The generalized Terwilliger algebra associated with a 2-homogeneous bipartite distance-regular graph

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Abstract

Let Γ denote a distance-regular graph with diameter $D \geq 3$ and valency $k \geq 3$. For every vertex x of Γ, there is an associated Terwilliger algebra $T(x)$. There exist well known relations in $T(x)$ called the triple product relations. In 2000, Egge introduced the generalized Terwilliger algebra $\mathcal{T}$ associated with Γ. The algebra $\mathcal{T}$ is defined by generators and relations, and the relations are analagous to the triple product relations. There is a surjective algebra homomorphism $\natural : \mathcal{T} \to T(x)$. If Γ is a complete graph, then $\natural$ is an isomorphism. If Γ is not a complete graph, then $\natural$ may or may not be an isomorphism, and in general the details are unknown. If $\natural$ is an isomorphism for all choices of vertex x, we say Γ is T-determined by the triple product relations. In this thesis, we investigate a family of bipartite distance-regular graphs, said to be 2-homogeneous, and we prove that these graphs are T-determined by the triple product relations. We obtain this result using an algebra $\mathcal{T}^{(D)}$ constructed from a pair of character algebras that are related in a certain way.

Dedication

To my son Henry, who always smiles at me when I set work down to come see him.

To my daughter Jane, who, despite my need to stop playing with her sometimes in order to work, still pinkie promises me that we are best friends forever.

To my wife Jackie, who has supported me at every turn, partnered with me in every decision, and sacrificed just as much as I have in order to help me realize this lifelong goal.

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Contents

Chapter 1

Introduction

Let $\Gamma = (X, R)$ denote a finite, undirected, connected graph, without loops or multiple edges, with vertex set X, edge set R, path-length function ∂, and diameter $D \geq 1$. For any vertex $x \in X$ and $0 \leq i \leq D$, let $\Gamma_i(x)$ denote the set of vertices in X that are at distance i from x.

We say that Γ is regular with valency k whenever $k = |\Gamma_1(x)|$ for all $x \in X$. We say that Γ is distance-regular whenever for any $0 \leq h, i, j \leq D$ and vertices $x, y \in X$ such that $\partial(x, y) = h$, the number $|\Gamma_i(x) \cap \Gamma_j(y)|$ is a constant depending only on h, i, j, but not on x and y [4]. We denote this constant by p^h_{ij} and call it an *intersection number of* Γ. If Γ is distance-regular, then Γ is regular with valency $k = p^0_{11}$.

Distance-regular graphs are heavily studied; see [2, 3, 4, 12, 21]. We give a few examples of distance-regular graphs. Recall the five platonic solids: tetrahedron, cube, octahedron, icosahedron, dodecahedron. For each of these, the 1-skeleton is a distance-regular graph. The 1-skeleton of the tetrahedron is the complete graph on 4 vertices. For $n \geq 1$, let K_n denote the complete graph on n vertices. Then K_n is distance-regular. We mentioned the cube. More generally, the hypercube $\mathcal{Q}_D$ of diameter D is distance-regular.

For the rest of this section, the following assumptions are in force. We assume Γ is distance-regular. To avoid trivialities, we assume $D \geq 3$ and $k \geq 3$.

Let A denote the adjacency matrix of our distance-regular graph Γ. The commutative subalgebra M of $\mathrm{Mat}_X(\mathbb{C})$ generated by A is called the Bose-Mesner algebra of Γ. For $0 \leq i \leq D$, let $A_i \in \mathrm{Mat}_X(\mathbb{C})$ denote the matrix with (x, y)-entry

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } \partial(x,y) = i; \\ 0 & \text{if } \partial(x,y) \neq i \end{cases} \qquad (x, y \in X).$$

We call A_i the ith distance matrix of Γ. The matrices $\{A_i\}_{i=0}^D$ form a basis for M. For $0 \leq i, j \leq D$,

$$A_i A_j = \sum_{h=0}^{D} p_{ij}^h A_h.$$

The algebra M has a second basis consisting of the primitive idempotents $\{E_i\}_{i=0}^D$. We have $\sum_{i=0}^D E_i = I$, where I is the identity matrix. Moreover,

$$E_i E_j = \delta_{ij} E_i \qquad (0 \leq i, j \leq D).$$

For $B, C \in \mathrm{Mat}_X(\mathbb{C})$, let $B \circ C$ denote the entry-wise product of B and C. The algebra M is closed under $\circ$ [4]. Thus, there exist scalars $q_{ij}^h \in \mathbb{C}$ $(0 \leq h, i, j \leq D)$ such that

$$E_i \circ E_j = |X|^{-1} \sum_{h=0}^{D} q_{ij}^h E_h \qquad (0 \leq i, j \leq D).$$

The scalars q_{ij}^h are called the Krein parameters of Γ [4].

For the rest of this section, fix a vertex x of Γ. For $0 \leq i \leq D$, let $E_i^* = E_i^*(x) \in \mathrm{Mat}_X(\mathbb{C})$ denote the diagonal matrix with (y, y)-entry

$$(E_i^*)_{yy} = \begin{cases} 1 & \text{if } \partial(x,y) = i; \\ 0 & \text{if } \partial(x,y) \neq i \end{cases} \qquad (y \in X).$$

We call E_i^* the ith dual primitive idempotent of Γ with respect to x. We have $\sum_{i=0}^D E_i^* = I$, and

$$E_i^* E_j^* = \delta_{ij} E_i^* \qquad (0 \leq i, j \leq D).$$

The matrices $\{E_i^*\}_{i=0}^D$ form a basis for a commutative subalgebra $M^* = M^*(x)$ of $\mathrm{Mat}_X(\mathbb{C})$. We call M^* the dual Bose-Mesner algbebra of Γ with respect to x.

For $0 \le i \le D$, let $A_i^* = A_i^*(x) \in \mathrm{Mat}_X(\mathbb{C})$ denote the diagonal matrix with (y, y)-entry

$$(A_i^*)_{yy} = |X|(E_i)_{xy} \qquad\qquad (y \in X).$$

We call A_i^* the ith dual distance matrix of Γ with respect to x. The matrices $\{A_i^*\}_{i=0}^D$ form a basis for M^* [21]. For $0 \le i, j \le D$,

$$A_i^* A_j^* = \sum_{h=0}^{D} q_{ij}^h A_h^*.$$

Let $T = T(x)$ denote the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by M and M^*. We call T the Terwilliger algebra of Γ with respect to x [38]. The algebra T is finite-dimensional, semi-simple, and noncommutative in general. Some notable papers about T are [6, 8, 9, 13, 17, 32, 35, 36, 38, 39, 40]. We mention some well-known relations in T. For $0 \le h, i, j \le D$,

$$\begin{aligned} E_h^* A_i E_j^* = 0 \qquad &\text{iff} \qquad p_{ij}^h = 0, \\ E_h A_i^* E_j = 0 \qquad &\text{iff} \qquad q_{ij}^h = 0. \end{aligned}$$

The above relations are called the triple product relations [38].

In [14], Eric Egge introduced the generalized Terwilliger algebra $\mathcal{T}$. The algebra $\mathcal{T}$ is defined by generators and relations. Roughly speaking, $\mathcal{T}$ is obtained from T by replacing M and M^* with more abstract algebras C and C^* called character algebras [2, 22]. The relations for $\mathcal{T}$ are analagous to the triple product relations.

We return our attention to the distance-regular graph Γ and the fixed vertex x. The algebras M, M^* are examples of character algebras. Consider the generalized Terwilliger algebra $\mathcal{T}$ constructed from $C = M$ and $C^* = M^*$. As we will see, there exists a surjective algebra homomorphism $\natural : \mathcal{T} \to T$. If the graph Γ is a complete graph, then $\natural$ is an isomorphism. If Γ is not a complete graph, then $\natural$ may or may not be an isomorphism, and in general the details are unknown.

In this thesis, we are interested in the case in which $\natural : \mathcal{T} \to T$ is an isomorphism for every choice of x. In order to discuss this case, we make a definition. We will say that Γ is *T-determined by the triple product relations* whenever $\natural : \mathcal{T} \to T$ is an isomorphism for all choices of x. Among our main results, we display a class of graphs that are T-determined by the triple product relations. We now describe this class of graphs.

We will be discussing distance-regular graphs that are bipartite. For the rest of this section, assume that Γ is bipartite. We say that Γ is 2-homogeneous whenever for $1 \leq i \leq D-1$ and vertices $x, y, z \in X$ such that $\partial(x,y) = 2$, $\partial(x,z) = i$, $\partial(y,z) = i$, the number $|\Gamma_1(x) \cap \Gamma_1(y) \cap \Gamma_{i-1}(z)|$ is a constant depending only on i and not on x, y, z.

In this thesis, we will show that if Γ is 2-homogeneous bipartite, then Γ is T-determined by the triple product relations. We have an additional main result that is more algebraic in nature. We will explain this additional result later in this section.

Concerning our results about 2-homogeneous bipartite graphs, it is convenient to consider the hypercube separately.

Assume that Γ is the hypercube $\mathcal{Q}_D$. In [15], Go showed that for all choices of vertex x, there is an algebra isomorphism

$$T \to \bigoplus_{r=0}^{\lfloor D/2 \rfloor} \mathrm{Mat}_{D+1-2r}(\mathbb{C}).$$

In [33], Schrijver described this isomorphism in detail. He used this description to give an improved upper bound on the number of binary codes of a fixed length. In [24], Miklavič used $\mathcal{Q}_D$ to give examples of Leonard triples. In [23], Levstein, Maldonado, and Penazzi studied $\mathcal{Q}_D$ from the point of view of symmetric D-tensors. In [20], Huang studied $\mathcal{Q}_D$ from the point of view of $U(\mathfrak{sl}_2)$-modules.

The following is our first main result.

Theorem 1. *The hypercube $\mathcal{Q}_D$ is T-determined by the triple product relations.*

Next we assume that our graph Γ is 2-homogeneous bipartite, but not a hypercube.

In [26], Nomura gave a classification for these graphs. In [7], Curtin displayed formulas for the eigenvalues of Γ in terms of a nonzero parameter q that is not a root of unity. In [11], Curtin showed that for all choices of vertex x, there is an algebra isomorphism

$$T \to \bigoplus_{r=0}^{\lfloor D/2 \rfloor} \mathrm{Mat}_{D+1-2r}(\mathbb{C}).$$

The following is our second main result.

Theorem 2. *Assume that* Γ *is 2-homogeneous and bipartite, but not a hypercube. Then* Γ *is* T*-determined by the triple product relations.*

Next we summarize our third main result. In our earlier discussion about the work of Curtin, we mentioned a parameter q. Generalizing Curtin's construction, we will define a pair of character algebras C, C^* parametrized by the diameter D and a nonzero $q \in \mathbb{C}$ that is not a root of unity. The generalized Terwilliger algebra constructed from C, C^* is denoted $\mathcal{T}^{(D)}$.

The following is our third main result.

Theorem 3. *For the above algebra* $\mathcal{T}^{(D)}$*, there exists an algebra isomorphism*

$$\mathcal{T}^{(D)} \to \sum_{r=0}^{\lfloor (D+1)/2 \rfloor} \mathrm{Mat}_{D+1-2r}(\mathbb{C}).$$

We will use Theorem 3 to prove Theorem 2.

This thesis is organized as follows. In Chapter 2, we review some background information. In Chapter 3, we prove Theorem 1. In Chapter 4, we prove Theorems 2, 3.

Chapter 2

Preliminaries

In this chapter, we review some preliminary information on notation, distance-regular graphs, totally bipartite tridiagonal pairs, idempotent systems, and character algebras.

2.1 Notation

For the remainder of this thesis, we use the following notations and conventions.

All algebras are assumed to be associative $\mathbb{C}$-algebras with 1.

Let z denote an indeterminate, and let $\mathbb{C}[z]$ denote the algebra consisting of polynomials in z with coefficients in $\mathbb{C}$.

If $\mathfrak{A}$ is an algebra and $\mathfrak{B} \subseteq \mathfrak{A}$ is a subalgebra, then $\mathfrak{A}/\mathfrak{B}$ denotes the quotient algebra. For $a, b \in \mathfrak{A}$, we say *a divides b in $\mathfrak{A}$* whenever there exists some $c \in \mathfrak{A}$ such that $ac = b$.

For integers i, j, we let

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j; \\ 0 & \text{if } i \neq j. \end{cases}$$

If V is any $\mathbb{C}$-vector space, then $I \in \operatorname{End}(V)$ denotes the identity endomorphism.

2.2 Distance-regular graphs

In this section, we review some definitions and results concerning distance-regular graphs and the Terwilliger algebra. For more information, we refer the reader to [4, 21, 38].

Suppose X is a nonempty finite set. Let $\mathrm{Mat}_X(\mathbb{C})$ denote the algebra consisting of the matrices with rows and columns indexed by X and entries in $\mathbb{C}$. Let $\mathbb{C}^X$ denote the vector space over $\mathbb{C}$ consisting of column vectors with coordinates indexed by X and entries in $\mathbb{C}$. The algebra $\mathrm{Mat}_X(\mathbb{C})$ acts on $\mathbb{C}^X$ by left multiplication. For any positive integer n, let $\mathrm{Mat}_n(\mathbb{C})$ denote the algebra consisting of square $n \times n$ matrices with entries in $\mathbb{C}$.

Let $\Gamma = (X, R)$ denote a finite, undirected, connected graph, without loops or multiple edges, with vertex set X, edge set R, and path-length distance function ∂. Let

$$D = \max\{\partial(x, y) \mid x, y \in X\}.$$

We call D the *diameter* of Γ. Vertices $x, y \in X$ are said to be *adjacent* whenever they form an edge. For any vertex $x \in X$, let $\Gamma(x)$ denote the set of vertices in X that are adjacent to x.

Define $A \in \mathrm{Mat}_X(\mathbb{C})$ with (x, y)-entry

$$A_{xy} = \begin{cases} 1 & \text{if } x \text{ and } y \text{ are adjacent;} \\ 0 & \text{if } x \text{ and } y \text{ are not adjacent} \end{cases} \qquad (x, y \in X).$$

We call A the *adjacency matrix of* Γ.

We say that Γ is *regular with valency* k whenever $k = |\Gamma(x)|$ for all $x \in X$.

For all $x \in X$ and for $0 \leq i \leq D$, let

$$\Gamma_i(x) = \{y \in X \mid \partial(x, y) = i\}.$$

Note that $\Gamma_1(x) = \Gamma(x)$.

We say that Γ is *distance-regular* whenever for any $0 \leq h, i, j \leq D$ and vertices $x, y \in X$

such that $\partial(x,y)=h$, the number $|\Gamma_i(x)\cap\Gamma_j(y)|$ is a constant depending only on h,i,j, but not on x and y. We denote this constant by p^h_{ij} and call it an *intersection number of* Γ. Observe that $p^h_{ij}=p^h_{ji}$ $(0\le h,i,j\le D)$.

For convenience, we let $c_i=p^i_{1\,i-1}$ $(1\le i\le D)$ and $b_i=p^i_{1\,i+1}$ $(0\le i\le D-1)$.

For the rest of this section, assume Γ is distance-regular. Note that Γ is regular with valency $k=p^0_{11}$ if $D\ge 1$.

We now recall the Bose-Mesner algebra of Γ. For $0\le i\le D$, define $A_i\in \mathrm{Mat}_X(\mathbb{C})$ with (x,y)-entry

$$(A_i)_{xy}=\begin{cases}1 & \text{if } \partial(x,y)=i;\\ 0 & \text{if } \partial(x,y)\neq i\end{cases} \qquad (x,y\in X).$$

We call A_i the *ith distance matrix of* Γ. Note that $A_0=I$, where $I\in\mathrm{Mat}_X(\mathbb{C})$ denotes the identity matrix. Note that $A_1=A$, provided that $D\ge 1$. The sum $\sum_{i=0}^{D}A_i=J$, where $J\in\mathrm{Mat}_X(\mathbb{C})$ denotes the matrix with every entry equal to 1. For notational convenience, define

$$A_i=0 \qquad (i<0 \text{ or } i>D). \tag{2.2.1}$$

By [4, p. 44],

$$A_iA_j=\sum_{h=0}^{D}p^h_{ij}A_h \qquad (0\le i,j\le D). \tag{2.2.2}$$

Thus $\{A_i\}_{i=0}^{D}$ forms a basis for a commutative subalgebra M of $\mathrm{Mat}_X(\mathbb{C})$. The matrix A generates M by [4, p. 127]. We call M the *Bose-Mesner algebra of* Γ. Note that M has dimension $D+1$.

Because A generates M, there exist polynomials $\{v_i\}_{i=0}^{D}$ in $\mathbb{C}[z]$ such that

$$A_i=v_i(A) \qquad (0\le i\le D).$$

We note that $v_i=1$ and $v_1=z$, provided that $D\ge 1$.

For $0\le i\le D$, define $k_i=p^0_{ii}$. For any $x\in X$, k_i is equal to the number of vertices at distance i from x. We call k_i the *ith valency of* Γ. Note that $k_0 = 1$. Moreover, $k_1 = k$ if $D \geq 1$.

The matrix A has $D+1$ distinct eigenvalues, because A generates M and M has dimension $D+1$. Denote these eigenvalues by $\theta_0 > \theta_1 > \cdots > \theta_D$. For $0 \leq i \leq D$, let $E_i \in \mathrm{Mat}_X(\mathbb{C})$ be the matrix which acts as I on the θ_i-eigenspace of A and as 0 on all other eigenspaces of A. The matrices $\{E_i\}_{i=0}^{D}$ form a basis for M, and

$$\sum_{i=0}^{D} E_i = I, \qquad\qquad E_i E_j = \delta_{ij} E_i \qquad\qquad (0 \leq i, j \leq D). \tag{2.2.3}$$

We call E_i the *primitive idempotent of* Γ *associated with* θ_i $(0 \leq i \leq D)$. Define

$$E_i = 0 \qquad\qquad (i < 0 \text{ or } i > D). \tag{2.2.4}$$

By [4, p. 45], we have $E_0 = |X|^{-1}J$. By construction,

$$A = \sum_{i=0}^{D} \theta_i E_i. \tag{2.2.5}$$

Next we recall the Krein parameters of Γ. For $B, C \in \mathrm{Mat}_X(\mathbb{C})$, let $B \circ C$ denote their entry-wise product. Note that

$$A_i \circ A_j = \delta_{ij} A_i \qquad\qquad (0 \leq i, j \leq D).$$

Consequently, M is closed under $\circ$. Because $\{E_i\}_{i=0}^{D}$ is a basis for M, there exist scalars $q_{ij}^h \in \mathbb{C}$ $(0 \leq h, i, j \leq D)$ such that

$$E_i \circ E_j = |X|^{-1} \sum_{h=0}^{D} q_{ij}^h E_h \qquad\qquad (0 \leq i, j \leq D). \tag{2.2.6}$$

The scalars q_{ij}^h are called the *Krein parameters of* Γ. By [4, p. 50], q_{ij}^h is real and nonnegative for $0 \leq h, i, j \leq D$.

For convenience, we let $c_i^* = q_{1\,i-1}^i$ $(1 \leq i \leq D)$ and $b_i^* = q_{1\,i+1}^i$ $(0 \leq i \leq D-1)$.

For $0 \leq i \leq D$, define $k_i^* = q_{ii}^0$. We call k_i^* the *ith dual valency of* Γ.

We next recall the dual Bose-Mesner algebras of Γ. Fix a vertex $x \in X$. For $0 \leq i \leq D$, let $E_i^* = E_i^*(x)$ denote the diagonal matrix in $\mathrm{Mat}_X(\mathbb{C})$ with (y, y)-entry

$$(E_i^*)_{yy} = \begin{cases} 1 & \text{if } \partial(x, y) = i; \\ \\ 0 & \text{if } \partial(x, y) \neq i \end{cases} \qquad (y \in X).$$

We call E_i^* the *ith dual primitive idempotent of* Γ *with respect to* x. Define

$$E_i^* = 0 \qquad (i < 0 \text{ or } i > D). \tag{2.2.7}$$

By construction,

$$\sum_{i=0}^{D} E_i^* = I, \qquad E_i^* E_j^* = \delta_{ij} E_i^* \qquad (0 \leq i, j \leq D).$$

It follows that $\{E_i^*\}_{i=0}^D$ forms a basis for a commutative subalgebra $M^* = M^*(x)$ of $\mathrm{Mat}_X(\mathbb{C})$. We call M^* the *dual Bose-Mesner algbebra of* Γ *with respect to* x.

For $0 \leq i \leq D$, we define a diagonal matrix $A_i^* = A_i^*(x) \in \mathrm{Mat}_X(\mathbb{C})$ with (y, y)-entry

$$(A_i^*)_{yy} = |X|(E_i)_{xy} \qquad (y \in X).$$

Note that $A_0^* = I$. For $D \geq 1$, we abbreviate $A^* = A_1^*$ and call this the *dual adjacency matrix of* Γ *with respect to* x. Define

$$A_i^* = 0 \qquad (i < 0 \text{ or } i > D). \tag{2.2.8}$$

By [21, Corollary 11.6], A^* generates M^*.

By (2.2.6),

$$A_i^* A_j^* = \sum_{h=0}^{D} q_{ij}^h A_h^* \qquad (0 \leq i, j \leq D).$$

By [21, Lemma 5.8], the matrices $\{A^*_i\}_{i=0}^D$ form a basis for M^*.

Because A^* generates M, there exist polynomials $\{v^*_i\}_{i=0}^D$ in $\mathbb{C}[z]$ such that

$$A^*_i = v^*_i(A^*) \qquad\qquad (0 \leq i \leq D).$$

We note that $v^*_0 = 1$ and $v^*_1 = z$.

We next recall the Terwilliger algebra of Γ. Let $T = T(x)$ denote the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by M and M^*. We call T the *Terwilliger algebra of* Γ *with respect to* x. We remark that T is sometimes called the subconstituent algebra.

For $D \geq 1$, since the matrices $\{E^*_i\}_{i=0}^D$ form a basis for M^*, there exist scalars $\theta^*_i \in \mathbb{C}$ $(0 \leq i \leq D)$ such that

$$A^* = \sum_{i=0}^{D} \theta^*_i E^*_i.$$

Because $\{A_i\}_{i=0}^D$ and $\{E_i\}_{i=0}^D$ each constitute bases for M, there must exist scalars $p_i(j), q_i(j) \in \mathbb{C}$ $(0 \leq i, j \leq D)$ such that

$$A_i = \sum_{j=0}^{D} p_i(j) E_j, \qquad\qquad E_i = |X|^{-1} \sum_{j=0}^{D} q_i(j) A_j \qquad\qquad (0 \leq i \leq D). \tag{2.2.9}$$

Similarly, there must exist scalars $p^*_i(j), q^*_i(j) \in \mathbb{C}$ $(0 \leq i, j \leq D)$ such that

$$A^*_i = \sum_{j=0}^{D} p^*_i(j) E^*_j, \qquad\qquad E^*_i = |X|^{-1} \sum_{j=0}^{D} q^*_i(j) A^*_j \qquad\qquad (0 \leq i \leq D).$$

By the construction of $\{E^*_i\}_{i=0}^D$ and $\{A^*_i\}_{i=0}^D$, we have

$$q_i(j) = p^*_i(j), \qquad\qquad p_i(j) = q^*_i(j) \qquad\qquad (0 \leq i, j \leq D). \tag{2.2.10}$$

For $D \geq 1$, we have

$$\theta_i = p_1(i), \qquad\qquad \theta^*_i = p^*_1(i) \qquad\qquad (0 \leq i \leq D). \tag{2.2.11}$$

Furthermore,

$$p_i(j) = v_i(\theta_j), \qquad p_i^*(j) = v_i^*(\theta_j^*) \qquad (0 \le i, j \le D).$$

By [4, Lemma 2.2.1],

$$p_0(j) = 1, \qquad q_0(j) = 1 \qquad (0 \le j \le D). \tag{2.2.12}$$

Define matrices $P, Q \in \mathrm{Mat}_{D+1}(\mathbb{C})$ with entries $P_{ij} = p_j(i)$ and $Q_{ij} = q_j(i)$ $(0 \le i, j \le D)$. Then P is the change of basis matrix from $\{E_i\}_{i=0}^D$ to $\{A_i\}_{i=0}^D$, and $|X|^{-1}Q$ is the change of basis matrix from $\{A_i\}_{i=0}^D$ to $\{E_i\}_{i=0}^D$. Moreover, P and $|X|^{-1}Q$ are inverses.

By (2.2.10), $|X|^{-1}P$ is the change of basis matrix from $\{A_i^*\}_{i=0}^D$ to $\{E_i^*\}_{i=0}^D$, and Q is the change of basis matrix from $\{E_i^*\}_{i=0}^D$ to $\{A_i^*\}_{i=0}^D$.

By [38, Lemma 3.2], the following hold for $0 \le h, i, j \le D$:

$$\begin{aligned} E_h^* A_i E_j^* = 0 \qquad &\text{iff} \qquad p_{ij}^h = 0, \\ E_h A_i^* E_j = 0 \qquad &\text{iff} \qquad q_{ij}^h = 0. \end{aligned}$$

The above statements are referred to as the *triple product relations.*

We conclude this section by recalling the notion of self-duality. We say that Γ is *self-dual* whenever $p_{ij}^h = q_{ij}^h$ $(0 \le h, i, j \le D)$. In this case, we have $k_i = k_i^*$ and $\theta_i = \theta_i^*$ $(0 \le i \le D)$, and we have $p_i(j) = q_i(j)$ $(0 \le i, j \le D)$. See [4, p. 49].

2.3 2-homogeneous bipartite distance-regular graphs

In this section, we review some definitions and results about 2-homogeneous bipartite distance-regular graphs.

Throughout this section, the following notation is in force. Let Γ denote a distance-regular graph with diameter D and valency k. Fix a vertex x of Γ. To avoid trivialities, assume $D \ge 3$ and $k \ge 3$.

Definition 2.3.1. We say Γ is *bipartite* if the intersection number $p^h_{ij} = 0$ whenever $h+i+j$ is odd $(0 \leq h, i, j \leq D)$.

Definition 2.3.2. (See [25, Section 1]) We say Γ is 2-homogeneous whenever for $1 \leq i \leq D-1$ and vertices x, y, z such that $\partial(x, y) = 2$, $\partial(x, z) = i$, $\partial(y, z) = i$, the number $|\Gamma_1(x) \cap \Gamma_1(y) \cap \Gamma_{i-1}(z)|$ is a constant depending only on i.

By [26], one example of a 2-homogeneous bipartite distance-regular graph is the hypercube $\mathcal{Q}_D$ of dimension D (see Definition 3.2.1). The case where $\Gamma = \mathcal{Q}_D$ is the main object of study in Chapter 3. For the remainder of this section, we are concerned with the case where Γ is not a hypercube.

For the rest of this section, assume Γ is bipartite, but not a hypercube. We recall some equivalent conditions for Γ being 2-homogeneous.

Proposition 2.3.3. (See [7, Theorem 35]) *The following (i)–(iii) are equivalent:*

(i) Γ *is 2-homogeneous;*

(ii) there exists a nonzero $q \in \mathbb{C}$ *that is not a root of unity such that*

$$\theta_i = \theta^*_i = \frac{(q^D + q^2)(q^{D-2i} - 1)}{q^{D-i}(q^2 - 1)} \qquad (0 \leq i \leq D);$$

(iii) there exists a nonzero $q \in \mathbb{C}$ *that is not a root of unity such that*

$$c_i = c^*_i = \frac{(q^D + q^2)(q^{2i} - 1)}{(q^D + q^{2i})(q^2 - 1)}, \qquad b_i = b^*_i = c_{D-i} \qquad (0 \leq i \leq D).$$

If (i)–(iii) hold, then the set of parameters q *that satisfy (ii) is equal to the set of parameters* q *that satisfy (iii).*

For the rest of this section, assume Γ is 2-homogeneous. Recall the Terwilliger algebra $T = T(x)$ of Γ.

Proposition 2.3.4. (See [11, Section 5]) *There exists an algebra isomorphsim*

$$T \to \bigoplus_{r=0}^{\lfloor D/2 \rfloor} \mathrm{Mat}_{D+1-2r}(\mathbb{C}).$$

Corollary 2.3.5. (See [11, Lemma 5.3]) *The dimension of the* $\mathbb{C}$*-vector space* T *is*

$$\dim(T) = \sum_{r=0}^{\lfloor D/2 \rfloor} (D+1-2r)^2.$$

2.4 Totally bipartite tridiagonal pairs

In this section, we review some definitions and results from [31] concerning totally bipartite tridiagonal pairs and totally bipartite tridiagonal systems. For more information, we refer the reader to [1, 5, 10, 16, 18, 19, 37].

For the rest of this section, let V denote a vector space over $\mathbb{C}$ with finite positive dimension. We recall the notion of a totally bipartite tridiagonal pair.

Definition 2.4.1. (See [31, Definition 3.1]) By a *totally bipartite tridiagonal pair* (or *TBT pair*) *on* V, we mean an ordered pair A, A^* of elements in $\mathrm{End}(V)$ that satisfy the following (i)–(iv):

(i) each of A, A^* is diagonalizable;

(ii) there exists an ordering $\{V_i\}_{i=0}^{D}$ of the eigenspaces of A such that

$$A^* V_i \subseteq V_{i-1} + V_{i+1} \qquad (0 \leq i \leq D), \tag{2.4.1}$$

where $V_{-1} = 0$ and $V_{D+1} = 0$;

(iii) there exists an ordering $\{V_i^*\}_{i=0}^{D^*}$ of the eigenspaces of A^* such that

$$A V_i^* \subseteq V_{i-1}^* + V_{i+1}^* \qquad (0 \leq i \leq D^*), \tag{2.4.2}$$

where $V_{-1}^* = 0$ and $V_{D^*+1}^* = 0$;

(iv) there does not exist a proper nonzero subspace $W \subset V$ such that $AW \subseteq W$ and $A^*W \subseteq W$.

We note that if A, A^* is a TBT pair on V, then so is A^*, A.

From now until the statement of Proposition 2.4.14, let A, A^* denote a TBT pair on V. An ordering of the eigenspaces of A is said to be *standard* whenever it satisfies (2.4.1). If the ordering $\{V_i\}_{i=0}^D$ is standard, then the ordering $\{V_{D-i}\}_{i=0}^D$ is also standard, and no further ordering is standard. Similar comments apply to A^*.

From now until the statement of Proposition 2.4.14, we fix a standard ordering $\{V_i\}_{i=0}^D$ of the eigenspaces of A and a standard ordering $\{V_i^*\}_{i=0}^{D^*}$ of the eigenspaces of A^*.

Lemma 2.4.2. (See [31, Corollary 4.11]) *We have $D = D^*$.*

We call D the *diameter* of the TBT pair A, A^*.

Definition 2.4.3. For $0 \leq i \leq D$, let E_i denote the element in $\mathrm{End}(V)$ such that $(E_i - I)V_i = 0$ and $E_iV_j = 0$ for $j \neq i$ $(0 \leq j \leq D)$. We call $\{E_i\}_{i=0}^D$ the *primitive idempotents of* A. For $0 \leq i \leq D$, let E_i^* denote the elment in $\mathrm{End}(V)$ such that $(E_i^* - I)V_i^* = 0$ and $E_i^*V_j^* = 0$ for $j \neq i$ $(0 \leq j \leq D)$. We call $\{E_i^*\}_{i=0}^D$ the *primitive idempotents of* A^*.

Lemma 2.4.4. (See [31, Section 2]) *The following (i)–(iii) hold:*

(i) $V_i = E_iV$ *and* $V_i^* = E_i^*V$ $(0 \leq i \leq D)$;

(ii) $E_iE_j = \delta_{ij}E_i$ *and* $E_i^*E_j^* = \delta_{ij}E_i^*$ $(0 \leq i, j \leq D)$;

(iii) $I = \sum_{i=0}^D E_i$ *and* $I = \sum_{i=0}^D E_i^*$.

Definition 2.4.5. (See [31, Definition 3.7]) By a *totally bipartite tridiagonal system* (or *TBT system*) *on* V, we mean a sequence

$$\Theta = (A; \{E_i\}_{i=0}^D; A^*; \{E_i^*\}_{i=0}^D)$$

of elements in $\mathrm{End}(V)$ that satisfy the following (i)–(iii):

(i) A, A^* is a TBT pair on V;

(ii) $\{E_i\}_{i=0}^D$ is a standard ordering of the primitive idempotents of A;

(iii) $\{E_i^*\}_{i=0}^D$ is a standard ordering of the primitive idempotents of A^*.

We say that the TBT pair A, A^* and the TBT system Θ are *associated.*

From now until the statement of Proposition 2.4.14, let $\Theta = (A; \{E_i\}_{i=0}^D; A^*; \{E_i^*\}_{i=0}^D)$ denote a TBT system on V. Note that the sequence

$$\Theta^* = (A^*; \{E_i^*\}_{i=0}^D; A; \{E_i\}_{i=0}^D)$$

is a TBT system on V. We call Θ^* the *dual of* Θ. We have $(\Theta^*)^* = \Theta$.

Going forward, we will use the following notational convention. For any object f attached to Θ, let f^* denote the corresponding object attached to Θ^*.

Let V' denote a vector space over $\mathbb{C}$ of finite positive dimension, and let

$$\Theta' = (A'; \{E_i'\}_{i=0}^D; A'^*; \{E_i'^*\}_{i=0}^D)$$

denote a TBT system on V'. By an *isomorphism of TBT systems from* Θ *to* Θ', we mean a $\mathbb{C}$-linear bijection $\psi : V \to V'$ such that

$$\begin{aligned} \psi A &= A'\psi, & \psi A^* &= A'^*\psi, & \\ \psi E_i &= E_i'\psi, & \psi E_i^* &= E_i'^*\psi & (0 \leq i \leq D). \end{aligned}$$

We say that Θ and Θ' are *isomorphic* whenever there exists an isomorphism of TBT systems from Θ to Θ'. We say that the TBT system Θ is *self-dual* whenever Θ and Θ^* are isomorphic.

Next we recall some definitions and results concerning the eigenspaces of A.

Definition 2.4.6. For $0 \leq i \leq D$, let θ_i denote the eigenvalue of A corresponding to V_i. We call the sequence $\{\theta_i\}_{i=0}^D$ the *eigenvalue sequence of* Θ. We call the sequence

$(\{\theta_i\}_{i=0}^D; \{\theta_i^*\}_{i=0}^D)$ the *eigenvalue array of* Θ.

Lemma 2.4.7. (See [31, Section 2]) *We have* $A = \sum_{i=0}^D \theta_i E_i$.

Lemma 2.4.8. (See [31, Corollary 4.11]) *We have* $\dim E_i V = 1$ $(0 \le i \le D)$. *Moreover,* $\dim V = D + 1$.

Definition 2.4.9. We say a basis $\{w_i\}_{i=0}^D$ for V is Θ-*standard* if it satisfies the following (i), (ii):

(i) $w_i \in E_i^* V$ $(0 \le i \le D)$;

(ii) $\sum_{r=0}^D w_r \in E_0 V$.

Lemma 2.4.10. (See [31, Definition 4.14 and Lemma 4.15]) *Let* $\{w_i\}_{i=0}^D$ *denote a basis for* V*. The basis* $\{w_i\}_{i=0}^D$ *is* Θ*-standard if and only if there exists a nonzero* $w \in E_0 V$ *such that* $w_i = E_i^* w$ *for* $0 \le i \le D$.

Lemma 2.4.11. *Let* $\{w_i\}_{i=0}^D$ *and* $\{w_i'\}_{i=0}^D$ *denote* Θ*-standard bases for* V*. Then there exists a nonzero* $\zeta \in \mathbb{C}$ *such that*

$$w_i' = \zeta w_i \qquad\qquad (0 \le i \le D).$$

Proof. Follows from Lemmas 2.4.8, 2.4.10. □

Lemma 2.4.12. (See [31, Lemmas 5.4, 11.7]) *Let* $\{w_i\}_{i=0}^D$ *denote a* Θ*-standard basis for* V*. With respect to this basis, the matrices representing* A *and* A^* *have the form*

$$A: \begin{pmatrix} 0 & b_0 & 0 & \dots & 0 & 0 \\ c_1 & 0 & b_1 & \dots & 0 & 0 \\ 0 & c_2 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 0 & b_{D-1} \\ 0 & 0 & 0 & \dots & c_D & 0 \end{pmatrix}, \qquad A^*: \ \mathrm{diag}(\theta_0^*, \theta_1^*, \dots, \theta_D^*).$$

The scalars $\{c_i\}_{i=1}^{D}$, $\{b_i\}_{i=0}^{D-1}$ *are defined by*

$$c_i = \frac{\theta_1\theta_i^* - \theta_0\theta_{i+1}^*}{\theta_{i-1}^* - \theta_{i+1}^*} \qquad (1 \le i \le D-1), \tag{2.4.3}$$

$$c_D = \theta_0, \tag{2.4.4}$$

$$b_i = c_{D-i} \qquad (0 \le i \le D-1). \tag{2.4.5}$$

We call $\{c_i\}_{i=1}^{D}$, $\{b_i\}_{i=0}^{D-1}$ the *intersection numbers of* Θ.

Lemma 2.4.13. (See [31, Lemma 5.3]) *The scalars* $\{c_i\}_{i=1}^{D}$, $\{b_i\}_{i=0}^{D-1}$ *are all nonzero.*

For notational convenience, let $c_{D+1} = 1$ and $b_{-1} = 1$. Let $i \in \mathbb{Z}$. Define $c_i = 0$ unless $1 \le i \le D+1$. Define $b_i = 0$ unless $-1 \le i \le D-1$.

We now relax the assumptions that A, A^* is a TBT pair and Θ is a TBT system. We classify the TBT systems up to isomorphism.

Proposition 2.4.14. (See [31, Theorem 11.1]) *Consider a sequence of scalars taken from* $\mathbb{C}$*:*

$$(\{\theta_i\}_{i=0}^{D}; \{\theta_i^*\}_{i=0}^{D}).$$

There exists a TBT system Θ *with eigenvalue array* $(\{\theta_i\}_{i=0}^{D}; \{\theta_i^*\}_{i=0}^{D})$ *if and only if the following (i)–(iii) hold:*

(i) $\theta_i \ne \theta_j, \theta_i^* \ne \theta_j^*$ *if* $i \ne j$ $(0 \le i, j \le D)$*;*

(ii) there exists $\beta \in \mathbb{C}$ *such that*

$$\theta_{i-1} - \beta\theta_i + \theta_{i+1} = 0, \qquad \theta_{i-1}^* - \beta\theta_i^* + \theta_{i+1}^* = 0 \qquad (1 \le i \le D-1);$$

(iii) $\theta_i + \theta_{D-i} = 0$, $\theta_i^* + \theta_{D-i}^* = 0$ $(0 \le i \le D)$*.*

In this case, Θ *is unique up to isomorphism of TBT systems.*

We conclude this section with a comment on self-duality.

Lemma 2.4.15. (See [31, Lemma 14.5]) *Let* $\Theta = (A; \{E_i\}_{i=0}^D; A^*; \{E_i^*\}_{i=0}^D)$ *denote a TBT system on* V *with eigenvalue array* $(\{\theta_i\}_{i=0}^D; \{\theta_i^*\}_{i=0}^D)$*. The TBT system* Θ *is self-dual if and only if* $\theta_i = \theta_i^*$ $(0 \le i \le D)$*. In this case,* $c_i = c_i^*$ $(1 \le i \le D)$*.*

2.5 The subalgebra generated by A

Throughout this section, the following notation is in force. Let D denote a nonnegative integer. Let V denote a vector space over $\mathbb{C}$ with dimension $D+1$. Let

$$\Theta = (A; \{E_i\}_{i=0}^D; A^*; \{E_i^*\}_{i=0}^D)$$

denote a TBT system on V.

In this section, we review some definitions and results concerning the subalgebra of $\mathrm{End}(V)$ generated by A.

Definition 2.5.1. Let M denote the subalgebra of $\mathrm{End}(V)$ generated by A. Note that M is commutative.

Lemma 2.5.2. *The elements* $\{E_i\}_{i=0}^D$ *form a basis for the* $\mathbb{C}$*-vector space* M*.*

Proof. Follows from Lemma 2.4.4. □

We just displayed a basis for M. Shortly, we will obtain another basis for M.

Definition 2.5.3. Let $\{v_i\}_{i=0}^{D+1}$ denote polynomials in $\mathbb{C}[z]$ such that

$$v_0 = 1, \qquad v_1 = \frac{z}{c_1},$$
$$z v_i = b_{i-1} v_{i-1} + c_{i+1} v_{i+1} \qquad (1 \le i \le D).$$

Lemma 2.5.4. *We have*

$$v_i = \frac{z v_{i-1} - b_{i-2} v_{i-2}}{c_i} \qquad (2 \le i \le D+1).$$

Proof. Follows from Definition 2.5.3. □

Lemma 2.5.5. (See [30, Lemma 4.25]) *The following (i), (ii) hold for* $0 \leq i \leq D+1$*:*

(i) the polynomial v_i *has degree* i*;*

(ii) the coefficient of z^i *in* v_i *is equal to* $\frac{1}{c_1 c_2 \cdots c_i}$.

Lemma 2.5.6. *The following (i), (ii) hold for* $0 \leq i \leq D+1$*:*

(i) if i *is even, then*

$$v_i \in \operatorname{span}\{z^j \mid 0 \leq j \leq D+1,\ j \text{ is even}\};$$

(ii) if i *is odd, then*

$$v_i \in \operatorname{span}\{z^j \mid 0 \leq j \leq D+1,\ j \text{ is odd}\}.$$

Proof. Follows from Definition 2.5.3, Lemmas 2.5.4, 2.5.5, and induction on i. □

Let $i \in Z$. For notational convenience, define $v_i = 0$ unless $0 \leq i \leq D+1$.

Definition 2.5.7. Let μ denote the following polynomial in $\mathbb{C}[z]$:

$$\mu = \prod_{i=0}^{D} (z - \theta_i).$$

Lemma 2.5.8. (See [2, Chapter III.1.3]) *The following (i), (ii) hold:*

(i) μ *is the minimal polynomial of* A*;*

(ii) $v_{D+1} = \frac{\mu}{c_1 c_2 \cdots c_D}$.

Definition 2.5.9. Let $A_i = v_i(A)$ $(0 \leq i \leq D+1)$.

Lemma 2.5.10. (See [30, Lemma 4.11]) *The following (i)–(iii) hold:*

(i) $\dim M = D+1$*;*

(ii) $\{A_i\}_{i=0}^{D}$ *form a basis for the* $\mathbb{C}$*-vector space* M*;*

(iii) $A_{D+1} = 0$.

We make some further observations about the elements $\{A_i\}_{i=0}^{D}$.

Lemma 2.5.11. *We have*

$$A_0 = I, \qquad\qquad A_1 = \frac{A}{c_1},$$
$$AA_i = b_{i-1}A_{i-1} + c_{i+1}A_{i+1} \qquad\qquad (1 \le i \le D).$$

Proof. Follows from Definitions 2.5.3 and 2.5.9. □

Corollary 2.5.12. *We have*

$$A_i = \frac{AA_{i-1} - b_{i-2}A_{i-2}}{c_i} \qquad\qquad (2 \le i \le D+1).$$

Proof. Follows from Lemma 2.5.11. □

Let $i \in \mathbb{Z}$. For notational convenience, define $A_i = 0$ unless $0 \le i \le D+1$.

We conclude this section with a comment.

Lemma 2.5.13. *Assume Θ is self-dual. Then $v_i = v_i^*$ $(0 \le i \le D+1)$ and $\mu = \mu^*$.*

Proof. Follows from Lemma 2.4.15 and Definition 2.5.3. □

2.6 Idempotent systems

In this section, we review some definitions and results from [28] concerning idempotent systems.

Throughout this section, the following notation is in force. Let D denote a nonnegative integer. Let V denote a vector space over $\mathbb{C}$ with dimension $D+1$.

Definition 2.6.1. By a *system of mutually orthogonal rank 1 idempotents in* End(V), we mean a sequence $\{E_i\}_{i=0}^{D}$ of elements in End(V) that satisfy the following (i), (ii) for $0 \le i, j \le D$:

(i) $E_i E_j = \delta_{ij} E_i$;

(ii) $\operatorname{rank}(E_i) = 1$.

Definition 2.6.2. (See [28, Definition 3.1]) By an *idempotent system on V*, we mean a sequence

$$(\{E_i\}_{i=0}^D; \{E_i^*\}_{i=0}^D)$$

of elements in $\operatorname{End}(V)$ that satisfy the following (i)–(iv):

(i) $\{E_i\}_{i=0}^D$ is a system of mutually orthogonal rank 1 idempotents in $\operatorname{End}(V)$;

(ii) $\{E_i^*\}_{i=0}^D$ is a system of mutually orthogonal rank 1 idempotents in $\operatorname{End}(V)$;

(iii) $E_0 E_i^* E_0 \neq 0$ $(0 \leq i \leq D)$;

(iv) $E_0^* E_i E_0^* \neq 0$ $(0 \leq i \leq D)$.

For the rest of this section, let $\Phi = (\{E_i\}_{i=0}^D; \{E_i^*\}_{i=0}^D)$ denote an idempotent system on V. Note that the sequence

$$\Phi^* = (\{E_i^*\}_{i=0}^D; \{E_i\}_{i=0}^D)$$

is an idempotent system on V. We call Φ^* the *dual of* Φ. We have $(\Phi^*)^* = \Phi$.

Going forward, we will use the following notational convention. For any object f attached to Φ, let f^* denote the corresponding object attached to Φ^*.

Let V' denote a vector space over $\mathbb{C}$ of finite positive dimension, and let

$$\Phi' = (\{E'_i\}_{i=0}^D; \{E'^*_i\}_{i=0}^D)$$

denote an idempotent system on V'. By an *isomorphism of idempotent systems from Φ to Φ'*, we mean a $\mathbb{C}$-linear bijection $\psi : V \to V'$ such that

$$\psi E_i = E'_i \psi, \qquad \psi E_i^* = E'^*_i \psi \qquad (0 \leq i \leq D).$$

We say that Φ and Φ' are isomorphic whenever there exists an isomorphism of TBT systems from Φ to Φ'.

Lemma 2.6.3. (See [28, Lemma 2.5]) *We have* $\sum_{i=0}^{D} E_i = I$.

Definition 2.6.4. (See [28, Definition 5.1]) We say that Φ is *symmetric* whenever there exists an antiautomorphism $\dagger$ of $\operatorname{End}(V)$ that fixes each of E_i, E_i^* $(0 \leq i \leq D)$.

For the rest of this section, assume the idempotent system Φ is symmetric.

Definition 2.6.5. Let M denote the subalgebra of $\operatorname{End}(V)$ generated by $\{E_i\}_{i=0}^{D}$.

Note that M is commutative, and $\{E_i\}_{i=0}^{D}$ form a basis for the $\mathbb{C}$-vector space M. Shortly, we will obtain another basis for M.

Definition 2.6.6. For $0 \leq i \leq D$, let $m_i = \operatorname{tr}(E_0^* E_i)$. Note that m_0 is nonzero (see [29, Lemma 7.2]). Let $\nu = m_0^{-1}$.

Note that by construction, $\nu = \nu^*$.

Lemma 2.6.7. (See [28, Lemma 6.3]) *There exists a unique* $\mathbb{C}$*-linear map* $\rho : M \to M^*$ *such that for* $Y \in M$,

$$Y E_0^* E_0 = Y^{\rho} E_0.$$

Definition 2.6.8. Let $A_i = \nu (E_i^*)^{\rho^*}$ $(0 \leq i \leq D)$.

Lemma 2.6.9. (See [28, Lemma 7.7]) *The elements* $\{A_i\}_{i=0}^{D}$ *form a basis for the* $\mathbb{C}$*-vector space* M.

Next we recall the intersection numbers and Krein parameters of Φ.

Definition 2.6.10. Because $\{A_i\}_{i=0}^{D}$ form a basis for M, there exist scalars p_{ij}^{h} $(0 \leq h, i, j \leq D)$ such that

$$A_i A_j = \sum_{h=0}^{D} p_{ij}^{h} A_h \qquad (0 \leq i, j \leq D).$$

We call the scalars p_{ij}^{h} the *intersection numbers of* Φ.

Let $h, i, j \in \mathbb{Z}$. For notational convenience, define $p^h_{ij} = 0$ unless $0 \leq h, i, j \leq D$.

Definition 2.6.11. For $0 \leq h, i, j \leq D$, abbreviate $q^h_{ij} = (p^h_{ij})^*$. We call the scalars q^h_{ij} the *Krein Parameters of* Φ.

Let $h, i, j \in \mathbb{Z}$. For notational convenience, define $q^h_{ij} = 0$ unless $0 \leq h, i, j \leq D$.

Definition 2.6.12. Let $k_i = p^0_{ii}$ $(0 \leq i \leq D)$.

Lemma 2.6.13. (See [28, Lemma 8.4]) *We have*

$$\nu = \sum_{i=0}^{D} k_i.$$

Lemma 2.6.14. (See [28, Lemma 10.11]) *We have*

$$k_h p^h_{ij} = k_i p^i_{jh} = k_j p^j_{hi} \qquad\qquad (0 \leq h, i, j \leq D).$$

We conclude this section with some definitions.

Definition 2.6.15. We say that Φ is *P-polynomial* whenever the following (i), (ii) hold for $0 \leq h, i, j \leq D$:

(i) $p^h_{ij} = 0$ whenever one of h, i, j is greater than the sum of the other two;

(ii) $p^h_{ij} \neq 0$ whenever one of h, i, j is equal to the sum of the other two.

Definition 2.6.16. We say that Φ is *Q-polynomial* whenever the following (i), (ii) hold for $0 \leq h, i, j \leq D$:

(i) $q^h_{ij} = 0$ whenever one of h, i, j is greater than the sum of the other two;

(ii) $q^h_{ij} \neq 0$ whenever one of h, i, j is equal to the sum of the other two.

We note that the idempotent system Φ is Q-polynomial if and only if the idempotent system Φ^* is P-polynomial.

2.7 Character algebras

In this section, we review some definitions and results concerning character algebras and character systems. In later sections, we will show how TBT systems, symmetric idempotent systems, and character systems are related.

We now define a character algebra. For more information on character algebras, see [2, 22]. For the rest of this section, let D denote a nonnegative integer.

Definition 2.7.1. By a *character algebra over* $\mathbb{C}$ *with diameter* D, we mean a sequence

$$(C; \{X_i\}_{i=0}^{D})$$

where C is a commutative $\mathbb{C}$-algebra and $\{X_i\}_{i=0}^{D}$ are elements in C that satisfy the following (i)–(iv):

(i) $X_0 = 1$;

(ii) $\{X_i\}_{i=0}^{D}$ form a basis for the $\mathbb{C}$-vector space C;

(iii) let p_{ij}^{h} $(0 \leq h, i, j \leq D)$ denote scalars such that

$$X_i X_j = \sum_{h=0}^{D} p_{ij}^{h} X_h \qquad (0 \leq i, j \leq D);$$

then there exist scalars $\{k_i\}_{i=0}^{D}$ such that

$$p_{ij}^{0} = \delta_{ij} k_i \qquad (0 \leq i, j \leq D);$$

(iv) the linear map $\pi_0 : C \to \mathbb{C}$ which satisfies $\pi_0(X_i) = k_i$ $(0 \leq i \leq D)$ is an algebra homomorphism.

We call the scalars k_i the *valencies of* $(C; \{X_i\})_{i=0}^{D}$. We call the scalars p_{ij}^{h} the *intersection numbers of* $(C; \{X_i\}_{i=0}^{D})$.

For the rest of this section, let $(C; \{X_i\}_{i=0}^{D})$ denote a character algebra over $\mathbb{C}$.

Let $h, i, j \in \mathbb{Z}$. For notational convenience, define $p^h_{ij} = 0$ unless $0 \leq h, i, j \leq D$.

Lemma 2.7.2. *We have* $p^h_{ij} = p^h_{ji}$ $(0 \leq h, i, j \leq D)$.

Proof. Follows from Definition 2.7.1 and since C is commutative. □

Definition 2.7.3. Let $\nu = \sum_{i=0}^{D} k_i$. We call ν the *size* of $(C; \{X_i\}_{i=0}^{D})$.

Lemma 2.7.4. (See [29, Definition 10.5]) *The scalar* ν *is nonzero.*

Next we recall the primitive idempotents of $(C; \{X_i\}_{i=0}^{D})$.

Proposition 2.7.5. (See [14, Section 2]) *There exists a basis* $\{E_i\}_{i=0}^{D}$ *for the* $\mathbb{C}$*-vector space* C *satisfying the following (i)–(iii):*

(i) $\sum_{i=0}^{D} E_i = X_0$;

(ii) $E_i E_j = \delta_{ij} E_i$ $(0 \leq i, j \leq D)$;

(iii) $E_0 = \nu^{-1} \sum_{i=0}^{D} X_i$.

The basis $\{E_i\}_{i=0}^{D}$ *is unique up to a permutation of* $E_1, E_2, \ldots, E_D$.

We call the elements $\{E_i\}_{i=0}^{D}$ the *primitive idempotents of* $(C; \{X_i\}_{i=0}^{D})$.

Definition 2.7.6. By a *character system over* $\mathbb{C}$ *with diameter* D, we mean a sequence

$$\Psi = (C; \{X_i\}_{i=0}^{D}; \{E_i\}_{i=0}^{D})$$

such that $(C; \{X_i\}_{i=0}^{D})$ is a character algebra over $\mathbb{C}$ and $\{E_i\}_{i=0}^{D}$ are the primitive idempotents of C. We say that the character algebra $(C, \{X_i\}_{i=0}^{D})$ and the character system Ψ are *associated.*

For the rest of this section, let $\Psi = (C; \{X_i\}_{i=0}^{D}; \{E_i\}_{i=0}^{D})$ denote a character system over $\mathbb{C}$. By the *intersection of numbers of* Ψ, we mean the intersection numbers of the character algebra $(C; \{X_i\}_{i=0}^{D})$. By the *valencies of* Ψ, we mean the valencies of $(C; \{X_i\}_{i=0}^{D})$.

Let $\Psi' = (C'; \{X'_i\}_{i=0}^D; \{E'_i\}_{i=0}^D)$ denote a character system over $\mathbb{C}$. By an *isomorphism of character systems from* Ψ *to* Ψ', we mean an algebra isomorphism $C \to C'$ that sends

$$X_i \mapsto X'_i, \qquad E_i \mapsto E'_i \qquad (0 \leq i \leq D).$$

We say that Ψ and Ψ' are *isomorphic* whenever there exists an isomorphism of character systems from Ψ to Ψ'.

Definition 2.7.7. For $0 \leq i, j \leq D$, let $p_i(j)$, $q_i(j)$ denote the scalars such that

$$X_i = \sum_{j=0}^{D} p_i(j) E_j, \qquad E_i = \nu^{-1} \sum_{j=0}^{D} q_i(j) X_j.$$

Lemma 2.7.8. *For* $0 \leq j \leq D$*, we have* $p_0(j) = 1$ *and* $q_0(j) = 1$*.*

Proof. Follows from Proposition 2.7.5 and Definition 2.7.7. □

Definition 2.7.9. Let $P \in \mathrm{Mat}_{D+1}(\mathbb{C})$ denote the matrix with ijth entry $p_j(i)$ $(0 \leq i, j \leq D)$. We call P the *first eigenmatrix of* Ψ. Let $Q \in \mathrm{Mat}_{D+1}(\mathbb{C})$ denote the matrix with ijth entry $q_j(i)$ $(0 \leq i, j \leq D)$. We call Q the *second eigenmatrix of* Ψ.

Note that P is the transition matrix from the basis $\{E_i\}_{i=0}^D$ to the basis $\{X_i\}_{i=0}^D$ for the $\mathbb{C}$-vector space C. Note that $\nu^{-1}Q$ is the transition matrix from the basis $\{X_i\}_{i=0}^D$ to the basis $\{E_i\}_{i=0}^D$ for the $\mathbb{C}$-vector space C. Hence

$$PQ = QP = \nu I. \tag{2.7.1}$$

Definition 2.7.10. We say the character system Ψ is *P-polynomial* whenever the following (i), (ii) hold for $0 \leq h, i, j \leq D$:

(i) $p^h_{ij} = 0$ whenever one of h, i, j is greater than the sum of the other two;

(ii) $p^h_{ij} \neq 0$ whenever one of h, i, j is equal to the sum of the other two.

We conclude this section with a comment.

Example 2.7.11. (See [2, Sections II.2, II.3]) *Let* Γ *denote a distance-regular graph of diameter* D*. Fix a vertex* x *of* Γ*. Let* M*,* M^**,* $\{A_i\}_{i=0}^D$*,* $\{A_i^*\}_{i=0}^D$*,* $\{E_i\}_{i=0}^D$*,* $\{E_i^*\}_{i=0}^D$ *be attached to* Γ *as explained in Section 2.2. Then the sequence* $(M; \{A_i\}_{i=0}^D; \{E_i\}_{i=0}^D)$ *is a character system over* $\mathbb{C}$*, and the sequence* $(M^*; \{A_i^*\}_{i=0}^D; \{E_i^*\}_{i=0}^D)$ *is a character system over* $\mathbb{C}$*.*

2.8 Dual character algebras

In this section, we review the notion of duality for character algebras and character systems.

Throughout this section, the following notation is in force. Let D denote a nonnegative integer. Let $\Psi = (C; \{X_i\}_{i=0}^D; \{E_i\}_{i=0}^D)$ denote a character system over $\mathbb{C}$.

Proposition 2.8.1. (See [29, Lemma 12.7]) *There exists, up to isomorphism of character systems, a unique character system* $\Psi^* = (C^*; \{X_i^*\}_{i=0}^D; \{E_i^*\}_{i=0}^D)$ *over* $\mathbb{C}$ *such that*

$$PP^* = P^*P = \nu I,$$

where P^* *is the first eigenmatrix of* Ψ^**.*

We call Ψ^* the *dual of* Ψ. We have $(\Psi^*)^* = \Psi$.

Going forward, we use the following notational convention. For any object f attached to Ψ, let f^* denote the corresponding object attached to Ψ^*.

We say that the character system Ψ is *self-dual* whenever $P = P^*$.

Remark 2.8.2. (See [14, Definition 3.2]) We have $\nu = \nu^*$.

Lemma 2.8.3. (See [14, Proposition 3.6]) *We have* $q_i(j) = p_i^*(j)$ $(0 \leq i, j \leq D)$*.*

Lemma 2.8.4. *Assume the character system* Ψ *is self-dual. Then*

$$p_i(j) = q_i(j) = p_i^*(j) = q_i^*(j) \qquad (0 \leq i, j \leq D).$$

Proof. Follows from Lemma 2.8.3 applied to Ψ and Ψ^* and the fact that $P = P^*$. □

Next we recall the Krein parameters of Ψ.

Definition 2.8.5. For $0 \leq h, i, j \leq D$, abbreviate $q^h_{ij} = (p^h_{ij})^*$. We call the scalars q^h_{ij} the *Krein parameters of* Ψ.

Let $h, i, j \in \mathbb{Z}$. For notational convenience, define $q^h_{ij} = 0$ unless $0 \leq h, i, j \leq D$.

Definition 2.8.6. We say that Ψ is *Q-polynomial* whenever the following (i), (ii) hold for $0 \leq h, i, j \leq D$:

(i) $q^h_{ij} = 0$ whenever one of h, i, j is greater than the sum of the other two;

(ii) $q^h_{ij} \neq 0$ whenever one of h, i, j is equal to the sum of the other two.

We note that the character system Ψ is Q-polynomial if and only if the character system Ψ^* is P-polynomial.

Lemma 2.8.7. (See [27, Theorem 6.2]) *Assume the character system* Ψ *is* P*-polynomial and* Q*-polynomial. Then the following (i), (ii) are equivalent:*

(i) Ψ *is self-dual;*

(ii) $p^h_{ij} = q^h_{ij}$ $(0 \leq h, i, j \leq D)$.

We conclude this section with a comment.

Example 2.8.8. (See [14, Example 3.3]) *With reference to Example 2.7.11, the character system* $(M^*; \{A^*_i\}^D_{i=0}; \{E^*_i\}^D_{i=0})$ *is isomorphic to the dual of the character system* $(M; \{A_i\}^D_{i=0}; \{E_i\}^D_{i=0})$.

2.9 TBT systems, idempotent systems, and character systems

In this section, we show how TBT systems, symmetric idempotent systems, and character systems are related.

Throughout this section, the following notation is in force. Let D denote a nonnegative integer. Let V denote a vector space over $\mathbb{C}$ with dimension $D+1$.

The next two results are about how TBT systems on V are related to symmetric idempotent systems on V.

Proposition 2.9.1. (See [28, Theorem 21.1]) *Let* $\Theta = (A; \{E_i\}_{i=0}^D; A^*; \{E^*_i\}_{i=0}^D)$ *denote a TBT system on* V*. The sequence*

$$\Phi_\Theta = (\{E_i\}_{i=0}^D; \{E^*_i\}_{i=0}^D)$$

is a symmetric idempotent system on V *that is* P*-polynomial and* Q*-polynomial.*

Referring to Proposition 2.9.1, we say that the TBT system Θ and the idempotent system Φ_Θ are *associated.*

Lemma 2.9.2. *Let* $\Theta = (A; \{E_i\}_{i=0}^D; A^*; \{E^*_i\}_{i=0}^D)$ *denote a TBT system on* V*, and let* Φ_Θ *denote the associated idempotent system. The following (i), (ii) hold:*

- *(i) the algebra* M *from Definition 2.5.1 is equal to the algebra* M *from Definition 2.6.5;*
- *(ii) for* $0 \le i \le D$*, the element* A_i *from Definition 2.5.9 is equal to the element* A_i *from Definition 2.6.8.*

Proof. (i) Follows from Definitions 2.5.1, 2.6.5 and Lemma 2.5.2.

(ii) Follows from [30, Definition 4.4 and Lemma 4.25]. □

We next consider how symmetric idempotent systems on V are related to character systems over $\mathbb{C}$ with diameter D.

Proposition 2.9.3. (See [29, Proposition 11.1]) *Let* $\Phi = (\{E_i\}_{i=0}^D; \{E^*_i\}_{i=0}^D)$ *denote a symmetric idempotent system on* V*. Let the algebra* M *be as in Definition 2.6.5, and let the elements* $\{A_i\}_{i=0}^D$ *be as in Definition 2.6.8. Then the sequence*

$$\Psi_\Phi = (M; \{A_i\}_{i=0}^D; \{E_i\}_{i=0}^D)$$

is a character system over $\mathbb{C}$*.*

Referring to Proposition 2.9.3, we say that the idempotent system Φ and the character system Ψ_Φ are *associated.*

Lemma 2.9.4. (See [29, Theorem 12.2] *Let* $\Phi = (\{E_i\}_{i=0}^D; \{E_i^*\}_{i=0}^D)$ *denote a symmetric idempotent system on* V*, and let* Ψ_Φ *denote the associated character system. Let* V' *denote a vector space over* $\mathbb{C}$ *with dimension* $D+1$*. Let* $\Phi' = (\{E_i'\}_{i=0}^D; \{E_i'^*\}_{i=0}^D)$ *denote a symmetric idempotent system on* V'*, and let* $\Psi_{\Phi'}$ *denote the associated character system. The following (i), (ii) are equivalent:*

(i) the idempotent systems Φ^* *and* Φ' *are isomorphic;*

(ii) the character systems Ψ_Φ^* *and* $\Psi_{\Phi'}$ *are isomorphic.*

Lemma 2.9.5. *Let* $\Phi = (\{E_i\}_{i=0}^D; \{E_i^*\}_{i=0}^D)$ *denote a symmetric idempotent system on* V*, and let* Ψ_Φ *denote the associated character system. The following (i), (ii) hold for* $0 \leq h, i, j \leq D$*:*

(i) the scalar p_{ij}^h *from Definition 2.6.10 is equal to the scalar* p_{ij}^h *from Definition 2.7.1;*

(ii) the scalar q_{ij}^h *from Definition 2.6.11 is equal to the scalar* q_{ij}^h *from Definition 2.8.5.*

Proof. (i) Follows from Definitions 2.6.10, 2.7.1.

(ii) Follows from Lemma 2.9.4 and (i). □

Lemma 2.9.6. *Let* $\Phi = (\{E_i\}_{i=0}^D; \{E_i^*\}_{i=0}^D)$ *denote a symmetric idempotent system on* V*, and let* Ψ_Φ *denote the associated character system. The following (i), (ii) hold:*

(i) for $0 \leq i \leq D$*, the scalar* k_i *from Definition 2.6.12 is equal to the scalar* k_i *from Definition 2.7.1;*

(ii) the scalar ν *from Definition 2.6.6 is equal to the scalar* ν *from Definition 2.7.3.*

Proof. (i) Follows from Definitions 2.6.12, 2.7.1 and Lemma 2.9.5.

(ii) Follows from Lemma 2.6.13, Definition 2.7.3, and (i). □

Lemma 2.9.7. *Let $\Phi = (\{E_i\}_{i=0}^D; \{E_i^*\}_{i=0}^D)$ denote a symmetric idempotent system on V, and let Ψ_Φ denote the associated character system. The following (i), (ii) hold:*

(i) Φ is P-polynomial if and only if Ψ_Φ is P-polynomial;

(ii) Φ is Q-polynomial if and only if Ψ_Φ is Q-polynomial.

Proof. Follows from Lemma 2.9.5. □

So far in this section, we have described how TBT systems on V are related to symmetric idempotent systems on V, and how symmetric idempotent systems on V are related to character systems over $\mathbb{C}$ with diameter D. From this description, we can see how TBT systems on V are related to character systems over $\mathbb{C}$ with diameter D. In the following result, we emphasize some aspects of this relationship.

Proposition 2.9.8. *Let $\Theta = (A; \{E_i\}_{i=0}^D; A^*; \{E_i^*\}_{i=0}^D)$ denote a TBT system on V. Let the algebra M be as in Definition 2.5.1, and let the elements $\{A_i\}_{i=0}^D$ be as in Definition 2.5.9. The sequence*

$$\Psi_\Theta = (M; \{A_i\}_{i=0}^D; \{E_i\}_{i=0}^D)$$

is a character system over $\mathbb{C}$ that is P-polynomial and Q-polynomial.

Proof. Follows from Propositions 2.9.1, 2.9.3 and Lemma 2.9.7. □

Referring to Proposition 2.9.8, we say that the TBT system Θ and the character system Ψ_Θ are *associated.*

For the rest of this section, let $\Theta = (A; \{E_i\}_{i=0}^D; A^*; \{E_i^*\}_{i=0})$ denote a TBT system on V, and let $\Psi_\Theta = (M; \{A_i\}_{i=0}^D; \{E_i\}_{i=0}^D)$ denote the associated character system. Our next goal is to show that if Θ is self-dual, then Ψ_Θ is self-dual.

Lemma 2.9.9. *We have $p_i(j) = v_i(\theta_j)$ $(0 \leq i, j \leq D)$.*

Proof. By Lemmas 2.4.4, 2.4.7 and Definition 2.5.9,

$$A_i = \sum_{r=0}^{D} v_i(\theta_r) E_r. \tag{2.9.1}$$

By Definition 2.7.7,

$$A_i = \sum_{r=0}^{D} p_i(r) E_r. \tag{2.9.2}$$

By Lemma 2.5.2, $\{E_r\}_{r=0}^{D}$ are linearly independent. Comparing coefficients in (2.9.1), (2.9.2), the result follows. □

Proposition 2.9.10. *Assume the TBT system* Θ *is self-dual. Then the character system* Ψ_Θ *is self-dual.*

Proof. By Lemmas 2.4.15, 2.5.13, 2.9.9,

$$p_i(j) = p_i^*(j) \qquad\qquad (0 \leq i, j \leq D).$$

The result follows. □

Chapter 3

The hypercube $\mathcal{Q}_D$

In this chapter, we investigate the generalized Terwilliger algebra associated with the hypercube (see Definition 3.2.1). The main goal of this chapter is to prove Theorem 1 from the introduction.

3.1 The generalized Terwilliger algebra associated with a distance-regular graph

In this section, we recall the generalized Terwilliger algebra and some related results from [14].

Throughout this section, the following notation is in force. Let $\Gamma = (X, R)$ denote a distance regular graph with diameter D. Fix a vertex $x \in X$. Recall the Terwilliger algebra $T = T(x)$ associated with x.

Definition 3.1.1. (See [14, Definition 4.1]) Let $\mathcal{T}$ denote the algebra with generators

$\{x_i\}_{i=0}^D, \{x_i^*\}_{i=0}^D$ and the following relations:

$$
\begin{array}{lll}
(T1) & x_0 = x_0^* = 1; & \\
(T2) & x_i x_j = \sum_{h=0}^{D} p_{ij}^h x_h & (0 \le i, j \le D); \\
(T2^*) & x_i^* x_j^* = \sum_{h=0}^{D} q_{ij}^h x_h^* & (0 \le i, j \le D); \\
(T3) & e_h^* x_i e_j^* = 0 \text{ if } p_{ij}^h = 0 & (0 \le h, i, j \le D); \\
(T3^*) & e_h x_i^* e_j = 0 \text{ if } q_{ij}^h = 0 & (0 \le h, i, j \le D).
\end{array}
$$

In the above lines, we define

$$e_i = |X|^{-1} \sum_{j=0}^{D} q_i(j) x_j, \qquad e_i^* = |X|^{-1} \sum_{j=0}^{D} q_i^*(j) x_j^* \qquad (0 \le i \le D). \tag{3.1.1}$$

The above p_{ij}^h, q_{ij}^h, $q_i(j)$, $q_i^*(j)$ are attached to Γ as explained in Section 2.2. We call $\mathcal{T}$ the *generalized Terwilliger algebra associated with* Γ.

For notational convenience, define

$$x_i = 0, \qquad x_i^* = 0, \qquad e_i = 0, \qquad e_i^* = 0 \qquad (i < 0 \text{ or } i > D). \tag{3.1.2}$$

Remark 3.1.2. If $D = 0$, then the algebra $\mathcal{T}$ is isomorphic to $\mathbb{C}$.

Remark 3.1.3. In [14, Definition 4.1], the relation (T1) is given as $x_0 = x_0^*$. In [14, Proposition 5.1], a proof is given that $x_0 = x_0^* = 1$. However the proof is not correct, and this can be seen by considering the case $D = 0$. Thus we adjusted our statement of $(T1)$ to incorporate the author's assumption that $x_0 = x_0^* = 1$.

Next we recall some results about $\mathcal{T}$.

Lemma 3.1.4. (See [14, p. 3]) *There exists an algebra homomorphism* $\natural : \mathcal{T} \to T$ *which* *sends*

$$x_i \mapsto A_i, \qquad x_i^* \mapsto A_i^*,$$
$$e_i \mapsto E_i, \qquad e_i^* \mapsto E_i^*$$

for $0 \leq i \leq D$. *Moreover,* $\natural$ *is surjective.*

The map $\natural$ in Lemma 3.1.4 is not an isomorphism in general. However we do have the following results.

Lemma 3.1.5. (See [14, Propositions 5.4 and 10.2]) *The following (i)–(iii) hold:*

(i) the elements $\{x_i\}_{i=0}^D$ *form a basis for a commutative subalgebra* $\mathcal{C}$ *of* $\mathcal{T}$*;*

(ii) the elements $\{e_i\}_{i=0}^D$ *form a basis for* $\mathcal{C}$*;*

(iii) the restriction of $\natural$ *to* $\mathcal{C}$ *induces an algebra isomorphism* $\mathcal{C} \to M$*.*

Lemma 3.1.6. (See [14, Propositions 5.7 and 10.3]) *The following (i)–(iii) hold:*

(i) the elements $\{x_i^*\}_{i=0}^D$ *form a basis for a commutative subalgebra* $\mathcal{C}^*$ *of* $\mathcal{T}$*;*

(ii) the elements $\{e_i^*\}_{i=0}^D$ *form a basis for* $\mathcal{C}^*$*;*

(iii) the restriction of $\natural$ *to* $\mathcal{C}^*$ *induces an algebra isomorphism* $\mathcal{C}^* \to M^*$*.*

Definition 3.1.7. We say Γ is *T-determined by the triple product relations* if $\natural : \mathcal{T} \to T$ is an algebra isomorphism for any choice of vertex x.

The next lemmas are consequences of Lemmas 3.1.5 and 3.1.6.

Lemma 3.1.8. (See [14, Propositions 5.3 and 5.6]) *The following (i), (ii) hold in* $\mathcal{T}$*:*

(i) $\sum_{i=0}^D e_i = 1$*;*

(ii) $\sum_{i=0}^D e_i^* = 1$.

Proof. Follows from (2.2.3) and Lemmas 3.1.5 and 3.1.6. $\square$

Lemma 3.1.9. (See [14, Propositions 5.3 and 5.6]) *For $0 \le i, j \le D$, the following (i), (ii) hold in $\mathcal{T}$:*

(i) $e_i e_j = \delta_{ij} e_i$;

(ii) $e_i^* e_j^* = \delta_{ij} e_j^*$.

Proof. Follows from (2.2.3) and Lemmas 3.1.5 and 3.1.6. □

Lemma 3.1.10. (See [14, Propositions 5.3 and 5.6]) *For $D \ge 1$, the following (i), (ii) hold in $\mathcal{T}$:*

(i) $x_1 = \sum_{i=0}^{D} \theta_i e_i$;

(ii) $x_1^* = \sum_{i=0}^{D} \theta_i^* e_i^*$.

Proof. Follows from (2.2.5) and Lemmas 3.1.5 and 3.1.6. □

Lemma 3.1.11. *The following (i), (ii) hold in $\mathcal{T}$:*

(i) $e_0 = |X|^{-1} \sum_{i=0}^{D} x_i$;

(ii) $e_0^* = |X|^{-1} \sum_{i=0}^{D} x_i^*$.

Proof. Follows from (2.2.10), (2.2.12), and (3.1.1). □

3.2 The hypercube $\mathcal{Q}_D$

In this section, we recall a family of distance-regular graphs called the hybercubes, and we review some results related to the associated Terwilliger Algebra. For more information, we refer the reader to [4, 15, 34].

Definition 3.2.1. Assume $D \ge 0$. Let $\mathcal{Q}_D$ denote the graph with vertex set X consisting of D-tuples $(a_1, a_2, \ldots, a_D)$ such that $a_i \in \{-1, 1\}$ $(1 \le i \le D)$. Two vertices are adjacent in $\mathcal{Q}_D$ whenever they differ in exactly one coordinate. The graph $\mathcal{Q}_D$ is called the *D-cube* or a *hypercube.*

The graph $\mathcal{Q}_D$ has 2^D vertices. It is well known that $\mathcal{Q}_D$ is 2-homogeneous bipartite and distance-regular (see [4, 26]).

From now through Lemma 3.2.17, we consider the distance-regular graph $\Gamma = \mathcal{Q}_D$ with $D \geq 1$. We now recall the intersection numbers of $\mathcal{Q}_D$.

Proposition 3.2.2. (See [34, p. 238]) *For $0 \leq h, i, j \leq D$, we have*

$$p^h_{ij} = \begin{cases} 0 & \textit{if } h+i+j \textit{ is odd;} \\ \binom{h}{i-\delta}\binom{D-h}{\delta} & \textit{if } h+i+j \textit{ is even,} \end{cases}$$

where $\delta = \frac{i+j-h}{2}$. In the lines above, we interpret $\binom{n}{m} = 0$ if $m < 0$ or $m > n$.

We mention some consequences of Proposition 3.2.2.

Corollary 3.2.3. *For $0 \leq i \leq D$, we have $k_i = \binom{D}{i}$.*

Proof. Follows from Proposition 3.2.2. □

Corollary 3.2.4. *For $0 \leq h, j \leq D$, we have*

$$p^h_{1j} = \begin{cases} D-h & \textit{if } h = j-1; \\ h & \textit{if } h = j+1; \\ 0 & \textit{if } h \neq j \pm 1. \end{cases}$$

Proof. Follows from Proposition 3.2.2. □

Corollary 3.2.5. *For $0 \leq i \leq D$, we have*

$$AA_i = (i+1)A_{i+1} + (D-i+1)A_{i-1},$$

where we recall (2.2.1).

Proof. Follows from (2.2.2) and Corollary 3.2.4. □

Motivated by Corollary 3.2.5, we now define some polynomials in $\mathbb{C}[z]$.

Definition 3.2.6. Let $\{v_i\}_{i=0}^{D+1}$ denote polynomials in $\mathbb{C}[z]$ such that $v_0 = 1$, $v_1 = z$, and

$$zv_i = (i+1)v_{i+1} + (D-i+1)v_{i-1} \qquad (1 \leq i \leq D).$$

We remark that the polynomial v_i has degree i and leading coefficient $\frac{1}{i!}$ $(0 \leq i \leq D+1)$.

Lemma 3.2.7. *We have* $v_i(A) = A_i$ $(0 \leq i \leq D)$ *and* $v_{D+1}(A) = 0$. *Furthermore,* $p_i(j) = v_i(\theta_j)$ $(0 \leq i, j \leq D)$.

Proof. By construction, $v_0(A) = I = A_0$ and $v_1(A) = A = A_1$. To see that $A_i = v_i(A)$ $(2 \leq i \leq D)$ and $v_{D+1}(A) = 0$, use induction and compare Corollary 3.2.5 with Definition 3.2.6.

To see that $v_i(\theta_j) = p_i(j)$, apply v_i to both sides of (2.2.5) and simplify with (2.2.3) to obtain

$$v_i(A) = \sum_{j=0}^{D} v_i(\theta_j)E_j.$$

Comparing the above with (2.2.9), it follows that $v_i(\theta_j) = p_i(j)$. □

We next consider the eigenvalues of $\mathcal{Q}_D$.

Lemma 3.2.8. (See [4, Proposition 9.2.1]) *For* $0 \leq i \leq D$ *we have* $\theta_i = D - 2i$.

The following definition is for notational convenience.

Definition 3.2.9. Define $\mu_D \in \mathbb{C}[z]$ by

$$\mu_D = \prod_{i=0}^{D} \big(z - (D-2i)\big). \qquad (3.2.1)$$

Lemma 3.2.10. *The minimal polynomial of A is equal to μ_D.*

Proof. Immediate from Lemma 3.2.8 and Definition 3.2.9. □

Lemma 3.2.11. *We have* $\mu_D = (D+1)!v_{D+1}$.

Proof. By Lemma 3.2.7, $v_{D+1}(A) = 0$. By Lemma 3.2.10, μ_D must divide v_{D+1} in $\mathbb{C}[z]$. Because both μ_D and v_{D+1} have degree $D+1$, one must be a scalar multiple of the other. Comparing the leading coefficients, the result follows. □

We have a comment.

Lemma 3.2.12. (See [4, p. 194]) *The hypercube $\mathcal{Q}_D$ is self-dual. In other words,* $p^h_{ij} = q^h_{ij}$ *($0 \le h, i, j \le D$).*

Corollary 3.2.13. *For $0 \le i \le D$, we have $k^*_i = \binom{D}{i}$.*

Proof. Follows from Corollary 3.2.3 and Lemma 3.2.12. □

Next we state some corollaries of Lemma 3.2.12. For the rest of this section, fix a vertex x of $\mathcal{Q}_D$, and let $T = T(x)$.

Corollary 3.2.14. *For $0 \le i \le D$, we have*

$$A^* A^*_i = (i+1) A^*_{i+1} + (D-i+1) A^*_{i-1},$$

where we recall (2.2.8)*.*

Proof. Similar to the proof of Corollary 3.2.5. □

Corollary 3.2.15. *We have $v_i(A^*) = A^*_i$ $(0 \le i \le D)$, and $v_{D+1}(A^*) = 0$. Furthermore, $p^*_i(j) = v_i(\theta^*_j)$ $(0 \le i, j \le D)$.*

Proof. Similar to the proof of Lemma 3.2.7. □

Corollary 3.2.16. *For $0 \le i \le D$ we have $\theta^*_i = D - 2i$.*

Proof. Follows from Lemma 3.2.12 and the discussion of self-duality at the end of Section 2.2. □

Lemma 3.2.17. *The minimal polynomial of A^* is equal to μ_D.*

Proof. Follows from Corollary 3.2.16. □

From now until the end of the section, we assume $\Gamma = \mathcal{Q}_D$ with $D \ge 0$.

In the next result, we consider the triples h, i, j such that p^h_{ij} and q^h_{ij} are nonzero.

Corollary 3.2.18. *For $0 \leq h, i, j \leq D$, the intersection number p^h_{ij} is nonzero if and only if the Krein parameter q^h_{ij} is nonzero if and only if the following (i)–(iii) hold:*

(i) none of h, i, j is greater than the sum of the other two;

(ii) $h + i + j \leq 2D$;

(iii) $h + i + j$ is even.

Proof. The case of $D = 0$ is trivial. The case of $D \geq 1$ follows upon inspection of Proposition 3.2.2 and Corollary 3.2.12. □

Definition 3.2.19. Let P_D denote the set consisting of the 3-tuples of integers (h, i, j) such that $0 \leq h, i, j \leq D$ which satisfy (i)–(iii) of Corollary 3.2.18.

Lemma 3.2.20. *For $0 \leq h, i, j \leq D$, the following (i)–(iii) are equivalent:*

(i) $p^h_{ij} \neq 0$;

(ii) $q^h_{ij} \neq 0$;

(iii) $(h, i, j) \in P_D$.

Proof. Compare Corollary 3.2.18 with Definition 3.2.19. □

We conclude this section with a brief definition and a comment about the Terwilliger algebra.

Definition 3.2.21. Assume x is a vertex of $\mathcal{Q}_D$. Let $T_D = T_D(x)$ denote the Terwilliger algebra of $\mathcal{Q}_D$ with respect to x.

Proposition 3.2.22. (See [15, Theorem 14.14]) *There exists an algebra isomorphism*

$$T_D \to \bigoplus_{r=0}^{\lfloor D/2 \rfloor} \mathrm{Mat}_{D+1-2r}(\mathbb{C}).$$

3.3 The generalized Terwilliger algebra associated with $\mathcal{Q}_D$

In Definition 3.1.1, we described the generalized Terwilliger algebra for a distance-regular graph. In this section, we consider this algebra for the graph $\mathcal{Q}_D$.

Definition 3.3.1. For $D \geq 0$, let $\mathcal{T}_D$ denote the generalized Terwilliger algebra associated with $\mathcal{Q}_D$.

By Remark 3.1.2, the algebra $\mathcal{T}_0$ is isomorphic to $\mathbb{C}$. For the rest of this section, we restrict our attention to the algebra $\mathcal{T}_D$ with $D \geq 1$.

We next observe some analogues of results from Section 3.2.

Lemma 3.3.2. *For $0 \leq i \leq D$ and with reference to* (3.1.2)*, the following (i), (ii) hold in $\mathcal{T}_D$:*

(i) $x_1 x_i = (i+1)x_{i+1} + (D-i+1)x_{i-1}$*;*

(ii) $x_1^* x_i^* = (i+1)x_{i+1}^* + (D-i+1)x_{i-1}^*$.

Proof. Follows from Lemmas 3.1.5 and 3.1.6 and Corollaries 3.2.5 and 3.2.14. □

Lemma 3.3.3. *For $0 \leq i \leq D$, the following (i), (ii) hold in $\mathcal{T}_D$:*

(i) $x_i = v_i(x_1)$*;*

(ii) $x_i^* = v_i(x_1^*)$.

Moreover, $\mathcal{T}_D$ is generated by x_1 and x_1^*.

Proof. Follows from Lemmas 3.1.5 and 3.1.6 and Corollaries 3.2.7 and 3.2.15. □

Lemma 3.3.4. *The polynomial μ_D is equal to the minimal polynomial of both x_1 and x_1^*.*

Proof. Follows from Lemmas 3.1.5, 3.1.6, 3.2.10, and 3.2.17. □

Now that we have Lemmas 3.3.3 and 3.3.4, some of the relations in Definition 3.1.1 become redundant, giving us the following, simpler presentation $\mathcal{T}_D$.

Proposition 3.3.5. *The algebra $\mathcal{T}_D$ is isomorphic to the algebra with* 1*, with generators x_1, x_1^* and the following relations:*

(1) $\mu_D(x_1) = 0$;

(2) $\mu_D(x_1^*) = 0$;

(3) $e_h^* x_i e_j^* = 0$ *if* $(h, i, j) \notin P_D \qquad (0 \leq h, i, j \leq D)$;

(4) $e_h x_i^* e_j = 0$ *if* $(h, i, j) \notin P_D \qquad (0 \leq h, i, j \leq D)$.

In the above lines, we define

$$\begin{aligned} x_0 &= 1, & x_0^* &= 1, & & \\ x_i &= v_i(x_1), & x_i^* &= v_i(x_1^*) & &(2 \leq i \leq D), \\ e_i &= 2^{-D} \sum_{j=0}^{D} q_i(j) x_j, & e_i^* &= 2^{-D} \sum_{j=0}^{D} q_i^*(j) x_j^* & &(0 \leq i \leq D). \end{aligned}$$

The above $q_i(j)$, $q_i^*(j)$ are attached to Γ as explained in Section 2.2.

Proof. Compare Definition 3.1.1 with Lemmas 3.3.3 and 3.3.4. □

We would like to provide another presentation for $\mathcal{T}_D$. To do this, we first define the following algebra.

Definition 3.3.6. Let $\mathcal{T}_D'$ denote the algebra with 1, with generators $e_0, \ldots, e_D, e_0^*, \ldots, e_D^*$ and the following relations:

(1) $\sum_{i=0}^{D} e_i = 1$;

(2) $\sum_{i=0}^{D} e_i^* = 1$;

(3) $e_h^* x_i e_j^* = 0$ if $(h, i, j) \notin P_D \qquad (0 \leq h, i, j \leq D)$;

(4) $e_h x_i^* e_j = 0$ if $(h, i, j) \notin P_D \qquad (0 \leq h, i, j \leq D)$.

In the above lines, we define

$$
\begin{aligned}
&x_0 = 1, && x^*_0 = 1, && && (3.3.1)\\
&x_1 = \sum_{i=0}^{D}(D-2i)e_i, && x^*_1 = \sum_{i=0}^{D}(D-2i)e^*_i, && && (3.3.2)\\
&x_i = v_i(x_1), && x^*_i = v_i(x^*_1) && (2 \le i \le D). && (3.3.3)
\end{aligned}
$$

We will soon show that the algebra $\mathcal{T}_D$ is isomorphic to $\mathcal{T}'_D$. We first give some lemmas about $\mathcal{T}'_D$.

Lemma 3.3.7. *For $0 \le i \le D$, the following (i), (ii) hold in $\mathcal{T}'_D$:*

(i) $e_i e_j = \delta_{ij} e_i$;

(ii) $e^*_i e^*_j = \delta_{ij} e^*_i$.

Proof. (i) First assume that $i \neq j$. The triple $(i, 0, j)$ violates Corollary 3.2.18 (i), thus $e_i x^*_0 e_j = 0$ by relation (4) of Definition 3.3.6. As $x^*_0 = 1$, we have $e_i e_j = 0$.

Next assume that $i = j$. We will show that $e_i^2 = e_i$. By relation (1) of Definition 3.3.6 and the previous paragraph,

$$e_i = e_i \sum_{\ell=0}^{D} e_\ell = e_i^2.$$

(ii) Similar to the proof of (i). □

Note that by (3.3.2) and Lemma 3.3.7, the following equations hold in $\mathcal{T}'_D$:

$$\big(x_1 - (D-2i)\big)e_i = 0, \qquad \big(x^*_1 - (D-2i)\big)e^*_i = 0 \qquad (0 \le i \le D). \qquad (3.3.4)$$

Lemma 3.3.8. *The following (i), (ii) hold in $\mathcal{T}'_D$:*

(i) $\mu_D(x_1) = 0$;

(ii) $\mu_D(x^*_1) = 0$.

Proof. (i) First note that the term $\big(z-(D-2i)\big)$ is a factor of the right-hand side of (3.2.1) $(0 \le i \le D)$. Thus by Definition 3.2.9 and (3.3.4),

$$\mu_D(x_1)e_i = 0 \qquad\qquad (0 \le i \le D).$$

Hence by relation (1) of Definition 3.3.6,

$$\mu_D(x_1) = \mu_D(x_1)\sum_{i=0}^{D} e_i = 0.$$

(ii) Similar to the proof of (i). □

Lemma 3.3.9. *For $0 \le i \le D$, the following (i), (ii) hold in $\mathcal{T}'_D$:*

(i) $e_i = 2^{-D}\sum_{j=0}^{D} q_i(j)x_j$;

(ii) $e^*_i = 2^{-D}\sum_{j=0}^{D} q^*_i(j)x^*_j$.

Proof. (i) By (3.3.3) and Lemma 3.3.7,

$$x_i = v_i(x_1) = \sum_{j=0}^{D} v_i(\theta_j)e_j.$$

Hence by Lemma 3.2.7,

$$x_i = \sum_{j=0}^{D} p_i(j)e_j. \tag{3.3.5}$$

The result follows from (3.3.5) and the fact that the matrices P and $2^{-D}Q$ from below (2.2.11) are inverses.

(ii) Similar to the proof of (i). □

We now show that the algebra $\mathcal{T}_D$ is isomorphic to $\mathcal{T}'_D$.

Proposition 3.3.10. *There exists a unique algebra isomorphism $\mathcal{T}_D \to \mathcal{T}'_D$ that sends*

$$x_1 \mapsto x_1, \qquad\qquad x^*_1 \mapsto x^*_1.$$

Moreover, this map sends

$$x_i \mapsto x_i, \qquad x_i^* \mapsto x_i^*,$$
$$e_i \mapsto e_i, \qquad e_i^* \mapsto e_i^*$$

for $0 \le i \le D$.

Proof. We will first show that there exists an algebra homomorphism $\sigma : \mathcal{T}_D \to \mathcal{T}'_D$ which sends $x_1 \mapsto x_1$ and $x_1^* \mapsto x_1^*$. We will then show that there exists an algebra homomorphism $\tau : \mathcal{T}'_D \to \mathcal{T}_D$ which sends $e_i \mapsto e_i$ and $e_i^* \mapsto e_i^*$ $(0 \le i \le D)$. We will next show that σ and τ are inverses, and hence algebra isomorphisms. We will last show that $\sigma(x_i) = x_i$, $\sigma(x_i^*) = x_i^*$, $\sigma(e_i) = e_i$, and $\sigma(e_i^*) = e_i^*$ $(0 \le i \le D)$.

We begin by showing that σ exists. For $0 \le i \le D$, let $f_i = 2^{-D} \sum_{j=0}^{D} q_i(j) x_j \in \mathcal{T}'_D$ and $f_i^* = 2^{-D} \sum_{j=0}^{D} q_i^*(j) x_j^* \in \mathcal{T}'_D$. To show that σ exists, it is sufficient to show that in $\mathcal{T}'_D$,

$$\mu_D(x_1) = 0, \qquad \mu_D(x_1^*) = 0, \tag{3.3.6}$$
$$f_h^* x_i f_j^* = 0, \qquad f_h x_i^* f_j = 0 \tag{3.3.7}$$

for $0 \le h, i, j \le D$ such that $(h, i, j) \notin P_D$.

Lemma 3.3.8 implies (3.3.6). Lemma 3.3.9 implies that $f_i = e_i$ and $f_i^* = e_i^*$ $(0 \le i \le D)$. Hence (3.3.7) follows by relations (3) and (4) of Definition 3.3.6.

We next show τ exists. Let $y_1 = \sum_{i=0}^{D} \theta_i e_i \in \mathcal{T}_D$ and $y_1^* = \sum_{i=0}^{D} \theta_i^* e_i^* \in \mathcal{T}_D$. To show that τ exists, it is sufficient to show that in $\mathcal{T}_D$,

$$\sum_{\ell=0}^{D} e_\ell = 1, \qquad \sum_{\ell=0}^{D} e_\ell^* = 1, \tag{3.3.8}$$
$$e_h^* v_i(y_1) e_j^* = 0, \qquad e_h v_i(y_1^*) e_j = 0 \tag{3.3.9}$$

for $0 \le h, i, j \le D$ such that $(h, i, j) \notin P_D$.

Lemma 3.1.8 implies (3.3.8). Lemma 3.1.10 implies that $x_1 = y_1$ and $x_1^* = y_1^*$. Hence $v_i(y_1) = x_i$ and $v_i(y_1^*) = x_i^*$ $(0 \leq i \leq D)$ by Lemma 3.3.3. Thus (3.3.9) follows by relations (3) and (4) of Proposition 3.3.5.

We now show that σ and τ are inverses. By Lemma 3.1.10, $\tau(x_1) = x_1$ and $\tau(x_1^*) = x_1^*$. Thus $\tau \circ \sigma : \mathcal{T}_D \to \mathcal{T}_D$ is the identity map. By Lemma 3.3.9, $\sigma(e_i) = e_i$ and $\sigma(e_i^*) = e_i^*$ $(0 \leq i \leq D)$. Thus $\sigma \circ \tau : \mathcal{T}_D' \to \mathcal{T}_D'$ is the identity map. Therefore σ and τ are inverses, and hence algebra isomorphisms.

By construction, $\sigma(x_1) = x_1$ and $\sigma(x_1^*) = x_1^*$. Thus $\sigma(x_i) = x_i$ and $\sigma(x_i^*) = x_i^*$ $(0 \leq i \leq D)$. As noted previously, $\sigma(e_i) = e_i$ and $\sigma(e_i^*) = e_i^*$ $(0 \leq i \leq D)$. This completes the proof. $\square$

For the rest of this chapter, we identify the algebras $\mathcal{T}_D$ and $\mathcal{T}_D'$ via the isomorphism in Proposition 3.3.10.

We next define a free algebra.

Definition 3.3.11. Let $\mathfrak{T}_D$ denote the free algebra with generators $\mathfrak{e}_0, \ldots, \mathfrak{e}_D, \mathfrak{e}_0^*, \ldots, \mathfrak{e}_d^*$. Define $\mathfrak{x}_0 = 1$, $\mathfrak{x}_1 = \sum_{i=0}^{D} \theta_i \mathfrak{e}_i$, and $\mathfrak{x}_i = v_i(\mathfrak{x}_1)$ $(2 \leq i \leq D)$. Similarly, define $\mathfrak{x}_0^* = 1$, $\mathfrak{x}_1^* = \sum_{i=0}^{D} \theta_i^* \mathfrak{e}_i^*$, and $\mathfrak{x}_i^* = v_i(x_1^*)$ $(2 \leq i \leq D)$.

For notational convenience, define

$$\mathfrak{x}_i = 0, \qquad \mathfrak{x}_i^* = 0, \qquad \mathfrak{e}_i = 0, \qquad \mathfrak{e}_i^* = 0 \qquad (i < 0 \text{ or } i > D). \tag{3.3.10}$$

Definition 3.3.12. Let $\mathfrak{S}_D$ denote the two-sided ideal of $\mathfrak{T}_D$ generated by the following (1)–(4).

1. $\sum_{i=0}^{D} \mathfrak{e}_i - 1$,
2. $\sum_{i=0}^{D} \mathfrak{e}_i^* - 1$,
3. $\mathfrak{e}_h^* \mathfrak{x}_i \mathfrak{e}_j^*$ such that $(h, i, j) \notin P_D \qquad (0 \leq h, i, j \leq D)$,
4. $\mathfrak{e}_h \mathfrak{x}_i^* \mathfrak{e}_j$ such that $(h, i, j) \notin P_D \qquad (0 \leq h, i, j \leq D)$.

Remark 3.3.13. Because $\mathfrak{T}_D$ is free, there exists an algebra homomorphism $\psi_D : \mathfrak{T}_D \to \mathcal{T}_D$ that sends

$$\mathfrak{e}_i \mapsto e_i, \qquad \mathfrak{e}_i^* \mapsto e_i^* \qquad (0 \leq i \leq D).$$

Comparing Definitions 3.3.6 and 3.3.12, it follows that the map ψ_D is surjective with kernel $\mathfrak{S}_D$, and that

$$\mathfrak{x}_i \mapsto x_i, \qquad \mathfrak{x}_i^* \mapsto x_i^* \qquad (0 \leq i \leq D).$$

To end this section, we define an algebra homomorphism that will be useful later in this chapter.

Definition 3.3.14. Consider the quotient algebra $\mathfrak{T}_D/\mathfrak{S}_D$. With reference to Remark 3.3.13, the algebra homomorphism ψ_D induces an algebra isomorphism $\mathfrak{T}_D/\mathfrak{S}_D \to \mathcal{T}_D$. We denote the inverse of this map by p_D.

3.4 The primary central idempotent of $\mathcal{T}_D$

We continue our discussion of the algebra $\mathcal{T}_D$ from Definition 3.3.1. In [14], Egge defines a certain element $u_0 \in \mathcal{T}_D$ called the primary central idempotent. Later in the chapter, we will use u_0 to compute the dimension of $\mathcal{T}_D$. In this section, we recall the definition of u_0 and develop some basic facts about it.

Lemma 3.4.1. (See [14, Propositions 11.1 and 11.4]) *For $D \geq 0$, following holds in $\mathcal{T}_D$:*

$$2^D \sum_{i=0}^{D} k_i^{-1} e_i^* e_0 e_i^* = 2^D \sum_{i=0}^{D} (k_i^*)^{-1} e_i e_0^* e_i. \tag{3.4.1}$$

This element is central and idempotent.

Definition 3.4.2. (See [14, Proposition 11.1]) Referring to Lemma 3.4.1, we define u_0 to be the common value expressed in (3.4.1). We call u_0 the *primary central idempotent of* $\mathcal{T}_D$.

Proposition 3.4.3. (See [14, Proposition 11.5 and Theorem 12.5]) *For* $D \geq 0$, *the following (i)–(iii) hold:*

(i) the sum $\mathcal{T}_D = \mathcal{T}_D u_0 + \mathcal{T}_D(1 - u_0)$ *is direct;*

(ii) $\mathcal{T}_D u_0$ *and* $\mathcal{T}_D(1 - u_0)$ *are both two-sided ideals of* $\mathcal{T}_D$*;*

(iii) the algebra $\mathcal{T}_D u_0$ *is isomorphic to* $\mathrm{Mat}_{D+1}(\mathbb{C})$.

Corollary 3.4.4. *For* $D \geq 0$, *the algebra* $\mathcal{T}_D$ *is isomorphic to the direct sum* $\mathrm{Mat}_{D+1}(\mathbb{C}) \oplus \mathcal{T}_D(1 - u_0)$.

Proof. Follows from Proposition 3.4.3. □

Corollary 3.4.5. *There exists an algebra isomorphism* $\mathcal{T}_D/\mathcal{T}_D u_0 \to \mathcal{T}_D(1-u_0)$ *that sends*

$$e_i + \mathcal{T}_D u_0 \mapsto e_i(1 - u_0), \qquad e_i^* + \mathcal{T}_D u_0 \mapsto e_i^*(1 - u_0) \qquad (0 \leq i \leq D).$$

Proof. Follows from Proposition 3.4.3 (i). □

We have some comments about u_0.

Lemma 3.4.6. *For* $D \geq 0$, *the following (i)–(iv) hold in* $\mathcal{T}_D$*:*

(i) $u_0 e_0 = e_0$*;*

(ii) $u_0 e_D = e_D$*;*

(iii) $u_0 e_0^* = e_0^*$*;*

(iv) $u_0 e_D^* = e_D^*$.

Proof. (i) By Lemma 3.1.9 (i), Lemma 3.1.11 (ii), Corollary 3.2.13, and Definition 3.4.2,

$$u_0e_0 = 2^D\bigg(\sum_{r=0}^{D}(k_r^*)^{-1}e_re_0^*e_r\bigg)e_0 = 2^De_0e_0^*e_0 = e_0\bigg(\sum_{i=0}^{D}x_i^*\bigg)e_0.$$

For $1 \leq i \leq D$, the triple $(0, i, 0)$ does not satisfy Corollary 3.2.18 (i), thus $e_0x_i^*e_0 = 0$ by relation (4) of Definition 3.3.6. Hence by relation (T1) of Definition 3.1.1,

$$e_0\bigg(\sum_{i=0}^{D}x_i^*\bigg)e_0 = e_0x_0^*e_0 = e_0^2 = e_0.$$

Therefore $u_0e_0 = e_0$.

(ii) By Lemma 3.1.9 (i), Lemma 3.1.11 (ii), Corollary 3.2.13, and Definition 3.4.2,

$$u_0e_D = 2^D\bigg(\sum_{r=0}^{D}(k_r^*)^{-1}e_re_0^*e_r\bigg)e_d = 2^De_De_0^*e_D = e_D\bigg(\sum_{i=0}^{D}x_i^*\bigg)e_D.$$

For $1 \leq i \leq D$, the triple (D, i, D) does not satisfy Corollary 3.2.18 (ii), thus $e_Dx_i^*e_D = 0$ by relation (4) of Definition 3.3.6. Hence by relation (T1) of Definition 3.1.1,

$$e_D\bigg(\sum_{i=0}^{D}x_i^*\bigg)e_D = e_Dx_0^*e_D = e_D^2 = e_D.$$

Therefore $u_0e_D = e_D$.

(iii) Similar to the proof of (i).

(iv) Similar to the proof of (ii). □

We finish this section with a comment about the case $D = 1$.

Proposition 3.4.7. *For $D = 1$, the element $u_0 = 1$. Moreover, the algebra $\mathcal{T}_1$ is isomorphic to* $\mathrm{Mat}_2(\mathbb{C})$.

Proof. Because $D = 1$, Lemmas 3.1.8 and 3.4.6 imply that

$$u_0 = u_0(e_0 + e_1) = e_0 + e_1 = 1.$$

Therefore the algebra $\mathcal{T}_1$ is isomorphic to $\mathrm{Mat}_2(\mathbb{C})$ by Proposition 3.4.3 part (iii). $\square$

3.5 $\mathcal{Q}_D$ is T-determined by the triple product relations

Recall Definition 3.1.7. In this section, we prove Theorem 1 from the introduction.

Recall the free algebra $\mathfrak{T}_D$ from Definition 3.3.11.

Definition 3.5.1. Assume $D \geq 2$. Let $\varphi_D : \mathfrak{T}_D \to \mathfrak{T}_{D-2}$ be the algebra homomorphism that sends

$$\begin{aligned} &\mathfrak{e}_0 \mapsto 0, && \mathfrak{e}_0^* \mapsto 0, && \\ &\mathfrak{e}_i \mapsto \mathfrak{e}_{i-1}, && \mathfrak{e}_i^* \mapsto \mathfrak{e}_{i-1}^* && (1 \leq i \leq D-1), \\ &\mathfrak{e}_D \mapsto 0, && \mathfrak{e}_D^* \mapsto 0. && \end{aligned}$$

Referring to Definition 3.5.1 and using (3.3.10), we see that φ_D sends

$$\mathfrak{e}_i \mapsto \mathfrak{e}_{i-1}, \qquad \mathfrak{e}_i^* \mapsto \mathfrak{e}_{i-1}^* \qquad (0 \leq i \leq D). \tag{3.5.1}$$

Definition 3.5.2. Assume $D \geq 2$. Let $\mathfrak{K}_D$ denote the two-sided ideal of $\mathfrak{T}_D$ generated by $\mathfrak{e}_0, \mathfrak{e}_D, \mathfrak{e}_0^*, \mathfrak{e}_d^*$.

Lemma 3.5.3. *Assume $D \geq 2$. The map φ_D is surjective with kernel $\mathfrak{K}_D$.*

Proof. Routine consequence of Definitions 3.5.1 and 3.5.2. $\square$

Lemma 3.5.4. *Assume $D \geq 2$. For $0 \leq i \leq D$, the following (i), (ii) hold:*

$$\text{(i)}\ \varphi_D(\mathfrak{x}_i) = \begin{cases} \mathfrak{x}_i & \text{if } i = 0 \text{ or } i = 1; \\ \mathfrak{x}_i - \mathfrak{x}_{i-2} & \text{if } 2 \leq i \leq D-2; \\ \frac{\mu_{D-2}(\mathfrak{x}_1)}{(D-1)!} - \mathfrak{x}_{i-2} & \text{if } i = D-1; \\ \frac{\mathfrak{x}_1 \mu_{D-2}(\mathfrak{x}_1)}{D!} - \mathfrak{x}_{i-2} & \text{if } i = D; \end{cases}$$

(ii) $$\varphi_D(\mathfrak{x}_i^*) = \begin{cases} \mathfrak{x}_i^* & \textit{if } i = 0 \textit{ or } i = 1; \\ \mathfrak{x}_i^* - \mathfrak{x}_{i-2}^* & \textit{if } 2 \le i \le D-2; \\ \frac{\mu_{D-2}(\mathfrak{x}_1^*)}{(D-1)!} - \mathfrak{x}_{i-2}^* & \textit{if } i = D-1; \\ \frac{\mathfrak{x}_1^* \mu_{D-2}(\mathfrak{x}_1^*)}{D!} - \mathfrak{x}_{i-2}^* & \textit{if } i = D. \end{cases}$$

Proof. (i) We begin with a comment. Note that by Definitions 3.2.6 and 3.3.11, the following holds in $\mathfrak{T}_D$:

$$j\mathfrak{x}_j = \mathfrak{x}_1\mathfrak{x}_{j-1} - (D-j+2)\mathfrak{x}_{j-2} \qquad (2 \le j \le D). \tag{3.5.2}$$

We now consider the cases for i.

First, assume $i = 0$. The result holds, because $\mathfrak{x}_0 = 1$. Next, assume $i = 1$. Then by Lemma 3.2.8, Definition 3.3.11, and Definition 3.5.1,

$$\begin{aligned} \varphi_D(\mathfrak{x}_1) &= \varphi_D\Big(\sum_{j=0}^{D}(D-2j)\mathfrak{e}_j\Big) \\ &= \sum_{j=1}^{D-1}(D-2j)\mathfrak{e}_{j-1} \\ &= \sum_{j=0}^{D-2}(D-2-2j)\mathfrak{e}_j \\ &= \mathfrak{x}_1. \end{aligned}$$

By (3.3.10), it is correct to say that φ_D sends $\mathfrak{x}_i \mapsto \mathfrak{x}_i - \mathfrak{x}_{i-2}$ for $i = 0$ and $i = 1$. This allows us to use induction for $2 \le i \le D-2$. We proceed by induction on i.

Assume $2 \le i \le D-2$. Setting $j = i$ in (3.5.2), applying φ_D to both sides, using induction, and dividing by i, we obtain

$$\varphi_D(\mathfrak{x}_i) = \frac{\mathfrak{x}_1(\mathfrak{x}_{i-1} - \mathfrak{x}_{i-3}) - (D-i+2)(\mathfrak{x}_{i-2} - \mathfrak{x}_{i-4})}{i}. \tag{3.5.3}$$

Using Definitions 3.2.6 and 3.3.11, we find that in $\mathfrak{T}_{D-2}$,

$$\mathfrak{x}_1\mathfrak{x}_{i-1} = i\mathfrak{x}_i + (D-i)\mathfrak{x}_{i-2}, \qquad \mathfrak{x}_1\mathfrak{x}_{i-3} = (i-2)\mathfrak{x}_{i-2} + (D-i+2)\mathfrak{x}_{i-4}. \tag{3.5.4}$$

In equation (3.5.3), we distribute terms in the numerator, then eliminate $\mathfrak{x}_1\mathfrak{x}_{i-1}$ and $\mathfrak{x}_1\mathfrak{x}_{i-3}$ via (3.5.4). This yields $\varphi_D(\mathfrak{x}_i) = \mathfrak{x}_i - \mathfrak{x}_{i-2}$.

Next, assume $i = D-1$. Setting $j = D-1$ in (3.5.2), applying φ_D to the result, using induction, and dividing by $D-1$ yields

$$\varphi_D(\mathfrak{x}_{D-1}) = \frac{\mathfrak{x}_1(\mathfrak{x}_{D-2} - \mathfrak{x}_{D-4}) - 3(\mathfrak{x}_{D-3} - \mathfrak{x}_{D-5})}{D-1}. \tag{3.5.5}$$

Using Definitions 3.2.6 and 3.3.11, we find that in $\mathfrak{T}_{D-2}$,

$$\mathfrak{x}_1\mathfrak{x}_{D-4} = (D-3)\mathfrak{x}_{D-3} + 3\mathfrak{x}_{D-5}. \tag{3.5.6}$$

In (3.5.5), we distribute terms in the numerator and eliminate $\mathfrak{x}_1\mathfrak{x}_{D-4}$ via (3.5.6). This yields

$$\varphi_D(\mathfrak{x}_{D-1}) = \frac{\mathfrak{x}_1\mathfrak{x}_{D-2} - \mathfrak{x}_{D-3}}{D-1} - \mathfrak{x}_{D-3}. \tag{3.5.7}$$

By Lemma 3.2.11,

$$\mathfrak{x}_1\mathfrak{x}_{D-2} - \mathfrak{x}_{D-3} = \frac{\mu_{D-2}(\mathfrak{x}_1)}{(D-2)!}. \tag{3.5.8}$$

We use (3.5.8) to eliminate the numerator in the right-hand side of (3.5.7). This yields

$$\varphi_D(\mathfrak{x}_{D-1}) = \frac{\mu_{D-2}(\mathfrak{x}_1)}{(D-1)!} - \mathfrak{x}_{D-3}.$$

For the rest of this proof, assume $i = D$. Setting $j = D$ in (3.5.2), applying φ_D to the result, using induction, and dividing by D yields

$$\varphi_D(\mathfrak{x}_D) = \frac{\mathfrak{x}_1\left(\frac{\mu_{D-2}(\mathfrak{x}_1)}{(D-1)!} - \mathfrak{x}_{D-3}\right) - 2(\mathfrak{x}_{D-2} - \mathfrak{x}_{D-4})}{D}. \tag{3.5.9}$$

Using Definitions 3.2.6 and 3.3.11, we find that in $\mathfrak{T}_{D-2}$,

$$\mathfrak{x}_1\mathfrak{x}_{D-3} = (D-2)\mathfrak{x}_{D-2} + 2\mathfrak{x}_{D-4}. \tag{3.5.10}$$

In (3.5.9), we distribute terms in the numerator and eliminate $\mathfrak{x}_1\mathfrak{x}_{D-3}$ via (3.5.10). This yields

$$\varphi_D(\mathfrak{x}_D) = \frac{\frac{\mathfrak{x}_1\mu_{D-2}(\mathfrak{x}_1)}{(D-1)!} - D\mathfrak{x}_{D-2}}{D} = \frac{\mathfrak{x}_1\mu_{D-2}(\mathfrak{x}_1)}{D!} - \mathfrak{x}_{D-2}.$$

(ii) Similar to the proof of (i). □

Recall the ideal $\mathfrak{S}_D \subseteq \mathfrak{T}_D$ from Definition 3.3.12. Our next general goal is to show that $\varphi_D(\mathfrak{S}_D) = \mathfrak{S}_{D-2}$. To do that, we will show that $\varphi_D(\mathfrak{S}_D) \subseteq \mathfrak{S}_{D-2}$ and $\mathfrak{S}_{D-2} \subseteq \varphi_D(\mathfrak{S}_D)$.

Lemma 3.5.5. *For $D \geq 2$, the following $(i), (ii)$ hold:*

(i) $\varphi_D\left(\sum_{i=0}^{D} \mathfrak{e}_i - 1\right) = \sum_{i=0}^{D-2} \mathfrak{e}_i - 1$;

(ii) $\varphi_D\left(\sum_{i=0}^{D} \mathfrak{e}^*_i - 1\right) = \sum_{i=0}^{D-2} \mathfrak{e}^*_i - 1$.

Proof. (i) By Definition 3.5.1,

$$\varphi_D\left(\sum_{i=0}^{D} \mathfrak{e}_i - 1\right) = \sum_{i=1}^{D-1} \mathfrak{e}_{i-1} - 1 = \sum_{i=0}^{D-2} \mathfrak{e}_i - 1.$$

(ii) Similar to the proof of (i). □

Lemma 3.5.6. *Assume $D \geq 2$. For $0 \leq h, i, j \leq D$ such that $(h, i, j) \notin P_D$, the following (i), (ii) hold:*

(i) $(h-1, i, j-1) \notin P_{D-2}$;

(ii) $(h-1, i-2, j-1) \notin P_{D-2}$.

Proof. We consider the three cases in Corollary 3.2.18. For convenience, we consider them in the order (ii), (iii), (i).

First, assume $h+i+j>2D$. Then

$$(h-1)+i+(j-1)>2(D-2),$$
$$(h-1)+(i-2)+(j-1)>2(D-2).$$

Thus (i) and (ii) hold.

Next, assume $h+i+j$ is odd. Then $(h-1)+i+(j-1)$ is odd and $(h-1)+(i-2)+(j-1)$ is odd. Thus (i) and (ii) hold.

For the rest of this proof, assume one of h,i,j is greater than the sum of the other two. This leaves two subcases:

$$i>h+j, \qquad\qquad i<|h-j|.$$

First, assume $i>h+j$. Then

$$i>(h-1)+(j-1),$$
$$i-2>(h-1)+(j-1).$$

Hence $h-1,i,j-1$ and $h-1,i-2,j-1$ fail the triangle inequality. Thus (i) and (ii) hold.

Lastly, assume $i<|h-j|$. Then

$$i<|(h-1)-(j-1)|,$$
$$i-2<|(h-1)-(j-1)|.$$

Hence $h-1,i,j-1$ and $h-1,i-2,j-1$ fail the triangle inequality. Thus (i) and (ii) hold.

□

Lemma 3.5.7. *Assume $D\geq 2$. For $0\leq h,i,j\leq D$ such that $(h,i,j)\notin P_D$, the following*

$(i), (ii)$ *hold:*

(i) $\varphi_D(\mathfrak{e}^*_h \mathfrak{x}_i \mathfrak{e}^*_j) \in \mathfrak{S}_{D-2}$*;*

(ii) $\varphi_D(\mathfrak{e}_h \mathfrak{x}^*_i \mathfrak{e}_j) \in \mathfrak{S}_{D-2}$*.*

Proof. (i) First note that if $h = 0$, $h = D$, $j = 0$, or $j = D$, then $\varphi_D(\mathfrak{e}^*_h \mathfrak{x}_i \mathfrak{e}^*_j) = 0$ by Definition 3.5.1. Thus for the remainder of this proof, we assume $1 \leq h, j \leq d - 1$.

We consider the cases from Lemma 3.5.4 (i).

First, assume $i = 0$ or $i = 1$. By Definition 3.5.1 and Lemma 3.5.4,

$$\varphi_D(\mathfrak{e}^*_h \mathfrak{x}_i \mathfrak{e}^*_j) = \mathfrak{e}^*_{h-1} \mathfrak{x}_i \mathfrak{e}^*_{j-1}.$$

By Lemma 3.5.6, $(h - 1, i, j - 1) \notin P_{D-2}$. Hence by Definition 3.3.12,

$$\mathfrak{e}^*_{h-1} \mathfrak{x}_i \mathfrak{e}^*_{j-1} \in \mathfrak{S}_{D-2}.$$

Next, assume $2 \leq i \leq D - 2$. Then

$$\varphi_D(\mathfrak{e}^*_h \mathfrak{x}_i \mathfrak{e}^*_j) = \mathfrak{e}^*_{h-1} \mathfrak{x}_i \mathfrak{e}^*_{j-1} - \mathfrak{e}^*_{h-1} \mathfrak{x}_{i-2} \mathfrak{e}^*_{j-1}.$$

By Lemma 3.5.6, $(h - 1, i, j - 1) \notin P_{D-2}$ and $(h - 1, i - 2, j - 1) \notin P_{D-2}$. Hence by Definition 3.3.12,

$$\mathfrak{e}^*_{h-1} \mathfrak{x}_i \mathfrak{e}^*_{j-1} - \mathfrak{e}^*_{h-1} \mathfrak{x}_{i-2} \mathfrak{e}^*_{j-1} \in \mathfrak{S}_{D-2}.$$

Next, assume $i = D - 1$. Then

$$\varphi_D(\mathfrak{e}^*_h \mathfrak{x}_i \mathfrak{e}^*_j) = \frac{\mathfrak{e}^*_{h-1} \mu_{D-2}(\mathfrak{x}_1) \mathfrak{e}^*_{j-1}}{(D-1)!} - \mathfrak{e}^*_{h-1} \mathfrak{x}_{i-2} \mathfrak{e}^*_{j-1}.$$

By Lemma 3.3.8, $\mu_{D-2}(\mathfrak{x}_1) \in \mathfrak{S}_{D-2}$. By Definition 3.3.12 and Lemma 3.5.6, $\mathfrak{e}^*_{h-1} \mathfrak{x}_{i-2} \mathfrak{e}^*_{j-1} \in \mathfrak{S}_{D-2}$. Thus

$$\frac{\mathfrak{e}^*_{h-1} \mu_{D-2}(\mathfrak{x}_1) \mathfrak{e}^*_{j-1}}{(D-1)!} - \mathfrak{e}^*_{h-1} \mathfrak{x}_{i-2} \mathfrak{e}^*_{j-1} \in \mathfrak{S}_{D-2}.$$

For the remainder of this proof, assume $i = D$. Then

$$\varphi_D(\mathfrak{e}_h^* \mathfrak{x}_i \mathfrak{e}_j^*) = \frac{\mathfrak{e}_{h-1}^* \mathfrak{x}_1 \mu_{D-2}(\mathfrak{x}_1) \mathfrak{e}_{j-1}^*}{d!} - \mathfrak{e}_{h-1}^* \mathfrak{x}_{i-2} \mathfrak{e}_{j-1}^*.$$

By Lemma 3.3.8, $\mu_{D-2}(\mathfrak{x}_1) \in \mathfrak{S}_{D-2}$. By Definition 3.3.12 and Lemma 3.5.6, $\mathfrak{e}_{h-1}^* \mathfrak{x}_{i-2} \mathfrak{e}_{j-1}^* \in \mathfrak{S}_{D-2}$. Thus

$$\frac{\mathfrak{e}_{h-1}^* \mathfrak{x}_1 \mu_{D-2}(\mathfrak{x}_1) \mathfrak{e}_{j-1}^*}{d!} - \mathfrak{e}_{h-1}^* \mathfrak{x}_{i-2} \mathfrak{e}_{j-1}^* \in \mathfrak{S}_{D-2}.$$

(ii) Similar to the proof of (i). □

We have now shown that $\varphi_D(\mathfrak{S}_D) \subseteq \mathfrak{S}_{D-2}$. Next we show that $\mathfrak{S}_{D-2} \subseteq \varphi_D(\mathfrak{S}_D)$. To that end, we include the following technical results.

Lemma 3.5.8. *Assume $D \geq 2$. The following (i), (ii) hold:*

(i) $\mu_{D-2}(\mathfrak{x}_1)(1 - \mathfrak{e}_0 - \mathfrak{e}_D) \in \mathfrak{S}_D$*;*

(ii) $\mu_{D-2}(\mathfrak{x}_1^*)(1 - \mathfrak{e}_0^* - \mathfrak{e}_d^*) \in \mathfrak{S}_D$.

Proof. (i) Observe that

$$\mu_{D-2}(\mathfrak{x}_1)(1 - \mathfrak{e}_0 - \mathfrak{e}_D) = \mu_{D-2}(\mathfrak{x}_1)\left(1 - \sum_{i=0}^{D} \mathfrak{e}_i\right) + \mu_{D-2}(\mathfrak{x}_1) \sum_{i=1}^{D-1} \mathfrak{e}_i. \tag{3.5.11}$$

By Definition 3.3.12, $1 - \sum_{i=0}^{D} \mathfrak{e}_i \in \mathfrak{S}_D$. Hence

$$\mu_{D-2}(\mathfrak{x}_1)\left(1 - \sum_{i=0}^{D} \mathfrak{e}_i\right) \in \mathfrak{S}_D. \tag{3.5.12}$$

By Lemma 3.2.8 and Definition 3.2.9,

$$\mu_{D-2}(\mathfrak{x}_1) = \prod_{i=1}^{D-1} (\mathfrak{x}_1 - \theta_i).$$

By (3.3.4) and Remark 3.3.13, $(\mathfrak{x}_1 - \theta_i)\mathfrak{e}_i \in \mathfrak{S}_D$ $(1 \le i \le D-1)$. Hence

$$\mu_{D-2}(\mathfrak{x}_1) \sum_{i=1}^{D-1} \mathfrak{e}_i \in \mathfrak{S}_D. \tag{3.5.13}$$

It follows from (3.5.11), (3.5.12), and (3.5.13) that

$$\mu_{D-2}(\mathfrak{x}_1)(1 - \mathfrak{e}_0 - \mathfrak{e}_D) \in \mathfrak{S}_D.$$

(ii) Similar to the proof of (i). □

Corollary 3.5.9. *Assume $D \ge 2$. The following (i), (ii) hold:*

(i) $\mu_{D-2}(\mathfrak{x}_1) \in \varphi_D(\mathfrak{S}_D)$;

(ii) $\mu_{D-2}(\mathfrak{x}_1^*) \in \varphi_D(\mathfrak{S}_D)$.

Proof. (i) By Lemma 3.5.8 (i), $\mu_{D-2}(\mathfrak{x}_1)(1 - \mathfrak{e}_0 - \mathfrak{e}_D) \in \mathfrak{S}_D$. By Definition 3.5.1 and Lemma 3.5.4,

$$\varphi_D\big(\mu_{D-2}(\mathfrak{x}_1)(1 - \mathfrak{e}_0 - \mathfrak{e}_D)\big) = \mu_{D-2}(\mathfrak{x}_1).$$

(ii) Similar to the proof of (i). □

Lemma 3.5.10. *Assume $D \ge 2$. For $0 \le h, i, j \le D-2$ such that $(h, i, j) \notin P_{D-2}$, either*

$$(h+1, i-2r, j+1) \notin P_D \qquad (0 \le r \le \lfloor i/2 \rfloor), \tag{3.5.14}$$

or

$$(h+1, i+2r, j+1) \notin P_D \qquad (1 \le r \le \lfloor (D-i)/2 \rfloor). \tag{3.5.15}$$

Proof. We consider the three cases in Corollary 3.2.18. For convenience, we consider these cases in order (ii), (iii), (i).

First, assume that $h + i + j > 2(D-2)$. Then

$$(h+1) + (i+2r) + (j+1) > 2D \qquad (1 \le r \le \lfloor (D-i)/2 \rfloor).$$

Thus (3.5.15) holds.

Next, assume that $h+i+j$ is odd. Then $(h+1)+(i-2r)+(j+1)$ is odd $(0 \le r \le \lfloor i/2 \rfloor)$. Thus (3.5.14) holds.

For the rest of this proof, assume that one of h, i, j is greater than the sum of the other two. This leaves two subcases:

$$i > h+j, \qquad\qquad i < |h-j|.$$

First, assume $i > h+j$. Then

$$i + 2r > (h+1) + (j+1) \qquad\qquad (1 \le r \le \lfloor (D-i)/2 \rfloor).$$

Hence $h+1, i+2r, j+1$ fail the triangle inequality $(1 \le r \le \lfloor (D-i)/2 \rfloor)$. Thus (3.5.15) holds.

Lastly, assume $i < |h-j|$. Then

$$i - 2r < |(h+1) - (j+1)| \qquad\qquad (0 \le r \le \lfloor i/2 \rfloor).$$

Hence $h+1, i-2r, j+1$ fail the triangle inequality $(0 \le r \le \lfloor i/2 \rfloor)$. Thus (3.5.14) holds. □

Lemma 3.5.11. *Assume $D \ge 2$. Then for $0 \le h, i, j \le D-2$ such that $(h,i,j) \notin P_{D-2}$, the following $(i), (ii)$ hold.*

(i) $\mathfrak{e}^*_h \mathfrak{x}_i \mathfrak{e}^*_j \in \varphi_D(\mathfrak{S}_D)$.

(ii) $\mathfrak{e}_h \mathfrak{x}^*_i \mathfrak{e}_j \in \varphi_D(\mathfrak{S}_D)$.

Proof. (i) We consider the two cases in Lemma 3.5.10.

First, assume (3.5.14) holds. Then by Definition 3.3.12,

$$\mathfrak{e}^*_{h+1} \mathfrak{x}_{i-2r} \mathfrak{e}^*_{j+1} \in \mathfrak{S}_D, \qquad\qquad (0 \le r \le \lfloor i/2 \rfloor).$$

By Definition 3.5.1 and Lemma 3.5.4,

$$\varphi_D\Bigg(\sum_{r=0}^{\lfloor i/2\rfloor} \mathfrak{e}^*_{h+1}\mathfrak{x}_{i-2r}\mathfrak{e}^*_{j+1}\Bigg) = \sum_{r=0}^{\lfloor i/2\rfloor-1} (\mathfrak{e}^*_h\mathfrak{x}_{i-2r}\mathfrak{e}^*_j - \mathfrak{e}^*_h\mathfrak{x}_{i-2r-2}\mathfrak{e}^*_j) + \mathfrak{e}^*_h\mathfrak{x}_{i-2\lfloor i/2\rfloor}\mathfrak{e}^*_j. \tag{3.5.16}$$

After expanding the sum and cancelling terms, the right-hand side of (3.5.16) becomes $\mathfrak{e}^*_h\mathfrak{x}_i\mathfrak{e}^*_j$. Thus $\mathfrak{e}^*_h\mathfrak{x}_i\mathfrak{e}^*_j \in \varphi_D(\mathfrak{S}_D)$.

For the rest of this proof, assume (3.5.15) holds. Then by Definition 3.3.12,

$$\mathfrak{e}^*_{h+1}\mathfrak{x}_{i+2r}\mathfrak{e}^*_{j+1} \in \mathfrak{S}_D \qquad\qquad (1 \le r \le \lfloor (D-i)/2\rfloor).$$

For notational convenience, define a polynomial $g \in \mathbb{C}[z]$ by

$$g = \begin{cases} \frac{1}{(D-1)!} & \text{if } D-i \text{ is odd;} \\ \\ \frac{z}{D!} & \text{if } D-i \text{ is even.} \end{cases}$$

We have defined g such that by Lemma 3.5.4,

$$\varphi_D(\mathfrak{x}_{i+2\lfloor (D-i)/2\rfloor}) = g(\mathfrak{x}_1)\mu_{D-2}(\mathfrak{x}_1) - \mathfrak{x}_{i+2\lfloor (D-i)/2\rfloor-2}.$$

Thus by Definition 3.5.1 and Lemma 3.5.4,

$$\begin{aligned} &\varphi_D\Bigg(\sum_{r=1}^{\lfloor (D-i)/2\rfloor} \mathfrak{e}^*_{h+1}\mathfrak{x}_{i+2r}\mathfrak{e}^*_{j+1}\Bigg) \\ &= \sum_{r=1}^{\lfloor (D-i)/2\rfloor-1} (\mathfrak{e}^*_h\mathfrak{x}_{i+2r}\mathfrak{e}^*_j - \mathfrak{e}^*_h\mathfrak{x}_{i+2r-2}\mathfrak{e}^*_j) + \mathfrak{e}^*_h g(\mathfrak{x}_1)\mu_{D-2}(\mathfrak{x}_1)\mathfrak{e}^*_j - \mathfrak{e}^*_h\mathfrak{x}_{i+2\lfloor (D-i)/2\rfloor-2}\mathfrak{e}^*_j. \end{aligned} \tag{3.5.17}$$

After expanding the sum and cancelling terms, the right-hand side of (3.5.17) becomes $-\mathfrak{e}^*_h\mathfrak{x}_i\mathfrak{e}^*_j + \mathfrak{e}^*_h g(\mathfrak{x}_1)\mu_{D-2}(\mathfrak{x}_1)\mathfrak{e}^*_j$. Hence

$$-\mathfrak{e}^*_h\mathfrak{x}_i\mathfrak{e}^*_j + \mathfrak{e}^*_h g(\mathfrak{x}_1)\mu_{D-2}(\mathfrak{x}_1)\mathfrak{e}^*_j \in \varphi_D(\mathfrak{S}_D). \tag{3.5.18}$$

By Corollary 3.5.9 (i) and the surjectivity of φ_D,

$$\mathfrak{e}_h^* g(\mathfrak{x}_1)\mu_{D-2}(\mathfrak{x}_1)\mathfrak{e}_j^* \in \varphi_D(\mathfrak{S}_D). \tag{3.5.19}$$

Therefore by (3.5.18) and (3.5.19), $\mathfrak{e}_h^*\mathfrak{x}_i\mathfrak{e}_j^* \in \varphi_D(\mathfrak{S}_D)$.

(ii) Similar to the proof of (i). □

We have $\mathfrak{S}_{D-2} \subseteq \varphi_D(\mathfrak{S}_D)$ by Lemmas 3.5.5 and 3.5.11 together with the surjectivity of φ_D.

Proposition 3.5.12. *Assume $D \geq 2$. Then $\varphi_D(\mathfrak{S}_D) = \mathfrak{S}_{D-2}$.*

Proof. We mentioned below Lemma 3.5.7 that $\varphi_D(\mathfrak{S}_D) \subseteq \mathfrak{S}_{D-2}$, and we mentioned below Lemma 3.5.11 that $\mathfrak{S}_{D-2} \subseteq \varphi_D(\mathfrak{S}_D)$. □

We next consider how φ_D induces an algebra homomorphism from $\mathcal{T}_D \to \mathcal{T}_{D-2}$. Recall Definitions 3.3.14 and 3.5.2.

Proposition 3.5.13. *Assume $D \geq 2$. Then there exists an algebra homomorphism $\varphi'_D : \mathcal{T}_D \to \mathcal{T}_{D-2}$ that sends*

$$\begin{aligned} &e_0 \mapsto 0, && e_0^* \mapsto 0, && \\ &e_i \mapsto e_{i-1}, && e_i^* \mapsto e_{i-1}^* && (1 \leq i \leq D-1), \\ &e_D \mapsto 0, && e_D^* \mapsto 0. && \end{aligned}$$

Moreover, φ'_D is surjective, and $\ker(\varphi'_D) = \psi_d(\mathfrak{K}_D)$.

Proof. We first consider the existence of φ'_D. By Lemma 3.3.13, Lemma 3.5.3, and Proposition 3.5.12, we have a surjective algebra homomorphism $\psi_{D-2} \circ \varphi_D : \mathfrak{T}_D \to \mathcal{T}_{D-2}$ with kernel equal to $\mathfrak{S}_D + \mathfrak{K}_D$. This map induces an algebra isomorphism from the quotient algebra $\mathfrak{T}_D/(\mathfrak{S}_D + \mathfrak{K}_D) \to \mathcal{T}_{D-2}$; we say this isomorphism is canonical.

Let $q : \mathfrak{T}_D/\mathfrak{S}_D \to \mathfrak{T}_D/(\mathfrak{S}_D + \mathfrak{K}_D)$ denote the quotient map, which we recall is an algebra homomorphism.

Recall the algebra isomorphism $p_D : \mathcal{T}_D \to \mathfrak{T}_D/\mathfrak{S}_D$ from Definition 3.3.14.

The following composition gives an algebra homomorphism from $\mathcal{T}_D \to \mathcal{T}_{D-2}$:

$$\varphi'_D : \quad \mathcal{T}_D \xrightarrow{p_D} \mathfrak{T}_D/\mathfrak{S}_D \xrightarrow{q} \mathfrak{T}_D/(\mathfrak{S}_D + \mathfrak{K}_D) \xrightarrow{can} \mathcal{T}_{D-2}. \tag{3.5.20}$$

We have shown that φ'_D exists. With reference to (3.1.2), one routinely check that φ'_D sends $e_i \mapsto e_{i-1}$ and $e^*_i \mapsto e^*_{i-1}$ $(0 \leq i \leq D)$.

We next show that φ'_D is surjective. This follows because each of the composition factors in (3.5.20) is surjective.

Lastly, we consider the kernel of φ'_D. Inspection of (3.5.20) shows that $\ker(\varphi'_D) = p_D^{-1}(\mathfrak{K}_D + \mathfrak{S}_D)$. By the construction of p_D, $p_D^{-1}(\mathfrak{K}_D + \mathfrak{S}_D) = \psi_D(\mathfrak{K}_D)$. Hence $\ker(\varphi'_D) = \psi_d(\mathfrak{K}_D)$. □

Proposition 3.5.14. *Assume $D \geq 2$. Then the ideal $\mathcal{T}_D u_0$ is equal to $\psi_D(\mathfrak{K}_D)$. Moreover, $\mathcal{T}_D u_0 = \ker(\varphi'_D)$.*

Proof. We first consider the first assertion. Because ψ_D is surjective, $\psi_D(\mathfrak{K}_D)$ is equal to the two-sided ideal of $\mathcal{T}_D$ generated by e_0, e_D, e^*_0, e^*_D. Thus by Lemma 3.4.6, the ideal $\psi_D(\mathfrak{K}_D) \subseteq \mathcal{T}_D u_0$.

Recall that $u_0 = 2^D \sum_{r=0}^{D} k_r^{-1} e^*_r e_0 e^*_r$. Thus

$$\mathcal{T}_D u_0 \subseteq \mathcal{T}_D e_0 \mathcal{T}_D \subseteq \psi_D(\mathfrak{K}_D).$$

This proves the first assertion.

The second assertion follows by the first, together with Proposition 3.5.13. □

Corollary 3.5.15. *Assume $D \geq 2$. Then there exists an algebra isomorphism $\mathcal{T}_D(1 -$*

$u_0) \to \mathcal{T}_{D-2}$ *that sends*

$$\begin{aligned} &e_0(1-u_0) \mapsto 0, && e^*_0(1-u_0) \mapsto 0, && \\ &e_i(1-u_0) \mapsto e_{i-1}, && e^*_i(1-u_0) \mapsto e^*_{i-1} && (1 \le i \le D-1), \\ &e_D(1-u_0) \mapsto 0, && e^*_D(1-u_0) \mapsto 0. && \end{aligned}$$

Proof. With reference to (3.1.2), Proposition 3.5.13 implies that there exists an induced algebra isomorphism $\mathcal{T}_D/\ker(\varphi'_D) \to \mathcal{T}_{D-2}$ which sends

$$e_i + \ker(\varphi'_D) \mapsto e_{i-1}, \qquad e^*_i + \ker(\varphi'_D) \mapsto e^*_{i-1} \qquad (0 \le i \le D).$$

By Proposition 3.5.14, we know that $\ker(\varphi'_D) = \mathcal{T}_D u_0$. Identifying the quotient algebra $\mathcal{T}_D/\mathcal{T}_D u_0$ with $\mathcal{T}_D(1-u_0)$ via the isomorphism in Corollary 3.4.5, the result follows. □

Corollary 3.5.16. *Assume $D \ge 2$. Then there exists an algebra isomorphism*

$$\mathcal{T}_D \to \mathrm{Mat}_{D+1}(\mathbb{C}) \oplus \mathcal{T}_{D-2}.$$

Proof. Follows from Corollaries 3.4.4 and 3.5.15. □

Proposition 3.5.17. *Assume $D \ge 0$. Then there exists an algebra isomorphism*

$$\mathcal{T}_D \to \bigoplus_{r=0}^{\lfloor D/2 \rfloor} \mathrm{Mat}_{D+1-2r}(\mathbb{C}).$$

Moreover, the algebra $\mathcal{T}_D$ is isomorphic to T_D.

Proof. We first consider the first assertion. We proceed by induction on D. The base cases of $D = 0$ and $D = 1$ are addressed in Remark 3.1.2 and Proposition 3.4.7. Now assume $D \ge 2$. By Corollary 3.5.16 and induction, we have algebra isomorphisms

$$\mathcal{T}_D \to \mathrm{Mat}_{D+1}(\mathbb{C}) \oplus \mathcal{T}_{D-2} \to \bigoplus_{r=0}^{\lfloor D/2 \rfloor} \mathrm{Mat}_{D+1-2r}(\mathbb{C}).$$

This completes the proof of the first assertion.

To prove the second assertion, compare this result to Proposition 3.2.22. □

We conclude with the main result of this chapter.

Theorem 3.5.18. *Assume $D \geq 0$. Then the graph $\mathcal{Q}_D$ is T-determined by the triple product relations.*

Proof. Fix a vertex x of $\mathcal{Q}_D$. By Proposition 3.5.17, $\mathcal{T}_D$ and T_D have the same dimension as $\mathbb{C}$-vector spaces. Because $\natural : \mathcal{T}_D \to T_D$ is a surjective algebra homomorphism between two algebras of the same dimension, it is an algebra isomorphism. The result follows from Definition 3.1.7. □

Chapter 4

2-homogeneous bipartite distance-regular graphs

In this chapter, motivated by Curtin's work on 2-homogeneous bipartite distance-regular graphs [7, 11], we define families of TBT pairs and character algebras with eigenvalues and intersection numbers analogous to those in Proposition 2.3.3 (ii), (iii). We then investigate the associated generalized Terwilliger algebras. Such a generalized Terwilliger algebra will be denoted $\mathcal{T}^{(D)}$. The main goals of this chapter are to prove Theorems 2, 3 from the introduction. As we will see, Theorem 2 follows as a consequence of Theorem 3, so we will prove Theorem 3 first.

4.1 The TBT system $\Theta^{(D)}$

Back in Section 2.4, we discussed TBT systems. In this section, we introduce a specific family of TBT systems.

Throughout this section, the following notation is in force. Let D denote a nonnegative integer. Let V denote a vector space over $\mathbb{C}$ with dimension $D+1$. Let $q \in \mathbb{C}$ denote a nonzero scalar that is not a root of unity.

Definition 4.1.1. Let

$$\theta_i^{(D)} = \frac{(q^D + q^2)(q^{D-2i} - 1)}{q^{D-i}(q^2 - 1)} \qquad (0 \le i \le D).$$

For notational convenience, abbreviate $\theta_i = \theta_i^{(D)}$ $(0 \le i \le D)$.

Our next goal is to show that the sequence $(\{\theta_i\}_{i=0}^D; \{\theta_i\}_{i=0}^D)$ satisfies the conditions of Proposition 2.4.14.

Lemma 4.1.2. *For $0 \le i, j \le D$, we have $\theta_i \ne \theta_j$ if $i \ne j$.*

Proof. By Definition 4.1.1,

$$\theta_i - \theta_j = \frac{(q^D + q^2)(q^i + q^{D-j})(q^{j-i} - 1)}{q^D(q^2 - 1)}. \tag{4.1.1}$$

In the right-hand side of (4.1.1), each factor is nonzero because $i \ne j$ and $q \ne 0$ and q is not a root of unity. The result follows. □

Lemma 4.1.3. *We have*

$$\theta_{i-1} - (q + q^{-1})\theta_i + \theta_{i+1} = 0 \qquad (1 \le i \le D - 1).$$

Proof. Follows from Definition 4.1.1. □

Lemma 4.1.4. *We have*

$$\theta_i + \theta_{D-i} = 0 \qquad (0 \le i \le D).$$

Proof. Follows from Definition 4.1.1. □

Proposition 4.1.5. *There exists a TBT system with eigenvalue array $(\{\theta_i\}_{i=0}^D; \{\theta_i\}_{i=0}^D)$.*

Proof. Follows from Proposition 2.4.14 and Lemmas 4.1.2–4.1.4. □

Definition 4.1.6. Let $\Theta^{(D)} = (A; \{E_i\}_{i=0}^D; A^*; \{E_i^*\}_{i=0}^D)$ denote the TBT system from Proposition 4.1.5. For notational convenience, abbreviate $\Theta = \Theta^{(D)}$.

Lemma 4.1.7. *The TBT system Θ is self-dual.*

Proof. Follows from Lemma 2.4.15. □

Next we describe the intersection numbers of Θ.

Lemma 4.1.8. *We have*

$$\begin{aligned} c_i &= \frac{(q^D+q^2)(q^{2i}-1)}{(q^D+q^{2i})(q^2-1)} & (1 \leq i \leq D), \qquad (4.1.2) \\ b_i &= \frac{(q^{D-i}+q^2)(q^{2(D-i)}-1)}{(q^D+q^{2(D-i)})(q^2-1)} & (0 \leq i \leq D-1). \end{aligned}$$

Proof. For $\{c_i\}_{i=1}^{D-1}$, evaluate the right-hand side of (2.4.3) using Definition 4.1.1. For c_D, evaluate the right-hand side of (2.4.4) using Definition 4.1.1. For $\{b_i\}_{i=0}^{D-1}$, evaluate the right-hand side of (2.4.5) using (4.1.2). □

Corollary 4.1.9. *We have $c_1 = 1$.*

Proof. Follows from Lemma 4.1.8. □

Recall the polynomials $\{v_i\}_{i=0}^{D+1}$ from Definition 2.5.3.

Corollary 4.1.10. *The following (i), (ii) hold:*

(i) $v_1 = z$;

(ii) $A_1 = A$.

Proof. (i) Follows from Definition 2.5.3 and Corollary 4.1.9.

(ii) Follows from Lemma 2.5.11 and Corollary 4.1.9. □

Lemma 4.1.11. *We have $c_D = c_i + b_i$ $(0 \leq i \leq D)$.*

Proof. Follows from Lemma 4.1.8. □

Lemma 4.1.12. *The following (i)–(iii) hold:*

(i) $c_i = c_i^*$ $(1 \leq i \leq D)$;

(ii) $v_i = v_i^*$ $(0 \leq i \leq D+1)$;

(iii) $\mu = \mu^*$.

Proof. Follows from Lemmas 2.4.15 and 2.5.13, Definition 2.7.7, and Lemma 4.1.7. □

4.2 The character system $\Psi^{(D)}$

Throughout this section, the following notation is in force. Let D denote a nonnegative integer. Let $q \in \mathbb{C}$ denote a nonzero scalar that is not a root of unity. Recall the TBT system

$$\Theta^{(D)} = (A; \{E_i\}_{i=0}^D; A^*; \{E_i^*\}_{i=0}^D)$$

from Definition 4.1.6. Recall the associated character system $\Psi_{\Theta^{(D)}} = (M; \{A_i\}_{i=0}^D; \{E_i\}_{i=0}^D)$ from Proposition 2.9.8. For notational convenience, abbreviate $\Psi^{(D)} = \Psi_{\Theta^{(D)}}$. We further abbreviate $\Psi = \Psi^{(D)}$ if the value of D is clear from the context.

In this section, we make some observations about Ψ.

Lemma 4.2.1. *The character system Ψ is self-dual.*

Proof. Follows from Proposition 2.9.10 and Lemma 4.1.7. □

Lemma 4.2.2. *We have $p_{ij}^h = q_{ij}^h$ $(0 \leq h, i, j \leq D)$.*

Proof. Follows from Lemmas 2.8.7, 4.2.1. □

We now introduce some notation. Let

$$\binom{j}{i}_c = \frac{c_j c_{j-1} \cdots c_{j-i+1}}{c_i c_{i-1} \cdots c_1} \qquad (0 \leq i \leq j \leq D). \tag{4.2.1}$$

We interpret

$$\binom{j}{0}_c = 1 \qquad (0 \leq j \leq D).$$

Lemma 4.2.3. *For $0 \leq i, j \leq D$,*

$$\binom{j}{i}_c = \binom{j}{j-i}_c.$$

Proof. Follows from (4.2.1). ☐

Lemma 4.2.4. (See [31, Definition 6.6]) *For $0 \leq i \leq D$,*

$$k_i = \binom{D}{i}_c.$$

Lemma 4.2.5. *We have $k_i = k_i^*$ $(0 \leq i \leq D)$.*

Proof. Follows from Lemmas 4.1.12, 4.2.4. ☐

Recall the scalars $p_i(j)$ and $q_i(j)$ from Definition 2.7.7.

Lemma 4.2.6. *For $0 \leq i, j \leq D$,*

$$p_i(j) = q_i(j) = p_i^*(j) = q_i^*(j).$$

Proof. Follows from Lemmas 2.8.4, 4.2.1. ☐

Lemma 4.2.7. *The following (i)–(iii) hold for $0 \leq i, j \leq D$:*

(i) $p_i(j) = p_j(i)$.

(ii) $p_i(D-j) = (-1)^i p_i(j)$.

(iii) $p_{D-i}(j) = (-1)^j p_i(j)$.

Proof. (i) Follows from Lemmas 4.2.5, 4.2.6 and [28, Lemma 13.5].

(ii) Follows from Lemmas 2.5.6, 2.9.9, 4.1.4.

(iii) Follows from (i) and (ii). ☐

Lemma 4.2.8. *For $0 \leq j \leq D$,*

$$p_D(j) = (-1)^j.$$

Proof. Follows from Lemmas 2.7.8, 4.2.7. □

Recall the polynomial $\mu \in \mathbb{C}[z]$ from Definition 2.5.7.

Lemma 4.2.9. *In the algebra* $\mathbb{C}[z]$*, the polynomial* μ *divides* $v_i v_j - \sum_{h=0}^{D} p^h_{ij} v_h$ *for* $0 \leq i, j \leq D$*.*

Proof. By Definitions 2.5.9, 2.6.10, we have

$$v_i(A)v_j(A) - \sum_{h=0}^{D} p^h_{ij} v_h(A) = 0.$$

The result follows from Lemma 2.5.8. □

4.3 The intersection numbers of Ψ

Throughout this section, the following notation is in force. Let D denote a nonnegative integer. Let $q \in \mathbb{C}$ denote a nonzero scalar that is not a root of unity. Recall the character system

$$\Psi = \Psi^{(D)} = (M; \{A_i\}_{i=0}^{D}; \{E_i\}_{i=0}^{D})$$

from the first paragraph of Section 4.2. In this section, we find the intersection numbers of Ψ in closed form.

Definition 4.3.1. Let the set $P^{(D)}$ consist of the 3-tuples of integers (h, i, j) such that $0 \leq h, i, j \leq D$ and the following (i)–(iii) hold:

(i) none of h, i, j is greater than the sum of the other two;

(ii) $h + i + j \leq 2D$;

(iii) $h + i + j$ is even.

For notational convenience, abbreviate $P = P^{(D)}$.

We will show that for $0 \leq h, i, j \leq D$, the intersection number $p^h_{ij} \neq 0$ if and only if $(h, i, j) \in P$. In this case, we will give an explicit formula for p^h_{ij}.

Lemma 4.3.2. *We have* $A_iA_D = A_{D-i}$ $(0 \leq i \leq D)$.

Proof. By Proposition 2.7.5, Definition 2.7.7, and Lemmas 4.2.7, 4.2.8,

$$A_iA_D = \Big(\sum_{j=0}^{D} p_i(j)E_j\Big)\Big(\sum_{j=0}^{D}(-1)^jE_j\Big) = \sum_{j=0}^{D}(-1)^jp_i(j)E_j = \sum_{j=0}^{D}p_{D-i}(j)E_j = A_{D-i}.$$

□

Lemma 4.3.3. *We have* $p^h_{ij} = p^h_{D-i\ D-j}$ $(0 \leq h, i, j \leq D)$.

Proof. By Lemmas 2.5.11, 4.3.2 and since M is commutative,

$$A_{D-i}A_{D-j} = A_iA_DA_jA_D = A_iA_DA_DA_j = A_iA_0A_j = A_iA_j.$$

The result follows from Definition 2.6.10. □

Lemma 4.3.4. *The following (i), (ii) hold for* $0 \leq i, j \leq D$*:*

(i) if $i + j$ *is even, then*

$$A_iA_j \in \operatorname{span}\{A_h \mid 0 \leq h \leq D,\ h \textit{ is even}\};$$

(ii) if $i + j$ *is odd, then*

$$A_iA_j \in \operatorname{span}\{A_h \mid 0 \leq h \leq D,\ h \textit{ is odd}\}.$$

Proof. Follows from Lemma 2.5.11, Corollary 2.5.12, and induction on i. □

Lemma 4.3.5. *For* $0 \leq h, i, j \leq D$*, the scalar* $p^h_{ij} = 0$ *if one of* h, i, j *is greater than the sum of the other two.*

Proof. By Proposition 2.9.8, the character system Ψ is P-polynomial. The result follows from Definition 2.7.10. □

Lemma 4.3.6. *For* $0 \leq h, i, j \leq D$*, the scalar* $p^h_{ij} = 0$ *if* $h + i + j > 2D$*.*

Proof. We have

$$h > D - i + D - j.$$

Hence by Lemma 4.3.5, $p^h_{D-i\ D-j} = 0$. The result follows from Lemma 4.3.3. $\square$

Lemma 4.3.7. *For $0 \le h, i, j \le D$, the scalar $p^h_{ij} = 0$ if $h + i + j$ is odd.*

Proof. Follows from Definition 2.6.10 and Lemma 4.3.4. $\square$

Proposition 4.3.8. *For $0 \le h, i, j \le D$, the scalar $p^h_{ij} = 0$ if $(h, i, j) \notin P$.*

Proof. Follows from Definition 4.3.1 and Lemmas 4.3.5–4.3.7. $\square$

Lemma 4.3.9. *For $0 \le h, j \le D$ and $2 \le i \le D$,*

$$p^h_{ij} = \frac{c_h p^{h-1}_{i-1\ j} + b_h p^{h+1}_{i-1\ j} - b_{i-2} p^h_{i-2\ j}}{c_i}. \tag{4.3.1}$$

Proof. By Corollary 2.5.12,

$$A_i A_j = \frac{(A A_{i-1} - b_{i-2} A_{i-2}) A_j}{c_i}. \tag{4.3.2}$$

Distribute A_j on the right-hand side of (4.3.2), then evaluate both sides using Definition 2.6.10. This yields

$$\sum_{r=0}^{D} p^r_{ij} A_r = \frac{\sum_{r=0}^{D} p^r_{i-1\ j} A A_r - b_{i-2} \sum_{r=0}^{D} p^r_{i-2\ \mathrm{j}} A_r}{c_i}. \tag{4.3.3}$$

By Lemma 2.5.10, the elements $\{A_r\}_{r=0}^{D}$ are linearly independent. Evaluate the right-hand side of (4.3.3) using Lemma 2.5.11 and compare coefficients. The result follows. $\square$

We now give the intersection numbers of $\Psi^{(D)}$ in closed form.

Theorem 4.3.10. *For $0 \le h, i, j \le D$, we have*

$$p^h_{ij} = \begin{cases} 0 & \text{if } (h, i, j) \notin P; \\ \binom{h}{i-\delta}_c \binom{D-h}{\delta}_c \frac{(q^{2D}+q^D)(q^{2i}+q^D)(q^{2j}+q^D)(q^{2(D-h)}+q^D)}{(q^{2\delta}+q^D)(q^{2(D-i+\delta)}+q^D)(q^{2(D-j+\delta)}+q^D)(q^{2(h+\delta)}+q^D)} & \text{if } (h, i, j) \in P, \end{cases}$$

where $\delta = \frac{i+j-h}{2}$.

Proof. We proceed by induction on i. The cases $i = 0$ and $i = 1$ are routine.

For the rest of this proof, assume $i \geq 2$. If $(h, i, j) \notin P$, then $p^h_{ij} = 0$ by Proposition 4.3.8. Assume $(h, i, j) \in P$. Evaluate the right-hand side of (4.3.1) using induction and Lemma 4.1.8. In this evaluation, it is convenient to treat the cases $h = j - i$ and $h = j + i$ separately. □

We emphasize one aspect of Theorem 4.3.10.

Corollary 4.3.11. *For* $0 \leq h, i, j \leq D$*, the following (i), (ii) are equivalent:*

(i) $p^h_{ij} \neq 0$*;*

(ii) $(h, i, j) \in P$*.*

Proof. Follows from Lemma 2.4.13, Theorem 4.3.10, and because $q \neq 0$ and q is not a root of unity. □

4.4 The generalized Terwilliger algebra associated with a character system

In this section, we review some definitions and results from [14] concerning the generalized Terwilliger algebra. A generalized Terwilliger algebra is constructed from a character system Ψ and its dual Ψ^*. In this section, we will not make any assumption about Ψ or Ψ^*. In later sections, we will assume that Ψ and Ψ^* come from the first paragraph of Section 4.2.

Throughout this section, the following notation is in force. Let D denote a nonnegative integer. Let $\Psi = (C; \{X_i\}_{i=0}^D; \{E_i\}_{i=0}^D)$ denote a character system over $\mathbb{C}$. Let $\Psi^* = (C^*; \{X^*_i\}_{i=0}^D; \{E^*_i\}_{i=0}^D)$ denote the dual of Ψ.

Definition 4.4.1. (See [14, Definition 4.1]) Let $\mathcal{T}$ denote the algebra with generators $\{x_i\}_{i=0}^D$, $\{x_i^*\}_{i=0}^D$ and the following relations:

$$
\begin{aligned}
&(T1) && x_0 = x_0^* = 1; && \\
&(T2) && x_i x_j = \sum_{h=0}^{D} p_{ij}^h x_h && (0 \le i, j \le D); \\
&(T2^*) && x_i^* x_j^* = \sum_{h=0}^{D} q_{ij}^h x_h^* && (0 \le i, j \le D); \\
&(T3) && e_h^* x_i e_j^* = 0 \text{ if } p_{ij}^h = 0 && (0 \le h, i, j \le D); \\
&(T3^*) && e_h x_i^* e_j = 0 \text{ if } q_{ij}^h = 0 && (0 \le h, i, j \le D).
\end{aligned}
$$

The elements $\{e_i\}_{i=0}^D$, $\{e_i^*\}_{i=0}^D$ are defined by

$$
e_i = \nu^{-1} \sum_{j=0}^{D} q_i(j) x_j, \qquad e_i^* = \nu^{-1} \sum_{j=0}^{D} q_i^*(j) x_j^* \qquad (0 \le i \le D).
$$

The above p_{ij}^h, q_{ij}^h, $q_i(j)$, $q_i^*(j)$, ν are attached to Ψ, Ψ^* as explained in Sections 2.7, 2.8. We call $\mathcal{T}$ the *generalized Terwilliger algebra associated with* Ψ.

Remark 4.4.2. Assume $D = 0$. Then the algebra $\mathcal{T}$ is isomorphic to $\mathbb{C}$.

For the rest of this section, assume $D \ge 1$.

Let $i \in \mathbb{Z}$. For notational convenience, define $x_i = 0$, $x_i^* = 0$, $e_i = 0$, $e_i^* = 0$ unless $0 \le i \le D$. Abbreviate $x = x_1$ and $x^* = x_1^*$.

Definition 4.4.3. Let $\mathcal{C}$ denote the subalgebra of $\mathcal{T}$ generated by $\{x_i\}_{i=0}^D$. Let $\mathcal{C}^*$ denote the subalgebra of $\mathcal{T}$ generated by $\{x_i^*\}_{i=0}^D$.

Lemma 4.4.4. (See [14, Section 5] *The following (i), (ii) hold:*

(i) there exists an algebra isomorphism $\phi : \mathcal{C} \to C$ *which sends*

$$
x_i \mapsto X_i, \qquad e_i \mapsto E_i \qquad (0 \le i \le D);
$$

(ii) there exists an algebra isomorphism $\phi^* : \mathcal{C}^* \to C^*$ *which sends*

$$x_i^* \mapsto X_i^*, \qquad e_i^* \mapsto E_i^* \qquad (0 \leq i \leq D).$$

Lemma 4.4.5. (See [14, Propositions 5.4, 5.7]) *The following (i), (ii) hold:*

(i) each of $\{x_i\}_{i=0}^D$, $\{e_i\}_{i=0}^D$ *is a basis for* $\mathcal{C}$*;*

(ii) each of $\{x_i^*\}_{i=0}^D$, $\{e_i^*\}_{i=0}^D$ *is a basis for* $\mathcal{C}^*$*.*

Lemma 4.4.6. (See [14, Proposition 5.3, 5.6]) *The following (i), (ii) hold:*

(i) $\sum_{i=0}^D e_i = 1$*;*

(ii) $\sum_{i=0}^D e_i^* = 1$*.*

Lemma 4.4.7. (See [14, Propositions 5.3, 5.6]) *The following (i), (ii) hold for* $0 \leq i, j \leq D$*:*

(i) $e_i e_j = \delta_{ij} e_i$*;*

(ii) $e_i^* e_j^* = \delta_{ij} e_j^*$*.*

Lemma 4.4.8. *The following (i), (ii) hold:*

(i) $e_0 = \nu^{-1} \sum_{i=0}^D x_i$*;*

(ii) $e_0^* = \nu^{-1} \sum_{i=0}^D x_i^*$*.*

Proof. Follows from Lemma 2.7.8 and Definition 4.4.1. □

Next we recall the central idempotent of $\mathcal{T}$.

Lemma 4.4.9. (See [14, Proposition 11.1]) *We have*

$$\nu \sum_{i=0}^D k_i^{-1} e_i^* e_0 e_i^* = \nu \sum_{i=0}^D (k_i^*)^{-1} e_i e_0^* e_i. \tag{4.4.1}$$

This common element is central in $\mathcal{T}$*.*

Definition 4.4.10. Referring to Lemma 4.4.9, let u denote the common element from (4.4.1). By [14, Proposition 11.4], $u^2 = u$. We call u the *primary central idempotent of* $\mathcal{T}$.

Lemma 4.4.11. (See [14, Proposition 11.2]) *The following (i), (ii) hold for* $0 \leq i \leq D$*:*

(i) $ue_i = \nu k_i^{-1} e_i e_0^* e_i$.

(ii) $ue_i^* = \nu (k_i^*)^{-1} e_i^* e_0 e_i^*$.

Lemma 4.4.12. (See [14, Corollary 11.3]) *The following (i), (ii) hold:*

(i) $ue_0 = e_0$*;*

(ii) $ue_0^* = e_0^*$.

Proposition 4.4.13. (See [14, Proposition 11.5 and Theorem 12.3]) *The following* (*i*)–(*iii*) *hold:*

(i) the sum $\mathcal{T} = \mathcal{T}u + \mathcal{T}(1-u)$ *is direct;*

(ii) $\mathcal{T}u$ *and* $\mathcal{T}(1-u)$ *are both two-sided ideals of* $\mathcal{T}$*;*

(iii) the algebra $\mathcal{T}u$ *is isomorphic to* $\mathrm{Mat}_{D+1}(\mathbb{C})$.

We emphasize some aspects of Proposition 4.4.13.

Corollary 4.4.14. *There exists an algebra isomorphism from* $\mathcal{T}$ *to the direct sum*

$$\mathrm{Mat}_{D+1}(\mathbb{C}) \oplus \mathcal{T}(1-u).$$

Proof. Follows from Proposition 4.4.13. □

We conclude this section with a comment about the quotient algebra $\mathcal{T}/\mathcal{T}u$.

Corollary 4.4.15. *There exists an algebra isomorphism* $\mathcal{T}/\mathcal{T}u \to \mathcal{T}(1-u)$ *that sends*

$$e_i + \mathcal{T}u \mapsto e_i(1-u), \qquad e_i^* + \mathcal{T}u \mapsto e_i^*(1-u) \qquad (0 \leq i \leq D).$$

Proof. Follows from Proposition 4.4.13. □

4.5 The generalized Terwilliger algebra associated with $\Psi^{(D)}$

In the previous section, we discussed the generalized Terwilliger algebra attached to a character system Ψ and its dual Ψ^*. In this section, we assume that Ψ and Ψ^* come from the first paragraph of Section 4.2.

Throughout this section, the following notation is in force. Let D denote a nonnegative integer. Let $q \in \mathbb{C}$ denote a nonzero scalar that is not a root of unity. Recall the character system

$$\Psi^{(D)} = (M; \{A_i\}_{i=0}^D; \{E_i\}_{i=0}^D)$$

from the first paragraph of Section 4.2. Recall from Lemma 4.2.1 that $\Psi^{(D)}$ is self-dual.

Definition 4.5.1. Let $\mathcal{T}^{(D)}$ denote the generalized Terwilliger algebra associated with the character system $\Psi^{(D)}$. For notational convenience, abbreviate $\mathcal{T} = \mathcal{T}^{(D)}$.

For the rest of this section, assume $D \geq 1$. We make some observations about $\mathcal{T}$.

Lemma 4.5.2. *With reference to Lemma 4.4.4, the following (i), (ii) hold:*

(i) ϕ *sends* $x \mapsto A$;

(ii) ϕ^* *sends* $x^* \mapsto A^*$.

Proof. Follows from Corollary 4.1.10 and Lemma 4.4.4. □

Lemma 4.5.3. *The following (i), (ii) hold:*

(i) $x = \sum_{i=0}^D \theta_i e_i$;

(ii) $x^* = \sum_{i=0}^D \theta_i e_i^*$.

Proof. Follows from Lemmas 2.4.7, 4.4.4, 4.5.2. □

Recall the polynomials $\{v_i\}_{i=0}^{D+1}$ from Definition 2.5.3.

Lemma 4.5.4. *The following (i), (ii) hold for* $0 \leq i \leq D$:

(i) $x_i = v_i(x)$;

(ii) $x^*_i = v_i(x^*)$.

Proof. Follows from Definition 2.5.9 and Lemmas 4.1.12, 4.4.4, 4.5.2. □

Let $i \in \mathbb{Z}$. Recall that $x_i = 0$, $x^*_i = 0$ unless $0 \leq i \leq D$.

Lemma 4.5.5. *The following (i), (ii) hold for* $1 \leq i \leq D$:

(i) $xx_i = b_{i-1}x_{i-1} + c_{i+1}x_{i+1}$;

(ii) $x^*x^*_i = b_{i-1}x^*_{i-1} + c_{i+1}x^*_{i+1}$.

Proof. Follows from Lemmas 2.5.11, 4.1.12, 4.4.4, 4.5.2. □

Corollary 4.5.6. *The following (i), (ii) hold for* $2 \leq i \leq D+1$*:*

(i) $x_i = \frac{xx_{i-1} - b_{i-2}x_{i-2}}{c_i}$;

(ii) $x^*_i = \frac{x^*x^*_{i-1} - b_{i-2}x^*_{i-2}}{c_i}$.

Proof. Follows from Lemma 4.5.5. □

Lemma 4.5.7. *The following (i), (ii) hold for* $0 \leq i \leq D$*:*

(i) $x_i x_D = x_{D-i}$;

(ii) $x^*_i x^*_D = x^*_{D-i}$.

Proof. Follows from Lemmas 4.3.2, 4.4.4. □

Recall the polynomial μ from Definition 2.5.7.

Lemma 4.5.8. *The following (i), (ii) hold:*

(i) μ *is the minimal polynomial of* x*;*

(ii) μ *is the minimal polynomial of* x^*.

Proof. Follows from Lemmas 2.5.8, 4.5.2. □

Lemma 4.5.9. *The following (i), (ii) hold:*

(i) $v_{D+1}(x) = 0$;

(ii) $v_{D+1}(x^*) = 0$.

Proof. Follows from Lemmas 2.5.8, 4.5.8. $\square$

Lemma 4.5.10. *The algebra* $\mathcal{T}$ *is generated by* x, x^*.

Proof. Follows from Definition 4.4.1 and Lemma 4.5.4. $\square$

Recall the set $P = P^{(D)}$ from Definition 4.3.1.

Lemma 4.5.11. *The following (i), (ii) hold:*

(i) $e^*_h x_i e^*_j = 0$ *if* $(h, i, j) \notin P$;

(ii) $e_h x^*_i e_j = 0$ *if* $(h, i, j) \notin P$.

Proof. Follows from Corollary 4.3.11 and Definition 4.4.1. $\square$

Recall the primary central idempotent $u \in \mathcal{T}$ from Definition 4.4.10. We make some observations about this element.

Lemma 4.5.12. *The following (i), (ii) hold:*

(i) $ue_D = e_D$;

(ii) $ue^*_D = e^*_D$.

Proof. (i) By Lemmas 4.2.4, 4.4.8, 4.4.11,

$$ue_D = \sum_{i=0}^{D} e_D x^*_i e_D. \tag{4.5.1}$$

By Definition 4.3.1 and Lemma 4.5.11, $e_D x^*_i e_D = 0$ for $1 \le i \le D$. Thus by (4.5.1),

$$ue_D = e_D x^*_0 e_D. \tag{4.5.2}$$

Evaluate the right-hand side of (4.5.2) using Definition 4.4.1 ($T1$) and Lemma 4.4.7. The result follows.

(ii) Similar to the proof of (i). $\square$

4.6 Alternate presentations for $\mathcal{T}$

Throughout this section, the following notation is in force. Let $D \geq 1$ denote an integer. Let $q \in \mathbb{C}$ denote a nonzero scalar that is not a root of unity. Recall the generalized Terwilliger algebra $\mathcal{T} = \mathcal{T}^{(D)}$ from Definitions 4.4.1, 4.5.1. In these definitions, $\mathcal{T}$ is presented by generators and relations. In this section, we describe two additional presentations for $\mathcal{T}$.

Recall the set $P = P^{(D)}$ from Definition 4.3.1.

Definition 4.6.1. Let $\mathcal{T}'$ denote the algebra with generators x, x^* and the following relations:

(1) $\mu(x) = 0$;

(2) $\mu(x^*) = 0$;

(3) $e^*_h x_i e^*_j = 0$ if $(h, i, j) \notin P$ $(0 \leq h, i, j \leq D)$;

(4) $e_h x^*_i e_j = 0$ if $(h, i, j) \notin P$ $(0 \leq h, i, j \leq D)$.

The elements $\{x_i\}_{i=0}^{D}$, $\{x^*_i\}_{i=0}^{D}$, $\{e_i\}_{i=0}^{D}$, $\{e^*_i\}_{i=0}^{D}$ are defined by

$$x_i = v_i(x), \qquad\qquad x^*_i = v_i(x^*),$$
$$e_i = \nu^{-1} \sum_{j=0}^{D} q_i(j) x_j, \qquad\qquad e^*_i = \nu^{-1} \sum_{j=0}^{D} q_i(j) x^*_j$$

for $0 \leq i \leq D$. The above μ, v_i, ν, $q_i(j)$ are attached to the generalized Terwilliger algebra $\mathcal{T}$ as explained in Sections 2.5, 2.7, 2.9, 4.4.

Shortly we will display an algebra isomorphism $\mathcal{T} \to \mathcal{T}'$ that sends

$$\begin{aligned} x_i &\mapsto x_i, & x_i^* &\mapsto x_i^*, \\ e_i &\mapsto e_i, & e_i^* &\mapsto e_i^* \end{aligned}$$

for $0 \leq i \leq D$.

Lemma 4.6.2. *The following (i), (ii) hold in $\mathcal{T}'$:*

(i) $x_0 = 1$*;*

(ii) $x_0^* = 1$.

Proof. Follows from Definitions 2.5.3, 4.6.1. □

Lemma 4.6.3. *The following (i), (ii) hold in $\mathcal{T}'$:*

(i) $x_1 = x$*;*

(ii) $x_1^* = x^*$.

Proof. Follows from Corollary 4.1.10 and Definition 4.6.1. □

Lemma 4.6.4. *The following (i), (ii) hold in $\mathcal{T}'$ for $0 \leq i, j \leq D$:*

(i) $x_i x_j = \sum_{h=0}^{D} p_{ij}^h x_h$*;*

(ii) $x_i^* x_j^* = \sum_{h=0}^{D} p_{ij}^h x_h^*$.

Proof. (i) By Lemma 4.2.9 and Definition 4.6.1,

$$v_i(x)v_j(x) - \sum_{h=0}^{D} p_{ij}^h v_h(x) = 0.$$

The result follows from Definition 4.6.1.

(ii) Similar to the proof of (i). □

Lemma 4.6.5. *The following (i), (ii) hold in $\mathcal{T}'$ for $0 \leq h, i, j \leq D$:*

(i) $e^*_h x_i e^*_j = 0$ *if* $p^h_{ij} = 0$*;*

(ii) $e_h x^*_i e_j = 0$ *if* $q^h_{ij} = 0$*.*

Proof. Follows from Lemma 4.2.2, Corollary 4.3.11, and Definition 4.6.1. □

Proposition 4.6.6. *There exists an algebra isomorphism* $\mathcal{T} \to \mathcal{T}'$ *that sends*

$$x_i \mapsto x_i, \qquad x^*_i \mapsto x^*_i,$$
$$e_i \mapsto e_i, \qquad e^*_i \mapsto e^*_i$$

for $0 \leq i \leq D$*.*

Proof. Consider the presentation for $\mathcal{T}$ from Definition 4.4.1. By Lemmas 4.2.6, 4.6.2, 4.6.4, 4.6.5 and Definition 4.6.1, there exists an algebra homomorphsim $\psi : \mathcal{T} \to \mathcal{T}'$ that sends $x_i \mapsto x_i$, $x^*_i \mapsto x^*_i$, $e_i \mapsto e_i$, $e^*_i \mapsto e^*_i$ for $0 \leq i \leq D$.

Consider the presentation for $\mathcal{T}'$ from Definition 4.6.1. By Lemmas 4.2.6, 4.5.4, 4.5.8, 4.5.11, 4.6.3 and Definition 4.4.1, there exists an algebra homomorphism $\psi' : \mathcal{T}' \to \mathcal{T}$ that sends $x_i \mapsto x_i$, $x^*_i \mapsto x^*_i$, $e_i \mapsto e_i$, $e^*_i \mapsto e^*_i$ for $0 \leq i \leq D$.

By construction, the algebra homomorphisms ψ and ψ' are inverses. Therefore they are algebra isomorphisms. □

For the rest of the paper, we identify the algebras $\mathcal{T}$ and $\mathcal{T}'$ via the isomorphism in Proposition 4.6.6.

Definition 4.6.7. Let $\mathcal{T}''$ denote the algebra with generators $\{e_i\}_{i=0}^D$, $\{e^*_i\}_{i=0}^D$ and the following relations:

(1) $\sum_{i=0}^D e_i = 1$;

(2) $\sum_{i=0}^D e^*_i = 1$;

(3) $e^*_h x_i e^*_j = 0$ if $(h, i, j) \notin P$ $(0 \leq h, i, j \leq D)$;

(4) $e_h x^*_i e_j = 0$ if $(h, i, j) \notin P$ $(0 \leq h, i, j \leq D)$.

The elements x, x^*, $\{x_i\}_{i=0}^D$, $\{x_i^*\}_{i=0}^D$ are defined by

$$x = \sum_{r=0}^{D} \theta_r e_r, \qquad x^* = \sum_{r=0}^{D} \theta_r e_r^*,$$
$$x_i = v_i(x), \qquad x_i^* = v_i(x^*) \qquad (0 \leq i \leq D).$$

The above θ_r, v_i are attached to the generalized Terwilliger algebra $\mathcal{T}$ as explained in Sections 2.4, 2.5, 2.9, 4.4.

Shortly we will display an algebra isomorphism $\mathcal{T} \to \mathcal{T}''$ that sends

$$x_i \mapsto x_i, \qquad x_i^* \mapsto x_i^*,$$
$$e_i \mapsto e_i, \qquad e_i^* \mapsto e_i^*$$

for $0 \leq i \leq D$.

Lemma 4.6.8. *The following (i), (ii) hold in $\mathcal{T}''$ for $0 \leq i, j \leq D$:*

(i) $e_i e_j = \delta_{ij} e_i$;

(ii) $e_i^* e_j^* = \delta_{ij} e_i^*$.

Proof. (i) First assume $i \neq j$. By Definitions 2.5.3, 4.6.7, $x_0^* = v_0(x^*) = 1$. By Definition 4.3.1 and since $i \neq j$, the triple $(i, 0, j) \notin P$. By these comments and Definition 4.6.7,

$$e_i e_j = e_i x_0^* e_j = 0.$$

Next assume $i = j$. By Definition 4.6.7 and the previous paragraph,

$$e_i = e_i \sum_{r=0}^{D} e_r = e_i^2.$$

(ii) Similar to the proof of (i). □

Lemma 4.6.9. *The following (i), (ii) hold in $\mathcal{T}''$:*

(i) $\mu(x) = 0$;

(ii) $\mu(x^*) = 0$.

Proof. (i) By Definition 4.6.7 and Lemma 4.6.8,

$$\mu(x) = \mu\biggl(\sum_{i=0}^{D} \theta_i e_i\biggr) = \sum_{i=0}^{D} \mu(\theta_i) e_i.$$

By Definition 2.5.7, $\mu(\theta_i) = 0$ for $0 \leq i \leq D$. The result follows.

(ii) Similar to the proof of (i). $\square$

Recall the first eigenmatrix P and the second eigenmatrix Q of Ψ.

Lemma 4.6.10. *The following (i), (ii) hold in* $\mathcal{T}''$ *for* $0 \leq i \leq D$*:*

(i) $e_i = \nu^{-1} \sum_{j=0}^{D} q_i(j) x_j$;

(ii) $e^*_i = \nu^{-1} \sum_{j=0}^{D} q_i(j) x^*_j$.

Proof. (i) By Lemmas 2.9.9, 4.6.8 and Definition 4.6.7,

$$x_i = \sum_{j=0}^{D} p_i(j) e_j.$$

The result follows from the fact that $PQ = QP = \nu I$.

(ii) Similar to the proof of (i), noting Lemma 4.2.6. $\square$

Proposition 4.6.11. *There exists an algebra isomorphism* $\mathcal{T} \to \mathcal{T}''$ *that sends*

$$x_i \mapsto x_i, \qquad x^*_i \mapsto x^*_i,$$
$$e_i \mapsto e_i, \qquad e^*_i \mapsto e^*_i$$

for $0 \leq i \leq D$.

Proof. Consider the presentation for $\mathcal{T}$ from Definition 4.6.1. By Definition 4.6.7 and Lemmas 4.6.9, 4.6.10, there exists an algebra homomorphism $\tau : \mathcal{T} \to \mathcal{T}''$ that sends $x_i \mapsto x_i$, $x^*_i \mapsto x^*_i$, $e_i \mapsto e_i$, $e^*_i \mapsto e^*_i$ for $0 \leq i \leq D$.

Consider the presentation for $\mathcal{T}''$ from Definition 4.6.7. By Lemmas 4.4.6, 4.5.3 and Definition 4.6.1, there exists an algebra homomorphism $\tau'' : \mathcal{T}'' \to \mathcal{T}$ that sends $x_i \mapsto x_i$, $x^*_i \mapsto x^*_i$, $e_i \mapsto e_i$, $e^*_i \mapsto e^*_i$ for $0 \leq i \leq D$.

By construction, the algebra homomorphisms τ and τ'' are inverses. Therefore they are algebra isomorphisms. □

For the rest of this paper, we identify the algebras $\mathcal{T}$ and $\mathcal{T}''$ via the isomorphism in Proposition 4.6.11.

We end this section with a comment about notation. We have been discussing the algebra $\mathcal{T} = \mathcal{T}^{(D)}$. In this discussion, we attached many items to $\mathcal{T}$, such as θ_i, c_i, v_i, etc. Going forward, we might write $\theta^{(D)}_i$, $c^{(D)}_i$, $v^{(D)}_i$, etc. in order to emphasize the origin of these items.

4.7 Comparing the polynomials $v^{(D)}_i$ and $v^{(D-2)}_i$

Throughout this section, the following notation is in force. Let $D \geq 2$ denote an integer. Let $q \in \mathbb{C}$ denote a nonzero scalar that is not a root of unity. Recall the generalized Terwilliger algebras $\mathcal{T}^{(D)}$ and $\mathcal{T}^{(D-2)}$ from Definitions 4.4.1, 4.5.1.

We motivate this section with some comments. In Section 4.8, we will display an algebra homomorphism $\flat^{(D)} : \mathcal{T}^{(D)} \to \mathcal{T}^{(D-2)}$. As we will see, $\flat^{(D)}$ sends $x^{(D)} \mapsto \alpha x^{(D-2)}$ and $x^{*(D)} \mapsto \alpha x^{*(D-2)}$, where $\alpha \in \mathbb{C}$ is a scalar we will define shortly. By Lemma 4.5.4,

$$\begin{aligned} x^{(D)}_i &= v^{(D)}_i(x^{(D)}), & x^{*(D)}_i &= v^{(D)}_i(x^{*(D)}) & & (0 \leq i \leq D), \\ x^{(D-2)}_i &= v^{(D-2)}_i(x^{(D-2)}), & x^{*(D-2)}_i &= v^{(D-2)}_i(x^{*(D-2)}) & & (0 \leq i \leq D-2). \end{aligned}$$

Thus for $0 \leq i \leq D$, $\flat^{(D)}$ sends $x^{(D)}_i \mapsto v^{(D)}_i(\alpha x^{(D-2)})$ and $x^*_i \mapsto v^{(D)}_i(\alpha x^{*(D-2)})$. Motivated by those comments, we investigate the polynomial $v^{(D)}_i(\alpha z)$.

Our goal in this section is to express the polynomial $v^{(D)}_i(\alpha z)$ in terms of the polynomials $\{v^{(D-2)}_j(z)\}_{j=0}^{D-1}$.

For $i \in \mathbb{Z}$, define

$$\alpha_i = \frac{q^D + q^{2i}}{q(q^{D-2} + q^{2i})}. \tag{4.7.1}$$

Abbreviate $\alpha = \alpha_1$. Note that for $i \in \mathbb{Z}$, α_i is nonzero and

$$\alpha_i^{-1} = \alpha_{D-1-i}. \tag{4.7.2}$$

We define the scalars $\{\gamma_i\}_{i\in\mathbb{Z}}$ such that $\gamma_0 = 1$ and

$$\gamma_i = \alpha_i \gamma_{i-1} \qquad (i \in \mathbb{Z}). \tag{4.7.3}$$

Note that $\gamma_i \neq 0$ for $i \in \mathbb{Z}$.

Lemma 4.7.1. *We have*

$$\theta_i^{(D)} = \alpha \theta_{i-1}^{(D-2)} \qquad (1 \leq i \leq D-1). \tag{4.7.4}$$

Proof. Evaluate (4.7.4) using Definition 4.1.1 and (4.7.1). □

Lemma 4.7.2. *We have*

$$c_i^{(D)} = \frac{\alpha}{\alpha_i} c_i^{(D-2)} \qquad (1 \leq i \leq D-2); \tag{4.7.5}$$

$$b_i^{(D)} = \alpha \alpha_{i-1} b_{i-2}^{(D-2)} \qquad (2 \leq i \leq D-1). \tag{4.7.6}$$

Proof. To verify (4.7.5), evaluate each side using Lemma 4.1.8 and (4.7.1).

(4.7.6) is verified in a similar fashion. □

The next two results are technical lemmas that we will use later in this section.

Lemma 4.7.3. *For* $2 \leq i \leq D$*,*

$$\alpha \gamma_{i-1} b_{i-2}^{(D-2)} - \alpha \gamma_{i-3} c_{i-2}^{(D-2)} - \gamma_{i-2} b_{i-2}^{(D)} = -\gamma_{i-2} c_i^{(D)}. \tag{4.7.7}$$

Proof. By Lemma 4.7.2 and (4.7.3),

$$\alpha\gamma_{i-1}b_{i-2}^{(D-2)} = \gamma_{i-2}b_i^{(D)}, \tag{4.7.8}$$

$$\alpha\gamma_{i-3}c_{i-2}^{(D-2)} = \gamma_{i-2}c_{i-2}^{(D)}. \tag{4.7.9}$$

By Lemma 4.1.11,

$$\gamma_{i-2}b_i^{(D)} - \gamma_{i-2}c_{i-2}^{(D)} - \gamma_{i-2}b_{i-2}^{(D)} = -\gamma_{i-2}c_i^{(D)}. \tag{4.7.10}$$

Evaluate the left-hand side of (4.7.7) using (4.7.8)–(4.7.10). The result follows. □

Let $i \in \mathbb{Z}$. Recall that $v_i^{(D-2)} = 0$ unless $0 \leq i \leq D-1$.

Lemma 4.7.4. *For* $2 \leq i \leq D$,

$$(\gamma_{i-4}b_{i-2}^{(D)} - \alpha\gamma_{i-3}b_{i-4}^{(D-2)})v_{i-4}^{(D-2)} = 0. \tag{4.7.11}$$

Proof. In (4.7.11), the expression on the left has two factors. If $i = 2$ or $i = 3$, then the second factor is 0. If $i \geq 4$, then the first factor is 0 by Lemma 4.7.2 and (4.7.3). The result follows. □

We are going to show that for $0 \leq i \leq D$,

$$v_i^{(D)}(\alpha z) = \begin{cases} \gamma_i v_i^{(D-2)} - \gamma_{i-2}v_{i-2}^{(D-2)} & \text{if } 0 \leq i \leq D-2; \\ \gamma_{D-3}(c_{D-1}^{(D)})^{-1}v_{D-1}^{(D-2)} - \gamma_{D-3}v_{D-3}^{(D-2)} & \text{if } i = D-1; \\ \alpha\gamma_{D-3}(c_{D-1}^{(D)}c_D^{(D)})^{-1}zv_{D-1}^{(D-2)} - \gamma_{D-2}v_{D-2}^{(D-2)} & \text{if } i = D. \end{cases}$$

In the following results, we treat the above three cases in turn.

Lemma 4.7.5. *For* $0 \leq i \leq D-2$,

$$v_i^{(D)}(\alpha z) = \gamma_i v_i^{(D-2)}(z) - \gamma_{i-2}v_{i-2}^{(D-2)}(z). \tag{4.7.12}$$

Proof. We proceed by induction on i. First assume that $i = 0$. Then (4.7.12) holds because both sides are equal to 1.

Next assume that $i = 1$. Then (4.7.12) holds because both sides are equal to αz.

For the rest of this proof, assume $i \geq 2$. By Lemma 2.5.4,

$$v_i^{(D)}(\alpha z) = \frac{\alpha z v_{i-1}^{(D)}(\alpha z) - b_{i-2}^{(D)} v_{i-2}^{(D)}(\alpha z)}{c_i^{(D)}}. \tag{4.7.13}$$

Evaluate the right-hand side of (4.7.13) using induction and Definition 2.5.3. This yields

$$\begin{aligned} v_i^{(D)}(\alpha z) = {} & \frac{\alpha\gamma_{i-1} c_i^{(D-2)} v_i^{(D-2)} + \big(\alpha\gamma_{i-1} b_{i-2}^{(D-2)} - \alpha\gamma_{i-3} c_{i-2}^{(D-2)} - \gamma_{i-2} b_{i-2}^{(D)}\big) v_{i-2}^{(D-2)}}{c_i^{(D)}} \\ & + \frac{\big(\gamma_{i-4} b_{i-2}^{(D)} - \alpha\gamma_{i-3} b_{i-4}^{(D-2)}\big) v_{i-4}^{(D-2)}}{c_i^{(D)}}. \end{aligned} \tag{4.7.14}$$

By Lemma 4.7.2 and (4.7.3),

$$\alpha\gamma_{i-1} c_i^{(D-2)} = \gamma_i c_i^{(D)}. \tag{4.7.15}$$

Evaluate the right-hand side of (4.7.14) using Lemmas 4.7.3, 4.7.4 and (4.7.15). The result follows. □

Lemma 4.7.6. *We have*

$$v_{D-1}^{(D)}(\alpha z) = \gamma_{D-3} (c_{D-1}^{(D)})^{-1} v_{D-1}^{(D-2)} - \gamma_{D-3} v_{D-3}^{(D-2)}.$$

Proof. By Lemma 2.5.4,

$$v_{D-1}^{(D)}(\alpha z) = \frac{\alpha z v_{D-2}^{(D)}(\alpha z) - b_{D-3}^{(D)} v_{D-3}^{(D)}(\alpha z)}{c_{D-1}^{(D)}}. \tag{4.7.16}$$

Evaluate the right-hand side of (4.7.16) using Lemma 4.7.5. This yields

$$v_{D-1}^{(D)}(\alpha z) = \frac{\alpha z(\gamma_{D-2} v_{D-2}^{(D-2)} - \gamma_{D-4} v_{D-4}^{(D-2)}) - b_{D-3}^{(D)}(\gamma_{D-3} v_{D-3}^{(D-2)} - \gamma_{D-5} v_{D-5}^{(D-2)})}{c_{D-1}^{(D)}}. \tag{4.7.17}$$

Distribute αz and $b_{D-3}^{(D)}$ on the right-hand side of (4.7.17), then evaluate using Definition 2.5.3. Recall that $c_{D-1}^{(D-2)} = 1$. This yields

$$\begin{aligned} v_{D-1}^{(D)}(\alpha z) = {} & \frac{\alpha\gamma_{D-2} v_{D-1}^{(D-2)} + (\alpha\gamma_{D-2} b_{D-3}^{(D-2)} - \alpha\gamma_{D-4} c_{D-3}^{(D-2)} - \gamma_{D-3} b_{D-3}^{(D)}) v_{D-3}^{(D-2)}}{c_{D-1}^{(D)}} \\ & + \frac{\big(\gamma_{D-5} b_{D-3}^{(D)} - \alpha\gamma_{D-4} b_{D-5}^{(D-2)}\big) v_{D-5}^{(D-2)}}{c_{D-1}^{(D)}}. \end{aligned} \tag{4.7.18}$$

By (4.7.2), (4.7.3),

$$\alpha\gamma_{D-2} = \gamma_{D-3}. \tag{4.7.19}$$

Evaluate the right-hand side of (4.7.18) using Lemmas 4.7.3, 4.7.4 and (4.7.19). The result follows. □

Lemma 4.7.7. *We have*

$$v_D^{(D)}(\alpha z) = \alpha\gamma_{D-3}(c_{D-1}^{(D)} c_D^{(D)})^{-1} z v_{D-1}^{(D-2)} - \gamma_{D-2} v_{D-2}^{(D-2)}.$$

Proof. By Lemma 2.5.4,

$$v_D^{(D)}(\alpha z) = \frac{\alpha z v_{D-1}^{(D)}(\alpha z) - b_{D-2}^{(D)} v_{D-2}^{(D)}(\alpha z)}{c_D^{(D)}}. \tag{4.7.20}$$

Evaluate the right-hand side of (4.7.20) using Lemmas 4.7.5, 4.7.6. This yields

$$\begin{aligned} v_D^{(D)}(\alpha z) = {} & \frac{\alpha z\big(\gamma_{D-3}(c_{D-1}^{(D)})^{-1} v_{D-1}^{(D)} - \gamma_{D-3} v_{D-3}^{(D-2)}\big)}{c_D^{(D)}} \\ & - \frac{b_{D-2}^{(D)}(\gamma_{D-2} v_{D-2}^{(D-2)} - \gamma_{D-4} v_{D-4}^{(D-2)})}{c_D^{(D)}}. \end{aligned} \tag{4.7.21}$$

Distribute αz and $b_{D-2}^{(D)}$ on the right-hand side of (4.7.21), then evaluate using Definition 2.5.3. This yields

$$v_D^{(D)}(\alpha z) = \frac{\alpha\gamma_{D-3}(c_{D-1}^{(D)})^{-1} z v_{D-1}^{(D)} - (\alpha\gamma_{D-3}c_{D-2}^{(D-2)} + \gamma_{D-2}b_{D-2}^{(D)})v_{D-2}^{(D-2)}}{c_D^{(D)}} + \frac{(\gamma_{D-4}b_{D-2}^{(D)} - \alpha\gamma_{D-3}b_{D-4}^{(D-2)})v_{D-4}^{(D-2)}}{c_D^{(D)}} \tag{4.7.22}$$

By Lemma 4.7.2 and (4.7.3),

$$\alpha\gamma_{D-3}c_{D-2}^{(D-2)} = \gamma_{D-2}c_{D-2}^{(D)}. \tag{4.7.23}$$

By Lemma 4.1.11,

$$\gamma_{D-2}c_{D-2}^{(D)} + \gamma_{D-2}b_{D-2}^{(D)} = \gamma_{D-2}c_D^{(D)}. \tag{4.7.24}$$

Evaluate the right-hand side of (4.7.22) using Lemma 4.7.4 and (4.7.23), (4.7.24). The result follows. □

We summarize the previous results.

Proposition 4.7.8. *For $0 \le i \le D$,*

$$v_i^{(D)}(\alpha z) = \begin{cases} \gamma_i v_i^{(D-2)} - \gamma_{i-2}v_{i-2}^{(D-2)} & \text{if } 0 \le i \le D-2; \\ \gamma_{D-3}(c_{D-1}^{(D)})^{-1} v_{D-1}^{(D-2)} - \gamma_{D-3}v_{D-3}^{(D-2)} & \text{if } i = D-1; \\ \alpha\gamma_{D-3}(c_{D-1}^{(D)}c_D^{(D)})^{-1} z v_{D-1}^{(D-2)} - \gamma_{D-2}v_{D-2}^{(D-2)} & \text{if } i = D. \end{cases}$$

Proof. Follows from Lemmas 4.7.5–4.7.7. □

4.8 An algebra homomorphism $\mathcal{T}^{(D)} \to \mathcal{T}^{(D-2)}$

Throughout this section, the following notation is in force. Let $D \ge 2$ denote an integer. Let $q \in \mathbb{C}$ denote a nonzero scalar that is not a root of unity. Recall the generalized Terwilliger algebras $\mathcal{T}^{(D)}$ and $\mathcal{T}^{(D-2)}$ from Definitions 4.4.1, 4.5.1. Recall the scalars

$\{\gamma_i\}_{i\in\mathbb{Z}}$ from (4.7.3).

In this section, we display a surjective algebra homomorphism $\flat^{(D)} : \mathcal{T}^{(D)} \to \mathcal{T}^{(D-2)}$ that sends

$$
\begin{aligned}
x_i^{(D)} &\mapsto \gamma_i x_i^{(D-2)} - \gamma_{i-2} x_i^{(D-2)}, & \qquad x_i^{*(D)} &\mapsto \gamma_i x_i^{*(D-2)} - \gamma_{i-2} x_i^{*(D-2)},\\
e_i^{(D)} &\mapsto e_{i-1}^{(D-2)}, & \qquad e_i^{*(D)} &\mapsto e_{i-1}^{*(D-2)}
\end{aligned}
$$

for $0 \le i \le D$. We are using the notational convention above Remark 4.4.2.

For notational convenience, define

$$
\begin{aligned}
e_i^{\flat} &= e_{i-1}^{(D-2)}, & \qquad e_i^{*\flat} &= e_{i-1}^{*(D-2)} & \qquad &(0 \le i \le D), & \qquad (4.8.1)\\
x^{\flat} &= \sum_{i=0}^{D} \theta_i^{(D)} e_i^{\flat}, & \qquad x^{*\flat} &= \sum_{i=0}^{D} \theta_i^{(D)} e_i^{*\flat}, & & & \qquad (4.8.2)\\
x_i^{\flat} &= v_i^{(D)}(x^{\flat}), & \qquad x_i^{*\flat} &= v_i^{(D)}(x^{*\flat}) & \qquad &(0 \le i \le D). & \qquad (4.8.3)
\end{aligned}
$$

Recall the scalar $\alpha = \alpha_1$ from (4.7.1).

Lemma 4.8.1. *The following (i), (ii) hold:*

(i) $x^{\flat} = \alpha x^{(D-2)}$*;*

(ii) $x^{*\flat} = \alpha x^{*(D-2)}$.

Proof. (i) Evaluate the right-hand side of (4.8.2) using Lemma 4.7.1 and (4.8.1), noting that $e_{-1}^{(D-2)} = 0$ and $e_{D-1}^{(D-2)} = 0$. This yields

$$x^{\flat} = \sum_{i=1}^{D-1} \alpha \theta_{i-1}^{(D-2)} e_{i-1}^{(D-2)}. \qquad (4.8.4)$$

Re-index the sum on the right-hand side of (4.8.4), then evaluate using Lemma 4.5.3. The result follows.

(ii) Similar to the proof of (i). $\square$

Lemma 4.8.2. *The following (i), (ii) hold for* $0 \le i \le D$*:*

(i) $x_i^\flat = \gamma_i x_i^{(D-2)} - \gamma_{i-2} x_i^{(D-2)}$;

(ii) $x_i^{*\flat} = \gamma_i x_i^{*(D-2)} - \gamma_{i-2} x_i^{*(D-2)}$.

Proof. (i) Follows from Lemmas 4.5.4, 4.8.1, Proposition 4.7.8, and (4.8.3). It is convenient to treat the cases $i = D-1$ and $i = D$ separately, noting that $v_{D-1}^{(D-2)}(x^{(D-2)}) = 0$ by Lemma 4.5.9.

(ii) Similar to the proof of (i). □

Lemma 4.8.3. *The following (i), (ii) hold:*

(i) $\sum_{i=0}^{D} e_i^\flat = 1$;

(ii) $\sum_{i=0}^{D} e_i^{*\flat} = 1$.

Proof. (i) By Lemma 4.4.6 and (4.8.1),

$$\sum_{i=0}^{D} e_i^\flat = \sum_{i=1}^{D-1} e_{i-1}^{(D-2)} = \sum_{i=0}^{D-2} e_i^{(D-2)} = 1.$$

(ii) Similar to the proof of (i). □

Lemma 4.8.4. *(Nicholson Hypercube Paper) For $0 \le h, i, j \le D$ such that $(h, i, j) \notin P^{(D)}$, the following (i), (ii) hold:*

(i) $(h-1, i, j-1) \notin P^{(D-2)}$;

(ii) $(h-1, i-2, j-1) \notin P^{(D-2)}$.

Lemma 4.8.5. *The following (i), (ii) hold for $0 \le h, i, j \le D$:*

(i) $e_h^{*\flat} x_i^\flat e_j^{*\flat} = 0$ *if* $(h, i, j) \notin P^{(D)}$;

(ii) $e_h^\flat x_i^{*\flat} e_j^\flat = 0$ *if* $(h, i, j) \notin P^{(D)}$.

Proof. (i) By (4.8.1) and Lemma 4.8.2,

$$e_h^{*\flat} x_i^\flat e_j^{*\flat} = \gamma_i e_{h-1}^{*(D-2)} x_i^{(D-2)} e_{j-1}^{*(D-2)} - \gamma_{i-2} e_{h-1}^{*(D-2)} x_{i-2}^{(D-2)} e_{j-1}^{*(D-2)}. \qquad (4.8.5)$$

By Lemmas 4.5.11, 4.8.4, the terms on the right-hand side of (4.8.5) are both equal to 0. The result follows.

(ii) Similar to the proof of (i). □

Proposition 4.8.6. *There exists a surjective algebra homomorphism* $\flat^{(D)} : \mathcal{T}^{(D)} \to \mathcal{T}^{(D-2)}$ *that sends*

$$\begin{aligned} x_i^{(D)} &\mapsto \gamma_i x_i^{(D-2)} - \gamma_{i-2} x_i^{(D-2)}, & x_i^{*(D)} &\mapsto \gamma_i x_i^{*(D-2)} - \gamma_{i-2} x_i^{*(D-2)}, \\ e_i^{(D)} &\mapsto e_{i-1}^{(D-2)}, & e_i^{*(D)} &\mapsto e_{i-1}^{*(D-2)} \end{aligned}$$

for $0 \le i \le D$.

Proof. Consider the presentation for $\mathcal{T}^{(D)}$ in Definition 4.6.7. By (4.8.1)–(4.8.3) and Lemmas 4.8.2, 4.8.3, 4.8.5, the algebra homomorphism $\flat^{(D)}$ exists.

By the construction of $\flat^{(D)}$, the elements $\{e_i^{(D-2)}\}_{i=0}^{D-2}$, $\{e_i^{*(D-2)}\}_{i=0}^{D-2}$ are contained in the image of $\mathcal{T}^{(D)}$ under $\flat^{(D)}$. By Definition 4.6.7, $\{e_i^{(D-2)}\}_{i=0}^{D-2}$, $\{e_i^{*(D-2)}\}_{i=0}^{D-2}$ generate $\mathcal{T}^{(D-2)}$. Therefore $\flat^{(D)}$ is surjective. □

We emphasize some aspects of Proposition 4.8.6.

Corollary 4.8.7. *The algebra homomorphism* $\flat^{(D)} : \mathcal{T}^{(D)} \to \mathcal{T}^{(D-2)}$ *sends*

$$e_0^{(D)} \mapsto 0, \qquad e_0^{*(D)} \mapsto 0, \qquad e_D^{(D)} \mapsto 0, \qquad e_D^{*(D)} \mapsto 0.$$

Proof. Follows from Proposition 4.8.6. □

4.9 The kernel of $\flat^{(D)}$

Throughout this section, the following notation is in force. Let $D \ge 2$ denote an integer. Let $q \in \mathbb{C}$ denote a nonzero scalar that is not a root of unity. Recall the surjective algebra homomorphism $\flat^{(D)} : \mathcal{T}^{(D)} \to \mathcal{T}^{(D-2)}$ from Proposition 4.8.6.

In this section, we describe the kernel of $\flat^{(D)}$.

Definition 4.9.1. Let $\mathcal{K}^{(D)}$ denote the two-sided ideal of $\mathcal{T}^{(D)}$ generated by $e_0^{(D)}$, $e_0^{*(D)}$, $e_D^{(D)}$, $e_D^{*(D)}$. In other words,

$$\mathcal{K}^{(D)} = \mathcal{T}^{(D)} e_0^{(D)} \mathcal{T}^{(D)} + \mathcal{T}^{(D)} e_0^{*(D)} \mathcal{T}^{(D)} + \mathcal{T}^{(D)} e_D^{(D)} \mathcal{T}^{(D)} + \mathcal{T}^{(D)} e_D^{*(D)} \mathcal{T}^{(D)}.$$

Shortly we will show that $\mathcal{K}^{(D)}$ is the kernel of the algebra homomorphism $\flat^{(D)}$.

Consider the quotient algebras $\mathcal{T}^{(D)}/\mathcal{K}^{(D)}$ and $\mathcal{T}^{(D)}/\ker(\flat^{(D)})$. By Corollary 4.8.7 and Definition 4.9.1, $\mathcal{K}^{(D)} \subseteq \ker(\flat^{(D)})$. Hence there exists a surjective algebra homomorphism $\iota : \mathcal{T}^{(D)}/\mathcal{K}^{(D)} \to \mathcal{T}^{(D)}/\ker(\flat^{(D)})$ that sends

$$y + \mathcal{K}^{(D)} \mapsto y + \ker(\flat^{(D)}) \qquad\qquad (y \in \mathcal{T}^{(D)}).$$

By Proposition 4.8.6, there exists an algebra isomorphism $can : \mathcal{T}^{(D)}/\ker(\flat^{(D)}) \to \mathcal{T}^{(D-2)}$ that sends

$$y + \ker(\flat^{(D)}) \mapsto \flat^{(D)}(y) \qquad\qquad (y \in \mathcal{T}^{(D)}).$$

Composing the previous two maps, we get a surjective algebra homomorphism

$$\overline{\flat^{(D)}} : \mathcal{T}^{(D)}/\mathcal{K}^{(D)} \xrightarrow{\iota} \mathcal{T}^{(D)}/\ker(\flat^{(D)}) \xrightarrow{can} \mathcal{T}^{(D-2)}. \tag{4.9.1}$$

We will show that $\overline{\flat^{(D)}}$ is an isomorphism. To do this, we will display the inverse of $\overline{\flat^{(D)}}$.

We will define an algebra homomorphism $\sharp^{(D-2)} : \mathcal{T}^{(D-2)} \to \mathcal{T}^{(D)}/\mathcal{K}^{(D)}$ and show that $\sharp^{(D-2)}$, $\overline{\flat^{(D)}}$ are inverses.

We bring in some notation. Going forward, for every $y \in \mathcal{T}^{(D)}$, let $\overline{y}$ denote the element $y + \mathcal{K}^{(D)} \in \mathcal{T}^{(D)}/\mathcal{K}^{(D)}$. We define some elements in $\mathcal{T}^{(D)}/\mathcal{K}^{(D)}$:

$$e_i^{\sharp} = \overline{e_{i+1}^{(D)}}, \qquad e_i^{*\sharp} = \overline{e_{i+1}^{*(D)}} \qquad (0 \le i \le D-2), \tag{4.9.2}$$

$$x^{\sharp} = \sum_{r=0}^{D-2} \theta_r^{(D-2)} e_r^{\sharp}, \qquad x^{*\sharp} = \sum_{r=0}^{D-2} \theta_r^{(D-2)} e_r^{*\sharp}, \tag{4.9.3}$$

$$x_i^{\sharp} = v_i^{(D-2)}(x^{\sharp}), \qquad x_i^{*\sharp} = v_i^{(D-2)}(x^{*\sharp}) \qquad (0 \le i \le D-2). \tag{4.9.4}$$

Let $i \in \mathbb{Z}$. For notational convenience, define $x^\sharp_i = 0$, $e^\sharp_i = 0$, $x^{*\sharp}_i = 0$, $e^{*\sharp}_i = 0$ unless $0 \le i \le D-2$.

Lemma 4.9.2. *We have*

$$\sum_{r=0}^{\lfloor D/2 \rfloor} \overline{x^{(D)}_{2r}} = 0, \qquad \sum_{r=0}^{\lfloor D/2 \rfloor} \overline{x^{*(D)}_{2r}} = 0, \tag{4.9.5}$$

$$\sum_{r=0}^{\lfloor (D-1)/2 \rfloor} \overline{x^{(D)}_{2r+1}} = 0, \qquad \sum_{r=0}^{\lfloor (D-1)/2 \rfloor} \overline{x^{*(D)}_{2r+1}} = 0. \tag{4.9.6}$$

Proof. Let R, R^* denote the respective sums in (4.9.5). Let S, S^* denote the respective sums in (4.9.6). By Lemmas 2.7.8, 4.2.6, 4.2.8 and Definition 4.4.1,

$$R + S = \overline{e^{(D)}_0}, \qquad R - S = \overline{e^{(D)}_D}.$$

By Definition 4.9.1, $\overline{e^{(D)}_0} = 0$ and $\overline{e^{(D)}_D} = 0$. By these comments, $R = 0$ and $S = 0$.

Similar comments show that $R^* = 0$ and $S^* = 0$. □

Recall the scalar $\alpha = \alpha_1$ from (4.7.1).

Lemma 4.9.3. *The following (i), (ii) hold in $\mathcal{T}^{(D)}/\mathcal{K}^{(D)}$:*

(i) $x^\sharp = \alpha^{-1}\overline{x^{(D)}}$;

(ii) $x^{*\sharp} = \alpha^{-1}\overline{x^{*(D)}}$.

Proof. (i) Evaluate the right-hand side of (4.9.3) using Lemma 4.7.1 and (4.9.2). This yields

$$x^\sharp = \sum_{i=0}^{D-2} \alpha^{-1}\theta^{(D)}_{i+1}\overline{e^{(D)}_{i+1}} = \alpha^{-1}\sum_{i=1}^{D-1}\theta^{(D)}_i \overline{e^{(D)}_i}.$$

By Definition 4.9.1, $\overline{e^{(D)}_0} = 0$ and $\overline{e^{(D)}_D} = 0$. By these comments,

$$x^\sharp = \alpha^{-1}\sum_{i=0}^{D}\theta^{(D)}_i \overline{e^{(D)}_i}. \tag{4.9.7}$$

The result follows from Lemma 4.5.3.

(ii) Similar to the proof of (i). □

Lemma 4.9.4. *The following (i), (ii) hold in* $\mathcal{T}^{(D)}/\mathcal{K}^{(D)}$*:*

(i) $v^{(D-2)}_{D-1}(x^\sharp) = 0$*;*

(ii) $v^{(D-2)}_{D-1}(x^{*\sharp}) = 0$.

Proof. (i) By Lemma 4.4.7 and (4.9.2),

$$e^\sharp_i e^\sharp_j = \delta_{ij} e^\sharp_i \qquad (0 \le i, j \le D). \tag{4.9.8}$$

By Definition 2.5.7 and (4.9.3), (4.9.8),

$$\mu^{(D-2)}(x^\sharp) = \sum_{i=0}^{D-2} \mu^{(D-2)}(\theta^{(D-2)}_i) e^\sharp_i = 0.$$

The result follows from Lemma 2.5.8.

(ii) Similar to the proof of (i). □

Recall the scalars $\{\gamma_i\}_{i\in\mathbb{Z}}$ from (4.7.3).

Lemma 4.9.5. *The following (i), (ii) hold in* $\mathcal{T}^{(D)}/\mathcal{K}^{(D)}$ *for* $0 \le i \le D$*:*

(i) $\overline{x^{(D)}_i} = \gamma_i x^\sharp_i - \gamma_{i-2} x^\sharp_{i-2}$*;*

(ii) $\overline{x^{*(D)}_i} = \gamma_i x^{*\sharp}_i - \gamma_{i-2} x^{*\sharp}_{i-2}$.

Proof. (i) Follows from Lemmas 4.5.4, 4.9.3, Proposition 4.7.8, and (4.9.4). It is convenient to treat the cases $i = D-1$ and $i = D$ separately, noting Lemma 4.9.4.

(ii) Similar to the proof of (i). □

Lemma 4.9.6. *For* $0 \le i \le D$*,*

$$\gamma_i^{-1} \sum_{r=0}^{\lfloor i/2 \rfloor} \overline{x^{(D)}_{i-2r}} = x^\sharp_i = -\gamma_i^{-1} \sum_{r=1}^{\lfloor (D-i)/2 \rfloor} \overline{x^{(D)}_{i+2r}},$$
$$\gamma_i^{-1} \sum_{r=0}^{\lfloor i/2 \rfloor} \overline{x^{*(D)}_{i-2r}} = x^{*\sharp}_i = -\gamma_i^{-1} \sum_{r=1}^{\lfloor (D-i)/2 \rfloor} \overline{x^{*(D)}_{i+2r}}.$$

Proof. For each of the above sums, evaluate each summand using Lemma 4.9.5. □

Lemma 4.9.7. *The following (i), (ii) hold in* $\mathcal{T}^{(D)}/\mathcal{K}^{(D)}$*:*

(i) $\sum_{i=0}^{D-2} e_i^{\sharp} = 1$*;*

(ii) $\sum_{i=0}^{D-2} e_i^{*\sharp} = 1$*.*

Proof. (i) By Definition 4.9.1, $\overline{e_0^{(D)}} = 0$, $\overline{e_D^{(D)}} = 0$ in $\mathcal{T}^{(D)}/\mathcal{K}^{(D)}$. Hence by Lemma 4.4.6 and (4.9.2),

$$\sum_{i=0}^{D-2} e_i^{\sharp} = \sum_{i=0}^{D-2} \overline{e_{i+1}^{(D)}} = \sum_{i=1}^{D-1} \overline{e_i^{(D)}} = \sum_{i=0}^{D} \overline{e_i^{(D)}} = 1.$$

(ii) Similar to the proof of (i). □

Lemma 4.9.8. *(Nicholson Hypercube Paper) For* $0 \leq h, i, j \leq D-2$ *such that* $(h, i, j) \notin P_{D-2}$*, then either*

$$(h+1, i-2r, j+1) \notin P_D \qquad (0 \leq r \leq \lfloor i/2 \rfloor),$$

or

$$(h+1, i+2r, j+1) \notin P_D \qquad (1 \leq r \leq \lfloor (D-i)/2 \rfloor).$$

Lemma 4.9.9. *The following (i), (ii) hold in* $\mathcal{T}^{(D)}/\mathcal{K}^{(D)}$ *for* $0 \leq h, i, j \leq D-2$*:*

(i) $e_h^{*\sharp} x_i^{\sharp} e_j^{*\sharp} = 0$ *if* $(h, i, j) \notin P_{D-2}$*;*

(ii) $e_h^{\sharp} x_i^{*\sharp} e_j^{\sharp} = 0$ *if* $(h, i, j) \notin P_{D-2}$*.*

Proof. (i) By (4.9.2) and Lemma 4.9.6,

$$\gamma_i^{-1} \sum_{r=0}^{\lfloor i/2 \rfloor} \overline{e_{h+1}^{*(D)} x_{i-2r}^{(D)} e_{j+1}^{*(D)}} = e_h^{*\sharp} x_i^{\sharp} e_j^{*\sharp} = -\gamma_i^{-1} \sum_{r=1}^{\lfloor (D-i)/2 \rfloor} \overline{e_{h+1}^{*(D)} x_{i+2r}^{(D)} e_{j+1}^{*(D)}}. \tag{4.9.9}$$

By Lemmas 4.5.11, 4.9.8, either all of the summands of the left-most sum or all of the summands of the right-most sum in (4.9.9) are equal to 0. The result follows.

(ii) Similar to (i). □

Proposition 4.9.10. *There exists an algebra homomorphism* $\sharp^{(D-2)} : \mathcal{T}^{(D-2)} \to \mathcal{T}^{(D)}/\mathcal{K}^{(D)}$ *that sends*

$$x_i^{(D-2)} \mapsto \gamma_i^{-1} \sum_{r=0}^{\lfloor i/2 \rfloor} \overline{x_{i-2r}^{(D)}}, \qquad x_i^{*(D-2)} \mapsto \gamma_i^{-1} \sum_{r=0}^{\lfloor i/2 \rfloor} \overline{x_{i-2r}^{*(D)}},$$

$$e_i^{(D-2)} \mapsto \overline{e_{i+1}^{(D)}}, \qquad e_i^{*(D-2)} \mapsto \overline{e_{i+1}^{*(D)}},$$

for $0 \leq i \leq D-2$. *Moreover,* $\sharp^{(D-2)}$ *and* $\overline{\flat^{(D)}}$ *are inverses.*

Proof. Consider the presentation for $\mathcal{T}^{(D-2)}$ in Definition 4.6.7. By (4.9.2)–(4.9.4) and Lemmas 4.9.6, 4.9.7, 4.9.9, the algebra homomorphism $\sharp^{(D-2)}$ exists. It follows from Proposition 4.8.6 and (4.9.1) that $\sharp^{(D-2)} = (\overline{\flat^{(D)}})^{-1}$. □

We mention some consequences of Proposition 4.9.10

Corollary 4.9.11. *The map* $\overline{\flat^{(D)}} : \mathcal{T}^{(D)}/\mathcal{K}^{(D)} \to \mathcal{T}^{(D-2)}$ *is an algebra isomorphism.*

Proof. Follows from Proposition 4.9.10. □

Corollary 4.9.12. *The map* $\iota : \mathcal{T}^{(D)}/\mathcal{K}^{(D)} \to \mathcal{T}^{(D)}/\ker(\flat^{(D)})$ *is an algebra isomorphism.*

Proof. Follows from (4.9.1), Corollary 4.9.11, and the fact that *can* : $\mathcal{T}^{(D)}/\ker(\flat^{(D)}) \to \mathcal{T}^{(D-2)}$ is an algebra isomorphism. □

Corollary 4.9.13. *We have* $\ker(\flat^{(D)}) = \mathcal{K}^{(D)}$.

Proof. Follows from Corollary 4.9.12. □

Recall the primary central idempotent $u^{(D)} \in \mathcal{T}^{(D)}$ from Definition 4.4.10. We make a comment about the two-sided ideal $\mathcal{T}^{(D)}u^{(D)}$ of $\mathcal{T}^{(D)}$.

Proposition 4.9.14. *We have* $\ker(\flat^{(D)}) = \mathcal{T}^{(D)}u^{(D)}$.

Proof. By Corollary 4.9.13, it is sufficient to prove that $\mathcal{K}^{(D)} = \mathcal{T}^{(D)}u^{(D)}$. By Definitions 4.4.10, 4.9.1, $\mathcal{T}^{(D)}u^{(D)} \subseteq \mathcal{K}^{(D)}$. By Lemmas 4.4.12, 4.5.12 and Definition 4.9.1, $\mathcal{K}^{(D)} \subseteq \mathcal{T}^{(D)}u^{(D)}$. The result follows. □

4.10 The algebra $\mathcal{T}^{(D)}$

Throughout this section, the following notation is in force. Let D denote a nonnegative integer. Let $q \in \mathbb{C}$ denote a nonzero scalar that is not a root of unity. Recall the generalized Terwilliger algebra $\mathcal{T}^{(D)}$ from Definitions 4.4.1, 4.5.1.

In this section, we prove Theorem 3 from the introduction. We treat the cases $D = 0$, $D = 1$, $D \geq 2$ separately.

The case $D = 0$ is addressed in Remark 4.4.2.

Next assume $D = 1$. Recall the primary central idempotent $u^{(1)} \in \mathcal{T}^{(1)}$ from Definition 4.4.10.

Lemma 4.10.1. *In $\mathcal{T}^{(1)}$, the element $u^{(1)} = 1$.*

Proof. By Lemmas 4.4.6, 4.4.12, 4.5.12,

$$u^{(1)} = u^{(1)}(e_0^{(1)} + e_1^{(1)}) = e_0^{(1)} + e_1^{(1)} = 1.$$

□

Proposition 4.10.2. *There exists an algebra isomorphism $\mathcal{T}^{(1)} \to \mathrm{Mat}_2(\mathbb{C})$.*

Proof. Follows from Corollary 4.4.14 and Lemma 4.10.1. □

Lemma 4.10.3. *Assume $D \geq 2$. Then there exists an algebra isomorphism*

$$\mathcal{T}^{(D)} \to \mathrm{Mat}_{D+1}(\mathbb{C}) \oplus \mathcal{T}^{(D-2)}.$$

Proof. Follows from Corollaries 4.4.14, 4.4.15 and Propositions 4.8.6, 4.9.14. □

We now prove Theorem 3 from the introduction.

Theorem 4.10.4. *For $D \geq 0$, there exists an algebra isomorphism*

$$\mathcal{T}^{(D)} \to \sum_{r=0}^{\lfloor (D+1)/2 \rfloor} \mathrm{Mat}_{D+1-2r}(\mathbb{C}).$$

Proof. Follows from Lemma 4.10.3 and induction on D. The base cases $D = 0$, $D = 1$ are settled in Remark 4.4.2 and Proposition 4.10.2. □

Corollary 4.10.5. *For $D \geq 0$,*

$$\dim(\mathcal{T}^{(D)}) = \sum_{r=0}^{\lfloor (D+1)/2 \rfloor} (D + 1 - 2r)^2.$$

Proof. Follows from Theorem 4.10.4. □

4.11 The generalized Terwilliger algebra associated with a 2-homogeneous bipartite distance-regular graph

In this section, we prove Theorem 2 from the introduction.

Throughout this section, the following notation is in force. Let Γ denote a 2-homogeneous bipartite distance-regular graph with diameter $D \geq 3$ and valency $k \geq 3$. Assume Γ is not a hypercube. Fix a vertex x of Γ. Recall the Terwilliger algebra $T = T(x)$ of Γ with respect to x. Recall the generalized Terwilliger algebra $\mathcal{T}$ associated with Γ from Definition 3.1.1. Let $q \in \mathbb{C}$ denote a nonzero scalar that is not a root of unity and that satisfies (ii), (iii) of Proposition 2.3.3. Recall the character system $\Psi^{(D)}$ from the first paragraph of Section 4.2. Recall the generalized Terwilliger algebra $\mathcal{T}^{(D)}$ from Definitions 4.4.1, 4.5.1.

Proposition 4.11.1. *With reference to Example 2.7.11, the following (i), (ii) hold:*

- *(i) the character system $(M; \{A_i\}_{i=0}^{D}; \{E_i\}_{i=0}^{D})$ is isomorphic to the character system $\Psi^{(D)}$;*
- *(ii) the character system $(M^*; \{A_i^*\}_{i=0}^{D}; \{E_i^*\}_{i=0}^{D})$ is isomorphic to the character system $\Psi^{*(D)}$.*

Proof. (i) Follows from Proposition 2.3.3, Lemmas 2.9.9, 4.1.8, and Definition 4.1.1.

(ii) Follows from (i) and Example 2.8.8. □

Proposition 4.11.2. *The generalized Terwilliger algebra $\mathcal{T}$ associated with Γ is algebra isomorphic to $\mathcal{T}^{(D)}$.*

Proof. Follows from Definition 4.5.1 and Proposition 4.11.1. $\square$

Corollary 4.11.3. *The generalized Terwilliger algebra $\mathcal{T}$ associated with Γ has dimension*

$$\dim(\mathcal{T}) = \sum_{r=0}^{\lfloor D/2 \rfloor} (D+1-2r)^2.$$

Proof. Follows from Corollary 4.10.5 and Proposition 4.11.2. $\square$

We now prove Theorem 2 from the introduction.

Theorem 4.11.4. *All 2-homogeneous bipartite distance-regular graphs with diameter at least 3 and valency at least 3 that are not hypercubes are T-determined by the triple product relations.*

Proof. By Lemma 3.1.4, the algebra homomorphism $\natural : \mathcal{T} \to T$ is surjective. By Corollaries 4.10.5, 4.11.3, $\dim(\mathcal{T}) = \dim(T)$. Therefore $\natural$ is an isomorphism. The result follows from Definition 3.1.7. $\square$

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