Differential Modules in Commutative Algebra

by
Maya Banks

A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
(Mathematics)

at the
University of Wisconsin-Madison
2024

Date of Final Oral Exam: 5/8/2024
The dissertation is approved by the following members of the Final Oral Committee:
Autumn Kent, Professor, Mathematics
Daniel Erman, Assistant Professor, Mathematics, University of Hawaiʻi at Mānoa
Michael Kemeny, Assistant Professor, Mathematics
Jordan Ellenberg, Professor, Mathematics

Differential Modules in Commutative Algebra

Maya Banks

Abstract

Differential modules generalize many familiar objects in commutative algebra. They arise naturally in connections with algebraic geometry and topology, and can also provide new insights and a fresh perspective on things as well-studied as free resolutions and complexes. In this thesis, we explore several homological and combinatorial properties of differential modules, comparing and contrasting with free resolutions. First, we study differential modules whose homology are a complete intersections by constructing a generalization of the Koszul complex. Second, we study differential modules as deformations of free complexes and obtain some results related to some rank conjectures in commutative algebra and algebraic topology. Finally, we study the graded Betti numbers of differential modules and propose a combinatorial theory mirroring that of Boij-Söderberg theory for minimal free resolutions. Throughout, we gain various insights on classical results and constructions in commutative algebra.

Dedication

In the last 10 months, thousands of students in Gaza have been murdered. Every university in Gaza has been destroyed. I submit this thesis with the solemn realization that this year in Gaza, no students are doing similarly and no degrees will be awarded. I dedicate this work to the Palestinian scholarship that will remain forever incomplete, the Palestinian degrees forever un-conferred, and the Palestinian lives brutally ended and forever altered by the genocidal Israeli state.

Declaration

I declare that the contents of this thesis reflect my own original research, except where otherwise indicated, and that all sources used have been properly cited. I further declare that I have not submitted the work contained herein as a thesis or similar document to any academic institution other than the University of Wisconsin–Madison, or in pursuit of any other degree.

Acknowledgements

First and foremost, thank you to my advisor Daniel Erman for generously sharing with me your guidance, support, and mathematical insight over the last six years. Thank you for your constructive, honest, and often extensive feedback, for imparting on me the importance of a well-chosen example, and above all for helping me to grow my confidence and sense of self as a mathematician.

Thanks as well to the other faculty, postdocs, and graduate students at UW who provided me with valuable mentorship and throughout graduate school, including but not limited to Michael Brown, Autumn Kent, Dima Arinkin, John Wiltshire-Gordon, Ola Sobieska, Juliette Bruce and Kit Newton. Thanks to the many members of the commutative algebra community beyond UW for professional, mathematical, and Macaulay2 support.

I also owe many thanks to my collaborators and throughout graduate school. Thank you Ola and Ritvik, and especially thanks to Keller, without whom many results in this thesis would never have seen the light of day.

This thesis would not have been possible without my friends in the math department. Thank you Colin, Alex, Alex, Yandi, Caitlyn, Caitlin, John, Boyana, Ivan, and everyone else, not just for your mathematical insights shared casually on the ninth floor or over drinks, or your late night study motivation, but above all for your camaraderie, your support, and your friendship.

Finally, immeasurable and innumerable thanks to my parents Judi and Matt, my sister Jess, and my partner James. Thank you and I love you.

Contents

Chapter 1

Introduction

1.1 Differential Modules

A *differential module* is a module equipped with a self-map that squares to zero. The reader will immediately note that this general definition encompasses a great many familiar mathematical objects such as resolutions and chain complexes, dg-algebras, pages of a spectral sequence, and more. Indeed, this simultaneous generalization is a key motivating feature of differential modules as objects of study: by investigating the structure and properties of differential modules, we can bring to light key similarities and differences between the various objects that they generalize. This helps us answer questions about which structures are actually crucial to some of our favorite results and constructions, which are merely convenient for the proof techniques, and which are simply superfluous.

Beyond this rather meta motivation, differential modules also arise as concrete objects in commutative algebra, algebraic geometry, and topology, and thus require the development of their own theoretical tools and techniques. A prime example of this in toric geometry comes in generalizing Koszul duality and the BGG correspondence to the nonstandard and multigraded setting, wherein one side of the correspondence involves differential modules rather than just complexes [BE21]. Another, to which we will return later on, is the Buchsbaum–Eisenbud–Horrocks conjecture and its connections to the Toral Rank Conjecture in algebraic topology [Car86; Hal85; BE77; Har79].

Let us consider an example of a *free differential module*—one where the underlying module is free. A free module endomorphism is given by a square matrix with entries in the ring, so this is how we will represent the differentials of free differential modules. Let $S = \mathbb{C}[x, y, z]$, the standard graded polynomial ring over the complex numbers. The module $M = S/(x, y, z) \simeq \mathbb{C}$ has minimal graded free resolution

$$\mathbf{F} = \quad 0 \longleftarrow S \xleftarrow{[x\ y\ z]} S(-1)^3 \xleftarrow{\begin{bmatrix} y & z & 0 \\ -x & 0 & z \\ 0 & -x & -y \end{bmatrix}} S(-2)^3 \xleftarrow{\begin{bmatrix} z \\ -y \\ x \end{bmatrix}} S(-3) \longleftarrow 0$$

We can consider $\mathbf{F}$ as a differential module by an operation called *folding*, where we collapse $\mathbf{F}$ into a single graded module and consider the differential on $\mathbf{F}$ as a single square-zero endomorphism.

$$\text{Fold}^0(\mathbf{F}) = \begin{matrix} S \\ \oplus \\ S(-1)^3 \\ \oplus \\ S(-2)^3 \\ \oplus \\ S(-3) \end{matrix}, \quad \begin{pmatrix} 0 & x & y & z & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & y & z & 0 & 0 \\ 0 & 0 & 0 & 0 & -x & 0 & z & 0 \\ 0 & 0 & 0 & 0 & 0 & -x & -y & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & z \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -y \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The "0" superscript indicates that the differential is a degree 0 map (this is true for the initial free resolution $\mathbf{F}$; since we introduced no degree shifts during folding, it remains true for the differential module). By imposing the appropriate degree shifts on the summands of the underlying module, we can use the folding operation to obtain from $\mathbf{F}$ a differential module whose differential is homogeneous of degree a for any $a \in \mathbb{Z}$. The differential will be represented by an identical matrix, but the degree shifts in the underlying module will change. For instance, in degrees 2 and -1 we get the following underlying modules upon applying the folding functor:

$$\mathrm{Fold}^2(\mathbf{F}) = \begin{matrix} S \\ \oplus \\ S(1)^3 \\ \oplus \\ S(2)^3 \\ \oplus \\ S(3) \end{matrix}, \qquad \mathrm{Fold}^{-1}(\mathbf{F}) = \begin{matrix} S \\ \oplus \\ S(-2)^3 \\ \oplus \\ S(-4)^3 \\ \oplus \\ S(-6) \end{matrix}$$

A recurring theme observed in the results in this thesis is that the degree of the differential module can have a substantial impact on various features of the differential module, including its numerical invariants. This is a stark departure from the world of graded free complexes, where we may always shift the degrees of individual free modules to assume that the degree of the differential is 0. For a differential module, the source and target of the differential are the same underlying module, so we can't shift degree on the source without similarly shifting the target.

We can also modify the differential to obtain new differential modules on the same underlying module by replacing some of the zeroes in the upper right corner of the matrix by nonzero elements of S in such a way that the matrix still squares to zero (a systematic exploration of this process is the content of Chapter 3). For example for each of the three folds of $\mathbf{F}$ given above, we could modify the differentials in as follows. In the examples below we use colors to indicate the block structure of the differential.

Example A (Degree 0)**.**

$$\begin{matrix} S \\ \oplus \\ S(-1)^3 \\ \oplus \\ S(-2)^3 \\ \oplus \\ S(-3) \end{matrix}, \quad \begin{pmatrix} 0 & x & y & z & x^2 & 0 & y^2 & z^3 \\ 0 & 0 & 0 & 0 & y & z & 0 & -y^2 \\ 0 & 0 & 0 & 0 & -x & 0 & z & 0 \\ 0 & 0 & 0 & 0 & 0 & -x & -y & -x^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & z \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -y \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Example B (Degree 2).

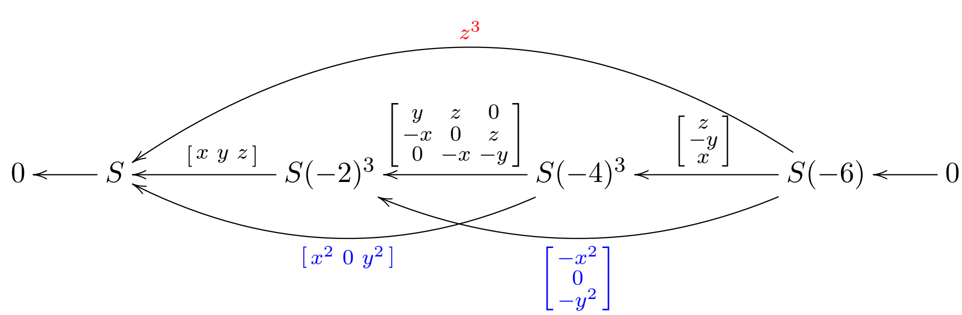
$$\begin{array}{c} S \\ \oplus \\ S(1)^3 \\ \oplus \\ S(2)^3 \\ \oplus \\ S(3) \end{array}, \quad \begin{pmatrix} 0 & x & y & z & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & y & z & 0 & 1 \\ 0 & 0 & 0 & 0 & -x & 0 & z & 0 \\ 0 & 0 & 0 & 0 & 0 & -x & -y & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & z \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -y \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Example C (Degree -1).

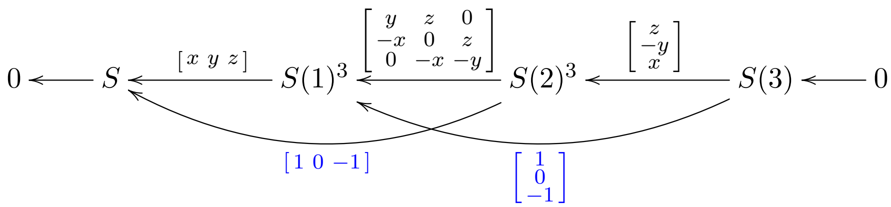
$$\begin{array}{c} S \\ \oplus \\ S(-2)^3 \\ \oplus \\ S(-4)^3 \\ \oplus \\ S(-6) \end{array}, \quad \begin{pmatrix} 0 & x & y & z & x^3 & 0 & y^3 & z^5 \\ 0 & 0 & 0 & 0 & y & z & 0 & -y^3 \\ 0 & 0 & 0 & 0 & -x & 0 & z & 0 \\ 0 & 0 & 0 & 0 & 0 & -x & -y & -x^3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & z \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -y \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Notice that the degree of the differential module determines the degrees of the elements that can be placed in various entries in the matrix, since a degree a differential module must have a differential that is homogeneous of degree a. Another way to visualize the three examples above is as the complex $\mathbf{F}$ with additional maps added.

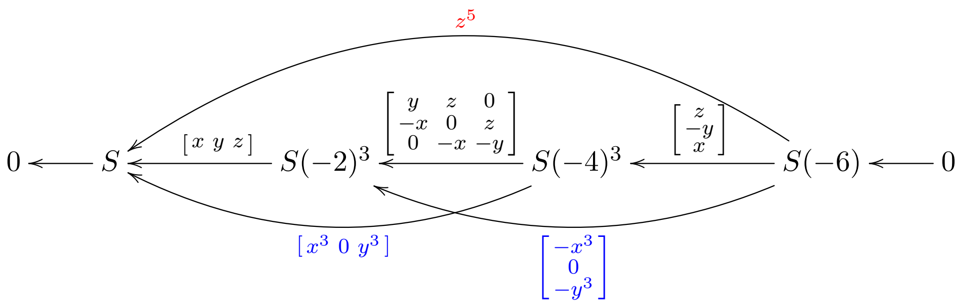
Example A.



Example B.



Example C.



Minimal free resolutions are objects that are both integral to and extremely well studied in commutative algebra and its connections with algebraic geometry. A central aim throughout this thesis is investigating how the well-known properties and behavior of minimal free resolutions lift to differential modules. For the remainder of this introduction, we explore this theme for the minimal free resolution **F** and the differential modules in Examples A, B, and C above.

1.2 Complete Intersections, Koszul Complexes, and DG Structures

The resolution **F** is a *Koszul complex* on the variables x, y, z. Koszul complexes are extremely well understood and are a valuable tool for generating examples of free complexes and minimal free resolutions of complete intersections, and for computing important invariants of modules (such as Koszul homology). Koszul complexes also have the structure of a *differential-graded (dg) algebra*. We can view **F** as an exterior algebra over $\mathbb{C}$: If $E = R^3$ with basis e_1, e_2, e_3, then **F** is equivalent as a dg algebra to $\bigwedge^\bullet E$ where the differential takes an element $\mathbf{a}$ to $\mathbf{a} \wedge (ze_1 - ye_2 + xe_3)$. In other words, the differential corresponds to multiplication by a certain exterior element of degree 1 (or -1 depending on the grading convention).

We can view the differentials in Examples A, B, and C above as each incorporating not just a single exterior element, but a sequence of compatible exterior elements.

Example A. Define exterior elements f_1, f_2, f_3 to be

$$f_1 = ze_1 - ye_2 + xe_3 \in \bigwedge^1 E$$
$$f_2 = -y^2 e_1 \wedge e_2 - x^2 e_2 \wedge e_3 \in \bigwedge^2 E$$
$$f_3 = z^3 e_1 \wedge e_2 \wedge e_3 \in \bigwedge^3 E.$$

Multiplication by f_i defines a map $\bigwedge^j E \to \bigwedge^{j+i} E$. Combining these three multiplication maps along with a sign convention defines the degree 0 differential on $\bigwedge^\bullet E$ given by the matrix in Example A above. Examples B and C can be obtained similarly, and we note that this construction is independent of the grading on S, using only the grading of the exterior algebra $\bigwedge^\bullet E$.

This construction is an example of what we define in Chapter 2 as a *Koszul differential module*. In general, this construction may break the dg algebra structure (i.e. the differential defined may not satisfy the *Leibniz rule* required for a dg algebra). Examples A and C can both be given the structure of a dg algebra, while there is no way to define such a structure on Example B. This is because Examples A and C happen to be isomorphic to $\mathrm{Fold}^0(\mathbf{F})$ and $\mathrm{Fold}^{-1}(\mathbf{F})$, respectively, and thus simply inherit their dg structure from $\mathbf{F}$. We show that a Koszul differential module can actually *only* be given a dg algebra structure if it is isomorphic to a fold of a complex with such a structure. In fact, Koszul differential modules *not* isomorphic to folds of such complexes can not even have the structure of a dg-*module* over the minimal free resolution of their homology. In other words, the theory of dg structures on minimal free resolutions is in some respect special to the case of complexes, requiring crucial structure that is lost upon passage to the category of differential modules.

Our study of Koszul differential modules is motivated in part by the question of somehow classifying differential modules that have a certain homology. Since the Koszul complex gives a minimal free resolution of a *complete intersection*, one goal of the Koszul differential module construction is to understand in general the differential modules whose homology is isomorphic to a complete intersection. All three of our running examples so far constitute Koszul differential modules with homolgoy $S/(x, y, z)$, but in general there are some additional assumptions needed to guarantee that a differential module with complete intersection homology is Koszul. The theory of Koszul differential modules built up in Chapter 2 constitutes a partial characterization of differential modules with complete intersection homology, and in Chapter 3 we take a totally different approach to obtain a much more general classification.

1.3 Deformations and Total Rank

We can also study the *Betti numbers* of $\mathbf{F}$, which are the ranks of the free modules appearing in the resolution. Buchsbaum–Eisenbud and Horrocks [BE77; Har79] asked whether the rank of the i^{th} free module in a minimal free resolution of M has rank at least $\binom{\operatorname{codim} M}{i}$, a question now referred to as the Buchsbaum–Eisenbud–Horrocks (BEH) Conjecture. The resolution $\mathbf{F}$ is a minimal free resolution of $S/(x, y, z)$, which has codimension 3, so we notice that $\mathbf{F}$ achieves the predicted bound. Walker [Wal17] and Walker–VandeBogert [VW24] proved a slightly weaker version of the BEH conjecture known as the Total Rank Conjecture, which states that the *sum* of the ranks of the modules appearing in the minimal free resolution of M is at least $2^{\operatorname{codim} M}$.

Carlsson linked the BEH Conjecture and the Total Rank Conjecture to a set of conjectures in algebraic topology, known as different versions of the *Toral* Rank Conjecture [Car86]. This conjecture posits that if a rank n torus—either $(\mathbb{Z}/p\mathbb{Z})^n$ or $(S^1)^n$—acts nicely (this means freely or semi-freely) on a nice topological space X (this means a finite CW complex or a smooth manifold), then the homology groups of X satisfy

$$\sum \operatorname{rank}_{\Bbbk} H_i(X; G) \geq 2^n$$

where $\Bbbk$ is either $\mathbb{Z}/p\mathbb{Z}$ or $\mathbb{Q}$ and G is $(\mathbb{Z}/p\mathbb{Z})^n$ or $(S^1)^n$, respectively. The p-group version of this conjecture is due to Carlsson, while the rational version was conjectured by Halperin [Hal85].

Carlsson posed a stronger algebraic conjecture that would imply the Toral Rank Conjecture. His original formulation was only for $\Bbbk = \mathbb{F}_2$, but we state it here for any $\Bbbk$, since it's just as false either way.

Conjecture 1.3.1. *If D is a free graded differential module of degree 1 over $S = \Bbbk[x_1, \ldots, x_n]$ with finite length homology, then* $\operatorname{rank}_S D \geq 2^n$.

Iyengar–Walker provided a counterexample to the above in the form of a free *complex* with smaller than expected total Betti numbers [IW18] (when the characteristic of the field is not equal to 2); however, their counterexample does not come from topology. The question of *which* differential modules satisfy the 2^n bound remains open, as does the question of what a tight lower bound on total rank of a differential module with finite length homology should be.

Let us return to **F** as our example. The homology of **F** is finite length, as is the homology of $\operatorname{Fold}^a(\mathbf{F})$ and each of the slightly altered differential modules above, all of which have total rank $8 = 2^3$. However, the differential in Example B is not minimal (i.e. it contains a unit). By performing a series of (nonobvious) row and column operations, we can find that Example B is isomorphic to a differential module with the same underlying module but differential given by the matrix

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & xy & xz & 0 & z-x \\ 0 & 0 & 0 & 0 & -y^2 & -yz & z-x & \\ 0 & 0 & 0 & 0 & -y^3 & -y^2z & yz & y^2 \\ 0 & 0 & 0 & 0 & xy^2 & xyz & -xz & -xy \end{pmatrix}$$

The upper left block is *trivial*, that is it contributes nothing to the homology of the differential module. Thus we have a differential module of rank 4 whose differential is given by the matrix

$$\begin{pmatrix} xy & xz & 0 & z-x \\ -y^2 & -yz & z-x & 0 \\ -y^3 & -y^2z & yz & y^2 \\ xy^2 & xyz & -xz & -xy \end{pmatrix} \tag{$*$}$$

and whose homology is still isomorphic to $\mathbb{C}$. On the other hand, notice that Examples A and C have no scalar entries in their differentials. This means we cannot use a similar process to obtain a differential module of smaller rank with the same homology.

In Chapter 3, we explore this phenomenom using deformation theoretic techniques to characterize differential modules with given homology (up to a certain type of quasi-isomorphism). The main result of this chapter is a geometric parameterization of *flagged quasi-isomorphism* classes of differential modules with a given homology. This parameterization allows us to compute explicit bounds on the dimension of the space of such differential modules. In particular, we have a criteria for checking whether any differential modules with homology M exist that are not quasi-isomorphic to the fold of the minimal free resolution of M. Among the consequences of our main results is the following:

Corollary 1.3.2. *Any graded differential module over* $S = \mathbb{C}[x_1, \ldots, x_n]$ *with homology* $\mathbb{C}$ *that has rank* $< 2^n$ *must have degree 2.*

This is a special case of a more general phenomenon:

Corollary 1.3.3. *If* M *is an* S*-module of finite length, then degree* a *differential modules with homology* M *and rank* $< 2^n$ *exist for only finitely many* $a \in \mathbb{Z}$*.*

1.4 Graded Betti Numbers and Boij-Söderberg Theory

Incorporating the graded data in $\mathbf{F}$, we can get a finer invariant: the *graded Betti numbers* of the module M, which we represent in a *Betti table*. A famous theory, first conceived by Boij-Söderberg [BS08] and first proven by Eisenbud, Schreyer, and Weyman [EFW11; ES09a; ES09b], combinatorially characterizes all possible Betti tables of finite length modules over S, up to rational multiple. This characterization comes in terms of what are called *pure resolutions*—resolutions where each homological degree contains a free module generated in only one internal degree (for instance, $\mathbf{F}$ is pure). In particular, the Betti tables of finite length modules over S span a rational cone whose extremal rays are generated by the Betti tables of pure resolutions of finite length modules.

Differential modules lack the data of a homological grading, but they still have a notion of a minimal free resolution [BE22], from which we can define the Betti numbers: The j^{th} Betti number is the number of generators of the minimal free resolution in degree j. We record the

Betti numbers in a 1 dimensional array, the *Betti vector*. For some examples, the respective Betti vectors of Examples A, B, and C are

$$A\colon (1^\circ, 3, 3, 1), \quad B\colon (2, 2, 0^\circ) \quad C\colon (1^\circ, 0, 3, 0, 3, 0, 1)$$

where the symbol $\circ$ denotes the 0^{th} entry. Note in particular the difference between Example B and the other two: Examples A and C are their own minimal free resolutions, so we can compute their Betti vectors by counting the number of generators in each degree. Example B is not minimal, but it's minimal free resolution has differential given by $(*)$ and underlying module $S(1)^2 \oplus S(2)^2$ (this is not obvious, but it would be if we were to actually trace through the row and column operations used to transform the original matrix given in Example B into the block matrix where $(*)$ appears).

For each degree a, the folding functor descends to a map on Betti tables, taking the Betti table $(\beta)_{i,j}$ of M to the Betti vector of the degree a fold of the minimal free resolution of M. We refer to this as "flattening" the Betti table, which should be visualized as collapsing the 2-dimensional Betti table to a 1-dimensional Betti vector along a line of a certain slope in the Betti table which depends on a. The Betti vectors of degree a differential modules over S with finite length homology also form a cone. While every Betti *table* flattens to a valid Betti *vector*, the converse is not true: not every Betti vector of a degree differential module with finite length homology can be obtained by flattening the Betti table of a finite length module over S. For example, while the Betti vectors of Example A and Example C are indeed flattenings of the Betti table of $\mathbf{F}$ (in degrees 0 and -1, respectively), the Betti vector of Example B cannot be obtained as the flattening of any Betti table in any degree (indeed, we can check this using the the lower bound on the total rank of a minimal free resolution–the sum of the entries in the Betti vector is less than 2^3, so this can't possibly be a flattening of a Betti table of a finite length module!).

In Chapter 4, we further investigate the combinatorics of the cone of Betti vectors of differential modules with finite length homology over S. We conjecture that while the flattening map is *not* an surjection onto the set of Betti vectors, it *is* a surjection onto extremal rays of the Betti cone (in other words, that all of the "new" Betti vectors appear in the interior of the Betti cone). We prove the conjecture for differential modules of strictly positive degree and for differential modules of degree 0 over $\mathbb{k}[t]$. Additionally we prove that there is a categorical pairing on the level of derived categories that allows us to relate Betti vectors of differential modules with cohomology of vector bundles on projective space.

1.5 What's to come

The remaining chapters of this thesis consists of three distinct papers which have been included in their entirety and may be read independently. These papers are included as they appear on the arXiv, with only minimal edits for formatting purposes. Each chapter contains its own introduction and background sections, thus there is redundancy in the preliminary content of each chapter. Chapters 2 and 3 consist of joint work with Keller VandeBogert.

Chapter 2

Differential Modules with Complete Intersection Homology

Abstract

Differential modules are natural generalizations of complexes. In this paper, we study differential modules with complete intersection homology, comparing and contrasting the theory of these differential modules with that of the Koszul complex. We construct a Koszul differential module that directly generalizes the classical Koszul complex and investigate which properties of the Koszul complex can be generalized to this setting.

2.1 Introduction

A *differential module* is a module equipped with a square-zero endomorphism. While initially introduced at least as far back as the classical treatise of Cartan and Eilenberg [CE16, Chapter 4], differential modules have become a topic of recent interest in commutative algebra motivated in part by their connections to the Buchsbaum–Eisenbud–Horrocks and Carlsson conjectures [ABI07; Car86; BE77; Har79], the BGG correspondence and Tate resolutions [BE21], and the representation theory of algebras ([Rou06], [RZ17]). Differential modules are a natural generalization of chain complexes, and their study can thus provide a novel perspective on familiar objects such as free resolutions. More generally, there is an ever-expanding literature focusing on the use of differential modules to provide new insight on old conjectures (see for instance [BD10], [ŞÜ19], and [IW18]), and also on the development of a general theory of differential modules for their own sake (see [Sta17], [Wei15], and [XYY15], and the references therein).

Our work is motivated in particular by recent work of Brown and Erman [BE22], in which they extend the notion of a *minimal free resolution* to differential modules. Moreover, they prove a theorem indicating that the classical theory of minimal free resolutions still plays a significant role in understanding the structure of minimal free resolutions of differential modules. In particular, they show that for a differential module D with homology $H(D)$, there is a *free flag* F whose structure and differential are partially controlled by the minimal free resolution of $H(D)$ and where there is a quasi-isomorphism $F \to D$ (see Theorem 3.2.11 for the precise statement). This result begs the question of whether properties of the minimal free resolution of the homology $H(D)$ can be 'lifted' to the free flag F. Brown and Erman examined this in the case where the homology is a Cohen-Macaulay codimension 2 algebra. We explore this question in the case where $H(D)$ is the quotient by an ideal generated by a regular sequence.

In the classical theory of minimal free resolutions, the Koszul complex is one of the most fundamental objects of study for the simple reason of its sheer ubiquity; it is well-known to be a minimal free resolution of ideals generated by regular sequences, but even for non-complete intersections, properties of the Koszul homology of an ideal make their appearance in relation to DG-algebra techniques, the study of Rees algebras, and in the construction of more subtle types of complexes. In this paper, we begin developing a parallel theory for differential modules by first constructing a differential module analog of the Koszul complex, directly generalizing the classical case. We also investigate which properties of the Koszul complex lift to the minimal free resolution of differential modules whose homology is a complete intersection. We ask three main questions, the first of which is the following:

Question 2.1.1. *For R a graded-local ring, what are the differential modules D whose homology is equal to the residue field $R/\mathfrak{m}$? More broadly, can we classify the differential modules with homology R/I a complete intersection?*

Example 2.1.2. Let $S = k[x_1, \ldots, x_n]$ for k a field. Let $D = S^4$ with differential given by the matrix

$$\begin{pmatrix} 0 & x_1 & x_2 & 0 \\ 0 & 0 & 0 & -x_2 \\ 0 & 0 & 0 & x_1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

This differential module has homology $S/(x_1, x_2)$. In fact, it is the differential module we obtain by taking the Koszul complex on the regular sequence (x_1, x_2) and viewing it as a differential module whose underlying module is the sum of the free modules in the Koszul complex and whose differential is the direct sum of the Koszul differentials. However, we can alter the differential by adding a nonzero entry to the top right corner without changing the homology of the differential module. That is, we get a family of differential modules D_f with the same underlying module and differential given by

$$\begin{pmatrix} 0 & x_1 & x_2 & f \\ 0 & 0 & 0 & -x_2 \\ 0 & 0 & 0 & x_1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Furthermore, by [BE22], every differential module with homology $S/(x_1, x_2)$ admits a quasi-isomorphism from some such D_f. However, not all choices of f yield nonisomorphic differential modules. For instance for $f \in (x_1, x_2)$ we can perform row and column operations to show that D_f is isomorphic to our original D, but this is not the case if (for instance) $f = 1$.

In the above example, we see the structure of the Koszul complex itself mirrored in the structure of the differential modules D_f. This motivates our next two questions, which have to do with how much of this structure is actually preserved when we pass from resolutions to differential modules.

Question 2.1.3. *The differential of the Koszul complex corresponds to a multiplication by a single element in a particular exterior algebra. For a differential module D with complete intersection homology, does every free flag resolution of the type described in [BE22] arise via a similar construction?*

Question 2.1.4. *The classical Koszul complex is well-known to admit the structure of a DG-algebra. Does the generalization of the Koszul complex to differential modules admit any kind of analogous structure?*

Our results give a partial answer to Question 2.1.1—while we do not give a classification of *all* differential modules with complete intersection homology, we do find constraints on such differential modules and prove results that simplify the classification question. We show that with some additional hypotheses, Question 2.1.3 can be answered affirmatively (see Theorem 2.4.9), but that absent these hypotheses we can construct examples of free flags with complete intersection homology that are not of the "expected" form. In contrast to the previous two questions, the answer to Question 2.1.4 seems to be a resounding "no", and indicates that generalizations of the classical notions of DG-algebra/module structures for differential modules will require much subtler formulations.

2.1.1 Results

Let R be a commutative graded local ring, D a module over R and $d : D \to D$ an R-module endomorphism that squares to 0. We define the homology of D to be $H(D) = \ker(d)/\operatorname{im} d$. If the underlying module D has the form $\bigoplus_{i\geq 0} F_i$ where F_i is free and the differential d satisfies that $d(F_j) \subseteq \bigoplus_{j<i} F_i$ then we call D a *free flag*. Note that any bounded below free complex can automatically be considered as a free flag. A core result of [BE22] states that any differential module D with finitely generated homology admits a quasi-isomorphism from a free flag (F, d) where

$$F_0 \xleftarrow{\delta} F_1 \xleftarrow{\delta} F_2 \leftarrow \cdots$$

is a minimal free resolution of $H(D)$ and the d restricts to δ when considered as a map $F_i \to F_{i-1}$. In this case, we can represent d via a block matrix

$$\begin{pmatrix} 0 & \delta & A_{2,0} & A_{3,0} & \cdots & A_{m,0} & \cdots \\ 0 & 0 & \delta & A_{3,1} & \cdots & A_{m,1} \cdots & \\ \vdots & & & \ddots & & \vdots & \\ 0 & 0 & 0 & 0 & \cdots & A_{m,m-2} & \cdots \\ 0 & 0 & 0 & 0 & \cdots & \delta & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \end{pmatrix}$$

where $A_{i,j} : F_i \to F_j$. We refer to free flags of this type as *anchored free flags*. Since every differential module admits an anchored free flag resolution, it is both convenient and reasonable to focus on differential modules of this form. To classify anchored free flag resolutions amounts to classifying the possibilities for $A_{i,j}$ that yield nonisomorphic differential modules. Our first result simplifies this task in the case where the homology is a complete intersection.

Theorem 2.1.5. *Let D be a free flag differential module with homology $H(D)$ a complete intersection where the differential d is given by a block matrix as above. Let F be the minimal free resolution of $H(D)$ considered as a differential module. Then $D \cong F$ if and only if $\operatorname{im} A_{i,0} \subseteq \operatorname{im} \delta$ for all $i \geq 2$.*

This result says that, in the case of a homology induced free flag with complete intersection homology, the property of being isomorphic to a minimal free resolution of the homology can be detected by only the first row of blocks in the differential. This further tells us that we can at least partially characterize the differential modules with given complete intersection homology R/I by the choices of nonzero maps $F_i \to R/I$.

We prove Theorem 2.1.5 in Section 2.3, along with a discussion of minimal free resolutions of differential modules with homology k. In particular, we also show via explicit construction that the total Betti numbers of a differential module with homology k may be strictly smaller than the sum of the Betti numbers of k.

In Section 2.4, we show how graded commutative algebras admitting divided powers can be used to construct differential modules. This allows us to construct a Koszul differential module– a family of differential modules generalizing the Koszul complex (see Construction 2.4.3). In much the same way as the Koszul complex provides a valuable source of examples in the study of minimal free resolutions, Koszul differential modules generate a large set of examples of free flag differential modules. Moreover, we prove the following theorem which shows that in certain cases we can guarantee that anchored free flags with complete intersection homology are isomorphic to the Koszul differential module.

Theorem 2.1.6 (See Theorem 2.4.9 for the more general statement)**.** *Let R be a Noetherian graded local ring with maximal ideal $\mathfrak{m}$, D a differential R-module with $H(D)$ a complete intersection and let $F = \bigoplus_{i\geq 0} F_i \to D$ be an anchored free flag resolution with differential d^F. If $\operatorname{im}(d^F) \cap F_0$ is generated by a regular sequence, then F is isomorphic to a Koszul differential module.*

Finally, in Section 2.5 we consider the existence of DG-module structures on free flag resolutions as posited in Question 2.1.4. This leads us to consider free flag resolutions that can be given the structure of a DG-module over the minimal free resolution of their homology. In the classical case of complexes, it is well-known that every free resolution admits the structure of a (possibly non-associative) DG-algebra structure, and hence the existence of such a structure is guaranteed. Our main result of this section is the following theorem which says that the existence of such a DG-module structure on an arbitrary free flag resolution is in fact a much rarer property.

Theorem 2.1.7. *Let F be an anchored free flag with complete intersection homology. If F admits the structure of a DG-module over the minimal free resolution of $H(F)$, then F is isomorphic to the Koszul complex considered as a differential module.*

This theorem gives an example of a property of free resolutions that does *not* generalize to the setting of differential modules. The inability to generalize this property tells us that the DG-algebra structure of free resolutions, at least for complete intersections, relies on structure that is unique to free resolutions rather than structure that can be extended to free flags. This is in contrast with properties that are successfully generalized to free flags in [ABI07] and [BE22].

2.2 Background

In this section, we introduce some background and notation on differential modules that will be used throughout the paper. This includes a straightforward reformulation of free flag differential modules without reference to matrices, which will be useful for avoiding complicated matrix computations; this formulation is implicit in the work of Avramov, Buchweitz, and Iyengar [ABI07], but we state it here since it will be used frequently throughout the paper.

Notation. Throughout this paper, R will denote a graded local ring. More precisely, $R = \bigoplus_{i \geq 0} R_i$ is an $\mathbb{N}$-graded Noetherian ring with R_0 local. Moreover, all differential modules throughout this paper will be assumed to have finitely generated homology.

Definition 2.2.1. A *differential module* (D, d) or (D, d^D) is an R-module D equipped with an R-endomorphism $d = d^D \colon D \to D$ that squares to 0. A differential module is $\mathbb{Z}$-graded of degree a if D is equipped with a $\mathbb{Z}$ grading over R such that $d\colon D \to D(a)$ is a graded map.

The *homology* of a differential module (D, d) is defined to be $\ker(d)/\operatorname{im}(d)$. If D is $\mathbb{Z}$-graded of degree a, then the homology is defined to be the quotient $\ker(d)/\operatorname{im}(d(-a))$.

A differential module is *free* if the underlying module D is a free R-module. The differential module D is *minimal* if $d \otimes k = 0$.

A morphism of differential modules $\phi\colon (D, d^D) \to (D', d^{D'})$ is a morphism of R-modules $D \to D'$ satisfying $d^{D'} \circ \phi = \phi \circ d^D$. Notice that morphisms of differential modules induce well-defined maps on homology in an identical fashion to the case of complexes. A morphism of differential modules is a *quasi-isomorphism* if the induced map on homology is a quasi-isomorphism.

The category of degree a differential R-modules will be denoted $\mathrm{DM}(a, R)$. The notation $\mathrm{DM}(R)$ will denote the full category of differential modules (without any concern for grading).

Remark 2.2.2. The collection of differential modules and their morphisms forms a category, denoted $\mathrm{DM}(R)$. Notice that a differential R-module (D, d) is equivalently a module over the ring $R[x]/(x^2)$, as mentioned in the introduction. In particular, the category $\mathrm{DM}(R)$ is equivalently the category of $R[x]/(x^2)$-modules, and as such $\mathrm{DM}(R)$ is an abelian category. The category $\mathrm{DM}(a, R)$ is also equivalently graded modules over $R[x]/(x^2)$ in the case that x is given degree

a.

The following definition will play an essential role throughout the paper, and it allows us to view the category of complexes as a subcategory of the category of differential modules, though it is important to note that this is *not* necessarily a full subcategory:

Definition 2.2.3. Given any complex F, there is a functor

$$\begin{aligned}\mathrm{Fold}\colon \mathrm{Com}(R) &\to \mathrm{DM}(R),\\ (F, d^F) &\mapsto \Big(\bigoplus_{i\in\mathbb{Z}} F_i, \bigoplus_{i\in\mathbb{Z}} d_i^F\Big),\\ \{\phi\colon F_\bullet \to G_\bullet\} &\mapsto \Big\{\bigoplus_{i\in\mathbb{Z}} \phi_i\colon\ \mathrm{Fold}(F_\bullet) \to \mathrm{Fold}(G_\bullet)\Big\}.\end{aligned}$$

The object $\mathrm{Fold}(F_\bullet)$ will often be referred to as the *fold* of the complex $F_\bullet$.

The following definition introduces *free flags*. These are a proper subclass of differential modules that still generalize complexes of free modules, and in general are better behaved than arbitrary differential modules. One way to think of free flags is as differential modules admitting a finite length filtration whose associated graded pieces are themselves free R-modules.

Definition 2.2.4. Let D be a differential module. Then D is a *free flag* if D admits a splitting $D = \bigoplus_{i\in\mathbb{Z}} F_i$, where each F_i is a free R-module, $F_i = 0$ for $i < 0$, and $d_D(F_i) \subseteq \bigoplus_{j<i} F_j$.

Given a free flag D with associated splitting $D = \bigoplus_{i\in\mathbb{Z}} F_i$, define $D^i \coloneqq \bigoplus_{j\leq i} F_j$. This will be referred to as the *flag filtration* on F. By definition of a free flag, one has $d_D(D^i) \subset D^{i-1}$, implying that the associated graded object associated to the flag filtration is a chain complex.

Associated to a free flag D, there are maps $A_{i,j}\colon F_i \to F_j$ induced by splitting the maps $d_D\colon F_i \to D^{i-1}$ with the isomorphism $\mathrm{Hom}(F_i, D^{i-1}) = \bigoplus_{j<i} \mathrm{Hom}(F_i, F_j)$.

A core theme of [ABI07] is that the flag structure on a differential module allows many proofs from classical homological algebra to be generalized to differential modules by substituting the homological grading for the grading induced by the flag filtration. As such, it is useful to pass from general differential modules to free flags, which we can do using the following definition.

Definition 2.2.5. For any differential module D, a *free flag resolution* F of D is a free flag F equipped with a quasi-isomorphism $F \to D$. A *minimal free resolution* of D is a quasi-isomorphism $M \to D$ that factors through a free flag resolution F such that $M \to F$ is a split injection and M is minimal.

Remark 2.2.6. Notice that if $F_\bullet \to M$ is a minimal free resolution of a module M, then $\mathrm{Fold}(F_\bullet) \to M$ (where M is viewed as having the 0 endomorphism) is a minimal free flag resolution of M (since $H(\mathrm{Fold}(F_\bullet)) = \bigoplus_{i \in \mathbb{Z}} H_i(F_\bullet)$). In general, there may be free flag resolutions of M that do not arise as the fold of a complex, and it is an interesting question as to when a free flag is isomorphic to the fold of some complex. We give a characterization of this property for certain classes of free flag resolutions in Section 3 which turns out to be quite effective in proving some general statements about free flag resolutions.

One way to think of free flags is as strictly upper triangular block matrices (as in the setup to Theorem 2.1.5). This can be useful for explicit computations and more matrix-theoretic methods, but we will also find it useful to think of free flags in a way that is not reliant on matrices. The following observation is a coordinate-free reformulation of the definition of a free flag that highlights the data of the maps $A_{i,j} \colon F_i \to F_j$ which determine the flag.

Observation 2.2.7. *A free flag is equivalently the data of a collection of free modules* $\{F_i \mid i \in \mathbb{Z}_{\geq 0}\}$ *and maps* $\{A_{i,j} \colon F_i \to F_j \mid j < i\}$, *such that for all* $j < i$, *one has the relation*

$$\sum_{j<k<i} A_{k,j} A_{i,k} = 0.$$

Remark 2.2.8. In order to distinguish the structure maps $A_{i,j}$ given in Observation 2.2.7, we will often use the more precise notation $A^D_{i,j}$ to specify that these maps determine the differential module D.

The theory of minimal free resolutions of arbitrary differential modules turns out to be quite subtle. However, the following result of Brown and Erman shows that the classical theory of minimal free resolutions of modules still plays an important role in understanding the homological properties of differential modules.

Theorem 2.2.9 ([BE22, Theorem 3.2])**.** *Let D be a differential module with finitely generated homology and $(F_\bullet, d) \to H(D)$ a minimal free resolution of $H(D)$. Then D admits a free flag resolution $\widetilde{F}$ where the underlying free module is $F_\bullet$ and where, in the notation of Observation 2.2.7, one has $A_{i,i-1} = d_i$ for all i.*

2.3 Anchored Free Flags and Folds of Complexes

In this section, we prove Theorem 2.3.6, which is our first structural result about free flag resolutions whose homology is a complete intersection. We make a note about terminology here:

Note. Throughout the paper, a "complete intersection" will be a quotient of a commutative Noetherian ring by a regular sequence. This is a slight loosening of the more standard definition, where it is assumed that the ambient ring is a regular ring. This decision is made for sake of conciseness of presentation.

Specifically, we consider a free flag resolution F as in Theorem 3.2.11 whose homology is a complete intersection, and we show that the question of whether or not F is trivial—i.e. where F is isomorphic to the fold of a Koszul complex—is entirely determined by an analysis of the "top row" of the differential. As an application, we then completely classify all differential modules D where $H(D)$ is isomorphic to the residue field k and R is regular.

We conclude the section with some interesting examples illustrating the subtlety of minimal free resolutions of differential modules. In particular, we show that if $R = k[x_1, \dots, x_n]$ is a standard graded polynomial ring over a field, then k viewed as a differential R-module in degree 2 has total Betti numbers *strictly* less than the total Betti numbers of k when viewed as an R-module in the usual fashion.

Definition 2.3.1. Let $\widetilde{F}$ be a free flag and $(F_\bullet, d) \to H(\widetilde{F})$ a minimal free resolution of $H(\widetilde{F})$. If $\widetilde{F}$ arises as in the statement of Theorem 3.2.11, then $\widetilde{F}$ will be called an *anchored free flag resolution*. The complex $(F_\bullet, d)$ is called the *anchor* of $\widetilde{F}$.

Remark 2.3.2. An anchored free flag resolution has differential with off-diagonal blocks coming from the minimal free resolution of the homology. These off-diagonal blocks can be thought of as "anchors" for the maps $A_{i,j}$ for $i - j \geq 2$; more precisely, we have complete freedom to choose the "higher-up" maps $A_{i,j}$ up to the constraint that these maps must still make the corresponding differential square to 0.

Conceptually, the following lemma shows that if one can perform column operations on the matrix representation of the differential of an anchored free flag to cancel a term $A_{i,0}$, then one can in fact perform column operations to cancel all other terms appearing along the associated diagonal.

Lemma 2.3.3. *Let D be an anchored free flag and assume $\operatorname{im} A^D_{i,0} \subset \operatorname{im} d_1$ for all $2 \leq i \leq m$ for some given m. Then D is isomorphic to a free flag D' satisfying $A^{D'}_{i,\ell} = 0$ for all $2 \leq i - \ell \leq m$.*

Proof. The assumption $\operatorname{im} A^D_{i,0} \subset \operatorname{im} d_1$ for all $2 \leq i \leq m$ implies that D is isomorphic to a differential module D' with the same underlying module determined by maps $\{A^{D'}_{i,j} \colon F_i \to F_j \mid i < j\}$, but satisfying $A^{D'}_{i,0} = 0$ for all $i \leq m$ and $A^{D'}_{i,i-1} = d_i$ for all i.

Let $2 \leq i - 1 \leq m$. Then there is the relation

$$\sum_{j<i} A^{D'}_{j,0} \cdot A^{D'}_{i,j} = 0.$$

Since $i \leq m + 1$ and $j < i$, one has that $A^{D'}_{j,0} = 0$ for each $j > 1$ appearing in the above equality. Thus $d_1 \circ A^{D'}_{i,1} = 0$, and exactness implies that $\operatorname{im} A^{D'}_{i,1} \subset \operatorname{im} d_2$ for each $i \leq m + 1$. Replacing D with D' and iterating this argument, the result follows. □

In particular, the above gives a criterion for D to be isomorphic to the fold of the minimal free resolution of its homology. One might hope that this is in fact an equivalence—that is, that any differential module isomorphic to the fold of the minimal free resolution of its homology can be identified in this way. This is in general not the case, as we see in the following example.

Example 2.3.4. Let $R = k[x_1, x_2]$ and E be a rank 2 free module on the basis e_1, e_2. Let

$D = \bigwedge^\bullet E$ be the free flag with differential

$$\begin{pmatrix} 0 & x_1^2 & x_1x_2 & x_1 \\ 0 & 0 & 0 & -x_2 \\ 0 & 0 & 0 & x_1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let $K_\bullet$ denote the minimal free resolution of $H(D)$

$$K_\bullet = \quad \bigwedge^0 E \xleftarrow{\begin{pmatrix} x_1^2 & x_1x_2 \end{pmatrix}} \bigwedge^1 E \xleftarrow{\begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}} \bigwedge^2 E$$

Then the morphism of differential modules $D \to \operatorname{Fold}(K_\bullet)$ induced by

$$1 \mapsto 1, \qquad e_1 \mapsto e_1, \quad e_2 \mapsto e_2, \qquad e_1 \wedge e_2 \mapsto e_1 \wedge e_2 + e_1,$$

is an isomorphism, even though $x_1 \notin (x_1^2, x_1x_2) = \operatorname{im} d_1^K$.

Note that the isomorphism described in the above example corresponds to performing *row* operations on the differential to cancel out the x_1 in the corner. In this case, we are able to cancel via row operations but not by column operations. This is explained by the lack of symmetry in the differentials of the minimal free resolution of $H(D)$. In fact, when the minimal free resolution of $H(D)$ is given by the Koszul complex—i.e. when $H(D)$ is a complete intersection—this scenario cannot arise, as we will show next.

First, notice that any morphism $\phi\colon (D, d) \to (D', d')$ of free flags with underlying free modules $\bigoplus_{i\in\mathbb{Z}} F_i$ and $\bigoplus_{i\in\mathbb{Z}} G_i$, respectively, decomposes as a direct sum of maps $\phi_{i,j}\colon F_i \to G_j$ for every $i, j \in \mathbb{Z}$.

Lemma 2.3.5. *Let D be a free flag anchored on the minimal free resolution of a finitely generated R-module M. Assume that there exists an isomorphism $\phi\colon D \to \operatorname{Fold}(F_\bullet)$, and assume that $\phi_{i-1,0}(d_i^F(F_i)) \subset \operatorname{im} d_1^F$ for all $i \geq 2$. Then $\operatorname{im} A_{i,0} \subset \operatorname{im} d_1^F$ for all $i \geq 2$.*

Proof. The map ϕ decomposes as a sum of maps of the form

$$\phi = \sum_{i,j=1}^{n} \phi_{i,j},$$

where $\phi_{i,j} \colon F_i \to F_j$ and n is the length of the flag (note $n = \infty$ is allowed here). We first claim that $\phi_{0,0} : F_0 \to F_0$ may be chosen to be the identity. To see this, notice that the fact that $\phi \circ d^D = d^{\mathrm{Fold}(F_\bullet)} \circ \phi$ implies that there are equalities

$$d_i^F \circ \phi_{0,i} = 0 \quad \text{for all } i \geq 1.$$

Since $F_\bullet$ is a resolution, it follows that $\phi_{0,i} = d_{i+1} \circ \phi'_{0,i+1}$ for some $\phi'_{0,i+1} : F_0 \to F_{i+1}$. On the other hand, let ψ denote the inverse of ϕ. By definition there is an equality

$$(*) \quad \phi_{0,0} \circ \psi_{0,0} + \phi_{1,0} \circ \psi_{0,1} + \cdots + \phi_{n,0} \circ \psi_{0,n} = \mathrm{id}_{F_0}.$$

Applying the functor $- \otimes_R k$ to $(*)$, the fact that $\operatorname{im} \phi_{0,i} \subset \mathfrak{m} F_i$ for each $i \geq 1$ (by the minimality assumption on $F_\bullet$) implies that $\phi_{0,0} \circ \psi_{0,0} \otimes_R k = \mathrm{id}_{F_0} \otimes_R k$. By Nakayama's lemma, the map $\phi_{0,0}$ is a surjective endomorphism of F_0 and hence an isomorphism. Changing bases as necessary, it is thus of no loss of generality to assume $\phi_{0,0}$ is the identity.

Now, let $f_i \in F_i$ be any element; the assumption that ϕ is a morphism of differential modules then yields:

$$\begin{aligned}
\phi(d^D(f_i)) &= \phi\Big(\sum_{0<j<i} A_{i,j}(f_i) \Big) + \phi \circ A_{i,0}(f_i) \\
&= \sum_{i,j=1}^{n} d_j^F(\phi_{i,j}(f_i)) \\
&= d^F(\phi(f_i)).
\end{aligned}$$

Comparing the above equality restricted to the direct summand F_0, one obtains the equality

$$d_1^F(\phi_{i,1}(f_i)) = A_{i,0}(f_i) + \sum_{0<j<i} \phi_{j,0}(A_{i,j}(f_i)). \tag{2.1}$$

Now, we proceed by induction on i to prove the desired statement. When $i = 2$, the above equality becomes

$$d_1^F(\phi_{2,1}(f_2)) = A_{2,0}(f_2) + \phi_{1,0}(d_2^F(f_2)).$$

The assumption that $\phi_{1,0}(d_2^F(f_2)) \in \operatorname{im} d_1^F$ implies that $A_{2,0}(F_2) \subset \operatorname{im} d_1^F$, and Lemma 2.3.3 implies that D may be replaced with a differential module satisfying $A_{i,i-2} = 0$ for all $i \geq 2$. Proceeding inductively, assume $i > 2$; by induction, we may assume that $A_{j,k} = 0$ for all $1 < j - k < i$. The equality (2.1) then reduces to

$$d_1^F(\phi_{i,1}(f_i)) = A_{i,0}(f_i) + \phi_{i-1,0}(d_i^F(f_i)),$$

and again the assumption $\phi_{i-1,0}(d_i^F(f_i)) \in \operatorname{im} d_1^F$ implies that $A_{i,0}(f_i) \in \operatorname{im} d_1^F$, and Lemma 2.3.3 allows us to replace D with a differential module satisfying $A_{j,k} = 0$ for all $j - k \leq i$. Iterating this argument, the result follows. □

The above holds, in particular, when $H(D)$ is a complete intersection.

Theorem 2.3.6. *Let D be an anchored free flag and assume that $H(D)$ is a complete intersection; that is, $H(D) \cong R/I$ where I is generated by a regular sequence. Then*

$$D \cong \operatorname{Fold}(F_\bullet) \quad \iff \quad \operatorname{im} A_{i,0}^D \subset \operatorname{im} d_1 \text{ for all } i \geq 2.$$

Proof. $\implies$: Let $H(D) = R/\mathfrak{a}$ where $\mathfrak{a}$ is generated by a regular sequence. By definition of the Koszul complex, the minimal free resolution of $R/\mathfrak{a}$ satisfies $d_i^F \otimes R/\mathfrak{a} = 0$ for all i, whence the assumption $\phi_{i-1,0}(d_i^F(F_i)) \subset \operatorname{im} d_1^F$ of Lemma 2.3.5 is trivially satisfied.

$\impliedby$: This implication holds without any further assumptions by Lemma 2.3.3. □

Under the perspective of free flags as upper triangular block matrices, this says that the property of D being isomorphic to the resolution of its homology is completely detectable via only the top row of the matrix. On the other hand, in the case where D is graded, the degrees of the entries of the top row may be deduced by using this grading. Adding this additional structure gives strong restrictions on the possibilities for differential modules D with complete intersection homology. In the most restrictive case, when $H(D) \cong k$, we have the following.

Corollary 2.3.7. *Assume R is a regular graded local ring. Let $D \in \mathrm{DM}(R, a)$ be an anchored free flag with $H(D) \cong k$. If $a \neq 2$, then $D \cong \mathrm{Fold}(K_\bullet)$, where $K_\bullet$ denotes the Koszul complex resolving k.*

Proof. Since D has degree a, the minimal free resolution of $H(D)$ is given by the Koszul complex $K_\bullet$ with the i^{th} free module with a degree shift by ia (so that the maps in the complex are all homogeneous of degree a). The maps $A_{i,j} \colon R(ia - a) \to R(ja - j + a)$ in the differential on D therefore have degree $(ja - j + a) - (ia - i)$. When $\deg A_{i,0} \neq 0$, we have $\operatorname{im} A_{i,0} \otimes k = 0$. On the other hand, $(ja - j + a) - (ia - i) = 0$ only when $a = 2$ and $i - j = 2$. Thus for $a \neq 2$ the result follows from Corollary 2.3.6. □

The above corollary implies that in degree $a \neq 2$, all differential R-modules with homology k have isomorphic anchored free flag resolutions, and that furthermore this resolution is minimal and isomorphic to the Koszul complex. However, this is not true for differential modules of degree 2. In general, any R-module may be viewed as a degree a differential module, for any integer a, and the following result shows that the homological invariants of $M \in \mathrm{DM}(a, R)$ can vary as the degree a varies.

Proposition 2.3.8. *Let $S := k[x_1, \ldots, x_n]$ be a standard graded polynomial ring over a field k, where $n \geq 2$. Then there exists a degree 2 free differential module D of rank 2^{n-1} with $H(D) \cong k$.*

Proof. We construct the differential module inductively. For $n = 2$, let $D = S^2$ with differential given by the matrix $\begin{pmatrix} x_1x_2 & -x_2^2 \\ x_1^2 & -x_1x_2 \end{pmatrix}$. One can check that D is a degree 2 differential module, and moreover that $H(D) \cong k$.

For $n \geq 3$, let

$$D_n := \mathrm{cone}(D_{n-1} \otimes_k k[x_n] \xrightarrow{x_n} D_{n-1} \otimes_k k[x_n]).$$

By the inductive hypothesis, there is an equality $H(D_{n-1} \otimes_k k[x_n]) \cong k \otimes_k k[x_n] = k[x_n]$, and by [ABI07, p. 1.2] the mapping cone induces an exact triangle

$$\begin{array}{ccc} k[x_n] \cong H(D_{n-1} \otimes_k k[x_n]) & \xrightarrow{\quad x_n \quad} & H(D_{n-1} \otimes_k k[x_n]) \cong k[x_n] \\ & \nwarrow \qquad \swarrow & \\ & H(D_n) & \end{array}$$

Since multiplication by x_n is injective, the above exact triangle degenerates to a short exact sequence:

$$0 \to k[x_n] \xrightarrow{x_n} k[x_n] \to H(D_n) \to 0,$$

in which case $H(D_n) \cong k$, as desired. Moreover, the rank of D_n is precisely $2 \cdot \operatorname{rank}(D_{n-1}) = 2 \cdot 2^{n-2} = 2^{n-1}$. Since the differential on D_n is given by the block matrix $\begin{pmatrix} -d \otimes_R S & x_n \operatorname{id} \\ 0 & d \otimes_R S \end{pmatrix}$, we can see furthermore that the degree of D_n is 2. $\square$

If we define the *Betti numbers* of a differential module as in [BE22], then the sum of the Betti numbers of D is equal to the rank of the minimal free resolution of D. We next show that the differential module constructed in 2.3.8 is actually a minimal free resolution. This will imply that the sum of the betti numbers of a degree 2 differential module with homology k may be at least as small as 2^{n-1}, strictly smaller than the total rank of a minimal free resolution with the same homology. Although this differential module is certainly free and minimal, it is not immediately obvious that it is a minimal free *resolution*, since this requires it to be a summand of a free flag resolution. To prove that this is indeed the case, we leverage the mapping cone structure of the differential module constructed in 2.3.8.

We will first need a lemma; in the following, recall that a differential module F is *contractible* if the identity map is homotopic to 0. A homotopy h for which $\operatorname{id}_F = d^F h + h d^F$ is called a *contracting homotopy*.

Lemma 2.3.9. *Let F and G be contractible differential modules with contracting homotopies h^F and h^G, respectively. If $\phi : F \to G$ is a morphism of differential modules satisfying $\phi \circ h^F =$*

$h^G \circ \phi$*, then the mapping cone* $\operatorname{cone}(\phi)$ *is contractible with contracting homotopy*

$$h^{\operatorname{cone}(\phi)} := \begin{pmatrix} -h^F & 0 \\ 0 & h^G \end{pmatrix}.$$

Proof. Recall that $\operatorname{cone}(\phi)$ has underlying free module isomorphic to $F \oplus G$ equipped with the differential whose block form is given by

$$d^{\operatorname{cone}(\phi)} = \begin{pmatrix} -d^F & 0 \\ -\phi & d^G \end{pmatrix}.$$

Using this, we compute:

$$\begin{aligned} d^{\operatorname{cone}(\phi)} h^{\operatorname{cone}(\phi)} + h^{\operatorname{cone}(\phi)} d^{\operatorname{cone}(\phi)} &= \begin{pmatrix} -d^F & 0 \\ -\phi & d^G \end{pmatrix} \begin{pmatrix} -h^F & 0 \\ 0 & h^G \end{pmatrix} + \begin{pmatrix} -h^F & 0 \\ 0 & h^G \end{pmatrix} \begin{pmatrix} -d^F & 0 \\ -\phi & d^G \end{pmatrix} \\ &= \begin{pmatrix} d^F h^F & 0 \\ \phi h^F & d^G h^G \end{pmatrix} + \begin{pmatrix} h^F d^F & 0 \\ -h^G \phi & h^G d^G \end{pmatrix} \\ &= \begin{pmatrix} \mathrm{id}_F & 0 \\ 0 & \mathrm{id}_G \end{pmatrix}. \end{aligned}$$

By definition, $h^{\operatorname{cone}(\phi)}$ is a contracting homotopy. □

Corollary 2.3.10. *Let* D_n *be the rank* 2^{n-1} *free differential module over* $k[x_1, \ldots, x_n]$ *defined in Proposition 2.3.8. Then* D_n *is its own minimal free resolution.*

Proof. Proceed by induction on n. For $n = 2$, the module $S \oplus S$ with differential $\begin{pmatrix} xy & x^2 \\ -y^2 & xy \end{pmatrix}$ is known to be its own minimal free resolution (see [BE22], Example 5.8). In particular, it is a minimal free summand of the free S-module $S \oplus S(1)^2 \oplus S(2)$ with differential

$$\begin{pmatrix} 0 & x & y & 1 \\ 0 & 0 & 0 & -y \\ 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let F^2 denote this free flag differential module, and notice that F^2 is anchored on the length 2

Koszul complex on variables x_1, x_2. For $n > 2$, define

$$F^n := \operatorname{cone}\left(x_n : F^{n-1} \otimes_k k[x_n] \to F^{n-1} \otimes_k k[x_n]\right).$$

By the inductive hypothesis, the free flag F^{n-1} is anchored on the Koszul complex for $x_1, \dots, x_{n-1}$. Since F^n is obtained by taking the mapping cone of multiplication by x_n, the differential module F^n is anchored on the Koszul complex for $x_1, \dots, x_n$.

Moreover, by induction the free flag F^{n-1} splits as a direct sum

$$F^{n-1} \cong D_{n-1} \oplus T^{n-1},$$

where T^{n-1} is some contractible differential $k[x_1, \dots, x_{n-1}]$-module and D_{n-1} denotes the differential module of Proposition 2.3.8. Combining this information, we find that there is an isomorphism:

$$\begin{aligned} F^n &= \operatorname{cone}\left(x_n : F^{n-1} \otimes_k k[x_n] \to F^{n-1} \otimes_k k[x_n]\right) \\ &= \underbrace{\operatorname{cone}(D_{n-1} \otimes_k k[x_n] \xrightarrow{x_n} D_{n-1} \otimes_k k[x_n])}_{=D_n} \oplus \underbrace{\operatorname{cone}(T^{n-1} \otimes_k k[x_n] \xrightarrow{x_n} T^{n-1} \otimes_k k[x_n])}_{:=T^n}. \end{aligned}$$

The image of any contractible object under an additive functor remains contractible, so the differential module $T^{n-1} \otimes_k k[x_n]$ is also contractible. Since scalar multiplication commutes with any choice of homotopy, the hypotheses of Lemma 2.3.9 are satisfied for the morphism $x_n : T^{n-1} \otimes_k k[x_n] \to T^{n-1} \otimes_k k[x_n]$. It thus follows from Lemma 2.3.9 that T^n is contractible and F^n splits as the direct sum of D_n and a contractible differential module. By definition, the differential module D_n is its own minimal free resolution. □

Note that because the differential module D constructed in Proposition 2.3.8 is not a free flag, it does not itself contradict the (disproven) conjecture of Avramov–Buchweitz–Iyengar that the rank of a free flag over R with finite length homology is at least $2^{\dim(R)}$ [ABI07, Conj. 5.3]. In fact, the example when $n = 2$ appears in [ABI07], and this construction directly generalizes their example.

2.4 A Generalization of the Koszul Complex for Differential Modules

In this section, we introduce a differential module analog of the Koszul complex and show that under certain hypotheses, all anchored free flags are isomorphic to this Koszul differential module. We also provide an example showing that in general not all such free flags with complete intersection homology are obtained by this Koszul differential module; to begin this section, we recall the definition of the Koszul complex that will be most convenient for our purposes.

Let R be a ring and E be a free R-module on basis $e_1, \dots, e_n$ and $\psi \colon E \to R$ any R-module homomorphism. The notation e_I will be shorthand for the basis element $e_{i_1} \wedge \cdots \wedge e_{i_k}$, where $I = \{i_1 < \cdots < i_k\}$ is an indexing set of the appropriate size. Recall that the Koszul complex can be constructed as the complex with the ith exterior power $\bigwedge^i E$ sitting in homological degree i and differential

$$e_{j_1,\dots,j_i} \mapsto \sum_{k=1}^{i} (-1)^{k+1} \psi(e_{j_k}) e_{j_1,\dots,\widetilde{j_k},\dots,j_i}.$$

Another way to view this map is as the composition

$$\begin{aligned} \bigwedge^i E &\xrightarrow{\text{comultiplication}} E \otimes \bigwedge^{i-1} E \\ &\xrightarrow{\psi \otimes 1} R \otimes \bigwedge^{i-1} E \cong \bigwedge^{i-1} E. \end{aligned}$$

This is equivalently described as multiplication by an element $f \in E^*$ (recall that $\bigwedge^\bullet E$ is a graded $\bigwedge^\bullet E^*$-module and vice versa). Choose a map $A_{i,0} \colon \bigwedge^i E \to R$ (the notation here is intentionally reminiscent of Observation 2.2.7). Notice that such a map is equivalently induced by multiplication by an element $f_i \in \bigwedge^i E^*$. This is because the pairing $\bigwedge^i E \otimes \bigwedge^{n-i} E \to \bigwedge^n E$ is perfect.

Our goal will now be to generalize the exterior algebra structure of the Koszul complex to define a "Koszul differential module". In order to make the general construction more clear, we illustrate with an example:

Example 2.4.1. Given an integer n, use the notation $[n] := \{1, \dots, n\}$. Let $R = k[x_I \mid I \subset$

$[4],\ I \neq \varnothing]$ and let f_i for $i = 1, \ldots, 4$ be the elements of $\bigwedge^i E^*$ induced by the maps

$$
\begin{array}{cccc}
E \to R & \bigwedge^2 E \to R & \bigwedge^3 E \to R & \bigwedge^4 E \to R \\
e_i \mapsto x_i & e_{ij} \mapsto \begin{cases} 0 & \text{if } i = 1, \\ x_{ij} & \text{otherwise.} \end{cases} & e_{ijk} \mapsto x_{ijk} & e_{1234} \mapsto x_{1234}.
\end{array}
$$

Define a free flag F whose underlying module is $\bigwedge^\bullet E$ and whose differential is given by the maps $A_{i,j} : \bigwedge^i E \to \bigwedge^j E$ defined by $A_{i,j}(g) = (-1)^{ij} f_{i-j} g$. To check that this indeed defines a differential module amounts to checking that for each $i > j$

$$\sum_{j<k<i} A_{k,j} A_{i,k} = \sum_{j<k<i} (-1)^{kj} (-1)^{ik} f_{k-j} f_{i-k} = 0$$

In this case, one just needs to verify the relations

$$f_1^2 = 0, \quad f_1 f_2 - f_2 f_1 = 0, \quad f_1 f_3 + f_3 f_1 + f_2^2 = 0.$$

We can generalize the construction in Example 2.4.1 to obtain a differential module in a similar way given a suitably chosen bialgebra. Recall that an algebra *admits divided powers* if the subalgebra generated by elements of even degree satisfies the axioms of a divided power algebra (for the definition of a divided power algebra see, for instance, [ABW82]). A canonical example of such an algebra (and the only example we will use in this paper) to keep in mind is the exterior algebra on a free module, where the elements of even degree are the divided power elements.

Proposition 2.4.2. *Let T denote any graded-cocommutative R-bialgebra such that T^* admits divided powers (where T^* denotes the graded dual). Given any $f_i \in T_i^*$, the notation $f_i \colon T_\ell \to T_{\ell-i}$ will denote the left-multiplication map. Assume either:*

i. $\operatorname{char} R = 2$*, or*

ii. $\operatorname{char} R \neq 2$ *and* $f_i \cdot f_j = 0$ *if both i and j are even.*

Define $A_{i,j} := (-1)^{ij} f_{i-j} \colon T_i \to T_j$. *Then the data*

$$\{T_i,\ A_{i,j} \colon T_i \to T_j \mid j < i,\ i \in \mathbb{Z}\}$$

determines a differential module.

Proof. One only needs to verify that $\sum_{j<k<i} A_{k,j} A_{i,k} = 0$ for all choices of i and j, and this is a straightforward computation. Assume that $i + j$ is odd; one computes:

$$\begin{aligned} \sum_{k=j+1}^{i-1} A_{k,j} \cdot A_{i,k} &= \sum_{k=j+1}^{(i+j-1)/2} (A_{k,j} A_{i,k} + A_{i+j-k,j} A_{i,i+j-k}) \\ &= \sum_{k=j+1}^{(i+j-1)/2} \Big((-1)^{jk+ik} + (-1)^{j(i+j-k)+i(i+j-k)+(i-k)(k-j)} \Big) f_{k-j} f_{i-k}. \end{aligned}$$

Thus it suffices to show that the coefficient

$$(-1)^{jk+ik} + (-1)^{j(i+j-k)+i(i+j-k)+(i-k)(k-j)}$$

is 0 if $i - k$ or $k - j$ is odd. Since the above expression is symmetric in i and j modulo 2, it is of no loss of generality to assume that $i \equiv_2 k + 1$. One computes:

$$jk + ik \equiv_2 jk, \quad \text{and}$$

$$j(i + j - k) + i(i + j - k) + (i - k)(k - j) \equiv_2 j(j + 1) + (k + 1)(j + 1) + j + k \equiv_2 jk + 1.$$

Thus the coefficient is 0 if $i - k$ and $k - j$ are not both even, and if they are both even, then $f_{i-k} f_{k-j} = 0$ or appears with coefficient 2, implying that these terms vanish as well.

If $i + j$ is even, then the computation is identical, with the only difference being that the term $f^2_{(i+j)/2}$ appears. If $\operatorname{char} k \neq 2$, then this term vanishes by assumption. If $\operatorname{char} k = 2$, then the assumption that T admits divided powers implies that $f^2_{(i+j)/2} = 2 f^{(2)}_{(i+j)/2} = 0$, so all terms again vanish. □

Notation. Let $f_i \colon T_i \to R$ for $i = 1, \ldots, n$ be a collection of maps induced by multiplication by

$f_i \in (T_i)^*$, where $T_\bullet$ is any graded-cocommutative bialgebra as in the statement of Proposition 2.4.2. The notation $K(f_1, \dots, f_n)$ will denote the (not necessarily differential) module induced by the data $\{T_i,\ (-1)^{ij} f_{i-j}\}$.

We can also construct a "generic" Koszul differential module; we will see that the theorems below give criteria for which free flags of the appropriate form are obtained as specializations on the generic Koszul differential module.

Construction 2.4.3 (Generic Koszul Differential Module)**.** Let $n \in \mathbb{N}$ and $A = \mathbb{Z}[x_I \mid I \subset [n],\ I \neq \varnothing]$. Let $E = \bigoplus_{i=1}^n Ae_i$ and let $f_i \in \bigwedge^i E^*$ be the generic maps

$$f_i = \sum_{|I|=i} x_I e_I^*.$$

Next, let I be the ideal generated the relations $f_i f_j = 0$ for $i,\ j$ both even. Then define $S := A/I$. Notice by construction $A/I \otimes K(f_1, \dots, f_n)$ is a differential module with homology isomorphic to $\mathbb{Z}$.

Use the notation $K^{\mathrm{gen}} := A/I \otimes K(f_1, \dots, f_n)$.

Remark 2.4.4. The relations induced by imposing the condition $f_i \cdot f_j = 0$ for i and j both even are in general quite complicated. The first case for which we obtain nontrivial equations is when $n = 4$ in Construction 2.4.3, in which case we are imposing the relation $f_2^{(2)} = 0$. Choosing bases, notice that f_2 is represented as a generic 4×4 skew symmetric matrix and the condition $f_2^{(2)} = 0$ means we are taking the quotient by the 4×4 pfaffian of this matrix representation.

Notice that in general, the ideal I appearing in Construction 2.4.3 is always generated by quadratic equations in the f_i.

Example 2.4.5. Assume $n = 6$ in the notation of Construction 2.4.3. Then the relations imposed come from setting

$$f_2^{(2)} = 0, \quad \text{and} \quad f_2 \cdot f_4 = 0.$$

The relations $f_2^{(2)} = 0$ are precisely the equations of the 4×4 pfaffians of the 6×6 matrix representation of the map f_2. The additional relation $f_2 \cdot f_4 = 0$ contributes, after choosing bases, the single quadratic equation

$$\sum_{\substack{I\subset[6],\\ |I|=2}} \operatorname{sgn}(I\subset[6])x_I x_{[6]\setminus I}=0.$$

It may be tempting to believe that all anchored free flags with complete intersection homology arise as specializations of Construction 2.4.3. We will see that this is true if the ring has characteristic 2, but the following example shows that this is not the case in general.

Example 2.4.6. Let $E=\bigoplus_{i=1}^4 Re_i$, where $R=k[x_1,\dots,x_4]$ and k is a field of characteristic $\neq 2$. Let F be the free flag defined by the following data:

$$A_{1,0}=A_{2,1}=A_{3,2}=A_{4,3}=x_1^3e_1^*+x_2^3e_2^*+x_3^3e_3^*+x_4^3e_4^*,$$

$$A_{2,0}=-A_{3,1}=A_{4,2}=x_1x_2e_{12}^*+x_2^2e_{34}^*,$$

$$A_{3,0}=-2x_1e_{134}^*,\quad A_{4,1}=A_{4,0}=0.$$

Under a choice of basis, we can write $F=R^1\oplus F^4\oplus R^6\oplus R^4\oplus R^1$ and express the differential d^F as a block matrix.

$$\begin{array}{c} \\ R^1\\ R^4\\ R^6\\ R^4\\ R^1\end{array}\begin{array}{c} \begin{array}{ccccc} R^1 & R^4 & R^6 & R^4 & R^1\end{array}\\ \begin{pmatrix} 0 & A_{1,0} & A_{2,0} & A_{3,0} & A_{4,0}\\ 0 & 0 & A_{2,1} & A_{3,1} & A_{4,1}\\ 0 & 0 & 0 & A_{3,2} & A_{4,2}\\ 0 & 0 & 0 & 0 & A_{4,3}\\ 0 & 0 & 0 & 0 & 0\end{pmatrix}\end{array}$$

The blocks $A_{i,i-1}$ on the first off-diagonal are the matrices appearing in the Koszul complex on $(x_1^3,x_2^3,x_3^3,x_4^3)$. The $A_{i,i-2}$ maps are given by the following:

$$A_{2,0} = \begin{pmatrix} x_1x_2 & 0 & 0 & 0 & 0 & x_2^2 \end{pmatrix} \quad A_{3,1} = \begin{pmatrix} 0 & 0 & -x_2^2 & 0 \\ 0 & 0 & 0 & -x_2^2 \\ -x_1x_2 & 0 & 0 & 0 \\ 0 & -x_1x_2 & 0 & 0 \end{pmatrix} \quad A_{4,2} = \begin{pmatrix} x_2^2 \\ 0 \\ 0 \\ 0 \\ 0 \\ x_1x_2 \end{pmatrix}$$

and we have $A_{3,0} = \begin{pmatrix} 0 & 0 & 2x_1 & 0 \end{pmatrix}$.

One can check that d^F squares to 0 so this data determines a well-defined free flag. However, $A_{3,0} \neq A_{4,1}$ since $A_{3,0}$ is given by multiplication by a nonzero element and $A_{4,1}$ is not. This means that the free flag induced by the above data is not of the form $K(f_1, f_2, f_3, f_4)$ for any choice of f_i.

The above example hinges on the observation that if we let $A_{1,0} = \begin{pmatrix} x_1^3 & x_2^3 & x_3^3 & x_4^3 \end{pmatrix}$ and $A_{4,3} = \begin{pmatrix} -x_4^3 & x_3^3 & -x_2^3 & x_1^3 \end{pmatrix}^T$ be the first and last matrices in the Koszul complex on $(x_1^3, x_2^3, x_3^3, x_4^3)$, then $A_{2,0}A_{4,2} + A_{3,0}A_{4,3} = 0$. The condition for F to be a differential module requires that $A_{1,0}A_{4,1} + A_{2,0}A_{4,2} + A_{3,0}A_{4,3} = 0$, which in this case forces $A_{4,1}$ to be 0 modulo $(x_1^3, x_2^3, x_3^3, x_4^3)$. This "forcing" occurred due to cancellation in $A_{2,0}A_{4,2} + A_{3,0}A_{4,3}$. Our next result shows that absent this cancellation, it is indeed the case that anchored free flags with complete intersection homology as in Construction 2.4.3 are Koszul differential modules.

This intuition is clarified in the following definition which presents a technical condition that will be needed as a hypothesis for the proof of Theorem 2.4.9.

Definition 2.4.7. Let R be a ring and $I \subset R$ an ideal. The ideal I is *completely Tor-independent* with respect to a family of ideals $J_1, \dots, J_\ell$ if for all $1 \leq i_1 < \cdots < i_k \leq \ell$, one has

$$\operatorname{Tor}^R_{>0}\left(\frac{R}{I}, \frac{R}{J_{i_1} + \cdots + J_{i_k}}\right) = 0.$$

Conceptually, Tor-independent objects do not have any nontrivial homological interaction; at the level of resolutions, this means that tensoring a resolution of a module M with a Tor-independent quotient ring preserves exactness.

The next theorem gives a partial characterization of anchored free flags, and is the main result of this section. The abridged version of this result states that if the homology $H(D)$ is sufficiently Tor-independent with respect to a subfamily of the ideals $\operatorname{im}(f_i\colon \bigwedge^i E \to R)$, then differential modules with complete intersection homology must arise as in Proposition 2.4.2.

Remark 2.4.8. Some remarks about the statement of Theorem 2.4.9 are necessary before the full statement: we can assume that the differential module D has free flag resolution anchored on a Koszul complex $K_\bullet$ resolving the complete intersection $H(D)$. Then the components of the flag differentials mapping to $K_0 = R$ are R-module homomorphisms $f_i : \bigwedge^i E \to R$. These are the elements $f_i : \bigwedge^i E \to R$ as written in the statement of the theorem, and the conclusion of the theorem is that D is isomorphic to the generalized Koszul flag $K(f_1, \dots, f_n)$ where the $f_i \in \bigwedge^i E^*$ arise as just mentioned.

Theorem 2.4.9. *Let D be a differential R-module with $H(D)$ a complete intersection, viewed as the cokernel of some map $f_1 : E \to R$ (where E is a free R-module). Let $F \to D$ be a free flag resolution anchored on the Koszul complex associated to $f_1 \in E^*$ and assume that* $\operatorname{im}(f_1\colon E \to R)$ *is completely Tor-independent with respect to the set*

$$\left\{ \operatorname{im}(f_i\colon \bigwedge^i E \to R) \mid i \text{ is even and } i \leq \operatorname{rank}(E)/2 \right\}.$$

Then F is isomorphic to the differential module of Proposition 2.4.2, where $T = \bigwedge^\bullet E$.

Proof. Proceed by induction on $i - j - 1$, where i, j are the indices of the component $A_{i,j}$ of the differential of D. When $i - j - 1 = 0$, this is the statement of Theorem 3.2.11 since it is of no loss of generality to assume that D is anchored on the Koszul complex resolving $H(D)$, associated to some element $f_1 \in E^*$. Let $e_1, \dots, e_n$ denote a basis for E. Assume now that $i - j - 1 \geq 1$. Inducting also on j, one may assume that for all $k > 0$, there is the equality $A_{i-k,j+1-k} = (-1)^{(i-k)(j-k)} f_{i-j-1}$ (the base case is for $k = j + 1$, which holds by assumption). Using this, one computes:

$$0 = \sum_{k=j+1}^{i-1} A_{k,j} A_{i,k}$$

$$(*) \quad = 2 \cdot \sum_{\substack{i-k,\ j-k \text{ even} \\ j<k\leq\lfloor(i+j)/2\rfloor}} f_{k-j} \cdot f_{i-k} + f_1\big((-1)^{(i+1)j+i+j} f_{i-j-1} - A_{i,j+1}\big).$$

If $i - j > \operatorname{rank} E$, then the equality $(*)$ reduces to $f_1\big((-1)^{(i+1)j+i+j} f_{i-j-1} - A_{i,j+1}\big) = 0$, and exactness of multiplication by f_1 implies that $(-1)^{(i+1)j+i+j} f_{i-j-1} - A_{i,j+1} \in \operatorname{im} f_1$. If $i-j \leq \operatorname{rank} E$, then notice that for each $j < k \leq \lfloor(i+j)/2\rfloor$, one has $i-k$ or $k-j \leq \operatorname{rank}(E)/2$. Define the ideal

$$\mathfrak{a} := \big(\operatorname{im}(f_\ell \colon \textstyle\bigwedge^\ell E \to R) \mid \ell \text{ is even and } \ell < i-j-1\big) \subset R.$$

Let I be any indexing set with $|I| = i-j-2$ and let $e_I^* \in \bigwedge^{i-j-2} E^*$ denote the basis element dual to $e_I \in \bigwedge^{i-j-2} E$. Multiplying the equality $(*)$ on the right by e_I^*, one obtains

$$f_1 \cdot (b \cdot e_I^*) \in \mathfrak{a}, \tag{2.2}$$

where $b := (-1)^{(i+1)j+i+j} f_{i-j-1} - A_{i,j+1}$. By the Tor-independence assumption, notice that if $K_\bullet$ denotes the Koszul complex induced by f_1, then the complex $K_\bullet \otimes_R R/\mathfrak{a}$ must remain exact. Combining this with (2.2) implies that $b \cdot e_I^*$ is a cycle in $K_\bullet \otimes_R R/\mathfrak{a}$, and hence by exactness there exists some $a \in \bigwedge^2 E$ such that

$$b \cdot e_I^* - f_1 \cdot a \in \mathfrak{a}E.$$

Multiplying the above on the right by e_i^* for $i = 1, \dots, n$, it follows that for all indexing sets J of size $i-j-1$, there exist elements $a_\ell^J \in \bigwedge^\ell E$ such that

$$b \cdot e_J^* = f_1 \cdot a_1^J + \underbrace{\sum_{\substack{\ell \text{ even}, \\ \ell<i-j-1}} f_\ell \cdot a_\ell^J}_{\in \mathfrak{a}}. \tag{2.3}$$

Recall that the identity map $\bigwedge^{i-j-1} E \to \bigwedge^{i-j-1} E$ is equivalently represented as right multiplication by the trace element $\sum_{|J|=i-j-1} e_J^* \otimes e_J$, whence:

$$\begin{aligned}
b &= b \cdot \Big(\sum_{|J|=i-j-1} e_J^* \otimes e_J \Big) \\
&= \sum_{|J|=i-j-1} be_J^* \otimes e_J \\
&= \sum_{|J|=i-j-1} (f_1 \cdot a_1^J) e_J + \sum_{|J|=i-j-1} \sum_{\substack{\ell \text{ even}, \\ \ell < i-j-1}} (f_\ell \cdot a_\ell^J) e_J \quad \text{(by 2.3)} \\
&= f_1 \cdot \Big(\sum_{|J|=i-j-1} a_1^J \cdot e_J \Big) + \sum_{\substack{\ell \text{ even}, \\ \ell < i-j-1}} f_\ell \cdot \Big(\sum_{|J|=i-j-1} a_\ell^J \cdot e_J \Big) \\
&\in \operatorname{im}\Big(f_1 \colon \bigwedge^{i-j} E \to \bigwedge^{i-j-1} E \Big) + \sum_{\substack{\ell \text{ even}, \\ \ell < i-j-1}} \operatorname{im}\Big(f_\ell \colon \bigwedge^{i-j-1+\ell} E \to \bigwedge^{i-j-1} E \Big).
\end{aligned}$$

Recalling that $b := (-1)^{(i+1)j+i+j} f_{i-j-1} - A_{i,j+1}$, this means that the differential component $A_{i,j+1}$ differs from $(-1)^{(i+1)j+i+j} f_{i-j-1}$ by multiplication by the elements f_ℓ, where $\ell < i-j-1$. It follows that one may perform row/column operations on the matrix representation of the square-zero endomorphism of D to ensure that $A_{i,j+1} = (-1)^{(i+1)j+i+j} f_{i-j-1}$. This completes the proof. □

The hypotheses of Theorem 2.4.9 are stated in a decent level of generality, but this is because we needed a condition that was general enough to capture a family of the special cases for which there existed an isomorphism to a Koszul flag. The following corollary makes explicit a list of common cases for which the hypotheses of Theorem 2.4.9 are satisfied:

Corollary 2.4.10. *The assumptions of Theorem 2.4.9 are satisfied in the following cases:*

i. The free module E satisfies $\operatorname{rank} E \leq 3$.

ii. $(R, \mathfrak{m})$ *is a Noetherian local ring and*

$$\operatorname{im}\Big(f_1 \colon E \to R \Big) + \sum_{\substack{\ell \leq \operatorname{rank}(E)/2 \\ \ell \text{ even}}} \operatorname{im}\Big(f_\ell \colon \bigwedge^{\ell} E \to R \Big)$$

is generated by a regular sequence contained in $\mathfrak{m}$.

iii. R *is a graded ring and*

$$\operatorname{im}\Big(f_1\colon E \to R\Big) + \sum_{\substack{\ell \leq \operatorname{rank}(E)/2 \\ \ell \text{ even}}} \operatorname{im}\Big(f_\ell\colon \bigwedge^{\ell} E \to R\Big)$$

is generated by a homogeneous regular sequence of positive degree.

iv. $R = k[x_1, \ldots, x_n]$ *and the image of each* $f_i\colon \bigwedge^i E \to R$ *for* $i = 1$ *and* $i \leq \operatorname{rank}(E)/2$ *even lie in polynomial rings in disjoint variables.*

In particular, if either:

i. $(R, \mathfrak{m})$ *is a Noetherian local ring and the first row of the matrix representation of the square-zero endomorphism of* D *generates a complete intersection contained in* $\mathfrak{m}$, *or*

ii. R *is a graded ring and the first row of the matrix representation of the square-zero endomorphism of* D *generates a homogeneous complete intersection of positive degree,*

then the assumptions of Theorem 2.4.9 are satisfied.

Interestingly, if we assume that R has characteristic 2, then the statement of Theorem 2.4.9 can be generalized significantly.

Theorem 2.4.11. *Assume* R *is a ring of characteristic* 2*. Let* F *be an anchored free flag with* $H(F)$ *a complete intersection. Then* F *isomorphic to the differential module of Proposition 2.4.2 for some choice of* $f_i \in \bigwedge^i E^*$, *where* $T = \bigwedge^\bullet E$.

Proof. Do the computation of the previous proof, but notice that all other extraneous terms cancel by the characteristic assumption (since one of the terms of the equation $(*)$ in the proof of Theorem 2.4.9 has coefficient 2). □

2.5 DG-module Structures on Free Flags

In this section, we study differential modules that can be given the structure of a DG-module over some DG-algebra. Our main motivation for considering this question is based on the philosophy that the homological properties of a differential module are tightly linked to those of the homology, as suggested by Theorem 3.2.11. One natural direction related to this question is the extent to which additional structure on the minimal free resolution of the homology can be "lifted" to the differential module. Our results here indicate that there are some very restrictive obstructions to lifting algebra structures to the level of differential modules.

It is evident that if a free resolution $F_\bullet$ of the homology $H(D)$ of an anchored free flag D admits the structure of an associative DG-algebra structure and $D \cong \mathrm{Fold}(F_\bullet)$, then the algebra structure on $F_\bullet$ can be transferred to a DG-module structure on D. We prove even further that if the homology $H(D)$ is a complete intersection, then this becomes an equivalence; more precisely: an anchored free flag D with $H(D)$ a complete intersection admits the structure of a DG-module over $K_\bullet$ *if and only if* $D \cong \mathrm{Fold}(K_\bullet)$, where $K_\bullet$ denotes the Koszul complex resolving $H(D)$.

We conclude the section with questions about DG-module structures on more general free flag resolutions. In particular, we know of no example of a free flag admitting a DG-module structure over the minimal free resolution of its homology that is *not* isomorphic to the fold of some complex, and are very interested in any such example.

Definition 2.5.1. A (graded commutative) *differential graded algebra* (F, d) (or *DG-algebra*) over a commutative Noetherian ring R is a complex of finitely generated free R-modules with differential d and with a unitary, associative multiplication $F \otimes_R F \to F$ satisfying

(a) $F_i F_j \subseteq F_{i+j}$,

(b) $d_{i+j}(f_i f_j) = d_i(f_i) f_j + (-1)^i f_i d_j(f_j)$,

(c) $f_i f_j = (-1)^{ij} f_j f_i$, and

(d) $f_i^2 = 0$ if i is odd,

where $f_k \in F_k$.

Remark 2.5.2. It is worth mentioning that this is a stricter definition of DG-algebra for the purposes of this paper, but there are more general definitions in the literature.

There does not exist a tensor product between arbitrary differential modules (or even free flags) that directly generalizes the tensor product of complexes, but, it is possible to construct such a product between a *complex* and a differential module.

Definition 2.5.3. Let $F_\bullet$ be a complex and D a differential module. Then the *box product* $F_\bullet \boxtimes_R D$ is defined to be the differential module with underlying module $\bigoplus_{i\in\mathbb{Z}} F_i \otimes_R D$ and differential

$$d^{F\boxtimes D}(f_i \otimes d) := d^F(f_i) \otimes d + (-1)^i f_i \otimes d^D(d).$$

Remark 2.5.4. This notion of a box product was introduced in the [ABI07, Subsection 1.9].

Definition 2.5.5. Let D be a differential module whose homology is a cyclic R-module and let $F_\bullet$ be a minimal free resolution of $H(D)$ admitting the structure of a DG-algebra. Then D is a DG-module over $F_\bullet$ if there exists a morphism of differential modules

$$p\colon F_\bullet \boxtimes_R D \to D$$

extending the R-module action on D. In such a case, the notation $f_i \cdot_D d := p(f_i \otimes d)$ will be used.

Remark 2.5.6. Sometimes the simpler notation $\cdot$ will be used over $\cdot_D$ when it is clear which product is being considered. It is important to note that there is almost no hope for an appropriate generalization of a DG-algebra even for general free flag resolutions. This is for at least two reasons: firstly, as already mentioned, there is no natural candidate for the tensor product of two differential modules, so one cannot employ a definition similar to Definition 2.5.5. Secondly, the "degree" of an element is not well-defined if it is induced by the flag filtration of arbitrary free flag $D = \bigoplus_{i\in\mathbb{Z}} F_i$, since any given $f_i \in F_i$ is contained in D^j for all $j \geq i$.

Observation 2.5.7. *If $F_\bullet$ is a complex admitting the structure of a DG-algebra, then* $\operatorname{Fold}(F_\bullet)$ *is a DG-module over $F_\bullet$.*

Proof. Just define the action on $\operatorname{Fold}(F_\bullet)$ via the product on $F_\bullet$. □

Observation 2.5.8. *Let $\phi\colon D \to D'$ be an isomorphism of differential modules and $F_\bullet$ a DG-algebra minimal free resolution of $H(D)$. If D' is a DG-module over $F_\bullet$, then D is a DG-module over $F_\bullet$ with the induced product:*

$$f_i \cdot_D d := \phi^{-1}\big(f_i \cdot_{D'} \phi(d)\big).$$

Moreover, ϕ becomes a morphism of DG-modules with this product.

Proof. The induced product is defined by making the following diagram commute:

$$\begin{array}{ccc} F_\bullet \boxtimes D & \xrightarrow{1\boxtimes\phi} & F_\bullet \boxtimes D' \\ \Big\downarrow{\scriptstyle \cdot_D} & & \Big\downarrow{\scriptstyle \cdot_{D'}} \\ D & \xleftarrow[\phi^{-1}]{} & D' \end{array}$$

□

Example 2.5.9. Let $R = k[x_1, x_2]$ and E be a rank 2 free module on the basis e_1, e_2. Let $D = \bigwedge^\bullet E$ be the free flag with differential

$$\begin{pmatrix} 0 & x_1 & x_2 & x_1^2 + x_2^2 \\ 0 & 0 & 0 & -x_2 \\ 0 & 0 & 0 & x_1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then D admits the structure of a DG-module over the Koszul complex $K_\bullet$ with the following product (any product not listed is understood to be 0):

$$1 \cdot_D d = d \text{ for all } d \in D, \quad e_1 \cdot_D 1 = e_1,\ e_2 \cdot_D 1 = e_2,$$

$$e_{12} \cdot_D 1 = e_{12} - x_1 e_1 - x_2 e_2,$$

$$e_1 \cdot_D e_2 = e_{12} - x_1 e_1 - x_2 e_2,$$

$$e_1 \cdot_D e_{12} = x_2 e_{12} - x_1 x_2 e_1 - x_2^2 e_2, \quad \text{and}$$

$$e_2 \cdot_D e_{12} = -x_1 e_{12} + x_1^2 e_1 + x_1 x_2 e_2.$$

Recall that by Theorem 2.3.6, the differential module D as in Example 2.5.9 is isomorphic to the fold of the Koszul complex on (x_1, x_2). In fact, the product given above is induced by this isomorphism. This construction works in general, that is if we have an isomorphism to a DG-algebra, we can obtain a DG-module structure in the same way. This leads to a string of immediate corollaries to Observation 2.5.8.

Corollary 2.5.10. *Let D be an anchored free flag and assume that $D \cong \operatorname{Fold}(F_\bullet)$ where $F_\bullet \to H(D)$ is a minimal free resolution. If $F_\bullet$ admits the structure of a DG-algebra, then D admits the structure of a DG-module over $F_\bullet$.*

Proof. This follows from Observation 2.5.8 and Observation 2.5.7. □

Corollary 2.5.11. *Let D be an anchored free flag and assume that the minimal free resolution $F_\bullet$ of $H(D)$ admits the structure of a DG-algebra. If the matrices $A_{i,0}$ satisfy $\operatorname{im} A_{i,0} \subset \operatorname{im} d_1$ for each $i \geq 2$, then D is a DG-module over $F_\bullet$.*

Proof. If $\operatorname{im} A_{i,0} \subset \operatorname{im} d_1$, then $D \cong \operatorname{Fold}(F_\bullet)$ by Lemma 2.3.3, so employ Corollary 2.5.10. □

Corollary 2.5.12. *Assume R is a regular graded local ring. Let D be a degree 0 anchored free flag with $H(D) \cong k$. Then D admits the structure of a DG-module over the minimal free resolution of k.*

Proof. The assumption that D has degree 0 implies that each matrix $A_{i,0}$ has entries in $\mathfrak{m}$, so employ Corollary 2.5.11. □

The above string of corollaries are all proved by reducing to the case that the differential module being considered may be realized as the folding of a DG-algebra resolution. The following theorem shows that this assumption is not only sufficient, but also *necessary* for free flags with complete intersection homology.

Theorem 2.5.13. *Let D be an anchored free flag with $H(D)$ a complete intersection. Let $K_\bullet$ denote the Kosul complex resolving $H(D)$. Then D is a DG-module over $K_\bullet$ if and only if $D \cong \operatorname{Fold}(K_\bullet)$.*

Proof. $\Longleftarrow$: This is clear by Observation 2.5.8.

$\Longrightarrow$: Choose $i := \min\{i > 1 \mid f_i \neq 0\}$, where $A_{i,0} =: f_i \in \bigwedge^i E^*$ are the defining data of components of the differential; if no such i exists, then $D \cong \operatorname{Fold}(K_\bullet)$ by Lemma 2.3.3 and there is nothing to prove, so assume i exists. Otherwise, Lemma 2.3.3 implies that we may assume $A_{k,j} = 0$ for all $k - j < i$. In particular, for all indexing sets I of size i, one has

$$d(e_I) = f_1(e_I) + f_i(e_I).$$

Assume that D has a DG-module structure over $K_\bullet$. By assumption $H(D) \cong R/\mathfrak{a}$ for some ideal $\mathfrak{a}$ that is generated by a regular sequence, and $f_1 \otimes R/\mathfrak{a} = 0$. One can choose the algebra structure such that $e_I \cdot e_J = e_I \wedge e_J + t_{i-1}$ for all $|I| + |J| \leq i$, where t_{i-1} is some element of $\bigoplus_{j \leq i-1} \bigwedge^j E$. This is because exactness of multiplication by f_1 forces $e_I \wedge e_J - p_{|I|+|J|}(e_I \cdot_D e_J)$ to be a boundary in the Koszul complex induced by f_1, where $p_{|I|+|J|} \colon D \to \bigwedge^{|I|+|J|} E$ denotes the projection onto the corresponding direct summand (and 0 is the only boundary with k-coefficients). Let e_I be any basis vector such that $f_i(e_I) \otimes R/\mathfrak{a} \neq 0$ (such an element must exist by selection of i), where $|I| = i$. Let ℓ be the first element of I and notice that:

$$\begin{aligned} f_1(e_I) + f_i(e_I) &= d(e_I) \\ &= d(e_\ell \cdot_D e_{I \backslash \ell} + t_{i-1}) \\ &= d(e_\ell) \cdot_D e_{I \backslash \ell} - e_\ell \cdot_D d(e_{I \backslash \ell}) + d(t_{i-1}). \end{aligned}$$

Tensoring the above relation with $R/\mathfrak{a}$, it follows that $f_i(e_I) \in \mathfrak{a}$, which is a contradiction to the assumptions. It follows that no DG-module structure can exist. $\square$

In view of Theorem 2.5.13, it follows that DG-module structures over the minimal free resolution of the homology are actually quite rare. Indeed, after running many examples it seems that the only time such a DG-module structure exists is if the free flag arises as the fold of the minimal free resolution, indicating that DG-module structures can be used to distinguish free flags that are in the isomorphism class of a complex. It is an interesting question as to whether there exists a family of structures, similar to a DG-module structure, that can be used to detect the isomorphism class of any anchored free flag. We conclude with the following (likely easier) question:

Question 2.5.14. *Does there exist an anchored free flag D admitting the structure of a module over the minimal free resolution of its homology $F_\bullet$ that is* not *isomorphic to* $\mathrm{Fold}(F_\bullet)$*?*

Chapter 3

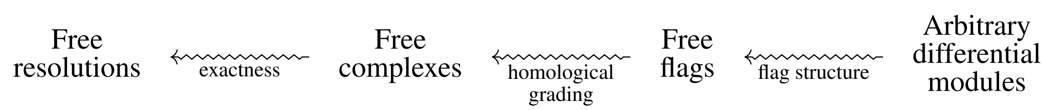
Differential Modules and Deformations of Free Complexes

Abstract

We develop a theory for deforming an arbitrary free complex into a differential module. The theory uses an iterative approach to parameterize the deformations and obstructions in terms of certain Ext groups, giving an algorithmic realization of a result of Brown-Erman. We apply this theory to study certain rigidity properties of free resolutions and related rank conjectures.

3.1 Introduction

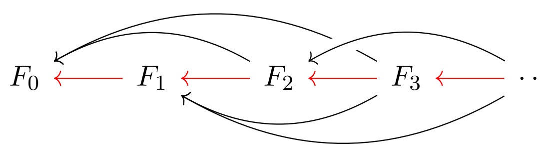
A differential module is a module equipped with a square-zero endomorphism. This square-zero map makes differential modules a natural generalizations of chain complexes—indeed, the category of differential modules is the category to which one sometimes must pass when the classical notion of $\mathbb{Z}$-graded complexes is not general enough. This phenomenon has been observed for instance in the multigraded analog of Koszul duality (see [HHW12], [BE21]) as well as in the literature on matrix factorizations, where it has been realized that one needs to "deform" complexes in order to represent certain functors on the category of matrix factorizations (see [OR20, Lemma 5.3.6]). Additionally, considering these more general objects can often shed new light on classical homological results by highlighting which properties of complexes are essential and which structure is extraneous. For instance, while arbitrary differential modules are quite general and lack the structure necessary to prove certain homological results, work of Avramov-Buchweitz-Iyengar [ABI07] shows that differential modules admitting the structure of a *flag* (see Definition 3.2.6) still possess many of the same properties as complexes. Differential modules with a flag structure are objects that lie somewhere between complexes—whose differentials respect a strict homological grading—and arbitrary differential modules. Work on the Buchsbaum-Eisenbud-Horrocks Conjecture on the Betti numbers of free resolutions and related conjectures of Halperin and Carlsson in algebraic topology [ABI07; IW18; Wal17] helps bring into focus a structural spectrum ranging from minimal free resolutions as the most specific objects to arbitrary differential modules as the most general. At each step, we see certain results that carry over to the next level of generality, while others fail.



The overarching goal motivating our work is to understand how the objects on the right-hand side of the picture related to the more well-studied objects on the left. Brown-Erman begin to examine this relationship in their work on minimal free resolutions of differential modules. They demonstrate a tight link between arbitrary differential modules and free resolutions by showing that (under suitable hypotheses on the ambient ring), every differential module admits a *minimal free resolution* that arises as a summand of a free flag whose structure is controlled by the minimal free resolution of the homology (see [BE22, Theorems 3.2 and 4.2]). These results demonstrate that the theory of arbitrary differential modules can be related back to the theory of minimal free resolutions, while at the same time raising more questions about the nature of this relationship—what properties of the homology can be 'lifted' to properties of the differential module? How much variation can there be in the behavior of differential modules with the same homology? What new geometric data (if any) do we add by passing from a minimal free resolution to a more general differential module with the same homology? To begin to answer these questions, we focus first on the following:

Question 3.1.1. *What are the possible differential modules with homology isomorphic to a given module? When are two such differential modules (quasi)isomorphic?*

The key results of Brown-Erman allow us to pass from arbitrary differential modules to a special class of free flags—those that are *anchored on* the minimal free resolution of their homology (see Definition 3.2.6)—so long as we work up to quasi-isomorphism. This reduces our question to identifying the possible free flags that are anchored on the minimal free resolution of their homology. A free flag may be pictured as a free complex with additional compatible maps that "go with the flow" of the complex's original differential but do not strictly respect the homological grading. The *anchor* of the flag is the underlying complex, shown in red in the following visualization.



Thus, the question at hand can be restated as follows:

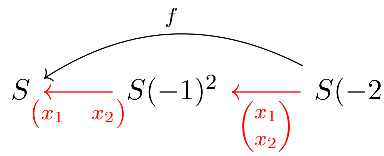
Question 3.1.2. *If we start with a minimal free resolution, what are all of the ways that we can add in additional maps so that the total differential still squares to zero* and *the homology remains unchanged?*

Even in a small case, this question can be subtle.

Example 3.1.3. Let $S = \Bbbk[x_1, \dots, x_n]$ and (D, ∂_f) be the differential module with underlying module $D = S \oplus S(-1)^2 \oplus S(-2)$ and differential given by the matrix

$$\partial_f = \begin{pmatrix} 0 & x_1 & x_2 & f \\ 0 & 0 & 0 & -x_2 \\ 0 & 0 & 0 & x_1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

where f is some homogeneous degree 2 polynomial. For all choices of f, the homology of (D, ∂_f) is isomorphic to $S/(x_1, x_2)$ and (D, ∂_f) is anchored on the minimal free resolution of its homology, which we can visualize like this:



It is not too hard to see that when $f \in (x_1, x_2)$, $D_f \simeq D_0$ since we can turn the matrix ∂_f into the matrix ∂_0 via elementary row and column operations. But when $f \notin (x_1, x_2)$ we get a differential module D_f that is not isomorphic to D_0, meaning that we have at least two different flagged isomorphism classes of free flags anchored on the minimal free resolution of $S/(x_1, x_2)$. Now, we ask: given two (nonzero) degree 2 polynomials f, g are D_f and D_g isomorphic? To answer this question amounts to checking for similarity of matrices over the polynomial ring in n variables, which is in general a hard problem. Note that it is easy to parameterize the set of free flags in the previous example that are anchored on the minimal free resolution of $S/(x_1, x_2)$ *if we don't care about isomorphism*—such free flags are parameterized by homogeneous degree 2 polynomials. However, even in this small example it is not clear how to account for isomorphism in such a parameterization.

One approach to this problem was taken in [BV22], where the authors construct a large family of differential modules anchored on the Koszul complex and prove that under certain hypotheses any differential module with complete intersection homology is quasi-isomorphic to one of these "Koszul differential modules". That being said, there are examples of free flags anchored on the Koszul complex that *do not* arise as a generalized Koszul differential module (see Example 4.7 of [BV22]), so this is not a complete characterization. Moreover, this approach leaves open the fundamental questions about the structure theory of differential modules mentioned above.

In this paper, we present a totally different approach for answering Question 3.1.1 that also seems to be a powerful tool for studying the uniform and asymptotic behavior of free flags with a fixed anchor. We furthermore recover and answer many of the results and questions posed in the work [BV22], and further clarify the deep connection between the classical homological theory of modules and that of differential modules. Our approach is motivated by deformation theory. Brown-Erman point out that a free flag can be deformed to its anchor by adjoining a variable and zeroing out all of the maps above the anchor [BE22, Remark 3.1]. We investigate the reverse direction of this observation: Given a free complex $\mathbf{F}$, we can deform $\mathbf{F}$ into a free flag by adding additional maps in a compatible way. We give a sense of the core idea below, reserving the full rigor and detail for later in the paper.

3.1.1 Example: A Deformation Theoretic Approach

Suppose we have a complex

$$F_0 \xleftarrow{\partial_{1,0}} F_1 \xleftarrow{\partial_{2,1}} F_2 \xleftarrow{\partial_{3,2}} F_3 \xleftarrow{\partial_{4,3}} F_4.$$

We consider this as a differential module with underlying module $F_0 \oplus F_1 \oplus F_2 \oplus F_3 \oplus F_4$ and differential given by the following block matrix.

$$\begin{pmatrix} 0 & \partial_{1,0} & 0 & 0 & 0 \\ 0 & 0 & \partial_{2,1} & 0 & 0 \\ 0 & 0 & 0 & \partial_{3,2} & 0 \\ 0 & 0 & 0 & 0 & \partial_{4,3} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We want to think about deforming this complex by iteratively adding maps to the upper right blocks of this matrix. To do this, we adjoin a new variable t to our ring and add maps $\partial_{i,i-2}$ with t as a coefficient to get the matrix

$$\begin{pmatrix} 0 & \partial_{1,0} & t\partial_{2,0} & 0 & 0 \\ 0 & 0 & \partial_{2,1} & t\partial_{3,1} & 0 \\ 0 & 0 & 0 & \partial_{3,2} & t\partial_{4,2} \\ 0 & 0 & 0 & 0 & \partial_{4,3} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Requiring that this matrix square to 0 mod t^2 imposes the relations

$$\partial_{1,0}\partial_{3,1} + \partial_{2,0}\partial_{3,2} = 0$$

$$\partial_{2,1}\partial_{4,2} + \partial_{3,1}\partial 4, 3 = 0$$

which characterize the possible choices for $\partial_{i,i-2}$. Next we add maps $\partial_{i,i-3}$ with a coefficient of t^2 to get the matrix

$$\begin{pmatrix} 0 & \partial_{1,0} & t\partial_{2,0} & t^2\partial_{3,0} & 0 \\ 0 & 0 & \partial_{2,1} & t\partial_{3,1} & t^2\partial_{4,1} \\ 0 & 0 & 0 & \partial_{3,2} & t\partial_{4,2} \\ 0 & 0 & 0 & 0 & \partial_{4,3} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Requiring this to square to 0 mod t^3 imposes a new relation

$$\partial_{1,0}\partial_{4,1} + \partial_{2,0}\partial_{4,2} + \partial_{3,0}\partial_{4,3} = 0.$$

This characterizes possible choices for $\partial_{3,0}$ and $\partial_{4,1}$, but it also introduces an obstruction: we may have chosen the $\partial_{i,i-2}$ maps in such a way that there is *no* compatible choice at the next step! For the final step, notice that any choice of map $t^3, \partial_{4,0}$ in the top right corner of the matrix will be compatible with the square-zero condition mod t^4. Furthermore any such matrix that squares to 0 mod t^4 actually squares to 0 on the nose, since the square doesn't have any entries with a power of t larger than 3 (in fact, no entry has a power of t larger than 2). Setting $t = 1$ we get a free flag, anchored on the complex we started with, whose differential is

$$\begin{pmatrix} 0 & \partial_{1,0} & \partial_{2,0} & \partial_{3,0} & \partial_{4,0} \\ 0 & 0 & \partial_{2,1} & \partial_{3,1} & \partial_{4,1} \\ 0 & 0 & 0 & \partial_{3,2} & \partial_{4,2} \\ 0 & 0 & 0 & 0 & \partial_{4,3} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

What is more, we will see later on that the homology remains unchanged at each step, and that any free flag anchored on the minimal resolution of its homology can be obtained via a similar process. One way to understand the possibilities for such a flag is thus to understand the possible deformations and obstructions at each step in the process outlined above. Our core question now becomes

Question 3.1.4. *What are all of the ways to deform a complex to a free flag? What are the* *obstructions to deforming, and when do these obstructions vanish?*

3.1.2 Results

The main results of this paper constitute a complete answer to Question 3.1.1 if we impose the condition that isomorphisms of free flags should be *flag-preserving* (See 3.2.8 for a discussion of this additional assumption) via a deformation-theoretic approach. We state our results here for differential modules with a given homology, but note that in Section 3.3 we actually prove the following results in slightly more generality.

Theorem 3.1.5. *Let M be a module over S with finite minimal free resolution* $\mathbf{F}$. *Every differential module with homology M is quasi-isomorphic to one that can be realized as a deformation of* $\mathbf{F}$. *Furthermore, both the deformations of* $\mathbf{F}$ *and the obstructions to deforming are represented by classes in* $\mathrm{Ext}_S^{\bullet}(M, M)$.

When we work over an algebraically closed field, this description of the deformations and obstructions yields a simple geometric description of the quasi-isomorphism classes of differential modules with homology M.

Theorem 3.1.6. *Assume $\Bbbk$ is algebraically closed and let $\mathcal{X}_a^M$ denote the set of degree a differential modules with homology M, up to flagged quasi-isomorphism. Then $\mathcal{X}_a^M$ has the structure of an algebraic variety over $\Bbbk$ whose dimension is bounded above by*

$$\sum_{i=2}^{\ell} \dim_{\Bbbk} \mathrm{Ext}_S^i(M, M)_{a-ia}$$

and below by

$$\sum_{i=2}^{\ell} \dim_{\Bbbk} \mathrm{Ext}_S^i(M, M)_{a-ia} - \sum_{i=4}^{\ell} \dim_{\Bbbk} \mathrm{Ext}_S^i(M, M)_{2a-ia}.$$

Corollary 3.1.7. *Assume $\Bbbk$ is algebraically closed and M is any S-module with minimal free resolution* $\mathbf{F}$. *If*

$$\mathrm{Ext}_S^i(M, M)_{a-ia} = 0$$

for all $2 \leq i \leq \mathrm{pdim}(M)$*, then every degree* a *differential module with homology* M *is quasi-isomorphic to* $\mathrm{Fold}^a(\mathbf{F})$ *(in other words,* M *is* a-rigid*; see Definition 3.4.1).*

Note that it is easy to write down a set of equations that cuts out the set of all free flags anchored on a fixed complex using only the condition that the differential squares to zero, but it is not at all clear how to write down a set of equations that also accounts for isomorphism class. Indeed, if we were to naively take the variety cut out by these equations and try to just mod out by isomorphism of differential modules, there is absolutely no indication, a priori, that the resulting object would be a variety. Our approach sidesteps this particular challenge by constructing $\mathcal{X}_a^M$ iteratively as a sequence of deformations, then showing that the isomorphism class of two such deformations at each step is controlled by higher Ext classes of the module M.

Our approach also yields several applications, one of which is that, for all but finitely many values of a, the study of degree a differential modules with finite length homology reduces to the study of minimal free resolutions of finite length S-modules. In particular:

Theorem 3.1.8. *Let* M *be a module of finite length. For* $|a| \gg 0$*, every free degree* a *differential module with homology* M *is quasi-isomorphic to the minimal free resolution of* M*.*

We prove this, as well as other applications and examples, in Section 3.4. More generally, we enumerate a list of conditions that is sufficient for the space parameterizing the isomorphism classes of free flags anchored on a fixed complex to consist of a single point, and in the process recover (and generalize) previous results proved in [BV22] with a completely different method. Finally, we discuss some applications of our work to rank conjectures for differential modules. We give some results about the existence of free flag differential modules whose total Betti number is strictly smaller than the sum of the Betti numbers of the homology; as it turns out, our characterization of the "deformation" terms gives a straightforward criterion in special cases for checking whether a differential module with unexpectedly small Betti numbers exists.

The paper is organized as follows. In Section 3.2, we establish the necessary background and conventions on (free flag) differential modules that will be essential for the rest of the paper. In Section 3.3 we consider an iterative process of constructing free flags with a fixed anchor and connect this process to the (non)triviality of certain cohomology classes of the associated endomorphism complex. After building up the necessary machinery, we prove our main result of the paper, Theorem 3.3.14, and note some interesting consequences for the derived category of differential modules. In Section 3.4, we apply the results established in Section 3.3 to understand multiple different aspects of free flag differential modules, including rigidity, Betti deficiency, and the connection between higher structure maps and the notion of systems of higher homotopies for matrix factorizations.

3.2 Anchored free flags

In this section, we introduce some background and notation on differential modules that will be used throughout the paper. From now on, unless otherwise specified, the notation S will denote any Noetherian graded-local ring, by which we mean an $\mathbb{N}$-graded ring (possibly concentrated in degree 0) whose degree 0 component is local with residue field $\mathbb{k}$. All modules considered throughout this paper will be finitely generated.

Let us first recall the notion of a *homogeneous* differential module:

Definition 3.2.1. A *differential module* (D, d) or (D, d^D) is an S-module D equipped with an S-endomorphism $d = d^D\colon D \to D$ that squares to 0. A differential module is graded of (internal) degree a if D is equipped with a grading over S such that $d\colon D \to D(a)$ is a graded map. The category of degree $\mathbf{a}$ differential S-modules will be denoted $\mathrm{DM}(S, a)$.

The *homology* of a differential module (D, d) is defined to be $\ker(d)/\operatorname{im}(d)$. If D is graded of degree a, then the homology is defined to be the quotient $\ker(d)/\operatorname{im}(d(-a))$.

A differential module is *free* if the underlying module D is a free S-module, and D is *minimal* if $d \otimes_S \mathbb{k} = 0$ (that is, the differential module D is minimal if its squarezero endomorphism is minimal).

A morphism of differential modules $\phi\colon (D, d^D) \to (D', d^{D'})$ is a morphism of S-modules $D \to D'$ satisfying $d^{D'} \circ \phi = \phi \circ d^D$. Notice that morphisms of differential modules induce well-defined maps on homology in an identical fashion to the case of complexes. A morphism of differential modules is a *quasi-isomorphism* if the induced map on homology is an isomorphism.

Example 3.2.2. Every homogeneous complex of S-modules may be viewed as a degree 0 differential module.

Remark 3.2.3. Differential modules are essentially the homologically ungraded analog of complexes, and as such they can vary much more wildly than complexes. The useful analogy here is the difference between graded rings versus ungraded rings: a differential S-module is equivalently a $S[t]/(t^2)$-module; on the other hand, the ring $S[t]/(t^2)$ may be viewed as a $\mathbb{Z}$-graded ring by assigning $\deg(t) = 1$. Complexes are thus equivalently described as $\mathbb{Z}$-graded $S[t]/(t^2)$-modules with respect to this grading, and homogeneous complexes are equivalently graded $S[t]/(t^2)$-modules where t is assigned the bidegree $(1, 0)$ (the first component denotes homological degree, and the second component internal degree).

Definition 3.2.4. Let S be a (positively) graded ring. The *degree a fold* is the functor:

$$\begin{aligned} \mathrm{Fold}_a \colon \ \mathrm{Ch}(S) &\to \mathrm{DM}(S, a) \\ (C_\bullet, \partial) &\mapsto \left(\bigoplus C_i(ia), \partial \right), \end{aligned}$$

where in the above $\mathrm{Ch}(S)$ is the category of homogeneous complexes of degree 0.

Remark 3.2.5. Notice that by construction the degree a fold of a homogeneous complex yields a differential module of degree a. In particular, the category of homogeneous chain complexes of S-modules not only embeds into the category of differential modules, but it also embeds into $\mathrm{DM}(S, a)$ for all $a \in \mathbb{Z}$. On the other hand, there is no clear relation between the categories $\mathrm{DM}(S, a)$ and $\mathrm{DM}(S, b)$ for different values of a and b; this is at least indicated by the fact that for a fixed module M, the Betti numbers of a differential module $D \in \mathrm{DM}(S, a)$ with $H(D) = M$ may change as a varies (see [BV22, Proposition 3.8]).

Next, we define the notion of a *free flag*. These can be thought of as a useful intermediary between chain complexes of free S-modules and general differential modules, and as we shall see are an important ingredient for generalizing free resolutions to the category of differential modules.

Definition 3.2.6. Let D be a differential module. Then D is a *free flag* if D admits a splitting $D = \bigoplus_{i\in\mathbb{Z}} F_i$, where each F_i is a free S-module, $F_i = 0$ for $i \leq 0$, and $d_D(F_i) \subseteq \bigoplus_{j<i} F_j$.

Every free flag is naturally filtered by its flag grading $D^0 = D_0 \subset D^1 \subset D^2 \subset \cdots$ by defining

$$D^i := \bigoplus_{j\leq i} F_j.$$

The spectral sequence associated to this filtration has E^0-page given by a complex with differentials $d_i^F : D^i/D^{i-1} = F_i \to D^{i-1}/D^{i-2} = F_{i-1}$. This complex is called the *anchor* of the free flag D.

A morphism of free flags $\phi : D \to D'$ is *flagged* or *flag-preserving* if $\phi(D^i) \subset D'^i$ for each $i \geq 0$. The *flagged quasi-isomorphism class* of an arbitrary differential module D is the set of differential modules D' quasi-isomorphic to D and such that there exists a flag-preserving quasi-isomorphism between free flag resolutions of D and D', respectively.

Remark 3.2.7. The squarezero endomorphism of the free flag may be represented as a strictly upper triangular block matrix, where the first off-diagonal blocks list the differentials of the anchor.

Remark 3.2.8. In general, an arbitrary morphism of free flags in the category of differential modules need not preserve the flag structure. Imposing this flagged assumption firstly ensures that kernels/cokernels/images of morphisms between free flags still have a flag structure, but there is an alternative interpretation of flagged differential modules for which flag-preserving morphisms arise naturally.

Let t be an indeterminate assigned homological degree 2, and consider the algebra $A := S[t]/(t^2)$. Recall that a (homologically indexed) A_∞-module D over A is a graded A-module equipped with maps $m_j^D : A_+^{\otimes j-1} \otimes_S D \to D$ of degree $j-2$ satisfying the *Stasheff identities* (see [Kel01]). Since $A_+ \cong S \cdot t$ is concentrated in homological degree 2, it follows that there are induced maps

$$d_j^D|_{D_i} := m_j^D \left(\underbrace{t \otimes \cdots \otimes t}_{j-1 \text{ copies}} \otimes - \right) |_{D_i} : D_i \to D_{i-j}.$$

Moreover, since A is being viewed as an A_∞-module with trivial higher multiplications, the Stasheff identities simplify significantly and we obtain the equalities

$$\sum_{j=1}^{i+1}(-1)^j d_j^D \circ d_{i-j+2}^D = 0 \quad \text{for all } i \geq 0.$$

Up to a sign convention (made precise in subsection 3.3.3) these identities equivalently define the data of a flag structure on the complex (D, m_1^D); thus a free flag is equivalently an A_∞-module over A whose underlying S-module is free. Morphisms $\phi : D \to D'$ of A_∞-modules are collections $\phi_j : A_+^{\otimes j-1} \otimes D \to D'$ of degree $j-1$ satisfying a similar set of Stasheff-type identities. Again, in the simpler setting that we are in these identities reduce to

$$\sum_{i+j=k} \psi_i \circ d_j^D = \sum_{i+j=k} (-1)^i d_j^D \circ \psi_i,$$

where $\psi_i := \phi_i(\underbrace{t \otimes \cdots \otimes t}_{i-1 \text{ times}} \otimes -) : D \to D[-i+1]$ is a morphism of the underlying modules for each $i \geq 1$. Thus, flag-preserving morphisms are a special case of the notion of A_∞-module morphisms, where the ambient A_∞-algebra is particularly simple.

The next proposition shows that "deforming" a resolution does not change homology, a fact that we will take advantage of later when studying quasi-isomorphism classes of differential modules.

Proposition 3.2.9. *Let D be a free flag and assume that the anchor of D is a free resolution of some module M. Then $H(D) \cong M$.*

Proof. Define

$$H(D)^i := \operatorname{Im}\left(H(D^i) \to H(D)\right).$$

By definition there is an equality

$$H(D) = \bigcup_{i \geq 0} H(D)^i$$

and by [ABI07, Section 2.6] a spectral sequence

$$E_i^2 = H_i(F) \implies H(D),$$

where F denotes the underlying anchor of D. Since $E_i^2 = 0$ for $i \neq 0$ by assumption, the spectral sequence degenerates on the second page and there are equalities

$$H(D)^0 = H_0(F), \quad H(D)^i = H(D)^{i-1} \quad \text{for } i > 0.$$

It follows that $H(D) = H_0(F) = M$. □

Remark 3.2.10. More generally, the above shows that if a free flag D is anchored on a complex with finite length homology, then $H(D)$ is a module of finite length (really, it suffices for any page of the associated spectral sequence to eventually have finite length homology; this is pointed out in [ABI07]).

To finish this section, we state a fundamental result of Brown-Erman. One way to interpret this statement is that if one is willing to work up to quasi-isomorphism, then the homological theory of differential modules is anchored on the classical theory of free resolutions of modules.

Theorem 3.2.11 ([BE22, Theorem 3.2])**.** *Let D be a differential module and $(F_\bullet, d) \to H(D)$ a minimal free resolution of $H(D)$. Then D admits a free flag resolution anchored on the complex F.*

Notation. For convenience, we write δ_j for the map $F \to F$ that, when restricted to F_i is equal to $\partial_{i,i-j}$ for each i (in general, we assume that $F_i = 0$ for $i < 0$ or $i > \ell$ so any maps to or from such F_i are defined to be the zero map).

Remark 3.2.12. Suppose that $\mathbf{F}_\bullet$ is a graded complex. Let $\mathbf{F}_\bullet^a$ be the complex obtained by twisting F_i by ia so that all maps are homogeneous of degree a. A free flag anchored on $\mathbf{F}_\bullet^a$ has underlying module $\bigoplus_i F_i(ai)$ and differential ∂ of degree a. For each $i > j$, ∂ restricts to a degree a map $\partial_{i,j} : F_i(ia) \to F_j(ja)$. It will sometimes be helpful to remember that a degree a map $F_i(ia) \to F_j(ja)$ is the equivalently a degree $a(j - i + 1)$ map $F_i \to F_j$.

3.3 Parameterizing Anchored Free Flags

3.3.1 Building Free flags

We describe a process for building an anchored free flag with a fixed homology M and prove that all free flags anchored on the minimal free resolution of M arise in this way.

To begin, we introduce the first main character of this construction, the *endomorphism complex*. We will find that all of the higher differentials of a free flag (and their obstructions) may be reinterpreted as elements of this complex.

Definition 3.3.1. Let $\mathbf{F}$ be a complex of S-modules (indexed homologically). The *endomorphism complex* $\operatorname{End}_S^\bullet(\mathbf{F})$ is the (cochain) complex with

$$\operatorname{End}_S^i(\mathbf{F}) := \prod_{n\in\mathbb{Z}} \operatorname{Hom}_S(\mathbf{F}_n, \mathbf{F}_{n-i}), \quad \text{and differential}$$

$$d^{\operatorname{End}_S^\bullet(\mathbf{F})}(\phi) := d^{\mathbf{F}} \circ \phi - (-1)^{|\phi|}\phi \circ d^{\mathbf{F}}.$$

Equipped with function composition, the complex $\operatorname{End}_S^\bullet(\mathbf{F})$ is naturally an associative DG-algebra. It is useful to note that

$$Z^i(\operatorname{End}_S^\bullet(\mathbf{F})) = \operatorname{Hom}_{\operatorname{Ch}(S)}(\mathbf{F}, \mathbf{F}[-i]),$$

$$B^i(\operatorname{End}_S^\bullet(\mathbf{F})) = \text{nullhomotopies } \mathbf{F} \to \mathbf{F}[-i].$$

If $\mathbf{F}$ is a free resolution of some S-module M, there is an isomorphism

$$H^i(\operatorname{End}_S^\bullet(\mathbf{F})) = \operatorname{Ext}_S^i(M, M).$$

The following notation will be tacitly used for the remainder of this section:

Notation. Let t denote any arbitrary indeterminate. The notation S_i will denote the ring

$$S_i := \frac{S[t]}{(t^i)}.$$

The following construction seeks to make precise the idea of iteratively deforming a complex

$\mathbf{F}$ to obtain a free flag anchored on $\mathbf{F}$. In doing this construction, one needs to work in the *truncated* polynomial ring S_i where i increases at each step, since this will allow us to guarantee that the resulting differential module still has squarezero endomorphism.

Construction 3.3.2. Let $\mathbf{F}$ be any homogeneous complex of free S-modules

$$\mathbf{F}: \quad F_0 \xleftarrow{\partial_{1,0}} F_1 \xleftarrow{\partial_{2,1}} \cdots \xleftarrow{\partial_{\ell,\ell-1}} F_\ell$$

with $(F, d_1) = \mathrm{Fold}_a(\mathbf{F})$.

One can iteratively build a free flag (F, ∂) anchored on $\mathbf{F}$ by filling in the matrix ∂ diagonal by diagonal, at the i^{th} step lifting from a differential module over S_{i-1} to one over S_i.

Suppose we have built up a differential module $(F \otimes S_{i-1}, d_{i-1}) \in \mathrm{DM}(S_{i-1}, a)$ anchored on $\mathbf{F} \otimes S_{i-1}$ where

$$d_{i-1} = \begin{pmatrix} 0 & \partial_{1,0} & t\partial_{2,0} & \cdots & t^{i-2}\partial_{i-1,0} & 0 & \cdots & 0 \\ 0 & 0 & \partial_{2,1} & t\partial_{3,1} & \cdots & t^{i-2}\partial_{i,1} & \cdots & 0 \\ 0 & 0 & 0 & \partial_{3,2} & t\partial_{4,2} & \cdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & \partial_{4,3} & \ddots & \cdots & t^{i-2}\partial_{\ell,\ell-i+1} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & \partial_{\ell-1,\ell-2} & t\partial_{\ell,\ell-2} \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \partial_{\ell,\ell-1} \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

If $(F \otimes_S S_{i-1}, d_{i-1})$ can be lifted to a differential module $(F \otimes_S S_i, d_i)$, we then lift by adding maps $\{t^{i-1}\partial_{j+i,j}\}_{j=0}^{\ell-i}$ to d_{i-1} above (adding on the next off-diagonal after the t^{i-2} terms). By the ℓ^{th} step we have a differential module $(F \otimes_S S_\ell, d_\ell) \in \mathrm{DM}(S_\ell, a)$. By setting $t = 1$, we obtain a differential module in $(F, \partial) \in \mathrm{DM}(S, a)$ anchored on $\mathbf{F}^a$.

We note that it will sometimes be more convenient to package together the maps $\{\partial_{j+i,j}\}_{j=0}^{\ell-i}$ as one element $\delta_i \in \mathrm{End}^i(\mathbf{F})$.

Remark 3.3.3. With notation as in the above construction, assign the variable t a homological grading by setting $\deg(t) = 1$ (and hence cohomological degree -1). The maps δ_j are a priori elements of $\mathrm{End}^j(\mathbf{F})$, and the process of rescaling yields the map $t^{j-1}\delta_j \in \mathrm{End}^1_{S_i}(\mathbf{F} \otimes_S S_i)$. In other words, this rescaling process forces all of the maps to be homogeneous of cohomological degree 1 and thus induces a $\mathbb{Z}$-grading on the differential module $(F \otimes_S S_i, d_i)$. This $\mathbb{Z}$-grading in fact agrees with the $\mathbb{Z}$-grading induced by the flag grading, but the introduction of the t-variable makes the bookkeeping much simpler. Similar rescaling tricks have also been used to induce $\mathbb{Z}$-gradings on $\mathbb{Z}/2\mathbb{Z}$-graded complexes (see for instance the discussion on page 2182 of [Bro+17]), and is intimately related to the idea of "unfolding" a $\mathbb{Z}/2\mathbb{Z}$-graded object.

Lemma 3.3.4. *Every free flag anchored on* $\mathbf{F}$ *may be obtained via Construction 3.3.2.*

Proof. Let $\mathbf{F} = \quad F_0 \xleftarrow{\partial_{1,0}} F_1 \xleftarrow{\partial_{2,1}} \cdots \xleftarrow{\partial_{\ell,\ell-1}} F_\ell \leftarrow 0$ be any complex of free S-modules. An anchored free flag F on $\mathbf{F}$ has differential

$$\partial = \begin{pmatrix} 0 & \partial_{1,0} & \partial_{2,0} & \cdots & \partial_{\ell,0} \\ 0 & 0 & \partial_{2,1} & \cdots & \partial_{\ell,1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \partial_{\ell,\ell-1} \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

Rescaling appropriately by powers of t to ∂, we obtain a differential module over $S[t]$ whose differential is

$$\partial' = \begin{pmatrix} 0 & \partial_{1,0} & t\partial_{2,0} & \cdots & t^{\ell-1}\partial_{\ell,0} \\ 0 & 0 & \partial_{2,1} & \cdots & t^{\ell-2}\partial_{\ell,1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \partial_{\ell,\ell-1} \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

and which specializes to (F, ∂) when $t = 1$. Note that for each i, $(F \otimes S_i, \partial')$ is a differential module over S_i, so this is the differential module we end up with if we run Construction 3.3.2 and at the i^{th} step pick the maps $\partial_{j,j-i}$ as our augmenting maps. □

The main result of this section is Theorem 3.1.5 parameterizing the lifts and obstructions in Construction 3.3.2 in terms of some cohomology groups of the endomorphism complex. We restate the theorem precisely in its full generality here.

Theorem 3.3.5. *The possible lifts of $F \otimes S_{i-1}, d_{i-1})$ to $\mathrm{DM}(S_i, a)$ as in Construction 3.3.2 are parameterized by $H^i(\mathrm{End}_S^\bullet \mathbf{F})_{a-ia}$. The obstructions to lifting correspond to classes in $H^{i+1}(\mathrm{End}_S^\bullet \mathbf{F})_{2a-ia}$.*

Proof. The proof is via the lemmas in the following subsection, which we combine to get the result thusly: In Lemma 3.3.8 we parameterize the lifts at the i^{th} step. We identify these lifts with the appropriate cohomology groups of the endomorphism complex of **F** in Lemma 3.3.9. We characterize the existence of lifts by the vanishing of the appropriate cohomology classes in Lemma 3.3.6. This yields the result. □

3.3.2 Parameterizing Lifts and Obstructions

Lemma 3.3.6. *The free flag $(F \otimes S_{i-1}, d_{i-1})$ can be lifted to a free flag $(F \otimes S_i, d_i)$ by adding maps $\{t^{i-1}\partial_{j+i,j}\}_{j=0}^{\ell-i}$ if and only if a certain map $\mathbf{F} \to \mathbf{F}[-i-1]$ is nullhomotopic (in other words, a certain class in $H^{i+1}(\mathrm{End}_S^\bullet(\mathbf{F}))$ is 0).*

Proof. Let d_i be the map obtained by adding the maps $t^{i-1}\delta_i$ to d_{i-1} as in Construction 3.3.2. This gives a differential module over $F \otimes_S S_i$ if and only if $d_i^2 = 0 \mod (t^{i-1})$. Assuming that $d_{i-1}^2 = 0$ mod t^{i-2}, this happens exactly when the following condition is satisfied:

$$\sum_{j=1}^{i} \delta_j \delta_{i+1-j} = 0$$

This can rewritten as

$$\delta_1\delta_i + \delta_i\delta_1 = -\sum_{j=2}^{i-1} \delta_j \delta_{i+1-j} \tag{3.1}$$

We claim that the right hand side of (3.1) defines a chain map $\mathbf{F} \to -\mathbf{F}[-i-1]$. This means that one must verify that there is an equality

$$\left(\sum_{j=2}^{i-1} \delta_j \delta_{i+1-j}\right) \delta_1 \stackrel{?}{=} \delta_1 \left(\sum_{j=2}^{i-1} \delta_j \delta_{i+1-j}\right).$$

By assumption $(F \otimes S_{i-1}, d_{i-1})$ is a differential module, which means that for any $2 \le p \le i-1$

there are equalities

$$\delta_p\delta_1 = -\sum_{k=1}^{p-1}\delta_k\delta_{p+1-k} \quad \text{and} \quad \delta_1\delta_p = -\sum_{\ell=2}^{p}\delta_\ell\delta_{p+1-\ell}.$$

Using this, we rewrite each of the sums above to get

$$\sum_{j=2}^{i-1}\sum_{k=1}^{i-j}\delta_j\delta_k\delta_{i+2-j-k} \stackrel{?}{=} \sum_{j=2}^{i-1}\sum_{\ell=2}^{j}\delta_\ell\delta_{j+1-\ell}\delta_{i+1-j}.$$

Switching the order of the sums on the right, we find

$$\sum_{j=2}^{i-1}\sum_{k=1}^{i-j}\delta_j\delta_k\delta_{i+2-j-k} \stackrel{?}{=} \sum_{\ell=2}^{i-1}\sum_{j=\ell}^{i-1}\delta_\ell\delta_{j+1-\ell}\delta_{i+1-j}.$$

Finally, reindex the righthand sum by first replacing j by $k+\ell-1$ to obtain

$$\sum_{j=2}^{i-1}\sum_{k=1}^{i-j}\delta_j\delta_k\delta_{i+2-j-k} \stackrel{?}{=} \sum_{\ell=2}^{i-1}\sum_{k=1}^{i-\ell}\delta_\ell\delta_k\delta_{i+2-\ell-k}.$$

Replacing ℓ on the right with j, we see that this is indeed an equality, so the right hand side of (3.1) is a chain map as claimed. This means that (3.1) is exactly the condition that the the maps $\{\partial_{j+i,j}\}$ are a nullhomotopy for this map $\mathbf{F} \to \mathbf{F}[-i-1]$. This chain map corresponds to a class in $H^{i+1}(\mathrm{End}_S^\bullet(\mathbf{F}))$. By Remark 3.2.12, the map δ_j has degree $a - ja$, so the chain map in (3.1) has degree $a - ia$ when considering the cohomology as a graded S-module. □

Remark 3.3.7. Viewing the endomorphism complex $A := \mathrm{End}_S^\bullet(F)$ as a DG S-algebra in the standard way, the condition (3.1) is (up to some sign changes which we will not get into) equivalent to the condition

$$d^A(\delta_i) = \sum_{j=2}^{i-1}(-1)^{j+1}\delta_j\delta_{i+1-j}.$$

This means that in the cohomology algebra $H^\bullet(A)$, all of the classes $\sum_{j=2}^{i-1}(-1)^{j+1}\delta_j\delta_{i+1-j}$ are trivial, and hence the Massey powers (in the sense of [Kra66, Section 3]) $\langle\delta_2\rangle^k$ are trivial for all $k \geq 2$. Since Massey powers are known to give well-defined cohomology classes, this yields an alternative proof that the expression on the righthand side of (3.1) is a morphism of complexes.

In a similar vein, the following lemma shows that the isomorphism classes of two different lifts as in Construction 3.3.2 are also parametrized by cohomology classes of $\mathrm{End}_S^\bullet(\mathbf{F})$:

Lemma 3.3.8. *If lifts exist as in Lemma 3.3.6, then the possible lifts are parameterized by*

$$\langle t^{i-1}\rangle H^i(\mathrm{End}_{S_i}^\bullet(\mathbf{F}\otimes_S S_i))_{a-ia}.$$

Proof. The statement will follow by proving

(1) the difference of any two lifts corresponds to a well-defined cohomology class on $\mathrm{End}_S^\bullet(\mathbf{F})$, and

(2) if this difference is 0 in cohomology (that is, the maps are homotopic), then the two lifts give isomorphic differential modules.

Suppose we have d_i and d_i' that are both lifts of d_{i-1} via maps $\{t^{i-1}\partial_{j+i,j}\}_{j=0}^{\ell-i}$ and $\{t^{i-1}\partial'_{j+i,j}\}_{j=0}^{\ell-i}$ respectively. Since both satisfy (3.1), their difference $\{t^{i-1}\partial_{j+i,j}-t^{i-1}\partial'_{j+i,j}\}_{j=0}^{\ell-i}$ defines a chain map

$$\mathbf{F}\otimes S_i\to -\mathbf{F}[-i]\otimes S_i$$

which by Remark 3.2.12 belongs to a class in $H^i(\mathrm{End}_{S_i}^\bullet(\mathbf{F}\otimes_S S_i))_{a-ia}$. Suppose this map is nullhomotopic. Then there exist $h_{j+i-1,j}:F_j\otimes S_i\to F_{j-i+1}\otimes S_i$ for each j so that

$$t^{i-1}\partial_{j+i,j}-t^{i-1}\partial'_{j+i,j}=t^{i-1}h_{j+i-1,j}\partial_{j+i,j+i-1}-t^{i-1}\partial_{j+1,j}h_{j+i,j+1}. \tag{3.2}$$

Let P_i is the matrix with 1s on the diagonal and whose entry in the p^{th} row and q^{th} column is $t^ih_{q,p}$ when $q-p=i$ and 0 otherwise (for instance see P_2 below).

$$P_2=\begin{pmatrix}1&0&t^2h_{2,0}&0&\dots&0&0\\0&1&0&t^2h_{3,1}&0&\dots&0\\0&0&1&0&t^2h_{4,2}&\dots&0\\\vdots&\vdots&\vdots&\ddots&\ddots&\ddots&\vdots\\0&0&0&\dots&1&0&t^2h_{\ell,\ell-2}\end{pmatrix}$$

By solving for $t^{i-1}\partial'_{j+i,j}$ in (3.2), we can write the differential d'_i as the matrix whose entries are

$$(d'_i)_{p,r} = \begin{cases} t^{r-p-1}\partial_{r,p} - t^{r-p-1}h_{r-1,p}\partial_{r,r-1} + t^{r-p-1}\partial_{p+1,p}h_{r,p+1} & \text{if } \; r-p=i \\ t^{r-p-1}\partial_{r,p} & \text{if } \; 1 \le r-p \le i-1 \\ 0 & \text{otherwise} \end{cases}$$

We claim that $d'_i = P_{i-1}^{-1} d_i P_{i-1} \pmod{t^i}$. Note that (mod t^i) the inverse P_{i-1}^{-1} is the matrix with entries

$$(P_{i-1})^{-1}_{p,k} = \begin{cases} -t^{i-1}h_{p,k} & \text{if } \; p-k=i-1 \\ 1 & \text{if } \; p-k=0 \\ 0 & \text{otherwise} \end{cases} .$$

Using this, we compute $P_{i-1}^{-1} d_i P_{i-1}$ entry-wise:

$$\begin{aligned} (P_{i-1}^{-1} d_i P_{i-1})_{p,r} &= \sum_{q=0}^{\ell}\sum_{k=0}^{\ell} (P_{i-1}^{-1})_{p,k}(d_i)_{k,q}(P_{i-1})_{q,r} \\ &= \sum_{k=0}^{\ell} (P_{i-1}^{-1})_{p,k}(d_i)_{k,r} + \sum_{k=0}^{\ell} (P_{i-1}^{-1})_{p,k}(d_i)_{k,r-i+1} t^{i-1} h_{r,r-i+1} \\ &= (d_i)_{p,r} + t^{i-1}(d_i)_{p,r-i+1}h_{r,r-i+1} - t^{i-1}h_{p+i-1,p}(d_i)_{p+i-1,r} \\ &\qquad - t^{2i-2}h_{p+i-1,p}(d_i)_{p+i-1,r-i+1}h_{r,r-i+1} \\ &= (d_i)_{p,r} + t^{i-1}(d_i)_{p,r-i+1}h_{r,r-i+1} - t^{i-1}h_{p+i-1,p}(d_i)_{p+i-1,r} \pmod{t^i} \\ &= \begin{cases} t^{i-1}\partial_{r,p} + t^{i-1}\partial_{r-i+1,p}h_{r,r-i+1} - t^{i-1}h_{p+i-1,p}\partial_{r,p+i-1} & \text{if } \; r-p=i \\ t^{r-p-1}\partial_{r,p} & \text{if } \; 1 \le r-p \le i-1 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Since $d'_i = P_{i-1}^{-1} d_i P_{i-1} \pmod{t^i}$, it follows that $(F \otimes_S S_i, d_i)$ and $(F \otimes_S S_i, d'_i)$ are isomorphic differential modules over S_i. Hence, the choices of lifts from S_{i-1} to S_i are parameterized up to isomorphism by a class in $\langle t^{i-1}\rangle H^i(\mathrm{End}^{\bullet}_S(\mathbf{F} \otimes_S S_i))_{a-ia}$. □

Lemma 3.3.9. *There is an isomorphism of* $\Bbbk$*-vector spaces*

$$\langle t^{i-1}\rangle H^i(\mathrm{End}^\bullet_{S_i}(\mathbf{F}\otimes_S S_i))_{a-ia} \simeq H^i(\mathrm{End}^\bullet_S(\mathbf{F}))_{a-ia}.$$

Proof. Let $\phi : S \to S_i$ be the natural inclusion of S-algebras; notice that $S_i \cong \bigoplus_{j=0}^{i-1} S\cdot t^j$ is a free S-module. In particular, the ideal $\langle t^{i-1}\rangle := (t^{i-1})/(t^i) \subset S_i$ is isomorphic to St^{i-1} and is hence a free S-module of rank 1. Since tensoring with a free S-module is exact and $\mathbf{F}$ is a complex of free S-modules, there is an isomorphism of S-modules

$$H^i(\mathrm{End}^\bullet_{S_i}(\mathbf{F}\otimes_S S_i)) \cong H^i(\mathrm{End}^\bullet_S(\mathbf{F}))\otimes_S S_i.$$

Multiplying the above by the ideal $\langle t^{i-1}\rangle \subset S_i$, it follows that

$$\begin{aligned}\langle t^{i-1}\rangle H^i(\mathrm{End}^\bullet_{S_i}(\mathbf{F}\otimes_S S_i)) &\simeq H^i(\mathrm{End}^\bullet_S(\mathbf{F}))\otimes_S \underbrace{\langle t^{i-1}\rangle S_i}_{\simeq S} \\ &\simeq H^i(\mathrm{End}^\bullet_S(\mathbf{F})).\end{aligned}$$

Restricting to homogeneous components yields the statement of the lemma. □

3.3.3 Flagged Isomorphism Classes and Homotopy Equivalence of Free Flag Data

The previous section proved Theorem 3.3.5, which we will use to classify flagged isomorphism classes of free flags on a fixed anchor. In order to do this, we will need to understand when two flags are determined by homotopic data (note that we have so far shown that homotopic data implies isomorphic free flags, but not the converse). In this section we study the relationship between the higher differentials of two flags, assuming that we know these flags are isomorphic via some flagged isomorphism. As it turns out, this assumption forces the defining data of both flags to be homotopic.

Notation. Let A be any $\mathbb{Z}$-graded algebra. For any $a \in A$, the notation $\overline{a}$ is defined as

$$\overline{a} := (-1)^{|a|+1}a, \quad \text{where } |a| := \deg a.$$

The following proposition is useful for translating between two equivalent perspectives on free flags that we have been employing implicitly in earlier sections, but we will need to make this identification precise in this section to ease the computations of Lemma 3.3.12.

Proposition 3.3.10. *Let* $\mathbf{F}$ *be any complex of* S*-modules and* $E := \operatorname{End}_S(\mathbf{F})$. *Define the map* $\tau : E \to E$ *as follows: given any* $\phi \in E$, *the map* $\tau(\phi)$ *is defined via*

$$\tau(\phi)(f) := (-1)^{|f|}\phi(f).$$

Then:

i. The map $\tau : E \to E$ *is an isomorphism of* S*-modules (that does* not *respect the algebra structure on* E*).*

ii. The map $\tau : E \to E$ *induces a bijection*

$$\left\{ \begin{smallmatrix} \textit{Tuples } (\delta_1,\ldots,\delta_n)\in E^2\times\cdots\times E^{n+1} \textit{ satisfying} \\ d^{\mathbf{F}}\delta_i+\delta_i d^{\mathbf{F}}=-\sum_{j=1}^{i-1}\delta_j\delta_{i-j} \textit{ for } i=1,\ldots,n \end{smallmatrix} \right\} \longleftrightarrow \left\{ \begin{smallmatrix} \textit{Tuples } (\delta_1,\ldots,\delta_n)\in E^2\times\cdots\times E^{n+1} \textit{ satisfying} \\ d^{E}(\delta_i)=-\sum_{j=1}^{i-1}\overline{\delta}_j\delta_{i-j} \textit{ for } i=1,\ldots,n \end{smallmatrix} \right\}.$$

Proof. **Proof of (i):** This is immediate from the fact that $\tau^2 = \mathrm{id}_E$, and the fact that τ does not respect the algebra structure is easily seen from the equality $\tau(\phi)\tau(\psi) = (-1)^{|\psi|}\phi\psi$.

Proof of (ii): Assume that $(\delta_1, \ldots, \delta_n) \in E^1 \times \cdots \times E^{n+1}$ is a tuple satisfying

$$d^{\mathbf{F}}\delta_i + \delta_i d^{\mathbf{F}} = -\sum_{j=1}^{i-1}\delta_j\delta_{i-j}.$$

Consider the tuple $(\tau(\delta_1), \ldots, \tau(\delta_n))$. Then:

$$\begin{aligned}
\sum_{j=1}^{i-1}\overline{\tau(\delta_j)}\tau(\delta_{i-j}) &= \sum_{j=1}^{i-1}(-1)^{j+1}\cdot(-1)^{i-j+1}\delta_j\delta_{i-j} \\
&= (-1)^i(d^{\mathbf{F}}\delta_i + \delta_i d^{\mathbf{F}}) \\
&= -\tau(d^{\mathbf{F}})\tau(\delta_i) + (-1)^{i+1}\tau(\delta_i)\tau(d^{\mathbf{F}}) = -d^{\operatorname{End}_S(\tau(\mathbf{F}))}(\tau(\delta_i)).
\end{aligned}$$

Since $\tau(\mathbf{F}) \cong \mathbf{F}$, the above computation shows that τ induces a bijection between the two sets of part (ii). □

Remark 3.3.11. For computational purposes, it is often more convenient to deal with tuples of data satisfying the identities

$$d^E(\delta_i) = -\sum_{j=1}^{i-1} \overline{\delta_j}\delta_{i-j},$$

since we are then able to leverage the fact that E is an associative DG-algebra, and in particular satisfies the graded Leibniz rule.

Lemma 3.3.12. *Let* $\mathbf{F}$ *be a complex of* S*-modules and* $E := \operatorname{End}_S(\mathbf{F})$*. Assume* $d^{\mathbf{F}} := \delta_0, \delta_1, \ldots, \delta_n$ *is any sequence of elements with* $\delta_i \in E^{i+1}$ *satisfying*

$$d^E(\delta_i) = -\sum_{j=1}^{i-1} \overline{\delta}_j\delta_{i-j}.$$

For any sequence of elements $\psi_1, \ldots, \psi_n$ *with* $\psi_\ell \in \operatorname{End}_S^\ell(\mathbf{F})$*, define the sequence* $d^{\mathbf{F}} := \delta_0', \delta_1', \ldots, \delta_n'$ *via*

$$\delta_i' := \delta_i + \sum_{\ell=1}^{i-1}(\overline{\psi_\ell}\delta_{i-\ell}' + \delta_{i-\ell}\psi_\ell) + d^E(\psi_i).$$

Then for all $i \geq 1$ *there are equalities:*

i. $d^E(\delta_i') = -\sum_{j=1}^{i-1} \overline{\delta}_j'\delta_{i-j}'$*, and*

ii. $\sum_{j=1}^{i-1} \overline{\delta}_j'\delta_{i-j}' = \sum_{j=1}^{i-1} \overline{\delta}_j\delta_{i-j} - \sum_{\ell=1}^{i-1} d^E(\overline{\psi_\ell}\delta_{i-\ell}' + \delta_{i-\ell}\psi_\ell)$.

Proof. Proceed by induction on i, with base case $i = 1$ being evident since $\delta_1' := \delta_1 + d^E(\psi_1)$. For $i > 1$, the equalities of (i) and (ii) are a consequence of the following computation:

$$\begin{aligned}
\sum_{j=1}^{i-1} \overline{\delta_j'}\delta_{i-j}' =& \sum_{j=1}^{i-1} \overline{\delta_j}\delta_{i-j}' + \sum_{\ell=1}^{i-1}(-1)^\ell d^E(\psi_\ell)\delta_{i-\ell} + \sum_{\ell=1}^{i-1}\sum_{j=\ell+1}^{i-1}\left(-\psi_\ell\overline{\delta_{i-\ell}'} - \overline{\delta}_{i-\ell}\overline{\psi_\ell}\right)\delta_{i-j}' \\
=& \sum_{j=1}^{i-1} \overline{\delta_j}\delta_{i-j} + \sum_{\ell=1}^{i-1} \overline{\delta}_{i-\ell}d^E(\psi_\ell) + \sum_{\ell=1}^{i-1}\sum_{j=1}^{i-\ell-1} \overline{\delta_j}\left(\overline{\psi_\ell}\delta_{i-\ell}' + \delta_{i-\ell}\psi_\ell\right) \\
&+ \sum_{\ell=1}^{i-1}(-1)^\ell d^E(\psi_\ell)\delta_{i-\ell} + \sum_{\ell=1}^{i-1}\sum_{j=\ell+1}^{i-1}\left(-\psi_\ell\overline{\delta_{i-\ell}'} - \overline{\delta}_{i-\ell}\overline{\psi_\ell}\right)\delta_{i-j}'
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{i-1} \overline{\delta}_j \delta_{i-j} + \sum_{\ell=1}^{i-1} \overline{\delta}_{i-\ell} d^E(\psi_\ell) + \sum_{\ell=1}^{i-1} \sum_{j=1}^{i-\ell-1} \overline{\delta}_j \delta_{i-j-\ell} \psi_\ell \\
&\quad + \sum_{\ell=1}^{i-1} (-1)^\ell d^E(\psi_\ell) \delta'_{i-\ell} - \sum_{\ell=1}^{i-1} \underbrace{\sum_{j=\ell+1}^{i-1}}_{\text{reindex: } j \mapsto j-\ell} \psi \overline{\delta}'_{j-\ell} \delta'_{i-j} \\
&= \sum_{j=1}^{i-1} \overline{\delta}_j \delta_{i-j} + \sum_{\ell=1}^{i-1} \overline{\delta}_{i-\ell} d^E(\psi_\ell) + \sum_{\ell=1}^{i-1} (-1)^\ell d^E(\psi_\ell) \delta'_{i-\ell} \\
&\quad - \sum_{\ell=1}^{i-1} \underbrace{d^E(\delta_{i-\ell})\psi_\ell}_{= d^E(\delta_{i-\ell}\psi_\ell) + \overline{\delta}_{i-\ell} d^E(\psi_\ell)} + \sum_{\ell=1}^{i-1} \underbrace{\psi_\ell d^E(\delta'_{i-\ell})}_{=(-1)^\ell d^E(\psi_\ell \delta'_{i-\ell}) - (-1)^\ell d^E(\psi_\ell) \delta'_{i-\ell}} \\
&= \sum_{j=1}^{i-1} \overline{\delta}_j \delta_{i-j} - \sum_{\ell=1}^{i-1} d^E(\overline{\psi_\ell} \delta'_{i-\ell} + \delta_{i-\ell} \psi_\ell) \\
&= - d^E \left(\delta_i + \sum_{\ell=1}^{i-1} (\overline{\psi_\ell} \delta'_{i-\ell} + \delta_{i-\ell} \psi_\ell) + d^E(\psi_i) \right) = -d^E(\delta'_i).
\end{aligned}
$$

□

Using Lemma 3.3.12, we can now show that flag-preserving isomorphisms of flagged differential modules induce homotopy equivalences between the defining data of these differential modules:

Theorem 3.3.13. *Assume that $\phi : D \to D'$ is a flag preserving isomorphism of free flags, where D and D' are determined by the data $\delta_1, \dots, \delta_n$ and $\delta'_1, \dots, \delta'_n$, respectively. Then for all $i = 1, \dots, n$ the data*

$$
\sum_{j=1}^{i-1} \delta_j \delta_{i-j} \quad \textit{and} \quad \sum_{j-1}^{i-1} \delta'_j \delta'_{i-j}
$$

are homotopic.

Proof. Write $\phi = \phi_0 + \phi_1 + \dots + \phi_n$, where $\phi_i \in \operatorname{Hom}^i_S(\mathbf{F}, \mathbf{F}')$ (note: $\mathbf{F}$ and $\mathbf{F}'$ are the anchors of D and D', respectively). Since ϕ is assumed to be an isomorphism, the map ϕ_0 must be a isomorphism of the underlying complexes $\mathbf{F}$ and $\mathbf{F}'$. Changing bases as necessary, it is of no loss of generality to assume $\mathbf{F} = \mathbf{F}'$ and $\phi_0 = \mathrm{id}_{\mathbf{F}}$.

With this, the assumption that ϕ is a morphism of flags yields the equality

$$\delta'_i = \delta_i + \sum_{\ell=1}^{i} (\delta_{i-\ell}\phi_\ell - \phi_\ell \delta'_{i-\ell}) \quad \text{for all } i = 1, \dots, n,$$

where the above summation uses the convention $\delta_0 := d^{\mathbf{F}}$. For each $\ell \geq 1$, define $\psi_\ell := (-1)^\ell \phi_\ell$ and observe that for any element $f \in \mathbf{F}_m$ there is a string of equalities:

$$\begin{aligned} \tau(\delta'_i)(f) &= (-1)^m \delta_i(f) + (-1)^m \sum_{\ell=1}^{i} (\delta_{i-\ell}\phi_\ell - \phi_\ell \delta'_{i-\ell}) \\ &= (-1)^m \delta_i(f) + \sum_{\ell=1}^{i-1} (\tau(\delta_{i-\ell})\psi_\ell(f) + \overline{\psi_\ell}\tau(\delta'_{i-\ell})(f)) + d^{\operatorname{End}_S(\tau(\mathbf{F})}(\tau(\psi_i)). \end{aligned}$$

It follows that via the automorphism τ, the data of a flag preserving isomorphism is equivalently the data of Lemma 3.3.12, which by part (ii) of Lemma 3.3.12 implies that the equations

$$\sum_{j=1}^{i-1} \delta_j \delta_{i-j} \quad \text{and} \quad \sum_{j-1}^{i-1} \delta'_j \delta'_{i-j}$$

are homotopic. □

3.3.4 Dimension Bounds

Using the material proved in subsections 3.3.1 and 3.3.3, we can show that for a fixed anchor $\mathbf{F}$, the flagged isomorphism classes of degree a free flags anchored on $\mathbf{F}$ are parametrized by a well-behaved geometric object:

Theorem 3.3.14. *Let S be a graded-local ring where the residue field $\Bbbk$ of S_0 is algebraically closed. The set of flagged isomorphism classes of free flags in $\operatorname{DM}(S, a)$ anchored on a free complex $\mathbf{F}$ is parameterized by a nonempty variety $\mathcal{X}_a^{\mathbf{F}}$ whose dimension satisfies the following bounds:*

$$\sum_{i=2}^{\ell} \dim_\Bbbk H^i(\operatorname{End}_S^\bullet(\mathbf{F}))_{a-ia} - \sum_{i=4}^{\ell} \dim_\Bbbk H^i(\operatorname{End}_S^\bullet(\mathbf{F}))_{2a-ia} \leq \dim \mathcal{X}_a^{\mathbf{F}} \leq \sum_{i=2}^{\ell} \dim_\Bbbk H^i(\operatorname{End}_S^\bullet(\mathbf{F}))_{a-ia}.$$

Moreover, $\dim \mathcal{X}_a^{\mathbf{F}} = 0$ *if and only if* $\mathcal{X}_a^{\mathbf{F}}$ *consists of a single point corresponding to the degree* a *fold* $\mathrm{Fold}^a(\mathbf{F})$.

Note. The degree of $2a - ia$ rather than $a - ia$ in the sum corresponding to the obstructions appears due to an indexing shift replacing $i + 1$ by i.

Proof. Throughout the proof, we will freely employ notation established in subsection 3.3.1, particularly Construction 3.3.2. The proof follows by inductively defining a family of algebraic varieties $\{X_i\}_{i \geq 1}$ using the constructions of 3.3.1 whose points will eventually correspond to all possible flagged isomorphism classes of free flags anchored on $\mathbf{F}$. When $i = 1$, the variety X_1 is defined to be the singleton corresponding to $\mathrm{Fold}^a(\mathbf{F})$.

For conciseness of notation, define $E_j^i := \mathrm{Cl}\left(\mathrm{Spec}(\mathrm{Sym}^*(H^i(\mathrm{End}_S(M))_j))\right)$ (where $\mathrm{Cl}(-)$ denotes the functor sending a $\Bbbk$-scheme to its closed points). Then for each $i \geq 1$, let

$$\nu_i : X_i \to E_{2a-ia}^{i+2},$$

be the map that sends the point corresponding to $F \otimes_S S_i$ to the chain map on the right hand side of equation (3.1). Since this map is polynomial in the coordinates of the underlying vector spaces, this is a morphism of (classical) varieties, and since $\Bbbk$ is algebraically closed this induces a morphism of $\Bbbk$-schemes.

By Lemma 3.3.6, the subset of X_i that can be lifted to a differential module in X_{i+1} is the fiber $K_i := \nu_i^{-1}(0)$ above the point corresponding to 0 in E_{2a-ia}^{i+2}. On the other hand, Lemma 3.3.8 implies that the flagged isomorphism classes of two lifts of the free flag $(F \otimes_S S_i, d_i)$ are parameterized by $E_{a-(i+1)a}^{i+1}$, in which case we set $X_{i+1} := K_i \times E_{a-(i+1)a}^{i+1}$ for $i \geq 1$. Notice that the points of X_{i+1} may explicitly be described as flagged isomorphism classes of free flags of the form $(F \otimes_S S_{i+1}, d_{i+1})$, obtained as in Construction 3.3.2. Inductively, this shows that each X_i is a variety and that there is an inclusion

$$X_i \subset X_1 \times E_{-a}^2 \times E_{-2a}^2 \times \cdots \times E_{a-ia}^i.$$

Define $\mathcal{X}_a^{\mathbf{F}}$ to be X_ℓ (recall that ℓ is the length of $\mathbf{F}$). Setting $i = \ell$ in the above inclusion and counting dimensions yields the upper bound on $\dim \mathcal{X}_a^{\mathbf{F}}$.

Next, we will establish the lower bound on $\dim \mathcal{X}_a^{\mathbf{F}}$. Define $V^{i+2} := \operatorname{im} \nu_i$. By the fiber dimension formula (see, for instance, [Har77, Exercise 3.22]) combined with the definition of X_i there is a string of (in)equalities

$$\begin{aligned}\dim K_i &\geq \dim X_i - \dim V^{i+2} \\ &= \dim K_{i-1} + \dim E^i_{a-ia} - \dim V^{i+2} \\ &\geq \dim K_{i-1} + \dim E^i_{a-ia} - \dim E^{i+2}_{2a-ia}.\end{aligned}$$

Setting $i = \ell$ and inducting downward, it follows that

$$\dim \underbrace{X_\ell}_{=\mathcal{X}_a^{\mathbf{F}}} \geq \sum_{i=2}^{\ell} \dim E^i_{a-ia} - \sum_{i=4} E^i_{2a-ia}$$

$$= \sum_{i=2}^{\ell} \dim_{\Bbbk} H^i(\operatorname{End}^\bullet_S(\mathbf{F}))_{a-ia} - \sum_{i=4}^{\ell} \dim_{\Bbbk} H^i(\operatorname{End}^\bullet_S(\mathbf{F}))_{2a-ia}.$$

This establishes the lower bound.

Finally, it remains to show that if $\dim \mathcal{X}_a^{\mathbf{F}} = 0$, then in fact $\mathcal{X}_a^{\mathbf{F}} = \{\operatorname{Fold}^a(\mathbf{F})\}$ is a single point. A priori, the assumption that $\dim \mathcal{X}_a^{\mathbf{F}} = 0$ implies that $\mathcal{X}_a^{\mathbf{F}}$ consists of a finite set of points. Suppose for sake of contradiction that $\mathcal{X}_a^{\mathbf{F}}$ contains another point corresponding to a flagged isomorphism class distinct from that of $\operatorname{Fold}^a(\mathbf{F})$, and denote the corresponding free flag by F. Choose any $\lambda \in \Bbbk^\times$. If the maps $\delta_2, \ldots, \delta_\ell$ denote the data defining the free flag F, then the data $\lambda\delta_2, \lambda^2\delta_3, \ldots, \lambda^{\ell-1}\delta_\ell$ defines another free flag, denoted F_λ, that is also anchored on $\mathbf{F}$.

We claim that $F_\lambda \ncong F$ in a flag-preserving way if λ is not a root of the polynomials $x^r - 1$ for $1 \leq r \leq \ell - 1$. Notice that this gives an immediate contradiction since the fact that $\Bbbk$ is infinite implies that $\mathcal{X}_a^{\mathbf{F}}$ must have infinitely many points (note that $\Bbbk$ is infinite by the algebraic closure assumption). To prove the claim, assume for sake of contradiction that there is a flag-preserving isomorphism $F_\lambda \cong F$ for λ as above. Denote by F^t_λ (resp. F^t) the deformed version of F_λ (resp. F) obtained by rescaling the maps δ_i by powers of t. Then the free flags F^t_λ and F^t must also be isomorphic, and hence $F^t_\lambda \otimes_S S_i \cong F^t \otimes_S S_i$ for all $i \geq 1$.

Choose $p := \min\{i \geq 2 \mid \delta_i \neq 0\}$. Observe that p is well-defined by the assumption that F is not in the isomorphism class of $\operatorname{Fold}^a(\mathbf{F})$. By the proof of (2) in Lemma 3.3.8, the fact that $F^t_\lambda \otimes_S S_{p+1} \cong F^t \otimes_S S_{p+1}$ implies that the difference $(\lambda^{p-1} - 1)\delta_p$ is nullhomotopic. By the assumptions on λ the scalar $\lambda^{p-1} - 1$ is nonzero, which implies that δ_p is nullhomotopic. Thus F may be replaced with an isomorphic free flag satisfying $\delta_2 = \delta_3 = \cdots = \delta_p = 0$. Iterating this argument, it follows that F has an isomorphic representative with $\delta_i = 0$ for all $i \geq 2$, implying that $F \cong \operatorname{Fold}^a(\mathbf{F})$; this contradiction yields the result. □

Remark 3.3.15. It is worth mentioning that the assumption of algebraic closure for the field $\Bbbk$ is only necessary for invoking the equivalence of categories between the classical notion of a variety and reduced integral $\Bbbk$-schemes (which is induced by taking closed points). In general, the results of 3.3.1 are totally independent of any assumptions on the base field $\Bbbk$ and hence always give rise to at least an algebraic parametrization of free flags with a fixed anchor.

Remark 3.3.16. The assumption that the isomorphisms are flag-preserving is essential for the well-definedness of the maps $\nu_i : X_i \to E^{i+2}_{2a-ia}$. That being said, it is not clear in general what the precise difference between flagged isomorphism classes and arbitrary (not necessarily flag-preserving) isomorphism classes is. As of yet we have not been able to find an example of a free flag anchored on a free resolution where the isomorphism class becomes larger once the flag-preserving assumption is dropped.

In view of Remark 3.3.16, we pose the following question:

Question 3.3.17. *Assume that there is an isomorphism $\phi : D \to D'$ of free flags anchored on a fixed complex $\mathbf{F}$. If $H^i(\operatorname{End}^\bullet_S(\mathbf{F})) = 0$ for all $i < 0$, then is it true that there is a* flag-preserving *isomorphism $D \to D'$?*

It turns out that we can also obtain dimension bounds for arbitrary (not necessarily homogeneous) differential modules with finite length homology:

Theorem 3.3.18. *Assume that $\Bbbk$ is algebraically closed and $\mathbf{F}$ has finite length homology. The set of flagged isomorphism classes of free flags in $\operatorname{DM}(S)$ anchored on $\mathbf{F}$ is parameterized by a* *nonempty variety $\mathcal{X}^{\mathbf{F}}$ whose dimension satisfies the bounds*

$$\dim_{\Bbbk} H^2(\operatorname{End}_S(\mathbf{F})) + \dim_{\Bbbk} H^3(\operatorname{End}_S(\mathbf{F})) \leq \dim \mathcal{X}^{\mathbf{F}} \leq \sum_{i=2}^{\ell} \dim_{\Bbbk} H^i(\operatorname{End}_S(\mathbf{F})).$$

Proof. Observe first that the constructions and results of subsection 3.3.1 are inherently independent of the homogeneity assumption, in which case we may employ the same argument as in Theorem 3.3.14. The finite length assumption ensures that all vector spaces involved are finite dimensional. □

3.4 Examples and Applications

In this section, we discuss some of the applications of our results including rigidity conditions for free complexes, total Betti numbers of differential modules, and matrix factorizations.

3.4.1 Rigid resolutions

In general, the quasi-isomorphism class of a degree a differential module is only controlled up to some choice of free flag anchored on a resolution (this is the content of [BE22]), but it is useful to know when one can guarantee that there are *no* nontrivial free flags anchored on a given complex. This is precisely the notion of a-rigidity:

Definition 3.4.1. A homogeneous complex $\mathbf{F}$ is *a-rigid* if the only flagged isomorphism class of a degree a free flag anchored on $\mathbf{F}$ is the degree a fold $\operatorname{Fold}^a(\mathbf{F})$. A homogeneous S-module M is a-rigid if some (equivalently every) homogeneous free resolution of M is a-rigid.

Remark 3.4.2. Another description of a-rigidity may be given as follows: consider the homology functor $H : \operatorname{DM}(S, a) \to S$-mod which sends a given differential module D to its homology $H(D)$. Then a given S-module M is a-rigid if the "fiber" above M is (essentially) contained inside the category of chain complexes, where $\operatorname{Ch}(S) \subset \operatorname{DM}(S, a)$ as in Remark 3.2.5.

Our results also allow us to generalize a result previously proved in [BV22], and in fact we can give a full characterization of a-rigid Artinian complete intersections:

Theorem 3.4.3. *Let $S = \Bbbk[x_1, \dots, x_n]$ ($n \geq 2$) be a standard-graded polynomial ring and $M = S/I$ be an Artinian complete intersection, where $I = (f_1, \dots, f_n)$ is generated by a homogeneous regular sequence with degrees $\underline{d} := (d_1 \leq d_2 \leq \cdots \leq d_n)$ (where $d_i := \deg f_i$). Then:*

i. If I is not generated by linear forms and

$$a < d_1 + d_2 + n - |\underline{d}| \quad \textit{or} \quad a > d_{n-1} + d_n$$

then S/I is a-rigid.

ii. If I is generated by linear forms, then S/I is a-rigid for all $a \neq 2$.

Remark 3.4.4. The hypotheses of Theorem 3.4.3 omit the case $n = 1$ for the simple reason that this case is trivial. If $n = 1$, then a free flag is equivalently a complex, so S/I is a-rigid for all $a \in \mathbb{Z}$. Likewise, if $n = 2$ then the flagged isomorphism classes of free flags are in direct bijection with the Ext classes $\mathrm{Ext}^2(S/I, S/I)$, so the statement really only needs to be proved for $n \geq 3$.

Remark 3.4.5. The quantity $|\underline{d}| - n$ is the *socle degree* of the complete intersection S/I, typically denoted $\mathrm{Socdeg}(S/I)$. This means that the bounds of Theorem 3.4.3(i) may be instead written as

$$a < d_1 + d_2 - \mathrm{Socdeg}(S/I) \quad \text{or} \quad a > d_{n-1} + d_n\}.$$

A quick example illustrating the statement of Theorem 3.4.3 may be useful before the proof:

Example 3.4.6. Suppose that I is a complete intersection with degree sequence $\underline{d} = (2, 2, 5, 7, 9)$ (where S/I is Artinian). Then

$$\mathrm{Socdeg}(S/I) = |\underline{d}| - n = 25 - 5 = 20,$$

$$d_1 + d_2 - \mathrm{Socdeg}(I) = 4 - 20 = -16, \quad \text{and} \quad d_{n-1} + d_n = 7 + 9 = 16.$$

Thus S/I is a-rigid for all $a < -16$ and all $a > 16$.

Proof of Theorem 3.4.3. We will actually prove a much more precise bound, then specialize to the case at hand. It is well-known that for a finite length complete intersection S/I over a polynomial ring, the maximal nonzero degree of S/I is precisely $|\underline{d}| - n$. Moreover, there is an isomorphism

$$\operatorname{Ext}^i_S(S/I, S/I) \cong S/I \otimes_S \bigwedge^{\bullet} V,$$

where $V = \bigoplus_{i=1}^n Se_i$ is a free S-module on basis elements e_i assigned internal degree $\deg(e_i) = -d_i$. For every $1 \le i \le n$, there is a direct sum decomposition

$$\bigwedge^i V = \bigoplus_{1 \le j_1 < \cdots < j_i \le n} Se_{j_1} \wedge \cdots \wedge e_{j_i},$$

in which case

$$\operatorname{Ext}^i_S(S/I, S/I) = \bigoplus_{1 \le j_1 < \cdots < j_i \le n} S/Ie_{j_1} \wedge \cdots \wedge e_{j_i}.$$

Let ℓ_i (respectively r_i) be the minimal (respectively maximal) internal degree of $\bigwedge^i V$. In order for $\operatorname{Ext}^i_S(S/I, S/I)_{a-ia}$ to be 0, there must be strict inequalities

$$a - ai > |\underline{d}| - n + r_i \quad \text{or} \quad a - ai < \ell_i. \tag{3.3}$$

Proof of (i): Notice that $r_i \le -(i-1)d_1 - d_2$ and $\ell_i \ge -(i-1)d_n - d_{n-1}$, in which case substituting these into the inequalities (3.4.1) implies that S/I will be a-rigid if the following inequalities hold for all $2 \le i \le n$:

$$a - ai > |\underline{d}| - n - (i-1)d_1 - d_2 \quad \text{or} \quad a - ai < -(i-1)d_n - d_{n-1}.$$

Rearranging yields the inequalities

$$a < d_1 + \frac{n - |\underline{d}| + d_2}{i-1}, \quad a > d_n + \frac{d_{n-1}}{i-1}.$$

The quantity $d_n + \frac{d_{n-1}}{i-1}$ is evidently maximal for $i = 2$, yielding $a > d_n + d_{n-1}$. On the other hand, we claim that $n - |\underline{d}| + d_2 \le 0$.

To see this, suppose for sake of contradiction that $|\underline{d}| < n + d_2$. Since $d_1 + (n-1)d_2 \leq |\underline{d}|$, it follows that $\frac{d_1}{n-2} + d_2 < \frac{n}{n-2}$. If $n \geq 4$, this implies that $d_2 < 2$ and hence $d_1 = d_2 = 1$. Likewise, if $n = 3$ then $d_1 + d_2 < 3$, again implying that $d_1 = d_2 = 1$ and hence $|\underline{d}| < n + 1$.

Since each $d_j > 0$, it follows that $|\underline{d}| = n$ and $d_j = 1$ for all $1 \leq j \leq n$. This contradicts the assumption that I is not generated by linear forms, so $n - |\underline{d}| + d_2 < 0$ and hence there is an inequality

$$d_1 + \frac{n - |\underline{d}| + d_2}{i-1} \geq d_1 + d_2 + n - |\underline{d}|.$$

It follows that if $a < d_1 + d_2 + n - |\underline{d}|$ or $a > d_{n-1} + d_n$, then S/I is a-rigid.

Proof of (ii): This proof is essentially identical but much simpler: if I is generated by linear forms, then $\operatorname{Ext}^i_S(S/I, S/I)$ is generated in internal degree $-i$ for all $2 \leq i \leq n$. This means that if S/I is a-rigid for some a, there is an equality $a - ai = -i$ for some $2 \leq i \leq n$. Rearranging this yields the equality $i = \frac{a}{a-1}$; since i must be an integer, it follows that $a = i = 2$. □

As it turns out, we can upgrade Theorem 3.4.3 to an equivalence:

Corollary 3.4.7. *The statement of Theorem 3.4.3 is an equivalence. More precisely: adopt notation and hypotheses as in Theorem 3.4.3. If*

$$d_1 + d_2 + n - |\underline{d}| \leq a \leq d_{n-1} + d_n$$

then S/I is not a-rigid.

Proof. The proof follows by constructing an explicit free flag anchored on the Koszul complex $K_\bullet$ resolving S/I that is not isomorphic to $\operatorname{Fold}(K_\bullet)$ for every a in the appropriate interval. To do this, it is necessary to make some initial observations. Note first that $(S/I)_j \neq 0$ for all $0 \leq j \leq \operatorname{Socdeg}(S/I)$, so for each j we may fix a nonzero element $m_j \in (S/I)_j$. With notation as in the proof of Theorem 3.4.3, consider the elements

$$m_j e_i \wedge e_{i+1} \in S/I \otimes_S \bigwedge^2 V, \quad 0 \leq j \leq \operatorname{Socdeg}(S/I),\ 1 \leq i \leq n-1.$$

Notice that each of these elements squares to 0, even when viewed in $\bigwedge^\bullet V$, and hence induce a well-defined differential module by choosing all higher structure maps (of degree > 2) to be 0. Moreover, notice that for all $1 \leq i \leq n-2$ there is an inequality

$$d_i + d_{i+1} \geq d_{i+1} + d_{i+2} - \mathrm{Socdeg}(S/I) \tag{3.4}$$

since rearranging the above expression yields the inequality

$$n - d_i \leq |\underline{d}| - d_{i+2}.$$

The righthand side of this inequality is at least $n-1$, and since $d_i \geq 1$ we also know that the lefthand side is at most $n-1$. With all of these ingredients, the proof concludes as follows: as i and j range through the values $1, \dots, n$ and $0, \dots, \mathrm{Socdeg}(S/I)$, respectively, the elements $m_j e_i \wedge e_{i+1} \in \mathrm{Ext}^2_S(S/I, S/I)_{-d_i - d_{i+1} + j}$ induce degree a differential modules for all $d_1 + d_2 - \mathrm{Socdeg}(S/I) \leq a \leq d_{n-1} + d_n$. The fact that a must range through all values in this interval follows from the inequality (3.4) along with a straightforward induction on i.

Finally, notice that each of the differential modules induced by the elements $m_j e_i \wedge e_{i+1}$ are not isomorphic to a fold of the Koszul complex, since Theorem 3.6 of [BV22] would imply that $m_j \in I$, a contradiction to the fact that m_j represents a nonzero element in $(S/I)_j$ by selection. $\square$

The following example should help to illustrate the proof of Corollary 3.4.7 more clearly:

Example 3.4.8. Let $S = \mathbb{k}[x_1, x_2, x_3]$ and $I = (x_1^2, x_2^2, x_3^3)$. The socle degree of S/I is 4, in which case Corollary 3.4.7 guarantees that we should be able to construct homogeneous free flags of degrees $\{0, 1, 2, 3, 4, 5\}$. To do this, choose elements:

$$1 \in (S/I)_0, \quad x_1 \in (S/I)_1, \quad x_1x_2 \in (S/I)_2, \quad x_1x_2x_3 \in (S/I)_3, \quad x_1x_2x_3^2 \in (S/I)_4$$

(note that the choice of these elements is arbitrary – any nonzero choices will do). We then consider the collection of elements in $\mathrm{Ext}^2_S(S/I, S/I)$:

$$\underbrace{x_1x_2x_3^2 e_1 \wedge e_2}_{\text{degree } 0}, \quad \underbrace{x_1x_2x_3 e_1 \wedge e_2}_{\text{degree } -1}, \quad \underbrace{x_1x_2 e_1 \wedge e_2}_{\text{degree } -2}, \quad \underbrace{x_1 e_1 \wedge e_2}_{\text{degree } -3}, \quad \underbrace{e_1 \wedge e_2}_{\text{degree } -4},$$

$$\underbrace{x_1x_2x_3^2e_2 \wedge e_3}_{\text{degree } -1}, \quad \underbrace{x_1x_2x_3e_2 \wedge e_3}_{\text{degree } -2}, \quad \underbrace{x_1x_2e_2 \wedge e_3}_{\text{degree } -3}, \quad \underbrace{x_1e_2 \wedge e_3}_{\text{degree } -4}, \quad \underbrace{e_2 \wedge e_3}_{\text{degree } -5}.$$

Notice that there is redundancy in the degrees of the differential modules induced by the Ext elements in this collection, but they still range through *all* values in the interval $\{0, 1, 2, 3, 4, 5\}$.

Remark 3.4.9. The construction of Corollary 3.4.7 actually generalizes the construction of [BV22, Proposition 3.8], since the differential modules constructed there are minimizations of the free flag induced by choosing $e_1 \wedge e_2 \in \operatorname{Ext}_S^2(\Bbbk, \Bbbk)_2$.

It is not difficult to see that the construction of Corollary 3.4.7 does not yield *all* possible free flags of a given degree, it is only a sufficiency result that guarantees nonrigidity within a given degree range. The following example helps illustrate this:

Example 3.4.10. Let $S = \Bbbk[x_1, \dots, x_4]$ and $I = (x_1^2, \dots, x_4^2)$, and assume that $\Bbbk$ has odd characteristic. Consider the elements

$$f_1 := x_1e_1 + x_2e_2 + x_3e_3 + x_4e_4, \quad \text{and}$$

$$f_2 := x_1x_2x_3x_4(e_1 \wedge e_2 + e_3 \wedge e_4) \in \bigwedge^2 V.$$

Notice that f_2 squares to 0 after descending to the Ext algebra, but it does *not* square to 0 when viewed as an element in the exterior algebra $\bigwedge^\bullet V$, since

$$(x_1x_2x_3x_4(e_1 \wedge e_2 + e_3 \wedge e_4))^2 = 2x_1^2x_2^2x_3^2x_4^2e_1 \wedge e_2 \wedge e_3 \wedge e_4 \neq 0.$$

Define

$$f_3 := x_1x_2x_3x_4\left(x_2x_3x_4e_2 \wedge e_3 \wedge e_4 - x_1x_3x_4e_1 \wedge e_3 \wedge e_4 + x_1x_2x_4e_1 \wedge e_2 \wedge e_4 - x_1x_2x_3e_1 \wedge e_2 \wedge e_3\right)$$

and notice that $f_1 \cdot f_3 - f_3 \cdot f_1 = f_2^2$. Since $f_2 \cdot f_3 + f_3 \cdot f_2 = 0$, it follows that the elements f_1, f_2, and f_3 induce a well-defined Koszul differential module as in [BV22, Proposition 4.2], and moreover this differential module is homogeneous of degree 0. It is not isomorphic to any of the free flags of Corollary 3.4.7, however.

Next, we consider the problem of understanding free flags anchored on complexes with finite length homology. Free flags anchored on arbitrary complexes are a little more subtle than being anchored on a free resolution, since we do not have complete control over how the homology changes as the higher structure maps are added iteratively. However, we do at least know that the flagged isomorphism classes are parametrized by modules concentrated in only finitely many internal degrees. This immediately yields:

Theorem 3.4.11. *Let* $\mathbf{F}$ *be any finite length complex of free* S*-modules with finite length homology. Then* $\mathbf{F}$ *is* a*-rigid for* $a \gg 0$*.*

Proof. The flagged isomorphism classes are determined by the size of the cohomology of the endomorphism complex $\operatorname{End}_S^\bullet(\mathbf{F})$. If $\mathbf{F}$ has finite length homology, the cohomology of $\operatorname{End}_S^\bullet(\mathbf{F})$ is 0 in sufficiently large degrees. □

Remark 3.4.12. The statement of Theorem 3.4.11 may be interpreted as a statement on the asymptotic behavior of free flags anchored on finite length complexes. In particular, it implies that "interesting" homogeneous deformations of homogeneous complexes with finite length homology are exceptional, in the sense that there is only a finite window where such deformations may occur.

3.4.2 Differential modules with small rank

In this section, we revisit the Rank Conjectures of Buchsbaum–Eisenbud–Horrocks [BE77; Har79], Halperin [Hal85] and Carlsson [Car86]. These conjectures, concerning lower bounds on the Betti numbers of free resolutions and toral rank of certain manifolds and CW complexes, formed part of the motivation for [ABI07]. In searching for a unifying conjecture that would imply both the algebraic and topological rank conjectures, they posited that conjectures on total rank of free complexes should also hold for differential modules admitting a free flag structure. While Walker proved the Total Rank Conjecture for minimal free resolutions (outside of characteristic 2) [Wal17], the more general conjecture was shown to be false outside of characteristic 2 even for *complexes* by Iyengar-Walker [IW18]. In light of these results, we are curious about where precisely counterexamples to the more general conjecture can occur. We explore this question by relating the total Betti number of a differential module to the total Betti numbers of its homology. To do this, we introduce the notion of *Betti-deficiency*, which can be viewed as a twist on a-rigidity. First, recall the definition of a Betti number for graded differential modules:

Definition 3.4.13. The j^{th} *Betti number* $\beta_j(D)$ of a $\mathbb{Z}$-graded differential module D is defined as

$$\beta_j(D) := \dim_{\Bbbk} H(F \otimes_S^{DM} \Bbbk)_j,$$

where F is any minimal free resolution of D (as in [BE22]). The *total Betti number* is

$$\beta(D) = \sum_{j \in \mathbb{Z}} \beta_j(D).$$

A differential module D is *Betti-deficient* is there is a strict inequality

$$\beta(D) < \beta(H(D)),$$

where $H(D)$ is being viewed as a differential module with 0 endomorphism.

The j^{th} Betti number of D counts the number of generators in degree j of a minimal free resolution of D. Likewise, the total Betti number counts the total number of generators (i.e. the total rank) of a minimal free resolution of D.

Remark 3.4.14. As noted in [BE22], Theorem 3.2.11 implies that there is always an inequality

$$\beta(D) \leq \beta(H(D)),$$

so Betti-deficient differential modules are the cases for which this inequality is strict.

Remark 3.4.15. Notice that if M is an a-rigid S-module, then there are *no* Betti-deficient differential modules with homology M. This is not an equivalence, however, and makes the notion of Betti-deficiency a slightly more subtle property to study.

Using the fact that any free flag resolution F of a differential module D may be chosen to be anchored on a minimal free resolution of $H(D)$, this means that a differential module is Betti-deficient if and only if the matrix representation of the endomorphism d^F has entries residing in the field $\Bbbk$. On the other hand, we know that the higher off-diagonal maps are equivalently described as elements of the endomorphism complex associated to the anchor of F. Combining these two facts yields the following:

Theorem 3.4.16. *Let M be an S-module with finite projective dimension. If there exists a Betti-deficient differential module $D \in \mathrm{DM}(S, a)$ with $H(D) = M$, then the Betti table for M must have two nonzero entries (in nonadjacent columns) lying on a line of slope $1 - a + \frac{a}{j}$ for some $j \geq 2$.*

Proof. Let M have minimal free resolution

$$\mathbf{F} : F_0 \leftarrow F_1 \leftarrow \cdots \leftarrow F_n$$

where $F_i = \bigoplus_{k \in \mathbb{Z}} S(-k)^{\beta_{i,k}}$. We will use the fact that the total Betti number of a differential module (D, ∂) is equal to the number of generators in a minimal free resolution of D, and that by [BE22, Theorems 3.2 and 4.2] we can assume that our minimal free resolution is a direct summand of an anchored free flag resolution of D. Using the minimization procedure in [BE22, Proposition 4.1], D has a minimal free resolution that is strictly smaller than its anchored free flag resolution exactly when the matrix representation of ∂ contains a unit. Such a unit corresponds to a degree 0 map

$$\begin{array}{ccc} F_i(ia) & \longrightarrow & F_{i-j}((i-j+1)a) \\ \| & & \| \\ \bigoplus_k S(ia-k)^{\beta_{i,k}} & & \bigoplus_\ell S((i-j+1)a-\ell)^{\beta_{i-j,\ell}} \end{array}$$

where $j \geq 2$. This means that for some i, j, k, ℓ with $\beta_{i,k}, \beta_{i-j,\ell} \neq 0$, there is an equality

$$ia - k = (i - j + 1)a - \ell$$

$$\ell - k = (1 - j)a.$$

In the Betti table for M, $\beta_{i,k}$ is located in column i and row $k - i$, and $\beta_{i-j,\ell}$ is located in column $i-j$ and row $\ell-(i-j)$, so $\beta_{i,k}$ and $\beta_{i-j,\ell}$ lie on a line of slope

$$\frac{\ell-(i-j)-(k-i)}{j}=\frac{\ell-k+j}{j}=\frac{(1-j)a+j}{j}=1-a+\frac{a}{j}.$$

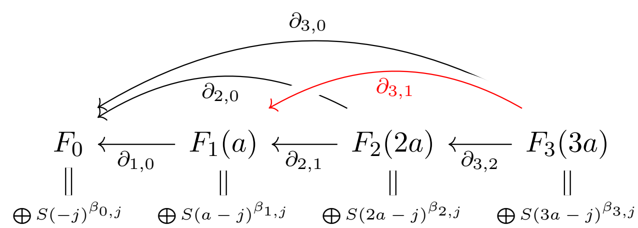
□

To more concretely illustrate the idea of Theorem 3.4.16 and its proof, suppose that M has minimal free resolution

$$F_0 \xleftarrow{\partial_{1,0}} F_1 \xleftarrow{\partial_{2,1}} F_2 \xleftarrow{\partial_{3,2}} F_3$$

and let D be a degree a differential module with homology M. If D is Betti-deficient, then the matrix representing the differential of D's minimal free resolution contains a unit. Suppose for example that the unit is in the red block in the matrix below, which we can visualize via the augmented complex on the right.

$$\begin{pmatrix} 0 & \partial_{1,0} & \partial_{2,0} & \partial_{3,0} \\ 0 & 0 & \partial_{2,1} & \partial_{3,1} \\ 0 & 0 & 0 & \partial_{3,2} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$



The presence of a unit entry in the degree a map $\partial_{3,1}$ means that there is a summand of $F_1(a)$ and a summand of $F_3(3a)$ that are generated in the degrees that differ by a, i.e some $\beta_{1,j}, \beta_{3,k} \neq 0$ for which $(3a-k)-(a-j)=a$. If, for example, $a=1$ one pair that would satisfy this is $\beta_{1,3}$ and $\beta_{3,4}$, in red in the Betti table below. The slope of line connecting these two entries is $\frac{1}{2}=1-\frac{1}{1}+\frac{1}{2}$ as in the theorem.

	$i:0$	1	2	3
$j-i:0$	$\beta_{0,0}$	$\beta_{1,1}$	$\beta_{2,2}$	$\beta_{3,3}$
1	$\beta_{0,1}$	$\beta_{1,2}$	$\beta_{2,3}$	$\beta_{3,4}$
2	$\beta_{0,2}$	$\beta_{1,3}$	$\beta_{2,4}$	$\beta_{3,5}$
3	$\beta_{0,3}$	$\beta_{1,4}$	$\beta_{2,5}$	$\beta_{3,6}$

In cases where we have a good understanding of the shape of the Betti table, we may obtain concrete numerical conditions for when a differential module may be Betti-deficient. For complete intersections for instance, we can write down a criterion in terms of the degrees of the generators of the ideal:

Corollary 3.4.17. *Let $I \subset S$ be minimally generated by a regular sequence $(f_1, \dots, f_\ell)$ where f_i has degree d_i. If there is a Betti-deficient differential module in $\mathrm{DM}(S,a)$ with homology S/I, then there are subsets $J_1, J_2 \subseteq \{1, \dots, \ell\}$ and $2 \leq j \leq \ell$ satisfying $|J_2| = |J_1| + j$ where*

$$\sum_{r \in J_1} d_r - \sum_{r \in J_2} d_r = a - aj.$$

Proof. The degrees of the i^{th} syzygies of S/I are $\sum_{r \in J} d_r$ where $|J| = i$, which means that in the i^{th} column of the Betti table the nonzero entries are located in rows $\left(\sum_{r \in J} d_r\right) - i$ for all subsets $J \subset \{1, \dots, \ell\}$ with $|J| = i$. Thus the slope of the line connecting two nonzero entries in columns $i, i+j$ for $j \geq 2$ is equal to

$$\frac{\left(\sum_{r \in J_1} d_r\right) - i - \left(\sum_{r \in J_2} d_r\right) + (i+j)}{j}$$

where $J_1, J_2 \subseteq \{1, \dots, \ell\}$, $|J_1| = i$, and $|J_2| = i + j$. Setting this equal to the slope given in Theorem 3.4.16 and simplifying yields the result.

□

Another class of Betti tables of which we have a good understanding are the *pure* Betti tables—those with only a single nonzero entry in each column.

Corollary 3.4.18. *Let $D \in \mathrm{DM}(S,a)$ be a differential module with homology M, where M has a pure resolution with degree sequence $(d_0, \dots, d_\ell)$. If M is Betti-deficient, then*

$$a = \frac{d_{i-j} - d_i}{1 - j}$$

for some $2 \leq j \leq i \leq \ell$.

Proof. The Betti table of M has nonzero entries β_{i,d_i} located in the i^{th} column and $(d_i - i)^{\text{th}}$ row. As in Theorem 3.4.16, if M is Betti-deficient then there are $\beta_{i,d_i}, \beta_{i-j,d_{i-j}} \neq 0$ with $j \geq 2$

satisfying

$$\begin{aligned} ia - d_i &= (i-j+1)a - d_{i-j} \\ d_{i-j} - d_i &= (1-j)a \\ a &= \frac{d_{i-j} - d_i}{1-j}. \end{aligned}$$

□

Setting $a = 0$ and recalling that the degree sequence for a pure resolution satisfies $d_0 < d_1 < \cdots < d_\ell$ immediately yields the following:

Corollary 3.4.19. *Let $D \in \mathrm{DM}(S, 0)$ be a differential module with homology M, where the graded minimal free resolution of M is pure. Then the total Betti number of D is equal to the sum of the Betti numbers of M (I.e. there are no Betti-deficient degree 0 differential modules with pure homology).*

3.4.3 Beyond free flags

We conclude with a curious observation connecting the idea of "higher maps" of a free flag with the notion of "systems of higher homotopies" coming from the matrix factorization literature.

Notice that the sequence of maps $\delta_1, \ldots, \delta_n$ in the definition of a free flag must satisfy the following equations in order for the endomorphism of the free flag to square to 0:

$$\sum_{j=1}^{i-1} \delta_j \circ \delta_{i-j} = 0.$$

One can abstract these equations to yield different types of differential modules:

Example 3.4.20. Let X denote a generic 5×5 skew-symmetric matrix and $R = k[X]$ its coordinate ring, where k is any field. Recall that for an indexing set $I = (i_1 < \cdots < i_\ell)$, the notation $\mathrm{Pf}_I(X)$ denotes the pfaffian of the submatrix of X formed by deleting rows and columns $i_1, \ldots, i_\ell$ from X. The ideal of submaximal pfaffians $\mathrm{Pf}(X)$ is a grade 3 Gorenstein ideal with minimal free resolutions of the form

$$F: \quad 0 \to R = F_3 \xrightarrow{d_1^t} R^5 = F_2 \xrightarrow{X} R^5 = F_1 \xrightarrow{d_1} R \to 0, \quad \text{where}$$

$$d_1 = \begin{pmatrix} \mathrm{Pf}_1(X) & -\mathrm{Pf}_2(X) & \dots & \mathrm{Pf}_5(X) \end{pmatrix}.$$

Let $e_1, \dots, e_5$, $f_1, \dots, f_5$, and g denote bases of F_1, F_2, and F_3, respectively. The complex F admits the structure of a graded-commutative DG-algebra extending the standard S-module structure with products (see [Avr98, Example 2.1.3])

$$e_i e_j = -e_j e_i = \sum_{k \neq i,j} (-1)^{i+j+k} \mathrm{Pf}_{ijk}(X) f_k, \quad (i < j)$$

$$e_i f_j = f_j e_i = \delta_{ij} g.$$

Consider the map that multiplies on the left by e_1, denoted ℓ_{e_1}. The matrix representation of ℓ_{e_1} restricted to each graded component may be computed using the above product formula as so:

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} : R^1 \to R^5, \quad \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_{45} & -x_{35} & x_{34} \\ 0 & -x_{45} & 0 & x_{25} & -x_{24} \\ 0 & x_{35} & -x_{25} & 0 & x_{23} \\ 0 & -x_{34} & x_{24} & -x_{23} & 0 \end{pmatrix} : R^5 \to R^5, \quad \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}^T : R^5 \to R^1.$$

Consider the following block matrix:

$$d^D := \begin{pmatrix} 0 & d_1 & 0 & 0 \\ \ell_{e_1} & 0 & X & 0 \\ 0 & -\ell_{e_1} & 0 & d_1^t \\ 0 & 0 & \ell_{e_1} & 0 \end{pmatrix}.$$

One can verify by direct computation that ${d^D}^2 = \mathrm{Pf}_1(X)\mathrm{id}_F$. In other words, the same process for constructing a free flag has yielded a "curved" (in the sense of, for instance, [KLN10]) differential module. Using the induced $\mathbb{Z}/2\mathbb{Z}$-grading to split F into its odd and even parts yields a nonminimal matrix factorization of $\mathrm{Pf}_1(X)$. In the language of matrix factorizations, the maps

$\delta_2, \ldots, \delta_n$ would be called a system of higher homotopies (in this case, there is only a single higher homotopy induced by left multiplication by e_1).

Chapter 4

Boij-Söderberg Conjectures for Differential Modules

Abstract

Boij-Söderberg theory gives a combinatorial description of the set of Betti tables belonging to finite length modules over the polynomial ring $S = \mathbb{k}[x_1, \ldots, x_n]$. We posit that a similar combinatorial description can be given for analogous numerical invariants of *graded differential S-modules*, which are natural generalizations of chain complexes. We prove several results that lend evidence in support of this conjecture, including a categorical pairing between the derived categories of graded differential S-modules and coherent sheaves on $\mathbb{P}^{n-1}$ and a proof of the conjecture in the case where $S = \mathbb{k}[t]$.

4.1 Introduction

Differential modules, modules equipped with an endomorphism squaring to zero, are natural generalizations of chain complexes. Initially appearing in more of a book-keeping capacity in [CE16], differential modules have increasingly become objects of interest in commutative algebra, algebraic geometry, and algebraic topology in particular as useful tools for generalizing classical results and conjectures [ŞÜ19; IW18; BE21; ABI07; BE22; BD10]. An overarching question in recent years can be distilled down to the following.

Question 4.1.1. *When can results about familiar objects be generalized to results about differential modules, and what insight does the ability or failure to generalize yield about the original results?*

Avramov-Buchweitz-Iyengar [ABI07] investigate this question for a special class of differential modules—*flag differential modules*—and find generalizations of several results in commutative algebra, including a generalization of the New Intersection Theorem of Hochster, Peskine, Roberts, and Szpiro [Hoc74; PS73; Rob89]. Brown and Erman give a generalization of the notion of a minimal free resolution [BE22], and Banks and VandeBogert construct a differential module generalization of the Koszul complex [BV22]. On the other hand, [BV22] also provide examples of the failure of certain notions to generalize nicely to the realm of differential modules, while Iyengar and Walker similarly show that bounds on the Betti numbers of a free resolution do not generalize to differential modules even in the case of free flags [IW18].

Our goal in this paper is to explore Question 4.1.1 for Boij-Söderberg theory, which concerns the combinatorial structure of the set of all possible Betti tables of modules over a given ring. Initially conjectured by Boij and Söderberg [BS08] for modules over the polynomial ring, this theory brought a new perspective to the study of Betti tables. The Boij-Söderberg conjectures were proven in the finite length case in 2008 by Eisenbud and Schreyer via a duality between free resolutions of graded modules over the polynomial ring and vector bundles on projective space [ES09a]. Their results describe the positive rational cone spanned by Betti tables belonging to modules over the polynomial ring and give a classification, up to rational multiple, of all such tables.

Remark 4.1.2. The original theory was developed in the finite length case and subsequently extended to other codimensions. For simplicity, we restrict ourselves here to the finite length case as well.

The results of [ES09a; BS12] were soon generalized and extended in various directions, such as [ES09b; Erm09; Flø10; FJK11; Ber+11; Ber+12; BS15; ES16; FLS18; FL18; SS21] as well as most recently in [IMW22]. In a particularly interesting extension of the original theory, Eisenbud and Erman [EE17] showed that the duality in [ES09a] does not actually depend on the structure of a free resolution—the duality holds for minimal free complexes with finite length homology more generally (a theme further explored by [IMW22]). This allowed the original Boij-Söderberg theory to be extended from free resolutions to the more general setting of free complexes, but it also showed that the foundations of the theory rested on a different level of structure than what was originally assumed and conjectured. In particular, one interpretation of this generalization is that we should think of Boij-Söderberg theory as *actually* being a theory about the numerics of minimal finite free complexes, rather than a theory about invariants of modules. This leads us to question what the "correct" level of generality for this theory really is. That is, how far can the original Boij-Söderberg theory extend, and what is the key structure on which the theory truly relies? Can Boij-Söderberg theory be further generalized beyond free complexes, and what new insight might such a generalization give us?

We posit a generalization of the Boij-Söderberg conjectures for differential modules and build evidence in support of this conjecture, including some partial results that shed new light on some of the key structures underlying the original theory, while simultaneously giving insight on what makes differential modules similar to and different from complexes and free resolutions.

Let $S = \Bbbk[x_1, \dots, x_n]$ for any field $\Bbbk$ and denote by $\mathrm{DM}(S, a)$ the category of differential modules of degree a over S, that is the graded modules D over S with a square 0 endomorphism $D \to D(a)$. For a differential module $D \in \mathrm{DM}(S, a)$, the *Betti vector* of D, denoted $\beta^{\mathrm{DM}}(D)$ is the vector whose i^{th} entry counts the number of degree i generators in a minimal free resolution of D (see Definition 4.2.7).

Let $\mathbb{V} = \bigoplus_{i,j \in \mathbb{Z}} \mathbb{Q}$ and $\mathbb{B} = \bigoplus_{j \in \mathbb{Z}} \mathbb{Q}$, so that Betti tables and Betti vectors are naturally elements of $\mathbb{V}$ and $\mathbb{B}$, respectively. We denote by $BS_{\text{mod}}(S) \subset \mathbb{V}$ and $BS_{\text{DM}}(S, a) \subset \mathbb{B}$ the rational cones of Betti tables of graded, finite length S-modules and Betti vectors of degree a differential S-modules with finite length homology.

The Betti vector of a differential module is an analogous numerical object to the Betti table of a minimal free resolution, but it represents a certain coarsening of the data of a Betti table. More precisely, for any $a \in \mathbb{Z}$ we may regard any S-complex as a differential module in $\mathrm{DM}(S, a)$

via what [BE22] call the *folding functor*, which we define in more detail in Section 2. As an operation on Betti tables, the folding functor "flattens" the Betti table of a minimal free resolution by forgetting all homological data and retaining only the internal grading. In other words, this flattening is a map $\mathbb{V} \to \mathbb{B}$ that maps Betti tables of minimal free resolutions of S-modules into $BS_{\mathrm{DM}}(S, a)$ for any a. We propose the following:

Conjecture 4.1.3. *This "flattening" gives a surjection of cones $BS_{mod}(S) \to BS_{DM}(S, a)$ for every a. More precisely Any differential module with finite length homology has a Betti vector that can be written as a positive rational combination of Betti vectors of differential modules whose homology have Betti tables that are extremal in $BS_{mod}(S)$.*

Before moving on to our results in support of this conjecture, we first consider a small example

Example 4.1.4. Consider the two variable case when $S = \Bbbk[x, y]$ and (D, ∂) is the degree 0 differential module where

$$D = S \oplus S(-1)^2 \oplus S(-2) \quad \text{and} \quad \partial = \begin{pmatrix} 0 & x & y & 0 \\ 0 & 0 & 0 & -y \\ 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

This differential module has homology $\Bbbk$ and Betti vector $\beta^{\mathrm{DM}}(D) = (1^\circ, 2, 1)$ where 1° indicates that the 1 is in the degree 0 position in the vector. In fact, by [BV22], every degree 0 differential module over $S = \Bbbk[x, y]$ with homology $\Bbbk$ has this Betti vector.

On the other hand, suppose we instead let (D, ∂) be a degree 2 differential module over S, with

$$D = S \oplus S(1)^2 \oplus S(2) \quad \text{and} \quad \partial = \begin{pmatrix} 0 & x & y & 0 \\ 0 & 0 & 0 & -y \\ 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Again, D has homology $\Bbbk$ and now its Betti vector is $\beta^{\mathrm{DM}}(D) = (1, 2, 1^\circ)$. On the other hand, let

(D', ∂') be the degree 2 differential module where

$$D' = S \oplus S(1)^2 \oplus S(2) \quad \text{and} \quad \partial = \begin{pmatrix} 0 & x & y & 1 \\ 0 & 0 & 0 & -y \\ 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

This differential module also has homology $\mathbb{k}$ but, unlike (D, ∂), (D', ∂') is not minimal. It has minimal free resolution

$$\begin{pmatrix} xy & x^2 \\ -y^2 & -xy \end{pmatrix}$$

and Betti vector $\beta^{\mathrm{DM}}(D') = (0, 2, 0^\circ)$.

We draw attention to three particular features of this example and the implications that they highlight. First, observe that $\beta^{\mathrm{DM}}(D')$ cannot be the Betti vector of a fold of minimal free resolution of a finite length, graded S-module since any such module must have total Betti numbers at least 4. In other words, while we conjecture that flattening gives a surjection on *cones*, this example shows that it certainly cannot be a surjection onto the actual semigroup of Betti vectors. The second observation is that we have two differential modules whose homologies are isomorphic, but whose minimal free resolutions–and thus their Betti vectors–are genuinely different. This behavior represents a stark departure from what we are used to in the realm of finite length, graded S-modules. Finally, we point out the differing behavior exhibited by the differential modules in $\mathrm{DM}(S, a)$ as a varies from 0 to 2.

This third point is actually a more general phenomenon that will continue to appear. When dealing with graded complexes, we are always free to twist the individual modules in the complex so that the differential is of degree 0. For differential modules however, the source and target of our differential are the same, so we can not twist one without twisting the other. This means that we are unable to control the degree of the differential in the way that we are used to, and must contend with differentials of nonzero degree. While a differential of nonzero degree may seem innocuous at first glance, we will see that this variance can actually have drastic impacts on the numerical invariants at hand. This phenomenon was observed as well in [BE22, Example 5.4] and

[BV22, Corollary 3.7, Proposition 3.8], and turns out to be a theme underpinning our results here as well. In general, it seems that while for degree 0 we may expect differential modules to behave more or less similarly to complexes in many regards, the same is not the case in more general degree. For Boij-Söderberg theory specifically, this behavior seems to support the idea that the ability to force the differential to be degree 0 is actually an indispensable piece of the structure on which the theory rests. More broadly, it seems that one answer to the question of what makes differential modules similar to or different from complexes lies in their degree.

4.1.1 Results

We split our results into three cases depending on whether the degree of the differential module is positive, negative, or zero. The positive degree case turns out to be much more straightforward—as well as much less reminiscent of the standard Boij-Söderberg theory—and we are able prove the following in full generality via quite elementary methods.

Theorem 4.1.5. *Conjecture 4.1.3 is true for differential modules of degree* $a > 0$.

We are able to prove this result by explicitly constructing all of the extremal rays of the Betti cone. The proof relies on a certain level of "cancellation" that can appear when passing from a differential module to its minimal free resolution–as demonstrated in Example 4.1.4. Our next result says essentially that when the degree of the differential module is not positive, there is a limit to this cancellation.

We prove the following which we show arises as a consequence of Avramov, Buchweitz, and Iyengar's Class Inequality [ABI07, Theorem 4.1].

Proposition 4.1.6. *Let* $F \in \mathrm{DM}(S, a)$ *be a free differential module with finite length homology, and assume that* $a \leq 0$*. Then the Betti vector* $\beta^{DM}(F)$ *has at least* $n + 1$ *nonzero entries.*

Recall that in standard Boij-Söderberg theory the extremal rays of the Betti cone come from the Betti tables of modules with *pure resolutions*, those with the fewest possible nonzero entries. Over $S = \Bbbk[x_1, \ldots, x_n]$, Betti tables of pure resolutions have exactly $n+1$ nonzero entries. Since folds of pure resolutions in degree $a \leq 0$ have Betti vectors with exactly $n + 1$ nonzero entries, Proposition 4.1.6 gives a hint that the extremal rays of $BS_{\text{mod}}(S)$ might remain extremal after flattening. This would mean that the extremal rays in $BS_{\text{DM}}(S,a)$ come from analogous objects as in the standard theory. This provides some motivation for Conjecture 4.1.3 in the $a \leq 0$ case as well as some hope that familiar methods may be applicable.

Indeed, when we restrict to the case $a = 0$, we are able to make use of familiar methods by generalizing the categorical pairing of Eisenbud and Erman [EE17]. The categorified duality in [EE17] clarified the pairing between Betti tables of finite length S-modules and cohomology tables of vector bundles on $\mathbb{P}^{n-1}$ which played a crucial role in the proof of the original Boij-Söderberg conjectures [ES09a]. In the original proof, Eisenbud and Schreyer show a sort of duality between $BS_{\text{mod}}(S)$ and the cone $C_{\text{vb}}(\mathbb{P}^{n-1})$ of cohomology tables of vector bundles on $\mathbb{P}^{n-1}$, which they then use to prove that the facet equations of the cone spanned by the pure Betti tables are nonnegative on the Betti table of any finite length S-module. Eisenbud and Erman then showed that there was in fact a pairing

$$BS_{\text{mod}}(S) \times C_{\text{vb}}(\mathbb{P}^{n-1}) \to BS_{\text{mod}}(\Bbbk[t])$$

which captured this duality.

We generalize this pairing to the setting of degree 0 differential modules and prove analogs of two key results from [EE17]. In particular, we construct a functor

$$\Phi\colon\ \mathrm{D}^{\mathrm{b}}_{\mathrm{DM}}(S,0) \times \mathrm{D}^{\mathrm{b}}(\mathbb{P}^{n-1}) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{DM}}(A,0)$$

where $A = \Bbbk[t]$, $\mathrm{D}^{\mathrm{b}}_{\mathrm{DM}}(S,0)$ and $\mathrm{D}^{\mathrm{b}}_{\mathrm{DM}}(A,0)$ denote the bounded derived categories of degree 0 differential modules over S and A, and $\mathrm{D}^{\mathrm{b}}(\mathbb{P}^{n-1})$ denotes the bounded derived category of coherent sheaves on $\mathbb{P}^{n-1}$. We show that Φ satisfies two crucial properties.

Theorem 4.1.7. *Let $F \in \mathrm{D}^{\mathrm{b}}_{\mathrm{DM}}(S,0)$ be free and $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(\mathbb{P}^{n-1})$. Then*

i. The Betti vector of $\Phi(F,\mathcal{E})$ depends only on the Betti vector of F and the absolute Hilbert function (see Definition 4.4.5) of $\mathcal{E}$.

ii. If $\tilde{F} \otimes \mathcal{E}$ is exact, then $\Phi(F,\mathcal{E})$ has finite length homology.

Thus our Φ also induces a pairing

$$BS_{\text{DM}}(S,0) \times C_{\text{vb}}(\mathbb{P}^{n-1}) \to BS_{\text{DM}}(A,0).$$

The existence of this pairing leads us to conjecture the following analog of the duality displayed in [EE17, Theorem 0.2].

Conjecture 4.1.8. *Let* $\mathbf{b} \in \mathbb{B}$. *Then the following are equivalent:*

i. $\mathbf{b} = \beta^{DM}(F)$ *for a degree 0 differential module* F *over* S *with finite length homology.*

ii. For every vector bundle $\mathcal{E}$ *on* $\mathbb{P}^{n-1}$, $\mathbf{b}$ *pairs with the absolute Hilbert function of* $\mathcal{E}$ *to give the Betti vector of a differential module in* $\text{DM}(A,0)$ *with finite length homology.*

Finally, we prove that Conjecture 4.1.3 holds for degree 0 differential modules over A, explicitly:

Theorem 4.1.9. *Every degree 0 differential modules over* $A = \Bbbk[t]$ *with finite length homology has a Betti vector that can be expressed as a positive rational combination of Betti vectors of folds of pure resolutions whose degree sequences form a chain.*

This result is part of what gives the categorical pairing its power. With a complete description of the extremal rays and facets of the Betti cone over A, the pairing Φ gives a machine for proving that a certain class of linear functionals are nonnegative on every Betti vector in $BS_{\text{DM}}(S,0)$ by reducing to $BS_{\text{DM}}(A,0)$ as a sort of "base case". Proving this nonnegativity is one of the main steps in a proof of Conjecture 4.3.7 in the general case. What ultimately remains standing in the way of a full proof of Conjecture 4.1.3 for degree 0 is a combinatorial description of the cone spanned by the pure Betti vectors. The authors of [ES09a] describe a simplicial fan structure which gives a facet description of the cone for finite length S-modules. While we do this for our base case of $BS_{\text{DM}}(A,0)$, we have not yet been able to replicate this for $BS_{\text{DM}}(S,0)$ more generally, though computational data enables us to conjecture that the extremal facets arise similarly as in [EE17]. Once such a description is proven, we can use the categorical pairing to prove that the equations of these facets are actually nonnegative on all Betti vectors of degree 0 differential modules with finite length homology.

In the case where $a < 0$, we are forced to abandon some of the methods that generalized nicely from complexes to degree 0 differential modules, such as the categorical pairing. However, the positive results we have obtained in the degree ≥ 0 cases lead us to believe that it is natural to extend Conjecture 4.1.3 to the negative degree case as well. That being said, in this setting, new methods appear to be needed to tackle the conjecture and computational data has thus far proved perplexing.

4.2 Background

For the rest of the paper, our convention will be to assume that R is graded-local with maximal ideal $\mathfrak{m}$ and residue field $\Bbbk = R/\mathfrak{m}$.

Definition 4.2.1. For a ring R, a *differential R-module* is a pair (D, ∂) where D is a module over R and $\partial\colon D \to D$ is an R-module endomorphism squaring to 0. In the case where D is graded, we may define, for any $a \in \mathbb{Z}$, a *degree a differential module* over R to be a differential module (D, ∂) where $\partial\colon D \to D(a)$ is homogeneous. We denote the category of degree a differential modules over R by $\mathrm{DM}(R, a)$. A *morphism* of differential modules is a module map that respects the differential, that is a map $f\colon (D, \partial) \to (D', \partial')$ satisfying $f\partial = \partial' f$.

Note that the data of a differential R-module (D, ∂) is equivalent to the data of a graded module over the ring $R[\varepsilon]/(\varepsilon^2)$ with $\deg(\varepsilon) = a$, where the action of ε corresponds to the action of the differential ∂. This implies that $\mathrm{DM}(R, a)$ is an abelian category.

Definition 4.2.2. For a differential module $(D, \partial) \in \mathrm{DM}(R, a)$, the *homology* of D is

$$H(D) = \ker(\partial)/\operatorname{Im}(\partial)(-a)$$

. A morphism of differential modules is a *quasi-isomorphism* if it induces an isomorphism on homology.

Differential modules are in a sense a generalization of chain complexes. In particular, any chain complex may be considered as a differential module where the underlying module is the direct sum of the modules in the complex and the differential is the differential from the complex. More precisely, we have a functor

$$\begin{aligned}\mathrm{Fold}_a\colon\ \mathrm{Ch}(R) &\to \mathrm{DM}(R,a)\\ (C_\bullet,\partial) &\mapsto \left(\bigoplus C_i(ia),\partial\right).\end{aligned}$$

In general, there is no way to consistently define a tensor product of two differential modules. However, we can define the tensor product of a differential module with a *complex* as follows. Given a differential module $D \in \mathrm{DM}(R,a)$ and a graded degree a complex $C_\bullet$ of R,R'-bimodules, the tensor product $D \otimes_R^{DM} C_\bullet \in \mathrm{DM}(R',a)$ is the differential module whose underlying module is $\mathrm{Fold}_a(D \otimes_R C_\bullet)$ and whose differential is given by

$$d \otimes c \mapsto d \otimes \partial^C(c) + (-1)^{\deg(c)} c \otimes \partial^D(d).$$

Since we may regard any R-module as a complex concentrated in degree 0, this defines a tensor product $D \otimes_R^{DM} N$ for any R-module N.

While a differential module is a generalization of a chain complex, passing from complexes to more general differential modules destroys the homological grading which is key to proving many results for complexes. However, we are able to regain some of the benefits of a homological grading for certain differential modules defined below.

Definition 4.2.3. A *flag* is a differential module (F,∂) of the form $F = \bigoplus F_i$ where

$$\partial(F_i) \subset \bigoplus_{j<i} F_j$$

. A flag F is *free/projective* if each of the F_i is free/projective. For a differential module F, the *free/projective class* of F is

$$\min\left\{ n \in \mathbb{N} \text{ such that } F \text{ admits a free/projective flag structure } F = \bigoplus_{i=0}^{n} F_i \right\}.$$

The free class of F over a ring R is denoted $\operatorname{freeclass}_R F$. If no flag structure on F exists, then we say $\operatorname{freeclass}_R F = \infty$

Theorem 4.2.4 ([ABI07] Class Inequality)**.** *Let D be a finitely generated differential module over the polynomial ring $S = \Bbbk[x_1, \dots, x_n]$. Then*

$$\operatorname{freeclass}_S D \geq \operatorname{height}(\operatorname{Ann} H(D)).$$

Given a (free) flag F, we can always extract certain complexes from F. First observe that a free flag F over R consists of the data of a family of free R-modules $\{F_i\}_{i\in\mathbb{N}}$ and a family of morphisms $\{A_{i,j} \colon F_i \to F_j\}_{i>j}$ such that

$$\sum_{i>j>k} A_{j,k} A_{i,j} = 0.$$

In particular, this requires that $A_{i,i-1}A_{i+1,i} = 0$ for all $i \geq 1$, so we there is a free complex

$$\mathbf{F} = \quad F_0 \xleftarrow{A_{1,0}} F_1 \xleftarrow{A_{2,1}} F_2 \leftarrow \cdots$$

Definition 4.2.5. In the setting detailed above, we say that the flag $\left(\bigoplus_{i\geq 0} F_i, \{A_{i,j}\}\right)$ is *anchored by* the complex $\mathbf{F}$. This definition was given in [BV22].

One main goal in the study of differential modules has been to generalize constructions and results for complexes to the differential module setting. Key progress in this endeavor was made by Brown and Erman [BE22] in their definition of free flag and minimal free resolutions of differential modules, which we reiterate here.

Definition 4.2.6. Let D be a differential module over a ring R. A *free flag resolution* of D is a quasi-isomorphism $F \to D$ where F is a free flag.

A result of [BE22] highlights an important link between a differential module D and the minimal free resolution of its homology. Their theorem says that *every* differential module has a free flag resolution anchored on the minimal free resolution of its homology. Following the terminology of [BV22], we call these free flag resolutions *anchored free flags*.

Definition 4.2.7. We say a differential R-module (M, ∂) is *minimal* if $\partial(M) \subseteq \mathfrak{m}M$. For a differential R-module D, *minimal free resolution* of D is a quasi-isomorphism $M \to D$ that factors through a splitting map $M \to F$ where F is a free flag resolution of D and M is minimal.

It is a further result of [BE22] that for differential modules admitting a finitely generated free flag resolution, minimal free resolutions exist, are finitely generated, and are unique up to differential module isomorphism.

We can define certain numerical invariants associated to a differential module. In particular, we have a parallel notion of the Betti numbers of graded differential module:

Definition 4.2.8. Let D be a differential R-module with minimal free resolution F and let N be an R-module. Then we define

$$\operatorname{Tor}^R_{DM}(D, N) := H(F \otimes^{DM}_R N).$$

The *Betti numbers* of D are then defined as

$$\beta^{\mathrm{DM}}_j(D) = \dim_{\Bbbk} \operatorname{Tor}^R_{DM}(D, \Bbbk)_j$$

Remark 4.2.9. One important observation is that, if M is a minimal free resolution of D, then $\beta^{\mathrm{DM}}_j(D)$ is equal to the number of generators of the degree j part of M as a $\Bbbk$-vector space (see [BE22, Remark 1.3(4)]).

Recall that a Betti diagram is called *pure* if it has at most one nonzero entry in each column, and that a resolution with a pure Betti diagram is called a *pure resolution*. We say that a Betti vector is pure if it is the Betti vector of the fold of a pure resolution.

Each pure Betti vector has an associated *degree sequence*, an increasing sequence of integers corresponding to the degrees of the nonzero entries in the Betti vector. We put a partial order on the set of degree sequences of length ℓ by saying that $\mathbf{a} = (a_0 < a_1 < \cdots < a_\ell) \leq \mathbf{b} = (b_0 < b_1 < \cdots < b_\ell)$ if $a_i \leq b_i$ for every i.

4.3 From Free Complexes to Differential Modules

In this section, we compare the theory of Betti tables of free resolutions and free complexes to that of Betti vectors of differential modules and propose a generalization of the Boij-Söderberg conjectures for differential modules.

Let $S = \mathbb{k}[x_1, \ldots, x_n]$ for any fixed field $\mathbb{k}$. We denote the space of *rational Betti vectors* by $\mathbb{B} = \bigoplus_{j \in \mathbb{Z}} \mathbb{Q}$. As in the case of the original Boij-Söderberg theory, our goal is to describe the rational cone spanned by the vectors arising as the Betti vectors of differential modules with finite length homology.

We begin by exploring how Betti vectors of differential modules over S are related to Betti tables of free S-complexes. First note that by Remark 4.2.9 it is enough to consider minimal free resolutions of differential modules, and by [BE22, Theorem 3.2, Theorem 4.2] we may assume that up to differential module isomorphism any minimal free resolution of a differential module D is a summand of an anchored free flag resolution $F \to D$ where F is the fold of the minimal free resolution of the homology $H(D)$. We thus have that $\beta_j^{\mathrm{DM}}(D)$ is at most the number of degree j generators of F, which is in turn the sum $\sum_i \beta_{i,j}(H(D))$.

Definition 4.3.1. Let $(\beta_{i,j})$ be the Betti table of a module M over S. The *degree a flattening* of $(\beta_{i,j})$ is the vector $\mathbf{b} \in \bigoplus_{j \in \mathbb{Z}} \mathbb{Q}$ defined by

$$\mathbf{b}_j = \sum_i \beta_{i,ai+j}$$

Remark 4.3.2. Combining the definitions of flattening and folding, we get that for $\mathbf{F}$ a minimal free resolution of M, the Betti vector of $\mathrm{Fold}_a(\mathbf{F})$ is the degree a flattening of the Betti table $(\beta_{i,j})$ of M.

Remark 4.3.2 says that for certain differential modules D, the Betti vector is equal to the flattening of the Betti table of the $H(D)$. While this is always the case when D is quasi-isomorphic to its homology, it is not true for all differential modules. We saw this in Example 4.1.4 with a degree 2 differential module, but such behavior also occurs in other degrees, as we see with the following.

Example 4.3.3. Let $R = \Bbbk[x, y]$ and let $D \in \mathrm{DM}(R, 0)$ have underlying module

$$R \oplus R(-2) \oplus R(-1)^2 \oplus R(-3)^2 \oplus R(-2) \oplus R(-4)$$

and differential given by the matrix

$$\begin{pmatrix} 0 & 0 & x & y & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x & y & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -y & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & x & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -y \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The homology $H(D)$ is isomorphic to $\Bbbk \oplus \Bbbk(-2)$ which has Betti table

	0	1	2
0:	1	2	1
1:	-	-	-
2:	1	2	1

This Betti table has degree 0 flattening $(1, 2, 2, 2, 1, 0, \ldots)$, but the differential module D is not minimal, so this is not the Betti vector of D. Rather, D has Betti vector $(1, 2, 0, 2, 1)$. This reflects some "cancellation" of the $R(-2)$ summands that occurs when passing from D to its minimal free resolution.

Remark 4.3.4. In fact, a careful analysis (which we omit here) shows that there is no finite length graded module over $\Bbbk[x, y]$ whose Betti table flattens to $(1, 2, 0, 2, 1)$. However, there is a free *complex* with finite length homology that gives this Betti vector. The complex in question has homology $\mathbb{k}$ in degree 0 and $\mathbb{k}(-2)$ in degree 1. Furthermore, the fold of this complex is isomorphic to the minimal free resolution of the differential module in the example. We note also that there *is* a module whose Betti table flattens to a rational multiple of the vector $(1, 2, 0, 2, 1)$, so this vector lives on a ray contained in the Betti cone for finite length S-modules.

Example 4.3.3 shows that there can be some difference between the Betti vector of a differential module and the Betti vector of the fold of the minimal free resolution of its homology. To understand the Betti cone of differential modules, we want to know how big this difference can be, that is, how much "cancellation" can occur. It turns out that the answer to this question depends substantially on the degree of the differential module, with the cases of positive degree and nonpositive degree exhibiting quite different properties. We thus divide the remainder of our discussion into those two cases.

4.3.1 Differential modules of degree $a \leq 0$

We begin with the nonpositive case, which we will see is the case that more closely resembles the behavior of complexes and resolutions. The following result says, in essence, that there is a limit to how much cancellation can occur in the Betti vector so long as the degree of the differential module is not positive.

Proposition 4.3.5. *Let $F \in \mathrm{DM}(S, a)$ be a free differential module with finite length homology for some $a \leq 0$. Then F is generated in at least $n + 1$ distinct degrees.*

Proof. Let (F, ∂) be a minimal, free, graded differential module generated in degrees $d_0, \ldots, d_\ell$ and let $H = H(F)$, the homology of F. Then the grading gives a natural flag structure $F = \bigoplus_i F_i$ where F_i is the piece of F generated in degree at least d_i. This defines a flag structure on F, since minimality and nonpositivity of a ensures that $\partial(F_i) \subset \bigoplus_{i<j} F_j$. Applying the Class Inequality (Theorem 4.2.4), $\ell \geq \operatorname{height} \operatorname{Ann}(H)$ which is equal to n by the finite length assumption. So F must be generated in at least $n + 1$ distinct degrees. □

Note that in the case of finite length modules over S, any Betti table with exactly $n + 1$ nonzero entries is the Betti table of a module with a pure resolution and, furthermore, Betti table is determined up to scalar multiple by the choice of nonzero entries. What we have shown for differential modules is weaker than this, but we conjecture that a similar phenomenon holds. That is, based on experimentation, we guess that the choice of nonzero entries determines the Betti vector up to scalar multiple, and that any Betti vector with exactly $n+1$ nonzero entries belongs to a differential module with pure homology. The reason this is nontrivial is that, due to the type of cancellation we have observed in Betti vectors, one could imagine a differential module D with non-pure homology H where some of the nonzero entries in the Betti table for H cancel out, causing the number of nonzero entries in $\beta^{\mathrm{DM}}(D)$ to be exactly $n+1$.

Remark 4.3.6. Proposition 4.3.5 relies very heavily on the degree ≤ 0 condition. We have already seen that for differential modules in degree $a > 0$ it may fail spectacularly. We will return to this case at the end of this section.

In degree ≤ 0, Proposition 4.3.5 tells us that differential modules with finite length homology cannot be generated in fewer degrees than minimal free complexes with finite length homology. In particular, their Betti vectors have at least as many nonzero entries as the degree 0 flattenings of Betti tables of pure resolutions. On the other hand, we know that any flattening of a Betti table of a finite length module over S exists as the Betti vector of a degree 0 differential module with finite length homology. This leads us to suspect a Boij-Söderberg type conjecture for differential modules.

Conjecture 4.3.7. *Every differential module over S with homology of finite length has a Betti vector that can be expressed as a positive rational combination of Betti vectors of folds of pure resolutions with finite length homology whose degree sequences form a chain.*

Conjecture 4.3.7 posits that the rational cone of Betti vectors of differential modules with finite length homology is the "flattening" of the cone of Betti tables of finite length modules over S. That is, *up to scalar multiple*, every Betti vector of a differential module with finite length homology can be obtained by taking the flattening the Betti table of a finite length module over S, as we saw in Example 4.3.3. We return to this example now in light of Conjecture 4.3.7 and compare the pure decomposition of the Betti vector to that of the Betti table of the homology.

Example 4.3.8. Let $R = \Bbbk[x,y]$ and (D,∂) be as in Example 4.3.3. The Betti table of the homology has pure decomposition

$$\begin{array}{cccc} & 0 & 1 & 2 \\ 0: & 1 & 2 & 1 \\ 1: & - & - & - \\ 2: & 1 & 2 & 1 \end{array} = \begin{array}{cccc} & 0 & 1 & 2 \\ 0: & 1 & 2 & 1 \\ 1: & - & - & - \\ 2: & - & - & - \end{array} + \begin{array}{cccc} & 0 & 1 & 2 \\ 0: & - & - & - \\ 1: & - & - & - \\ 2: & 1 & 2 & 1 \end{array}$$

while the pure decomposition of $\beta^{\mathrm{DM}}(D)$ is

$$\beta^{\mathrm{DM}}(D) = \frac{1}{2}(2^\circ, 3, 0, 1, 0) + \frac{1}{2}(0^\circ, 1, 0, 3, 2)$$

On the other hand, consider the differential module (D', ∂') where $D' = D$ is the same underlying module, but where ∂' is now given by the matrix

$$\begin{pmatrix} 0 & 0 & x & y & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x & y & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -y & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & x & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -y \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(this matrix is *almost* the same, except that the 1 has been replaced with a 0). The homology of (D', ∂') is the same as the homology of (D, ∂), but the Betti vector is

$$\beta^{\mathrm{DM}}(D') = (1^\circ, 2, 2, 2, 1) = (1^\circ, 2, 1, 0, 0) + (0^\circ, 0, 1, 2, 1).$$

In particular, the pure decomposition of the Betti vector of a differential module does not necessarily come from the pure decomposition of the Betti table of its homology.

4.3.2 Differential modules with positive degree

We now return to the case where our differential modules have positive degree. As we will see, this case demonstrates behavior that differs starkly from that of $\mathrm{DM}(S, a)$ for $a \leq 0$, so much so that we may handle it with methods that are much more direct and elementary. We begin with an example of Remark 4.3.6.

Example 4.3.9. Let $R = \Bbbk[x, y]$ and consider the degree 1 differential module R^4 with differential

$$\partial = \begin{pmatrix} 0 & x & y & 0 \\ 0 & 0 & 0 & -y \\ 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 \end{pmatrix} : R^4 \to R^4(1).$$

This differential module is a minimal free flag, so its Betti numbers are the number of generators in each degree. That is, its Betti vector is $(4, 0, 0, \ldots)$.

One consequence of Example 4.3.9 is that the rational cone of Betti vectors of degree 2 differential modules over $\Bbbk[x, y]$ with finite length homology includes the entire positive orthant. It turns out that this is not an isolated phenomenon.

Proposition 4.3.10. *For any $a > 0$, the rational cone of Betti vectors of degree a differential modules over S with finite length homology is equal to the positive orthant in $\mathbb{B}$.*

Proof. It suffices to show that the standard basis vectors are in the cone, i.e. that every standard basis vector is a rational multiple of the Betti vector of some $D \in \mathrm{DM}(S, a)$ with finite length homology. To show this, we show that we can get multiples of basis vectors as the degree a folds of certain minimal free complexes.

Let $\mathbf{K}$ be the koszul complex on $(x_1^a, x_2^a, \ldots, x_n^a)$, and define K to be $\mathrm{Fold}_a(\mathbf{K}) \in \mathrm{DM}(S, a)$. Then K has underlying module S^{2^n} and has minimal differential, so its Betti vector is $(2^n, 0, 0, \ldots)$. By twisting appropriately, we obtain a differential module whose Betti vector has 2^n in the i^{th} entry and 0 elsewhere. □

Remark 4.3.11. Any resolution may be folded into a degree a differential module for any a, and the resulting Betti vector varies for different values of a. We can picture the vectors coming from pure resolutions as moving around while a varies. For positive a, some of these vectors "flare out" as far as possible resulting in a cone that fills the entire positive orthant, losing some of its combinatorial structure.

As an immediate corollary of Proposition 4.3.10, we obtain the following:

Theorem 4.3.12. *Conjecture 4.3.7 is true for* $a > 0$.

Proof. We will prove this by constructing the extremal rays of the cone explicitly. By Proposition 4.3.10, extremal rays of the Betti cone are spanned by the standard basis vectors $\mathbf{e}_i$. The ray generated by e_i contains the Betti vector of $\mathrm{Fold}_a(\mathbf{K}(x_1^a, \dots, x_n^a)[-i])$. Since the Koszul complex on $(x_1^a, \dots, x_n^a)$ is pure of type $(0, a, 2a, \dots, na)$ with finite length homology, the Betti cone is spanned by Betti vectors of differential modules whose homology have pure resolutions. Furthermore, the shifted degree sequences of these resolutions form a chain. □

4.4 Categorified Duality

In this section, we restrict to the case where $a = 0$. We give a generalization of the categorical pairing described in [EE17]. Our version is a pairing between the derived category of differential S-modules and the derived category of coherent sheaves on $\mathbb{P}^{n-1}$ and takes values in the derived category of differential A-modules where $A = \Bbbk[t]$.

Definition 4.4.1. By formally inverting all quasi-ismorphisms in $\mathrm{DM}(S, a)$, we form the *derived category of differential S-modules*, denoted $\mathrm{D_{DM}}(S, a)$. The subcategory of $\mathrm{D_{DM}}(S, a)$ whose objects have finitely generated homology is the *bounded derived category of differential S-modules*, denoted $\mathrm{D^b_{DM}}(S, a)$.

Notation. We denote by $\mathrm{DM}(\mathbb{P}^{n-1})$ the category of sheaves of differential $\mathcal{O}_{\mathbb{P}^{n-1}}$-modules, and denote the corresponding derived and bounded derived categories in the same way as above. We denote the bounded derived category of sheaves of (not necessarily differential) $\mathcal{O}_{\mathbb{P}^{n-1}}$-modules by $\mathrm{D^b}(\mathbb{P}^{n-1})$.

A key ingredient of the categorical pairing is the derived pushforard of a differential module, which we now describe here in some detail, following the same idea as [BE21, pp. 11-12].

Definition 4.4.2 (Derived pushforward of a differential module)**.** Let $(\mathcal{D}, \delta)$ be a representative of a class in $\mathrm{D}^{\mathrm{b}}_{\mathrm{DM}}(\mathbb{P}^{n-1}_A)$. Take $\pi\colon \mathbb{P}^{n-1}_A = \mathbb{P}^{n-1} \otimes \mathbb{A}^1 \to \mathbb{A}^1$ to be projection.

We exploit the equivalence in [BE21, Remark 2.3] between the category of differential modules over a ring R and the category of 1-periodic R-complexes. Taking the expansion of $(\mathcal{D}, \delta)$ we get a 1-periodic complex $\mathbf{Ex}(\mathcal{D}) =$

$$\cdots \leftarrow \mathcal{D} \overset{\delta}{\leftarrow} \mathcal{D} \overset{\delta}{\leftarrow} \mathcal{D} \leftarrow \cdots$$

Now we compute the derived pushforward $R\pi_*$ of this complex. In each position we take the Čech resolution of $\mathcal{D}$ with respect to the usual affine cover of $\mathbb{P}^{n-1}$ which gives a bicomplex

$$\begin{array}{ccccccccc}
\cdots & \xleftarrow{\pm\delta} & \mathcal{C}^{n-1} & \xleftarrow{\pm\delta} & \mathcal{C}^{n-1} & \xleftarrow{\pm\delta} & \mathcal{C}^{n-1} & \xleftarrow{\pm\delta} & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \xleftarrow{\delta} & \vdots & \xleftarrow{\delta} & \vdots & \xleftarrow{\delta} & \vdots & \xleftarrow{\delta} & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \xleftarrow{-\delta} & \mathcal{C}^{1} & \xleftarrow{-\delta} & \mathcal{C}^{1} & \xleftarrow{-\delta} & \mathcal{C}^{1} & \xleftarrow{-\delta} & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \xleftarrow{\delta} & \mathcal{C}^{0} & \xleftarrow{\delta} & \mathcal{C}^{0} & \xleftarrow{\delta} & \mathcal{C}^{0} & \xleftarrow{\delta} & \cdots
\end{array}$$

where the vertical arrows are the maps in the Čech resolution and the horizontal arrows are induced by the differential δ (and thus still square to 0). We apply π_* to this complex and totalize. Since the Čech complex is 0 after n steps, each diagonal of the bicomplex above contains one copy each of $\mathcal{C}^0, \mathcal{C}^1, \ldots, \mathcal{C}^{n-1}$, so this yields a 1-periodic complex

$$\cdots \leftarrow \bigoplus_{i=0}^{n-1} \pi_*(\mathcal{C}^i) \leftarrow \bigoplus_{i=0}^{n-1} \pi_*(\mathcal{C}^i) \leftarrow \bigoplus_{i=0}^{n-1} \pi_*(\mathcal{C}^i) \leftarrow \cdots$$

We consider this as a sheaf of differential $\mathcal{O}_{\mathbb{A}^1}$-modules, and the class represented by this object is what we define to be $R\pi_*(\mathcal{D})$.

Remark 4.4.3. The degree 0 condition is necessary in this definition of $R\pi_*$. In particular, we need the total complex after taking the Čech resolution to be one-periodic, but this fails to be the case if the twists of $\mathcal{D}$ are not the same at each spot, since twisting a sheaf can genuinely change its cohomology.

With this definition in hand, we now define our categorical pairing. Let $\varphi\colon S \to S \otimes_{\Bbbk} A$ be the homomorphism defined by $x_i \mapsto tx_i$. This defineds an S-module structure on $S \otimes_{\Bbbk} A = S[t]$ which in turn yields an action of $\mathbb{P}^{n-1}$ on $\mathbb{P}^{n-1}_A$. Using this action, we define for $F \in \mathrm{DM}(S)$, a bigraded sheaf of differential $\mathcal{O}_{\mathbb{P}^{n-1}_A}$-modules $\Sigma F = \tilde{F} \otimes^{DM}_{\mathbb{P}^{n-1}} \mathcal{O}_{\mathbb{P}^{n-1}_A}$, where the tensor product is taken using the structure given by φ. If we write the differential ∂ on F as a matrix, the differental $\Sigma\partial$ on ΣF is given by the matrix obtained from ∂ by replacing every entry f by $t^{\deg(f)} f$. The purpose of this step is that it will allow us to push forward to A while still retaining all of the graded data of our differential module over S.

Definition 4.4.4. For $F \in \mathrm{D}^{\mathrm{b}}_{\mathrm{DM}}(S, 0)$ and $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(\mathbb{P}^{n-1})$, we define the functor

$$\Phi\colon\ \mathrm{D}^{\mathrm{b}}_{\mathrm{DM}}(S, 0) \times \mathrm{D}^{\mathrm{b}}(\mathbb{P}^{n-1}) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{DM}}(A, 0)$$

by

$$\Phi(F, \mathcal{E}) = R\pi_* \left(\Sigma F \otimes^{DM}_{\mathbb{P}^{n-1}_A} (\mathcal{E} \boxtimes \mathcal{O}_{\mathbb{A}^1}) \right).$$

where π is the projection $\mathbb{P}^{n-1} \times \mathbb{A}^1 \to \mathbb{A}^1$.

Definition 4.4.5. Let $\gamma(\mathcal{E})$ denote the vector whose j^{th} entry is the sum of the ranks of the hypercohomology modules of $\mathcal{E}(j)$. We define the *absolute Hilbert function* of $\mathcal{E}$ to be the function $j \mapsto \gamma_j(\mathcal{E})$. This should be thought of as a cousin of the Hilbert polynomial of $\mathcal{E}$, but where the Euler characteristic is replaced by the sum $\sum_{i \geq 0} h^i(\mathcal{E}(j))$.

Theorem 4.4.6. *$\beta^{DM}(\Phi(F, \mathcal{E}))$ depends only on $\beta^{DM}(F)$ and $\gamma(\mathcal{E})$, in particular the Betti vector of $\Phi(F, \mathcal{E})$ is given by the formula*

$$\beta^{DM}_j(\Phi(F, \mathcal{E})) = \beta^{DM}_j(F)\gamma_{-j}(\mathcal{E})$$

Proof. Replacing F with its minimal free resolution, we may assume that up to quasi-isomorphism that its underlying module is $F = \bigoplus_{j \in \mathbb{Z}} S(-j)^{\beta^{\mathrm{DM}}(F)_j}$. Let $\mathcal{F} = \Sigma F \otimes^{DM}_{\mathbb{P}^{n-1}_A} (\mathcal{E} \boxtimes \mathcal{O}_{\mathbb{A}^1})$. We can rewrite the underlying module of $\mathcal{F}$ as

$$\begin{aligned}\mathcal{F} &= \Sigma F \otimes_{\mathbb{P}^{n-1}_A} (\mathcal{E} \boxtimes \mathcal{O}_{\mathbb{A}^1}) \\ &= \left(\bigoplus_{j\in\mathbb{Z}} \mathcal{O}_{\mathbb{P}^{n-1}_A}(-j,-j)^{\beta^{\mathrm{DM}}(F)_j} \right) \otimes_{\mathbb{P}^{n-1}_A} (\mathcal{E} \boxtimes A) \\ &= \bigoplus_{j\in\mathbb{Z}} (\mathcal{O}_{\mathbb{P}^{n-1}}(-j) \boxtimes A(-j))^{\beta^{\mathrm{DM}}(F)_j} \otimes_{\mathbb{P}^{n-1}_A} (\mathcal{E} \boxtimes A) \\ &= \bigoplus_{j\in\mathbb{Z}} \left(\mathcal{E}(-j)^{\beta^{\mathrm{DM}}(F)_j} \boxtimes A(-j) \right)\end{aligned}$$

By Definition 4.4.2, to compute $R\pi_*(\mathcal{F})$ we expand $\mathcal{F}$ to a one-periodic complex, resolve it with a Čech resolution at each step, then compute π_*. Letting $\mathcal{C}^i(-)$ denote the i^{th} sheaf in the Čech resolution using the standard affine cover of $\mathbb{P}^{n-1}$, we get the following bicomplex

$$\begin{array}{ccccccc} \cdots \xleftarrow{\pm\delta} & \bigoplus_{j\in\mathbb{Z}} \mathcal{C}^{n-1}(\mathcal{E}(-j)^{\beta^{\mathrm{DM}}(F)_j}) \boxtimes A(-j) & \xleftarrow{\pm\delta} & \bigoplus_{j\in\mathbb{Z}} \mathcal{C}^{n-1}(\mathcal{E}(-j)^{\beta^{\mathrm{DM}}(F)_j}) \boxtimes A(-j) & \xleftarrow{\pm\delta} \cdots \\ & \uparrow & & \uparrow & \\ & \vdots & & \vdots & \\ & \uparrow & & \uparrow & \\ \cdots \xleftarrow{-\delta} & \bigoplus_{j\in\mathbb{Z}} \mathcal{C}^{1}(\mathcal{E}(-j)^{\beta^{\mathrm{DM}}(F)_j}) \boxtimes A(-j) & \xleftarrow{-\delta} & \bigoplus_{j\in\mathbb{Z}} \mathcal{C}^{1}(\mathcal{E}(-j)^{\beta^{\mathrm{DM}}(F)_j}) \boxtimes A(-j) & \xleftarrow{-\delta} \cdots \\ & \uparrow & & \uparrow & \\ \cdots \xleftarrow{\delta} & \bigoplus_{j\in\mathbb{Z}} \mathcal{C}^{0}(\mathcal{E}(-j)^{\beta^{\mathrm{DM}}(F)_j}) \boxtimes A(-j) & \xleftarrow{\delta} & \bigoplus_{j\in\mathbb{Z}} \mathcal{C}^{0}(\mathcal{E}(-j)^{\beta^{\mathrm{DM}}(F)_j}) \boxtimes A(-j) & \xleftarrow{\delta} \cdots \end{array}$$

Now we apply π_*, but since every sheaf in the bicomplex is a box tensor of a sheaf on $\mathbb{P}^{n-1}$ with a twist of the structure sheaf on $\mathbb{A}^1$ and π is a proper map, applying π_* to $\mathcal{C}^i(\mathcal{E}(-j)^{\beta^{\mathrm{DM}}(F)_j}) \boxtimes A(-j)$ amounts to taking global sections of $\mathcal{C}^i(\mathcal{E}(-j)^{\beta^{\mathrm{DM}}(F)_j})$ tensored with the appropriate twists of A. This turns each column into a direct sum of complexes $\Gamma\left(\mathbb{P}^{n-1}, \mathcal{C}^\bullet(\mathcal{E}(-j))\right) \otimes_{\mathbb{k}} A(-j)$. Since $\Gamma\left(\mathbb{P}^{n-1}, \mathcal{C}^\bullet(\mathcal{E}(-j))\right)$ is a complex of $\mathbb{k}$-vector spaces, the vertical maps in the bicomplex are split. By [EFS03, Lemma 3.5], the totalization of the bicomplex is homotopic to

$$\cdots \leftarrow \bigoplus_{j\in\mathbb{Z}} H^{tot}(\mathcal{E}(-j)^{\beta^{\mathrm{DM}}(F)_j}) \otimes_{\Bbbk} A(-j) \leftarrow \bigoplus_{j\in\mathbb{Z}} H^{tot}(\mathcal{E}(-j)^{\beta^{\mathrm{DM}}(F)_j}) \otimes_{\Bbbk} A(-j) \leftarrow \cdots$$

where H^{tot} denotes the sum of the hypercohomology modules. The differential is a sum of maps $H^i \to H^{(j<i)}$, where each map comes from taking repeated compositions of the horizontal map δ with the section that splits the vertical map. In particular, this complex is minimal and 1-periodic. The minimal free differential A-module that this corresponds to is a representative of the class $R\pi_*(\mathcal{F}) \in \mathrm{D}^{\mathrm{b}}_{\mathrm{DM}}(A, 0)$, and we can compute its Betti vector by counting the number of generators of the underlying module in each degree. We have

$$R\pi_*(\mathcal{F}) \simeq \bigoplus_{j\in\mathbb{Z}} H^{tot}(\mathcal{E}(-j)^{\beta^{\mathrm{DM}}(F)_j}) \otimes_{\Bbbk} A(-j) = \bigoplus_{j\in\mathbb{Z}} A(-j)^{\beta^{\mathrm{DM}}(F)_j \gamma(\mathcal{E})_{-j}}$$

So the Betti vector of $\Phi(F, \mathcal{E})$ has j^{th} entry

$$\beta^{\mathrm{DM}}_j(\Phi(F, \mathcal{E})) = \beta^{\mathrm{DM}}_j(F)\gamma_{-j}(\mathcal{E}).$$

□

Corollary 4.4.7. *Let $F \in \mathrm{D}^{\mathrm{b}}_{\mathrm{DM}}(S)$ and $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(\mathbb{P}^{n-1})$. Then*

i. $\beta^{DM}_j(F) = \beta^{DM}_j(\Phi(F, \mathcal{O}_{\mathbb{P}^{n-1}}(j))$.

ii. $\gamma_j(\mathcal{E}) = \beta^{DM}_{-j}(\Phi(S(j), \mathcal{E}))$.

Lemma 4.4.8. *As sheaves of differential $\mathcal{O}_{\mathbb{P}^{n-1}_{\Bbbk(t)}}$-modules, there is an isomorphism*

$$\left(\Sigma F \otimes_{\mathbb{P}^{n-1}_A} (\mathcal{E} \boxtimes \mathcal{O}_{\mathbb{A}^1})\right) \otimes_A \Bbbk(t) \simeq \left(\tilde{F} \otimes_{\mathbb{P}^{n-1}_{\Bbbk}} \mathcal{E}\right) \otimes_{\Bbbk} \Bbbk(t).$$

Proof. We have a commutative diagram of rings

$$\begin{array}{ccc} S & \xrightarrow{\varphi} & S \otimes_{\Bbbk} A \\ {\scriptstyle 1\otimes_{\Bbbk}\Bbbk(t)}\downarrow & & \downarrow{\scriptstyle 1\otimes_A \Bbbk(t)} \\ S \otimes_{\Bbbk} \Bbbk(t) & \xrightarrow[\tilde{\varphi}]{} & S \otimes_{\Bbbk} \Bbbk(t) \end{array} \qquad \begin{array}{l} \varphi\colon x_i \mapsto x_i \otimes t \\ \tilde{\varphi}\colon x_i \mapsto x_i \otimes t \end{array}$$

where the map $\tilde{\varphi}$ is invertible. This induces a diagram

$$\begin{array}{ccc} \mathrm{DM}(\mathbb{P}^{n-1}_{\Bbbk}) & \xrightarrow{\varphi} & \mathrm{DM}(\mathbb{P}^{n-1}_{A}) \\ {\scriptstyle -\otimes_{\Bbbk}\Bbbk(t)}\downarrow & & \downarrow{\scriptstyle -\otimes_A \Bbbk(t)} \\ \mathrm{DM}(\mathbb{P}^{n-1}_{\Bbbk(t)}) & \xrightarrow[\tilde{\varphi}]{} & \mathrm{DM}(\mathbb{P}^{n-1}_{\Bbbk(t)}) \end{array}$$

Unraveling definitions, we have $\varphi\left(\tilde{F} \otimes_{\mathbb{P}^{n-1}} \mathcal{E}\right) = \Sigma F \otimes_{\mathbb{P}^{n-1}_A} (\mathcal{E} \boxtimes \mathcal{O}_{\mathbb{A}^1})$ as differential modules—that is, φ is a map of modules that is compatible with the differential (if we consider the differentials to be given by matrices, the map φ just replaces each entry f with $t^{\deg f} f$). By commutativity of the diagram, the claim follows since $\tilde{\varphi}$ is invertible. $\square$

Theorem 4.4.9. *If $\tilde{F} \otimes \mathcal{E}$ is exact then $\Phi(F, \mathcal{E})$ has finite length homology.*

Proof. First we note that the conditions of exactness and finite length are not altered by the grading on A, so in what follows we will always forget the A-grading. Because flat pullback commutes with proper pushforward [Sta22], we have a commutative diagram

$$\begin{array}{ccc} \mathrm{D}^{\mathrm{b}}_{\mathrm{DM}}(\mathbb{P}^{n-1}_{A}) & \xrightarrow{R\pi_*} & \mathrm{D}^{\mathrm{b}}_{\mathrm{DM}}(A) \\ {\scriptstyle -\otimes_{A}\Bbbk(t)}\downarrow & & \downarrow{\scriptstyle -\otimes_{\Bbbk} \Bbbk(t)} \\ \mathrm{D}^{\mathrm{b}}_{\mathrm{DM}}(\mathbb{P}^{n-1}_{\Bbbk(t)}) & \xrightarrow{R\pi_*} & \mathrm{D}^{\mathrm{b}}_{\mathrm{DM}}(\Bbbk(t)) \end{array}$$

The differential module $\Phi(F, \mathcal{E})$ is the image under $R\pi_*$ of the sheaf $\mathcal{F} = \Sigma F \otimes_{\mathbb{P}^{n-1}_A} (\mathcal{E} \boxtimes \mathcal{O}_{\mathbb{A}^1})$. On the other hand, taking the tensor product $\mathcal{F} \otimes_A \Bbbk(t)$ gives $\left(\Sigma F \otimes_{\mathbb{P}^{n-1}_A} (\mathcal{E} \boxtimes \mathcal{O}_{\mathbb{A}^1})\right) \otimes_A \Bbbk(t)$. By Lemma 4.4.8, this is isomorphic to $(\tilde{F} \otimes \mathcal{E}) \otimes_{\Bbbk} \Bbbk(t)$. But since $\tilde{F} \otimes \mathcal{E}$ is exact, $(\tilde{F} \otimes \mathcal{E}) \otimes_{\Bbbk} \Bbbk(t)$ is 0 in the derived category. Thus $\Phi(F, \mathcal{E}) \otimes_{\Bbbk} \Bbbk(t)$ must be 0 in $\mathrm{D}^{\mathrm{b}}_{\mathrm{DM}}(\Bbbk(t))$. Since inverting t yields a differential module with no homology, it therefore follows that $\Phi(F, \mathcal{E})$ has finite length homology. $\square$

The implication of Theorems 4.4.6 and 4.4.9 is that the categorical pairing Φ descends to a pairing on cones

$$BS_{\text{DM}}(S,0)\times C_{\text{vb}}(\mathbb{P}^{n-1})\to BS_{\text{DM}}(A,0)$$

where $C_{\text{vb}}(\mathbb{P}^{n-1})$ is the cone of cohomology vectors of vector bundles on $\mathbb{P}^{n-1}$. In Section 4.5, we will describe the exterior facets of $BS_{\text{DM}}(A,0)$ as linear functionals on $\mathbb{B}$, and in Section 4.6 we will use the base case of $BS_{\text{DM}}(A,0)$ alongside some computational data to outline a strategy for how this pairing on cones could be used to prove Conjecture 4.3.7 in full for degree 0 differential modules.

We finish this section with a concrete example to demonstrate how Φ works.

Example 4.4.10. Let $S=\Bbbk[x,y]$ and let (F,∂) be the fold of the Koszul complex on the variables, so

$$F=S\oplus S(-1)^2\oplus S(-2),\quad \partial=\begin{pmatrix}0&x&y&0\\0&0&0&-y\\0&0&0&x\\0&0&0&0\end{pmatrix}$$

and let $\mathcal{E}=\mathcal{O}_{\mathbb{P}^1}$. Then ΣF has underlying sheaf $\mathcal{O}_{\mathbb{P}^1_A}\oplus\mathcal{O}_{\mathbb{P}^1_A}(-1,-1)^2\oplus\mathcal{O}_{\mathbb{P}^1_A}(-2,-2)$ with differential

$$\Sigma\partial=\begin{pmatrix}0&tx&ty&0\\0&0&0&-ty\\0&0&0&tx\\0&0&0&0\end{pmatrix}.$$

Tensoring with $\mathcal{E}\boxtimes\mathcal{O}_{\mathbb{A}^1}$ does not change the underlying sheaf or differential since $\mathcal{E}=\mathcal{O}_{\mathbb{P}^1}$.

Now computing $R\pi_*$, we get a differential A-module whose underying module is

$$\bigoplus_{j\in\mathbb{Z}}A(-j)^{\beta^{\text{DM}}(F)_j\gamma(\mathcal{E})_{-j}}=A\oplus A(-2)$$

since $\mathcal{O}_{\mathbb{P}^1}(-1)$ has no cohomology. To compute the differential on $\Phi(F,\mathcal{E})$, we use [EFS03, Lemma 3.5], which tells us that our differential should look like

$$\begin{pmatrix}d_{0,0}&d_{1,0}\\0&d_{1,1}\end{pmatrix}$$

where $d_{0,0} : A \to A$ and $d_{1,1} : A(-2) \to A(-2)$ are induced by the pieces of $\Sigma\partial$ that map $\mathcal{O}_{\mathbb{P}^1_A} \to \mathcal{O}_{\mathbb{P}^1_A}$ and $\mathcal{O}_{\mathbb{P}^1_A}(-2) \to \mathcal{O}_{\mathbb{P}^1_A}(-2)$ and are therefore 0 since ΣF was flag. By 4.4.9 the homology is finite length, so the map $d_{1,0} : A(-2) \to A$ must be of the form at^2 for some unit $a \in A$, so our differential is

$$\begin{pmatrix} 0 & at^2 \\ 0 & 0 \end{pmatrix}$$

4.5 Differential Modules Over $\mathbb{k}[t]$

In this section we investigate in detail the case of degree 0 differential modules over the polynomial ring in one variable. In particular, we prove Conjecture 4.3.7 for differential modules over $\mathbb{k}[t]$ with finite length homology. When combined with the results of Section 4.4, this gives a base case for the theory in $n > 1$ variables.

Theorem 4.5.1. *Every degree 0 differential modules over $A = \mathbb{k}[t]$ with finite length homology has a Betti vector that can be expressed as a positive rational combination of Betti vectors of folds of pure resolutions whose degree sequences form a chain.*

To prove Theorem 4.5.1, we will define three cones in $\mathbb{B}$. The first is the cone B of Betti vectors of differential modules with finite length homology. The second is the cone C spanned by Betti vectors of folds of pure resolutions. We will show these two cones are equal via a third cone T defined by the nonnegativity of the following linear functionals:

$$\left\{ \tau_j \colon \beta \mapsto -\beta_j + \sum_{i \neq j} \beta_i \quad \text{and} \quad \sigma_j \colon \beta \mapsto \beta_j \quad \text{for} \quad j \in \mathbb{Z} \right\} \tag{4.1}$$

Let $\mathbb{B}_{(p,q)}$ be the subspace of vectors $\mathbf{b}$ such that $b_n = 0$ for $n > p$ or $n < p$, i.e. all nonzero entries occur between the indices p and q. We set $B_{(p,q)} = \mathbb{B}_{(p,q)} \cap B, C_{(p,q)} = \mathbb{B}_{(p,q)} \cap C$, and $T_{(p,q)} = \mathbb{B}_{(p,q)} \cap T$. We first show that for any $p, q, T_{(p,q)} \subseteq C_{(p,q)}$ by giving an explicit algorithm for decomposing vectors in T as positive rational sums of Betti vectors of folds of pure resolutions.

Algorithm 1 Simplicial Decomposition Algorithm

Input: vector $\mathbf{b} = (b_p, b_{p+1}, \ldots, b_q) \in T$
Output: decomposition of $\mathbf{b}$ as rational combination of pure Betti vectors $c\mathbf{e}_{a,b}$

```
L := empty list
PHASE I:
while τ_i(b) > 0 for all p ≤ i ≤ q do
    Let k, ℓ be the first two indices such that b_k, b_ℓ ≠ 0
    Set j to be the an index where τ_j(b) is minimal
    c := min{b_k, b_ℓ, (1/2)τ_j · b}
    b := b − c e_{k,ℓ}
    add (c, {k, l}) to L
end while
PHASE II:
if τ_j(b) = 0 then
    for i = p, ..., q do
        if b_i ≠ 0 then
            add (b_i, {i, j}) to L
        end if
        b := b − b_i e_{i,j}
    end for
end if
    return L
```

We demonstrate the above algorithm with an example:

Example 4.5.2. Take $\mathbf{b} = (3, 4, 2, 5)$ and assume our nonzero window is between indicies 0 and 3. We can quickly see that $\tau_i(\mathbf{b}) > 0$ for $i = 0, \ldots, 3$, so we proceed with 'Phase I', which is essentially a greedy algorithm. We take $(3, 4, 2, 5) - (1, 1, 0, 0) = (2, 3, 2, 5)$, which is our new $\mathbf{b}$. Again we check that $\tau_i(\mathbf{b}) > 0$ for each i. Since the first two entries are still nonzero, we take our new $\mathbf{b}$ to be $(2, 3, 2, 5) - (1, 1, 0, 0) = (1, 2, 2, 5)$. Now $\tau_3(\mathbf{b}) = 0$, so we move on to 'Phase II', which tells us to decompose $(1, 2, 2, 5)$ as $(1, 0, 0, 1) + 2(0, 1, 0, 1) + 2(0, 0, 1, 1)$. This gives us a final decomposition of $\mathbf{b}$ as

$$\mathbf{b} = 2(1, 1, 0, 0) + (1, 0, 0, 1) + 2(0, 1, 0, 1) + 2(0, 0, 1, 1).$$

The correctness of the decomposition algorithm is proved in the following lemmas.

Lemma 4.5.3. *If $\tau_j\mathbf{b} = 0$ for some j and $b_n = 0$ for $k < n < l$ and $n < k$, then Phase II of the algorithm gives a decomposition of $\mathbf{b}$ as a rational combination of pure Betti vectors.*

Proof. By definition, $\tau_j \mathbf{b} = 0$ if and only if $b_j = \sum_{i \neq j} b_i$, so $\mathbf{b} = \sum_{i \neq j} b_i \mathbf{e}_{\{i,j\}}$. □

The next two lemmas combine to say that after each iteration of the while loop in Phase I of the algorithm, the vector $\mathbf{b}$ remains in the cone T.

Lemma 4.5.4. *Let* $\mathbf{b} \in T$ *and* $c = \min\{b_k, b_\ell, \frac{1}{2}\tau_j \cdot b\}$ *where* j *is an index such that* $\tau_j \mathbf{b}$ *is minimal. Let* $\mathbf{b}' = \mathbf{b} - c\mathbf{e}_{\{k,l\}}$. *Then* $\tau_i \mathbf{b}' \geq 0$ *for all* i.

Proof. By definition,

$$\begin{aligned} \tau_i \mathbf{b}' &= \tau_i \mathbf{b} - c\tau_i \mathbf{e}_{\{k,l\}} \\ &= \max \begin{cases} \tau_i \mathbf{b} - b_k \tau_i \mathbf{e}_{\{k,l\}} \\ \tau_i \mathbf{b} - b_l \tau_i \mathbf{e}_{\{k,l\}} \\ \tau_i \mathbf{b} - (\frac{1}{2}\tau_j \mathbf{b})\tau_i \mathbf{e}_{\{k,l\}} \end{cases} \end{aligned}$$

Note that $\tau_i \mathbf{e}_{\{k,l\}} = 0$ if $i = k, l$ and $\tau_i \mathbf{e}_{\{k,l\}} = 2$ otherwise, so we have

$$\tau_i \mathbf{b}' = \begin{cases} \tau_i \mathbf{b} & \text{if } i = k, l \\ \max\{\tau_i \mathbf{b} - 2b_k, \tau_i \mathbf{b} - 2b_l, \tau_i \mathbf{b} - \tau_j \mathbf{b}\} & \text{if } i \neq k, l \end{cases}$$

By assumption, $\tau_i \mathbf{b} \geq 0$, so in the case where $i = k, l$ we're done. In the case where $i \neq k, l$, $\tau_i \mathbf{b} - \tau_j \mathbf{b} \geq 0$ since j is an index where $\tau_j \mathbf{b}$ is minimal. Since $\tau_i \mathbf{b} - \tau_j \mathbf{b} \geq 0$, we must have that

$$\max\{\tau_i \mathbf{b} - 2b_k, \tau_i \mathbf{b} - 2b_l, \tau_i \mathbf{b} - \tau_j \mathbf{b}\} \geq 0.$$

□

Lemma 4.5.5. *Let* $\mathbf{b} \in T$ *and* $c = \min\{b_k, b_\ell, \frac{1}{2}\tau_j \cdot b\}$ *where* j *is an index such that* $\tau_j \mathbf{b}$ *is minimal. Let* $\mathbf{b}' = \mathbf{b} - c\mathbf{e}_{\{k,l\}}$. *Then* $\sigma_i \mathbf{b}' \geq 0$ *for all* i.

Proof.

$$\sigma_i \mathbf{b}' = \sigma_i \mathbf{b} - c\sigma_i \mathbf{e}_{\{k,l\}} = \begin{cases} b_i - c & \text{if } i = k, l \\ b_i & \text{if } i \neq k, l \end{cases}$$

By assumption, $b_i \geq 0$, so in the case where $i \neq k, l$ $\sigma_i \mathbf{b}' \geq 0$. If $i = k, l$,

$$\sigma_i \mathbf{b}' = b_i - c = \max\{b_i - b_k, b_i - b_l, b_i - \frac{1}{2}\tau_j \mathbf{b}\}.$$

If $i = k$, then $b_i - b_k = 0$ and if $i = l$ then $b_i - b_l = 0$, so in eather case $\sigma_i \mathbf{b}' = \max\{b_i - b_k, b_i - b_l, b_i - \frac{1}{2}\tau_j \mathbf{b}\} \geq 0$. □

We have seen already that Phase II of the algorithm terminates with a decomposition by pure vectors, so to show that the algorithm as a whole terminates we just need the following:

Lemma 4.5.6. *Phase I of the algorithm terminates, that is, after a finite number of iterations,* $\mathbf{b}$ *will satisfy* $\tau_j \mathbf{b} = 0$ *for some* j.

Proof. First notice that if we multiply $\mathbf{b}$ by a constant, the only impact on the algorithm is that at each step the choice of c will be multiplied by said constant, no other aspect of the decomposition will be altered. Multiplying by a constant, we may assume that $b_i \in \mathbb{Z}$ for all i and, furthermore, that $\sum b_i$ is even. This implies that $\frac{1}{2}\tau_i \mathbf{b} \in \mathbb{Z}$ for all i,so all entries of $\mathbf{b}$ are integers throughout every step of the algorithm.

We proceed by induction on the number of nonzero entries of $\mathbf{b}$. If all entries are 0, the result is trivial. Exactly one nonzero entry is impossible by the assumption that $\tau_i \mathbf{b} \geq 0$ for all i. If b_k and b_ℓ are the only two nonzero entries of $\mathbf{b}$, then $\tau_k \mathbf{b} = \tau_\ell \mathbf{b} = 0$. Suppose that $\mathbf{b}$ has more than 2 nonzero entries and take $c = \min\{b_k, b_l, \frac{1}{2}\tau_j \mathbf{b}\}$ where j is an index such that $\tau_j \mathbf{b}$ is minimal as in the algorithm. If $c = b_k$ or b_l, then replacing $\mathbf{b}$ by $\mathbf{b} - c\mathbf{e}_{\{k,l\}}$ results in either $b_l = 0$ or $b_k = 0$, so the number of nonzero entries decreases and we may apply the induction hypothesis to get the result.

Now consider the case where $c = \frac{1}{2}\tau_j\mathbf{b}$ and let $\mathbf{b}' = \mathbf{b} - c\mathbf{e}_{\{k,l\}}$. Then we have

$$\tau_i\mathbf{b}' = \tau_i\mathbf{b} - \left(\frac{1}{2}\tau_j\mathbf{b}\right)\tau_i\mathbf{e}_{\{k,l\}} = \begin{cases} \tau_i\mathbf{b} & \text{if } i = k, l \\ \tau_i\mathbf{b} - \tau_j\mathbf{b} & \text{if } i \neq k, l \end{cases}.$$

If $j \neq k, l$ then $\tau_j\mathbf{b}' = \tau_j\mathbf{b} - \tau_j\mathbf{b} = 0$ and we're done.

If $j = k$ or l, then $\tau_i\mathbf{b}' = \tau_i\mathbf{b} - \tau_j\mathbf{b}$, in particular $\tau_i\mathbf{b}$ strictly decreases for all $i \neq k, l$ every iteration. Since $\tau_i\mathbf{b} \in \mathbb{Z}$ for all i, after enough iterations there will be some $i \neq k, l$ for which $\tau_i\mathbf{b}' < \tau_j\mathbf{b}$. Since in each iteration, j is picked so that τ_j is minimal, this means that in the subsequent iteration of the algorithm we will be in the case where $j \neq k, l$, so the result follows. □

The above lemmas combine to show that for any $p < q$, we have containment $T_{(p,q)} \subseteq C_{(p,q)}$, i.e. that the cone spanned by the pure Betti vectors is contained in the cone defined by the facet equations in (4.1). The following proposition says that additionally any vector in T can be decomposed via pure Betti vectors whose degree sequences *form a chain*.

Proposition 4.5.7. *The degree sequences in L form a chain.*

Proof. First we consider the two phases of the algorithm separately. In Phase I, the degree sequence added to the list at each iteration is (k, ℓ) where k, ℓ are the first two indices of nonzero entries in $\mathbf{b}$. In order for the values of k, ℓ to change at the next iteration, one of b_k or b_ℓ must become zero, meaning that the first two indices of nonzero entries in $\mathbf{b}$ give a degree sequence strictly greater than the previous (k, ℓ). In Phase II, j is fixed and i increases, so the degree sequences (i, j) for $i < j$ and (j, i) for $i > j$ form a chain.

What remains to prove is that the first degree sequence added to L in Phase II is greater than the last degree sequence added to L in Phase I. Let (k, ℓ) be the final degree sequence added in Phase I. This means that for all $i < k$ and all $k < i < \ell$, $b_i = 0$ since k and ℓ are the first two nonzero indices. If $\tau_j \cdot \mathbf{b} = 0$ then either $j > \ell$ or $\mathbf{b} = 0$. In the latter case, no more degree sequences are added, so we're done. In the former case, the first degree sequence added in Phase II is either (i, j) for $i = k$ or ℓ, in which case we have $(i, j) > (k, \ell)$ or $i > \ell$ in which case we also have $(i,j) > (k,\ell)$. □

We can now prove Theorem 4.5.1 by establishing the equality of the three cones B, C, and T.

Proof of Theorem 4.5.1. The containment $C \subseteq B$ follows from the existence of differential modules with pure Betti vectors.

By the simplicial decomposition algorithm and subsequent lemmas, we have $T_{(a,b)} \subseteq C_{(a,b)}$. Since any vector in $\mathbb{B}$ has finitely many nonzero entries, this gives the containment $T \subseteq C$.

To show the containment $B \subseteq T$, we will prove that the functionals τ_j and σ_j are nonnegative on the Betti vector $\beta^{\mathrm{DM}}(D)$ for any graded, degree 0 differential module D with finite length homology over $\Bbbk[t]$. Let $D \in \mathrm{DM}(A, 0)$ have finite length homology H. Since H has finite length over a PID, it is torsion. Thus H is of the form $\bigoplus A(-p)/(t^q)$, so it suffices to check nonnegativity for $H = A(-p)/(t^q)$. In this case, H has minimal free resolution $A(-p-q) \xrightarrow{t^q} A(-p) \to 0$. By [BE22, Theorem 3.2], D has a free flag resolution F of the form $A(-p-q) \oplus A(-p)$ with differential $\begin{pmatrix} 0 & t^q \\ 0 & 0 \end{pmatrix}$, which is minimal. Thus $\beta^{\mathrm{DM}}(D)_i = 1$ for $i = p, p+q$ and 0 otherwise, so $\tau_j \beta^{\mathrm{DM}}(D) = 0$ or 2 and $\sigma_j \beta^{\mathrm{DM}}(D) = 0$ or 1.

This establishes the containments $B \subseteq T \subseteq C$ and $C \subseteq B$. Thus the cone B of Betti vectors of differential modules with finite length homology over A is equal to the cone C spanned by the pure Betti vectors.

Furthermore, by Proposition 4.5.7, every Betti vector of a differential module with finte length homology has a decomposition by pure Betti vectors whose degree sequences form a chain. □

Remark 4.5.8. For Betti diagrams of finite length modules, every Betti table has a *unique* decomposition by pure Betti tables whose degree sequences form a chain. For differential modules however, the Betti vector my have a nonunique decomposition even after insisting that the pure vectors come from a chain of degree sequences. As an easy example, consider the Betti vector $(1,1,1,1)$. It can be decomposed either as $(1,1,0,0) + (0,0,1,1), (1,0,1,0) + (0,1,0,1),$ or $(1,0,0,1) + (0,1,1,0)$. Of these three decompositions, the first two both correspond to degree sequences that form a chain. Our algorithm would yield the first decomposition, corresponding to the chain of degree sequences $(0,1) < (2,3)$. The second decomposition corresponds to the chain

$(0,2) < (1,3)$.

4.6 Predictions for Differential Modules Over S

We conjecture that Theorem 4.5.1 gives a potential base case for the general theory in n variables. Because the τ and σ functionals are non-negative on differential modules in $\mathrm{D}^{\mathrm{b}}_{\mathrm{DM}}(A,0)$ with finite length homology, Theorem 4.4.9 implies that they are non-negative on $\beta^{\mathrm{DM}}(\Phi(F,\mathcal{E}))$ for any coherent sheaf $\mathcal{E}$ and any differential module $F \in \mathrm{D}^{\mathrm{b}}_{\mathrm{DM}}(S,0)$ with finite length homology. In the original theory, similar machinery provided a method for proving nonnegativity of facet equations since the facet equations of the cone spanned by the pure Betti tables came from cohomology tables of vector bundles on $\mathbb{P}^n$. Due to the apparent similarity between the original theory and the generalization to differential modules in the degree 0 case, we expect that something similar might hapen here as well. This illuminates a strategy for proving Conjecture 4.3.7 in general. The idea is that once we know the exterior facets of the cone spanned by Betti vectors of differential modules with pure homology, this will enable us to prove non-negativity of the facet equations on the Betti vectors of differential modules in $\mathrm{DM}(S,0)$, provided the facet equations come from cohomology tables of sheaves on $\mathbb{P}^{n-1}$ as in the original theory. Proving a facet description has remained elusive thus far, but we can use Macaulay2 to compute facet descriptions of the cone in finite dimensional windows.

For instance, let's examine the case where $S = \Bbbk[x,y,z]$ and look at the cone spanned by Betti vectors of differential modules with homology that is pure with degree sequence between $(0,1,2,3)$ and $(4,5,6,7)$. A Fourier Motzkin elimination computation gives the columns of the following matrix as a subset of the linear functionals in the facet description of the cone:

$$\begin{pmatrix} & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & 15 & 15 & -10 & 5 & 6 & 15 & 2 & 3 & 5 & 1 & 8 & 3 & \cdots \\ \cdots & 10 & 10 & 6 & 2 & -3 & -8 & 0 & 1 & 0 & 0 & 3 & 0 & \cdots \\ \cdots & -6 & 6 & 3 & 0 & 1 & 3 & 1 & 0 & 3 & 0 & 0 & 1 & \cdots \\ \cdots & 3 & -3 & 1 & 1 & 0 & 0 & 1 & 0 & -4 & 1 & 1 & 0 & \cdots \\ \cdots & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 3 & -3 & 0 & 3 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 1 & 0 & 2 & -3 & 0 & 6 & 3 & 8 & \cdots \\ \cdots & 0 & 0 & 1 & 2 & 3 & 3 & 5 & 6 & 5 & 10 & 8 & -15 & \cdots \\ \cdots & 1 & 1 & 3 & -5 & 6 & 8 & -9 & 10 & 12 & 15 & -15 & 24 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix} \tag{4.2}$$

One might recognize the entries of each column as the ranks of the sheaf cohomology modules of *supernatural* vector bundles on $\mathbb{P}^2$ (see [ES09b]) appearing with a single negative entry in each column. For instance, twists of the line bundle $\mathcal{O}_{\mathbb{P}^2}$ have cohomology modules whose ranks are given by the entries in the first three columns of (4.2), with some shift. If we look at the sign pattern, we notice that there is a single negative entry in each column, which is exactly what we saw in the facet equations for $BS_{\text{DM}}(A, 0)$, where the facets were given by taking inner product with vectors of the form $\tau_j = (\ldots, 1, 1, -1, 1, \ldots)$, with the negative entry occurring in the j^{th} entry. What is more, we see this same phenomenon repeated for all examples computed.

This leads us to a conjecture that, similarly to [EE17] the facets $BS_{\text{DM}}(S, 0)$ come from combining the cohomology tables of certain vector bundles on $\mathbb{P}^{n-1}$ with the facets for $BS_{\text{DM}}(A, 0)$.

Conjecture 4.6.1. *The exterior facets of the cone of Betti vectors of degree 0 differential modules with finite length homology over S are given by the vanishing of linear functionals that arise as the composition*

$$\mathrm{D}^{\mathrm{b}}_{\mathrm{DM}}(S) \xrightarrow{\Phi(-,\mathcal{E})} \mathrm{D}^{\mathrm{b}}_{\mathrm{DM}}(A) \xrightarrow{f} \mathbb{Z}$$

where $\mathcal{E}$ is a supernatural vector bundle on $\mathbb{P}^{n-1}$ and f is one of the linear functionals τ_j or σ_i defined in (4.1).

To see this in action, consider the first column of (4.2), and suppose we index so that the -6

is the entry in position 0. The vector

$$(\ldots, 15, 10, 6, 3, 1, 0, 0, 1, \ldots)$$

is equal to $\gamma(\mathcal{O}_{\mathbb{P}^{n-1}}(-5))$. Dot product with this vector defines a linear functional that sends $\beta^{\mathrm{DM}}(F)$ to $\beta^{\mathrm{DM}}(\Phi(F, \mathcal{O}_{\mathbb{P}^{n-1}}(-5)))$. Composing this with τ_0 as defined in (4.1) gives the linear function that is dot product with the first column of (4.2).

Bibliography

[ABW82] Kaan Akin, David A. Buchsbaum, and Jerzy Weyman. “Schur functors and Schur complexes”. In: *Adv. in Math.* 44.3 (1982), pp. 207–278. ISSN: 0001-8708. DOI: 10.1016/0001-8708(82)90039-1. URL: https://doi.org/10.1016/0001-8708(82)90039-1.

[Avr98] Luchezar L Avramov. “Infinite free resolutions”. In: *Six lectures on commutative algebra*. Springer, 1998, pp. 1–118.

[ABI07] Luchezar L. Avramov, Ragnar-Olaf Buchweitz, and Srikanth Iyengar. “Class and rank of differential modules”. In: *Invent. Math.* 169.1 (2007), pp. 1–35. ISSN: 0020-9910. DOI: 10.1007/s00222-007-0041-6. URL: https://doi.org/10.1007/s00222-007-0041-6.

[BV22] Maya Banks and Keller VandeBogert. “Differential Modules with Complete Intersection Homology”. In: *arXiv preprint arXiv:2203.15017* (2022).

[Ber+11] Christine Berkesch et al. “Shapes of free resolutions over a local ring”. In: *Mathematische Annalen* 354.3 (Nov. 2011), pp. 939–954. DOI: 10.1007/s00208-011-0760-2.

[Ber+12] Christine Berkesch et al. “The cone of Betti diagrams over a hypersurface ring of low embedding dimension”. In: *Journal of Pure and Applied Algebra* 216.10 (Oct. 2012), pp. 2256–2268. DOI: 10.1016/j.jpaa.2012.03.007.

[BS15] Mats Boij and Gregory G. Smith. “Cones of Hilbert Functions”. In: *International Mathematics Research Notices* 2015.20 (2015), pp. 10314–10338. DOI: 10.1093/imrn/rnu265. URL: https://doi.org/10.1093/imrn/rnu265.

[BS08] Mats Boij and Jonas Söderberg. “Graded Betti numbers of Cohen-Macaulay modules and the multiplicity conjecture”. In: *Journal of the London Mathematical Society* 78.1 (Apr. 2008), pp. 85–106. ISSN: 0024-6107. DOI: 10.1112/jlms/jdn013. URL: http://dx.doi.org/10.1112/jlms/jdn013.

[BS12] Mats Boij and Jonas Söderberg. “Betti numbers of graded modules and the multiplicity conjecture in the non-Cohen–Macaulay case”. In: *Algebra & Number Theory* 6.3 (2012), pp. 437–454.

[BD10] Adam Boocher and Justin W DeVries. “On the Rank of Multigraded Differential Modules”. In: *arXiv preprint arXiv:1011.2167* (2010).

[BE22] Michael Brown and Daniel Erman. “Minimal free resolutions of differential modules”. In: *Transactions of the American Mathematical Society* 375.10 (2022), pp. 7509–7528.

[Bro+17] Michael Brown et al. "Adams operations on matrix factorizations". In: *Algebra & Number Theory* 11.9 (2017), pp. 2165–2192.

[BE21] Michael K. Brown and Daniel Erman. "Tate resolutions on toric varieties". In: *To Appear, Journal of the European Mathematical Society, arXiv preprint arXiv:2108.03345* (2021).

[BE77] David A Buchsbaum and David Eisenbud. "Algebra structures for finite free resolutions, and some structure theorems for ideals of codimension 3". In: *American Journal of Mathematics* 99.3 (1977), pp. 447–485.

[Car86] Gunnar Carlsson. "Free $\mathbb{Z}/2k$-actions and a problem in commutative algebra". In: *Transformation Groups Poznań 1985*. Ed. by Stefan Jackowski and Krzysztof Pawałowski. Berlin, Heidelberg: Springer Berlin Heidelberg, 1986, pp. 79–83. ISBN: 978-3-540-47097-7.

[CE16] Henry Cartan and Samuel Eilenberg. *Homological Algebra (PMS-19), Volume 19*. Princeton university press, 2016.

[EE17] David Eisenbud and Daniel Erman. "Categorified duality in Boij–Söderberg theory and invariants of free complexes". In: *Journal of the European Mathematical Society* 19.9 (2017), pp. 2657–2695. ISSN: 1435-9855. DOI: `10.4171/jems/725`. URL: `http://dx.doi.org/10.4171/JEMS/725`.

[EFS03] David Eisenbud, Gunnar Fløystad, and Frank-Olaf Schreyer. "Sheaf cohomology and free resolutions over exterior algebras". In: *Transactions of the American Mathematical Society* 355.11 (July 2003), pp. 4397–4426. DOI: `10.1090/s0002-9947-03-03291-4`.

[EFW11] David Eisenbud, Gunnar Fløystad, and Jerzy Weyman. "The existence of equivariant pure free resolutions". In: *Annales de l'institut Fourier* 61.3 (2011), pp. 905–926. ISSN: 1777-5310. DOI: `10.5802/aif.2632`. URL: `http://dx.doi.org/10.5802/aif.2632`.

[ES09a] David Eisenbud and Frank-Olaf Schreyer. "Betti numbers of graded modules and cohomology of vector bundles". In: *J. Amer. Math. Soc.* 22.3 (2009), pp. 859–888. ISSN: 0894-0347.

[ES09b] David Eisenbud and Frank-Olaf Schreyer. *Cohomology of Coherent Sheaves and Series of Supernatural Bundles*. 2009. arXiv: `0902.1594 [math.AG]`.

[Erm09] Daniel Erman. "The semigroup of Betti diagrams". In: *Algebra & Number Theory* 3.3 (May 2009), pp. 341–365. ISSN: 1937-0652. DOI: `10.2140/ant.2009.3.341`. URL: `http://dx.doi.org/10.2140/ant.2009.3.341`.

[ES16] Daniel Erman and Steven V Sam. "Supernatural analogues of Beilinson monads". In: *Compositio Mathematica* 152.12 (Nov. 2016), pp. 2545–2562. DOI: `10.1112/s0010437x16007703`. URL: `https://doi.org/10.1112s0010437x16007703`.

[Flø10] Gunnar Fløystad. "The linear space of Betti diagrams of multigraded artinian modules". In: *Mathematical Research Letters* 17.5 (2010), pp. 943–958. DOI: `10.4310/mrl.2010.v17.n5.a11`.

[FJK11] Gunnar Fløystad, Trygve Johnsen, and Andreas Leopold Knutsen, eds. *Combinatorial Aspects of Commutative Algebra and Algebraic Geometry*. Springer Berlin Heidelberg, 2011. DOI: 10.1007/978-3-642-19492-4.

[FL18] Nicolas Ford and Jake Levinson. "Foundations of Boij–Söderberg theory for Grassmannians". In: *Compositio Mathematica* 154.10 (Sept. 2018), pp. 2205–2238. DOI: 10.1112/s0010437x18007418. URL: https://doi.org/10.1112/s0010437x18007418.

[FLS18] Nicolas Ford, Jake Levinson, and Steven Sam. "Towards Boij–Söderberg theory for Grassmannians : the case of square matrices". In: *Algebra & Number Theory* 12.2 (May 2018), pp. 285–303. DOI: 10.2140/ant.2018.12.285. URL: https://doi.org/10.2140/ant.2018.12.285.

[Hal85] Stephen Halperin. "Aspects of topology: Rational homotopy and torus actions". In: 1985.

[Har77] Robin Hartshorne. *Algebraic geometry*. Graduate Texts in Mathematics, No. 52. New York: Springer-Verlag, 1977, pp. xvi+496. ISBN: 0-387-90244-9.

[Har79] Robin Hartshorne. "Algebraic vector bundles on projective spaces: A problem list". In: *Topology* 18.2 (1979), pp. 117–128. ISSN: 0040-9383. DOI: https://doi.org/10.1016/0040-9383(79)90030-2. URL: https://www.sciencedirect.com/science/article/pii/0040938379900302.

[HHW12] F. T. Hawwa, J. William Hoffman, and Haohao Wang. "Koszul duality for multigraded algebras". In: *Eur. J. Pure Appl. Math.* 5.4 (2012), pp. 511–539.

[Hoc74] Melvin Hochster. "The equicharacteristic case of some homological conjectures on local rings". In: *Bulletin of the American Mathematical Society* 80.4 (1974), pp. 683–686. DOI: bams/1183535698. URL: https://doi.org/.

[IW18] Srikanth B Iyengar and Mark E Walker. "Examples of finite free complexes of small rank and small homology". In: *Acta Mathematica* 221.1 (2018), pp. 143–158.

[IMW22] Srikanth B. Iyengar, Linquan Ma, and Mark E. Walker. *Lim Ulrich sequences and Boij-Söderberg cones*. 2022. DOI: 10.48550/ARXIV.2209.03498. URL: https://arxiv.org/abs/2209.03498.

[Kel01] Bernhard Keller. "INTRODUCTION TO A-INFINITY ALGEBRAS AND MODULES". In: *Homology, Homotopy and Applications* 3.1 (2001), pp. 1–35.

[KLN10] Bernhard Keller, Wendy Lowen, and Pedro Nicolás. "On the (non) vanishing of some "derived" categories of curved dg algebras". In: *Journal of Pure and Applied Algebra* 214.7 (2010), pp. 1271–1284.

[Kra66] David Kraines. "Massey higher products". In: *Transactions of the American Mathematical Society* 124.3 (1966), pp. 431–449.

[OR20] Alexei Oblomkov and Lev Rozansky. "Soergel bimodules and matrix factorizations". In: *arXiv preprint arXiv:2010.14546* (2020).

[PS73] Christian Peskine and Lucien Szpiro. "Dimension projective finie et cohomologie locale". fr. In: *Publications Mathématiques de l'IHÉS* 42 (1973), pp. 47–119. URL: http://www.numdam.org/item/PMIHES_1973__42__47_0/.

[RZ17] Claus Michael Ringel and Pu Zhang. "Representations of quivers over the algebra of dual numbers". In: *Journal of Algebra* 475 (2017), pp. 327–360.

[Rob89] Paul Roberts. "Intersection Theorems". In: *Commutative Algebra*. Ed. by M. Hochster, C. Huneke, and J. D. Sally. New York, NY: Springer New York, 1989, pp. 417–436. ISBN: 978-1-4612-3660-3.

[Rou06] Raphaël Rouquier. "Representation dimension of exterior algebras". In: *Inventiones mathematicae* 165.2 (2006), pp. 357–367.

[ŞÜ19] Berrin Şentürk and Özgün Ünlü. "Carlsson's rank conjecture and a conjecture on square-zero upper triangular matrices". In: *Journal of Pure and Applied Algebra* 223.6 (2019), pp. 2562–2584.

[Sta22] The Stacks project authors. *The Stacks project*. https://stacks.math.columbia.edu. 2022.

[Sta17] Torkil Stai. "Differential modules over quadratic monomial algebras". In: *Algebras and Representation Theory* 20.5 (2017), pp. 1239–1247.

[SS21] Alessandro De Stefani and Ilya Smirnov. "Decomposition of graded local cohomology tables". In: *Mathematische Zeitschrift* 297.1 (2021), pp. 1–24. DOI: 10.1007/s00209-020-02494-9. URL: https://doi.org/10.1007/s00209-020-02494-9.

[VW24] Keller VandeBogert and Mark E. Walker. *The Total Rank Conjecture in Characteristic Two*. 2024. arXiv: 2305.09771 [math.AC].

[Wal17] Mark E Walker. "Total Betti numbers of modules of finite projective dimension". In: *Annals of Mathematics* 186.2 (2017), pp. 641–646.

[Wei15] Jiaqun Wei. "Gorenstein homological theory for differential modules". In: *Proceedings of the Royal Society of Edinburgh Section A: Mathematics* 145.3 (2015), pp. 639–655.

[XYY15] Huabo Xu, Shilin Yang, and Hailou Yao. "Gorenstein theory for n-th differential modules". In: *Periodica Mathematica Hungarica* 71.1 (2015), pp. 112–124.