Projective geometries, Grassmann graphs, and a generalization of the Askey-Wilson relations

by

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Abstract

This thesis is about projective geometries, Grassmann graphs, and a generalization of the Askey-Wilson relations. The thesis contains three main results. The projective geometry and its corresponding Grassmann graphs are defined as follows. Let $\mathbb{F}_q$ denote a finite field with q elements. Fix an integer $n \geqslant 1$. Let $\mathcal{V}$ denote an n-dimensional vector space over $\mathbb{F}_q$. Let the set P consist of the subspaces of $\mathcal{V}$. The set P, together with the inclusion partial order, is a poset called a projective geometry. For $1 \leqslant k \leqslant n-1$ the Grassmann graph $J_q(n,k)$ is defined as follows. The vertex set X of $J_q(n,k)$ consists of the k-dimensional subspaces of $\mathcal{V}$. Two vertices x, y are adjacent whenever $x \cap y$ has dimension $k-1$. Our first main result concerns how to use $J_q(n,k)$ to potentially recover P. Pick distinct $x, y \in X$. The geometry P contains the elements x, y, $x \cap y$, $x+y$. Define

$$\mathcal{B}_{xy} = \{z \in X \mid \partial(z,x) = 1,\ \partial(z,y) = \partial(x,y)+1\},$$
$$\mathcal{C}_{xy} = \{z \in X \mid \partial(z,x) = 1,\ \partial(z,y) = \partial(x,y)-1\},$$

where ∂ is the path-length distance function of $J_q(n,k)$. We consider a Euclidean space E of dimension $(q^n-1)/(q-1)-1$. We turn E into a Euclidean representation of $J_q(n,k)$ associated with the second largest eigenvalue of the adjacency matrix. We represent the elements $x, y, x \cap y, x+y$ and the sets $\mathcal{B}_{xy}, \mathcal{C}_{xy}$ as vectors in E. We write the vector representations of $x \cap y, x+y$ as linear combinations of the vector representations of $x, y, \mathcal{B}_{xy}, \mathcal{C}_{xy}$; this is our first main result. For our second main result, we consider the stabilizer $\mathrm{Stab}(x,y)$ of x, y in $GL(\mathcal{V})$. We find the orbits of the $\mathrm{Stab}(x,y)$-action on the local graph of x. As we will see, there are five orbits. These orbits form an equitable partition. We compute the corresponding structure constants; this is our second main result. In our third main result, we display two matrices $A, A^* \in \mathrm{Mat}_P(\mathbb{C})$ that satisfy a generalization of the Askey-Wilson relations. We define a matrix $A \in \mathrm{Mat}_P(\mathbb{C})$ as follows. For $u, v \in P$, the (u,v)-entry of A is 1 if each of u, v covers $u \cap v$, and 0 otherwise. Fix $y \in P$ with $\dim y = k$. We define a diagonal matrix $A^* \in \mathrm{Mat}_P(\mathbb{C})$ as follows. For $u \in P$, the (u,u)-entry of A^* is $q^{\dim(u \cap y)}$. We show that

$$A^2A^* - \big(q+q^{-1}\big)AA^*A + A^*A^2 - \mathcal{Y}\big(AA^* + A^*A\big) - \mathcal{P}A^* = \Omega A + G,$$
$$A^{*2}A - \big(q+q^{-1}\big)A^*AA^* + AA^{*2} = \mathcal{Y}A^{*2} + \Omega A^* + G^*,$$

where $\mathcal{Y}, \mathcal{P}, \Omega, G, G^*$ are matrices in $\mathrm{Mat}_P(\mathbb{C})$ that commute with each of A, A^*. We give precise formulas for $\mathcal{Y}, \mathcal{P}, \Omega, G, G^*$.

Dedication

I would like to dedicate this thesis to my family, Gisun Seong, Ranmi Kim, and Dr. Chris Seong. It is their endless support that helped me become who I am today.

I would also like to dedicate this thesis to all my relatives, especially my grandparents, Kyungsik Seong and Jungja Kim, who would be delighted, more than anyone else, to see their grandchild become a Doctor of Philosophy.

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Contents

Chapter 1

Introduction

This thesis is about projective geometries, Grassmann graphs, and a generalization of the Askey-Wilson relations. The thesis contains three main results, which we will describe shortly. To set the stage, we briefly define the projective geometry and its corresponding Grassmann graphs. Let $\mathbb{F}_q$ denote a finite field with q elements. Fix an integer $n \geqslant 1$. Let $\mathcal{V}$ denote an n-dimensional vector space over $\mathbb{F}_q$. Let the set $P = P_q(n)$ consist of the subspaces of $\mathcal{V}$. The set P, together with the inclusion partial order, is a poset called a projective geometry. For $0 \leqslant \ell \leqslant n$, let the set P_ℓ consist of the ℓ-dimensional subspaces of $\mathcal{V}$. For $1 \leqslant k \leqslant n-1$ the Grassmann graph $J_q(n,k)$ is defined as follows. The vertex set of $J_q(n,k)$ is P_k. Two vertices $x, y \in P_k$ are adjacent whenever $x \cap y \in P_{k-1}$.

For the rest of this section, we abbreviate $\Gamma = J_q(n,k)$ and $X = P_k$. To avoid trivialities, we always assume $n > 2k \geqslant 6$.

We now describe our first main result. This result concerns how to use Γ to potentially recover P. Pick distinct vertices $x, y \in X$. The geometry P contains the elements x, y, $x \cap y$, $x + y$. Our goal is to describe $x \cap y$ and $x + y$ using only the graph structure of Γ.

To achieve this goal we construct a Euclidean representation of Γ in the sense of [7, Lecture 12]. We will use the notation $[m] = (q^m - 1)/(q-1)$ for an integer m. Let E denote a Euclidean space that has dimension $[n]-1$ and inner product $\langle\ ,\ \rangle$. We represent the elements of P_1 as vectors in E as follows. For $s \in P_1$, we define the vector $\widehat{s} \in E$ such that the following (i)–(iv) are satisfied:

(i) $E = \operatorname{Span}\{\widehat{s} \mid s \in P_1\}$;

(ii) for $s \in P_1$, $\left\|\widehat{s}\right\|^2 = [n] - 1$;

(iii) for distinct $s, t \in P_1$, $\left\langle \widehat{s}, \widehat{t} \right\rangle = -1$;

(iv) $\displaystyle\sum_{s \in P_1} \widehat{s} = 0$.

Next, we represent the elements of P as vectors in E as follows. For $u \in P$ we define $\widehat{u} \in E$

as

$$\widehat{u} = \sum_{\substack{s \in P_1 \\ s \subseteq u}} \widehat{s}.$$

We show that for $u, v \in P$,

$$\langle \widehat{u}, \widehat{v} \rangle = [n][h] - [i][j],$$

where $i = \dim u$, $j = \dim v$, $h = \dim(u \cap v)$.

Let ∂ denote the path-length distance function of Γ. We show that for $x, y \in X$,

$$\langle \widehat{x}, \widehat{y} \rangle = [n][k-i] - [k]^2,$$

where $i = \partial(x, y)$. For $x \in X$, let $\Gamma(x)$ denote the set of vertices in Γ that are adjacent to x. We show that

$$\sum_{z \in \Gamma(x)} \widehat{z} = \theta_1 \widehat{x}$$

where $\theta_1 = q^2[k-1][n-k-1] - 1$. The scalar θ_1 is the second largest eigenvalue of the adjacency matrix of Γ. We show that the vectors $\{\widehat{x} \mid x \in X\}$ span E. Using the above facts, we show that the vectors $\{\widehat{x} \mid x \in X\}$ give a Euclidean representation of Γ. Using the $GL(\mathcal{V})$-action on P_1 we turn E into a $GL(\mathcal{V})$-module.

Pick $x, y \in X$ such that $1 < \partial(x, y) < k$, and consider the stabilizer $\mathrm{Stab}(x, y)$ of x, y in $GL(\mathcal{V})$. Let $\mathrm{Fix}(x, y)$ denote the subspace of E consisting of the vectors that are fixed by every element of $\mathrm{Stab}(x, y)$. We show that the vectors

$$\widehat{x}, \qquad \widehat{y}, \qquad \widehat{x \cap y}, \qquad \widehat{x + y} \tag{1.1}$$

form a basis for $\mathrm{Fix}(x, y)$.

Using x and y we construct two vectors $B_{xy}, C_{xy} \in E$ as follows. Define the sets

$$\mathcal{B}_{xy} = \{z \in \Gamma(x) \mid \partial(y, z) = \partial(x, y) + 1\},$$
$$\mathcal{C}_{xy} = \{z \in \Gamma(x) \mid \partial(y, z) = \partial(x, y) - 1\}.$$

Define the vectors

$$B_{xy} = \sum_{z \in \mathcal{B}_{xy}} \widehat{z}, \qquad C_{xy} = \sum_{z \in \mathcal{C}_{xy}} \widehat{z}.$$

We show that the vectors

$$\widehat{x}, \qquad \widehat{y}, \qquad B_{xy}, \qquad C_{xy} \tag{1.2}$$

form a basis for $\mathrm{Fix}(x, y)$.

We find all the inner products between

(i) pairs of vectors in (1.1);

(ii) pairs of vectors in (1.2);

(iii) a vector in (1.1) and a vector in (1.2).

Using these inner products, we obtain the transition matrices between the bases (1.1) and (1.2). This yields

$$\begin{aligned}
\widehat{x \cap y} &= \frac{[k-i][n-k-1]}{q^{k-1}[n-2k]}\hat{x} + \frac{[k-i]}{q^{k-i+1}[i-1][n-2k]}\hat{y} \\
&\qquad + \frac{-1}{q^{k+i}[n-2k]}B_{xy} + \frac{-[k-i]}{q^{k}[i-1][n-2k]}C_{xy}, \\
\widehat{x + y} &= \frac{-[k-1][n-k-i]}{q^{k-i-1}[n-2k]}\hat{x} + \frac{-[n-k-i]}{q^{k-2i+1}[i-1][n-2k]}\hat{y} \\
&\qquad + \frac{1}{q^{k}[n-2k]}B_{xy} + \frac{[n-k-i]}{q^{k-i}[i-1][n-2k]}C_{xy},
\end{aligned}$$

where $i = \partial(x, y)$. The above equations are our first main result.

We now describe our second main result. Pick $x, y \in X$ such that $1 < \partial(x, y) < k$. We consider the orbits of the $\mathrm{Stab}(x, y)$-action on $\Gamma(x)$. As we will see, there are five orbits, two of which are $\mathcal{B}_{xy}$ and $\mathcal{C}_{xy}$. We now describe the other three orbits.

Define the set

$$\mathcal{A}_{xy} = \{z \in \Gamma(x) \mid \partial(y, z) = \partial(x, y)\}.$$

We partition the set $\mathcal{A}_{xy}$ into the following three sets:

$$\begin{aligned}
\mathcal{A}^{+}_{xy} &= \{z \in \mathcal{A}_{xy} \mid z + x + y \supsetneq x + y,\ z \cap x \cap y = x \cap y\}, \\
\mathcal{A}^{0}_{xy} &= \{z \in \mathcal{A}_{xy} \mid z + x + y = x + y,\ z \cap x \cap y = x \cap y\}, \\
\mathcal{A}^{-}_{xy} &= \{z \in \mathcal{A}_{xy} \mid z + x + y = x + y,\ z \cap x \cap y \subsetneq x \cap y\}.
\end{aligned}$$

We show that the sets

$$\mathcal{A}^{+}_{xy}, \qquad \mathcal{A}^{0}_{xy}, \qquad \mathcal{A}^{-}_{xy}$$

are orbits of the $\mathrm{Stab}(x, y)$-action on $\Gamma(x)$. Hence,

$$\mathcal{B}_{xy}, \qquad \mathcal{C}_{xy}, \qquad \mathcal{A}^{+}_{xy}, \qquad \mathcal{A}^{0}_{xy}, \qquad \mathcal{A}^{-}_{xy} \tag{1.3}$$

are the five orbits of the $\mathrm{Stab}(x, y)$-action on $\Gamma(x)$. By construction, (1.3) is a partition of $\Gamma(x)$ that is equitable in the sense of [5, p. 159]. We call this partition the y-partition of $\Gamma(x)$.

Define the vectors

$$A^{+}_{xy} = \sum_{z \in \mathcal{A}^{+}_{xy}} \hat{z}, \qquad A^{0}_{xy} = \sum_{z \in \mathcal{A}^{0}_{xy}} \hat{z}, \qquad A^{-}_{xy} = \sum_{z \in \mathcal{A}^{-}_{xy}} \hat{z}. \tag{1.4}$$

Recall the subspace $\mathrm{Fix}(x, y)$ of E. We show that $A^{+}_{xy}, A^{0}_{xy}, A^{-}_{xy}$ are contained in $\mathrm{Fix}(x, y)$. We write each vector in (1.4) as a linear combination of the vectors in (1.1) and also the vectors in (1.2). We find the inner products between:

(i) any vector in (1.4) and any vector in the basis (1.1);

(ii) any vector in (1.4) and any vector in the basis (1.2);

(iii) any pair of vectors in (1.4).

We mentioned that the y-partition of $\Gamma(x)$ is equitable. We compute the corresponding structure constants using the inner products that involve $B_{xy}, C_{xy}, A^+_{xy}, A^0_{xy}, A^-_{xy}$. In the table below, for each orbit $\mathcal{O}$ in the header column, and each orbit $\mathcal{N}$ in the header row, the $(\mathcal{O},\mathcal{N})$-entry gives the number of vertices in $\mathcal{N}$ that are adjacent to a given vertex in $\mathcal{O}$. Write $i = \partial(x,y)$.

	$\mathcal{B}_{xy}$	$\mathcal{C}_{xy}$	$\mathcal{A}^+_{xy}$	$\mathcal{A}^0_{xy}$	$\mathcal{A}^-_{xy}$
$\mathcal{B}_{xy}$	$q^{i+1}[k-i]$ $+q^{i+1}[n-k-i]-q-1$	0	$q[i]$	0	$q[i]$
$\mathcal{C}_{xy}$	0	$2q[i-1]$	$q^{i+1}[n-k-i]$	$(q-1)\big(2[i]-1\big)$	$q^{i+1}[k-i]$
$\mathcal{A}^+_{xy}$	$q^{i+1}[k-i]$	$[i]$	$q[n-k]-q-1$	$(q-1)[i]$	0
$\mathcal{A}^0_{xy}$	0	$2[i]-1$	$q^{i+1}[n-k-i]$	$(q-1)\big(2[i]-1\big)-1$	$q^{i+1}[k-i]$
$\mathcal{A}^-_{xy}$	$q^{i+1}[n-k-i]$	$[i]$	0	$(q-1)[i]$	$q[k]-q-1$

The table above is our second main result.

We now describe our third main result. This result concerns a generalization of the Askey-Wilson relations. To motivate our result, we first describe the Askey-Wilson relations. Let $\mathcal{U}$ denote a nonzero vector space of finite dimension over an arbitrary field. A pair of linear maps $A : \mathcal{U} \to \mathcal{U}$ and $A^* : \mathcal{U} \to \mathcal{U}$ is called a Leonard pair whenever each of A, A^* acts on an eigenbasis of the other one in an irreducible tridiagonal fashion. In this case, A and A^* satisfy a pair of relations of the form

$$A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^* + A^*A) - \varrho A^* = \gamma^* A^2 + \omega A + \eta I,$$
$$A^{*2}A - \beta A^*AA^* + AA^{*2} - \gamma^*(A^*A + AA^*) - \varrho^* A = \gamma A^{*2} + \omega A^* + \eta^* I,$$

where $\beta, \gamma, \gamma^*, \varrho, \varrho^*, \omega, \eta, \eta^*$ are appropriate scalars; see [9, Theorem 1.5]. The above relations are called the Askey-Wilson relations.

We return our attention to the projective geometry P. We will display two matrices $A, A^* \in \mathrm{Mat}_P(\mathbb{C})$ that satisfy a generalization of the Askey-Wilson relations. Define $A \in \mathrm{Mat}_P(\mathbb{C})$ as follows. For $u, v \in P$, the (u,v)-entry of A is

$$A_{u,v} = \begin{cases} 1 & \text{if each of } u, v \text{ covers } u \cap v, \\ 0 & \text{otherwise.} \end{cases}$$

Fix $y \in X$. Define a diagonal matrix $A^* \in \mathrm{Mat}_P(\mathbb{C})$ as follows. For $u \in P$, the (u,u)-entry of A^* is

$$\big(A^*\big)_{u,u} = q^{\dim(u \cap y)}.$$

We show that

$$A^2A^* - \big(q+q^{-1}\big)AA^*A + A^*A^2 - \mathcal{Y}\big(AA^* + A^*A\big) - \mathcal{P}A^* = \Omega A + G, \qquad (1.5)$$

$$A^{*2}A - \big(q+q^{-1}\big)A^*AA^* + AA^{*2} = \mathcal{Y}A^{*2} + \Omega A^* + G^*, \qquad (1.6)$$

where $\mathcal{Y}, \mathcal{P}, \Omega, G, G^*$ are matrices in $\mathrm{Mat}_P(\mathbb{C})$ that commute with each of A, A^*. We give precise formulas for $\mathcal{Y}, \mathcal{P}, \Omega, G, G^*$ in the main body of the thesis. The relations (1.5), (1.6) are our third main result.

The thesis is organized as follows. In Chapter 2 we present some basic facts about the Grassmann graph Γ and the projective geometry P. In Chapter 3 we discuss the recovery of P from Γ. In Chapter 4 we find the orbits of the $\mathrm{Stab}(x, y)$-action on $\Gamma(x)$. We also find the corresponding structure constants. In Chapter 5 we discuss a generalization of the Askey-Wilson relations. Chapter 6 is an appendix on some linear algebra facts and relations that are used throughout the thesis.

Chapter 2

Preliminaries

2.1 Distance-regular graph

Let $\Gamma = (X, \mathcal{E})$ denote a finite undirected graph that is connected, without loops or multiple edges, with vertex set X, edge set $\mathcal{E}$, and path-length distance function ∂. Two vertices are said to be adjacent whenever they form an edge. The diameter D of Γ is defined as $D = \max\{\partial(x,y) \mid x, y \in X\}$. For $x \in X$, define the set $\Gamma(x) = \{y \in X \mid \partial(x,y) = 1\}$. The subgraph induced on $\Gamma(x)$ is called the *local graph* of x.

We say that Γ is *regular with valency* κ whenever $|\Gamma(x)| = \kappa$ for all $x \in X$. We say that Γ is *distance-regular* whenever for all non-negative integers h, i, j and all $x, y \in X$ such that $\partial(x,y) = h$, the cardinality of the set $\{z \in X \mid \partial(x,z) = i, \partial(y,z) = j\}$ depends only on h, i, j. This cardinality is denoted by $p^h_{i,j}$. If Γ is distance-regular, then Γ is regular with valency $\kappa = p^0_{1,1}$. For the rest of this section, we assume that Γ is distance-regular with diameter $D \geqslant 3$.

Define

$$b_i = p^i_{1,i+1} \ (0 \leqslant i < D), \qquad a_i = p^i_{1,i} \ (0 \leqslant i \leqslant D), \qquad c_i = p^i_{1,i-1} \ (0 < i \leqslant D).$$

Note that $b_0 = \kappa$, $a_0 = 0$, $c_1 = 1$. Also note that

$$b_i + a_i + c_i = \kappa \qquad (0 \leqslant i \leqslant D), \tag{2.1}$$

where $c_0 = 0$ and $b_D = 0$.

We call b_i, a_i, c_i the *intersection numbers of* Γ.

By the *eigenvalues of* Γ we mean the eigenvalues of the adjacency matrix of Γ. Since Γ is distance-regular, by [2, p. 128], Γ has $D+1$ eigenvalues; we denote these eigenvalues by

$$\theta_0 > \theta_1 > \cdots > \theta_D.$$

By [2, p. 129], $\theta_0 = \kappa$.

By the *spectrum of* Γ we mean the set of ordered pairs $\{(\theta_i, m_i)\}_{i=0}^{D}$, where $\{\theta_i\}_{i=0}^{D}$ are the eigenvalues of Γ and m_i the dimension of the θ_i-eigenspace $(0 \leqslant i \leqslant D)$.

2.2 Grassmann graph $J_q(n,k)$

We now define the Grassmann graph $J_q(n,k)$. Let $\mathbb{F}_q$ denote a finite field with q elements, and let n, k denote positive integers such that $n > k$. Let $\mathcal{V}$ denote an n-dimensional vector space over $\mathbb{F}_q$. The Grassmann graph $J_q(n,k)$ has vertex set X consisting of the k-dimensional subspaces of $\mathcal{V}$. Vertices x, y of $J_q(n,k)$ are adjacent whenever $x \cap y$ has dimension $k-1$.

According to [2, p. 268], the graphs $J_q(n,k)$ and $J_q(n,n-k)$ are isomorphic. Without loss of generality, we may assume $n \geqslant 2k$. Under this assumption, the diameter of $J_q(n,k)$ is equal to k. (See [2, Theorem 9.3.3].) The case $n = 2k$ is somewhat special, so throughout the thesis we assume that $n > 2k$.

For the rest of the thesis, we assume that Γ is the Grassmann graph $J_q(n,k)$ with $k \geqslant 3$.

In what follows, we will use the notation

$$[m] = \frac{q^m - 1}{q - 1} \qquad (m \in \mathbb{Z}). \tag{2.2}$$

By [2, Theorem 9.3.2], the valency of Γ is

$$\kappa = q[k][n-k]. \tag{2.3}$$

By [2, Theorem 9.3.3], the intersection numbers of Γ are

$$b_i = q^{2i+1}[k-i][n-k-i], \qquad c_i = [i]^2 \qquad (0 \leqslant i \leqslant k). \tag{2.4}$$

By [2, Theorem 9.3.3], the eigenvalues of Γ are

$$\theta_i = q^{i+1}[k-i][n-k-i] - [i] \qquad (0 \leqslant i \leqslant k). \tag{2.5}$$

The given ordering of the eigenvalues is known to be Q-polynomial in the sense of [2, p. 135].

2.3 The projective geometry $P_q(n)$

To study the graph Γ, it is helpful to view its vertex set X as a subset of a certain poset $P_q(n)$, which is defined as follows.

Definition 2.3.1 Let the poset $P_q(n)$ consist of the subspaces of $\mathcal{V}$, together with the partial order given by inclusion. This poset $P_q(n)$ is called the *projective geometry*.

For the rest of the thesis, we abbreviate $P = P_q(n)$. In this section we present some lemmas about the poset P.

For $0 \leqslant \ell \leqslant n$, let the set P_ℓ consist of the ℓ-dimensional subspaces of $\mathcal{V}$. Note that $X = P_k$. Also note that $P_0 = \{0\}$ and $P_n = \{\mathcal{V}\}$. We have a partition

$$P = \bigcup_{\ell=0}^{n} P_\ell. \tag{2.6}$$

For $u, v \in P$, we say that v *covers* u whenever $v \supseteq u$ and $\dim v = \dim u + 1$.

Lemma 2.3.2 [1, p. 47] *For $u, v \in P$ we have*

$$\dim u + \dim v = \dim(u \cap v) + \dim(u + v).$$

Lemma 2.3.3 *Let $u, v \in P$. Let the subset $\mathcal{R} \subseteq \mathcal{V}$ form a basis for $u \cap v$. Extend the basis $\mathcal{R}$ to a basis $\mathcal{R} \cup \mathcal{S}$ for u, and extend the basis $\mathcal{R}$ to a basis $\mathcal{R} \cup \mathcal{T}$ for v. Then $\mathcal{R} \cup \mathcal{S} \cup \mathcal{T}$ forms a basis for the subspace $u + v$.*

Proof. Since the set $\mathcal{R} \cup \mathcal{S}$ is a basis for u and the set $\mathcal{R} \cup \mathcal{T}$ is a basis for v, the set $\mathcal{R} \cup \mathcal{S} \cup \mathcal{T}$ spans $u + v$.

By Lemma 2.3.2,

$$\dim(u + v) = \dim u + \dim v - \dim(u \cap v) = |\mathcal{R}| + |\mathcal{S}| + |\mathcal{T}|.$$

The result follows. □

Lemma 2.3.4 [2, p. 269] *For $x, y \in X$ the dimension of $x \cap y$ is $k - \partial(x, y)$.*

Lemma 2.3.5 *For $x, y \in X$ the dimension of $x + y$ is $k + \partial(x, y)$.*

Proof. Routine using Lemmas 2.3.2, 2.3.4. □

Lemma 2.3.6 *Let $x, y, z \in X$ satisfy $\partial(x, y) = \partial(x, z) + \partial(z, y)$. Then $x \cap y \subseteq z \subseteq x + y$.*

Proof. Routine from linear algebra. □

Definition 2.3.7 For $u \in P$ define the set

$$\Omega(u) = \{s \in P_1 \mid s \subseteq u\}.$$

Note that $\Omega(\mathcal{V}) = P_1$.

For $u, v \in P$ we have

$$\Omega(u) \cap \Omega(v) = \Omega(u \cap v). \tag{2.7}$$

The following notation will be useful. For an integer $m \geqslant 0$, define

$$[m]^! = [m][m-1] \cdots [2][1].$$

We interpret $[0]^! = 1$.

For integers $0 \leqslant r \leqslant m$, define

$$\begin{bmatrix} m \\ r \end{bmatrix} = \frac{[m]^!}{[r]^![m-r]^!}.$$

Lemma 2.3.8 [2, Theorem 9.3.2] *For integers $0 \leqslant r \leqslant m$, $\begin{bmatrix} m \\ r \end{bmatrix}$ is equal to the number of r-dimensional subspaces of a given element in P_m.*

By Lemma 2.3.8, we find that for all $u \in P$,

$$\left|\Omega(u)\right| = [m], \tag{2.8}$$

where $u \in P_m$.

Letting $u = \mathcal{V}$ in (2.8), we get

$$|P_1| = \left|\Omega(\mathcal{V})\right| = [n].$$

Recall the partition in (2.6). We now present another partition of P. Pick $y \in X$. For $0 \leqslant i \leqslant k$ and $0 \leqslant j \leqslant n-k$, define

$$P_{i,j} = \left\{u \in P \mid \dim(u \cap y) = i, \dim u = i + j\right\}.$$

We have a partition

$$P = \bigcup_{i=0}^{k} \bigcup_{j=0}^{n-k} P_{i,j}. \tag{2.9}$$

For notational convenience, for integers r, s we define $P_{r,s}$ to be empty unless $0 \leqslant r \leqslant k$ and $0 \leqslant s \leqslant n-k$.

In the diagram below, we illustrate the set P and the subsets $P_{i,j}$ $(0 \leqslant i \leqslant k,\ 0 \leqslant j \leqslant n-k)$.

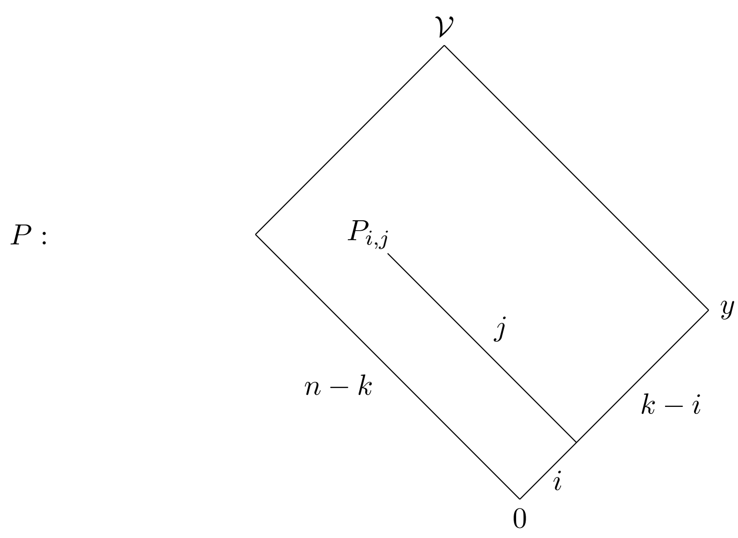

Figure 2.1: The projective geometry P and the location of $P_{i,j}$.

We now describe how the sets

$$P_\ell \qquad\qquad (0 \leqslant \ell \leqslant n)$$

are related to the sets

$$P_{i,j} \qquad\qquad (0 \leqslant i \leqslant k, \;\; 0 \leqslant j \leqslant n-k).$$

For $0 \leqslant \ell \leqslant n$ we have a partition

$$P_\ell = \bigcup_{i,j} P_{i,j}, \tag{2.10}$$

where the union is over the ordered pairs (i, j) such that $0 \leqslant i \leqslant k$ and $0 \leqslant j \leqslant n-k$ and $i + j = \ell$.

Earlier, we described the covering relation on P. Next, we give a refinement of the covering relation.

Lemma 2.3.9 *Let $u, v \in P$ such that v covers u. Write*

$$u \in P_{i,j}, \qquad\qquad v \in P_{r,s}.$$

Then either (i) $r = i+1$ *and* $s = j$, *or* (ii) $r = i$ *and* $s = j+1$.

Proof. By linear algebra,

$$u \cap v \in P_{a,b},$$

where $a \leqslant \min\{i, r\}$ and $b \leqslant \min\{j, s\}$. Since v covers u, we have $u \subseteq v$ and $r+s = i+j+1$. Since $u \cap v = u \in P_{i,j}$, we have $i \leqslant r$ and $j \leqslant s$. The result follows. □

Definition 2.3.10 Referring to Lemma 2.3.9, we say that v */-covers* u whenever (i) holds, and v *\\-covers* u whenever (ii) holds.

We illustrate Definition 2.3.10 using the diagrams below.

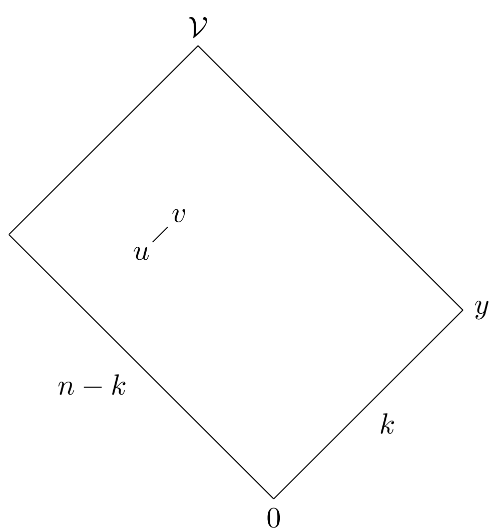

Figure 2.2A: v /-covers u.

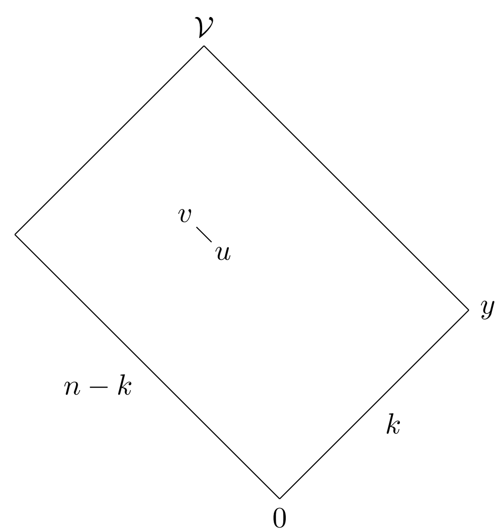

Figure 2.2B: v \-covers u.

Lemma 2.3.11 [10, Lemma 5.1] *For* $0 \leqslant i \leqslant k$ *and* $0 \leqslant j \leqslant n-k$*, the following* (i)–(iv) *hold:*

(i) *each element in* $P_{i,j}$ /*-covers exactly* $q^j[i]$ *elements;*

(ii) *each element in* $P_{i,j}$ $\backslash$*-covers exactly* $[j]$ *elements;*

(iii) *each element in* $P_{i,j}$ *is* /*-covered by exactly* $[k-i]$ *elements;*

(iv) *each element in* $P_{i,j}$ *is* $\backslash$*-covered by exactly* $q^{k-i}[n-k-j]$ *elements.*

We now comment on the symmetries of P. Recall that the general linear group $GL(\mathcal{V})$ consists of the invertible $\mathbb{F}_q$-linear maps from $\mathcal{V}$ to $\mathcal{V}$. The action of $GL(\mathcal{V})$ on $\mathcal{V}$ induces a permutation action of $GL(\mathcal{V})$ on the set P. This permutation action respects the partial order on P. The orbits of the action are P_ℓ for $0 \leqslant \ell \leqslant n$.

Lemma 2.3.12 *For* $x, y \in X$ *and* $\sigma \in GL(\mathcal{V})$*, we have* $\partial(x,y) = \partial(\sigma(x), \sigma(y))$.

Proof. Immediate from Lemma 2.3.4 and the fact that the σ-action preserves dimension. □

Chapter 3

Using a Grassmann graph to recover the underlying projective geometry

In this chapter we discuss how one could potentially recover the underlying projective geometry P from a given Grassmann graph $J_q(n,k)$. We represent the elements of P in some Euclidean space E. Pick $x, y \in X$ such that $1 < \partial(x,y) < k$. We define the stabilizer $\mathrm{Stab}(x,y)$ in $GL(\mathcal{V})$ and some subspace $\mathrm{Fix}(x,y)$ in E. We find two bases for $\mathrm{Fix}(x,y)$, namely the geometric basis and the combinatorial basis. We display the transition matrices between the two bases. We also present many inner products that involve the basis vectors. We finish the chapter by discussing the connection between the recovery of P and the uniqueness problem for the Grassmann graphs.

3.1 Representing P using a Euclidean space E

In this section we represent the elements of P as vectors in a Euclidean space. We will do this in two stages. In the first stage we consider the elements of P_1.

Let E denote a Euclidean space with dimension $[n]-1$ and bilinear form $\langle\ ,\ \rangle$. Recall the notation $\|\nu\|^2 = \langle \nu, \nu \rangle$ for any $\nu \in E$. We define a function

$$\begin{aligned} P_1 &\to E \\ s &\mapsto \widehat{s} \end{aligned} \tag{3.1}$$

that satisfies the following conditions (C1) − (C4):

(C1) $E = \mathrm{Span}\{\widehat{s} \mid s \in P_1\}$;

(C2) for $s \in P_1$, $\|\widehat{s}\|^2 = [n] - 1$;

(C3) for distinct $s, t \in P_1$, $\left\langle \widehat{s}, \widehat{t} \right\rangle = -1$;

(C4) $\sum_{s \in P_1} \widehat{s} = 0$.

Next, we extend the function (3.1) to a function

$$\begin{aligned} P &\to E \\ u &\mapsto \widehat{u} \end{aligned} \tag{3.2}$$

such that for all $u \in P$,

$$\widehat{u} = \sum_{s \in \Omega(u)} \widehat{s}. \tag{3.3}$$

Note that $\widehat{u} = 0$ if $u \in P_0$ or $u \in P_n$.

In (C4), we gave a linear dependence on $\{\widehat{s}\}_{s \in P_1}$. Next we show that (C4) is essentially the only linear dependence among $\{\widehat{s}\}_{s \in P_1}$.

We use the following notation. For sets $\mathcal{R} \subseteq \mathcal{S}$, define $\mathcal{S} \backslash \mathcal{R}$ to be the complement of $\mathcal{R}$ in $\mathcal{S}$.

Lemma 3.1.1 *Given real numbers* $\{\alpha_s\}_{s \in P_1}$ *the following are equivalent:*

(i) $0 = \sum_{s \in P_1} \alpha_s \widehat{s}$;

(ii) α_s *is independent of* s *for* $s \in P_1$.

Proof. (i)⇒(ii) Pick $t \in P_1$. Referring to (C4), multiply each side by α_t to obtain

$$0 = \sum_{s \in P_1} \alpha_t \widehat{s}. \tag{3.4}$$

Subtract (3.4) from the equation in (i) to obtain

$$0 = \sum_{s \in P_1 \backslash \{t\}} (\alpha_s - \alpha_t) \widehat{s}.$$

By Lemma 6.1.1 in the appendix, the vectors $\{\widehat{s}\}_{s \in P_1 \backslash \{t\}}$ form a basis for E, so they are linearly independent. Hence $\alpha_s - \alpha_t = 0$ for all $s \in P_1$. The result follows.

(ii)⇒(i) Immediate from (C4). □

Lemma 3.1.2 *The Euclidean space* E *becomes a* $GL(\mathcal{V})$*-module such that for all* $u \in P$ *and* $\sigma \in GL(\mathcal{V})$,

$$\sigma\big(\widehat{u}\big) = \widehat{\sigma(u)}.$$

Proof. It suffices to show that there exists a group homomorphism $\Theta : GL(\mathcal{V}) \to GL(E)$ such that for all $u \in P$ and $\sigma \in GL(\mathcal{V})$,

$$\Theta(\sigma)\big(\widehat{u}\big) = \widehat{\sigma(u)}. \tag{3.5}$$

Let $\sigma \in GL(\mathcal{V})$. We first show that there exists a unique $\Theta(\sigma) \in GL(E)$ such that (3.5) holds for all $u \in P_1$.

By construction, the dimension of E is $|P_1| - 1$.

Note that σ acts as a permutation on P_1. By this along with (C1) and (C4), we obtain

$$E = \text{Span}\left\{ \widehat{\sigma(s)} \;\middle|\; s \in P_1 \right\}, \qquad \sum_{s \in P_1} \widehat{\sigma(s)} = \sum_{s \in P_1} \widehat{s} = 0.$$

By these comments and Lemma 6.1.2 in the appendix, there exists a unique $\Theta(\sigma) \in GL(E)$ that satisfies (3.5) for all $u \in P_1$.

Next we show that $\Theta(\sigma)$ satisfies (3.5) for all $u \in P$. By (3.3),

$$\Theta(\sigma)\big(\widehat{u}\big) = \Theta(\sigma)\left(\sum_{s \in \Omega(u)} \widehat{s}\right) = \sum_{s \in \Omega(u)} \Theta(\sigma)\big(\widehat{s}\big) = \sum_{s \in \Omega(u)} \widehat{\sigma(s)} = \widehat{\sigma(u)}.$$

Next we show that Θ is a group homomorphism. For $\sigma, \tau \in GL(\mathcal{V})$, we show that $\Theta(\sigma)\Theta(\tau) = \Theta(\sigma\tau)$.

For $u \in P$ we have

$$\Theta(\sigma)\Theta(\tau)\big(\widehat{u}\big) = \Theta(\sigma)\Big(\widehat{\tau(u)}\Big) = \widehat{\sigma\tau(u)} = \Theta(\sigma\tau)\big(\widehat{u}\big).$$

Therefore, $\Theta(\sigma)\Theta(\tau) = \Theta(\sigma\tau)$.

We have shown that there exists a unique group homomorphism $\Theta : GL(\mathcal{V}) \to GL(E)$ that satisfies (3.5) for all $\sigma \in GL(\mathcal{V})$ and $u \in P$. Consequently, E becomes a $GL(\mathcal{V})$-module such that $\sigma\big(\widehat{u}\big) = \widehat{\sigma(u)}$ for all $u \in P$ and $\sigma \in GL(\mathcal{V})$. □

Lemma 3.1.3 *For $\mu, \nu \in E$ and $\sigma \in GL(\mathcal{V})$,*

$$\langle \mu, \nu \rangle = \big\langle \sigma(\mu), \sigma(\nu) \big\rangle.$$

Proof. In view of (C1), we may assume without loss that $\mu = \widehat{s}$ and $\nu = \widehat{t}$ where $s, t \in P_1$. The result is a routine consequence of (C2) and (C3). □

Lemma 3.1.4 *Pick a vector $\mu \in E$ and write*

$$\mu = \sum_{s \in P_1} \alpha_s \widehat{s}. \tag{3.6}$$

For $\sigma \in GL(\mathcal{V})$ that fixes μ, $\alpha_s = \alpha_{\sigma(s)}$ for all $s \in P_1$.

Proof. Referring to (3.6), apply σ^{-1} to each side and evaluate the result using Lemma 3.1.2 to get

$$\mu = \sum_{s \in P_1} \alpha_s \widehat{\sigma^{-1}(s)}. \tag{3.7}$$

We make a change of variables. In (3.7), replace s by $\sigma(s)$ to get

$$\mu = \sum_{s \in P_1} \alpha_{\sigma(s)} \widehat{s}. \tag{3.8}$$

Subtract (3.6) from (3.8) to obtain

$$0 = \sum_{s \in P_1} (\alpha_{\sigma(s)} - \alpha_s) \widehat{s}.$$

By Lemma 3.1.1, the scalar $\alpha_{\sigma(s)} - \alpha_s$ is independent of $s \in P_1$. Denote this common value by β. Then

$$\sum_{s \in P_1} \left(\alpha_{\sigma(s)} - \alpha_s\right) = \sum_{s \in P_1} \beta = \beta |P_1|. \tag{3.9}$$

Note that

$$\sum_{s \in P_1} \left(\alpha_{\sigma(s)} - \alpha_s\right) = \sum_{s \in P_1} \alpha_s - \sum_{s \in P_1} \alpha_s = 0. \tag{3.10}$$

By (3.9), (3.10) and the fact that $|P_1|$ is nonzero, we obtain $\beta = 0$. The result follows. □

Corollary 3.1.5 *Pick a vector $\mu \in E$ and write*

$$\mu = \sum_{s \in P_1} \alpha_s \widehat{s}.$$

For a subgroup H of $GL(\mathcal{V})$ the following are equivalent:

(i) *every element of H fixes μ;*

(ii) *for $s, t \in P_1$ that are contained in the same H-orbit, $\alpha_s = \alpha_t$.*

Proof. Immediate from Lemma 3.1.4. □

Our next general goal is to restate Corollary 3.1.5 using a different point of view.

Definition 3.1.6 Let $\mathcal{S}$ denote a subset of P_1. By the *characteristic vector of* $\mathcal{S}$ we mean the vector

$$\sum_{s \in \mathcal{S}} \widehat{s}.$$

Note that this characteristic vector is contained in E.

Lemma 3.1.7 *For a subset $\mathcal{S} \subseteq P_1$ the sum of the following vectors is zero:*

(i) *the characteristic vector of $\mathcal{S}$;*

(ii) *the characteristic vector of the set $P_1 \backslash \mathcal{S}$.*

Proof. Immediate from (C4). □

Lemma 3.1.8 *Pick a vector $\mu \in E$ and a subgroup H of $GL(\mathcal{V})$. Then the following are equivalent:*

(i) *every element of H fixes μ;*

(ii) *μ is contained in the span of the characteristic vectors of the H-orbits in P_1.*

Proof. Immediate from Corollary 3.1.5. □

3.2 The stabilizer of an element in P

In this section we pick an element in P and consider its stabilizer in $GL(\mathcal{V})$. We describe the orbits of the stabilizer acting on P. We also consider how the stabilizer acts on E.

For $u \in P$, let $\mathrm{Stab}(u)$ denote the subgroup of $GL(\mathcal{V})$ consisting of the elements that fix u. We call $\mathrm{Stab}(u)$ the *stabilizer of u in $GL(\mathcal{V})$*. Note that $\mathrm{Stab}(0) = GL(\mathcal{V})$ and $\mathrm{Stab}(\mathcal{V}) = GL(\mathcal{V})$.

Lemma 3.2.1 *For $u, v, v' \in P$ the following are equivalent:*

(i) $\dim v = \dim v'$ *and* $\dim(u \cap v) = \dim(u \cap v')$*;*

(ii) *the subspaces v and v' are contained in the same orbit of the* $\mathrm{Stab}(u)$*-action on P.*

Proof. (i)⇒(ii) We display an element $\sigma \in \mathrm{Stab}(u)$ that sends $v \mapsto v'$.

Let the subset $\mathcal{R} \subseteq \mathcal{V}$ form a basis for $u \cap v$. Extend the basis $\mathcal{R}$ to a basis $\mathcal{R} \cup \mathcal{S}$ for u. Extend the basis $\mathcal{R}$ to a basis $\mathcal{R} \cup \mathcal{T}$ for v. By Lemma 2.3.3, $\mathcal{R} \cup \mathcal{S} \cup \mathcal{T}$ is a basis for $u + v$. Extend the basis $\mathcal{R} \cup \mathcal{S} \cup \mathcal{T}$ to a basis $\mathcal{R} \cup \mathcal{S} \cup \mathcal{T} \cup \mathcal{Q}$ for $\mathcal{V}$.

Let the subset $\mathcal{R}' \subseteq \mathcal{V}$ form a basis for $u \cap v'$. Extend the basis $\mathcal{R}'$ to a basis $\mathcal{R}' \cup \mathcal{S}'$ for u. Extend the basis $\mathcal{R}'$ to a basis $\mathcal{R}' \cup \mathcal{T}'$ for v'. By Lemma 2.3.3, $\mathcal{R}' \cup \mathcal{S}' \cup \mathcal{T}'$ is a basis for $u + v'$. Extend the basis $\mathcal{R}' \cup \mathcal{S}' \cup \mathcal{T}'$ to a basis $\mathcal{R}' \cup \mathcal{S}' \cup \mathcal{T}' \cup \mathcal{Q}'$ for $\mathcal{V}$.

By linear algebra, there exists $\sigma \in GL(\mathcal{V})$ that sends $\mathcal{R} \mapsto \mathcal{R}'$, $\mathcal{S} \mapsto \mathcal{S}'$, $\mathcal{T} \mapsto \mathcal{T}'$, $\mathcal{Q} \mapsto \mathcal{Q}'$. By construction, σ is contained in $\mathrm{Stab}(u)$ and sends $v \mapsto v'$.

(ii)⇒(i) Let $\sigma \in \mathrm{Stab}(u)$ send $v \mapsto v'$. By linear algebra, σ sends $u \cap v \mapsto u \cap v'$. The result follows since dimensions are left invariant under the σ-action. □

Corollary 3.2.2 *For $u \in P$ the following hold.*

(i) *If $u \neq 0$ and $u \neq \mathcal{V}$, then the* $\mathrm{Stab}(u)$*-action on P_1 has two orbits, $\Omega(u)$ and $P_1 \backslash \Omega(u)$.*

(ii) *If $u = 0$ or $u = \mathcal{V}$, then the* $\mathrm{Stab}(u)$*-action on P_1 has a single orbit.*

Proof. Use Lemma 3.2.1. □

For $u \in P$, let $\mathrm{Fix}(u)$ denote the subspace of E consisting of the vectors that are fixed by every element of $\mathrm{Stab}(u)$. Note that $\mathrm{Fix}(0) = 0$ and $\mathrm{Fix}(\mathcal{V}) = 0$.

Lemma 3.2.3 *For $u \in P$ the subspace* $\mathrm{Fix}(u)$ *is spanned by $\hat{u}$.*

Proof. First assume that $u = 0$ or $u = \mathcal{V}$. Then the result holds since $\widehat{u} = 0$. Next assume that $u \neq 0$ and $u \neq \mathcal{V}$. By Corollaries 3.1.5 and 3.2.2, the subspace $\mathrm{Fix}(u)$ is spanned by the characteristic vectors of $\Omega(u)$ and $P_1 \backslash \Omega(u)$. By Lemma 3.1.7 and Lemma 6.1.1 in the appendix, the characteristic vector of $\Omega(u)$ forms a basis for $\mathrm{Fix}(u)$. The result follows. □

3.3 A Euclidean representation of Γ

In this section we recall the notion of a Euclidean representation of Γ. We then show that the Euclidean space E, together with the restriction of the map (3.2) to X is a Euclidean representation of Γ.

Definition 3.3.1 [7, Lecture 12] By a *Euclidean representation of* Γ, we mean a pair (F, ψ) such that F is a nonzero Euclidean space, and $\psi : X \to F$ is a map that satisfies the following (i)–(iii):

(i) F is spanned by $\{\psi(x) \mid x \in X\}$;

(ii) for all $x, y \in X$, the inner product $\langle \psi(x), \psi(y) \rangle$ depends only on $\partial(x, y)$;

(iii) there exists $\vartheta \in \mathbb{R}$ such that for all $x \in X$,

$$\sum_{z \in \Gamma(x)} \psi(z) = \vartheta \psi(x).$$

We have some comments about Definition 3.3.1. Let (F, ψ) denote a Euclidean representation of Γ. It is shown in [7, Lecture 13] that the scalar ϑ in Definition 3.3.1(iii) is an eigenvalue of Γ. We call ϑ the *associated eigenvalue*. For any eigenvalue θ of Γ, it is shown in [7, Lecture 13] how the corresponding eigenspace gives a Euclidean representation of Γ that is associated with θ.

Recall the Euclidean space E from Section 3.1. Our next goal is to show that the Euclidean space E, together with the restriction of the map (3.2) to X is a Euclidean representation of Γ.

Lemma 3.3.2 *For $u, v \in P$ we have*

$$\langle \widehat{u}, \widehat{v} \rangle = [n][h] - [i][j],$$

where $u \in P_i$, $v \in P_j$, and $u \cap v \in P_h$.

Proof. By (2.8),

$$|\Omega(u)| = [i], \qquad |\Omega(v)| = [j]. \tag{3.11}$$

In view of (3.3), write

$$\langle \widehat{u}, \widehat{v} \rangle = \left\langle \sum_{s \in \Omega(u)} \widehat{s}, \sum_{t \in \Omega(v)} \widehat{t} \right\rangle = \sum_{s \in \Omega(u) \cap \Omega(v)} \|\widehat{s}\|^2 + \sum_{\substack{s \in \Omega(u), t \in \Omega(v) \\ s \neq t}} \langle \widehat{s}, \widehat{t} \rangle. \tag{3.12}$$

By (2.7) and (2.8),

$$\left|\Omega(u) \cap \Omega(v)\right| = \left|\Omega(u \cap v)\right| = [h]. \tag{3.13}$$

By (3.11) and (3.13),

$$\left|\left\{(s,t) \in \Omega(u) \times \Omega(v) \;\middle|\; s \neq t\right\}\right| = \left|\Omega(u)\right|\left|\Omega(v)\right| - \left|\Omega(u \cap v)\right| = [i][j] - [h]. \tag{3.14}$$

The result follows from (3.12)–(3.14), along with (C2), (C3). □

Corollary 3.3.3 *For $u \in P$ we have*

$$\left\|\widehat{u}\right\|^2 = q^i[i][n-i],$$

where $u \in P_i$.

Proof. Set $u = v$ in Lemma 3.3.2. □

Lemma 3.3.4 *For $x, y \in X$ we have*

$$\left\langle \widehat{x}, \widehat{y} \right\rangle = [n][k-i] - [k]^2,$$

where $i = \partial(x, y)$.

Proof. Immediate from Lemmas 2.3.4, 3.3.2. □

Corollary 3.3.5 *For $x \in X$,*

$$\left\|\widehat{x}\right\|^2 = q^k[k][n-k].$$

Proof. Set $i = 0$ in Lemma 3.3.4. □

Lemma 3.3.6 *For $x \in X$,*

$$\sum_{z \in \Gamma(x)} \widehat{z} = \theta_1 \widehat{x},$$

where θ_1 is from (2.5).

Proof. Note that the set $\Gamma(x)$ is invariant under $\mathrm{Stab}(x)$. Hence, the vector

$$\sum_{z \in \Gamma(x)} \widehat{z} \tag{3.15}$$

is fixed by every element of $\mathrm{Stab}(x)$. Therefore, the vector (3.15) is contained in $\mathrm{Fix}(x)$. By Lemma 3.2.3, the subspace $\mathrm{Fix}(x)$ is spanned by $\widehat{x}$. Hence there exists $\alpha \in \mathbb{R}$ such that

$$\sum_{z \in \Gamma(x)} \widehat{z} = \alpha \widehat{x}. \tag{3.16}$$

It remains to show that $\alpha = \theta_1$.

Referring to (3.16), we take the inner product of each side with $\widehat{x}$ to obtain

$$\sum_{z\in\Gamma(x)} \langle \widehat{x}, \widehat{z} \rangle = \alpha \|\widehat{x}\|^2. \tag{3.17}$$

By (2.1), we have $\kappa = |\Gamma(x)|$; refer to (2.3) for the value of κ. In (3.17), we evaluate $\langle \widehat{x}, \widehat{z} \rangle$ using Lemma 3.3.4 and $\|\widehat{x}\|^2$ using Corollary 3.3.5 and solve the resulting equation for α. Evaluate the result and simplify using (2.5) to obtain $\alpha = \theta_1$. □

Lemma 3.3.7 *The vector space E is spanned by $\{\widehat{x} \mid x \in X\}$.*

Proof. In view of (C1), it suffices to show that $\widehat{s} \in \mathrm{Span}\{\widehat{x} \mid x \in X\}$ for every $s \in P_1$. Pick $s \in P_1$. There exists $y \in X$ that contains s. By Lemma 3.1.2, the vector

$$\sum_{\sigma\in \mathrm{Stab}(s)} \sigma(\widehat{y}) \tag{3.18}$$

is contained in $\mathrm{Span}\{\widehat{x} \mid x \in X\}$. Next we show that (3.18) is a nonzero scalar multiple of $\widehat{s}$. The vector (3.18) is fixed by every element of $\mathrm{Stab}(s)$. Hence, the vector (3.18) is contained in $\mathrm{Fix}(s)$. By Lemma 3.2.3, there exists $\beta \in \mathbb{R}$ such that (3.18) is equal to $\beta\widehat{s}$. We will show that the scalar β is nonzero. Take the inner product between (3.18) and $\widehat{s}$. Using Lemmas 3.1.2, 3.3.2 and the fact that $s \cap \sigma(y) = s$ for all $\sigma \in \mathrm{Stab}(s)$, we get

$$\beta\|\widehat{s}\|^2 = \sum_{\sigma\in \mathrm{Stab}(s)} \Big\langle \sigma(\widehat{y}), \widehat{s} \Big\rangle = \sum_{\sigma\in \mathrm{Stab}(s)} \Big([n] - [k]\Big) = |\mathrm{Stab}(s)|\Big([n] - [k]\Big) \neq 0.$$

Hence $\beta \neq 0$. We have shown that the vector (3.18) is a nonzero scalar multiple of $\widehat{s}$. We mentioned earlier that (3.18) is in $\mathrm{Span}\{\widehat{x} \mid x \in X\}$. By these comments, $\widehat{s} \in \mathrm{Span}\{\widehat{x} \mid x \in X\}$ and the result follows. □

By Lemmas 3.3.4, 3.3.6, 3.3.7 the Euclidean space E, together with the restriction of the map (3.2) to X gives a Euclidean representation of Γ. This representation is associated with the eigenvalue θ_1.

We finish this section with a remark. By [2, Theorem 9.3.3], $GL(\mathcal{V})$ acts on Γ in a distance-transitive fashion. Applying [2, Prop. 4.1.11], we get that the Euclidean space E is irreducible as a $GL(\mathcal{V})$-module.

3.4 The stabilizer of two elements in P

In this section we pick two distinct elements in P and consider their stabilizer in $GL(\mathcal{V})$. We describe the orbits of this stabilizer acting on P_1.

Pick distinct $u, v \in P$ such that $0 \neq u \neq \mathcal{V}$ and $0 \neq v \neq \mathcal{V}$. Let $\mathrm{Stab}(u, v)$ denote the subgroup of $GL(\mathcal{V})$ consisting of the elements that fix both u and v. We call $\mathrm{Stab}(u, v)$ the *stabilizer of u and v in $GL(\mathcal{V})$*.

We will describe the action of $\mathrm{Stab}(u, v)$ on P_1. There are six cases to consider:

Case	description
1	$u \cap v \neq 0,\ u \nsubseteq v,\ v \nsubseteq u,\ u + v \neq \mathcal{V}$
2	$u \cap v \neq 0,\ u \nsubseteq v,\ v \nsubseteq u,\ u + v = \mathcal{V}$
3	$u \cap v = 0,\ u \nsubseteq v,\ v \nsubseteq u,\ u + v \neq \mathcal{V}$
4	$u \cap v = 0,\ u \nsubseteq v,\ v \nsubseteq u,\ u + v = \mathcal{V}$
5	$u \subseteq v$
6	$v \subseteq u.$

Lemma 3.4.1 *In the table below, we give the orbits for the action of* $\mathrm{Stab}(u, v)$ *on* P_1*.*

Case	orbits of $\mathrm{Stab}(u, v)$ on P_1
1	$\Omega(u \cap v)$, $\Omega(u) \backslash \Omega(u \cap v)$, $\Omega(v) \backslash \Omega(u \cap v)$, $\Omega(u+v) \backslash \big(\Omega(u) \cup \Omega(v)\big)$, $P_1 \backslash \Omega(u+v)$
2	$\Omega(u \cap v)$, $\Omega(u) \backslash \Omega(u \cap v)$, $\Omega(v) \backslash \Omega(u \cap v)$, $P_1 \backslash \big(\Omega(u) \cup \Omega(v)\big)$
3	$\Omega(u)$, $\Omega(v)$, $\Omega(u+v) \backslash \big(\Omega(u) \cup \Omega(v)\big)$, $P_1 \backslash \Omega(u+v)$
4	$\Omega(u)$, $\Omega(v)$, $P_1 \backslash \big(\Omega(u) \cup \Omega(v)\big)$
5	$\Omega(u)$, $\Omega(v) \backslash \Omega(u)$, $P_1 \backslash \Omega(v)$
6	$\Omega(v)$, $\Omega(u) \backslash \Omega(v)$, $P_1 \backslash \Omega(u)$

Proof. First assume Case 1. Observe that the five sets displayed in the table are fixed by the elements of $\mathrm{Stab}(u, v)$. Hence each set is a disjoint union of orbits of $\mathrm{Stab}(u, v)$.

We now show that each set forms a single orbit of $\mathrm{Stab}(u, v)$. We start with $\Omega(u \cap v)$.

Pick distinct $s, s' \in \Omega(u \cap v)$. We show that there exists $\sigma \in \mathrm{Stab}(u, v)$ that sends $s \mapsto s'$.

Let η denote a nonzero vector in s. Note that the set $\{\eta\}$ is a basis for s; extend this basis to a basis $\{\eta\} \cup \mathcal{R}$ for $u \cap v$. Extend the basis $\{\eta\} \cup \mathcal{R}$ for $u \cap v$ to a basis $\{\eta\} \cup \mathcal{R} \cup \mathcal{T}$ for u. Extend the basis $\{\eta\} \cup \mathcal{R}$ for $u \cap v$ to a basis $\{\eta\} \cup \mathcal{R} \cup \mathcal{P}$ for v. By Lemma 2.3.3, $\{\eta\} \cup \mathcal{R} \cup \mathcal{T} \cup \mathcal{P}$ is a basis for $u + v$. Extend the basis $\{\eta\} \cup \mathcal{R} \cup \mathcal{T} \cup \mathcal{P}$ for $u + v$ to a basis $\{\eta\} \cup \mathcal{R} \cup \mathcal{T} \cup \mathcal{P} \cup \mathcal{Q}$ for $\mathcal{V}$.

Let η' denote a nonzero vector in s'. Note that the set $\{\eta'\}$ is a basis for s'; extend this basis to a basis $\{\eta'\} \cup \mathcal{R}'$ for $u \cap v$. Note that $\{\eta'\} \cup \mathcal{R}' \cup \mathcal{T}$ is a basis for u. Note that $\{\eta'\} \cup \mathcal{R}' \cup \mathcal{P}$ is a basis for v. By Lemma 2.3.3, $\{\eta'\} \cup \mathcal{R}' \cup \mathcal{T} \cup \mathcal{P}$ is a basis for $u + v$. Note that $\{\eta'\} \cup \mathcal{R}' \cup \mathcal{T} \cup \mathcal{P} \cup \mathcal{Q}$ is a basis for $\mathcal{V}$.

By linear algebra, there exists $\sigma \in GL(\mathcal{V})$ that sends $\eta \mapsto \eta'$, $\mathcal{R} \mapsto \mathcal{R}'$ and acts as the identity on each of $\mathcal{T}$, $\mathcal{P}$, $\mathcal{Q}$. By construction, σ is contained in $\mathrm{Stab}(u, v)$ and sends $s \mapsto s'$.

Next we show that the set $\Omega(u) \backslash \Omega(u \cap v)$ is a single orbit of $\mathrm{Stab}(u, v)$.

Pick distinct $s, s' \in \Omega(u) \backslash \Omega(u \cap v)$. We show that there exists $\sigma \in \mathrm{Stab}(u, v)$ that sends $s \mapsto s'$.

Let $\mathcal{R} \subseteq \mathcal{V}$ form a basis for $u \cap v$. Let η denote a nonzero vector in s. Note that $\{\eta\}$ is a basis for s. The subspace s is not contained in $u \cap v$, so $\{\eta\} \cup \mathcal{R}$ is a basis for $s + (u \cap v)$. Extend the basis $\{\eta\} \cup \mathcal{R}$ for $s + (u \cap v)$ to a basis $\{\eta\} \cup \mathcal{R} \cup \mathcal{T}$ for u. Extend the basis $\mathcal{R}$ for $u \cap v$ to a basis $\mathcal{R} \cup \mathcal{P}$ for v. By Lemma 2.3.3, $\{\eta\} \cup \mathcal{R} \cup \mathcal{T} \cup \mathcal{P}$ is a basis for $u + v$. Extend the basis $\{\eta\} \cup \mathcal{R} \cup \mathcal{T} \cup \mathcal{P}$ for $u + v$ to a basis $\{\eta\} \cup \mathcal{R} \cup \mathcal{T} \cup \mathcal{P} \cup \mathcal{Q}$ for $\mathcal{V}$.

Let η' denote a nonzero vector in s'. Note that $\{\eta'\}$ is a basis for s'. The subspace s' is not contained in $u \cap v$, so $\{\eta'\} \cup \mathcal{R}$ is a basis for $s' + (u \cap v)$. Extend the basis $\{\eta'\} \cup \mathcal{R}$ for $s' + (u \cap v)$ to a basis $\{\eta'\} \cup \mathcal{R} \cup \mathcal{T}'$ for u. By Lemma 2.3.3, $\{\eta'\} \cup \mathcal{R} \cup \mathcal{T}' \cup \mathcal{P}$ is a basis for $u + v$. Note that $\{\eta'\} \cup \mathcal{R} \cup \mathcal{T}' \cup \mathcal{P} \cup \mathcal{Q}$ is a basis for $\mathcal{V}$.

By linear algebra, there exists $\sigma \in GL(\mathcal{V})$ that sends $\eta \mapsto \eta'$, $\mathcal{T} \mapsto \mathcal{T}'$ and acts as the identity on each of $\mathcal{R}$, $\mathcal{P}$, $\mathcal{Q}$. By construction, σ is contained in $\mathrm{Stab}(u, v)$ and sends $s \mapsto s'$.

By symmetry, the set $\Omega(v) \backslash \Omega(u \cap v)$ is a single orbit of $\mathrm{Stab}(u, v)$.

Next we show that the set $\Omega(u + v) \backslash \big(\Omega(u) \cup \Omega(v)\big)$ is a single orbit of $\mathrm{Stab}(u, v)$.

Pick distinct $s, s' \in \Omega(u+v) \backslash \big(\Omega(u) \cup \Omega(v)\big)$. We show that there exists $\sigma \in \mathrm{Stab}(u, v)$ that sends $s \mapsto s'$.

Let η denote a nonzero vector in s and let η' denote a nonzero vector in s'. By linear algebra, there exist sets $\mathcal{R}, \mathcal{T}, \mathcal{P}$ such that $\mathcal{R}$ is a basis for $u \cap v$, $\mathcal{R} \cup \mathcal{T}$ is a basis for u, $\mathcal{R} \cup \mathcal{P}$ is a basis for v, and $\eta = r + t + p$ for $r \in \mathcal{R}$, $t \in \mathcal{T}$, $p \in \mathcal{P}$.

Similarly, there exist sets $\mathcal{R}', \mathcal{T}', \mathcal{P}'$ such that $\mathcal{R}'$ is a basis for $u \cap v$, $\mathcal{R}' \cup \mathcal{T}'$ is a basis for u, $\mathcal{R}' \cup \mathcal{P}'$ is a basis for v, and $\eta' = r' + t' + p'$ for $r' \in \mathcal{R}'$, $t' \in \mathcal{T}'$, $p' \in \mathcal{P}'$.

By linear algebra, there exists $\sigma \in GL(\mathcal{V})$ that sends $\mathcal{R} \mapsto \mathcal{R}'$, $\mathcal{T} \mapsto \mathcal{T}'$, $\mathcal{P} \mapsto \mathcal{P}'$ such that $r \mapsto r'$, $t \mapsto t'$, $p \mapsto p'$. By construction, σ is contained in $\mathrm{Stab}(u, v)$ and sends $s \mapsto s'$.

Next we show that the set $P_1 \backslash \Omega(u + v)$ is a single orbit of $\mathrm{Stab}(u, v)$.

Pick distinct $s, s' \in P_1 \backslash \Omega(u + v)$. We show that there exists $\sigma \in \mathrm{Stab}(u, v)$ that sends $s \mapsto s'$.

Let $\mathcal{P} \subseteq \mathcal{V}$ form a basis for $u + v$. Let η denote a nonzero vector in s. Note that $\{\eta\}$ is a basis for s. The subspace s is not contained in $u + v$, so $\{\eta\} \cup \mathcal{P}$ is a basis for $s + u + v$. Extend the basis $\{\eta\} \cup \mathcal{P}$ for $s + u + v$ to a basis $\{\eta\} \cup \mathcal{P} \cup \mathcal{Q}$ for $\mathcal{V}$.

Let η' denote a nonzero vector in s'. Note that $\{\eta'\}$ is a basis for s'. The subspace s' is not contained in $u + v$, so $\{\eta'\} \cup \mathcal{P}$ is a basis for $s' + u + v$. Extend the basis $\{\eta'\} \cup \mathcal{P}$ for $s' + u + v$ to a basis $\{\eta'\} \cup \mathcal{P} \cup \mathcal{Q}'$ for $\mathcal{V}$.

By linear algebra, there exists $\sigma \in GL(\mathcal{V})$ that sends $\eta \mapsto \eta'$, $\mathcal{Q} \mapsto \mathcal{Q}'$ and acts as the identity on $\mathcal{P}$. By construction, σ is contained in $\mathrm{Stab}(u, v)$ and sends $s \mapsto s'$.

We have proved the result for Case 1. For the remaining cases the proof is similar, and omitted. □

3.5 The stabilizer of two elements in P; action on E

In this section we pick two distinct elements in P and consider their stabilizer in $GL(\mathcal{V})$. We consider the action of this stabilizer on the Euclidean space E.

Pick distinct $u, v \in P$ such that $0 \neq u \neq \mathcal{V}$ and $0 \neq v \neq \mathcal{V}$. Let $\mathrm{Fix}(u, v)$ denote the subspace of E consisting of the vectors that are fixed by every element of $\mathrm{Stab}(u, v)$. We give a basis for $\mathrm{Fix}(u, v)$. In what follows, we refer to the cases in Lemma 3.4.1.

Lemma 3.5.1 *In the table below, we display a basis for* $\mathrm{Fix}(u, v)$.

Case	basis for $\mathrm{Fix}(u,v)$
1	$\widehat{u \cap v}, \quad \hat{u}, \quad \hat{v}, \quad \widehat{u+v}$
2	$\widehat{u \cap v}, \quad \hat{u}, \quad \hat{v}$
3	$\hat{u}, \quad \hat{v}, \quad \widehat{u+v}$
4	$\hat{u}, \quad \hat{v}$
5	$\hat{u}, \quad \hat{v}$
6	$\hat{u}, \quad \hat{v}$

Proof. We first assume Case 1. By Lemma 3.1.8, the subspace $\mathrm{Fix}(u, v)$ is spanned by the characteristic vectors of the $\mathrm{Stab}(u, v)$-orbits on P_1. These orbits are given in Lemma 3.4.1. Their characteristic vectors are

$$\widehat{u \cap v}, \qquad \hat{u} - \widehat{u \cap v}, \qquad \hat{v} - \widehat{u \cap v}, \qquad \widehat{u+v} - \hat{u} - \hat{v} + \widehat{u \cap v}, \qquad -\widehat{u+v}. \tag{3.19}$$

The five vectors in (3.19) sum to 0. By Lemma 3.1.1, any four of these vectors are linearly independent, and hence form a basis for $\mathrm{Fix}(u, v)$. In particular, the following vectors form a basis for $\mathrm{Fix}(u, v)$:

$$\widehat{u \cap v}, \qquad \hat{u} - \widehat{u \cap v}, \qquad \hat{v} - \widehat{u \cap v}, \qquad -\widehat{u+v}.$$

Adjusting the basis above, the result follows.

We have proved the result for Case 1. For the remaining cases the proof is similar, and omitted. $\square$

Definition 3.5.2 Pick distinct $u, v \in P$. By the *geometric basis for* $\mathrm{Fix}(u, v)$, we mean the basis displayed in Lemma 3.5.1.

In the next result, we consider how Lemma 3.5.1 looks for the case in which $u, v \in X$.

Theorem 3.5.3 *Pick distinct $x, y \in X$. In the table below, we display the geometric basis for* $\mathrm{Fix}(x, y)$.

Case	geometric basis for $\mathrm{Fix}(x,y)$
$1 \leqslant \partial(x,y) < k$	$\widehat{x}, \quad \widehat{y}, \quad \widehat{x \cap y}, \quad \widehat{x+y}$
$\partial(x,y) = k$	$\widehat{x}, \quad \widehat{y}, \quad \widehat{x+y}$

Proof. Since x and y are distinct, $x \nsubseteq y$ and $y \nsubseteq x$. Since $n > 2k$, we have $x + y \neq \mathcal{V}$.

We first assume that $1 \leqslant \partial(x,y) < k$. Since $\partial(x,y) < k$, we have $x \cap y \neq 0$. The result follows from Case 1 of Lemma 3.5.1.

Next we assume that $\partial(x,y) = k$. By this assumption, $x \cap y = 0$. The result follows from Case 3 of Lemma 3.5.1. □

Pick distinct $x, y \in X$. Our next general goal is to use the graph Γ to find another basis for $\mathrm{Fix}(x,y)$, called the combinatorial basis. We will focus on the case $1 < \partial(x,y) < k$. The cases $\partial(x,y) = 1$ and $\partial(x,y) = k$ are more involved and require different methods; these cases will be handled in a future paper.

3.6 The subspace $\mathrm{Fix}(x,y)$ and its combinatorial basis

Pick $x, y \in X$ such that $1 < \partial(x,y) < k$. In this section we use the graph Γ to construct two vectors B_{xy} and C_{xy}. In Section 3.8, we will show that the following vectors form a basis for $\mathrm{Fix}(x,y)$:

$$\widehat{x}, \qquad \widehat{y}, \qquad B_{xy}, \qquad C_{xy}. \tag{3.20}$$

The basis (3.20) will be called the combinatorial basis. We will formally define this basis in Section 3.8.

To define the vectors B_{xy} and C_{xy}, we first consider two orbits of the $\mathrm{Stab}(x,y)$-action on P.

Definition 3.6.1 For $x, y \in X$ such that $1 < \partial(x,y) < k$, define

$$\mathcal{B}_{xy} = \{z \in \Gamma(x) \mid \partial(y,z) = \partial(x,y) + 1\},$$
$$\mathcal{C}_{xy} = \{z \in \Gamma(x) \mid \partial(y,z) = \partial(x,y) - 1\}.$$

Observe that $|\mathcal{B}_{xy}| = b_i$ and $|\mathcal{C}_{xy}| = c_i$, where $i = \partial(x,y)$.

Our next goal is to show that the sets $\mathcal{B}_{xy}$ and $\mathcal{C}_{xy}$ are orbits of $\mathrm{Stab}(x,y)$.

Lemma 3.6.2 *For $x, y \in X$ such that $1 < \partial(x,y) < k$, the set $\mathcal{C}_{xy}$ is an orbit of the* $\mathrm{Stab}(x,y)$*-action on P.*

Proof. By Lemma 2.3.12, the set $\mathcal{C}_{xy}$ is a disjoint union of orbits of $\mathrm{Stab}(x,y)$. We now show that this union consists of a single orbit. Let $z, z' \in \mathcal{C}_{xy}$. We display an element $\sigma \in \mathrm{Stab}(x,y)$ that sends $z \mapsto z'$.

By Lemma 2.3.6,

$$x \cap y \subseteq z \subseteq x + y, \qquad\qquad x \cap y \subseteq z' \subseteq x + y.$$

Let the set $\mathcal{R} \subseteq \mathcal{V}$ form a basis for $x \cap y$. Extend the basis $\mathcal{R}$ for $x \cap y$ to a basis $\mathcal{R} \cup \mathcal{S}$ for $x \cap z$. Extend the basis $\mathcal{R}$ for $x \cap y$ to a basis $\mathcal{R} \cup \mathcal{T}$ for $z \cap y$. By Lemma 2.3.3, $\mathcal{R} \cup \mathcal{S} \cup \mathcal{T}$ is a basis for z. Extend the basis $\mathcal{R} \cup \mathcal{S}$ for $x \cap z$ to a basis $\mathcal{R} \cup \mathcal{S} \cup \mathcal{P}$ for x. Extend the basis $\mathcal{R} \cup \mathcal{T}$ for $z \cap y$ to a basis $\mathcal{R} \cup \mathcal{T} \cup \mathcal{Q}$ for y. By Lemma 2.3.3, $\mathcal{R} \cup \mathcal{S} \cup \mathcal{T} \cup \mathcal{P} \cup \mathcal{Q}$ is a basis for $x + y$. Extend the basis $\mathcal{R} \cup \mathcal{S} \cup \mathcal{T} \cup \mathcal{P} \cup \mathcal{Q}$ for $x + y$ to a basis $\mathcal{R} \cup \mathcal{S} \cup \mathcal{T} \cup \mathcal{P} \cup \mathcal{Q} \cup \mathcal{W}$ for $\mathcal{V}$.

Extend the basis $\mathcal{R}$ for $x \cap y$ to a basis $\mathcal{R} \cup \mathcal{S}'$ for $x \cap z'$. Extend the basis $\mathcal{R}$ for $x \cap y$ to a basis $\mathcal{R} \cup \mathcal{T}'$ for $z' \cap y$. By Lemma 2.3.3, $\mathcal{R} \cup \mathcal{S}' \cup \mathcal{T}'$ is a basis for z'. Extend the basis $\mathcal{R} \cup \mathcal{S}'$ for $x \cap z'$ to a basis $\mathcal{R} \cup \mathcal{S}' \cup \mathcal{P}'$ for x. Extend the basis $R \cup \mathcal{T}'$ for $z' \cap y$ to a basis $\mathcal{R} \cup \mathcal{T}' \cup \mathcal{Q}'$ for y. By Lemma 2.3.3, $\mathcal{R} \cup \mathcal{S}' \cup \mathcal{T}' \cup \mathcal{P}' \cup \mathcal{Q}'$ is a basis for $x + y$. Note that $\mathcal{R} \cup \mathcal{S}' \cup \mathcal{T}' \cup \mathcal{P}' \cup \mathcal{Q}' \cup \mathcal{W}$ is a basis for $\mathcal{V}$.

By linear algebra, there exists $\sigma \in GL(\mathcal{V})$ that sends $\mathcal{S} \mapsto \mathcal{S}'$, $\mathcal{T} \mapsto \mathcal{T}'$, $\mathcal{P} \mapsto \mathcal{P}'$, $\mathcal{Q} \mapsto \mathcal{Q}'$ and acts as the identity on each of $\mathcal{R}$, $\mathcal{W}$. By construction, σ is contained in $\mathrm{Stab}(x, y)$ and sends $z \mapsto z'$. The result follows. □

Lemma 3.6.3 *Let $x, y, x', y' \in X$ satisfy $1 < \partial(x, y) = \partial(x', y') < k$. Let $z \in \mathcal{C}_{xy}$ and $z' \in \mathcal{C}_{x'y'}$. Then there exists an element in $GL(\mathcal{V})$ that sends $x \mapsto x'$, $y \mapsto y'$, $z \mapsto z'$.*

Proof. Since Γ is distance-transitive, we may assume without loss that $x = x'$ and $y = y'$. The result follows from Lemma 3.6.2. □

Lemma 3.6.4 *For $x, y \in X$ such that $1 < \partial(x, y) < k$, the set $\mathcal{B}_{xy}$ is an orbit of the* $\mathrm{Stab}(x, y)$*-action on P.*

Proof. By Lemma 2.3.12, the set $\mathcal{B}_{xy}$ is a disjoint union of orbits of $\mathrm{Stab}(x, y)$. We now show that this union consists of a single orbit. Let $z, z' \in \mathcal{B}_{xy}$. By construction, $x \in \mathcal{C}_{zy}$ and $x \in \mathcal{C}_{z'y}$. By Lemma 3.6.3, there exists an element in $\mathrm{Stab}(x, y)$ that sends $z \mapsto z'$. The result follows. □

Definition 3.6.5 For $x, y \in X$ such that $1 < \partial(x, y) < k$, define the vectors

$$B_{xy} = \sum_{z \in \mathcal{B}_{xy}} \hat{z}, \qquad\qquad C_{xy} = \sum_{z \in \mathcal{C}_{xy}} \hat{z}.$$

Lemma 3.6.6 *For $x, y \in X$ such that $1 < \partial(x, y) < k$, the subspace* $\mathrm{Fix}(x, y)$ *contains B_{xy} and C_{xy}.*

Proof. Pick $\sigma \in \mathrm{Stab}(x, y)$. The map σ fixes the sets $\mathcal{B}_{xy}$ and $\mathcal{C}_{xy}$. The result follows. □

3.7 The inner products involving the geometric basis and the combinatorial basis

Pick $x, y \in X$ such that $1 < \partial(x, y) < k$. In this section we compute the inner products between:

(i) any two vectors in the geometric basis for $\mathrm{Fix}(x, y)$;

(ii) any vector in the geometric basis for $\mathrm{Fix}(x, y)$ and any vector in the set (3.20);

(iii) any two vectors in the set (3.20).

We start with case (i).

Lemma 3.7.1 *For $x, y \in X$ such that $1 < \partial(x, y) < k$, we have*

$$\left\langle \widehat{x}, \widehat{x \cap y} \right\rangle = q^k [k-i][n-k],$$
$$\left\langle \widehat{y}, \widehat{x \cap y} \right\rangle = q^k [k-i][n-k],$$

where $i = \partial(x, y)$.

Proof. Routine using Lemmas 2.3.4, 3.3.2 and linear algebra. □

Lemma 3.7.2 *For $x, y \in X$ such that $1 < \partial(x, y) < k$, we have*

$$\left\langle \widehat{x}, \widehat{x + y} \right\rangle = q^{k+i} [k][n-k-i],$$
$$\left\langle \widehat{y}, \widehat{x + y} \right\rangle = q^{k+i} [k][n-k-i],$$

where $i = \partial(x, y)$.

Proof. Routine using Lemmas 2.3.5, 3.3.2 and linear algebra. □

Lemma 3.7.3 *For $x, y \in X$ such that $1 < \partial(x, y) < k$, we have*

$$\left\| \widehat{x \cap y} \right\|^2 = q^{k-i} [k-i][n-k+i],$$
$$\left\langle \widehat{x \cap y}, \widehat{x + y} \right\rangle = q^{k+i} [k-i][n-k-i],$$
$$\left\| \widehat{x + y} \right\|^2 = q^{k+i} [k+i][n-k-i],$$

where $i = \partial(x, y)$.

Proof. Routine using Lemmas 2.3.4, 2.3.5, 3.3.2 and linear algebra. □

Theorem 3.7.4 *Let $x, y \in X$ satisfy $1 < \partial(x, y) < k$. In the following table, for each vector u in the header column, and each vector v in the header row, the (u, v)-entry of the table gives the inner product $\langle u, v \rangle$. Write $i = \partial(x, y)$.*

$\langle\ ,\ \rangle$	$\widehat{x}$	$\widehat{y}$	$\widehat{x \cap y}$	$\widehat{x + y}$
$\widehat{x}$	$q^k[k][n-k]$	$[n][k-i]-[k]^2$	$q^k[k-i][n-k]$	$q^{k+i}[k][n-k-i]$
$\widehat{y}$	$[n][k-i]-[k]^2$	$q^k[k][n-k]$	$q^k[k-i][n-k]$	$q^{k+i}[k][n-k-i]$
$\widehat{x \cap y}$	$q^k[k-i][n-k]$	$q^k[k-i][n-k]$	$q^{k-i}[k-i][n-k+i]$	$q^{k+i}[k-i][n-k-i]$
$\widehat{x + y}$	$q^{k+i}[k][n-k-i]$	$q^{k+i}[k][n-k-i]$	$q^{k+i}[k-i][n-k-i]$	$q^{k+i}[k+i][n-k-i]$

Proof. Combine Corollary 3.3.5 and Lemmas 3.3.4, 3.7.1–3.7.3. □

Next we compute the inner products for case (ii).

Lemma 3.7.5 *For $x, y \in X$ such that $1 < \partial(x, y) < k$, we have*

$$\left\langle B_{xy}, \widehat{x} \right\rangle = q^{2i+1}[k-i][n-k-i]\Big([n][k-1] - [k]^2\Big), \tag{3.21}$$

$$\left\langle C_{xy}, \widehat{x} \right\rangle = [i]^2\Big([n][k-1] - [k]^2\Big), \tag{3.22}$$

where $i = \partial(x, y)$.

Proof. We first prove (3.21). In the left equation of Definition 3.6.5, take the inner product of each side with $\widehat{x}$. Evaluate the result using Lemma 3.3.4 and (2.4) to get (3.21).

We have now verified (3.21). (3.22) is obtained in a similar fashion. □

Lemma 3.7.6 *For $x, y \in X$ such that $1 < \partial(x, y) < k$, we have*

$$\left\langle B_{xy}, \widehat{y} \right\rangle = q^{2i+1}[k-i][n-k-i]\Big([n][k-i-1] - [k]^2\Big), \tag{3.23}$$

$$\left\langle C_{xy}, \widehat{y} \right\rangle = [i]^2\Big([n][k-i+1] - [k]^2\Big), \tag{3.24}$$

where $i = \partial(x, y)$.

Proof. We first prove (3.23). In the left equation of Definition 3.6.5, take the inner product of each side with $\widehat{y}$. Evaluate the result using Lemma 3.3.4 and (2.4) to get (3.23).

We have now verified (3.23). (3.24) is obtained in a similar fashion. □

Lemma 3.7.7 *For $x, y \in X$ such that $1 < \partial(x, y) < k$, we have*

$$\left\langle B_{xy}, \widehat{x \cap y} \right\rangle = q^{2i+1}[k-i][n-k-i]\Big([n][k-i-1] - [k-i][k]\Big), \tag{3.25}$$

$$\left\langle B_{xy}, \widehat{x + y} \right\rangle = q^{2i+1}[k-i][n-k-i]\Big([n][k-1] - [k][k+i]\Big), \tag{3.26}$$

where $i = \partial(x, y)$.

Proof. We first prove (3.25). Using the left equation of Definition 3.6.5, we obtain

$$\left\langle B_{xy}, \widehat{x \cap y} \right\rangle = \sum_{z \in \mathcal{B}_{xy}} \left\langle \hat{z}, \widehat{x \cap y} \right\rangle.$$

Let $z \in \mathcal{B}_{xy}$. By the definition of $\mathcal{B}_{xy}$, we have $\partial(z, y) = i+1$. By Lemma 2.3.4, $\dim(z \cap y) = k - i - 1$. Also, $\partial(z, y) = \partial(z, x) + \partial(x, y)$, so by Lemma 2.3.6, $x \supseteq z \cap y$. Hence

$$\dim(z \cap x \cap y) = \dim(z \cap y) = k - i - 1.$$

Using Lemma 3.3.2 we obtain

$$\left\langle \hat{z}, \widehat{x \cap y} \right\rangle = [n][k - i - 1] - [k - i][k].$$

Use the value of b_i in (2.4) to obtain (3.25).

Next we prove (3.26). Using the left equation of Definition 3.6.5, we obtain

$$\left\langle B_{xy}, \widehat{x + y} \right\rangle = \sum_{z \in \mathcal{B}_{xy}} \left\langle \hat{z}, \widehat{x + y} \right\rangle.$$

Let $z \in \mathcal{B}_{xy}$. Recall that $\partial(z, y) = i + 1$. By Lemma 2.3.5, $\dim(z + y) = k + i + 1$. Also recall that $\partial(z, y) = \partial(z, x) + \partial(x, y)$, so by Lemma 2.3.6, $x \subseteq z + y$. Hence

$$\dim(z + x + y) = \dim(z + y) = k + i + 1.$$

By Lemma 2.3.2,

$$\dim\big(z \cap (x + y)\big) = \dim z + \dim(x + y) - \dim(z + x + y) = k - 1.$$

Using Lemma 3.3.2 we obtain

$$\left\langle \hat{z}, \widehat{x + y} \right\rangle = [n][k - 1] - [k][k + i].$$

Use the value of b_i in (2.4) to obtain (3.26). $\square$

Lemma 3.7.8 *For $x, y \in X$ such that $1 < \partial(x, y) < k$, we have*

$$\left\langle C_{xy}, \widehat{x \cap y} \right\rangle = q^k [i]^2 [k - i][n - k], \tag{3.27}$$

$$\left\langle C_{xy}, \widehat{x + y} \right\rangle = q^{k+i} [i]^2 [k][n - k - i], \tag{3.28}$$

where $i = \partial(x, y)$.

Proof. We first prove (3.27). Using the right equation of Definition 3.6.5, we obtain

$$\left\langle C_{xy}, \widehat{x \cap y} \right\rangle = \sum_{z \in \mathcal{C}_{xy}} \left\langle \hat{z}, \widehat{x \cap y} \right\rangle.$$

Let $z \in \mathcal{C}_{xy}$. By the definition of $\mathcal{C}_{xy}$, we have $\partial(z, y) = i-1$. Since $\partial(x, y) = \partial(x, z)+\partial(z, y)$, we have $z \supseteq x \cap y$ by Lemma 2.3.6. Hence

$$\dim(z \cap x \cap y) = \dim(x \cap y) = k - i.$$

Using Lemma 3.3.2 we obtain

$$\left\langle \hat{z}, \widehat{x \cap y} \right\rangle = [n][k-i] - [k][k-i] = q^k [k-i][n-k].$$

Use the value of c_i in (2.4) to obtain (3.27).

Next we prove (3.28). Using the right equation of Definition 3.6.5, we obtain

$$\left\langle C_{xy}, \widehat{x + y} \right\rangle = \sum_{z \in \mathcal{C}_{xy}} \left\langle \hat{z}, \widehat{x + y} \right\rangle.$$

Let $z \in \mathcal{C}_{xy}$. Recall that $\partial(x, y) = \partial(x, z) + \partial(z, y)$, so by Lemma 2.3.6, $z \subseteq x + y$. Hence

$$z \cap (x + y) = z.$$

Therefore,

$$\dim\big(z \cap (x + y)\big) = \dim z = k.$$

Using Lemma 3.3.2 we obtain

$$\left\langle \hat{z}, \widehat{x + y} \right\rangle = [n][k] - [k][k+i] = q^{k+i}[k][n-k-i].$$

Use the value of c_i in (2.4) to obtain (3.28). □

Theorem 3.7.9 *Let $x, y \in X$ satisfy $1 < \partial(x, y) < k$. In the following table, for each vector u in the header column, and each vector v in the header row, the (u, v)-entry of the table gives the inner product $\langle u, v \rangle$. Write $i = \partial(x, y)$.*

$\langle\ ,\ \rangle$	$\hat{x}$	$\hat{y}$	$\widehat{x \cap y}$	$\widehat{x + y}$
$\hat{x}$	$q^k[k][n-k]$	$[n][k-i]-[k]^2$	$q^k[k-i][n-k]$	$q^{k+i}[k][n-k-i]$
$\hat{y}$	$[n][k-i]-[k]^2$	$q^k[k][n-k]$	$q^k[k-i][n-k]$	$q^{k+i}[k][n-k-i]$
B_{xy}	$q^{2i+1}[k-i][n-k-i]\cdot\big([n][k-1]-[k]^2\big)$	$q^{2i+1}[k-i][n-k-i]\cdot\big([n][k-i-1]-[k]^2\big)$	$q^{2i+1}[k-i][n-k-i]\cdot([n][k-i-1]-[k-i][k])$	$q^{2i+1}[k-i][n-k-i]\cdot([n][k-1]-[k][k+i])$
C_{xy}	$[i]^2\big([n][k-1]-[k]^2\big)$	$[i]^2\big([n][k-i+1]-[k]^2\big)$	$q^k[i]^2[k-i][n-k]$	$q^{k+i}[i]^2[k][n-k-i]$

Proof. Combine Corollary 3.3.5 and Lemmas 3.3.4, 3.7.1, 3.7.2, 3.7.5–3.7.8. □

Next we compute the inner products for case (iii). To do this, we first write the vectors B_{xy} and C_{xy} as linear combinations in the geometric basis for $\mathrm{Fix}(x, y)$.

For $1 < i < k$ let M_i denote the matrix of inner products in Theorem 3.7.4.

Lemma 3.7.10 *For $1 < i < k$ the inverse of the matrix M_i is given by*

$$M_i^{-1} = \frac{1}{q^{k-i}(q-1)[i]^2[n]} \begin{pmatrix} q^i & 1 & -q^i & -1 \\ 1 & q^i & -q^i & -1 \\ -q^i & -q^i & \frac{q^i[k]-[i]}{[k-i]} & 1 \\ -1 & -1 & 1 & \frac{q^i[n-k]-[i]}{q^{2i}[n-k-i]} \end{pmatrix}. \tag{3.29}$$

Proof. Routine. □

Lemma 3.7.11 *For $x, y \in X$ such that $1 < \partial(x, y) < k$, we have*

$$B_{xy} = q^{2i}[k-i][n-k-i]\widehat{x} - q^{2i}[n-k-i]\widehat{x \cap y} - q^i[k-i]\widehat{x+y}, \tag{3.30}$$

$$C_{xy} = q[i-1]^2\widehat{x} + q^{i-1}\widehat{y} + q^i[i-1]\widehat{x \cap y} + [i-1]\widehat{x+y}, \tag{3.31}$$

where $i = \partial(x, y)$.

Proof. Write

$$B_{xy} = \alpha\widehat{x} + \beta\widehat{y} + \gamma\widehat{x \cap y} + \delta\widehat{x+y}, \tag{3.32}$$

$$C_{xy} = \alpha'\widehat{x} + \beta'\widehat{y} + \gamma'\widehat{x \cap y} + \delta'\widehat{x+y}, \tag{3.33}$$

for $\alpha, \beta, \gamma, \delta, \alpha', \beta', \gamma', \delta' \in \mathbb{R}$.

In each of (3.32) and (3.33), we take the inner product of either side with each of $\widehat{x}, \widehat{y}, \widehat{x \cap y}$, $\widehat{x+y}$ to obtain

$$M_i \begin{pmatrix} \alpha & \alpha' \\ \beta & \beta' \\ \gamma & \gamma' \\ \delta & \delta' \end{pmatrix} = \begin{pmatrix} \langle B_{xy}, \widehat{x}\rangle & \langle C_{xy}, \widehat{x}\rangle \\ \langle B_{xy}, \widehat{y}\rangle & \langle C_{xy}, \widehat{y}\rangle \\ \langle B_{xy}, \widehat{x \cap y}\rangle & \langle C_{xy}, \widehat{x \cap y}\rangle \\ \langle B_{xy}, \widehat{x+y}\rangle & \langle C_{xy}, \widehat{x+y}\rangle \end{pmatrix}.$$

Use the table in Theorem 3.7.9 and the matrix (3.29) to obtain

$$\begin{pmatrix} \alpha & \alpha' \\ \beta & \beta' \\ \gamma & \gamma' \\ \delta & \delta' \end{pmatrix} = \begin{pmatrix} q^{2i}[k-i][n-k-i] & q[i-1]^2 \\ 0 & q^{i-1} \\ -q^{2i}[n-k-i] & q^i[i-1] \\ -q^i[k-i] & [i-1] \end{pmatrix}. \tag{3.34}$$

The result follows. □

Lemma 3.7.12 *For $x, y \in X$ such that $1 < \partial(x,y) < k$, we have*

$$\big\langle B_{xy}, B_{xy}\big\rangle = q^{4i+2}[k-i][n-k-i]\Big(q^{k-i-2}[n]\Big([k-i]+[n-k-i]\Big) + [k-i][n-k-i]\Big([n][k-2]-[k]^2\Big)\Big),$$

where $i = \partial(x,y)$.

Proof. Using (3.30),

$$\big\langle B_{xy}, B_{xy}\big\rangle = q^{2i}[k-i][n-k-i]\big\langle B_{xy}, \widehat{x}\big\rangle - q^{2i}[n-k-i]\Big\langle B_{xy}, \widehat{x\cap y}\Big\rangle - q^{i}[k-i]\Big\langle B_{xy}, \widehat{x+y}\Big\rangle.$$

Evaluate the above equation using (3.21), (3.25), (3.26) to obtain the result. □

Lemma 3.7.13 *For $x, y \in X$ such that $1 < \partial(x,y) < k$, we have*

$$\big\langle B_{xy}, C_{xy}\big\rangle = q^{2i+1}[k-i][n-k-i][i]^2\Big([n][k-2]-[k]^2\Big),$$

where $i = \partial(x,y)$.

Proof. Using (3.31),

$$\big\langle B_{xy}, C_{xy}\big\rangle = q[i-1]^2\big\langle B_{xy}, \widehat{x}\big\rangle + q^{i-1}\big\langle B_{xy}, \widehat{y}\big\rangle + q^{i}[i-1]\Big\langle B_{xy}, \widehat{x\cap y}\Big\rangle + [i-1]\Big\langle B_{xy}, \widehat{x+y}\Big\rangle.$$

Evaluate the above equation using (3.21), (3.23), (3.25), (3.26) to obtain the result. □

Lemma 3.7.14 *For $x, y \in X$ such that $1 < \partial(x,y) < k$, we have*

$$\big\langle C_{xy}, C_{xy}\big\rangle = [i]^2\Big(q^{k-2}[n]\Big(2q[i-1]+q+1\Big) + [i]^2\Big([n][k-2]-[k]^2\Big)\Big),$$

where $i = \partial(x,y)$.

Proof. Using (3.31),

$$\big\langle C_{xy}, C_{xy}\big\rangle = q[i-1]^2\big\langle C_{xy}, \widehat{x}\big\rangle + q^{i-1}\big\langle C_{xy}, \widehat{y}\big\rangle + q^{i}[i-1]\Big\langle C_{xy}, \widehat{x\cap y}\Big\rangle + [i-1]\Big\langle C_{xy}, \widehat{x+y}\Big\rangle.$$

Evaluate the above equation using (3.22), (3.24), (3.27), (3.28) to obtain the result. □

Theorem 3.7.15 *Let $x, y \in X$ satisfy $1 < \partial(x,y) < k$. In the following table, for each vector u in the header column, and each vector v in the header row, the (u,v)-entry of the table gives the inner product $\langle u, v\rangle$. Write $i = \partial(x,y)$.*

$\langle\, ,\, \rangle$	$\widehat{x}$	$\widehat{y}$	B_{xy}	C_{xy}
$\widehat{x}$	$q^k[k][n-k]$	$[n][k-i]-[k]^2$	$q^{2i+1}[k-i][n-k-i]\cdot\big([n][k-1]-[k]^2\big)$	$[i]^2\big([n][k-1]-[k]^2\big)$
$\widehat{y}$	$[n][k-i]-[k]^2$	$q^k[k][n-k]$	$q^{2i+1}[k-i][n-k-i]\cdot\big([n][k-i-1]-[k]^2\big)$	$[i]^2\big([n][k-i+1]-[k]^2\big)$
B_{xy}	$q^{2i+1}[k-i][n-k-i]\cdot\big([n][k-1]-[k]^2\big)$	$q^{2i+1}[k-i][n-k-i]\cdot\big([n][k-i-1]-[k]^2\big)$	$q^{4i+2}[k-i][n-k-i]\cdot\big(q^{k-i-2}[n]([k-i]+[n-k-i])+[k-i][n-k-i]\cdot([n][k-2]-[k]^2)\big)$	$q^{2i+1}[k-i][n-k-i]\cdot[i]^2\big([n][k-2]-[k]^2\big)$
C_{xy}	$[i]^2\big([n][k-1]-[k]^2\big)$	$[i]^2\big([n][k-i+1]-[k]^2\big)$	$q^{2i+1}[k-i][n-k-i]\cdot[i]^2\big([n][k-2]-[k]^2\big)$	$[i]^2\big(q^{k-2}[n](2q[i-1]+q+1)+[i]^2([n][k-2]-[k]^2)\big)$

Proof. Combine Corollary 3.3.5 and Lemmas 3.3.4, 3.7.5, 3.7.6, 3.7.12–3.7.14. □

3.8 The combinatorial basis for $\mathrm{Fix}(x,y)$

Pick $x, y \in X$ such that $1 < \partial(x,y) < k$. In this section we prove that the vectors in (3.20) form a basis for the subspace $\mathrm{Fix}(x,y)$. We formally define the combinatorial basis for $\mathrm{Fix}(x,y)$. We also present the transition matrices between the geometric basis and the combinatorial basis for $\mathrm{Fix}(x,y)$.

Theorem 3.8.1 *For $x, y \in X$ such that $1 < \partial(x,y) < k$, the vectors in* (3.20) *form a basis for* $\mathrm{Fix}(x,y)$.

Proof. We first find the matrix of coefficients when we write the vectors in (3.20) as linear combinations in the geometric basis for $\mathrm{Fix}(x,y)$. From Lemma 3.7.11, we routinely obtain the following matrix of coefficients:

$$\begin{pmatrix} 1 & 0 & q^{2i}[k-i][n-k-i] & q[i-1]^2 \\ 0 & 1 & 0 & q^{i-1} \\ 0 & 0 & -q^{2i}[n-k-i] & q^i[i-1] \\ 0 & 0 & -q^i[k-i] & [i-1] \end{pmatrix}, \tag{3.35}$$

where $i = \partial(x,y)$.

It suffices to show that the determinant of this matrix is nonzero.

The determinant is equal to $-q^{k+i}[i-1][n-2k]$. We have $[i-1] \neq 0$ since $1 < i < k$. Also, $[n-2k] \neq 0$ since $n > 2k$. Hence the determinant of the matrix (3.35) is nonzero. The result follows. □

Definition 3.8.2 Let $x, y \in X$ satisfy $1 < \partial(x,y) < k$. By the *combinatorial basis for* $\mathrm{Fix}(x,y)$, we mean the basis formed by the vectors in (3.20).

Next we give the transition matrices between the geometric basis and the combinatorial basis for $\mathrm{Fix}(x,y)$. Throughout this chapter we will use the convention described in [4, p. 352] for transition matrices.

Theorem 3.8.3 *Let $x, y \in X$ satisfy $1 < \partial(x,y) < k$. Write $i = \partial(x,y)$. The transition matrix from the geometric basis to the combinatorial basis for* $\mathrm{Fix}(x,y)$ *is equal to the matrix* (3.35).

The transition matrix from the combinatorial basis to the geometric basis for $\mathrm{Fix}(x,y)$ *is equal to*

$$\begin{pmatrix} 1 & 0 & \frac{[k-i][n-k-1]}{q^{k-1}[n-2k]} & \frac{-[k-1][n-k-i]}{q^{k-i-1}[n-2k]} \\ 0 & 1 & \frac{[k-i]}{q^{k-i+1}[i-1][n-2k]} & \frac{-[n-k-i]}{q^{k-2i+1}[i-1][n-2k]} \\ 0 & 0 & \frac{-1}{q^{k+i}[n-2k]} & \frac{1}{q^{k}[n-2k]} \\ 0 & 0 & \frac{-[k-i]}{q^{k}[i-1][n-2k]} & \frac{[n-k-i]}{q^{k-i}[i-1][n-2k]} \end{pmatrix}. \tag{3.36}$$

Proof. The first assertion is immediate from the construction of the matrix (3.35). For the second assertion, take the inverse of the matrix (3.35) to obtain the matrix (3.36). □

Theorem 3.8.4 *For $x, y \in X$ such that $1 < \partial(x,y) < k$, we have*

$$\begin{aligned} \widehat{x \cap y} = & \frac{[k-i][n-k-1]}{q^{k-1}[n-2k]}\hat{x} + \frac{[k-i]}{q^{k-i+1}[i-1][n-2k]}\hat{y} \\ & + \frac{-1}{q^{k+i}[n-2k]}B_{xy} + \frac{-[k-i]}{q^{k}[i-1][n-2k]}C_{xy}, \\ \widehat{x+y} = & \frac{-[k-1][n-k-i]}{q^{k-i-1}[n-2k]}\hat{x} + \frac{-[n-k-i]}{q^{k-2i+1}[i-1][n-2k]}\hat{y} \\ & + \frac{1}{q^{k}[n-2k]}B_{xy} + \frac{[n-k-i]}{q^{k-i}[i-1][n-2k]}C_{xy}, \end{aligned}$$

where $i = \partial(x,y)$.

Proof. Routine from the matrix (3.36). □

3.9 The subspace $\overline{\mathrm{Fix}}(x,y)$; swapping x and y

Pick $x, y \in X$ such that $1 < \partial(x,y) < k$. In this section we consider an element $\sigma \in GL(\mathcal{V})$ that swaps x and y. We show that the action of σ on $\mathrm{Fix}(x,y)$ has eigenvalues -1 and 1. We describe the corresponding eigenspaces. The eigenspace that corresponds to the eigenvalue 1 will be denoted by $\overline{\mathrm{Fix}}(x,y)$. We construct a basis for $\overline{\mathrm{Fix}}(x,y)$, called the geometric basis; this basis is derived from the geometric basis for $\mathrm{Fix}(x,y)$. We construct another basis for $\overline{\mathrm{Fix}}(x,y)$, called the combinatorial basis; this basis is derived from the combinatorial basis for $\mathrm{Fix}(x,y)$. We give the transition matrices between the geometric basis and the combinatorial basis for $\overline{\mathrm{Fix}}(x,y)$.

Lemma 3.9.1 *For $x, y \in X$ such that $1 < \partial(x,y) < k$, there exists $\sigma \in GL(\mathcal{V})$ that swaps x and y.*

Proof. Immediate from the fact that the graph Γ is distance-transitive. □

Lemma 3.9.2 *Let $x, y \in X$ satisfy $1 < \partial(x, y) < k$. Pick $\sigma \in GL(\mathcal{V})$ that swaps x and y. The following* (i)–(iii) *hold:*

(i) $\mathrm{Fix}(x, y)$ *is invariant under σ;*

(ii) *on* $\mathrm{Fix}(x, y)$*, we have* $\sigma^2 = \mathrm{id}$ *and* $\sigma \neq \pm\mathrm{id}$*, where* id *is the identity element of* $GL(\mathcal{V})$*;*

(iii) *the restriction of σ to* $\mathrm{Fix}(x, y)$ *is diagonalizable with eigenvalues* -1 *and* 1.

Proof. (i) Immediate from the construction of σ.

(ii) The first assertion is immediate from the construction of σ. We now prove the second assertion. Note that $\hat{x} - \hat{y}$ is a nonzero vector in $\mathrm{Fix}(x, y)$ such that

$$\sigma\big(\hat{x} - \hat{y}\big) = \hat{y} - \hat{x}.$$

Hence, $\sigma \neq \mathrm{id}$ on $\mathrm{Fix}(x, y)$. Note that $\hat{x} + \hat{y}$ is a nonzero vector in $\mathrm{Fix}(x, y)$ such that

$$\sigma\big(\hat{x} + \hat{y}\big) = \hat{x} + \hat{y}.$$

Hence, $\sigma \neq -\mathrm{id}$ on $\mathrm{Fix}(x, y)$. The result follows.

(iii) Immediate from (ii) of this lemma. □

We now find bases for the eigenspaces corresponding to the eigenvalues listed in Lemma 3.9.2(iii).

Lemma 3.9.3 *Let $x, y \in X$ satisfy $1 < \partial(x, y) < k$. Pick $\sigma \in GL(\mathcal{V})$ that swaps x and y. For the action of σ on* $\mathrm{Fix}(x, y)$ *the following hold:*

(i) *the eigenspace with eigenvalue -1 has a basis $\hat{x} - \hat{y}$;*

(ii) *the following vectors form a basis for the eigenspace with eigenvalue* 1.

$$\hat{x} + \hat{y}, \qquad \widehat{x \cap y}, \qquad \widehat{x + y} \tag{3.37}$$

Proof. For the action of σ on $\mathrm{Fix}(x, y)$, the eigenspace with eigenvalue -1 contains the vector $\hat{x} - \hat{y}$, and the eigenspace with eigenvalue 1 contains the three vectors in (3.37).

Recall the geometric basis for $\mathrm{Fix}(x, y)$ given in Theorem 3.5.3. Adjusting this basis, we find that the following vectors form a basis for $\mathrm{Fix}(x, y)$:

$$\hat{x} - \hat{y}, \qquad \hat{x} + \hat{y}, \qquad \widehat{x \cap y}, \qquad \widehat{x + y}.$$

The result follows. □

For $x, y \in X$ such that $1 < \partial(x, y) < k$, the two eigenspaces described in Lemma 3.9.3 are independent of the choice of $\sigma \in GL(\mathcal{V})$ that swaps x and y.

Definition 3.9.4 For $x, y \in X$ such that $1 < \partial(x,y) < k$, define $\overline{\text{Fix}}(x,y)$ to be the eigenspace for the eigenvalue 1 described in Lemma 3.9.3(ii).

Note that $\overline{\text{Fix}}(x,y)$ has a basis (3.37).

Definition 3.9.5 Let $x, y \in X$ satisfy $1 < \partial(x,y) < k$. By the *geometric basis for* $\overline{\text{Fix}}(x,y)$, we mean the basis (3.37).

Pick $x, y \in X$ such that $1 < \partial(x,y) < k$. We just introduced the geometric basis for $\overline{\text{Fix}}(x,y)$. We now use the combinatorial basis for $\text{Fix}(x,y)$ to find another basis for $\overline{\text{Fix}}(x,y)$. To find this basis, we recall the balanced set condition.

Lemma 3.9.6 [6, Theorem 3.3] *For $x, y \in X$ such that $1 < \partial(x,y) < k$,*

$$B_{xy} - B_{yx} = \zeta\big(\widehat{x} - \widehat{y}\big), \qquad C_{xy} - C_{yx} = \xi\big(\widehat{x} - \widehat{y}\big),$$

where

$$\zeta = q^{2i}[k-i][n-k-i], \qquad \xi = q[i][i-2], \qquad i = \partial(x,y).$$

Definition 3.9.7 For $x, y \in X$ such that $1 < \partial(x,y) < k$, define

$$\overline{B_{xy}} = B_{xy} - \zeta\widehat{x}, \qquad \overline{C_{xy}} = C_{xy} - \xi\widehat{x},$$

where ζ, ξ are from Lemma 3.9.6.

By Lemma 3.9.6,

$$\overline{B_{xy}} = \overline{B_{yx}}, \qquad \overline{C_{xy}} = \overline{C_{yx}}. \tag{3.38}$$

Lemma 3.9.8 *For $x, y \in X$ such that $1 < \partial(x,y) < k$, the subspace $\overline{\text{Fix}}(x,y)$ contains the vectors $\overline{B_{xy}}$, $\overline{C_{xy}}$.*

Proof. By Theorem 3.8.1 the subspace $\text{Fix}(x,y)$ contains the vectors $\overline{B_{xy}}$ and $\overline{C_{xy}}$. Let $\sigma \in GL(\mathcal{V})$ swap x and y. By construction, $\sigma\left(\overline{B_{xy}}\right) = \overline{B_{yx}}$ and $\sigma\left(\overline{C_{xy}}\right) = \overline{C_{yx}}$. By (3.38), the vectors $\overline{B_{xy}}$ and $\overline{C_{xy}}$ are fixed by σ. The result follows. □

Lemma 3.9.9 *For $x, y \in X$ such that $1 < \partial(x,y) < k$,*

$$\begin{aligned}\overline{B_{xy}} &= -q^{2i}[n-k-i]\widehat{x\cap y} - q^{i}[k-i]\widehat{x+y},\\ \overline{C_{xy}} &= q^{i-1}(\widehat{x}+\widehat{y}) + q^{i}[i-1]\widehat{x\cap y} + [i-1]\widehat{x+y},\end{aligned}$$

where $i = \partial(x,y)$.

Proof. Routine using Lemma 3.7.11 and Definition 3.9.7. □

Theorem 3.9.10 *For $x, y \in X$ such that $1 < \partial(x,y) < k$, the following vectors form a basis for $\overline{\text{Fix}}(x,y)$:*

$$\widehat{x} + \widehat{y}, \qquad \overline{B_{xy}}, \qquad \overline{C_{xy}}. \tag{3.39}$$

Proof. We first find the matrix of coefficients when we write the vectors in (3.39) as linear combinations in the geometric basis for $\overline{\mathrm{Fix}}(x,y)$. From Lemma 3.9.9, we routinely obtain the following matrix of coefficients:

$$\begin{pmatrix} 1 & 0 & q^{i-1} \\ 0 & -q^{2i}[n-k-i] & q^{i}[i-1] \\ 0 & -q^{i}[k-i] & [i-1] \end{pmatrix}, \tag{3.40}$$

where $i = \partial(x,y)$.

It suffices to show that the determinant of this matrix is nonzero.

The determinant is equal to $-q^{k+i}[i-1][n-2k]$. We have $[i-1] \neq 0$ since $1 < i < k$. Also, $[n-2k] \neq 0$ since $n > 2k$. Hence the determinant of the matrix (3.40) is nonzero. The result follows. □

Definition 3.9.11 Let $x, y \in X$ satisfy $1 < \partial(x,y) < k$. By the *combinatorial basis for* $\overline{\mathrm{Fix}}(x,y)$, we mean the basis formed by the vectors in (3.39).

Next we give the transition matrices between the geometric basis and the combinatorial basis for $\overline{\mathrm{Fix}}(x,y)$.

Theorem 3.9.12 *Let $x, y \in X$ satisfy $1 < \partial(x,y) < k$. Write $i = \partial(x,y)$. The transition matrix from the geometric basis to the combinatorial basis for $\overline{\mathrm{Fix}}(x,y)$ is equal to the matrix* (3.40).

The transition matrix from the combinatorial basis to the geometric basis for $\overline{\mathrm{Fix}}(x,y)$ is equal to

$$\begin{pmatrix} 1 & \frac{[k-i]}{q^{k-i+1}[i-1][n-2k]} & \frac{-[n-k-i]}{q^{k-2i+1}[i-1][n-2k]} \\ 0 & \frac{-1}{q^{k+i}[n-2k]} & \frac{1}{q^{k}[n-2k]} \\ 0 & \frac{-[k-i]}{q^{k}[i-1][n-2k]} & \frac{[n-k-i]}{q^{k-i}[i-1][n-2k]} \end{pmatrix}. \tag{3.41}$$

Proof. The first assertion is immediate from the construction of the matrix (3.40). For the second assertion, take the inverse of the matrix (3.40) to obtain the matrix (3.41). □

Theorem 3.9.13 *For $x, y \in X$ such that $1 < \partial(x,y) < k$, we have*

$$\widehat{x \cap y} = \frac{[k-i]}{q^{k-i+1}[i-1][n-2k]}\left(\widehat{x} + \widehat{y}\right) + \frac{-1}{q^{k+i}[n-2k]}\overline{B_{xy}} + \frac{-[k-i]}{q^{k}[i-1][n-2k]}\overline{C_{xy}},$$

$$\widehat{x + y} = \frac{-[n-k-i]}{q^{k-2i+1}[i-1][n-2k]}\left(\widehat{x} + \widehat{y}\right) + \frac{1}{q^{k}[n-2k]}\overline{B_{xy}} + \frac{[n-k-i]}{q^{k-i}[i-1][n-2k]}\overline{C_{xy}},$$

where $i = \partial(x,y)$.

Proof. Routine from the matrix (3.41). □

3.10 The subspace $\mathrm{Fix}(x \cap y, x + y)$

Pick distinct $x, y \in X$ such that $1 < \partial(x, y) < k$. In this section we describe the subspace $\mathrm{Fix}(x \cap y, x + y)$. We show that the subspace $\mathrm{Fix}(x \cap y, x + y)$ is contained in $\overline{\mathrm{Fix}}(x, y)$. We construct a basis for $\mathrm{Fix}(x \cap y, x + y)$, called the combinatorial basis; this basis is derived from the combinatorial basis for $\overline{\mathrm{Fix}}(x, y)$. We give the transition matrices between the geometric basis and the combinatorial basis for $\mathrm{Fix}(x \cap y, x + y)$. At the end of the section, we describe the orthogonal complement of $\mathrm{Fix}(x \cap y, x + y)$ in $\overline{\mathrm{Fix}}(x, y)$.

Theorem 3.10.1 *For $x, y \in X$ such that $1 < \partial(x, y) < k$, the following vectors form a basis for* $\mathrm{Fix}(x \cap y, x + y)$*:*

$$\widehat{x \cap y}, \qquad \widehat{x + y}. \tag{3.42}$$

Proof. Since $\partial(x, y) < k$, we have $x \cap y \neq 0$. Note that $x \cap y \subseteq x + y$. Since $n > 2k$, we have $x + y \neq \mathcal{V}$. The result follows from Case 5 of Lemma 3.5.1. □

In view of Definition 3.5.2, the basis (3.42) is the geometric basis for $\mathrm{Fix}(x \cap y, x + y)$.

Corollary 3.10.2 *For $x, y \in X$ such that $1 < \partial(x, y) < k$, the subspace* $\mathrm{Fix}(x \cap y, x + y)$ *is contained in* $\overline{\mathrm{Fix}}(x, y)$*.*

Proof. The geometric basis for $\mathrm{Fix}(x \cap y, x + y)$ is a subset of (3.37). The vectors in (3.37) form a basis for $\overline{\mathrm{Fix}}(x, y)$. The result follows. □

In Theorem 3.10.1, we gave a basis for $\mathrm{Fix}(x \cap y, x + y)$. Our next goal is to use the combinatorial basis for $\overline{\mathrm{Fix}}(x, y)$ to find another basis for $\mathrm{Fix}(x \cap y, x + y)$.

Definition 3.10.3 For $x, y \in X$ such that $1 < \partial(x, y) < k$, define

$$\overset{\smile}{B_{xy}} = \overline{B_{xy}}, \qquad \overset{\smile}{C_{xy}} = \overline{C_{xy}} - q^{i-1}\big(\hat{x} + \hat{y}\big),$$

where $i = \partial(x, y)$.

Lemma 3.10.4 *For $x, y \in X$ such that $1 < \partial(x, y) < k$,*

$$\overset{\smile}{B_{xy}} = -q^{2i}[n-k-i]\widehat{x \cap y} - q^{i}[k-i]\widehat{x + y},$$
$$\overset{\smile}{C_{xy}} = q^{i}[i-1]\widehat{x \cap y} + [i-1]\widehat{x + y},$$

where $i = \partial(x, y)$.

Proof. Routine using Lemma 3.9.9 and Definition 3.10.3. □

Lemma 3.10.5 *For $x, y \in X$ such that $1 < \partial(x, y) < k$, the subspace* $\mathrm{Fix}(x \cap y, x + y)$ *contains the following vectors:*

$$\overset{\smile}{B_{xy}}, \qquad \overset{\smile}{C_{xy}}. \tag{3.43}$$

Proof. Referring to Lemma 3.10.4, the vectors $\overset{\smile}{B_{xy}}$ and $\overset{\smile}{C_{xy}}$ are linear combinations in (3.42). The result follows from Theorem 3.10.1. □

Theorem 3.10.6 *For $x, y \in X$ such that $1 < \partial(x,y) < k$, the two vectors in* (3.43) *form a basis for* $\mathrm{Fix}(x \cap y, x+y)$.

Proof. We first find the matrix of coefficients when we write the vectors in (3.43) as linear combinations in the geometric basis for $\mathrm{Fix}(x \cap y, x+y)$. From Lemma 3.10.4, we routinely obtain the following matrix of coefficients:

$$\begin{pmatrix} -q^{2i}[n-k-i] & q^i[i-1] \\ -q^i[k-i] & [i-1] \end{pmatrix}, \tag{3.44}$$

where $i = \partial(x,y)$.

It suffices to show that the determinant of this matrix is nonzero.

The determinant is equal to $-q^{k+i}[i-1][n-2k]$. We have $[i-1] \neq 0$ since $1 < i < k$. Also, $[n-2k] \neq 0$ since $n > 2k$. Hence the determinant of the matrix (3.44) is nonzero. The result follows. □

Definition 3.10.7 Let $x, y \in X$ satisfy $1 < \partial(x,y) < k$. By the *combinatorial basis for* $\mathrm{Fix}(x \cap y, x+y)$, we mean the basis formed by the vectors in (3.43).

Next we display the transition matrices between the geometric basis and the combinatorial basis for $\mathrm{Fix}(x \cap y, x+y)$.

Theorem 3.10.8 *Let $x, y \in X$ satisfy $1 < \partial(x,y) < k$. Write $i = \partial(x,y)$. The transition matrix from the geometric basis to the combinatorial basis for* $\mathrm{Fix}(x \cap y, x+y)$ *is equal to the matrix* (3.44).

The transition matrix from the combinatorial basis to the geometric basis for $\mathrm{Fix}(x \cap y, x+y)$ *is equal to*

$$\begin{pmatrix} \frac{-1}{q^{k+i}[n-2k]} & \frac{1}{q^k[n-2k]} \\ \frac{-[k-i]}{q^k[i-1][n-2k]} & \frac{[n-k-i]}{q^{k-i}[i-1][n-2k]} \end{pmatrix}. \tag{3.45}$$

Proof. The first assertion is immediate from the construction of the matrix (3.44). For the second assertion, take the inverse of the matrix (3.44) to obtain the matrix (3.45). □

Theorem 3.10.9 *For $x, y \in X$ such that $1 < \partial(x,y) < k$, we have*

$$\widehat{x \cap y} = \frac{-1}{q^{k+i}[n-2k]}\widetilde{B_{xy}} + \frac{-[k-i]}{q^k[i-1][n-2k]}\widetilde{C_{xy}},$$
$$\widehat{x + y} = \frac{1}{q^k[n-2k]}\widetilde{B_{xy}} + \frac{[n-k-i]}{q^{k-i}[i-1][n-2k]}\widetilde{C_{xy}},$$

where $i = \partial(x,y)$.

Proof. Routine from the matrix (3.45). □

Recall from Corollary 3.10.2 that the subspace $\mathrm{Fix}(x \cap y, x + y)$ is contained in $\overline{\mathrm{Fix}}(x, y)$. Let $\mathrm{Fix}(x \cap y, x + y)^{\perp}$ denote the orthogonal complement of $\mathrm{Fix}(x \cap y, x + y)$ in $\overline{\mathrm{Fix}}(x, y)$. Our next goal is to find a basis for $\mathrm{Fix}(x \cap y, x + y)^{\perp}$. We will express this basis in terms of the geometric basis for $\overline{\mathrm{Fix}}(x, y)$ and also the combinatorial basis for $\overline{\mathrm{Fix}}(x, y)$.

Lemma 3.10.10 *For $x, y \in X$ such that $1 < \partial(x, y) < k$, the subspace $\mathrm{Fix}(x \cap y, x + y)^{\perp}$ has dimension 1.*

Proof. By Lemma 3.9.3(ii), the subspace $\overline{\mathrm{Fix}}(x, y)$ has dimension 3. By Theorem 3.10.1, the subspace $\mathrm{Fix}(x \cap y, x + y)$ has dimension 2. The result follows. □

Lemma 3.10.11 *For $x, y \in X$ such that $1 < \partial(x, y) < k$, the subspace $\mathrm{Fix}(x \cap y, x + y)^{\perp}$ has a basis*

$$\left(q^i + 1\right)\left(\widehat{x} + \widehat{y}\right) - 2q^i\widehat{x \cap y} - 2\widehat{x + y}, \tag{3.46}$$

where $i = \partial(x, y)$.

Proof. Using the table in Theorem 3.7.4 one routinely verifies that the vector (3.46) is contained in $\mathrm{Fix}(x \cap y, x + y)^{\perp}$. The three vectors in (3.37) are linearly independent, so the vector (3.46) is nonzero. The result follows from Lemma 3.10.10. □

Lemma 3.10.12 *Referring to* Lemma 3.10.11*, the vector* (3.46) *is equal to*

$$\frac{1}{[i-1]}\Big(\left(q^{i-1} + 1\right)[i]\left(\widehat{x} + \widehat{y}\right) - 2\overline{C_{xy}}\Big),$$

where $i = \partial(x, y)$.

Proof. Apply the transition matrix (3.41) to the vector (3.46). □

3.11 Comments about the uniqueness problem for Γ

Consider the following question: is the Grassmann graph $J_q(n, k)$ uniquely determined up to isomorphism by its intersection numbers? In this section, we comment on how this problem relates to the main result of this chapter.

Consider the Grassmann graph $\Gamma = J_q(n, k)$ with $n > 2k \geqslant 6$. Let Γ' denote a distance-regular graph that has the same intersection numbers as Γ. Note that Γ' and Γ have the same eigenvalues. Recall the vertex set X for Γ, and let X' denote the vertex set for Γ'.

Let E' denote a Euclidean space of dimension $[n] - 1$. By [7, Lecture 13], there exists a Euclidean representation (E', ψ) of Γ' associated with the eigenvalue θ_1. We normalize this representation such that

$$\left\|\psi(x')\right\|^2 = q^k[k][n-k]$$

for all $x' \in X'$.

Let us attempt to recover the projective geometry P from Γ'. To motivate things, we first consider Γ. For $x, y \in X$ such that $1 < \partial(x, y) < k$, we defined the sets $\mathcal{B}_{xy}, \mathcal{C}_{xy}$ in

Definition 3.6.1. In Definition 3.6.5, we used $\mathcal{B}_{xy}, \mathcal{C}_{xy}$ to define the vectors B_{xy}, C_{xy} in E. In Theorem 3.8.4, we wrote each vector $\widehat{x \cap y}, \widehat{x + y}$ as a linear combination of

$$\hat{x}, \qquad \hat{y}, \qquad B_{xy}, \qquad C_{xy}.$$

We now talk about the graph Γ'; in the Euclidean space E', let us attempt to mimic the vectors $\widehat{x \cap y}, \widehat{x + y}$. Let $x', y' \in X'$ satisfy $\partial\big(x', y'\big) = \partial(x, y)$. Define the sets $\mathcal{B}_{x'y'}, \mathcal{C}_{x'y'}$ similarly to Definition 3.6.1. Define the vectors $B_{x'y'}, C_{x'y'}$ similarly to Definition 3.6.5. For notational convenience, write $i = \partial(x', y')$.

Motivated by the first equation in Theorem 3.8.4, we bring in the vector

$$\begin{aligned} &\frac{[k-i][n-k-1]}{q^{k-1}[n-2k]}\psi(x') + \frac{[k-i]}{q^{k-i+1}[i-1][n-2k]}\psi(y') \\ &\qquad + \frac{-1}{q^{k+i}[n-2k]}B_{x'y'} + \frac{-[k-i]}{q^k[i-1][n-2k]}C_{x'y'}. \end{aligned} \tag{3.47}$$

This is the vector in E' that will mimic $\widehat{x \cap y}$. For motivational purposes, we denote the vector (3.47) by $\psi\big(x' \cap y'\big)$.

Motivated by the second equation in Theorem 3.8.4, we bring in the vector

$$\begin{aligned} &\frac{-[k-1][n-k-i]}{q^{k-i-1}[n-2k]}\psi(x') + \frac{-[n-k-i]}{q^{k-2i+1}[i-1][n-2k]}\psi(y') \\ &\qquad + \frac{1}{q^{k}[n-2k]}B_{x'y'} + \frac{[n-k-i]}{q^{k-i}[i-1][n-2k]}C_{x'y'}. \end{aligned} \tag{3.48}$$

This is the vector in E' that will mimic $\widehat{x + y}$. For motivational purposes, we denote the vector (3.48) by $\psi\big(x' + y'\big)$.

Problem 3.11.1 Try to show that for all $z' \in X'$, the inner product

$$\Big\langle \psi\big(x' \cap y'\big), \psi\big(z'\big) \Big\rangle \tag{3.49}$$

is equal to one of the values

$$[n][k-i-\ell] - [k-i][k] \qquad (0 \leqslant \ell \leqslant k-i). \tag{3.50}$$

By Lemma 3.3.2, this will happen if Γ' is isomorphic to Γ.

Problem 3.11.2 We define a binary relation on X' called partner. For $z', w' \in X'$, we say that z', w' are partners whenever

$$\Big\langle \psi\big(x' \cap y'\big), \psi\big(z'\big) \Big\rangle = \Big\langle \psi\big(x' \cap y'\big), \psi\big(w'\big) \Big\rangle.$$

By construction, partner is an equivalence relation. Try to show that the partner equivalence classes form an equitable partition of X'. If Γ' is isomorphic to Γ then this will happen because the partner equivalence classes are the orbits of $\mathrm{Stab}\big(x' \cap y'\big)$.

Problem 3.11.3 Consider the set of vertices $z' \in X'$ such that the inner product (3.49) is equal to (3.50) with $\ell = 0$. Try to show that the subgraph of Γ' induced on this set, is geodesically closed and has diameter i. If Γ' is isomorphic to Γ then this will happen because the set will consist of the vertices in X' that contain $x' \cap y'$.

Chapter 4

Some orbits of a two-vertex stabilizer in a Grassmann graph

Pick $x, y \in X$ such that $1 < \partial(x, y) < k$. Recall the stabilizer $\mathrm{Stab}(x, y)$ in $GL(\mathcal{V})$. In this chapter, we investigate the orbits of $\mathrm{Stab}(x, y)$ acting on $\Gamma(x)$. As we will see, there are five orbits. We already mentioned two of the orbits, namely $\mathcal{B}_{xy}$ and $\mathcal{C}_{xy}$. We now describe the other three orbits.

4.1 The y-partition of $\Gamma(x)$

Definition 4.1.1 For $x, y \in X$ such that $1 < \partial(x, y) < k$, define

$$\mathcal{A}_{xy} = \big\{z \in \Gamma(x) \mid \partial(y, z) = \partial(x, y)\big\}.$$

Observe that

$$\big|\mathcal{A}_{xy}\big| = a_i, \qquad\qquad i = \partial(x, y).$$

Definition 4.1.2 For $x, y \in X$ such that $1 < \partial(x, y) < k$, define the vector

$$A_{xy} = \sum_{z \in \mathcal{A}_{xy}} \hat{z}.$$

The set $\mathcal{A}_{xy}$ turns out to be the disjoint union of three orbits of the $\mathrm{Stab}(x, y)$-action on $\Gamma(x)$. Our next general goal is to describe these three orbits.

Definition 4.1.3 For $x, y \in X$ such that $1 < \partial(x, y) < k$, define

$$\mathcal{A}^{+}_{xy} = \big\{z \in \mathcal{A}_{xy} \mid z + x + y \supsetneq x + y, z \cap x \cap y = x \cap y\big\},$$
$$\mathcal{A}^{0}_{xy} = \big\{z \in \mathcal{A}_{xy} \mid z + x + y = x + y, z \cap x \cap y = x \cap y\big\},$$
$$\mathcal{A}^{-}_{xy} = \big\{z \in \mathcal{A}_{xy} \mid z + x + y = x + y, z \cap x \cap y \subsetneq x \cap y\big\}.$$

We are going to show that the three sets in Definition 4.1.3 are orbits of $\mathrm{Stab}(x, y)$. First we have a few remarks.

Lemma 4.1.4 *For $x, y \in X$ such that $1 < \partial(x,y) < k$, the set $\mathcal{A}_{xy}$ is the disjoint union of the sets $\mathcal{A}_{xy}^{+}, \mathcal{A}_{xy}^{0}, \mathcal{A}_{xy}^{-}$.*

Proof. By linear algebra, the set $\{z \in \mathcal{A}_{xy} \mid z + x + y \supsetneq x + y,\ z \cap x \cap y \subsetneq x \cap y\}$ is empty. The result follows. □

Lemma 4.1.5 *For $x, y \in X$ such that $1 < \partial(x,y) < k$,*

$$|\mathcal{A}_{xy}^{+}| = q^{i+1}[i][n-k-i], \qquad |\mathcal{A}_{xy}^{0}| = (q-1)[i]^2, \qquad |\mathcal{A}_{xy}^{-}| = q^{i+1}[i][k-i], \tag{4.1}$$

where $i = \partial(x,y)$.

Proof. Routine from counting. □

Observe that the values in (4.1) depend only on $\partial(x,y)$.

Definition 4.1.6 We refer to Lemma 4.1.5. For $1 < i < k$, define

$$a_i^{+} = |\mathcal{A}_{xy}^{+}|, \qquad a_i^{0} = |\mathcal{A}_{xy}^{0}|, \qquad a_i^{-} = |\mathcal{A}_{xy}^{-}|, \tag{4.2}$$

where $i = \partial(x,y)$.

Note that $a_i^{+} + a_i^{0} + a_i^{-} = a_i$ for $1 < i < k$.

Our next goal is to show that $\mathcal{A}_{xy}^{0}$ is an orbit of $\mathrm{Stab}(x,y)$.

Lemma 4.1.7 *For $x, y \in X$ such that $1 < \partial(x,y) < k$, let $z \in \mathcal{A}_{xy}^{0}$. Then*

$$x \cap y \subseteq (z + x) \cap y.$$

Moreover,

$$\dim(x \cap y) + 1 = \dim\big((z + x) \cap y\big).$$

Proof. Routine from the definition of $\mathcal{A}_{xy}^{0}$ and linear algebra. □

Lemma 4.1.8 *For $x, y \in X$ such that $1 < \partial(x,y) < k$, let $z \in \mathcal{A}_{xy}^{0}$. Then there exist vectors*

$$\mu \in (z + x) \cap y, \qquad \eta \in z, \qquad \nu \in x$$

such that

$$\mu \notin x \cap y, \qquad \eta \notin z \cap x, \qquad \nu \notin z \cap x,$$
$$\mu = \eta + \nu.$$

Proof. Pick $\mu \in (z + x) \cap y$ such that $\mu \notin x \cap y$. Note that $\mu \in z + x$. Also note that $\mu \notin x$ and $\mu \notin z$.

Hence, μ is a linear combination of some nonzero vector $\eta \in z$ and some nonzero vector $\nu \in x$. We assume without loss that $\mu = \eta + \nu$.

Assume that $\eta \in z \cap x$. Then $\mu = \eta + \nu \in x$, which is a contradiction. Hence, $\eta \notin z \cap x$.

Assume that $\nu \in z \cap x$. Then $\mu = \eta + \nu \in z$, which is a contradiction. Hence, $\nu \notin z \cap x$. The result follows. □

Lemma 4.1.9 *For $x, y \in X$ such that $1 < \partial(x,y) < k$, let $z \in \mathcal{A}^0_{xy}$. Let the vectors μ, η, ν be from Lemma 4.1.8. Then*

$$z + \mathbb{F}\mu = z + x, \qquad z + \mathbb{F}\nu = z + x, \qquad (z \cap x) + \mathbb{F}\eta = z, \tag{4.3}$$
$$x + \mathbb{F}\mu = z + x, \qquad x + \mathbb{F}\eta = z + x, \qquad (z \cap x) + \mathbb{F}\nu = x, \tag{4.4}$$
$$(x \cap y) + \mathbb{F}\mu = (z + x) \cap y. \tag{4.5}$$

Moreover, for each equation in (4.3), (4.4), (4.5) *the sum on the left is direct.*

Proof. Immediate from linear algebra. □

Lemma 4.1.10 *For $x, y \in X$ such that $1 < \partial(x,y) < k$, the set $\mathcal{A}^0_{xy}$ is an orbit of the* $\mathrm{Stab}(x,y)$*-action on* $\Gamma(x)$.

Proof. By Lemma 3.2.1, the set $\mathcal{A}^0_{xy}$ is a disjoint union of orbits of $\mathrm{Stab}(x,y)$. We now show that $\mathcal{A}^0_{xy}$ is a single orbit.

Let $z, z' \in \mathcal{A}^0_{xy}$. It suffices to show that there exists $\sigma \in \mathrm{Stab}(x,y)$ that sends $z \mapsto z'$.

Let the vectors μ, η, ν be from Lemma 4.1.8. Let the subset $\mathcal{R} \subseteq \mathcal{V}$ form a basis for $x \cap y$. Extend the basis $\mathcal{R}$ for $x \cap y$ to a basis $\mathcal{R} \cup \mathcal{S}$ for $z \cap x$. By the third equation in (4.3), $\mathcal{R} \cup \mathcal{S} \cup \{\eta\}$ forms a basis for z. By the third equation in (4.4), $\mathcal{R} \cup \mathcal{S} \cup \{\nu\}$ forms a basis for x. By (4.5), $\mathcal{R} \cup \{\mu\}$ forms a basis for $(z + x) \cap y$. By the first equation in (4.4), $\mathcal{R} \cup \mathcal{S} \cup \{\mu, \nu\}$ forms a basis for $z + x$. Extend the basis $\mathcal{R} \cup \{\mu\}$ for $(z + x) \cap y$ to a basis $\mathcal{R} \cup \mathcal{Q} \cup \{\mu\}$ for y. By Lemma 2.3.3, $\mathcal{R} \cup \mathcal{S} \cup \mathcal{Q} \cup \{\mu, \nu\}$ forms a basis for $x + y$. Extend the basis $\mathcal{R} \cup \mathcal{S} \cup \mathcal{Q} \cup \{\mu, \nu\}$ for $x + y$ to a basis $\mathcal{R} \cup \mathcal{S} \cup \mathcal{Q} \cup \mathcal{W} \cup \{\mu, \nu\}$ for $\mathcal{V}$.

Recall the element $z' \in \mathcal{A}^0_{xy}$. Consider the corresponding vectors μ', η', ν' from Lemma 4.1.8. Extend the basis $\mathcal{R}$ for $x \cap y$ to a basis $\mathcal{R} \cup \mathcal{S}'$ for $z' \cap x$. By the third equation in (4.3), $\mathcal{R} \cup \mathcal{S}' \cup \{\eta'\}$ forms a basis for z'. By the third equation in (4.4), $\mathcal{R} \cup \mathcal{S}' \cup \{\nu'\}$ forms a basis for x. By (4.5), $\mathcal{R} \cup \{\mu'\}$ forms a basis for $(z' + x) \cap y$. By the first equation in (4.4), $\mathcal{R} \cup \mathcal{S}' \cup \{\mu', \nu'\}$ forms a basis for $z' + x$. Extend the basis $\mathcal{R} \cup \{\mu'\}$ for $(z' + x) \cap y$ to a basis $\mathcal{R} \cup \mathcal{Q}' \cup \{\mu'\}$ for y. By Lemma 2.3.3, $\mathcal{R} \cup \mathcal{S}' \cup \mathcal{Q}' \cup \{\mu', \nu'\}$ forms a basis for $x + y$. Extend the basis $\mathcal{R} \cup \mathcal{S}' \cup \mathcal{Q}' \cup \{\mu', \nu'\}$ for $x + y$ to a basis $\mathcal{R} \cup \mathcal{S}' \cup \mathcal{Q}' \cup \mathcal{W}' \cup \{\mu', \nu'\}$ for $\mathcal{V}$.

By linear algebra, there exists $\sigma \in GL(\mathcal{V})$ that sends $\mathcal{S} \mapsto \mathcal{S}'$, $\mathcal{Q} \mapsto \mathcal{Q}'$, $\mathcal{W} \mapsto \mathcal{W}'$, $\mu \mapsto \mu'$, $\nu \mapsto \nu'$ and acts as the identity on $\mathcal{R}$. By construction, σ is contained in $\mathrm{Stab}(x,y)$ and sends $z \mapsto z'$. The result follows. □

Lemma 4.1.11 *For $x, y \in X$ such that $1 < \partial(x,y) < k$, the sets $\mathcal{A}^+_{xy}, \mathcal{A}^-_{xy}$ are orbits of the* $\mathrm{Stab}(x,y)$*-action on* $\Gamma(x)$.

Proof. Similar to Lemma 4.1.10. □

Theorem 4.1.12 *For $x, y \in X$ such that $1 < \partial(x,y) < k$, the following sets are orbits of the* $\mathrm{Stab}(x,y)$*-action on* $\Gamma(x)$*:*

$$\mathcal{B}_{xy}, \qquad \mathcal{C}_{xy}, \qquad \mathcal{A}^+_{xy}, \qquad \mathcal{A}^0_{xy}, \qquad \mathcal{A}^-_{xy}. \tag{4.6}$$

Furthermore, these orbits form a partition of $\Gamma(x)$.

Proof. For the first assertion, combine Lemmas 3.6.2, 3.6.4, 4.1.10, 4.1.11. The second assertion is immediate from Lemma 4.1.4 and the fact that the disjoint union of $\mathcal{B}_{xy}, \mathcal{C}_{xy}, \mathcal{A}_{xy}$ is equal to $\Gamma(x)$. □

Definition 4.1.13 Let $x, y \in X$ satisfy $1 < \partial(x, y) < k$, and consider the partition of $\Gamma(x)$ given in (4.6). By construction, this partition is equitable in the sense of [5, p. 159]. We call this partition the *y-partition of* $\Gamma(x)$.

4.2 The vectors $A^+_{xy}, A^0_{xy}, A^-_{xy}$

Pick $x, y \in X$ such that $1 < \partial(x, y) < k$. Recall the sets $\mathcal{A}^+_{xy}, \mathcal{A}^0_{xy}, \mathcal{A}^-_{xy}$ from Definition 4.1.3. Recall the Euclidean space E from Section 3.1. In this section we use these sets to define some vectors $A^+_{xy}, A^0_{xy}, A^-_{xy}$ in the Euclidean space E. We show that $A^+_{xy}, A^0_{xy}, A^-_{xy}$ are contained in $\mathrm{Fix}(x, y)$. We write $A^+_{xy}, A^0_{xy}, A^-_{xy}$ in terms of the geometric basis for $\mathrm{Fix}(x, y)$ and also the combinatorial basis for $\mathrm{Fix}(x, y)$.

Definition 4.2.1 For $x, y \in X$ such that $1 < \partial(x, y) < k$, define the vectors

$$A^+_{xy} = \sum_{z \in \mathcal{A}^+_{xy}} \hat{z}, \qquad A^0_{xy} = \sum_{z \in \mathcal{A}^0_{xy}} \hat{z}, \qquad A^-_{xy} = \sum_{z \in \mathcal{A}^-_{xy}} \hat{z}. \tag{4.7}$$

Note that $A^+_{xy}, A^0_{xy}, A^-_{xy}$ are contained in E.

By Lemma 4.1.4, $A_{xy} = A^+_{xy} + A^0_{xy} + A^-_{xy}$.

Lemma 4.2.2 *For* $x, y \in X$ *such that* $1 < \partial(x, y) < k$, *the vectors* $A^+_{xy}, A^0_{xy}, A^-_{xy}$ *are contained in* $\mathrm{Fix}(x, y)$.

Proof. Pick $\sigma \in \mathrm{Stab}(x, y)$. Since $\mathcal{A}^+_{xy}, \mathcal{A}^0_{xy}, \mathcal{A}^-_{xy}$ are orbits of the $\mathrm{Stab}(x, y)$-action on $\Gamma(x)$, the map σ fixes $\mathcal{A}^+_{xy}, \mathcal{A}^0_{xy}, \mathcal{A}^-_{xy}$. The result follows. □

Recall the geometric basis for $\mathrm{Fix}(x, y)$ from Theorem 3.5.3. Our next goal is to write $A^+_{xy}, A^0_{xy}, A^-_{xy}$ in terms of the geometric basis for $\mathrm{Fix}(x, y)$. To do this, we find inner products that involve the vectors $A^+_{xy}, A^0_{xy}, A^-_{xy}$. Recall the matrix M_i of inner products given in Theorem 3.7.4.

Lemma 4.2.3 *For* $x, y \in X$ *such that* $1 < \partial(x, y) < k$, *we have*

$$\left\langle A^+_{xy}, \hat{x} \right\rangle = q^{i+1}[i][n-k-i]\Big([n][k-1] - [k]^2\Big), \tag{4.8}$$

$$\left\langle A^0_{xy}, \hat{x} \right\rangle = (q-1)[i]^2\Big([n][k-1] - [k]^2\Big), \tag{4.9}$$

$$\left\langle A^-_{xy}, \hat{x} \right\rangle = q^{i+1}[i][k-i]\Big([n][k-1] - [k]^2\Big), \tag{4.10}$$

where $i = \partial(x, y)$.

Proof. We first prove (4.8). Using the first equation in (4.7), we obtain

$$\left\langle A^{+}_{xy}, \widehat{x} \right\rangle = \sum_{z \in \mathcal{A}^{+}_{xy}} \langle \widehat{z}, \widehat{x} \rangle. \tag{4.11}$$

Pick $z \in \mathcal{A}^{+}_{xy}$. By the definition of $\mathcal{A}^{+}_{xy}$ and Lemma 3.3.4,

$$\langle \widehat{z}, \widehat{x} \rangle = [n][k-1] - [k]^2. \tag{4.12}$$

By the above comments,

$$\left\langle A^{+}_{xy}, \widehat{x} \right\rangle = |\mathcal{A}^{+}_{xy}| \Big([n][k-1] - [k]^2 \Big). \tag{4.13}$$

In (4.13), we evaluate $|\mathcal{A}^{+}_{xy}|$ using (4.1); this yields (4.8).

We have now verified (4.8). Equations (4.9) and (4.10) are obtained in a similar fashion. □

Lemma 4.2.4 *For $x, y \in X$ such that $1 < \partial(x,y) < k$, we have*

$$\left\langle A^{+}_{xy}, \widehat{y} \right\rangle = q^{i+1}[i][n-k-i]\Big([n][k-i] - [k]^2 \Big), \tag{4.14}$$

$$\left\langle A^{0}_{xy}, \widehat{y} \right\rangle = (q-1)[i]^2\Big([n][k-i] - [k]^2 \Big), \tag{4.15}$$

$$\left\langle A^{-}_{xy}, \widehat{y} \right\rangle = q^{i+1}[i][k-i]\Big([n][k-i] - [k]^2 \Big), \tag{4.16}$$

where $i = \partial(x,y)$.

Proof. We first prove (4.14). Using the first equation in (4.7), we obtain

$$\left\langle A^{+}_{xy}, \widehat{y} \right\rangle = \sum_{z \in \mathcal{A}^{+}_{xy}} \langle \widehat{z}, \widehat{y} \rangle. \tag{4.17}$$

Pick $z \in \mathcal{A}^{+}_{xy}$. By the definition of $\mathcal{A}^{+}_{xy}$ and Lemma 3.3.4,

$$\langle \widehat{z}, \widehat{y} \rangle = [n][k-i] - [k]^2. \tag{4.18}$$

By the above comments,

$$\left\langle A^{+}_{xy}, \widehat{y} \right\rangle = |\mathcal{A}^{+}_{xy}| \Big([n][k-i] - [k]^2 \Big). \tag{4.19}$$

In (4.19), we evaluate $|\mathcal{A}^{+}_{xy}|$ using (4.1); this yields (4.14).

We have now verified (4.14). Equations (4.15) and (4.16) are obtained in a similar fashion.
□

Lemma 4.2.5 *For $x, y \in X$ such that $1 < \partial(x,y) < k$, we have*

$$\left\langle A_{xy}^{+}, \widehat{x \cap y} \right\rangle = q^{k+i+1}[i][n-k-i][k-i][n-k], \tag{4.20}$$

$$\left\langle A_{xy}^{0}, \widehat{x \cap y} \right\rangle = q^{k}(q-1)[i]^2[k-i][n-k], \tag{4.21}$$

$$\left\langle A_{xy}^{-}, \widehat{x \cap y} \right\rangle = q^{i+1}[i][k-i]\Big([n][k-i-1] - [k-i][k]\Big), \tag{4.22}$$

where $i = \partial(x,y)$.

Proof. We first prove (4.20). Using the first equation in (4.7), we obtain

$$\left\langle A_{xy}^{+}, \widehat{x \cap y} \right\rangle = \sum_{z \in \mathcal{A}_{xy}^{+}} \left\langle \hat{z}, \widehat{x \cap y} \right\rangle. \tag{4.23}$$

Pick $z \in \mathcal{A}_{xy}^{+}$. By the definition of $\mathcal{A}_{xy}^{+}$ and Lemma 3.3.2,

$$\left\langle \hat{z}, \widehat{x \cap y} \right\rangle = [n][k-i] - [k][k-i] = q^k[k-i][n-k]. \tag{4.24}$$

By the above comments,

$$\left\langle A_{xy}^{+}, \widehat{x \cap y} \right\rangle = \left|\mathcal{A}_{xy}^{+}\right| q^k[k-i][n-k]. \tag{4.25}$$

In (4.25), we evaluate $\left|\mathcal{A}_{xy}^{+}\right|$ using (4.1); this yields (4.20).

We have now verified (4.20). Equation (4.21) is obtained in a similar fashion.

Next we prove (4.22). Using the last equation in (4.7), we obtain

$$\left\langle A_{xy}^{-}, \widehat{x \cap y} \right\rangle = \sum_{z \in \mathcal{A}_{xy}^{-}} \left\langle \hat{z}, \widehat{x \cap y} \right\rangle. \tag{4.26}$$

Pick $z \in \mathcal{A}_{xy}^{-}$. By the definition of $\mathcal{A}_{xy}^{-}$ and Lemma 2.3.2,

$$\dim(z \cap x \cap y) = k-i-1. \tag{4.27}$$

By (4.27) and Lemma 3.3.2,

$$\left\langle \hat{z}, \widehat{x \cap y} \right\rangle = [n][k-i-1] - [k-i][k]. \tag{4.28}$$

By the above comments,

$$\left\langle A_{xy}^{-}, \widehat{x \cap y} \right\rangle = \left|\mathcal{A}_{xy}^{-}\right|\Big([n][k-i-1] - [k-i][k]\Big). \tag{4.29}$$

In (4.29), we evaluate $\left|\mathcal{A}_{xy}^{-}\right|$ using (4.1); this yields (4.22). □

Lemma 4.2.6 *For $x, y \in X$ such that $1 < \partial(x,y) < k$, we have*

$$\left\langle A^+_{xy}, \widehat{x+y} \right\rangle = q^{i+1}[i][n-k-i]\Big([n][k-1] - [k][k+i]\Big), \tag{4.30}$$

$$\left\langle A^0_{xy}, \widehat{x+y} \right\rangle = q^{k+i}(q-1)[i]^2[k][n-k-i], \tag{4.31}$$

$$\left\langle A^-_{xy}, \widehat{x+y} \right\rangle = q^{k+2i+1}[i][k-i][k][n-k-i], \tag{4.32}$$

where $i = \partial(x,y)$.

Proof. We first prove (4.30). Using the first equation in (4.7), we obtain

$$\left\langle A^+_{xy}, \widehat{x+y} \right\rangle = \sum_{z \in \mathcal{A}^+_{xy}} \left\langle \widehat{z}, \widehat{x+y} \right\rangle. \tag{4.33}$$

Pick $z \in \mathcal{A}^+_{xy}$. By the definition of $\mathcal{A}^+_{xy}$ and Lemma 2.3.2,

$$\dim\big(z \cap (x+y)\big) = k-1. \tag{4.34}$$

By (4.34) and Lemma 3.3.2,

$$\left\langle \widehat{z}, \widehat{x+y} \right\rangle = [n][k-1] - [k][k+i]. \tag{4.35}$$

By the above comments,

$$\left\langle A^+_{xy}, \widehat{x+y} \right\rangle = \left|\mathcal{A}^+_{xy}\right|\Big([n][k-1] - [k][k+i]\Big). \tag{4.36}$$

In (4.36), we evaluate $\left|\mathcal{A}^+_{xy}\right|$ using (4.1); this yields (4.30).

Next we prove (4.31). Using the second equation in (4.7), we obtain

$$\left\langle A^0_{xy}, \widehat{x+y} \right\rangle = \sum_{z \in \mathcal{A}^0_{xy}} \left\langle \widehat{z}, \widehat{x+y} \right\rangle. \tag{4.37}$$

Pick $z \in \mathcal{A}^0_{xy}$. By the definition of $\mathcal{A}^0_{xy}$,

$$z \cap (x+y) = z. \tag{4.38}$$

By (4.38) and Lemma 3.3.2,

$$\left\langle \widehat{z}, \widehat{x \cap y} \right\rangle = [n][k] - [k][k+i] = q^{k+i}[k][n-k-i]. \tag{4.39}$$

By the above comments,

$$\left\langle A^0_{xy}, \widehat{x+y} \right\rangle = \left|\mathcal{A}^0_{xy}\right| q^{k+i}[k][n-k-i]. \tag{4.40}$$

In (4.40), we evaluate $\left|\mathcal{A}^0_{xy}\right|$ using (4.1); this yields (4.31).

We have now verified (4.31). Equation (4.32) is obtained in a similar fashion. □

Theorem 4.2.7 *Let $x, y \in X$ satisfy $1 < \partial(x,y) < k$. In the following table, for each vector u in the header column, and each vector v in the header row, the (u,v)-entry of the table gives the inner product $\langle u, v\rangle$. Write $i = \partial(x,y)$.*

$\langle\ ,\ \rangle$	A^+_{xy}	A^0_{xy}	A^-_{xy}
$\widehat{x}$	$q^{i+1}[i][n-k-i]\big([n][k-1]-[k]^2\big)$	$(q-1)[i]^2\big([n][k-1]-[k]^2\big)$	$q^{i+1}[i][k-i]\big([n][k-1]-[k]^2\big)$
$\widehat{y}$	$q^{i+1}[i][n-k-i]\big([n][k-i]-[k]^2\big)$	$(q-1)[i]^2\big([n][k-i]-[k]^2\big)$	$q^{i+1}[i][k-i]\big([n][k-i]-[k]^2\big)$
$\widehat{x \cap y}$	$q^{k+i+1}[i][n-k-i][k-i][n-k]$	$q^k(q-1)[i]^2[k-i][n-k]$	$q^{i+1}[i][k-i]([n][k-i-1]-[k-i][k])$
$\widehat{x + y}$	$q^{i+1}[i][n-k-i]([n][k-1]-[k][k+i])$	$q^{k+i}(q-1)[i]^2[k][n-k-i]$	$q^{k+2i+1}[i][k-i][k][n-k-i]$

Proof. Combine Lemmas 4.2.3–4.2.6. □

In the next result, we write $A^+_{xy}, A^0_{xy}, A^-_{xy}$ in terms of the geometric basis for $\mathrm{Fix}(x,y)$.

Theorem 4.2.8 *For $x, y \in X$ such that $1 < \partial(x,y) < k$, we have*

$$A^+_{xy} = q^{i+1}[n-k-i][i-1]\widehat{x} + q^{2i}[n-k-i]\widehat{x\cap y} - [i]\widehat{x+y}, \tag{4.41}$$
$$A^0_{xy} = \big(q^i[i-1]-[i]\big)\widehat{x} - q^{i-1}\widehat{y} + q^{2i-1}\widehat{x\cap y} + q^{i-1}\widehat{x+y}, \tag{4.42}$$
$$A^-_{xy} = q^{i+1}[k-i][i-1]\widehat{x} - q^i[i]\widehat{x\cap y} + q^i[k-i]\widehat{x+y}, \tag{4.43}$$

where $i = \partial(x,y)$.

Proof. Write

$$A^+_{xy} = \alpha\widehat{x} + \beta\widehat{y} + \gamma\widehat{x\cap y} + \delta\widehat{x+y}, \tag{4.44}$$
$$A^0_{xy} = \alpha'\widehat{x} + \beta'\widehat{y} + \gamma'\widehat{x\cap y} + \delta'\widehat{x+y}, \tag{4.45}$$
$$A^-_{xy} = \alpha''\widehat{x} + \beta''\widehat{y} + \gamma''\widehat{x\cap y} + \delta''\widehat{x+y}, \tag{4.46}$$

for $\alpha, \beta, \gamma, \delta, \alpha', \beta', \gamma', \delta', \alpha'', \beta'', \gamma'', \delta'' \in \mathbb{R}$.

Let N_i denote the matrix of inner products from Theorem 4.2.7.

In each of (4.44), (4.45), (4.46) we take the inner product of either side with each of $\widehat{x}, \widehat{y}, \widehat{x\cap y}, \widehat{x+y}$ to obtain

$$M_i \begin{pmatrix} \alpha & \alpha' & \alpha'' \\ \beta & \beta' & \beta'' \\ \gamma & \gamma' & \gamma'' \\ \delta & \delta' & \delta'' \end{pmatrix} = N_i.$$

The matrix M_i is invertible by Lemma 3.7.10, so

$$\begin{pmatrix} \alpha & \alpha' & \alpha'' \\ \beta & \beta' & \beta'' \\ \gamma & \gamma' & \gamma'' \\ \delta & \delta' & \delta'' \end{pmatrix} = M_i^{-1} N_i.$$

Using Lemma 3.7.10 and matrix multiplication we obtain

$$M_i^{-1} N_i = \begin{pmatrix} q^{i+1}[n-k-i][i-1] & q^i[i-1]-[i] & q^{i+1}[k-i][i-1] \\ 0 & -q^{i-1} & 0 \\ q^{2i}[n-k-i] & q^{2i-1} & -q^i[i] \\ -[i] & q^{i-1} & q^i[k-i] \end{pmatrix}.$$

The result follows. □

Fix $x, y \in X$ such that $1 < \partial(x,y) < k$. Our next goal is to write $A^+_{xy}, A^0_{xy}, A^-_{xy}$ in terms of the combinatorial basis for $\mathrm{Fix}(x,y)$.

Theorem 4.2.9 *For $x, y \in X$ such that $1 < \partial(x,y) < k$, we have*

$$\begin{aligned} A^+_{xy} = & \frac{[k-1][n-k-i][n-k]}{q^{k-i-1}[n-2k]}\widehat{x} + \frac{[k][n-k-i]}{q^{k-2i+1}[i-1][n-2k]}\widehat{y} \\ & - \frac{[n-k]}{q^k[n-2k]}B_{xy} - \frac{[k][n-k-i]}{q^{k-i}[i-1][n-2k]}C_{xy}, \end{aligned} \tag{4.47}$$

$$A^0_{xy} = -[i]\widehat{x} - \frac{q^{i-1}[i]}{[i-1]}\widehat{y} + \frac{q^{i-1}}{[i-1]}C_{xy}, \tag{4.48}$$

$$\begin{aligned} A^-_{xy} = & -\frac{[k-i][k][n-k-1]}{q^{k-i-1}[n-2k]}\widehat{x} - \frac{[k-i][n-k]}{q^{k-2i+1}[i-1][n-2k]}\widehat{y} \\ & + \frac{[k]}{q^k[n-2k]}B_{xy} + \frac{[k-i][n-k]}{q^{k-i}[i-1][n-2k]}C_{xy}, \end{aligned} \tag{4.49}$$

where $i = \partial(x,y)$.

Proof. We first prove (4.47). In the equation (4.41), eliminate $\widehat{x \cap y}$ and $\widehat{x+y}$ using Lemma 3.8.4 and simplify the result.

We have now verified (4.47). Equations (4.48) and (4.49) are obtained in a similar fashion. □

4.3 Some inner products involving the y-partition of $\Gamma(x)$

Pick $x, y \in X$ such that $1 < \partial(x,y) < k$. In this section we calculate the inner products between the vectors $\widehat{x}, \widehat{y}, B_{xy}, C_{xy}, A^+_{xy}, A^0_{xy}, A^-_{xy}$.

Theorem 4.3.1 *Let $x, y \in X$ satisfy $1 < \partial(x,y) < k$. In the following table, for each vector u in the header column, and each vector v in the header row, the (u,v)-entry of the table gives the inner product $\langle u, v \rangle$. Write $i = \partial(x,y)$.*

$\langle\ ,\ \rangle$	A^+_{xy}	A^0_{xy}	A^-_{xy}
$\widehat{x}$	$q^{i+1}[i][n-k-i]\big([n][k-1]-[k]^2\big)$	$(q-1)[i]^2\big([n][k-1]-[k]^2\big)$	$q^{i+1}[i][k-i]\big([n][k-1]-[k]^2\big)$
$\widehat{y}$	$q^{i+1}[i][n-k-i]\big([n][k-i]-[k]^2\big)$	$(q-1)[i]^2\big([n][k-i]-[k]^2\big)$	$q^{i+1}[i][k-i]\big([n][k-i]-[k]^2\big)$
B_{xy}	$q^{2i+2}[i][k-i][n-k-i] \Big(\big(q^i[n-k-i]-1\big)\big([n][k-2]-[k]^2\big) +\big([n][k-1]-[k]^2\big)\Big)$	$q^{2i+1}(q-1)[k-i][n-k-i] [i]^2\Big([n][k-2]-[k]^2\Big)$	$q^{2i+2}[i][k-i][n-k-i] \Big(\big(q^i[k-i]-1\big)\big([n][k-2]-[k]^2\big) +\big([n][k-1]-[k]^2\big)\Big)$
C_{xy}	$q^{i+1}[n-k-i][i]^2 \Big(q^{k-2}[n]+[i]\big([n][k-2]-[k]^2\big)\Big)$	$(q-1)[i]^2\Big(q^{k-2}[n](2[i]-1) +[i]^2\big([n][k-2]-[k]^2\big)\Big)$	$q^{i+1}[k-i][i]^2 \Big(q^{k-2}[n]+[i]\big([n][k-2]-[k]^2\big)\Big)$

Proof. The entries in the first two rows are immediate from Theorem 4.2.7.

Next we calculate the inner product $\Big\langle B_{xy}, A^+_{xy}\Big\rangle$.

Using (4.41),

$$\Big\langle B_{xy}, A^+_{xy}\Big\rangle = q^{i+1}[n-k-i][i-1]\langle B_{xy}, \widehat{x}\rangle + q^{2i}[n-k-i]\Big\langle B_{xy}, \widehat{x\cap y}\Big\rangle - [i]\Big\langle B_{xy}, \widehat{x+y}\Big\rangle.$$

In the above equation, evaluate the right-hand side using Lemma 3.7.9.

We have now calculated the inner product $\Big\langle B_{xy}, A^+_{xy}\Big\rangle$. For the other inner products the calculations are similar, and omitted. □

Theorem 4.3.2 *Let $x, y \in X$ satisfy $1 < \partial(x,y) < k$. In the following table, for each vector u in the header column, and each vector v in the header row, the (u,v)-entry of the table gives the inner product $\langle u, v\rangle$. Write $i = \partial(x,y)$.*

$\langle\ ,\ \rangle$	A^+_{xy}	A^0_{xy}	A^-_{xy}
A^+_{xy}	$q^{2i+2}[i][n-k-i]\Big(q^{k-i-2}[n][n-k] +[i][n-k-i]\big([n][k-2]-[k]^2\big)\Big)$	$q^{i+1}(q-1)[n-k-i][i]^2 \Big(q^{k-2}[n]+[i]\big([n][k-2]-[k]^2\big)\Big)$	$q^{2i+2}[k-i][n-k-i] [i]^2\big([n][k-2]-[k]^2\big)$
A^0_{xy}	$q^{i+1}(q-1)[n-k-i][i]^2 \Big(q^{k-2}[n]+[i]\big([n][k-2]-[k]^2\big)\Big)$	$(q-1)[i]^2\Big(q^{k-2}[n]\Big(2(q-1)[i]+1\Big) +(q-1)[i]^2\big([n][k-2]-[k]^2\big)\Big)$	$q^{i+1}(q-1)[k-i][i]^2 \Big(q^{k-2}[n]+[i]\big([n][k-2]-[k]^2\big)\Big)$
A^-_{xy}	$q^{2i+2}[k-i][n-k-i] [i]^2\big([n][k-2]-[k]^2\big)$	$q^{i+1}(q-1)[k-i][i]^2 \Big(q^{k-2}[n]+[i]\big([n][k-2]-[k]^2\big)\Big)$	$q^{2i+2}[i][k-i]\Big(q^{k-i-2}[n][k] +[i][k-i]\big([n][k-2]-[k]^2\big)\Big)$

Proof. We will calculate the inner product $\Big\langle A^+_{xy}, A^+_{xy}\Big\rangle$.

Using (4.41),

$$\Big\langle A^+_{xy}, A^+_{xy}\Big\rangle = q^{i+1}[n-k-i][i-1]\Big\langle A^+_{xy}, \widehat{x}\Big\rangle + q^{2i}[n-k-i]\Big\langle A^+_{xy}, \widehat{x\cap y}\Big\rangle - [i]\Big\langle A^+_{xy}, \widehat{x+y}\Big\rangle.$$

In the above equation, evaluate the right-hand side using Theorem 4.2.7.

We have now calculated the inner product $\left\langle A^+_{xy}, A^+_{xy}\right\rangle$. For the other inner products the calculations are similar, and omitted. □

4.4 Some combinatorics and algebra involving the y-partition of $\Gamma(x)$

Let $x, y \in X$ satisfy $1 < \partial(x,y) < k$. In (4.6) we partitioned $\Gamma(x)$ into five orbits for $\mathrm{Stab}(x,y)$. In this section, we describe the edges between pairs of orbits in this partition. In this description we use a 5 by 5 matrix. We find the eigenvalues of this matrix. For each eigenvalue we display a row eigenvector and column eigenvector.

Theorem 4.4.1 *Let $x, y \in X$ satisfy $1 < \partial(x,y) < k$, and consider the orbits of* $\mathrm{Stab}(x,y)$ *on $\Gamma(x)$. Referring to the table below, for each orbit $\mathcal{O}$ in the header column, and each orbit $\mathcal{N}$ in the header row, the $(\mathcal{O},\mathcal{N})$-entry gives the number of vertices in $\mathcal{N}$ that are adjacent to a given vertex in $\mathcal{O}$. Write $i = \partial(x,y)$.*

	$\mathcal{B}_{xy}$	$\mathcal{C}_{xy}$	$\mathcal{A}^+_{xy}$	$\mathcal{A}^0_{xy}$	$\mathcal{A}^-_{xy}$
$\mathcal{B}_{xy}$	$q^{i+1}[k-i]+q^{i+1}[n-k-i]-q-1$	0	$q[i]$	0	$q[i]$
$\mathcal{C}_{xy}$	0	$2q[i-1]$	$q^{i+1}[n-k-i]$	$(q-1)(2[i]-1)$	$q^{i+1}[k-i]$
$\mathcal{A}^+_{xy}$	$q^{i+1}[k-i]$	$[i]$	$q[n-k]-q-1$	$(q-1)[i]$	0
$\mathcal{A}^0_{xy}$	0	$2[i]-1$	$q^{i+1}[n-k-i]$	$(q-1)(2[i]-1)-1$	$q^{i+1}[k-i]$
$\mathcal{A}^-_{xy}$	$q^{i+1}[n-k-i]$	$[i]$	0	$(q-1)[i]$	$q[k]-q-1$

Proof. We will verify the $(\mathcal{B}_{xy},\mathcal{B}_{xy})$-entry of the table. Pick a vertex $w \in \mathcal{B}_{xy}$. Let # denote the number of vertices in $\mathcal{B}_{xy}$ that are adjacent to w. Note that # is independent of the choice of w, because the partition (4.6) is equitable.

We now compute #. By construction, each vertex in $\mathcal{B}_{xy}$ is at distance at most 2 from w. Using Lemma 3.3.4 and Corollary 3.3.5, we obtain

$$\left\langle \hat{w}, B_{xy}\right\rangle = \sum_{z\in\mathcal{B}_{xy}} \left\langle \hat{w}, \hat{z}\right\rangle = q^k[k][n-k] + \#\Big([n][k-1]-[k]^2\Big) + \Big(|\mathcal{B}_{xy}| - \# - 1\Big)\Big([n][k-2]-[k]^2\Big). \tag{4.50}$$

By construction,

$$\left\langle B_{xy}, B_{xy}\right\rangle = |\mathcal{B}_{xy}|\left\langle \hat{w}, B_{xy}\right\rangle. \tag{4.51}$$

We now evaluate (4.51). The left-hand side is evaluated using the (B_{xy}, B_{xy})-entry in the table of Lemma 3.7.15. The right-hand side is evaluated using (4.50) and $b_i = |\mathcal{B}_{xy}|$; the value of b_i is given in (2.4).

After evaluating (4.51), we solve the resulting equation for #; this yields the $(\mathcal{B}_{xy}, \mathcal{B}_{xy})$-entry of the table. The other entries are obtained in a similar fashion. □

Definition 4.4.2 For $1 < i < k$ let $\mathcal{M}_i$ denote the 5×5 matrix in Theorem 4.4.1.

Note that $\mathcal{M}_i$ is not symmetric. We now give the transpose $\mathcal{M}_i^t$.

Lemma 4.4.3 *For $1 < i < k$, we have $\mathcal{M}_i^t = D\mathcal{M}_i D^{-1}$, where $D = \operatorname{diag}(b_i, c_i, a_i^+, a_i^0, a_i^-)$. Recall b_i, c_i from (2.4) and a_i^+, a_i^0, a_i^- from (4.2).*

Proof. Immediate. □

Our next goal is to find the eigenvalues of $\mathcal{M}_i$. For each eigenvalue we display a row eigenvector and a column eigenvector.

Lemma 4.4.4 *For $1 < i < k$, the eigenvalues of the matrix $\mathcal{M}_i$ are*

$$a_1, \qquad q[n-k]-q-1, \qquad q[k]-q-1, \qquad -1, \qquad -q-1,$$

where $a_1 = q[k] + q[n-k] - q - 1$.

Proof. Routine. □

Lemma 4.4.5 *For $1 < i < k$ we consider the matrix $\mathcal{M}_i$. In the table below, for each eigenvalue of $\mathcal{M}_i$, we display a corresponding row eigenvector and column eigenvector. Recall b_i, c_i from (2.4) and a_i^+, a_i^0, a_i^- from (4.2).*

Eigenvalue of $\mathcal{M}_i$	*corresponding row eigenvector*	*corresponding column eigenvector*
a_1	$(b_i, c_i, a_i^+, a_i^0, a_i^-)$	$(1,1,1,1,1)^t$
$q[n-k]-q-1$	$(a_i^+, -c_i, -a_i^+, -a_i^0, qc_i)$	$(qc_i, -a_i^-, -a_i^-, -a_i^-, qc_i)^t$
$q[k]-q-1$	$(a_i^-, -c_i, qc_i, -a_i^0, -a_i^-)$	$(qc_i, -a_i^+, qc_i, -a_i^+, -a_i^+)^t$
-1	$(0,1,0,-1,0)$	$(0, q-1, 0, -1, 0)^t$
$-q-1$	$(q, 1, -q, q-1, -q)$	$(qc_i, b_i, -a_i^-, b_i, -a_i^+)^t$

Proof. Routine. □

Remark 4.4.1 [3, 10] For $x \in X$ the spectrum of the local graph $\Gamma(x)$ is given in the table below. Recall $a_1 = q[k] + q[n-k] - q - 1$.

Eigenvalue	Multiplicity
a_1	1
$q[n-k]-q-1$	$[k]-1$
$q[k]-q-1$	$[n-k]-1$
-1	$(q-1)[k][n-k]$
$-q-1$	$q^2[k-1][n-k-1]$

Chapter 5

A generalization of the Askey-Wilson relations using a projective geometry

Recall the projective geometry P. In this chapter we define some matrices $A, A^* \in \mathrm{Mat}_P(\mathbb{C})$ that satisfy a generalization of the Askey-Wilson relations. To define the matrices A, A^*, we bring in some subalgebra $\mathcal{H}$ of $\mathrm{Mat}_P(\mathbb{C})$. We discuss the irreducible $\mathcal{H}$-submodules in the standard $\mathrm{Mat}_P(\mathbb{C})$-module.

5.1 An algebra $\mathcal{H}$

For the rest of the chapter, write $h = n - k$. Note that $\dim \mathcal{V} = h + k$. Let $\mathrm{Mat}_P(\mathbb{C})$ denote the $\mathbb{C}$-algebra consisting of the matrices with rows and columns indexed by P and all entries in $\mathbb{C}$. Let V denote the $\mathbb{C}$-vector space consisting of the column vectors with rows indexed by P and all entries in $\mathbb{C}$. The algebra $\mathrm{Mat}_P(\mathbb{C})$ acts on V by left multiplication. The $\mathrm{Mat}_P(\mathbb{C})$-module V is called *standard*. For $u \in P$, define the column vector $\widehat{u} \in V$ that has u-entry 1 and all other entries 0. The vectors $\{\widehat{u} \mid u \in P\}$ form a basis for V.

Definition 5.1.1 For $0 \leqslant \ell \leqslant h + k$ define a diagonal matrix $E^*_\ell \in \mathrm{Mat}_P(\mathbb{C})$ as follows. For $u \in P$, the (u, u)-entry of E^*_ℓ is

$$\left(E^*_\ell\right)_{u,u} = \begin{cases} 1 & \text{if } u \in P_\ell, \\ 0 & \text{if } u \notin P_\ell. \end{cases}$$

Recall the partition (2.6). We have

$$I = \sum_{\ell=0}^{h+k} E^*_\ell,$$

$$E^*_i E^*_j = \delta_{i,j} E^*_i \qquad\qquad (0 \leqslant i, j \leqslant h + k).$$

For $0 \leqslant \ell \leqslant h + k$,

$$E^*_\ell V = \mathrm{Span}\{\widehat{u} \mid u \in P_\ell\}.$$

The following sum is direct:

$$V = \sum_{\ell=0}^{h+k} E^*_\ell V.$$

For the rest of this chapter, we fix $y \in P_k$.

Definition 5.1.2 For $0 \leqslant i \leqslant k$ and $0 \leqslant j \leqslant h$, define a diagonal matrix $E^*_{i,j} \in \mathrm{Mat}_P(\mathbb{C})$ as follows. For $u \in P$, the (u,u)-entry of $E^*_{i,j}$ is

$$\big(E^*_{i,j}\big)_{u,u} = \begin{cases} 1 & \text{if } u \in P_{i,j}, \\ 0 & \text{if } u \notin P_{i,j}. \end{cases}$$

For notational convenience, for integers r, s we define $E^*_{r,s} = 0$ unless $0 \leqslant r \leqslant k$ and $0 \leqslant s \leqslant h$.

For $0 \leqslant i \leqslant k$ and $0 \leqslant j \leqslant h$, we have

$$E^*_{i,j} V = \mathrm{Span}\{\hat{u} \mid u \in P_{i,j}\}.$$

Recall the partition (2.9). The following sum is direct:

$$V = \sum_{i=0}^{k} \sum_{j=0}^{h} E^*_{i,j} V.$$

Recall the partition (2.10). For $0 \leqslant \ell \leqslant h+k$,

$$E^*_\ell = \sum_{i,j} E^*_{i,j},$$

$$E^*_\ell V = \sum_{i,j} E^*_{i,j} V, \qquad \text{(direct sum)}$$

where the sums are over the ordered pairs (i,j) such that $0 \leqslant i \leqslant k$ and $0 \leqslant j \leqslant h$ and $i + j = \ell$.

Next we define a subalgebra $\mathcal{K}$ of $\mathrm{Mat}_P(\mathbb{C})$.

Definition 5.1.3 [10, Definition 6.1] The matrices

$$E^*_{i,j} \qquad (0 \leqslant i \leqslant k, \ \ 0 \leqslant j \leqslant h)$$

form a basis for a commutative subalgebra of $\mathrm{Mat}_P(\mathbb{C})$. Denote this subalgebra by $\mathcal{K}$.

Our next goal is to describe a generating set for $\mathcal{K}$. For the rest of the thesis, $q^{1/2}$ denotes the positive square root of q.

Definition 5.1.4 We define diagonal matrices $K_1, K_2 \in \mathrm{Mat}_P(\mathbb{C})$ as follows. For $u \in P$ the (u,u)-entry is

$$\big(K_1\big)_{u,u} = q^{\frac{k}{2}-i}, \qquad \big(K_2\big)_{u,u} = q^{j-\frac{h}{2}},$$

where $u \in P_{i,j}$.

Note that K_1, K_2 are invertible. By [10, Prop. 6.3], the algebra $\mathcal{K}$ is generated by $K_1^{\pm 1}, K_2^{\pm 1}$.

Lemma 5.1.5 *For $0 \leqslant i \leqslant k$ and $0 \leqslant j \leqslant h$, the subspace $E^*_{i,j}V$ is a common eigenspace for $K_1^{\pm 1}, K_2^{\pm 1}$. The corresponding eigenvalues are given in the table below.*

Element in $\mathcal{K}$	*Eigenvalue corresponding to $E^*_{i,j}V$*
K_1	$q^{\frac{k}{2}-i}$
K_1^{-1}	$q^{i-\frac{k}{2}}$
K_2	$q^{j-\frac{h}{2}}$
K_2^{-1}	$q^{\frac{h}{2}-j}$

Proof. Immediate from Definition 5.1.4. □

Next we define some matrices in $\mathrm{Mat}_P(\mathbb{C})$ that will be useful. Recall the notion of /-cover and \-cover from Definition 2.3.10.

Definition 5.1.6 We define matrices $L_1, L_2, R_1, R_2 \in \mathrm{Mat}_P(\mathbb{C})$ as follows. For $u, v \in P$ their (u, v)-entries are

$$\left(L_1\right)_{u,v} = \begin{cases} 1 & \text{if } v \text{ /-covers } u, \\ 0 & \text{if } v \text{ does not /-cover } u, \end{cases}$$

$$\left(L_2\right)_{u,v} = \begin{cases} 1 & \text{if } v \text{ \textbackslash-covers } u, \\ 0 & \text{if } v \text{ does not \textbackslash-cover } u, \end{cases}$$

$$\left(R_1\right)_{u,v} = \begin{cases} 1 & \text{if } u \text{ /-covers } v, \\ 0 & \text{if } u \text{ does not /-cover } v, \end{cases}$$

$$\left(R_2\right)_{u,v} = \begin{cases} 1 & \text{if } u \text{ \textbackslash-covers } v, \\ 0 & \text{if } u \text{ does not \textbackslash-cover } v. \end{cases}$$

Note that $R_1 = L_1^t$ and $R_2 = L_2^t$, where t is the matrix-transpose.

Lemma 5.1.7 *For $v \in P$,*

$$L_1\hat{v} = \sum_{v \text{ /-covers } u} \hat{u}, \qquad L_2\hat{v} = \sum_{v \text{ \textbackslash-covers } u} \hat{u},$$

$$R_1\hat{v} = \sum_{u \text{ /-covers } v} \hat{u}, \qquad R_2\hat{v} = \sum_{u \text{ \textbackslash-covers } v} \hat{u}.$$

Proof. Immediate from Definition 5.1.6. □

Lemma 5.1.8 *For* $0 \leqslant i \leqslant k$ *and* $0 \leqslant j \leqslant h$,

$$L_1 E^*_{i,j} V \subseteq E^*_{i-1,j} V, \qquad L_2 E^*_{i,j} V \subseteq E^*_{i,j-1} V, \tag{5.1}$$

$$R_1 E^*_{i,j} V \subseteq E^*_{i+1,j} V, \qquad R_2 E^*_{i,j} V \subseteq E^*_{i,j+1} V. \tag{5.2}$$

Proof. Immediate from Lemma 5.1.7. □

Definition 5.1.9 Let $\mathcal{H}$ denote the subalgebra of $\mathrm{Mat}_P(\mathbb{C})$ generated by L_1, L_2, R_1, R_2, $K_1^{\pm 1}$, $K_2^{\pm 1}$.

By construction, the vector space V is an $\mathcal{H}$-module. Let W denote an $\mathcal{H}$-submodule in V. We say that W is *irreducible* whenever W does not contain an $\mathcal{H}$-submodule besides 0 or W. Note that $\mathcal{H}$ is closed under the conjugate-transpose map. Therefore the $\mathcal{H}$-module V is a direct sum of irreducible $\mathcal{H}$-modules.

Lemma 5.1.10 [10, Lemma 8.2, 8.5, 8.11] *Let* W *denote an irreducible* $\mathcal{H}$*-module. Then there exist integers* α, β, ρ *such that the following* (i), (ii) *hold:*

(i)

$$0 \leqslant \rho, \qquad 0 \leqslant \alpha \leqslant \frac{k-\rho}{2}, \qquad 0 \leqslant \beta \leqslant \frac{h-\rho}{2}; \tag{5.3}$$

(ii) *for* $0 \leqslant i \leqslant k$ *and* $0 \leqslant j \leqslant h$,

$$\dim E^*_{i,j} W = \begin{cases} 1 & \text{if } \alpha \leqslant i \leqslant k-\rho-\alpha,\ \rho+\beta \leqslant j \leqslant h-\beta, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 5.1.11 Let W denote an irreducible $\mathcal{H}$-module. Referring to Lemma 5.1.10, we call the triple (α, β, ρ) the *type of* W.

Lemma 5.1.12 [3, p. 133] *For each triple* (α, β, ρ) *of integers that satisfy (5.3), there exists an irreducible* $\mathcal{H}$*-module of type* (α, β, ρ)*. This* $\mathcal{H}$*-module is unique up to isomorphism of* $\mathcal{H}$*-modules.*

We illustrate Lemma 5.1.10 using the diagram below.

$P:$

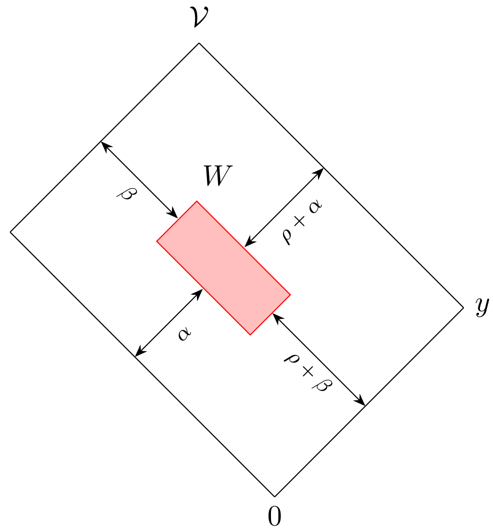

Figure 5.1: The irreducible $\mathcal{H}$-module W is represented by the red rectangle.

Our next goal is to describe the action of the $\mathcal{H}$-generators on the irreducible $\mathcal{H}$-modules. For the rest of this section, let W denote an irreducible $\mathcal{H}$-module of type (α, β, ρ).

Lemma 5.1.13 *For $\alpha \leqslant i \leqslant k - \rho - \alpha$ and $\rho + \beta \leqslant j \leqslant h - \beta$, the subspace $E^*_{i,j}W$ is a common eigenspace for $K_1^{\pm 1}, K_2^{\pm 1}$. The corresponding eigenvalues are given in the table below.*

Element in $\mathcal{H}$	*Eigenvalue corresponding to $E^*_{i,j}W$*
K_1	$q^{\frac{k}{2}-i}$
K_1^{-1}	$q^{i-\frac{k}{2}}$
K_2	$q^{j-\frac{h}{2}}$
K_2^{-1}	$q^{\frac{h}{2}-j}$

Proof. Immediate from Lemma 5.1.5. □

Lemma 5.1.14 *For $\alpha \leqslant i \leqslant k - \rho - \alpha$ and $\rho + \beta \leqslant j \leqslant h - \beta$,*

$$L_1 E^*_{i,j}W \subseteq E^*_{i-1,j}W, \qquad L_2 E^*_{i,j}W \subseteq E^*_{i,j-1}W, \tag{5.4}$$

$$R_1 E^*_{i,j}W \subseteq E^*_{i+1,j}W, \qquad R_2 E^*_{i,j}W \subseteq E^*_{i,j+1}W. \tag{5.5}$$

Proof. Immediate from Lemma 5.1.8. □

The following result is a variation on [10, Prop. 8.7].

Lemma 5.1.15 *The irreducible $\mathcal{H}$-module W has a basis*

$$w_{i,j} \qquad (\alpha \leqslant i \leqslant k-\rho-\alpha, \quad \rho+\beta \leqslant j \leqslant h-\beta) \tag{5.6}$$

with the following properties. For $\alpha \leqslant i \leqslant k-\rho-\alpha$ and $\rho+\beta \leqslant j \leqslant h-\beta$,

$$w_{i,j} \in E^*_{i,j}W,$$

$$L_1 w_{i,j} = q^{\frac{\rho+\alpha+\beta+i+j-1}{2}}[k-\rho-\alpha-i+1]w_{i-1,j}, \tag{5.7}$$

$$L_2 w_{i,j} = q^{k-\frac{\rho+\alpha-\beta+i-j+1}{2}}[h-\beta-j+1]w_{i,j-1}, \tag{5.8}$$

$$R_1 w_{i,j} = q^{-\frac{\rho-\alpha+\beta+i-j}{2}}[i-\alpha+1]w_{i+1,j}, \tag{5.9}$$

$$R_2 w_{i,j} = q^{\frac{\rho+\alpha+\beta-i-j}{2}}[j-\rho-\beta+1]w_{i,j+1}. \tag{5.10}$$

In the above lines, we interpret $w_{r,s} = 0$ unless $\alpha \leqslant r \leqslant k-\rho-\alpha$ and $\rho+\beta \leqslant s \leqslant h-\beta$.

Proof. Let

$$w'_{n,m} \qquad (0 \leqslant n \leqslant k-2\alpha-\rho, \quad 0 \leqslant m \leqslant h-2\beta-\rho)$$

denote the basis from [10, Prop. 8.7]. Note that

$$w'_{n,m} \in E^*_{\alpha+n,\, \rho+\beta+m}W.$$

Define

$$w_{i,j} = q^{-\frac{(i-\alpha)(j-\rho-\beta)}{2}} w'_{i-\alpha,\, j-\rho-\beta} \qquad (\alpha \leqslant i \leqslant k-\rho-\alpha, \quad \rho+\beta \leqslant j \leqslant h-\beta). \tag{5.11}$$

Then we have

$$w_{i,j} \in E^*_{i,j}W.$$

By Lemma 5.1.10(ii), the vectors in (5.11) form a basis for W. Equations (5.7)–(5.10) follow from (5.11) and [10, Prop. 8.7]. □

Definition 5.1.16 A basis for W that satisfies Lemma 5.1.15 will be called *standard*.

Lemma 5.1.17 *For $\alpha \leqslant i \leqslant k-\rho-\alpha$ and $\rho+\beta \leqslant j \leqslant h-\beta$, the subspace $E^*_{i,j}W$ is invariant under each of*

$$L_1R_1, \qquad R_1L_1, \qquad L_2R_2, \qquad R_2L_2.$$

The corresponding eigenvalues are given in the table below.

Element in $\mathcal{H}$	*Eigenvalue corresponding to $E^*_{i,j}W$*
L_1R_1	$q^{\alpha+j}[i-\alpha+1][k-\rho-\alpha-i]$
R_1L_1	$q^{\alpha+j}[i-\alpha][k-\rho-\alpha-i+1]$
L_2R_2	$q^{k+\beta-i}[j-\rho-\beta+1][h-\beta-j]$
R_2L_2	$q^{k+\beta-i}[j-\rho-\beta][h-\beta-j+1]$

Proof. The first assertion follows from (5.4), (5.5). For the second assertion, combine (5.7)–(5.10). □

5.2 Matrices F^0, F^+, F^- and their action on irreducible $\mathcal{H}$-modules

In this section we define some matrices F^0, F^+, F^- in $\mathcal{H}$, and describe their action on the irreducible $\mathcal{H}$-modules.

Definition 5.2.1 Define matrices $F^0, F^+, F^- \in \mathrm{Mat}_P(\mathbb{C})$ as follows. For $u, v \in P$ their (u, v)-entries are

$$\left(F^0\right)_{u,v} = \begin{cases} 1 & \text{if } u+v \text{ /-covers both } u \text{ and } v, \text{ which } \backslash\text{-cover } u \cap v, \\ 0 & \text{otherwise,} \end{cases}$$

$$\left(F^+\right)_{u,v} = \begin{cases} 1 & \text{if } u+v \ \backslash\text{-covers both } u \text{ and } v, \\ 0 & \text{otherwise,} \end{cases}$$

$$\left(F^-\right)_{u,v} = \begin{cases} 1 & \text{if both } u \text{ and } v \text{ /-cover } u \cap v, \\ 0 & \text{otherwise.} \end{cases}$$

Our next goal is to show that F^0, F^+, F^- are contained in $\mathcal{H}$. To do this, we write each of F^0, F^+, F^- in terms of the generators of $\mathcal{H}$.

Lemma 5.2.2 *The matrix F^0 satisfies*

$$F^0 = L_1R_1 - R_1L_1 + (q-1)^{-1}\Big(q^{\frac{h+k}{2}} K_1^{-1}K_2 - q^{\frac{k}{2}}K_1 - q^{\frac{h}{2}}K_2 + I\Big) \tag{5.12}$$

$$= R_2L_2 - L_2R_2 + (q-1)^{-1}\Big(q^{\frac{h+k}{2}} K_1K_2^{-1} - q^{\frac{k}{2}}K_1 - q^{\frac{h}{2}}K_2 + I\Big). \tag{5.13}$$

Proof. We first prove (5.12). For $u, v \in P$ we calculate the (u, v)-entry of the right-hand side of (5.12). By Lemma 2.3.11(iii), the (u, v)-entry of L_1R_1 is

$$\left(L_1R_1\right)_{u,v} = \begin{cases} [k-i] & \text{if } u = v \text{ and } u \in P_{i,j} \ (0 \leqslant i \leqslant k,\ 0 \leqslant j \leqslant h), \\ 1 & \text{if } u+v \text{ /-covers } u \text{ and } u+v \text{ /-covers } v, \\ 0 & \text{otherwise.} \end{cases} \tag{5.14}$$

By Lemma 2.3.11(i), the (u, v)-entry of R_1L_1 is

$$\left(R_1L_1\right)_{u,v} = \begin{cases} q^j[i] & \text{if } u = v \text{ and } u \in P_{i,j} \ (0 \leqslant i \leqslant k,\ 0 \leqslant j \leqslant h), \\ 1 & \text{if } u \text{ /-covers } u \cap v \text{ and } v \text{ /-covers } u \cap v, \\ 0 & \text{otherwise.} \end{cases} \tag{5.15}$$

By Definition 5.1.4 the matrix

$$(q-1)^{-1}\Big(q^{\frac{h+k}{2}} K_2K_1^{-1} - q^{\frac{k}{2}}K_1 - q^{\frac{h}{2}}K_2 + I\Big) \tag{5.16}$$

is diagonal. The (u, u)-entry of (5.16) is equal to $q^j[i] - [k-i]$, where $u \in P_{i,j}$. By these comments, the (u, v)-entry of the right-hand side of (5.12) is equal to the (u, v)-entry of the left-hand side. The result follows.

We have proved (5.12). Equation (5.13) follows from (5.12) and Lemma 6.2.4 in the appendix. □

Lemma 5.2.3 *The matrix F^+ satisfies*

$$F^+ = L_2R_2 - q^{\frac{k}{2}}(q-1)^{-1}K_1\Big(q^{\frac{h}{2}}K_2^{-1} - I\Big). \tag{5.17}$$

Proof. For $u, v \in P$ we calculate the (u, v)-entry of the right-hand side of (5.17). By Lemma 2.3.11(iv), the (u, v)-entry of L_2R_2 is

$$\big(L_2R_2\big)_{u,v} = \begin{cases} q^{k-i}[h-j] & \text{if } u = v \text{ and } u \in P_{i,j}\ (0 \leqslant i \leqslant k,\ 0 \leqslant j \leqslant h), \\ 1 & \text{if } u+v \text{ \\-covers } u \text{ and } u+v \text{ \\-covers } v, \\ 0 & \text{otherwise.} \end{cases} \tag{5.18}$$

By Definition 5.1.4 the matrix

$$q^{\frac{k}{2}}(q-1)^{-1}K_1\Big(q^{\frac{h}{2}}K_2^{-1} - I\Big) \tag{5.19}$$

is diagonal. The (u, u)-entry of (5.19) is equal to $q^{k-i}[h-j]$, where $u \in P_{i,j}$. By these comments, the (u, v)-entry on the right-hand side of (5.17) is equal to the (u, v)-entry on the left-hand side. The result follows. □

Lemma 5.2.4 *The matrix F^- satisfies*

$$F^- = R_1L_1 - q^{\frac{h}{2}}(q-1)^{-1}\Big(q^{\frac{k}{2}}K_1^{-1} - I\Big)K_2. \tag{5.20}$$

Proof. For $u, v \in P$ we calculate the (u, v)-entry of the right-hand side of (5.20). By Definition 5.1.4 the matrix

$$q^{\frac{h}{2}}(q-1)^{-1}\Big(q^{\frac{k}{2}}K_1^{-1} - I\Big)K_2 \tag{5.21}$$

is diagonal. The (u, u)-entry of (5.21) is $q^j[i]$, where $u \in P_{i,j}$. By these comments and (5.15), the (u, v)-entry on the right-hand side of (5.20) is equal to the (u, v)-entry on the left-hand side. The result follows. □

Lemma 5.2.5 *The matrices F^0, F^+, F^- are contained in $\mathcal{H}$.*

Proof. Immediate from Lemmas 5.2.2–5.2.4. □

Next we write $L_1R_1, R_1L_1, L_2R_2, R_2L_2$ in terms of $F^0, F^+, F^-, K_1^{\pm 1}, K_2^{\pm 1}$.

Lemma 5.2.6 *The following* (5.22)–(5.25) *hold:*

$$L_1R_1 = F^0 + F^- + (q-1)^{-1}\Big(q^{\frac{k}{2}}K_1 - I\Big); \tag{5.22}$$

$$R_1L_1 = F^- + q^{\frac{h}{2}}(q-1)^{-1}\Big(q^{\frac{k}{2}}K_1^{-1} - I\Big)K_2; \tag{5.23}$$

$$L_2R_2 = F^+ + q^{\frac{k}{2}}(q-1)^{-1}K_1\Big(q^{\frac{h}{2}}K_2^{-1} - I\Big); \tag{5.24}$$

$$R_2L_2 = F^0 + F^+ + (q-1)^{-1}\Big(q^{\frac{h}{2}}K_2 - I\Big). \tag{5.25}$$

Proof. Routine from Lemmas 5.2.2–5.2.4. □

Next we define a matrix F.

Definition 5.2.7 Define

$$F = F^0 + F^+ + F^-. \tag{5.26}$$

Lemma 5.2.8 *We have*

$$F = L_1R_1 + L_2R_2 - (q-1)^{-1}\Big(q^{\frac{h+k}{2}}K_1K_2^{-1} - I\Big), \tag{5.27}$$

$$= R_1L_1 + R_2L_2 - (q-1)^{-1}\Big(q^{\frac{h+k}{2}}K_1^{-1}K_2 - I\Big). \tag{5.28}$$

Proof. Equation (5.27) follows from (5.22), (5.24), (5.26). Equation (5.28) follows from (5.27) and Lemma 6.2.4 in the appendix. □

The following result gives a combinatorial interpretation of F.

Lemma 5.2.9 *The matrix F has the following entries. For $u, v \in P$, the (u, v)-entry is*

$$F_{u,v} = \begin{cases} 1 & \textit{if each of } u, v \textit{ covers } u \cap v \textit{ and } \dim(u \cap y) = \dim(v \cap y), \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. This is a routine consequence of Lemma 5.2.8. □

Next we describe the action of F^0, F^+, F^-, F on the irreducible $\mathcal{H}$-modules. Recall the standard module V.

Lemma 5.2.10 *For $0 \leqslant i \leqslant k$ and $0 \leqslant j \leqslant h$,*

$$F^0E^*_{i,j}V \subseteq E^*_{i,j}V, \qquad F^+E^*_{i,j}V \subseteq E^*_{i,j}V, \tag{5.29}$$

$$F^-E^*_{i,j}V \subseteq E^*_{i,j}V, \qquad FE^*_{i,j}V \subseteq E^*_{i,j}V. \tag{5.30}$$

Proof. For F^0 the result is immediate from (5.12), Definition 5.1.4, Lemma 5.1.8. For F^+ and F^- the proofs are similar, and omitted. For F, the result follows from (5.26), (5.29), and the left equation of (5.30). □

For the rest of this section, let W denote an irreducible $\mathcal{H}$-module of type (α, β, ρ).

Lemma 5.2.11 *For $\alpha \leqslant i \leqslant k - \rho - \alpha$ and $\rho + \beta \leqslant j \leqslant h - \beta$, the subspace $E^*_{i,j}W$ is invariant under each of F^0, F^+, F^-, F.*

Proof. Immediate from Lemma 5.2.10. □

Definition 5.2.12 For $\alpha \leqslant i \leqslant k - \rho - \alpha$ and $\rho + \beta \leqslant j \leqslant h - \beta$ let

$$a^0_{i,j}(W), \qquad a^+_{i,j}(W), \qquad a^-_{i,j}(W), \qquad a_{i,j}(W)$$

denote the eigenvalues of F^0, F^+, F^-, F respectively that correspond to $E^*_{i,j}W$.

By construction we have

$$a_{i,j}(W) = a^0_{i,j}(W) + a^+_{i,j}(W) + a^-_{i,j}(W).$$

Theorem 5.2.13 *For $\alpha \leqslant i \leqslant k-\rho-\alpha$ and $\rho+\beta \leqslant j \leqslant h-\beta$,*

$$a^0_{i,j}(W) = q^{k-i}[j-\rho]-[j], \tag{5.31}$$

$$a^+_{i,j}(W) = q^{k-i}\Big(q^{\beta+1}[j-\rho-\beta][h-\beta-j]-[\beta]\Big), \tag{5.32}$$

$$a^-_{i,j}(W) = q^{j}\Big(q^{\alpha+1}[i-\alpha][k-\rho-\alpha-i]-[\alpha]\Big). \tag{5.33}$$

Proof. We prove (5.31). Pick $w \in E^*_{i,j}W$. By (5.12),

$$F^0 w = \bigg(L_1R_1 - R_1L_1 + (q-1)^{-1}\Big(q^{\frac{h+k}{2}}K_2K_1^{-1} - q^{\frac{k}{2}}K_1 - q^{\frac{h}{2}}K_2 + I\Big)\bigg)w. \tag{5.34}$$

Evaluate the right-hand side of (5.34) using the L_1R_1, R_1L_1-entries of the table in Lemma 5.1.17 and the eigenvalues in Lemma 5.1.13. The result follows.

We have proved (5.31). The proofs for (5.32) and (5.33) are similar, and omitted. □

5.3 The center of $\mathcal{H}$

In this section we describe the center of $\mathcal{H}$.

The following result is a variation on [10, Theorem 9.3].

Theorem 5.3.1 *The center of $\mathcal{H}$ is generated by the following three elements:*

$$\Omega_0 = q^{-\frac{h+k}{2}}\Big((q-1)F^0K_1^{-1}K_2^{-1} + q^{\frac{h}{2}}K_1^{-1} + q^{\frac{k}{2}}K_2^{-1} - K_1^{-1}K_2^{-1}\Big), \tag{5.35}$$

$$\Omega_1 = q^{-\frac{h}{2}}\left(qF^0K_2^{-1} + (q-1)F^-K_2^{-1} + \frac{q^{\frac{k}{2}+1}K_1K_2^{-1} + q^{\frac{h+k}{2}+1}K_1^{-1} - qK_2^{-1}}{q-1}\right) - \frac{q}{q-1}I, \tag{5.36}$$

$$\Omega_2 = q^{-\frac{k}{2}}\left(qF^0K_1^{-1} + (q-1)F^+K_1^{-1} + \frac{q^{\frac{h}{2}+1}K_1^{-1}K_2 + q^{\frac{h+k}{2}+1}K_2^{-1} - qK_1^{-1}}{q-1}\right) - \frac{q}{q-1}I. \tag{5.37}$$

Proof. We refer to $\Lambda_0, \Lambda_1, \Lambda_2$ from [10, Section 9]. We routinely verify that

$$\Omega_0 = \Lambda_0^{-1}, \qquad \Omega_1 = q^{\frac{k-1}{2}}(q-1)\Lambda_1 - \frac{q+1}{q-1}I, \qquad \Omega_2 = q^{\frac{h-1}{2}}(q-1)\Lambda_2 - \frac{q+1}{q-1}I.$$

By [10, Theorem 9.3], the elements $\Lambda_0, \Lambda_1, \Lambda_2$ generate the center of $\mathcal{H}$. The result follows. □

Corollary 5.3.2 *For $0 \leqslant i \leqslant k$ and $0 \leqslant j \leqslant h$,*

$$\Omega_0 E^*_{i,j}V \subseteq E^*_{i,j}V, \qquad \Omega_1 E^*_{i,j}V \subseteq E^*_{i,j}V, \qquad \Omega_2 E^*_{i,j}V \subseteq E^*_{i,j}V.$$

Proof. Immediate from (5.35)–(5.37). □

Next we express F^0, F^+, F^- in terms of $\Omega_0, \Omega_1, \Omega_2, K_1^{\pm 1}, K_2^{\pm 1}$.

Lemma 5.3.3 *We have*

$$F^0 = (q-1)^{-1}\Big(q^{\frac{h+k}{2}}\Omega_0 K_1 K_2 - q^{\frac{k}{2}}K_1 - q^{\frac{h}{2}}K_2 + I\Big), \tag{5.38}$$

$$F^+ = (q-1)^{-1}\left(q^{\frac{k}{2}}\Omega_2 - (q-1)^{-1}\left(q^{\frac{h+k}{2}+1}\Big(\Omega_0 K_2 + K_2^{-1}\Big) - 2q^{\frac{k}{2}+1}I\right)\right)K_1, \tag{5.39}$$

$$F^- = (q-1)^{-1}\left(q^{\frac{h}{2}}\Omega_1 - (q-1)^{-1}\left(q^{\frac{h+k}{2}+1}\Big(\Omega_0 K_1 + K_1^{-1}\Big) - 2q^{\frac{h}{2}+1}I\right)\right)K_2. \tag{5.40}$$

Proof. Combine (5.35)–(5.37). □

We now find the action of $\Omega_0, \Omega_1, \Omega_2$ on the irreducible $\mathcal{H}$-modules.

Theorem 5.3.4 *Let W denote an irreducible $\mathcal{H}$-module of type (α, β, ρ). Then each of Ω_0, Ω_1, Ω_2 acts on W as a scalar multiple of the identity. The scalars are given in the table below.*

Central element in $\mathcal{H}$	*Scalar corresponding to W*
Ω_0	$q^{-\rho}$
Ω_1	$q[k-\rho-\alpha] + [\alpha]$
Ω_2	$q[h-\rho-\beta] + [\beta]$

Proof. We prove the Ω_0-entry of the table. For $\alpha \leqslant i \leqslant k-\rho-\alpha$ and $\rho+\beta \leqslant j \leqslant h-\beta$, pick a nonzero vector $w \in E^*_{i,j}W$. It suffices to show that

$$\Omega_0 w = q^{-\rho}w.$$

In view of (5.35), combine (5.31) and Lemma 5.1.13. The result follows.

We have proved the Ω_0-entry of the table. The proofs for the other entries are similar, and omitted. □

We finish this section with a comment.

Lemma 5.3.5 *The matrices F^0, F^+, F^- mutually commute.*

Proof. All the terms on the right-hand sides of (5.38)–(5.40) mutually commute. The result follows. □

5.4 A subalgebra $\overline{\mathcal{H}}$

In this section we define a subalgebra $\overline{\mathcal{H}}$ of $\mathcal{H}$. We find some matrices that are contained in $\overline{\mathcal{H}}$. We describe the action of these matrices on the irreducible $\mathcal{H}$-modules.

Definition 5.4.1 Define

$$\overline{\mathcal{H}} = \sum_{\ell=0}^{h+k} E_\ell^* \mathcal{H} E_\ell^*.$$

Note that $\overline{\mathcal{H}}$ is a subalgebra of $\mathcal{H}$.

Our next goal is to find some matrices that are contained in $\overline{\mathcal{H}}$. Note that any diagonal matrix in $\mathcal{H}$ is contained in $\overline{\mathcal{H}}$.

Lemma 5.4.2 *The matrices $K_1^{\pm 1}, K_2^{\pm 1}$ are contained in $\overline{\mathcal{H}}$.*

Proof. The result follows from the fact that $K_1^{\pm 1}, K_2^{\pm 1}$ are diagonal. □

Definition 5.4.3 Define

$$R = L_1 R_2, \qquad\qquad L = L_2 R_1. \tag{5.41}$$

By Lemma 6.2.2 in the appendix,

$$R = R_2 L_1, \qquad\qquad L = R_1 L_2. \tag{5.42}$$

Note that $R, L \in \mathcal{H}$. Also note that $R = L^t$.

Lemma 5.4.4 *For $u, v \in P$, the (u, v)-entries of R, L are*

$$R_{u,v} = \begin{cases} 1 & \text{if } u \ \backslash\text{-covers } u \cap v \text{ and } v \ /\text{-covers } u \cap v, \\ 0 & \text{otherwise,} \end{cases}$$

$$L_{u,v} = \begin{cases} 1 & \text{if } u \ /\text{-covers } u \cap v \text{ and } v \ \backslash\text{-covers } u \cap v, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Immediate from (5.41). □

Lemma 5.4.5 *For $0 \leqslant i \leqslant k$ and $0 \leqslant j \leqslant h$,*

$$R E_{i,j}^* V \subseteq E_{i-1,j+1}^* V, \qquad\qquad L E_{i,j}^* V \subseteq E_{i+1,j-1}^* V. \tag{5.43}$$

Proof. Combine (5.1), (5.2), (5.41). □

Lemma 5.4.6 *The matrices R, L are contained in $\overline{\mathcal{H}}$.*

Proof. Recall that $R, L \in \mathcal{H}$. By (5.43),

$$R = \sum_{\ell=0}^{h+k} E_\ell^* R E_\ell^*, \qquad\qquad L = \sum_{\ell=0}^{h+k} E_\ell^* L E_\ell^*.$$

The result follows. □

Next we bring in F^0, F^+, F^-, F.

Lemma 5.4.7 *The matrices F^0, F^+, F^-, F are contained in $\overline{\mathcal{H}}$.*

Proof. Recall from Lemma 5.2.5 that $F^0, F^+, F^-, F \in \mathcal{H}$. By Lemma 5.2.10,

$$F^0 = \sum_{\ell=0}^{h+k} E_\ell^* F^0 E_\ell^*, \qquad F^+ = \sum_{\ell=0}^{h+k} E_\ell^* F^+ E_\ell^*, \tag{5.44}$$

$$F^- = \sum_{\ell=0}^{h+k} E_\ell^* F^- E_\ell^*, \qquad F = \sum_{\ell=0}^{h+k} E_\ell^* F E_\ell^*. \tag{5.45}$$

The result follows. □

Next we define a matrix A.

Definition 5.4.8 Define $A \in \mathrm{Mat}_P(\mathbb{C})$ as follows. For $u, v \in P$ the (u, v)-entry of A is

$$A_{u,v} = \begin{cases} 1 & \text{if each of } u, v \text{ covers } u \cap v, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 5.4.9 *We have*

$$A = R + L + F. \tag{5.46}$$

Proof. Combine (5.26), (5.41) and Definition 5.4.8. The result follows from linear algebra. □

Lemma 5.4.10 *We have*

$$A = \big(L_1 + L_2\big)\big(R_1 + R_2\big) - (q-1)^{-1}\Big(q^{\frac{h+k}{2}} K_1 K_2^{-1} - I\Big), \tag{5.47}$$

$$= \big(R_1 + R_2\big)\big(L_1 + L_2\big) - (q-1)^{-1}\Big(q^{\frac{h+k}{2}} K_1^{-1} K_2 - I\Big). \tag{5.48}$$

Proof. Equation (5.47) follows from (5.27), (5.41), (5.46). Equation (5.48) follows from (5.28), (5.41), (5.46). □

Lemma 5.4.11 *The matrix A is contained in $\overline{\mathcal{H}}$.*

Proof. Immediate from (5.46). □

Lemma 5.4.12 *For $0 \leqslant i \leqslant k$ and $0 \leqslant j \leqslant h$,*

$$AE_{i,j}^* V \subseteq E_{i+1,j-1}^* V + E_{i,j}^* V + E_{i-1,j+1}^* V. \tag{5.49}$$

Proof. In view of (5.46), combine (5.43) and the right equation in (5.45). □

Next we define a matrix A^*.

Definition 5.4.13 Define
$$A^* = q^{\frac{k}{2}} K_1^{-1}.$$

Lemma 5.4.14 *For $u \in P$ the (u,u)-entry of A^* is*
$$\left(A^*\right)_{u,u} = q^i,$$
where $u \in P_{i,j}$.

Proof. Immediate from Definitions 5.1.4, 5.4.13. □

Lemma 5.4.15 *The matrix A^* is contained in $\overline{\mathcal{H}}$.*

Proof. By Definition 5.4.13, the matrix A^* is diagonal, and contained in $\mathcal{H}$. The result follows. □

Next we bring in $\Omega_0, \Omega_1, \Omega_2$.

Lemma 5.4.16 *The matrices $\Omega_0, \Omega_1, \Omega_2$ are contained in $\overline{\mathcal{H}}$.*

Proof. For each of (5.35)–(5.37), we observe that every term on the right hand side is contained in $\overline{\mathcal{H}}$. The result follows. □

5.5 The action of R, L, A, A^* on the irreducible $\mathcal{H}$-modules

In this section we describe the action of R, L, A, A^* on the irreducible $\mathcal{H}$-modules. For the rest of this section, let W denote an irreducible $\mathcal{H}$-module of type (α, β, ρ).

Lemma 5.5.1 *For $\alpha \leqslant i \leqslant k - \rho - \alpha$ and $\rho + \beta \leqslant j \leqslant h - \beta$,*
$$RE^*_{i,j}W \subseteq E^*_{i-1,j+1}W, \qquad LE^*_{i,j}W \subseteq E^*_{i+1,j-1}W. \tag{5.50}$$

Proof. Immediate from (5.43). □

Recall from (5.6) a standard basis for W:
$$w_{i,j} \qquad (\alpha \leqslant i \leqslant k - \rho - \alpha,\ \rho + \beta \leqslant j \leqslant h - \beta).$$

Lemma 5.5.2 *For $\alpha \leqslant i \leqslant k - \rho - \alpha$ and $\rho + \beta \leqslant j \leqslant h - \beta$,*
$$Rw_{i+1,j-1} = c_{i,j}(W) w_{i,j}, \qquad Lw_{i-1,j+1} = b_{i,j}(W) w_{i,j}, \tag{5.51}$$
where
$$c_{i,j}(W) = q^{\rho+\alpha+\beta}[j - \rho - \beta][k - \rho - \alpha - i], \tag{5.52}$$
$$b_{i,j}(W) = q^{k-\rho-i+j+1}[i - \alpha][h - \beta - j]. \tag{5.53}$$

Proof. By (5.50), the equations in (5.51) hold for some $c_{i,j}(W), b_{i,j}(W) \in \mathbb{C}$. We now prove (5.52). In view of the left equation of (5.41), combine (5.7) and (5.10). The result follows. We have proved (5.52). The proof of (5.53) is similar, and omitted. □

Next we describe the action of A, A^* on W.

Lemma 5.5.3 *For $\alpha \leqslant i \leqslant k-\rho-\alpha$ and $\rho+\beta \leqslant j \leqslant h-\beta$,*

$$AE^*_{i,j}W \subseteq E^*_{i+1,j-1}W + E^*_{i,j}W + E^*_{i-1,j+1}W.$$

Proof. Immediate from (5.49). □

Lemma 5.5.4 *For $\alpha \leqslant i \leqslant k-\rho-\alpha$ and $\rho+\beta \leqslant j \leqslant h-\beta$,*

$$Aw_{i,j} = b_{i+1,j-1}(W)w_{i+1,j-1} + a_{i,j}(W)w_{i,j} + c_{i-1,j+1}(W)w_{i-1,j+1}.$$

In the above equation, we interpret

$$w_{r,s} = 0, \qquad b_{r,s}(W) = 0, \qquad c_{r,s}(W) = 0,$$

unless $\alpha \leqslant r \leqslant k-\rho-\alpha$ and $\rho+\beta \leqslant s \leqslant h-\beta$.

Proof. In view of (5.46), combine (5.51) and the definition of $a_{i,j}(W)$. □

Lemma 5.5.5 *For $\alpha \leqslant i \leqslant k-\rho-\alpha$ and $\rho+\beta \leqslant j \leqslant h-\beta$, the subspace $E^*_{i,j}W$ is invariant under A^*. The corresponding eigenvalue is q^i.*

Proof. Immediate from Lemma 5.4.14. □

5.6 A generalization of the Askey-Wilson relations

Recall the matrix A from Definition 5.4.8 and the matrix A^* from Definition 5.4.13. In this section, we show that A, A^* satisfy a pair of relations that generalize the Askey-Wilson relations [9, Theorem 1.5].

Theorem 5.6.1 *The matrices A, A^* satisfy*

$$A^2A^* - \left(q+q^{-1}\right)AA^*A + A^*A^2 - \mathcal{Y}\left(AA^* + A^*A\right) - \mathcal{P}A^* = \Omega A + G, \tag{5.54}$$
$$A^{*2}A - \left(q+q^{-1}\right)A^*AA^* + AA^{*2} = \mathcal{Y}A^{*2} + \Omega A^* + G^*, \tag{5.55}$$

where

$$\begin{aligned}
\mathcal{Y} &= q^{\frac{h+k}{2}}\left(K_1K_2^{-1} + K_1^{-1}K_2\right) - q^{-1}(q-1)I,\\
\mathcal{P} &= q(q-1)^{-2}\Big(\mathcal{Y}^2 - q^{h+k-2}(q+1)^2 I\Big),\\
\Omega &= -\,q^{\frac{h+k}{2}-1}K_1^{-1}K_2\Big((q-1)\Omega_1 + (q+1)I\Big) - q^{k-1}\Big((q-1)\Omega_2 + (q+1)I\Big),\\
G &= -\,(q-1)^{-1}\Bigg(q^{\frac{h+k}{2}-1}\Big(qK_1^{-1}K_2\mathcal{Y} - q^{\frac{h+k}{2}}(q+1)I\Big)\Omega_1\\
&\qquad + q^{k-1}\Big(q\mathcal{Y} - q^{\frac{h+k}{2}}(q+1)K_1^{-1}K_2\Big)\Omega_2\Bigg) - (q+1)(q-1)^{-2}\cdot\\
&\qquad\qquad \Bigg(\Big(q^{\frac{h+k}{2}}K_1^{-1}K_2 + q^kI\Big)\mathcal{Y} - q^{\frac{h+k}{2}-1}(q+1)\Big(q^kK_1^{-1}K_2 + q^{\frac{h+k}{2}}I\Big)\Bigg),\\
G^* &= q^{\frac{h+3k}{2}-1}(q+1)\Omega_0K_1^{-1}K_2.
\end{aligned}$$

Proof. To verify (5.54), (5.55) we eliminate A^* using Definition 5.4.13. We use (5.26), (5.38)–(5.40), (5.46) to write A in terms of $R, L, K_1^{\pm 1}, K_2^{\pm 1}, \Omega_0, \Omega_1, \Omega_2$. We evaluate the result using the relations in the appendix. □

Lemma 5.6.2 *The matrices $\mathcal{Y}, \mathcal{P}, \Omega, G, G^*$ are central in $\overline{\mathcal{H}}$.*

Proof. Observe that each of $K_1^{-1}K_2$, $K_1K_2^{-1}$, $\Omega_0, \Omega_1, \Omega_2$ is central in $\overline{\mathcal{H}}$. The result follows. □

Note 5.6.1 The relations (5.54), (5.55) are generalizations of the Askey-Wilson relations that appear in [9, Theorem 1.5].

Next we describe the action of $\mathcal{Y}, \mathcal{P}, \Omega, G, G^*$ on the irreducible $\mathcal{H}$-modules.

Theorem 5.6.3 *Let W denote an irreducible H-module of type (α, β, ρ). Then for $\alpha \leqslant i \leqslant k-\rho-\alpha$ and $\rho+\beta \leqslant j \leqslant h-\beta$, the subspace $E^*_{i,j}W$ is invariant under each of $\mathcal{Y}, \mathcal{P}, \Omega, G, G^*$. The corresponding eigenvalues are given in the table below.*

Element in $\overline{\mathcal{H}}$	*Eigenvalue corresponding to $E^*_{i,j}W$*
$\mathcal{Y}$	$q^{h+k-\ell} + q^{\ell} - 1 + q^{-1}$
$\mathcal{P}$	$q(q-1)^{-2}\Big(\big(q^{h+k-\ell} + q^{\ell} - 1 + q^{-1}\big)^2 - q^{h+k-2}(q+1)^2\Big)$
Ω	$-\big(q^{h+k-\rho-\beta} + q^{k+\beta-1} + q^{k+\ell-\rho-\alpha} + q^{\ell+\alpha-1}\big)$
G	$(q-1)^{-1}\Big(\big(q^{k-\rho-\alpha+1}+q^{\alpha}\big)\big(q^{\ell-1}[h+k-\ell]-q^{\ell}[\ell]\big) - \big(q^{h-\rho-\beta+1}+q^{\beta}\big)\big(q^{k}[h+k-\ell]-q^{k-1}[\ell]\big)\Big)$
G^*	$q^{k+\ell-\rho-1}(q+1)$

In the above table, we write $\ell = i + j$.

Proof. Combine Lemma 5.1.13 and Theorem 5.3.4 and the definitions of $\mathcal{Y}, \mathcal{P}, \Omega, G, G^*$. $\square$

5.7 Parameters ν, μ, d, e

Let W denote an irreducible $\mathcal{H}$-module of type (α, β, ρ). In some earlier papers in the literature, the action of $\overline{\mathcal{H}}$ on $E^*_k W$ is often expressed in terms of ν, μ, d, e where

$$\begin{aligned}
\nu &= \min\{j \mid E^*_{k-j,j} W \neq 0,\ 0 \leqslant j \leqslant k\},\\
\mu &= \min\{i + j \mid E^*_{i,j} W \neq 0,\ 0 \leqslant i \leqslant k,\ 0 \leqslant j \leqslant k\},\\
d &= \big|\{j \mid E^*_{k-j,j} W \neq 0,\ 0 \leqslant j \leqslant k\}\big| - 1,
\end{aligned}$$

and e is an auxiliary parameter. (See [3], [8, Example 6.1].) The parameters ν, μ, d are often called the endpoint, dual endpoint, and diameter, respectively. In this section, we describe how to convert from α, β, ρ to ν, μ, d, e.

There are three cases to consider:

(C1) $\beta - \alpha \leqslant 0$;

(C2) $0 < \beta - \alpha \leqslant h - k$;

(C3) $h - k < \beta - \alpha$.

We illustrate the cases (C1)–(C3) using the diagrams below.

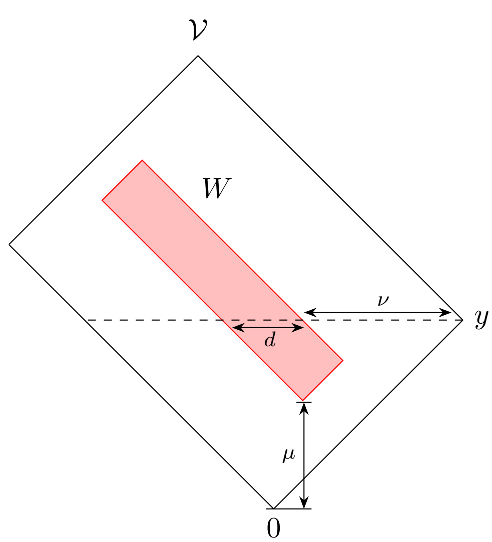

Figure 5.2A: Case (C1)

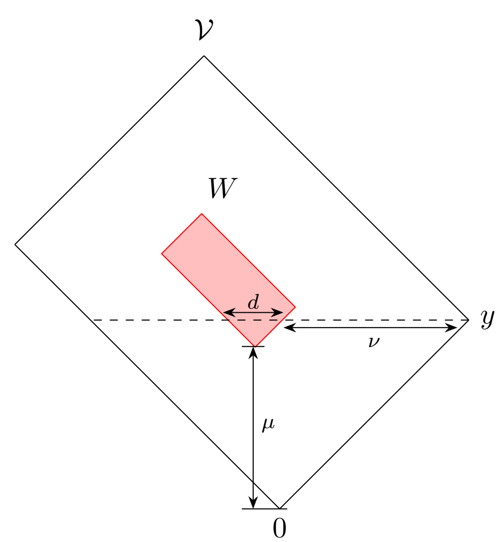

Figure 5.2B: Case (C2)

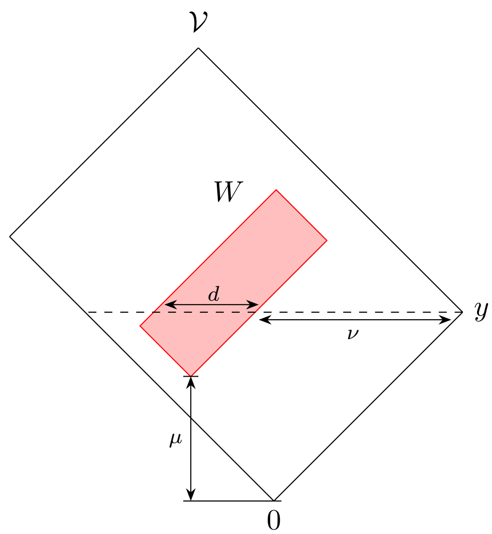

Figure 5.2C: Case (C3)

Lemma 5.7.1 [3, p. 129] *Below we express* α, β, ρ *in terms of* ν, μ, d, e.

In case (C1),

$$\alpha = \nu - e, \qquad \beta = \mu - \nu, \qquad \rho = e.$$

In case (C2),

$$\alpha = \mu - \nu, \qquad \beta = \frac{\mu - e}{2}, \qquad \rho = \nu - \frac{\mu - e}{2}.$$

In case (C3),

$$\alpha = \mu - \nu, \qquad \beta = k - h + \nu - e, \qquad \rho = h - k + e.$$

Chapter 6

Appendix

6.1 Linear algebra

In this section, we derive some linear algebra facts that are used in the main body of the thesis.

Lemma 6.1.1 *Let $\mathcal{W}$ denote a vector space of finite positive dimension δ. Suppose the vectors $\{\mu_i\}_{i=0}^{\delta}$ span $\mathcal{W}$ and sum to 0. Then any δ vectors from $\{\mu_i\}_{i=0}^{\delta}$ form a basis for $\mathcal{W}$.*

Proof. It suffices to show that the vectors $\{\mu_i\}_{i=1}^{\delta}$ form a basis for $\mathcal{W}$.

Since the vectors $\{\mu_i\}_{i=0}^{\delta}$ sum to 0, μ_0 is in the span of $\{\mu_i\}_{i=1}^{\delta}$. Hence,

$$\text{Span}\{\mu_i\}_{i=1}^{\delta} = \text{Span}\{\mu_i\}_{i=0}^{\delta} = W.$$

The vector space $\mathcal{W}$ has dimension δ, so by linear algebra, the vectors $\{\mu_i\}_{i=1}^{\delta}$ form a basis for $\mathcal{W}$. The result follows. □

Lemma 6.1.2 *Let $\mathcal{W}$ denote a vector space of finite positive dimension δ. Suppose the vectors $\{\mu_i\}_{i=0}^{\delta}$ span $\mathcal{W}$ and satisfy*

$$\sum_{i=0}^{\delta} \mu_i = 0. \tag{6.1}$$

Suppose the vectors $\{\nu_i\}_{i=0}^{\delta}$ span $\mathcal{W}$. Then the following are equivalent.

(i) *There exists $\sigma \in GL(\mathcal{W})$ such that*

$$\sigma(\mu_i) = \nu_i \qquad (0 \leqslant i \leqslant \delta). \tag{6.2}$$

(ii) *The following sum holds:*

$$\sum_{i=0}^{\delta} \nu_i = 0. \tag{6.3}$$

Moreover, if (i) *and* (ii) *hold, then the map* σ *is unique.*

Proof. (i) $\Rightarrow$ (ii) Apply σ to each side of (6.1), and evaluate the result using (6.2).

(ii) $\Rightarrow$ (i) By (6.1) and Lemma 6.1.1, the vectors $\{\mu_i\}_{i=1}^{\delta}$ form a basis for $\mathcal{W}$.

By (6.3) and Lemma 6.1.1, the vectors $\{\nu_i\}_{i=1}^{\delta}$ form a basis for $\mathcal{W}$. By linear algebra, there exists a map $\sigma \in GL(\mathcal{W})$ such that

$$\sigma(\mu_i) = \nu_i \qquad (1 \leqslant i \leqslant \delta). \tag{6.4}$$

Using (6.1), (6.3), (6.4), we obtain

$$\sigma(\mu_0) = \nu_0.$$

By these comments, (i) holds. We have shown the equivalence of (i) and (ii).

Assume that (i) and (ii) hold. We now show that the map σ is unique. Let the map $\sigma' \in GL(\mathcal{W})$ satisfy (6.2).

Using (6.2), we obtain

$$\big(\sigma - \sigma'\big)(\mu_i) = 0 \qquad (0 \leqslant i \leqslant \delta).$$

The vectors $\{\mu_i\}_{i=0}^{\delta}$ span $\mathcal{W}$, so $\sigma - \sigma' = 0$. Therefore $\sigma = \sigma'$. □

6.2 Relations between the generators of $\mathcal{H}$

Recall the algebra $\mathcal{H}$ from Definition 5.1.9. In this section, we recall from [10] the relations between the generators of $\mathcal{H}$.

Lemma 6.2.1 [10, Lemma 7.4] *The following* (i)–(viii) *hold:*

(i) $K_1L_1 = qL_1K_1$;

(ii) $K_1L_2 = L_2K_1$;

(iii) $qK_1R_1 = R_1K_1$;

(iv) $K_1R_2 = R_2K_1$;

(v) $K_2L_1 = L_1K_2$;

(vi) $qK_2L_2 = L_2K_2$;

(vii) $K_2R_1 = R_1K_2$;

(viii) $K_2R_2 = qR_2K_2$.

Lemma 6.2.2 [10, Lemma 7.5] *The following* (i)–(iv) *hold:*

(i) $L_1R_2 = R_2L_1$;

(ii) $L_2R_1 = R_1L_2$;

(iii) $qL_1L_2 = L_2L_1$;

(iv) $R_1R_2 = qR_2R_1$.

Lemma 6.2.3 [10, Lemma 7.6] *The following* (i)–(iv) *hold.*

(i) $R_1^2L_1 - (q+1)R_1L_1R_1 + qL_1R_1^2 = -q^{\frac{h+k}{2}-1}(q+1)K_1^{-1}K_2R_1$;

(ii) $qR_2^2L_2 - (q+1)R_2L_2R_2 + L_2R_2^2 = -q^{\frac{h+k}{2}}(q+1)K_1K_2^{-1}R_2$;

(iii) $qL_1^2R_1 - (q+1)L_1R_1L_1 + R_1L_1^2 = -q^{\frac{h+k}{2}}(q+1)K_1^{-1}K_2L_1$;

(iv) $L_2^2R_2 - (q+1)L_2R_2L_2 + qR_2L_2^2 = -q^{\frac{h+k}{2}-1}(q+1)K_1K_2^{-1}L_2$.

Lemma 6.2.4 [10, Lemma 7.7] *The generators of $\mathcal{H}$ satisfy*

$$L_1R_1 - R_1L_1 + L_2R_2 - R_2L_2 = q^{\frac{h+k}{2}}(q-1)^{-1}(K_1K_2^{-1} - K_1^{-1}K_2).$$

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