LEONARD TRIPLES ASSOCIATED WITH THE ANTICOMMUTATOR SPIN ALGEBRA

By

George Brown

A dissertation submitted in partial fulfillment of the
requirements for the degree of

Doctor of Philosophy
(Mathematics)

at the

UNIVERSITY OF WISCONSIN – MADISON

2012

Date of final oral examination: August 13, 2012

The dissertation is approved by the following members of the Final Oral Committee:

Professor R. A. Brualdi, Professor Emeritus, Mathematics

Professor J. Ellenberg, Professor, Mathematics

Professor T. Millar, Professor, Mathematics

Professor P. Terwilliger, Professor, Mathematics

Professor P. M. Wood, Assistant Professor, Mathematics

Abstract

This thesis is about four classes of objects: Leonard pairs, Leonard triples, the finite-dimensional irreducible modules for the anticommutator spin algebra $\mathcal{A}$, and a family of distance-regular graphs. Let $\mathbb{K}$ denote an algebraically closed field of characteristic zero. Let V denote a vector space over $\mathbb{K}$ with finite positive dimension. A Leonard pair on V is an ordered pair of linear transformations in $\mathrm{End}(V)$ such that for each of these transformations there exists a basis for V with respect to which the matrix representing that transformation is diagonal and the matrix representing the other transformation is irreducible tridiagonal. Whenever these tridiagonal matrices are bipartite, the Leonard pair is said to be totally bipartite. A mild weakening of the bipartite assumption yields a type of Leonard pair said to be totally almost bipartite. A Leonard pair is said to be totally B/AB whenever it is totally bipartite or totally almost bipartite. The notion of a Leonard triple and the corresponding notion of totally B/AB are similarly defined. There are families of Leonard pairs and Leonard triples said to have Bannai/Ito type. The Leonard pairs and Leonard triples of interest to us are the ones that are totally B/AB and of Bannai/Ito type.

Let $\mathcal{A}$ denote the unital associative $\mathbb{K}$-algebra defined by generators x, y, z and relations

$$xy + yx = 2z, \qquad yz + zy = 2x, \qquad zx + xz = 2y.$$

The algebra $\mathcal{A}$ has a presentation involving generators x, y and relations

$$x^2y + 2xyx + yx^2 = 4y, \qquad y^2x + 2yxy + xy^2 = 4x.$$

In this thesis we obtain the following results. We classify up to isomorphism the totally B/AB Leonard pairs of Bannai/Ito type. We classify up to isomorphism the totally B/AB Leonard triples of Bannai/Ito type. We classify up to isomorphism the finite-dimensional irreducible $\mathcal{A}$-modules. We show that these three classes of objects are essentially in one-to-one correspondence, and describe these correspondences in detail. We then display examples of totally B/AB Leonard triples of Bannai/Ito type in the Terwilliger algebras of certain distance-regular graphs. Specifically, totally bipartite Leonard triples of Bannai/Ito type will show up in the even-diameter hypercubes and totally almost bipartite Leonard triples of Bannai/Ito type will show up in the antipodal quotients of odd-diameter hypercubes.

Acknowledgements

I would like to thank my advisor Professor Paul Terwilliger for his contributions to this thesis. His expert guidance and constant encouragement made this thesis possible in the first place.

I have enjoyed being part of Team Terwilliger, whose ranks include Sarah Bockting-Conrad, Alison Gordon, Ed Hanson, Ali Godjali, Jae-Ho Lee, Gabriel Pretel and Boyd Worawannotai. May the team continue growing.

I am grateful to my mother, Harriet Fell and my sister Tova Brown, for helping to instill in me a love for mathematics from an early age and for helping to keep that love alive today. This thesis is for the memory of my father, Sheldon Brown, who I wish was still around to see it.

Contents

Chapter 1

Introduction

Throughout this thesis, $\mathbb{K}$ denotes an algebraically closed field of characteristic zero.

We now recall the definition of a Leonard pair. To do this, we use the following terms. A square matrix B is said to be *tridiagonal* whenever each nonzero entry lies on either the diagonal, the subdiagonal, or the superdiagonal. Assume B is tridiagonal. Then B is said to be *irreducible* whenever each entry on the subdiagonal or superdiagonal is nonzero.

Definition 1.1 [13, Definition 1.1] Let V denote a vector space over $\mathbb{K}$ with finite positive dimension. By a *Leonard pair* on V we mean an ordered pair of linear transformations $A : V \to V$, $A^* : V \to V$ which satisfy the conditions (i), (ii) below.

(i) There exists a basis for V with respect to which the matrix representing A is diagonal and the matrix representing A^* is irreducible tridiagonal.

(ii) There exists a basis for V with respect to which the matrix representing A^* is diagonal and the matrix representing A is irreducible tridiagonal.

The *diameter* of the Leonard pair A, A^* is defined to be one less than the dimension of V.

If A, A^* is a Leonard pair on V then so is A^*, A.

We will be considering two families of Leonard pairs said to be totally bipartite and totally almost bipartite. Before defining these families, we first review a few concepts. Let V denote a vector space over $\mathbb{K}$ with finite positive dimension. By a *decomposition* of V we mean a sequence of one-dimensional subspaces of V whose direct sum is V. For any basis $\{v_i\}_{i=0}^d$ for V, the sequence $\{\mathbb{K}v_i\}_{i=0}^d$ is a decomposition of V; the decomposition $\{\mathbb{K}v_i\}_{i=0}^d$ is said to *correspond* to the basis $\{v_i\}_{i=0}^d$. Given a decomposition $\{V_i\}_{i=0}^d$ of V, for $0 \leq i \leq d$ pick $0 \neq v_i \in V_i$. Then $\{v_i\}_{i=0}^d$ is a basis for V which corresponds to $\{V_i\}_{i=0}^d$.

Let A, A^* denote a Leonard pair on V. A basis for V is called *standard* whenever it satisfies Definition 1.1(i). Observe that, given a decomposition $\{V_i\}_{i=0}^d$ of V, the following (i), (ii) are equivalent.

(i) There exists a standard basis for V which corresponds to $\{V_i\}_{i=0}^d$.

(ii) Every basis for V which corresponds to $\{V_i\}_{i=0}^d$ is standard.

We say that the decomposition $\{V_i\}_{i=0}^d$ is *standard* whenever (i), (ii) hold. Observe that if the decomposition $\{V_i\}_{i=0}^d$ is standard, then so is $\{V_{d-i}\}_{i=0}^d$ and no other decomposition of V is standard.

For any nonnegative integer d let $\mathrm{Mat}_{d+1}(\mathbb{K})$ denote the $\mathbb{K}$-algebra consisting of all $d+1$ by $d+1$ matrices that have entries in $\mathbb{K}$. We index the rows and columns by $0, 1, \ldots, d$.

Let $B \in \mathrm{Mat}_{d+1}(\mathbb{K})$ be tridiagonal. We say that B is *bipartite* whenever $B_{ii} = 0$ for $0 \leq i \leq d$.

Definition 1.2 A Leonard pair A, A^* is said to be *bipartite* whenever the matrix representing A from Definition 1.1(ii) is bipartite. The Leonard pair A, A^* is said to be *dual* *bipartite* whenever the Leonard pair A^*, A is bipartite. The Leonard pair A, A^* is said to be *totally bipartite* whenever it is bipartite and dual bipartite.

Let $B \in \mathrm{Mat}_{d+1}(\mathbb{K})$ be tridiagonal. We say that B is *almost bipartite* whenever exactly one of $B_{0,0}, B_{d,d}$ is nonzero and $B_{ii} = 0$ for $1 \leq i \leq d-1$.

Definition 1.3 A Leonard pair A, A^* is said to be *almost bipartite* whenever the matrix representing A from Definition 1.1(ii) is almost bipartite. The Leonard pair A, A^* is said to be *dual almost bipartite* whenever the Leonard pair A^*, A is almost bipartite. The Leonard pair A, A^* is said to be *totally almost bipartite* whenever it is almost bipartite and dual almost bipartite.

The notion of a Leonard triple was introduced by Brian Curtin in [5]. We recall the definition.

Definition 1.4 [5, Definition 1.2] Let V denote a vector space over $\mathbb{K}$ with finite positive dimension. By a *Leonard triple* on V we mean an ordered triple of linear transformations $A : V \to V$, $A^* : V \to V$, $A^\varepsilon : V \to V$ which satisfy the conditions (i)–(iii) below.

(i) There exists a basis for V with respect to which the matrix representing A is diagonal and the matrices representing A^* and A^ε are irreducible tridiagonal.

(ii) There exists a basis for V with respect to which the matrix representing A^* is diagonal and the matrices representing A^ε and A are irreducible tridiagonal.

(iii) There exists a basis for V with respect to which the matrix representing A^ε is diagonal and the matrices representing A and A^* are irreducible tridiagonal.

The *diameter* of the Leonard triple A, A^*, A^ε is defined to be one less than the dimension of V.

Definition 1.5 In Definition 1.4 we defined a Leonard triple A, A^*, A^ε. In that definition we mentioned six tridiagonal matrices. The Leonard triple A, A^*, A^ε is said to be *totally bipartite* (resp. *totally almost bipartite*) whenever each of the six tridiagonal matrices is bipartite (resp. almost bipartite).

For notational convenience, we say that a Leonard pair or Leonard triple is totally B/AB whenever it is either totally bipartite or totally almost bipartite.

For any Leonard triple, any two of the three form a Leonard pair. We say that these Leonard pairs are *associated* with the Leonard triple. The Leonard triple is totally bipartite if and only if all of the associated Leonard pairs are totally bipartite. The Leonard triple is totally almost bipartite if and only if all of the associated Leonard pairs are totally almost bipartite.

In [13], Terwilliger classified the Leonard pairs up to isomorphism. By that classification, the isomorphism classes of Leonard pairs fall naturally into thirteen families: q-Racah, q-Hahn, dual q-Hahn, q-Krawtchouk, dual q-Krawtchouk, affine q-Krawtchouk, quantum q-Krawtchouk, Racah, Hahn, dual Hahn, Krawtchouk, Bannai/Ito and orphan. For each integer $d \geq 3$ these families partition the isomorphism classes of Leonard pairs that have diameter d. It remains an open problem to classify the Leonard triples up to isomorphism. However, in [5], Curtin classified a family of Leonard triples said to be *modular*.

We say that a Leonard triple is of Bannai/Ito type whenever all of its associated

Leonard pairs are of Bannai/Ito type. Leonard pairs of Bannai/Ito type arise in conjunction with the Bannai/Ito polynomials. These polynomials were introduced in [2, pp. 271–273] by Bannai and Ito. In [17], Tsujimoto, Vinet and Zhedanov studied the Bannai/Ito polynomials in conjunction with Dunkl shift operators and representations of Jordan algebras. Totally B/AB Leonard pairs and Leonard triples also appear in the literature. In [18], Miklavič studied totally bipartite Leonard triples associated with some representations of the Lie algebra $\mathfrak{sl}_2$ constructed using hypercubes. The Leonard pairs associated with these Leonard triples are of Krawtchouk type. In [8], Havlíček, Klimyk and Pošta displayed representations of the nonstandard q-deformed cyclically symmetric algebra $U'_q(\mathfrak{so}_3)$. These representations yield both totally bipartite and totally almost bipartite Leonard triples. The Leonard pairs associated with these Leonard triples are of q-Racah type.

The Leonard pairs and Leonard triples of interest to us are the ones that are totally B/AB and of Bannai/Ito type. To describe these Leonard pairs and Leonard triples, we consider a $\mathbb{K}$-algebra $\mathcal{A}$ defined by generators x, y, z and relations

$$xy + yx = 2z, \qquad yz + zy = 2x, \qquad zx + xz = 2y. \tag{1.1}$$

The algebra $\mathcal{A}$ has an alternate presentation using generators x, y and relations

$$x^2y + 2xyx + yx^2 = 4y, \qquad y^2x + 2yxy + xy^2 = 4x.$$

The algebra $\mathcal{A}$ has appeared previously in the literature [1]. In [1, Section 1], Arik and Kayserilioglu introduced an algebra involving the relations (1.1). They called this the anticommutator spin algebra and studied it in conjunction with fermionic quantum systems and the angular momentum algebra. We say more about Arik and Kayserilioglu's results after Theorem 2.36.

The present thesis is about how the following are related: (i) Totally B/AB Leonard pairs of Bannai/Ito type; (ii) Totally B/AB Leonard triples of Bannai/Ito type; (iii) Finite-dimensional irreducible $\mathcal{A}$-modules, (iv) Even-diameter hypercubes and the antipodal quotients of odd-diameter hypercubes. We now summarize our main results. We classify up to isomorphism the totally B/AB Leonard pairs of Bannai/Ito type. We classify up to isomorphism the totally B/AB Leonard triples of Bannai/Ito type. We classify up to isomorphism the finite-dimensional irreducible $\mathcal{A}$-modules. We show that these three classes of objects are essentially in one-to-one correspondence. The correspondence is described as follows. Let V denote a finite-dimensional irreducible $\mathcal{A}$-module. Then the actions of x, y (resp. x, y, z) on V form a totally B/AB Leonard pair (resp. Leonard triple) of Bannai/Ito type. Conversely, let A, A^* (resp. A, A^*, A^ε) denote a totally B/AB Leonard pair (resp. Leonard triple) of Bannai/Ito type with diameter at least 3 and let V denote the underlying vector space. Then there exists an irreducible $\mathcal{A}$-module structure on V and nonzero scalars ξ, ξ^* (resp. $\xi, \xi^*, \xi^\varepsilon$) such that A, A^* (resp. A, A^*, A^ε) act on V as $\xi x, \xi^* y$ (resp. $\xi x, \xi^* y, \xi^\varepsilon z$) respectively.

We now summarize our results in greater detail. We first describe the algebra $\mathcal{A}$. As part of this description, we display an action of the symmetric group S_4 on $\mathcal{A}$ as a group of automorphisms. We then classify up to isomorphism the finite-dimensional irreducible $\mathcal{A}$-modules. Let V denote a finite-dimensional irreducible $\mathcal{A}$-module. We describe how twisting V via an element of S_4 affects the isomorphism class of V. We obtain the eigenvalues and corresponding primitive idempotents for the actions of x, y, z on V. We use twisting via the S_4-action to simplify the calculations. We display six bases for V. With respect to each of these bases the matrix representing one of x, y, z is diagonal and the matrices representing the other two are irreducible tridiagonal. We display the matrices representing the actions of x, y, z on V with respect to each of the six bases. From this, we show that x, y act on V as a totally B/AB Leonard pair of Bannai/Ito type and x, y, z act on V as a totally B/AB Leonard triple of Bannai/Ito type.

Next we classify up to isomorphism the totally B/AB Leonard pairs of Bannai/Ito type. To avoid trivialities, we assume the diameter is at least 3. To obtain this classification, we use the Askey-Wilson relations for a Leonard pair A, A^* described by Terwilliger and Vidunas [16]. For the case in which A, A^* is totally B/AB and of Bannai/Ito type, we show that the Askey-Wilson relations take the form

$$A^2A^* + 2AA^*A + A^*A^2 = \varrho A^*, \qquad A^{*2}A + 2A^*AA^* + AA^{*2} = \varrho^* A,$$

where $\varrho, \varrho^* \in \mathbb{K}$ are nonzero. Using these relations, we show that for every totally B/AB Leonard pair A, A^* on V of Bannai/Ito type with diameter at least 3, there exist nonzero scalars $\xi, \xi^* \in \mathbb{K}$ and an $\mathcal{A}$-module structure on V such that A, A^* act as $\xi x, \xi^* y$ respectively. From the preceding paragraphs, we obtain a correspondence between finite-dimensional irreducible $\mathcal{A}$-modules and totally B/AB Leonard pairs of Bannai/Ito type. Using this correspondence we obtain our classification of the totally B/AB Leonard pairs of Bannai/Ito type.

Next we classify up to isomorphism the totally B/AB Leonard triples of Bannai/Ito type. Again we assume the diameter is at least 3. To obtain this classification, we use some results of Nomura and Terwilliger [11] concerning linear transformations that are tridiagonal with respect to both eigenbases of a Leonard pair A, A^*. For the case in which A, A^* is associated with a totally B/AB Leonard triple A, A^*, A^ε of Bannai/Ito type, we use these results to show that

$$\zeta^{\varepsilon}(AA^* + A^*A) = A^{\varepsilon}, \qquad \zeta(A^*A^{\varepsilon} + A^{\varepsilon}A^*) = A, \qquad \zeta^*(A^{\varepsilon}A + AA^{\varepsilon}) = A^*,$$

where $\zeta, \zeta^*, \zeta^{\varepsilon} \in \mathbb{K}$ are nonzero. Using these relations, we show that for every totally B/AB Leonard triple A, A^*, A^{ε} on V of Bannai/Ito type with diameter at least 3, there exist nonzero scalars $\xi, \xi^*, \xi^{\varepsilon} \in \mathbb{K}$ and an $\mathcal{A}$-module structure on V such that A, A^*, A^{ε} act as $\xi x, \xi^* y, \xi^{\varepsilon} z$ respectively. From the preceding paragraphs, we obtain a correspondence between finite-dimensional irreducible $\mathcal{A}$-modules and totally B/AB Leonard triples of Bannai/Ito type. Using this correspondence we obtain our classification of the totally B/AB Leonard triples of Bannai/Ito type.

Next we display examples of totally B/AB Leonard triples in the context of distance-regular graphs. In [6], Go displayed an action of the Lie algebra $\mathfrak{sl}_2$ on the standard modules for hypercubes. In [18], Miklavič used Go's results to display examples totally bipartite Leonard triples of Krawtchouk type. We introduce an operator, called a skew operator, that gives $\mathcal{A}$-module structures to finite-dimensional $\mathfrak{sl}_2$-modules. Using skew operators, we display an action of the algebra $\mathcal{A}$ on the primary modules of even-diameter hypercubes and the antipodal quotients of odd-diameter hypercubes similar to the $\mathfrak{sl}_2$-action from [6]. From the even-diameter hypercubes we obtain totally bipartite Leonard triples of Bannai/Ito type. From the antipodal quotients of odd-diameter hypercubes we obtain totally almost bipartite Leonard triples of Bannai/Ito type.

The thesis is organized as follows. In Chapter 2, we define the anticommutator spin algebra $\mathcal{A}$ and classify its finite-dimensional irreducible modules. We display an action of S_4 on $\mathcal{A}$ as a group of automorphisms show how twisting a finite-dimensional irreducible $\mathcal{A}$-module via an element of S_4 affects the isomorphism class of that module.

In Chapter 3, we describe the matrices representing the actions of x, y, z on each finite-dimensional irreducible $\mathcal{A}$-module with respect to six different bases. We show that the generators of x, y, z act as a Leonard triple on every finite-dimensional irreducible $\mathcal{A}$-module. In Chapter 4, we classify the totally B/AB Leonard pairs of Bannai/Ito type and the totally B/AB Leonard triples of Bannai/Ito type and show how they correspond to finite-dimensional irreducible $\mathcal{A}$-modules. In Chapter 5 we recall the Lie algebra $\mathfrak{sl}_2$. We introduce the notion of a skew operator and use this operator to give $\mathcal{A}$-module structures to every finite-dimensional $\mathfrak{sl}_2$-module structure. In Chapter 6 we recall the notion of a distance-regular graph. We describe the Terwilliger algebras of the even-diameter hypercubes and the antipodal quotients of odd-diameter hypercubes. We display an $\mathcal{A}$-action on the standard module of each graph in these two families. From this we display examples of totally B/AB Leonard triples of Bannai/Ito type.

Chapter 2

The anticommutator spin algebra

2.1 The algebra $\mathcal{A}$ and its automorphisms

We now define the $\mathbb{K}$-algebra $\mathcal{A}$.

Definition 2.1 [1, Section 1] Let $\mathcal{A}$ denote the unital associative algebra over $\mathbb{K}$ with generators x, y, z and relations

$$\begin{aligned} xy + yx &= 2z, & (2.1) \\ yz + zy &= 2x, & (2.2) \\ zx + xz &= 2y. & (2.3) \end{aligned}$$

We refer to the algebra $\mathcal{A}$ as the *anticommutator spin algebra.*

Note that $\mathcal{A}$ is generated by any two of x, y, z. This yields the following two-generator presentation of $\mathcal{A}$.

Lemma 2.2 *The algebra $\mathcal{A}$ has a presentation involving generators x, y and relations*

$$\begin{aligned} x^2y + 2xyx + yx^2 &= 4y, & (2.4) \\ y^2x + 2yxy + xy^2 &= 4x. & (2.5) \end{aligned}$$

Proof: Rewrite relations (2.2), (2.3) by eliminating z using line (2.1). $\square$

Lemma 2.3 *Any algebra automorphism of $\mathcal{A}$ that fixes at least two of x, y, z is the identity.*

Proof: Since any two of x, y, z generate $\mathcal{A}$, any automorphism that fixes at least two of x, y, z must fix all of $\mathcal{A}$. $\square$

Each permutation of x, y, z extends to a unique algebra automorphism of $\mathcal{A}$; this can be checked using relations (2.1)–(2.3). This gives an action of the symmetric group S_3 on $\mathcal{A}$ as a group of automorphisms. There are also algebra automorphisms of $\mathcal{A}$ that change the sign of two of x, y, z while preserving the third; this gives an action of the Klein-four group K_4 on $\mathcal{A}$ as a group of automorphisms.

In a moment we will show how the S_3 and K_4 actions interact, but first it will be useful to establish that these actions are faithful.

Definition 2.4 Let $\mathbb{I}$ denote the set consisting of the symbols $0, x, y, z$.

Lemma 2.5 *For $n \in \mathbb{I}$ there exists a unique algebra homomorphism $f_n : \mathcal{A} \to \mathbb{K}$ satisfying*

n	$f_n(x)$	$f_n(y)$	$f_n(z)$
0	1	1	1
x	1	-1	-1
y	-1	1	-1
z	-1	-1	1

Moreover, f_n is surjective.

Proof: One verifies that f_n exists through routine calculation using Definition 2.1. Also f_n is unique since $\mathcal{A}$ is generated by x, y, z. Observe f_n is nonzero and hence surjective. $\square$

Lemma 2.6 *The elements* $x, y, z, 1$ *are linearly independent in the* $\mathbb{K}$*-vector space* $\mathcal{A}$*.*

Proof: Let $a, b, c, d \in \mathbb{K}$ satisfy $ax + by + cz + d = 0$. For each $n \in \mathbb{I}$, we apply f_n to this equation and get

$$\begin{aligned} a+b+c+d &= 0, \\ a-b-c+d &= 0, \\ -a+b-c+d &= 0, \\ -a-b+c+d &= 0. \end{aligned}$$

The coefficient matrix of the above system of equations is non-singular, so the unique solution is $a = b = c = d = 0$. Therefore $x, y, z, 1$ are linearly independent. $\square$

Corollary 2.7 $\pm x, \pm y, \pm z$ *are mutually distinct elements of* $\mathcal{A}$*.*

Proof: Immediate from Lemma 2.6. $\square$

Recall the S_3 and K_4 actions from below Definition 2.1.

Corollary 2.8 S_3 *and* K_4 *act faithfully on* $\mathcal{A}$*.*

Proof: By Corollary 2.7, S_3 and K_4 act faithfully on the set $\{\pm x, \pm y, \pm z\}$, so they act faithfully on $\mathcal{A}$. $\square$

We remark that, in Section 6.2, we will classify up to isomorphism the finite-dimensional irreducible $\mathcal{A}$-modules. The solutions to this classification include four infinite classes, corresponding to almost bipartite Leonard triples. The $\mathcal{A}$-modules in these classes are indexed by a nonnegative integer called the diameter. The f_n from Lemma 2.5 come from the $\mathcal{A}$-modules of diameter 0 in these classes.

Lemma 2.9 *Let σ denote an automorphism of $\mathcal{A}$ that fixes each of x, y, z up to sign. Then σ must change the sign of an even number of x, y, z.*

Proof: By Lemma 2.3, if σ fixes any two of x, y, z it must fix all three, so σ cannot change the sign of exactly one of x, y, z. Also, σ cannot change the sign of all three of x, y, z because, if it did, we could compose it with a non-identity element of K_4 to get an automorphism that changes the sign of exactly one of x, y, z. The result follows. $\square$

Let $\mathrm{Aut}(\mathcal{A})$ denote the set consisting of all automorphisms of $\mathcal{A}$ and note that $\mathrm{Aut}(\mathcal{A})$ forms a group under composition. Let G denote the subgroup of $\mathrm{Aut}(\mathcal{A})$ that fixes the set $\{\pm x, \pm y, \pm z\}$. Let S denote the subgroup of $\mathrm{Aut}(\mathcal{A})$ that fixes the set $\{x, y, z\}$ and let K denote the subgroup of $\mathrm{Aut}(\mathcal{A})$ that fixes each of x, y, z up to sign. Observe that S and K are both subgroups of G.

Corollary 2.8 gives an injection of groups $S_3 \hookrightarrow \mathrm{Aut}(\mathcal{A})$ and, by construction, the image of this injection is S. Similarly, Corollary 2.8 gives an injection of groups $K_4 \hookrightarrow \mathrm{Aut}(\mathcal{A})$. By Lemma 2.9 and the definition of the K_4-action, the image of this injection is K. Since $S, K \subseteq G$, this gives group injections, $S_3 \hookrightarrow G$, $K_4 \hookrightarrow G$ whose images are S, K respectively.

It will turn out that G is isomorphic to S_4 and that G is a semi-direct product $K \rtimes S$.

Proposition 2.10 $G = K \rtimes S$.

Proof: By [7, Proposition 11.2], it suffices to show $S \cap K = \{1_G\}$, $K \triangleleft G$ and $G = KS$. By construction, $S \cap K = \{1_G\}$. By definition the elements of G permute $\pm x, \pm y, \pm z$. We define a binary relation $\sim$ on the set $\{\pm x, \pm y, \pm z\}$ such that $u \sim v$ if and only if $u = \pm v$. Observe that $\sim$ is an equivalence relation. Moreover, observe that the elements of G permute the three equivalence classes of $\sim$, resulting in a group homomorphism $\varphi : G \to S_3$. The kernel of this homomorphism is K, so $K \triangleleft G$. Furthermore, the composition $S \hookrightarrow G \to S_3$ is an isomorphism $S \to S_3$, so $G = KS$. By these comments $G = K \rtimes S$. □

Our next goal is to show that G is isomorphic to S_4.

Definition 2.11 For $n \in \mathbb{I}$, define $h_n \in \mathcal{A}$ as follows:

$$\begin{aligned} h_0 &= x + y + z, & h_x &= x - y - z, \\ h_y &= -x + y - z, & h_z &= -x - y + z. \end{aligned} \tag{2.6}$$

Lemma 2.12 *We have*

$$x = \frac{h_0 + h_x}{2}, \qquad y = \frac{h_0 + h_y}{2}, \qquad z = \frac{h_0 + h_z}{2}.$$

Moreover, the algebra $\mathcal{A}$ *is generated by* $\{h_n\}_{n \in \mathbb{I}}$.

Proof: Routine. □

Let $\tilde{G}$ denote the group of all permutations of $\mathbb{I}$ and observe $\tilde{G}$ is isomorphic to S_4.

Proposition 2.13 *There exists a group isomorphism* $G \to \tilde{G}, \sigma \mapsto \tilde{\sigma}$ *such that* $\sigma(h_n) = h_{\tilde{\sigma}(n)}$ *for all* $n \in \mathbb{I}$.

Proof: We first show that G fixes the set $\{h_n\}_{n \in \mathbb{I}}$. Since G is generated by S and K it suffices to show that S and K fix $\{h_n\}_{n \in \mathbb{I}}$. We check that this is the case for S by the construction below Lemma 2.9. We check that this is the case for K by the construction below Lemma 2.9 along with Lemma 2.9 itself. Since G fixes the set $\{h_n\}_{n \in \mathbb{I}}$, there is a unique group homomorphism $G \to \tilde{G}, \sigma \mapsto \tilde{\sigma}$ such that $\sigma(h_n) = h_{\tilde{\sigma}(n)}$ for all $n \in \mathbb{I}$. The action of G on $\{h_n\}_{n \in \mathbb{I}}$ is faithful in view of Lemma 2.12. The homomorphism is an isomorphism since each of $G, \tilde{G}$ have cardinality 24. $\square$

Corollary 2.14 *The group G is isomorphic to S_4.*

Proof: G is isomorphic to $\tilde{G}$ by Proposition 2.13 and $\tilde{G}$ is isomorphic to S_4 by construction. $\square$

We just established a group isomorphism $G \to \tilde{G}$. We have subgroups $S, K \subseteq G$. We now consider what this isomorphism does to the elements of S and K. To this end, let $\tilde{S}$ denote the subgroup of $\tilde{G}$ consisting of the elements that fix 0. Let $\tilde{K}$ denote the unique normal subgroup of $\tilde{G}$ of order 4. Note that $\tilde{K}$ consists of

$$(0x)(yz), \qquad (0y)(zx), \qquad (0z)(xy),$$

together with the identity.

Lemma 2.15 *With respect to the group isomorphism $G \to \tilde{G}$ from Proposition 2.13, the image of S is $\tilde{S}$. Moreover, let $\sigma \in S$. Recall that σ permutes the elements x, y, z of $\mathcal{A}$. Then $\tilde{\sigma}$ permutes the elements x, y, z of $\mathbb{I}$ in the corresponding way.*

Proof: First we show how σ acts on h_0.

$$\begin{aligned}\sigma(h_0) &= \sigma(x+y+z)\\ &= \sigma(x)+\sigma(y)+\sigma(z)\\ &= x+y+z\\ &= h_0,\end{aligned}$$

so $\tilde{\sigma}$ fixes 0. Let a, b, c denote distinct elements of $\{x, y, z\}$. Then

$$\begin{aligned}\sigma(h_a) &= \sigma(a-b-c)\\ &= \sigma(a)-\sigma(b)-\sigma(c),\end{aligned}$$

so $\tilde{\sigma}(a) = \sigma(a)$ when $a \in \{x, y, z\}$. The image of S is $\tilde{S}$ by the definition of $\tilde{S}$. $\square$

Lemma 2.16 *With respect to the isomorphism $G \to \tilde{G}$ from Proposition 2.13, the image of K is $\tilde{K}$. Given a non-identity element $\sigma \in K$, recall that σ fixes one of x, y, z and changes the sign of the other two. Let a, b, c denote distinct elements of $\{x, y, z\}$ such that σ fixes a and changes the sign of b and c. Now, viewing a, b, c as elements of $\mathbb{I}$, then $\tilde{\sigma}$ is $(0, a)(b, c)$.*

Proof: $\sigma, \tilde{\sigma}$ are both involutions, so $\tilde{\sigma}$ is a composition of disjoint 2-cycles. It is therefore sufficient to show how $\tilde{\sigma}$ acts on 0 and b.

$$\begin{aligned}\sigma(h_0) &= \sigma(a+b+c)\\ &= \sigma(a)+\sigma(b)+\sigma(c)\\ &= a-b-c\\ &= h_a,\end{aligned}$$

so $\tilde{\sigma}$ switches 0 and a.

$$\begin{aligned} \sigma(h_b) &= \sigma(-a+b-c) \\ &= -\sigma(a)+\sigma(b)-\sigma(c) \\ &= -a-b+c \\ &= h_c, \end{aligned}$$

so $\tilde{\sigma}$ switches b and c. The image of K is $\tilde{K}$ by the definition of $\tilde{K}$. $\square$

2.2 The finite-dimensional irreducible $\mathcal{A}$-modules

In this section we classify the finite-dimensional irreducible $\mathcal{A}$-modules up to isomorphism. This classification is given in Theorem 2.36.

We adopt the following conventions. Let V denote a vector space over $\mathbb{K}$. By $\mathrm{End}(V)$ we mean the $\mathbb{K}$-algebra of linear transformations from V to V. Let $B \in \mathrm{End}(V)$. By an *eigenvalue* of B we mean a root of the minimal polynomial of B. For an eigenvalue θ of B, the *eigenspace for B associated with θ* is the subspace $\{v \in V | B.v = \theta v\}$. B is *diagonalizable* whenever V is spanned by its eigenspaces.

Definition 2.17 Let V denote an $\mathcal{A}$-module. For $\lambda \in \mathbb{K}$, we define $V(\lambda) = \{v \in V | x.v = \lambda v\}$.

Lemma 2.18 *Let V denote an $\mathcal{A}$-module. Then $(y+z).V(\lambda) \subseteq V(2-\lambda)$ and $(y-z).V(\lambda) \subseteq V(-2-\lambda)$. Moreover, $y.V(\lambda) \subseteq V(2-\lambda)+V(-2-\lambda)$ and $z.V(\lambda) \subseteq V(2-\lambda)+V(-2-\lambda)$.*

Proof: Let $v \in V(\lambda)$. Using Definition 2.1 we find that $(y+z).v \in V(2-\lambda)$ and $(y-z).v \in V(-2-\lambda)$. The first two assertions follow from this. The last two assertions follow from the first two and the observation that each of y and z is a linear combination of $y+z, y-z$. $\square$

We define functions $f : \mathbb{K} \to \mathbb{K}$ and $g : \mathbb{K} \to \mathbb{K}$ such that $f(\lambda) = 2 - \lambda$ and $g(\lambda) = -2 - \lambda$ for all $\lambda \in \mathbb{K}$. Observe $f(f(\lambda)) = \lambda$ and $g(g(\lambda)) = \lambda$ for all $\lambda \in \mathbb{K}$, so f and g are permutations of $\mathbb{K}$. Note that f has a single orbit of size 1, namely $\{1\}$ and all other orbits have size 2. Similarly, g has a single orbit of size 1, namely $\{-1\}$ and all other orbits have size 2.

We make an observation.

Lemma 2.19 *The sum of the elements in an orbit of f is equal to the size of the orbit. The sum of the elements of an orbit of g is equal to -1 times the size of the orbit.*

Definition 2.20 Given a set L of elements of $\mathbb{K}$, we say that L is *closed* whenever $f(L) \subseteq L$ and $g(L) \subseteq L$.

Lemma 2.21 *Let L denote a nonempty closed subset of $\mathbb{K}$. Then L has infinitely many elements.*

Proof: We assume L has finite cardinality n and obtain a contradiction. Because L is closed, it can be partitioned into orbits of f. By Lemma 2.19, the sum of the elements in L is n. Similarly, L can be partitioned into orbits of g. By Lemma 2.19, the sum of the elements in L is $-n$. This implies $n = -n$, so $n = 0$. But L is nonempty, a contradiction. The result follows. $\square$

Definition 2.22 We say that two distinct elements of $\mathbb{K}$ are *adjacent* whenever they are in the same f-orbit or the same g-orbit. A set $L \subseteq \mathbb{K}$ is said to be *connected* whenever the following (i), (ii) hold.

(i) L is nonempty.

(ii) For any partition of L into nonempty subsets M_1 and M_2 there exist $\mu \in M_1$ and $\sigma \in M_2$ such that μ and σ are adjacent.

Lemma 2.23 *Let V denote a finite-dimensional irreducible $\mathcal{A}$-module. Then the action of x on V is diagonalizable. Moreover, the set $L = \{\lambda \in \mathbb{K} | V(\lambda) \neq 0\}$ is connected.*

Proof: Since V is nonzero and finite-dimensional and since the ground field $\mathbb{K}$ is algebraically closed there exists a nonzero vector in V that is an eigenvector for x. Therefore $V(\lambda) \neq 0$ where λ is the corresponding eigenvalue. So L is nonempty.

Let M_1, M_2 denote a partition of L such that M_1 is nonempty and no element of M_1 is adjacent to any element of M_2. Define $W = \sum_{\mu \in M_1} V(\mu)$. Then W is closed under the action of $\mathcal{A}$ by Lemma 2.18, and nonzero because M_1 is nonempty and $V(\lambda) \neq 0$ for all $\lambda \in M_1$.

Since the $\mathcal{A}$-module V is irreducible, we have $V = W$. It follows that $M_1 = L$ and M_2 is empty, so L is connected. Furthermore, we have $V = \sum_{\mu \in L} V(\mu)$, so the action of x on V is diagonalizable. $\square$

We will continue discussing the finite-dimensional irreducible $\mathcal{A}$-modules after a comment.

Lemma 2.24 *Let L denote a finite and connected subset of $\mathbb{K}$ with cardinality $d+1$. Then there is an ordering $\{\theta_i\}_{i=0}^{d}$ of the elements of L such that θ_i, θ_{i+1} are adjacent for $0 \leq i \leq d-1$.*

Proof: We will construct an ordering $\{\theta_i\}_{i=0}^{d}$ of the elements of L. Assume $d \geq 1$; otherwise, the result is trivial. By definition, L is finite and nonempty. Therefore, by Lemma 2.21, L is not closed, so there must be an element $\theta_0 \in L$ such that either $f(\theta_0) \notin L$ or $g(\theta_0) \notin L$. Exactly one of $f(\theta_0), g(\theta_0)$ is in L or else the sets $\{\theta_0\}$ and $L \setminus \{\theta_0\}$ will violate Definition 2.22(ii). If $f(\theta_0) \in L$ define $\{\theta_i\}_{i=0}^{d}$ to be the first $d+1$ elements of the sequence

$$\theta_0, \quad f(\theta_0), \quad g(f(\theta_0)), \quad f(g(f(\theta_0))), \quad g(f(g(f(\theta_0)))), \ldots$$

If $g(\theta_0) \in L$ define $\{\theta_i\}_{i=0}^{d}$ to be the first $d+1$ elements of the sequence

$$\theta_0, \quad g(\theta_0), \quad f(g(\theta_0)), \quad g(f(g(\theta_0))), \quad f(g(f(g(\theta_0)))), \ldots$$

We claim that $\{\theta_i\}_{i=0}^{d}$ is an ordering of the elements of L. Of the integers $0, 1, \ldots, d$, let c denote the maximal one such that $\{\theta_i\}_{i=0}^{c}$ are mutually distinct and in L. We show that $c = d$. Let $M_1 = \{\theta_i\}_{i=0}^{c}$ and $M_2 = L \setminus M_1$. Then M_1, M_2 is a partition of L and no element of M_1 is adjacent to an element of M_2. By Definition 2.22(ii), one of M_1, M_2 is empty. By construction M_1 is nonempty so M_2 is empty and $M_1 = L$. Therefore $c = d$, thus proving the claim. By construction θ_i, θ_{i+1} are adjacent for $0 \leq i \leq d-1$. The result follows. □

Corollary 2.25 *Let V denote a finite-dimensional irreducible $\mathcal{A}$-module. Then there* *is an ordering* $\{\theta_i\}_{i=0}^d$ *of the eigenvalues for the action of* x *on* V *such that* θ_i, θ_{i+1} *are adjacent for* $0 \leq i \leq d-1$.

Proof: Immediate from Lemmas 2.23 and 2.24. $\square$

Let V denote a finite-dimensional irreducible $\mathcal{A}$-module. An ordering $\{\theta_i\}_{i=0}^d$ of elements of $\mathbb{K}$ will be called *standard* whenever θ_i, θ_{i+1} are adjacent for $0 \leq i \leq d-1$. Note that if the ordering $\{\theta_i\}_{i=0}^d$ is standard then so is the ordering $\{\theta_{d-i}\}_{i=0}^d$. When we display our Leonard pairs and Leonard triples it will turn out that the eigenvalues for the action of x on a standard decomposition of V form a standard ordering of the eigenvalues.

Let $\{\theta_i\}_{i=0}^d$ denote a standard ordering of eigenvalues for the action of x on V. For $d \geq 1$,

$$\theta_i = (-1)^i(\theta_0 - 2\varepsilon i) \qquad (0 \leq i \leq d), \tag{2.7}$$

where $\varepsilon = 1$ if $\theta_1 = f(\theta_0)$ and $\varepsilon = -1$ if $\theta_1 = g(\theta_0)$. Note that, for $d = 0$, equation (2.7) holds for $\varepsilon = \pm 1$.

We now consider how an element in $\{\theta_i\}_{i=0}^d$ could be adjacent to an element of $\mathbb{K}$ not among $\{\theta_i\}_{i=0}^d$. Recall that if $\lambda, \mu \in \mathbb{K}$ are adjacent then either $\lambda = f(\mu)$ or $\lambda = g(\mu)$. First assume that $d = 0$. Then θ_0 is adjacent to a number other than θ_0 because $f(\theta_0) \neq g(\theta_0)$. Next assume that $d \geq 1$. By construction, θ_j is adjacent only to $\theta_{j-1}, \theta_{j+1}$ for $1 \leq j \leq d-1$.

Lemma 2.26 *With the above notation, assume* $d \geq 1$. *The following table holds.*

ε	θ_0	Values for f and g	θ_0 is adjacent to
1	-1	$f(\theta_0)=\theta_1, g(\theta_0)=\theta_0$	*only* θ_1
1	$\neq -1$	$f(\theta_0)=\theta_1, g(\theta_0)\notin\{\theta_i\}_{i=0}^d$	θ_1 *and an element of* $\mathbb{K}\setminus\{\theta_i\}_{i=0}^d$
-1	1	$f(\theta_0)=\theta_0, g(\theta_0)=\theta_1$	*only* θ_1
-1	$\neq 1$	$f(\theta_0)\notin\{\theta_i\}_{i=0}^d, g(\theta_0)=\theta_1$	θ_1 *and an element of* $\mathbb{K}\setminus\{\theta_i\}_{i=0}^d$

Define $\varepsilon' = (-1)^{d-1}\varepsilon$ *and note that* $\varepsilon' = 1$ *if* $\theta_{d-1} = f(\theta_d)$ *and* $\varepsilon' = -1$ *if* $\theta_{d-1} = g(\theta_d)$. *Then the following table holds.*

ε'	θ_d	Values for f and g	θ_d is adjacent to
1	-1	$f(\theta_d)=\theta_{d-1}, g(\theta_d)=\theta_d$	*only* θ_{d-1}
1	$\neq -1$	$f(\theta_d)=\theta_{d-1}, g(\theta_d)\notin\{\theta_i\}_{i=0}^d$	θ_{d-1} *and an element of* $\mathbb{K}\setminus\{\theta_i\}_{i=0}^d$
-1	1	$f(\theta_d)=\theta_d, g(\theta_d)=\theta_{d-1}$	*only* θ_{d-1}
-1	$\neq 1$	$f(\theta_d)\notin\{\theta_i\}_{i=0}^d, g(\theta_d)=\theta_{d-1}$	θ_{d-1} *and an element of* $\mathbb{K}\setminus\{\theta_i\}_{i=0}^d$

Proof: We first show the first table holds. Rows $1, 3$: immediate.

Row 2: By construction $f(\theta_0) = \theta_1$. We now show that $g(\theta_0) \notin \{\theta_i\}_{i=0}^d$. By way of contradiction, assume $g(\theta_0) \in \{\theta_i\}_{i=0}^d$. Then there exists an integer i with $0 \leq i \leq d$ such that $g(\theta_0) = \theta_i$. By (2.7), the definition of g, and the fact that $\varepsilon = 1$, we have

$$-2 - \theta_0 = (-1)^i(\theta_0 - 2i). \tag{2.8}$$

First assume i is odd. Then (2.8) reduces to $i = -1$, a contradiction. Next assume i is even. Then (2.8) reduces to $\theta_0 = i - 1$. We now show that $i = 0$. Assume not. Then, by (2.7) with $i - 1$ we find that $\theta_{i-1} = \theta_0$ but $i - 1 \neq 0$ since i is even. This contradicts the fact that $\{\theta_i\}_{i=0}^d$ are distinct. Therefore $i = 0$ so $\theta_0 = -1$, a contradiction. We have now shown that $g(\theta_0) \notin \{\theta_i\}_{i=0}^d$. It follows that θ_0 is adjacent to θ_1 and an element of $\mathbb{K} \setminus \{\theta_i\}_{i=0}^d$.

Row 4: similar to row 2.

To obtain Table 2, apply Table 1 to the standard ordering $\{\theta_{d-i}\}_{i=0}^{d}$ of eigenvalues for the action of x on V. $\square$

We will be discussing five classes of $\mathcal{A}$-modules. The first class will be denoted $B(d)$ (B for "bipartite"). The other four will be denoted $AB(d, n)$ with $n \in \mathbb{I}$ (AB for "almost bipartite"). It will become clear in Section 3.3 why we use these terms. We now introduce the first of these classes.

Lemma 2.27 *Let d denote a nonnegative even integer. There exists an $\mathcal{A}$-module V with basis $\{v_i\}_{i=0}^{d}$ on which x, y, z act as follows. For $0 \leq i \leq d$,*

$$
\begin{aligned}
x.v_i &= (-1)^i (d-2i) v_i, && (2.9)\\
y.v_i &= (d-i+1) v_{i-1} + (i+1) v_{i+1}, && (2.10)\\
z.v_i &= (-1)^{i-1} (d-i+1) v_{i-1} + (-1)^i (i+1) v_{i+1}, && (2.11)
\end{aligned}
$$

where $v_{-1} = 0$ and $v_{d+1} = 0$. The $\mathcal{A}$-module V is irreducible. An $\mathcal{A}$-module isomorphic to V is said to have type $B(d)$.

Proof: One can show that V is an $\mathcal{A}$-module by routine calculation using Definition 2.1. We now show that V is irreducible. Let W denote a nonzero $\mathcal{A}$-submodule of V. We claim that for $0 \leq i \leq d-1$, if $v_i \in W$ then $v_{i+1} \in W$. Let i be given and assume $v_i \in W$. Adding (2.10) to $(-1)^i$ times (2.11), we find $(y + (-1)^i z).v_i = 2(i+1)v_{i+1}$. Because $2(i+1)$ is nonzero, we have $v_{i+1} \in W$ as desired. A similar argument shows that, for $1 \leq i \leq d$, if $v_i \in W$ then $v_{i-1} \in W$.

We now show that there exists an integer j $(0 \leq j \leq d)$ such that $v_j \in W$. For notational convenience define $\theta_i = (-1)^i(d-2i)$ for $0 \leq i \leq d$ and consider the following elements of $\mathcal{A}$:

$$e_i = \prod_{j \neq i} \frac{x - \theta_j 1}{\theta_i - \theta_j} \qquad (0 \leq i \leq d). \tag{2.12}$$

Using (2.9), we obtain $e_i.v_j = \delta_{ij} v_j$ for $0 \leq i, j \leq d$. Here δ_{ij} denotes the Kronecker delta. Recall that $\{v_i\}_{i=0}^d$ is a basis for V. Let $v = c_0 v_0 + c_1 v_1 + \cdots + c_d v_d$ denote a nonzero vector in W. Since v is nonzero there exists j $(0 \leq j \leq d)$ such that c_j is nonzero. Then $e_j.v = c_j v_j$ is a nonzero scalar multiple of v_j, so $v_j \in W$. By this and our preliminary comments we find that $W = V$. $\square$

Remark 2.28 For d odd, an $\mathcal{A}$-module V as in Lemma 2.27 exists, but it is not irreducible. Indeed, we have a direct sum of $\mathcal{A}$-modules $V = V_1 + V_2$ where $V_1 = \text{span}\{v_i + v_{d-i}\}_{i=0}^d$ and $V_2 = \text{span}\{v_i - v_{d-i}\}_{i=0}^d$.

Lemma 2.29 *Let d denote a nonnegative integer. There exists an $\mathcal{A}$-module V with basis $\{v_i\}_{i=0}^d$ on which x, y, z act as follows. For $0 \leq i \leq d$,*

$$x.v_i = (-1)^{d+i}(2d-2i+1)v_i, \tag{2.13}$$

$$y.v_i = (-1)^d(2d-i+2)v_{i-1} + (-1)^d(i+1)v_{i+1}, \tag{2.14}$$

$$z.v_i = (-1)^{i-1}(2d-i+2)v_{i-1} + (-1)^i(i+1)v_{i+1}, \tag{2.15}$$

where $v_{-1} = 0$ and $v_{d+1} = v_d$. The $\mathcal{A}$-module V is irreducible. An $\mathcal{A}$-module isomorphic to V is said to have type $AB(d, 0)$.

Proof: Similar to the proof of Lemma 2.27. □

Lemma 2.30 *Let d denote a nonnegative integer. There exists an $\mathcal{A}$-module V with basis $\{v_i\}_{i=0}^{d}$ on which x, y, z act as follows. For $0 \leq i \leq d$,*

$$\begin{aligned} x.v_i &= (-1)^{d+i}(2d-2i+1)v_i, && (2.16)\\ y.v_i &= (-1)^{d+1}(2d-i+2)v_{i-1} + (-1)^{d+1}(i+1)v_{i+1}, && (2.17)\\ z.v_i &= (-1)^{i}(2d-i+2)v_{i-1} + (-1)^{i+1}(i+1)v_{i+1}, && (2.18) \end{aligned}$$

where $v_{-1} = 0$ and $v_{d+1} = v_d$. The $\mathcal{A}$-module V is irreducible. An $\mathcal{A}$-module isomorphic to V is said to have type $AB(d, x)$.

Proof: Similar to the proof of Lemma 2.27. □

Lemma 2.31 *Let d denote a nonnegative integer. There exists an $\mathcal{A}$-module V with basis $\{v_i\}_{i=0}^{d}$ on which x, y, z act as follows. For $0 \leq i \leq d$,*

$$\begin{aligned} x.v_i &= (-1)^{d+i+1}(2d-2i+1)v_i, && (2.19)\\ y.v_i &= (-1)^{d}(2d-i+2)v_{i-1} + (-1)^{d}(i+1)v_{i+1}, && (2.20)\\ z.v_i &= (-1)^{i}(2d-i+2)v_{i-1} + (-1)^{i+1}(i+1)v_{i+1}, && (2.21) \end{aligned}$$

where $v_{-1} = 0$ and $v_{d+1} = v_d$. The $\mathcal{A}$-module V is irreducible. An $\mathcal{A}$-module isomorphic to V is said to have type $AB(d, y)$.

Proof: Similar to the proof of Lemma 2.27. □

Lemma 2.32 *Let d denote a nonnegative integer. There exists an $\mathcal{A}$-module V with basis $\{v_i\}_{i=0}^d$ on which x, y, z act as follows. For $0 \leq i \leq d$,*

$$\begin{aligned} x.v_i &= (-1)^{d+i+1}(2d-2i+1)v_i, & (2.22) \\ y.v_i &= (-1)^{d+1}(2d-i+2)v_{i-1} + (-1)^{d+1}(i+1)v_{i+1}, & (2.23) \\ z.v_i &= (-1)^{i-1}(2d-i+2)v_{i-1} + (-1)^{i}(i+1)v_{i+1}, & (2.24) \end{aligned}$$

where $v_{-1} = 0$ and $v_{d+1} = v_d$. The $\mathcal{A}$-module V is irreducible. An $\mathcal{A}$-module isomorphic to V is said to have type $AB(d, z)$.

Proof: Similar to the proof of Lemma 2.27. □

Definition 2.33 Let V denote a finite-dimensional irreducible $\mathcal{A}$-module from Lemmas 2.27–2.32. We define the *diameter* of V to be one less than the dimension of V. Thus $\mathcal{A}$-modules of types $B(d)$ and $AB(d, n)$ have diameter d.

Definition 2.34 An $\mathcal{A}$-module V is said to have *type* B when there exists an even integer $d \geq 0$ such that V is of type $B(d)$. The module is said to have *type* AB when there exists an integer $d \geq 0$ and $n \in \mathbb{I}$ such that V is of type $AB(d, n)$.

We comment on Definition 2.34. We will explain in Section 3.3 that on an $\mathcal{A}$-module of type B, the generators x, y, z act as a totally bipartite Leonard triple and on an $\mathcal{A}$-module of type AB, the generators x, y, z act as a totally almost bipartite Leonard triple.

Our goal for the rest of this section is to show that every finite-dimensional irreducible $\mathcal{A}$-module is isomorphic to exactly one $\mathcal{A}$-module from Lemmas 2.27–2.32. As the next result shows, we can distinguish between the five families using the traces of the x, y, z actions.

Theorem 2.35 *Let V denote an $\mathcal{A}$-module contained in one of the five families from Lemmas 2.27–2.32. Then the traces of x, y, z on V are given in the following table.*

	$\mathrm{tr}(x)$	$\mathrm{tr}(y)$	$\mathrm{tr}(z)$
$B(d)$	0	0	0
$AB(d,0)$	$(-1)^d(d+1)$	$(-1)^d(d+1)$	$(-1)^d(d+1)$
$AB(d,x)$	$(-1)^d(d+1)$	$(-1)^{d+1}(d+1)$	$(-1)^{d+1}(d+1)$
$AB(d,y)$	$(-1)^{d+1}(d+1)$	$(-1)^d(d+1)$	$(-1)^{d+1}(d+1)$
$AB(d,z)$	$(-1)^{d+1}(d+1)$	$(-1)^{d+1}(d+1)$	$(-1)^d(d+1)$

Proof: Routine. □

Theorem 2.36 *Every finite-dimensional irreducible $\mathcal{A}$-module is isomorphic to exactly one of the modules from Lemmas 2.27–2.32.*

Proof: We first claim that the modules from Lemmas 2.27–2.32 are mutually non-isomorphic. To do this we refer to the table from Theorem 2.35. If two such $\mathcal{A}$-modules have different values of d, then they have different dimensions and are therefore non-isomorphic. If they have the same value of d, but come from different rows of the table, then they must differ on the traces of at least one of x, y, z and are therefore non-isomorphic. The claim follows.

Let V denote a finite-dimensional irreducible $\mathcal{A}$-module. We will show that V is isomorphic to a module from Lemmas 2.27–2.32. Let $\{\theta_i\}_{i=0}^{d}$ denote a standard ordering of the eigenvalues for the action of x on V. Recall that the ordering $\{\theta_{d-i}\}_{i=0}^{d}$ is also standard.

By Lemma 2.21, there exists an integer r $(0 \leq r \leq d)$ such that θ_r is adjacent to an element of $\mathbb{K}$ not among $\{\theta_i\}_{i=0}^{d}$. By the observation above Lemma 2.26, $r = 0$ or $r = d$. Replacing $\{\theta_i\}_{i=0}^{d}$ with $\{\theta_{d-i}\}_{i=0}^{d}$ as necessary, we may assume, without loss of generality, that $r = 0$.

Now θ_0 is adjacent to an element of $\mathbb{K}$ not among $\{\theta_i\}_{i=0}^{d}$. Recall this number is either $2 - \theta_0$ or $-2 - \theta_0$. When $d \geq 1$, let ε be as below line (2.7). For notational convenience we define ε for $d = 0$. In this case if $\theta_0 = \pm 1$ we define $\varepsilon = \theta_0$ and if $\theta_0 \neq \pm 1$ we define $\varepsilon = 1$. For all values of d, $-2\varepsilon - \theta_0$ is not among $\{\theta_i\}_{i=0}^{d}$. Therefore $V(-2\varepsilon - \theta_0) = 0$. By this and Lemma 2.18 we have $(y - \varepsilon z).V(\theta_0) = 0$.

We have that θ_i satisfies (2.7) for $0 \leq i \leq d$. For notational convenience, we define θ_i by the equation (2.7) for all integers $i \geq 0$.

Let $0 \neq w_0 \in V(\theta_0)$. We have $(y - \varepsilon z).w_0 = 0$. We define vectors $\{w_i\}_{i \geq 1}$ recursively by

$$w_i = \frac{\varepsilon}{2i}(y + (-1)^{i-1}\varepsilon z).w_{i-1} \qquad i \geq 1. \tag{2.25}$$

By Lemma 2.18, $w_i \in V(\theta_i)$ for $i \geq 0$. In (2.25) we replace i with $i + 1$ and rearrange the terms to get

$$(y + (-1)^{i}\varepsilon z).w_i = 2\varepsilon(i + 1)w_{i+1} \qquad i \geq 0. \tag{2.26}$$

We claim that, for $i \geq 0$,

$$(y - (-1)^{i}\varepsilon z).w_i = 2\varepsilon(\varepsilon\theta_0 - i + 1)w_{i-1}, \tag{2.27}$$

where $w_{-1} = 0$. We do this using induction on i. First assume $i = 0$. Then (2.27) holds since both sides are equal to 0. Now assume $i \geq 1$. Using (2.2) we check that

$(y+z)^2-(y-z)^2=4x$. This implies $(y+\varepsilon z)^2-(y-\varepsilon z)^2=\varepsilon 4x$, so

$$((y+\varepsilon z)^2-(y-\varepsilon z)^2).w_{i-1}=\varepsilon 4x.w_{i-1}. \tag{2.28}$$

As we evaluate (2.28), we consider two cases:

Case 1 (i is even): By (2.26), we have $(y-\varepsilon z).w_{i-1}=2\varepsilon i w_i$ and $(y+\varepsilon z).w_{i-2}=2\varepsilon(i-1)w_{i-1}$. By (2.27) and induction we have $(y+\varepsilon z).w_{i-1}=2\varepsilon(\varepsilon\theta_0-i+2)w_{i-2}$. By (2.7) we have $x.w_{i-1}=(2\varepsilon(i-1)-\theta_0)w_{i-1}$. By these comments and (2.28) we routinely obtain (2.27).

Case 2 (i is odd): By (2.26), we have $(y+\varepsilon z).w_{i-1}=2\varepsilon i w_i$ and $(y-\varepsilon z).w_{i-2}=2\varepsilon(i-1)w_{i-1}$. By (2.27) and induction we have $(y-\varepsilon z).w_{i-1}=2\varepsilon(\varepsilon\theta_0-i+2)w_{i-2}$. By (2.7) we have $x.w_{i-1}=(\theta_0-2\varepsilon(i-1))w_{i-1}$. By these comments and (2.28) we routinely obtain (2.27).

We have now verified (2.27). We next claim that, for $i\geq 0$,

$$x.w_i = (-1)^i(\theta_0-2\varepsilon i)w_i, \tag{2.29}$$

$$y.w_i = \varepsilon(\varepsilon\theta_0-i+1)w_{i-1}+\varepsilon(i+1)w_{i+1}, \tag{2.30}$$

$$z.w_i = (-1)^{i-1}(\varepsilon\theta_0-i+1)w_{i-1}+(-1)^i(i+1)w_{i+1}. \tag{2.31}$$

Adding (2.26), (2.27) and dividing the result by 2 we get (2.30). Adding $(-1)^i\varepsilon$ times (2.26) to $(-1)^{i-1}\varepsilon$ times (2.27) and dividing the result by 2 we get (2.31). Combining the fact that $w_i\in V(\theta_i)$ with (2.7) we obtain (2.29). The claim follows.

By (2.29)–(2.31), $\mathrm{span}\{w_i\}_{i\geq 0}$ is closed under the actions of x,y,z, and hence all of $\mathcal{A}$. Because the $\mathcal{A}$-module V is irreducible and $w_0\neq 0$, we have $\mathrm{span}\{w_i\}_{i\geq 0}=V$.

We now show there exists a nonnegative integer t such that $w_t=0$. By construction, the sequences $\{\theta_{2i}\}_{i\geq 0}$ and $\{\theta_{2i+1}\}_{i\geq 0}$ are arithmetic progressions, so $\{\theta_i\}_{i\geq 0}$ has an infinite number of distinct elements. Since V is finite-dimensional, there must be a nonnegative integer i such that θ_i is not among $\{\theta_j\}_{j=0}^{d}$. Observe $V(\theta_i) = 0$ so $w_i = 0$.

Assume $w_t = 0$. We now show that $t \geq d + 1$. Assume $t \leq d$. By (2.26) $w_i = 0$ for all $i \geq t$. Therefore $V = \text{span}\{w_i\}_{i=0}^{t-1}$. By this, and the fact that $\{\theta_i\}_{i=0}^{d}$ are distinct, we have that $V(\theta_d) = 0$, a contradiction. Therefore $t \geq d + 1$.

Let c denote the smallest integer such that $c \geq d$ and $w_{c+1} = 0$. Then $V = \text{span}\{w_i\}_{i=0}^{c}$. Setting $i = c + 1$ in (2.27) and using $w_{c+1} = 0$, we get $2\varepsilon(\varepsilon\theta_0 - c)w_c = 0$, but 2ε and w_c are nonzero, so $\varepsilon\theta_0 - c = 0$. This means $\theta_0 = \varepsilon c$. In particular θ_0 is an integer, so, by (2.7), θ_i are integers for all $i \geq 0$.

Also by (2.7), $\{\theta_i\}_{i\geq 0}$ are either all even or all odd. We now consider these two subcases separately.

Case 1 ($\{\theta_i\}_{i\geq 0}$ are even): Since θ_d is even, it is not equal to ± 1. By rows 2 and 4 of the second table from Lemma 2.26, θ_d is adjacent to an element of $\mathbb{K}$ not among $\{\theta_i\}_{i=0}^{d}$. Therefore θ_{d+1} is not among $\{\theta_i\}_{i=0}^{d}$. This means $w_{d+1} = 0$, so $c \leq d$. By this and the fact that $c \geq d$, we have $c = d$.

From this we draw two conclusions. First of all, using $\theta_0 = \varepsilon c$, we find $\theta_0 = \varepsilon d$. Secondly, we find that the vectors $\{w_i\}_{i=0}^{d}$ form a basis for V. If $\varepsilon = 1$, we define $v_i = w_i$ for $0 \leq i \leq d$. If $\varepsilon = -1$, we define $v_i = (-1)^i w_{d-i}$. In both cases $\{v_i\}_{i=0}^{d}$ is a basis for V. Combining the construction of $\{v_i\}_{i=0}^{d}$, (2.29)–(2.31) and $\theta_0 = \varepsilon d$, we obtain (2.9)–(2.11).

Case 2 ($\{\theta_i\}_{i\geq 0}$ are odd): Recall $\theta_0 = \varepsilon c$ so c is odd. Therefore there exists an integer $k \geq 0$ such that $c = 2k + 1$. We show that $k = d$. By (2.7) and since c is odd we have $\theta_i = \theta_{c-i}$ for $0 \leq i \leq c$. In this equation we set $i = k$ to get $\theta_k = \theta_{k+1}$. Because $\{\theta_i\}_{i=0}^{d}$ are distinct, $k \geq d$.

This implies $c \geq 2d+1 > d$, so $V(\theta_{d+1}) \neq 0$. By Lemma 2.26 rows 5–8, we have $\theta_d = \pm 1$. By (2.7) with $i = d$, we get $\theta_d = (-1)^d(\varepsilon c - 2\varepsilon d)$, so $c - 2d = \pm 1$. This means k is either d or $d-1$, but $k \geq d$. Therefore $k = d$ and hence $c = 2d+1$, so $\theta_i = \varepsilon(-1)^i(2d - 2i + 1)$.

We now have $V = \text{span}\{w_i\}_{i=0}^{2d+1}$. Let $V_0 = \text{span}\{w_i + w_{c-i}\}_{i=0}^{d}$ and $V_1 = \text{span}\{w_i - w_{c-i}\}_{i=0}^{d}$. Observe by construction that $V = V_0 + V_1$ and by (2.29)–(2.31), V_0 and V_1 are closed under the action of $\mathcal{A}$. By these comments and the fact that the $\mathcal{A}$-module V is irreducible and the fact that $V = V_0 + V_1$, either $V_0 = 0$ and $V_1 = V$, or $V_1 = 0$ and $V_0 = V$. In the former case, we define $\delta = -1$ and in the latter case, we define $\delta = 1$. Then $w_i = \delta w_{c-i}$ for $0 \leq i \leq c$ and the vectors $\{w_i\}_{i=0}^{d}$ form a basis for V. Let $v_i = \delta^i w_i$ for $0 \leq i \leq c$. Then $\{v_i\}_{i=0}^{d}$ is a basis for V and $v_i = v_{c-i}$ for $0 \leq i \leq c$.

Using (2.29)–(2.31) and the definition of $\{v_i\}_{i=0}^{d}$, we determine the actions of x, y, z on $\{v_i\}_{i=0}^{d}$ for the different values of ε, δ. Comparing these actions with the data from Lemmas 2.29–2.32, we find that the $\mathcal{A}$-module V is in the isomorphism class displayed in the table below.

	$(-1)^d\delta = 1$	$(-1)^d\delta = -1$
$(-1)^d\varepsilon = 1$	$AB(d, 0)$	$AB(d, x)$
$(-1)^d\varepsilon = -1$	$AB(d, y)$	$AB(d, z)$

□

We comment on Theorem 2.36. In [1], Arik and Kayserilioglu defined the anticommutator spin algebra as a complex unital associative algebra with generators J_1, J_2, J_3 and relations

$$\{J_1, J_2\} = J_3, \qquad \{J_2, J_3\} = J_1, \qquad \{J_3, J_1\} = J_2, \tag{2.32}$$

where $\{A, B\} = AB + BA$. They denoted this algebra $ACSA$. Comparing equations (2.1)–(2.3) and (2.32), we see that, when $\mathbb{K} = \mathbb{C}$, there is an algebra isomorphism $\mathcal{A} \to ACSA$ that sends $x \mapsto 2J_3, y \mapsto 2J_1, z \mapsto 2J_2$. Arik and Kayserilioglu claimed to classify up to isomorphism all finite-dimensional irreducible representations of $ACSA$. However, their result is incorrect; they only found three types of representations instead of the five described in Lemmas 2.27–2.32. What Arik and Kayserilioglu actually classified were the possible eigenvalue sequences for the action of J_3 in a finite-dimensional irreducible representation. But the distinct isomorphism classes $AB(d, 0)$ and $AB(d, x)$ yield the same eigenvalue sequence for the action of J_3. Similarly, the distinct isomorphism classes $AB(d, y)$ and $AB(d, z)$ yield the same eigenvalue sequence for the action of J_3.

We include a result for later use.

Lemma 2.37 *Let V denote an $\mathcal{A}$-module contained in one of the five families from Lemmas 2.27–2.32. Then for $n \in \mathbb{I}$, the trace of h_n on V is given on the following table.*

	$\mathrm{tr}(h_0)$	$\mathrm{tr}(h_x)$	$\mathrm{tr}(h_y)$	$\mathrm{tr}(h_z)$
$B(d)$	0	0	0	0
$AB(d,0)$	$3(-1)^d(d+1)$	$(-1)^{d+1}(d+1)$	$(-1)^{d+1}(d+1)$	$(-1)^{d+1}(d+1)$
$AB(d,x)$	$(-1)^{d+1}(d+1)$	$3(-1)^d(d+1)$	$(-1)^{d+1}(d+1)$	$(-1)^{d+1}(d+1)$
$AB(d,y)$	$(-1)^{d+1}(d+1)$	$(-1)^{d+1}(d+1)$	$3(-1)^d(d+1)$	$(-1)^{d+1}(d+1)$
$AB(d,z)$	$(-1)^{d+1}(d+1)$	$(-1)^{d+1}(d+1)$	$(-1)^{d+1}(d+1)$	$3(-1)^d(d+1)$

Proof: Apply Theorem 2.35 to Definition 2.11. □

2.3 The G-action on the $\mathcal{A}$-modules

Recall the subgroup $G \subseteq$Aut$(\mathcal{A})$ from below Lemma 2.9. Let V denote a finite-dimensional irreducible $\mathcal{A}$-module. In this section we show what happens when we twist V via an element of G.

Definition 2.38 Let V denote an $\mathcal{A}$-module. For $\sigma \in$Aut$(\mathcal{A})$ there exists an $\mathcal{A}$-module structure on V, called V *twisted via* σ that behaves as follows: for all $a \in \mathcal{A}, v \in V$, the vector $a.v$ computed in V twisted via σ coincides with the vector $\sigma^{-1}(a).v$ computed in the original $\mathcal{A}$-module V. Sometimes we abbreviate ${}^{\sigma}V$ for V twisted via σ. Observe that Aut$(\mathcal{A})$ acts on the set of $\mathcal{A}$-modules, with σ sending V to ${}^{\sigma}V$ for all $\sigma \in$Aut$(\mathcal{A})$ and every $\mathcal{A}$-module V. Observe that V and ${}^{\sigma}V$ have the same dimension and that ${}^{\sigma}V$ is irreducible if and only if V is irreducible.

In Section 6.2 we described the set of isomorphism classes of finite-dimensional irreducible $\mathcal{A}$-modules. From Definition 2.38, G acts on this set. We now investigate this G-action. Recall from Definition 2.4 that $\mathbb{I}$ consists of the symbols $0, x, y, z$.

Theorem 2.39 *Let V denote a finite-dimensional irreducible $\mathcal{A}$-module of diameter d and let $\sigma \in G$. Then the following (i), (ii) hold.*

(i) *Assume V is of type $B(d)$. Then ${}^{\sigma}V$ is of type $B(d)$.*

(ii) *Assume V is of type $AB(d, n)$ for some $n \in \mathbb{I}$. Then ${}^{\sigma}V$ is of type $AB(d, \tilde{\sigma}(n))$,*

where $\tilde{\sigma}$ is from Proposition 2.13.

Proof: For $m \in \mathbb{I}$, the action of h_m on ${}^{\sigma}V$ coincides with the action of $\sigma^{-1}(h_m)$ on V. Therefore, the trace of h_m on ${}^{\sigma}V$ is equal to the trace of $\sigma^{-1}(h_m)$ on V. We evaluate the table from Lemma 2.37 using this and Proposition 2.13. The result follows. $\square$

By Theorem 2.39(i) the isomorphism class $B(d)$ is stabilized by everything in G. For $n \in \mathbb{I}$ we now determine the stabilizer in G of the isomorphism class of type $AB(d, n)$. Recall the subgroups $K, S \subseteq G$ from below Lemma 2.9.

Definition 2.40 Recall that the group K consists of the automorphisms of $\mathcal{A}$ that fix each of x, y, z up to sign. Recall that $|K| = 4$ by Lemma 2.9. We define a bijection $\mathbb{I} \to K, n \mapsto \rho_n$ as follows. The automorphism ρ_0 is the identity element of K. For each nonzero $n \in \mathbb{I}$, by Lemma 2.9, there exists a unique element of K that fixes n and changes the sign of the other two elements of $\{x, y, z\}$. We denote this element of K by ρ_n.

Recall the group $\tilde{G}$ of permutations of

$mathbbI$ and the isomorphism $G \to \tilde{G}$ from Proposition 2.13. Note that, for nonzero $n \in \mathbb{I}$, $\tilde{\rho}_n = (0n)(ml)$ where m, l are the remaining nonzero elements of $\mathbb{I}$.

Lemma 2.41 *Let $n \in \mathbb{I}$ and let V denote a finite-dimensional irreducible $\mathcal{A}$-module of type $AB(d, n)$. Then, for $\sigma \in G$, the following (i)–(iii) are equivalent.*

(i) *${}^{\sigma}V$ is of type $AB(d, n)$.*

(ii) *$\tilde{\sigma}$ fixes n.*

(iii) *$\sigma \in \rho_n S \rho_n^{-1}$, where ρ_n is from Definition 2.40.*

Proof: (i)$\Leftrightarrow$(ii): Follows from Proposition 2.13 and Theorem 2.39.

(ii)⇔(iii): First assume that $n = 0$, so that ρ_n is the identity. Then $\rho_n S \rho_n^{-1} = S$. By Lemma 2.15, $\tilde{S}$ consists of the permutations of $\mathbb{I}$ that fix 0. Now assume $n \neq 0$. Then, by Lemma 2.15 and the note after Definition 2.40, we check that $\tilde{\rho}_n \tilde{S} \tilde{\rho}_n^{-1}$ consists of the permutations of $\mathbb{I}$ that fix n. Therefore $\sigma \in \rho_n S \rho_n^{-1}$ if and only if $\tilde{\sigma}$ fixes n. $\square$

Chapter 3

$\mathcal{A}$-modules and Leonard triples

3.1 The primitive idempotents

In this section we determine the eigenvalues for the actions of x, y, z on a finite-dimensional irreducible $\mathcal{A}$-module, and we define the corresponding primitive idempotents.

Definition 3.1 Let V denote a vector space over $\mathbb{K}$ with positive finite dimension and let $b : V \to V$ denote a diagonalizable linear transformation. Let $\{V_i\}_{i=0}^{d}$ denote an ordering of the eigenspaces of b. For $0 \leq i \leq d$ let θ_i denote the eigenvalue for b associated with V_i and define $e_i \in \mathrm{End(V)}$ such that $(e_i - I)V_i = 0$ and $e_i V_j = 0$ for $j \neq i$ $(0 \leq j \leq d)$. Here I denotes the identity of $\mathrm{End}(V)$. We call e_i the *primitive idempotent of* b corresponding to θ_i. Observe that

(i) $\sum_{i=0}^{d} e_i = I$,

(ii) $e_i e_j = \delta_{ij} e_i \qquad (0 \leq i, j \leq d)$,

(iii) $a e_i = \theta_i e_i \qquad (0 \leq i \leq d)$,

(iv) $e_i V = V_i \qquad (0 \leq i \leq d)$.

Note that

$$e_i = \prod_{j \neq i} \frac{b - \theta_j I}{\theta_i - \theta_j} \qquad (0 \leq i \leq d). \tag{3.1}$$

We will now determine the eigenvalues for the actions of x, y, z on a finite-dimensional irreducible $\mathcal{A}$-module V. To do this, we will first determine the eigenvalues for the action of x, y, z when V is of type $B(d)$ or $AB(d, 0)$. Then we will determine the eigenvalues for the actions of x, y, z when V is of type $AB(d, n)$ for nonzero $n \in \mathbb{I}$.

Proposition 3.2 *Let V denote a finite-dimensional irreducible $\mathcal{A}$-module of type $B(d)$ or $AB(d, 0)$. For each of x, y, z the action on V is diagonalizable. The eigenvalues for this action are given in the table below.*

	$B(d)$	$AB(d, 0)$
x	$(-1)^i(d-2i)$	$(-1)^{d+i}(2d-2i+1)$
y	$(-1)^i(d-2i)$	$(-1)^{d+i}(2d-2i+1)$
z	$(-1)^i(d-2i)$	$(-1)^{d+i}(2d-2i+1)$

In the above table, the integer i runs from 0 to d.

Proof: If V is of type $B(d)$, then by Lemma 2.27, the action of x on V is diagonalizable with the desired eigenvalues. If V is of type $AB(d, 0)$, then by Lemma 2.29 the action of x on V is diagonalizable with the desired eigenvalues. We have now verified our assertions for x.

We now verify our assertions for y, z. To that end, let a denote one of y, z. Pick an element $\sigma \in S$ such that $\sigma(a) = x$. By Theorem 2.39 and Lemma 2.41, the twisted module ${}^{\sigma}V$ is of the same type as V. By Definition 2.38, the action of x on ${}^{\sigma}V$ coincides with the action of $\sigma^{-1}(x) = a$ on the untwisted module V. Therefore the actions are both diagonalizable and have the same eigenvalues. $\square$

Proposition 3.3 *Fix a nonzero* $n \in \mathbb{I}$ *and let* V *denote a finite-dimensional irreducible* $\mathcal{A}$*-module of type* $AB(d,n)$*. For each of* x, y, z *the action on* V *is diagonalizable. The eigenvalues for this action are given in the table below.*

	$AB(d,x)$	$AB(d,y)$	$AB(d,z)$
x	$(-1)^{d+i}(2d-2i+1)$	$(-1)^{d+i+1}(2d-2i+1)$	$(-1)^{d+i+1}(2d-2i+1)$
y	$(-1)^{d+i+1}(2d-2i+1)$	$(-1)^{d+i}(2d-2i+1)$	$(-1)^{d+i+1}(2d-2i+1)$
z	$(-1)^{d+i+1}(2d-2i+1)$	$(-1)^{d+i+1}(2d-2i+1)$	$(-1)^{d+i}(2d-2i+1)$

In the above table, the integer i *runs from* 0 *to* d*.*

Proof: Recall the automorphism $\rho = \rho_n$ of $\mathcal{A}$ from Definition 2.40. By Theorem 2.39 and the note at the end of Definition 2.40 we find that the twisted module ${}^{\rho}V$ is of type $AB(d,0)$. Let a denote one of x, y, z and note that, by prop 3.2, the action of a on ${}^{\rho}V$ is diagonalizable with eigenvalues $\{(-1)^{d+i}(2d-2i+1)\}_{i=0}^{d}$. By Definition 2.38, the action of $\rho(a)$ on the untwisted module V is diagonalizable with eigenvalues $\{(-1)^{d+i}(2d-2i+1)\}_{i=0}^{d}$. By Definition 2.40, $\rho(a) = a$ when $n = a$ and $\rho(a) = -a$ when $n \neq a$. The result follows. □

Definition 3.4 Let V denote a finite-dimensional irreducible $\mathcal{A}$-module of type $B(d)$ or $AB(d,0)$. For a among x, y, z and $0 \leq i \leq d$, let θ_i^a denote the i^{th} eigenvalue of a on V from the table in prop 3.2. We define e_i^a to be the primitive idempotent associated with θ_i^a, for the action of a on V.

Definition 3.5 For nonzero $n \in \mathbb{I}$, let V denote a finite-dimensional irreducible $\mathcal{A}$-module of type $AB(d,n)$. For a among x, y, z and $0 \leq i \leq d$, let θ_i^a denote the i^{th}

eigenvalue of a on V from the table in prop 3.3. We define e_i^a to be the primitive idempotent associated with θ_i^a, for the action of a on V.

Recall the notion of standard order from below Corollary 2.25.

Lemma 3.6 *Let a be among x, y, z. With respect to Definitions 3.4, 3.5, the ordering $\{\theta_i^a\}_{i=0}^d$ is standard.*

Proof: Use the tables in props 3.2, 3.3. $\square$

We now present two slightly technical results that will be used in later sections.

Lemma 3.7 *Let V denote a finite-dimensional irreducible $\mathcal{A}$-module of type $B(d)$. Pick an element $\sigma \in S$. Pick a among x, y, z. Then $\sigma(e_i^a) = e_i^{\sigma(a)}$ for $0 \leq i \leq d$.*

Proof: The idempotents $e_i^a, e_i^{\sigma(a)}$ are found using (3.1). By prop 3.2, $\theta_i^a = \theta_i^{\sigma(a)}$. The result follows. $\square$

We set some notation for later use. Let $0 \neq a \in \mathbb{I}$. We define the function $\widehat{a} : \mathbb{I} \to \mathbb{K}$ to by $\widehat{a}(n) = 1$ for $n \in \{0, a\}$ and $\widehat{a}(n) = -1$ for $n \in \mathbb{I} \setminus \{0, a\}$.

Lemma 3.8 *Let V denote a finite-dimensional irreducible $\mathcal{A}$-module of type $AB(d, n)$ and let ρ_n be as in Definition 2.40. Pick an element $\sigma \in S$ and let $\tau = \rho_n \sigma \rho_n^{-1}$. Pick a among x, y, z. Then $\tau(e_i^a) = e_i^{\sigma(a)}$ for $0 \leq i \leq d$.*

Proof: The idempotents $e_i^a, e_i^{\sigma(a)}$ are found using (3.1). We have $\tau(a) = \widehat{a}(n)\widehat{\sigma(a)}(n)\sigma(a)$ by Definition 2.40 and $\theta_i^a = \widehat{a}(n)\widehat{\sigma(a)}(n)\theta_i^{\sigma(a)}$ by props 3.2, 3.3. The result follows. $\square$

3.2 Six bases for V

Let V denote a finite-dimensional irreducible $\mathcal{A}$-module. In this section we will display six bases for V with respect to which the matrices representing x, y, z are attractive. To begin, we will look at the basis for V provided in Lemmas 2.27–2.32.

Lemma 3.9 *Let V denote a finite-dimensional irreducible $\mathcal{A}$-module. Let $\{v_i\}_{i=0}^d$ denote the basis for V from Lemmas 2.27–2.32. Then the following (i)–(iii) hold.*

(i) $v_i \in e_i^x V \qquad (0 \leq i \leq d)$.

(ii) *Let* $v = \sum_{i=0}^d v_i$. *Then* $v \in e_0^y V$.

(iii) $v_i = e_i^x v \qquad (0 \leq i \leq d)$.

Proof: (i) Follows from equations (2.9), (2.16), (2.19), (2.22) and Propositions 3.2, 3.3.

(ii) Follows from equations (2.10), (2.17), (2.20), (2.23) and Propositions 3.2, 3.3.

(iii) By part (ii), $e_i v = \sum_{j=0}^d e_i^x v_j$. Since $v_j \in e_j^x V$ we have $e_i^x v_j = \delta_{ij} v_i$. The result follows. $\square$

Lemma 3.10 *Let V denote a finite-dimensional irreducible $\mathcal{A}$-module. Pick a, b among x, y, z with $a \neq b$. Then the action of $e_i^a e_0^b$ on V is nonzero for $0 \leq i \leq d$.*

Proof: Observe $e_i^x e_0^y V$ is nonzero because it contains the nonzero vector v_i from Lemma 3.9. Now, let $\sigma \in S$ denote the unique automorphism of $\mathcal{A}$ such that $\sigma(a) = x$ and $\sigma(b) = y$. Let $\rho \in K$ denote the identity if V is of type $B(d)$ and ρ_n if V is of type $AB(d, n)$. Let $\tau = \rho\sigma\rho^{-1}$. By Lemma 2.41, the $\mathcal{A}$-modules V and ${}^\tau V$ are isomorphic, so the action of $e^x_i e^y_0$ on ${}^\tau V$ is nonzero. By Lemmas 3.7, 3.8, the action of $e^x_i e^y_0$ on ${}^\tau V$ coincides with the action of $e^a_i e^b_0$ on V. Therefore $e^a_i e^b_0 V \neq 0$ as desired. $\square$

We now obtain six bases for V.

Theorem 3.11 *Let V denote a finite-dimensional irreducible $\mathcal{A}$-module of diameter d. Pick a, b among x, y, z with $a \neq b$. Then, for $0 \neq v^b \in e^b_0 V$ and $0 \leq i \leq d$, $e^a_i v^b$ is nonzero and therefore a basis for $e^a_i V$. Moreover, the sequence $\{e^a_i v^b\}_{i=0}^d$ is a basis for V.*

Proof: We have $\dim(e^b_0 V) = 1$ and $0 \neq v^b \in e^b_0 V$, so v^b spans $e^b_0 V$. Therefore $e^a_i v^b$ spans $e^a_i e^b_0 V$. Now $e^a_i v^b \neq 0$ in view of Lemma 3.10. $\square$

3.3 The matrices representing x, y, z with respect to the six bases

Let V denote a finite-dimensional irreducible $\mathcal{A}$-module. In Theorem 3.11 we displayed six bases for V. In this section we will display the matrices representing x, y, z with respect to these bases.

Lemma 3.12 *Let V denote a finite-dimensional irreducible $\mathcal{A}$-module of type $B(d)$. Let a, b, c denote a permutation of x, y, z. For $0 \leq i \leq d$, the following equations hold on V:*

$$a e^a_i e^b_0 = (-1)^i (d - 2i) e^a_i e^b_0, \qquad (3.2)$$

$$b e^a_i e^b_0 = (d - i + 1) e^a_{i-1} e^b_0 + (i + 1) e^a_{i+1} e^b_0, \qquad (3.3)$$

$$ce_i^a e_0^b \quad = \quad (-1)^{i-1}(d-i+1)e_{i-1}^a e_0^b + (-1)^i (i+1) e_{i+1}^a e_0^b. \tag{3.4}$$

Here $e_{-1}^a = 0$ *and* $e_{d+1}^a = 0$.

Proof: For the case $(a, b, c) = (x, y, z)$ the equations (3.2)–(3.4) are reformations of (2.9)–(2.11) in light of Lemma 3.9. The remaining cases follow from Lemma 3.7. $\square$

Theorem 3.13 *Let* V *denote a finite-dimensional irreducible* $\mathcal{A}$*-module of type* $B(d)$*. Pick* a, b *among* x, y, z *with* $a \neq b$ *and recall the basis* $\{e_i^a v^b\}_{i=0}^d$ *from Theorem 3.11. With respect to this basis, the matrices representing* x, y, z *are described below. Let* c *denote the element of* $\{x, y, z\}$ *other than* a, b*. The matrices are*

$$a : \mathrm{diag}(d, 2-d, d-4, \ldots, 4-d, d-2, -d),$$

$$b : \begin{pmatrix} 0 & d & & & & & & \\ 1 & 0 & d-1 & & & & \mathbf{0} & \\ & 2 & 0 & d-2 & & & & \\ & & 3 & \cdot & \cdot & & & \\ & & & \cdot & \cdot & \cdot & & \\ & & & & \cdot & \cdot & 2 & \\ & \mathbf{0} & & & & d-1 & 0 & 1 \\ & & & & & & d & 0 \end{pmatrix},$$

$$c : \left(\begin{array}{cccccccc}
0 & d & & & & & & \\
1 & 0 & 1-d & & & & \mathbf{0} & \\
 & -2 & 0 & d-2 & & & & \\
 & & 3 & \cdot & \cdot & & & \\
 & & & \cdot & \cdot & \cdot & & \\
 & & & & \cdot & \cdot & 2 & \\
 & \mathbf{0} & & & & d-1 & 0 & -1 \\
 & & & & & & -d & 0
\end{array}\right).$$

Proof: The actions for a, b, c on $\{e_i^a v^b\}_{i=0}^d$ are found by applying equations (3.2)–(3.4) to v^b and recalling that $e_0^b v^b = v^b$. $\square$

Lemma 3.14 *Let V denote a finite-dimensional irreducible $\mathcal{A}$-module of type $AB(d,n)$. Let a, b, c denote a permutation of x, y, z. For $0 \leq i \leq d$, the following equations hold on V:*

$$a e_i^a e_0^b = \widehat{a}(n)(-1)^{d+i}(2d-2i+1)e_i^a e_0^b, \tag{3.5}$$

$$b e_i^a e_0^b = \widehat{b}(n)(-1)^{d}(2d-i+2)e_{i-1}^a e_0^b + \widehat{a}(n)(-1)^{d}(i+1)e_{i+1}^a e_0^b, \tag{3.6}$$

$$c e_i^a e_0^b = \widehat{c}(n)(-1)^{i-1}(2d-i+2)e_{i-1}^a e_0^b + \widehat{c}(n)(-1)^{i}(i+1)e_{i+1}^a e_0^b. \tag{3.7}$$

Here $e_{-1}^a = 0$ and $e_{d+1}^a = e_d^a$. We are using the hat notation from above Lemma 3.8.

Proof: For the case $(a,b,c) = (x,y,z)$, the equations (3.5)–(3.7) are reformations of (2.13)–(2.24) in light of Lemma 3.9. The remaining cases follow from Lemma 3.8. $\square$

Theorem 3.15 *Let V denote a finite-dimensional irreducible $\mathcal{A}$-module of type $AB(d,n)$. Pick a,b among x,y,z with $a \neq b$ and recall the basis $\{e_i^a v^b\}_{i=0}^d$ from Theorem 3.11. With respect to this basis, the matrices representing x,y,z are described below. Let c denote the element of $\{x,y,z\}$ other than a,b. The matrices are*

$$a : \widehat{a}(n)\mathrm{diag}((-1)^d(2d+1), \ldots, 9, -7, 5, -3, 1),$$

$$b : \widehat{b}(n)(-1)^d \begin{pmatrix} 0 & 2d+1 & & & & & & \\ 1 & 0 & 2d & & & & \mathbf{0} & \\ & 2 & 0 & 2d-1 & & & & \\ & & 3 & \cdot & \cdot & & & \\ & & & \cdot & \cdot & \cdot & & \\ & & & & \cdot & \cdot & d+3 & \\ & \mathbf{0} & & & & d-1 & 0 & d+2 \\ & & & & & & d & d+1 \end{pmatrix},$$

$$c : \widehat{c}(n) \begin{pmatrix} 0 & 2d+1 & & & & & & \\ 1 & 0 & -2d & & & & \mathbf{0} & \\ & -2 & 0 & 2d-1 & & & & \\ & & 3 & \cdot & \cdot & & & \\ & & & \cdot & \cdot & \cdot & & \\ & & & & \cdot & \cdot & (-1)^{d-2}(d+3) & \\ & \mathbf{0} & & & & (-1)^{d-2}(d-1) & 0 & (-1)^{d-1}(d+2) \\ & & & & & & (-1)^{d-1}d & (-1)^d(d+1) \end{pmatrix}.$$

Proof: The actions for a, b, c on $\{e_i^a v^b\}_{i=0}^d$ are found by applying equations (3.5)–(3.7) to v^b and recalling that $e_0^b v^b = v^b$. $\square$

Theorem 3.16 *Let V denote a finite-dimensional irreducible $\mathcal{A}$-module. Then the actions of x, y, z on V form a Leonard triple. If V is of type B, then the Leonard triple is totally bipartite, and if V is of type AB, then the Leonard triple is totally almost bipartite.*

Proof: Use Definitions 1.4, 1.5 and the data from Theorems 3.13, 3.15. $\square$

Corollary 3.17 *Let V denote a finite-dimensional irreducible $\mathcal{A}$-module. For any nonzero scalars $\xi, \xi^*, \xi^\varepsilon$ in $\mathbb{K}$, let $A = \xi x$, $A^* = \xi^* y$, $A^\varepsilon = \xi^\varepsilon z$ Then the actions of A, A^*, A^ε form a Leonard triple. If V is of type B, then the Leonard triple is totally bipartite, and if V is of type AB, then the Leonard triple is totally almost bipartite.*

Proof: Immediate. $\square$

Chapter 4

Totally B/AB Leonard pairs and Leonard triples of Bannai/Ito type

4.1 Totally B/AB Leonard pairs of Bannai/Ito type

In Theorem 3.16 and Corollary 3.17 we displayed totally B/AB Leonard triples arising from finite-dimensional irreducible $\mathcal{A}$-modules. In this section we classify the Leonard pairs associated with these Leonard triples. We show that they correspond to a family of totally B/AB Leonard pairs said to have Bannai/Ito type. Using this correspondence we classify the totally B/AB Leonard pairs of Bannai/Ito type with diameter at least 3.

Notation 4.1 Let V denote a vector space over $\mathbb{K}$ with finite positive dimension. Let A, A^* denote a Leonard pair on V. Let $\{v_i\}_{i=0}^d$ denote a basis for V with respect to which A is diagonal and A^* is irreducible tridiagonal. Let $\{v_i^*\}_{i=0}^d$ denote a basis for V with respect to which A^* is diagonal and A is irreducible tridiagonal. For $0 \leq i \leq d$, let θ_i denote the eigenvalue for A associated with v_i and let θ_i^* denote the eigenvalue for A^* associated with v_i^*.

Lemma 4.2 [16, Theorem 1.5] *With reference to Notation 4.1, there exists a sequence of scalars* $\beta, \gamma, \gamma^*, \varrho, \varrho^*, \omega, \eta, \eta^*$ *taken from* $\mathbb{K}$ *such that both*

$$\begin{aligned} A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^* + A^*A) - \varrho A^* &= \gamma^* A^2 + \omega A + \eta I, & (4.1) \\ A^{*2}A - \beta A^*AA^* + AA^{*2} - \gamma^*(A^*A + AA^*) - \varrho^* A &= \gamma A^{*2} + \omega A^* + \eta^* I. & (4.2) \end{aligned}$$

The sequence is uniquely determined by the pair A, A^* *provided the diameter is at least* 3*.*

The equations (4.1), (4.2) are known as the *Askey-Wilson* relations.

Lemma 4.3 [13, Theorem 1.9(v)] *With reference to Notation 4.1, the expressions*

$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}, \qquad \frac{\theta^*_{i-2} - \theta^*_{i+1}}{\theta^*_{i-1} - \theta^*_i} \tag{4.3}$$

are equal and independent of i *for* $2 \leq i \leq d - 1$*.*

Definition 4.4 [16, Definition 4.2] Given scalars β, γ, ϱ in $\mathbb{K}$ we define a two-variable polynomial

$$P(\lambda, \mu) = \lambda^2 - \beta\lambda\mu + \mu^2 - \gamma(\lambda + \mu) - \varrho.$$

Given scalars $\beta, \gamma^*, \varrho^*$ in $\mathbb{K}$ we define a two-variable polynomial

$$P^*(\lambda, \mu) = \lambda^2 - \beta\lambda\mu + \mu^2 - \gamma^*(\lambda + \mu) - \varrho^*.$$

We introduce further notation.

Notation 4.5 With reference to Notation 4.1, for $0 \leq i \leq d$, let a_i (resp. a^*_i) denote the (i, i)-entries for the matrix representing A (resp. A^*) with respect to the basis $\{v^*_i\}_{i=0}^d$ (resp. $\{v_i\}_{i=0}^d$).

We obtain some formulae involving $\{a_i\}_{i=0}^d, \{a_i^*\}_{i=0}^d$.

Lemma 4.6 [16, Corollary 5.2] *Let* $\beta, \gamma, \gamma^*, \varrho, \varrho^*, \omega, \eta, \eta^*$ *denote scalars in* $\mathbb{K}$*. Then with reference to Notation 4.1, 4.5 and Definition 4.4, the following (i), (ii) are equivalent.*

(i) *The sequence* $\beta, \gamma, \gamma^*, \varrho, \varrho^*, \omega, \eta, \eta^*$ *satisfies (4.1) and (4.2).*

(ii) *For* $1 \leq i \leq d$ *both*

$$P(\theta_{i-1}, \theta_i) = 0, \qquad P^*(\theta^*_{i-1}, \theta^*_i) = 0, \tag{4.4}$$

and for $0 \leq i \leq d$ *both*

$$\begin{aligned} a_i^* P(\theta_i, \theta_i) &= \gamma^* \theta_i^2 + \omega \theta_i + \eta, & (4.5)\\ a_i P^*(\theta_i^*, \theta_i^*) &= \gamma \theta_i^{*2} + \omega \theta_i^* + \eta^*. & (4.6) \end{aligned}$$

Let the Leonard pair A, A^* be from Notation 4.1. Observe that A, A^* is bipartite (resp. dual bipartite) if and only if a_i (resp. a_i^*) is equal to 0 for $0 \leq i \leq d$. Similarly, A, A^* is almost bipartite (resp. dual almost bipartite) if and only if exactly one of a_0, a_d (resp. a_0^*, a_d^*) is nonzero and a_i (resp. a_i^*) is equal to 0 for $1 \leq i \leq d-1$.

Lemma 4.7 *With reference to Notation 4.1, the following (i), (ii) hold.*

(i) *Suppose* A, A^* *is bipartite. Then* $\theta_i = -\theta_{d-i}$ *for* $0 \leq i \leq d$.

(ii) *Suppose* A, A^* *is dual bipartite. Then* $\theta_i^* = -\theta^*_{d-i}$ *for* $0 \leq i \leq d$.

Proof: (i) Recall the bases $\{v_i\}_{i=0}^d$ and $\{v_i^*\}_{i=0}^d$ for V from Notation 4.1. Let $s^* \in \mathrm{End}(V)$ be defined by $s^*.v_i^* = (-1)^i v_i^*$ for $0 \leq i \leq d$. By construction, s^* is invertible, so $\{s^*.v_i\}_{i=0}^d$ is a basis for V. Because the matrix representing A with respect to the basis

$\{v_i^*\}_{i=0}^d$ is bipartite tridiagonal, we have $As^* = -s^*A$. Recall that v_i is an eigenvector for A with eigenvalue θ_i for $0 \leq i \leq d$. From these facts, $s^*.v_i$ is an eigenvector for A with eigenvalue $-\theta_i$ for $0 \leq i \leq d$. Therefore the matrix representing A with respect to the basis $\{s^*.v_i\}_{i=0}^d$ is diagonal. Because the matrix representing A^* with respect to the basis $\{v_i^*\}_{i=0}^d$ is diagonal, we have $A^*s^* = s^*A^*$. Recall that the matrix representing A^* with respect to the basis $\{v_i\}_{i=0}^d$ is irreducible tridiagonal. From these facts, the matrix representing A^* with respect to the basis $\{s^*.v_i\}_{i=0}^d$ is irreducible tridiagonal. Therefore $\{s^*.v_i\}_{i=0}^d$ is a standard basis for V and $\{\mathbb{K}s^*.v_i\}_{i=0}^d$ is a standard decomposition of V. Recall that $\{\mathbb{K}v_i\}_{i=0}^d$ and $\{\mathbb{K}v_{d-i}\}_{i=0}^d$ are the only decompositions of V that are standard with respect to the Leonard pair A, A^*. Therefore $\{\mathbb{K}s^*.v_i\}_{i=0}^d$ is equal to either $\{\mathbb{K}v_i\}_{i=0}^d$ or $\{\mathbb{K}v_{d-i}\}_{i=0}^d$. By applying A to the bases $\{v_i\}_{i=0}^d$, $\{v_i^*\}_{i=0}^d$ and $\{s^*.v_i\}_{i=0}^d$ for V, we routinely find that the decompositions $\{\mathbb{K}s^*.v_i\}_{i=0}^d, \{\mathbb{K}v_{d-i}\}_{i=0}^d$ coincide. It follows that $\theta_i = -\theta_{d-i}$ for $0 \leq i \leq d$, as desired.

(ii) Similar. $\square$

Recall the Askey-Wilson relations from lines (4.1), (4.2). We now refine these relations in the case in which A, A^* is totally B/AB.

Theorem 4.8 *With reference to Notation 4.1, assume A, A^* is totally bipartite. Then the scalars $\gamma, \gamma^*, \omega, \eta, \eta^*$ from Lemma 4.2 are all zero provided the diameter $d \geq 2$.*

Proof: By construction, the scalar a_i^* from Notation 4.5 is equal to zero for $0 \leq i \leq d$. By this and Lemma 4.6, the left-hand side of (4.5) is equal to zero. Note that the right-hand side of (4.5) involves a quadratic polynomial $\gamma^*\lambda^2 + \omega\lambda + \eta$. For this polynomial, θ_i is a root for $0 \leq i \leq d$. Because $d \geq 2$, this polynomial has at least three distinct roots and is therefore zero. Therefore $\gamma^* = \omega = \eta = 0$. By a similar argument using equation (4.6), we find that $\gamma = \eta^* = 0$. $\square$

Theorem 4.9 *With reference to Notation 4.1, assume A, A^* is totally almost bipartite. Then the scalars $\gamma, \gamma^*, \omega, \eta, \eta^*$ from Lemma 4.2 are all zero provided the diameter $d \geq 3$.*

Proof: By construction, the scalar a_i^* from Notation 4.5 is equal to zero for $1 \leq i \leq d-1$ and exactly one of a_0^*, a_d^* is nonzero. Replacing $\{a_i^*\}_{i=0}^d$ by $\{a_{d-i}^*\}_{i=0}^d$ as necessary, we may assume, without loss of generality, that $a_0^* = 0$. Then $a_i^* = 0$ for $0 \leq i \leq d-1$. By this and Lemma 4.6, the left-hand side of (4.5) is equal to zero for $0 \leq i \leq d-1$. Recall that the right-hand side of (4.5) involves a quadratic polynomial $\gamma^*\lambda^2 + \omega\lambda + \eta$. For this polynomial, θ_i is a root for $0 \leq i \leq d-1$. Because $d \geq 3$, this polynomial has at least three distinct roots and is therefore zero. Therefore $\gamma^* = \omega = \eta = 0$. By a similar argument using equation (4.6), we find that $\gamma = \eta^* = 0$. $\square$

We will show that, when A, A^* is totally B/AB, the scalars ϱ, ϱ^* are nonzero provided the diameter is not too small. The following Lemma will be used to show this.

Lemma 4.10 *With reference to Notation 4.1 and Lemma 4.2, the following (i), (ii) hold.*

(i) *Suppose the parameters γ, ϱ from Lemma 4.2 are zero. Then $\{\theta_i\}_{i=0}^d$ is a geometric progression. Let q denote the common value of θ_i/θ_{i-1}. Then $q + q^{-1} = \beta$, where β is from Lemma 4.2.*

(ii) *Suppose the parameters* γ^*, ϱ^* *from Lemma 4.2 are zero. Then* $\{\theta^*_i\}_{i=0}^d$ *is a geometric progression. Let* q *denote the common value of* $\theta^*_i/\theta^*_{i-1}$. *Then* $q + q^{-1} = \beta$, *where* β *is from Lemma 4.2.*

Proof: (i) Let $r \in \mathbb{K}$ denote a solution to $r + r^{-1} = \beta$. Substituting $r + r^{-1}$ for β in the left-hand equation of (4.4), and setting $\gamma = \varrho = 0$, we find that, for $1 \leq i \leq d$,

$$\begin{aligned} 0 &= \theta_{i-1}^2 - (r + r^{-1})\theta_{i-1}\theta_i + \theta_i^2 \\ &= (\theta_i - r\theta_{i-1})(\theta_i - r^{-1}\theta_{i-1}), \end{aligned}$$

so $\theta_i = r\theta_{i-1}$ or $\theta_i = r^{-1}\theta_{i-1}$. Since $\{\theta_i\}_{i=0}^d$ are mutually distinct, either $\theta_i = r\theta_{i-1}$ for1 $\leq i \leq d$ or $\theta_i = r^{-1}\theta_{i-1}$ for $1 \leq i \leq d$. In the former case, set $q = r$ and in the latter case set $q = r^{-1}$. The result follows.

(ii) Similar. ☐

Lemma 4.11 *With reference to Notation 4.1, assume* A, A^* *is totally bipartite and the diameter* $d \geq 2$. *Then the scalars* ϱ, ϱ^* *from Lemma 4.2 are nonzero.*

Proof: By way of contradiction, assume that at least one of ϱ, ϱ^* is zero. Without loss of generality, we may assume $\varrho = 0$. By Lemma 4.7, $\theta_0 = -\theta_d$ and $\theta_1 = -\theta_{d-1}$. By Theorem 4.8 and Lemma 4.10(i), there exists a nonzero scalar q such that $q = \theta_i/\theta_{i-1}$ for $1 \leq i \leq d$ and $q + q^{-1} = \beta$, where β is from Lemma 4.2. Observe $\theta_i = q^i\theta_0$ for $0 \leq i \leq d$. By these comments, $\theta_0 = -q^d\theta_0$ and $q\theta_0 = -q^{d-1}\theta_0$. We may now argue $\theta_2 = q^2\theta_0 = -q^d\theta_0 = \theta_0$ for a contradiction. The result follows. ☐

Lemma 4.12 *With reference to Notation 4.1, assume* A, A^* *is totally almost bipartite and* $d \geq 3$. *Then at least one of* $P(\theta_0, \theta_0), P(\theta_d, \theta_d)$ *is zero and at least one of* $P^*(\theta^*_0, \theta^*_0), P^*(\theta^*_d, \theta^*_d)$ *is zero.*

Proof: By Theorem 4.9, the right-hand sides of equations (4.5), (4.6) equal zero for $0 \leq i \leq d$. By construction, one of a_0, a_d is nonzero. If $a_0 \neq 0$ then $P(\theta_0, \theta_0) = 0$ and if $a_d \neq 0$ then $P(\theta_d, \theta_d) = 0$. Similarly, one of a^*_0, a^*_d is nonzero. If $a^*_0 \neq 0$ then $P^*(\theta^*_0, \theta^*_0) = 0$ and if $a^*_d \neq 0$ then $P^*(\theta^*_d, \theta^*_d) = 0$. □

Lemma 4.13 *With reference to Notation 4.1, assume* A, A^* *is totally almost bipartite and the diameter* $d \geq 3$. *Then the scalars* ϱ, ϱ^* *from Lemma 4.2 are nonzero.*

Proof: By way of contradiction, assume that at least one of ϱ, ϱ^* is zero. Without loss of generality, we may assume $\varrho = 0$. By Lemma 4.12, at least one of $P(\theta_0, \theta_0), P(\theta_d, \theta_d)$ is zero. Replacing $\{\theta_i\}_{i=0}^d$ with $\{\theta_{d-i}\}_{i=0}^d$ as necessary, we may assume, without loss of generality, that $P(\theta_d, \theta_d) = 0$. Evaluating using Definition 4.4 and Theorem 4.8, we find $(2 - \beta)\theta_d^2 = 0$. Therefore, either $\beta = 2$ or $\theta_d = 0$. In the first case, the θ_i are equal for $0 \leq i \leq d$, a contradiction. In the second case, Lemma 4.10(i) is contradicted. The result follows. □

Theorem 4.14 *Let* A, A^* *denote a totally B/AB Leonard pair. Then there exists a sequence of scalars* $\beta, \varrho, \varrho^*$ *in* $\mathbb{K}$ *with* ϱ, ϱ^* *nonzero such that both*

$$A^2A^* - \beta AA^*A + A^*A^2 = \varrho A^*, \tag{4.7}$$

$$A^{*2}A - \beta A^*AA^* + AA^{*2} = \varrho^* A. \tag{4.8}$$

Proof: Note that equations (4.7), (4.8) are what we get upon setting $\gamma, \gamma^*, \omega, \eta, \eta^*$ equal to zero in equations (4.1), (4.2). First assume $d \geq 3$. Then (4.7), (4.8) hold by Theorems 4.8, 4.9 and Lemmas 4.11, 4.13. Now assume $d \leq 2$. We routinely verify the assertion using Lemmas 4.6 and 4.12. $\square$

In [15, Example 5.14] a Leonard pair is said to be of *Bannai/Ito type* whenever the common value of (4.3) is equal to -1. When this occurs, the parameter β from Lemma 4.2 is equal to -2 and the relations (4.7), (4.8) become

$$A^2A^* + 2AA^*A + A^*A^2 = \varrho A^*, \tag{4.9}$$

$$A^{*2}A + 2A^*AA^* + AA^{*2} = \varrho^* A. \tag{4.10}$$

When ϱ, ϱ^* are equal to 4, equations (4.9), (4.10) become equations (2.4), (2.5). Consequently the Leonard pairs associated with the Leonard triple from Theorem 3.16 are of Bannai/Ito type.

Note that, for $d \geq 3$, the scalar β from Lemma 4.2 is uniquely determined and is equal to the scalar β from Theorem 4.14. However, when $d \leq 2$, β not unique. In this case it is conceivable that equations (4.1), (4.2) are satisfied with $\beta = -2$ and equations (4.7), (4.8) are satisfied with $\beta \neq -2$, but equations (4.9), (4.10) are not satisfied. Because of this, some of the following theorems assume $d \geq 3$.

Theorem 4.15 *Let V denote a finite-dimensional irreducible $\mathcal{A}$-module and let ξ, ξ^* denote nonzero scalars in $\mathbb{K}$. Then $\xi x, \xi^* y$ act on V as a Leonard pair of Bannai/Ito type. If V is of type B then the Leonard pair is totally bipartite. If V is of type AB then the Leonard pair is totally almost bipartite.*

Proof: Immediate. □

Theorem 4.16 *Let V denote a vector space over $\mathbb{K}$ with finite dimension at least 4 and let A, A^* denote a totally B/AB Leonard pair on V of Bannai/Ito type. Then there exists an irreducible $\mathcal{A}$-module structure on V and nonzero scalars ξ, ξ^* such that x, y act on V as $A\xi^{-1}, A^*\xi^{*-1}$ respectively. If A, A^* is totally bipartite then the $\mathcal{A}$-module V is of type B and if A, A^* is totally almost bipartite then the $\mathcal{A}$-module V is of type AB. The $\mathcal{A}$-module structure of V is uniquely determined by the scalars ξ, ξ^* and each of the scalars ξ, ξ^* is unique up to sign.*

Proof: Since $\mathbb{K}$ is algebraically closed, there exist scalars ξ, ξ^* in $\mathbb{K}$ such that $4\xi^2 = \varrho$, $4\xi^{*2} = \varrho^*$. Because the scalars ϱ, ϱ^* are nonzero, the scalars ξ, ξ^* are nonzero. By equations (4.9), (4.10), we find that $A\xi^{-1}, A^*\xi^{*-1}$ satisfy equations (2.4), (2.5). Therefore there exists an $\mathcal{A}$-module structure such that x, y act on V as $A\xi^{-1}, A^*\xi^{*-1}$ respectively.

The proof that the $\mathcal{A}$-module V is irreducible is similar to the proof of Lemma 2.27. By Theorems 2.35, 2.36, we find that the $\mathcal{A}$-module V is of type B whenever A, A^* is totally bipartite and of type AB whenever A, A^* is totally almost bipartite.

Given scalars $\xi, \xi^* \in \mathbb{K}$, there is at most one $\mathcal{A}$-module structure on V such that A, A^* act as $\xi x, \xi^* y$ respectively. Because $\varrho = 4\xi^2$ and $\varrho^* = 4\xi^{*2}$ the choices of ξ, ξ^* are each unique up to sign. □

Theorem 4.16 implies the following result that may be of independent interest.

Corollary 4.17 *Let A, A^* denote a totally bipartite (resp. totally almost bipartite)*

Leonard pair of Bannai/Ito type with diameter at least 3*. Let* V *denote the underlying vector space. Then there exists a linear transformation* $A^{\varepsilon} \in \mathrm{End}(V)$ *such that* A, A^*, A^{ε} *is a totally bipartite (resp. totally almost bipartite) Leonard triple.*

Proof: Let ξ, ξ^* be as in Theorem 4.16 and let V be given the corresponding $\mathcal{A}$-module structure. Let $\xi^{\varepsilon} \in \mathbb{K}$ be nonzero and let A^{ε} denote the action on V of $\xi^{\varepsilon} z$. By Corollary 3.17, A, A^*, A^{ε} is a totally bipartite (resp. totally almost bipartite) Leonard triple as desired. □

We now classify the totally B/AB Leonard pairs of Bannai/Ito type with diameter $d \geq 3$. We will be using the notion of isomorphism of Leonard pairs. For a precise definition, see [14, Definition 3.4].

Theorem 4.18 *Let* d *denote an integer at least* 3 *and let* ϱ, ϱ^* *denote scalars in* $\mathbb{K}$*. Then the following (i), (ii) are equivalent.*

- (i) *There exists a totally bipartite Leonard pair* A, A^* *of Bannai/Ito type with diameter* d *that satisfies equations (4.9), (4.10).*
- (ii) *The integer* d *is even and the scalars* ϱ, ϱ^* *are nonzero.*

Moreover, assume (i), (ii) hold. Then the Leonard pair A, A^* *is unique up to isomorphism.*

Proof: (ii)⇒(i): Let V denote a finite-dimensional irreducible $\mathcal{A}$-module of type $B(d)$. Let ξ, ξ^* in $\mathbb{K}$ satisfy $4\xi^2 = \varrho$ and $4\xi^{*2} = \varrho^*$. Let A, A^* denote the actions on V of $\xi x, \xi^* y$ respectively. Then, by Theorem 4.15, A, A^* is a totally bipartite Leonard pair of Bannai/Ito type with diameter d that satisfies equations (4.9), (4.10).

(i)⇒(ii): Let V denote the vector space underlying A, A^*. By Theorem 4.16, there exists an $\mathcal{A}$-module structure on V of type B and nonzero scalars ξ, ξ^* such that A, A^* act as $\xi x, \xi^* y$ respectively. The dimension of V is $d+1$, so V is of type $B(d)$. By this and Theorem 2.36, d is even. We routinely find that $\varrho = 4\xi^2$ and $\varrho^* = 4\xi^{*2}$, so ϱ, ϱ^* are nonzero.

Now assume (i), (ii) hold. We show the Leonard pair A, A^* is unique up to isomorphism. Let B, B^* denote a totally bipartite Leonard pair of Bannai/Ito type with diameter d that satisfies equations (4.9), (4.10). We show the Leonard pairs A, A^* and B, B^* are isomorphic. Let V denote the vector space underlying A, A^* and let W denote the vector space underlying B, B^*. By Theorem 4.16, there exist scalars ξ, ξ^* in $\mathbb{K}$ and an $\mathcal{A}$-module structure on V such that A, A^* act on V as $\xi x, \xi^* y$ respectively. Similarly, there exist scalars $\xi', \xi^{*\prime}$ in $\mathbb{K}$ and an $\mathcal{A}$-module structure on W such that B, B^* act on W as $\xi' x, \xi^{*\prime} y$ respectively. The $\mathcal{A}$-modules V, W are both of type $B(d)$ and hence isomorphic. By Theorem 4.16 the scalars ξ, ξ^* are unique up to sign, as are the scalars $\xi', \xi^{*\prime}$. Moreover, both $4\xi^2, 4\xi'^2$ are equal to ϱ and both $4\xi^{*2}, 4\xi^{*\prime 2}$ are equal to ϱ^*. Changing the signs of ξ, ξ^* as necessary, we may assume, without loss of generality, that $\xi = \xi'$ and $\xi^* = \xi^{*\prime}$. Let $\phi : V \to W$ denote an isomorphism of $\mathcal{A}$-modules. Then $\phi \circ A = \xi(\phi \circ x) = \xi(x \circ \phi) = B \circ \phi$ and $\phi \circ A^* = \xi^*(\phi \circ y) = \xi^*(y \circ \phi) = B^* \circ \phi$ on V. These equations show the Leonard pairs A, A^* and B, B^* are isomorphic. □

As a consequence of Lemma 4.2 and Theorem 4.18, one obtains the following Corollary.

Corollary 4.19 *There is a bijection between the set of isomorphism classes of totally* *bipartite Leonard pairs of Bannai/Ito type with diameter at least* 3 *and sequences* d, ϱ, ϱ^* *of scalars such that* $d \geq 3$ *is an even integer and* ϱ, ϱ^* *are nonzero.*

Theorem 4.20 *Let* d *denote an integer at least* 3 *and let* τ, τ^* *denote scalars in* $\mathbb{K}$. *Then the following (i), (ii) are equivalent.*

(i) *There exists a totally almost bipartite Leonard pair* A, A^* *of Bannai/Ito type with diameter* d, $\mathrm{tr}(A) = \tau$ *and* $\mathrm{tr}(A^*) = \tau^*$.

(ii) *The scalars* τ, τ^* *are nonzero.*

Moreover, assume (i), (ii) hold. Then the Leonard pair A, A^* *is unique up to isomorphism.*

Proof: (ii)⇒(i): Let V denote a finite-dimensional irreducible $\mathcal{A}$-module of type $AB(d, 0)$. Let A, A^* denote the actions on V of $\tau(-1)^d(d+1)^{-1}x, \tau^*(-1)^d(d+1)^{-1}y$ respectively. Then, by Theorem 4.15, A, A^* is a totally almost bipartite Leonard pair of Bannai/Ito type with diameter d. By Theorem 2.35, $\mathrm{tr}(A) = \tau$ and $\mathrm{tr}(A^*) = \tau^*$.

(i)⇒(ii): Immediate from Definition 1.5.

Now assume (i), (ii) hold. We show the Leonard pair A, A^* is unique up to isomorphism. Let B, B^* denote a totally almost bipartite Leonard pair of Bannai/Ito type with diameter d such that $\mathrm{tr}(B) = \tau$ and $\mathrm{tr}(B^*) = \tau^*$. We show the Leonard pairs A, A^* and B, B^* are isomorphic. Let V denote the vector space underlying A, A^* and let W denote the vector space underlying B, B^*. By Theorem 4.16, there exist scalars ξ, ξ^* in $\mathbb{K}$ and an $\mathcal{A}$-module structure on V such that A, A^* act on V as $\xi x, \xi^* y$ respectively. Similarly, there exist scalars $\xi', \xi^{*\prime}$ in $\mathbb{K}$ and an $\mathcal{A}$-module structure on W such that B, B^* act on W as $\xi' x, \xi^{*\prime} y$ respectively. The $\mathcal{A}$-module V is of type $AB(d, n)$ and the $\mathcal{A}$-module W is of type $AB(d, n')$ for some $n, n' \in \mathbb{I}$. By Theorem 2.35 together with $\mathrm{tr}(A) = \mathrm{tr}(B)$ and $\mathrm{tr}(A^*) = \mathrm{tr}(B^*)$, we obtain $\xi = \pm\xi'$ and $\xi^* = \pm\xi^{*\prime}$, with equality if and only if $n = n'$. By Theorem 4.16, our choice of scalars ξ, ξ^* was unique up to sign. Changing the signs of ξ, ξ^* as necessary, we may assume, without loss of generality, that $\xi = \xi'$, $\xi^* = \xi^{*\prime}$, and hence $n = n'$. Then the $\mathcal{A}$-modules V and W are isomorphic. Let $\phi : V \to W$ denote an isomorphism of $\mathcal{A}$-modules. Then $\phi \circ A = \xi(\phi \circ x) = \xi(x \circ \phi) = B \circ \phi$ and $\phi \circ A^* = \xi^*(\phi \circ y) = \xi^*(y \circ \phi) = B^* \circ \phi$ on V. These equations show the Leonard pairs A, A^* and B, B^* are isomorphic. $\square$

As a consequence of Theorem 4.20 and the fact that every linear transformation has a unique trace, one obtains the following Corollary.

Corollary 4.21 *There is a bijection between the set of isomorphism classes of totally almost bipartite Leonard pairs of Bannai/Ito type with diameter at least* 3 *and sequences* d, τ, τ^* *of scalars such that* $d \geq 3$ *is an integer and* τ, τ^* *are nonzero.*

We comment on why Theorems 4.18 and 4.20 use different parameters. Let A, A^* denote a totally almost bipartite Leonard pair of Bannai/Ito type. Let the scalars τ, τ^* be from from Theorem 4.20 and let the scalars ϱ, ϱ^* be from equations (4.9), (4.10). Then

$$\varrho = \frac{4\tau^2}{(d+1)^2}, \qquad \varrho^* = \frac{4\tau^{*2}}{(d+1)^2}.$$

Given an integer d at least 3 and nonzero scalars ϱ, ϱ^*, the scalars τ, τ^* that satisfy the above equation are each unique up to sign. Therefore, there are exactly 4 isomorphism classes of totally almost bipartite Leonard pairs of Bannai/Ito type with diameter d that satisfy equations (4.9), (4.10).

In Theorem 4.28 we display a correspondence between totally B/AB Leonard triples of Bannai/Ito type and $\mathcal{A}$-modules. To do this, we present further results about Leonard pairs. Let A, A^* denote a Leonard pair.

Lemma 4.22 *With reference to Notation 4.1, let A, A^* be totally B/AB and of Bannai/Ito type with diameter $d \geq 3$. Then the elements*

$$A^2A^*, \qquad AA^*A, \qquad A^*A^2, \tag{4.11}$$

are linearly independent.

Proof: Let s, t, u be scalars in $\mathbb{K}$ satisfying $sA^2A^* + tAA^*A + uA^*A^2 = 0$. We show that each of s, t, u is zero. With reference to Notation 4.1, let E_i^* denote the primitive idempotent corresponding to θ_i^* for $0 \leq i \leq d$. Then the following hold:

$$sE_0^*A^2A^*E_0^* + tE_0^*AA^*AE_0^* + uE_0^*A^*A^2E_0^* = 0, \tag{4.12}$$

$$sE_0^*A^2A^*E_2^* + tE_0^*AA^*AE_2^* + uE_0^*A^*A^2E_2^* = 0, \tag{4.13}$$

$$sE_2^*A^2A^*E_0^* + tE_2^*AA^*AE_0^* + uE_2^*A^*A^2E_0^* = 0. \tag{4.14}$$

With respect to the basis $\{v_i^*\}_{i=0}^d$ from Notation 4.1, the matrix representing A^* is diagonal with (i,i)-entry θ_i^* for $0 \leq i \leq d$. For $0 \leq i \leq d$, the matrix representing E_i^* has (i,i)-entry 1 and all other entries zero. With respect to Notation 4.5, the matrix representing A is irreducible tridiagonal with (i,i) entry a_i for $0 \leq i \leq d$. The matrix representing A is either bipartite or almost bipartite. Therefore at most one of a_0, a_d is nonzero and $a_i = 0$ for $1 \leq i \leq d-1$. Reversing the order of the eigenvalues as necessary we may assume, without loss of generality, that $a_0 = 0$. Based on this information, we routinely find that equations (4.12)–(4.14) reduce to

$$\begin{aligned}
(s\theta^*_0 + t\theta^*_1 + u\theta^*_0)E^*_0A^2E^*_0 &= 0, && (4.15)\\
(s\theta^*_2 + t\theta^*_1 + u\theta^*_0)E^*_0A^2E^*_2 &= 0, && (4.16)\\
(s\theta^*_0 + t\theta^*_1 + u\theta^*_2)E^*_2A^2E^*_0 &= 0. && (4.17)
\end{aligned}$$

Moreover, $E^*_0A^2E^*_0, E^*_0A^2E^*_2, E^*_2A^2E^*_0$ are all nonzero, resulting in the following equations:

$$\begin{aligned}
s\theta^*_0 + t\theta^*_1 + u\theta^*_0 &= 0, && (4.18)\\
s\theta^*_2 + t\theta^*_1 + u\theta^*_0 &= 0, && (4.19)\\
s\theta^*_0 + t\theta^*_1 + u\theta^*_2 &= 0. && (4.20)
\end{aligned}$$

We view (4.18)–(4.20) as a system of linear equations in the indeterminates s, t, u. The determinant of the coefficient matrix is $-\theta^*_1(\theta^*_0 - \theta^*_2)^2$. Because $\{\theta^*_i\}_{i=0}^d$ are distinct, we have $\theta^*_0 - \theta^*_2 \neq 0$. Combining Theorem 4.16, the eigenvalue data for y from Propositions 3.2, 3.3 and the fact that $d \geq 3$, we find that $\theta^*_1 \neq 0$. From this, we routinely find that $s = 0, t = 0, u = 0$ is the only solution to the system (4.18)–(4.20). Therefore, (4.11) are linearly independent as desired. $\square$

With reference to Notation 4.1, let E_i, E^*_i denote the primitive idempotents corresponding to θ_i, θ^*_i respectively for $0 \leq i \leq d$. Let $\mathcal{X}$ denote the $\mathbb{K}$-subspace of V consisting of the $X \in \mathrm{End}(V)$ such that both

$$\begin{aligned}
E_iXE_j &= 0 \qquad \text{if } |i-j| > 1, && (4.21)\\
E^*_iXE^*_j &= 0 \qquad \text{if } |i-j| > 1, && (4.22)
\end{aligned}$$

for $0 \leq i, j \leq d$. Observe that, if the Leonard pair A, A^* is associated with a Leonard triple A, A^*, A^ε, then $A^\varepsilon \in \mathcal{X}$.

Lemma 4.23 [11, Theorem 1.5] *The space $\mathcal{X}$ is spanned by*

$$I, A, A^*, AA^*, A^*A. \tag{4.23}$$

Moreover, (4.23) is a basis for $\mathcal{X}$ provided $d \geq 2$.

4.2 Totally B/AB Leonard triples of Bannai/Ito type

In Section 4.1, we classified the Leonard pairs arising from finite-dimensional irreducible $\mathcal{A}$-modules. In this section, we classify the Leonard triples arising from finite-dimensional irreducible $\mathcal{A}$-modules. We show that they correspond to a family of totally B/AB Leonard triples said to have Bannai/Ito type. From this correspondence we classify the totally B/AB Leonard triples of Bannai/Ito type with diameter at least 3.

Notation 4.24 Let V denote a vector space over $\mathbb{K}$ with finite positive dimension. Let A, A^*, A^ε denote a Leonard triple on V. Let $\{v_i\}_{i=0}^d$ denote a basis for V under which A is diagonal and A^*, A^ε are irreducible tridiagonal. Let $\{v_i^*\}_{i=0}^d$ denote a basis for V under which A^* is diagonal and A^ε, A are irreducible tridiagonal. Let $\{v_i^\varepsilon\}_{i=0}^d$ denote a basis for V under which A^ε is diagonal and A, A^* are irreducible tridiagonal. For $0 \leq i \leq d$, let θ_i denote the eigenvalue for A associated with v_i, let θ_i^* denote the eigenvalue for A^* associated with v_i^* and let θ_i^ε denote the eigenvalue for A^ε associated with v_i^ε.

Definition 4.25 We say that a Leonard triple A, A^*, A^ε is of *Bannai/Ito type* whenever all of the associated Leonard pairs are of Bannai/Ito type.

Lemma 4.26 *Let A, A^*, A^ε denote a Leonard triple. If any of the six Leonard pairs associated with A, A^*, A^ε is of Bannai/Ito type, then the Leonard triple A, A^*, A^ε is of Bannai/Ito type.*

Proof: Assume otherwise. If the Leonard pair A, A^* is of Bannai/Ito type then so is A^*, A. Therefore we may assume, without loss of generality, that the Leonard pair A, A^* is of Bannai/Ito type and the Leonard pair A, A^ε is not of Bannai/Ito type. With reference to Notation 4.24, consider the common value of $(\theta_{i-2} - \theta_{i+1})/(\theta_{i-1} - \theta_i)$ for $2 \leq i \leq d-1$. Because A, A^* is of Bannai/Ito type, that common value is equal to -1. Because A, A^ε is not of Bannai/Ito type, that same common value is not equal to -1. This is a contradiction, and the result follows. $\square$

Theorem 4.27 *Let V denote a finite-dimensional irreducible $\mathcal{A}$-module and let A, A^*, A^ε denote the Leonard triple from Corollary 3.17. Then A, A^*, A^ε is of Bannai/Ito type. If V is of type B then A, A^*, A^ε is totally bipartite. If V is of type AB then A, A^*, A^ε is totally almost bipartite.*

Proof: Immediate. $\square$

Theorem 4.28 *Let V denote a vector space over $\mathbb{K}$ with finite dimension at least 4 and let A, A^*, A^ε denote a totally B/AB Leonard triple on V of Bannai/Ito type. Then there exists an irreducible $\mathcal{A}$-module structure on V and nonzero scalars $\xi, \xi^*, \xi^\varepsilon$ in $\mathbb{K}$ such that x, y, z act as $A\xi^{-1}, A^*\xi^{*-1}, A^\varepsilon\xi^{\varepsilon-1}$ respectively. If A, A^*, A^ε is totally bipartite then the $\mathcal{A}$-module V is of type B and if A, A^*, A^ε is totally almost bipartite then the $\mathcal{A}$-module*

V is of type AB. The $\mathcal{A}$-module structure of V is uniquely determined by the scalars $\xi, \xi^*, \xi^\varepsilon$ and the scalars $\xi, \xi^*, \xi^\varepsilon$ are unique up to changing the sign of two of them.

Proof: We first claim there exist scalars $\zeta_1, \zeta_2, \zeta_1^*, \zeta_2^*, \zeta_1^\varepsilon, \zeta_2^\varepsilon \in \mathbb{K}$ such that

$$\begin{aligned} \zeta_1^\varepsilon AA^* + \zeta_2^\varepsilon A^*A &= A^\varepsilon, && (4.24)\\ \zeta_1 A^*A^\varepsilon + \zeta_2 A^\varepsilon A^* &= A, && (4.25)\\ \zeta_1^* A^\varepsilon A + \zeta_2^* AA^\varepsilon &= A^*. && (4.26) \end{aligned}$$

To prove the claim, we first show that line (4.24) holds. By Lemma 4.23 and the fact that $d \geq 3$, there exist unique scalars $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \in \mathbb{K}$ such that

$$A^\varepsilon = \alpha_1 I + \alpha_2 A + \alpha_3 A^* + \alpha_4 AA^* + \alpha_5 A^*A. \tag{4.27}$$

In equation (4.27) set $\zeta_1^\varepsilon = \alpha_4$ and $\zeta_2^\varepsilon = \alpha_5$. We show that $\alpha_1, \alpha_2, \alpha_3$ are equal to zero. We first show that α_1, α_2 are equal to zero. Consider the matrix B^ε representing A^ε with respect to the basis $\{v_i\}_{i=0}^d$ from Notation 4.24. By construction, B^ε is irreducible tridiagonal and either bipartite or almost bipartite. The matrices representing A^*, AA^*, A^*A are also tridiagonal and either bipartite or almost bipartite. Therefore, $B^\varepsilon_{i,i} = 0$ for $1 \leq i \leq d-1$. By (4.27), we have that, for $1 \leq i \leq d-1$,

$$B^\varepsilon_{i,i} = \alpha_1 + \alpha_2 \theta_i.$$

Because $d \geq 3$ and $\{\theta_i\}_{i=1}^{d-1}$ are distinct, we have that α_1, α_2 are both equal to zero. The proof that $\alpha_3 = 0$ is similar, using the matrix representing A^ε with respect to the basis $\{v_i^*\}_{i=0}^d$ from Notation 4.24. Therefore equation (4.24) holds. Equations (4.25), (4.26) are similar and the claim follows.

We now refine the relations (4.24)–(4.26). We claim that there exist nonzero scalars $\zeta, \zeta^*, \zeta^\varepsilon \in \mathbb{K}$ such that

$$\begin{aligned} \zeta^\varepsilon(AA^* + A^*A) &= A^\varepsilon, && (4.28)\\ \zeta(A^*A^\varepsilon + A^\varepsilon A^*) &= A, && (4.29)\\ \zeta^*(A^\varepsilon A + AA^\varepsilon) &= A^*. && (4.30) \end{aligned}$$

Substituting the left-hand side of equation (4.24) for A^ε in equation (4.26), we find that

$$\zeta_1^\varepsilon \zeta_2^* A^2 A^* + (\zeta_1^\varepsilon \zeta_1^* + \zeta_2^\varepsilon \zeta_2^*) AA^*A + \zeta_2^\varepsilon \zeta_1^* A^* A^2 = A^*. \tag{4.31}$$

Equations (4.9) and (4.31) both express A^* as a linear combination of (4.11). By Lemma 4.22, we have

$$\begin{aligned} \varrho \zeta_1^\varepsilon \zeta_2^* &= 1, && (4.32)\\ \varrho(\zeta_1^\varepsilon \zeta_1^* + \zeta_2^\varepsilon \zeta_2^*) &= 2, && (4.33)\\ \varrho \zeta_2^\varepsilon \zeta_1^* &= 1. && (4.34) \end{aligned}$$

By equation (4.32), we have $\zeta_1^\varepsilon \neq 0$ and by equation (4.34), we have $\zeta_2^\varepsilon \neq 0$. Solving equations (4.32) and (4.34) for ζ_2^*, ζ_1^* respectively and substituting into equation (4.33), we get $\zeta_1^\varepsilon(\zeta_2^\varepsilon)^{-1} + \zeta_2^\varepsilon(\zeta_1^\varepsilon)^{-1} = 2$. Therefore $\zeta_1^\varepsilon = \zeta_2^\varepsilon$ and both are nonzero. Let ζ^ε denote the common value of $\zeta_1^\varepsilon, \zeta_2^\varepsilon$. Then equation (4.28) holds. Equations (4.29), (4.30) are similar and the second claim follows.

Since $\mathbb{K}$ is algebraically closed and $\zeta, \zeta^*, \zeta^\varepsilon$ are nonzero, there exist $\xi, \xi^*, \xi^\varepsilon$ such that $\xi^2 = (4\zeta^*\zeta^\varepsilon)^{-1}$, $\xi^{*2} = (4\zeta^\varepsilon\zeta)^{-1}$ and $\xi^{\varepsilon 2} = (4\zeta\zeta^*)^{-1}$. The choices for $\xi, \xi^*, \xi^\varepsilon$ are unique up to sign and $\xi\xi^*\xi^\varepsilon = \pm(8\zeta\zeta^*\zeta^\varepsilon)^{-1}$. Choose $\xi, \xi^*\xi^\varepsilon$ such that $\xi\xi^*\xi^\varepsilon = (8\zeta\zeta^*\zeta^\varepsilon)^{-1}$. We have $\xi, \xi^*, \xi^\varepsilon \neq 0$. By equations (4.28)–(4.30) we have that $A\xi^{-1}, A^*\xi^{*-1}, A^\varepsilon\xi^{\varepsilon-1}$ satisfy equations (2.1)–(2.3). Therefore there exists an $\mathcal{A}$-module structure such that x, y, z act as $A\xi^{-1}, A^*\xi^{*-1}, A^\varepsilon\xi^{\varepsilon-1}$ respectively.

The proof that the $\mathcal{A}$-module V is irreducible is similar to the proof of Lemma 2.27. By Theorems 2.35, 2.36, we find that, if A, A^*, A^ε is totally bipartite then V is of type B and if A, A^*, A^ε is totally almost bipartite then V is of type AB.

Given scalars $\xi, \xi^*, \xi^\varepsilon \in \mathbb{K}$, there is at most one $\mathcal{A}$-module structure on V such that A, A^*, A^ε act as $\xi x, \xi^* y, \xi^\varepsilon z$ respectively. Because $\xi^2 = (4\zeta^*\zeta^\varepsilon)^{-1}$, $\xi^{*2} = (4\zeta^\varepsilon\zeta)^{-1}$, $\xi^{\varepsilon 2} = (4\zeta\zeta^*)^{-1}$ and $\xi\xi^*\xi^\varepsilon = (8\zeta\zeta^*\zeta^\varepsilon)^{-1}$, the choices of ξ, ξ^* are unique up to sign change and ξ^ε is uniquely determined by ξ, ξ^*. □

Corollary 4.29 *Let A, A^*, A^ε denote a totally B/AB Leonard triple of Bannai/Ito type with diameter $d \geq 3$. Then there exist unique scalars $\nu, \nu^*, \nu^\varepsilon \in \mathbb{K}$ such that*

$$AA^* + A^*A = \nu^\varepsilon A^\varepsilon, \tag{4.35}$$

$$A^*A^\varepsilon + A^\varepsilon A^* = \nu A, \tag{4.36}$$

$$A^\varepsilon A + AA^\varepsilon = \nu^* A^*. \tag{4.37}$$

In Theorem 4.28 we assume that $d \geq 3$. To see that this assumption is necessary, we show that, for $d = 2$, the Theorem is false. By [18, Theorems 10.1(i), 10.2(ii), 10.4(iii)] with $d = 2$,

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \qquad A^* = \begin{pmatrix} 0 & 2 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \end{pmatrix}, \qquad A^\varepsilon = \begin{pmatrix} 0 & -2\mathbf{i} & 0 \\ \mathbf{i} & 0 & -\mathbf{i} \\ 0 & 2\mathbf{i} & 0 \end{pmatrix}$$

is a Leonard triple with diameter 2. Observing [18, Theorems 10.1(ii),(iii), 10.2(i),(iii), 10.4(i), (ii)] we find that the Leonard triple is totally bipartite, and we routinely find that each Leonard pair obtained from this Leonard triple satisfies equations (4.9), (4.10) with $\varrho = 4$ and $\varrho^* = 4$, and is hence of Bannai/Ito type. However, there are no scalars $\zeta, \zeta^*, \zeta^\varepsilon$ that satisfy equation (4.28). Therefore, there is no $\mathcal{A}$-module structure as described in Theorem 4.28.

We now classify the totally B/AB Leonard triples of Bannai/Ito type with diameter $d \geq 3$. We will be using the notion of isomorphism of Leonard triples. For a precise definition, see [5, Definition 8.2].

Theorem 4.30 *Let d denote an integer at least 3 and let $\nu, \nu^*, \nu^\varepsilon$ denote scalars in $\mathbb{K}$. Then the following (i), (ii) are equivalent.*

(i) *There exists a totally bipartite Leonard triple A, A^*, A^ε of Bannai/Ito type with diameter d that satisfies equations (4.35)–(4.37).*

(ii) *The integer d is even and the scalars $\nu, \nu^*, \nu^\varepsilon$ are nonzero.*

Moreover, assume (i), (ii) hold. Then the Leonard triple A, A^*, A^ε is unique up to isomorphism.

Proof: (ii)⇒(i): Let V denote a finite-dimensional irreducible $\mathcal{A}$-module of type $B(d)$. Let $\xi, \xi^*, \xi^\varepsilon$ in $\mathbb{K}$ satisfy $\xi^2 = \frac{1}{4}\nu^*\nu^\varepsilon$, $\xi^{*2} = \frac{1}{4}\nu^\varepsilon\nu$, $\xi^{\varepsilon 2} = \frac{1}{4}\nu\nu^*$ and $\xi\xi^*\xi^\varepsilon = \frac{1}{8}\nu\nu^*\nu^\varepsilon$. Let A, A^*, A^ε denote the actions on V of $\xi x, \xi^* y, \xi^\varepsilon z$ respectively. Then, by Theorem 4.27, A, A^*, A^ε is a totally bipartite Leonard triple of Bannai/Ito type with diameter d that satisfies equations (4.35)–(4.37).

(i)⇒(ii): By Theorem 4.28, there exists an $\mathcal{A}$-module structure on V of type B and nonzero scalars $\xi, \xi^*, \xi^\varepsilon$ such that A, A^*, A^ε act as $\xi x, \xi^* y, \xi^\varepsilon z$ respectively. The dimension of V is $d + 1$, so V is of type $B(d)$. By this and Theorem 2.36, d is even.

We routinely find that $\nu = \frac{1}{2}\xi^{-1}\xi^*\xi^\varepsilon$, $\nu^* = \frac{1}{2}\xi^{*-1}\xi^\varepsilon\xi$ and $\nu^\varepsilon = \frac{1}{2}\xi^{\varepsilon-1}\xi\xi^*$ so $\zeta, \zeta^*, \zeta^\varepsilon$ are nonzero.

Now assume (i), (ii) hold. We show the Leonard triple A, A^*, A^ε is unique up to isomorphism. Let B, B^*, B^ε denote a totally bipartite Leonard triple of Bannai/Ito type with diameter d that satisfies equations (4.35)–(4.37). We show the Leonard triples A, A^*, A^ε and B, B^*, B^ε are isomorphic. Let V denote the vector space underlying A, A^*, A^ε and let W denote the vector space underlying B, B^*, B^ε. By Theorem 4.28, there exist scalars $\xi, \xi^*, \xi^\varepsilon$ in $\mathbb{K}$ and an $\mathcal{A}$-module structure on V such that A, A^*, A^ε act on V as $\xi x, \xi^* y, \xi^\varepsilon z$ respectively. Similarly, there exist scalars $\xi', \xi^{*\prime}, \xi^{\varepsilon\prime}$ in $\mathbb{K}$ and an $\mathcal{A}$-module structure on W such that B, B^*, B^ε act on W as $\xi' x, \xi^{*\prime} y, \xi^{\varepsilon\prime} z$ respectively. The $\mathcal{A}$-modules V, W are both of type $B(d)$ and hence isomorphic. By Theorem 8.16 the scalars ξ, ξ^* are unique up to sign as are the scalars $\xi', \xi^{*\prime}$. Moreover, the scalar ξ^ε is uniquely determined by ξ, ξ^* and the scalar $\xi^{\varepsilon\prime}$ is uniquely determined by $\xi', \xi^{*\prime}$. Moreover, both ξ^2, ξ'^2 are equal to $\frac{1}{4}\nu^*\nu^\varepsilon$, both $\xi^{*2}, \xi^{*\prime 2}$ are equal to $\frac{1}{4}\nu^\varepsilon\nu$, both $\xi^{\varepsilon 2}, \xi^{\varepsilon\prime 2}$ are equal to $\frac{1}{4}\nu\nu^*$ and both $\xi\xi^*\xi^\varepsilon, \xi'\xi^{*\prime}\xi^{\varepsilon\prime}$ are equal to $\frac{1}{8}\nu\nu^*\nu^\varepsilon$. Changing the signs of $\xi, \xi^*, \xi^\varepsilon$ as necessary, we may assume, without loss of generality, that $\xi = \xi'$, $\xi^* = \xi^{*\prime}$ and $\xi^\varepsilon = \xi^{\varepsilon\prime}$. Let $\phi : V \to W$ denote an isomorphism of $\mathcal{A}$-modules. Then $\phi \circ A = \xi(\phi \circ x) = \xi(x \circ \phi) = B \circ \phi$, $\phi \circ A^* = \xi^*(\phi \circ y) = \xi^*(y \circ \phi) = B^* \circ \phi$ and $\phi \circ A^\varepsilon = \xi^\varepsilon(\phi \circ z) = \xi^\varepsilon(z \circ \phi) = B^\varepsilon \circ \phi$ on V. These equations show the Leonard triples A, A^*, A^ε and B, B^*, B^ε are isomorphic. $\square$

As a consequence of Lemma 4.29 and Theorem 4.30, one obtains the following Corollary.

Corollary 4.31 *There is a bijection between the set of isomorphism classes of totally bipartite Leonard triples of Bannai/Ito type with diameter at least* 3 *and sequences* $d, \nu, \nu^*, \nu^\varepsilon$ *of scalars such that* $d \geq 3$ *is an even integer and* $\nu, \nu^*, \nu^\varepsilon$ *are nonzero.*

Theorem 4.32 *Let* d *denote an integer at least* 3 *and let* $\tau, \tau^*, \tau^\varepsilon$ *denote scalars in* $\mathbb{K}$. *Then the following (i), (ii) are equivalent.*

(i) *There exists a totally almost bipartite Leonard triple* A, A^*, A^ε *of Bannai/Ito type with diameter* d, $\operatorname{tr}(A) = \tau$, $\operatorname{tr}(A^*) = \tau^*$ *and* $\operatorname{tr}(A^\varepsilon) = \tau^\varepsilon$.

(ii) *The scalars* $\tau, \tau^*, \tau^\varepsilon$ *are nonzero.*

Moreover, assume (i), (ii) hold. Then the Leonard triple A, A^*, A^ε *is unique up to isomorphism.*

Proof: (ii)⇒(i): Let V denote a finite-dimensional irreducible $\mathcal{A}$-module of type $AB(d, 0)$. Let A, A^*, A^ε denote the actions of $\tau(-1)^d(d+1)^{-1}x, \tau^*(-1)^d(d+1)^{-1}y, A^\varepsilon = \tau^\varepsilon(-1)^d(d+1)^{-1}z$ respectively. Then, by Theorem 4.15, A, A^*, A^ε is a totally almost bipartite Leonard triple of Bannai/Ito type with diameter d. By Theorem 2.35, $\operatorname{tr}(A) = \tau$, $\operatorname{tr}(A^*) = \tau^*$ and $\operatorname{tr}(A^\varepsilon) = \tau^\varepsilon$.

(i)⇒(ii): Immediate from Definition 1.5.

Now assume (i), (ii) hold. We show the Leonard triple A, A^*, A^ε is unique up to isomorphism. Let B, B^*, B^ε denote a totally almost bipartite Leonard triple of Bannai/Ito type with diameter d such that $\operatorname{tr}(B) = \tau$, $\operatorname{tr}(B^*) = \tau^*$ and $\operatorname{tr}(B^\varepsilon) = \tau^\varepsilon$. We show the Leonard triples A, A^*, A^ε and B, B^*, B^ε are isomorphic. Let V denote the vector space underlying A, A^*, A^ε and let W denote the vector space underlying B, B^*, B^ε. By Theorem 4.28, there exist scalars $\xi, \xi^*, \xi^\varepsilon$ in $\mathbb{K}$ and an $\mathcal{A}$-module structure on V such that

A, A^*, A^ε act on V as $\xi x, \xi^* y, \xi^\varepsilon z$ respectively. Similarly, there exist scalars $\xi', \xi^{*\prime}, \xi^{\varepsilon\prime}$ in $\mathbb{K}$ and an $\mathcal{A}$-module structure on W such that B, B^*, B^ε act on W as $\xi' x, \xi^{*\prime} y, \xi^{\varepsilon\prime} z$ respectively. The $\mathcal{A}$-module V is of type $AB(d,n)$ and the $\mathcal{A}$-module W is of type $AB(d,n')$ for some $n, n' \in \mathbb{I}$. By Theorem 2.35 together with $\operatorname{tr}(A) = \operatorname{tr}(B)$, $\operatorname{tr}(A^*) = \operatorname{tr}(B^*)$ and $\operatorname{tr}(A^\varepsilon) = \operatorname{tr}(B^\varepsilon)$, we obtain $\xi = \pm\xi'$, $\xi^* = \pm\xi^{*\prime}$ and $\xi^\varepsilon = \pm\xi^{\varepsilon\prime}$, with equality if and only if $n = n'$. By Theorem 4.28, our choice of scalars ξ, ξ^* was unique up to sign and our choice of ξ^ε was determined by ξ, ξ^*. Changing the signs of $\xi, \xi^*, \xi^\varepsilon$ as necessary, we may assume, without loss of generality, that $\xi = \xi'$, $\xi^* = \xi^{*\prime}$, $\xi^\varepsilon = \xi^{\varepsilon\prime}$ and hence $n = n'$. Then the $\mathcal{A}$-modules V and W are isomorphic. Let $\phi : V \to W$ denote an isomorphism of $\mathcal{A}$-modules. Then $\phi \circ A = \xi(\phi \circ x) = \xi(x \circ \phi) = B \circ \phi$, $\phi \circ A^* = \xi^*(\phi \circ y) = \xi^*(y \circ \phi) = B^* \circ \phi$ and $\phi \circ A^\varepsilon = \xi^\varepsilon(\phi \circ z) = \xi^\varepsilon(z \circ \phi) = B^\varepsilon \circ \phi$ on V. These equations show the Leonard triples A, A^*, A^ε and B, B^*, B^ε are isomorphic. $\square$

As a consequence of Theorem 4.32 and the fact that every linear transformation has a unique trace, one obtains the following Corollary.

Corollary 4.33 *There is a bijection between the set of isomorphism classes of totally almost bipartite Leonard triples of Bannai/Ito type with diameter at least* 3 *and sequences* $d, \tau, \tau^*, \tau^\varepsilon$ *of scalars such that* $d \geq 3$ *is an integer and* $\tau, \tau^*, \tau^\varepsilon$ *are nonzero.*

We comment on why Theorems 4.30 and 4.32 use different parameters. Let A, A^*, A^ε denote a totally bipartite Leonard triple of Bannai/Ito type. Let the scalars $\tau, \tau^*, \tau^\varepsilon$ be from Theorem 4.32 and let the scalars $\nu, \nu^*, \nu^\varepsilon$ be from equations (4.35)–(4.37). Then

$$\nu = \frac{(-1)^d 2\tau^*\tau^\varepsilon}{(d+1)\tau}, \qquad \nu^* = \frac{(-1)^d 2\tau^\varepsilon\tau}{(d+1)\tau^*}, \qquad \nu^\varepsilon = \frac{(-1)^d 2\tau\tau^*}{(d+1)\tau^\varepsilon}.$$

Given an integer d at least three and nonzero scalars $\nu, \nu^*, \nu^\varepsilon$, the scalars $\tau, \tau^*, \tau^\varepsilon$ that satisfy the above equation are unique up to changing the sign of an even number of them. Therefore, there are exactly 4 isomorphism classes of totally almost bipartite Leonard triples of Bannai/Ito type with diameter d that satisfy equations (4.35)–(4.37).

4.3 Normalized Leonard triples

Observe that, given a vector space V over $\mathbb{K}$ of finite positive dimension, Theorems 4.27, 4.28 provided a correspondence between $\mathcal{A}$-modules on V and totally B/AB Leonard triples on V of Bannai/Ito type. However, this is not a bijection as multiple Leonard triples may correspond to the same $\mathcal{A}$-module. In this section, we introduce the notion of a normalized Leonard triple which will allow us to obtain a bijection.

Definition 4.34 Let V denote a vector space over $\mathbb{K}$ with positive finite dimension and let A, A^*, A^ε denote a totally B/AB Leonard triple on V of Bannai/Ito type. We say the Leonard triple A, A^*, A^ε is *normalized* whenever the scalars $\nu, \nu^*, \nu^\varepsilon$ from equations (4.35)–(4.37) are all equal to 2.

We now describe the relationship between finite-dimensional irreducible $\mathcal{A}$-modules and normalized totally B/AB Leonard triples of Bannai/Ito type.

Proposition 4.35 *Let d denote a nonnegative integer and let V denote a finite-dimensional irreducible $\mathcal{A}$-module of diameter d. Then the actions of x, y, z on V form a normalized totally B/AB Leonard triple of Bannai/Ito type with diameter d. If the $\mathcal{A}$-module V is of type B, then the Leonard triple x, y, z is bipartite. If the $\mathcal{A}$-module V is of type AB, then the Leonard triple x, y, z is almost bipartite.*

Proof: Immediate. □

Proposition 4.36 *Let V denote a vector space over $\mathbb{K}$ with finite dimension $d+1 \geq 4$ and let A, A^*, A^ε denote a normalized totally B/AB Leonard triple on V of Bannai/Ito type. Then there exists a unique $\mathcal{A}$-module structure on V such that x, y, z act on V as A, A^*, A^ε. If A, A^*, A^ε is totally bipartite then V is of type B. If A, A^*, A^ε is totally almost bipartite then V is of type AB. Moreover, the Leonard triple A, A^*, A^ε and the $\mathcal{A}$-module both have diameter d.*

Proof: Immediate. □

Remark 4.37 *Let V denote a vector space over $\mathbb{K}$ with finite dimension at least 4. Combining Propositions 4.35 and 4.36, we obtain a bijection between the set of irreducible $\mathcal{A}$-modules on V and the set of normalized totally B/AB Leonard triples on V. Under this bijection the $\mathcal{A}$-modules on V of type B correspond to the normalized totally bipartite Leonard triples on V and the $\mathcal{A}$-modules on V of type AB correspond to the normalized totally almost bipartite Leonard triples on V. Moreover, the bijection preserves diameter.*

We now describe how normalized totally B/AB Leonard triples of Bannai/Ito type relate to general totally B/AB Leonard triples of Bannai/Ito type.

Lemma 4.38 *Let V denote a vector space over $\mathbb{K}$ with finite dimension at least 4 and let A, A^*, A^ε denote a totally B/AB Leonard triple on V of Bannai/Ito type. Let $\xi, \xi^*, \xi^\varepsilon$ denote scalars in $\mathbb{K}$ and let $\nu, \nu^*, \nu^\varepsilon$ denote the scalars from equations (4.35)–(4.37).*

Then the Leonard triple $\xi A, \xi^* A^*, \xi^\varepsilon A^\varepsilon$ is normalized if and only if the following hold.

$$\xi^2 = 4(\nu^*\nu^\varepsilon)^{-1}, \qquad \xi^{*2} = 4(\nu^\varepsilon\nu)^{-1}, \qquad \xi^{\varepsilon 2} = 4(\nu\nu^*)^{-1}, \qquad \xi\xi^*\xi^\varepsilon = 8(\nu\nu^*\nu^\varepsilon)^{-1}.$$

Proof: Routine. □

Proposition 4.39 *Let V denote a vector space over $\mathbb{K}$ with finite dimension at least* 4 *and let A, A^*, A^ε denote a totally B/AB Leonard triple on V of Bannai/Ito type. Then there exist exactly four sequences of nonzero scalars $\xi, \xi^*, \xi^\varepsilon \in \mathbb{K}$ such that the Leonard triple $\xi A, \xi^* A^*, \xi^\varepsilon A^\varepsilon$ is normalized. Whenever the Leonard triple A, A^*, A^ε is totally bipartite, the four resulting normalized Leonard triples are isomorphic. Whenever the Leonard triple A, A^*, A^ε is totally almost bipartite, the four resulting normalized Leonard triples are mutually nonisomorphic.*

Proof: The existence of exactly four sequences of scalars $\xi, \xi^*, \xi^\varepsilon$ is a routine consequence of Lemma 4.38.

Assume the Leonard triple A, A^*, A^ε is totally bipartite. Then the four resulting normalized Leonard triples are totally bipartite and have the same diameter. By this, Theorem 4.30 and Definition 4.34 one routinely finds that the four resulting normalized Leonard triples are isomorphic as desired.

Now assume the Leonard triple A, A^*, A^ε is totally almost bipartite. By comparing the traces of the resulting normalized Leonard triples one routinely finds that they are mutually nonisomorphic as desired. □

Observe that each totally almost bipartite Leonard triple of Bannai/Ito type can be scaled to four mutually nonisomorphic normalized Leonard triples. We now elaborate on the differences between these four Leonard triples. Recall the set $\mathbb{I} = \{0, x, y, z\}$.

Definition 4.40 Let d denote a nonnegative integer, let V denote a vector space over $\mathbb{K}$ of dimension $d+1$ and let A, A^*, A^ε denote a totally almost bipartite Leonard triple on V of Bannai/Ito type. Let $n \in \mathbb{I}$. We say the Leonard triple A, A^*, A^ε is *n-normalized* whenever the traces of A, A^*, A^ε agree with the following table.

	$\mathrm{tr}(A)$	$\mathrm{tr}(A^*)$	$\mathrm{tr}(A^\varepsilon)$
$n = 0$	$(-1)^d(d+1)$	$(-1)^d(d+1)$	$(-1)^d(d+1)$
$n = x$	$(-1)^d(d+1)$	$(-1)^{d+1}(d+1)$	$(-1)^{d+1}(d+1)$
$n = y$	$(-1)^{d+1}(d+1)$	$(-1)^d(d+1)$	$(-1)^{d+1}(d+1)$
$n = z$	$(-1)^{d+1}(d+1)$	$(-1)^{d+1}(d+1)$	$(-1)^d(d+1)$

Lemma 4.41 *Let A, A^*, A^ε denote a totally almost bipartite Leonard triple of Bannai/Ito type. Then the Leonard triple A, A^*, A^ε is normalized if and only if there exists $n \in \mathbb{I}$ such that the Leonard triple A, A^*, A^ε is n-normalized.*

Proof: Immediate from Theorem 2.35 and Definition 4.40. □

Proposition 4.42 *Let V denote a vector space over $\mathbb{K}$ with finite dimension at least 4 and let A, A^*, A^ε denote a totally almost bipartite Leonard triple on V of Bannai/Ito type. Then, for each $n \in \mathbb{I}$, there exists a unique sequence of nonzero scalars $\xi, \xi^*, \xi^\varepsilon \in \mathbb{K}$ such that the Leonard triple $\xi A, \xi^* A^*, \xi^\varepsilon A^\varepsilon$ is n-normalized. These constitute the four sequences $\xi, \xi^*, \xi^\varepsilon$ from Proposition 4.39.*

Proof: Immediate ☐

Proposition 4.43 *Under the bijection from Remark 4.37, the $\mathcal{A}$-modules on V of type $AB(d,n)$ correspond to the n-normalized totally almost bipartite Leonard triples on V.*

Proof: Immediate. ☐

Chapter 5

$\mathfrak{sl}_2$-modules and $\mathcal{A}$-modules

5.1 The Lie algebra $\mathfrak{sl}_2$

In [6], Go displayed a relationship between the hypercube Q_D and the finite-dimensional modules for the Lie algebra $\mathfrak{sl}_2$. In [18], Miklavič used this relationship to display certain totally bipartite Leonard triples of Kroutchouk type. We will use this information to obtain similar results involving the algebra $\mathcal{A}$ and totally B/AB Leonard triples of Bannai/Ito type. First we recall the Lie algebra $\mathfrak{sl}_2$. We will use a version of $\mathfrak{sl}_2$ favored by certain physicists (See [19]).

Definition 5.1 [19, Proposition 1.21] We define $\mathfrak{sl}_2$ to be the Lie algebra over $\mathbb{K}$ with basis X, Y, Z and Lie bracket

$$\begin{aligned} [X, Y] &= 2\mathbf{i}Z, && (5.1) \\ [Y, Z] &= 2\mathbf{i}X, && (5.2) \\ [Z, X] &= 2\mathbf{i}Y. && (5.3) \end{aligned}$$

The finite-dimensional irreducible $\mathfrak{sl}_2$-modules are well-known (see [9, Theorem, p. 33]). Up to isomorphism, there is one irreducible $\mathfrak{sl}_2$-module for each positive finite dimension. Miklavič showed that, on a $(d+1)$-dimensional $\mathfrak{sl}_2$-module V, X, Y, Z act as a bipartite Leonard triple of diameter d.

Lemma 5.2 [18, Lemma 9.1, Theorems 10.1–10.4] *Let d denote a nonnegative integer and let V denote a finite-dimensional irreducible $\mathfrak{sl}_2$-module of dimension $d+1$. Then the following hold:*

(i) *There exists a basis $\{v_i\}_{i=0}^d$ for V on which X, Y, Z act as follows. For $0 \leq i \leq d$:*

$$\begin{aligned} X.v_i &= (d-i+1)v_{i-1} + (i+1)v_{i+1} && (5.4) \\ Y.v_i &= (d-2i)v_i && (5.5) \\ Z.v_i &= \mathbf{i}(d-i+1)v_{i-1} - \mathbf{i}(i+1)v_{i+1} && (5.6) \end{aligned}$$

where $v_{-1} = 0$ and $v_{d+1} = 0$.

(ii) *There exists a basis $\{w_i\}_{i=0}^d$ for V on which X, Y, Z act as follows. For $0 \leq i \leq d$:*

$$\begin{aligned} X.w_i &= \mathbf{i}(d-i+1)w_{i-1} - \mathbf{i}(i+1)w_{i+1} && (5.7) \\ Y.w_i &= (d-i+1)w_{i-1} + (i+1)w_{i+1} && (5.8) \\ Z.w_i &= (d-2i)w_i && (5.9) \end{aligned}$$

where $v_{-1} = 0$ and $v_{d+1} = 0$.

The $\mathfrak{sl}_2$-module V from Lemma 5.2 is said to have *diameter d*.

Lemma 5.3 *Let V denote an irreducible $\mathfrak{sl}_2$-module with diameter d and let $\{w_i\}_{i=0}^d$ denote the basis from Lemma 5.2(ii). Then $Y.\sum_{i=0}^d w_i = d\sum_{i=0}^d w_i$.*

Proof: Routine. □

5.2 From $\mathfrak{sl}_2$-modules to $\mathcal{A}$-modules

In this section we will demonstrate a relationship between $\mathcal{A}$-modules and $\mathfrak{sl}_2$-modules. We introduce a type of operator called a skew operator. We use skew operators to give $\mathcal{A}$-module structure to each finite-dimensional $\mathfrak{sl}_2$-module.

Definition 5.4 By a *skew operator* we mean a $\mathbb{K}$-linear operator σ that acts on finite-dimensional $\mathfrak{sl}_2$-modules and satisfies the following relations:

$$\sigma^2 = I, \tag{5.10}$$

$$\sigma X = -X\sigma, \qquad \sigma Y = Y\sigma, \qquad \sigma Z = -Z\sigma. \tag{5.11}$$

Here, X, Y, Z are from Definition 5.1.

Lemma 5.5 *Let σ be a skew operator. Then $X, \sigma Y, -\sigma \mathbf{i} Z$ satisfy relations (2.1)–(2.3).*

Proof: Routine. $\square$

Lemma 5.6 *Let σ be a skew operator. Then $-\sigma$ is a skew operator. Furthermore every skew operator acts as either σ or $-\sigma$ on any given finite-dimensional irreducible $\mathfrak{sl}_2$-module.*

Proof: Routine calculation shows that $-\sigma$ is a skew operator. Now let $\check{\sigma}$ be a skew operator and let V be a finite-dimensional irreducible $\mathfrak{sl}_2$-module. By (5.11) $\sigma\check{\sigma}$ commutes with each of X, Y, Z. Since the $\mathfrak{sl}_2$-module V is irreducible, there exists $c \in \mathbb{K}$ such that $\sigma\hat{s} = cI$ on V. Therefore $\check{\sigma} = c\sigma$ in view of (5.10). Also by (5.10) $(c\sigma)^2 = \check{\sigma}^2 = I$, so

$c = \pm 1$. $\square$

We have been discussing skew operators. However we have not yet shown that a skew operator exists. We now show that a skew operator exists by constructing one. To construct our operator, we define two other operators, called h and k. Our skew operator will be equal to hk. We will make use of the following lemma.

Lemma 5.7 *Let V be a finite-dimensional $\mathfrak{sl}_2$-module. Then $\frac{\mathbf{i}Y-X}{2}, \frac{\mathbf{i}Y+X}{2}$ are nilpotent on V.*

Proof: Finite-dimensional $\mathfrak{sl}_2$-modules are completely reducible. Therefore we may assume without loss of generality that V is irreducible. Let d be the diameter of V and let $\{w_i\}_{i=0}^d$ be the basis for V from Lemma 5.2(ii). Then the matrix representing $\frac{\mathbf{i}Y-X}{2}$ With regards to $\{w_i\}_{i=0}^d$ is upper triangular with all diagonal entries zero and the matrix representing $\frac{\mathbf{i}Y+X}{2}$ is lower triangular with zeroes on the diagonal. The result followes. $\square$

Definition 5.8 Let h denote the $\mathbb{K}$-linear operator that acts on finite-dimensional $\mathfrak{sl}_2$-modules as follows:

$$h = \exp\left(\frac{\mathbf{i}Y - X}{2}\right) \exp\left(\frac{\mathbf{i}Y + X}{2}\right) \exp\left(\frac{\mathbf{i}Y - X}{2}\right).$$

We recall a fact about Lie algebras. Let $\mathfrak{g}$ a Lie algebra and let $r, s \in \mathfrak{g}$ where r is nilpotent on all finite-dimensional $\mathfrak{g}$-modules. Then $\exp(r)s\exp(r)^{-1} = \exp(r)s\exp(-r) =$exp ad$(r)s$.

Lemma 5.9 *Let h be as in Definition 5.8. The following holds for $\xi \in \mathfrak{sl}_2$.*

$$h\xi h^{-1} = \exp \operatorname{ad}\left(\frac{\mathbf{i}Y - X}{2}\right) \exp \operatorname{ad}\left(\frac{\mathbf{i}Y + X}{2}\right) \exp \operatorname{ad}\left(\frac{\mathbf{i}Y - X}{2}\right) \xi. \tag{5.12}$$

Proof: Routine using Definition 5.8 and the above comment. □

Lemma 5.10 *The following (i), (ii) hold.*

(i) *The matrix representing the action of* $\exp \operatorname{ad}(\frac{\mathbf{i}X-Y}{2})$ *on* $\mathfrak{sl}_2$ *With regards to the basis* X, Y, Z *is*

$$\begin{pmatrix} 1/2 & \mathbf{i}/2 & -1 \\ \mathbf{i}/2 & 3/2 & \mathbf{i} \\ 1 & -\mathbf{i} & 1 \end{pmatrix}.$$

(ii) *The matrix representing the action of* $\exp \operatorname{ad}(\frac{\mathbf{i}X+Y}{2})$ *on* $\mathfrak{sl}_2$ *With regards to the basis* X, Y, Z *is*

$$\begin{pmatrix} 1/2 & -\mathbf{i}/2 & -1 \\ -\mathbf{i}/2 & 3/2 & -\mathbf{i} \\ 1 & \mathbf{i} & 1 \end{pmatrix}.$$

Proof: Routine calculation using equations (5.1)–(5.3). □

Lemma 5.11 *Let* X, Y, Z *be as in Definition 5.1. Then the element* h *from Definition 5.8 satisfies the following relations.*

$$hX = -Xh \qquad hY = Yh \qquad hZ = -Zh \tag{5.13}$$

Proof: Evaluate each of $hXh^{-1}, hYh^{-1}, hZh^{-1}$ using Lemmas 5.9, 5.10. □

A consequence of equation (5.13) is that h satisfies equation (5.11). We now consider h^2. To compute h^2 we look at how h acts on finite-dimensional $\mathfrak{sl}_2$-modules. We now examine this action in detail.

Lemma 5.12 *Let d denote a nonnegative integer and let V denote a $\mathfrak{sl}_2$-module of diameter d. Pick an integer i with $0 \leq i \leq d$. Let v denote an eigenvector for X with eigenvalue $d-2i$. Then v is an eigenvector for h with eigenvalue $(-1)^i\mathbf{i}^d$.*

Proof: Recall the basis $\{v_i\}_{i=0}^d$ for V from Lemma 5.2(i) and recall that, for $0 \leq i \leq d$, v_i is an eigenvector for X with eigenvalue $d-2i$. By Lemma 5.11, h commutes with Y. Therefore $\{v_i\}_{i=0}^d$ is an eigenbasis for h. By Lemma 5.2(i) the action of X on V is irreducible bipartite tridiagonal With regards to the basis $\{v_i\}_{i=0}^d$. By Lemma 5.11, $hX = -Xh$. Therefore there exists a scalar $c \in \mathbb{K}$ such that, for $0 \leq i \leq d$, v_i is an eigenvector for h with eigenvalue $(-1)^i c$.

We will now show that $c = \mathbf{i}^d$. Recall the basis $\{w_i\}_{i=0}^d$ for V from Lemma 5.2(ii). By Lemma 5.3 and the results from the previous paragraph, $h\sum_{i=0}^d w_i = c\sum_{i=0}^d w_i$. Through routine calculation one finds that, With regards to the basis $\{w_i\}_{i=0}^d$, the matrices representing $\exp(\frac{\mathbf{i}Y-X}{2})$, $\exp(\frac{\mathbf{i}Y+X}{2})$ are upper triangular. Moreover, for $0 \leq i \leq j \leq d$ the (i,j)-entry of the matrix representing $\exp(\frac{\mathbf{i}Y-X}{2})$ is $\binom{d-i}{d-j}\mathbf{i}^{j-i}$ and the (i,j)-entry of the matrix representing $\exp(\frac{\mathbf{i}Y+X}{2})$ is $\binom{i}{d-j}\mathbf{i}^{i-j}$. From this we conclude, for $0 \leq j \leq d$ the (d,j)-entry of the matrix representing h with regards to the basis $\{w_i\}_{i=0}^d$ is

$$\begin{aligned}
\sum_{k=0}^{j} \mathbf{i}^{d-k}\binom{d}{d-k}\mathbf{i}^{j-k}\binom{d-k}{d-j} &= \mathbf{i}^{d+j}\sum_{k=0}^{j}(-1)^{2k}\binom{d}{d-k}\binom{d-k}{d-j} \\
&= \mathbf{i}^{d+j}\sum_{k=0}^{j}(-1)^{2k}\frac{d!(d-k)!}{k!(d-k)!(d-j)!(j-k)!}
\end{aligned}$$

$$
\begin{aligned}
&= \mathbf{i}^{d+j}\sum_{k=0}^{j}(-1)^{2k}\frac{d!j!}{k!j!(d-j)!(j-k)!} \\
&= \mathbf{i}^{d+j}\binom{d}{j}\sum_{k=0}^{j}(-1)^{2k}\binom{j}{k} \\
&= \delta_{j0}\mathbf{i}^{d}
\end{aligned}
$$

Therefore, when $h\sum_{i=0}^{d} v_i$ is expressed as a linear combination of $\{w_i\}_{i=0}^{d}$, the coefficient of w_d must be $\mathbf{i}^d$, so $c = \mathbf{i}^d$. The result follows. $\square$

Corollary 5.13 *Let h be from Definition 5.8. Let d denote a nonnegative integer and let V denote a $\mathfrak{sl}_2$-module of diameter d. Then $h^2 = (-1)^d$ holds on V.*

Proof: Immediate. $\square$

We see from Corollary 5.13 that, although h satisfies equation (5.11), it does not satisfy equation (5.10) and is therefore not a skew operator. To construct our skew operator, we introduce the operator k.

Lemma 5.14 *There exists a unique $\mathbb{K}$-linear operator k acting on finite-dimensional $\mathfrak{sl}_2$-modules such that k acts as 1 on each odd-dimensional irreducible submodule and as $-\mathbf{i}$ on each even-dimensional irreducible submodule.*

Proof: Follows from the fact that every finite-dimensional $\mathfrak{sl}_2$-module is completely irreducible. $\square$

Lemma 5.15 *Let k denote the operator from Lemma 5.14. Then k commutes with each of X, Y, Z.*

Proof: Immediate. □

We are now ready to define the operator s.

Definition 5.16 We define $s = hk$ where h is from Definition 5.8 and k is from Lemma 5.14.

Theorem 5.17 *The operator s from Definition 5.16 is a skew operator.*

Proof: Equation (5.10) is satisfied by Corollary 5.13 and Lemma 5.14. Equations (5.11) are satisfied by equations (5.13) and Lemma 5.14. □

Theorem 5.18 *Let V be a finite-dimensional $\mathfrak{sl}_2$-module. Let s be as in Definition 5.16. Then the following (i), (ii) hold.*

(i) *There exists an $\mathcal{A}$-module structure on V for which x, y, z act as $X, sY-, s\mathbf{i}Z$ respectively.*

(ii) *There exists an $\mathcal{A}$-module structure on V for which x, y, z act as $X, -sY, s\mathbf{i}Z$ respectively.*

Proof: (i) Immediate from Lemma 5.5 and Theorem 5.17.

(ii) Immediate from Lemmas 5.5, 5.6 and Theorem 5.17. □

By the *first* (resp. *second*) $\mathcal{A}$-module structure on V we mean the $\mathcal{A}$-module structure on V from Theorem 5.18(i) (resp. Theorem 5.18(ii)).

Theorem 5.18 gives $\mathcal{A}$-module structures to every finite-dimensional $\mathfrak{sl}_2$-module. When the $\mathfrak{sl}_2$-module is irreducible, it is not guaranteed that the corresponding $\mathcal{A}$-modules will be irreducible. We now give necessary and sufficient conditions for the $\mathcal{A}$-module to be irreducible. When the $\mathcal{A}$-module isn't irreducible, we will describe how it decomposes into irreducible $\mathcal{A}$-modules. Moreover, we will show that every finite-dimensional irreducible $\mathcal{A}$-module can be obtained from an $\mathfrak{sl}_2$-module by this procedure.

Theorem 5.19 *Let d denote a nonnegative even integer and let V be a $(d+1)$-dimensional irreducible $\mathfrak{sl}_2$-module. Then both $\mathcal{A}$-module structures on V are irreducible and of type $B(d)$.*

Proof: Let $\{v_i\}_{i=0}^d$ be the basis for V from Lemma 5.2(i). For the first $\mathcal{A}$-module structure, observe the action of $X, sY, -s\mathbf{i}Z$ on $\{v_i\}_{i=0}^d$ coincides with the action of x, y, z on the basis $\{e_i^y v^x\}_{i=0}^d$ from Theorem 3.13. For the second $\mathcal{A}$-module structure, observe the action of $X, -sY, s\mathbf{i}Z$ on $\{v_{d-i}\}_{i=0}^d$ coincides with the action of x, y, z on the basis $\{e_i^y v^x\}_{i=0}^d$ from Theorem 3.13. □

Theorem 5.20 *Let $d = 2\delta + 1$ denote a nonnegative odd integer and let V be an irreducible $\mathfrak{sl}_2$-module with diameter d. Then, under both $\mathcal{A}$-module structures, V is the direct sum of the $\mathcal{A}$-submodules $V^+ = \mathrm{span}\{v_i + v_{d-i}\}_{i=0}^{\delta}$ and $V^- = \mathrm{span}\{(-1)^i(v_i - v_{d-i})\}_{i=0}^{\delta}$ where $\{v_i\}_{i=0}^d$ is the basis from Lemma 5.2(i). Under each $\mathcal{A}$-module structure, the $\mathcal{A}$-submodules V^+, V^- are irreducible. Moreover, the following (i),(ii) hold.*

(i) *Under the first $\mathcal{A}$-module structure on V, the $\mathcal{A}$-submodule V^+ is of type $AB(\delta, 0)$ and the $\mathcal{A}$-submodule V^- is of type $AB(\delta, y)$.*

(ii) *Under the second $\mathcal{A}$-module structure on V, the $\mathcal{A}$-submodule V^+ is of type $AB(\delta, z)$ and the $\mathcal{A}$-submodule V^- is of type $AB(\delta, x)$.*

Proof: (i) The action of $sX, Y, s\mathbf{i}Z$ on $\{v_i + v_{d-i}\}_{i=0}^{\delta}$ coincides with the action of x, y, z on the basis $\{e_i^y v^x\}_{i=0}^d$ from Theorem 3.13 with $n = 0$ and the action of $X, sY, -s\mathbf{i}Z$ on $\{(-1)^i(v_i - v_{d-i})\}_{i=0}^{\delta}$ coincides with the action of x, y, z on the basis $\{e_i^y v^x\}_{i=0}^d$ from Theorem 3.13 with $n = y$.

(ii) The action of $-sX, Y, -s\mathbf{i}Z$ on $\{v_i + v_{d-i}\}_{i=0}^{\delta}$ coincides with the action of x, y, z on the basis $\{e_i^y v^x\}_{i=0}^d$ from Theorem 3.13 with $n = z$ and the action of $X, -sY, s\mathbf{i}Z$ on $\{(-1)^i(v_i - v_{d-i})\}_{i=0}^{\delta}$ coincides with the action of x, y, z on the basis $\{e_i^y v^x\}_{i=0}^d$ from Theorem 3.13 with $n = x$. $\square$

Chapter 6

Distance-regular graphs

6.1 Distance-regular graph preliminaries

In this section we review some definitions and basic results concerning distance-regular graphs. For the remainder of the chapter we will work over the field $\mathbb{C}$. Observe that, since $\mathbb{C}$ is an algebraically closed field of characteristic zero, every result from chapters 1–5 applies when $\mathbb{K} = \mathbb{C}$.

Let $\mathcal{X}$ denote a nonempty finite set. Let $\mathrm{Mat}_{\mathcal{X}}(\mathbb{C})$ denote the $\mathbb{C}$-algebra of matrices with entries in $\mathbb{C}$ and with rows and columns indexed by $\mathcal{X}$. Let $V = \mathbb{C}^{\mathcal{X}}$ denote the vector space over $\mathbb{C}$ consisting of column vectors with entries in $\mathbb{C}$ and rows indexed by $\mathcal{X}$. We observe $\mathrm{Mat}_{\mathcal{X}}(\mathbb{C})$ acts on V by left multiplication. We refer to V as the *standard module* of $\mathrm{Mat}_{\mathcal{X}}(\mathbb{C})$. For $\mathbf{x} \in \mathcal{X}$, we the vector in V indexed by $\mathbf{x}$ is denoted $\hat{\mathbf{x}}$. Let $L \in \mathrm{End}(V)$. Given $\mathbf{N} \in \mathrm{Mat}_{\mathcal{X}}(\mathbb{C})$, we say $\mathbf{N}$ *represents* L whenever $L\hat{\mathbf{x}} = \sum_{\mathbf{y}\in\mathcal{X}} \mathbf{N}_{\mathbf{yx}}\hat{\mathbf{y}}$ for all $\mathbf{x} \in \mathcal{X}$.

Let $\Gamma = (\mathcal{X}, \mathcal{R})$ denote a finite, undirected, connected graph, without loops or multiple edges, with vertex set $\mathcal{X}$, edge set $\mathcal{R}$, path-length distance function ∂ and diameter $D = \max\{\partial(\mathbf{x}, \mathbf{y}) | \mathbf{x}, \mathbf{y} \in \mathcal{X}\}$. For a vertex $\mathbf{x} \in \mathcal{X}$ and an integer $i \geq 0$ let $\Gamma_i(\mathbf{x})$ denote the set of vertices at distance i from a. For an integer $k \geq 0$ we say Γ is *regular with valency* k whenever $|\Gamma_1(\mathbf{x})| = k$ for all $\mathbf{x} \in \mathcal{X}$. We say Γ is *distance-regular* whenever for all integers $0 \leq h, i, j \leq D$ and all $\mathbf{x}, \mathbf{y} \in \mathcal{X}$ with $\partial(\mathbf{x}, \mathbf{y}) = h$ the number p^h_{ij}, defined to be $|\Gamma_i(\mathbf{x}) \cap \Gamma_j(\mathbf{y})|$, is independent of $\mathbf{x}, \mathbf{y}$. The constants p^h_{ij} are known as the *intersection numbers* of Γ. From now on we assume Γ is distance-regular with $D \geq 1$. For convenience, set $c_i = p^i_{1,i-1}$ for $1 \leq i \leq D$, $a_i = p^i_{1i}$ for $0 \leq i \leq D$, $b_i = p^i_{1,i+1}$ for $0 \leq i \leq D-1$, $k_i = p^0_{ii}$ for $0 \leq i \leq D$, $c_0 = 0$ and $b_D = 0$. We observe that Γ is regular with valency $k = k_1 = b_0$ and that $c_i + a_i + b_i = k$ for $0 \leq i \leq d$. By [3, p. 127] the following (i), (ii) hold for $0 \leq h, i, j \leq D$.

(i) $p^h_{ij} = 0$ if one of h, i, j is greater than the sum of the other two.

(ii) $p^h_{ij} \neq 0$ if one of h, i, j is equal to the sum of the other two.

We now recall the Bose-Mesner algebra of Γ. For $0 \leq i \leq D$ let $\mathbf{A}_i$ denote the matrix in $\mathrm{Mat}_{\mathcal{X}}(\mathbb{C})$ with entries

$$(\mathbf{A}_i)_{\mathbf{xy}} = \begin{cases} 1 & \text{if } \partial(\mathbf{x}, \mathbf{y}) = i, \\ 0 & \text{if } \partial(\mathbf{x}, \mathbf{y}) \neq i, \end{cases} \qquad (\mathbf{x}, \mathbf{y} \in \mathcal{X}).$$

We abbreviate $\mathbf{A} = \mathbf{A}_1$ and call this the *adjacency matrix of* Γ. Let M denote the subalgebra of $\mathrm{Mat}_{\mathcal{X}}(\mathbb{C})$ generated by $\mathbf{A}$. By [3, p. 44], $\{\mathbf{A}_i\}_{i=0}^D$ is a basis for M. We call M the *Bose-Mesner algebra of* Γ. Observe that M is commutative and semi-simple. By [3, p. 45] there exists a basis $\{\mathbf{E}_i\}_{i=0}^D$ for M such that

$$\mathbf{E}_0 = |\mathcal{X}|^{-1}\mathbf{J}, \tag{6.1}$$

$$\sum_{i=0}^{d} \mathbf{E}_i = \mathbf{I}, \tag{6.2}$$

$$\mathbf{E}_i\mathbf{E}_j = \delta_{ij} \qquad (0 \leq i, j \leq D), \tag{6.3}$$

where $\mathbf{I}$ and $\mathbf{J}$ denote the identity and the all-ones matrix of $\mathrm{Mat}_{\mathcal{X}}(\mathbb{C})$, respectively and where δ_{ij} denotes the Kronecker delta. For convenience we define $E_i = 0$ whenever $i < 0$

or $i > d$. The matrices $\{E_i\}_{i=0}^D$ are known as the *primitive idempotents of* Γ, and $\mathbf{E}_0$ is called the *trivial* idempotent. We recall the eigenvalues of Γ. Since $\{E_i\}_{i=0}^D$ is a basis for M, there exist scalars $\{\theta_i\}_{i=0}^D$ in $\mathbb{C}$ such that

$$\mathbf{A} = \sum_{i=0}^{D} \theta_i \mathbf{E}_i. \tag{6.4}$$

Combining this with equations (6.1) and (6.3) we find that $\mathbf{A}\mathbf{E}_i = \mathbf{E}_i\mathbf{A} = \theta_i\mathbf{E}_i$ for $0 \le i \le D$ and $\theta_0 = k$. Observe that $\{\theta_i\}_{i=0}^D$ are mutually distinct since $\mathbf{A}$ generates M. We refer to θ_i as the *eigenvalue of* Γ *associated with* $\mathbf{E}_i$. For $0 \le i \le D$, let m_i denote the rank of $\mathbf{E}_i$. We call m_i the *multiplicity* of θ_i.

By equations (6.2), (6.3),

$$V = \sum_{i=0}^{D} \mathbf{E}_i V \qquad \text{(direct sum)}. \tag{6.5}$$

By linear interpolation,

$$\mathbf{E}_i = \prod_{j \neq i} \frac{\mathbf{A} - \theta_j \mathbf{I}}{\theta_i - \theta_j} \qquad (0 \le i \le D). \tag{6.6}$$

We now recall the Q-polynomial property. Note that $\mathbf{A}_i \circ \mathbf{A}_j = \delta_{ij}\mathbf{A}_i$ for $0 \le i, j \le D$, where $\circ$ is the entry-wise product. Therefore M is closed under $\circ$. Thus there exist $q_{ij}^h \in \mathbb{C}$ for $0 \le h, i, j \le D$ such that

$$\mathbf{E}_i \circ \mathbf{E}_j = |\mathcal{X}|^{-1} \sum_{h=0}^{d} q_{ij}^h \mathbf{E}_h \qquad (0 \le i, j \le D).$$

The scalars q_{ij}^h are called the *Krein parameters* of Γ. The ordering $\{\theta_i\}_{i=0}^D$ of eigenvalues for Γ is said to be *Q-polynomial* whenever the following (i), (ii) hold.

(i) $q_{ij}^h = 0$ if one of h, i, j is greater than the sum of the other two.

(ii) $q_{ij}^h \neq 0$ if one of h, i, j is equal to the sum of the other two.

The graph Γ is said to be *Q-polynomial* whenever there exists a Q-polynomial ordering $\{\theta_i\}_{i=0}^{D}$ of the eigenvalues of Γ.

6.2 The Terwilliger algebra

In this section we recall the dual Bose-Mesner algebra and the Terwilliger algebra of Γ. For the rest of this section fix $\mathbf{x} \in \mathcal{X}$. For $0 \leq i \leq D$ let $\mathbf{E}_i^* = \mathbf{E}_i^*(\mathbf{x})$ denote the diagonal matrix in $\mathrm{Mat}_{\mathcal{X}}(\mathbb{C})$ with entries

$$(\mathbf{E}_i^*)_{\mathbf{yy}} = \begin{cases} 1 & \text{if } \partial(\mathbf{x},\mathbf{y}) = i, \\ 0 & \text{if } \partial(\mathbf{x},\mathbf{y}) \neq i, \end{cases} \qquad (\mathbf{y} \in \mathcal{X}).$$

We call $\mathbf{E}_i^*$ the i^{th} *dual idempotent of* Γ *With regards to* $\mathbf{x}$. We observe

$$\sum_{i=0}^{d} \mathbf{E}_i^* = \mathbf{I}, \tag{6.7}$$

$$\mathbf{E}_i^*\mathbf{E}_j^* = \delta_{ij}\mathbf{E}_i^* \qquad (0 \leq i, j \leq D), \tag{6.8}$$

By construction $\{\mathbf{E}_i^*\}_{i=0}^{D}$ is linearly independent. Let $M^* = M^*(\mathbf{x})$ denote the subalgebra of $\mathrm{Mat}_{\mathcal{X}}(\mathbb{C})$ spanned by $\{\mathbf{E}_i^*\}_{i=0}^{D}$. We call M^* the *dual Bose-Mesner algebra of* Γ *With regards to* $\mathbf{x}$. We observe M^* is commutative and semi-simple.

Assume Γ is Q-polynomial with regards to the ordering $\{\theta_i\}_{i=0}^{D}$ of eigenvalues of Γ. For $0 \leq i \leq D$, let $\mathbf{A}_i^* = \mathbf{A}_i^*(\mathbf{x})$ denote the diagonal matrix in $\mathrm{Mat}_{\mathcal{X}}(\mathbb{C})$ with $(\mathbf{y}, \mathbf{y})$-entry

$$(\mathbf{A}_i^*)_{\mathbf{yy}} = |\mathcal{X}|(\mathbf{E}_i)_{\mathbf{xy}} \qquad (\mathbf{y} \in \mathcal{X}). \tag{6.9}$$

We call $\mathbf{A}^*$ the i^{th} *dual distance matrix of* Γ *With regards to* $\mathbf{x}$. The matrix $\mathbf{A}_1^*$ is often denoted $\mathbf{A}^*$ and referred to as the *dual adjacency matrix of* Γ *with regards to* $\mathbf{x}$. By [12, Lemma 3.11], M^* is generated by $\mathbf{A}^*$. We recall the dual eigenvalues of Γ. Since

$\{\mathbf{E}_i^*\}_{i=0}^D$ is a basis for M^* there exist scalars $\{\theta_i^*\}_{i=0}^D$ in $\mathbb{C}$ such that

$$\mathbf{A}^* = \sum_{i=0}^{D} \theta_i \mathbf{E}_i^*. \tag{6.10}$$

Combining this with equation (6.8) we find that $\mathbf{A}^*\mathbf{E}_i^* = \mathbf{E}_i^*\mathbf{A}^* = \theta_i^*\mathbf{E}_i^*$ for $0 \leq i \leq D$. The scalars are mutually distinct since $\mathbf{A}^*$ generates M^*. Note that θ_i^* is an eigenvalue of $\mathbf{A}^*$ and $\mathbf{E}_i^*V$ is the corresponding eigenspace for $0 \leq i \leq D$. By equations (6.7), (6.8),

$$V = \sum_{i=0}^{D} \mathbf{E}_i^* V \qquad \text{(direct sum)}. \tag{6.11}$$

We call the sequence $\{\theta_i^*\}_{i=0}^D$ the *dual eigenvalue sequence of* Γ. Observe that for $0 \leq i \leq D$ the rank of $\mathbf{E}_i^*$ is k_i. Therefore k_i is the multiplicity with which θ_i^* appears as an eigenvalue of $\mathbf{A}^*$.

By linear interpolation,

$$\mathbf{E}_i^* = \prod_{j \neq i} \frac{\mathbf{A}^* - \theta_j^* \mathbf{I}}{\theta_i^* - \theta_j^*} \qquad (0 \leq i \leq d). \tag{6.12}$$

By [12, Lemma 3.2] the following (i), (ii) hold for $0 \leq h, j \leq D$.

(i) $\mathbf{E}_j^*\mathbf{A}\mathbf{E}_h^* = 0$ if and only if $p_{ij}^h = 0$.

(ii) $\mathbf{E}_j\mathbf{A}^*\mathbf{E}_h = 0$ if and only if $q_{ij}^h = 0$.

Let $T = T(\mathbf{x})$ denote the subalgebra of $\mathrm{Mat}_{\mathcal{X}}(\mathbb{C})$ generated by M and M^*. We call T the *Terwilliger algebra of* Γ *With regards to* x. [12, Definition 3.3]

By a T*-module* we mean a subspace W of V such that $\mathbf{B}W \subseteq W$ for all $\mathbf{B} \in T$. Let W denote a T-module. Then W is said to be *irreducible* whenever W is nonzero and W contains no T-modules other than 0 and W.

By [12, Lemma 3.4(ii)] V decomposes into a direct sum of irreducible T-modules. Let W denote an irreducible T-module. By [12, Lemma 3.4(iii)] W is the direct sum of the nonvanishing $\mathbf{E}_i W$ $(0 \leq i \leq D)$ and the direct sum of the nonvanishing $\mathbf{E}^*_i V$ $(0 \leq i \leq D)$. By the *endpoint* of W we mean $\min\{i|0 \leq i \leq D, \mathbf{E}^*_i W \neq 0\}$. By the *diameter* of W we mean $|\{i|0 \leq i \leq D, \mathbf{E}^*_i W \neq 0\}| - 1$. By the *dual endpoint* of W we mean $\min\{i|0 \leq i \leq D, \mathbf{E}_i W \neq 0\}$. By the *dual diameter* of W we mean $|\{i|0 \leq i \leq D, \mathbf{E}_i W \neq 0\}| - 1$. By [10, Lemma 4.5] the diameter and the dual diameter of W coincide. Let r and r^* denote the endpoint and the dual endpoint of W, respectively, and let d denote the diameter of W. By [12, Lemma 3.9, Lemma 3.12] the following (i), (ii) hold for $0 \leq i \leq D$.

(i) $\mathbf{E}_i W \neq 0$ if and only if $r^* \leq i \leq r^* + d$.

(ii) $\mathbf{E}^*_i W \neq 0$ if and only if $r \leq i \leq r + d$.

Let W denote an irreducible T-module. By [12, Lemma 3.9, Lemma 3.12] the following (i), (ii) are equivalent.

(i) $\dim(\mathbf{E}_i W) \leq 1$ for $0 \leq i \leq D$.

(ii) $\dim(\mathbf{E}^*_i W) \leq 1$ for $0 \leq i \leq D$.

In this case W is called *thin*.

6.3 The hypercube Q_D

In this section we recall the hypercube graph and some of its basic properties. Let D denote a positive integer, and let $\{0, 1\}^D$ denote the set of sequences $\{t_i\}_{i=1}^D$ there $t_i \in \{0, 1\}$ for $1 \leq i \leq D$. Let Q_D denote the graph with vertex set $\mathcal{X} = \{0, 1\}^D$, and where two vertices are adjacent if and only if they differ in exactly one coordinate. We call Q_D the *D-cube* or a *hypercube.* The graph Q_D is connected and for $\mathbf{x}, \mathbf{y} \in \mathcal{X}$ the distance $\partial(\mathbf{x}, \mathbf{y})$ is the number of coordinates at which $\mathbf{x}$ and $\mathbf{y}$ differ. In particular the diameter of Q_D equals D. The graph Q_D is bipartite with bipartition $\mathcal{X} = \mathcal{X}^+ \cup \mathcal{X}^-$, where $\mathcal{X}^+$ (resp. $\mathcal{X}^-$) is the set of vertices of Q_D with an even (resp. odd) number of positive coordinates.

By [2, p. 304] Q_D is distance-regular with intersection numbers

$$a_i = 0, \qquad b_i = D - i, \qquad c_i = i, \qquad k_i = \binom{D}{i}, \qquad (0 \leq i \leq D). \tag{6.13}$$

For $0 \leq i \leq D$, $D - 2i$ is an eigenvalue with multiplicity $\binom{D}{i}$. The Graph Q_D is Q-polynomial With regards to the ordering $\{D - 2i\}_{i=0}^{D}$ of eigenvalues. When D is odd, this is the only Q-polynomial ordering of eigenvalues. By [2, p. 305] D is even, $\{(-1)^i(D - 2i)\}_{i=0}^{D}$ is also a Q-polynomial ordering of eigenvalues.

Notation 6.1 Let D denote a nonnegative integer. Let Q_D denote the hypercube with diameter D. Let V denote the standard module for Q_D. Let M denote the Bose-Mesner algebra of Q_D. Fix a vertex $\mathbf{x}$ of Q_D, let $M^* = M^*(\mathbf{x})$ denote the dual Bose-Mesner algebra of Q_D With regards to p and let $T = T(p)$ denote the Terwilliger algebra of Q_D With regards to $\mathbf{x}$. For $0 \leq i \leq D$, let $\mathbf{A}_i$ (resp. $\mathbf{A}_i^*$) denote the i^{th} distance matrix (resp. dual distance matrix) for Q_D With regards to the Q-polynomial ordering $\{D - 2i\}_{i=0}^{D}$ of eigenvalues with adjacency matrix $\mathbf{A} = \mathbf{A}_1$ and dual adjacency matrix $\mathbf{A}^* = \mathbf{A}_1^*$. For $0 \leq i \leq D$, let $\mathbf{E}_i$ (resp. $\mathbf{E}_i^*$) denote the i^{th} primitive idempotent for $\mathbf{A}$ (resp. $\mathbf{A}^*$) With regards to the Q-polynomial ordering $\{D - 2i\}_{i=0}^{D}$ of eigenvalues. For $0 \leq i \leq D$, when D is even, let $\mathbf{B}_i$ (resp. $\mathbf{F}_i$) denote the i^{th} dual distance matrix (resp. primitive idempotent of $\mathbf{A}$) associated with the Q-polynomial ordering $\{(-1)^i(D - 2i)\}_{i=0}^{D}$ of eigenvalues with dual distance matrix $\mathbf{B} = \mathbf{B}_1$.

The following results about the hypercube Q_D will be useful later.

Lemma 6.2 *With regards to Notation 6.1, the following (i), (ii) hold.*

(i) $(\mathbf{E}_{D-i})_{\mathbf{yz}} = (-1)^{\partial(\mathbf{y},\mathbf{z})}(\mathbf{E}_i)_{\mathbf{yz}} \qquad (\mathbf{y},\mathbf{z} \in \mathcal{X})$.

(ii) $\mathbf{A}^*_{D-i}$ *is diagonal with* $(\mathbf{A}^*_{D-i})_{\mathbf{yy}} = (-1)^{\partial(\mathbf{x},\mathbf{y})}(\mathbf{A}^*_i)_{\mathbf{yy}} \qquad (\mathbf{y} \in \mathcal{X})$.

Proof: (i) Let $\mathbf{S} \in \mathrm{Mat}_{\mathcal{X}}(\mathbb{C})$ denote the diagonal matrix whose $(\mathbf{y},\mathbf{y})$-entry is 1 whenever $\mathbf{y} \in \mathcal{X}^+$ and -1 whenever $\mathbf{y} \in \mathcal{X}^-$. Observe that the vector $v \in \mathbb{C}^{\mathcal{X}}$ is an eigenvector for Q_D with eigenvalue θ if and only if $\mathbf{S}.v$ is an eigenvector for Q_D with eigenvalue $-\theta$. The result follows routinely from this and the fact that $\mathbf{E}_i, \mathbf{E}_{D-i}$ project to the eigenspaces of Q_D with eigenvalues $D-2i$ and $2i-D$ respectively.

(ii) Immediate from (i) and Equation (6.9). □

By [6, (35)], the matrix $\mathbf{A}^*$ is diagonal with $(\mathbf{y},\mathbf{y})$-entry $D-2i$ where $i = \partial(\mathbf{x},\mathbf{y})$. Combining this result with Lemma 6.2(ii) we find that $\mathbf{A}^*_{D-1}$ is diagonal with $(\mathbf{y},\mathbf{y})$-entry $(-1)^i(D-2i)$ where $i = \partial(\mathbf{x},\mathbf{y})$.

6.4 $\mathfrak{sl}_2$-modules and hypercubes

In this section, we recall a result from Go relating hypercubes to $\mathfrak{sl}_2$-modules.

Lemma 6.3 [6, Theorem 13.2] *With regard to Notation 6.1 there exists a unique $\mathfrak{sl}_2$-module structure on V such that the generators X, Y act as $\mathbf{A}, \mathbf{A}^*$ respectively.*

Observe that X, Y generate $\mathfrak{sl}_2$ and $\mathbf{A}, \mathbf{A}^*$ generate T. Therefore the $\mathfrak{sl}_2$-module structure on V from Lemma 6.3 induces a surjective homomorphism of $\mathbb{C}$-algebras from $U(\mathfrak{sl}_2) \to T$. As a consequence we obtain the following Lemma.

Lemma 6.4 *With regard to Notation 6.1 , let V have the $\mathfrak{sl}_2$-module structure from Lemma 6.3 and let $W \subseteq V$ denote an irreducible T-module. Then W is irreducible as an $\mathfrak{sl}_2$-module.*

6.5 The antipodal quotient $\tilde{Q}_D$ of Q_D

In this section we recall the antipodal quotient of the hypercube and some of its basic properties. The hypercube is an antipodal 2-cover meaning that, for every vertex $\mathbf{y} \in \mathcal{X}$, there exists a unique vertex $\mathbf{y}' \in \mathcal{X}$ such that $\partial(\mathbf{y}, \mathbf{y}') = D$. The vertex $\mathbf{y}'$ is called the *antipode* of $\mathbf{y}$. Define the binary relation $\sim$ on $\mathcal{X}$ by $\mathbf{y} \sim \mathbf{z}$ whenever either $\mathbf{y} = \mathbf{z}$ or $\mathbf{y}' = \mathbf{z}$. Observe that $\sim$ is an equivalence relation. For every $\mathbf{y} \in \mathcal{X}$, the corresponding equivalence class will be denoted $[\mathbf{y}]$. Let $\tilde{\mathcal{X}}$ denote the set of equivalence classes of $\sim$. We define the graph $\tilde{Q}_D$ as follows. The vertex set is $\mathcal{X}$. Given $\mathbf{u}, \mathbf{v} \in \tilde{\mathcal{X}}$, $\mathbf{u}, \mathbf{v}$ are said to be adjacent in $\tilde{Q}_D$ if and only if there exist vertices $\mathbf{y}, \mathbf{z} \in \mathcal{X}$ such that $\mathbf{y} \in \mathbf{u}$, $\mathbf{z} \in \mathbf{v}$ and $\mathbf{x}, \mathbf{y}$ are adjacent in Q_D.

By [2, p. 306] $\tilde{Q}_D$ is distance-regular with diameter $\mathcal{D}$ where $\mathcal{D} = \lfloor \frac{D}{2} \rfloor$. Observe that, when D is even, $D = 2\mathcal{D}$ and, when D is odd, $D = 2\mathcal{D} + 1$. The intersection numbers of $\tilde{Q}_D$ are

$$a_i = 0, \qquad b_i = D - i, \qquad c_i = i, \qquad k_i = \binom{D}{i}, \qquad (0 \leq i \leq \mathcal{D} - 1), \tag{6.14}$$

$$a_{\mathcal{D}} = 0, \qquad b_{\mathcal{D}} = 0, \qquad c_{\mathcal{D}} = D, \qquad k_{\mathcal{D}} = \frac{1}{2}\binom{D}{\mathcal{D}}, \qquad (D \text{ even}), \tag{6.15}$$

$$a_{\mathcal{D}} = \mathcal{D} + 1, \qquad b_{\mathcal{D}} = 0, \qquad c_{\mathcal{D}} = \mathcal{D}, \qquad k_{\mathcal{D}} = \binom{D}{\mathcal{D}}, \qquad (D \text{ odd}). \tag{6.16}$$

For $0 \leq i \leq \mathcal{D}$, $D - 4i$ is an eigenvalue for $\tilde{Q}_D$ with multiplicity $\binom{2D}{2i}$. The graph $\tilde{Q}_D$ is Q-polynomial With regards to the ordering $\{D - 4i\}_{i=0}^{\mathcal{D}}$ of eigenvalues. When D is even, this is the only Q-polynomial ordering of eigenvalues. By [2, p. 305], when D is odd, $\{(-1)^i(D-2i)\}_{i=0}^{\mathcal{D}}$ is also a Q-polynomial ordering of the eigenvalues for $\tilde{Q}_D$. This is the ordering that we will be concerned with in this thesis.

Notation 6.5 Let the integer D, the graph Q_D and the vertex p be from Notation 6.1 and let $D = 2\mathcal{D}+1$ be odd. Let $\tilde{Q}_D$ denote the antipodal quotient of Q_D. Let $\tilde{V}$ denote the standard module for $\tilde{Q}_D$. Let $\tilde{M}$ denote the Bose-Mesner algebra of $\tilde{Q}_D$. Let $\tilde{M}^* = \tilde{M}^*([\mathbf{x}])$ denote the dual Bose-Mesner algebra of $\tilde{Q}_D$ With regards to $[\mathbf{x}]$ and let $\tilde{T} = \tilde{T}([\mathbf{x}])$ denote the Terwilliger algebra of $\tilde{Q}_D$ With regards to $[\mathbf{x}]$. For $0 \le i \le \mathcal{D}$, let $\tilde{\mathbf{A}}_i$ (resp. $\tilde{\mathbf{B}}_i$) denote the i^{th} distance matrix (resp. dual distance matrix) for $\tilde{Q}_D$ With regards to the Q-polynomial ordering $\{(-1)^i(D-2i)\}_{i=0}^{\mathcal{D}}$ of eigenvalues with adjacency matrix $\tilde{\mathbf{A}} = \tilde{\mathbf{A}}_1$ and dual adjacency matrix $\tilde{\mathbf{B}} = \tilde{\mathbf{B}}_1$. For $0 \le i \le D$, let $\tilde{\mathbf{E}}_i$ (resp. $\tilde{\mathbf{F}}_i$) denote the i^{th} primitive idempotent for $\tilde{\mathbf{A}}$ (resp. $\tilde{\mathbf{B}}$) With regards to the Q-polynomial ordering $\{(-1)^i(D-2i)\}_{i=0}^{\mathcal{D}}$ of eigenvalues.

6.6 $\mathcal{A}$-modules and Q_D

In Section 5.2, we displayed $\mathcal{A}$-module structures to finite-dimensional $\mathfrak{sl}_2$-modules. Go displayed $\mathfrak{sl}_2$-module structures to the primary module of the hypercube Q_D. In this section we display an $\mathcal{A}$-module structure on the primary module of the hypercube Q_D. We first recall the following result about irreducible T-modules.

Lemma 6.6 [6, Theorems 6.3, 8.1] *With regards to Notation 6.1, let W denote an irreducible T-module with endpoint r. Then the following (i)–(iv) hold.*

(i) *r satisfies $0 \le r \le D/2$.*

(ii) *W has diameter $D - 2r$.*

(iii) *W had dual endpoint r.*

(iii) *W is thin.*

For the following Lemma, recall the operators h from Definition 5.8, k from Lemma 5.14 and s from Definition 5.16.

Lemma 6.7 *With regards to Notation 6.1, let V be endowed with the $\mathfrak{sl}_2$-module structure from Lemma 6.3. Then the following (i)–(iii) hold.*

(i) *The matrix representing the action of h on V is diagonal with $(\mathbf{y}, \mathbf{y})$-entry $(-1)^i \mathbf{i}^D$ for each vertex $\mathbf{y}$ of Q_D where $i = \partial(\mathbf{x}, \mathbf{y})$.*

(ii) *The matrix representing the action of k on V is I whenever D is even and $-\mathbf{i}I$ whenever D is odd.*

(iii) *The matrix representing the action of s on V is diagonal with $(\mathbf{y}, \mathbf{y})$-entry $(-1)^{\lfloor \frac{D}{2} \rfloor + i}$ for each vertex $\mathbf{y}$ of Q_D where $i = \partial(\mathbf{x}, \mathbf{y})$.*

Proof: (i) By Lemma 6.3(i), the action of Y on V coincides with the action of $\mathbf{A}^*$ on V. By the comment at the end of Section 6.3, the matrix $\mathbf{A}^*$ is diagonal with $(\mathbf{y}, \mathbf{y})$-entry $d - 2i$. The result follows from this information along with Lemma 5.12.

(ii) By Lemma 6.6(ii), the dimension of every irreducible T-module has the same parity. By this and Lemma 5.14, k acts on V as 1 whenever D is even and as $-\mathbf{i}$ whenever D is odd, as desired.

(iii) Immediate from Definition 5.16 and parts (i), (ii). □

Lemma 6.8 *With regards to Notation 6.1, let V be endowed with the $\mathfrak{sl}_2$-module structure from Lemma 6.3. Let $\varepsilon = 1$ when D is congruent to 0 or 1 modulo 4 and $\varepsilon = -1$ when D is congruent to 2 or 3 modulo 4. Then the following (i),(ii) hold.*

(i) *In the first $\mathcal{A}$-module structure on V the generators x, y act on V as $\mathbf{A}, \varepsilon\mathbf{A}^*_{D-1}$ respectively.*

(ii) *In the second $\mathcal{A}$-module structure on V the generators x, y act on V as $\mathbf{A}, -\varepsilon\mathbf{A}^*_{D-1}$ respectively.*

Proof: (i) By Lemma 5.5, the generators x, y act on V as sX, Y respectively. By Lemma 6.3(i), the generators X, Y act as $\mathbf{A}, \mathbf{A}^*$ respectively. The matrix representing $s\mathbf{A}^*$ is diagonal with $(\mathbf{y}, \mathbf{y})$-entry $(-1)^{\mathcal{D}+i}(D - 2i)$ where $i = \partial(\mathbf{y}, \mathbf{y})$ and $\mathcal{D} = \lfloor \frac{D}{2} \rfloor$. By the comment at the end of Section 6.3, the matrix representing $\mathbf{A}^*_{D-1}$ is diagonal with $(\mathbf{y}, \mathbf{y})$-entry $(-1)^i(D - 2i)$ where $i = \partial(\mathbf{x}, \mathbf{y})$. The result follows.

(ii) Immediate from part (i) and Lemma 5.5. $\square$

Definition 6.9 With regards to Notation 6.1, let V be given an $\mathcal{A}$-module structure. We say the $\mathcal{A}$-module structure on V is *positive* whenever x, y act on V as $\mathbf{A}, \mathbf{A}^*_{D-1}$ respectively. We say the $\mathcal{A}$-module structure on V is *negative* whenever x, y act on V as $\mathbf{A}, -\mathbf{A}^*_{D-1}$ respectively

We now show the existence and uniqueness of positive and negative $\mathcal{A}$-module structures.

Lemma 6.10 *With regards to Notation 6.1, there exists a unique positive $\mathcal{A}$-module structure on V and a unique negative $\mathcal{A}$-module structure on V.*

Proof: By Lemma 6.8, one finds examples of positive and negative $\mathcal{A}$-module structures on V as given in the following table.

	D congruent to 0 or 1, mod 4	D congruent to 2 or 3 mod 4
positive	first	second
negative	second	first

The $\mathcal{A}$-module structures are unique because x, y generate $\mathcal{A}$. □

We will describe the irreducible $\mathcal{A}$-submodules of the positive $\mathcal{A}$-module structure on V.

Theorem 6.11 *With regards to notation 6.1, assume D is even. The following (i), (ii) hold*

(i) *There is a unique $\mathcal{A}$-module structure on V such that the generators x, y act as $\mathbf{A}, \mathbf{B}$ respectively.*

(ii) *Let W denote an irreducible T-module with endpoint r. Then W is an irreducible $\mathcal{A}$-module of type $B(D-2r)$.*

Proof: (i) Immediate from Lemma 6.10.

(ii) By Lemma 6.6(ii),(iv), W has dimension $D-2r+1$. The result follows from this information and Theorem 5.19. □

In Theorem 6.16 we will describe the irreducible $\mathcal{A}$-modules when D is odd. To do this, we introduce some useful results.

Lemma 6.12 [4, Lemma 4.1] *With regards to Notation 6.1, the following (i), (ii) hold.*

(i) $\mathbf{A}_D\hat{\mathbf{y}} = \hat{\mathbf{y}}' \qquad (\mathbf{y} \in \mathcal{X})$.

(ii) $\mathbf{A}_D : V \to V$ *is an isomorphism of vector spaces.*

Lemma 6.13 [4, Lemma 4.2] *With regards to Notation 6.1,* $\mathbf{A}_D^2 = \mathbf{I}$.

Lemma 6.14 [4, Lemma 4.3] *With regards to Notation 6.1, define*

$$V_+ = \mathrm{span}\{\hat{\mathbf{y}} + \hat{\mathbf{y}}' | \mathbf{y} \in \mathcal{X}\}, \qquad\qquad V_- = \mathrm{span}\{\hat{\mathbf{y}} - \hat{\mathbf{y}}' | \mathbf{y} \in \mathcal{X}\}.$$

Then the following (i)–(v) hold.

(i) $V = V_+ + V_-$ *(direct sum).*

(ii) $(\mathbf{A}_D - \mathbf{I})V_+ = 0$.

(iii) $(\mathbf{A}_D + \mathbf{I})V_- = 0$.

(iv) $(\mathbf{A}_D + \mathbf{I})V = V_+$.

(v) $(\mathbf{A}_D - \mathbf{I})V = V_-$.

Lemma 6.15 [4, Lemma 8.3] *With regards to Notation 6.1, let* V_+, V_- *be as in Lemma 6.14. Then the following (i), (ii) hold.*

(i) $V_+ = \sum_{i \text{ even}} \mathbf{E}_i V$ *(direct sum).*

(ii) $V_- = \sum_{i \text{ odd}} \mathbf{E}_i V$ *(direct sum).*

Theorem 6.16 *With regards to Notation 6.1, assume* $D = 2\mathcal{D} + 1$ *is odd. Let* V *be endowed with the* $\mathcal{A}$*-module structure from Lemma 6.10. Then the following (i), (ii) hold*

(i) *The subspaces* V_+, V_- *of* V *from Lemma 6.14 are* $\mathcal{A}$*-submodules of* V*.*

(ii) *Let* W *denote an irreducible* T*-module with endpoint* r*. Then* W *is a direct sum of two irreducible* $\mathcal{A}$*-modules* $(W \cap V_+), (W \cap V_-)$*. The space* $W \cap V_+$ *is an irreducible* $\mathcal{A}$*-module of type* $AB(\mathcal{D} - r, n)$ *where* n *is given in the following table.*

n	$\mathcal{D}$ *even*	$\mathcal{D}$ *odd*
r *even*	0	z
r *odd*	x	y

Moreover $W \cap V_-$ *is an irreducible* $\mathcal{A}$*-module of type* $AB(\mathcal{D} - r, n)$ *where* n *is given in the following table.*

n	$\mathcal{D}$ *even*	$\mathcal{D}$ *odd*
r *even*	y	x
r *odd*	z	0

Proof: (i) One routinely finds that

$$\mathbf{A}V_+ \subseteq V_+, \qquad \mathbf{A}^*_{D-1}V_+ \subseteq V_+, \tag{6.17}$$

$$\mathbf{A}V_+ \subseteq V_-, \qquad \mathbf{A}^*_{D-1}V_+ \subseteq V_-. \tag{6.18}$$

The result follows from the above equations, Lemma 6.10 and the fact that x, y generate $\mathcal{A}$.

(ii) By Lemma 6.4, W is irreducible as an $\mathfrak{sl}_2$-module and, by Lemma 6.6(ii), W has diameter $D - 2r$. By Theorem 5.20 and the fact that $D - 2r$ is odd, one finds that W is the direct sum of two irreducible $\mathcal{A}$-modules each with diameter $\mathcal{D} - r$. By Theorem 5.20 and Lemma 6.8, the two irreducible modules are of type $AB(\mathcal{D} - r, 0)$ and $AB(\mathcal{D} - r, x)$

when $\mathcal{D}-r$ is even and they are of type $AB(\mathcal{D}-r,y)$ and $AB(\mathcal{D}-r,z)$ when $\mathcal{D}-r$ is odd. By Lemmas 6.15, 6.6(iii), the $\mathcal{A}$-modules $W\cap V_+$ and $W\cap V_-$ each have diameter $\mathcal{D}-r$ and, by Lemma 6.14, their direct sum is W. Therefore $W\cap V_+$ and $W\cap V_-$ are the two irreducible $\mathcal{A}$-submodules of W from Theorem 5.20. By line (2.7) and Lemmas 6.15, 6.6(iii), the action of A on V_+ has trace $(-1)^r(\mathcal{D}-r+1)$ and the action of A on V_- had trace $(-1)^{r+1}(\mathcal{D}-r+1)$. Comparing these traces to the traces for x in Lemmas 2.29–2.32, one routinely obtains the result as desired. $\square$

6.7 $\mathcal{A}$-modules and $\tilde{Q}_D$

In Section 6.6, we displayed an $\mathcal{A}$-module structure on the primary module of the hypercube Q_D. In this section we will display an $\mathcal{A}$-module structure for the standard module of the antipodal quotient $\tilde{Q}_D$ and we will describe the irreducible $\mathcal{A}$-modules when D is odd. Before this we describe the irreducible $\tilde{T}$-modules.

Definition 6.17 With regards to Notation 6.1, 6.5, let $\psi : V \to \tilde{V}$ denote the vector space homomorphism sending $\hat{\mathbf{y}} \mapsto \widehat{[\mathbf{y}]}$.

Lemma 6.18 [4, Lemma 10.4] *With regards to Notation 6.5, let ψ be from Definition 6.17. Then the following (i), (ii) hold.*

(i) $V_- = \mathrm{Ker}\psi$.

(ii) *Let φ denote the restriction of ψ to V_+. Then $\varphi : V_+ \to \tilde{V}$ is an isomorphism of vector spaces.*

Lemma 6.19 [4, Corollary 11.2] *With regards to Notation 6.1, 6.5, the map* $W \mapsto \psi(W)$ *is a bijection from the set of* T*-modules to the set of* $\tilde{T}$*-modules.*

Lemma 6.20 *With regards to Notation 6.1, 6.5, let* W *denote an irreducible* $\tilde{T}$*-module with endpoint* r*. Then the following (i)–(iii) hold.*

(i) W *has diameter* $\mathcal{D} - r$*.*

(ii) W *has dual endpoint* r*.*

(iii) W *is thin.*

Proof: Follows from Lemmas 6.6, 6.15, 6.18, 6.19. □

Theorem 6.21 *With regards to Notation 6.5 assume* $D = 2\mathcal{D} + 1$ *is odd. Then the following (i), (ii) hold.*

(i) *There is a unique* $\mathcal{A}$*-module structure on* $\tilde{V}$ *such that the generators* x, y *act as* $\tilde{\mathbf{A}}, \tilde{\mathbf{B}}$ *respectively.*

(ii) *Let* W *denote an irreducible* $\tilde{T}$*-module with endpoint* r*. Then* W *is an irreducible* $\mathcal{A}$*-module of type* $AB(\mathcal{D} - r, n)$ *where* n *given in the following table.*

n	$\mathcal{D}$ *even*	$\mathcal{D}$ *odd*
r *even*	0	z
r *odd*	x	y

Proof: (i) With regards to Notation 6.1 let the vector space homomorphism φ be from Lemma 6.18(ii). Recall that φ is an isomorphism of vector spaces from V_+ to $\tilde{V}$. One routinely finds that $\varphi \circ \mathbf{A} \circ \varphi^{-1} = \tilde{\mathbf{A}}$ and $\varphi \circ \mathbf{A}^*_{D-1} \circ \varphi^{-1} = \tilde{\mathbf{B}}$. By this and Lemma 6.10 one finds the existence of an $\mathcal{A}$-module structure on $\tilde{V}$ such that the generators x, y act as $\tilde{\mathbf{A}}, \tilde{\mathbf{B}}$ respectively. The $\mathcal{A}$-module structure is unique because x, y generate $\mathcal{A}$.

(ii) By construction and Lemma 6.18(ii), the vector space isomorphism $\varphi : V_+ \to \tilde{V}$ is an isomorphism of $\mathcal{A}$-modules. Therefore the $\mathcal{A}$-module $\varphi^{-1}(W) \subseteq V_+$ has diameter $\mathcal{D} - r$. The isomorphism class of $\varphi^{-1}(W)$ and hence of W, can be routinely computed using the above information along with the data from Theorem 6.16. The result follows. $\square$

6.8 Leonard triples from Q_D and $\tilde{Q}_D$

In Section 6.6 (resp. Section 6.7) we found that every irreducible T-module (resp. $\tilde{T}$-module) induces a finite-dimensional irreducible $\mathcal{A}$-module. In this section, we show that every irreducible T-module (resp. $\tilde{T}$-module) induces a totally bipartite (resp. totally almost bipartite) Leonard triple of Bannai/Ito type. We this section we describe the Leonard triples arising from our construction.

In Lemma 6.3, Go displayed an $\mathfrak{sl}_2$-module structure on V on which X, Y act as $\mathbf{A}, \mathbf{A}^*$ respectively. In [18], Miklavič introduced a matrix $\mathbf{A}^\varepsilon \in T$ that acts on V as Z under this $\mathfrak{sl}_2$-module structure. In Section 6.6 we displayed an $\mathcal{A}$-module structure on V (resp. $\tilde{V}$) on which $\mathbf{A}, \mathbf{B}$ (resp. $\tilde{\mathbf{A}}, \tilde{\mathbf{B}}$) act as x, y respectively. In this section we introduce the matrix $\mathbf{C} \in T$(resp. $\tilde{\mathbf{C}} \in \tilde{T}$) that act on V (resp. $\tilde{V}$) as z under these $\mathcal{A}$-module structures.

We first define the matrix C.

Definition 6.22 With regards to Notation 6.1, let $\mathbf{C} \in \mathrm{Mat}_{\mathcal{X}}(\mathbb{C})$ denote the matrix representing the action of z on V under the positive $\mathcal{A}$-module structure.

Proposition 6.23 *With regards to Notation 6.1, the matrix* $\mathbf{C}$ *from Definition 6.22 is contained in* T.

Proof: Immediate from equation (2.1) and Lemma 6.10. $\square$

Definition 6.24 Given a graph Γ with vertex set $\mathcal{X}$ and edge set $\mathcal{R}$, a matrix $\mathbf{N} \in \mathrm{Mat}_{\mathcal{X}}(\mathbb{C})$ is said to be a *weighted adjacency matrix* for Γ whenever the $(\mathbf{y}, \mathbf{z})$-entry is nonzero for all adjacent $b, c \in \mathcal{X}$ and zero for all nonadjacent $\mathbf{y}, \mathbf{z} \in \mathcal{X}$.

Proposition 6.25 *With regards to Notation 6.1, the matrix* C *from Definition 6.22 is a weighted adjacency matrix for* Q_D*. For adjacent* $\mathbf{y}, \mathbf{z} \in \mathcal{X}$*, the* $(\mathbf{y}, \mathbf{z})$*-entry of* $\mathbf{C}$ *is* $(-1)^i$ *where* i *is the minimum of* $\partial(\mathbf{x}, \mathbf{y})$ *and* $\partial(\mathbf{x}, \mathbf{z})$.

Proof: Routine calculation using equation (2.1) and Theorem 6.11. $\square$

We now define the matrix $\tilde{\mathbf{C}}$.

Definition 6.26 With regards to Notation 6.5, let $\tilde{\mathbf{C}} \in \mathrm{Mat}_{\tilde{\mathcal{X}}}(\mathbf{C})$ denote the matrix representing the action of z on $\tilde{V}$ under the $\mathcal{A}$-module structure from Theorem 6.21.

Proposition 6.27 *With regards to Notation 6.5, the matrix* $\tilde{\mathbf{C}}$ *from Definition 6.26 is contained in* $\tilde{T}$.

Proof: Immediate from equation (2.1) and Theorem 6.21(i). ☐

We have a few comments about Lemma 6.14. Recall that, when D is odd, the space V_+ is a T-submodule of V. By this and Proposition 6.23, V_+ is closed under the action of $\mathbf{C}$.

Proposition 6.28 *With regards to Notations 6.1, 6.5, let D be odd and let $\varphi : V \rightarrow \tilde{V}$ be from Lemma 6.18(ii). Then $\tilde{\mathbf{C}} = \varphi \circ \mathbf{C} \circ \varphi^{-1}$ on V.*

Proof: By construction, φ is an isomorphism of $\mathcal{A}$-modules from V_+ under the positive $\mathcal{A}$-module structure to $\tilde{V}$ under the $\mathcal{A}$-module structure from Theorem 6.21. The result follows from this information along with Definitions 6.22, 6.26. ☐

Proposition 6.29 *With regards to Notation 6.5, the matrix $\tilde{\mathbf{C}}$ from Definition 6.26 is a weighted adjacency matrix for $\tilde{Q}_D$. For adjacent $\mathbf{u}, \mathbf{v} \in \tilde{\mathcal{X}}$, the $(\mathbf{u}, \mathbf{v})$-entry of $\tilde{\mathbf{C}}$ is $(-1)^i$ where i is the minimum of $\partial([\mathbf{x}], \mathbf{u})$ and $\partial([\mathbf{x}], \mathbf{v})$.*

Proof: Routine calculation using Definitions 6.17. 6.22, 6.26 and Proposition 6.28. ☐

From this correspondence we obtain the following results.

Theorem 6.30 *With regards to Notation 6.1, let W denote an irreducible T-module. Then the actions of $\mathbf{A}, \mathbf{B}, \mathbf{C}$ on W form a totally bipartite Leonard triple of Bannai/Ito type.*

Proof: Immediate from Theorem 4.27. ☐

Theorem 6.31 *With regards to Notation 6.5, let W denote an irreducible $\tilde{T}$-module. Then the actions of $\tilde{\mathbf{A}}, \tilde{\mathbf{B}}, \tilde{\mathbf{C}}$ on W form a totally almost bipartite Leonard triple of Bannai/Ito type.*

Proof: Immediate from Theorem 4.27. □

We have found that every irreducible T-module (resp. $\tilde{T}$-module) induces a totally bipartite (resp. totally almost bipartite) Leonard triple of Bannai/Ito type. Previously, we classified the totally B/AB Leonard triples of Bannai/Ito type. In this section we describe the Leonard triples arising from our construction.

Theorem 6.32 *With regards to Notation 6.1, let D be even and let $\mathbf{C}$ be from Definition 6.22. Let W denote an irreducible T-module with endpoint r. Then the Leonard triple formed by the actions actions of $\mathbf{A}, \mathbf{B}, \mathbf{C}$ on W is normalized with diameter $D - 2r$.*

Proof: By Theorem 6.11 and Definition 6.22, W is an irreducible $\mathcal{A}$-module of type $B(D-2r)$ on which x, y, z act as $\mathbf{A}, \mathbf{B}, \mathbf{C}$ respectively. By this and Proposition 4.35, the Leonard triple formed by the actions actions of $\mathbf{A}, \mathbf{B}, \mathbf{C}$ on W is normalized with diameter $D - 2r$. □

Theorem 6.33 *With regards to Notations 6.1, 6.5, let $\tilde{\mathbf{C}}$ be from Definition 6.26. Let W denote an irreducible $\tilde{T}$-module with endpoint r. Then the Leonard triple formed by the actions of $\tilde{\mathbf{A}}, \tilde{\mathbf{B}}, \tilde{\mathbf{C}}$ on W is n-normalized with diameter $\mathcal{D} - r$ where n is given in the following table.*

n	$\mathcal{D}$ *even*	$\mathcal{D}$ *odd*
r *even*	0	z
r *odd*	x	y

Proof: By Theorem 6.21 and Definition 6.26, W is an irreducible $\mathcal{A}$-module of type $AB(\mathcal{D}-r, n)$ on which x, y, z act as $\tilde{\mathbf{A}}, \tilde{\mathbf{B}}, \tilde{\mathbf{C}}$ respectively. By this, Propositions 4.35 and 4.43, the Leonard triple formed by the actions actions of $\mathbf{A}, \mathbf{B}, \mathbf{C}$ on W is n-normalized with diameter $D - 2r$. $\square$

Bibliography

[1] M. Arik and U. Kayserilioglu, *The anticommutator spin algebra, its representations and quantum group invariance*, Int. J. Mod. Phys., 18 (2003), pp. 5039–5046.

[2] E. Bannai and T. Ito, *Algebraic Combinatorics I: Association Schemes*, Benjamin & Cummings, Mento Park, CA, 1984.

[3] A. E. Brouwer, A. M. Cohen, and A. Neumaier, *Distance-Regular Graphs*, Springer Verlag, New York, NY, 1989.

[4] B. V. C. Collins, *The terwilliger algebra of an almost-bipartite distance-regular graph and its antipodal 2-cover*, Discrete Mathematics, 216 (2000), pp. 35–69.

[5] B. Curtin, *Modular Leonard triples*, Linear Algebra Appl., 424 (2007), pp. 510–539.

[6] J. T. Go, *The Terwilliger algebra of the hypercube*, Europ. J. Combinatorics, 23 (2002), pp. 399–429.

[7] P A. Grillet, *Abstract Algebra*, Second Edition, Springer Science+Business Media, LLC, 2007.

[8] M. Havlíček, A. U. Klimyk, and S. Pošta, *Representations of the cyclically symmetric q-deformed algebra* $\mathfrak{so}_q(3)$, Journal of Mathematical Physics, 20 (1990), pp. 2135–2161.

[9] J. E. Humphries, *Introduction to Lie Algebras and Representation Theory*, Springer Science+Business Media Inc., 1972.

[10] T. Ito, K. Tanabe, and P. Terwilliger, *Some algebra related to p- and q-polynomial association schemes*, DIMACS Series in Discrete Mathematics and Theoretical Computer Science, 56 (2001), pp. 167–192.

[11] K. Nomura and P. Terwilliger, *Linear transformations that are tridiagonal with respect to both eigenbases of a leonard pair*, Linear Algebra Appl., 420 (2007), pp. 198–207.

[12] P. Terwilliger, *The subconstituent algebra of an association scheme, Part III*, J. Algebraic Combin., 2 (1993), pp. 177–210.

[13] ——, *Two linear transformations each tridiagonal with respect to an eigenbasis of the other*, Linear Algebra Appl., 330 (2001), pp. 149–203.

[14] ——, *Leonard pairs and the q-Racah polynomials*, Linear Algebra Appl., 387 (2004), pp. 235–276.

[15] ——, *Two linear transformations each tridiagonal with respect to an eigenbasis of the other; comments on the parameter array*, Designs, Codes and Cryptography., (2005).

[16] P. Terwilliger and R. Vidunas, *Leonard pairs and the Askey-Wilson relations*, Journal of Algebra and Its Applications, 2 (2004), pp. 411–426.

[17] S. Tsujimoto, L. Vinet, and A.S. Zhedanov, *Dunkl shift operators and Bannai-Ito polynomials*, (2011).

[18] Š. Miklavič, *Leonard triples and hypercubes*, J. Algebraic Combin., 28 (2008), pp. 397–424.

[19] E. Zeidler, *Quantum Field Theory III: Gauge Theory: A Bridge between Mathematicians and Physicists*, Springer; 1st Edition, New York, 2011.