NUMERICAL METHODS FOR MULTISCALE HYPERBOLIC AND KINETIC EQUATIONS

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Abstract

Hyperbolic and kinetic equations often have parameters that vary considerably over the region. In certain asymptotic regimes where the parameter is very small, the standard hyperbolic or kinetic solvers break down because of the prohibitive computational cost. This thesis explores two efficient methods — *Domain Decomposition* methods and *Asymptotic Preserving (AP)* methods for these problems.

The first part aims at constructing a domain decomposition formulation for the Jin-Xin relaxation system [58] with two-scale relaxations, which is a prototype for more general physical problems such as phase transitions, river flows, kinetic theories etc.. We propose the interface condition based on the sign of the characteristic speed at the interface. A rigorous analysis on the L^2 error estimate is presented, based on the Laplace Tranform, for the linear case with an optimal convergence rate. For the nonlinear case, using standard compactness argument, we are able to prove the asymptotic convergence of the solution of the original relaxation system to the unique entropy weak solution of the domain decomposition system. The interface condition is derived rigorously by matched asymptotic analysis for a general flux with an extension to the case when a standing shock is sticking to the interface.

The second part focuses on the development of AP methods for kinetic equations in the high field regime where both the collision and field effect dominate the evolution. The stiff force term poses extra numerical challenges as apposed to the stiff collision term which has been well-studied in the hydrodynamic regime. We first consider the Vlasov-Poisson-Fokker-Planck system used in electrostatic plasma and astrophysics. The

AP scheme is constructed based on the combination of two stiff terms so as to use the symmetric discretization in [59]. The semiconductor Boltzmann equation is considered next. By penalizing the collision term by a classical BGK operator and treating the force term implicitly, we are able to overcome the exceptional difficulty that no specific expression of the local equilibrium is available. The distribution function is still shown to converge to the high field limit, which guarantees the capturing of the asymptotics without numerically resolving the small parameter.

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Chapter 1

Introduction

Multiscale description that hybridizes the microscopic (or mesoscopic) kinetic and macroscopic hydrodynamic models is necessary in many physical situations. Such problems are usually described by equations that contain small parameters, such as relaxation time or mean free path. In the asymptotic regimes where only the averaged or macroscopic quantities are important, tremendous computational difficulties arise because the small parameters are prohibitively expensive to be resolved. Considerable attributes have been paid to the design of efficient and robust numerical methods for this kind of problems, among which the most popular ones are *Domain Decomposition* methods and *Asymptotic Preserving (AP)* methods.

This thesis, mainly composed of two parts, explores both approaches. The first part aims at developing a domain decomposition method for a semilinear hyperbolic systems with two-scale relaxations. The efficiency of such method has been mathematically justified for both linear (Chapter 2) and nonlinear cases (Chapter 3). The second part is devoted to designing asymptotic preserving methods for kinetic equations in the high field regime (Chapter 4 and Chapter 5).

Hyperbolic systems with relaxations are used to characterize many physical problems. Important examples include gas not in thermodynamic equilibrium [98], phase transitions with small transition time [66, 91], viscoelasticity with vanishing memories

[86, 100], and kinetic theories [16], among many others. In general, relaxation phenomena happen when a stable equilibrium state is perturbed, resulting in a set of rate equations where the local source term is amplified by the reciprocal of the relaxation time ϵ. In the regimes where ϵ is very small, the asymptotic expansion of it gives the local equilibrium that leads to the conservation laws. Such regimes pose huge computational challenges since one needs to numerically resolve the small scales which can be too costly. A domain decomposition method, that couples the hyperbolic relaxation system in the non-equilibrium regime with the associated conservation laws in the equilibrium regime is computationally competitive.

Our goal in the first part of this thesis is to derive an interface condition to bridge the equations in two regimes. The major difficulty is that a boundary layer might be generated when approximating the relaxation system by the conservation laws. Thus a way to avoid resolving the boundary layer while still preserving the outer profile is desirable. Here we consider instead a simplified model, the well-known Jin-Xin relaxation system [58] as a prototype for its intrinsic difficulties and its relevance in applications. Inspired by [41], we propose the interface condition which depends on the sign of the characteristic speed at the interface by adopting the boundary layer analysis [55]. This new domain decomposition system, completely decoupled, can be solved numerically using any high resolution method. Moreover, it can be easily extended to more complicated cases such as dynamic interface.

To show that our domain decomposition system is asymptotically close to the original two-scale hyperbolic relaxation system is not an easy task. For the linear case where the flux takes the form $f(u) = \lambda u$, we are able to represent the solutions explicitly by the Laplace Transform, thus deriving the stiff well-posedness of the system as well as the L^2 error estimate with the optimal convergence rate. The asymptotic error is shown to depend on the smaller relaxation time, and the boundary and interface layer effects. For the nonlinear case, by introducing a regularized system that smooth out the discontinuous relaxation parameter, we are able to show the existence of the unique entropy weak solution to the original hyperbolic relaxation system by compactness argument. A special trick here is that we cannot use the classical L^1 contraction principle to get a uniform BV estimate since the relaxation parameter is space dependent. Therefore we refer to the invariance w.r.t. time and get the estimate by working on a system obtained by taking the derivative w.r.t. time of the original one. We also show the strong convergence in $L^\infty\big((0,T), L^1_{\text{loc}}(R)\big)$ to a weak solution as the smaller parameter goes to zero thanks to the uniform estimates in *sup* and BV norms. Finally by a matched asymptotic analysis, we prove in a rigorous way that for a general nonlinear flux, the traces of u at the interface are linked by the well-known Bardos-Lerous-Nédélec condition, which is indeed an extension of what we have in [55] by taking into account the situation when a standing shock sticks to the interface.

Despite its efficiency and accuracy, the domain decomposition method suffers from severe difficulties in where to put the interface and how to propose the interface condition especially when the governing equations are complicated. Thus a new approach, the asymptotic preserving schemes, that pioneered by Jin [53, 54], has attracted considerable interests recently. The main idea of these schemes is to preserve the discrete analog of the continuous asymptotic limits from the microscopic to the macroscopic models. A distinctive feature is that only a microscopic solver is needed everywhere at the cost of the macroscopic level. These schemes have been well studied for kinetic equations that have a hydrodynamic or diffusive scalings [36, 59, 35, 51, 43, 44, 37]. However, there is little result for the system with a high field scaling where both the field effect and collision are dominant in the evolution process. The second part of this thesis is to develop numerical methods for such systems, with emphasis on the connections to their high field limit.

We first consider the Vlasov-Poisson-Fokker-Planck (VPFP) system, which is the kinetic description of the Brownian motion of a large system of particles in a surrounding bath. Such system arises for instance in the electrostatic plasma where the interactions are Coulomb force and in galaxies where massive particles interact through gravitational force. Its high field scaling, measured by the ratio of the two important physical quantities—the mean free path and the Debye length, leads asymptotically to a nonlinear convection equation for the macroscopic mass density as the ratio goes to zero [78, 45].

Besides the usual stability constraint posed by the small parameter in the collision term like the most kinetic equations in the hydrodynamic regime, new difficulties arise in the high field regime here. One comes from the stiff force term and the other comes from the diffusive nature of the Fokker-Planck operator. We propose an asymptotic preserving method [57] based on the observation that the two stiff terms can be combined into one term that shares the same structure as the classical Fokker-Planck operator, so that a symmetric discretization [59] can be applied. Therefore, both stiff terms can be treated implicitly simultaneously, while only a symmetric tri-diagonal matrix has to be inverted. This scheme also offers favorable properties such as mass conservation, positivity preservation and uniform stability.

We next consider the semiconductor Boltzmann equation which describes the semiclassical evolution of the electron distribution function f. The collision here is a linear integral function of f in the low electron densities, describing the interactions of electrons with themselves and lattice imperfections; while it takes a nonlinear integral form when the electron density is high. Unfortunately, there is no symmetric combination of the two stiff terms as we did for the VPFP system. Even worse, there is no explicit expression of the local equilibrium, which makes the existing AP method [33, 56, 106] hard to implement. We adopt the penalization idea introduced by Filbet and Jin [36] and, inspired by the fact that functions that share the same conserved quantities with the exact local equilibrium can be used as candidates for penalty, we only penalize the collision term by a BGK operator that conserves mass, and treat the stiff force term implicitly by the spectral method. This idea is applicable to both cases above. The compensate for the "wrong Maxwellian" penalty is merely an extra term depending on the time step in the asymptotic error.

The rest of the thesis is organized as follows.

1.1 Chapter Organization

In Chapter 2, after a brief review of the Jin-Xin relaxation system and its properties, we focus on the initial boundary value problem of it in the upper half plane. Following the asymptotic expansion [103], we are able to reveal the explicit expression of the boundary layer profile. The interface condition is then proposed relying on this information. By use of the Laplace Transform, we prove the stiff well-posedness and asymptotic convergence for the derived domain decomposition system when the flux in the associated conservation law is linear. An optimal convergence rate is obtained as well. The corresponding numerical algorithms are given and tested on several examples in the end.

The analytical results for the nonlinear flux are gathered in Chaper 3. By introducing a regularized system, we first show the well-posedness of the original two-scale hyperbolic relaxation system. At the same time, we construct a priori estimate of the solution in the sup_x and BV_x norm, as well as a local continuity result in time. After that, a strong convergence in $L^\infty\big((0,T), L^1_{\text{loc}}(R)\big)$ to a entropy weak solution of the limit system is obtained through standard compactness argument. Finally by matched asymptotic analysis, we derive an autonomous ordinary differential equation for the inner solution, which together with some fine estimation, implies the interface condition. This is an extension to the one constructed in Chapter 2.

Chapter 4 aims at designing an asymptotic preserving scheme for Vlasov-Poisson-Fokker-Planck system in the high field regime. Starting from the system, we first give a brief review of the formal derivation of the high field limit as well as the underlying physics. Based on the key observation that two stiff terms can be treated together as a Fokker-Planck operator, we give the first order scheme using the symmetric discretization introduced in [59]. A mathematical justification of the properties of the scheme is obtained as well. The scheme is extended to second order. Extensive numerical examples are given to verify the asymptotic property, uniform accuracy, and its efficiency for problems with mixing scales and the Riemann initial data.

A similar numerical issue for the semiconductor Boltzmann equation is investigated in Chapter 5, with quite different solutions. Asymptotic property of the scheme is proved to be of order of the small parameter with an extra term depending on the time step. Such a property has been verified by several numerical examples for both degenerate and nondegenerate cases. Examples with the Riemann initial data and mixing scales are also given to check the performance of the scheme. A physically more realistic example is also presented in the end with our new observation.

Chapter 2

A semilinear hyperbolic system with two-scale relaxations

2.1 Introduction

Consider the hyperbolic system

$$\begin{cases} u_t^\epsilon + v_x^\epsilon = 0, & \text{(2.1a)} \\ v_t^\epsilon + u_x^\epsilon = -\dfrac{1}{\epsilon(x)}(v^\epsilon - f(u^\epsilon)), & \text{(2.1b)} \end{cases}$$

where $\epsilon(x)$ is the relaxation time and $f(x)$ satisfies the sub-characteristic condition:

$$|f'(x)| < 1. \tag{2.2}$$

The problem is posed for $x \in [-L, L]$ and $t > 0$ with initial data

$$u^\epsilon(x, 0) = u_0(x),\ \ v^\epsilon(x, 0) = v_0(x), \tag{2.3}$$

and the order of the relaxation time varies considerably over the domain $[-L, L]$. In this chapter, we consider the case when $\epsilon(x)$ is given by:

$$\epsilon(x) = 1, \quad x \in [-L, 0); \qquad \epsilon(x) = \epsilon, \quad x \in (0, L], \tag{2.4}$$

where $\epsilon \ll 1$ is a small parameter. For the boundary condition, we simply choose the Dirichlet condition for u, i.e:

$$u^\epsilon(x_L, t) = b_L(t),\ \ u^\epsilon(x_R, t) = b_R(t). \tag{2.5}$$

More general boundary conditions can also be analyzed by the method of the present chapter. The initial data and boundary data are required to be compatible, i.e., $b_1(0) = u_0(x_L),\ b_2(0) = u_0(x_R)$.

Since the relaxation time is small in the region $(0, L]$, numerical computation of this system becomes very costly. On the other hand, in $(0, L]$, the solution is, to leading order in ϵ, governed by the equilibrium equation

$$u_t + f(u)_x = 0\,, \tag{2.6}$$

which can be more efficiently solved numerically. Thus a domain decomposition method, which couples the relaxation system (2.1) for $x \in [-L, 0)$, where $\epsilon(x) = O(1)$, with the equilibrium equation (2.6) for $x \in (0, L]$, is computationally competitive. Interface conditions at $x = 0$ must be provided for this coupling.

System (2.1) was first proposed by Jin-Xin [58] for numerical purpose, which supplies a new and powerful approximation to equilibrium conservation law (2.6). There have been many works concerning the asymptotic convergence of the relaxation systems (2.1) to the corresponding conservation laws (2.6) as the relaxation time tends to zero. Most of the results dealt with the Cauchy problem. In particular, Natalini [75] gave a rigorous proof that the solution to Cauchy problem (2.1) with initial condition (2.3) converges strongly in $C([0,\infty), L^1_{loc}(\mathbb{R}))$ to the unique entropy solution of (2.6) when $\epsilon \to 0$. See also [76] for a review in this direction, and results for larger systems [7] and on more general hyperbolic systems with relaxations [21, 65, 77, 11].

In the presence of physical boundary conditions, Kriess and some others first gave the suitability of boundary conditions for linear hyperbolic systems when the source term is not stiff, see, for examples [64, 50, 73, 85]. Wang and Xin [99] later gave a similar result of the system (2.1)–(2.3) with boundary condition (2.5). They proved that when the initial and boundary data satisfy a strict version of the subcharacteristic condition (2.2), the solution of the relaxation system converges as $\epsilon \to 0$ to a unique weak solution of the conservation law (2.6) which satisfies the boundary-entropy condition. [104] and [103] then gave an explicit necessary and sufficient condition (the so-called "Stiff Kriess Condition") on the boundary that guarantees the uniform well-posedness of the IBVP, and also revealed the boundary layer structures. [103, 105] dealt with the linear cases while [104] considered the nonlinear one. For the convergence of the relaxation scheme, one can refer to [19, 94].

Domain decomposition methods connecting kinetic equation and its hydrodynamic or diffusion limit have received a lot of attention in the past 20 years. This project is strongly motivated by [41]. Others can refer to [5, 95, 10, 32, 107, 62, 63, 29, 28, 31, 93, 46, 60]. A thorough study on the problem of this chapter provides a better understanding of the more general coupling problem of kinetic and hydrodynamic equations, since indeed the Jin-Xin relaxation system (2.1) can be viewed as a discrete-velocity kinetic model, while (2.6) resembles some important features of hydrodynamic (compressible Euler) equations.

Relaxation systems themselves are important in many physical situations, such as kinetics theories [16], gases not in thermodynamic equilibrium [98], phase transitions with small transition time [66], river flows, traffic flows and more general waves [100].

In this chapter, we give a domain decomposition method for system (2.1)–(2.4) by providing the interface condition at $x = 0$. The interface condition depends on the sign of $f'(u)$ at the interface. When $f'(u(0,t)) < 0$, there will be an interface layer in u around $x = 0^+$ when approximating the original system (2.1) by (2.6), then one can solve (2.6) in the right region first and then transfers the value of $v(0,t)$ to the left as one boundary condition for (2.1) in the left region, see (2.12)–(2.13). On the other hand, when $f'(u(0,t)) < 0$, one just uses $v(0,t) = f(u(0,t))$ as one boundary condition for (2.1) in the left region, and solves it first, then uses the value $u(0,t)$ as the boundary condition for (2.6) in the right region. The details are given in (2.14)–(2.15). For the linear case, i.e., $f(u) = \lambda u$, where $|\lambda| < 1$ a constant, we first prove the stiff well-posedness of the original system (2.1) in Theorem 2.3 in the sense that the L^2 norm of the solution is controlled by the L^2 norm of the initial and boundary data. Then we prove the asymptotic convergence in Theorem 2.4 to show that the difference between the solution to our domain decomposition system and the solution to the original system is asymptotically small. Sharp error estimates are also given.

This domain decomposition can be directly extended to more general cases, such as the coupling of multiple regions, $f'(u(0,t))$ changing sign in time, ϵ depending on both time and space [27], and more complicated cases such as when the equilibrium equation is a hyperbolic system instead of the scalar conservation law, and in higher space dimensions. Some details are given in section 2.6.

This chapter is organized as follows. In section 2.2 we show the formal expansion of the initial boundary value problem (2.1) in the upper half plane $\{x > 0,\ t > 0\}$ in which the boundary layer may exist. We also refer to the theorems in [104] which validate this expansion. Section 2.3 is devoted to present the domain decomposition method, and the corresponding interface condition is given. We then prove the stiff well-posedness and asymptotic convergence for the linear case. The theorems are proved in two parts: one for homogeneous initial data (section 2.4) and the other the inhomogeneous one (section 2.5). For the homogeneous one, we simply use the Laplace Transform to obtain the solution, while for the inhomogeneous case, we construct several auxiliary systems to decompose the solution into two parts, one generated by the initial data, and the other by the interface condition. With this decomposition, we are able to use some existing results for the Cauchy problem to avoid the difficulties raised by the Laplace Transform. Finally in section 2.6, we present the corresponding numerical algorithms and some extensions of the domain decomposition method, and finally give some numerical examples to validate the theoretical analysis.

2.2 The local equilibrium limit

In this section, we recall the asymptotic analysis proposed in [104]. Here we only consider the boundary layer effect, and let

$$v_0(x) = f(u_0(x))$$

in order to avoid the initial layer effect. When $x \in [0, L]$ where ϵ is small, one can use the hyperbolic conservation law (2.6) to approximate the relaxation system. Away from $x = 0$ and $t = 0$, use the expansion

$$\begin{aligned} u^\epsilon(x,t) &\sim u^0(x,t) + \epsilon u^1(x,t) + \epsilon^2 u^2(x,t) + \ldots, \\ v^\epsilon(x,t) &\sim v^0(x,t) + \epsilon v^1(x,t) + \epsilon^2 v^2(x,t) + \ldots, \end{aligned}$$

then matching the orders of ϵ, one obtains:

$$\begin{aligned} v^0 &= f(u^0), \\ \partial_t u^0 + \partial_x v^0 &= 0, \\ \partial_t v^0 + \partial_x u^0 &= -(v^1 - f'(u^0)u^1). \end{aligned} \tag{2.7}$$

$$\cdots$$

Thus the leading order of the expansion gives

$$\partial_t u^0 + \partial_x f(u^0) = 0,\ v^0 = f(u^0), \tag{2.8}$$

which is the equilibrium limit (the zero relaxation limit) (2.6).

Near $x = 0$, introduce the stretched variable $\zeta = x/\epsilon$, and write the asymptotic expansion of $u^\epsilon(x,t)$ as

$$u^\epsilon(x,t) \sim u^0(x,t) + \epsilon u^1(x,t) \ + \ldots + \Gamma_u^0(\zeta,t) + \epsilon\Gamma_u^1(\zeta,t) + \ldots,$$

$$v^\epsilon(x,t) \sim v^0(x,t) + \epsilon v^1(x,t) + \ldots + \Gamma_v^0(\zeta,t) + \epsilon\Gamma_v^1(\zeta,t) + \ldots,$$

here Γ_u^0, Γ_v^0, Γ_u^1, Γ_v^1, ..., depending on ζ and t, are the boundary layer correctors near $x = 0$. Apply this ansatz to (2.1), and expand the nonlinear term $f(u^\epsilon)$ near $x = 0$ as

$$\begin{aligned} f(u^\epsilon) &= f(u^0(x,t) + \Gamma_u^0(\zeta,t) + \epsilon u^1(x,t) + \epsilon\Gamma_u^1(\zeta,t) + ...) \\ &= f(u^0(0,t) + \epsilon\zeta\partial_x u^0(0,t) + ... + \Gamma_u^0(\zeta,t) + \epsilon u^1(x,t) + \epsilon\Gamma_u^1(\zeta,t) + ...) \\ &= f(u^0(0,t) + \Gamma_u^0(\zeta,t)) + \epsilon f'(u^0(0,t) + \Gamma_u^0(\zeta,t))(\zeta\partial_x u^0(0,t) + u^1(0,t) + \Gamma_u^1(\zeta,t)) + \epsilon^2 ... \end{aligned}$$

where the second equality comes from the relation $x = \epsilon\zeta$. By using (2.7) and (2.8) one has the equation to the leading order $O(\frac{1}{\epsilon})$

$$\partial_\zeta \Gamma_v^0 = 0, \tag{2.9}$$

$$\partial_\zeta \Gamma_u^0 = -(v^0(0,t) + \Gamma_v^0 - f(\Gamma_u^0 + u^0(0,t))). \tag{2.10}$$

(2.9) implies $\Gamma_v^0 \equiv 0$ because the boundary layer $\Gamma_v^0(\zeta,0)$ should decay as $\zeta \to 0$. Also, (2.10) can be written as

$$(\Gamma_u^0)_\zeta = -(v^0(0,t) - f(u^0(0,t) + \Gamma_u^0)) \simeq f'(u^0(0,t))\Gamma_u^0(\zeta,t),$$

thus one gets the behavior of the boundary layer in u:

$$\Gamma_u^0(\zeta, t) = \exp(f'(u^0(0,t))\zeta)\Gamma_u^0(0,t). \tag{2.11}$$

Since the boundary layer has to decay exponentially fast, one needs $f'(u^0(0,t)) < 0$. In other words, if $f'(u^0(0,t)) < 0$, there will be a boundary layer, otherwise there will not be a boundary layer.

The above analysis was rigorously validated in [104].

2.3 A domain decomposition method

In section 2.2, one sees that when ϵ goes to 0, the hyperbolic system (2.1) can be approximated by the equilibrium equation (2.6) that does not have any stiff term. But the interface condition that connects the two regions should be provided. In this section, we will give the detailed algorithm that approximates the solution of the two-scale problem. We will consider the case with $f'(u(0,t)) < 0$ and $f'(u(0,t)) > 0$ separately.

2.3.1 $f'(u(0,t)) < 0$

In this case, there will be an interface layer in u near the interface $x = 0$, so one can not simply use u obtained from $(0, L]$ to solve (2.6) in domain $[-L, 0)$. Instead we can use the information of v at $x = 0$ directly from the equation in $(0, L]$ since there is no $O(1)$ interface layer in v. Here is the coupling algorithm.

- **Step 1.** For $x \in (0, L]$, solve

$$\begin{cases} u^r_t + f(u^r)_x = 0, & (2.12a) \\ v^r(x,t) = f(u^r(x,t)), & (2.12b) \\ u^r(x,0) = u_0(x), & (2.12c) \\ u^r(L,t) = b_R(t). & (2.12d) \end{cases}$$

Note in this case one can solve (2.12) first to get $v^r(0,t)$, and then solve (2.13).

- **Step 2.** For $x \in [-L, 0)$, solve

$$\begin{cases} u^l_t + v^l_x = 0, & (2.13a) \\ v^l_t + u^l_x = -(v^l - f(u^l)), & (2.13b) \\ u^l(x,0) = u_0(x),\ \ v^l(x,0) = v_0(x), & (2.13c) \\ u^l(-L,t) = b_L(t), & (2.13d) \\ v^l(0,t) = v^r(0,t), & (2.13e) \end{cases}$$

where $v^r(0,t)$ is obtained from Step 1.

2.3.2 $f'(u(0,t)) > 0$

In this case, at the interface $x = 0$ there is no $O(1)$ interface layer in u and v. In other words, u and v are in local equilibrium $v = f(u)$, and we can just use this as the interface condition. We give the following algorithm.

- **Step 1.** For $x \in [-L, 0)$, solve

$$\begin{cases} u^l_t + v^l_x = 0, & \text{(2.14a)} \\ v^l_t + u^l_x = -(v^l - f(u^l)), & \text{(2.14b)} \\ u^l(x,0) = u_0(x),\ v^l(x,0) = v_0(x), & \text{(2.14c)} \\ u^l(-L,t) = b_L(t), & \text{(2.14d)} \\ f(u^l(0,t)) = v^l(0,t); & \text{(2.14e)} \end{cases}$$

- **Step 2.** For $x \in (0, L]$, solve

$$\begin{cases} u^r_t + f(u^r)_x = 0, & \text{(2.15a)} \\ v^r(x,t) = f(u^r(x,t)), & \text{(2.15b)} \\ u^r(x,0) = u_0(x), & \text{(2.15c)} \\ u^r(0,t) = u^l(0,t), & \text{(2.15d)} \end{cases}$$

  where $u^l(0,t)$ is obtained from Step 1.

Remark 2.1. *In this case there will be a boundary layer in u near $x = L^-$, which is why in Theorem 2.4 that the convergence rate is $O(\epsilon)$.*

In both cases, we define the solution to the domain decomposition system as follows:

$$\begin{cases} u(x,t) = u^l(x,t),\ v(x,t) = v^l(x,t), \quad (x,t) \in [-L,0) \times [0,T], & \text{(2.16a)} \\ u(x,t) = u^r(x,t),\ v(x,t) = v^r(x,t), \quad (x,t) \in (0,L] \times [0,T]. & \text{(2.16b)} \end{cases}$$

Remark 2.2. *If $f'(u(0,t))$ changes sign at the interface, one can check the sign of $f'(u(0,t))$ at the current time step, and then use either (2.12)–(2.13) or (2.14)–(2.15) to continue to the next step. More general cases, such as time-dependent ϵ or higher space dimensions, are discussed in section 2.6.3.*

The detailed numerical implementation of this domain decomposition method is given in section 2.6.

Now we state the main theorems in this chapter about the stiff well-posedness of the original relaxation system and asymptotic convergence of our domain decomposition system.

Theorem 2.3. *Let $U^\epsilon = (u^\epsilon, v^\epsilon)^T$ be the solution of the original system (2.1). If $u_0(x), v_0(x), b_L(t),\ b_R(t) \in L^2$, and $U_0(\pm L) = 0$, $b_L(0) = b_R(0) = 0$, then the solution to the original system (2.1), with variable $\epsilon(x)$ given in (2.4), is stiffly well-posed in the sense:*

$$\begin{aligned}&\int_0^T \int_{-L}^L |U^\epsilon(x,t)|^2 dxdt + \int_0^T |U^\epsilon(-L,t)|^2 dt + \int_0^T |U^\epsilon(L,t)|^2 dt \\ &\le K_T \left[\int_0^T |b_L(t)|^2 dt + \int_0^T |b_R(t)|^2 dt + \int_{-L}^L |U_0(x)|^2 dx \right],\end{aligned}$$

where K_T is a positive constant independent of ϵ. Moreover, if $u_0(x)$, $v_0(x)$, $b_L(t)$ and $b_R(t)$ are continuous, then the solution U^ϵ is continuous in x.

Theorem 2.4. *Assume $b_L(t), b_R(t) \in L^2(\mathbf{R}^+)$, $U_0(\pm L) = 0$, $U_0(x) \in H^3([-L, L])$ and $U_0(0) = U_0'(0) = U_0''(0) = 0$, then there exists a unique solution $U = (u, v)^T$ of the domain decomposition system (2.12)–(2.13) or (2.14)–(2.15) such that*

$$\int_{-L}^L \int_0^\infty |U^\epsilon - U|^2 e^{-2\alpha t} dtdx \to 0$$

as $\epsilon \to 0$ for any $\alpha > 0$. Moreover, if we assume $b_L(t),\ b_R(t) \in H^2(\mathbb{R}^+)$, $b_L(0) = b_L'(0) =$

$b_R(0) = b'_R(0) = 0$, and $U'_0(\pm L) = 0$, then

$$\begin{aligned}
&\int_{-L}^{L}\int_0^{\infty} |U^\epsilon - U|^2 e^{-2\alpha t}\, dtdx \\
\le\ & O(1)\epsilon\|b_L\|_{L^2}^2 + O(1)\epsilon\|b_R\|_{L^2}^2 + O(1)\epsilon^2\|b_L\|_{H^2}^2 \\
& + O(1)\epsilon^2\|b_R\|_{H^2}^2 + O(1)\epsilon\|v_0 - \lambda u_0\|_{L^2[0,L]}^2 \\
& + \begin{cases} O(1)\epsilon^2\|U_0\|_{H^3}^2, & \text{for } \lambda > 0, \\ O(1)\epsilon\|U_0\|_{L^2}^2 + O(1)\epsilon^2\|U_0\|_{H^3}^2, & \text{for } \lambda < 0. \end{cases}
\end{aligned}$$

Remark 2.5. *(1) In the $\lambda < 0$ case, there is an interface layer near $x = 0^+$, while in the $\lambda > 0$ case, there is a boundary layer near $x = L^-$, so in both cases, the optimal convergence rate due to the boundary data is $O(1)\epsilon$, which is where the terms $O(1)\epsilon\|b_L(t)\|_{L^2}^2 + O(1)\epsilon\|b_R(t)\|_{L^2}^2$ come from.*

(2) The lower convergence rate in the case of $\lambda < 0$ is due to the presence of an interface layer near $x = 0^+$ generated by the initial data.

(3) $O(1)\epsilon\|v_0 - \lambda u_0\|_{L^2[0,L]}^2$ comes from the initial layer in v.

2.4 Error estimate for the domain decomposition method in the linear case: the homogeneous initial data

In this and the next sections, we will give a rigorous justification of the domain decomposition method for linear problems, where $f(u) = \lambda u$, for $|\lambda| < 1$ a constant. We first represent the exact solution to the original system (2.1)–(2.4) by the Laplace Transform, and then study the stiff wellposedness and the asymptotic convergence followed by direct calculations.

Denote

$$U^\epsilon = \begin{pmatrix} u^\epsilon \\ v^\epsilon \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 0 \\ \lambda & -1 \end{pmatrix}.$$

Here we consider system (2.1) with zero initial data (2.3), i.e., $u_0(x) = 0$, $v_0(x) = 0$ and nonzero boundary data (2.5). In this case one can focus on the boundary layer effects and avoid the interactions between the initial and boundary layers.

2.4.1 Solution by the Laplace Transform

When (2.1) is linear, i.e., $f(u) = \lambda u$, one can find the exact solution of (2.1)–(2.4) by the Laplace Transform. Let

$$\hat{U}^\epsilon(x,\xi) = \mathcal{L}(U^\epsilon) = \int_0^\infty e^{-\xi t} U^\epsilon(x,t)dt, \quad \text{Re}(\xi) > 0.$$

Here $\xi = \alpha + i\beta$, then $\mathcal{L}(\partial_t U^\epsilon) = \xi \hat{U}^\epsilon - U^\epsilon(x,0) = \xi \hat{U}^\epsilon(x,\xi)$. With the homogeneous initial condition, system (2.1)–(2.5) becomes

$$\partial_x \hat{U}^\epsilon = \frac{1}{\epsilon(x)} A^{-1}(S - \epsilon(x)\xi I)\hat{U}^\epsilon = \frac{1}{\epsilon(x)} M(\epsilon(x)\xi)\hat{U}^\epsilon, \tag{2.17}$$

$$\hat{u}^\epsilon(-L,\xi) = \hat{b}_L(\xi), \ \ \hat{u}^\epsilon(L,\xi) = \hat{b}_R(\xi), \tag{2.18}$$

where matrix

$$M(\xi) = A^{-1}(S - \epsilon(x)\xi I) \tag{2.19}$$

has two eigenvalues

$$\mu_\pm(\xi) = \frac{\lambda \pm \sqrt{\lambda^2 + 4\xi(1+\xi)}}{2}, \tag{2.20}$$

and two corresponding eigenvectors

$$\begin{pmatrix} 1 \\ \frac{\mu_\mp(\xi)}{1+\xi} \end{pmatrix} = \begin{pmatrix} 1 \\ g_\mp(\xi) \end{pmatrix}. \tag{2.21}$$

Thus the solution of (2.17) (2.18) can be written as:

$$\begin{cases} \hat{U}^\epsilon(x,\xi) = c_1 e^{\mu_-(\xi)x}\begin{pmatrix} 1 \\ g_+(\xi)\end{pmatrix} + c_2 e^{\mu_+(\xi)x}\begin{pmatrix} 1 \\ g_-(\xi)\end{pmatrix} & \text{for } x<0,\ \epsilon(x)=1; \\ \\ \hat{U}^\epsilon(x,\xi) = c_3 e^{\mu_-(\epsilon\xi)\frac{x}{\epsilon}}\begin{pmatrix} 1 \\ g_+(\epsilon\xi)\end{pmatrix} + c_4 e^{\mu_+(\epsilon\xi)\frac{x}{\epsilon}}\begin{pmatrix} 1 \\ g_-(\epsilon\xi)\end{pmatrix} & \text{for } x>0,\ \epsilon(x)=\epsilon, \end{cases} \tag{2.22}$$

where the coefficient c_1, c_2, c_3, c_4 are determined by the boundary conditions:

$$c_1 e^{-\mu_-(\xi)L} + c_2 e^{-\mu_+(\xi)L} = \hat{b}_L(\xi), \tag{2.23}$$

$$c_3 e^{\mu_-(\xi\epsilon)\frac{L}{\epsilon}} + c_4 e^{\mu_+(\xi\epsilon)\frac{L}{\epsilon}} = \hat{b}_R(\xi). \tag{2.24}$$

By continuity at the interface, one has

$$c_1 + c_2 = c_3 + c_4, \tag{2.25}$$

$$c_1 g_+(\xi) + c_2 g_-(\xi) = c_3 g_+(\epsilon\xi) + c_4 g_-(\epsilon\xi). \tag{2.26}$$

From (2.23)–(2.26), one sees that c_1–c_4 are uniquely determined. Denote

$$c_3 = Ec_1 + Fc_2, \tag{2.27}$$

$$c_4 = Gc_1 + Hc_2, \tag{2.28}$$

where

$$E = \frac{g_+(\xi) - g_-(\epsilon\xi)}{g_+(\epsilon\xi) - g_-(\epsilon\xi)},\ F = \frac{g_-(\xi) - g_-(\epsilon\xi)}{g_+(\epsilon\xi) - g_-(\epsilon\xi)},\ G = \frac{g_+(\xi) - g_+(\epsilon\xi)}{g_-(\epsilon\xi) - g_+(\epsilon\xi)},\ H = \frac{g_-(\xi) - g_+(\epsilon\xi)}{g_-(\epsilon\xi) - g_+(\epsilon\xi)}.$$

Plugging (2.27)–(2.28) into (2.23)–(2.24), one has

$$c_1 = \frac{\hat{b}_R(\xi) e^{-\mu_+(\xi)L} - \hat{b}_L(\xi)(Fe^{\mu_-(\epsilon\xi)\frac{L}{\epsilon}} + He^{\mu_+(\epsilon\xi)\frac{L}{\epsilon}})}{(Ee^{\mu_-(\epsilon\xi)\frac{L}{\epsilon}} + Ge^{\mu_+(\epsilon\xi)\frac{L}{\epsilon}})e^{-\mu_+(\xi)L} - (Fe^{\mu_-(\epsilon\xi)\frac{L}{\epsilon}} + He^{\mu_+(\epsilon\xi)\frac{L}{\epsilon}})e^{-\mu_-(\xi)L}}, \tag{2.29}$$

$$c_2 = \frac{\hat{b}_R(\xi) e^{-\mu_-(\xi)L} - \hat{b}_L(\xi)(Ee^{\mu_-(\epsilon\xi)\frac{L}{\epsilon}} + Ge^{\mu_+(\epsilon\xi)\frac{L}{\epsilon}})}{(Fe^{\mu_-(\epsilon\xi)\frac{L}{\epsilon}} + He^{\mu_+(\epsilon\xi)\frac{L}{\epsilon}})e^{-\mu_-(\xi)L} - (Ee^{\mu_-(\epsilon\xi)\frac{L}{\epsilon}} + Ge^{\mu_+(\epsilon\xi)\frac{L}{\epsilon}})e^{-\mu_+(\xi)L}}. \tag{2.30}$$

2.4.2 Stiff well-posedness

We first summarize some properties of the eigenvalues $\mu_\pm(\xi)$ in (2.20) and $g_\pm(\xi)$ appeared in the eigenvector in (2.21), which will be heavily used hereafter. We then prove the stiff well-posedness stated in Theorem 2.3.

First, we give some bounds on $\mu_\pm(\xi)$.

Lemma 2.6. *Under the subcharacteristic condition $|\lambda| < 1$, one has*

$$(1)\ |\lambda|(1+2\alpha) \le Re\ \sqrt{\lambda^2 + 4\xi(1+\xi)} \le 1+2\alpha,\ \textit{for } Re\ (\xi) = \alpha \ge 0; \tag{2.31}$$

$$(2)\ Re\ \mu_+(\xi) > 0,\ \ Re\ \mu_-(\xi) < 0; \tag{2.32}$$

$$(3)\ \textit{when } \lambda < 0,\ 2Re\ \mu_-(\epsilon\xi) \le -2|\lambda|,\ 2Re\ \mu_+(\epsilon\xi) \ge -2\epsilon\lambda\alpha; \tag{2.33}$$

$$\textit{when } \lambda > 0,\ 2Re\ \mu_-(\epsilon\xi) \le -2\epsilon\lambda\alpha,\ 2Re\ \mu_+(\epsilon\xi) \ge 2\lambda. \tag{2.34}$$

For the proof of the lemma, please refer to [103].

Now we give bounds and asymptotic behavior of $g_\pm(\epsilon\xi)$.

Lemma 2.7. *Under the subcharacteristic condition $|\lambda| < 1$, one has*

(1) For $\lambda > 0$, $g_-(\epsilon\xi) = O(1)\epsilon\xi$, and $0 < C_1 \le |g_+(\epsilon\xi)| \le C_2$, here C_1 and C_2 are two positive constants, and $g_+(\epsilon\xi) - \lambda = O(1)\epsilon\xi$;

(2) For $\lambda < 0$, $g_+(\epsilon\xi) = O(1)\epsilon\xi$, and $0 < C_3 \le |g_-(\epsilon\xi)| \le C_4$, here C_3 and C_4 are two positive constants, and $g_-(\epsilon\xi) - \lambda = O(1)\epsilon\xi$;

(3) $g_\pm(\xi) - g_\pm(\epsilon\xi)$, $g_+(\xi) - g_-(\epsilon\xi)$, $g_+(\epsilon\xi) - g_-(\xi)$ are uniformly bounded with respect to both ϵ and ξ, and they are bounded away from zero for $Re\xi = \alpha > 0$.

Proof. (1). When $\lambda > 0$, from definition (2.21), one sees

$$g_-(\epsilon\xi) = \frac{\lambda - \sqrt{\lambda^2 + 4\epsilon\xi(1+\epsilon\xi)}}{2(1+\epsilon\xi)} = \frac{-2\epsilon\xi}{\lambda + \sqrt{\lambda^2 + 4\epsilon\xi(1+\epsilon\xi)}} = O(1)\epsilon\xi,$$

and

$$g_+(\epsilon\xi) = \frac{\lambda + \sqrt{\lambda^2 + 4\epsilon\xi(1+\epsilon\xi)}}{2(1+\epsilon\xi)}.$$

In order to prove that $g_+(\epsilon\xi)$ is uniformly bounded with respect to $\epsilon\xi$, and the denominator is nonzero, one just needs to check what happens when $|\epsilon\xi|$ goes to 0 or ∞. Let $\epsilon\xi = re^{i\theta}$, one sees that when $|\epsilon\xi| \to 0$, i.e., when $r \to 0$, $|g_+(\epsilon\xi)| \to \lambda$; when $|\epsilon\xi| \to \infty$, i.e., when $|r| \to \infty$, one has

$$|g_+(\epsilon\xi)| \to \left| \frac{\sqrt{\lambda^2 + 4re^{i\theta}(1+re^{i\theta})}}{2(1+re^{i\theta})} \right| \to (\cos^2 2\theta + \sin^4\theta)^{\frac{1}{4}},$$

which is bounded and nonzero. Moreover,

$$g_+(\epsilon\xi) - \lambda = \frac{2(1-\lambda^2)\epsilon\xi}{\sqrt{\lambda^2 + 4\epsilon\xi(1+\epsilon\xi)} + \lambda(1+2\epsilon\xi)} = O(1)\epsilon\xi.$$

(2). When $\lambda < 0$, similarly one has

$$g_+(\epsilon\xi) = \frac{\lambda + \sqrt{\lambda^2 + 4\epsilon\xi(1+\epsilon\xi)}}{2(1+\epsilon\xi)} = \frac{-2\epsilon\xi}{\lambda - \sqrt{\lambda^2 + 4\epsilon\xi(1+\epsilon\xi)}} = O(1)\epsilon\xi,$$

and

$$g_-(\epsilon\xi) = \frac{\lambda - \sqrt{\lambda^2 + 4\epsilon\xi(1+\epsilon\xi)}}{2(1+\epsilon\xi)}.$$

In the same way as in (1), one can prove that $g_-(\epsilon\xi)$ is uniformly bounded in $\epsilon\xi$.

(3). Note

$$g_+(\xi) - g_-(\epsilon\xi) = \frac{\lambda\xi(\epsilon-1) + (1+\epsilon\xi)\sqrt{\lambda^2 + 4\xi(1+\xi)} + (1+\xi)\sqrt{\lambda^2 + 4\epsilon\xi(1+\epsilon\xi)}}{(1+\xi)(1+\epsilon\xi)}.$$

Let $\xi = re^{\theta}$, then when $\epsilon \to 0$, and $|r| \to 0$, one has $|g_+(\xi) - g_-(\epsilon\xi)| \to 2|\lambda|$. When $\epsilon \to 0$, $|r| \to \infty$, and $\epsilon|r| \to 0$, one has

$$|g_+(\xi) - g_-(\epsilon\xi)| \to \left| \frac{\lambda + \sqrt{\lambda^2 + 4\xi(1+\xi)}}{1+\xi} \right| \to \frac{1}{2}(\cos^2 2\theta + \sin^4\theta)^{\frac{1}{4}},$$

which is bounded and nonzero. When $\epsilon \to 0$, $|r| \to \infty$, and $\epsilon|r| \to \infty$, one can still prove that $|g_+(\xi) - g_-(\epsilon\xi)|$ is uniformly bounded away from 0, but the detailed calculation will be omitted. Similarly, one can prove the same result for $g_+(\epsilon\xi) - g_-(\xi)$. As for $g_+(\xi) - g_+(\epsilon\xi)$, notice

$$g_+(\xi) - g_+(\epsilon\xi) = \frac{\lambda\xi(\epsilon - 1) + (1 + \epsilon\xi)\sqrt{\lambda^2 + 4\xi(1 + \xi)} - (1 + \xi)\sqrt{\lambda^2 + 4\epsilon\xi(1 + \epsilon\xi)}}{(1 + \xi)(1 + \epsilon\xi)},$$

then following the same procedure as above, it is not hard to check that it is uniformly bounded as $\epsilon \to 0$, and $|\xi| \to \infty$. Moreover, when $\epsilon \to 0$, $|\xi| \to \alpha$,

$$|g_+(\xi) - g_+(\epsilon\xi)| \to \left|\frac{-\lambda\alpha + \sqrt{\lambda^2 + 4\alpha(1 + \alpha)} - (1 + \alpha)\lambda}{1 + \alpha}\right|,$$

which is nonzero, one can arrive at the same conclusion for $g_-(\xi) - g_-(\epsilon\xi)$. □

Remark 2.8. *(1) We will fix $Re\xi = \alpha > 0$ from now on.*

(2) From definition (2.22), one sees that, when $\lambda > 0$, by (2.34), there is a boundary layer near $x = L$, and on the other hand, when $\lambda < 0$, by (2.33), there is an interface layer near $x = 0$. This observation will play an important role in subsequent proofs.

Now we prove the theorem about the stiff well-posedness.

Proof. We use solution (2.22) of the original system (2.1) given by the Laplace Transform. Consider the integral:

$$\begin{aligned}\int_{-L}^{L} dx \int_{-\infty}^{\infty} |\hat{U}^\epsilon(x,\xi)|^2 d\beta \quad = \quad & \int_{-L}^{0} e^{2Re\mu_-(\xi)x} dx \int_{-\infty}^{\infty} |c_1|^2(1 + |g_+(\xi)|^2) d\beta \\ & + \int_{-L}^{0} e^{2Re\mu_+(\xi)x} dx \int_{-\infty}^{\infty} |c_2|^2(1 + |g_-(\xi)|^2) d\beta \\ & + \int_{0}^{L} e^{2Re\mu_-(\epsilon\xi)\frac{x}{\epsilon}} dx \int_{-\infty}^{\infty} |c_3|^2(1 + |g_+(\epsilon\xi)|^2) d\beta \\ & + \int_{0}^{L} dx \int_{-\infty}^{\infty} |c_4 e^{\mu_+(\epsilon\xi)\frac{x}{\epsilon}}|^2(1 + |g_-(\epsilon\xi)|^2) d\beta.\end{aligned}$$

By Lemma 2.7 one sees E, F, G, and H in (2.27)–(2.28) are uniformly bounded away from 0. And from (2.29), (2.30), (2.27) and (2.28) one gets

$$c_1,\ c_2,\ c_3,\ c_4 = O(1)(\hat{b}_L(\xi) + \hat{b}_R(\xi)),$$

and moreover, from (2.24),

$$e^{\mu_+(\epsilon\xi)\frac{L}{\epsilon}} c_4 = (\hat{b}_R(\xi) - c_3 e^{\mu_-(\epsilon\xi)\frac{L}{\epsilon}}), \tag{2.35}$$

so $e^{\mu_+(\epsilon\xi)\frac{L}{\epsilon}} c_4 = O(1) e^{\mu_-(\epsilon\xi)\frac{L}{\epsilon}} \hat{b}_L(\xi) + O(1)\hat{b}_R(\xi)$. Therefore

$$\int_{-L}^{L} dx \int_{-\infty}^{\infty} |\hat{U}^\epsilon(x,\xi)|^2 d\beta \le O(1) \int_{-\infty}^{\infty} \left(|\hat{b}_L(\xi)|^2 + |\hat{b}_R(\xi)|^2 \right) d\beta. \tag{2.36}$$

Then by Parseval's identity:

$$\int_0^\infty e^{-2\alpha t} |U^\epsilon(x,t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{U}^\epsilon(x, \alpha + i\beta)|^2 d\beta, \tag{2.37}$$

the stiff well-posedness, as stated in Theorem 2.3, now follows. □

2.4.3 Asymptotic convergence and error estimates

Next we turn to the question of the asymptotic convergence and error estimate stated in Theorem 2.4. To prove the theorem, still we compare the analytical solution of the domain decomposition problem (2.12)–(2.15) with the original problem given in section 4.1 with the help of the Laplace Transform.

Proof. Consider the case $\lambda < 0$ first. The solution of (2.12) is

$$u^r(x,t) = \begin{cases} 0, & x - L \le \lambda t, \\ b_R\left(t + \frac{1}{\lambda}(L-x)\right), & x - L \ge \lambda t,\ 0 \le x \le L. \end{cases}$$

Using the Laplace Transform, it becomes

$$\hat{u}^r(x,\xi) = \hat{b}_R(\xi)e^{\frac{\xi}{\lambda}(L-x)}, \quad \text{for } x > 0. \tag{2.38}$$

The solution of (2.13) is

$$\hat{U}^l(x,\xi) = d_1 e^{\mu_-(\xi)x}\begin{pmatrix} 1 \\ g_+(\xi) \end{pmatrix} + d_2 e^{\mu_+(\xi)x}\begin{pmatrix} 1 \\ g_-(\xi) \end{pmatrix}, \tag{2.39}$$

where d_1 and d_2 are determined by

$$d_1 e^{-\mu_-(\xi)L} + d_2 e^{-\mu_+(\xi)L} = \hat{b}_L(\xi), \tag{2.40}$$

$$d_1 g_+(\xi) + d_2 g_-(\xi) = \lambda \hat{b}_R(\xi) e^{\frac{\xi}{\lambda}L}. \tag{2.41}$$

Now compare the first expression of (2.22) with (2.39), and the second with (2.38) respectively. For $x \in [0,L]$, using (2.22) and (2.38), one gets for the first component u

$$\begin{aligned}
&\int_0^L dx \int_{-\infty}^{\infty} |\hat{u}^r - \hat{u}^\epsilon|^2 d\beta = \int_0^L dx \int_{-\infty}^{\infty} |c_3 e^{\mu_-(\epsilon\xi)\frac{x}{\epsilon}} + c_4 e^{\mu_+(\epsilon\xi)\frac{x}{\epsilon}} - \hat{b}_R e^{\frac{\xi}{\lambda}(L-x)}|^2 d\beta \\
\leq & \int_0^L dx \int_{-\infty}^{\infty} |c_3(e^{\mu_-(\epsilon\xi)\frac{x}{\epsilon}} - e^{\mu_+(\epsilon\xi)\frac{x-L}{\epsilon}} e^{\mu_-(\epsilon\xi)\frac{L}{\epsilon}})|^2 d\beta + \int_0^L dx \int_{-\infty}^{\infty} |\hat{b}_R(\xi)|^2 |e^{\mu_+(\epsilon\xi)\frac{x-L}{\epsilon}} - e^{\frac{\xi}{\lambda}(L-x)}|^2 d\beta \\
= & \ I_1 + I_2.
\end{aligned}$$

Here the first inequality was derived by substituting c_4 in (2.35). For I_1, it is easy to see:

$$I_1 \leq O(1)\int_{-\infty}^{\infty} |c_3(\xi)|^2 d\beta \left(\int_0^L e^{2\mathrm{Re}\mu_-(\epsilon\xi)\frac{x}{\epsilon}} dx + e^{2\mathrm{Re}\mu_-(\epsilon\xi)\frac{L}{\epsilon}} \int_0^L e^{2\mathrm{Re}\mu_+(\epsilon\xi)\frac{x-L}{\epsilon}} dx \right).$$

Then by (2.33) one gets the estimate for I_1 as:

$$\begin{aligned}
I_1 &\leq O(1)\epsilon \int_{-\infty}^{\infty} |c_3(\xi)|^2 d\beta \\
&= O(1)\epsilon \int_{-\infty}^{\infty} (|\hat{b}_L|^2 + |\hat{b}_R|^2) d\beta \leq O(1)\epsilon (\|b_L\|_{L^2}^2 + \|b_R\|_{L^2}^2). \qquad (2.42)
\end{aligned}$$

Note here in I_1, the term that contains $e^{\mu_-(\epsilon\xi)\frac{x}{\epsilon}}$ is the result of interface layer, which drops the L^2 convergence rate down to $\epsilon^{\frac{1}{2}}$.

For I_2, notice

$$\begin{aligned}&\int_0^L |e^{\mu_+(\epsilon\xi)\frac{x-L}{\epsilon}} - e^{\frac{\xi}{\lambda}(L-x)}|^2 dx\\ =\;& \int_0^L |e^{\mu_+(\epsilon\xi)\frac{-x}{\epsilon}} - e^{\frac{\xi}{\lambda}x}|^2 dx \le \int_0^\infty |e^{\mu_+(\epsilon\xi)\frac{-x}{\epsilon}} - e^{\frac{\xi}{\lambda}x}|^2 dx = O(1)\left|\frac{\mu_+(\epsilon\xi)}{\epsilon} + \frac{\xi}{\lambda}\right|^2 \end{aligned} \tag{2.43}$$

one has

$$\begin{aligned} I_2 &\le \int_{-\infty}^{\infty} O(1)\left|\frac{\mu_+(\epsilon\xi)}{\epsilon} + \frac{\xi}{\lambda}|^2|\hat{b}_R(\xi)\right|^2 d\beta = O(1)\epsilon^2 \int_{-\infty}^{\infty} |\xi|^4|\hat{b}_R(\xi)|^2 d\beta \\ &\le O(1)\epsilon^2\|b_R\|_{H^2}^2. \end{aligned} \tag{2.44}$$

Here we use the fact

$$\frac{\mu_+(\epsilon\xi)}{\epsilon} + \frac{\xi}{\lambda} = \frac{2\epsilon\xi^2(1-\lambda^2)}{\lambda(\lambda^2 + 2\epsilon\xi - \lambda\sqrt{\lambda^2 + 4\epsilon\xi(1+\epsilon\xi)})} = O(1)\epsilon\xi^2, \tag{2.45}$$

and also we assume that $b_R(t) \in H^2(\mathbb{R}^+)$ and $b_R(t)$ satisfies the compatibility condition $b_R(0) = b'_R(0) = 0$. Adding I_1 and I_2 yields

$$\int_0^L dx \int_{-\infty}^{\infty} |\hat{u}^l - \hat{u}^\epsilon|^2 d\beta \le O(1)\epsilon(\|b_L\|_{L^2}^2 + \|b_R\|_{L^2}^2) + O(1)\epsilon^2\ \|b_R\|_{H^2}^2\ . \tag{2.46}$$

When $x \in [-L, 0]$ the difference between (2.22) and (2.39) is the difference between the coefficients, i.e.

$$\int_{-L}^0 dx \int_{-\infty}^{\infty} |\hat{U}^l - \hat{U}^\epsilon|^2 d\beta \;=\; O(1)\int_{-\infty}^{\infty} (|d_1 - c_1|^2 + |d_2 - c_2|^2) d\beta.$$

Compare (2.23)–(2.26) with (2.40)–(2.41), one finds

$$|c_1 - d_1| = O(1)\epsilon\xi(\hat{b}_L + \hat{b}_R), \quad |c_2 - d_2| = O(1)\epsilon\xi(\hat{b}_L + \hat{b}_R),$$

after using Lemma 2.7 and some basic calculations. The details are omitted.

Therefore,

$$\begin{aligned}\int_{-L}^{0} dx \int_{-\infty}^{\infty} |\hat{U}^l - \hat{U}^\epsilon|^2 d\beta &\le O(1)\epsilon^2 \int_{-\infty}^{\infty} (|\xi \hat{b}_L(\xi)|^2 + |\xi \hat{b}_R(\xi)|^2) d\xi \\ &\le O(1)\epsilon^2 (\|b_L\|_{H^1}^2 + \|b_R\|_{H^1}^2). \end{aligned} \tag{2.47}$$

Here we used the assumption that $b_L(t) \in H^1(\mathbb{R}^+)$, and $b_L(t)$ satisfies $b_L(0) = 0$. Now we are done with the $\lambda < 0$ case.

For $\lambda > 0$, the proof is similar. First the solution to (2.14) is

$$\hat{U}^l(x,\xi) = k_1 e^{\mu_-(\xi)x} \begin{pmatrix} 1 \\ g_+(\xi) \end{pmatrix} + k_2 e^{\mu_+(\xi)x} \begin{pmatrix} 1 \\ g_-(\xi) \end{pmatrix}, \ -L \le x \le 0, \tag{2.48}$$

where k_1 and k_2 are determined by

$$k_1 e^{-\mu_-(\xi)L} + k_2 e^{-\mu_+(\xi)L} = \hat{b}_L(\xi), \tag{2.49}$$

$$k_1(\lambda - g_+(\xi)) + k_2(\lambda - g_-(\xi)) = 0. \tag{2.50}$$

When $0 \le x \le L$, the solution to (2.15) is

$$u^r(x,t) = \begin{cases} 0, & \lambda t \le x \le L, \\ u^l(0, t - \frac{x}{\lambda}), & 0 \le x \le \lambda t, 0 \le x \le L. \end{cases}$$

So after using the Laplace Transform, one gets:

$$\hat{u}^r(x,\xi) = e^{-\xi\frac{x}{\lambda}} \hat{u}^l(0^-,\xi) = e^{-\xi\frac{x}{\lambda}}(k_1 + k_2). \tag{2.51}$$

Now compare (2.48) and (2.51) with (2.22). The difference between (2.48) and the first expression of (2.22) is again the difference between the coefficients. Thus

$$\int_{-L}^{0} dx \int_{-\infty}^{\infty} |\hat{U}^l - \hat{U}^\epsilon|^2 d\beta = O(1) \int_{-\infty}^{\infty} (|k_1 - c_1|^2 + |k_2 - c_2|^2) d\beta.$$

Comparing (2.23)–(2.26) with (2.49) (2.50), one finds

$$|c_1 - k_1| = O(1)\epsilon\xi\hat{b}_L, \ |c_2 - k_2| = O(1)\epsilon\xi\hat{b}_L.$$

Therefore,

$$\begin{aligned} \int_{-L}^{0} dx \int_{-\infty}^{\infty} |\hat{U}^l - \hat{U}^\epsilon|^2 d\beta &\leq O(1)\epsilon^2 \int_{-\infty}^{\infty} |\xi\hat{b}_L(\xi)|^2 d\xi \\ &\leq O(1)\epsilon^2 \|b_L\|_{H^1}^2. \end{aligned} \tag{2.52}$$

The difference between (2.51) and the second expression of (2.22) is estimated as follows:

$$\begin{aligned} & \int_0^L dx \int_{-\infty}^{\infty} |\hat{u}^r - \hat{u}^\epsilon|^2 d\beta \\ = & \int_0^L dx \int_{-\infty}^{\infty} |c_3 e^{\mu_-(\epsilon\xi)\frac{x}{\epsilon}} + c_4 e^{\mu_+(\epsilon\xi)\frac{x}{\epsilon}} - (k_1 + k_2) e^{-\xi\frac{x}{\lambda}}|^2 d\beta \\ = & \int_0^L dx \int_{-\infty}^{\infty} |c_3 (e^{\mu_-(\epsilon\xi)\frac{x}{\epsilon}} - e^{\mu_+(\epsilon\xi)\frac{x-L}{\epsilon}} e^{\mu_-(\epsilon\xi)\frac{L}{\epsilon}}) + \hat{b}_R e^{\mu_+(\epsilon\xi)\frac{x-L}{\epsilon}} - (k_1 + k_2) e^{-\xi\frac{x}{\lambda}}|^2 d\beta \\ \leq & J_1 + J_2 + J_3. \end{aligned}$$

To get the second equality, we again use (2.35). First,

$$\begin{aligned} J_1 &= \int_{-\infty}^{\infty} |\hat{b}_R(\xi)|^2 d\beta \int_0^L e^{2\mathrm{Re}\mu_+(\epsilon\xi)\frac{x-L}{\epsilon}} dx \\ &\leq O(1)\epsilon \|b_R(t)\|_{L^2}^2, \end{aligned} \tag{2.53}$$

where the inequalities (2.32), (2.33) and (2.34) were used. For J_2, one has:

$$\begin{aligned} J_2 &= \int_0^L dx \int_{-\infty}^{\infty} \left| [c_3 - (k_1 + k_2)] e^{\mu_-(\epsilon\xi)\frac{x}{\epsilon}} - c_3 e^{\mu_+(\epsilon\xi)\frac{x-L}{\epsilon}} e^{\mu_-(\epsilon\xi)\frac{L}{\epsilon}} \right|^2 \\ &\leq \int_0^L dx \int_{-\infty}^{\infty} |c_3 - k_1 - k_2|^2 e^{2Re\mu_-(\epsilon\xi)\frac{x}{\epsilon}} d\beta + O(1)\epsilon \int_{-\infty}^{\infty} |c_3|^2 d\beta. \end{aligned}$$

Since $c_3 + c_4 = c_1 + c_2 = k_1 + k_2 + O(1)\epsilon\xi\hat{b}_L(\xi)$, $c_4 = e^{-\mu_+(\epsilon\xi)\frac{L}{\epsilon}}(\hat{b}_R(\xi) - c_3 e^{\mu_-(\epsilon\xi)\frac{L}{\epsilon}})$, one has $|c_3 - k_1 - k_2|^2 = O(1)\epsilon^2|\xi\hat{b}_L(\xi)|^2$. Therefore,

$$J_2 \leq O(1)\epsilon^2 \|b_L\|_{H^1}^2 + O(1)\epsilon \|b_L\|_{L^2}^2. \tag{2.54}$$

Note here the convergence rate is ϵ, which is caused by the boundary layer effect of $e^{\mu_+(\epsilon\xi)\frac{x-L}{\epsilon}}$ in J_1 and J_2. The remaining part J_3 is

$$\begin{aligned} J_3 &= \int_0^L dx \int_{-\infty}^{\infty} |(k_1+k_2)e^{\mu_-(\epsilon\xi)\frac{x}{\epsilon}} - (k_1+k_2)e^{-\xi\frac{x}{\lambda}}|^2 d\beta \\ &\leq O(1)\int_0^{\infty} |e^{\mu_-(\epsilon\xi)\frac{x}{\epsilon}} - e^{-\xi\frac{x}{\lambda}}|^2 dx \int_{-\infty}^{\infty} |k_1+k_2|^2 d\beta \\ &\leq O(1)\epsilon^2 \|b_L\|_{H^2}^2. \end{aligned} \tag{2.55}$$

The calculation here is similar to (2.44). In total, one gets

$$\int_0^L dx \int_{-\infty}^{\infty} |\hat{u}^r - \hat{u}^\epsilon|^2 d\beta \leq O(1)\epsilon(\|b_L\|_{L^2}^2 + \|b_R\|_{L^2}^2) + O(1)\epsilon^2 \ \|b_L\|_{H^2}^2 \ . \tag{2.56}$$

To this end, we have proved Theorem 2.4 with zero initial data. □

Remark 2.9. *Here we jump to the estimation of the convergence rate, and omit the steps to prove the uniform convergence stated in Theorem 2.4, which is easily obtained by dominated convergence theorem.*

2.5 Error estimate for the domain decomposition method in the linear case: the inhomogeneous initial data

The case with inhomogeneous initial data is much more complicated. For clarity, we consider instead the Cauchy problem here, that is, $x \in (-\infty, \infty)$ instead of $[-L, L]$. A new idea here is to construct some related initial value problem and make use of the existing results about the Cauchy problem [103] to overcome the difficulties arisen in the Laplace Transform. With these two results, the problem with both boundary and initial data is straightforward, and details will be omitted.

2.5.1 Solution by the Laplace Transform

Again, we solve system (2.1) with $L = \infty$ by the Laplace Transform. Then (2.1) and (2.3) become:

$$\partial_x \hat{U}^\epsilon = \frac{1}{\epsilon(x)} M(\epsilon(x)\xi)\hat{U}^\epsilon + A^{-1}U_0(x), \tag{2.57}$$

where M is defined in (2.19). Then the general solution is:

$$\begin{cases} \hat{U}^\epsilon(x,\xi) = e^{M(\xi)x}(\hat{U}_L + \int_0^x e^{-M(\xi)y} A^{-1}U_0(y)dy) & \text{for } x<0,\ \epsilon(x)=1; \\ \\ \hat{U}^\epsilon(x,\xi) = e^{M(\epsilon\xi)\frac{x}{\epsilon}}(\hat{U}_R + \int_0^x e^{-M(\epsilon\xi)\frac{y}{\epsilon}} A^{-1}U_0(y)dy) & \text{for } x>0,\ \epsilon(x)=\epsilon, \end{cases} \tag{2.58}$$

where one can denote $e^{M(\xi)x}$ by

$$e^{M(\xi)x} = e^{\mu_+(\xi)x}\Phi_+(\xi) + e^{\mu_-(\xi)x}\Phi_-(\xi), \tag{2.59}$$

and $\Phi_\pm$ are defined by:

$$\Phi_+(\xi) = \frac{1}{g_+(\xi)-g_-(\xi)} \begin{pmatrix} 1 \\ \\ g_-(\xi) \end{pmatrix} (g_+(\xi), -1), \tag{2.60}$$

$$\Phi_-(\xi) = \frac{1}{g_+(\xi)-g_-(\xi)} \begin{pmatrix} 1 \\ \\ g_+(\xi) \end{pmatrix} (-g_-(\xi),\ 1). \tag{2.61}$$

Then (2.58) can be rewritten as:

$$\begin{cases} \hat{U}^\epsilon(x,\xi) = e^{\mu_+(\xi)x}\Phi_+(\xi)(\hat{U}_L(\xi) + \int_0^x e^{-\mu_+(\xi)y} A^{-1}U_0(y)dy) \\ \\ \quad +e^{\mu_-(\xi)x}\Phi_-(\xi)(\hat{U}_L(\xi) + \int_0^x e^{-\mu_-(\xi)y} A^{-1}U_0(y)dy) & \text{for } x<0,\ \epsilon(x)=1; \\ \\ \hat{U}^\epsilon(x,\xi) = e^{\mu_+(\epsilon\xi)\frac{x}{\epsilon}}\Phi_+(\epsilon\xi)(\hat{U}_R(\xi) + \int_0^x e^{-\mu_+(\epsilon\xi)\frac{y}{\epsilon}} A^{-1}U_0(y)dy) \\ \\ \quad +e^{\mu_-(\epsilon\xi)\frac{x}{\epsilon}}\Phi_-(\epsilon\xi)(\hat{U}_R(\xi) + \int_0^x e^{-\mu_-(\epsilon\xi)\frac{y}{\epsilon}} A^{-1}U_0(y)dy) & \text{for } x>0,\ \epsilon(x)=\epsilon. \end{cases} \tag{2.62}$$

Here $\hat{U}_L(\xi) = \begin{pmatrix} \hat{u}_L(\xi) \\ \hat{v}_L(\xi) \end{pmatrix}$ and $\hat{U}_R(\xi) = \begin{pmatrix} \hat{u}_R(\xi) \\ \hat{v}_R(\xi) \end{pmatrix}$ are two vectors independent of x, and defined by the boundary condition and interface conditions as follows.

First, when $x \to \infty$, $\hat{U}^\epsilon(x,\xi) \to 0$, one gets

$$(g_+(\epsilon\xi),\ -1) \begin{pmatrix} \hat{u}_R(\xi) \\ \hat{v}_R(\xi) \end{pmatrix} + \int_0^\infty e^{-\mu_+(\epsilon\xi)\frac{y}{\epsilon}} (g_+(\epsilon\xi),\ -1) \begin{pmatrix} v_0 \\ u_0 \end{pmatrix} (y) dy = 0,$$

that is,

$$g_+(\epsilon\xi)\hat{u}_R(\xi) - \hat{v}_R(\xi) + \int_0^\infty e^{-\mu_+(\epsilon\xi)\frac{y}{\epsilon}} (v_0(y)g_+(\epsilon\xi) - u_0(y))dy = 0. \tag{2.63}$$

When $x \to -\infty$, $\hat{U}^\epsilon(x,\xi) \to 0$, thus

$$(-g_-(\xi),\ 1) \begin{pmatrix} \hat{u}_L(\xi) \\ \hat{v}_L(\xi) \end{pmatrix} + \int_0^{-\infty} e^{-\mu_-(\xi)y} (-g_-(\xi),\ 1) \begin{pmatrix} v_0 \\ u_0 \end{pmatrix} (y) dy = 0,$$

that is,

$$-g_-(\xi)\hat{u}_L(\xi) + \hat{v}_L(\xi) + \int_0^{-\infty} e^{-\mu_-(\xi)y} (-v_0(y)g_-(\xi) + u_0(y))dy = 0. \tag{2.64}$$

Then by continuity, $\Phi_+(\xi)\hat{U}_L + \Phi_-(\xi)\hat{U}_L = \Phi_+(\epsilon\xi)\hat{U}_R + \Phi_-(\epsilon\xi)\hat{U}_R$, it is easy to get:

$$\hat{u}_L = \hat{u}_R,\ \hat{v}_L = \hat{v}_R. \tag{2.65}$$

Plugging (2.63) - (2.65) into (2.62), one ends up with a simplified version of (2.62):

$$\begin{aligned}\hat{U}^\epsilon(x,\xi) &= \frac{1}{g_+(\epsilon\xi)-g_-(\epsilon\xi)}\left\{\begin{pmatrix}1\\ g_-(\epsilon\xi)\end{pmatrix}\int_x^\infty e^{\mu_+(\epsilon\xi)\frac{x-y}{\epsilon}}(u_0(y)-v_0(y)g_+(\epsilon\xi))dy\right.\\ &+\begin{pmatrix}1\\ g_+(\epsilon\xi)\end{pmatrix}\int_0^x e^{\mu_-(\epsilon\xi)\frac{x-y}{\epsilon}}(u_0(y)-v_0(y)g_-(\epsilon\xi))dy\\ &\left.+\begin{pmatrix}1\\ g_+(\epsilon\xi)\end{pmatrix}e^{\mu_-(\epsilon\xi)\frac{x}{\epsilon}}(\hat{v}_R(\xi)-\hat{u}_R(\xi)g_-(\epsilon\xi))\right\}, \quad \text{for } x>0; \qquad (2.66)\end{aligned}$$

and

$$\begin{aligned}\hat{U}^\epsilon(x,\xi) &= \frac{1}{g_+(\xi)-g_-(\xi)}\left\{\begin{pmatrix}1\\ g_+(\xi)\end{pmatrix}\int_{-\infty}^x e^{\mu_-(\xi)(x-y)}(u_0(y)-v_0(y)g_-(\xi))dy\right.\\ &+\begin{pmatrix}1\\ g_-(\xi)\end{pmatrix}\int_x^0 e^{\mu_+(\xi)(x-y)}(u_0(y)-v_0(y)g_+(\xi))dy\\ &\left.+\begin{pmatrix}1\\ g_-(\xi)\end{pmatrix}e^{\mu_+(\xi)x}(-\hat{v}_L(\xi)+\hat{u}_L(\xi)g_+(\xi))\right\}, \quad \text{for } x<0. \qquad (2.67)\end{aligned}$$

2.5.2 The stiff well-posedness

Due to the nonzero initial data, it is hard to estimate the L^2 norm of the solution from the expression (2.66)–(2.67). So we take a detour to look at the initial value problem with initial data supported in the right (or left) half plane. For this initial value problem, one can solve it by the Fourier Transform, thus avoid the difficulties caused by the Laplace Transform. Without loss of generality, we consider $x>0$ here. The $x<0$ case is the same. First we have the following lemma.

Lemma 2.10. *Assume* $U^\epsilon_{IVP} = \begin{pmatrix} u^\epsilon_{IVP} \\ v^\epsilon_{IVP} \end{pmatrix}$ *is the solution to*

$$\begin{cases} u^\epsilon_t + v^\epsilon_x = 0, & (2.68a) \\ v^\epsilon_t + u^\epsilon_x = -\dfrac{1}{\epsilon}(v^\epsilon - \lambda u^\epsilon), & (2.68b) \\ u^\epsilon(x,0) = u_0(x), v^\epsilon(x,0) = v_0(x), & (2.68c) \end{cases}$$

here u_0 *and* v_0 *are supported in* $[0,\infty)$*. Then the solution, after the Laplace Transform, is*

$$\begin{aligned} \hat{U}^\epsilon_{IVP}(x,\xi) &= \frac{1}{g_+(\epsilon\xi) - g_-(\epsilon\xi)} \left\{ \begin{pmatrix} 1 \\ g_-(\epsilon\xi) \end{pmatrix} \int_x^\infty e^{\mu_+(\epsilon\xi)\frac{x-y}{\epsilon}} \left(u_0(y) - v_0(y) g_+(\epsilon\xi)\right) dy \right. \\ &\left. + \begin{pmatrix} 1 \\ g_+(\epsilon\xi) \end{pmatrix} \int_0^x e^{\mu_-(\epsilon\xi)\frac{x-y}{\epsilon}} \left(u_0(y) - v_0(y) g_-(\epsilon\xi)\right) dy \right\}, \end{aligned} \tag{2.69}$$

and the following inequality holds:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\hat{U}^\epsilon_{IVP}(x,\xi)|^2 dx d\beta \le O(1) \int_0^\infty |U_0(x)|^2 dx. \tag{2.70}$$

Proof. First solution (2.69) is obtained in the same way as (2.66), so we will omit the details. Then if the Fourier Transform w.r.t x is used instead of the Laplace Transform w.r.t t in this case, one gets [103]

$$\int_{-\infty}^{\infty} |U^\epsilon_{IVP}(x,t)|^2 dx \le O(1) \int_0^\infty |U_0(x)|^2 dx, \quad \forall t > 0. \tag{2.71}$$

Integrating with respect to t gives

$$\int_0^\infty dt \int_{-\infty}^{\infty} e^{-2\alpha t} |U^\epsilon_{IVP}(x,t)|^2 dx \le O(1) \int_0^\infty |U_0(x)|^2 dx.$$

Then by Parseval's identity (2.37), one can prove the inequality. For more details, see [103]. □

One also needs to estimate $\int_0^\infty e^{-\mu_+(\epsilon\xi)\frac{y}{\epsilon}}(u_0(y)-v_0(y)g_+(\epsilon\xi))dy$ and $\int_0^{-\infty} e^{-\mu_-(\xi)y}$ $(u_0(y)-v_0(y)g_-(\xi))dy$ which appear in (2.63) and (2.64) respectively. The estimates of these two integrals are similar by using the energy estimate. So we only estimate the first integral here.

Lemma 2.11. *Let*

$$\hat{w}_{IBVP}(\epsilon\xi) = \int_0^\infty e^{-\mu_+(\epsilon\xi)\frac{y}{\epsilon}}(u_0(y)-v_0(y)g_+(\epsilon\xi))dy, \tag{2.72}$$

then

$$\int_{-\infty}^{\infty} |\hat{w}_{IBVP}(\epsilon\xi)|^2 d\beta \le O(1)\int_0^\infty |U_0(x)|^2 dx. \tag{2.73}$$

Proof. The idea of the proof follows that in [103]. We construct the following initial boundary value problem on the right half plane $x > 0$. Later one can see that $\hat{w}_{IBVP}(\epsilon\xi)$ can be expressed by the Laplace Transform of the boundary value of the following problem, thus can be bounded by the initial data. This is the key motivation of constructing the following system:

$$\begin{cases} u_t^\epsilon + v_x^\epsilon = 0, & (2.74a) \\ v_t^\epsilon + u_x^\epsilon = -\dfrac{1}{\epsilon}(v^\epsilon - \lambda u^\epsilon), & (2.74b) \\ u^\epsilon(x,0) = u_0(x), v^\epsilon(x,0) = v_0(x), & (2.74c) \\ B_u u^\epsilon(0,t) + B_v v^\epsilon(0,t) = 0. & (2.74d) \end{cases}$$

Here B_u and B_v are two constants that satisfy the so-called Stiff Kreiss Condition (SKC) [103]: $\frac{B_u}{B_v} \notin [-1, \frac{\lambda+|\lambda|}{2}]$. The Laplace Transform of the solution to this system can be written as:

$$\begin{aligned} \hat{U}^\epsilon_{IBVP}(x,\xi) &= e^{\mu_+(\epsilon\xi)\frac{x}{\epsilon}}\Phi_+(\epsilon\xi)\left[\hat{U}^\epsilon_{IBVP}(0,\xi) + \int_0^x e^{-\mu_+(\epsilon\xi)\frac{y}{\epsilon}} A^{-1}U_0(y)dy\right] \\ &\quad + e^{\mu_-(\epsilon\xi)\frac{x}{\epsilon}}\Phi_-(\epsilon\xi)\left[\hat{U}^\epsilon_{IBVP}(0,\xi) + \int_0^x e^{-\mu_-(\epsilon\xi)\frac{y}{\epsilon}} A^{-1}U_0(y)dy\right], \end{aligned} \tag{2.75}$$

where $\hat{U}^{\epsilon}_{IBVP}(0,\xi) = \begin{pmatrix} \hat{u}^{\epsilon}_{IBVP} \\ \hat{v}^{\epsilon}_{IBVP} \end{pmatrix}$ satisfies

$$\begin{cases} B_u \hat{u}^{\epsilon}_{IBVP}(0,\xi) + B_v \hat{v}^{\epsilon}_{IBVP}(0,\xi) = 0, & (2.76a) \\ \Phi_+(\epsilon\xi)\left(\hat{U}^{\epsilon}_{IBVP}(0,\xi) + \int_0^{\infty} e^{-\mu_+(\epsilon\xi)\frac{y}{\epsilon}} A^{-1} U_0(y)dy\right) = 0. & (2.76b) \end{cases}$$

From definition (2.72), the second condition (2.76b) can be written as

$$g_+(\epsilon\xi)\hat{u}^{\epsilon}_{IBVP}(0,\xi) - \hat{v}^{\epsilon}_{IBVP}(0,\xi) = \hat{w}_{IBVP}(\epsilon\xi),$$

thus

$$\hat{U}^{\epsilon}_{IBVP}(0,\xi) = \frac{\hat{w}_{IBVP}(\epsilon\xi)}{B_u + B_v g_+(\epsilon\xi)} \begin{pmatrix} B_v \\ -B_u \end{pmatrix}. \tag{2.77}$$

Now the energy estimate can be used to get the upper bound of $\int_0^T |U_{IBVP}(0,t)|^2 dt$. Let $H = \begin{pmatrix} 1 & -\lambda \\ -\lambda & 1 \end{pmatrix}$, multiply (2.74) by $e^{-2\alpha t}U^T H$, and integrate over $[0,T]\times[0,\infty)$, one has (here we omit the subscription and superscription for a while)

$$\begin{aligned} & \frac{1}{2}\int_0^{\infty} (U,HU)(x,T)e^{-2\alpha T}dx + \alpha\int_0^T\!\!\int_0^{\infty} (U,HU)(x,t)e^{-2\alpha t}dxdt \\ & +\frac{1}{\epsilon}\int_0^T\!\!\int_0^{\infty} (v-\lambda u)^2 e^{-2\alpha t}dxdt + \frac{1}{2}\int_0^T (\lambda u^2 - 2uv + \lambda v^2)(0,t)e^{-2\alpha t}dt \\ = \;& \frac{1}{2}\int_0^{\infty} (U_0(x), HU_0(x))dx. \end{aligned}$$

One needs to choose the boundary condition such that $\lambda u(0,t)^2 - 2u(0,t)v(0,t) + \lambda v(0,t)^2 \geq c|U(0,t)|^2$, where c is a bounded constant. Later we will show that this kind of boundary condition exits and it is a subclass of SKC. Then one can get

$$\int_0^T |U^{\epsilon}_{IBVP}(0,t)|^2 e^{-2\alpha t}dt \leq O(1)\int_0^{\infty} |U_0(x)|^2 dx. \tag{2.78}$$

Let $T \to \infty$, then

$$\int_0^{\infty} |U^{\epsilon}_{IBVP}(0,t)|^2 e^{-2\alpha t} dt \le O(1) \int_0^{\infty} |U_0(x)|^2 dx. \tag{2.79}$$

By Parseval's identity and (2.77) (2.79), one obtains (2.73). As for the boundary condition, there are plenty of choices. Any B_u and B_v that satisfy

$$\begin{aligned}
&\frac{B_u}{B_v} > -\frac{1}{\lambda}(1-\sqrt{1-\lambda^2}) \text{ or } \frac{B_u}{B_v} < -\frac{1}{\lambda}(1+\sqrt{1-\lambda^2}), \quad \text{for } \lambda > 0,\\
&-\frac{1}{\lambda}(1-\sqrt{1-\lambda^2}) < \frac{B_u}{B_v} < -\frac{1}{\lambda}(1+\sqrt{1-\lambda^2}), \quad \text{for } \lambda < 0,\\
&\frac{B_u}{B_v} > 0, \quad \text{for } \lambda = 0,
\end{aligned}$$

will work, and it is not hard to see it is a subclass of the SKC. □

Similarly, we have the following corollary.

Corollary 2.12. *Let*

$$\hat{w}_{IBVP2}(\xi) = \int_0^{-\infty} e^{-\mu_-(\xi)y}(u_0(y) - v_0(y)g_-(\xi))dy, \tag{2.80}$$

then

$$\int_{-\infty}^{\infty} |\hat{w}_{IBVP2}(\xi)|^2 d\beta \le O(1) \int_{-\infty}^{0} |U_0(x)|^2 dx. \tag{2.81}$$

Now we go back to the proof of Theorem 2.3 of the stiff well-posedness with nonzero initial data and when the problem is set in $(-\infty,\infty)$ instead of $[-L,L]$.

Proof. When $x > 0$, from the solution (2.66) one gets

$$\begin{aligned}
&\int_0^{\infty} dx \int_{-\infty}^{\infty} |\hat{U}^{\epsilon}(x,\xi)|^2 d\beta \le \int_0^{\infty} dx \int_{-\infty}^{\infty} |\hat{U}^{\epsilon}_{IVP}|^2 d\beta \\
&+ \int_0^{\infty} dx \int_{-\infty}^{\infty} \left| \begin{pmatrix} 1 \\ g_+(\epsilon\xi) \end{pmatrix} e^{\mu_-(\epsilon\xi)\frac{x}{\epsilon}} (v_R - g_-(\epsilon\xi)u_R) \right|^2 \frac{1}{|g_+(\epsilon\xi) - g_-(\epsilon\xi)|^2} d\beta \\
= \; & I_1 + I_2.
\end{aligned} \tag{2.82}$$

By (2.70) I_1 can be estimated as:

$$I_1 \leq O(1)\int_0^{\infty} |U_0(x)|^2 dx. \tag{2.83}$$

As for I_2, since $\frac{1}{|g_+(\epsilon\xi)-g_-(\epsilon\xi)|}$ is uniformly bounded, one has

$$I_2 \leq O(1)\int_0^{\infty} dx \int_{-\infty}^{\infty} e^{2Re\mu_-(\epsilon\xi)\frac{x}{\epsilon}} \left[|v_R|^2 + O(1)|u_R|^2\right] d\beta.$$

Then by (2.63), (2.64), (2.72) and (2.80), one obtains:

$$\begin{aligned} g_+(\epsilon\xi)u_R - v_R &= \hat{w}_{IBVP}, \\ -g_-(\xi)u_R - v_R &= \hat{w}_{IBVP2}. \end{aligned}$$

Thus $u_R = O(1)\hat{w}_{IBVP}(\epsilon\xi) + O(1)\hat{w}_{IBVP2}(\xi)$, $v_R = O(1)\hat{w}_{IBVP}(\epsilon\xi) + O(1)\hat{w}_{IBVP2}(\xi)$. Finally by Lemma 2.11 and Corollary 2.12,

$$I_2 \leq -O(1)\frac{\epsilon}{2Re\mu_-(\epsilon\xi)}\int_{-\infty}^{\infty} |U_0(x)|^2 dx. \tag{2.84}$$

Then by (2.33) (2.34), one sees that $\int_0^{\infty} dx \int_{-\infty}^{\infty} |\hat{U}^{\epsilon}(x,\xi)|^2 d\beta$ is uniformly bounded. In the same way, one can prove

$$\int_{-\infty}^{0} dx \int_{-\infty}^{\infty} |\hat{U}^{\epsilon}(x,\xi)|^2 d\beta \leq O(1)\int_{-\infty}^{\infty} |U_0(x)|^2 dx. \tag{2.85}$$

Till now we have proved the stiff well-posedness of the original system stated in Theorem 2.3. □

2.5.3 The asymptotic convergence and error estimates

Next we will prove the asymptotic convergence and error estimates. The first step is still using the Laplace Transform to represent the exact solution. We will consider the case $\lambda < 0$ first. Consider the domain decomposition system (2.12)–(2.13) with $L = \infty$. The case when L is finite is the same but with two more extra terms coming from the boundary which can be analyzed in the same way as follows.

First in comparing the solution to the domain decomposition system (2.12)–(2.13) with the original system (2.1), in order to avoid the difficulties caused by the Laplace Transform, we need the help of the following lemma, which compares the initial value problem (2.68) with its reduced system:

$$\begin{cases} u_t^0 + \lambda u_x^0 = 0, & (2.86a) \\ u^0(x,0) = u_0(x). & (2.86b) \end{cases}$$

Here we assume $u_0(x)$ is supported on $[0, \infty)$.

Lemma 2.13. *Let U_{IVP}^{ϵ} and U_{IVP}^{0} be the solution of relaxation problem (2.68) and equilibrium problem (2.86) respectively, then*

$$\int_0^{\infty} dx \int_{-\infty}^{\infty} |\hat{U}_{IVP}^{\epsilon} - \hat{U}_{IVP}^{0}|^2 d\beta \le O(1)\epsilon^2 \|U_0\|_{H^2}^2 + O(1)\epsilon \|v_0 - \lambda u_0\|_{L^2[0,\infty)}^2. \tag{2.87}$$

Proof. The proof is based on the Fourier Transform, and one can refer to [103] for details. □

Now we are ready to prove Theorem 2.4 about the asymptotic convergence of the domain decomposition system.

Proof. When $x > 0$, the solution is $u^r(x,t) = u_0(x - \lambda t)$. After the Laplace Transform, one gets:

$$\hat{u}^r(x,\xi) = -\frac{1}{\lambda} \int_x^{\infty} u_0(y) e^{-\frac{\xi}{\lambda}(x-y)} dy, \tag{2.88}$$

$$\hat{v}^r = \lambda \hat{u}^r. \tag{2.89}$$

For $x < 0$, the solution to (2.13) can be represented as

$$\begin{aligned}\hat{U}^l(x,\xi) &= e^{\mu_+(\xi)x}\Phi_+(\xi)(\hat{D}(\xi) + \int_0^x e^{-\mu_+(\xi)y}A^{-1}U_0(y)dy) \\ &\quad + e^{\mu_-(\xi)x}\Phi_-(\xi)(\hat{D}(\xi) + \int_0^x e^{-\mu_-(\xi)y}A^{-1}U_0(y)dy). \qquad (2.90)\end{aligned}$$

Here $\hat{D}(\xi) = \begin{pmatrix} \hat{D}_u(\xi) \\ \hat{D}_v(\xi) \end{pmatrix}$ is determined by:

$$(-g_-(\xi)\ 1)\begin{pmatrix} \hat{D}_u(\xi) \\ \hat{D}_v(\xi) \end{pmatrix} + \int_0^{-\infty} e^{-\mu_-(\xi)y}(-g_-(\xi)\ 1)\begin{pmatrix} v_0 \\ u_0 \end{pmatrix}(y)dy = 0,$$

$$\frac{1}{g_+(\xi) - g_-(\xi)}\Big[(\hat{D}_u(\xi)g_+(\xi) - \hat{D}_v(\xi))g_-(\xi) + (\hat{D}_v(\xi) - \hat{D}_u(\xi)g_-(\xi))g_+(\xi)\Big] = -\int_0^\infty u_0(y)e^{\frac{\xi}{\lambda}y}dy,$$

where the second equation is simplified as

$$\hat{D}_v(\xi) = -\int_0^\infty u_0(y)e^{\frac{\xi}{\lambda}y}dy. \qquad (2.91)$$

Now one can compare the difference of (2.88) and (2.66) on the right domain. Since the solution to (2.86) is (2.88), and part of (2.66) is (2.69), one has

$$\begin{aligned}&\int_0^\infty dx \int_{-\infty}^\infty |\hat{u}^\epsilon - \hat{u}^r|^2 d\beta \le \int_0^\infty dx \int_{-\infty}^\infty |\hat{u}^\epsilon_{IVP} - \hat{u}^0_{IVP}|^2 d\beta \\ &+ \int_0^\infty dx \int_{-\infty}^\infty \left|\frac{1}{g_+(\epsilon\xi) - g_-(\epsilon\xi)}\right|^2 (1 + |g_+(\epsilon\xi)|^2)|e^{\mu_-(\epsilon\xi)\frac{x}{\epsilon}}(\hat{v}_R(\xi) - g_-(\epsilon\xi)\hat{u}_R(\xi))|^2 d\beta \\ =\ & \mathbb{I}_1 + \mathbb{I}_2,\end{aligned}$$

$$\begin{aligned}\mathbb{I}_1 &\le O(1)\epsilon^2\|U_0\|^2_{H^2} + O(1)\epsilon\|v_0 - \lambda u_0\|^2_{L^2[0,\infty)}, \qquad (2.92) \\ \mathbb{I}_2 &\le O(1)\int_0^\infty e^{2\mathrm{Re}\mu_-(\epsilon\xi)\frac{x}{\epsilon}}dx \int_{-\infty}^\infty |\hat{v}_R(\xi) - g_-(\epsilon\xi)\hat{u}_R(\xi)|^2 d\beta \le O(1)\epsilon\|U_0\|^2_{L^2}. \qquad (2.93)\end{aligned}$$

The calculation of the last inequality is the same as (2.84). Notice here that the term that contains $e^{2\mathrm{Re}\mu_-(\epsilon\xi)\frac{x}{\epsilon}}$ is due to the interface layer, since the initial data can induce an interface layer at the interface in this case.

Now compare the solution on the left domain, (2.90) with (2.62). The difference comes from the difference in coefficients, thus

$$\begin{aligned}
&\int_{-\infty}^{0} dx \int_{-\infty}^{\infty} |\hat{U}^l - \hat{U}^\epsilon|^2 d\beta \\
\leq\ & O(1) \int_{-\infty}^{0} dx \int_{-\infty}^{\infty} |e^{\mu_+(\xi)x}|^2 (1 + |g_-(\xi)|^2)[g_+(\xi)(\hat{D}_u - \hat{u}_L) - (\hat{D}_v - \hat{v}_L)]^2 d\beta \\
&+ \int_{-\infty}^{0} dx \int_{-\infty}^{\infty} |e^{\mu_-(\xi)x}|^2 (1 + |g_+(\xi)|^2)[-g_-(\xi)(\hat{D}_u - \hat{u}_L) + (\hat{D}_v - \hat{v}_L)]^2 d\beta.
\end{aligned}$$

By boundary conditions (2.64) and (2.91), the second term vanishes, so

$$\int_{-\infty}^{0} dx \int_{-\infty}^{\infty} |\hat{U}^l - \hat{U}^\epsilon|^2 d\beta \leq O(1) \int_{-\infty}^{0} dx \int_{-\infty}^{\infty} e^{2Re\mu_+(\xi)x} (|\hat{D}_u - \hat{u}_L|^2 + |\hat{D}_v - \hat{v}_L|^2) d\beta.$$

Next compare the parameters derived in the original system (2.63)–(2.65) with those of the domain decomposition method (2.91) (2.91), one gets

$$\begin{aligned}
&\int_{-\infty}^{0} dx \int_{-\infty}^{\infty} |\hat{U}^l - \hat{U}^\epsilon|^2 d\beta = O(1) \int_{-\infty}^{\infty} (|\hat{D}_u - \hat{u}_L|^2 + |\hat{D}_v - \hat{v}_L|^2) d\beta \\
=\ & O(1) \int_{-\infty}^{\infty} |\hat{D}_v - \hat{v}_L|^2 d\beta \\
=\ & O(1) \int_{-\infty}^{\infty} \left| - \int_0^\infty e^{\frac{\xi}{\lambda}y} u_0(y) dy - \frac{-\hat{w}_{IBVP} + \frac{g_+(\epsilon\xi)}{g_-(\xi)} \hat{w}_{IBVP2}}{1 - \frac{g_+(\epsilon\xi)}{g_-(\xi)}} \right|^2 d\beta \\
\leq\ & O(1) \int_{-\infty}^{\infty} \left| \int_0^\infty (u_0(y) - v_0(y) g_+(\epsilon\xi)) e^{-\mu_+(\epsilon\xi)\frac{y}{\epsilon}} dy - \int_0^\infty u_0(y) e^{\frac{\xi}{\lambda}y} dy \right|^2 d\beta \\
&+ O(1) \int_{-\infty}^{\infty} \left| g_+(\epsilon\xi) \int_0^{-\infty} (u_0(y) - v_0(y) g_-(\xi)) e^{-\mu_-(\xi)y} dy \right|^2 d\beta \\
&+ O(1) \int_{-\infty}^{\infty} \left| g_+(\epsilon\xi) \int_0^\infty u_0(y) e^{\frac{\xi}{\lambda}y} dy \right|^2 d\beta \\
=\ & \mathbb{J}_1 + \mathbb{J}_2 + \mathbb{J}_3.
\end{aligned}$$

We begin with the simplest part $\mathbb{J}_3$. First note that $g(\epsilon\xi)$ is uniformly bounded (see Lemma 2.7), and $\int_0^\infty u_0(y) e^{\frac{\xi}{\lambda}y} dy$ can be considered as Laplace transform of $u_0(y)$, so by Parseval's identity, the integral $\int_{-\infty}^{\infty} \left| \int_0^\infty u_0(y) e^{\frac{\xi}{\lambda}y} dy \right|^2 d\beta$ is uniformly bounded, then by dominated convergence theorem, $\mathbb{J}_3 \to 0$ as $\epsilon \to 0$. Moreover, since when $\lambda < 0$, $g_+(\epsilon\xi) = O(1)\epsilon\xi$, thus

$$\mathbb{J}_3 \leq O(1)\epsilon^2 \int_{-\infty}^{\infty} \left| \xi \int_0^{\infty} u_0(y) e^{\frac{\xi}{\lambda} y} dy \right|^2 d\beta.$$

If the compatibility condition on $u_0(y)$ is assumed such that $u_0(0) = 0$, $\int_0^\infty \xi u_0(y) e^{\frac{\xi}{\lambda} y} dy$ can be considered as the Laplace transform to $u_0'(y)$, so

$$\mathbb{J}_3 \leq O(1)\epsilon^2 \int_{-\infty}^{\infty} |\mathcal{L}(u_0'(y))(\xi)|^2 d\beta \leq O(1)\epsilon^2 \int_0^{\infty} |u_0'(y)|^2 dy. \tag{2.94}$$

Next we look at $\mathbb{J}_2$. Similar to $\mathbb{J}_3$, one will first get

$$\mathbb{J}_2 \leq O(1)\epsilon^2 \int_{-\infty}^{\infty} \left| \xi \int_0^{-\infty} (u_0(y) - v_0(y) g_-(\xi)) e^{-\mu_-(\xi) y} dy \right|^2 d\beta.$$

By recalling (2.80) and integration by parts, one gets

$$\hat{w}_{IBVP2} = -\frac{1}{\mu_-(\xi)} \int_0^{-\infty} e^{-\mu_-(\xi) y} (-u_0' + g_-(\xi) v_0') dy, \tag{2.95}$$

where the compatibility conditions $u_0(0) = 0$ and $v_0(0) = 0$ are used. Since $-\mu_-(\xi) = \mu_+(\xi) - 2\lambda$, one has

$$(\mu_+(\xi) - 2\lambda)\hat{w}_{IBVP2} = \int_0^{-\infty} e^{-\mu_-(\xi) y} (-u_0' + g_-(\xi) v_0') dy.$$

Notice when $\lambda < 0$, $\mu_+(\xi) = -\frac{\xi}{g_-(\xi)}$, thus

$$-\xi \hat{w}_{IBVP2} = 2\lambda g_-(\xi) \hat{w}_{IBVP2} + g_-(\xi) \int_0^{-\infty} (-u_0'(y) + g_-(\xi) v_0'(y)) dy.$$

Therefore, the following estimate holds:

$$\int_{-\infty}^{\infty} |\xi \hat{w}_{IBVP2}|^2 d\beta \leq O(1) \int_{-\infty}^{\infty} \left| \int_0^{-\infty} e^{-\mu_-(\xi) y} (u_0' - g_-(\xi) v_0')(y) dy \right|^2 d\beta + O(1) \int_{-\infty}^{\infty} |\hat{w}_{IBVP2}(\xi)|^2 d\beta. \tag{2.96}$$

The integral with respect to y on the right hand side is similar to $\hat{w}_{IBVP2}$ in (2.80), except to change u_0 and v_0 to u_0' and v_0' respectively. So one has

$$\int_{-\infty}^{\infty} |\xi \hat{w}_{IBVP2}|^2 d\beta \leq O(1) \int_0^{\infty} |U_0'(x)|^2 dx + O(1) \int_0^{\infty} |U_0(x)|^2 dx. \tag{2.97}$$

Therefore,

$$\mathbb{J}_2 \leq O(1)\epsilon^2 \int_0^{\infty} |U_0'(x)|^2 dx. \tag{2.98}$$

Now we turn to $\mathbb{J}_1$. First using $g_+(\epsilon\xi) \sim O(1)\epsilon\xi$ gives

$$\mathbb{J}_1 \leq O(1) \int_{-\infty}^{\infty} \epsilon^2 \left| \xi \int_0^{\infty} v_0 e^{-\mu_+(\epsilon\xi)\frac{y}{\epsilon}} dy \right|^2 d\beta + O(1) \int_{-\infty}^{\infty} \left| \int_0^{\infty} u_0(y)(e^{-\mu_+(\epsilon\xi)\frac{y}{\epsilon}} - e^{\frac{\xi}{\lambda}y}) dy \right|^2 d\beta. \tag{2.99}$$

Notice in (2.72) if one exchanges u_0 and v_0 and let $u_0 \equiv 0$, then use (2.72) (2.96), and similar to (2.97), one will get:

$$\int_{-\infty}^{\infty} \left| \xi \int_0^{\infty} v_0 e^{-\mu_+(\epsilon\xi)\frac{y}{\epsilon}} dy \right|^2 d\beta \leq O(1) \int_0^{\infty} |v_0'(x)|^2 dx. \tag{2.100}$$

On the other hand, for the term $\int_{-\infty}^{\infty} \left| \int_0^{\infty} u_0(y)(e^{-\mu_+(\epsilon\xi)\frac{y}{\epsilon}} - e^{\frac{\xi}{\lambda}y}) dy \right|^2 d\beta$, integration by parts w.r.t. y three times, and assume the compatibility conditions $u_0(0) = u_0'(0) = u_0''(0)$, it becomes

$$\begin{aligned}
&\int_{-\infty}^{\infty} \left| \int_0^{\infty} u_0(y)(e^{-\mu_+(\epsilon\xi)\frac{y}{\epsilon}} - e^{\frac{\xi}{\lambda}y}) dy \right|^2 d\beta \\
= &\int_{-\infty}^{\infty} \left| \left(e^{\frac{\xi}{\lambda}y} \left(\frac{\lambda}{\xi}\right)^3 - e^{-\frac{\mu_+(\epsilon\xi)}{\epsilon}} \left(-\frac{\epsilon}{\mu_+(\epsilon\xi)} \right)^3 \right) u_0'''(y) dy \right|^2 d\beta \\
= &\int_{-\infty}^{\infty} \left| \int_0^{\infty} \left(e^{\frac{\xi}{\lambda}y} \left(\frac{\lambda}{\xi}\right)^3 - e^{\frac{\xi}{\lambda}y} \left(\frac{-\epsilon}{\mu_+(\epsilon\xi)}\right)^3 + e^{\frac{\xi}{\lambda}y} \left(\frac{-\epsilon}{\mu_+(\epsilon\xi)}\right)^3 - e^{-\frac{\mu_+(\epsilon\xi)}{\epsilon}} \left(\frac{-\epsilon}{\mu_+(\epsilon\xi)}\right)^3 \right) u_0'''(y) dy \right|^2 d\beta \\
\leq &\int_{-\infty}^{\infty} \left| \left(\left(\frac{\lambda}{\xi}\right)^3 - \left(\frac{-\epsilon}{\mu_+(\epsilon\xi)}\right)^3 \right) \int_0^{\infty} e^{\frac{\xi}{\lambda}y} u_0'''(y) dy \right|^2 d\beta + \int_{-\infty}^{\infty} \left| \left(\frac{\epsilon}{\mu_+(\epsilon\xi)}\right)^3 \int_0^{\infty} (e^{\frac{\xi}{\lambda}y} - e^{-\mu_+(\epsilon\xi)\frac{y}{\epsilon}}) u_0'''(y) dy \right|^2 d\beta \\
= &\ \mathbb{L}_1 + \mathbb{L}_2
\end{aligned}$$

For $\mathbb{L}_1$, notice that

$$\frac{\lambda}{\xi}-\frac{\epsilon}{\mu_+(\epsilon\xi)}=\frac{2\lambda\epsilon\xi+\lambda+\sqrt{\lambda^2+4\epsilon\xi(1+\epsilon\xi)}}{2\xi(1+\epsilon\xi)}=\frac{\lambda\epsilon}{1+\epsilon\xi}+\frac{\epsilon}{\lambda-\sqrt{\lambda^2+4\epsilon\xi(1+\epsilon\xi)}}=O(1)\epsilon$$

by Lemma 2.6, and

$$\frac{\epsilon}{\mu_+(\epsilon\xi)}=-\frac{1}{\xi g_-(\epsilon\xi)}=O(1)\frac{1}{\xi}, \tag{2.101}$$

by Lemma 2.7, one can estimate $\mathbb{L}_1$ as

$$\mathbb{L}_1 \leq O(1)\epsilon^2\int_{-\infty}^{\infty}\left|\frac{1}{\xi}\int_0^\infty u_0^{'''}(y)e^{\frac{\xi}{\lambda}}dy\right|^2 d\beta \leq O(1)\epsilon^2\ \|u_0^{'''}\|_{L^2}\ . \tag{2.102}$$

In term $\mathbb{L}_2$, use Cauchy-Schwartz inequality for integral w.r.t. y, one obtains

$$\begin{aligned}\mathbb{L}_2 \ &\leq\ O(1)\int_{-\infty}^{\infty}\left|\frac{1}{\xi}\right|^6\int_0^\infty\left|e^{-\mu_+(\epsilon\xi)\frac{y}{\epsilon}}-e^{\frac{\xi}{\lambda}y}\right|^2 dy\int_0^\infty|u_0^{'''}(y)|^2dyd\beta \\ &\leq\ O(1)\ \|u_0^{'''}\|_{L^2}\int_{-\infty}^{\infty}\epsilon^2\left|\frac{1}{\xi}\right|^2 d\beta\ \|u_0^{'''}\|_{L^2}=O(1)\epsilon^2,\end{aligned} \tag{2.103}$$

where the second inequality using the fact in (2.43) and (2.45). Therefore, one arrives at the estimation for $\mathbb{J}_3$:

$$\mathbb{J}_3 \leq O(1)\epsilon^2\|u_0(y)\|_{H^3}^2+O(1)\epsilon^2\|v_0(y)\|_{H^1}^2, \tag{2.104}$$

In summary,

$$\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}|\hat{U}^\epsilon-\hat{U}|^2d\beta \leq O(1)\epsilon\|v_0-\lambda u_0\|_{L^2}^2+O(1)\epsilon\|U_0\|_{L^2[0,\infty)}^2+O(1)\epsilon^2\|U_0\|_{H^3}^2. \tag{2.105}$$

The case with $\lambda>0$ is rather similar, but there is no interface layer at $x=0$, so one will find the term that contains $\|U_0\|_{L^2}^2$ will have a convergence rate $O(1)\epsilon^2$ instead of $O(1)\epsilon$.

□

2.6 Domain-decomposition based numerical schemes and numerical experiments

We use Δt and Δx to represent the time step and mesh size respectively, u_j^n to denote u at time $n\Delta t$ and position $j\Delta x$. Let $M = T/\Delta t$, and $N = L/\Delta x$. We use the upwind scheme to the Riemann invariants $u \pm v$ to solve the left part (2.13) or (2.14), and use the Godunov scheme to solve the equilibrium equation in (2.12) or (2.15).

2.6.1 The numerical scheme

Case I: $f'(u_0^n) < 0$, $\forall n \geq 0$

- **Step 1. Discretization of (2.12) on the right domain.**

  For $j = 0, 1, ..., N$, $n = 0, 1, ..., M$, solve

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{F_{j+\frac{1}{2}}^n - F_{j-\frac{1}{2}}^n}{\Delta x} = 0, \tag{2.106}$$

$$v_j^{n+1} = f(u_j^{n+1}), \tag{2.107}$$

$$u_j^0 = u_0(x_j), \quad v_j^0 = v_0(x_j), \tag{2.108}$$

$$u_N^n = b_R(t^n); \tag{2.109}$$

  where $F_{j+\frac{1}{2}}^n = f(R(0, u_j^n, u_{j+1}^n))$, $F_{j-\frac{1}{2}}^n = f(R(0, u_{j-1}^n, u_j^n))$, and $R(0, \zeta, \eta)$, the

Riemann solver, is defined as:

$$R(0,\zeta,\eta)=\begin{cases}\zeta, & \text{if } f'(\zeta),\ f'(\eta)\le 0,\\ \eta, & \text{if } f'(\zeta),\ f'(\eta)\ge 0,\\ \zeta, & \text{if } f'(\zeta)>0>f'(\eta),\ s>0,\\ \eta, & \text{if } f'(\zeta)>0>f'(\eta),\ s<0,\\ f'^{-1}(0), & \text{otherwise.}\end{cases}$$

where $s=\frac{f(\zeta)-f(\eta)}{\zeta-\eta}$ is the shock speed.

- **Step 2. Discretization of (2.13) on the left domain.**

  For $j=-N,...,-1,0,\ n=0,1,...,M,$ let the Riemann invariants $P_j^n=u_j^n+v_j^n, Q_j^n=u_j^n-v_j^n$, and solve

$$\frac{P_j^{n+1}-P_j^n}{\Delta t}+\frac{P_j^n-P_{j-1}^n}{\Delta x}=-(v_j^n-f(u_j^n)), \tag{2.110}$$

$$\frac{Q_j^{n+1}-Q_j^n}{\Delta t}-\frac{Q_{j+1}^n-Q_j^n}{\Delta x}=(v_j^n-f(u_j^n)), \tag{2.111}$$

$$P_j^0=u_0(x_j)+v_0(x_j),\quad Q_j^0=u_0(x_j)-v_0(x_j), \tag{2.112}$$

$$u_{-N}^{n+1}=b_L(t^{n+1}),\quad v_0^{n+1}\ \text{obtained from right by (2.107).} \tag{2.113}$$

Case II: $f'(u_o^n)>0,\ \forall n\ge 0$

- **Step 1. Discretization of (2.14) on the left domain.**

  For $j=-N,....-1,0,\ n=0,1,...,M,$ let the Riemann invariants $P_j^n=u_j^n+v_j^n,\ Q_j^n=u_j^n-v_j^n$, then solve

$$\frac{P_j^{n+1} - P_j^n}{\Delta t} + \frac{P_j^n - P_{j-1}^n}{\Delta x} = -(v_j^n - f(u_j^n)), \tag{2.114}$$

$$\frac{Q_j^{n+1} - Q_j^n}{\Delta t} - \frac{Q_{j+1}^n - Q_j^n}{\Delta x} = (v_j^n - f(u_j^n)) \tag{2.115}$$

$$P_j^0 = u_0(x_j) + v_0(x_j), \quad Q_j^0 = u_0(x_j) - v_0(x_j), \tag{2.116}$$

$$u_{-N}^{n+1} = b_L(t^{n+1}), \tag{2.117}$$

$$P_0^{n+1} = u_0^{n+1} + f(u_0^{n+1}); \tag{2.118}$$

- **Step 2. Discretization of (2.12) on the left domain.**

  For $j = 1, ..., N$, $n = 0, 1, ..., M$, solve

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{F_{j+\frac{1}{2}}^n - F_{j-\frac{1}{2}}^n}{\Delta x} = 0, \tag{2.119}$$

$$u_j^0 = u_0(x_j), \quad v_j^0 = v_0(x_j), \tag{2.120}$$

$$u_0^{N+1} \text{ obtained from left,} \tag{2.121}$$

  where $F_{j+\frac{1}{2}}^n$ and $F_{j-\frac{1}{2}}^n$ are defined as in *Case I*. To solve for u_0^{n+1}, since (2.114) is an explicit scheme for P^{n+1}, we first use it to get P_0^{n+1}, and then use Newton iteration for (2.118) to get u_0^{n+1}.

2.6.2 Coupling of multiple regions

The previous method for two regions can be easily extended to three or more regions with different scales. For example, consider the coupling that consists of equilibrium (where $\epsilon(x)$ is small) region on the left, relaxation (where $\epsilon(x)$ is of $O(1)$) in the middle, and equilibrium region on the right, that is,

$$\epsilon(x) = \epsilon, \ x \in [-L, x_1); \quad \epsilon(x) = 1, \ x \in [x_1, x_2); \quad \epsilon(x) = \epsilon, \ x \in [x_2, L],$$

where $x_1 < x_2$. Consider the case $f'(u(x_1,t)) < 0$ and $f'(u(x_2,t)) > 0$ for $t \leq T$. The other cases can be treated similarly. Our algorithm will solve the middle region $[x_1, x_2)$ first with interface condition $v = f(u)$ at x_1 and x_2, and then solve the left and right regions. To be more specific, one can follow the following steps.

- **Step 1.** For $j = N_1 + 1, ...N_2$, $n = 0, 1, ...M$ that correspond to the middle region $[x_1, x_2)$, solve the equations (2.114)–(2.116) for Riemann invariants $P_j^n = u_j^n + v_j^n$, $Q_j^n = u_j^n - v_j^n$ with boundary conditions at x_1 and x_2 respectively given by

$$P_{N_1+1}^{n+1} = u_{N_1+1}^{n+1} + f(u_{N_1+1}^{n+1}); \quad P_{N_2}^{n+1} = u_{N_2}^{n+1} + f(u_{N_2}^{n+1}). \tag{2.122}$$

  Notice here one needs to use Newton's iteration at both boundary points to get $u_{N_1+1}^{n+1}$ and $u_{N_2}^{n+1}$ from $P_{N_1+1}^{n+1}$ and $P_{N_2}^{n+1}$ respectively using (2.122).

- **Step 2.** For $j = 0, ...N_1$, $n = 0, 1, ...M$, one is in the left region $[-L, x_1)$, solve (2.106) and (2.108) with boundary value $u_{N_1+1}^{n+1}$ got from the previous step.

- **Step 3.** For $j = N_2 + 1, ...N$, $n = 0, 1, ...M$, solve (2.119) and (2.120) with the boundary value $u_{N_2}^{n+1}$ obtained from step 1.

In summary, near the interface, if there is a boundary layer in the equilibrium region, then solve the equilibrium equation first and then pass on to the relaxation regions through the value of v; on the other hand, if there is no boundary layer, then one can always take $v = f(u)$ as the interface condition and solve the relaxation region first. In any situation, the system can be completely decoupled in different regions, and one can always find an appropriate order to solve them.

2.6.3 More general cases

- If $f'(u)$ changes sign at interface, one can check the sign of $f'(u)$ at the current time step, and then use either (2.106)-(2.113) or (2.114)-(2.121) to continue to the next step.

- If ϵ also depends on t, so the interface may be dynamic, then one needs to adaptively adjust the interface location (see for example [27]) and then use the domain decomposition method.

- In higher space dimension, if the interface is a curve or surface, we simply use the Cartesian grids and extend the $1d$ method to higher dimensions using dimension-by-dimension technique. This will result a first order error due to the grid effect. A more sophisticated method would use an interface aligned mesh or immersed interface method [70]. We will not elaborate on this since it is out of the scope of the paper.

2.6.4 Numerical examples

The first two examples are given to validate our domain decomposition system numerically. Therefore we focus on the behavior of l^1 error with a changing ϵ (we only change ϵ for $x > 0$, while for $x < 0$, let $\epsilon = 1$). Here we use $\Delta x = 10^{-3}$, $\Delta t = 2.5 \times 10^{-4}$ in the regime $\Delta x,\ \Delta t \ll \epsilon$, and run the algorithm to $T = 0.2$. We change ϵ from 0.05 to 0.0025, then calculate the error

$$U_{l^1} = \max_{0\le n\le M} \sum_{j=0}^{N} |(u^\epsilon)_j^n - u_j^n|\Delta x, \quad V_{l^1} = \max_{0\le n\le M} \sum_{j=0}^{N} |(v^\epsilon)_j^n - v_j^n|\Delta x.$$

Here $(u^\epsilon)_j^n$ and $(v^\epsilon)_j^n$ are obtained by directly solving the original system (2.1)–(2.5).

Example 2.1. Let $f(u^\epsilon) = \frac{1}{4}(e^{-u^\epsilon} - 1)$ in (2.1), with initial condition $u^\epsilon(x, 0) = \sin(\pi x)^3$, and boundary condition $u(-1, t) = u(1, t) = 0$. In this case, $f'(u) < 0$, so there will be an interface layer at the interface $x = 0$. The left picture in Figure 1 gives the $\log(error)$ versus $\log(\epsilon)$. One can see that the convergence rate is $O(\epsilon)$.

Example 2.2. Now we consider the case $f'(u) > 0$. Let $f(u^\epsilon) = \frac{1}{4}(e^{u^\epsilon} - 1)$, initial condition $u^\epsilon(x, 0) = \sin(\pi x)^3$, and boundary condition $u(-1, t) = u(1, t) = 0$. Still one sees that the convergence rate is $O(\epsilon)$, as shown in the right picture in Figure 1.

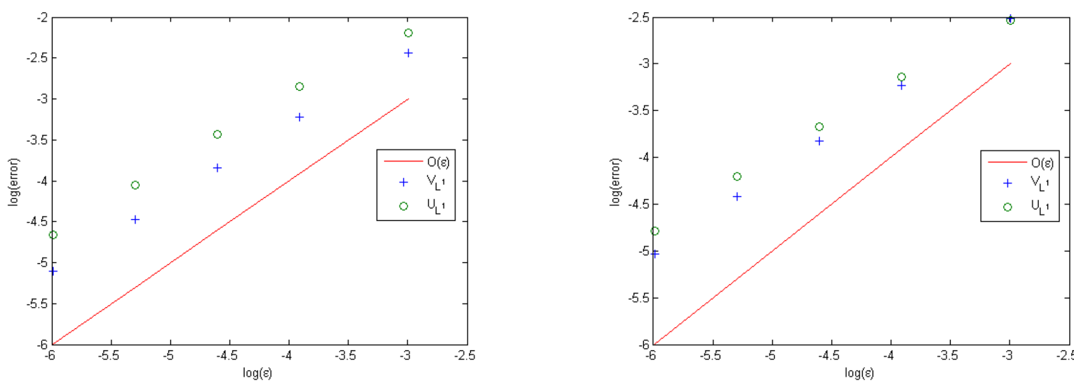

Figure 1: Convergence rate for Example 2.1 when there is a boundary layer at the interface and for Example 2.2 when there is no boundary layer at the interface.

Next we will compare our domain decomposition method using the underresolved mesh with the original relaxation system. Let $\epsilon = 0.002$ be fixed for $x > 0$. The relaxation system is solved by fine mesh ($\Delta x,\ \Delta t \ll \epsilon$) to serve as the reference solution to (2.1)–(2.5), which are referred to as "analytical" solutions in the Figures 2 and 3.

Example 2.3. The set up is the same as example 2.1. The solutions are plotted at $T = 0.5$. In this case, there is an interface layer in u at $x = 0$, as one can see from Figure 2. In comparison, one can see that the relaxation system solved with a relatively large mesh size ($\Delta x,\ \Delta t \gg \epsilon$), referred to as "under-relax" in Figures 2 and 3, gives poor results at the interface which results in larger numerical errors away from the interface. The error becomes smaller if the mesh size is reduced (yet still underresolved). On the other hand, the domain decomposition method gives more accurate approximation even when the mesh size is large ($\Delta x,\ \Delta t \gg \epsilon$).

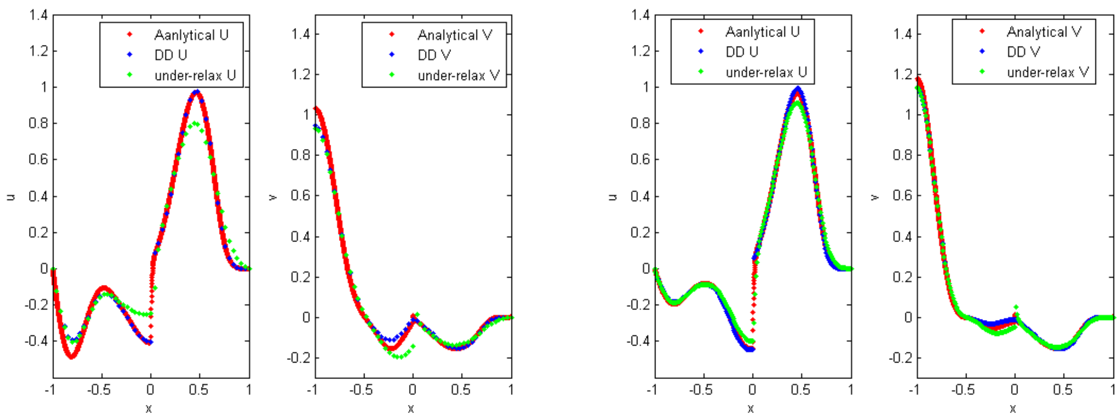

Figure 2: Example 2.3. We use $\Delta x = 0.04$, $\Delta t = 0.02$ (left), and $\Delta x = 0.01$, $\Delta t = 0.005$ (right).

Example 2.4. The set up is the same as example 2.2. The results at $T = 0.6$ are plotted in Figure 3. Similar to example 2.3, one can find that the relaxation system is better approximated with the decreasing of the mesh size, while the domain decomposition method gives good approximation even with the large mesh size compared to ϵ.

Example 2.5. Let $f(u^\epsilon)$ be the same as in example 2.2, but consider the Riemann initial data:

$$u^\epsilon(x,0) = \begin{cases} -1, & \text{if } -1 \le x \le -0.2; \\ 1, & \text{if } -0.2 < x \le 1. \end{cases}$$

In this case a contact discontinuity formed at the left hand side will propagate across the interface to the right. Let $\Delta x = 0.02$, $\Delta t = 0.01$. From Figure 4, one will see

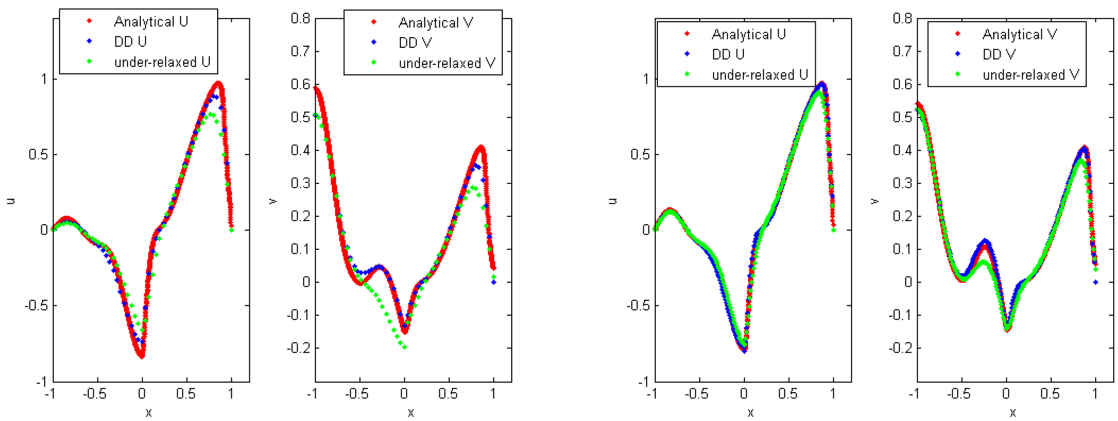

Figure 3: Example 2.4. We use $\Delta x = 0.04$, $\Delta t = 0.02$ (left), and $\Delta x = 0.01$, $\Delta t = 0.005$ (right).

that, before the contact discontinuity passes through the interface, there is not much difference between the under-resolved solution of the relaxation system and the domain decomposition solution, but after that the domain decomposition method has an obvious advantage in producing more accurate results. The results are given at different times to show the dynamics of the solution.

Example 2.6. Let $f(u^\epsilon)$ be the same as in example 2.1, and consider the following Riemann initial data:

$$u^\epsilon(x,0) = \begin{cases} 1, & \text{if } -1 \le x \le 0.2; \\ -1, & \text{if } 0.2 < x \le 1. \end{cases}$$

Here we use $\Delta x = 0.02$ and $\Delta t = 0.01$. In this case, a shock forms at the right region and propagates to the left region. From Figure 5, one can see that, when the shock crosses the interface, the domain decomposition method gives spurious solution at the interface. This is because our interface layer analysis assumes that the solution is smooth, yet here the interaction between the interface layer and shock complicates the problem, thus our domain decomposition system may not be valid here.

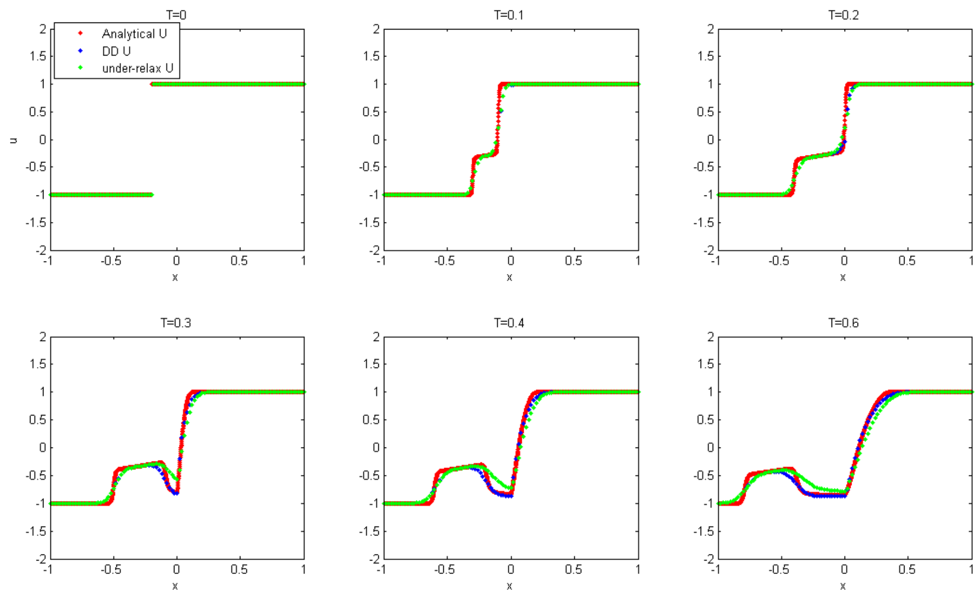

Figure 4: Example 2.5, a contact discontinuity passing through the interface.

Example 2.7. Let $f(u^\epsilon)$ be the same as in example 2.1, and consider the following Riemann initial data:

$$u^\epsilon(x,0) = \begin{cases} -1, & \text{if } -1 \le x \le 0.2; \\ 1, & \text{if } 0.2 < x \le 1. \end{cases}$$

With this initial data, a rarefaction wave forms in the right region, and propagates across the interface to the left. We still let $\Delta x = 0.02$ and $\Delta t = 0.01$, and the solutions are plotted at different times in Figure 6. One can see that, unlike a shock, the domain decomposition method gives a good approximation when the rarefaction wave crosses the interface.

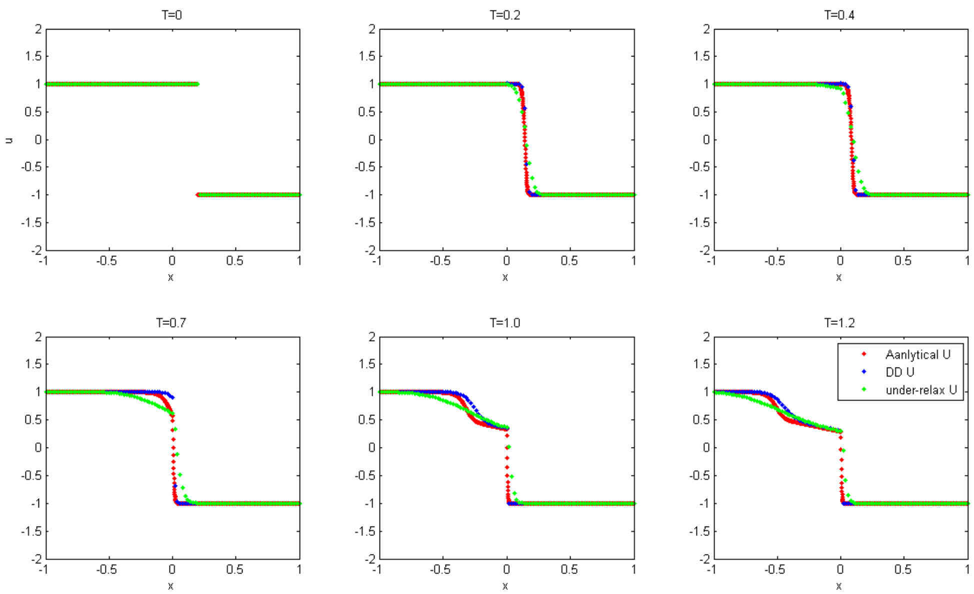

Figure 5: Example 2.6, a shock from the right region passing through the interface.

2.7 Concluding remarks

In this chapter, a domain decomposition method is presented and analyzed on a semilinear hyperbolic system with multiple relaxation times. In the region where the relaxation time is small, an asymptotic equilibrium equation is used for computational efficiency which is coupled with the original relaxation system on the other part of the region through an interface condition. A rigorous analysis establishes the well-posedness and error estimate in terms of the relaxation time on this domain decomposition method, and numerical results are presented to study the performance of this method.

This is a prototype model for the more general coupling of kinetic and hydrodynamic equations which are competitive multiscale computational methods using multi-physics, thus a deep mathematical understanding of this simpler model problem will shed light on the more general physical problems.

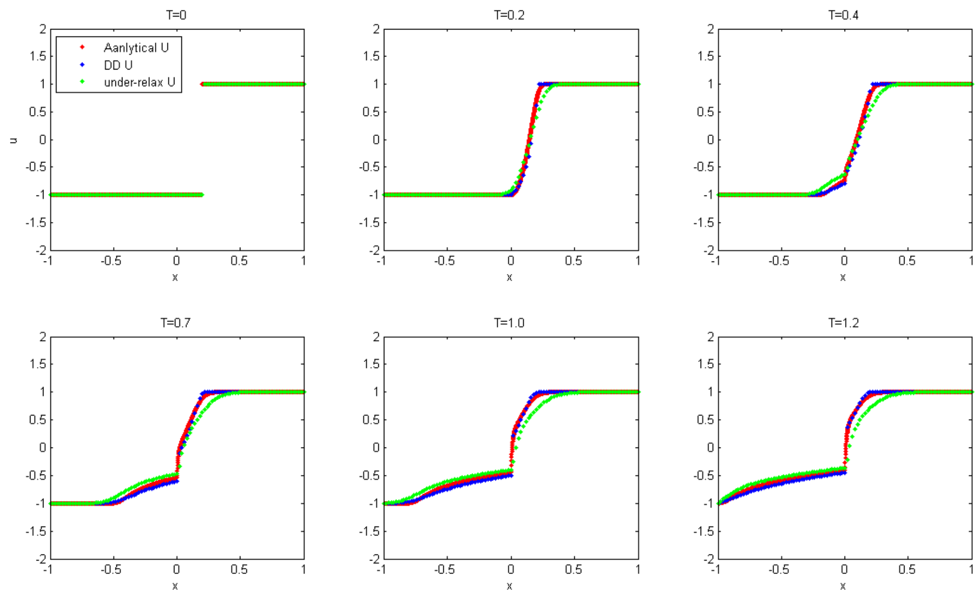

Figure 6: Example 2.7, rarefaction wave

There are still remaining problems to be studied. Among them we mention the problem of shock passing through the interface, nonlinear hyperbolic systems with relaxation, and the error estimate on the numerical schemes based on such a domain decomposition method.

Chapter 3

Asymptotic convergence of a hyperbolic relaxation system to a domain decomposition system: the nonlinear case

3.1 Introduction

In this chapter, we would like to extend the the asymptotic convergence result in the last chapter to the nonlinear case. Recall the original two-scale hyperbolic relaxation system (2.1)

$$\begin{cases} u_t^\epsilon + v_x^\epsilon = 0, & \text{(3.1a)} \\ v_t^\epsilon + a^2 u_x^\epsilon = -\lambda(x,\epsilon)(v^\epsilon - f(u^\epsilon)), & \text{(3.1b)} \end{cases}$$

where $\lambda(\epsilon, x)$ is the reciprocal of the relaxation time and takes the form

$$\lambda(x,\epsilon) = \begin{cases} 1, \ x < 0, \\ \frac{1}{\epsilon}, \ x > 0, \ \epsilon \ll 1. \end{cases} \tag{3.2}$$

The initial data is assumed to be well-prepared

$$u_0(x) \in L^\infty(R) \cap \mathrm{BV}(R), \tag{3.3}$$

$$v_0(x) = f(u_0(x)) \quad \text{so that } v_0 \in L^\infty(R) \cap \mathrm{BV}(R), \tag{3.4}$$

and the sub-characteristic condition is specified as follows [75]

$$a > M(N_0), \ \text{where } N_0 = \max\big(\, \| u_0 \|_{L^\infty(R)}, \| f(u_0) \|_{L^\infty(R)} \,\big), \tag{3.5}$$

$$M(N_0) = \sup_{|\xi| \le B(N_0)} |f'(\xi)|, \ B(N_0) = 2N_0 + F(2N_0),$$
$$F(N_0) = \sup_{|\xi| < N_0} |f(\xi)|.$$

We will show in this chapter that for general nonlinear flux f, the correct coupling obtained at the limit of (3.1) is as follows.

$$\begin{cases} \partial_t u + \partial_x v = 0, \ t > 0, \ x < 0 & \text{(3.6a)} \\ \partial_t v + a^2 \partial_x u = f(u) - v, & \text{(3.6b)} \\ u(0, x) = u_0(x), \ v(0, x) = v_0(x), & \text{(3.6c)} \\ u(t, 0^-) = \mathcal{U}(t, 0), \ v(t, 0^-) = v(t, 0^+) & \text{(3.6d)} \end{cases}$$

$$\begin{cases} a^2 \dfrac{d}{dy} \mathcal{U}(t, y) = f(\mathcal{U}(t, y)) - v(t, 0^-), \ y > 0 & \text{(3.7a)} \\ \mathcal{U}(t, 0) = u(t, 0^-) & \text{(3.7b)} \end{cases}$$

$$\begin{cases} \partial_t u + \partial_x f(u) = 0, \ t > 0, \ x > 0 & \text{(3.8a)} \\ v(t, x) = f(u(t, x)), & \text{(3.8b)} \\ u(0, x) = u_0(x), & \text{(3.8c)} \\ u(t, 0^+) \overset{BLN}{:=} u(t, 0^-) & \text{(3.8d)} \end{cases}$$

where (3.8d) is in the sense of the well-known Bardos-Lerous-Nédélec condition

$$\operatorname{sgn}\big(u(t,0^+) - u(t,0^-)\big)\big(f(u(t,0^+)) - f(k)\big) \leq 0, \text{ for all k between } u(t,0^-) \text{ and } u(t,0^+). \tag{3.9}$$

Here $\mathcal{U}$ is the inner solution defined in (3.73). Notice that the limit system verifies the equality of the flux at the interface

$$v(t,0^-) = v(t,0^+) = f(u(t,0^+)). \tag{3.10}$$

As already pointed out in the last chapter, the naive connection in $u(t,0^-) = u(t,0^+)$ is not correct in the case when there is an interface layer, therefore (3.8d) is needed. In fact, (3.6d) and (3.8d) are nothing but an extension of (2.13e), (2.14e) and (2.15d) to a more general flux. Namely, when there is no shock sticking to the interface, i.e., $u(t,0^+) = \mathcal{U}(t,+\infty)$, then

- if $f'(\mathcal{U}(t,+\infty)) = f'(u(t,0^+)) > 0$, then (3.7) only has trivial solution, therefore $u(t,0^-) = u(t,0^+)$, which implies $v(t,0^-) = f(u(t,0^+)) = f(u(t,0^-))$, the one that has been used in (2.14e);
- if $f'(\mathcal{U}(t,+\infty)) = f'(u(t,0^+)) \leq 0$, then (3.7) has non trivial but strictly monotone solution $\mathcal{U}$, and we only need $v(t,0^+) = v(t,0^-)$ as a connection between (3.6) and (3.8), the same as in (2.13e). And moreover, the difference between $u(t,0^+)$ and $u(t,0^-)$ satisfies (3.9).

It is also important to point out that there is no need to explicitly solve the inner problem, as observed in [55] and also [96] on the coupling of kinetic and fluid equations. We will assume throughout the chapter that the flux function $f \in C^1(R)$ with $f(0) = 0$, and no genuine nonlinearity is needed.

The rest of the chapter is organized as follows. In the next section, we will establish the well-posedness of the original two-scale hyperbolic system (3.1) as well as a priori estimates for the solution which make allows to pass rigorously to the limit. Then the strong convergence to the unique entropy weak solution of (3.6) and (3.8) is obtained in section 3.3. Section 3.4 is devoted to the derivation of the interface condition (3.6d) and (3.8d) via matched asymptotic analysis. And in the end, some concluding remarks are given in section 3.5.

3.2 Well-posedness of the original two-scale hyperbolic system

3.2.1 The regularized system

Existence and stability of the solution to the Cauchy problem (3.1)–(3.4) will rely on a suitable regularization of the equations and the data. Let $\rho(x)$ be a non-negative symmetric kernel with

$$\rho \in C_c^\infty(R), \;\; \rho \geq 0, \;\; \mathrm{supp}(\rho) \subset [-1,1], \;\; \int_R \rho(x)dx = 1, \tag{3.11}$$

and consider the sequence of mollifiers $\{\rho_\delta\}_{\delta>0}$ generated by ρ

$$\rho_\delta(x) = \frac{1}{\delta}\rho\left(\frac{x}{\delta}\right), \quad x \in R.$$

The discontinuous relaxation coefficient in (3.2) is given the following classical regularization

$$\Lambda(x,\epsilon,\delta) = \left(\rho_\delta * \lambda(\cdot,\epsilon)\right)(x)$$

with the property $\min(1, \frac{1}{\epsilon}) \leq \Lambda(x, \epsilon, \delta) \leq \max(1, \frac{1}{\epsilon})$. More precisely

$$\Lambda(x, \epsilon, \delta) = \begin{cases} \frac{1}{\epsilon}, \ x \geq \delta, \\ \text{smooth transition}, \ -\delta < x < \delta, \\ 1, \ x \leq -\delta, \end{cases}$$

so that for any given fixed $\epsilon > 0$, $\| \Lambda(\cdot, \epsilon, \delta) - \lambda(\cdot, \epsilon) \|_{L^1(R)}$ stays uniformly bounded w.r.t. $\delta > 0$ with the following well-known property

$$\lim_{\delta \to 0} \| \Lambda(\cdot, \epsilon, \delta) - \lambda(\cdot, \epsilon) \|_{L^1(R)} = 0. \tag{3.12}$$

The forthcoming analysis will heavily make use of (3.12) which we stress to hold for any given fixed $\epsilon > 0$. The main guideline is to recover uniform estimates in ϵ when sending the regularization parameter δ to 0 while keeping ϵ fixed. The initial data is regularized as follows. Consider a smooth function

$$\psi \in C_c^\infty(R^+), \ \ 0 \leq \psi(y) \leq 1, \ \ \psi(y) = \begin{cases} 1, \ \ 0 \leq y \leq 1, \\ 0, \ \ y \geq 2, \end{cases} \quad \sup|\psi'(y)| \leq 1, \tag{3.13}$$

and define a sequence of smooth truncating functions $\{\psi^\delta\}_{\delta > 0}$ setting

$$\psi^\delta(x) = \psi(\delta |x|), \ \ x \in R.$$

The initial data u_0 in (3.3) is smoothed and truncated according to

$$u_0^\delta(x) = \psi^\delta(x) \ (\rho_\delta * u_0)(x)$$

so that $u_0^\delta \in C_c^\infty(R)$ for all $\delta > 0$. Recall that (see Guisti [40] for a proof)

$$\rho_\delta * u_0 \to u_0 \text{ in } L^1_{loc}(R) \text{ as } \delta \text{ goes to 0 with } \lim_{\delta \to 0^+} \mathrm{TV}(\rho_\delta * u_0) \leq \mathrm{TV}(u_0), \tag{3.14}$$

so that we easily deduce from the properties of the truncation function stated in (3.13)

$$u_0^\delta \to u_0 \ \text{ in } L^1_{loc}(R) \text{ as } \delta \to 0 \text{ with } \ \| u_0^\delta \|_{L^\infty(R)} \leq \| u_0 \|_{L^\infty(R)}, \tag{3.15}$$

while

$$\mathrm{TV}(u_0^\delta) = \| \partial_x u_0^\delta \|_{L^1(R)} \leq \| \rho_\delta * u_0 \|_{L^1(R)} + \mathrm{TV}(\rho_\delta * u_0) \leq ||u_0||_{L^\infty(R)} + TV(\rho_\delta * u_0). \quad (3.16)$$

Hence, the sequence $\{u_0^\delta\}_{\delta>0}$ has uniformly bounded total variation and sup norm. The initial data v_0 is also regularized and truncated in a well-prepared manner

$$v_0^\delta = f(u_0^\delta), \quad \text{with } v_0^\delta \text{ uniformly bounded in } L^\infty(R) \cap \mathrm{BV}(R).$$

In view of (3.15)–(3.16), we clearly have that $v_0^\delta \to v_0 = f(u_0)$ in $L^1_{loc}(R)$ as $\delta \to 0$. Equipped with these regularizations, we propose the following regularized Cauchy problem

$$\begin{cases} \partial_t u^{\epsilon,\delta} + \partial_x v^{\epsilon,\delta} = 0, & (3.17a) \\ \partial_t v^{\epsilon,\delta} + a^2 \partial_x u^{\epsilon,\delta} = -\Lambda(x,\epsilon,\delta)(v^{\epsilon,\delta} - f(u^{\epsilon,\delta})), & (3.17b) \end{cases}$$

with initial data

$$u^{\epsilon,\delta}(0,x) = u_0^\delta(x), \;\; v^{\epsilon,\delta}(0,x) = f(u_0^\delta(x)). \quad (3.18)$$

The local existence theory for the smooth solution to Cauchy problem (3.17)–(3.18) is classical, and we refer the reader to Protter-Weinberger [84] for instance for the following result. To simplify the notations, we omit the superscripts ϵ and δ when these refer to fixed parameters.

Theorem 3.1. *Given initial data u_0 and v_0 in $C^1(R)$ that vanish outside the interval $[-M, M]$ for $M > 0$, there exists a unique classical solution (u, v) to the Cauchy problem (3.17) defined on a maximal time interval $[0, T_c)$. If the maximal time T_c is finite, then necessarily :*

$$\lim_{t \to T_c} \sup_{x \in R} \big(|u(t,x)| + |v(t,x)| \big) = +\infty. \quad (3.19)$$

The solution (u,v) *belongs to* $C^1\big([0,T_c)\times R\big)$ *and vanishes outside* $\bigcup_{t\in[0,T_c)}[-(M+at),M+at]$.

Under the sub-characteristic condition (3.5), the global in time existence of the classical solution $(u^{\epsilon,\delta},v^{\epsilon,\delta})$ of (3.17)–(3.18) for any given $\delta>0$ and $\epsilon>0$ is guaranteed by the following result due to Natalini [75].

Proposition 1. *Under the sub-characteristic condition (3.5), the classical solution* $(u^{\epsilon,\delta},v^{\epsilon,\delta})$ *of (3.17)–(3.18) is bounded in sup norm for all time, uniformly w.r.t.* ϵ *and* δ *with*

$$|u^{\epsilon,\delta}(t,x)|\le B(N_0),\quad |v^{\epsilon,\delta}(t,x)|\le aB(N_0),\quad \textit{for } (t,x)\in(0,\infty)\times R. \tag{3.20}$$

The smooth non-homogeneity in the space variable x in the relaxation parameter $\Lambda(x,\epsilon,\delta)>0$ does not affect the proof given by Natalini [75], exactly the same steps apply.

3.2.2 Existence and stability of the solution

The main results of this section ensures the existence and stability for the original Cauchy problem (3.1)–(3.4).

Theorem 3.2. *Given well-prepared initial data* u_0 *and* v_0 *satisfying the assumption (3.3)-(3.4). Assume the sub-characteristic condition (3.5). Then for any given fixed parameter* $\epsilon>0$ *and time* $T>0$*, the sequence* $\big\{(u^{\epsilon,\delta},v^{\epsilon,\delta})\big\}_{\delta>0}$ *of classical solutions of the regularized problem (3.17)-(3.18) converges as* δ *goes to zero to a unique weak solution* (u^ϵ,v^ϵ) *of the Cauchy problem (3.1)–(3.4) in* $L^\infty\big((0,T),L^1_{loc}(R)\big)$*. This weak solution satisfies the following a priori estimates, for a real constant* $C>0$ *independent* *of* ϵ*:*

$$(i) \quad \| u^\epsilon(t,\cdot) \|_{L^\infty(R)} \leq B(N_0), \;\; \| v^\epsilon(t,\cdot) \|_{L^\infty(R)} \leq aB(N_0), \tag{3.21}$$

$$(ii) \;\; TVu^\epsilon(t,\cdot) \leq C, \;\; TVv^\epsilon(t,\cdot) \leq C, \tag{3.22}$$

while for all compact set $K \subset R$

$$(iii) \quad \int_K |u^\epsilon(t_2,x) - u^\epsilon(t_1,x)|dx \leq C|t_2 - t_1|, \;\; 0 \leq t_1 \leq t_2 \tag{3.23}$$

$$\int_K |v^\epsilon(t_2,x) - v^\epsilon(t_1,x)|dx \leq C|t_2 - t_1|. \tag{3.24}$$

The proof of this statement heavily relies on the use of the characteristic variables

$$r_\pm(t,x) = a\; u(t,x) \pm v(t,x) \tag{3.25}$$

where we temporarily skip the small parameters in the notations for simplicity. Equipped with this customary change of variable, the relaxation system (3.1) or (3.17) are given in the convenient diagonal form:

$$\begin{cases} \partial_t r_- - a\partial_x r_- = -\Lambda(x,\epsilon,\delta)G(r_-,r_+), & \text{(3.26a)} \\ \partial_t r_+ + a\partial_x r_+ = \;\; \Lambda(x,\epsilon,\delta)G(r_-,r_+), & \text{(3.26b)} \end{cases}$$

where

$$G(r_-,r_+) = f\left(\frac{r_- + r_+}{2a}\right) - \frac{r_+ - r_-}{2}. \tag{3.27}$$

At the corner stone of the analysis, the sub-characteristic condition (3.5) (see Katsoulakis-Tzavaras [61], Natalini [75] for instance) makes the mapping G quasi-monotone in the sense that

$$\partial_{r_-} G(r_-,r_+) < 0, \;\; \partial_{r_+} G(r_-,r_+) > 0 \tag{3.28}$$

for any given pair (r_-, r_+) verifying $|r_-+r_+|/2a < B(N_0)$ so that $|f'((r_-+r_+)/2a))| < a$ holds with a prescribed according to (3.5). In addition, there exists locally a unique C^1 curve $(r_+, h(r_+))$ of equilibria, i.e. verifying

$$G(h(r_+), r_+) = 0 \tag{3.29}$$

for all r_+ in R such that $|r_+ + h(r_+)|/2a < B(N_0)$ so that (3.5) holds true. For all the r_+ under consideration,

$$h(r_+) \text{ is strictly increasing,} \tag{3.30}$$

and we have $h(0) = 0$. It is worth observing that the (well-defined) unique solution l of $l + h(l) = 2ak$ for any given k with $|k| < B(N_0)$ verifies the identity $l - h(l) = 2f(k)$.

Let us also recall the L^1 contraction property which will be heavily used later on. Given two classical solutions of the equations (3.26), then the differences $r_- - \bar{r_-}$ and $r_+ - \bar{r}_+$ obey, once respectively multiplied by $\mathrm{sgn}(r_- - \bar{r}_-)$ and $\mathrm{sgn}(r_+ - \bar{r}_+)$:

$$\partial_t|r_- - \bar{r}_-| - a\partial_x|r_- - \bar{r}_-| = -\Lambda(x,\epsilon,\delta)\big(G(r_-,r_+) - G(\bar{r}_-,\bar{r}_+)\big)\mathrm{sgn}(r_- - \bar{r}_-),$$

$$\partial_t|r_+ - \bar{r}_+| + a\partial_x|r_+ - \bar{r}_+| = +\Lambda(x,\epsilon,\delta)\big(G(r_-,r_+) - G(\bar{r}_-,\bar{r}_+)\big)\mathrm{sgn}(r_+ - \bar{r}_+).$$

Thus adding these two identities gives

$$\begin{aligned}\partial_t\big(|r_+ - \bar{r}_+| + |r_- - \bar{r}_-|\big) + a\partial_x\big(|r_+ - \bar{r}_+| - |r_- - \bar{r}_-|\big)\\ = \Lambda(x,\epsilon,\delta)\big(G(r_-,r_+) - G(\bar{r}_-,\bar{r}_+)\big)\big(\mathrm{sgn}(r_+ - \bar{r}_+) - \mathrm{sgn}(r_- - \bar{r}_-)\big) \le 0,\end{aligned} \tag{3.31}$$

thanks to the quasi-monotonicity property (3.28). A first consequence of these inequalities is

Proposition 2. *Under the assumption of Theorem 3.2, the limit solution* (u^ϵ, v^ϵ) *satisfies for all* $\epsilon > 0$ *the following entropy like inequalities expressed in terms of the characteristic variables*

$$\iint_{R_t^+ \times R} \Big[\big(|r_+^\epsilon - l| + |r_-^\epsilon - h(l)|\big)\partial_t \varphi + a\big(|r_+^\epsilon - l| - |r_-^\epsilon - h(l)|\big)\partial_x \varphi \Big] dtdx \geq 0. \quad (3.32)$$

for any given non negative test function φ *in* $\mathcal{C}_c^1(R^+ \times R)$ *and all* l *in* R *such that* $|l + h(l)|/2a < B(N_0)$.

A second consequence is the following L^1 contraction property (see also [61], [75]).

Proposition 3. *Assume the sub-characteristic condition (3.5). Let* $(u^{\epsilon,\delta}, v^{\epsilon,\delta})$ *and* $(\bar{u}^{\epsilon\delta}, \bar{v}^{\epsilon\delta})$ *be two classical solutions of the equations (3.17) with initial data* (u_0^δ, v_0^δ) *and* $(\bar{u}_0^\delta, \bar{v}_0^\delta)$, *that vanish outside the cone* $\bigcup_{t\geq 0}[-(M + at), (M + at)]$. *Then for all time* $t > 0$, *the associated characteristic variables obey the following inequality*

$$\int_{-M}^{M} \big(\, |r_+^{\epsilon\delta} - \bar{r}_+^{\epsilon\delta}| + |r_-^{\epsilon\delta} - \bar{r}_-^{\epsilon\delta}|\, \big)(t,x)dx \leq \int_{-(M+at)}^{M+at} \big(\, |(r_+)_0^\delta - (\bar{r}_+)_0^\delta| + |(r_-)_0^\delta - (\bar{r}_-)_0^\delta|\, \big)(x)dx. \quad (3.33)$$

In the case of a problem invariant by translation in x, the above L^1 contraction principle is well known to imply a uniform BV estimate for the solution to (3.17) as long as the initial data (u_0, v_0) is chosen in BV (see [75] for instance). However, the dependence of the relaxation coefficient $\Lambda(x, \epsilon, \delta)$ in the space variable obviously prevents the classical solution of the regularized equation (3.17) from being translation invariant in x. Thus the expected uniform BV estimate can no longer be inferred from (3.33), neither can it be derived from the direct differentiation w.r.t. x of the governing equations (3.17) (see [92]) since in the limit $\delta \to 0$, $\partial_x \Lambda(x, \epsilon, \delta)$ concentrates in a Dirac mass at the interface $x = 0$. Instead, we can take advantage of the invariance w.r.t. time variable of the classical solutions of (3.17) and we prove hereafter that uniform BV estimates can be inferred from it. The key estimates are gathered in the following statement.

Proposition 4. *Under the assumption of Theorem 3.2, the classical solution* $(u^{\epsilon,\delta}, v^{\epsilon,\delta})$ *of the regularized Cauchy problem (3.17)–(3.18) verifies, for any given* $\epsilon > 0$ *and* $\delta > 0$*,*

$$(i)\quad \| u^{\epsilon,\delta}(t,\cdot) \|_{L^\infty(R)} \leq B(N_0), \ \ \| v^{\epsilon,\delta}(t,\cdot) \|_{L^\infty(R)} \leq aB(N_0), \tag{3.34}$$

$$(ii)\quad \| \partial_t u^{\epsilon,\delta}(t,\cdot) \|_{L^1(R)} \leq C, \| \partial_t v^{\epsilon,\delta}(t,\cdot) \|_{L^1(R)} \leq C, \tag{3.35}$$

$$(iii)\ \ TV v^{\epsilon}(t,\cdot) \leq C, \tag{3.36}$$

$$(iv)\ \ TV_{\{x<0\}}\big(u^{\epsilon,\delta}(t,\cdot)\big) \leq C\big(1 + \| \lambda(\cdot,\epsilon) - \Lambda(\cdot,\epsilon,\delta) \|_{L^1(R)} \big), \tag{3.37}$$

$$TV_{\{x>0\}}\big(u^{\epsilon,\delta}(t,\cdot)\big) \leq C\big(1 + \| \lambda(\cdot,\epsilon) - \Lambda(\cdot,\epsilon,\delta) \|_{L^1(R)} \big), \tag{3.38}$$

for some constant $C > 0$ *independent of* ϵ *and* δ*.*

Proof. (i) (3.34) is nothing but the estimate (3.20) stated in Propositon 1.

(ii) Deriving the uniform L^1 estimate for the time derivative of the classical solutions $(u^{\epsilon,\delta}, v^{\epsilon,\delta})$ relies on differentiating system (3.17) w.r.t. time. Define

$$s_-^{\epsilon,\delta}(t,x) \;=\; a\partial_t u^{\epsilon,\delta}(t,x) - \partial_t v^{\epsilon,\delta}(t,x), \tag{3.39}$$

$$s_+^{\epsilon,\delta}(t,x) \;=\; a\partial_t u^{\epsilon,\delta}(t,x) + \partial_t v^{\epsilon,\delta}(t,x), \tag{3.40}$$

then they solve the system

$$\begin{cases} \partial_t s_-^{\epsilon,\delta} - a\partial_x s_-^{\epsilon,\delta} = -\Lambda(x,\epsilon,\delta) R(s_-^{\epsilon,\delta}, s_+^{\epsilon,\delta}), & \text{(3.41a)} \\ \partial_t s_+^{\epsilon,\delta} + a\partial_x s_+^{\epsilon,\delta} = \Lambda(x,\epsilon,\delta) R(s_-^{\epsilon,\delta}, s_+^{\epsilon,\delta}), & \text{(3.41b)} \end{cases}$$

where

$$R(s_-, s_+) = f'(u)\frac{s_- + s_+}{2a} - \frac{s_+ - s_-}{2}, \tag{3.42}$$

again omitting the superscripts ϵ, δ for simplicity. Under the sub-characteristic condition (3.5), R obeys the quasi-monotonicity property

$$\partial_{s_-} R(s_-, s_+) < 0, \;\; \partial_{s_+} R(s_-, s_+) > 0 \tag{3.43}$$

for any given $\epsilon > 0$ and $\delta > 0$. Similar steps to those used in the derivation of the L^1 contraction principle from quasi-monotonicity yields

$$\int_{|x|<M} \left(|s_+^{\epsilon,\delta}(t,x)| + |s_-^{\epsilon,\delta}(t,x)|\right) dx \leq \int_{|x|<M+at} \left(|s_+^{\epsilon,\delta}(0,x)| + |s_-^{\epsilon,\delta}(0,x)|\right) dx \tag{3.44}$$

for any given real number $M > 0$ so that the initial data $s_+^{\epsilon,\delta}(0,x)$, $s_-^{\epsilon,\delta}(0,x)$ vanish for all x with $|x| \geq M$. Next observe from the governing equations (3.17) expressed at time $t = 0$

$$\begin{cases} \partial_t u^{\epsilon,\delta}(0,x) = -\dfrac{d}{dx} v_0^\delta(x), & \text{(3.45a)} \\ \partial_t v^{\epsilon,\delta}(0,x) = -a^2 \dfrac{d}{dx} u_0^\delta(x) + \Lambda(x,\epsilon,\delta)(f(u_0^\delta)(x) - v_0^\delta(x)), & \text{(3.45b)} \end{cases}$$

then by the choice of the well-prepared initial data $v_0^\delta = f(u_0^\delta)$, we get

$$\| \partial_t u^{\epsilon,\delta}(0,\cdot) \|_{L^1(R)} \leq \| f'(u_0^\delta) \|_{L^\infty(R)} \, \mathrm{TV}(u_0^\delta), \tag{3.46}$$

$$\| \partial_t v^{\epsilon,\delta}(0,\cdot) \|_{L^1(R)} \leq a^2 \mathrm{TV}(u_0^\delta) \tag{3.47}$$

so as to infer the bound

$$\max\Big(\| s_+^{\epsilon,\delta}(0,\cdot) \|_{L^1(R)}, \| s_-^{\epsilon,\delta}(0,\cdot) \|_{L^1(R)} \Big) \leq a\big(\| f'(u_0^\delta) \|_{L^\infty(R)} + a\big) \mathrm{TV}(u_0^\delta) \leq C,$$

where $C > 0$ is independent of δ. We therefore deduce from the L^1 contraction property (3.44)

$$\| s_-^{\epsilon,\delta}(t,\cdot) \|_{L^1(R)} \leq C, \;\; \| s_+^{\epsilon,\delta}(t,\cdot) \|_{L^1(R)} \leq C.$$

These uniform estimates clearly imply (3.35).

(iii) The expected BV estimate (3.36) of $v^{\epsilon,\delta}$ immediately follows from the identity

$$\partial_x v^{\epsilon,\delta}(t,x) = -\partial_t u^{\epsilon,\delta}(t,x).$$

(iv) Let us now establish the next uniform BV estimate (3.38) in two steps. First define

$$\xi^{\epsilon,\delta}(t,x) = f(u^{\epsilon,\delta}(t,x)) - v^{\epsilon,\delta}(t,x), \tag{3.48}$$

it clearly solves

$$\partial_t \xi^{\epsilon,\delta} + \Lambda(x,\epsilon,\delta)\xi^{\epsilon,\delta} = a^2 \partial_x u^{\epsilon,\delta} - f'(u^{\epsilon,\delta})\partial_x v^{\epsilon,\delta}. \tag{3.49}$$

Now recast this equation as follows

$$\partial_t \xi^{\epsilon,\delta} + \lambda(x,\epsilon)\xi^{\epsilon,\delta} = a^2 \partial_x u^{\epsilon,\delta} - f'(u^{\epsilon,\delta})\partial_x v^{\epsilon,\delta} + \Theta^{\epsilon,\delta} \tag{3.50}$$

where by definition

$$\Theta^{\epsilon,\delta}(t,x) = (\lambda(x,\epsilon) - \Lambda(x,\epsilon,\delta))(f(u^{\epsilon,\delta}) - v^{\epsilon,\delta}). \tag{3.51}$$

Observe that

$$\begin{aligned} \int_R |\Theta^{\epsilon,\delta}(t,x)|dx \ &\leq \ \|\,(f(u^{\epsilon,\delta}) - v^{\epsilon,\delta})(t,.)\,\|_{L^\infty(R)}\|\,\lambda(.,\epsilon) - \Lambda(.,\epsilon,\delta)\,\|_{L^1(R)} \\ &\leq \ C\,\|\,\lambda(.,\epsilon) - \Lambda(.,\epsilon,\delta)\,\|_{L^1(R)} \end{aligned} \tag{3.52}$$

for some constant $C > 0$ independent of ϵ and δ. Let us again emphasize that for any given fixed $\epsilon > 0$, the L^1 distance $\|\,\lambda(\cdot,\epsilon) - \Lambda(\cdot,\epsilon,\delta)\,\|_{L^1(R)}$ stays uniformly bounded w.r.t. $\delta > 0$ and in addition vanishes in the limit $\delta \to 0$ (see (3.12)). Now focusing on positive values of x so that $\lambda(x,\epsilon) = \frac{1}{\epsilon}$, we infer the pointwise in x inequality:

$$\begin{aligned}\partial_t \int_0^t e^{\frac{s}{\epsilon}} |\xi^{\epsilon,\delta}(s,x)| ds \;\; &\leq \;\; e^{\frac{t}{\epsilon}} \Big\{ \left|\Theta^{\epsilon,\delta}(t,x)\right| + \| f'(u^{\epsilon,\delta})(t,.) \|_{L^\infty(R)} \left|\partial_x v^{\epsilon,\delta}(t,x)\right| \Big\} \\ &\qquad + a^2 e^{\frac{t}{\epsilon}} \left|\partial_x u^{\epsilon,\delta}(t,x)\right|, \quad x > 0,\end{aligned}$$

which implies once integrated in time

$$\begin{aligned}\left|\xi^{\epsilon,\delta}(t,x)\right| \; &\leq e^{-\frac{t}{\epsilon}} |\xi^{\epsilon,\delta}(0,x)| + \int_0^t e^{\frac{s-t}{\epsilon}} \left\{ |\Theta^{\epsilon,\delta}(s,x)| + \| f'(u^{\epsilon,\delta})(s,.) \|_{L^\infty(R)} \, |\partial_x v^{\epsilon,\delta}(s,x)| \right\} ds \\ &\qquad + a^2 \int_0^t e^{\frac{s-t}{\epsilon}} \left|\partial_x u^{\epsilon,\delta}(s,x)\right| ds, \quad x > 0, \qquad (3.53)\end{aligned}$$

where $\xi^{\epsilon,\delta}(0,x) = f(u^{\epsilon,\delta}(0,x)) - v^{\epsilon,\delta}(0,x) = 0$ since the initial data is well-prepared. We next propose the following bootstrap argument. Rewrite the equation (3.17b) as follows

$$a^2 \partial_x u^{\epsilon,\delta} = -\partial_t v^{\epsilon,\delta} + \lambda(x,\epsilon)(f(u^{\epsilon,\delta}) - v^{\epsilon,\delta}) - \Theta^{\epsilon,\delta}(x,t), \qquad (3.54)$$

with $\Theta^{\epsilon,\delta}$ defined in (3.51). One easily infers the rough estimate

$$\int_0^\infty \left|\partial_x u^{\epsilon,\delta}(t,x)\right| dx \leq \frac{1}{a^2} \int_0^\infty \left(|\partial_t v^{\epsilon,\delta}| + |\Theta^{\epsilon,\delta}| \right)(t,x) dx + \frac{1}{\epsilon a^2} \int_0^\infty |\xi^{\epsilon,\delta}|(t,x) dx, \quad (3.55)$$

valid for all time $t > 0$. Introduce

$$g^{\epsilon,\delta}(t) = \int_0^\infty \left|\partial_x u^{\epsilon,\delta}(t,x)\right| dx \qquad (3.56)$$

then plugging the estimate (3.53) in (3.55) yields

$$\begin{aligned}g^{\epsilon,\delta}(t) \leq \; &\frac{1}{a^2} \, \| \partial_t v^{\epsilon,\delta}(t,.) \|_{L^1(R)} + \frac{1}{a^2} \, \| \Theta^{\epsilon,\delta}(t,.) \|_{L^1(R)} + \frac{1}{\epsilon} \int_0^t e^{\frac{s-t}{\epsilon}} g^{\epsilon,\delta}(s) ds \\ &+ \frac{1}{\epsilon a^2} \int_0^t e^{\frac{s-t}{\epsilon}} \Big(\| \Theta^{\epsilon,\delta}(s,\cdot) \|_{L^1(R)} + \| f'(u^{\epsilon,\delta}(s,\cdot)) \|_{L^\infty(R)} \| \partial_x v^{\epsilon,\delta}(s,\cdot) \|_{L^1(R)} \Big) ds.\end{aligned}$$

Observe that the estimates (ii) and (3.52) ensure

$$\| \partial_t v^{\epsilon,\delta}(t,\cdot) \|_{L^1(R)} + \| \Theta^{\epsilon,\delta}(t,\cdot) \|_{L^1(R)} \leq C \Big(1 + \| \lambda(\cdot,\epsilon) - \Lambda(\cdot,\epsilon,\delta) \|_{L^1(R)} \Big)$$

where the constant $C > 0$ is independent of t, ϵ and δ. Similarly, one may infer the estimate

$$\begin{aligned}
&\frac{1}{\epsilon}\int_0^t e^{\frac{s-t}{\epsilon}}\Big(\|\,\Theta^{\epsilon,\delta}(s,\cdot)\,\|_{L^1(R)}+\|\,f'(u^{\epsilon,\delta}(s,\cdot))\,\|_{L^\infty(R)}\|\,\partial_x v^{\epsilon,\delta}(s,\cdot)\,\|_{L^1(R)}\Big)ds\\
\le\;& C\Big(1+\|\,\lambda(\cdot,\epsilon)-\Lambda(\cdot,\epsilon,\delta)\,\|_{L^1(R)}\Big)\Big(\frac{1}{\epsilon}\int_0^t e^{\frac{s-t}{\epsilon}}ds\Big)\\
\le\;& C\Big(1+\|\,\lambda(\cdot,\epsilon)-\Lambda(\cdot,\epsilon,\delta)\,\|_{L^1(R)}\Big). \qquad (3.57)
\end{aligned}$$

We thus arrive at

$$g^{\epsilon,\delta}(t) \;\le C\Big(1+\|\,\lambda(\cdot,\epsilon)-\Lambda(\cdot,\epsilon,\delta)\,\|_{L^1(R)}\Big)+\tfrac{1}{\epsilon}\textstyle\int_0^t e^{\frac{s-t}{\epsilon}}g^{\epsilon,\delta}(s)ds, \qquad (3.58)$$

which by the Gronwall's inequality implies that for all time $t > 0$

$$\mathrm{TV}_{\{x>0\}}\big(u^{\epsilon,\delta}(t,\cdot)\big)=g^{\epsilon,\delta}(t)\le C\Big(\,\|\,\lambda(\cdot,\epsilon)-\Lambda(\cdot,\epsilon,\delta)\,\|_{L^1(R)}+1\Big)e^{\frac{1}{\epsilon}\int_0^t e^{\frac{s-t}{\epsilon}}ds}, \qquad (3.59)$$

with $\frac{1}{\epsilon}\int_0^t e^{\frac{s-t}{\epsilon}}ds \le 1$.

At last, we establish the estimate on the total variation of $u^{\epsilon,\delta}$ in the left half line $\{x < 0\}$. We proceed using similar steps to those developed above. Consider again the equilibrium gap function $\xi^{\epsilon,\delta}$ defined in (3.48) and its governing equation (3.50). Now consider negative values of x so that $\lambda(x,\epsilon) = 1$ to get the analogue of (3.53) but for the half line $\{x < 0\}$:

$$\begin{aligned}
\big|\xi^{\epsilon,\delta}(t,x)\big| \;\le\;& e^{-t}|\xi^{\epsilon,\delta}(0,x)|+\int_0^t e^{s-t}\left\{|\Theta^{\epsilon,\delta}(s,x)|+\|\,f'(u^{\epsilon,\delta})(s,.)\,\|_{L^\infty(R)}\;|\partial_x v^{\epsilon,\delta}(s,x)|\right\}ds\\
&+a^2\int_0^t e^{s-t}\left|\partial_x u^{\epsilon,\delta}(s,x)\right|ds, \quad x<0, \qquad (3.60)
\end{aligned}$$

so that defining

$$g^{\epsilon,\delta}(t)=\int_{-\infty}^0\left|\partial_x u^{\epsilon,\delta}(t,x)\right|dx \qquad (3.61)$$

yields again from (3.54) but expressed for negative values of x a similar estimate to (3.58). The Gronwall's inequality again gives the expected conclusion. Details are left to the reader. □

Now we are ready to prove the main theorem in this section.

Proof the Theorem 3.2. First observe from the estimates (iv) established in Proposition 4 that, for any given $\delta > 0$

$$\begin{aligned} \mathrm{TV}_R\big(u^{\epsilon,\delta}(t,\cdot)\big) &= \mathrm{TV}_{\{x<0\}}\big(u^{\epsilon,\delta}(t,\cdot)\big) + \mathrm{TV}_{\{x>0\}}\big(u^{\epsilon,\delta}(t,\cdot)\big) + |u^{\epsilon,\delta}(t,0^+) - u^{\epsilon,\delta}(t,0^-)| \\ &\leq C\big(1+ \| \lambda(\cdot,\epsilon) - \Lambda(\cdot,\epsilon,\delta) \|_{L^1(R)} \big), \end{aligned} \tag{3.62}$$

since $u^{\epsilon,\delta}(t,0^-) = u^{\epsilon,\delta}(t,0^+)$, for some positive constant C independent of ϵ and δ. Now let ϵ be fixed, the estimate (3.62) ensures that $\{u^{\epsilon,\delta}\}_{\delta>0}$ stays uniformly in $\mathrm{BV}(R_x)$ for all time $t \in (0,T)$ with $T > 0$ given, while being uniformly bounded in sup-norm. The well-known Helly's Theorem asserts that for any given compact K_x in R_x, the canonical embedding of $L^1(K_x) \cap BV(R_x)$ is compact in $L^1(K_x)$. So let $\{s_n\}_{n\in N}$ be a countable dense subset of $[0,T]$. For each time s_k, a classical diagonal extraction procedure in space gives the existence of an extracted subsequence, still labeled $\big\{\big(u^{\epsilon,\delta}(s_k,\cdot), v^{\epsilon,\delta}(s_k,\cdot)\big)\big\}_{\delta>0}$ which converges in $L^1_{\mathrm{loc}}(R_x)$ to some limit $(u^\epsilon(s_k,\cdot), v^\epsilon(s_k,\cdot))$ with bounded sup-norm as δ goes to 0. By another diagonal extraction procedure in time, we can assume that the same subsequence $\big\{(u^{\epsilon,\delta}(s_j,\cdot), v^{\epsilon,\delta}(s_j,\cdot))\big\}_{\delta>0}$ converges to $(u^\epsilon(s_j,\cdot), v^\epsilon(s_j,\cdot))$ at each time s_j up to some relabeling. To reach any given time $t \in (0,T)$, we partition $[0,T]$ into N subintervals (t_i, t_{i+1}) where t_i are selected from the dense subset $\{s_n\}_{n\in N}$ so that $|t_{i+1} - t_i| < \eta$ for given $\eta > 0$. Let us first show that $\{u^{\epsilon,\delta}(t,\cdot)\}_{\delta>0}$ is a Cauchy sequence in $L^\infty\big((0,T), L^1_{\mathrm{loc}}(R)\big)$. Consider any given pair δ_1, $\delta_2 > 0$, then for any $t \in (0,T)$, there exists t_i such that $|t_i - t| \leq \eta$ and we write

$$\| u^{\epsilon,\delta_2}(t,\cdot) - u^{\epsilon,\delta_1}(t,\cdot) \|_{L^1_{\text{loc}(R_x)}}$$
$$\leq \| u^{\epsilon,\delta_2}(t,\cdot) - u^{\epsilon,\delta_2}(t_i,\cdot) \|_{L^1_{\text{loc}(R_x)}} + \| u^{\epsilon,\delta_2}(t_i,\cdot) - u^{\epsilon,\delta_1}(t_i,\cdot) \|_{L^1_{\text{loc}(R_x)}} + \| u^{\epsilon,\delta_1}(t_i,\cdot) - u^{\epsilon,\delta_1}(t,\cdot) \|_{L^1_{\text{loc}(R_x)}} .$$

The term in the middle comes from a Cauchy sequence and goes to 0 as δ_1, δ_2 go to 0. As for the first and third terms, observe that for any given $K \subset R$,

$$\int_K |u^{\epsilon,\delta}(t,x) - u^{\epsilon,\delta}(t_i,x)| dx \leq \int_{t_i}^t \Big(\int_K |\partial_t u^{\epsilon,\delta}(t,x)| dx \Big) dt \leq C|t_i - t| \leq C\eta \qquad (3.63)$$

for some constant $C > 0$ independent of $\epsilon > 0$ and of K in view of estimate (3.35). We therefore get

$$\| u^{\epsilon,\delta_2}(t,\cdot) - u^{\epsilon,\delta_1}(t,\cdot) \|_{L^1(R)} \leq 2C\eta + \| u^{\epsilon,\delta_2}(t_i,\cdot) - u^{\epsilon,\delta_1}(t_i,\cdot) \|_{L^1(R)} . \qquad (3.64)$$

Since η can be made arbitrarily small, this shows that $\{u^{\epsilon,\delta}(t,\cdot)\}_{\delta>0}$ is a uniform Cauchy sequence in $t \in (0,T)$. The same arguments apply to the sequence $\{v^{\epsilon,\delta}(t,\cdot)\}_{\delta>0}$. Hence we proved that there exists an extracted subsequence, denoted by $\big\{ \big(u^{\epsilon,\delta}(t,\cdot), v^{\epsilon,\delta}(t,\cdot) \big) \big\}_{\delta>0}$, which is uniformly bounded in sup norm and which converges in $L^\infty\big((0,T), L^1_{\text{loc}}(R)\big)$ and almost everywhere to some limit $\big(u^\epsilon(t,\cdot), v^\epsilon(t,\cdot)\big)$. Let us conclude by showing that all extracted subsequences actually converge to the same limit which proves in turn uniqueness. Start from the L^1 contraction principle (3.33), one has for all time $t \in (0,T)$ and $M > 0$

$$\int_{-M}^{M} \big(|r_+^{\epsilon,\delta} - \bar{r}_+^{\epsilon,\delta}| + |r_-^{\epsilon,\delta} - \bar{r}_-^{\epsilon,\delta}| \big)(t,x) dx \leq \int_{-(M+at)}^{M+at} \big(|(r_+)_0^\delta - (\bar{r}_+)_0^\delta| + |(r_+)_0^\delta - (\bar{r}_+)_0^\delta| \big)(x) dx.$$

The above convergence results assert that there exists an extracted subsequence $(r_+^{\epsilon,\delta}, r_-^{\epsilon,\delta})$ (resp. $(\bar{r}_+^{\epsilon,\delta}, \bar{r}_-^{\epsilon,\delta})$) which converges to some limit $(r_+^\epsilon, r_-^\epsilon)$ (resp. $(\bar{r}_+^\epsilon, \bar{r}_-^\epsilon)$) in $L^\infty((0,T), L^1_{\text{loc}}(R_x))$

as δ goes to 0. These limits are seen to satisfy

$$\int_{-M}^{M} \big(\, |r_+^\epsilon - \bar{r}_+^\epsilon| + |r_-^\epsilon - \bar{r}_-^\epsilon| \,\big)(t,x)dx \le \int_{-(M+at)}^{M+at} \big(\, |r_+^0 - \bar{r}_+^0| + |r_-^0 - \bar{r}_-^0| \,\big)(x)dx,$$

which gives the expected uniqueness property. Let us now characterize the limit (u^ϵ, v^ϵ). First observe from the inequality (3.31) that for any given non-negative test function $\varphi \in C_c^1\big((0,\infty)\times R\big)$

$$\iint_{R_t^+\times R}\Big[\big(|r_+^{\epsilon,\delta} - \bar{r}_+^{\epsilon,\delta}| + |r_-^{\epsilon,\delta} - \bar{r}_-^{\epsilon,\delta}|\big)\partial_t\varphi + a\big(|r_+^{\epsilon,\delta} - \bar{r}_+^{\epsilon,\delta}| - |r_-^{\epsilon,\delta} - \bar{r}_-^{\epsilon,\delta}|\big)\partial_x\varphi\Big]dtdx \ge 0.$$

Let us now choose well-prepared initial data $\bar{u}_0$ and $\bar{v}_0$ with compact support such that $(\bar{r}_+)_0^\delta = \psi_\delta(x)l$ and $(\bar{r}_-)_0^\delta = \psi_\delta(x)h(l)$ for any given $l \in R$ with $|l+h(l)|/2a < B(N_0)$. Observe that $u^\epsilon(t,x) = \frac{l+h(l)}{2a}$, $v^\epsilon(t,x) = \frac{l-h(l)}{2}$ trivially solve the Cauchy problem (3.1) so that by uniqueness we have in the limit $\delta \to 0$: $\bar{r}_+^\epsilon(t,x) = l$ and $\bar{r}_-^\epsilon(t,x) = h(l)$. Therefore we have proved (3.32) for all the l under consideration. An additional characterization of the limit (u^ϵ, v^ϵ) comes as follows. Let us consider test function $\varphi \in C_c^1\big((0,\infty)\times R\big)$, then the weak form of (3.17) reads

$$\begin{cases} \displaystyle\iint_{R_t^+\times R_x} u^{\epsilon,\delta}\varphi_t + v^{\epsilon,\delta}\varphi_x dxdt = 0, & \text{(3.65a)} \\ \displaystyle\iint_{R_t^+\times R_x} v^{\epsilon,\delta}\varphi_t + a^2u^{\epsilon,\delta}\varphi_x - \Lambda(x,\epsilon,\delta)(f(u^{\epsilon,\delta}) - v^{\epsilon,\delta})\varphi dxdt = 0. & \text{(3.65b)} \end{cases}$$

Notice that since f is smooth with $f(0) = 0$ and $\{u^{\epsilon,\delta}\}$ stays uniformly bounded in sup norm, one has $f(u^{\epsilon,\delta}(t,x))| \le C|u^{\epsilon,\delta}|$. Moreover, $f(u^{\epsilon,\delta}(t,x)) \to f(u^\epsilon(t,x))$ a.e. in t and x. Therefore by the Lebesgue's dominated convergence theorem, $f(u^{\epsilon,\delta}(t,x)) \to f(u^\epsilon(t,x))$ in $L^1_{\text{loc}}(R_t^+\times R_x)$. Now passing to the limit in (3.65), notice that

$$\begin{aligned} &\iint_{R_t^+\times R_x} \Lambda(x,\epsilon,\delta)(f(u^{\epsilon,\delta}) - v^{\epsilon,\delta})\varphi dxdt \\ =& \iint_{R_t^+\times R_x} (\Lambda(x,\epsilon,\delta) - \lambda(x,\epsilon))(f(u^{\epsilon,\delta}) - v^{\epsilon,\delta})\varphi dxdt + \iint_{R_t^+\times R_x} \lambda(x,\epsilon)(f(u^{\epsilon,\delta}) - v^{\epsilon,\delta})\varphi dxdt \end{aligned}$$

and by (3.12) and the sup norm estimate (3.34), the limit (u^ϵ, v^ϵ) indeed solves

$$\begin{cases} \int u^\epsilon \varphi_t + v^\epsilon \varphi_x dxdt = 0, & (3.66a) \\ \int v^\epsilon \varphi_t + a^2 u^\epsilon \varphi_x - \lambda(x,\epsilon)(f(u^\epsilon) - v^\epsilon)\varphi dxdt = 0, & (3.66b) \end{cases}$$

for any given test function φ in $C_c^1\big((0,\infty)\times R\big)$.

Let us conclude this section by proving the expected a priori estimates, namely (ii) and (iii) in Theorem 3.2. The L^1_{loc} continuity in time estimates (iii) are direct consequences of their counter parts verified in (3.63) by $\{u^{\epsilon,\delta}\}_{\delta>0}$ for any given $\epsilon > 0$, with some constant independent of ϵ. Concerning the BV estimate, let us first observe from the estimate (iv) in Proposition 4 that focuses on the right half line R_x^+. For any given $\varphi \in C_c^1(R_x^+)$ with $\|\varphi\|_{L^\infty(R)} \leq 1$,

$$\int_R u^{\epsilon,\delta}(t,x)\partial_x\varphi dx \leq \sup_{\varphi\in C_c^1(R_x^+)} \int_R u^{\epsilon,\delta}\partial_x\varphi dx = \text{TV}(u^{\epsilon,\delta}) \leq C(1+\|\lambda(\cdot,\epsilon) - \Lambda(\cdot,\epsilon,\delta)\|_{L^1(R)})$$

sending δ to 0 with $\epsilon > 0$ kept fixed yields

$$\int_{R_x^+} u^\epsilon(t,x)\partial_x\varphi dx \leq C$$

with the same uniform constant C above which does not depend on ϵ. We therefore conclude that

$$\text{TV}_{\{x>0\}}\big(u^\epsilon(t,\cdot)\big) = \sup_{\varphi\in C_c^1(R_x^+), ||\varphi||_{L^\infty(R)}\leq 1} \int_{R^+} u^\epsilon(t,x)\partial_x\varphi dx \leq C.$$

Similarly, one can derive a uniform BV estimate in the open set R_x^-

$$\text{TV}_{\{x<0\}}\big(u^\epsilon(t,\cdot)\big) = \sup_{\varphi\in C_c^1(R_-^+), ||\varphi||_{L^\infty(R)}\leq 1} \int_{R^-} u^\epsilon\partial_x\varphi \leq C$$

for some constant $C > 0$ independent of ϵ. Then $u^\epsilon(t,\cdot)$ having finite total variation in R_x^- and R_x^+, does admit left and right traces at $x = 0$. We may therefore write

$$\text{TV}_R\big(u^\epsilon(t,\cdot)\big) = \text{TV}_{\{x<0\}}\big(u^\epsilon(t,\cdot)\big) + \text{TV}_{\{x>0\}}\big(u^\epsilon(t,\cdot)\big) + \big|u^\epsilon(t,0^+) - u^\epsilon(t,0^-)\big|$$

with $|u^\epsilon(t,0^+) - u^\epsilon(t,0^-)| \leq 2 \parallel u^\epsilon(t,\cdot) \parallel_{L^\infty(R)} \leq C$ in view of (3.34). So we conclude that $\{u^\epsilon\}_{\epsilon>0}$ has uniform in ϵ bounded total variation. The corresponding uniform estimate for $\{v^\epsilon\}_{\epsilon>0}$ follows from the companion estimate (iii) in Proposition 4. This concludes the proof of the main theorem in this section.

□

3.3 Strong convergence in $L^\infty\big((0,T), L^1_{\mathbf{loc}}(R)\big)$

In this section, we want to show the limit behavior when sending the relaxation parameter ϵ to 0.

Theorem 3.3. *Given any initial condition u_0, v_0 satisfying (3.3)–(3.4) and assume the sub characteristic condition (3.5), the sequence (u^ϵ, v^ϵ) converges to a unique limit (u, v) in $L^\infty\big((0,T), L^1_{loc}(R)\big)$ for all $T > 0$, where u and v have bounded sup norm and bounded total variation. The limit solves for any test function $\varphi \in C^1_c((0,\infty) \times R^-_x)$:*

$$(a) \qquad \int_{R^+_t \times R^-_x} u\varphi_t + v\varphi_x dxdt = 0, \tag{3.67}$$

$$\int_{R^+_t \times R^-_x} v\varphi_t + a^2 u\varphi_x - (f(u) - v)\varphi dxdt = 0, \tag{3.68}$$

while $v(t,x) = f(u(t,x))$ a.e. $t > 0$, $x > 0$ and for any given non-negative test function $\varphi \in C^1_c((0,\infty) \times R^+_x)$

$$(b) \qquad \int_{R^+_t \times R^+_x} |u-k|\varphi_t + sgn(u-k)(f(u) - f(k))\varphi_k dxdt \geq 0, \;\; \varphi \geq 0, \tag{3.69}$$

for all $k \in R$ with $|k| < B(N_0)$.

In addition, the following L^1 contraction principle holds

$$\frac{1}{2a} \| (r_+ - \bar{r}_+)(t,.) \|_{L^1_{loc(R^-_x)}} + \frac{1}{2a} \| (r_- - \bar{r}_-)(t,\cdot) \|_{L^1_{loc(R^-_x)}} + \| (u - \bar{u})(t,.) \|_{L^1_{loc(R^+_x)}} \leq \| u_0 - \bar{u}_0 \|_{L^1_{loc(R_x)}}, \tag{3.70}$$

and the estimates (i)-(iii) in Theorem 3.2 are satisfied for (u,v) *as well.*

To begin with, we need the following technical lemma.

Lemma 3.4. *For a.e.* $t>0$*, we have*

$$\|(f(u^\epsilon)-v^\epsilon)(t,\cdot)\|_{L^1(R_x^+)}\leq C\epsilon \tag{3.71}$$

where $C>0$ *is independent of* ϵ*.*

Proof. (3.17b) gives

$$\begin{aligned}
&\int_{R_x^+}\lambda(x,\epsilon)\left|f(u^{\epsilon,\delta})-v^{\epsilon,\delta}\right|(t,x)dx\\
\leq\ &\|\partial_t v^{\epsilon,\delta}(t,\cdot)\|_{L^1(R)}+a^2\mathrm{TV}(u^{\epsilon,\delta}(t,\cdot))+\|f(u^{\epsilon,\delta})-v^{\epsilon,\delta}\|_{L^\infty(R)}\|\lambda(\cdot,\epsilon)-\Lambda(\cdot,\epsilon,\delta)\|_{L^1(R)}\\
\leq\ &C\big(1+\|\lambda(\cdot,\epsilon)-\Lambda(\cdot,\epsilon,\delta)\|_{L^1(R)}\big)
\end{aligned}$$

for some uniform constant C independent of ϵ or δ in view of estimates (i)-(iv) in Proposition 4. Then using the strong convergence argument given before, sending δ to 0 with $\epsilon>0$ fixed yields

$$\frac{1}{\epsilon}\int_{R_x^+}|f(u^\epsilon)-v^\epsilon|(t,x)dx\leq C.$$

□

Now we go back to prove the main theorem in this section.

Proof of Theorem 3.3. For any given $T>0$, the uniform estimates (i)-(iii) in Theorem 3.2 allow to prove that there exists an extracted subsequence $\{(u^\epsilon,v^\epsilon)\}_{\epsilon>0}$ that has uniform bounded sup norm in x which converges in $L^\infty\big((0,T),L^1_{\text{loc}}(R)\big)$ and a.e. in t, x to some limit (u,v) as ϵ goes to 0 using identical steps as those developed in the proof of Theorem 3.2. Now given two initial data u_0 and $\bar{u}_0$ with corresponding well-prepared v_0 and $\bar{v}_0$. First observe that the sequence $(r_-^\epsilon, r_+^\epsilon)_{\epsilon>0}$ (resp. $(\bar{r}_-^\epsilon, \bar{r}_+^\epsilon)_{\epsilon>0}$) converges in $L^\infty\big((0,T), L^1_{\text{loc}}(R)\big)$ for any given $T > 0$ to (r_-, r_+) (resp. $\bar{r}_-$, $\bar{r}_+$). Lemma 3.4 implies that $v(t,x) = f(u(t,x))$ a.e. $t > 0$, $x > 0$, so that $r_-(t,x) = h(r_+(t,x))$ (resp. $\bar{r}_-(t,x) = h(\bar{r}_+(t,x))$). Recall the L^1 contraction principle (3.33) and passing to the limit $\epsilon \to 0$ yields

$$\int_{-M}^0 \Big(|r_+ - \bar{r}_+| + |r_- - \bar{r}_-|\Big)(t,x)dx + \int_0^M \Big(|r_+ - \bar{r}_+| + |h(r_+) - h(\bar{r}_+)|\Big)(t,x)dx$$
$$\leq \int_{|x|<M+at} \Big(|r_+^0 - \bar{r}_+^0| + |h(r_+^0) - h(\bar{r}_+^0)|\Big)(x)dx$$

since the initial data is well-prepared. Now by the increasing monotonicity property of h, one has

$$(r_+ - \bar{r}_+)(h(r_+) - h(\bar{r}_+)) \geq 0$$

so that

$$|r_+ - \bar{r}_+| + |h(r_+) - h(\bar{r}_+)| = |r_+ - \bar{r}_+ + h(r_+) - h(\bar{r}_+)| = 2a|u - \bar{u}|.$$

Therefore we can write

$$\int_0^M \Big(|r_+ - \bar{r}_+| + |h(r_+) - h(\bar{r}_+)|\Big)(t,x)dx = 2a\int_0^M |u - \bar{u}|(t,x)dx,$$

and correspondingly

$$\int_{|x|\leq M+at} \Big(|r_+^0 - \bar{r}_+^0| + |h(r_+^0) - h(\bar{r}_+^0)|\Big)(x)dx = 2a\int_{|x|\leq M+at} |u_0 - \bar{u}_0|(x)dx.$$

We thus have for any given $M > 0$ and time $t > 0$

$$\int_{-M}^0 \Big(|r_+ - \bar{r}_+| + |r_- - \bar{r}_-|\Big)(t,x)dx + 2a\int_0^M |u - \bar{u}|(t,x)dx \leq 2a\int_{|x|\leq M+at} |u_0 - \bar{u}_0|(x)dx,$$

which proves the L^1_{loc} contraction principle (3.70). This principle implies uniqueness of the limit (u, v). Proving that this limit satisfies (3.67)–(3.68) follows from (3.66) using routine arguments already sketched in the proof of Theorem 3.2 and considering the test function $\varphi \in C_c^1\big((0,\infty) \times R_x^-\big)$. Let us at last derive (3.69) from inequality (3.32) focusing on the non-negative test function φ with compact support in R_x^+. Taking the limit $\epsilon \to 0$ provides

$$\int_{R_t^+ \times R_x^+} \big(|r_+ - l| + |h(r_+) - h(l)|\big)\varphi_t + a\big(|r_+ - l| - |h(r_+) - h(l)|\big)\varphi_x dtdx \geq 0$$

for any given $l \in R$ such that $k = \frac{l+h(l)}{2a}$ verifies $|k| < B(N_0)$. Observe that $\frac{l-h(l)}{2} = f(k)$. The increasing property met by h ensures

$$|r_+ - l| + |h(r_+) - h(l)| = |r_+ + h(r_+) - (l + h(l))| = 2a|u - k|$$

together with

$$|r_+ - l| - |h(r_+) - h(l)| = \text{sgn}(r_+ - l)(r_+ - l - h(r_+) + h(l)) = 2\text{sgn}(u-k)(f(u) - f(k)) \tag{3.72}$$

since $(u - k)(r_+ - l) \geq 0$. This yields the expected Kružkov entropy inequalities. □

3.4 Matched asymptotic analysis

So far we only showed that the solution of the original system (3.1) converges to the weak solution of (3.67)–(3.69). However, since the test function φ vanishes in a neighborhood of the interface, we missed the information at $x = 0$. In this section, we want to derive the interface condition by matched asymptotic analysis in a rigorous way, which is the generalization of the one used in the domain decomposition system (2.12)–(2.15).

Since a relaxation layer may develop at the interface, we propose to reveal its structure in the limit ϵ goes to zero using a blow-up technique (see [96] in a related setting). Fix $\epsilon > 0$ and $\delta > 0$, define the fast variable $y = \frac{x}{\epsilon}$ so that $x = \epsilon y$ and let

$$\mathcal{U}^{\epsilon,\delta}(t,y) = u^{\epsilon,\delta}(t,\epsilon y), \qquad \mathcal{V}^{\epsilon,\delta}(t,y) = v^{\epsilon,\delta}(t,\epsilon y), \qquad y \in R. \tag{3.73}$$

Observe that $\mathcal{U}^{\epsilon,\delta}(t,.)$ and $u^{\epsilon,\delta}(t,.)$ (resp. $\mathcal{V}^{\epsilon,\delta}(t,.)$ and $v^{\epsilon,\delta}(t,.)$) have the same sup norm and total variation, so that Proposition 4 ensures

$$\|\mathcal{U}^{\epsilon,\delta}(t,\cdot)\|_{L^\infty(R)} \leq C, \ \ \mathrm{TV}\big(\mathcal{U}^{\epsilon,\delta}(t,\cdot)\big) \leq C\big(1 + \|\lambda(\cdot,\epsilon) - \Lambda(\cdot,\epsilon,\delta)\|_{L^1(R)}\big), \tag{3.74}$$

$$\|\mathcal{V}^{\epsilon,\delta}(t,\cdot)\|_{L^\infty(R)} \leq C, \ \ \mathrm{TV}\big(\mathcal{V}^{\epsilon,\delta}(t,\cdot)\big) \leq C, \tag{3.75}$$

for some constant $C > 0$ independent of ϵ and δ. Since again $\mathcal{U}^{\epsilon,\delta}$ and $u^{\epsilon,\delta}$ have identical sup norm, the following sub-characteristic condition holds uniformly in ϵ and δ :

$$|f'(\mathcal{U}^{\epsilon,\delta})| < a, \tag{3.76}$$

with a prescribed according to (3.5). Hence, the quasi-monotone property (3.28) and the monotonicity of h expressed in (3.30) apply for all the values of $\mathcal{U}^{\epsilon,\delta}$ under consideration. This will play an important role hereafter.

The rescaled profiles $\mathcal{U}^{\epsilon,\delta}$ and $\mathcal{V}^{\epsilon,\delta}$ are easily seen to be the C^1 solutions of

$$\begin{cases} \epsilon\partial_t\mathcal{U}^{\epsilon,\delta} + \partial_y\mathcal{V}^{\epsilon,\delta} = 0, \quad t > 0;\ y \in R, & \text{(3.77a)} \\ \epsilon\partial_t\mathcal{V}^{\epsilon,\delta} + a^2\partial_y\mathcal{U}^{\epsilon,\delta} = \epsilon\Lambda(\epsilon y,\epsilon,\delta)(f(\mathcal{U}^{\epsilon,\delta}) - \mathcal{V}^{\epsilon,\delta}), & \text{(3.77b)} \end{cases}$$

for any given positive ϵ and δ. In view of the estimates (3.74)-(3.75), sending δ to 0 with ϵ fixed and then letting ϵ go to 0, the sequence $\big\{\mathcal{U}^{\epsilon,\delta}(t,\cdot), \mathcal{V}^{\epsilon,\delta}(t,\cdot)\big\}_{\epsilon,\delta>0}$ can be shown to converge to some limit $(\mathcal{U}(t,\cdot), \mathcal{V}(t,\cdot))$ in $L^1_{\mathrm{loc}}(R_y)$ for any given $t > 0$. Clearly $(\mathcal{U}(t,\cdot), \mathcal{V}(t,\cdot))$ have bounded sup norm and bounded total variation. They will be referred hereafter to as the inner relaxation layer or the inner solution for short, while $(u(t,\cdot),\ v(t,\cdot))$ will be called the outer solution.

Let us first establish the following results only concerned with the inner solutions.

Lemma 3.5. *For a.e.* $t > 0$*, the inner solution* $(\mathcal{U}(t,\cdot),\ \mathcal{V}(t,\cdot))$ *is Lipschitz continuous in* y *and admits bounded asymptotic limit* $(\mathcal{U}(t,\pm\infty),\ \mathcal{V}(t,\pm\infty))$*. It verifies*

$$\mathcal{V}(t,y) = \mathcal{V}(t,+\infty) = \mathcal{V}(t,-\infty),\ \ y \in R, \tag{3.78}$$

with

$$\mathcal{U}(t,y) = \mathcal{U}(t,-\infty), \quad y < 0, \tag{3.79}$$

and

$$\begin{cases} a^2 d_y \mathcal{U}(t,y) = f(\mathcal{U}(t,y)) - \mathcal{V}(t,-\infty), \quad y > 0, \\ \mathcal{U}(t,0) = \mathcal{U}(t,-\infty). \end{cases} \tag{3.80}$$

Moreover, we have :

$$f(\mathcal{U}(t,+\infty)) = \mathcal{V}(t,-\infty). \tag{3.81}$$

If at time $t > 0$*, the solution* $\mathcal{U}(t,\cdot)$ *to (3.80) is not locally constant in the half line* $\{y > 0\}$*, then it must be strictly monotone for* $y > 0$ *with*

$$f'(\mathcal{U}(t,+\infty)) \leq 0. \tag{3.82}$$

Observe that the time t acts as a parameter for the inner solution. The asymptotic limits $\mathcal{U}(t,\pm\infty)$ and $\mathcal{V}(t,\pm\infty)$ will be determined in the forthcoming matching analysis with the left and right traces at $x = 0$ of the outer solutions $u(t,x)$ and $v(t,x)$.

Proof. The weak form of (3.77) reads

$$\begin{cases} \displaystyle\iint_{R_t^+\times R_y} \epsilon\, \mathcal{U}^{\epsilon,\delta}\partial_t\varphi + \mathcal{V}^{\epsilon,\delta}\partial_y\varphi \; dtdy = 0, & (3.83a) \\ \displaystyle\iint_{R_t^+\times R_y} \epsilon\, \mathcal{V}^{\epsilon,\delta}\partial_t\varphi + a^2\mathcal{U}^{\epsilon,\delta}\partial_y\varphi - \epsilon\Lambda(\epsilon y,\epsilon,\delta)(f(\mathcal{U}^{\epsilon,\delta}) - \mathcal{V}^{\epsilon,\delta})\varphi \; dtdy = 0, & (3.83b) \end{cases}$$

for any given test function $\varphi \in C_c^1\big((0,\infty)\times R_y\big)$. We have

$$\lim_{\delta\to 0} \epsilon\Lambda(\epsilon y,\epsilon,\delta) := \alpha(y,\epsilon) = \begin{cases} \epsilon, & y<0, \\ 1, & y>0, \end{cases} \tag{3.84}$$

so that for any given time $t>0$ the uniform sup norm and total variation estimates (3.74)-(3.75) clearly ensure that in the limit $\delta\to 0$, ϵ fixed, and then $\epsilon\to 0$ the inner solution $(\mathcal{U},\mathcal{V})$ verifies

$$\begin{cases} \displaystyle\iint_{R_t^+\times R_y} \mathcal{V}(t,y)\partial_y\varphi(t,y) \; dtdy = 0, & (3.85a) \\ \displaystyle\iint_{R_t^+\times R_y} a^2\mathcal{U}(t,y)\partial_y\varphi(y) - \alpha(y,0)(f(\mathcal{U}(t,y) - \mathcal{V}(t,y))\varphi(t,y) \; dtdy = 0. & (3.85b) \end{cases}$$

Choosing test function $\varphi(t,y) = \phi(t)\psi(y)$ for $\phi \in C_c^1((0,\infty))$ and $\psi \in C_c^1(R_y)$ yields for a.e. $t>0$

$$\begin{cases} \displaystyle\int_{R_y} \mathcal{V}(t,y)\psi'(y) \; dy = 0, & (3.86a) \\ \displaystyle\int_{R_y} a^2\mathcal{U}(t,y)\psi'(y) - \alpha(y,0)(f(\mathcal{U}(t,y)) - \mathcal{V}(t,y))\psi(y) \; dy = 0, & (3.86b) \end{cases}$$

since ϕ can be chosen arbitrarily. Obviously $\mathcal{V}(t,y)$ is constant in y for a.e. $t>0$ and thus (3.78) holds. Then choosing ψ with compact support in R_y^-, (3.84) immediately gives that $\mathcal{U}(t,.)$ also stays constant in the half line $\{y<0\}$

$$\mathcal{U}(t,y) = \mathcal{U}(t,-\infty) \quad y<0. \tag{3.87}$$

Then choosing ψ with compact support in R_y^+, one easily infers that $\mathcal{U}(t,.)$ is a classical solution of the ordinary differential equation

$$a^2 \frac{d}{dy}\mathcal{U}(t,y) = f(\mathcal{U}(t,y)) - \mathcal{V}(t,-\infty), \quad y > 0. \tag{3.88}$$

To derive the required initial data $\mathcal{U}(t,0^+)$, we choose at last $\psi \in C_c^1(R_y)$ and argue the property that $\mathcal{U}(t,y)$ is constant with $\alpha(y,0) = 0$ for $y < 0$ while being a smooth solution of (3.88) for $y > 0$. Integrations by part in (3.86b) which we recast as follows

$$\int_{R_y^-} a^2\mathcal{U}(t,y)\psi'(y)\ dy + \int_{R_y^+} a^2\mathcal{U}(t,y)\psi'(y) - \alpha(y,0)(f(\mathcal{U}(t,y)) - \mathcal{V}(t,y))\psi(y)\ dy = 0,$$

then resumes to

$$a^2(\mathcal{U}(t,0^+) - \mathcal{U}(t,0^-))\ \psi(0) = 0,$$

namely in view of (3.87)

$$\mathcal{U}(t,0^+) = \mathcal{U}(t,0^-) = \mathcal{U}(t,-\infty). \tag{3.89}$$

This identifies the initial data of the Cauchy problem (3.88) and proves by the way the Lip continuity property of $\mathcal{U}(t,.)$ in the fast variable y.

Let us prove that the solution $\mathcal{U}(t,\cdot)$ of the ODE Cauchy problem (3.88) with (3.89) is either trivial, that is $\mathcal{U}(t,y) = \mathcal{U}(t,-\infty)$ for all $y > 0$ (and thus all y in R), or strictly monotone in the half line $\{y > 0\}$. Indeed assume that $d_y\mathcal{U}(t,y)$ vanishes for some $y^\star > 0$ so that $\mathcal{U}(t,y^\star)$ is a critical point of (3.85), *i.e.* $f(\mathcal{U}(t,y^\star)) - \mathcal{V}(t,-\infty) = 0$. But classical properties of scalar autonomous ODE problem ensure that a critical point cannot be achieved for finite $y > 0$, so that if $y^\star$ is finite, necessarily $\mathcal{U}(t,y)$ stays constant for all y. Conversely assume the inner solution to be non-trivial, then it is necessarily strictly monotone for all finite $y > 0$. To conclude, let us prove in the latter case that

$$\lim_{y\to+\infty} d_y\mathcal{U}(t,y) = 0. \tag{3.90}$$

This will imply that $\mathcal{U}(t,+\infty)$ is a critical point of the ODE (3.88), that is :

$$f(\mathcal{U}(t,+\infty)) = \mathcal{V}(t,+\infty) = \mathcal{V}(t,-\infty). \tag{3.91}$$

By the Hartman-Grobman's Theorem [47], we further observe that $\mathcal{U}(t,+\infty)$ cannot be an unstable critical point, namely $f'(\mathcal{U}(t,+\infty)) > 0$ cannot hold, so that the last claim of Lemma 3.5

$$f'(\mathcal{U}(t,+\infty)) \leq 0,$$

must be valid. To derive (3.90), let us differentiate the smooth ODE (3.88) w.r.t. y to get the following representation formula for the derivative

$$d_y\mathcal{U}(t,y) = d_y\mathcal{U}(t,0^+)\ \mathcal{L}(y), \quad \mathcal{L}(y) \equiv exp\Big(\frac{1}{a^2}\int_0^y f'(\mathcal{U}(t,s))ds\Big),$$

where we have $|d_y\mathcal{U}(t,0^+)| > 0$ in the case of a non-trivial inner solution. Let us prove that the uniform boundedness for $y > 0$ of $\mathcal{U}(t,y)$ necessarily requires

$$\lim_{y\to+\infty} \mathcal{L}(y) = 0, \tag{3.92}$$

which in turn implies (3.90). First observe that since $\mathcal{U}(t,s)$ monotonically reaches a finite limit $\mathcal{U}(t,+\infty)$ as s goes to $+\infty$ while by assumption, $f(u)$ admits a finite number of inflection points, necessarily $f'(\mathcal{U}(t,s))$ keeps a constant sign for large enough values of s. Consequently, $\mathcal{L}(y)$ admits a limit as y goes to $+\infty$ which may be null or not. To rise a contradiction, assume that there exists some strictly positive constant $\eta > 0$ with the property that

$$\mathcal{L}(y) > \eta > 0, \quad \text{for all } y > 0. \tag{3.93}$$

In such a case, one would infer that $(\mathcal{U}(t,+\infty) - \mathcal{U}(t,0))d_y\mathcal{U}(t,y) > (\mathcal{U}(t,+\infty) - \mathcal{U}(t,0))d_y\mathcal{U}(t,0^+)\eta$, since by the monotonicity property $(\mathcal{U}(t,+\infty) - \mathcal{U}(t,0))d_y\mathcal{U}(t,0^+) >$

0, and hence $(\mathcal{U}(t,+\infty)-\mathcal{U}(t,0))(\mathcal{U}(t,y)-\mathcal{U}(t,0)) > (\mathcal{U}(t,+\infty)-\mathcal{U}(t,0))d_y\mathcal{U}(t,0^+)\eta\, y$. This linear growth estimate clearly rises a contradiction with the uniform boundedness of the inner solution $\mathcal{U}(t,y)$ so that (3.93) cannot hold true. In other words, (3.92) must be valid. This concludes the proof. □

We now derive the following matching conditions to link the inner solution $(\mathcal{U},\mathcal{V})$ with the outer solution (u,v).

Proposition 5. *For a.e.* $t>0$, $\mathcal{V}$ *and* v *perfectly match*

$$\mathcal{V}(t,y) = v(t,0^-) = v(t,0^+), \quad \textit{for all } y \in R. \tag{3.94}$$

$\mathcal{U}$ *and* u *are linked according to*

$$\mathcal{U}(t,y) = u(t,0^-), \quad y<0. \tag{3.95}$$

while, defining $R_\pm(t,y) = a\mathcal{U}(t,y) \pm \mathcal{V}(t,y)$, *the following inequalities hold*

$$\frac{1}{2}\Big(|R_+(t,y)-l| - |R_-(t,y)-h(l)|\Big) \geq sgn(u(t,0^+)-k)(f(u(t,0^+))-f(k)), \quad y>0, \tag{3.96}$$

for any given $l \in R$ *such that* $k = (l+h(l))/2a$ *verifies* $|k| < ||u_0||_{L^\infty(R)}$.

The proof is postponed at the end of this section. Observe that the matching condition (3.94) together with the identity $\mathcal{V}(t,+\infty) = f(\mathcal{U}(t,+\infty)$ stated in (3.81) actually ensures

$$f(\mathcal{U}(t,+\infty)) = v(t,0^+) = v(t,0^-). \tag{3.97}$$

By contrast to $\mathcal{U}(t,-\infty)$ and $u(t,0^-)$, $\mathcal{U}(t,+\infty)$ and $u(t,0^+)$ may not match, but a first consequence of the entropy like inequalities (3.96) is

Corollary 3.6. *For a.e. $t > 0$, the following Kružkov inequalities hold*

$$sgn\big(\mathcal{U}(t,+\infty)-k\big)\big(f(\mathcal{U}(t,+\infty))-f(k)\big) \geq sgn(u(t,0^+)-k)(f(u(t,0^+))-f(k)), \quad (3.98)$$

for all $k \in \lfloor \mathcal{U}(t,+\infty), u(t,0^+)\rceil$. In particular, we have :

$$f(\mathcal{U}(t,+\infty)) = f(u(t,0^+)) = v(t,0^-). \quad (3.99)$$

For any given real numbers a and b, $\lfloor a,b\rceil$ denotes the interval $(\min(a,b),\max(a,b))$. Rephrasing (3.98)–(3.99), if $\mathcal{U}(t,+\infty)$ and $u(t,0^+)$ differ, they are actually separated by an entropy satisfying standing shock for the equilibrium scalar conservation law (2.6).

Remark 3.7. *If $\mathcal{U}(t,+\infty) \neq u(t,0^+)$. In the case of a genuinely non-linear flux f (either strictly convex or concave), the Kružkov entropy inequalities (3.98) are known to be equivalent to the following requirement (see for instance [24]) :*

$$f'(u(t,0^+)) < 0 < f'(\mathcal{U}(t,+\infty)). \quad (3.100)$$

As a consequence, $\mathcal{U}(t,+\infty)$ would be an asymptoticly unstable critical point in the sense of Hartman-Grobman Theorem [47], which cannot be, hence the inner solution is necessarily trivial

$$\mathcal{U}(t,y) = u(t,0^-) \qquad \text{for all } y \in R. \quad (3.101)$$

The standing shock thus sticks at the corresponding interface, separating directly $u(t,0^-)$ from $u(t,0^+)$. The situation may turn more exotic in the case of a general non-linear flux f, an entropy satisfying standing shock verifies in general

$$f'(u(t,0^+)) \leq 0 \leq f'(\mathcal{U}(t,+\infty)), \quad (3.102)$$

where both inequalities may be strict or not and, simultaneously or not. Necessarily, if the inner layer is non-trivial, the critical point $\mathcal{U}(t,+\infty)$ cannot be hyperbolic (this would require $f'(\mathcal{U}(t,+\infty)) \leq 0$) and we must have $f'(\mathcal{U}(t,+\infty)) = 0$. Standing shock and non trivial relaxation layer may well coexist for general fluxes.

To further explore the matching conditions in between $\mathcal{U}$ and u in the setting of a general flux function f, let us state another consequence of the entropy inequalities (3.96) :

Corollary 3.8. *For a.e. $t > 0$, the following inequalities are met :*

$$sgn\big(u(t,0^+) - k\big)\big(f(u(t,0^+)) - f(k)\big) \leq 0, \tag{3.103}$$

for all k in $\lfloor u(t,0^-), \mathcal{U}(t,+\infty)\rceil$.

As a consequence of Corollaries 3.6 and 3.8, the most important outcome of the matching analysis is the last result of this section :

Proposition 6. *For a.e. $t > 0$, left and right traces of the outer solution $u(t,x)$ at $x = 0$ obey :*

$$sgn\big(u(t,0^+) - u(t,0^-)\big)\big(f(u(t,0^+)) - f(k)\big) \leq 0, \quad k \in \lfloor u(t,0^-), u(t,0^+)\rceil, \tag{3.104}$$

while

$$v(t,0^-) = f(u(t,0^+)). \tag{3.105}$$

Rephrasing this statement, the traces of the outer solution $u(t,x)$ at $x = 0$ are linked by the so-called Bardos-Leroux-Nédélec boundary condition [6] expressed for the equilibrium scalar conservation law (2.6). With this respect, the detailed knowledge of the inner solution $\mathcal{U}$ can be bypassed.

Let us establish the proposed statements. We begin with :

Proof of the Proposition 5. Let $(r_-^{\epsilon,\delta}, r_+^{\epsilon,\delta})$ denote the solution of the Cauchy problem (3.26) in diagonal form, with initial data $(r_\pm)_0^\delta = au_0^\delta \pm v_0^\delta$. Then for any given $l \in R$ with $|(l + h(l))/2a| \leq ||u_0||_{L^\infty(R)}$, let us introduce the smooth truncated initial data $(r_+)_{0,l}^\delta(x) = \psi_\delta(x)l$ and $(r_-)_{0,l}^\delta(x) = \psi_\delta(x)h(l)$ (note that $\psi_\delta(x)(\rho_\delta * l)(x) = \psi_\delta(x)l$) and the corresponding unique solution $(r_{-,l}^{\epsilon,\delta}, r_{+,l}^{\epsilon,\delta})$. Observe that for fixed $\epsilon > 0$, $\lim_{\delta\to 0} r_{+,l}^{\epsilon,\delta} = l$ and $\lim_{\delta\to 0} r_{-,l}^{\epsilon,\delta} = h(l)$. The two solutions under consideration obey the entropy like inequality (3.31)

$$\partial_t\big(|r_+^{\epsilon,\delta} - r_{+,l}^{\epsilon,\delta}| + |r_-^{\epsilon,\delta} - r_{-,l}^{\epsilon,\delta}|\big) + a\partial_x\big(|r_+^{\epsilon,\delta} - r_{+,l}^{\epsilon,\delta}| - |r_-^{\epsilon,\delta} - r_{-,l}^{\epsilon,\delta}|\big) \leq 0. \tag{3.106}$$

To condensate the notations, let

$$p_l^{\epsilon,\delta} = |r_+^{\epsilon,\delta} - r_{+,l}^{\epsilon,\delta}| + |r_-^{\epsilon,\delta} - r_{-,l}^{\epsilon,\delta}|, \quad q_l^{\epsilon,\delta} = a\left(|r_+^{\epsilon,\delta} - r_{+,l}^{\epsilon,\delta}| - |r_-^{\epsilon,\delta} - r_{-,l}^{\epsilon,\delta}|\right)$$

so that (3.106) reads

$$\partial_t p_l^{\epsilon,\delta} + \partial_x q_l^{\epsilon,\delta} \leq 0. \tag{3.107}$$

Let $\epsilon > 0$ be given and $y > 0$ be fixed, consider any $b > 0$ satisfying

$$0 < \epsilon y < b. \tag{3.108}$$

For any given time $T > 0$, let us multiply (3.107) by any given non-negative test function $\varphi(t)$ in $C_c^1\big((0,T)\big)$ and integrate for $(t,x) \in [0,T] \times [\epsilon y, b]$ to infer under the ordering condition (3.108) :

$$\int_0^T\int_{\epsilon y}^b -p_l^{\epsilon,\delta}(t,x)\varphi'(t)dtdx + \int_0^T \big(q_l^{\epsilon,\delta}(t,b) - q_l^{\epsilon,\delta}(t,\epsilon y)\big)\varphi(t)dt \leq 0. \tag{3.109}$$

Let us define $R_\pm^{\epsilon,\delta}(t,y) = a\mathcal{U}^{\epsilon,\delta}(t,y) \pm \mathcal{V}^{\epsilon,\delta}(t,y) = r_\pm^{\epsilon,\delta}(t,\epsilon y)$ and correspondingly $R_{\pm,l}^{\epsilon,\delta}(t,y) = r_{\pm,l}^{\epsilon,\delta}(t,\epsilon y)$. Setting $\mathcal{Q}_l^{\epsilon,\delta}(t,y) = a\left(|R_+^{\epsilon,\delta} - R_{+,l}^{\epsilon,\delta}| - |R_-^{\epsilon,\delta} - R_{-,l}^{\epsilon,\delta}|\right)(t,y)$, we have the identity $\mathcal{Q}_l^{\epsilon,\delta}(t,y) = q_l^{\epsilon,\delta}(t,\epsilon y)$. Hence changing the sign in (3.109) gives :

$$\int_0^T\int_{\epsilon y}^b p_l^{\epsilon,\delta}(t,x)\varphi'(t)dtdx + \int_0^T \big(\mathcal{Q}_l^{\epsilon,\delta}(t,y) - q_l^{\epsilon,\delta}(t,b)\big)\varphi(t)dt \geq 0.$$

Let $\eta > \epsilon y$ and average the above inequality (valid for all $b > \epsilon y$) for $b \in (\eta, 2\eta)$ to get

$$\frac{1}{\eta}\int_0^T\int_\eta^{2\eta}\int_{\epsilon y}^b p_l^{\epsilon,\delta}(t,x)\varphi'(t)dtdbdx + \int_0^T \left(\mathcal{Q}_l^{\epsilon,\delta}(t,y) - \frac{1}{\eta}\int_\eta^{2\eta} q_l^{\epsilon,\delta}(t,b)db\right)\varphi(t)dt \geq 0.$$

Let us observe that

$$\left|\frac{1}{\eta}\int_0^T\int_\eta^{2\eta}\int_{\epsilon y}^b p_l^{\epsilon,\delta}(t,x)\varphi'(t)dtdbdx\right|$$
$$\leq \sup_{0\leq t\leq T} \| p_l^{\epsilon,\delta}(t,\cdot)\|_{L^\infty(R)}\|\varphi'\|_{L^1(0,T)}\ \frac{1}{\eta}\int_\eta^{2\eta}(b-\epsilon y)db \leq C(\frac{3}{2}\eta - \epsilon y)$$

for some uniform constant $C > 0$ in ϵ and δ thanks to the corresponding uniform sup norm estimates satisfied by the solutions $(u^{\epsilon,\delta}, v^{\epsilon,\delta})$ and $(u_l^{\epsilon,\delta}, v_l^{\epsilon,\delta})$ of the regularized problem (3.17). For $\epsilon > 0$ fixed, passing to the limit $\delta \to 0$ and then to the limit $\epsilon \to 0$ yields

$$\int_0^T \varphi(t)\left(\mathcal{Q}_l(t,y) - \frac{1}{\eta}\int_\eta^{2\eta} q_l(t,b)db\right)dt \geq -\frac{3}{2}C\eta. \tag{3.110}$$

Observe that this inequality now holds for any $\eta > 0$ since the ordering condition (3.108) resumes to $b > 0$. In (3.110), we have

$$\mathcal{Q}_l(t,y) = a\left(|R_+(t,y) - l| - |R_-(t,y) - h(l)|\right)$$

with $R_\pm(t,y) = a\mathcal{U}(t,y) \pm \mathcal{V}(t,y)$ while

$$q_l(t,b) = a\left(|r_+(t,b) - l| - |h(r_+(t,b)) - h(l)|\right)$$

since all the b under consideration are positive, *i.e.* we deal with the equilibrium half line $\{x > 0\}$ with the property that $r_-(t,x) = h(r_+(t,x))$. Arguments based on the monotonicity of h and already developed in the proof of Theorem 3.3 (see (3.72)) ensure the identity

$$q_l(t,b) = 2a\ \mathrm{sgn}\big(u(t,b) - k\big)\left(f(u(t,b)) - f(k)\right), \quad k = \frac{l + h(l)}{2a}.$$

Passing to the limit $\eta \to 0$ in (3.110) yields (recall that $u(t,b)$ has bounded total variation and thus admits a right trace at $x = 0$)

$$\int_0^T \varphi(t) \left(\mathcal{Q}_l(t,y) - q_l(t,0^+) \right) dt \geq 0.$$

This inequality is valid for any given non-negative test function φ in $C_c^1((0,T))$, so that we deduce the inequality (3.96). Let us now prove that the next matching condition $\mathcal{V}(t,y) = v(t,0^+)$ for $y > 0$ and a.e. $t > 0$. To this aim, we start from the equation $\partial_t u^{\epsilon,\delta} + \partial_x v^{\epsilon,\delta} = 0$ and repeat the same arguments as previously to get

$$\int_0^T \varphi(t) \left(\mathcal{V}(t,y) - \frac{1}{\eta} \int_\eta^{2\eta} v(t,b) db \right) = 0, \quad y > 0, \text{ a.e. } t > 0$$

for all $\eta > 0$ and all test function $\varphi \in C_c^1((0,T))$. Sending $\eta \to 0$ yields the expected result. To derive the condition $\mathcal{U}(t,y) = u(t,0^-)$ when $y < 0$, we proceed *mutatis mutandis* the same way choosing $\epsilon > 0$, some fixed $y < 0$ and negative real number b satisfying the ordering condition $b < \epsilon y < 0$ and we apply the above steps to the equation $\partial_t v^{\epsilon,\delta} + a^2 \partial_x u^{\epsilon,\delta} = f(u^{\epsilon,\delta}) - v^{\epsilon,\delta}$ to get

$$\begin{aligned}
&\int_0^T \left(\mathcal{U}^{\epsilon,\delta}(t,y) - \frac{1}{|\eta|} \int_{-2|\eta|}^{-|\eta|} u^{\epsilon,\delta}(t,b) db \right) \varphi(t) dt \\
&+ \frac{1}{|\eta| a^2} \int_0^T \int_{-2|\eta|}^{-|\eta|} \int_{\epsilon y}^{b} \Big(v^{\epsilon,\delta}(t,x) \varphi'(t) + \left(f(u^{\epsilon,\delta}(t,x)) - v^{\epsilon,\delta}(t,x) \right) \varphi(t) \Big) dx dt db = 0
\end{aligned}$$

for any given test function $\varphi \in C_c^1((0,T))$. Uniform sup norm estimates for $u^{\epsilon,\delta}$ and $v^{\epsilon,\delta}$ again apply to prove that the second term vanishes in the limit $\delta \to 0$ then $\epsilon \to 0$ and $\eta \to 0$, while the first term gives rise to

$$\int_0^T \left(\mathcal{U}(t,y) - u(t,0^-) \right) \varphi(t) dt = 0, \quad y < 0.$$

This implies the expected result. At last to prove the condition $\mathcal{V}(t,y) = v(t,0^+)$ for $y > 0$ (hence the continuity property $v(t,0^-) = v(t,0^+)$) we proceed similarly but with the equation $\partial_t u^{\epsilon,\delta} + \partial_x v^{\epsilon,\delta} = 0$. Details are left to the reader. $\square$

Let us now turn establishing Corollaries 3.6 and 3.8.

Proof of Corollary 3.6. Sending y to $+\infty$ in (3.96), one has for any given $l \in R$ such that $k = (l + h(l))/2a$ verifies $|k| < ||u_0||_{L^\infty(R)}$:

$$\frac{1}{2}\Big(|R_+(t,+\infty)-l|-|R_-(t,+\infty)-h(l)|\Big) \geq \operatorname{sgn}(u(t,0^+)-k)(f(u(t,0^+))-f(k)). \quad (3.111)$$

In particular, these inequalities are valid restricting attention to values of l so that k belongs to $\lfloor \mathcal{U}(y,+\infty), u(t,0^+)\rceil$. This will suffice to our purpose. Note from (3.81) that $R_\pm(t,+\infty) = a\mathcal{U}(t,+\infty) \pm f(\mathcal{U}(t,+\infty))$, thus the identity $R_-(t,+\infty) = h(R_+(t,+\infty))$ holds. Rephrasing arguments developed in the course of Theorem 3.3 (see (3.72)), the left hand side of the inequality (3.111) is seen to boil down to $\operatorname{sgn}\big(\mathcal{U}(t,+\infty) - k\big)\big(f(\mathcal{U}(t,+\infty)) - f(k)\big)$ for all the k under consideration, which is nothing but (3.96). It then suffices to choose successively $k = \mathcal{U}(t,+\infty)$ and $k = u(t,0^+)$ to get (3.97). □

Proof of Corollary 3.8. Let us first recall that $R_\pm(t,y) = a\mathcal{U}(t,y) \pm f(\mathcal{U}(t,+\infty))$ for a.e. $t > 0$. Now given y fixed, choose $l_y = a\mathcal{U}(t,y) + f(\mathcal{U}(t,y))$ in (3.96) with the property $h(l_y) = a\mathcal{U}(t,y) - f(\mathcal{U}(t,y))$. Observe that :

$$|R_+(t,y) - l_y| = |f(\mathcal{U}(t,y)) - f(\mathcal{U}(t,+\infty))|,$$
$$|R_-(t,y) - h(l_y)| = |f(\mathcal{U}(t,y)) - f(\mathcal{U}(t,+\infty))|,$$

so that we arrive at

$$0 \geq \operatorname{sgn}(u(t,0^+) - \mathcal{U}(t,y))(f(u(t,0^+)) - f(\mathcal{U}(t,y))) \quad (3.112)$$

since $k_y = (l_y + h(l_y))/2a = \mathcal{U}(t,y)$. Notice that (3.112) is true for any given $y > 0$, so it implies (3.103) for all k in between $\mathcal{U}(t,0)$ and $\mathcal{U}(t,+\infty)$ with $\mathcal{U}(t,0) = u(t,0^-)$ by (3.95). □

The proof of Proposition 6 will rely on the following technical result.

Lemma 3.9. *With the notations of Proposition 5, for a.e. $t > 0$, the entropy like inequalities*

$$sgn(\mathcal{U}(t,+\infty) - k)\big(f(\mathcal{U}(t,+\infty)) - f(k)\big) \leq 0, \tag{3.113}$$

are satisfied for all $k \in \lfloor u(t,0^-), \mathcal{U}(t,+\infty)\rceil$.

Proof. Recall that $R_\pm(t,s)$ solve for all $s > 0$

$$\pm a\ d_s R_\pm(t,s) = \pm G(R_-, R_+)(t,s),$$

so that, the quasi-monotone property (3.28) ensures for all $l \in R$ with $|l + h(l)|/2a < ||u_0||_{L^\infty(R)}$

$$\frac{1}{2} d_s \Big(|R_+ - l| - |R_- - h(l)| \Big) \leq 0, \quad s > 0$$

since the sub-characteristic condition (3.76) is satisfied. Integrating the above inequality for s in $[y,Y]$ with $y < Y$ yields

$$\frac{1}{2}\Big(|R_+(t,Y) - l| - |R_-(t,Y) - h(l)|\Big) \leq \frac{1}{2}\Big(|R_+(t,y) - l| - |R_-(t,y) - h(l)|\Big).$$

Sending Y to $+\infty$ gives the following inequality

$$\mathrm{sgn}(\mathcal{U}(t,+\infty) - k)\big(f(\mathcal{U}(t,+\infty) - f(k))\big) \leq \frac{1}{2}\Big(|R_+(t,y) - l| - |R_-(t,y) - h(l)|\Big),$$

using identical steps to those developed in the proof of Corollary 3.6. Choosing at last for any given $y > 0$, $l_y = a\mathcal{U}(t,y) + f(\mathcal{U}(t,y))$ like in the course of the proof of Corollary 3.8 yields the required conclusion. □

Proof of Proposition 6. Clearly, a proof is needed only when $u(t,0^-)$ and $u(t,0^+)$ are distinct, we thus assume $u(t,0^-) \neq u(t,0^+)$. First, observe that in the case of a trivial relaxation layer, namely $\mathcal{U}(t,+\infty) = u(t,0^-)$, the inequalities (3.98) assert that $u(t,0^-)$ and $u(t,0^+)$ are connected by a standing shock for the conservation law (2.6). Hence the Bardos-Leroux-Nédélec condition (3.104) applies by definition. Second, assume the inner layer to be non-trivial, $u(t,0^-) \neq \mathcal{U}(t,+\infty)$, but with $\mathcal{U}(t,+\infty) = u(t,0^+)$. Then the inequalities (3.113) immediately give the required result thanks to the mononicity in y of the inner solution $\mathcal{U}(t,y)$ since this property readily implies

$$sgn(u(t,0^+) - u(t,0^-)) = sgn(\mathcal{U}(t,+\infty) - u(t,0^+)) = sgn(\mathcal{U}(t,+\infty) - k),$$

for all $k \in \lfloor u(t,0^-), \mathcal{U}(t,+\infty)\rceil$.

We are thus left with the case of three distinct values $u(t,0^-)$, $\mathcal{U}(t,+\infty)$ and $u(t,0^+)$. The key step is to show that the entropy inequalities (3.103) and (3.113) actually ensure a natural ordering in between these three distinct values. We start assuming that the monotone in y function $\mathcal{U}(t,y)$ is increasing and we intend to prove

$$u(t,0^-) < \mathcal{U}(t,+\infty) < u(t,0^+). \tag{3.114}$$

In that aim, observe that the inequalities (3.113) give

$$f(\mathcal{U}(t,+\infty)) \leq f(k), \quad k \in [u(t,0^-), \mathcal{U}(t,+\infty)]$$

since again due to monotonicity, $sgn(\mathcal{U}(t,+\infty) - k) = +1$ for all the k under consideration. But $\mathcal{U}(t,+\infty)$ and $u(t,0^+)$ verify $f(\mathcal{U}(t,+\infty)) = f(u(t,0^+))$, we thus get

$$f(u(t,0^+)) \leq f(k), \quad k \in [u(t,0^-), \mathcal{U}(t,+\infty)]. \tag{3.115}$$

The inequalities (3.103) in turn ensure that necessarily $sgn(u(t,0^+) - k) = +1$ for all the k under consideration, and in particular for $k = \mathcal{U}(t,+\infty)$. Hence the proposed ordering (3.114). To conclude, we notice after Oleinik (see [24] for instance) that in the case $\mathcal{U}(t,y) < u(t,0^+)$, the entropy inequalities (3.98) are equivalent to the geometrical property that the chord in between $\mathcal{U}(t,y)$ and $u(t,0^+)$ must stay below the graph of $f(u)$, this just reads $f(u(t,0^+)) \leq f(k)$ for all $k \in [\mathcal{U}(t,+\infty), u(t,0^+)]$, that is to say:

$$f(u(t,0^+) \leq f(k)), \quad k \in [u(t,0^-), u(t,0^+)]. \tag{3.116}$$

This is nothing but the expected condition (3.104) since again by (3.114) $sgn(u(t,0^+) - u(t,0^-)) = +1$.

The case of a decreasing inner solution $\mathcal{U}(t,.)$ can be treated by a straightforward adaptation of the steps we have proposed, recalling after Oleinik that an entropy satisfying shock with $\mathcal{U}(t,y) > u(t,0^+)$ comes with the property that the chord joining $u(t,0^+)$ to $\mathcal{U}(t,y)$ must stay above the graph of $f(u)$. Details are left to the reader. This concludes the proof. □

3.5 Concluding remarks

The solution of the original two-scale hyperbolic relaxation system (3.1)–(3.4) is proved to converge as the smaller relaxation time vanishes to the unique solution of the domain decomposition system (3.6)–(3.8). The interface condition is derived in a rigorous way by matched asymptotic analysis, and result in a form that is the well-known Bardos-Lerous-Nédélec condition. This is an extension of that in the last chapter by allowing a standing shock sticking to the interface.

Chapter 4

Vlasov-Poisson-Fokker-Planck system

4.1 Introduction

The Vlasov-Poisson-Fokker-Planck (VPFP) system is the kinetic description of the Brownian motion of a large system of particles in a surrounding bath. For example, in electrostatic plasma, when the interactions between the electrons and a surrounding bath through Coulomb force are taken into account, the time evolution of the electron distribution function $f:\ (t,x,v)\in\mathbb{R}_+\times\mathbb{R}^{\mathbb{N}}\times\mathbb{R}^{\mathbb{N}}\to\mathbb{R}_+$ solves the VPFP system, under the action of a self-consistent potential ϕ:

$$\begin{cases} \partial_t f + v\cdot\nabla_x f - \dfrac{q}{m_e}\nabla_x\phi\cdot\nabla_v f = \dfrac{1}{\tau_e}\mathcal{L}_{\mathcal{FP}}(f), & (4.1a)\\ -\triangle_x\,\phi = \dfrac{q}{\epsilon_0}(\rho - h(x)), & (4.1b)\end{cases}$$

where ϵ_0 is the vacuum permittivity, q and m_e are elementary charge and mass of the electrons, and τ_e is the relaxation time due to the collisions of the particles with the surrounding bath. The function $h(x)$ is a given positive background charge, and one can assume the *global neutrality relation*

$$\int_{\mathbb{R}^{\mathbb{N}}}\int_{\mathbb{R}^{\mathbb{N}}} f(x,v)\,dxdv = \int_{\mathbb{R}^{\mathbb{N}}} h(x)\,dx. \tag{4.2}$$

$\rho(t,x)$ is the density of electrons given by

$$\rho(t,x) = \int_{\mathbb{R}^{\mathbb{N}}} f(t,x,v)dv.$$

$\mathcal{L}_{\mathcal{FP}}(f)$ is the Fokker-Planck operator

$$\mathcal{L}_{\mathcal{FP}}(f) = \nabla_v \cdot (vf + \mu_e \nabla_v f),$$

where $\sqrt{\mu_e} = \sqrt{\frac{k_B T_{th}}{m_e}}$ is the thermal velocity, k_B is the Planck constant, and T_{th} is the temperature of the bath. Two important physical quantities that characterize the particle system are the mean free path $l_e = \sqrt{\mu_e}\tau_e$, which is the average distance traveled by a particle between two successive collisions, and the Debye length $\Lambda = \sqrt{\frac{\epsilon_0 k_B T_{th}}{q^2 \mathcal{N}}}$, which is the typical distance over which significant charge separation can occur. Here $\mathcal{N}$ is the typical value for the concentration of the particles. Another application of the VPFP system is in galaxies where massive particles interacting through gravitational force. The main difference is that the force is attractive, so in (4.1b) we have $\triangle_x \phi = \frac{q}{\epsilon_0}(\rho - h(x))$ instead, and the physical meanings of the constants are different.

The existence and uniqueness of the weak and classical solutions of the VPFP and related systems have been well studied. Degond [25] first showed the existence of a global-in-time smooth solution for the Vlasov-Fokker-Planck equation in one and two space dimensions in electrostatic case, and also proved the convergence of the solution to that of the Vlasov-Poisson equations when the diffusion coefficient goes to zero. Later on, Bouchut [8, 9] extended the result to three dimensions when the electric field was coupled through a Poisson equation, and the results were given in both electrostatic and gravitational cases. Zheng and Majda [108] gave the existence of a global weak solution of the VPFP system from a new prospect, where by allowing the initial data to be measure-valued, it includes some physically interesting case such as electron sheets. For more results, one can refer to [14, 15, 97].

If the mean free path of the electrons is much smaller than the Debye length, then system (4.1) can be written in the dimensionless form as

$$\begin{cases} \partial_t f + v \cdot \nabla_x f - \dfrac{1}{\epsilon} \nabla_x \phi \cdot \nabla_v f = \dfrac{1}{\epsilon} \mathcal{P}_{non}(f), & \text{(4.3a)} \\ - \triangle_x \phi = \rho - h, & \text{(4.3b)} \end{cases}$$

where $\epsilon = \left(\frac{l_e}{\Lambda}\right)^2$, the ratio between the mean free path and the Debye length, and $\mathcal{P}_{non}$ is the nondimensionalized Fokker-Planck operator:

$$\mathcal{P}_{non} = \nabla_v \cdot (v f_\epsilon + \nabla_v f_\epsilon).$$

Under this scaling, the limiting process $\epsilon \to 0$ is the so-called high-field limit which is different from the low-field limit (or named as parabolic limit), in which the diffusion dominates the behavior, see [83] for example. The high field limit was first introduced in [82] in which it gave a fluid approximation to the semiconductor Boltzmann equation for high electric fields. Later some numerical simulations of this kinetic model and high-field model were performed in [17].

Now one can formally derive the limit equation. First integrating (4.3a) over $\mathbb{R}^N$, one gets

$$\partial_t \rho + \nabla_x \cdot j = 0, \tag{4.4}$$

where $j = \int_{\mathbb{R}^N} v f(t,x,v) dv$. Then multiplying (4.3a) by v and integrating over $\mathbb{R}^N$ to get

$$\epsilon(\partial_t j + \nabla_x \cdot q) + \rho \nabla_x \phi + j = 0, \tag{4.5}$$

where $q = \int_{\mathbb{R}^N} v \otimes v f(t,x,v) dv$. Let $\epsilon \to 0$ in (4.5), one obtains

$$j = -\rho \nabla_x \phi. \tag{4.6}$$

Then plugging it into (4.4) to get the high field limit equation

$$\begin{cases} \partial_t \rho - \nabla_x \cdot (\rho \nabla_x \phi) = 0, & (4.7a) \\ - \triangle_x \phi = \rho - h(x). & (4.7b) \end{cases}$$

This formal analysis can be made rigorous. In [78], it was first proved that in one dimension (for both space and velocity) the solution of (4.1) converges to (4.7) when $\epsilon \to 0$. It was also shown that the limit system has a smooth global-in-time solution in electrostatic case and a local-in-time solution in gravitational case. The results were extended to multidimension in the electrostatic case in [45].

Efforts have been devoted to numerically solving the VPFP system, see for instance, [48, 49, 101, 102]. All these schemes use a particle method, random or deterministic, to treat the convective part and deal with the Fokker-Planck operator by reconstructing the distribution function via a field-free Fokker-Planck kernel. These methods are efficient but only have first order accuracy. Another approach was given in [88] using a finite difference method, with implicit time discretization. Although this method is free of the constraint $\triangle t \sim \triangle v^2$, it has to invert a nonsymmetric matrix which is the main difficulty in higher dimension.

Unlike the previous works which intend to capture the behavior of the Vlasov-Poisson system such as Landau damping when the diffusion effect is rather weak, our goal is to develop a scheme that is efficient in the high field regime. The numerical difficulties arise in two ways. The first one is the stiff coefficient in the forcing term containing the electric potential. An explicit method would require that $\triangle t \sim \min(\triangle x, \ \epsilon \triangle v)$ which becomes too expensive when ϵ is small. The other one is the diffusive nature of the Fokker-Planck operator, which poses the constraint $\triangle t \sim O(\epsilon \triangle v^2)$. Instead of treating the forcing term and Fokker-Planck operator separately, our idea is to combine both stiff terms and propose a time implicit method to overcome these two difficulties simultaneously. The combined term still has the form of a Fokker-Planck operator, with a Maxwellian that depends on $\nabla_x \phi$, and it is treated implicitly in the same way as [59] so that only a *symmetric* tri-diagonal matrix has to be inverted. This induces an Asymptotic Preserving(AP) method, as characterized by Jin in [53]. See [54] for a review. It allows large time steps and coarse meshes in the regime $\epsilon \ll 1$. This method can be extended to higher dimension directly.

The rest of the chapter is organized as follows. In section 4.2 we give the first order scheme and prove some properties of it such as positivity, stability, mass and asymptotic preservation. A second order scheme is given at the end of this section. Section 4.3 is devoted to numerically validate the properties of the scheme. By comparing with the explicit scheme on a resolved mesh, we show that our scheme is efficient in capturing the high field limit. At last, some concluding remarks are given in section 4.4.

4.2 An AP scheme for the VPFP system in the high field regime

A standard explicit scheme for the VPFP system requires time step $\triangle t \sim \min(\epsilon \triangle v^2, \triangle x)$ due to the stiffness of the forcing term and collision term contained $\frac{1}{\epsilon}$ and diffusive nature of the Fokker-Planck operator. In order to avoid this constraint, we propose the following scheme which is based on an implicit treatment of the combined stiff terms.

We first combine the two stiff terms in (4.3a), $\frac{1}{\epsilon}\nabla_x \phi_\epsilon \cdot \nabla_v f$ and $\frac{1}{\epsilon}\mathcal{P}_{non}$. In this way, we will not change the property of Fokker-Planck operator, but can treat the two stiff terms simultaneously.

An equivalent form of the VPFP system reads

$$\begin{cases} \partial_t f + v \cdot \nabla_x f = \frac{1}{\epsilon}\mathcal{P}(f), & (4.8a) \\ -\triangle_x \phi = \rho - h, & (4.8b) \end{cases}$$

where

$$\mathcal{P}(f) = \nabla_v \cdot \left[e^{-\frac{|v+\nabla_x \phi|^2}{2}} \nabla_v \left(e^{\frac{|v+\nabla_x \phi|^2}{2}} f \right) \right]. \tag{4.9}$$

In this form, one can introduce

$$M = e^{-\frac{|v+\nabla_x \phi|^2}{2}}, \tag{4.10}$$

and one will see that formally f goes to

$$f_{eq} = \frac{\rho}{(2\pi)^{\frac{N}{2}}} M = \frac{\rho}{(2\pi)^{\frac{N}{2}}} e^{-\frac{|v+\nabla_x \phi|^2}{2}}$$

when pushing ϵ to 0. This is the so-called "local Maxwellian", and it is easy to check that the limit ρ indeed solves the high field limit equation (4.7), see for example [45].

4.2.1 The first order scheme

The time discretization of the first order scheme reads

$$\frac{f^{n+1} - f^n}{\triangle t} + v \cdot \nabla_x f^n = \frac{1}{\epsilon} P(f^{n+1}), \tag{4.11}$$

where the operator P is the discrete version of operator $\mathcal{P}$, and it is treated in the same way as [59]. In fact, there are several methods about how to discretize the Fokker-Planck operator, such as [20, 34, 67]. Here we choose the method of [59] because it gives a symmetric matrix which is not only easy to invert, but also has some good properties such as negative definiteness. From now on, denote $f(x_i, v_j, t^n)$ by $f^n_{i,j}$, where

$0 \leq i \leq N_x$, $0 \leq j \leq N_v$, and N_x and N_v are the number of mesh points in x and v directions respectively. We briefly state the discretization as follows. Let

$$\tilde{P}(g) = \frac{1}{\sqrt{M}} \nabla_v \cdot \left(M \nabla_x \left(\frac{g}{\sqrt{M}} \right) \right), \tag{4.12}$$

then it relates to P as

$$P(f) = \sqrt{M} \tilde{P} \left(\frac{f}{\sqrt{M}} \right). \tag{4.13}$$

The discretization of $\tilde{P}$ is straightforward, and the one dimensional version takes the form

$$\begin{aligned} &(\tilde{P}g)_j \\ = &\frac{1}{\triangle v^2 \sqrt{M_j}} \left(\sqrt{M_j M_{j+1}} \left[\left(\frac{g}{\sqrt{M}} \right)_{j+1} - \left(\frac{g}{\sqrt{M}} \right)_j \right] - \sqrt{M_j M_{j-1}} \left[\left(\frac{g}{\sqrt{M}} \right)_j - \left(\frac{g}{\sqrt{M}} \right)_{j-1} \right] \right) \\ = &\frac{1}{\triangle v^2} \left(g_{j+1} - \frac{\sqrt{M_{j+1}} + \sqrt{M_{j-1}}}{\sqrt{M_j}} g_j + g_{j-1} \right). \end{aligned} \tag{4.14}$$

Similarly, one can extend it to higher dimension with no extra efforts. Therefore (4.11) becomes

$$\frac{f^{n+1} - f^n}{\triangle t} + v \cdot \nabla_x f^n = \frac{1}{\epsilon} \sqrt{M^{n+1}} \tilde{P} \left(\frac{f^{n+1}}{\sqrt{M^{n+1}}} \right). \tag{4.15}$$

Now we can summarize the algorithm for the first order method. Given f^n, ρ^n and ϕ^n at time t^n.

- **Step 1.** Approximate the transport term $v \cdot \nabla_x f^n$ in (4.15) by a first order upwind method or second order high resolution method.

- **Step 2.** Sum (4.15) over discrete v, note that the right hand side will be zero (see(4.16)), so ρ^{n+1} can be obtained explicitly in this step.

- **Step 3.** Solve (4.3b) by any Poisson solver, say, fast Fourier Transform for periodic case, to get ϕ^{n+1}. Then calculate M^{n+1} via (4.10).

- **Step 4.** Plug M^{n+1} into (4.15), one ends up with a linear system for f^{n+1}, invert the system by the conjugate gradient method to get f^{n+1}.

4.2.2 Some properties of the scheme

In this section, we show that in one space dimension, the first order scheme has some good properties under the hyperbolic CFL condition, which is not restrictive at all.

Mass conservation

The original system preserves mass, so it is desirable to have this property numerically. Observe that

$$\begin{aligned}
&\sum_j \sqrt{M_j}\tilde{P}\left(\frac{f}{\sqrt{M}}\right)_j \\
=\ & \frac{1}{\triangle v^2}\left(\sum_j \sqrt{M_j M_{j+1}}\left[\left(\frac{f}{M}\right)_{j+1} - \left(\frac{f}{M}\right)_j\right] - \sum_j \sqrt{M_{j-1}M_j}\left[\left(\frac{f}{M}\right)_{j-1} - \left(\frac{f}{M}\right)_j\right]\right) \\
=\ & 0,
\end{aligned} \tag{4.16}$$

the conservation of mass follows if a conservative scheme is used for the convection term $v\partial_x f$.

Positivity preservation

Plugging (4.14) into (4.15) and with the upwind discretization on $\partial_x f$, the first order scheme reads

$$\begin{aligned}
&\frac{f_{i,j}^{n+1} - f_{i,j}^n}{\Delta t} + \max(v_j,\ 0)\frac{f_{i,j}^n - f_{i-1,j}^n}{\triangle x} + \min(v_j,\ 0)\frac{f_{i+1,j}^n - f_{i,j}^n}{\triangle x} \\
=& \frac{\sqrt{M_{i,j}^{n+1}}}{\epsilon \triangle v^2}\left(\sqrt{M_{i,j+1}^{n+1}}\left[\left(\frac{f}{M}\right)_{i,j+1}^{n+1} - \left(\frac{f}{M}\right)_{i,j}^{n+1}\right] + \sqrt{M_{i,j-1}^{n+1}}\left[\left(\frac{f}{M}\right)_{i,j-1}^{n+1} - \left(\frac{f}{M}\right)_{i,j}^{n+1}\right]\right)
\end{aligned} \tag{4.17}$$

We use the maximum principle argument. If at time t^n, $f_{i,j}^n$ is positive for all $0 \le i \le N_x$, $0 \le j \le N_v$, and assume $\left(\frac{f}{M}\right)_{k,l}^{n+1} = \min_{i,j} \left(\frac{f}{M}\right)_{i,j}^{n+1}$ where $0 \le k \le N_x$, $0 \le l \le N_v$. Then from (4.17), one has

$$\begin{aligned} f_{k,l}^{n+1} &= f_{k,l}^n \left(1 - v_l^+ \frac{\triangle t}{\triangle x} + v_l^- \frac{\triangle t}{\triangle x}\right) + v_l^+ \frac{\triangle t}{\triangle x} f_{k-1,l}^n - v_l^- \frac{\triangle t}{\triangle x} f_{k+1,l}^n \\ &+ \frac{\sqrt{M_{k,l}^{n+1}}}{\epsilon \triangle v^2} \left(\sqrt{M_{k,l+1}^{n+1}} \left[\left(\frac{f}{M}\right)_{k,l+1}^{n+1} - \left(\frac{f}{M}\right)_{k,l}^{n+1} \right] + \sqrt{M_{k,l-1}^{n+1}} \left[\left(\frac{f}{M}\right)_{k,l-1}^{n+1} - \left(\frac{f}{M}\right)_{k,l}^{n+1} \right] \right), \end{aligned}$$

where $v_l^+ = \max(v_l, 0) \ge 0$, $v_l^- = \min(v_l, 0) \le 0$. Under the CFL condition $\max_j |v_j| \frac{\triangle t}{\triangle x} \le 1$, it is easy to see that the right hand side of the above expression is positive. Note that M is always positive, so $\min_{i,j} \left(\frac{f}{M}\right)_{i,j}^{n+1}$ is positive, which implies that $f_{i,j}^{n+1}$ is positive for all $0 \le i \le N_x$, $0 \le j \le N_v$.

Stability

Having the properties of positivity and mass conservation, stability directly follows. Consider l^1 norm $\| f^n \|_{l^1} = \sum_{i,j} |f_{i,j}^n|$, then one has

$$\| f^{n+1} \|_{l^1} = \sum_{i,j} |f_{i,j}^{n+1}| = \sum_{i,j} f_{i,j}^{n+1} = \sum_{i,j} f_{i,j}^n = \| f^n \|_{l^1}, \tag{4.18}$$

where the second equality comes from positivity, and the third equality is a consequence of the mass conservation.

Asymptotic preservation

Following the idea in [44], define the discrete entropy

$$H_{i,j}^n = \sum_j f_{i,j} \log \left(\frac{f}{M}\right)_{i,j}^n, \tag{4.19}$$

where $H_{i,j} = H(x_i, v_j)$, and for the time being we will omit the subscript i and superscript n without any ambiguity. Then it is not hard to show the following inequality:

$$\begin{aligned}
&\sum_j P(f_j) \log\left(\frac{f}{M}\right)_j \\
=& \sum_j \sqrt{M_j}\tilde{P}\left(\frac{f_j}{\sqrt{M_j}}\right) \log\left(\frac{f}{M}\right)_j \\
=& \frac{1}{\triangle v^2}\sum_j \sqrt{M_j M_{j+1}}\left[\left(\frac{f}{M}\right)_{j+1} - \left(\frac{f}{M}\right)_j\right] \log\left(\frac{f}{M}\right)_j \\
& \qquad -\frac{1}{\triangle v^2}\sum_j \sqrt{M_j M_{j-1}}\left[\left(\frac{f}{M}\right)_j - \left(\frac{f}{M}\right)_{j-1}\right] \log\left(\frac{f}{M}\right)_j \\
=& \frac{1}{\triangle v^2}\left(\sum_j \sqrt{M_j M_{j+1}}\left[\left(\frac{f}{M}\right)_{j+1} - \left(\frac{f}{M}\right)_j\right]\left[\log\left(\frac{f}{M}\right)_j - \log\left(\frac{f}{M}\right)_{j+1}\right]\right) \quad (4.20) \\
\leq& \; 0.
\end{aligned}$$

And from the last equality, every term in the summation is no greater than 0, so

$$\sum_j P(f_j) \log\left(\frac{f}{M}\right)_j = 0 \;\Rightarrow\; \left(\frac{f}{M}\right)_j \text{ is independent of } j,$$

or $f_j = CM_j,\ \forall j$, where C is a function independent of j (or v). And by mass conservation, $C = \frac{\rho}{(2\pi)^{\frac{N}{2}}}$. Then from (4.11), one has

$$\epsilon\left[\sum_j \left(\frac{f_j^{n+1} - f_j^n}{\triangle t} + v\partial_x f_j^n\right) \log\left(\frac{f}{M}\right)_j^{n+1}\right] = \sum_j P(f_j^{n+1}) \log\left(\frac{f}{M}\right)_j^{n+1}, \qquad (4.21)$$

so $\epsilon \to 0$ implies $\sum_j P(f_j^{n+1}) \log\left(\frac{f}{M}\right)_j^{n+1} \to 0$, thus $f_j^{n+1} \to \frac{\rho^{n+1}}{(2\pi)^{\frac{N}{2}}} M_j^{n+1}$, $\forall j$.

Now go back to the scheme (4.15), summation over j gives

$$\frac{\rho^{n+1} - \rho^n}{\triangle t} + \sum_j v_j \cdot \partial_x f_j^n = 0. \qquad (4.22)$$

The above argument says f^n converges to $\frac{\rho^n}{(2\pi)^{\frac{N}{2}}} M^n$ as $\epsilon \to 0$, so

$$\begin{aligned}
\sum_j v_j \partial_x f_j^n &= \partial_x \left[\sum_j (v_j + \partial_x \phi^n - \partial_x \phi^n) e^{-\frac{|v_j + \partial_x \phi^n|^2}{2}} \frac{\rho^n}{(2\pi)^{\frac{N}{2}}} \right] \\
&= -\partial_x \left[\sum_j \partial_x \phi^n e^{-\frac{|v_j + \partial_x \phi^n|^2}{2}} \frac{\rho^n}{(2\pi)^{\frac{N}{2}}} \right] \\
&= -\partial_x (\rho^n \partial_x \phi^n) \sum_j \frac{1}{(2\pi)^{\frac{N}{2}}} e^{-\frac{|v_j + \partial_x \phi^n|^2}{2}}, \qquad (4.23)
\end{aligned}$$

and one can see that $\sum_j \frac{1}{(2\pi)^{\frac{N}{2}}} e^{-\frac{|v_j + \partial_x \phi^n|^2}{2}}$ approximates 1 with a second order accuracy in v, plugging (4.23) into (4.22) one gets a time consistent semidiscretized form of the limit equation (4.7), thus justifying the correct high field limit in the time discrete case.

Remark 4.1. *In fact, for the space homogeneous case ($\partial_x f = 0$), one can show that the entropy decays from the inequality (4.20). Note that in this case M does not change with time. Multiply (4.11) with* $\log \left(\frac{f_j^{n+1}}{M_j} \right)$ *and summing over j, and by (4.20) one has*

$$\sum_j f_j^{n+1} \log \left(\frac{f_j^{n+1}}{M_j} \right) - \sum_j f_j^n \log \left(\frac{f_j^{n+1}}{M_j} \right) = \frac{1}{\epsilon} \sum_j P(f_j^{n+1}) \log(\frac{f_j^{n+1}}{M_j}) \le 0,$$

or equivalently,

$$\sum_j f_j^{n+1} \log \left(\frac{f_j^{n+1}}{M_j} \right) - \sum_j f_j^n \log \left(\frac{f_j^n}{M_j} \right) + \sum_j f_j^n \left[\log \left(\frac{f_j^n}{M_j} \right) - \log \left(\frac{f_j^{n+1}}{M_j} \right) \right] \le 0$$

Thus

$$\begin{aligned}
\sum_j f_j^{n+1} \log \left(\frac{f_j^{n+1}}{M_j} \right) - \sum_j f_j^n \log \left(\frac{f_j^n}{M_j} \right) &= \sum_j f_j^n \log \left(\frac{f_j^{n+1}}{f_j^n} - 1 + 1 \right) \\
&\le \sum_j f_j^n \left(\frac{f_j^{n+1}}{f_j^n} - 1 \right) = \sum_j (f_j^{n+1} - f_j^n) = 0,
\end{aligned}$$

where the inequality comes from the inequality $\log(1 + x) \le x$*, and the last equality is the result of mass conservation.*

4.2.3 A second order scheme

Using backward difference formula for time discretization [43], the second order scheme in one space dimension is given by

$$\frac{3f^{n+1} - 4f^n + f^{n-1}}{2 \triangle t} + 2v\partial_x f^n - v\partial_x f^{n-1} = \frac{1}{\epsilon}\sqrt{M^{n+1}}\tilde{P}(\frac{f^{n+1}}{\sqrt{M^{n+1}}}). \tag{4.24}$$

For space discretization, we use the MUSCL scheme [68], i.e.,

$$v_j \cdot \partial_x f = v_j \frac{f_{i+\frac{1}{2},j} - f_{i-\frac{1}{2},j}}{\triangle x}, \tag{4.25}$$

and $f_{i+\frac{1}{2},j}$ takes the form

$$v_j > 0, \qquad f_{i+\frac{1}{2},j} = f_{i,j} + \frac{1}{2}\phi(\theta_{i+\frac{1}{2}})(f_{i+1,j} - f_{i,j}); \tag{4.26}$$

$$v_j < 0, \qquad f_{i+\frac{1}{2},j} = f_{i+1,j} - \frac{1}{2}\phi(\theta_{i+\frac{1}{2}})(f_{i+1,j} - f_{i,j}), \tag{4.27}$$

where $\theta_{i+\frac{1}{2}}$ is the smooth indicator, and ϕ is the slope limiter function, say, the minmod limiter [69]

$$\phi(\theta) = \max\{0, \ \min\{1, \theta\}\}. \tag{4.28}$$

4.3 Numerical Examples

In order to avoid some difficulties that might be introduced by boundaries, we will consider periodic boundary condition in x-direction. Our simulations will be for the electrostatic case, for which a global-in-time smooth solution exists.

4.3.1 The order of convergence

This section is devoted to check the order of accuracy of the schemes (4.15) and (4.24). Consider the VPFP system in $1d_x \times 1d_v$. Take the equilibrium initial data

$$\rho^0(x) = \frac{\sqrt{2\pi}}{2}(2+\cos(2\pi x)), \quad f^0(x,v) = \frac{\rho^0(x)}{\sqrt{2\pi}} e^{-\frac{|v+\phi_x^0|^2}{2}}, \tag{4.29}$$

where $x \in [0,1]$, $v \in [-6,6]$. ϕ^0 is the solution to (4.3b) with

$$h(x) = \frac{\sqrt{2\pi}}{1.2661} e^{\cos(2\pi x)}, \tag{4.30}$$

and satisfies the periodic boundary condition $\phi^0(0) = \phi^0(1)$.

We take $N_v = 64$ as the number of grid points in $v-$direction, and take space grid points $N_x = 32,\ 64,\ 128,\ 256,\ 512$ respectively. Choose time step $\triangle t = \triangle x/8$ to satisfy the CFL condition $\triangle t \le \triangle x/\max_j |v_j|$ in transport part. The output time is $T_{max} = 0.125$. Check the relative error in l^1 norm

$$e_{\triangle x} = \max_{t\in(0,T_{max})} \frac{\| f_{\triangle x}(t) - f_{2\triangle x}(t) \|_1}{\| f^0 \|_1},$$

where $f_{\triangle x}$ is the numerical solution calculated from a grid of size $\triangle x$. If $e_{\triangle x} \le C\triangle x^k$ for all $0 < \triangle x \ll 1$, then the scheme is said to be k-th order accurate.

The l^1 error of the first order and second order methods are presented in Figure 7. The order of accuracy is shown to be first and second in space and time uniformly with respect to ϵ. (The error in v is spectrally small, see [59], so it will not contribute much to the errors.)

4.3.2 The asymptotic preserving property

In this section, we want to show that no matter whether the initial data is in equilibrium, the first order method (4.15) and second order method (4.24) will push f towards the

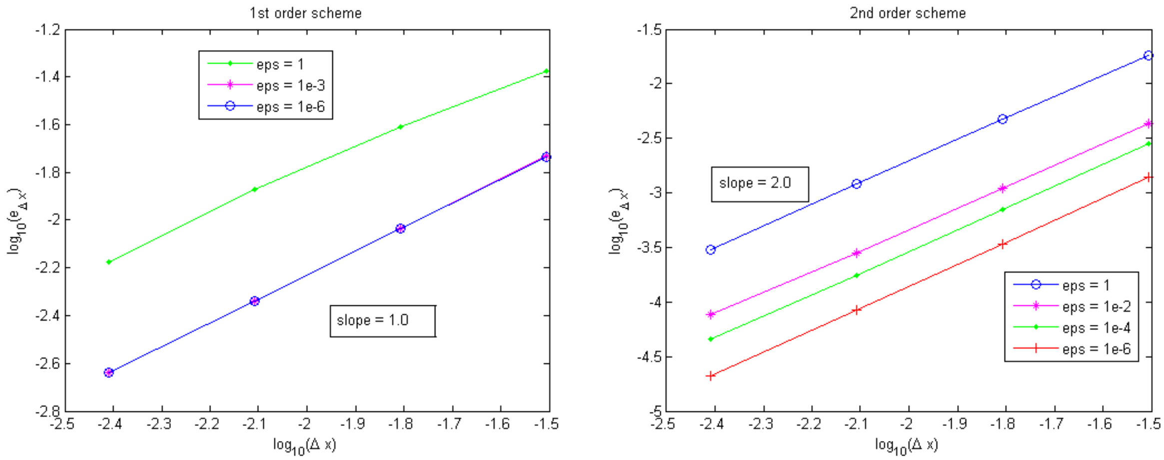

Figure 7: The l^1 errors of the first order scheme (left) and second order scheme (right).

local Maxwellian in one step, and this is exactly the strong AP property defined in [36].

For the equilibrium initial data, we take the same one as in previous section (4.29) . For nonequilibrium initial data we take the following "double peak" function

$$\rho^0(x) = \frac{\sqrt{2\pi}}{2}(2+\cos(2\pi x)), \quad f^0(x,v) = \frac{\rho^0(x)}{\sqrt{2\pi}}\left(e^{-\frac{|v+1.5|^2}{2}} + e^{-\frac{|v-1.5|^2}{2}}\right), \tag{4.31}$$

and let $h(x) = \frac{5.0132}{1.2661}e^{\cos(2\pi x)}$ which satisfies the neutrality condition. We show the time evolution of the "distance" between f and equilibrium $M^{eq} = \frac{\rho}{\sqrt{2\pi}}e^{-\frac{|v+\phi_x|^2}{2}}$ with respect to different ϵ.

$$\| f - M^{eq} \|_1 = \sum_{i,j} |f_{i,j} - M^{eq}_{i,j}| \bigtriangleup x \bigtriangleup v.$$

Figure 8 gives the time evolution of $\| f - M^{eq} \|_1$ for different ϵ, which shows that $f^n - (M^{eq})^n = O(\epsilon)$ for all $n \geq 1$ whether the initial condition is in equilibrium or not. This validates that the first order scheme is indeed AP.

For the second order scheme, we have similar results, see Figure 9.

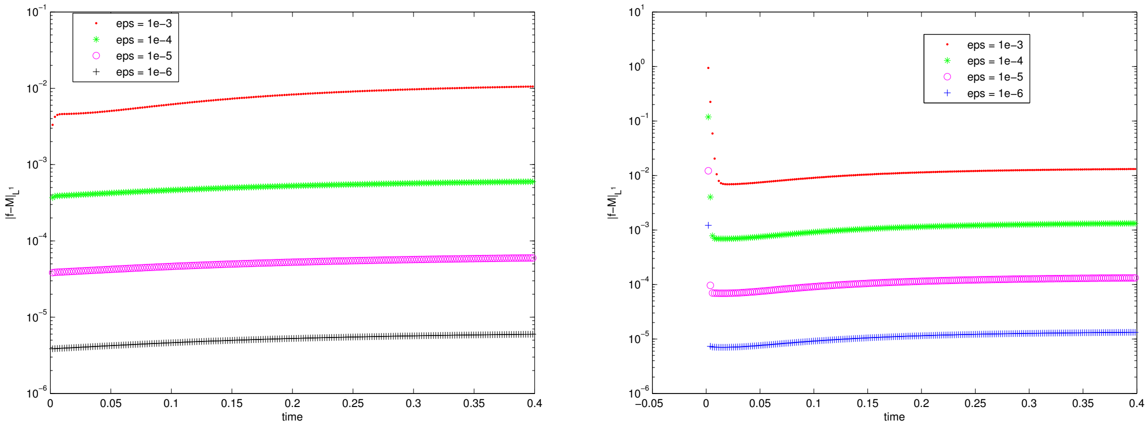

Figure 8: The time evolution of $\| f - M^{eq} \|_1$ for different ϵ with equilibrium initial data (left) and nonequilibrium initial data (right) using the first order scheme. The mesh sizes are $N_v = 64$, $N_x = 64$, $\triangle t = \triangle x/8$.

4.3.3 Mixing regimes

Now we test our scheme in mixing regimes, where ϵ varies in space by several orders of magnitude. For example, take ϵ to be

$$\epsilon(x) = \begin{cases} \epsilon_0 + \frac{1}{2}(\tanh(5 - 10x) + \tanh(5 + 10x)) & x \leq 0.3; \\ \epsilon_0 & x > 0.3, \end{cases}$$

where $\epsilon_0 = 0.001$, so that it contains both the kinetic and high field regimes. The initial data is given by

$$\rho^0(x) = \frac{\sqrt{2\pi}}{6}(2 + \sin(2\pi x)), \quad f^0(x, v) = \frac{\rho^0(x)}{\sqrt{2\pi}} e^{-\frac{|v + \phi^0_x|^2}{2}}, \tag{4.32}$$

where $x \in [-1, 1]$ and ϕ^0 is the solution to (4.3b) with

$$h(x) = \frac{1.6711}{1.2661} e^{\cos(2\pi x)}, \tag{4.33}$$

and satisfies the periodic boundary condition $\phi^0(-1) = \phi^0(1)$.

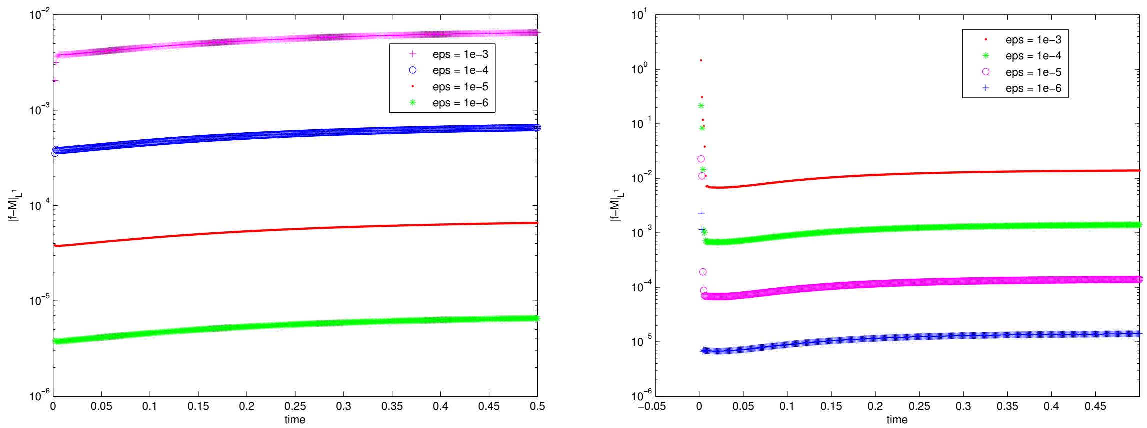

Figure 9: The time evolution of $\| f - M^{eq} \|_1$ for different ϵ with equilibrium initial data (left) and nonequilibrium initial data (right) using the second order scheme. The mesh size are $N_v = 64$, $N_x = 64$, $\triangle t = \triangle x/15$.

In this test we compare the second order scheme (4.24) with the explicit scheme which uses the second order Runge-Kutta discretization in time and MUSCL scheme for space discretization. In our scheme, we take $N_x = 100$ and $\triangle t = \triangle x/15 = 0.00125$, while in explicit scheme, we take $N_x = 2000$ and $\triangle t = \min\{\frac{\triangle x}{\max|v|}, \epsilon_0 \Delta x, \epsilon_0 \Delta v^2\}/5 = 7.0313\, e{-6}$. The shape of ρ at three different times are presented in Figure 10, and one can see that our new second order scheme gives a good approximation to the "reference" solution obtained by the explicit method with much smaller mesh size and time step.

4.3.4 A Riemann problem

Now we apply our second order method to the $1 - D$ Riemann problem:

$$\begin{cases} (\rho_l, h_l) = (1/8, 1/2), \qquad 0 \le x < 1/4; & (4.34a) \\ (\rho_m, h_m) = (1/2, 1/8), \qquad 1/4 \le x < 3/4; & (4.34b) \\ (\rho_r, h_r) = (1/8, 1/2), \qquad 3/4 \le x \le 1. & (4.34c) \end{cases}$$

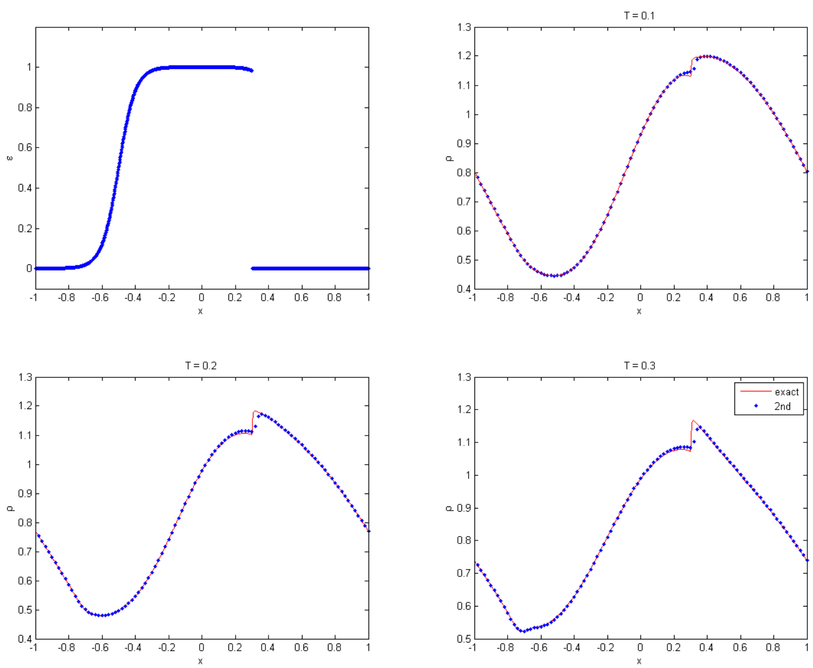

Figure 10: The mixing regime problem. The solid line is computed by an explicit method with refined mesh and serves as the "reference" solution. The dots are obtained by the new second order scheme.

Let ϕ initially be the solution to $-\triangle_x \phi = \rho - h$, and $f = \frac{\rho}{\sqrt{2\pi}} e^{-\frac{|x+\nabla_x \phi|^2}{2}}$. Again periodic boundary condition in x direction is applied.

For our second order scheme, we take $N_x = 100$ and $\triangle t = \triangle x/15 = 0.00125$. In comparison, we use the second order Runge-Kutta scheme with MUSCL scheme in space, and take $N_x = 2000$ and $\triangle t = \min\{\frac{\triangle x}{\max|v|}, \epsilon_0 \triangle x, \epsilon_0 \triangle v^2\}/5 = 7.0313\, e-6$. We compute the macroscopic variable ρ, ϕ and flux $j(t,x) = \int_{\mathcal{R}} v f(t,x,v) dv$. Figure 11 shows that the results obtained by our second order scheme agrees very well with the "reference" solution obtained by the explicit scheme with refined mesh.

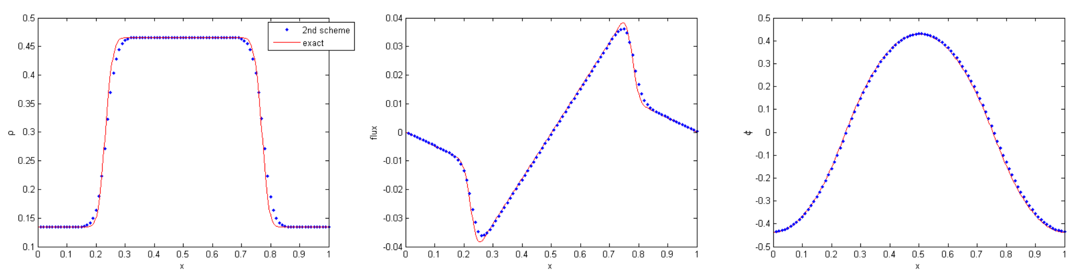

Figure 11: The comparison of density, flux and potential for a Riemann problem at time t = 0.2 between the under-resolved solution by the second order scheme (dots) and resolved solution by the explicit second order Runge-Kutta scheme (solid line).

4.4 Concluding remarks

An asymptotic-preserving scheme for the Vlasov-Poisson-Fokker-Planck system in the high field regime has been introduced in this chapter. The main idea is to combine the two stiff terms, $\frac{1}{\epsilon}\nabla_x \phi_\epsilon \cdot \nabla_v f_\epsilon$ and $\frac{1}{\epsilon}\mathcal{P}_{non}$ together into a modified form of the Fokker-Planck operator, which contains the information of the potential. Then we can use the method developed in [59] to discretize this modified collision operator, and resulting in an implicit scheme that only needs to invert a symmetric system, which can be solved by the conjugate gradient method. This scheme shares some good properties: it conserves mass, preserves positivity, is stable and asymptotic preserving. A uniformly second order scheme is also available here. Some numerical experiments are carried out to test the performance of the scheme.

Chapter 5

The semiconductor Boltzmann equation

5.1 Introduction

In the semiconductor kinetic theory, the semi-classical evolution of the electron distribution function $f(t,x,v)$, in the parabolic band approximation, solves the kinetic equation:

$$\partial_t f + v \cdot \nabla_x f - \frac{q}{m_e} E \cdot \nabla_v f = \mathcal{Q}(f), \quad t > 0, x \in \mathbb{R}^{d_x}, v \in \mathbb{R}^{d_v}, \tag{5.1}$$

where q and m_e are positive elementary charge and effective mass of electrons, $E(t,x)$ is the electric field. The collision operator $\mathcal{Q}$ can be decomposed into three effects

$$\mathcal{Q} = \mathcal{Q}_{el} + \mathcal{Q}_{inel} + \mathcal{Q}_{ee}, \tag{5.2}$$

where $\mathcal{Q}_{el}$ and $\mathcal{Q}_{inel}$ describe the interactions between the electrons and the lattice imperfections, with the first one caused by ionized impurities and elastic part of the phonon collisions (or called crystal vibrations) and the second one by inelastic part of the phonon collisions. $\mathcal{Q}_{ee}$ characterizes the correlations between electrons themselves. For low electron densities, the general form of $\mathcal{Q}$ is [74]

$$\mathcal{Q}(f) = \int_{\mathbb{R}^{d_v}} \big(s(v', v) f(t,x,v') - s(v,v') f(t,x,v)\big) dv', \tag{5.3}$$

where s is the transition probability depending on the specific scattering mechanism described above, and satisfies the principle of detailed balance

$$s(v', v)M(v') = s(v, v')M(v), \tag{5.4}$$

where

$$M(v) = \left(\frac{2\pi K_B T}{m}\right)^{-\frac{d_v}{2}} e^{-\frac{v^2}{2v_{th}^2}} \tag{5.5}$$

is the Maxwellian, v_{th} is the thermal velocity related to the lattice temperature T through $v_{th}^2 = \frac{K_B T}{m_e}$ and K_B is the Boltzmann constant. The null space of $\mathcal{Q}$ in (5.3) is spaned by this Maxwellian (5.5).

When the electron density is high, one should take Pauli's exclusion principle into account, and the collision operator $\mathcal{Q}$ becomes

$$\mathcal{Q}_{deg}(f) = \int_{\mathbb{R}^{N_v}} \big(s(v', v)f'(1-f) - s(v, v')f(1-f')\big)dv', \tag{5.6}$$

which is referred to as the degenerate case. Here f and f' are shorthanded notations for $f(t, x, v)$ and $f(t, x, v')$ respectively.

In principle, the electric field is produced self-consistently by the electrons moving in a fixed ion background with doping profile $h(x)$ throngh

$$\nabla_x(\varepsilon(x)\nabla_x\Phi) = \rho(x) - h(x), \qquad E = -\nabla_x\Phi, \tag{5.7}$$

where $\rho(x)$ is the electron density, Φ is the electrostatic potential and $\varepsilon(x)$ is the permittivity of the material.

The numerical computation of electron transport in semiconductors through the Boltzmann equation (BE) (5.1) is usually too costly for practical purposes since it involves the resolution of a problem rested on 7-dimensional time and space. Several macroscopic models based on the diffusion approximation were derived. The classical drift-diffusion (DD) [87] model was introduced, with the assumption that all the scatterings in $\mathcal{Q}$ are strong and that the electron temperature relaxes to the lattice temperature at the microscopic time scale. The connection between BE and DD model has been well understood physically and mathematically [42, 81]. The case of the Fermi-Dirac statistics was investigated in [42] as well. However, in most situations, the momentum relaxation occurs much faster than temperature relaxation, thus results in an intermediate state at which the electrons have reached a local equilibrium with a different temperature other than the lattice temperature. The time evolution of this state is described by the Energy-Transport (ET) model, which is a system of diffusion equations for the electron density and energy. This model can be viewed as an augmented drift-diffusion model, and is derived asymptotically under the scaling that both the elastic $\mathcal{Q}_{el}$ and electron-electron $\mathcal{Q}_{ee}$ collisions are dominant [4]. Another model is the Spherical Harmonic Expansion (SHE) model which is obtained based on the observation that in some cases the electron-electron collision cannot constitute one of the dominant scattering mechanisms [89, 90]. This model, the only dominant collision mechanism of which is $\mathcal{Q}_{el}$, can be considered as a diffusion equation in the extended space: position and energy. In fact, the ET model was usually derived through the SHE model by taking the limit on the scaled electron-electron collision mean free path [26]. See also [30] for the new and simpler derivation of the ET model directly through the Boltzmann equation. [3] outlines a hierarchy between various macroscopic models as well as shows the macroscopic limit that links the two successive steps within the hierarchy.

However, due to the rapid progress in miniaturization of semiconductor devices, the standard drift diffusion models break down in some regime of hot electron transport.

This regime concerns the physical situations where both the electric effects and collisions are dominant, which is called the high field regime. After rescaling of the variables, equation (5.1) can be written as

$$\partial_t f + v \cdot \nabla_x f - \frac{1}{\epsilon} E \cdot \nabla_v f = \frac{1}{\epsilon} \mathcal{Q}(f), \quad t > 0, x \in \mathbb{R}^{d_x}, v \in \mathbb{R}^{d_v}, \tag{5.8}$$

where ϵ is the ratio between the mean free path and the typical length scale. It was first studied by Frosali *et.al* [39, 38], and later by Poupaud [82] for the nondegenerate case, where the limiting equation is a linear convection equation for the mass density with the convection proportional to the electric field. It also gives a necessary condition for the limit equation to embrace a unique solution, while if such a condition is not satisfied, a travelling wave solution will exist which is the so-called runaway phenomenon. When the electrostatic potential is obtained through the Poisson equation, [18] derives the high field limit for the BGK-type collsion, and also reveals the boundary layer behavior when bounded domain is considered. The high field asymptotics for the degenerate case was carried out in [1], where the limit equation is a nonlinear convection equation for the macroscopic density which has a local in time regular solution. It was revisited in [2] where the convergence to entropy solutions and existence of shock profiles for the limit nonlinear conservation law were considered.

Considerable literature has been devoted to the design of efficient and accurate numerical methods for (5.1), such as [52, 12, 13, 22], to name just a few. This scheme becomes inefficient in the high field regime. Only recently, schemes efficient in the high field regime started to emerge [57, 23]. In this chapter, we are interested in designing a numerical method that automatically becomes a macroscopic solver for the high field limit equation when sending the small parameter ϵ to 0, which is the so called *Asymptotic*

Preserving (AP) property put forward by Jin [53].

As one can see, when ϵ is small, two terms of equation (5.8) become stiff and explicit schemes are subject to severe stability constraints. Implicit schemes allow larger time steps and mesh sizes, but it is usually expensive due to the prohibitive computational cost required by inverting a large algebraic system, even in the non-degenerate case where the collision operator is linear. Another remarkable difficulty is that there is no specific form of the *local equilibrium* M_h in the high field regime, which makes the modern asymptotic preserving methods such as [23, 33, 106] very hard to implement. To overcome the first difficulty, we follow the idea in [36] by penalizing the non-symmetric stiff term by a BGK operator which is much easier to treat implicitly. To overcome the second difficulty, inspired by the observation in [35] that one needs not to use the exact local equilibrium as a penalization but a "good" approximation of it might be enough, we only penalize the collision term by a 'classical' BGK operator with the Maxwellian defined in (5.5) instead of the real Maxwellian for the high field limit, and leave the stiff force term alone implicitly, at the cost of a weaker AP property

$$f^n - M_h^n = O(\triangle t + \epsilon) \text{ for } n \geq N \text{ and any initial data,} \tag{5.9}$$

where f^n denotes f at discrete time t^n.

The rest of the chapter is organized as follows. In the next section we give a brief review of the scalings in the high field regime and the corresponding macroscopic limit. Section 5.3 is devoted to the new scheme, as well as a derivation of its asymptotic property. Then we present several numerical examples to test the efficiency, accuracy and asymptotic property of the schemes in section 5.4. We also point out an open question and our understanding in the end. At last, some concluding remarks are given in section 5.5.

5.2 Scalings and the high field limit

Since the transition probability in (5.3) satisfies the detailed balance principle, it is convenient to introduce a new function

$$\phi(v,v') = \frac{s(v',v)}{M(v)}, \quad \text{so that } \phi(v,v') = \phi(v',v). \tag{5.10}$$

Then the collision $\mathcal{Q}$ reads

$$\mathcal{Q}(f) = \int_{\mathbb{R}^{d_v}} \phi(v,v')\big(M(v)f(t,x,v') - M(v')f(t,x,v)\big)dv'. \tag{5.11}$$

Following [82], and also Chapter 2 in [74], introduce the rescaled variables:

$$\tilde{x} = \frac{x}{L}, \quad \tilde{t} = \frac{t}{T}, \quad \tilde{v} = \frac{v}{v_{th}},$$

where L and T are reference length and time. By the dimension argument, the collision term should be proportional to the reciprocal of a characteristic time, thus we define an average relaxation time τ and the rescaled collision $\tilde{\mathcal{Q}}$

$$\frac{1}{\tau} = \int_{\mathbb{R}^{\mathbb{N}}} \phi(v,v')M(v)M(v')dvdv', \qquad \tilde{\mathcal{Q}} = \tau\mathcal{Q}.$$

Note here that for the degenerate case the definition of τ is a bit different but similar. The mean free path now can be defined as $l = \tau v_{th}$. Next define the thermal voltage U_{th} and the rescaled electric field $\tilde{E}$ as

$$U_{th} = \frac{m_e v_{th}^2}{q}, \qquad \tilde{E} = \frac{E}{E_0},$$

where E_0 is a reference field. Then the Boltzmann equation (5.1) takes the form

$$\frac{\tau}{T}\,\partial_{\tilde{t}} f + \frac{\tau v_{th}}{L}\,\tilde{v}\cdot\nabla_{\tilde{x}} f - \frac{\tau v_{th}}{U_{th}} E_0\,\tilde{E}\cdot\nabla_{\tilde{v}} f = \tilde{\mathcal{Q}}. \tag{5.12}$$

Now introduce the dimensionless parameter $\epsilon = \frac{l}{L}$ and consider the high field scalings

$$E_0 = \frac{U_{th}}{l}, \quad T = \frac{\tau}{\epsilon},$$

(5.12) becomes

$$\partial_t f + v \cdot \nabla_x f - \frac{1}{\epsilon} E \cdot \nabla_v f = \frac{1}{\epsilon} Q(f), \tag{5.13}$$

where we have dropped the tilde for convenience.

5.2.1 The high field limit: the nondegenerate case

In (5.13), when ϵ vanishes, the limiting equation is a linear convection equation for the macroscropic particle density with a convection proportional to the scaled electric field. That is,

$$f(t, x, v) \to \rho(t, x) F_{E(t,x)}(v), \tag{5.14}$$

where $F_{E(t,x)}(v)$ is the solution to

$$\int_{\mathbb{R}^{d_v}} F_E(v) dv = 1, \quad E \cdot \nabla_v F_E + \mathcal{Q}(F_E) = 0, \quad F_E \geq 0; \tag{5.15}$$

while the equation for the macroscopic density ρ is obtained by integrating (5.13) w.r.t. v

$$\partial_t \rho(t, x) + \int_{\mathbb{R}^{\mathbb{N}}} v \cdot \nabla_x f = 0, \tag{5.16}$$

and then passing to the limit to get

$$\partial_t \rho(t, x) + \nabla_x \cdot (\rho(t, x) \sigma(E(t, x))) = 0, \quad \sigma(E) = \int_{\mathbb{R}^{d_v}} v F_E(v) dv. \tag{5.17}$$

Not all $\mathcal{Q}$ gives a unique solution of (5.15). Poupaud [82] gave a criteria for the transition probability s in the following theorem.

Theorem 5.1. *[82] Assume that the collision cross-section $\phi(v, v') > 0$ satisfies $\phi(v, v') \in W^{1,\infty}(\mathbb{R}^{2d_v})$, then the collision frequency*

$$\nu(v) = \int_{\mathbb{R}^{\mathbb{N}}} s(v, v')dv' = \int_{\mathbb{R}^{\mathbb{N}}} \phi(v, v')M(v')dv' \tag{5.18}$$

is bounded and positive. If it further satisfies

$$\int_0^\infty \nu(v + \eta E)d\eta = +\infty, \quad a.e., \tag{5.19}$$

and the initial data $f(0, x, v) = f^0(x, v)$ solves $E \cdot \nabla_v f^0(x, v) - \mathcal{Q}(f^0)(x, v) = 0$ a.e., then the solution to (5.13) converges to ρF_E in the following sense: $\exists$ a positive constant C_T that depends on the initial data such that for any time $t \leq T$, the following inequality

$$\| f(t, \cdot, \cdot) - \rho(t, \cdot)F_{E(t,\cdot)}(\cdot) \|_{L^1(\mathbb{R}^{d_x} \times \mathbb{R}^{d_v})} \leq C_T \ \epsilon$$

holds.

Remark 5.2. *Equation (5.17) together with (5.15) can be regarded as the first order approximation of (5.13) which resembles the hydrodynamic approximation of the Boltzmann equation by the Euler equations. (5.17) that rules out all the diffusion effect, is nothing but Ohm's law. If one goes further to the second order approximation, a new drift diffusion equation can be derived, which again resembles the Navier-Stokes approximation of the Boltzmann equation.*

Remark 5.3. *The above result is obtained for the case where electrical field is given. The analytical result for the case when the electrical field is self-consistent through the Poisson equation is only derived by Cercignani, Gamba and Levermore in [18] for the BGK collision operator, while for general collision it is still open.*

5.2.2 The high field limit: the degenerate case

Similar to the nondegenerate case, the transition probability $s(v, v')$ in (5.6) also satisfies the principle of detailed balance [80], so it can be reformulated in the same way as (5.11),

$$\mathcal{Q}_{deg}(f)(t,x,v) = \int_{\mathbb{R}^{\mathbb{N}}} \phi(v',v)\Big(M(v)f(t,x,v')\big(1-f(t,x,v)\big)-M(v')f(t,x,v)\big(1-f(t,x,v')\big)\Big)dv', \tag{5.20}$$

where $M(v)$ and $\phi(v', v)$ are defined the same as before in (5.5) and (5.10). The null space of $\mathcal{Q}_{deg}(f)(t, x, v)$ is spaned by the Fermi-Dirac distribution

$$M_{FD} = \frac{1}{1 + e^{\frac{m_e v^2}{2K_B T} - \frac{\mu}{K_B T}}}, \tag{5.21}$$

where T is the lattice temperature and μ is the electron Fermi energy. The dimensionless form of the degenerate case is the same as (5.13), except that the collision $\mathcal{Q}$ is replaced by $\mathcal{Q}_{deg}$.

Assume B is either the Brillouin zone or the whole space $\mathbb{R}^{d_v}$. When sending ϵ to 0, f can no longer be decoupled into two functions with one depending on x and t and the other on v seperately because of the nonlinearity of the collision operator, instead one has, under the hypothesis that $\phi \in W^{2,\infty}(B^2)$ and $\phi_0 \leq \phi(v, v') \leq \phi_1$ for some positive constant ϕ_0 and ϕ_1,

$$f \to F(\rho(t,x), E(t,x))(v)$$

where $F(\rho, E)(v)$ is the unique solution in space $D_E = \{F \in L^1(B);\ E \cdot \nabla_v F \in L^1(B)\}$ such that $0 \leq F \leq 1$ and

$$E \cdot \nabla_v F - \mathcal{Q}_{deg}(F) = 0, \quad \int_{\mathbb{R}^{\mathbb{N}}} F(t,x,v)dv = \rho(t,x). \tag{5.22}$$

Moreover, the mapping

$$(\rho, E) \mapsto F(\rho, E) \tag{5.23}$$

from $\mathbb{R}^+ \times \mathbb{R}^{d_x}$ to $L^1(B)$ is C^2 differentiable. Then the macroscopic density ρ solves

$$\partial_t \rho(t,x) + \nabla_x \Big(j(\rho(t,x); E(t,x)) \Big) = 0, \;\; \rho(0,x) = \int_{\mathbb{R}^{\mathbb{N}}} f^0(x,v)dv, \tag{5.24}$$

where $j(\rho; E) = \int_{\mathbb{R}^{\mathbb{N}}} vF(\rho, E)(v)dv$. This result was proved in [1] for a given $E(x) \in \mathbb{R}^{d_x}$ on the time intervals such that the limit solution is regular.

Remark 5.4. *Although there is no such condition like (5.19) to insure the existence of the limit solution, the hypothesis that $\phi(v, v')$ should be uniformly bounded from below and above already implies it.*

Remark 5.5. *Due to the nonlinearity of the flux function in (5.24), only the existence and uniqueness of a local in time regular solutions were available and shock might be generated later [2]. This is different from the nondegenerate case, where the limit equation (5.17) is linear in ρ, thus a unique global in time solution exists.*

5.3 A numerical scheme for the semiconductor Boltzmann equation

To design an asymptotic preserving method, one usually needs to treat the two stiff terms – the force term and collision term implicitly. However, this would bring new difficulties to invert the algebraic system originated by the non-symmetric difference operator and the collision operator. In the last chapter when the collision is of the Fokker-Planck type, these two terms were combined and rewrote into one symmetric operator in velocity space. But unfortunately, this strategy cannot be implemented here because no symmetric combination of the two is available. Another remarkable difficulty is that one cannot write down the local equilibrium M_h in high filed limit explicitly, thus we cannot use the existent asymptotic preserving method for kinetic equation in the hydrodynamic regime [23], nor can we use the even-odd decomposition [56] due to the fact that one cannot derive a "non-stiff" force term. Here we adopt the penalization idea introduced by Filbet and Jin [36]. In addition, inspired by the fact that functions that share the same conserved quantities with the exact local equilibrium can be used as candidates for penalty [35], we will only penalize the collision term by a BGK operator which conserves mass, and treat the stiff force term implicitly by the spectral method. To better illustrate our idea, we begin with the simplest case which is the so-called "time relaxation model".

Here for the sake of simplicity, we will explain our idea in the one dimensional case. The generalization to the multidimensional case can be done in a straightforward manner simply using the dimension-by-dimension discretization. Denote $f(x_l, v_m, t^n)$ by f_{lm}^n, where $0 \leq l \leq N_x$ and $0 \leq m \leq N_v$, and N_x and N_v are the numbers of mesh points in x and v directions respectively.

5.3.1 The nondegenerate isotropic case

In the low density approximation, if one only considers the collisions with background impurities, the collision opertor can be approximated by a linear relaxation time operator [18, 74]:

$$\mathcal{Q} = \int Mf' - M'f dv' = M\rho - f, \tag{5.25}$$

which is the simplest case with $\phi(v', v) = 1$ in (5.11). This is usually called the "time-relaxation" model.

In this model, one can directly treat both stiff terms implicitly. The first order scheme reads

$$\frac{f^{n+1}-f^n}{\triangle t} + v\cdot\nabla_x f^n - \frac{1}{\epsilon}E\cdot\nabla_v f^{n+1} = \frac{1}{\epsilon}(M\rho^{n+1}-f^{n+1}), \tag{5.26}$$

and we use the spectral discretization for the stiff force term. The scheme can be implemented as follows

- Step 1. Integrate (5.26) over v, note that the two stiff terms vanish, and one ends up with an explicit semidiscrete scheme for ρ^{n+1}:

  $$\frac{\rho^{n+1}-\rho^n}{\triangle t} + \nabla_x\cdot\int_{\mathbb{R}^{\mathbb{N}}} v f^n dv = 0;$$

  - Step 1.1. If the electrical field is obtained through (5.7), then solve it by any Poisson solver such as the spectral method to get E^{n+1}.

- Step 2. Approximate the transport term $v\cdot\nabla_x f^n$ in (5.26) by a non-oscillatory high resolution shock-capturing method.

- Step 3. Use the spectral discretization for the stiff force term, i.e., (5.26) can be reformulated into

  $$\left[1+\frac{\triangle t}{\epsilon}-\frac{\triangle t}{\epsilon}E\cdot\nabla_v\right]f^{n+1} = f^n - \triangle t v\cdot\nabla_x f^n + \frac{\triangle t}{\epsilon}M\rho^{n+1},$$

  then take discrete Fourier Transform w.r.t. v on both sides, one has

  $$\left[1+\frac{\triangle t}{\epsilon}-i\frac{\triangle t}{\epsilon}E\cdot k\right]\hat{f}^{n+1} = \mathscr{F}\left(f^n - \triangle t v\cdot\nabla_x f^n + \frac{\triangle t}{\epsilon}M\rho^{n+1}\right), \tag{5.27}$$

  where $\hat{f}$ and $\mathscr{F}(f)$ denote the discrete Fourier Transform of f w.r.t. v.

- Step 4. Use the inverse Fourier transform on $\hat{f}^{n+1}$ to get f^{n+1}.

Remark 5.6. *Although the force term $\frac{E}{\epsilon} \cdot \nabla_v f$ contains a derivative, which looks "more stiff" than the collision term, one cannot just treat it implicitly and leave the collision explicit. This can be seen from the simple Fourier analysis on the toy model*

$$f_t + \frac{E}{\epsilon} f_v = -\frac{1}{\epsilon} f, \tag{5.28}$$

with the discretization

$$\frac{f^{n+1} - f^n}{\triangle t} + \frac{E}{\epsilon} \partial_v f^{n+1} = -\frac{1}{\epsilon} f^n,$$

where $f^n(v)$ denotes $f(t^n, v)$. Applying the Fourier Transform on $f^n(v)$ w.r.t. v to get $\hat{f}^n(k)$, then one has

$$\left|\hat{f}^{n+1}(k)\right|^2 = \frac{\left(1 - \frac{\triangle t}{\epsilon}\right)^2}{1 + \left(\frac{\triangle t}{\epsilon} E \cdot k\right)^2} \left|\hat{f}^n(k)\right|^2,$$

note that for stability the coefficient on the right hand side needs to be less than one for all values of k, that is $\frac{\triangle t}{\epsilon} < 2$, thus $\triangle t$ must be dependent of ϵ.

Since both of the stiff terms are treated fully implicitly, we have the following strong AP property.

Proposition 7. *Let*

$$\| f^n(x, \cdot) \|_{L^2(\mathbb{R}^{d_v})} = \sqrt{\int_{\mathbb{R}^{\mathbb{N}}} f(t^n, x, v)^2 dv}, \tag{5.29}$$

then in the regime $\triangle t \gg \epsilon$ we have

$$\|f^n - M_h^n\|_{L^2(\mathbb{R}^{d_v})} \leq \alpha^n \|f^0 - M_h^0\|_{L^2(\mathbb{R}^{d_v})} + O(\epsilon) \quad \text{with } \alpha < C\epsilon < 1 \text{ and } C \text{ independent of } \epsilon, \tag{5.30}$$

where $M_h^n = \rho^n F_E$ is the local equilibrium in the high field regime, with F_E being the solution to the limit equation (5.15) with $\mathcal{Q} = \rho M - f$.

Proof. Assume all functions are smooth and in $L^2(\mathbb{R}^{d_v})$. Since M_h satisfies $-E\cdot\nabla_v M_h^{n+1} = \mathcal{Q}(M_h^{n+1}) = \rho^{n+1}M - M_h^{n+1}$, a simple manipulation of scheme (5.26) gives

$$\left(1+\frac{\triangle t}{\epsilon}-\frac{\triangle t}{\epsilon}E\cdot\nabla_v\right)(f^{n+1}-M_h^{n+1}) = (f^n-M_h^n)-(M_h^{n+1}-M_h^n)-\triangle t v\cdot\nabla_x f^n. \quad (5.31)$$

Now take the Fourier Transform w.r.t. v on both sides, (5.31) reformulates to

$$\begin{aligned} &\hat{f}^{n+1}-\hat{M}_h^{n+1} \\ =\ &\frac{\hat{f}^n-\hat{M}_h^n}{1+\frac{\triangle t}{\epsilon}-i\frac{\triangle t}{\epsilon}E\cdot k}-\frac{\hat{M}_h^{n+1}-\hat{M}_h^n}{1+\frac{\triangle t}{\epsilon}-i\frac{\triangle t}{\epsilon}E\cdot k}-\frac{\triangle t}{1+\frac{\triangle t}{\epsilon}-i\frac{\triangle t}{\epsilon}E\cdot k}\mathscr{F}(v\cdot\nabla_x f^n). \quad (5.32)\end{aligned}$$

Let

$$G=\frac{1}{1+\frac{\triangle t}{\epsilon}-i\frac{\triangle t}{\epsilon}E\cdot k}, \quad (5.33)$$

take the L^2 norm on both sides, one has, by Minkowski inequality

$$\begin{aligned} &\| \hat{f}^{n+1}-\hat{M}_h^{n+1} \|_{L^2} \\ \le\ &\| G(\hat{f}^n-\hat{M}_h^n) \|_{L^2} + \left\| G\left((\hat{M}_h^{n+1}-\hat{M}_h^n)+\triangle t\mathscr{F}(v\cdot\nabla_x f^n)\right)\right\|_{L^2}. \quad (5.34)\end{aligned}$$

Notice that $\left|\hat{M}_h^{n+1}-\hat{M}_h^n\right| \le C_1\triangle t(\hat{M}_h)_t$ and $\left\|G\triangle t\right\|_{L^\infty}\le C_2\epsilon$ for $\triangle t\gg\epsilon$, where C_1 and C_2 are two constants independent of $\triangle t$ and ϵ. Then (5.34) becomes

$$\| \hat{f}^{n+1}-\hat{M}_h^{n+1} \|_{L^2} \le \| G(\hat{f}^n-\hat{M}_h^n) \|_{L^2} + C\epsilon. \quad (5.35)$$

Since $\| G \|_{L^\infty}\le C_3\epsilon < 1$ with C_3 independent of ϵ for $\triangle t\gg\epsilon$, applying Parseval's indentity, one has

$$\|f^{n+1}-M_h^{n+1}\|_{L^2(\mathbb{R}^{d_v})} \le \alpha\|f^n-M_h^n\|_{L^2(\mathbb{R}^{d_v})} + O(\epsilon). \quad (5.36)$$

□

5.3.2 The nondegenerate anisotropic case

This section is devoted to the nondegenerate anisotropic case. Recall that the collision operator takes the form

$$\mathcal{Q}(f) = \int_{\mathbb{R}^{N_v}} \phi(v, v')\big(M(v)f(t, x, v') - M(v')f(t, x, v)\big)dv' = \mathcal{Q}^+(f) - \nu(v)f. \quad (5.37)$$

Although $\mathcal{Q}$ is linear in f, due to the non-symmetric nature of the transition probability $s(v', v)$, treating it implicitly as what we did in the last section will make it difficult to invert, especially in higher dimensions. To overcome this difficulty, we adopt the idea introduced by Filbet and Jin in [36] by penalizing the collision term by a BGK operator, the simple structure of which makes it easy to be treated implicitly. Thus the first order scheme reads

$$\frac{f^{n+1} - f^n}{\triangle t} + v \cdot \nabla_x f^n - \frac{1}{\epsilon} E \cdot \nabla_v f^{n+1} = \frac{1}{\epsilon}\mathcal{Q}(f^n) - \frac{\lambda}{\epsilon}(\rho^n M - f^n) + \frac{\lambda}{\epsilon}(\rho^{n+1} M - f^{n+1}), \quad (5.38)$$

where M is defined in (5.5). Then (5.38) has the similar implicit structure as (5.26), thus one can solve it by the same steps introduced in section 5.3.1, yielding a scheme that is implicit but can be implemented explicitly.

Notice that in [36], the penalty is the local equilibrium of the collision operator, which will drive f to the right Maxwellian if treated implicitly. However, as it has been mentioned, there is no explicit form of the "high field equilibrium" which is the solution to $E \cdot \nabla_v f = \mathcal{Q}(f)$, so we instead penalize the equation by the Maxwellian of the collision term, and this will indeed force f to the right local equilibrium by the following proposition. The cost of this "wrong Maxwellian" penalty is the extra $\triangle t$ error in (5.39). It is interesting to point out that this result is similar to the observation in [35] where the authors use the classical Maxwellian instead of the quantum one to penalize the quantum Boltzmann collision operator, and get the right asymptotic property.

Proposition 8. *In (5.29), if $\mathcal{Q}$ takes the form of $\mathcal{Q} = \rho M - f$ and $\lambda > \frac{1}{2}$, then*

$$\|f^n - M_h^n\|_{L^2(\mathbb{R}^{d_v})} \leq \alpha^n \|f^0 - M_h^0\|_{L^2(\mathbb{R}^{d_v})} + O(\epsilon + \triangle t) \quad \textit{with } \alpha < 1, \tag{5.39}$$

where $M_h^n = \rho^n F_E$ is the local equilibrium in the high field regime, with F_E being the solution to the limit equation (5.15) with $\mathcal{Q}$ defined as above.

The proof is very similar to the one for Proposition 9 in the next section and is omitted here.

Remark 5.7 (Choice of λ)**.** *For the general collision, λ should be chosen to satisfy $\lambda > \max_v \mu(v)$, where ν is the collision frequency defined in (5.18). One can also refer to [106] for positivity concern.*

Remark 5.8. *This method can be easily extended to case with non-parabolic energy diagram such as Kane's model [12, 13, 22] since the convection term is treated explicitly.*

5.3.3 The degenerate case

When the quantum effect is taken into account, the collision operator becomes nonlinear. Nevertheless, this can be dealt with in the same way as in section 5.3.2 at the same cost. Again inspired by [35], we use the classical Boltzmann distribution instead of the Fermi-Dirac distribution (5.21) as the penalty to avoid the complicated nonlinear solver for the Fermi-energy in (5.21) from mass density ρ. Since the $\triangle t$ error will be inevitable in the asymptotic property as we have seen in the last section, this change of penalty will only introduce new error of $O(\triangle t)$. Similar to (5.38), the first order scheme takes the form

$$\frac{f^{n+1} - f^n}{\triangle t} + v \cdot \nabla_x f^n - \frac{1}{\epsilon} E \cdot \nabla_v f^{n+1} = \frac{1}{\epsilon} \mathcal{Q}_{deg}(f^n) - \frac{\lambda}{\epsilon}(\rho^n M - f^n) + \frac{\lambda}{\epsilon}(\rho^{n+1} M - f^{n+1}), \tag{5.40}$$

where $\mathcal{Q}_{deg}$ is defined in (5.6) and M is the same as (5.5). In practice, similar to Remark 5.7, λ is chosen to be $\max_v \int_{\mathbb{R}^{\mathbb{N}}} \phi(v', v) M(v')(1 - f(v'))dv'$.

Because of the nonlinearity of $\mathcal{Q}_{deg}$, it is not easy to check the asymptotic property analytically. Instead we check it for the case where $\mathcal{Q}_{deg}$ is replaced by the "quantum BGK" operator "$M_{FD} - f$" in the following proposition.

Proposition 9. *Assume the solutions are smooth and in $L^2(\mathbb{R}^{d_v})$. If $\lambda > \frac{1}{2}$, then the scheme (5.40) with $\mathcal{Q}_{deg}$ replaced by "$M_{FD} - f$" has the asymptotic property*

$$\|f^n - M_{qh}^n\|_{L^2(\mathbb{R}^{d_v})} \leq \alpha^n \|f^0 - M_{qh}^0\|_{L^2(\mathbb{R}^{d_v})} + O(\epsilon + \triangle t) \tag{5.41}$$

with $0 < \alpha < 1$, where M_{qh} is the solution to the high field limit equation

$$-E \cdot \nabla_v M_{qh} = M_{FD} - M_{qh}, \ \int_{\mathbb{R}^{\mathbb{N}}} M_{qh} dv = \int_{\mathbb{R}^{\mathbb{N}}} f dv = \int_{\mathbb{R}^{\mathbb{N}}} M_{FD} dv = \rho. \tag{5.42}$$

Proof. Since M_{qh} satisfies $-E \cdot \nabla_v M_{qh}^{n+1} = M_{FD}^{n+1} - M_{qh}^{n+1}$, the scheme (5.40) becomes

$$\begin{aligned}
& \left(1 + \frac{\lambda \triangle t}{\epsilon} - \frac{\triangle t}{\epsilon} E \cdot \nabla_v\right)(f^{n+1} - M_{qh}^{n+1}) \\
= \ & \left(1 + \frac{(\lambda - 1) \triangle t}{\epsilon}\right)(f^n - M_{qh}^n) - \left(1 + \frac{(\lambda - 1) \triangle t}{\epsilon}\right)(M_{qh}^{n+1} - M_{qh}^n) \\
& + \frac{\lambda \triangle t}{\epsilon}(\rho^{n+1} - \rho^n) M - \frac{\triangle t}{\epsilon}(M_{FD}^{n+1} - M_{FD}^n) - \triangle t v \cdot \nabla_x f^n.
\end{aligned} \tag{5.43}$$

After taking the Fourier Transform w.r.t. v on both sides, it reformulates to

$$\begin{aligned}
& \hat{f}^{n+1} - \hat{M}_{qh}^{n+1} \\
= & \frac{1 + \frac{(\lambda-1)\triangle t}{\epsilon}}{1 + \frac{\lambda \triangle t}{\epsilon} - i\frac{\triangle t}{\epsilon} E \cdot k}\big(\hat{f}^n - \hat{M}_{qh}^n\big) - \frac{1 + \frac{(\lambda-1)\triangle t}{\epsilon}}{1 + \frac{\lambda \triangle t}{\epsilon} - i\frac{\triangle t}{\epsilon} E \cdot k}\big(\hat{M}_{qh}^{n+1} - \hat{M}_{qh}^n\big) \\
& + \frac{\frac{\lambda \triangle t}{\epsilon}}{1 + \frac{\lambda \triangle t}{\epsilon} - i\frac{\triangle t}{\epsilon} E \cdot k}(\rho^{n+1} - \rho^n)\hat{M} - \frac{\frac{\triangle t}{\epsilon}}{1 + \frac{\lambda \triangle t}{\epsilon} - i\frac{\triangle t}{\epsilon} E \cdot k}\big(\hat{M}_{FD}^{n+1} - \hat{M}_{FD}^n\big) \\
& - \frac{\triangle t}{1 + \frac{\lambda \triangle t}{\epsilon} - i\frac{\triangle t}{\epsilon} E \cdot k}\mathscr{F}\big(v \cdot \nabla_x f^n\big).
\end{aligned} \tag{5.44}$$

Let

$$G_1 = \frac{1 + \frac{(\lambda-1)\triangle t}{\epsilon}}{1 + \frac{\lambda \triangle t}{\epsilon} - i\frac{\triangle t}{\epsilon} E \cdot k}, \tag{5.45}$$

take the L^2 norm for (5.44), and apply the same procedure as in Proposition 7, we have

$$\| \hat{f}^{n+1} - \hat{M}_{qh}^{n+1} \|_{L^2} \leq \| G_1 \|_{L^\infty} \| \hat{f}^n - \hat{M}_{qh}^n \|_{L^2} + O(\triangle t + \epsilon), \tag{5.46}$$

where the $O(\triangle t)$ terms come from the second, third and forth terms in (5.44) and form major difference compared to (5.32). If $\lambda > \frac{1}{2}$, $\| G_1 \|_{L^\infty} \leq \alpha < 1$, the result (5.41) then follows. □

Remark 5.9. *To get better asymptotic property than (5.39) and (5.41), we would like to extend the scheme to second order. Follow the idea in [51], using backward difference formula in time and MUSCL scheme [68] in space, we have*

$$\frac{3f^{n+1} - 4f^n + f^{n-1}}{2 \triangle t} + 2v \cdot \partial_x f^n - v \cdot \partial_x f^{n-1} + \frac{1}{\epsilon} E^{n+1} \cdot \partial_v f^{n+1}$$
$$= \frac{2}{\epsilon}\mathcal{Q}(f^n) - \frac{1}{\epsilon}\mathcal{Q}(f^{n-1}) - \frac{2\lambda}{\epsilon}(\rho^n M - f^n) + \frac{\lambda}{\epsilon}(\rho^{n-1} M - f^{n-1}) + \frac{\lambda}{\epsilon}(\rho^{n+1} M - f^{n+1}). \tag{5.47}$$

However, since the stiff terms contain a first derivative, it poses a very restrictive bound on λ for stability (here one condition we derived is $|\nabla_v \mathcal{Q}(f)| \leq \lambda \leq \min(3, \frac{5}{2} + \frac{\epsilon}{\triangle t})|\nabla_v \mathcal{Q}(f)|$) which might not be applicable in general cases, a better second order discretization in time is needed in the future work.

5.4 Numerical examples

In this section, we perform several numerical tests for the semiconductor Boltzmann equations with different collisions and in different asymptotic regimes. In the one dimensional examples, we use the following settings unless otherwise specified. The computational domain for x and v is $[0, L_x] \times [-L_v, L_v] = [0, 1] \times [-8, 8]$ with $N_x = 128$ grid points in x direction and $N_v = 32$ in v direction. The time step is chosen to be $\triangle t = \frac{\triangle x}{10}$ to satisfy the CFL condition $\triangle t \leq \frac{\triangle x}{\max_j |v_j|}$ in the transport part. Periodic boundary conditions in x will be used to avoid any difficulties that might be generated by boundary. The "M" is the absolute Maxwellian

$$M(v) = \frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2}}. \tag{5.48}$$

The permittivity $\varepsilon(x)$ in the Poisson equation (5.7) is taken to be $\varepsilon(x) \equiv 1$.

5.4.1 The time relaxation model

We first test the numerical method presented in section 5.3.1 for the simplest time relaxation model (5.13) with (5.25). The initial condition is taken as

$$\rho^0(x) = \frac{\sqrt{2\pi}}{2} \left(2 + \cos(2\pi x)\right), \quad \text{and } f^0(x, v) = \rho^0(x) M(v), \tag{5.49}$$

which is not at the local equilibrium. The electric field $E(t, x)$ satisfies the Poisson equation (5.7) with the doping profile

$$h(x) = \frac{\int_0^{L_x} \rho(x) dx}{1.2611} e^{\cos(2\pi x)}. \tag{5.50}$$

We show the time evolution of the asymptotic error defined as

$$errorAP^n = \sum_{l,m} |E_l \cdot \nabla_v f_l^n + M\rho_l^n - f_{lm}^n| \triangle x \triangle v, \tag{5.51}$$

where the derivative w.r.t. v is calculated by the spectral method. Figure 12 gives the error with ϵ decreasing by $\frac{1}{10}$ each time, which shows that the asymptotic error is of order ϵ, thus verifies the results in Proposition 7.

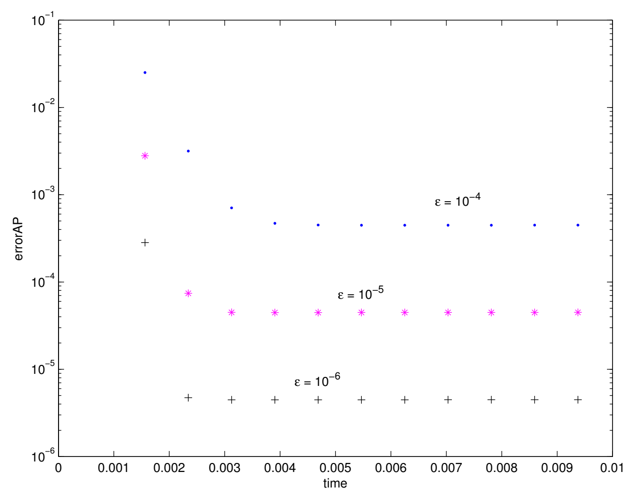

Figure 12: The time relaxation model coupled with the Poisson equation for the electric field. The time evolution of asymptotic error (5.51) for different ϵ with nonequilibrium initial data using the first order scheme in section 5.3.1.

5.4.2 The nondegenerate anisotropic case

In this section, we consider the nondegenerate anisotropic case with collision cross-section defined as

$$\phi(v, v') = 1 + e^{-(v-v')^2}, \tag{5.52}$$

and the initial condition is chosen the same as (5.49).

- Asymptotic property

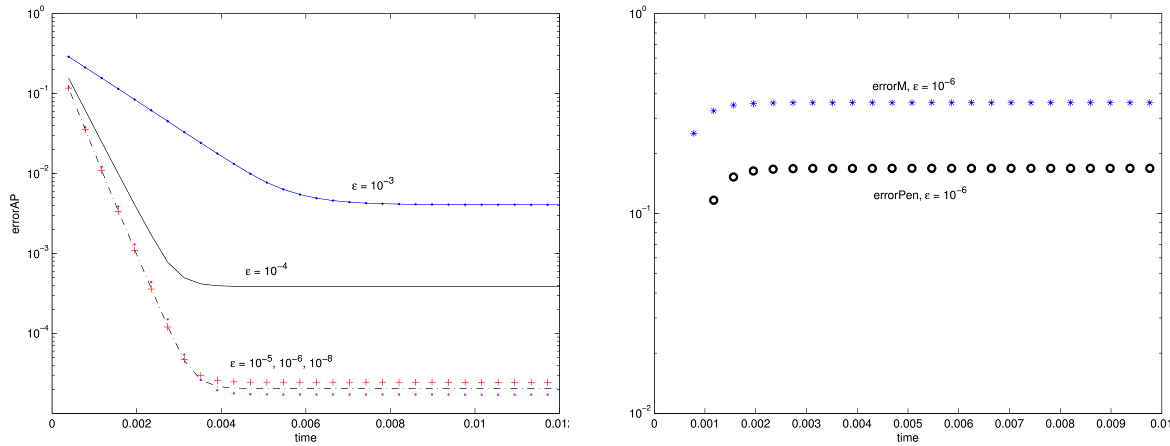

Figure 13: The nondegenerate anisotropic model with a fixed electrical field. The time evolution of asymptotic error (5.53) for different ϵ with nonequilibrium initial data (left), and a test of other errors (5.54) and (5.55) in comparison (right).

Consider fixed $E = 0.2$ at the moment. Figure 13 gives the relations between ϵ and the asymptotic error defined as

$$errorAP^n = \sum_{l,m} \left| E_l \cdot \nabla_v f_l^n + \mathcal{Q}(f)_{l,m}^n \right| \triangle x \triangle v, \tag{5.53}$$

where the derivative w.r.t. v is again calculated by the spectral method. The initial data is away from the equilibrium. It can be seen in Figure 13 that when ϵ is relatively large, the error is dominated by ϵ. However, when ϵ is small enough, the time step $\triangle t = 3.9063e-4$ will play a role so that the error will not decrease with the ϵ. The first order scheme is better performed asymptotically than we expected in Proposition 8 as the error observed in Figure 13 is smaller than $O(\triangle t)$.

To show that our scheme does not push f to the wrong Maxwellian, in Figure 13 we also plot the following two errors. One is defined as

$$errorPen^n = \sum_{l,m} \left| E_l \cdot \nabla_v f_l^n + \lambda(\rho M - f) \right| \triangle x \triangle v \tag{5.54}$$

to show that our penalization will not affect the asymptotic property. The other is the distance between f and the Maxwellian of the collision

$$errorM^n = \sum_{l,m} \left| f_{l,m}^n - \rho_l^n M \right| \triangle x \triangle v, \tag{5.55}$$

which is to show that our implicit treatment of the stiff force term necessarily accounts for the right asymptotic limit. It is shown that both errors stays large when ϵ is small, which means f will not be driven to either cases above when sending ϵ to 0.

- A Riemann problem

Consider the Riemann initial data to test the efficiency of the method:

$$\begin{cases} (\rho_l, h_l) = (1/8, 1/2), & 0 \le x < 1/4; \quad (5.56a) \\ (\rho_m, h_m) = (1/2, 1/8), & 1/4 \le x < 3/4; \quad (5.56b) \\ (\rho_r, h_r) = (1/8, 1/2), & 3/4 \le x \le 1. \quad (5.56c) \end{cases}$$

Initially $f^0(x,v) = \frac{\rho}{\sqrt{2\pi}} e^{-\frac{v^2}{2}}$ and let E be the solution of $-\nabla_x E = \rho - h$. Again periodic boundary condition in x direction is applied. ϵ is fixed to be 10^{-3}. For reference solution, we use the explicit second order Runge-Kutta discretization in time and MUSCL scheme for space discretization, with $N_x = 1024$, $N_v = 64$ and $\triangle t = \min(\triangle x/10, \epsilon \triangle v)/4 = 2.4414e-05$. Define the flux and energy as the first and second moments of f:

$$\text{flux} = \int_{-L_v}^{L_v} f v dv, \;\; \text{energy} = \int_{-L_v}^{L_v} f v^2 dv. \tag{5.57}$$

From Figure 14, one sees a good match between our solution and the reference solution.

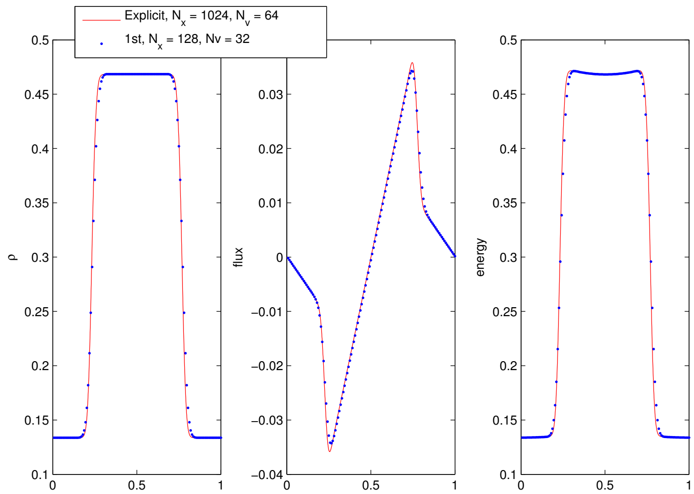

Figure 14: The plot of density, flux and energy at time $t = 0.2$ of the anisotropic nondegenerate case with (5.52) with E obtained by Poisson equation. The initial data is given in (5.56).

5.4.3 The degenerate case

In this section, we consider the degenerate case where the collision $\mathcal{Q}_{deg}$ is defined as (5.6).

- Asymptotic property

The initial condition is taken as

$$\rho^0(x) = \frac{\sqrt{2\pi}}{4}(2 + \cos(2\pi x)), \quad \text{and } f^0(x, v) = \rho^0(x) M(v) \tag{5.58}$$

to satisfy $0 \le f \le 1$. The electrical field E is obtained through the Poisson equation $-\nabla_x E = \rho - h$ with h given by (5.50). Again we compare the asymptotic error (5.53) with $\mathcal{Q}$ replaced by $\mathcal{Q}_{deg}$ for different orders of ϵ. As in the non-degenerate anisotropic case, the error is first dominated by ϵ and then by $\triangle t^\beta$ when ϵ is small enough, which

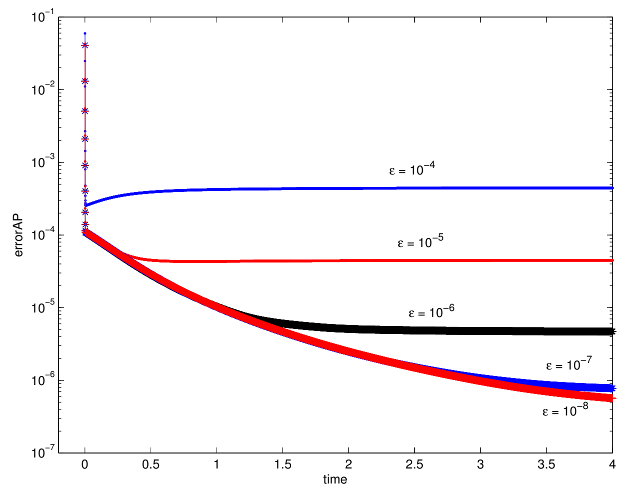

Figure 15: The degenerate isotropic model coupled with the Poisson equation. The time evolution of asymptotic error (5.53) with $\mathcal{Q}$ replaced by $\mathcal{Q}_{deg}$ for different ϵ with nonequilibrium initial data using the first order scheme in section 5.3.1.

is the same as was shown in section 5.3.3 (in section 5.3.3, β is shown to be 1, but numerically we get better result with $\beta > 1$), see Figure 15 where $\triangle t = 3.9063e - 4$.

- Mixing scales

To test the ability of our scheme for mixing scales, consider ϵ taking the following form:

$$\epsilon(x) = \begin{cases} \epsilon_0 + \frac{1}{2}(\tanh(5 - 10x) + \tanh(5 + 10x)) & x \leq 0.3; \\ \epsilon_0 & x > 0.3, \end{cases} \tag{5.59}$$

where $\epsilon_0 = 0.001$ so that it contains both the kinetic and high field regimes, see Figure 16. The initial condition is taken to be

$$f^0(x) = \frac{1}{6}(2 + \sin(\pi x))e^{-\frac{1}{2}v^2}. \tag{5.60}$$

Consider the anisotropic scattering where $\phi(v, v')$ is taken the same form as in (5.52). E is calculated through Poisson equation (5.7) with h given by (5.50). We use the second

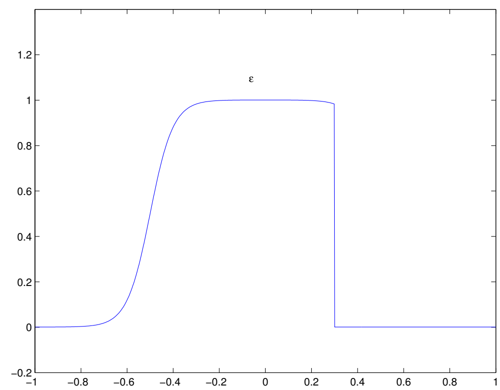

Figure 16: ϵ defined in (5.59).

order Runge-Kutta time discretization with the MUSCL scheme on a refined mesh to get the reference solution. Good agreements of these two solutions can be observed in Figure 17.

5.4.4 The electron-phonon interaction model

In this section, we consider a physically more realistic model, the electron-phonon interaction model, where the transition probability is

$$s(v, v') = K_0 \delta\left(\frac{v'^2}{2} - \frac{v^2}{2}\right) + K\left[(n_q + 1)\delta\left(\frac{v'^2}{2} - \frac{v^2}{2} + \hbar\omega_p\right) + n_q \delta\left(\frac{v'^2}{2} - \frac{v^2}{2} - \hbar\omega_p\right)\right], \tag{5.61}$$

and n_q given by $n_q = \frac{1}{e^{\frac{\hbar\omega_p}{K_B T_L}} - 1}$ is the occupation number of phonons. Here $\hbar$ is the planck constant, K_B is the Boltzmann constant, ω_p is the constant phonon frequency, T_L is the lattice temperature, K and K_0 are two constants for the material.

The singular nature of $s(v, v')$ makes the collision hard to compute numerically, but the cylindrical symmetry of s make it possible to use polar coordinates so that the

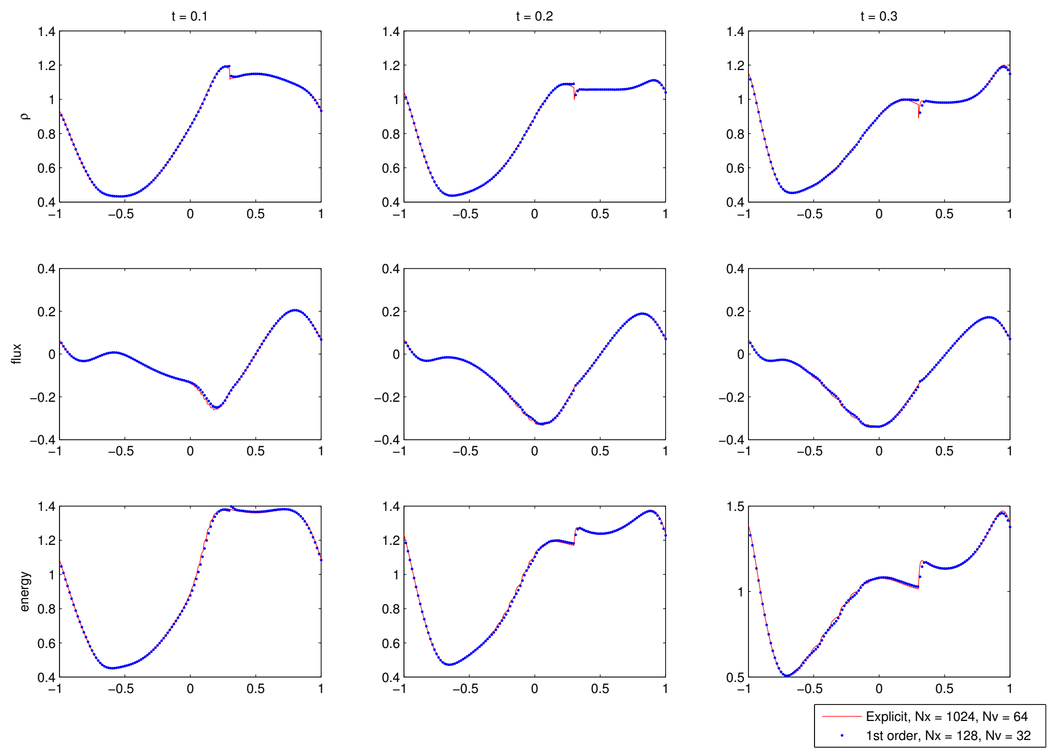

Figure 17: The degenerate anisotropic model coupled with the Poisson equation. Consider the mixing regimes with ϵ given in (5.59). Compare the first order scheme (5.26) on coarse mesh with an explicit method on refined mesh. We plot the macroscopic density, flux and energy at different times.

singularity in the delta function can be removed and the dimension of integral can be decreased by one [12, 13, 22]. However, this trick is not easy to be implemented here since we treat the stiff force term implicitly, and changing to polar coordinates will make it harder to invert. Instead, we use the spectral method [79] which can also remove the singularity.

In this numerical example, assume $d_x = 1$ and $d_v = 2$. Recall that the collision (5.3) can be written as

$$Q = Q^+(f)(t,x,v) - \nu(v)f(t,x,v). \tag{5.62}$$

Similar to [79], we restrict f on the domain $D_v = [-L_v, L_v]^2$ and extend it periodically to the whole domain. L_v is chosen such that the support of f is $\text{supp}(f) \subset B(0,R) = B_R$ and $L_v = 2R$. Similar to [79], approximate f by truncated Fourier serious

$$f(v) \approx \sum_{k=-Nv/2+1}^{N_v/2} \hat{f}_k e^{i\frac{\pi}{L_v}k\cdot v}, \qquad \hat{f}_k = \frac{1}{(2L_v)^2}\int_{D_v} f(v) e^{-i\frac{\pi}{L_v}k\cdot v} dv, \tag{5.63}$$

then $Q^+(f)$ is computed as follows

$$\begin{aligned} Q^+(f) &= \int_{B_R} S(v',v) f(t,x,v') dv' \\ &= \sum_{k=-Nv/2+1}^{N_v/2} \hat{f}_k \int_{B_R} e^{i\frac{\pi}{L_v}k\cdot v'} \left[(n_q+1)K\delta\left(\frac{1}{2}v^2 - \frac{1}{2}v'^2 + \hbar w_p\right)\right. \\ &\qquad \left. + n_q K\delta\left(\frac{1}{2}v^2 - \frac{1}{2}v'^2 - \hbar w_p\right) + K_0\delta\left(\frac{1}{2}v^2 - \frac{1}{2}v'^2\right)\right] dv'. \end{aligned} \tag{5.64}$$

Let $\xi' = \frac{1}{2}v'^2$, then change of variable $v' = \sqrt{2\xi'}(\cos\theta', \sin\theta')$ leads to

$$\begin{aligned} Q^+(f) &= \sum_{k=-Nv/2+1}^{N_v/2} \hat{f}_k \left[(n_q+1)K \int_0^{2\pi} e^{i|k|\sqrt{2(\xi+\hbar w_p)}\cos\theta' \frac{\pi}{L_v}} d\theta' \chi_{\xi+\hbar w_p \le \frac{1}{2}R^2} \right. \\ &\qquad + n_q K \int_0^{2\pi} e^{i|k|\sqrt{2(\xi-\hbar w_p)}\cos\theta' \frac{\pi}{L_v}} d\theta' \chi_{0\le\xi-\hbar w_p \le \frac{1}{2}R^2} \\ &\qquad \left. + K_0 \int_0^{2\pi} e^{i|k|\sqrt{2\xi}\cos\theta' \frac{\pi}{L_v}} d\theta' \chi_{\xi \le \frac{1}{2}R^2} \right] \\ &= \sum_{k=-Nv/2+1}^{N_v/2} \hat{f}_k B(|k|, |v|), \end{aligned} \tag{5.65}$$

with

$$\begin{aligned} B(|k|,|v|) = 2\pi &\left[(n_q+1)K J_0\left(\sqrt{2(\xi+\hbar w_p)}|k|\frac{\pi}{L_v} \right) \chi_{\xi+\hbar w_p \le \frac{1}{2}R^2} \right. \\ &\left. + K n_q J_0\left(\sqrt{2(\xi-\hbar w_p)}|k|\frac{\pi}{L_v} \right) \chi_{0\le\xi-\hbar w_p\le\frac{1}{2}R^2} + K_0 J_0\left(\sqrt{2\xi}|k|\frac{\pi}{L_v} \right) \chi_{\xi\le\frac{1}{2}R^2} \right], \end{aligned} \tag{5.66}$$

where J_0 is the Bessel function of order 0

$$J_0(\alpha) = \frac{1}{2\pi}\int_0^{2\pi} e^{i\alpha\cos\theta} d\theta.$$

In the same way, the collision frequency $\nu(v)$ can be computed as

$$\begin{aligned} \nu(v) &= \int_{B(0,L_v)} s(v,v')dv \\ &= 2\pi \left[K(n_q+1)\chi_{0\le\xi-\hbar w_p\le\frac{1}{2}R^2} + K n_q \chi_{\xi+\hbar w_p\le\frac{1}{2}R^2} + K_0 \chi_{\xi\le\frac{1}{2}R^2} \right]. \end{aligned} \tag{5.67}$$

Now let $L_v = 8$, $x \in [0,1]$, $K_B T_L = \frac{1}{2}$, $\hbar w_p = 1$, $K = 0$, and $K_0 = 1/5\pi$. Then the $2D$ Maxwellian is $M = \frac{1}{\left(\sqrt{\frac{\pi}{2K_B T_L}}\right)^2} e^{-\frac{v^2}{2K_B T_L}} = \frac{1}{\pi}e^{-v^2}$. We test two situations. One is for the pure high field regime with fixed $\epsilon = 10^{-3}$ and the initial data taking the form of (5.49) with M replaced by $\frac{1}{\pi}e^{-v^2}$. The macroscopic quantities at time $t = 0.2$ are given in Figure 18. The other is for mixing regimes problem, as ϵ defined the same as (5.59) but on the space interval $[0,1]$ and initial condition taken (5.60). To get better accuracy,

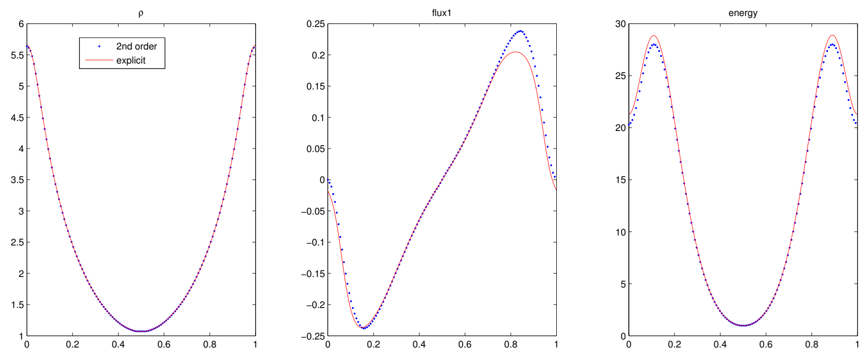

Figure 18: The macroscopic quantities for the electron-phonon interaction model with smooth initial data (5.49) and $\epsilon = 10^{-3}$: mass density (ρ), fluxes in v_1 (flux1) and v_2 (flux2) directions, and energy at time $T = 0.2$. $\epsilon = 10^{-3}$. Solid line: explicit method with $N_x = 1024$, $N_v = 32$. Dots: second order scheme (5.47) with $N_x = 128$, $N_v = 32$.

we use (5.47) and choose $\lambda = \max_v \nu(v)$ in this case which does not violate the stability constraint. See Figure 19 for the time evolution of the macroscopic quantities. The reference solution is calculated by the forward Euler method with second order slope limiter method for space discretization on a much finer mesh.

It can be checked that the collision frequency (5.67) meets the condition (5.19), but $\phi(v, v') = \frac{s(v,v')}{M(v')}$ does not belong to $W^{1,\infty}(\mathbb{R}^4)$ as assumed in Theorem 5.1. To our knowledge, no result is available numerically or analytically for the existence of the high field limit in this situation. From our numerical experiment, it seems to indicate that in this case, the solution does exist since our schemes capture it well in Figure 19. This is the first attempt to treat this problem in the high field regime, and we would like to put the designing of a fast efficient scheme in the future as well as the approximation of the runaway phenomenon that might be generated in this case.

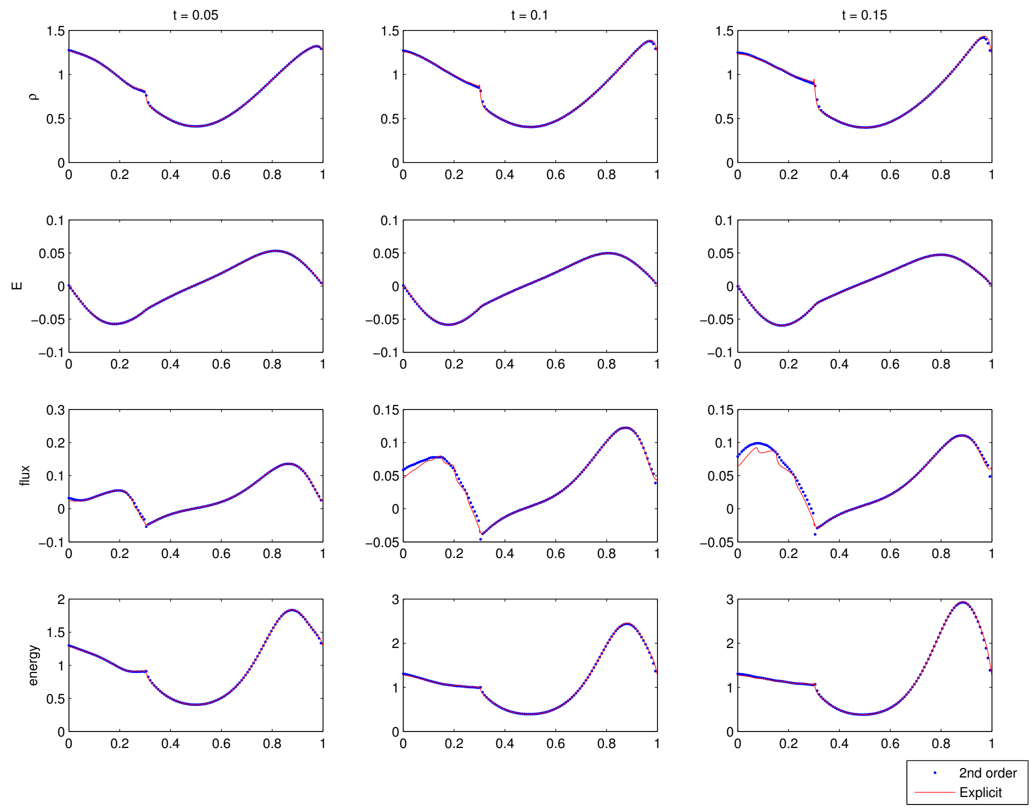

Figure 19: The time evolution of macroscopic quantities in electron-phonon interaction model in mix regimes (5.59) with initial data (5.60): mass density, electric field, flux in v_1 direction, and energy. Solid line: an explicit method with $N_x = 1024$, $N_v = 32$. Dots: the second order scheme (5.47) with $N_x = 128$, $N_v = 32$.

5.5 Concluding remarks

A numerical scheme for the semiconductor Boltzmann equation efficient in the high field regime has been introduced in this chapter. One difficulty in this problem is that there is no explicit form for the local equilibrium, which makes the asymptotic preserving methods hard to design. Our main idea is to penalize the collision term by a BGK operator and treat the stiff force term implicitly by the spectral method. The price to pay for this "wrong Maxwellian" penalization is merely the $\triangle t$ error in the asymptotic property (5.39) and (5.41). We also test our scheme to a physically more realistic model in the end, and give our explanation to an open question.

Chapter 6

Conclusion

6.1 The hyperbolic relaxation system

For the hyperbolic system with two-scale relaxations, it is much more efficient to compute the approximate scalar conservation law in the regime where the relaxation time is small. Motivated by [41], we propose a domain decomposition method for it. The interface condition is obtained according to the sign of the characteristic speed at the interface via formal asymptotic analysis. The derived system, completely decoupled, is proved to be asymptotically close to the original hyperbolic relaxation system. When the flux in the associated scalar conservation law is linear, we are able to represent the solution explicitly by the Laplace Transform so as to derive the asymptotic error of the domain decomposition system in L^2 sense. The idea of the proof follows that in [103] with new contributions on constructing auxiliary systems. On the other hand, for the general nonlinear flux, by use of the classical compactness argument, we first prove the existence and stability of the solution to the original system as well as its strong convergence to the entropy weak solution of the new decoupled system. Then by matched asymptotic analysis, we redo the derivation of the interface condition in a rigorous way, obtaining an extension of it to the case when there is a standing shock sticking to the interface.

Although we have a deep mathematical understanding of the simple 2×2 prototype model, there is still a long way to reach the more general coupling of the kinetic and hydrodynamic equations. An interesting attempt is to consider the coupling of two hyperbolic systems with different number of equations, for instance, the coupling between the Broadwell model and the corresponding hydrodynamic limit. Difficulties arise here because there are incoming and outgoing waves in both equilibrium and non-equilibrium regimes, making the resulting domain decomposition system impossible to be decoupled. And one characteristic speed of the Broadwell model being zero further complicates the boundary layer analysis [72, 71]. Another situation is untouched either, that is when shock passes through the interface, the boundary layer and shock layer may have some interesting interactions that is worth thinking about.

6.2 The high field limit

We construct numerical schemes for the Vlasov-Poisson-Fokker-Planck system and the semiconductor Boltzmann equation efficient in the high field regime. Numerical challenges come form the two stiff terms: the collision term and the force term. For the first system, we treat both terms implicitly at the same time based on the observation that the combination of them share the same structure as the Fokker-Planck operator, hence the symmetric discretization in [59] can be applied here. Such method is proved to have all the expected properties: mass conservation, positivity preservation, asymptotic preservation and stability. An extension to the second order is also obtained. For the second equation, another distinctive difficulty is that no explicit expression of the equilibrium is available. To overcome it, we adopt the idea from [36] by penalizing the collision term by a classical 'BGK' operator and treat the force term implicitly by the spectral method. This idea is applicable to both non-degenerate and degenerate cases. A compensate for the wrong "Maxwelliam" is merely the extra $O(\triangle t)$ error in the asymptotic property, the same conclusion as observed in [35].

The methods for the VPFP system is well-established, but for the semiconductor Boltzmann equation, there is some room to improve. A second order scheme that upgrade the $O(\triangle t)$ error to $O(\triangle t^2)$ is imperative. As mentioned in Chapter 5, we are trapped in the stability requirement for the magnitude of the penalty parameter, and we would like to find a good second order method with good stability in the near future. For the electron-phonon interaction model in section 5.4.4, we are interested in exploring further in constructing a fast efficient solver for the singular collision operator and a numerical approximation of the potential runaway phenomenon as well.

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