On A_∞-Algebras and Partition Functions

By

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Abstract

Moduli spaces of curves of genus g with n marked points, $\mathcal{M}_{g,n}$, are one of the main objects of study in algebraic geometry. Kontsevich has described the construction of a class, called partition function, in $H^*(\mathcal{M}_{g,n})^{\Sigma_n}$, using even cyclic A_∞-algebras and ribbon graphs with orientation. Igusa has given an explicit construction of the partition function using an orientation defined by Conant and Vogtmann. Amorim and Tu have given a similar construction for odd cyclic A_∞-algebras and ribbon graphs using a so-called twisted orientation. In this thesis we define a new notion of orientation for ribbon graphs and A_∞-algebras of arbitrary parity, and we show that it matches with the one of Conant-Vogtmann (in the even case) and with the one of Amorim-Tu (in the odd case). We also give a graphical representation of the orientation which allows us to draw the ribbon graph in a canonical way, and this leads to a canonical expression of the partition function.

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Chapter 1

Introduction

Moduli spaces of smooth and compact curves of genus g with n marked points, $\mathcal{M}_{g,n}$, are one of the main objects of study in algebraic geometry. They have a rich history going back to Riemann, but in their modern form were introduced by Deligne and Mumford [4]. The topology of these moduli spaces is related to Teichmüller theory. Mapping class groups act on Teichmüller spaces and the quotient is homeomorphic to moduli space of curves.

Ribbon graphs are graphs with a cyclic order of half edges given at each vertex. Strebel [13] and Penner [10] have discovered that there is a close relation between ribbon graphs and the geometry of moduli spaces of curves. If one thickens each edge of a ribbon graph into a ribbon, one gets an oriented surface with a boundary. This surface will have genus g and n boundary components. Strebel, Penner and Kontsevich have explained the relation between the homology of $\mathcal{M}_{g,n}$ and $R_{g,n}$, a chain complex spanned by ribbon graphs of genus g and n boundary components, with a particular differential d.

A_∞-algebras were first defined by Stasheff [12] in 1963. They feature prominently in the theory of Mirror Symmetry. They allow to give a finite dimensional model for the derived category of any complex compact manifold [7]. Further endowed with an extra structure, a pairing with respect to which the operations are cyclic, A_∞- algebras correspond to Calabi-Yau manifolds.

Kontsevich has realized that there is a close relation between cyclic A_∞-algebras and the cohomology of ribbon graphs. He was able to give a definition of a so-called partition function, a function which takes as an input a cyclic A_∞-algebra and a ribbon graph and gives us a number. The precise definition requires one to specify extra data, an orientation of the ribbon graph. This notion depends on the parity of the A_∞-algebra. In the even case the orientation we need is given by Conant and Vogtmann [3]. In this case Igusa gave an explicit formula for the partition function [5], using the orientation defined by Conant and Vogtmann. In the odd A_∞-algebra case the orientation, called twisted orientation, is given by Amorim and Tu [1].

The purpose of this thesis is to give a unified interpretation of the orientation given in [3] and the twisted orientation given in [1]. We will give a graphical explanation of the orientation that helps us understand Kontsevich's partition function in both contexts.

Given a cyclic A_∞-algebra, a ribbon graph and 4 choices over the graph (order the edges, orient the edges, order the vertices and pick a starting half edge at each vertex) we will give a canonical representation of the ribbon graph, for which we can recover a canonical expression for Kontsevich's partition function. In the even A_∞-algebra case this recovers the orientation of Conant and Vogtmann [3]. In the odd A_∞-algebra case we get the twisted orientation of Amorim and Tu [1].

Chapter 2

Signs with Graphical Conventions

2.1 The Symmetric Monoidal Structure on Graded Vector Spaces

Let k be a field. A $\mathbb{Z}$-graded vector space V is a vector space together with a decomposition

$$V = \bigoplus_{i \in \mathbb{Z}} V^i$$

where the elements $v \in V^i$ will be said to be homogeneous elements of degree i. The degree of such v is denoted by $|v| = i$. For any integer n, $V[n]$ denotes the graded vector space with shifted grading $V[n]^i = V^{n+i}$ and the underlying space V.

The category of graded vector spaces over the field k equipped with the usual tensor product $\otimes$ and a natural isomorphism between $V \otimes W$ and $W \otimes V$, forms a symmetric monoidal category, denoted by $(grVect_k, \otimes)$.

2.2 Koszul Rule of Signs

Koszul rule of signs gives us the natural isomorphism between $V \otimes W$ and $W \otimes V$ for V and W in $grVect_k$ as the twisting map

$$tw(v \otimes w) = (-1)^{|v||w|} w \otimes v$$

on homogeneous elements $v \in V^i$ and $w \in W^j$. This rule extends to V and W by linearity. Note that $tw \circ tw$ is the identity map, i.e. tw is an involution.

A linear map of degree n between graded vector spaces is a linear map $f : V \to W$ such that $f(V^i) \subseteq W^{i+n}$ for all $i \in \mathbb{Z}$. We use the same notation for the degree as before, $|f| = n$.

Let V, W, A and B be graded vector spaces, $f : V \to W$ and $g : A \to B$ be linear maps of degree m and n, respectively. Then $f \otimes g : V \otimes A \to W \otimes B$ is a linear map satisfying the Koszul rule of signs

$$(f \otimes g)(v \otimes w) = (-1)^{|g||v|} f(v) \otimes g(w)$$

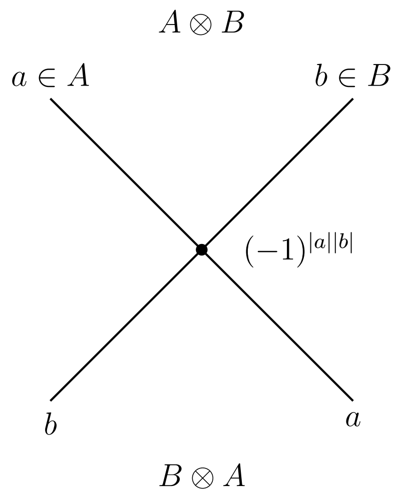
for homogeneous elements $v \in V^i$ and $w \in W^j$.

2.3 Graphical Representations

We will give a graphical representation for maps between tensor products of graded vector spaces. All the graphical representations are read from top to bottom. Let us describe the fundamental pieces of the diagrams and what operations they represent.

In the above notation whenever we switch two consecutive elements (morphisms or elements in the vector space) we get a sign depending on the product of their degrees.

We can represent this interchange by using crossing strands. This crossing will give us a sign depending on the signs of the incoming edges.

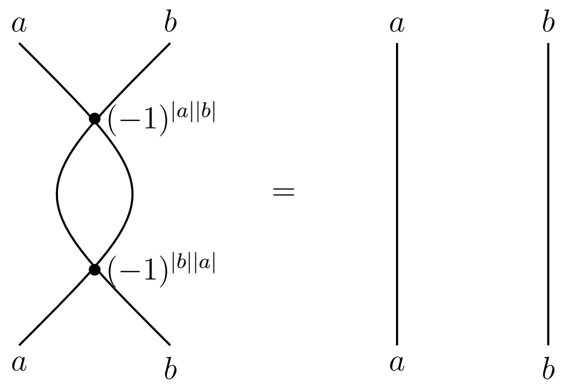


2.4 Shifted Signs

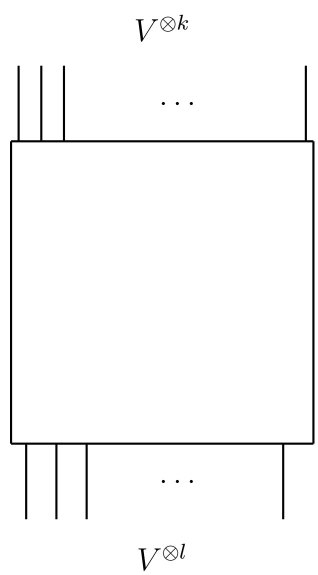
Another convention that is used for the signs is the shifted sign convention. Let us have a homogeneous degree d element $v \in V$, i.e. $v \in V^d$. If we consider this v in the shifted graded vector space $V[1]$ it will have degree $d-1$. This follows from the fact that $V[1]^{d-1} = V^d$. We will denote the shifted degree of v by $|v|'$. We have $|v|' = |v| - 1$.

2.5 More on Graphical Representations

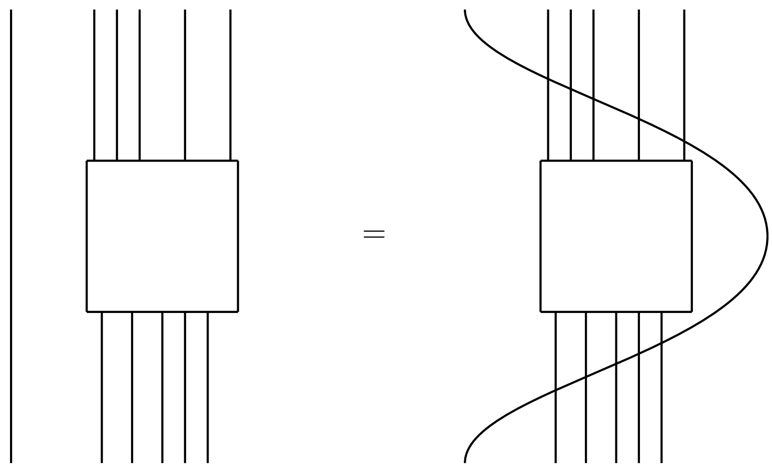
If we cross two strands twice, both crossings will give us the same sign. We can represent this fact by the equivalence of the following two diagrams



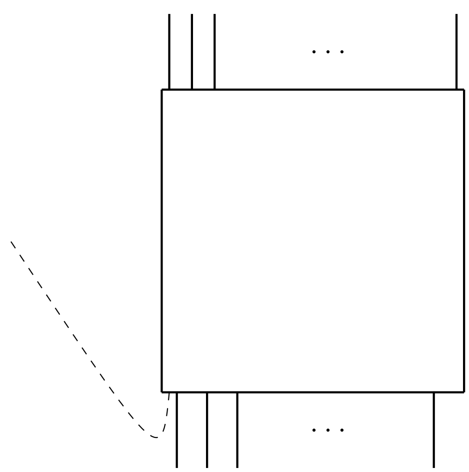
We can also represent a multi-tensor map $f : V^{\otimes k} \to V^{\otimes l}$ by the following diagram



In the above diagram if we have k inputs, the edges above the box, and l outputs, the edges below the box, it represents a multi-tensor map $V^{\otimes k} \to V^{\otimes l}$. For multi-tensor maps of even degree we have an additional equivalence of the following two diagrams



The above relation will help us move strands around even maps and help us disentangle bigger pictures. We want to be able to do something similar for odd degree multi-tensor maps. To deal with that problem we will add a dashed line to the box as below (see, for example, Sheridan [11])

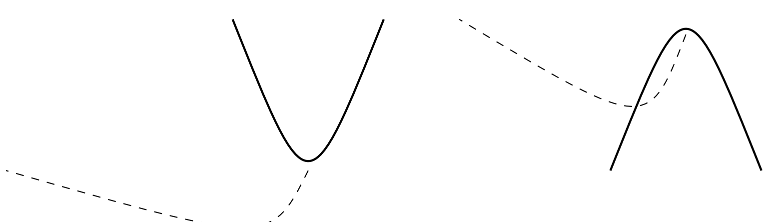


The dashed line is considered to be a purely odd degree line and makes our diagram even total degree. One end of the dashed line is connected to our odd diagram. The other end is free. As a convention we will assume that the free ends will go to the left of the picture. When we have multiple dashed lines we will assume that their free ends are kept in a predefined order. Note that intersection of two dashed lines always introduces a negative sign.

2.6 Odd and Even Pairings

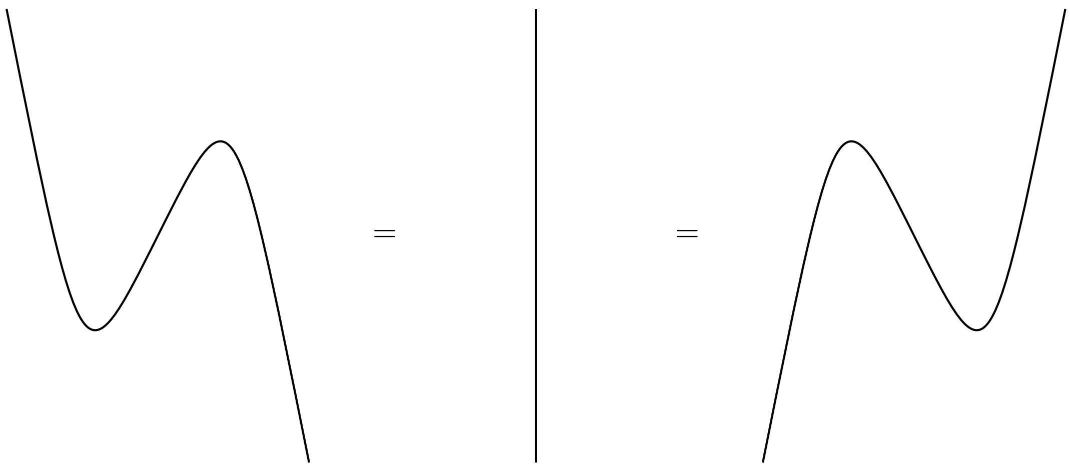
We have two more fundamental pieces of our graphical representations, cup and cap. We will assume that our vector space V is endowed with a perfect pairing. Cup represents the pairing (inner product) $\langle -, - \rangle : V \otimes V \to k$ on V defined over the field k. Let V be finite dimensional with basis $v_1, \ldots, v_n \in V$. This will give us an isomorphism $\phi : V \to V^*$. Let v_i^* be the dual of v_i. Let us also define $v^i = \phi^{-1}(v_i^*)$. Cap represents the map $k \to V \otimes V : 1 \mapsto \sum_i v_i \otimes v^i$. Let us note that $\langle v^i, v_i \rangle = 1$ and $\langle v^i, v_j \rangle = 0$ for $i \neq j$. The following diagrams represent even degree cup and cap, respectively

Similarly for the degree reasons, we will add dashed lines for the odd degree maps as below.



We will only discuss homogeneous pairings.

Theorem 2.1 *The following three diagrams are equivalent*



Proof:

Let us show that the diagram on the left is equivalent to identity (the diagram in the middle).

Let us read the diagram on the left from top to bottom. First we have the input, $x \in V$. We can write the coordinates of x in both basis as $x = \sum_i a_i v_i$ and $x = \sum_i b_i v^i$. Note that $\langle v^i, x \rangle = a_i$ and $\langle x, v_j \rangle = b_j$. First thing that we come across is the cap. That gives us the map

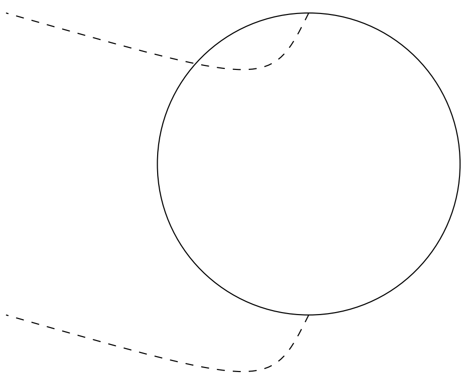
$$\begin{aligned} id \otimes cap : V \to V \otimes V \otimes V : x \mapsto & \quad x \otimes \sum_i v_i \otimes v^i \\ = & \quad \sum_i x \otimes v_i \otimes v^i \end{aligned}$$

As a second and final operation that we see in our diagram we have the cup product of the left 2 terms.

$$\begin{aligned}\langle -,- \rangle \cdot id : V \otimes V \otimes V \to V : \sum_i x \otimes v_i \otimes v^i \mapsto & \quad \sum_i \langle x, v_i \rangle \cdot v^i \\ = & \quad \sum_i b_i \cdot v^i \\ = & \quad x\end{aligned}$$

Hence the diagram on the left corresponds to an identity map. We can give a similar argument for the diagram on the right. This gives us freedom to remove kinks from strands.

Remark. Let V be a vector space with odd and even degree parts, $V = V_{odd} \oplus V_{even}$. If cup and cap are both odd the following picture gives us the super-dimension of V.



Here the dashed lines intersect our graph only once and since they have degree one we might get a sign depending on the solid edge. If we read the diagram from top to bottom, we first have a cap then an intersection with a dashed line and finally a cup. This corresponds to

$$\begin{aligned}1 \mapsto \sum_i v_i \otimes v^i \mapsto \sum_i (-1)^{|v_i|} v_i \otimes v^i \mapsto \sum_i (-1)^{|v_i|} \langle v_i, v^i \rangle \quad & = \quad \sum_i (-1)^{|v_i|} \\ & = \quad \dim V_{even} - \dim V_{odd} \\ & = \quad 0\end{aligned}$$

This is a confirmation of the fact that our pairings are defined in the correct way. Doing the evaluation over a circle will give us the super dimension.

Chapter 3

A_∞-Algebras

3.1 Definition

A_∞-algebras were invented by Stasheff [12] at the beginning of sixties. There are two sign conventions (due to Cho [2]) when defining A_∞-algebras, one using the regular signs and the other using shifted signs.

Definition 3.1 *Let k be a field. An A_∞-algebra (with no shifting) over k is a $\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z}$-graded vector space with graded structure maps $m_n^{ns} : A^{\otimes n} \to A$ of degree $2-n$ satisfying the following relation for each $k \geq 1$*

$$\sum_{r+s+t=k} (-1)^{\varepsilon_1} m_{r+1+t}^{ns}(x_1, \ldots, x_r, m_s^{ns}(x_{r+1}, \ldots, x_{r+s}), x_{r+s+1}, \ldots, x_k) = 0$$

where $\varepsilon_1 = (r+1)(s+1) + s(|x_1| + \ldots + |x_r|)$

Definition 3.2 *Let k be a field. An A_∞-algebra over k is a $\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z}$-graded vector space with graded structure maps $m_n : A[1]^{\otimes n} \to A[1]$ of degree 1 satisfying the following relation for each $k \geq 1$*

$$\sum_{r+s+t=k} (-1)^{\varepsilon_2} m_{r+1+t}(x_1, \ldots, x_r, m_s(x_{r+1}, \ldots, x_{r+s}), x_{r+s+1}, \ldots, x_k) = 0$$

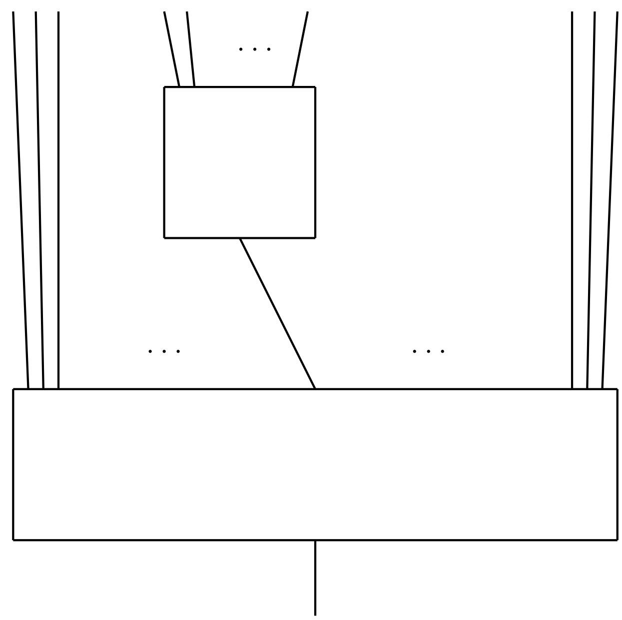
where $\varepsilon_2 = |x_1|' + \ldots + |x_r|'$ and $|x_i|' = |x_i| + 1$ is the shifted sign.

Remark 3.3 *Given an A_∞-algebra with structure maps satisfying the non-shifted convention in definition 3.1, we can obtain an A_∞-algebra with shifted sign convention in definition 3.2 by replacing m_k's by m_k^{ns} as follows.*

$$m_k(x_1, \ldots, x_k) = (-1)^{\sum_{i=1}^{k-1}(k-i)|x_i|} m_k^{ns}(x_1, \ldots, x_k)$$

We can represent both defining equations for the A_∞-algebras by using the graphical representations from the previous chapter.

$$0 \quad = \sum_{k-inputs} \quad \pm$$



In the above picture we have k inputs fixed. We apply two multiplications as seen above so that we get one output. The summation is over all ways to apply two such graded structure maps, m_k's.

From now on, unless otherwise noted, we will assume our A_∞-algebras are defined using the shifted sign convention.

3.2 Properties

Some immediate results from the definition:

- We have $m_1 \circ m_1 = 0$ and (A, m_1) is a differential complex.
- In the first convention if m_1^{ns} and m_2^{ns} are the only nontrivial maps we get a dg-algebra with m_1^{ns} as the differential and m_2^{ns} as the product. This follows from the fact that $m_1^{ns} m_2^{ns} = m_2^{ns}(m_1^{ns} \otimes 1 + 1 \otimes m_1^{ns})$.
- In the first convention if only nontrivial product is m_2^{ns} then A is an associative graded algebra.
- In general, m_2 is associative up to homotopy, given by m_3.

3.3 Examples

1. A basic example of an A_∞ algebra is as follows. Let the underlying space be $\mathbb{Z}/2\mathbb{Z}$-graded 1-dimensional $\mathbb{C}^1$ concentrated in even degrees. Let $m_{2n+1} = 0$ for all n and even degree maps are arbitrary linear maps. [8]

2. Let B be an ordinary algebra and $N \geq 1$ integer. Let ε be an indeterminate of degree $2 - N$. Let $A = B[\varepsilon]/(\varepsilon^2)$. First let us put a trivial A_∞-structure on A by setting $m_n = 0$ for $n \neq 2$ and m_2 is the multiplication in B. Let $c : B^{\otimes N} \to B$ be any linear map. If c is also a Hochschild cocycle for B we can define a deformed A_∞-structure on A as follows [6]

$$m'_n = \begin{cases} m_n & \text{if } n \neq N, \\ m_n + \varepsilon c & \text{if } n = N. \end{cases}$$

3. Let X be a complex compact manifold and let $\mathcal{E}$ be a vector bundle on X. Then

$$Ext_X^*(\mathcal{E},\mathcal{E}) := \oplus_{n\in\mathbb{Z}} Ext^n(\mathcal{E},\mathcal{E})$$

can be given a natural A_∞ structure. $A := Hom_X(\mathcal{E},\mathcal{E})$ is a dg-algebra and hence an A_∞-algebra. Then $HA = Ext_X^*(\mathcal{E},\mathcal{E})$ has an induced A_∞-structure. Here $m_1 = 0$, m_2 is Yoneda product and higher m_k's are given via Homotopy Perturbation Lemma.

3.4 Cyclic A_∞-Algebras

We will be interested in A_∞-algebras equipped with an extra structure. An A_∞-algebra is called a cyclic A_∞-algebra, if it is equipped with a non-degenerate pairing (inner product) $\langle , \rangle : A[1] \otimes A[1] \to k$ that satisfies the following conditions

$$\begin{aligned} \langle m_k(x_1,\ldots,x_k), x_{k+1}\rangle &= (-1)^{K(x)}\langle x_1, m_k(x_2,\ldots,x_{k+1})\rangle \\ \langle a,b\rangle &= -(-1)^{|a|'|b|'}\langle b,a\rangle \end{aligned}$$

where $K(x)$ is the Koszul sign that we have defined in the chapter 1, given by $K(x) = |x_1|'(|x_2|' + \ldots + |x_{k+1}|')$. The first equation is the cyclicity condition. The second equation means our pairing is skew symmetric. Note that, we have used the shifted sign convention above. We can also define the pairing in the non-shifted case by the following

$$\langle a,b\rangle^{ns} = (-1)^{|a|}\langle a,b\rangle$$

Using the skew-symmetricity we can get the symmetricity condition for the non-shifted setting as below

$$
\begin{aligned}
\langle a,b\rangle^{ns} &= (-1)^{|a|}\langle a,b\rangle \\
&= (-1)^{|a|+|a|'|b|'+1}\langle b,a\rangle \\
&= (-1)^{|a|+(|a|+1)(|b|+1)+1}\langle b,a\rangle \\
&= (-1)^{|a||b|+2|a|+|b|+2}\langle b,a\rangle \\
&= (-1)^{|a||b|}(-1)^{|b|}\langle b,a\rangle \\
\langle a,b\rangle^{ns} &= (-1)^{|a||b|}\langle b,a\rangle^{ns}
\end{aligned}
$$

We will also use the notation $c_{k+1}(x_1,\ldots,x_{k+1}) = \langle m_k(x_1,\ldots,x_k), x_{k+1}\rangle$.

We will define an A_∞-algebra to be odd if the degree of the pairing is odd and even if the degree of the pairing is even. If A is an odd A_∞-algebra this means that $\langle a,b\rangle = 0$ for homogeneous $a,b\in A$ when $|a|\equiv|b| \mod 2$. For an even A_∞-algebra A we get that $\langle a,b\rangle = 0$ for homogeneous $a,b\in A$ when $|a|\not\equiv|b| \mod 2$. The parity of the pairing does not depend on if we use shifted signs or not.

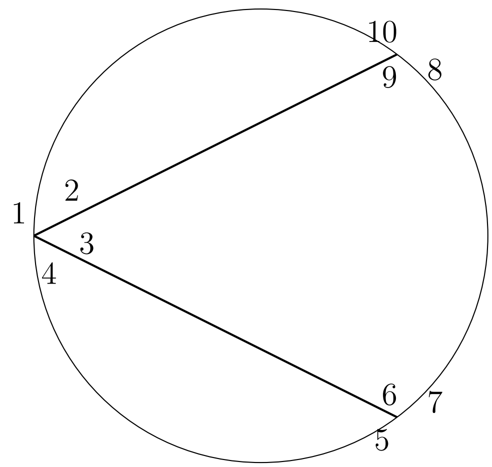
Chapter 4

Ribbon Graphs

4.1 Definition

A ribbon graph is a finite connected graph equipped with a cyclic ordering on the half edges incident to each vertex, see [5]. More precisely, it's a graph $\Gamma = (V, E, i)$ consisting of a finite set of vertices $V = V(\Gamma)$, finite set of edges $E = E(\Gamma)$ and an incidence map $i : E \to (V \times V)/\Sigma_2$ from each edge to unordered pairs of vertices of that edge, together with a cyclic ordering on the set of half-edges incident to each vertex in V. We will denote the set of all half edges of a graph Γ by H or $H(\Gamma)$. The valency of a vertex $v \in V$, denoted $\mathrm{val}(v)$, is the number of half edges incident to that edge.

Ribbon graphs can be described using permutation records. To specify a ribbon graph Γ, we will specify two permutations σ, τ on $H(\Gamma)$. Here σ is an arbitrary permutation and τ is an involution with no fixed points. Under this representation the vertices $V(\Gamma)$ corresponds to the cycles of σ and the edges $E(\Gamma)$ corresponds to the cycles of τ. Then the faces $F(\Gamma)$ of our graph corresponds to the cycles of $\sigma\tau$. We will illustrate the last fact in the example below.

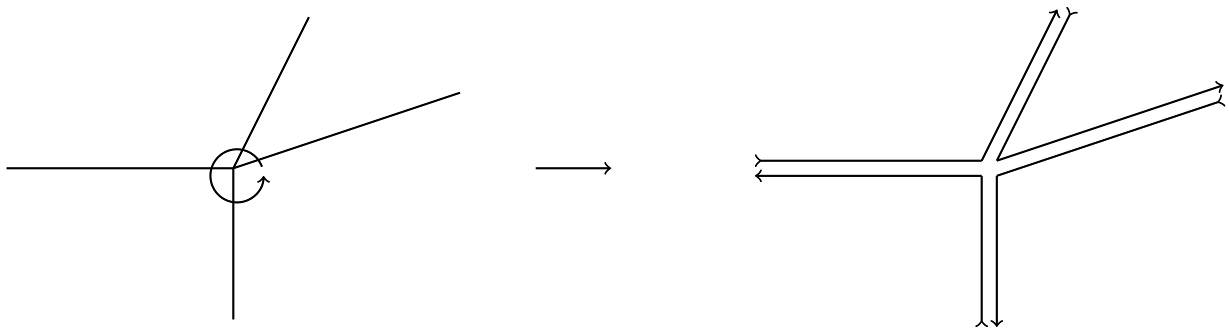


Example: In the picture above we can write $\sigma = (1\ 2\ 3\ 4)(5\ 6\ 7)(8\ 9\ 10)$ and $\tau = (1\ 10)(2\ 9)(3\ 6)(4\ 5)(7\ 8)$. This gives us the 4 faces of the graph above as the cycles in $\sigma\tau = (1\ 8\ 5)(2\ 10)(3\ 7\ 9)(4\ 6)$

We can also represent ribbon graphs on a plane as a graph drawn in a way that the half edges at each vertex are in a clockwise order. This may require us to draw it in a way that edges intersect outside of the vertices.

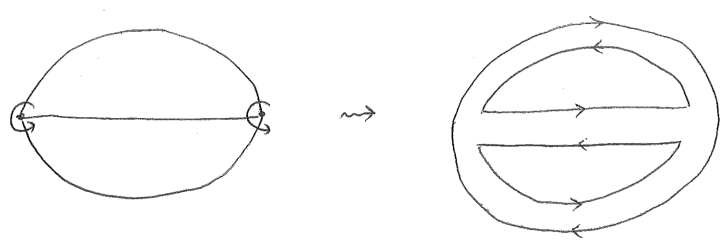
For every ribbon graph we may associate to it a topological surface with boundary. We construct this surface by first replacing half edges at each vertex with thin strips (ribbons) where the orientation of the boundary of the strips are determined by the orientation of the vertex. After connecting each half-edge with its corresponding half-edge in a way that the orientations match, we get a connected oriented surface with boundary. The resulting surface might no longer be planar. Let g be the genus of the resulting surface and let n be the number of boundary components.

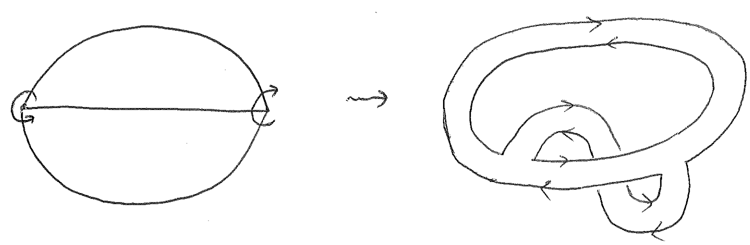
The orientation at the boundary is the same as the orientation at the vertex



We replace the vertex on the left with the surface on the right

The cyclic ordering at each vertex is an important part of the ribbon graph data as you can see in the following 2 examples.





Both pictures have the same underlying graph with different orientations at the vertices. First example has genus $g = 0$ with $n = 3$ boundary components, meanwhile the second surface has genus $g = 1$ with $n = 1$ boundary component.

Since the resulting surface is oriented we can calculate its Euler characteristic and get

$$
\begin{aligned}
|V| - |E| + |F| &= \chi \\
|V| - |E| + n &= 2 - 2g \qquad (4.1)
\end{aligned}
$$

A stable ribbon graph is a ribbon graph where the valency of each vertex is at least 3. From now on we will only consider the ribbon graphs that are stable.

Theorem 4.1 *For each $g, n \geq 0$ there are only finitely many stable ribbon graphs with genus g and n boundary components.*

Proof. Since the graph is stable, we have $\mathrm{val}(v) \geq 3$ for any $v \in V$. For each vertex there are at least 3 half edges, i.e. $|H| \geq 3|V|$. Since for each edge there are exactly 2 half edges we have $2|E| = |H|$. Hence we get $|V| \leq \frac{2}{3}|E|$ or $-|E| \leq -\frac{3}{2}|V|$. From the equation 4.1 we get

$$
\begin{aligned}
|V| - |E| &= 2 - 2g - n \\
|V| - \frac{3}{2}|V| &\geq 2 - 2g - n \\
-\frac{1}{2}|V| &\geq 2 - 2g - n \\
|V| &\leq 4g + 2n - 4
\end{aligned}
$$

and also

$$
\begin{aligned}
|E| &= |V| + 2g + n - 2 \\
|E| &\leq 6g + 3n - 6
\end{aligned}
$$

So for a fixed (g, n) we have an upper bound on the number of vertices and edges. This means we only have finitely many such graphs.

4.2 Orientation of Graphs

For a real vector space V, the determinant is defined as $\det(V) := \bigwedge^{top} V$, the wedge product in degree $\dim V$. The orientation on V can be thought of as a unit vector in $\det(V)$. For any real vector space V we have 2 possible orientations, represented by 1 or -1 on $\det(V)$. If V is endowed with a basis, orientation of V corresponds to choosing an order of basis elements, up to even permutations.

For a set S we denote by $\mathbb{R}S$ the real vector space with S as its basis. Let us denote the set of two half edges contained in an edge e by $H(e)$.

Definition 4.2 *[3] An orientation or on a graph Γ is a unit vector in*

$$\det \mathbb{R}V(\Gamma) \otimes \bigotimes_{e \in E(\Gamma)} \det \mathbb{R}H(e),$$

i.e. the orientation on a graph Γ is determined by first ordering the vertices of Γ and then ordering the half edges in each edge of Γ, up to even changes of this data.

Remark. Conant and Voigtman have proved in [3] that a ribbon graph Γ has a natural orientation if all its vertices have odd valence. Since we have

$$\sum_{v \in V(\Gamma)} \mathrm{val}(v) = |H| = 2|E|$$

having all vertices with odd valence means that $|V|$ must be even.

Definition 4.3 *[3] Given a graph Γ and an edge $e \in E(\Gamma)$, which is not a loop, we can define Γ/e as the graph obtained by collapsing the edge e. If an orientation on Γ is given as orienting the edges and ordering vertices so that the source of e is the second from last and its target is the last vertex, the induced orientation on Γ/e is given by taking the coalesced vertex to be the last vertex and letting the remaining edges and vertices to be oriented and ordered as before.*

4.3 The Differential d

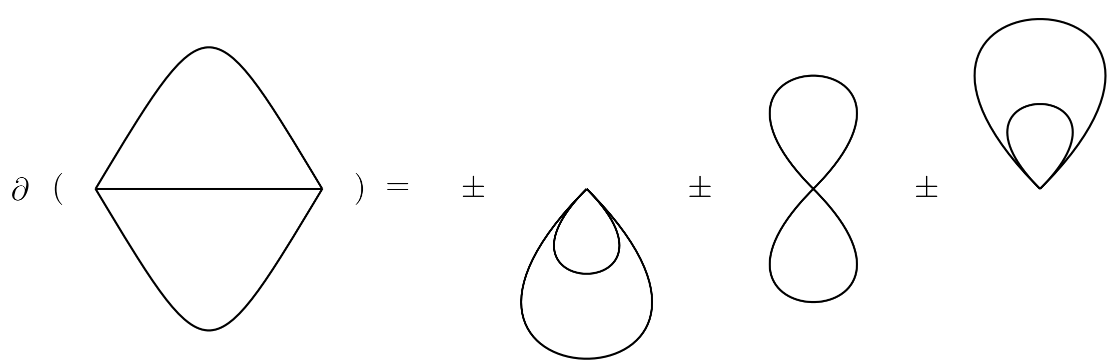
Definition 4.4 *[5] A graph Γ has codimension n if it has obtained from a trivalent graph by collapsing n edges.*

Let $\mathcal{G}^n$ be the free abelian group generated by all isomorphism classes $[\Gamma]$ of connected oriented ribbon graphs Γ of codimension n, modulo the relation $-[\Gamma] = [-\Gamma]$ where $-\Gamma$ is Γ with opposite orientation. If $[\Gamma]$ has an orientation reversing automorphism this implies $2[\Gamma] = 0$. We can define the boundary operator $\partial : \mathcal{G}^n \to \mathcal{G}^{n+1}$ by

$$\partial[\Gamma] = \sum_e [\Gamma/e]$$

where $e \in E[\Gamma]$ is not a loop.

Example:



When we apply ∂ to the graph on the left above, we have 3 choices for the edge to collapse. We do all three and add the resulting graphs. We used $\pm$ because the orientation of the graph is ignored for now.

Theorem 4.5 *The map ∂ is a boundary map, i.e. $\partial^2 = 0$.*

Proof. Let Γ be our graph and let the order of vertices in $or(\Gamma)$ be given as $v_1, \ldots, v_n$. When we apply ∂ twice, we are collapsing two edges. We either collapse two edges with no common vertex or two edges that share only one vertex. If both edges are between same two vertices, after applying the first ∂ the other edge becomes a loop and we can not apply ∂ to a loop. Since ∂ is done over all edges, both possible pairs of edges $\{e, f\}$ to be deleted appears twice, once as (e, f) and once as (f, e).

Case 1: Let us pick two edges e_1 going from v_a to v_b and e_2 going from v_c to v_d where all a, b, c, d are different. Without loss of generality assume that $a < b$ and $c < d$. First we need to change the list of vertices from $v_1, \ldots, v_n$ into $v_1, \ldots, \hat{v_a}, \ldots, v_n, v_a$ and then into $v_1, \ldots, \hat{v_a}, \ldots, \hat{v_b} \ldots, v_n, v_a, v_b$. Here by hat we represent the fact that the vertex is no longer in its original position. In the first step v_a goes through $n - a$ other vertices and v_b goes through $n - b$ other vertices. This gives us a sign $(-1)^{(n-a)+(n-b)} = (-1)^{a+b}$. Then we can collapse e_1 and get a new vertex v_1'. Here the order of vertices are given in $or(\Gamma/e_1)$ be given as $v_1, \ldots, \hat{v_a}, \ldots, \hat{v_b} \ldots, v_n, v_1'$ and we have

$$or(\Gamma/e_1) = (-1)^{a+b} or(\Gamma)$$

Note that we now have $n - 1$ vertices. Now let us collapse e_2 in Γ/e_1. By the same argument as above we get a new set of vertices $v_1, \ldots, v_n, v_1', v_2'$ with the new orientation in the graph given as

$$or((\Gamma/e_1)/e_2) = (-1)^{c+d} or(\Gamma/e_1) = (-1)^{a+b+c+d} or(\Gamma)$$

If we first collapse e_2 and then e_1, similarly we get a new set of vertices in the orientation as $v_1, \ldots, v_n, v_2', v_1'$. Note that the last two elements are switched this time. If we put them in the original order we get an extra negative sign.

$$or((\Gamma/e_2)/e_1) = -(-1)^{a+b+c+d} or(\Gamma)$$

Hence the same graph that we get by collapsing edges e_1 and e_2 appears twice but with opposite signs. The summation over all non intersecting pairs of edges in ∂^2 gives 0.

Case 2: Now let us pick two edges e_1 going from v_a to v_b and e_2 going from v_b to v_c. Without loss of generality assume that $a < b < c$. As in the first case if we delete e_1 we get a new vertex v_1' where the order of vertices are now $v_1, \ldots, v_n, v_1'$. The new orientation is as before $or(\Gamma/e_1) = (-1)^{a+b} or(\Gamma)$. But now e_2 goes from v_1' to v_c. Since v_1' is already in the last place we only need to move v_c into the last position. Since we have $n-1$ vertices we get a new sign $(-1)^{n-1-c}$. Now we have the order of vertices as $v_1, \ldots, v_n, v_0'$ where v_0' is the vertex that we get by collapsing e_1 and e_2. We have

$$or((\Gamma/e_1)/e_2) = (-1)^{n-1-c} or(\Gamma/e_1) = (-1)^{a+b+n-1-c} or(\Gamma)$$

If we first collapse e_2 we get the order of vertices as $v_1, \ldots, v_n, v_2'$ with $or(\Gamma/e_2) = (-1)^{b+c} or(\Gamma)$. The edge e_1 now goes from v_a to v_2'. To collapse e_1 first we need to move v_a into last position. We get $v_1, \ldots, v_n, v_2', v_a$ and the sign $(-1)^{n-1-a}$. Then we need to switch the last two vertices to get the correct orientation on e_1 and get another negative sign. This gives

$$or((\Gamma/e_2)/e_1) = (-1)^{n-1-a+1} or(\Gamma/e_2) = (-1)^{n-a+b+c} or(\Gamma)$$

which is the negative of the sign that we get by collapsing e_1 first. Hence we get the same graph with opposite orientations. The summation over all edges that intersect only at a vertex gives 0 under ∂^2.

Hence we have $\partial^2 = 0$.

The compactly supported dual of this complex is the graph cohomology complex given as follows

Definition 4.6 *[5] $\mathcal{G}_n$ is the additive group of all homomorphisms $f : \mathcal{G}^n \to \mathbb{Z}$ so that $f[\Gamma] \neq 0$ for only finitely many $[\Gamma]$. Thus $\mathcal{G}_n$ is generated by duals $[\Gamma]^*$ of generators of $\mathcal{G}^n$. The boundary map $d : \mathcal{G}_n \to \mathcal{G}_{n-1}$ is given in terms of these dual generators by*

$$d[\Gamma]^* = \sum l_i [\Gamma_i]^*$$

where l_i is equal to the number of edges e in Γ_i so that $\Gamma_i/e \cong \Gamma$ minus the number of edges e in Γ_i so that $\Gamma_i/e \cong -\Gamma$. The sum is over a basis for $\mathcal{G}_{n-1}$.

Remark Let $\mathrm{Hom}^+(\Gamma_i, \Gamma)$ be the set of morphisms $f : \Gamma_i \to \Gamma$ so that the orientation of Γ agrees with the orientation induced from Γ_i by f. Similarly let $\mathrm{Hom}^-(\Gamma_i, \Gamma)$ be such morphisms where the orientations disagree. We can write the coefficient l_i above as

$$l_i = \frac{|\,\mathrm{Hom}^+(\Gamma_i, \Gamma)| - |\,\mathrm{Hom}^-(\Gamma_i, \Gamma)|}{|\,\mathrm{Aut}(\Gamma)|} \in \mathbb{Z}$$

The l_i is the number of left equivalence classes of morphisms $f : \Gamma_i \to \Gamma$. If we define r_i to be the number of right equivalence classes of morphisms $f : \Gamma_i \to \Gamma$ we get a similar formula

$$r_i = \frac{|\,\mathrm{Hom}^+(\Gamma_i, \Gamma)| - |\,\mathrm{Hom}^-(\Gamma_i, \Gamma)|}{|\,\mathrm{Aut}(\Gamma_i)|} \in \mathbb{Z}$$

Note that $r_i = \dfrac{|\,\mathrm{Aut}(\Gamma)|}{|\,\mathrm{Aut}(\Gamma_i)|} l_i$ If we also define $\langle \Gamma \rangle := |\,\mathrm{Aut}(\Gamma)|[\Gamma]^*$ we can write the differential as

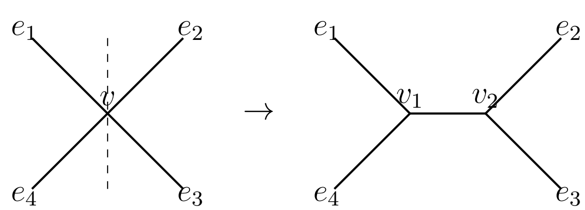
$$\begin{aligned}
d\langle\Gamma\rangle &= d(|\operatorname{Aut}(\Gamma)|[\Gamma]^*) \\
&= |\operatorname{Aut}(\Gamma)| d[\Gamma]^* \\
&= |\operatorname{Aut}(\Gamma)| \sum l_i [\Gamma_i]^* \\
&= |\operatorname{Aut}(\Gamma)| \sum l_i \frac{\langle\Gamma_i\rangle}{|\operatorname{Aut}(\Gamma_i)|} \\
&= |\operatorname{Aut}(\Gamma)| \sum l_i \frac{\langle\Gamma_i\rangle}{|\operatorname{Aut}(\Gamma_i)|} \\
&= \sum \frac{|\operatorname{Aut}(\Gamma)|}{|\operatorname{Aut}(\Gamma_i)|} l_i \langle\Gamma_i\rangle \\
d\langle\Gamma\rangle &= \sum r_i \langle\Gamma_i\rangle
\end{aligned}$$

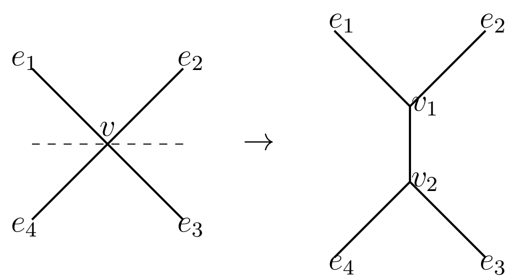
Definition 4.7 *[5] Let $\mathcal{G}_*^{\mathbb{Z}}$ denote the subcomplex of $\mathcal{G}_*$ generated by the elements $\langle\Gamma\rangle$. We call $\mathcal{G}_*^{\mathbb{Z}}$ the integral subcomplex of $\mathcal{G}_*$*

We can give a pictorial description of d at each vertex. For an oriented ribbon graph Γ, each vertex v of valence n splits into 2 vertices v_1, v_2 connected by an edge. Hence we have $\operatorname{val}(v_1) + \operatorname{val}(v_2) = n + 2$. Note that our ribbon graphs are stable and $\operatorname{val}(v_i) \geq 3$. We can think of this splitting as a choice of a diagonal in a convex n-gon. Hence we have $\dfrac{n(n-3)}{2}$ different splittings at this vertex.

If our vertex v is trivalent there is no possible splitting where both v_1 and v_2 will be at least trivalent.

For a vertex of valency 4 we have 2 possible splittings as shown below.





Theorem 4.8 *The map d is a boundary map, i.e. $d^2 = 0$.*

Proof. This follows from the fact that d is adjoint to ∂. We get

$$d^2 = (\partial^\dagger)^2 = (\partial^2)^\dagger = 0$$

Let $R^{or}_{g,n}$ be the abelian group generated by oriented ribbon graphs with genus g, n boundary components and differential d. The main result relating ribbon graphs and the homology of moduli spaces of curves was obtained by Strebel and Penner

Theorem 4.9 *[13, 10]* $H^*(R^{or}_{g,n}) \cong H_*(\mathcal{M}_{g,n})^{\Sigma_n}$

Chapter 5

Partition Functions and Graphical Orientations

5.1 Definition

In the previous two chapters, we have talked about two different topics, A_∞-algebras and ribbon graphs. In this chapter we will show how both concepts help us define *Partition Functions*. In [8] and [9] Kontsevich explains how to get a cocycle on the associative graph cohomology complex $\mathcal{G}_*$, from a finite dimensional $\mathbb{Z}/2\mathbb{Z}$-graded A_∞-algebra A. Igusa gave an explicit construction [5] of this cocycle using the orientation that we have defined in the previous chapter [3].

Let $A = A_0 \oplus A_1$ be a $\mathbb{Z}/2\mathbb{Z}$-graded cyclic A_∞-algebra (also called superalgebra) with nondegenerate even scalar product $\langle , \rangle^{ns} : A \otimes A \to R$. Note that $\langle a, b\rangle^{ns} = 0$ if $|a| + |b| = 1$ and $\langle a, b\rangle^{ns} = (-1)^{|a|}\langle b, a\rangle^{ns}$. The second condition is the symmetry condition for even pairings in [5]. Then Igusa gives a partition function

$$Z_A : \mathcal{G}_*^{\mathbb{Z}} \to R$$

that is defined on any generator $\langle \Gamma \rangle$ with orientation *or* as follows

$$Z_A\langle\Gamma\rangle = \epsilon_1 \sum_{states} \prod_i \langle m_{n_i}(x_{i1}, \ldots, x_{in_i}), x_{i0}\rangle^{ns} \epsilon_2 \prod_j \langle \bar{y_j^*}, y_j^*\rangle^{ns}$$

Now let us explain the terms in the above expression. In order to write it, first we need to make some choices.

- We first need to choose a basis $\{y_j\}$ for A.
- We will order vertices of Γ. Let us denote the ordered vertices by $v_1, v_2, \ldots, v_n$.
- We will pick an orientation of each edge. This is the same as writing each edge e_i as an ordered pair of its half edges $(h_i, \bar{h_i})$.
- We will pick a starting half edge at each vertex v_i and denote that half edge by e_{i0}. This will allow us to write half edges incident to v_i in a clockwise order as $e_{i0}, e_{i1}, \ldots, e_{in_i}$ where the valency of v_i is $n_i + 1$

A state is an assignment of basis elements of A to each half edge e_{ij}. The summation is over all possible states. First product is over all vertices v_i. The second product is over all edges e_j where y_j and $\bar{y_j}$ are the basis elements assigned to the half edges $(h_j, \bar{h_j})$ of e_j. y_j^* is the dual of y_j. Here we have oriented the edge e_j from h_j to $\bar{h_j}$.

Then we can define ϵ_2 to be the sign that we get by permuting the odd half edges in $e_{10}, e_{11}, \ldots, e_{1n_1}, e_{20}, e_{21}, \ldots, e_{2n_2}, e_{30}, \ldots$ into $h_0, \bar{h_0}, h_1, \bar{h_1}, \ldots$. To be more precise, we first ignore the half edges in both orderings that are labeled by even elements of the algebra and then set ϵ_2 to be the sign of the permutation of the remaining half edges with odd basis elements. In the expression $h_0, \bar{h_0}, h_1, \bar{h_1}, \ldots$ it may look like we have also picked an order on the edges. But this is not important. To be able to get a nonzero element from the second pairing in the formula we need h_j to $\bar{h_j}$ to have the same parity. If we switch the order of two edges then one of the edges will move through both half edges of the other edge and in total will get no sign.

Finally, ϵ_1 is the sign that transforms the orientation *or* of the graph into $sign(v_1, e_{10}, e_{1n_1}, \ldots, e_{11}, v_2, e_{20}, e_{2n_2}, \ldots e_{21} \ldots)$.

Now let's show that the partition function is well defined. Since we are summing over all states each basis element and its dual appears only once. Hence the formula is independent of the choice of basis. Now let us show that transposing vertices v_1 and v_2 changes both ϵ_1 and ϵ_2 by a factor of $(-1)^{n_1 n_2}$, hence doesn't effect Z_A. To get the sign change in ϵ_1 we need to find the sign that we get by switching $v_1, e_{10}, e_{1n_1}, \ldots, e_{11}, v_2, e_{20}, e_{2n_2}, \ldots e_{21} \ldots$ into $v_2, e_{20}, e_{2n_2}, \ldots e_{21}, v_1, e_{10}, e_{1n_1}, \ldots, e_{11} \ldots$. This is given by the permutation that switches the block $v_1, e_{10}, e_{1n_1}, \ldots, e_{11}$ of length $n_1 + 2$ and the block $v_2, e_{20}, e_{2n_2}, \ldots e_{21}$ of length $n_2 + 2$. This changes ϵ_1 by a factor of $(-1)^{(n_1+2)(n_2+2)} = (-1)^{n_1 n_2}$, as we wanted.

Before we look at ϵ_2, let us look at the total degree of all half edges in a given vertex. We need $\langle m_{n_i}(x_{i1}, \ldots, x_{in_i}), x_{i0} \rangle$ to be nonzero, i.e. both terms in the pairing must have the same degree modulo 2. Since $\deg m_{n_i} \equiv n_i \mod 2$ we get

$$\begin{aligned} n_i + \deg x_{i1} + \ldots + \deg x_{in_i} &\equiv \deg x_{i0} \mod 2 \\ \deg x_{i0} + \deg x_{i1} + \ldots + \deg x_{in_i} &\equiv n_i \mod 2 \end{aligned}$$

This means for a vertex of valence $n_i + 1$, total degree of all half edges coming from that vertex must be $n_i \mod 2$, otherwise we get 0 in Z_A.

To get the sign change in ϵ_2 we need to find the sign that we get by permuting the odd elements in $e_{11}, \ldots e_{1n_1}, e_{10}, e_{21}, \ldots e_{2n_2}, e_{20}, \ldots$ into $e_{21}, \ldots e_{2n_2}, e_{20}, e_{11}, \ldots e_{1n_1}, e_{10}, \ldots$. Here we are switching the blocks $e_{11}, \ldots, e_{1n_1}, e_{10}$ that corresponds to the half edges coming out of v_1 and $e_{21} \ldots e_{2n_2}, e_{20}$ the half edges coming out of v_2. The blocks have corresponding degrees n_1 and n_2 modulo 2. Hence we get a factor of $(-1)^{n_1 n_2}$ as we wanted.

Finally, let us cyclically permute the half edges $x_1, \ldots, x_n, x_0$ of the vertex v with valency $n+1$ into $x_0, x_1, \ldots, x_n$. This changes ϵ_1 by a factor of $(-1)^n$. To get the new ϵ_2 we need to move x_0 through $x_1, \ldots, x_n$. Hence we change ϵ_2 by a factor of $(-1)^{|x_0|\sum_{i=1}^n |x_i|} = (-1)^{|x_0|(n-|x_0|)}$. We also get an extra factor of $(-1)^{n+|x_0|+|x_0|n}$ (Koszul sign) from the cyclicity condition. Hence the total sign change is

$$(-1)^{n+|x_0|(n-|x_0|)+n+|x_0|+|x_0|n} = (-1)^{n+n|x_0|-|x_0|+n+|x_0|+|x_0|n} = 1$$

The partition function Z_A is well defined.

Theorem 5.1 *[9] Z_A is a cocycle on $\mathcal{G}_*$.*

Example: [8] Let us define an A_∞-algebra A over $\mathbb{Q}$. We take $A = \mathbb{Q}$ and we set $m_{odd} = 0$ and m_{2k} is the multiplication by x_k for a fixed choice of rational numbers $x_i \in \mathbb{Q}$. Let us also set $\langle a, b\rangle = ab$. Since states correspond to assigning basis vectors on each half edge, we only have one state where we assign 1 to each half edge. Then our partition function has only one term, depending on x_i's.

The odd case:

In [1] Amorim and Tu gave a construction of the partition function, also called Kontsevich class, for odd A_∞-algebras. Their construction is analogous to that of Igusa, but they needed to use a different orientation, which they called a twisted orientation. Instead of ordering the vertices and orienting the edges, a twisted orientation just orders the edges. In the next sections we will describe a more general definition of an orientation of a ribbon graph, which will work for both odd and even A_∞-algebras.

5.2 Understanding the Signs from Graphical Representations

In the definition of $Z_A\langle\Gamma\rangle$ we get the sign ϵ_2 by ordering the vertices and orienting the edges. This is exactly the information that we have from the orientation *or* of Γ. Let us extend the orientation data of Γ in a more general way. Given a ribbon graph Γ, we will make the following four choices.

1. Order the edges
2. Orient the edges
3. Order the vertices
4. Choose a starting half edge at each vertex

Once these choices are made, this will lead to a canonical expression, similar to the one of Igusa.

We will show that in the even pairing case items 1 and 4 do not change the sign in our graphical convention, thus recovering the orientation of Conant and Vogtmann. For the odd pairing case only item 1 will introduce a sign. This gives us the *twisted orientation* used in [1].

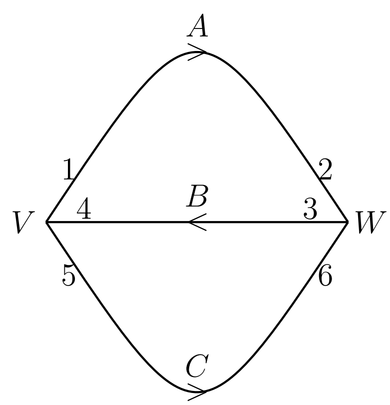
Theorem 5.2

- *In the even case only items* 2 *(Orient the edges) and* 3 *(Order the vertices) give a sign. This gives us the orientation of* Γ*.*

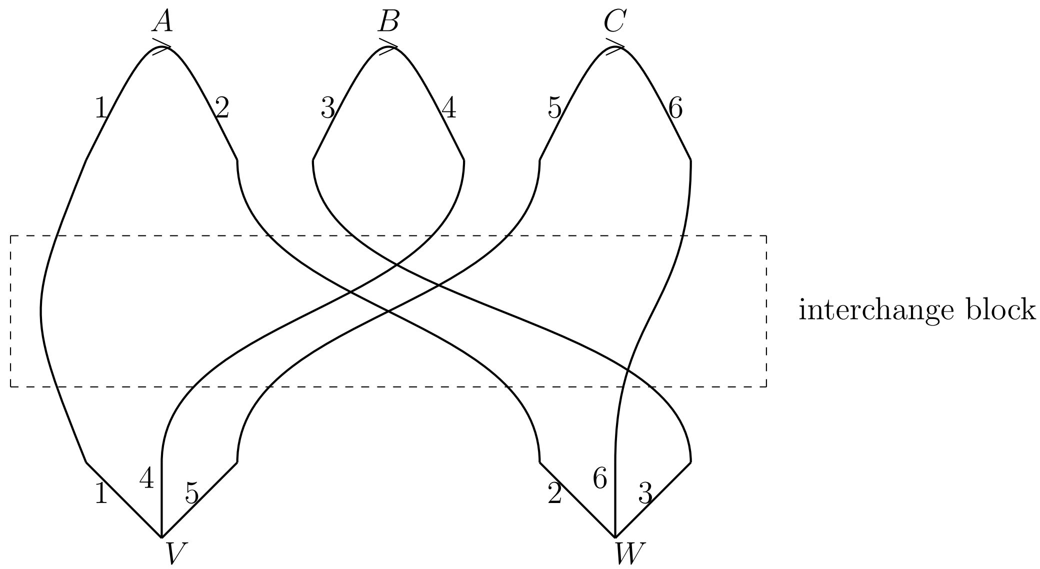
- *In the odd case only item* 1 *(Order the edges) gives a sign. This corresponds to the twisted orientation of* Γ.

We will give a graphical proof of this theorem in the following 2 sections.

Before we analyze even and odd cases separately, let us give a pictorial description of the partition function below. Making all the choices above allows us to draw our ribbon graph in a canonical way. We will first order and orient edges on top. Then we will order the vertices below, pick a starting half edge and then follow the clockwise orientation. Then the edges and vertices are connected using an interchange block which introduces a sign. Let us show this over an example.



In the ribbon graph above, edges are ordered A, B, C, orientations are given by arrows, vertices are ordered V, W and starting half edges are picked to be 1 for V and 2 for W. After picking the starting half edge we follow the clockwise orientation.

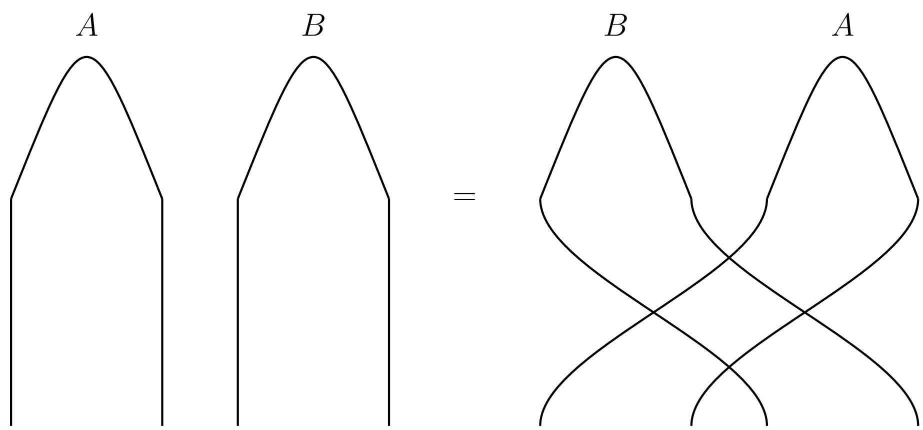


The picture above is uniquely determined by the four choices we made. It is correct for even operations. When we use an odd operation we will add dashed lines as described in the first chapter. The interchange block gives us the sign ϵ_2 in Igusa's definition. Note that we get a sign only when we intersect two odd degree elements.

5.3 Even Pairing

In this chapter we will look at all 4 items that have been described in the previous section. We will show that only items 2 and 3 will give us a sign. That will be equivalent to the orientation notion defined by Conant and Vogtmann [3].

1. Order of edges doesn't matter.



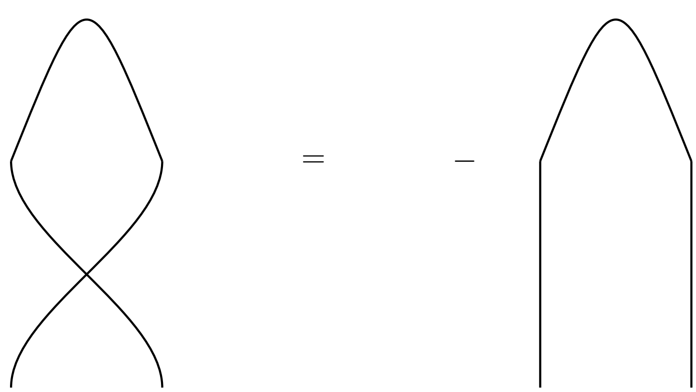
In the even pairing, cap is an even operation. When we switch two consecutive edges we are moving one even operations through another. This will not introduce a sign.

2. Changing the edge orientation introduces a sign.

Let $e_1, \dots, e_n$ be a basis of A and $e^1, \dots, e^n \in A$ be the dual basis so that $\langle e_i, e^j \rangle = 1$ if $i = j$ and equal to 0 otherwise. We have $e_i^\vee = e^i$. We also have by the skew-symmetricity

$$\begin{aligned} 1 = \langle e_i, e^i \rangle &= -(-1)^{|e_i|'|e^i|'} \langle e^i, e_i \rangle \\ 1 &= \langle e^i, -(-1)^{|e_i|'|e^i|'} e_i \rangle \end{aligned}$$

This means that $(e^i)^\vee = -(-1)^{|e_i|'|e^i|'} e_i$. Note that $e^1, \dots, e^n$ is also a basis of A. Thus we have the graphical relation.

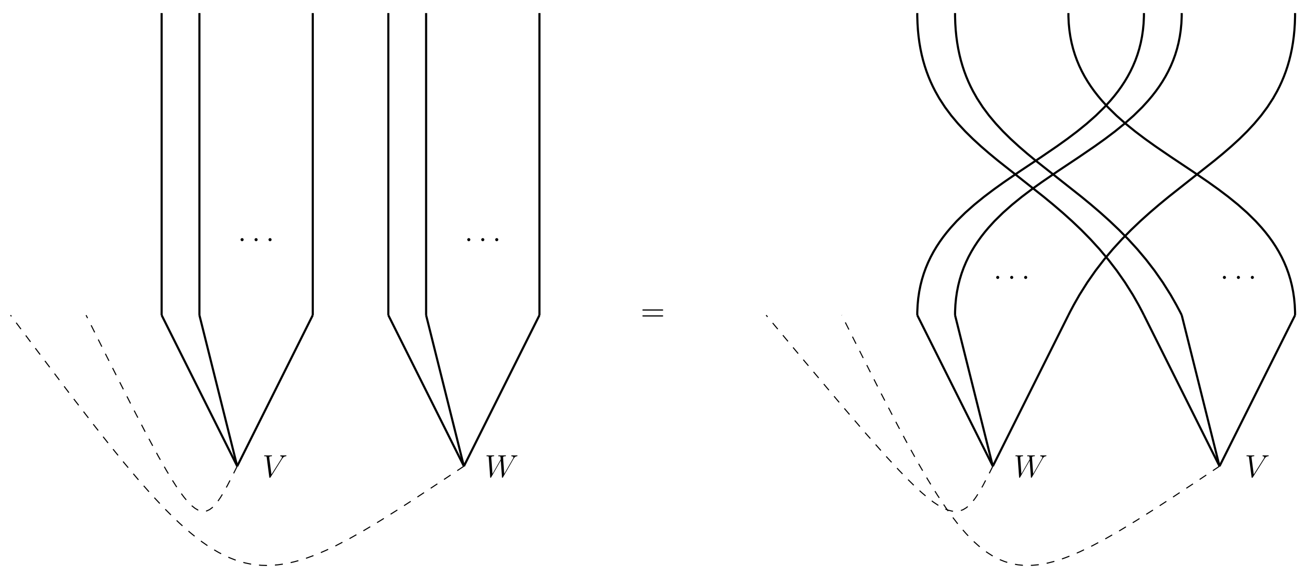


The cap (with the sign) on the right corresponds to $-\sum_i e_i \otimes e^i$ and the graph on the left corresponds to $\sum_i (-1)^{|e_i|'|e^i|'} e^i \otimes e_i$ and we have

$$\begin{aligned} -\sum_i e_i \otimes e^i &= -\sum_i e^i \otimes (e^i)^\vee \\ &= -\sum_i e^i \otimes -(-1)^{|e_i|'|e^i|'} e_i \\ &= \sum_i (-1)^{|e_i|'|e^i|'} e^i \otimes e_i \end{aligned}$$

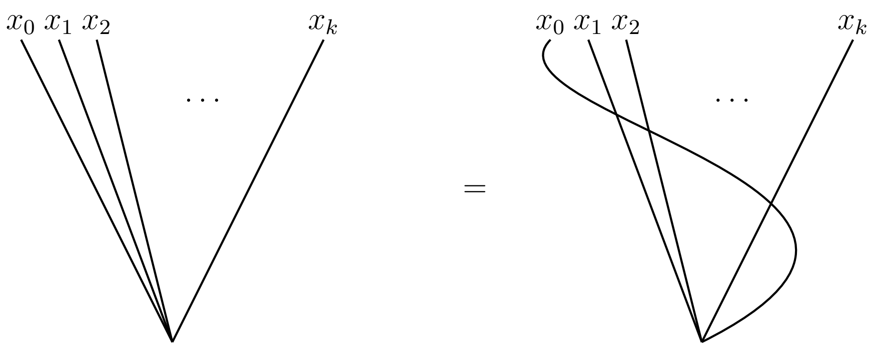
3. Changing the order of 2 consecutive vertices introduces a sign.

A vertex corresponds to $c_{k+1}(x_0, \ldots, x_k) = \langle m_k(x_0, \ldots, x_{k-1}), x_k \rangle$. To get a nonzero value from the vertex we need to have the shifted degrees as $|x_k|' \equiv |m_k(x_0, \ldots, x_{k-1})|'$ mod 2. This means that $|x_k|' \equiv 1 + \sum_{i=0}^{k-1} |x_i|' \mod 2$ or $\sum_{i=0}^{k} |x_i|' \equiv 1 \mod 2$. Since c_k is an odd operation we will add a dashed line (of order 1) at the vertex to make it even.



We can go from the graph on the left to right to by switching two even operations. This doesn't give us any sign. We will also intersect the dashed lines to preserve their original ordering. That will give us a negative sign, if we do not draw the dashed lines.

4. Choosing a starting half edge at a vertex doesn't matter.



The graph on the left corresponds to $\langle m_k(x_0, \dots, x_{k-1}), x_k \rangle$ and the graph on the right corresponds to $(-1)^{K(x)} \langle m_k(x_1, \dots, x_k), x_0 \rangle$ where $K(x)$ is the Koszul sign given by $K(x) = |x_0|'(|x_1|' + \dots + |x_k|')$. The equality of the two graphs exactly corresponds to the cyclicity condition. Note that we didn't use the fact that we have an even pairing. The same argument holds for the odd case.

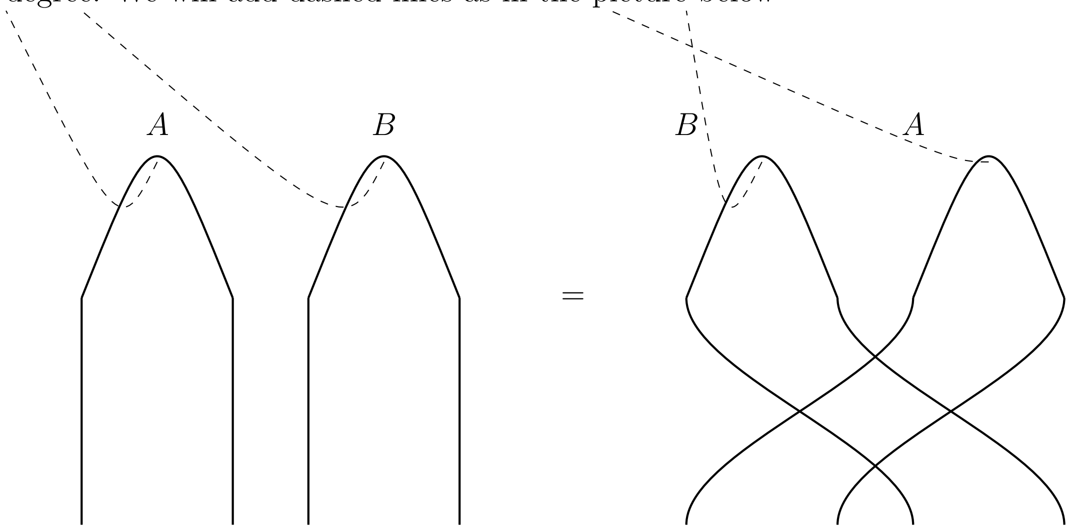
We conclude that of the choices we have made, only choices 2 and 3 matter. Thus we recover the notion of orientation due to Conant and Vogtmann, which is ordering the vertices and orienting the edges.

5.4 Odd Pairing

Now assume that the algebra has an odd pairing. We have already shown in the previous section that choosing a starting edge doesn't give us a sign. Let us look at the other 3 pieces of orientation data. Only the order of edges will give us a sign. That is the twisted orientation defined by Amorim and Tu [1].

1. Switching two edges introduces a sign.

Let us look at two consecutive edges. Since our pairing is odd, the cap will have odd degree. We will add dashed lines as in the picture below



With the added dashed lines, operations are even. We can move them freely without getting a sign. We will also keep the order of dashed lines. That will give us an intersection of dashed lines. Hence we get a sign when we do not draw the dashed lines.

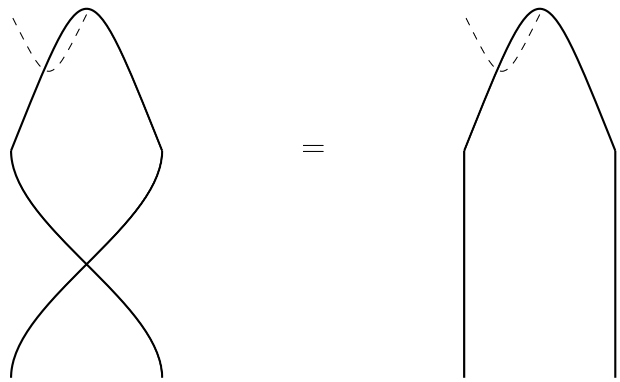
2. Changing the orientation doesn't matter.

Just like the even case, let's pick $e_1, \ldots, e_n$, a basis of A and $e^1, \ldots, e^n \in A$ as the dual basis so that $\langle e_i, e^j \rangle = 1$ if $i = j$ and equal to 0 otherwise.. But this time we have

$$\begin{aligned} 1 = \langle e_i, e^i \rangle &= -(-1)^{|e_i|'|e^i|'} \langle e^i, e_i \rangle \\ 1 &= -\langle e^i, e_i \rangle \\ 1 &= \langle e^i, -e_i \rangle \end{aligned}$$

since $|e_i|' \equiv |e^i|' + 1 \mod 2$. This means that $(e^i)^\vee = -e_i$. Note that $e^1, \ldots, e^n$ is also a basis of A.

We represent this graphically as the equality,



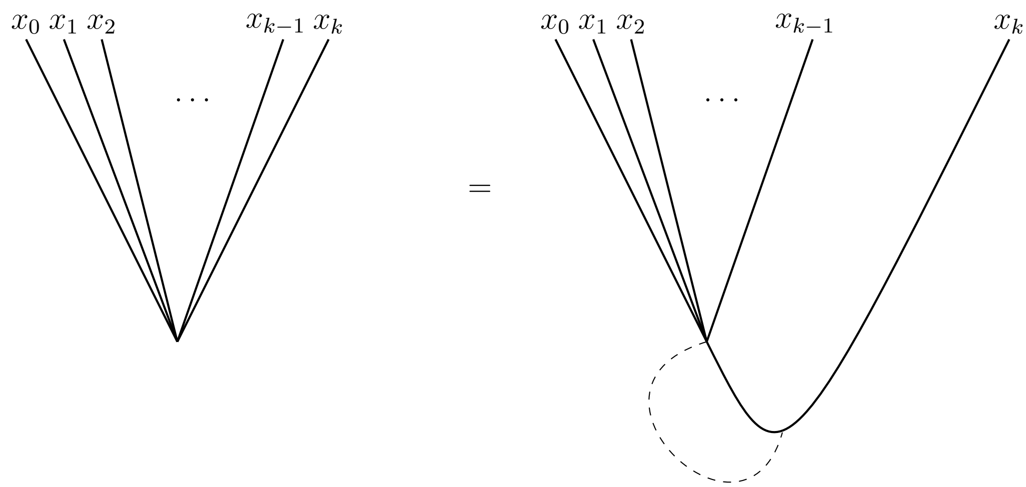
The cap on the right corresponds to $\sum_i (-1)^{|e_i|'} e_i \otimes e^i$ and the graph on the left corresponds to $\sum_i (-1)^{|e_i|'}(-1)^{|e_i|'|e^i|'} e^i \otimes e_i = \sum_i (-1)^{|e_i|'} e^i \otimes e_i$ and we have

$$\begin{aligned}
\sum_i (-1)^{|e_i|'} e_i \otimes e^i &= \sum_i (-1)^{|e^i|'} e^i \otimes (e^i)^\vee \\
&= \sum_i (-1)^{|e^i|'} e^i \otimes (-e_i) \\
&= \sum_i (-1)^{|e^i|'+1} e^i \otimes e_i \\
&= \sum_i (-1)^{|e_i|'} e^i \otimes e_i
\end{aligned}$$

as we wanted

3. Order of the vertices doesn't matter.

Since the vertices and cups are odd in this case both operations require a dashed line. Hence we can put a dashed line at both and connect them as in the picture below.



This makes all the vertices into even operations, which allows us to move and order them freely.

We conclude that only choice 1 matters, thus recovering the orientation data of Amorim and Tu.

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