \tau_{f,n}, \rho\tau_{1,1}, \ldots, \rho\tau_{f,n}\}$, where $\tau_{i,j}|_L = \tau_i$.

Definition 3.2. *A CM type Φ has Φ_0-signature $(a_1, b_1, \ldots, a_f, b_f)$ if Φ consists of a_i many of the $\{\tau_{i,1}, \ldots, \tau_{i,n}\}$ and b_i many of the $\{\rho\tau_{i,1}, \ldots, \rho\tau_{i,n}\}$.*

By the definition of a CM type, we have that $a_i + b_i = n$ for each i. And similarly to the unitary case, we can specify a CM type by $S_1, \ldots, S_f$, where each $S_i \subseteq \{1, 2, \ldots n\}$. I.e. if Φ is a CM type corresponding to $S_1, \ldots, S_f$, then $k \in S_i$ if and only if $\rho\tau_{i,k} \in \Phi$. So if each S_i has size b_i, then the corresponding CM type has Φ_0-signature $(n-b_1, b_1, \ldots, n-b_f, b_f)$.

If Φ is a CM type of LK, and Φ_0, Φ_1 are two different CM types of L, then the Φ_0-signature of Φ is different from the Φ_1-signature of Φ. Let $(a_1, b_1, \ldots, a_f, b_f)$ be the Φ_0-signature of Φ. Then for every i such that $\tau_i \notin \Phi_0 \cap \Phi_1$, we swap a_i and b_i to determine the Φ_1-signature of Φ.

For example, if $E = kF$ is a unitary CM field, there are only two CM types of the imaginary quadratic field k, we will call them Φ_1 and Φ_ρ with $\Phi_1 \cap \Phi_\rho = \varnothing$. If Φ is a CM type of E with Φ_1-signature (a, b), then the Φ_ρ-signature of Φ is (b, a).

Alternatively, we could also define the signature of a CM type Φ of LK as an unordered set of unordered pairs of numbers. I.e. if the Φ_0-signature of Φ is $(a_1, b_1, \ldots, a_f, b_f)$, we could define the signature of Φ as $\{\{a_1, b_1\}, \ldots, \{a_f, b_f\}\}$. This definition is independent of the choice of Φ_0. However, we are interested in extending the Colmez Conjecture from L to LK, and it will be useful for us to have Φ_0 explicit in our notation.

3.3 An Imprimitive $(n-2, 2)$ Theorem

In this imprimitive setting, we would like to generalize the $(n-2, 2)$ theorem of the previous chapter. To have any hope of extending the Colmez Conjecture from L to LK, we will need to make some assumptions about the Colmez Conjecture on L.

Theorem 3.3. *Using the notation in the beginning of this chapter, let LK be an L-imprimitive CM field for a CM field L. Let Φ_0 be a CM type of L, and assume that* *the Colmez Conjecture is true for* Φ_0*. Let* Φ *be a CM type of* LK*. If the Colmez Conjecture is true for all CM types* $\Phi_{T_1,\ldots,T_f}$*, for subsets* $T_1, \ldots, T_f \subseteq \{1, 2, \ldots, n\}$ *such that* $\#T_1 + \#T_2 + \cdots \#T_f \leq 2$*, then the Colmez Conjecture is true for* Φ*.*

Just as in the unitary case, we will prove the above theorem by explicitly writing A_Φ in terms of simpler CM types. That is, the above theorem is a direct corollary of the following result.

Theorem 3.4. *Let* LK *be an* L*-imprimitive CM field for a CM field* L*. Let* Φ_0 *be a CM type of* L*, and let* Φ *be a CM type of* LK*. Then, the class function appearing in the Colmez Conjecture,* A^0_Φ *can be written as a sum*

$$A^0_\Phi = \sum c_{T_1,\ldots,T_f} A^0_{\Phi_{T_1,\ldots,T_f}}$$

where $c_{T_1,\ldots,T_f} \in \mathbb{Q}$ *are explicit constants and* $\Phi_{T_1,\ldots,T_f}$ *are CM types corresponding to subsets* $T_1, \ldots, T_f \subseteq \{1, 2, \ldots, n\}$ *such that* $\#T_1 + \#T_2 + \cdots \#T_f \leq 2$*. See equations* (3.1), (3.2), (3.3) *for the explicit calculation of* A_Φ *in terms of these simpler CM types.*

The 2 in the above theorem is in some sense "the same" 2 as in the $(n-2, 2)$ theorem. Throughout this calculation, we can see that this 2 seems to come from the quadratic nature of the convolution involved in the definition of A_Φ.

Proof. Since Φ is a CM type, Φ corresponds to a set of subsets $S_1, \ldots, S_f \subseteq \{1, 2, \ldots, n\}$. Note that this correspondence depends on the CM type Φ_0.

Our first step will be explicitly calculating A_Φ^0. To be more precise about the correspondence between Φ and $S_1, \ldots, S_f$, we have

$$\begin{aligned}\Phi &= \tau_{1,1} + \cdots + \tau_{1,n} + \cdots + \tau_{f,1} + \cdots + \tau_{f,n} \\ &+ (\rho - 1)\left(\sum_{i \in S_1} \tau_{1,i} + \cdots + \sum_{i \in S_f} \tau_{f,i}\right) \\ &= \Phi_0 + (\rho - 1)\left(\sum_{i \in S_1} \tau_{1,i} + \cdots + \sum_{i \in S_f} \tau_{f,i}\right).\end{aligned}$$

Next, we need to extend Φ to $(LK)^c$, the Galois closure of LK. Let $H = \mathrm{Gal}((LK)^c/LK)$. Then,

$$\begin{aligned}\Phi^c &= \tau_{1,1}H + \cdots + \tau_{1,n}H + \cdots + \tau_{f,1}H + \cdots + \tau_{f,n}H \\ &+ (\rho - 1)\left(\sum_{i \in S_1} \tau_{1,i}H + \cdots + \sum_{i \in S_f} \tau_{f,i}H\right) \\ &= \Phi_0^c + (\rho - 1)\left(\sum_{i \in S_1} \tau_{1,i}H + \cdots + \sum_{i \in S_f} \tau_{f,i}H\right).\end{aligned}$$

where Φ_0^c denotes the extension of Φ_0 from L to $(LK)^c$.

Now to calculate $\widetilde{\Phi}$, we need to invert every element showing up in Φ^c, which gives

$$\widetilde{\Phi} = \widetilde{\Phi_0^c} + (\rho - 1)\left(\sum_{j \in S_1} H\tau_{1,j}^{-1} + \cdots + \sum_{i \in S_f} H\tau_{f,i}^{-1}\right).$$

Next we calculate $A_\Phi := \frac{1}{[(LK)^c:\mathbb{Q}]}\Phi^c\widetilde{\Phi^c}$,

$$A_\Phi = \frac{1}{2fn}\Phi_0^c\widetilde{\Phi_0^c} + (\rho-1)\left(\frac{1}{2fn}\right)\Bigg(\Phi_0^c\Bigg(\sum_{j\in S_1} H\tau_{1,j}^{-1} + \cdots + \sum_{j\in S_f} H\tau_{f,j}^{-1}\Bigg)$$

$$+\Bigg(\sum_{i\in S_1}\tau_{1,i}H + \cdots + \sum_{i\in S_f}\tau_{f,i}H\Bigg)\widetilde{\Phi_0^c}$$

$$-2(\#H)\Bigg(\sum_{i\in S_1}\tau_{1,i} + \cdots + \sum_{i\in S_f}\tau_{f,i}\Bigg)H\Bigg(\sum_{j\in S_1}\tau_{1,j}^{-1} + \cdots + \sum_{j\in S_f}\tau_{f,j}^{-1}\Bigg)\Bigg).$$

We will rewrite A_Φ in terms of simpler CM types. Suppose each S_i has size b_i. To simplify notation, if $T_i \subseteq \{1,2,\ldots,n\}$, we will write Φ_{T_i} to indicate the CM type corresponding to the set of subsets $\varnothing,\cdots,T_i,\cdots,\varnothing$ and A_{T_i} to indicate the function $A_{\Phi_{T_i}}$.

First, we will take A_Φ and subtract away all of the CM types corresponding to the 2 element subsets of $S_1,\ldots,S_f$. More precisely, we have

$$A_\Phi - \sum_{\{c_1,d_1\}\subseteq S_1} A_{\{c_1,d_1\}} - \ldots - \sum_{\{c_f,d_f\}\subseteq S_f} A_{\{c_f,d_f\}}$$

$$= \left(1 - \binom{b_1}{2} - \ldots - \binom{b_f}{2}\right) A_{\Phi_0}$$

$$+ (\rho-1)\left(\frac{1}{2fn}\right)\left(\sum_{k=1}^{f}(2-b_k)\left[\Phi_0^c\left(\sum_{j\in S_k} H\tau_{k,j}^{-1}\right) + \left(\sum_{i\in S_k}\tau_{k,i}H\right)\widetilde{\Phi_0^c}\right]\right.$$

$$-2(\#H)\Bigg((2-b_1)\sum_{i\in S_1}\tau_{1,i}H\tau_{1,i}^{-1} + \sum_{i\in S_1,j\in S_2}\tau_{1,i}H\tau_{2,j}^{-1} + \cdots + \sum_{i\in S_1,j\in S_f}\tau_{1,i}H\tau_{f,j}^{-1}$$

$$+\sum_{i\in S_2,j\in S_1}\tau_{2,i}H\tau_{1,j}^{-1} + (2-b_2)\sum_{i\in S_2}\tau_{2,i}H\tau_{2,i}^{-1} + \cdots + \sum_{i\in S_2,j\in S_f}\tau_{2,i}H\tau_{f,j}^{-1}$$

$$+\cdots$$

$$\left.+\sum_{i\in S_f,j\in S_1}\tau_{f,i}H\tau_{1,j}^{-1} + \cdots + \sum_{i\in S_f,j\in S_{f-1}}\tau_{f,i}H\tau_{f-1,j}^{-1} + (2-b_f)\sum_{i\in S_f}\tau_{f,i}H\tau_{f,i}^{-1}\Bigg)\right).$$

Next, we will subtract away all of the CM types coming from elements of $S_i \times S_j$ with $i < j$, giving

$$\begin{aligned}
&A_\Phi - \sum_{\{c_1,d_1\}\subseteq S_1} A_{\{c_1,d_1\}} - \ldots - \sum_{\{c_f,d_f\}\subseteq S_f} A_{\{c_f,d_f\}} - \sum_{1\le i<j\le f}\ \sum_{(\alpha_i,\beta_j)\in S_i\times S_j} A_{\{\alpha_i\},\{\beta_j\}} \\
&\quad = \left(1 - \binom{b_1}{2} - \ldots - \binom{b_f}{2} - \sum_{1\le i<j\le f} b_i b_j\right) A_{\Phi_0} \\
&\quad + (\rho - 1)\left(\frac{2 - b_1 - \cdots - b_f}{2fn}\right)\left(\sum_{k=1}^{f} \Phi_0^c\left(\sum_{j\in S_k} H\tau_{k,j}^{-1}\right) + \left(\sum_{i\in S_k}\tau_{k,i}H\right)\widetilde{\Phi_0^c}\right. \\
&\quad \left.- 2(\#H)\left(\sum_{i\in S_1}\tau_{1,i}H\tau_{1,i}^{-1} + \cdots + \sum_{i\in S_f}\tau_{f,i}H\tau_{f,i}^{-1}\right)\right).
\end{aligned}$$

Now, the quantity on the right hand side of the above expression can be written in terms of CM types coming from 1 element subsets of each of the S_i, together with the empty set.

That is to say, our above calculations show that

$$A_\Phi = \sum_{\{c_1,d_1\}\subseteq S_1} A_{\{c_1,d_1\}} + \ldots + \sum_{\{c_f,d_f\}\subseteq S_f} A_{\{c_f,d_f\}} + \sum_{1\le i<j\le f}\ \sum_{(\alpha_i,\beta_j)\in S_i\times S_j} A_{\{\alpha_i\},\{\beta_j\}} \tag{3.1}$$

$$+ (2 - b_1 - \cdots - b_f)\left(\sum_{i_1\in S_1} A_{\{i_1\}} + \cdots + \sum_{i_f\in S_f} A_{\{i_f\}}\right) \tag{3.2}$$

$$+ \left(1 - \binom{b_1}{2} - \cdots - \binom{b_f}{2} - \sum_{1\le i<j\le f} b_i b_j - (2 - b_1 - \cdots - b_f)(b_1 + \cdots + b_f)\right) A_{\Phi_0}. \tag{3.3}$$

□

3.4 Further Imprimitive Results

Ideally, we would like to prove a version of the Average Colmez Conjecture in the imprimitive setting. Currently, this result is out of reach, but in this section, we record some partial results towards this goal.

First, let us consider the Galois equivalence. Note that the only CM type of Φ_0-signature $(n, 0, n, 0, \dots, n, 0)$ is the extension of Φ_0 to LK. Since the Galois group $\mathrm{Gal}((LK)^c/\mathbb{Q})$ acts transitively on $\mathrm{Hom}(LK, \mathbb{C})$, we have that all CM types of Φ_0-signature

$$(n-1, 1, n, 0, \dots, n, 0), (n, 0, n-1, 1, \dots, n, 0), \dots, (n, 0, n, 0, \dots, n-1, 1)$$

are equivalent. That is to say, all CM types corresponding to sets $S_1, \dots, S_f$ with $\#S_1 + \cdots + \#S_f = 1$ are equivalent.

Before we calculate the A^0_Φ for such a CM type, we need the following calculations on group characters.

Lemma 3.5. *Consider the following set up of fields.*

$$\begin{tikzcd}
M \arrow[dr, dash, "H"] \arrow[dd, dash, "H_1"'] & \\
& L \arrow[dl, dash, "2"] \\
F \arrow[dr, dash] & \\
& \mathbb{Q}
\end{tikzcd}$$

Let $G = \mathrm{Gal}(M/\mathbb{Q})$ and suppose there is a ρ in the center of G such that $\mathrm{Gal}(L/F) = \langle \rho \rangle$. Let $\chi_{L/F} : H_1 \to \{\pm 1\}$ be the non-trivial map factoring through H. That is to say,

$\chi_{L/F}(H) = 1$ *and* $\chi_{L/F}(\rho H) = -1$. *Let* $\pi = \mathrm{Ind}_{H_1}^{G}(\chi_{L/F})$. *Then,*

$$\chi_\pi = (1-\rho) \sum_{\widetilde{g} \in G/H_1} \widetilde{g} H \widetilde{g}^{-1}.$$

Proof. Let f denote the function on the right hand side of the statement of the lemma. For $s \in G$, we have

$$\begin{aligned}
\chi_\pi(s) &= \frac{1}{\# H_1} \sum_{t \in G, tst^{-1} \in H_1} \chi_{L/F}(tst^{-1}) \\
&= \frac{1}{\# H_1} \left(\#\{t \in G : tst^{-1} \in H\} - \#\{t \in G : tst^{-1} \in \rho H\} \right).
\end{aligned}$$

On the other hand, let $g_1, \ldots, g_n$ be a set of simultaneous left and right cosets for H_1 in G. Then,

$$f(s) = \#\{i : g_i s g_i^{-1} \in H\} - \#\{i : g_i s g_i^{-1} \in \rho H\}.$$

Now if $g_i s g_i^{-1} \in H$, then $(hg_i)s(hg_i)^{-1} \in H$ for any $h \in H_1$ (here we use that ρ is in the center of G). Similarly, if $g_i s g_i^{-1} \in \rho H$, then $(hg_i)s(hg_i)^{-1} \in \rho H$ for any $h \in H_1$. Thus, the result follows. $\square$

Now, we'd like to calculate A^0_Φ for simple CM types. In terms of Φ_0-signature, the simplest CM type is the extension of Φ_0 to LK. This is the only CM type with signature $(n, 0, \ldots, n, 0)$. As Φ_0 is an arbitrary CM type of L, it is perhaps too much to hope for to give a decomposition of $A^0_{\Phi_0}$ into irreducible characters of $\mathrm{Gal}((LK)^c/\mathbb{Q})$. However, we can use $A^0_{\Phi_0}$ as a tool in describing the more complicated class functions.

Now, we turn to the calculation of A^0_Φ for a CM type with $\# S_1 + \cdots + \# S_f = 1$. We can prove the Colmez Conjecture for these CM types.

Theorem 3.6. *Let LK be an imprimitive CM field containing a CM field L. Let Φ_0 be a CM type of L and assume the Colmez Conjecture is true for Φ_0. Then, the Colmez*

Conjecture holds for a CM type $\Phi_{(i,j)}$ of LK, where

$$\Phi_{(i,j)} = \Phi_0 + (\rho - 1)\tau_{i,j}.$$

See equation (3.4) *for an explicit calculation of $A_{\Phi_{(i,j)}}$.*

Proof. We will prove this by explicit calculation. We calculate $A_{\Phi_{(i,j)}}$, and then notice that the Colmez Conjecture is known for the class functions that arise. First, we extend to the Galois closure via we have that

$$\Phi^c_{(i,j)} = \Phi^c_0 + (\rho - 1)\tau_{i,j}H,$$

and so

$$\widetilde{\Phi^c_{(i,j)}} = \widetilde{\Phi^c_0} + (\rho - 1)H\tau^{-1}_{i,j}.$$

Now we can calculate $A_{\Phi_{(i,j)}} = \frac{1}{[(LK)^c:\mathbb{Q}]}\Phi^c_{(i,j)}\widetilde{\Phi^c_{(i,j)}}$,

$$\begin{aligned} A_{\Phi_{(i,j)}} &= \frac{1}{[(LK)^c:\mathbb{Q}]}(\Phi^c_0 + (\rho-1)\tau_{i,j}H)(\widetilde{\Phi^c_0} + (\rho-1)H\tau^{-1}_{i,j}) \\ &= A_{\Phi_0} + (\rho-1)\frac{1}{2fn(\#H)}\left(\tau_{i,j}H\widetilde{\Phi^c_0} + \Phi^c_0 H\tau^{-1}_{i,j} - 2(\#H)\tau_{i,j}H\tau^{-1}_{i,j}\right). \end{aligned}$$

Since all CM types with $\#S_1 + \cdots + \#S_f = 1$ are equivalent, we can sum the above expression over all (i,j) to obtain

$$\begin{aligned} fnA_{\Phi_{(i,j)}} &= fnA_{\Phi_0} + (\rho-1)\frac{1}{2fn(\#H)}\sum_{i,j}\left(\tau_{i,j}H\widetilde{\Phi^c_0} + \Phi^c_0 H\tau^{-1}_{i,j} - 2(\#H)\tau_{i,j}H\tau^{-1}_{i,j}\right) \\ &= fnA_{\Phi_0} + (\rho-1)\frac{1}{2fn(\#H)}\left(\Phi^c_0\widetilde{\Phi^c_0} + \Phi^c_0\widetilde{\Phi^c_0} - 2(\#H)\sum_{i,j}\tau_{i,j}H\tau^{-1}_{i,j}\right) \\ &= fnA_{\Phi_0} - 2(1-\rho)A_{\Phi_0} + \frac{1}{fn}\chi_{\mathrm{Ind}^G_{\mathrm{Gal}((LK)^c/K)}(\chi_{LK/K})}. \end{aligned}$$

Therefore,

$$A_{\Phi_{(i,j)}} = A_{\Phi_0} - \frac{2}{fn}(1-\rho)A_{\Phi_0} + \frac{1}{(fn)^2}\chi_{\mathrm{Ind}^G_{\mathrm{Gal}((LK)^c/K)}(\chi_{LK/K})}.$$

And from here, we can pass to the class function in a straightforward way,

$$A^0_{\Phi_{(i,j)}} = A^0_{\Phi_0} - \frac{2}{fn}(1-\rho)A^0_{\Phi_0} + \frac{1}{(fn)^2}\chi_{\mathrm{Ind}^G_{\mathrm{Gal}((LK)^c/K)}(\chi_{LK/K})}.$$

We would like to write the decomposition of $A^0_{\Phi_{i,j}}$ in terms of irreducible characters of G. Unfortunately as Φ_0 is an arbitrary CM type of L, we have no chance of explicitly determining such characters, so we rely on the fact that we are assuming the Colmez Conjecture holds for Φ_0.

By Proposition 2.1 in Yang-Yin [27], we know that the Colmez Conjecture holds for the character $\chi_{LK/K}$.

We would like to say something about the expression $(1-\rho)A^0_{\Phi_0}$. Suppose we write

$$A_{\Phi_0^0} = \sum_C a_C[C] = a_1\chi_1 + \sum_{\pi\neq 1} a_\pi\chi_\pi,$$

where C ranges through conjugacy classes of $\mathrm{Gal}(L^c/\mathbb{Q})$ and π ranges through irreducible representations of $\mathrm{Gal}(L^c/\mathbb{Q})$, and χ_1 represents the trivial representation. Moreover, we have that $\rho A^0_{\Phi_0}(g) = A^0_{\Phi_0}(\rho g)$.

In [9], Colmez shows that every $\chi_\pi \neq 1$ that shows up in the decomposition of $A^0_{\Phi_0}$ is an odd character, i.e. $\chi_\pi(\rho g) = -\chi_\pi(g)$.

Therefore,

$$\rho A^0_{\Phi_0} = a_1\chi_1 - \sum_{\pi\neq 1} a_\pi\chi_\pi = 2a_1\chi_1 - A^0_{\Phi_0},$$

and so

$$(1-\rho)A^0_{\Phi_0} = 2A^0_{\Phi_0} - 2a_1\chi_1.$$

Therefore,

$$A^0_{\Phi_{(i,j)}} = (1 - \frac{4}{fn})A^0_{\Phi_0} + \frac{4}{fn}a_1\chi_1 + \frac{1}{(fn)^2}\chi_{\mathrm{Ind}^G_{\mathrm{Gal}((LK)^c/K)}(\chi_{LK/K})}. \tag{3.4}$$

$\square$

The remaining CM types from Theorem 3.3 are those CM types with $\#S_1 + \cdots \#S_f = 2$. These CM types come in two forms, based on whether or not there is an i with $S_i = 2$. We hope to address these remaining calculations in future work.

Chapter 4

Theta Lifting in Signature $(1,1)$

4.1 Theta Lifting Introduction

In this chapter, we will turn our focus from results on the Colmez Conjecture to theta lifting. The main result is the calculation of an integral of a Borcherds product on a signature $(1,1)$ quadratic space, extending the work of Kudla [12], Schofer [19], Ehlen [11], and Bruinier-Yang [6].

In the first section of this chapter, we set up all the notation we will use throughout. In section 2, we introduce Eisenstein series and the Siegel-Weil formula. Section 3 will go through the main calculation of the integral, and section 4 contains technical details justifying some of the steps used in section 3.

Let $D > 0$ be a fundamental discriminant, and consider the $\mathbb{Q}$ quadratic space $V = \mathbb{Q}(\sqrt{D})$ with quadratic form $q(z) = z\overline{z}$. Then, V is a signature $(1,1)$ quadratic space. Let $H = \mathrm{GSpin}(V) = \mathbb{G}_{m,k}$ where $k = \mathbb{Q}(\sqrt{D})$ (viewing $\mathbb{Q}(\sqrt{D})$ as a field rather than a quadratic space), and then $\mathrm{SO}(V) = \mathbb{G}^1_{m,k}$.

We will use the symmetric space $\mathbb{D}$, the space of positive definite lines of $V(\mathbb{R})$. For $z \in \mathbb{D}$ a positive definite line in $V(\mathbb{R})$, and $x \in V(\mathbb{R})$, let $(x,x)_z$ denote the majorant associated to z. That is,

$$(x,x)_z = (x_z, x_z) - (x_{z^\perp}, x_{z^\perp}).$$

With this majorant, we define the Gaussian as $\varphi_\infty(x,z) = e^{-\pi(x,x)_z}$. This Gaussian will be used in the infinite component of our theta function.

For K a compact subgroup of $H(\mathbb{A}_f)$, let X_K denote the space

$$X_K = H(\mathbb{Q})\backslash \left(\mathbb{D} \times H(\mathbb{A}_f)/K\right).$$

Let L be an even lattice in V with dual lattice L'. Let $S(V(\mathbb{A}_f))$ be the space of Schwartz functions on $V(\mathbb{A}_f)$. Inside of $S(V(\mathbb{A}_f))$, we have the subspace S_L, which is the space of Schwartz functions supported on $L' \otimes \widehat{\mathbb{Z}}$ and constant on the cosets of $L \otimes \widehat{\mathbb{Z}}$. The space S_L is finite dimensional with basis $\{\phi_\mu\}_{\mu \in L'/L}$, where ϕ_μ is the characteristic function of the coset $\mu + L \otimes \widehat{\mathbb{Z}}$.

For $\tau \in \mathbb{H}, z \in \mathbb{D}, h \in H(\mathbb{A}_f)$, and $\varphi_f \in S(V(\mathbb{A}_f))$, we define the theta function as

$$\theta(\tau, z, h, \varphi_f) := \sum_{x \in V(\mathbb{Q})} \omega(g_\tau)\Big(\varphi_\infty(-,z) \otimes \omega(h)\varphi_f\Big)(x).$$

In the above expression for θ, $g_\tau \in \mathrm{SL}_2(\mathbb{R})$ such that $g_\tau \cdot i = \tau$, and ω is the Weil Representation.

Associated to the lattice L we have the theta function $\theta_L : \mathbb{H} \times \mathbb{D} \times H(\mathbb{A}_f) \to S_L$ defined by

$$\theta_L(\tau, z, h) = \sum_{\mu \in L'/L} \theta(\tau, z, h, \phi_\mu)\phi_\mu.$$

The final input into the theta lift is $F : \mathbb{H} \to S_L$ a weakly holomorphic vector valued modular form with representation ρ_L. We will use the Fourier expansion of F, which we write as,

$$\begin{aligned} F(\tau) &= \sum_{\mu \in L'/L} F_\mu(\tau)\phi_\mu \\ &= \sum_\mu \sum_{m \in \mathbb{Q}} c_\mu(m) q^m \phi_\mu. \end{aligned}$$

With all of this, we can define the theta lift, first introduced by Borcherds in [3]. For τ, z, h, F as above, let

$$\Phi(z,h,F) = \underset{\sigma=0}{\mathrm{CT}}\left\{ \lim_{t\to\infty} \int_{\mathcal{F}_t} \sum_{\mu} F_\mu(\tau)\theta(\tau,z,h,\phi_\mu)v^{-\sigma-2}dudv \right\},$$

where $\mathcal{F}_t = \{\tau = u + vi \in \mathbb{H} : |u| \leq \frac{1}{2}, |\tau| \geq 1, \text{ and } v \leq t\}$ is the fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$ up to a height $v \leq t$, and by $\underset{\sigma=0}{\mathrm{CT}}$ we mean the constant term in the Laurent expansion at $\sigma = 0$.

The above is the general definition of the theta lift. However for our purposes, the integral with converge at the value $\sigma = 0$ and hence we need not worry about the Laurent expansion in terms of σ.

The main goal of this chapter is to use the Siegel-Weil formula to compute the integral $\frac{1}{\mathrm{vol}(X_K)} \int_{X_K} \Phi(z,h,F)\, dzdh$ in terms of an Eisenstein series.

4.2 Eisenstein Series and the Siegel-Weil Formula

In this section, we introduce the Eisenstein series and state the Siegel-Weil formula, which will be the connection between theta series and Eisenstein series. But first, we introduce some notation.

For $x \in \mathbb{Q}^*\backslash\mathbb{A}^*$, let $\chi_V(x) = (x, D)_\mathbb{A}$ be the adelic quadratic character. For $a \in \mathbb{A}^*, b \in \mathbb{A}$, let $m(a), n(b) \in \mathrm{SL}_2(\mathbb{A})$ denote the matrices $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ respectively.

For $s \in \mathbb{C}$, let $I(s, \chi_V)$ be the principal series representation

$$I(s,\chi_V) = \left\{ \Phi : \mathrm{SL}_2(\mathbb{A}) \to \mathbb{C} : \Phi(n(b)m(a)g, s) = \chi_V(a)|a|_\mathbb{A}^{s+1}\Phi(g,s) \right\}.$$

There is an action of $\mathrm{SL}_2(\mathbb{A})$ on $I(s, \chi_V)$ by right translation. We also have an $\mathrm{SL}_2(\mathbb{A})$

intertwining map

$$\lambda : S(V(\mathbb{A})) \to I(0, \chi_V)$$
$$\varphi \mapsto \lambda(\varphi)(g) := (\omega(g)\varphi)(0).$$

We call a section $\Phi(g,s) \in I(s,\chi_V)$ standard if the restriction of Φ to the subgroup $\mathrm{SO}_2(\mathbb{R})\,\mathrm{SL}_2(\widehat{\mathbb{Z}})$ is independent of s. For a Schwartz function $\varphi \in S(V(\mathbb{A}))$, there is a unique extension of $\lambda(\varphi)$ to a standard section $\lambda(\varphi, s) \in I(s,\chi_V)$ such that $\lambda(\varphi,0) = \lambda(\varphi)$.

Let $P \leq \mathrm{SL}_2(\mathbb{Q})$ be the subgroup of upper triangular matrices. For $g \in \mathrm{SL}_2(\mathbb{A})$, $s \in \mathbb{C}$, and Φ a standard section, define the Eisenstein series $E(g,s,\Phi)$ by

$$E(g,s,\Phi) = \sum_{\gamma \in P\backslash \mathrm{SL}_2(\mathbb{Q})} \Phi(\gamma g, s).$$

This Eisenstein series converges for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, and moreover it has a meromorphic continuation to $\mathbb{C}$.

For our use, we will use a slightly more explicit Eisenstein series. For an integer ℓ. let χ_ℓ be the character of $\mathrm{SO}_2(\mathbb{R})$ of weight ℓ. That is,

$$\mathrm{SO}_2(\mathbb{R}) \xrightarrow{\chi_\ell} \mathbb{C}^*$$
$$\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \mapsto e^{i\ell\theta}.$$

Let $\Phi^\ell_\infty(g,s)$ be the standard section of $\mathrm{SL}_2(\mathbb{R})$ induced by χ_ℓ. Then, for $\phi_f \in S(V(\mathbb{A}_f))$, let $E(\tau, s, \phi, \ell)$ denote the Eisenstein series

$$E(\tau,s,\phi,\ell) := v^{\frac{-\ell}{2}} E(g_\tau, s, \Phi^\ell_\infty \otimes \lambda(\phi_f)).$$

The connection between Eisenstein series and the theta series from the previous section is due to the Siegel-Weil formula. For a much more thorough discussion, we refer the reader to Chapter 4 of [12]. Originally, the Siegel-Weil formula related the sum of theta functions associated to quadratic forms with classical Eisenstein Series. The adelic version below is the greatly generalized version due to Kudla.

Theorem 4.1 (Siegel-Weil)**.** *Let* φ_∞ *For* $g \in \mathrm{SL}_2(\mathbb{A}), \varphi_f \in S(V(\mathbb{A}_f))$*, we have*

$$E(g_\tau, 0, \varphi_\infty \otimes \varphi) = \frac{1}{\mathrm{vol}(X_K)} \int_{X_K} \theta(\tau, z, h, \varphi_f)\, d\mu(z, h).$$

For the proof of this theorem, see Theorem 2.1 in [12].

4.3 Evaluation of the Integral

This section contains the main theorem of this chapter of the thesis

Theorem 4.2. *Using all of the notation set up in the previous sections of this chapter, we have*

$$\frac{1}{\mathrm{Vol}(X_K)} \int_{X_K} \Phi(z, h, F)\, d\mu(z, h) = \lim_{t \to \infty} -2 \left(\textit{Constant Term} \sum_{\mu} F_\mu(\tau) E(\tau, 0, \phi_\mu, 2) \right) \Bigg|_{v=t}.$$

Proof. The main idea of this proof is to exchange the order of integration, use the Siegel-Weil theorem to integrate the theta function, and then use Stokes' theorem with the lowering operators to evaluate the remaining integral. These ideas were set forth by [12], where Kudla works out the corresponding integral for a signature $(n, 2)$ quadratic space.

In the next section, we will show that we can switch the order of integration and plug in $\sigma = 0$ to determine the theta lift. That is, in the next section, we will show that the following manipulation is valid,

$$\int_{X_K} \Phi(z,h,F)\, d\mu(z,h) = \int_{X_K} \underset{\sigma=0}{\mathrm{CT}} \left\{ \lim_{t\to\infty} \int_{\mathcal{F}_t} \sum_{\mu} F_\mu(\tau)\theta(\tau,z,h,\phi_\mu) v^{-\sigma-2}\, dudv \right\} d\mu(z,h)$$
$$= \lim_{t\to\infty} \int_{\mathcal{F}_t} \int_{X_K} \sum_{\mu} F_\mu(\tau)\theta(\tau,z,h,\phi_\mu) v^{-2}\, d\mu(z,h) dudv.$$

Next, we note that each $F_\mu(\tau)$ is independent of z, h and so we can simplify the above integral and then use the Siegel-Weil formula,

$$\int_{X_K} \Phi(z,h,F)\, d\mu(z,h) = \lim_{t\to\infty} \int_{\mathcal{F}_t} \sum_{\mu} F_\mu(\tau) \int_{X_K} \theta(\tau,z,h,\phi_\mu)\, d\mu(z,h) v^{-2}\, dudv$$
$$= \lim_{t\to\infty} \int_{\mathcal{F}_t} \sum_{\mu} F_\mu(\tau) E(\tau,0,\phi_\mu,0) v^{-2}\, dudv.$$

Next, we use the lowering operator for Eisenstein series. This is an operator taking an Eisenstein series of weight k to an Eisenstein series of weight $k-2$. For a thorough discussion on the Lie Algebra background of this operator, see [12]. However, for our purposes, we will only need to discuss how this operator behaves on the upper half plane. For any Schwarz function φ, we have that

$$\frac{\partial}{\partial \tau} \Big\{ 4iE(\tau,0,\varphi,2) \Big\} = E(\tau,0,\varphi,0) v^{-2}.$$

First, using the lowering operator relation and the fact that $dudv = \dfrac{1}{2i} d\overline{\tau} d\tau$, we have that

$$\int_{\mathcal{F}_t} \sum_{\mu} F_\mu(\tau) E(\tau,0,\phi_\mu,0) v^{-2}\, dudv = 2\int_{\mathcal{F}_t} d\left(\sum_{\mu} F_\mu(\tau) E(\tau,0,\phi_\mu,2) d\tau \right).$$

From this, we can apply Stokes' theorem

$$2\int_{\mathcal{F}_t} d\left(\sum_{\mu} F_\mu(\tau) E(\tau,0,\phi_\mu,2) d\tau \right) = 2\int_{\partial\mathcal{F}_t} \sum_{\mu} F_\mu(\tau) E(\tau,0,\phi_\mu,2)\, d\tau.$$

Finally, with the modularity of F_μ and $E(\tau, 0, \phi_\mu, 2)$ we can make cancellations along different boundary components of $\partial\mathcal{F}_t$. Doing this leaves

$$\begin{aligned}2\int_{\partial\mathcal{F}_t}\sum_\mu F_\mu(\tau)E(\tau,0,\phi_\mu,2)\,d\tau &= 2\int_{\frac{1}{2}+it}^{-\frac{1}{2}+it}\sum_\mu F_\mu(\tau)E(\tau,0,\phi_\mu,2)du\\ &= -2\left(\text{Constant Term}\sum_\mu F_\mu(\tau)E(\tau,0,\phi_\mu,2)\right)\Bigg|_{v=t}.\end{aligned}$$

□

This ends the main proof of this chapter of the thesis. We believe that these ideas can be used to give interesting formulas for the integrals of certain modular forms along geodesics associated to real quadratic fields.

4.4 Technical Details on Absolute Convergence

The purpose of this section is to show that the following integral absolutely converges, justifying the exchange of integration of the previous section,

$$\int_{X_K}\lim_{t\to\infty}\int_{\mathcal{F}_t}\sum_\mu F_\mu(\tau)\theta(\tau,z,h,\phi_\mu)v^{-2}\,dudvd\mu(z,h).$$

First, we note that the region $\mathcal{F}_t$ can be split into two pieces, $\mathcal{F}_t = \mathcal{F}_1 + \mathcal{B}_t$, where $\mathcal{B}_t = \{\tau = u + iv : \frac{-1}{2} \le u \le \frac{1}{2}, 1 \le v \le t\}$. Since $\mathcal{F}_1$ is a compact region, absolute convergence of the integral is guaranteed, so we can replace $\mathcal{F}_t$ by $\mathcal{B}_t$ in the integral.

So we would like to prove absolute convergence of the integral

$$\int_{X_K}\int_1^\infty\int_{-1/2}^{1/2}\sum_\mu F_\mu(\tau)\theta(\tau,z,h,\phi_\mu)v^{-2}\,dudvd\mu(z,h).$$

Let $C(v,z,h) = v^{\frac{-1}{2}} \int_{-1/2}^{1/2} \sum_{\mu} F_\mu(\tau)\theta(\tau,z,h,\phi_\mu)du$. If we use the Fourier expansions of both θ and $\sum_\mu F_\mu$, we get that

$$C(v,z,h) = \sum_{\mu} \sum_{m\in\mathbb{Q}} c_\mu(-m) \sum_{\substack{x\in V(\mathbb{Q})\\ Q(x)=m}} \phi_\mu(h^{-1}x)e^{2\pi v(x_{z^\perp},x_{z^\perp})}.$$

Since ϕ_μ is an indicator function, we can bound $C(v,z,h)$ by

$$C(v,z,h) \leq \sum_{\mu} \sum_{m\in\mathbb{Q}} c_\mu(-m) \sum_{\substack{x\in hL'\\ Q(x)=m}} e^{4\pi v Q(x_{z^\perp})}.$$

Let $L_z = \mathbb{Q}z \cap L$ and similarly, let $L_{z^\perp} = \mathbb{Q}z^\perp \cap L$. For $x \in (hL_z)' \oplus (hL_{z^\perp})'$, write $x = x_z + x_{z^\perp}$ for the associated decomposition. Let $\widetilde{\ell} \in V(\mathbb{R})$ such that $(hL_{z^\perp}) = \mathbb{Z}\widetilde{\ell}$.

Then we have

$$\begin{aligned}|C(v,z,h)| &\leq \sum_{\mu} \sum_{m\in\mathbb{Q}} |c_\mu(-m)| \sum_{\substack{x\in(hL_z)'\oplus(hL_{z^\perp})'\\ Q(x)=m}} e^{4\pi v Q(x_{z^\perp})} \\ &= \sum_{t=0}^{\infty} a_t(m)e^{4\pi v t^2 Q(\widetilde{\ell})}.\end{aligned}$$

where

$$a_t(m) = \left|\{x = x_z + x_{z^\perp} \in (hL_z)' \oplus (hL_{z^\perp})' : Q(x) = m, x_{z^\perp} = \pm t\widetilde{\ell}\}\right|.$$

Let's look at $a_t(m)$ for a moment. For $x = x_z + x_{z^\perp}$ in the set being counted by $a_t(m)$, we have that $Q(x) = Q(x_z) + Q(x_{z^\perp})$, i.e. $m = Q(x_z) + t^2 Q(\widetilde{\ell})$. Since $Q(x_z) = m - t^2 Q(\widetilde{\ell})$ and z is a 1-dimensional space, there are at most two possibilities for x_z. So we have that $a_t(m) \leq 2$. Moreover, for a fixed $t \in \mathbb{N}$, we have that $a_t(m) = 0$ for m sufficiently negative. We can be a little more precise and say that for a fixed $t \in \mathbb{N}$, there are $O(t^2)$ many non-zero values of $a_t(m)$.

Below, we have the following bounds for $|C(v,z,h)|$, where the rearranging of sums is valid since all terms are positive,

$$\begin{aligned}|C(v,z,h)| &\leq \sum_{\mu}\sum_{m\in\mathbb{Q}}|c_\mu(-m)|\sum_{t=0}^{\infty}a_t(m)e^{4\pi vt^2Q(\widetilde{\ell})}\\ &=\sum_{\mu}\sum_{m\in\mathbb{Q}}\sum_{t=0}^{\infty}|c_\mu(-m)|a_t(m)e^{4\pi vt^2Q(\widetilde{\ell})}\\ &=\sum_{\mu}\sum_{t=0}^{\infty}\sum_{m\in\mathbb{Q}}|c_\mu(-m)|a_t(m)e^{4\pi vt^2Q(\widetilde{\ell})}\\ &=\sum_{\mu}\sum_{t=0}^{\infty}e^{4\pi vt^2Q(\widetilde{\ell})}\sum_{m\in\mathbb{Q}}|c_\mu(-m)a_t(m)|.\end{aligned}$$

It will be important to look at the sum $\sum_{m\in\mathbb{Q}}|c_\mu(-m)|a_t(m)$. For each μ and t, it is a finite sum. Since there are only finitely many negative Fourier coefficients

$$\sum_{\mu}\sum_{m\geq 0}|c_\mu(-m)|a_t(m)$$

is a constant that is independent of t. For the negative values of m, we use a result of [5]. There exists a constant $C>0$ such that $c_\mu(-m)=O(e^{C\sqrt{|m|}})$ as $m\to-\infty$. Therefore,

$$\sum_{\mu}\sum_{m\in\mathbb{Q}}|c_\mu(-m)|a_t(m)=O(e^{Ct}).$$

Finally, we turn to absolute convergence of the integral

$$\int_{X_K}\int_1^\infty C(v,z,h)v^{-3/2}dvd\mu(z,h).$$

To show the absolute convergence of the above integral, we need to show convergence of I, where

$$I:=\int_{X_K}\int_1^\infty |C(v,z,h)|v^{-3/2}dvd\mu(z,h).$$

We can bound I by

$$\begin{aligned}
I &\leq \int_{X_K}\int_1^\infty \sum_\mu \sum_{m\in\mathbb{Q}} \sum_{\substack{x\in V(\mathbb{Q})\\ Q(x)=m}} e^{4\pi v Q(x_{z^\perp})}|c_\mu(-m)v^{-3/2}|dv d\mu(z,h) \\
&\leq \int_{X_K}\int_1^\infty \sum_\mu \sum_{m\in\mathbb{Q}} |c_\mu(-m)| \sum_{\substack{x\in(hL_z)'\oplus(hL_{z^\perp})'\\ Q(x)=m}} e^{4\pi v Q(x_{z^\perp})} v^{-3/2} dv d\mu(z,h) \\
&= \int_{X_K}\int_1^\infty \sum_\mu \sum_{m\in\mathbb{Q}} |c_\mu(-m)| \sum_{t=0}^\infty a_t(m) e^{4\pi v t^2 Q(\widetilde{\ell})} v^{-3/2} dv d\mu(z,h) \\
&= \int_{X_K}\int_1^\infty \sum_\mu \sum_{t=0}^\infty e^{4\pi v t^2 Q(\widetilde{\ell})} v^{-3/2} \left(\sum_{m\in\mathbb{Q}} |c_\mu(-m) a_t(m)\right) dv d\mu(z,h) \\
&<< \int_{X_K}\int_1^\infty \sum_\mu v^{-3/2} \sum_{t=0}^\infty t^2 e^{Ct+4\pi v t^2 Q(\widetilde{\ell})} dv d\mu(z,h).
\end{aligned}$$

To study the asymptotics (in terms of v) of the inner sum $\sum_{t=0}^\infty t^2 e^{Ct+4\pi v t^2 Q(\widetilde{\ell})}$, we can look at the integral (and recall that $Q(\widetilde{\ell}) < 0$)

$$\int_0^\infty t^2 e^{4\pi v t^2 Q(\widetilde{\ell})} dt = \frac{Q(\widetilde{\ell})^{\frac{-3}{2}} v^{\frac{-3}{2}}}{32\pi}.$$

Using this, we get

$$\begin{aligned}
I &<< \int_{X_K}\int_1^\infty v^{-3} Q(\widetilde{\ell})^{-3/2} dv d\mu(z,h) \\
&<< \int_{X_K} Q(\widetilde{\ell})^{-3/2} d\mu(z,h).
\end{aligned}$$

As a function of z and h, $Q(\widetilde{\ell})^{-3/2}$ is a non-vanishing continuous function on the compact space X_K. Therefore, the integral $\int_{X_K} Q(\widetilde{\ell})^{-3/2} d\mu(z,h)$ converges and so we have absolute convergence of our original integral.

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