Kudla-Rapoport conjecture for Krämer models

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Abstract

The Siegel-Weil formula is a beautiful identity that relates weighted sum of theta series with Eisenstein series. This classical formula has a geometric version relating geometric theta series with Eisenstein series, where the geometric theta series is certain generating series of special cycles on unitary/orthogonal Shimura varieties. Much deeper is the arithmetic Siegel-Weil formula, which is an analogous story in the integral model of unitary/orthogonal Shimura varieties.

In this thesis, we study a local arithmetic Siegel-Weil formula: the Kudla-Rapoport conjecture for the Krämer models of unitary Rapoport–Zink spaces at ramified places. We conjectured and proved a precise identity between the arithmetic intersection numbers of special cycles on Krämer models and modified derived local densities of hermitian forms. Combining our formula with the analogous formula over unramified primes, we can prove an arithmetic Siegel–Weil formula for unitary Shimura varieties, which relates intersection of special cycles on unitary Shimura vartieties over imaginary quadratic fields with derivative of Eisenstein series. This thesis is based on several joint works with Chao Li, Yousheng Shi and Tonghai Yang.

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Contents

Chapter 1

Introduction

1.0.1 Background

Consider the quadratic form given by $Q(x) = x_1^2 + \cdots + x_m^2$, and $r_m(n) = \#\{(x_1, \cdots, x_m) \in \mathbb{Z}^m \mid Q(x_1, \cdots, x_m) = n\}$. Lagrange's famous four-square theorem claims that $r_4(n) \neq 0$ for any $n \in \mathbb{N}^+$. Then a natural question to ask is that can we find a formula for $r_4(n)$? It turns out that in order to achieve this, it is very helpful to consider the generating series $\theta_4(\tau) := \sum_{i=0}^{\infty} r_4(n) q^n$ instead of $r_4(n)$ alone. One can show that $\theta_4(\tau)$ is a modular form of weight 2 and level 4. In fact, the space of modular form of weight 2 and level 4 is generated by two explicitly constructed Eisenstein series. A comparison of Fourier coefficients shows that $\theta_2(\tau)$ is a scalar multiple of one of the Eisenstein series. More specifically,

$$\theta_4(\tau) = 1 + 8 \sum_{n=1}^{\infty} \Big(\sum_{\substack{0<d|n \\ 4\nmid d}} d \Big) q^n.$$

As a result, we may recover Jacobi's beautiful formula:

$$r_4(n) = 8 \Big(\sum_{\substack{0<d|n \\ 4\nmid d}} d \Big).$$

In general, an explicit formula for $r_m(n)$ is not known for $m > 8$. However, Siegel discovered an average formula (over the genus class of $Q_m(x)$) in 1930s ([Sie35]) and extended it to the indefinite integral quadratic forms later ([Sie51]). More explicitly, we have the following formula:

$$\sum_{[[Q]]} \theta_Q(\tau) = \text{special value of Siegel Eisenstein series,} \tag{1.1}$$

where the summation is over the genus class of a positive definite integral quadratic form Q, and θ_Q is the theta series associated with Q.

Weil interpreted Siegel's work in terms of Weil representation and extended it to all quadratic and hermitian forms (within the convergence range) in 1960s ([Wei65]) as follows. To simplify the exposition, we mainly focus on the unitary case. Let F be a CM number field with maximal total real subfield F_0 and (V, h) be a non-degenerate hermitian space over F of dimension n with Witt index r. Let $H = \mathrm{U}(V)$ and $G = \mathrm{U}(m, m)$ so that (G, H) is a reductive dual pair. Fix a character χ of $\mathbb{A}_F^\times / F^\times$ such that $\chi|_{\mathbb{A}^\times} = \varepsilon_{F/F_0}^n$ where ε_{F/F_0} is the quadratic character of $\mathbb{A}^\times / F_0^\times$ associated to F/F_0 via class field theory. Then we have an action of $H(\mathbb{A}) \times G(\mathbb{A})$ on the space of Schwartz function $\mathscr{S}(V^m(\mathbb{A}))$ via the Weil representation ω.

Now we define Siegel Eisenstein series. We use P to denote the Siegel parabolic subgroup of G and let K be the standard maximal compact subgroup of $G(\mathbb{A})$. For $\varphi \in \mathscr{S}(V^m(\mathbb{A}))$, define a holomorphic section Φ_s of $\mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} \chi |\det|_{\mathbb{A}_F}^s$ by

$$\Phi_s(g) = |a(g)|_{\mathbb{A}_F}^{s-s_0} \omega(g)\varphi(0).$$

Here $s_0 = (n-m)/2$ and $|a(g)|_{\mathbb{A}_F} = |\det(a)|_{\mathbb{A}_F}$ for some $a \in \mathrm{GL}_m(\mathbb{A}_{F_0})$ so that $g = \begin{pmatrix} a & * \\ 0 & {}^t\bar{a}^{-1} \end{pmatrix} \cdot k$ for some $k \in K$. Then the Siegel Eisenstein series

$$E(g, s, \varphi) := \sum_{\gamma \in P(Q) \backslash G(Q)} \Phi_s(\gamma g)$$

is absolutely convergent if $\mathrm{Re}(s) > m/2$, and has a meromorphic continuation to the whole s-plane if φ is K-finite.

On the other hand, we define the theta function

$$\Theta(g, h; \varphi) = \sum_{x \in V^n(F)} \omega(g, h)\varphi(x)$$

for each φ, and consider its integral

$$I(g, \varphi) = \int_{H(F)\backslash H(\mathbb{A})} \Theta(g, h; \varphi) dh$$

if it is absolutely convergent. Here we normalize the Haar measure dh on $H(\mathbb{A})$ so that $\mathrm{vol}(H(F)\backslash H(\mathbb{A})) = 1$. In [Wei65], Weil showed that the theta integral $I(g, \varphi)$ is absolutely convergent for all φ if and only if $n - r > m$ or (V, h) is anisotropic (the so called convergent range). Moreover he proved that if $n > 2m$, then

$$I(g, \varphi) = E(g, s_0, \varphi).$$

In particular, both sides of this identity are absolutely convergent in this case. In general, we need to regularize the theta integral and identity it with certain residue of Eisenstein series, which is called regularized Siegel-Weil formula. See [KR88b, KR88a, KR94, Ike96, Ich04, GQT14] for more details.

There is also a geometric analogue of the above story. For simplicity, we again focus on the unitary case.

Fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and fix a CM type $\Phi \subset \mathrm{Hom}(F, \overline{\mathbb{Q}}) = \mathrm{Hom}(F, \mathbb{C})$ of F. We also identify the CM type Φ with the set of archimedean places of F_0. Let V be an F/F_0-hermitian space of dimension n. Assume the signatures of V are $\{(n-1, 1)_{\phi_0}, (n, 0)_{\phi \in \Phi - \{\phi_0\}}\}$ for some distinguished element $\phi_0 \in \Phi$. Then there is a unitary Shimura variety X of dimension $n - 1$ associated with V. There is a special cycle $Z(T)$ of codimension r on X associated with a $r \times r$ positive definite hermitian matrix $T \in$

$\mathrm{Herm}_n(F)$. This definition can be extended to all $r\times r$ semi-definite T to form a geometric analogue of the classical theta series valued in $\mathrm{CH}^r(X)$:

$$\theta_r^{geo}(\tau) := \sum_{T\geq 0} Z(T)q^T,\ q^T = e^{2\pi i \mathrm{tr}(T\tau)},$$

where τ lies in the hermitian upper half plane $\mathbb{H}_r$. Here $\mathbb{H}_r$ is the domain consisting of all complex $r\times r$ matrices τ such that $i({}^t\bar{\tau}-\tau)$ is positive definite.

Kudla conjectured that the orthogonal analogue of $\theta_r^{geo}(\tau)$ is a Siegel modular form valued in $\mathrm{CH}^r(X)$ in [Kud97, Kud04], which was proved due to various authors' work (see [Bor99],[Zha09] and [BWR15]). Yifeng Liu formulated the unitary case in [Liu11], where he also proved the case $r=1$ and the formal modularity. For more detail, see the beautiful survey article of Chao Li [Li23].

When X is compact and $r=n-1$, $Z(T)\in \mathrm{CH}^{n-1}(X)$ and we can consider $\deg(Z(T))$ in this case. Taking the degree of θ_{n-1}^{geo}, we may propose the following geometric Siegel-Weil formula:

$$\deg(\theta_{n-1}^{geo}) = \text{special value of unitary Eisenstein series.} \tag{1.2}$$

The orthogonal analogue of this formula was proved in [Kud97] and the unitary case should follow from a similar argument. In fact, taking degree of $Z(T)$ factors through the cycle class map. Then Kudla-Millson theory [KM90] gives an explicit construction of a representative of $\mathrm{cl}(Z(T))$ where cl denotes the cycle class map. When X is non-compact, we expect similar results as above should still hold although some modification is needed. It is still an open question to find the correct modification in general.

This is not the end of the story. Much deeper is its arithmetic version. Assume X has an integral model $\mathcal{X}$ over O_E where E is the reflex field. In reality, we often need to consider its toroidal compactification. However, the main results of this thesis do not involve the boundary components so we ignore this issue here. For $\tau\in\mathbb{H}_r$, we can write it as $\tau = \mathsf{x}+i\mathsf{y}$ where y is a $r\times r$ positive definite hermitian matrix. For such y, we can define a Kudla's Green current for $Z(T)$ depending on y. Given the Green current, we seek to define $\widehat{\mathcal{Z}}(T,\mathsf{y}) \in \widehat{\mathrm{CH}}^r(\mathcal{X})$ whose restriction on generic fiber recovers $Z(T)$. Then we may consider an arithmetic analogue of $\theta^{geo}(\tau)$:

$$\widehat{\theta}^{ar}_r(\tau,\mathsf{y}) := \sum_{T\geq 0} \widehat{\mathcal{Z}}(T,\mathsf{y})q^T,\ q^T = e^{2\pi i \mathrm{tr}(T\tau)}.$$

Similarly as before, this is conjectured to be modular. Although the cases $r \geq 2$ are still wide open, a version of this for the case $r = 1$ was proved in [BHK$^+$20] (for $n > 2$), which has been applied in various places successfully. We remark that this modularity theorem is one of the main ingredients of Wei Zhang's proof of arithmetic fundamental lemma in [Zha21]. In an ongoing work with Yousheng Shi and Tonghai Yang, we will extend this modularity theorem to the case $n = 2$.

In another extreme case $r = n$, $\widehat{\theta}^{ar}_r(\tau,\mathsf{y})$ is an arithmetic cycle of virtual dimension 0 and we can take arithmetic degree. Then one expects an arithmetic Siegel-Weil formula roughly of the following form:

$$\widehat{\deg}(\widehat{\theta}^{ar}_n) \text{``}=\text{''} \text{derivative of incoherent unitary Eisenstein series.} \tag{1.3}$$

In fact, it is a highly challenging task to make (1.3) precise. However, when T is non-degenerate, there is a well-defined intersection number $\mathrm{Int}(T)$ so that we expect

$$\mathrm{Int}(T)q^T \text{``}=\text{''} E'_T(\tau,0) \tag{1.4}$$

where $E_T(\tau,s)$ is the T-th derivative of certain incoherent Eisenstein series.

When T is non-degenerate but not positive definite, the intersection happens at the archimedean places. We refer [Liu11, GS19, BY20] for more details. In fact, [GS19] even treats the case when T is degenerate. When T is positive definite of rank n, the integral special cycle $\mathcal{Z}(T)$ is supported on the supersingular locus over a finite prime p which is either inert or ramified in E. Because of this, it turns out that (1.4) can be reduced to a local arithmetic Siegel-Weil formula by p-adic uniformization. For $\mathrm{U}(n-1,1)$–Shimura varieties with hyperspecial level at an *unramified* place, Kudla–Rapoport [KR11] conjectured a precise local arithmetic Siegel-Weil formula, now known as the *Kudla–Rapoport conjecture*. It is a precise identity between the arithmetic intersection number of special cycles on unitary Rapoport–Zink spaces (the geometric side) and the *central derivative* of local representation densities of hermitian forms (the analytic side). The original Kudla-Rapoport conjecture was recently proved by Li-Zhang [LZ22a] via Fourier transform and an ingenious induction.

1.0.2 Kudla-Rapoport conjecture over ramified primes

Now a natural question and important question is to formulate and prove an analogue of the Kudla–Rapoport conjecture over a *ramified* prime. There are two well-studied unitary Rapoport–Zink spaces over ramified primes with different level structures. One is the *exotic smooth model* which has good reduction, and the other one is the *Krämer model* which has bad reduction. The analogue of Kudla–Rapoport conjecture was formulated and proved for the even dimensional exotic smooth model by Li-Liu in [LL22] using a strategy similar to [LZ22a].

However, the situation is more complicated for the Krämer model— due to the bad reduction of the model, it is expected that one needs to modify the analytic side of the conjecture by certain nontrivial linear combination of *central values* of local representation densities. Such phenomenon was first observed by Kudla–Rapoport [KR00] via explicit computation in their proof of the orthogonal local arithmetic Siegel-Weil formula for the Drinfeld p-adic half plane.

In [Shi22], Shi obtained a Kudla-Rapoport type formula for the Krämer model associated with a 2-dimensional nonsplit hermitian space. In this case, there happens to be no modification on the analytic side. In a joint work with Shi and Yang [HSY23b], we studied the case for a 2-dimensional split hermitian space using an exceptional isomorphism between the corresponding unitary Rapoport-Zink space and Drinfel p-adic half plane. The analytic side in this case needs nontrivial modification.

Inspired by the results obtained in [Shi22, HSY23b], we formulated *the Kudla–Rapoport conjecture for Krämer models* in another joint work with Shi and Yang [HSY23a], where we give a *precise and conceptual* modification for the analytic side. We also proved the conjecture for the case $n = 3$ via explicit calculation. It is a new construction on the geometric side called difference cycle which makes the explicit calculation possible. The difference cycle is motivated by the primitive local density from the analytic side and has surprisingly simple property which eventually simplifies the calculation significantly.

Finally, in a joint work with Li, Shi and Yang [HLSY22], we resolved the conjecture positively in full generality via a new method using partial Fourier transform and induction.

As a first application, we relax the local assumptions in the arithmetic Siegel–Weil formula for $\mathrm{U}(n-1,1)$–Shimura varieties by allowing Krämer models at ramified places. The main theorem should also be useful to relax the local assumptions at ramified places in the arithmetic inner product formula [LL21, LL22] and its p-adic avatar by Disegni–Liu [DL22].

1.0.3 The precise formulation of Kudla–Rapoport conjecture for Krämer models

Now we give more details about the formulation of Kudla–Rapoport conjecture for Krämer models. Let p be an odd prime. Let F_0 be a finite extension of $\mathbb{Q}_p$ with residue field $\kappa = \mathbb{F}_q$. Let F be a ramified quadratic extension of F_0. Let π be a uniformizer of F such that $\mathrm{Tr}_{F/F_0}(\pi) = 0$. Then $\pi_0 := \pi^2$ is a uniformizer of F_0. Let $\breve{F}$ be the completion of the maximal unramified extension of F. Let O_F and $O_{\breve{F}}$ be the ring of integers of F and $\breve{F}$ respectively.

First, we introduce the geometric side of the conjecture. Let $n \geq 2$ be an integer. To define the Krämer model of the unitary Rapoport–Zink space, we fix a (principally polarized) supersingular hermitian O_F-modules $\mathbb{X}$ of signature $(1, n-1)$ over $\bar{\kappa}$ (Definition

2.1.1). The *Krämer model* $\mathcal{N} = \mathcal{N}_n$ is the formal scheme over $\mathrm{Spf}\, O_{\breve{F}}$ parameterizing hermitian formal O_F-modules X of signature $(1, n-1)$ within the quasi-isogeny class of $\mathbb{X}$, together with a rank 1 filtration $\mathcal{F} \subset \mathrm{Lie} X$ satisfying the Krämer condition (Definition 2.1.2). The space $\mathcal{N}$ is locally of finite type, and semistable of relative dimension $n-1$ over $\mathrm{Spf}\, O_{\breve{F}}$. There are two choices of the framing object $\mathbb{X}$ (up to quasi-isogeny), giving rise to two non-isomorphic (resp. isomorphic) spaces $\mathcal{N}$ when n is even (resp. odd) (§2.1.2).

Let $\mathbb{Y}$ be the framing hermitian O_F-modules of signature $(0,1)$ over $\bar{\kappa}$ defined as in Definition 2.1.1. The *space of quasi-homomorphisms* $\mathbb{V} = \mathbb{V}_n := \mathrm{Hom}_{O_F}(\mathbb{Y}, \mathbb{X}) \otimes_{O_F} F$ carries a natural F/F_0-hermitian form, which makes $\mathbb{V}$ a non-degenerate F/F_0-hermitian space of dimension n (§2.1.2). The two choices of the framing object $\mathbb{X}$ exactly correspond to the two isomorphism classes of $\mathbb{V}$, classified by $\chi(\mathbb{V}) := \chi((-1)^{\frac{n(n-1)}{2}} \det(\mathbb{V})) \in \{\pm 1\}$, where $\chi : F_0^\times \to \{\pm 1\}$ is the quadratic character associated to F/F_0. For any subset $L \subset \mathbb{V}$, the *special cycle* $\mathcal{Z}(L)$ (§2.2.1) is a closed formal subscheme of $\mathcal{N}$, over which each quasi-homomorphism $x \in L$ deforms to homomorphisms.

Let $L \subset \mathbb{V}$ be an O_F-lattice (of full rank n). We will associate to L two integers: the *arithmetic intersection number* $\mathrm{Int}(L)$ and the *modified derived local density* $\partial\mathrm{Den}(L)$.

Definition 1.0.1. *Let $L \subset \mathbb{V}$ be an O_F-lattice. Let $x_1, \ldots, x_n$ be an O_F-basis of L. Define the* arithmetic intersection number

$$\mathrm{Int}(L) := \chi(\mathcal{N}, \mathcal{O}_{\mathcal{Z}(x_1)} \otimes^{\mathbb{L}} \cdots \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}(x_n)}) \in \mathbb{Z}, \tag{1.5}$$

where $\mathcal{O}_{\mathcal{Z}(x_i)}$ denotes the structure sheaf of the special divisor $\mathcal{Z}(x_i)$, $\otimes^{\mathbb{L}}$ denotes the derived tensor product of coherent sheaves on $\mathcal{N}$, and χ denotes the Euler–Poincaré characteristic (Definition 2.2.6). By Howard [How19, Corollary D]), we know that $\mathrm{Int}(L)$ *is independent of the choice of the basis $x_1, \ldots, x_n$ and hence is a well-defined invariant of L itself.*

We remark that virtually we regard the intersection between $\mathcal{Z}(x_1), \ldots, \mathcal{Z}(x_n)$ as a 0-cycle. However in general they do not intersect properly. We use derived tensor in the definition of our intersection number here to remedy this.

Now we introduce the analytic side of the conjecture. For M another hermitian O_F-lattice (of arbitrary rank), denote by $\mathrm{Herm}_{L,M}$ the O_{F_0}-scheme of hermitian O_F-module homomorphisms from L to M (Definition 4.1.1) and define its *local density* to be

$$\mathrm{Den}(M,L) := \lim_{d\to+\infty} \frac{|\mathrm{Herm}_{L,M}(O_{F_0}/\pi_0^d)|}{q^{N\cdot d_{L,M}}},$$

where $d_{L,M}$ is the dimension of $\mathrm{Herm}_{L,M} \otimes_{O_{F_0}} F_0$. Let H be the standard hyperbolic hermitian O_F-lattice of rank 2 (given by the hermitian matrix $\left(\begin{smallmatrix} 0 & \pi^{-1} \\ -\pi^{-1} & 0 \end{smallmatrix}\right)$). It is well-known that there exists a *local density polynomial* $\mathrm{Den}(M,L,X) \in \mathbb{Q}[X]$ such that for any integer $k \geq 0$,

$$\mathrm{Den}(M,L,q^{-2k}) = \mathrm{Den}(H^k \oplus M, L). \tag{1.6}$$

When M has also rank n and $\chi(M) = -\chi(L)$, we have $\mathrm{Den}(M,L) = 0$ (Lemma 5.3.6) and in this case we write

$$\mathrm{Den}'(M,L) := -2 \cdot \left.\frac{\mathrm{d}}{\mathrm{d}X}\right|_{X=1} \mathrm{Den}(M,L,X).$$

Define the (normalized) *derived local density*

$$\mathrm{Den}'(L) := \frac{\mathrm{Den}'(I_n,L)}{\mathrm{Den}(I_n,I_n)} \in \mathbb{Q}. \tag{1.7}$$

Here I_n is the unimodular lattice of rank n with $\chi(I_n) = -\chi(L)$. Recall that a hermitian O_F-lattice L is *unimodular*.

In the thesis, we also use $\alpha(M,L,X)$ to denote $\mathrm{Den}(M,L,X)$. However, we use $\alpha'(M,L)$ to denote $-\left.\frac{\mathrm{d}}{\mathrm{d}X}\right|_{X=1} \alpha(M,L,X)$.

The naive analogue of the Kudla–Rapoport conjecture for Krämer model should be

$$\mathrm{Int}(L) \overset{?}{=} \mathrm{Den}'(L).$$

However, as explained in [HSY23a] this naive analogue does not hold for the following reasons. On the one hand, by definition $\mathrm{Int}(L)$ vanishes unless L is integral (i.e., $L \subset L^\sharp$), while $\mathrm{Den}'(L)$ does not vanish for non-integral lattices L which are dual to vertex lattices. More precisely, recall that an integral O_F-lattice $\Lambda \subset \mathbb{V}$ is called a *vertex lattice (of type t)* if $\Lambda^\sharp/\Lambda$ is a κ-vector space (of dimension t). For a vertex lattice $\Lambda \subset \mathbb{V}$ of type $t > 0$, we have $\Lambda^\sharp$ is non-integral so $\mathrm{Int}(\Lambda^\sharp) = 0$, while $\mathrm{Den}'(\Lambda^\sharp) \neq 0$ in general (see e.g. (5.38)). In general, we define the type $t(L)$ of L to be the number of positive fundamental invariants of L (see §1.0.6).

To account for these discrepancies, we will define $\partial\mathrm{Den}(L)$ by modifying $\mathrm{Den}'(L)$ with a linear combination of the (normalized) *local densities* (Corollary 5.3.7)

$$\mathrm{Den}_t(L) := \frac{\mathrm{Den}(\Lambda_t^\sharp, L)}{\mathrm{Den}(\Lambda_t^\sharp, \Lambda_t^\sharp)} \in \mathbb{Z}. \tag{1.8}$$

Here $\Lambda_t \subset \mathbb{V}$ is a vertex lattice of type t (in particular $\chi(\Lambda_t^\sharp) = \chi(L)$). Recall that the possible vertex type t is given by any even integer such that $0 \leq t \leq t_{\max}$, where

$$t_{\max} = \begin{cases} n, & \text{if } n \text{ even}, \chi(\mathbb{V}) = +1, \\ n-1, & \text{if } n \text{ odd}, \\ n-2, & \text{if } n \text{ even}, \chi(\mathbb{V}) = -1. \end{cases}$$

Definition 1.0.2. *Let $L \subset \mathbb{V}$ be an O_F-lattice. Define the* modified derived local density *(Corollary 5.5.2)*

$$\partial\mathrm{Den}(L) := \mathrm{Den}'(L) + \sum_{j=1}^{t_{\max}/2} c_{2j} \cdot \mathrm{Den}_{2j}(L) \in \mathbb{Z}. \tag{1.9}$$

Here the coefficients $c_{2j} \in \mathbb{Q}$ are chosen to satisfy

$$\partial\mathrm{Den}(\Lambda_{2i}^\sharp) = 0, \quad 1 \leq i \leq t_{\max}/2, \tag{1.10}$$

which turns out to be a linear system in $(c_2, c_4, \ldots, c_{t_{\max}})$ with a unique solution (Theorem

4.2.1).

The main goal of this thesis is to prove the following local arithmetic Siegel-Weil formula, settling the main conjecture of [HSY23a]. We will prove this theorem in §5.7.

Theorem 1.0.3 (Kudla–Rapoport conjecture for Krämer models). *Let $L \subset \mathbb{V}$ be an O_F-lattice. Then*

$$\mathrm{Int}(L) = \partial\mathrm{Den}(L).$$

1.0.4 Global application

Now we describe the global application of the local arithmetic Siegel-Weil formula. For brevity and clarity of exposition we restrict our attention to the case when F is an imaginary quadratic field. We further assume $F = \mathbb{Q}(\sqrt{d_F})$ where $d_F \equiv 1 \mod 8$ (essentially we simply need 2 splits in F). We remark here that our results can be applied to the case when F is a general CM field given correct local assumptions. Let $\mathcal{M}_{(1,n-1)}$ be the moduli functor over $\operatorname{Spec} O_F$ such that for an O_F-scheme S, $\mathcal{M}_{(1,n-1)}(S)$ parametrizes isomorphism classes of principally polarized abelian schemes $A_{/S}$ of relative dimension n with an action $\iota : O_F \to \mathrm{End}(A)$, a compatible polarization $\lambda : A \to A^\vee$ and a filtration $\mathcal{F}_A \subset \mathrm{Lie}A$ which satisfies the signature $(1, n-1)$ condition: O_F acts on $\mathrm{Lie}A/\mathcal{F}_A$ via the structure map $s : O_F \to \mathcal{O}_S$ and acts on $\mathcal{F}_A$ via conjugation of s. Let V be a hermitian vector space over F of signature $(n-1, 1)$ containing a self-dual lattice L. The lattice L determines an open and closed substack

$$\mathcal{M} \subset \mathcal{M}_{(0,1)} \times_{\operatorname{Spec} O_F} \mathcal{M}_{(1,n-1)}$$

which is an integral model of a unitary Shimura variety. For a point in $\mathcal{M}(S)$ (S an O_F-scheme), i.e., a pair $(E, \iota_0, \lambda_0) \in \mathcal{M}_{(0,1)}(S)$, $(A, \iota, \lambda, \mathcal{F}_A) \in \mathcal{M}_{(1,n-1)}(S)$, define the locally free O_F-module

$$V'(E, A) = \mathrm{Hom}_{O_F}(E, A).$$

It is equipped with the hermitian form $h'(x,y) = \iota_0^{-1}(\lambda_0^{-1}\circ y^\vee \circ \lambda \circ x)$. For a $m\times m$ nonsingular hermitian matrix T with values in O_F, let $\mathcal{Z}(T)$ be the stack of collections $(E,\iota_0,\lambda_0,A,\iota,\lambda,\mathcal{F}_A,\mathbf{x})$ such that $(E,\iota_0,\lambda_0,A,\iota,\lambda,\mathcal{F}_A)\in\mathcal{M}(S)$, $\mathbf{x}\in V'(E,A)^m$ with $h'(\mathbf{x},\mathbf{x}) = T$. Then $\mathcal{Z}(T)$ is representable by a Deligne-Mumford stack which is finite and unramified over $\mathcal{M}$ ([KR14b, Proposition 2.9]). When $t\in\mathbb{Z}_{>0}$, each component of $\mathcal{Z}(t)$ can be viewed as a divisor by [How15, Proposition 3.2.3]. In general, $\mathcal{Z}(T)$ does not necessarily have the expected codimension which is the rank of T.

Let $\mathbb{V}=\{\mathbb{V}_p\}$ be the incoherent hermitian space over $\mathbb{A}_F$ associated to V. Alternatively, it is the positive definite hermitian space over $\mathbb{A}_F$ such that $\mathbb{V}_\ell\cong V_\ell$ for all finite ℓ. It is "incoherent" in the sense that it does not come from a global hermitian space. For a nonsingular hermitian matrix T of rank n with values in O_F, Let V_T be the hermitian space with gram matrix T. Define

$$\mathrm{Diff}(T,\mathbb{V}) \coloneqq \{p \text{ a place of } \mathbb{Q} \mid \mathbb{V}_p \text{ is not isomorphic to } (V_T)_p\}. \tag{1.11}$$

Then $\mathcal{Z}(T)$ is empty if $|\mathrm{Diff}(T,\mathbb{V})| = \{\infty\}$ or $|\mathrm{Diff}(T,\mathbb{V})|>1$.

If $\mathrm{Diff}(T,\mathbb{V})=\{\infty\}$, then T is non-singular but not positive definite. Although $\mathcal{Z}(T)$ is empty in this case, there is contribution at ∞ by Green current. We have an arithmetic 0-cycle $\widehat{\mathcal{Z}}(T,\mathsf{y}) \coloneqq (0,\mathrm{Gr}(T,\mathsf{y})$ where $\mathrm{Gr}(T,\mathsf{y})$ is a Green current defined as in [Liu11] and [GS19] with the parameter y being a positive definite hermitian matrix y of rank n (which will be imaginary part of τ) , see for example [LZ22a, §15.3]. Define the arithmetic degree

$$\widehat{\deg}(\widehat{\mathcal{Z}}(T,\mathsf{y})) \coloneqq \frac{1}{2}\int_{\mathcal{M}(\mathbb{C})} \mathrm{Gr}(T,\mathsf{y}) \tag{1.12}$$

If $\mathrm{Diff}(T,\mathbb{V})=\{p\}$ for an inert or ramified prime p in F, then T is positive definite and the support of $\mathcal{Z}(T)$ is on the supersingular locus of $\mathcal{M}$ over $\operatorname{Spec}\mathbb{F}_p$. It is this case where Kudla-Rapoport conjecture will be a major ingredient. Consider the arithmetic cycle $\widehat{\mathcal{Z}}(T,\mathsf{y}) \coloneqq (\mathcal{Z}(T),0)$. Let e be the ramification index of $F_p/\mathbb{Q}_p$. Virtually, we are considering $\widehat{\mathcal{Z}}(T, \mathsf{y})$ as an arithmetic 0-cycle. Hence we consider the arithmetic degree

$$\widehat{\deg}(\widehat{\mathcal{Z}}(T, \mathsf{y})) = \chi(\mathcal{Z}(T), \mathcal{O}_{\mathcal{Z}(t_1)} \otimes^{\mathbb{L}} \cdots \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}(t_n)}) \cdot \log p^{2/e}, \tag{1.13}$$

where $\otimes^{\mathbb{L}}$ stands for derived tensor product on the category of coherent sheaves on $\mathcal{M}$, χ is Euler-characteristic and t_i $(1 \leq i \leq n)$ are the diagonal entries of T. We remark here that (1.13) can be reduced to the intersection number (1.5) on the Rapoport-Zink space $\mathcal{N}$ over p by p-adic uniformization. As in the local situation, $\mathcal{Z}(t_1), \ldots, \mathcal{Z}(t_n)$ do not intersect properly in general and we take derived tensor to fix this.

Now we consider the analytic side. Let $\varphi_L = \mathbf{1}_{\widehat{L}^n} \in \mathscr{S}(\mathbb{V}_f^n)$. Associated to $\varphi := \varphi_L \otimes \varphi_\infty \in \mathscr{S}(\mathbb{V}^n)$, where φ_∞ is the Gaussian function, we define a classical *incoherent Eisenstein series* on the hermitian upper half space $\tau = \mathsf{x} + i\mathsf{y} \in \mathbb{H}_n$:

$$E(\tau, s, \varphi) := \chi_\infty(\det(a))^{-1} \det(\mathrm{y})^{-n/2} \cdot E\left(g_\tau, s, \varphi\right), \quad g_\tau := n(\mathrm{x})m(a) \in G_n(\mathbb{A}), \tag{1.14}$$

where $a \in \mathrm{GL}_n\left(F_\infty\right)$ such that $\mathrm{y} = a^t \bar{a}$.

Here $E(\tau, s, \varphi)$ has a meromorphic continuation and a functional equation relating $s \leftrightarrow -s$ and $E(\tau, 0, \varphi) = 0$ by the incoherence. Therefore, we consider its *central derivative*

$$\mathrm{Eis}'(\tau, \varphi_L) := \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} E(\tau, s, \varphi).$$

Analogous to the local situation, we need to modify $\mathrm{Eis}'(\tau, \varphi_L)$ by central values of coherent Eisenstein series. For a ramified place p in F, let ${}^p\mathbb{V}$ be the *coherent* hermitian space over $\mathbb{A}_F$ nearby $\mathbb{V}$ at p, namely $({}^p\mathbb{V})_\ell \simeq \mathbb{V}_\ell$ exactly for all places $\ell \neq p$. Recall that we call a hermitian O_{F_p}-lattice Λ a vertex lattice of type t if $\pi\Lambda^\sharp \subset \Lambda \subset \Lambda^\sharp$ and $[\Lambda^\sharp : \Lambda] = t$ where π is a uniformizer of F_p. For any vertex lattice $\Lambda_{t,p} \subset ({}^p\mathbb{V})_p$ of type t, the Schwartz function $\varphi_L^p \otimes \mathbf{1}_{(\Lambda_{t,p}^\sharp)^n} \in \mathscr{S}(({}^p\mathbb{V})^n)$ gives a classical *coherent Eisenstein series* $E(\tau, s, \varphi_L^p \otimes \mathbf{1}_{(\Lambda_{t,p}^\sharp)^n})$ defined similarly as in (1.14). Here $\varphi_L^p = \otimes_{\ell \neq p} \varphi_{L_\ell}$. Analogous to (1.8), define the (normalized) *central values*

$$ {}^{v}\mathrm{Eis}_t(\tau,\varphi_L) := \frac{\mathrm{vol}(K_p)}{\mathrm{vol}(K_{\Lambda^\sharp_{t,p}})} \cdot E(\tau, 0, \varphi^p \otimes \mathbf{1}_{(\Lambda^\sharp_{t,p})^n}). \tag{1.15} $$

Here $K_{\Lambda^\sharp_{t,p}} \subset \mathrm{U}({}^p\mathbb{V})(F_{0,p})$ is the stabilizer of $\Lambda^\sharp_{t,p}$, and the volumes are taken with respect to the Haar measures on $\mathrm{U}(V)(F_{0,p})$ and $\mathrm{U}({}^p\mathbb{V})(F_{0,p})$ as defined in [LL21, Definition 3.8]. Let $\mathcal{V}_{\mathrm{ram}}$ be the set of places of F_0 ramified in F. For a $p \in \mathcal{V}_{\mathrm{ram}}$ in F, analogous to (1.9), define the linear combination

$$ {}^{p}\mathrm{Eis}(\tau,\varphi_L) := \sum_{j=1}^{t_{\max,p}/2} c_{2j,p} \cdot {}^{p}\mathrm{Eis}_{2j}(\tau,\varphi_L), \tag{1.16} $$

where $t_{\max,p}$ and $c_{2j,p}$ are the numbers $t_{\max}$ and c_{2j} respectively in (1.9) for the local hermitian space $({}^p\mathbb{V})_p$ over the ramified extension $F_p/\mathbb{Q}_p$. Define the *modified central derivative*

$$ \partial\mathrm{Eis}(\tau,\varphi_L) := \mathrm{Eis}'(\tau,\varphi_L) + (-1)^n \sum_{p\in\mathcal{V}_{\mathrm{ram}}} {}^{p}\mathrm{Eis}(\tau,\varphi_L), \tag{1.17} $$

which is unitary modular form for $\mathrm{U}(n,n)$. We fix the canonical unramified additive character $\psi : \mathbb{A}_{\mathbb{Q}}/\mathbb{Q} \to \mathbb{C}^\times$ with $\psi_\infty(x) = e^{2\pi i x}$. Then $\partial\mathrm{Eis}(\tau,\varphi_L)$ has a Fourier expansion as follows:

$$ \partial\mathrm{Eis}(\tau,\varphi_L) = \sum_{T\in\mathrm{Herm}_n(F)} \partial\mathrm{Eis}_T(\tau,\varphi_L). \tag{1.18} $$

The following result asserts an identity between the arithmetic degrees of special cycles and the nonsingular Fourier coefficients of the modified central derivative of the incoherent Eisenstein series, whose proof is essentially reduced to Theorem 1.0.3 and the unramified version proved in [LZ22a].

Theorem 1.0.4 (Arithmetic Siegel–Weil formula: nonsingular terms)**.** *Let $F = \mathbb{Q}(\sqrt{d_F})$ with $d_F \equiv 1 \mod 8$ and L be the unimodular lattice we fixed before. We have*

$$ \widehat{\deg}(\widehat{\mathcal{Z}}(T,\mathsf{y}))q^T = c_K \cdot \partial\mathrm{Eis}_T(\tau,\varphi_L), $$

where $q^T := \psi_\infty(\mathrm{Tr}(T\tau))$, c_K *is a nonzero constant independent of* T *and* φ_L*.*

Notice that $\widehat{\deg}(\widehat{\mathcal{Z}}(T, \mathsf{y}))$ does not depend on y when T is positive definite. We also remark that it is an open and very challenging question to extend this identity to singular terms.

1.0.5 Organization of this thesis

The rest of this thesis is organized as follows.

We will give the background material in Chapter two, including the definiton of Rapoport-Zink space, special cycles and their basic properties which will be used freely in later chapters.

In Chapter three, we focus on the case $n = 2$. Using the exceptional isomorphism between $\mathcal{N}_2$ and Drinfeld upper half plane (after base change and blow up) and Grothendieck-Messing theory, we write down the local equations of special divisors. Based on this, we prove the conjecture for the case $\mathbb{V}$ is 2-dimensional split hermitian space via explicit calculation. This chapter is based on the joint work with Shi and Yang ([HSY23b]).

In Chapter four, we explain how to precisely formulate the modified conjecture. We also prove conjecture for $n = 3$ using induction and explicit computation. We will define the difference cycle, which is motivated by the primitive local density from the analytic side, and show that it has some surprisingly simple property. Accordingly, we will establish some induction formula for the local density. Finally we compare the geometric side and analytic side to prove the conjecture inductively. This chapter is based on the joint work with Shi and Yang ([HSY23a]).

Finally, in Chapter five, we prove the conjecture in full generality via a new method even in the $n = 2, 3$ case. The proof relies on partial Fourier transform and induction. On the geometric side, we can use linear invariance to avoid Tate conjecture for Deligne-Lusztig variety. On the analytic side, one of the major novelty is a decomposition formula for local density and a surprisingly simple formula for (derived) primitive local density. A more detailed summary will be given at the beginning of the chapter. This chapter is based on the joint work with Li, Shi and Yang [HLSY22].

1.0.6 Notation and terminology

In this thesis, a lattice means a hermitian O_F-lattice without explicit mentioning. Unless otherwise stated, L always means a non-degenerate lattice of rank n with a hermitian form $(\ ,\ )$.

- We say a sublattice $L^\flat$ of a hermitian space is non-degenerate if the restriction of the hermitian form to it is non-degenerate.

- We define $L^\sharp$ to be the dual lattice of L with respect to the hermitian form $(\ ,\ )$. If $L \subset L^\sharp$, we say L is integral.

- Following [LL22, Definition 2.11], for a lattice L with hermitian form $(\ ,\ )$, we say that a basis $\{\ell_1, \ldots, \ell_n\}$ of L is a normal basis (which always exists by [LL22, Lemma 2.12]) if its moment matrix $T = ((\ell_i, \ell_j))_{1\le i,j\le n}$ is conjugate to

$$\left(\beta_1\pi^{2b_1}\right) \oplus \cdots \oplus \left(\beta_s\pi^{2b_s}\right) \oplus \begin{pmatrix} 0 & \pi^{2c_1+1} \\ -\pi^{2c_1+1} & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & \pi^{2c_t+1} \\ -\pi^{2c_t+1} & 0 \end{pmatrix}$$

  by a permutation matrix, for some $\beta_1, \ldots, \beta_s \in O_{F_0}^\times$ and $b_1, \ldots, b_s, c_1, \ldots, c_t \in \mathbb{Z}$. Moreover, we define its (unitary) fundamental invariants $(a_1, \cdots, a_n)$ to be the unique nondecreasing rearrangement of $(2b_1, \cdots, 2b_s, 2c_1+1, \cdots, 2c_t+1)$.

- We define the type $t(L)$ of L to be the number of positive fundamental invariants of L. We use $r(L)$ to denote the rank of L and call L a *full type* lattice if $r(L) = t(L)$.

- We define the valuation of L to be $\mathrm{val}(L) := \sum_{i=1}^n a_i$, where $(a_1, \cdots, a_n)$ are the fundamental invariants of L. For $x \in L$, we define $\mathrm{val}(x) = \mathrm{val}((x,x))$, where $\mathrm{val}(\pi_0) = 1$.

- For a hermitian space $\mathbb{V}$, we let $\mathbb{V}^{?i} := \{x \in \mathbb{V} \mid \mathrm{val}(x) ? i\}$ where ? can be $\geq$, $\leq$ or $=$.

- For a ring R, we use $\langle \ell_1, \cdots, \ell_n \rangle_R$ to denote $\mathrm{Span}_R\{\ell_1, \cdots, \ell_n\}$. When $R = O_F$, we simply write $\langle \ell_1, \cdots, \ell_n \rangle$. We use L_F to denote $L \otimes_{O_F} F$.

- For a hermitian lattice of rank n, we define its sign as

$$\chi(L) \coloneqq \chi((-1)^{\frac{n(n-1)}{2}} \det(L)) = \pm 1$$

  where χ is the quadratic character of $F_0^\times$ associated to F/F_0. Without explicit mentioning, we always use ϵ to denote $\chi(L)$.

- Let I_m^ϵ denote a unimodular lattice of rank m with $\chi(I_m^\epsilon) = \epsilon$. We also simply denote a unimodular lattice of rank m by I_m if we do not need to remember its sign or its sign is clear in the context. In particular, when we consider $\mathrm{Den}'(I_n, L)$, we mean $I_n = I_n^{-\epsilon}$.

- We call a sublattice $N \subset M$ primitive in M if $\dim_{\mathbb{F}_q} \overline{N} = r(N)$, where $\overline{N} = (N + \pi M)/\pi M$. We also use $\overline{L}$ to denote $L \otimes_{O_F} O_F/(\pi)$.

- For two lattices L, L' of same rank, let $n(L', L) = \#\{L'' \subset L_F \mid L \subset L'', L'' \cong L'\}$.

- We let $\delta_{\mathrm{odd}}(n) = 1$ if n is an odd integer and $\delta_{\mathrm{odd}}(n) = 0$ if n is an even integer.

Chapter 2

Rapoport-Zink space and special cycle

2.1 Rapoport-Zink space

We denote $\bar{a}$ the Galois conjugate of $a \in F$ over F_0. Denote by $\mathrm{Nilp}\, O_{\breve{F}}$ be the category of $O_{\breve{F}}$-schemes S such that π is locally nilpotent on S. For such an S, denote its special fiber $S \times_{\mathrm{Spf}\, \mathcal{O}_{\breve{F}}} \mathrm{Spec}\, \bar{\kappa}$ by $\bar{S}$. Let $\sigma \in \mathrm{Gal}(\breve{F}_0/F_0)$ be the Frobenius element. We fix an injection of rings $i_0 : O_{F_0} \to \mathcal{O}_{\breve{F}_0}$ and an injection $i : O_F \to O_{\breve{F}}$ extending i_0. Denote by $\bar{i} : O_F \to O_{\breve{F}}$ the map $a \mapsto i(\bar{a})$.

2.1.1 RZ spaces

Let $S \in \mathrm{Nilp}\, O_{\breve{F}}$. A p-divisible strict O_{F_0}-module over S is a p-divisible group over S with an O_{F_0}-action whose induced action on its Lie algebra is via $O_{F_0} \xrightarrow{i_0} O_{\breve{F}_0} \to \mathcal{O}_S$.

Definition 2.1.1. *A formal hermitian O_F-module of dimension n over S is a triple (X, ι, λ) where X is a supersingular p-divisible strict O_{F_0}-module over S of dimension n and F_0-height $2n$ (supersingular means the O_{F_0}-relative Dieudonné module of X at each geometric point of S has slope $\frac{1}{2}$), $\iota : O_F \to \mathrm{End}(X)$ is an O_F-action and $\lambda : X \to X^\vee$ is a principal polarization in the category of strict O_{F_0}-modules such that the Rosati involution* *induced by λ is the Galois conjugation of F/F_0 when restricted on O_F.*

Definition 2.1.2. *Fix a formal hermitian O_F-module $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ of dimension n over $\bar{\kappa}$. The moduli space $\mathcal{N}_n$ is the functor such that $\mathcal{N}_n(S)$ for any $S \in \mathrm{Nilp}\, O_{\breve{F}}$ is the set of isomorphism classes of quintuples $(X, \iota, \lambda, \rho, \mathcal{F})$ such that*

1. *(X, ι, λ) is a formal hermitian O_F-module over S.*
2. *$\rho : X \times_S \bar{S} \to \mathbb{X} \times_{\mathrm{Spec}\, \bar{\kappa}} \bar{S}$ is a quasi-isogeny of formal O_F-modules of height 0.*
3. *$\mathcal{F}$ satisfies Krämer's ([Krä03]) signature condition: it is a local direct summand of $\mathrm{Lie} X$ of rank $n-1$ as an $\mathcal{O}_S$-module such that O_F acts on $\mathcal{F}$ by $O_F \xrightarrow{\bar{i}} O_{\breve{F}} \to \mathcal{O}_S$ and acts on $\mathrm{Lie} X/\mathcal{F}$ by $O_F \xrightarrow{i} O_{\breve{F}} \to \mathcal{O}_S$.*

An isomorphism between two such quintuples $(X, \iota, \lambda, \rho, \mathcal{F})$ and $(X', \iota', \lambda', \rho', \mathcal{F}')$ is an isomorphism $\alpha : X \to X'$ such that $\rho' \circ (\alpha \times_S \bar{S}) = \rho$, $\alpha^*(\lambda')$ is a $O_{F_0}^{\times}$-multiple of λ and $\alpha_*(\mathcal{F}) = \mathcal{F}'$.

Notice that $\mathcal{N}_n$ is a relative Rapoport-Zink space in the sense of [Mih22]. By [How19, Proposition 2.2], $\mathcal{N}_n$ is representable by a flat formal scheme of relative dimension $n-1$ over $\mathrm{Spf}\, O_{\breve{F}}$. We drop the subscript n in $\mathcal{N}_n$ when there is no ambiguity.

Although we mainly study $\mathcal{N}_n$, we also need another moduli space proposed in [Pap00].

Definition 2.1.3. *Fix a formal hermitian O_F-module $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ of dimension n over $\bar{\kappa}$. The moduli space $\mathcal{N}_n$ is the functor such that $\mathcal{N}_n^{\mathrm{Pap}}(S)$ for any $S \in \mathrm{Nilp}\, O_{\breve{F}}$ is the set of isomorphism classes of quintuples $(X, \iota, \lambda, \rho)$ such that*

1. *(X, ι, λ) is a formal hermitian O_F-module over S.*
2. *$\rho : X \times_S \bar{S} \to \mathbb{X} \times_{\mathrm{Spec}\, \bar{\kappa}} \bar{S}$ is a quasi-isogeny of formal O_F-modules of height 0.*
3. *$\iota : O_F \to \mathrm{End}(X)$ is an O_F-action on X satisfying Kottwitz condition:*

$$\mathrm{char}(\iota(\pi)|\mathrm{Lie}X) = (T - \pi)(T + \pi)^{n-1}.$$

By [Krä03] (see also [Shi22, Proposition 2.7]), the natural forgetful functor $\Phi : \mathcal{N}_n \to \mathcal{N}_n^{\mathrm{Pap}}$ forgetting $\mathcal{F}$ is the blow up of $\mathcal{N}_n^{\mathrm{Pap}}$ along its singular locus Sing. For each point $\Lambda \in \mathrm{Sing}$, its inverse image $\Phi^{-1}(\Lambda)$ is an exceptional divisor Exc_Λ isomorphic to $\mathbb{P}_k^{n-1}$.

2.1.2 Associated hermitian spaces

For a strict O_{F_0}-module X over $\bar{\kappa}$, let $M(X)$ be the O_{F_0}-relative Dieudonné module of X. Let $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ be the framing object as in Definition 2.1.2, and $N = M(\mathbb{X}) \otimes_{O_{F_0}} F_0$ be its rational relative Dieudonne module. Then N is a $2n$-dimensional $\breve{F}_0$-vector space equipped with a σ-linear operator $\mathbf{F}$ and a σ^{-1}-linear operator $\mathbf{V}$. The O_F-action $\iota_{\mathbb{X}} : O_F \to \mathrm{End}(\mathbb{X})$ induces on N an O_F-action commuting with $\mathbf{F}$ and $\mathbf{V}$. We still denote this induced action by $\iota_{\mathbb{X}}$ and denote $\iota_{\mathbb{X}}(\pi)$ by Π. The principal polarization of $\mathbb{X}$ induces a skew-symmetric $\breve{F}_0$-bilinear form $\langle\, ,\, \rangle$ on N satisfying

$$\langle \mathbf{F}x, y\rangle = \langle x, \mathbf{V}y\rangle^\sigma, \quad \langle \iota(a)x, y\rangle = \langle x, \iota(\bar{a})y\rangle,$$

for any $x, y \in N$, $a \in O_F$. Then N is an n-dimensional $\breve{F}$-vector space equipped with a $\breve{F}/\breve{F}_0$-hermitian form $(\, ,\,)$ defined by (see [Shi18, (2.6)])

$$(x, y) := \delta(\langle \Pi x, y\rangle + \pi\langle x, y\rangle), \tag{2.1}$$

where δ is a fixed element in $\mathcal{O}_{\breve{F}_0}^\times$ such that $\sigma(\delta) = -\delta$. We can use the relation

$$\langle x, y\rangle = \frac{1}{2\delta}\mathrm{Tr}_{\breve{F}/\breve{F}_0}(\pi^{-1}(x, y)). \tag{2.2}$$

to recover $\langle\, ,\, \rangle$. Let $\tau := \Pi\mathbf{V}^{-1}$ and $C := N^\tau$. Then C is an F-vector space of dimension n and $N = C \otimes_{F_0} \breve{F}_0$. The restriction of $(\, ,\,)$ to C is a F/F_0-hermitian form which we still denote by $(\, ,\,)$. There are two choices of $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ up to quasi-isogenies preserving the polarization on the nose, according to the sign $\epsilon = \chi(C)$ of C. Here $\chi : F_0^\times \to \{\pm 1\}$ is the character associated to the quadratic extension F/F_0 and we define the sign of C as

$$\chi(C) := \chi((-1)^{n(n-1)/2}\det(C)).$$

When n is odd, two different choices of ϵ give us isomorphic moduli spaces. When n is even, two different choices of ϵ give us two sets of non-isomorphic moduli spaces. See [Shi18, Remark 2.16] and [RTW14, Remark 4.2].

Fix a formal hermitian O_F-module $(\mathbb{Y}, \iota_{\mathbb{Y}}, \lambda_{\mathbb{Y}})$ of dimension 1 over $\operatorname{Spec}\bar{\kappa}$. Define

$$\mathbb{V}_n = \mathrm{Hom}_{O_F}(\mathbb{Y}, \mathbb{X}) \otimes \mathbb{Q}. \tag{2.3}$$

We drop the subscript n of $\mathbb{V}_n$ unless we need to specify the dimension. The vector space $\mathbb{V}$ is equipped with a hermitian form $(\ ,\ )_{\mathbb{V}}$ such that for any $x, y \in \mathbb{V}$

$$(x, y)_{\mathbb{V}} = \lambda_{\mathbb{Y}}^{-1} \circ y^{\vee} \circ \lambda_{\mathbb{X}} \circ x \in \mathrm{End}(\mathbb{Y}) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\sim} F \tag{2.4}$$

where $y^{\vee}$ is the dual quasi-homomorphism of y. The hermitian spaces $(\mathbb{V}, (\ ,\ )_{\mathbb{V}})$ and $(C, (,))$ are related by the F-linear isomorphism

$$b : \mathbb{V} \to C, \quad x \mapsto x(e) \tag{2.5}$$

where e is a generator of τ-fixed points of the O_{F_0}-relative Dieudonné module $M(\mathbb{Y})$. The relative Dieudonné module $M(\mathbb{Y})$ is equipped with a hermitian form $(\ ,\ )_{\mathbb{Y}}$ such that $(e, e)_{\mathbb{Y}} \in O_{F_0}^{\times}$. By [Shi18, Lemma 3.6], we have

$$(x, x)_{\mathbb{V}} \cdot (e, e)_{\mathbb{Y}} = (b(x), b(x)). \tag{2.6}$$

Here the bilinear form $(,)_{\mathbb{Y}}$ is the analogue of the form $(,)$ in (2.1) defined on the rational relative Dieudonné module of $\mathbb{Y}$. By scaling the polarization $\lambda_{\mathbb{Y}}$ by a factor in $O_{F_0}^{\times}$ we can assume that

$$(e,e)_{\mathbb{Y}}=1,$$

so $\mathbb{V}$ and C are isomorphic as hermitian spaces. When the context is clear we often drop the subscript $\mathbb{V}$ in $(\ ,\ )_{\mathbb{V}}$.

Note that, when n is odd, $\mathcal{N}_n$ does not depends on $\chi(\mathbb{V})$. However, when n is even, different choices of the framing object with different $\chi(\mathbb{V})$ will give non-isomorphic $\mathcal{N}_n$. In this case, we will use $\mathcal{N}_{n,\epsilon}$ to denote the Rapoport-Zink space given by a framing object $(\mathbb{X},\iota_{\mathbb{X}},\lambda_{\mathbb{X}})$ with $\chi(\mathbb{V})=\epsilon$.

2.1.3 Bruhat-Tits stratification of $\mathcal{N}^{\mathrm{Pap}}_{red}$

In this subsection, we review the description of the reduced locus of $\mathcal{N}^{\mathrm{Pap}}$ in [RTW14]. We remark that although the results of [RTW14] treat the case $F_0=\mathbb{Q}_p$, it extends to the category of strict formal O_{F_0}-modules using relative Dieudonné theory. This will be used heavily in Chapter 3 and Chapter 4. We also mention that there is a similar story for the reduced locus of $\mathcal{N}$ which will be established in Chapter 5. From now, we use $\mathcal{N}^{\mathrm{Pap}}_{\mathrm{red}}$ to denote the reduced locus of $\mathcal{N}^{\mathrm{Pap}}$.

For an $\mathcal{O}_{\breve{F}}$-lattice M of N, define $M^\sharp$ to be the dual lattice of M with respect to the form $(,)_{\mathbb{X}}$. Recall the following results.

Proposition 2.1.4. *([RTW14, Proposition 2.2 and 2.4]) Let $\mathcal{N}(k)$ be the set of $\mathcal{O}_{\breve{F}}$-lattices*

$$\mathcal{N}(k)=\{M\subset C\otimes_F\breve{F}\mid M^\sharp=M,\,\pi\tau(M)\subset M\subset\pi^{-1}\tau(M),\,\dim_k(M+\tau(M))/M\leq 1\}.$$

Then the map

$$\mathcal{N}^{\mathrm{Pap}}(k)\to\mathcal{N}(k),\quad x=(X,\iota,\lambda,\rho)\mapsto M(x)=\rho(M(X))\subset N$$

is a bijection.

We say a lattice $\Lambda\subset C$ is a vertex lattice if $\pi\Lambda\subseteq\Lambda^\sharp\subseteq\Lambda$ and we call $t=\dim_{\mathbb{F}_q}(\Lambda/\Lambda^\sharp)$

the type of Λ (in Chapter 5, we will use a slightly different convention of vertex lattice for convenience). We denote the set of vertex lattices (resp. of type t) by $\mathcal{V}$ (resp. $\mathcal{V}^t$). Using the isomorphism of hermitian spaces (2.5), we often identify Λ with $b^{-1}(\Lambda)$ and use the same notation to denote both. We say two vertex lattice Λ_1 and Λ_2 are neighbours if $\Lambda_1 \subset \Lambda_2$ or $\Lambda_2 \subset \Lambda_1$. Then we can define a simplicial complex $\mathcal{L}$ as follows. When n is odd or when n is even and C is non-split, then an r-simplex is formed by $\Lambda_0, \dots, \Lambda_r$ if any two members of this set are neighbours. When n is even and C is split, we refer to discussion before [RTW14, 3.4] for the definition of $\mathcal{L}$. We also use $\mathcal{L}_{n,\epsilon}$ to denote $\mathcal{L}$ if C has dimension n and $\chi(C) = \epsilon$. Again when n is odd, $\mathcal{L}_{n,1} = \mathcal{L}_{n,-1}$, hence we use $\mathcal{L}_n$ to denote it.

By results in Sections 4 and 6 of loc. cit., to each $\Lambda \in \mathcal{V}^t$ we can associate a Deligne-Lusztig varieties $\mathcal{N}_\Lambda^{\mathrm{Pap}}$ and $\mathcal{N}_\Lambda^{\mathrm{Pap},\circ}$ of dimension $t/2$, such that

$$\mathcal{N}_\Lambda^{\mathrm{Pap}}(k) = \{M \in \mathcal{N}(k) \mid M \subset \Lambda^\sharp \otimes_{O_F} \mathcal{O}_{\breve{F}}\},$$

and

$$\mathcal{N}_\Lambda^{\mathrm{Pap},\circ}(k) = \{M \in \mathcal{N}(k) \mid \Lambda^\sharp(M) = \Lambda^\sharp\}.$$

Here $\Lambda(M)$ is the minimal hermitian dual of a vertex lattice such that $\Lambda(M) \otimes_{O_F} \mathcal{O}_{\breve{F}}$ contains M which always exists by [RTW14, Proposition 4.1]. By Theorem 1.1 of loc.cit., we know that

$$\mathcal{N}_\Lambda^{\mathrm{Pap}} := \bigsqcup_{\Lambda' \in \mathcal{L}, \Lambda' \subseteq \Lambda} \mathcal{N}_{\Lambda'}^{\mathrm{Pap},\circ},$$

and

$$\mathcal{N}_{\mathrm{red}}^{\mathrm{Pap}} = \bigsqcup_{\Lambda \in \mathcal{L}} \mathcal{N}_\Lambda^{\mathrm{Pap},\circ}$$

where each $\mathcal{N}_\Lambda^{\mathrm{Pap}}$ is a closed subvariety of $\mathcal{N}_{\mathrm{red}}^{\mathrm{Pap}}$. By loc. cit., we also know that

$$\mathcal{N}_\Lambda^{\mathrm{Pap}} \cap \mathcal{N}_{\Lambda'}^{\mathrm{Pap}} = \begin{cases} \mathcal{N}_{\Lambda\cap\Lambda'}^{\mathrm{Pap}}, & \text{if } \Lambda \cap \Lambda' \in \mathcal{V}, \\ \emptyset & \text{otherwise.} \end{cases}$$

For a lattice $L \subset \mathbb{V}$, define

$$\mathcal{V}(L) := \{\Lambda \in \mathcal{V} \mid L \subseteq \Lambda^\sharp\}, \text{ and } \mathcal{V}^t(L) := \{\Lambda \in \mathcal{V}^t \mid L \subseteq \Lambda^\sharp\}. \tag{2.7}$$

When $L = \mathrm{Span}\{\mathbf{x}\}$ we also denote $\mathcal{V}(L)$ (resp. $\mathcal{V}^t(L)$) by $\mathcal{V}(\mathbf{x})$ (resp. $\mathcal{V}^t(\mathbf{x})$). For any subset S of $\mathcal{V}$, we define $\mathcal{L}(S)$ to be the subcomplex of $\mathcal{L}$ such that a simplex is in $\mathcal{L}(S)$ if and only if every vertex in it is in S. For a lattice L of $\mathcal{V}$ and $\mathbf{x} \in C$, define

$$\mathcal{L}(L) = \mathcal{L}(\mathcal{V}(L)). \tag{2.8}$$

When $L = \mathrm{Span}\{\mathbf{x}\}$ we also denote $\mathcal{L}(L)$ by $\mathcal{L}(\mathbf{x})$.

We have a similar result for the reduced locus of $\mathcal{N}_{\mathrm{red}}$ which will be studied and used in Chapter 5. Here we only give brief review and refer the reader to Chapter 5 for details.

By Proposition 5.1.17 below, $\mathcal{N}_\Lambda = \mathrm{Exc}_\Lambda$ for type 0 lattices Λ. The reduced locus $\mathcal{N}_{\mathrm{red}}$ has a decomposition (see Theorem 5.1.19)

$$\mathcal{N}_{\mathrm{red}} = \bigcup_\Lambda \mathcal{N}_\Lambda,$$

where the union is over all vertex lattices. The reduced subscheme $\mathcal{Z}(L)_{\mathrm{red}}$ is a union of Bruhat-Tits strata (see Proposition 5.1.20)

$$\mathcal{Z}(L)_{\mathrm{red}} = \bigcup_{L subset \Lambda^\sharp} \mathcal{N}_\Lambda. \tag{2.9}$$

We remark that $\mathcal{N}_\Lambda$ is a desingularization of $\mathcal{N}_\Lambda^{\mathrm{Pap}}$. In particular, $\mathcal{N}_\Lambda$ is regular.

For vertex lattices of type 0, we define Exc_Λ following the idea of [How19, Appendix].

Definition 2.1.5. *The exceptional divisor* Exc *of* $\mathcal{N}$ *is the set of all points* $z = (X, \iota, \lambda, \rho, \mathcal{F}) \in \mathcal{N}(\bar{\kappa})$ *such that the action*

$$\iota : O_F \to \mathrm{End}(\mathrm{Lie} X)$$

factor through $O_F \xrightarrow{i} O_{\breve{F}} \to \bar{\kappa}$ *where* $O_{\breve{F}} \to \bar{\kappa}$ *is the quotient map. For a vertex lattice* Λ *in* C *of type* 0*, define* Exc_Λ *to be the set of all points* $z = (X, \iota, \lambda, \rho, \mathcal{F}) \in \mathrm{Exc}$ *such that* $\rho(M(X)) = \Lambda \otimes_{O_F} O_{\breve{F}}$*. Both* Exc *and* Exc_Λ *are closed subset of* $\mathcal{N}$ *and we endow them the structure of reduced schemes over* $\bar{\kappa}$*.*

The following is a refinement of [How19, Proposition A.2].

Lemma 2.1.6. *Each* Exc_Λ *is a Cartier divisor of* $\mathcal{N}$ *isomorphic to* $\mathbb{P}^{n-1}_{\bar{\kappa}}$*. The scheme* Exc *is a disjoint union of* Exc_Λ *over all type* 0 *lattices* Λ *in* C*.*

Proof. Let $z = (X, \iota, \lambda, \rho, \mathcal{F}) \in \mathrm{Exc}(\bar{\kappa})$ and $M = \rho(M(X)) \subset N$. Then the action of $\iota(\pi)$ on $\mathrm{Lie} X$ is trivial. Hence $\Pi M \subset \mathbf{V} M$ as $\mathrm{Lie} X = M/\mathbf{V} M$. Since $\dim_{\bar{\kappa}}(M/\mathbf{V}M) = \dim_{\bar{\kappa}}(M/\Pi M) = n$, we know $\mathbf{V}M = \Pi M$ which is equivalent to $\tau(M) = M$. By [RTW14, Proposition 4.1], $M = \Lambda \otimes_{O_F} O_{\breve{F}}$ for some vertex lattice Λ. Since M is unimodular, Λ is of type 0. Hence $z \in \mathrm{Exc}_\Lambda(\bar{\kappa})$. Moreover for any $\bar{\kappa}$-algebra R, every rank $n-1$ locally direct summand of $\mathrm{Lie} X_R$ satisfies Krämer's signature condition as in Definition 2.1.2 and determines a point of $\mathrm{Exc}_\Lambda(R)$ uniquely. So we get an isomorphism $\mathbb{P}^{n-1}_{\bar{\kappa}} \to \mathrm{Exc}_\Lambda$. Since $\mathcal{N}$ is regular and Exc_Λ has codimension 1, Exc_Λ is a Cartier divisor in $\mathcal{N}$. By looking at $\rho(M(X))$, it is clear that $\mathrm{Exc}_\Lambda \cap \mathrm{Exc}_{\Lambda'}(k) = \emptyset$ if $\Lambda \neq \Lambda'$. Hence Exc is a disjoint union of over all type 0 lattices Λ. ∎

2.2 Special cycles and intersection numbers

2.2.1 Special cycles

We fix a canonical lift $(\mathcal{G}, \iota_{\mathcal{G}}, \lambda_{\mathcal{G}})$ of $(\mathbb{Y}, \iota_{\mathbb{Y}}, \lambda_{\mathbb{Y}})$ to $O_{\breve{F}}$ in the sense of [Gro86] such that the action of O_F on $\mathrm{Lie}\mathcal{G}$ is via the inclusion $\bar{i}$. Such a lift is unique up to isomorphism by [How19, Proposition 2.1].

Definition 2.2.1. *For an O_F-lattice L of $\mathbb{V}$, define $\mathcal{Z}(L)$ to be the subfunctor of $\mathcal{N}$ such that $\mathcal{N}(S)$ is the set of isomorphism classes of tuples $(X, \iota, \lambda, \rho, \mathcal{F}) \in \mathcal{N}(S)$ such that for any $x \in L$ the quasi-homomorphism*

$$\rho^{-1} \circ x \circ \rho_{\mathcal{G}} : \mathbb{Y} \times_{\operatorname{Spec} \bar{\kappa}} \bar{S} \to X \times_S \bar{S}$$

entends to a homomorphism $\mathcal{G} \times_{\operatorname{Spf} \mathcal{O}_{\breve{F}}} S \to X$. By Grothendieck-Messing theory $\mathcal{Z}(L)$ is a closed formal subscheme of $\mathcal{N}$. For $x \in \mathbb{V}$, we let $\mathcal{Z}(x) := \mathcal{Z}(L)$ when $L = \langle x \rangle$.

We can similarly define special cycles on $\mathcal{N}^{\mathrm{Pap}}$ and denote it as $\mathcal{Z}^{\mathrm{Pap}}(L)$. If we want to emphasize the special cycle $\mathcal{Z}(L)$ is on $\mathcal{N}_{n,\epsilon}$ (resp. $\mathcal{N}_{n,\epsilon}^{\mathrm{Pap}}$), we will denote it as $\mathcal{Z}_{n,\epsilon}(L)$ (resp. $\mathcal{Z}_{n,\epsilon}^{\mathrm{Pap}}(L)$).

Definition 2.2.2. *For an O_F-lattice $L \subset \mathbb{V}$, define $\tilde{\mathcal{Z}}(L)$ to be the strict transform of $\mathcal{Z}^{\mathrm{Pap}}(L)$ under the blow up $\mathcal{N}_n^{\mathrm{Kra}} \to \mathcal{N}_n^{\mathrm{Pap}}$.*

2.2.2 Horizontal and vertical part

A formal scheme X over $\operatorname{Spf} O_{\breve{F}}$ is called horizontal (resp. vertical) if it is flat over $\operatorname{Spf} \mathcal{O}_{\breve{F}}$ (resp. π is locally nilpotent on $\mathcal{O}_X$). For a formal scheme X over $\operatorname{Spf} O_{\breve{F}}$, its horizontal part $X_{\mathscr{H}}$ is canonically defined by the ideal sheaf $\mathcal{O}_{X,\mathrm{tor}}$ of torsion sections on $\mathcal{O}_X$. If X is noetherian, there exists a $m \in \mathbb{Z}_{>0}$ such that $\pi^m \mathcal{O}_{X,\mathrm{tor}} = 0$. We define the vertical part $X_{\mathscr{V}} \subset X$ to be the closed formal subscheme defined by the ideal sheaf $\pi^m \mathcal{O}_X$. Since $\mathcal{O}_{X,\mathrm{tor}} \cap \pi^m \mathcal{O}_X = \{0\}$, we have the following decomposition by primary decomposition

$$X = X_{\mathscr{H}} \cup X_{\mathscr{V}} \tag{2.10}$$

as a union of horizontal and vertical formal subschemes. Notice that the horizontal part $X_{\mathscr{H}}$ is canonically defined while the vertical part $X_{\mathscr{V}}$ depends on the choice of m.

Lemma 2.2.3. *For a lattice $L^\flat \subset \mathbb{V}$ of rank greater or equal to $n-1$ with non-degenerate hermitian form, $\mathcal{Z}(L^\flat)$ is noetherian.*

Proof. First we know that $\mathcal{Z}(L)$ is locally noetherian since it is a closed formal subscheme of $\mathcal{N}$ which is locally noetherian. Since the rank of L is greater or equal to $n-1$, the number of vertex lattices Λ containing L is finite. By (2.9), we know that $\mathcal{Z}(L)_{\mathrm{red}}$ is a closed subset in finitely many irreducible components of $\mathcal{N}_{\mathrm{red}}$. Since each irreducible component of $\mathcal{N}_{\mathrm{red}}$ is quasi-compact, we know that $\mathcal{Z}(L)$ is quasi-compact, hence noetherian. □

Lemma 2.2.4. *For a rank $n-1$ lattice $L^\flat \subset \mathbb{V}$ with non-degenerate hermitian form, $\mathcal{Z}(L)_{\mathscr{V}}$ is supported on the reduced locus $\mathcal{N}_{\mathrm{red}}$ of $\mathcal{N}$, i.e., $\mathcal{O}_{\mathcal{Z}(L)_{\mathscr{V}}}$ is annihilated by a power of the ideal sheaf of $\mathcal{N}_{\mathrm{red}}$.*

Proof. We remark here that $\mathcal{N}_{\mathrm{red}}$ is exactly the supersingular locus of $\mathcal{N}$. Hence the proof of the lemma is the same as that of [LZ22a, Lemma 5.1.1]. □

2.2.3 Derived special cycles

For a locally noetherian formal scheme X together with a formal subscheme Y, denote by $K_0^Y(X)$ the Grothendieck group of finite complexes of coherent locally free $\mathcal{O}_X$-modules acyclic outside Y. For such a complex $A^\bullet$, denote by $[A^\bullet]$ the element in $K_0^Y(X)$ represented by it. We use $K_0(X)$ to denote $K_0^X(X)$. Let $K_0'(Y)$ be the Grothendieck group of coherent sheaves of $\mathcal{O}_Y$-modules on Y. We have a group homomorphism $K_0^Y(X) \to K_0'(Y)$ which is an isomorphism when X is regular.

Denote by $\mathrm{F}^i K_0^Y(X)$ the codimension i filtration on $K_0^Y(X)$ and $\mathrm{Gr}^i K_0^Y(X)$ its i-th graded piece. When X is regular, we have a cup product $\cdot$ on $K_0^Y(X)_{\mathbb{Q}}$ defined by tensor product of complexes. Under the identification $K_0^Y(X) \xrightarrow{\sim} K_0'(Y)$, the cup product is nothing but derived tensor product:

$$[A] \cdot [B] = [A \otimes^{\mathbb{L}}_{\mathcal{O}_X} B].$$

When X is a scheme, the cup product satisfies ([SABK94, Section I.3, Theorem 1.3])

$$\mathrm{F}^i K_0^Y(X)_{\mathbb{Q}} \cdot \mathrm{F}^j K_0^Y(X)_{\mathbb{Q}} \subset \mathrm{F}^{i+j} K_0^Y(X)_{\mathbb{Q}}. \tag{2.11}$$

It is expected that (2.11) is also true when X is a formal scheme, see [Zha21, (B.3)], however we do not need this fact in this thesis. Throughout the thesis, we assume $X = \mathcal{N}$ unless stated otherwise.

Let $K_0'(Y)$ be the Grothendieck group of coherent sheaves of $\mathcal{O}_Y$-modules on Y. When X is regular we have the following isomorphism

$$K_0^Y(X) \cong K_0'(Y). \tag{2.12}$$

In particular, $K_0(X) \cong K_0'(X)$.

Recall that for $x \in \mathbb{V}$, $\mathcal{Z}(x)$ is a Cartier divisor ([How19, Proposition 4.3]).

Definition 2.2.5. *Let $L \subset \mathbb{V}$ be a rank r lattice with a basis $\{x_1, \ldots, x_r\}$. Define ${}^{\mathbb{L}}\mathcal{Z}(L)$ to be*

$$[\mathcal{O}_{\mathcal{Z}(x_1)} \otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{N}} \cdots \otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{N}} \mathcal{O}_{\mathcal{Z}(x_r)}] \in K_0^{\mathcal{Z}(L)}(\mathcal{N}) \tag{2.13}$$

where $\otimes^{\mathbb{L}}$ is the derived tensor product of complexes of coherent locally free sheaves on $\mathcal{N}$. By [How19, Corollary C], ${}^{\mathbb{L}}\mathcal{Z}(L)$ is independent of the choice of the basis $\{x_1, \ldots, x_r\}$.

Definition 2.2.6. *When L has rank n, we define the intersection number*

$$\mathrm{Int}(L) = \chi(\mathcal{N}, {}^{\mathbb{L}}\mathcal{Z}(L)), \tag{2.14}$$

where χ is the Euler characteristic.

Lemma 2.2.7. *When L is a rank n lattice in $\mathbb{V}$, $\mathcal{Z}(L)$ is a proper scheme over* $\operatorname{Spf} \mathcal{O}_{\breve{F}}$. *In particular,* $\mathrm{Int}(L)$ *is finite.*

Proof. By Lemma 2.2.4 $\mathcal{Z}(L)_{\mathscr{V}}$ is a scheme. We show that $\mathcal{Z}(L)_{\mathscr{H}}$ is empty. If not, there exists $z \in \mathcal{Z}(L)(\mathcal{O}_K)$ for some finite extension K of $\breve{F}$. Let X be the corresponding formal hermitian O_F-module of signature $(1, n-1)$ over $\mathcal{O}_K$. Since L has rank n and $\mathcal{G}$ has signature $(0,1)$, this would imply that X has signature $(0, n)$, which is a contradiction. Hence $\mathcal{Z}(L)$ is a scheme. Since $\mathcal{Z}(L)_{\mathrm{red}}$ is contained in finitely many irreducible components of $\mathcal{N}_{\mathrm{red}}$ and each irreducible component is proper over $\operatorname{Spec} \bar{\kappa}$, it follows that $\mathcal{Z}(L)$ is proper over $\operatorname{Spf} \mathcal{O}_{\breve{F}}$. The finiteness of $\mathrm{Int}(L)$ then follows from the same argument before [Zha21, (B.4)]. $\square$

Chapter 3

Kudla-Rapoport conjecture for $\mathrm{U}(1,1)$ over ramified primes

In this chapter, we treat the case $\mathbb{V}$ is 2-dimensional split hermitian space. Using the exceptional isomorphism between $\mathcal{N}_2$ and Drinfeld upper half plane (after base change and blow up) and Grothendieck-Messing theory, we write down the local equations of special divisors. Based on this, we prove the conjecture for the case $\mathbb{V}$ is 2-dimensional split hermitian space via explicit calculation. This chapter is based on the joint work with Shi and Yang ([HSY23b]).

Our goal is to prove the following theorem.

Theorem 3.0.1. *Let L be the unimodular O_F-hermitian lattice of rank 2 as in the beginning of the introduction with Gram matrix S, and let $T = T(\mathbf{x}_1, \mathbf{x}_2) \in \mathrm{Herm}_2(F)$ with $\mathbf{x}_i \in \mathbb{V}$ being independent. Then $\alpha(L, T) = 0$, and*

$$\mathrm{Int}(L_{\mathbf{x}_1,\mathbf{x}_2}) = 2\frac{\alpha'(L,T)}{\alpha(L,S)} - \frac{2q^2}{q^2-1}\frac{\alpha(H,T)}{\alpha(L,S)}.$$

3.1 Preliminary

Following [BC91] and [KR14a], we briefly review several moduli functors in this section for later use.

3.1.1 Framing objects and their Dieudonné modules

Throughout the thesis, by relative Dieudonné module we mean relative O_{F_0}-Dieudonné module in the sense of [RZ96, 3.56], which is a special case of the O_{F_0}-display studied in [ACZ16]. Let $\mathbb{Y}$ be the unique 1 dimensional formal O_{F_0}-module of relative height 2 over $\bar{\kappa}$ as in the introduction. Then its relative Dieudonné module (see page 4 of [KR14a] or [RZ96, Proposition 3.56]) $M(\mathbb{Y})$ is a free $O_{\breve{F}_0}$-module of rank 2. We can choose a basis $\{e, f\}$ of $M(\mathbb{Y})$ such that

$$Fe = f,\ Ff = \pi_0 e,\ Ve = f,\ Vf = \pi_0 e,\ \langle e, f\rangle = \delta. \tag{3.1}$$

With respect to this basis, we have

$$\mathrm{End}(M(\mathbb{Y})) \cong \left\{ \begin{pmatrix} a & b\pi_0 \\ b^\sigma & a^\sigma \end{pmatrix} \mid a, b \in O_E \right\} = O_B. \tag{3.2}$$

In particular

$$\iota_{\mathbb{Y}}(\pi) = \begin{pmatrix} 0 & \pi_0 \\ 1 & 0 \end{pmatrix}. \tag{3.3}$$

The framing object $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ in the introduction can be explicitly realized as follows. Let $\mathbb{X} = \mathbb{Y} \times \mathbb{Y}$ and identify $\mathrm{End}(\mathbb{X})$ with $M_2(O_B)$. Then we can define an action of O_B on $\mathbb{X}$ via

$$\iota_{\mathbb{X}}(b) = \begin{pmatrix} b & 0 \\ 0 & \pi b \pi^{-1} \end{pmatrix}. \tag{3.4}$$

Finally, we identify $\mathbb{X}^\vee = \mathbb{X}$ and define the principal polarization

$$\lambda_{\mathbb{X}} = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right). \tag{3.5}$$

Then the Rosati involution associated to $\lambda_{\mathbb{X}}$ induces the involution $b \mapsto b^* = \pi b^\iota \pi^{-1}$ on B. This triple $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ is the basic Drinfeld triple defined in [KR14a, Proposition 1.1]. We obtain the basic framing object $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ from the above framing object by restricting $\iota_{\mathbb{X}}$ on O_F. One can check that the hermitian form on $\mathbb{L} = \mathrm{Hom}_{O_F}(\mathbb{Y}, \mathbb{X})$ induced by $\lambda_{\mathbb{X}}$ is **isotropic** as required in this thesis (see Lemma 3.1.2).

We choose a basis $\{e_0, f_0, e_1, f_1\}$ of $M(\mathbb{X})$, the relative Dieudonné module of $\mathbb{X}$, such that $\{e_i, f_i\}$ $(i = 0, 1)$ is the basis of $M(\mathbb{Y})$ for the i-th copy of $\mathbb{Y}$. Then we have

$$Fe_i = f_i,\ Ff_i = \pi_0 e_i,\ Ve_i = f_i,\ Vf_i = \pi_0 e_i. \tag{3.6}$$

$$\iota_{\mathbb{X}}(\pi)e_i = f_i,\ \iota_{\mathbb{X}}(\pi)f_i = \pi_0 e_i,\ \iota_{\mathbb{X}}(\delta)e_i = (-1)^i \delta e_i,\ \iota_{\mathbb{X}}(\delta)f_i = (-1)^{i+1}\delta f_i. \tag{3.7}$$

Remark 3.1.1. *In some literature (e.g. [BC91] and [RTW14]), the operator $\iota_{\mathbb{X}}(\pi)$ is denoted as Π. in this thesis, since the action of O_F on the relative Dieudonné module (later Cartier module) is unambiguous, we will mostly write $\iota_{\mathbb{X}}(a)$ (later $\iota(a)$) simply as a for $a \in O_F$.*

We also need to consider the O_E-action on $M(\mathbb{X})$ obtained by restricting $\iota_{\mathbb{X}}$ on O_E. The grading on $M(\mathbb{X})$ as is

$$M_0 = M(\mathbb{X})_0 = \mathrm{span}_{O_{\breve{F}_0}}\{e_0, f_1\},\ M_1 = M(\mathbb{X})_1 = \mathrm{span}_{O_{\breve{F}_0}}\{e_1, f_0\}. \tag{3.8}$$

Let $N = N(\mathbb{X}) \coloneqq M(\mathbb{X}) \otimes \mathbb{Q}$. The principal polarization $\lambda_{\mathbb{X}}$ induces an alternating form $\langle , \rangle$ on N such that

$$\langle Fx, y\rangle = \langle x, Vy\rangle^{\sigma}, \tag{3.9}$$
$$\langle ax, y\rangle = \langle x, \bar{a}y\rangle, a \in O_F.$$

In terms of the $\breve{F}_0$-basis $\{e_0, f_0, e_1, f_1\}$, we have

$$\langle e_0, f_1\rangle = \langle e_1, f_0\rangle = \delta, \tag{3.10}$$

and all the other pairings between the basis vectors are 0. Following [KR14a], we can define a hermitian form $(,)$ on N by

$$(x, y) = \delta[\langle \pi x, y\rangle + \pi\langle x, y\rangle]. \tag{3.11}$$

Let $\tau = \pi V^{-1}$. Then $C := N^{\tau} = \mathrm{span}_F\{e_0, e_1\}$ is a hermitian F-vector space with $C \otimes_F \breve{F} = N$. Moreover,

$$(e_0, e_1) = -\delta^2,\ (e_0, e_0) = (e_1, e_1) = 0. \tag{3.12}$$

So $(C, (,))$ is isotropic, satisfying the assumption of this chapter. There is a similarly defined hermitian form $(,)_{\mathbb{Y}}$ on $N(\mathbb{Y})^{\tau}$ given by

$$(x, y)_{\mathbb{Y}} = \delta[\langle \pi x, y\rangle + \pi\langle x, y\rangle]. \tag{3.13}$$

Here $\langle e, f\rangle = \delta$ as in (3.1).

Recall $\mathbb{L} = \mathrm{Hom}_{O_F}(\mathbb{Y}, \mathbb{X}) \cong \mathrm{Hom}_{O_F}(M(\mathbb{Y}), M(\mathbb{X}))$ and $\mathbb{V} = \mathbb{L} \otimes_{O_F} F$. For $\mathbf{x} \in \mathbb{L}$, we will also use $\mathbf{x}$ to denote the corresponding homomorphism between relative Dieudonné modules when there is no confusion. Now $\mathbf{x}(e) \in M(\mathbb{X})^{\tau}$ since $\tau(e) = e$ by (3.2), (3.3) and $\mathbf{x}$ commutes with π and V. This way, we obtain a map from $\mathbb{L}$ to $M(\mathbb{X})^{\tau}$ by sending $\mathbf{x} \in \mathbb{V}$ to $\mathbf{x}(e)$.

Lemma 3.1.2. *([Shi18, Lemma 3.3]) We can identity $\mathbb{L}$ with $M(\mathbb{X})^{\tau}$ by sending $\mathbf{x} \in \mathbb{L}$ to $\mathbf{x}(e)$. Moreover,*

$$h(\mathbf{x}, \mathbf{x})(e, e)_{\mathbb{Y}} = (\mathbf{x}(e), \mathbf{x}(e)). \tag{3.14}$$

According to (3.13), $(e, e)_{\mathbb{Y}} = -\delta^2$. *In particular, $h(,)$ is also isotropic since $(,)$ is isotropic as we showed previously.*

Because of Lemma 3.1.2, we will often identify $\mathbb{V}$ with C, and $\mathbb{L}$ with $M(\mathbb{X})^{\tau}$, via $\mathbf{x} \mapsto \mathbf{x}(e)$. The hermitian form is shifted by a factor $-\delta^2$.

3.1.2 Moduli space $\mathcal{M}$

We first recall the following result of Kramer ([Krä03]).

Proposition 3.1.3. *$\mathcal{N}^{\mathrm{Kra}}$ is representable by a formal scheme over* $\mathrm{Spf}\, O_{\breve{F}}$ *which is regular and has semi-stable reduction. The natural forgetful map*

$$\Phi : \mathcal{N}^{\mathrm{Kra}} \to \mathcal{N}_{(1,1)}$$
$$(X, \iota, \lambda, \rho, \mathcal{F}_X) \mapsto (X, \iota, \lambda, \rho)$$

is an isomorphism outside the singular locus Sing *of* $\mathcal{N}_{(1,1)}$*, and* $\Phi^{-1}(x)$ *for* $x \in \mathrm{Sing}$ *is an exceptional divisor which is isomoprhic to the projective line* $\mathbb{P}^1_{\bar{\kappa}}$.

By [Shi22, Proposition 2.7], $\mathcal{N}^{\mathrm{Kra}}$ is the blow up of $\mathcal{N}_{(1,1)}$ along Sing.

As mentioned in the introduction, the formal scheme $\mathcal{N}_{(1,1)}$ descends to a formal scheme $\mathcal{N}$ over $O_{\breve{F}_0}$ (since the Kottwitz determinant condition is defined over $O_{\breve{F}_0}$). A beautiful result of Kudla and Rapoport ([KR14a]) asserts that $\mathcal{N}$ is actually isomorphic to the modular functor $\mathcal{M}$ which is represented by the base change $\breve{\Omega}$ of the formal completion of the Drinfeld's upper half plane. The observation is a key ingredient in obtaining the main results of this thesis. We now briefly review $\mathcal{M}$ and $\breve{\Omega}$ in this subsection and next one.

A formal O_B-module over an O_{F_0}-algebra R is a formal O_{F_0}-module X over R with an action $\iota_B : O_B \to \mathrm{End}(X)$ extending the action of O_{F_0}. X is called **special** if the action of $O_E \subset O_B$ makes Lie$X$ a free $R \otimes_{O_{F_0}} O_E$-module of rank one. The R-module $\mathrm{Lie}(X)$ is $\mathbb{Z}/2\mathbb{Z}$-graded by the action of O_E:

$$(\mathrm{Lie}X)_0 = \{x \in \mathrm{Lie}(X) \mid \iota_{O_B}(a)m = am \text{ for all } a \in O_E\},$$
$$(\mathrm{Lie}X)_1 = \{x \in \mathrm{Lie}(X) \mid \iota_{O_B}(a)m = \sigma(a)m \text{ for all } a \in O_E\}.$$

Then X is special if $\mathrm{Lie}(X)_i$ is a free R-module of rank one. Over $\bar{\kappa}$, there is a unique special formal O_B-module up to O_B-linear isogeny, which is $(\mathbb{X}, \iota_{\mathbb{X}})$ in Section 3.1.1.

Definition 3.1.4. *We define a moduli functor $\mathcal{M}$ on $\mathrm{Nilp}_{O_{\breve{F}_0}}$ that associates S with the set of isomorphism classes of triples (X, ι_B, ρ), where*

- *(X, ι_B) is a special formal O_B-module of dimension 2 and relative height 4 over S.*
- *ρ is a O_B-linear quasi-isogeny*

$$\rho : X \times_S \bar{S} \to \mathbb{X} \times_{\mathrm{Spec}\, \bar{\kappa}} \bar{S}$$

  of height 0. Here $\bar{S}$ is the special fiber of S.

The following is a result of Drinfeld that shows the automatic existence of polarizations on special formal O_B-modules (see Proposition 1.1 of [KR14a]).

Proposition 3.1.5 (Drinfeld)**.** *Let $\pi \in O_B$ be a uniformizer such that $\pi^2 = \pi_0$, and consider the involution $b \to b^* = \pi b' \pi^{-1}$ of B, where $b \to b'$ denotes the main involution.*

(a) On $\mathbb{X}$ there exists a principal polarization $\lambda^0_{\mathbb{X}} : \mathbb{X} \xrightarrow{\sim} \mathbb{X}^\vee$ with associated Rosati involution $b \to b^*$. Furthermore, $\lambda^0_{\mathbb{X}}$ is unique up to a factor in $O_{F_0}^\times$.

(b) Fix $\lambda^0_{\mathbb{X}}$ as in (a). Let $(X, \iota, \rho) \in \mathcal{M}(S)$, where $S \in \mathrm{Nilp}_{O_{\breve{F}_0}}$. On X there exists a unique principal polarization $\lambda^0_X : X \xrightarrow{\sim} X^\vee$ making the following diagram commutative,

$$\begin{array}{ccc} X \times_S \bar{S} & \xrightarrow{\lambda^0_X} & X^\vee \times_S \bar{S} \\ \downarrow \rho & & \uparrow \rho^\vee \\ \mathbb{X} \times_{\mathrm{Spec}\, \bar{\kappa}} \bar{S} & \xrightarrow{\lambda^0_{\mathbb{X}}} & \mathbb{X}^\vee \times_{\mathrm{Spec}\, \bar{\kappa}} \bar{S} \end{array}$$

Theorem 3.1.6. *([KR14a, Theorem 1.2]) Assume that $p \neq 2$. The morphism of functors on $\mathrm{Nilp}_{O_{\breve{F}_0}}$ given by $(X, \iota_B, \rho) \to (X, \iota, \lambda^0_X, \rho)$ induces an isomorphism of formal schemes*

$$\eta_F : \mathcal{M} \xrightarrow{\sim} \mathcal{N}.$$

Here ι is the restriction of ι_B to O_F and λ^0_X is the principal polarization given by Proposition 3.1.5.

3.1.3 Drinfeld upper half plane and its formal completion

For convenience of the reader and to set up notation for the rest of the thesis, we briefly review the well-known facts about the Drinfeld upper half place Ω, its formal completion $\widehat{\Omega}$, and its base change $\breve{\Omega}$ to $W = O_{\breve{F}_0}$, following [BC91].

Recall that the Bruhat-Tits tree $\mathcal{B}(\mathrm{PGL}_2(F_0))$ consists of vertices and edges. The vertices are given by the homothety classes $[\Lambda]$ of O_{F_0}-lattices in F_0^2, and the edges are given by pairs $([\Lambda], [\Lambda'])$ of the homothety classes such that $\pi_0\Lambda' \subset \Lambda \subset \Lambda'$ for suitable choices of lattices Λ and Λ' in their homothety classes. We then say Λ and Λ' are adjacent. We use $\mathbb{P}_{[\Lambda]}$ to denote the projective line over O_{F_0} associated to Λ depending on its homothety class. Let

$$\Omega_{[\Lambda]} = \mathbb{P}_{[\Lambda]} - \mathbb{P}_{[\Lambda]}(\kappa)$$

be the projective line with the $q+1$ rational κ-points removed. When $([\Lambda], [\Lambda'])$ is an edge, $\Lambda \pmod{\pi_0}$ gives a κ-rational point in $\mathbb{P}_{[\Lambda']}$. We write $\mathbb{P}_{[\Lambda,\Lambda']}$ for the blow up of $\mathbb{P}_{[\Lambda']}$ at this point, which is isomorphic to the blow up of $\mathbb{P}_{[\Lambda]}$ at the rational point determined by Λ'. We write $\Omega_{[\Lambda,\Lambda']}$ for the complement of the nonsingular rational points of the special fiber of $\mathbb{P}_{[\Lambda,\Lambda']}$. There is an open embedding $\Omega_{[\Lambda]} \hookrightarrow \Omega_{[\Lambda,\Lambda']}$.

Define $\widehat{\Omega}_{[\Lambda]}$ and $\widehat{\Omega}_{[\Lambda,\Lambda']}$ to be the formal completion of $\Omega_{[\Lambda]}$ and $\Omega_{[\Lambda,\Lambda']}$ along their special fibers respectively. For two different edges with a common vertex $[\Lambda]$, we can glue $\widehat{\Omega}_{[\Lambda,\Lambda']}$ and $\widehat{\Omega}_{[\Lambda,\Lambda'']}$ along $\widehat{\Omega}_{[\Lambda]}$. This gives the formal scheme $\widehat{\Omega}$ by glueing over $\mathcal{B}(\mathrm{PGL}_2(F_0))$. The generic fiber of $\widehat{\Omega}$ is the Drinfeld p-adic half space $\Omega = \mathbb{P}^1(\mathbb{C}_p) - \mathbb{P}^1(F_0)$, where $\mathbb{C}_p$ is the completion of $\bar{F}_0$. Define $\breve{\Omega} = \widehat{\Omega} \times_{\mathrm{Spf}\, O_{F_0}} \mathrm{Spf}\, O_{\breve{F}_0}$. Similarly define $\breve{\Omega}_{[\Lambda]}$ and $\breve{\Omega}_{[\Lambda,\Lambda']}$. Drinfeld's well-known result (see [Dri76] and [BC91]) asserts that $\mathcal{M}$ is represented by $\breve{\Omega}$. The following proposition will be used in Section 4.

Proposition 3.1.7. *(Deligne functor) ([BC91, Propositions 4.2, 4.4])*

(a) $\widehat{\Omega}_\Lambda$ represents the functor that associates an O_{F_0}-algebra $R \in \mathrm{Nilp}_{O_{F_0}}$ the collection of isomorphism classes of pairs $(\mathcal{L}, \alpha)$, where $\mathcal{L}$ is a free R-module of rank 1 and $\alpha : \Lambda \to \mathcal{L}$ is a homomorphism of O_{F_0}-modules satisfying the condition:

- *for all* $x \in \operatorname{Spec}(R/\pi_0 R)$, *the map* $\alpha(x) : \Lambda/\pi_0\Lambda \to \mathcal{L} \otimes_R k(x)$ *is injective.*

(b) $\widehat{\Omega}_{[\Lambda,\Lambda']}$ *represents the functor that associates an* O_{F_0}*-algebra* $R \in \mathrm{Nilp}_{O_{F_0}}$ *the collection of isomorphism classes of commutative diagrams:*

$$\begin{tikzcd} \pi_0\Lambda' \arrow[r, hook] \arrow[d, "\alpha'/\pi_0"] & \Lambda \arrow[r, hook] \arrow[d, "\alpha"] & \Lambda' \arrow[d, "\alpha'"] \\ \mathcal{L}' \arrow[r, "c'"] & \mathcal{L} \arrow[r, "c"] & \mathcal{L}' \end{tikzcd}$$

where $\mathcal{L}$ *and* $\mathcal{L}'$ *are free* R*-modules of rank 1,* α *and* α' *are the homomorphisms of* $\mathcal{O}_{F_0}$*-modules,* c *and* c' *are homomorphisms of* R*-modules, satisfying the conditions: for all* $x \in \operatorname{Spec}(R/\pi_0 R)$, *one has*

- *ker*$(\alpha'(x)\colon \Lambda'/\pi_0\Lambda' \to \mathcal{L}' \otimes_R k(x)) \subset \Lambda/\pi_0\Lambda'$,
- *ker*$(\alpha(x)\colon \Lambda/\pi_0\Lambda \to \mathcal{L} \otimes_R k(x)) \subset \pi_0\Lambda'/\pi_0\Lambda$.

We now describe the explicit equations of $\widehat{\Omega}_{[\Lambda]}$ and $\widehat{\Omega}_{[\Lambda,\Lambda']}$. Let $\{e_1, e_2\}$ be a basis of Λ. Then we have

$$\widehat{\Omega}_{[\Lambda]} = (\mathbb{P}_{[\Lambda]} - \mathbb{P}_{[\Lambda]}(k))^\wedge = \operatorname{Spf} O_{F_0}[T, (T^q - T)^{-1}]^\wedge, \tag{3.15}$$

where $T = \frac{X_0}{X_1}$ and X_i is the coordinate of $\mathbb{P}_{[\Lambda]}$ with respect to the basis $\{e_1, e_2\}$ and "hat" indicates completion along the special fiber.

Without loss of generality, we can assume Λ' has a basis $\{e_1, \pi_0^{-1} e_2\}$. Then we have

$$\widehat{\Omega}_{[\Lambda,\Lambda']} = \operatorname{Spf} O_{F_0}[T_0, T_1, (T_0^{q-1} - 1)^{-1}, (T_1^{q-1} - 1)^{-1}]^\wedge/(T_0T_1 - \pi_0). \tag{3.16}$$

In this case, the open immersions $\widehat{\Omega}_{[\Lambda]} \hookrightarrow \widehat{\Omega}_{[\Lambda,\Lambda']}$ and $\widehat{\Omega}_{[\Lambda']} \hookrightarrow \widehat{\Omega}_{[\Lambda,\Lambda']}$ are induced by

$$T_0 \mapsto T,\ T_1 \mapsto \pi_0 \cdot T^{-1}, \tag{3.17}$$

and

$$T_0 \mapsto \pi_0 \cdot T^{-1},\ T_1 \mapsto T. \tag{3.18}$$

3.2 Special fiber of special cycle

In this section, we study the support of the special cycles. Essentially we only need the Pappas model $\mathcal{N}_{(1,1)}$ (see Proposition 3.2.5).

3.2.1 Special fiber of $\mathcal{N}_{(1,1)}$.

Since $O_F/(\pi) = O_{F_0}/(\pi_0) = \kappa$, we have $\mathcal{N}_{(1,1)}/\bar{\kappa} \cong \mathcal{N}/\bar{\kappa}$. We briefly review the structure of $\mathcal{N}_{(1,1)}/\bar{\kappa}$ following [KR14a].

Recall that $C = N^\tau, \tau = \pi V^{-1}$. For an O_F-lattice Λ_F in C, set

$$\Lambda_F^\sharp = \{x \in C \mid (x, \Lambda_F) \subset O_F\}.$$

Similarly, for a lattice $O_{\breve{F}}$-lattice $M \subset N$, set

$$M^\sharp = \{x \in N \mid (x, M) \subset O_{\breve{F}}\}.$$

For an O_F-lattice Λ in C, Λ is called a vertex lattice of type t if $\pi\Lambda \subset \Lambda^\sharp \overset{t}{\subset} \Lambda$, meaning $[\Lambda : \Lambda^\sharp] = t$. In our case $t = 0$ or 2 by Lemma 3.2 of [RTW14].

Remark 3.2.1. *In the rest of this chapter, we will often use the subscript* 0 *or* 2 *to indicate the type of a vertex* O_F*-lattice. For example* Λ_0 *often stands for a vertex lattice of type* 0.

Let $\mathcal{B} = \mathcal{B}(\mathrm{PU}(C))$ be the Bruhat-Tits tree of $\mathrm{PU}(C)$—the projective unitary group of C. Its vertices correspond to vertex O_F-lattices Λ_2 of type 2, and its edges correspond to vertex O_F-lattices Λ_0 of type 0. Two vertices Λ_2 and Λ_2' are connected by an edge Λ_0, or adjacent, if and only if $\Lambda_2 \cap \Lambda_2' = \Lambda_0$. The following lemma is easy to check (recall that C is isotropic) and is left to the reader.

Lemma 3.2.2. *Let* Λ_0 *be a vertex* O_F*-lattice of type* 0*. There is an* O_F*-basis* $\{w_0, w_1\}$ *of* Λ_0 *with Gram matrix* $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$*. There are exactly two vertex* O_F*-lattices of type* 2 *containing* Λ_0*:*

$$\Lambda_2 = O_F\pi^{-1}w_0 + O_F w_1, \quad \textit{and} \quad \Lambda_2' = O_F w_0 + O_F \pi^{-1} w_1.$$

There are $q+1$ adjacent (type 2) vertices of Λ_2 in $\mathcal{B}$, and they are

$$\Lambda_\infty = O_F\pi^{-2}w_0 + O_F\pi w_1, \quad \Lambda_k = O_F w_0 + O_F\pi^{-1}(k\pi^{-1}w_0 + w_1)$$

where k runs through the representatives of $O_{F_0}/(\pi_0)$.

Recall the following results.

Proposition 3.2.3. *([RTW14, Proposition 2.2] and [KR14a, Lemma 3.2]) Let $\mathcal{L}(N)$ be the set of $O_{\breve{F}}$-lattices*

$$\{M \subset N \mid \pi_0 M \subset VM \overset{2}{\subset} M, \quad M^\sharp = M, \quad \dim_{\bar{\kappa}} VM/(VM \cap \pi M) \leq 1, \}.$$

Then the map

$$\mathcal{N}_{(1,1)}(\bar{\kappa}) \to \mathcal{L}(N), \quad x = (X, \iota, \lambda, \rho) \mapsto M(x) := \rho(M(X)) \subset N$$

is a bijection. Moreover, for $M = M(x) \in \mathcal{L}(N)$, we have

1. *If M is τ-stable, then M is of the form $M = \Lambda_0 \otimes_{O_F} O_{\breve{F}}$ for some vertex O_F-lattice Λ_0 of type 0 in C,*
2. *If M is not τ-stable, then*

$$M + \tau M = \Lambda_2 \otimes_{O_F} O_{\breve{F}}$$

for some vertex O_F-lattice Λ_2 of type 2 in C.

Proposition 3.2.4. *As in the introduction, we use $\mathbb{P}_{\bar{\Lambda}_2}$ to denote the projective line over $\bar{\kappa}$ associated to $\Lambda_2 \otimes_{O_F} \bar{\kappa}$.*

(1) For every vertex O_F-lattice Λ_2 of type 2, there is a closed immersion over $\bar{\kappa}$

$$i_{\bar{\Lambda}_2} : \mathbb{P}_{\bar{\Lambda}_2} \to \mathcal{N}_{(1,1)}/\bar{\kappa},$$

which is given on $\bar{\kappa}$-points as follows: it sends a line $l \subset \Lambda_2 \otimes_{O_F} \bar{\kappa}$ to its preimage under the projection $\Lambda_2 \otimes \mathcal{O}_{\breve{F}} \to (\Lambda_2/\pi\Lambda_2) \otimes_{O_F} \bar{\kappa}$. Moreover, $i_{\Lambda_2}(l)$ is τ-invariant if and only if l is κ-rational, i.e., l is the extension of a line in $\Lambda_2/\pi\Lambda_2$.

(2) Let $\mathbb{P}_{\bar{\Lambda}_2}$ denote also its image in $\mathcal{N}_{(1,1)}/\bar{\kappa}$ under i_{Λ_2}. Then

$$\mathcal{N}_{(1,1)}/\bar{\kappa} = \bigcup_{\Lambda_2} \mathbb{P}_{\bar{\Lambda}_2},$$

where the union is over all vertex lattices of type 2. Two such lines $\mathbb{P}_{\bar{\Lambda}_2}$ and $\mathbb{P}_{\bar{\Lambda}'_2}$ intersect if and only if Λ_2 and Λ'_2 are connected by an edge, i.e., $\Lambda_2 \cap \Lambda'_2 = \Lambda_0$ is a vertex O_F-lattice of type 0. In such a case, the two projective lines intersect at exactly one κ-rational point, denoted by pt_{Λ_0}. Such a point is called a superspecial point in $\mathcal{N}_{(1,1)}$.

(3) The singular locus Sing *of $\mathcal{N}_{(1,1)}$ consists of all the superspecial pt_{Λ_0} with Λ_0 running through all vertex O_F-lattices of type 0.*

Proof. For the proof of (1), see [KR14a, Lemma 3.3, Proposition 3.4]. For (2), if a point x lies in the intersection between $\mathbb{P}_{\bar{\Lambda}_2}$ and $\mathbb{P}_{\bar{\Lambda}'_2}$, then $M(x) \subset \Lambda_2 \otimes_{O_F} O_{\breve{F}} \cap \Lambda'_2 \otimes_{O_F} O_{\breve{F}}$ by the description of $i_{\bar{\Lambda}_2}$ in (1). Together with the fact $M(x) = M(x)^\sharp$, this implies $\Lambda_2 \cap \Lambda'_2 = \Lambda_0$, where Λ_0 is some vertex lattice of type 0. The converse is straightforward. Part (3) is immediate from part (2) and the description of the singular locus of $\mathcal{N}_{(1,1)}$ by local model (see for example [Pap00, Theorem 4.5]). $\square$

Recall that the blow up $\Phi : \mathcal{N}^{\mathrm{Kra}} \to \mathcal{N}_{(1,1)}$ is an isomorphism outside the singular locus Sing.

Proposition 3.2.5. *(1) For a type 2 lattice Λ_2, the Zariski closure of $\Phi^{-1}(\mathbb{P}_{\bar{\Lambda}_2} \setminus Sing)$ is a projective line over $\bar{\kappa}$ which we still denote by $\mathbb{P}_{\bar{\Lambda}_2}$.*

(2) For a vertex lattice Λ_0 of type 0, $\Phi^{-1}(pt_{\Lambda_0})$ is a projective line over $\bar{\kappa}$ which we denote by Exc_{Λ_0}.

(3) On the special fiber of $\mathcal{N}^{\mathrm{Kra}}$, two different lines $\mathbb{P}_{\bar{\Lambda}_2}$ and $\mathbb{P}_{\bar{\Lambda}'_2}$ never intersect, and two different lines Exc_{Λ_0} and $\mathrm{Exc}_{\Lambda'_0}$ never intersect. $\mathbb{P}_{\bar{\Lambda}_2}$ and Exc_{Λ_0} intersect at one point if $\Lambda_0 \subset \Lambda_2$, otherwise they do not intersect.

Proof. The above facts will be clear after the discussion in Section 3.3.3. $\square$

A superspecial point pt_{Λ_0} belongs to $\mathcal{Z}(\mathbf{x})$ if and only if $\mathrm{Exc}_{\Lambda_0} \subseteq \mathcal{Z}^{\mathrm{Kra}}(\mathbf{x})$. So in the rest of the section, we only need to study $\mathcal{Z}(\mathbf{x})$.

3.2.2 Bruhat-Tits trees of special cycles: rank 1 case

We let

$$\mathcal{T}(\mathbf{x}) = \{\Lambda_2 \text{ is of type } 2 \mid \mathbb{P}_{\bar{\Lambda}_2} \cap \mathcal{Z}(\mathbf{x}) \neq \emptyset\} \cup \{\Lambda_0 \text{ is of type } 0 \mid pt_{\Lambda_0} \in \mathcal{Z}(\mathbf{x})\}.$$

We will also view a lattice in $\mathcal{T}(\mathbf{x})$ as a vertex or an edge in $\mathcal{B}$ depending on whether it is of type 2 or 0. We will see that $\mathcal{T}(\mathbf{x})$ is a tree by Corollaries 3.2.11 and 3.2.13.

Lemma 3.2.6. *Let $\mathbf{x} \in \mathbb{V}$. Then*

$$\mathcal{Z}(\mathbf{x})(\bar{\kappa}) \cap \mathbb{P}_{\bar{\Lambda}_2}(\bar{\kappa}) = \begin{cases} \mathbb{P}_{\bar{\Lambda}_2}(\bar{\kappa}), & \text{if } \mathbf{x}(e) \in \pi\Lambda_2, \\ \text{a single point}, & \text{if } \mathbf{x}(e) \in \Lambda_2 \setminus \pi\Lambda_2, \\ \emptyset, & \text{if } \mathbf{x}(e) \notin \Lambda_2. \end{cases}$$

In particular, $\Lambda_2 \in \mathcal{T}(\mathbf{x})$ if and only if $\mathbf{x}(e) \in \Lambda_2$.

Proof. Let $x = (X, \iota, \lambda, \rho) \in \mathcal{N}_{(1,1)}(\bar{\kappa})$, let $M(x) = \rho(M(X)) \subset N$ as in Proposition 3.2.3. Then

$$\begin{aligned} x \in \mathcal{Z}(\mathbf{x})(\bar{\kappa}) &\iff \mathbf{x}(M(\mathbb{Y})) \subset M(x) \\ &\iff \mathbf{x}(e) \in M(x),\ \mathbf{x}(f) \in M(x) \\ &\iff \mathbf{x}(e) \in M(x) \end{aligned}$$

since $\mathbf{x}(e) \in M(x)$ implies $\mathbf{x}(f) = \mathbf{x}(Ve) \in VM(x) \subset M(x)$.

On the other hand,

$$x \in \mathbb{P}_{\bar{\Lambda}_2}(\bar{\kappa}) \iff \pi\Lambda_2 \otimes_{\mathcal{O}_F} \mathcal{O}_{\breve{F}} \subset M(x) \subset \Lambda_2 \otimes_{\mathcal{O}_F} \mathcal{O}_{\breve{F}}.$$

If $\mathbf{x}(e) \in \pi\Lambda_2$, then $\mathbf{x}(e) \in M(x)$ for any x such that $\pi\Lambda_2 \subset M(x) \subset \Lambda_2$. This implies $\mathbb{P}_{\bar{\Lambda}_2}(\bar{\kappa}) \subset \mathcal{Z}(\mathbf{x})(\bar{\kappa})$.

If $\mathbf{x}(e) \in \Lambda_2 \setminus \pi\Lambda_2$, then the image of $\mathbf{x}(e)$ is contained in exactly one line in $\bar{\Lambda}_{F,2}$ and thus gives a single point in $\mathbb{P}^1_{\bar{\Lambda}_2}(\bar{\kappa})$.

Finally, if $\mathbf{x}(e) \notin \Lambda_2$, it can not lie in any sub-lattice of $\Lambda_2 \otimes_{\mathcal{O}_F} \mathcal{O}_{\breve{F}}$.

□

From now on, fix $0 \neq \mathbf{x} \in \mathbb{V}$, and let $b = \mathbf{x}(e)$, $q(b) = (b, b) = -\delta^2 h(\mathbf{x}, \mathbf{x})$.

Lemma 3.2.7. *Assume $q(b) \neq 0$. Then there exist a unique vertex O_F-lattice Λ_b of type 0 such that $\pi^{-\mathrm{ord}_{\pi_0}(q(b))} b \in \Lambda_b \setminus \pi\Lambda_b$.*

Proof. Replacing b by $\pi^{-\mathrm{ord}_{\pi_0}(q(b))} b$ if necessary, we may assume $\mathrm{ord}_{\pi_0}(q(b)) = 0$, i.e., $(b, b) = u_0 \in \mathcal{O}_{F_0}^\times$. Write $(Fb)^\perp = Fc$, then $C = Fb \oplus Fc$. Since $q(b)$ is a unit, by [Jac62, Proposition 4.2], every vertex lattice of type 0 containing b has the orthogonal decomposition

$$\Lambda = O_F b + \Lambda_1, \quad \Lambda_1 = \Lambda \cap Fc = O_F b'$$

for some $b' = rc$. Λ is a vertex lattice of type 0 if and only if $q(b') = r\bar{r}q(c)$ is a unit. Such an r exists and is unique up to a unit in $O_F^\times$. So the vertex O_F-lattice Λ_b of type 0 such that $b \in \Lambda_b \setminus \pi\Lambda_b$ exists and is unique. □

Definition 3.2.8. *Assume that each edge in $\mathcal{B}$ has length 1. Let Λ and Λ' be two vertex O_F-lattices. The distance $d(\Lambda, \Lambda')$ is defined to be*

1. *the distance between the two vertices in $\mathcal{B}$ if both are of type 2,*
2. *the distance between the vertex Λ and the midpoint of the edge Λ' in $\mathcal{B}$ if Λ is of type 2 and Λ' is of type 0,*
3. *the distance between the midpoints of the edges Λ and Λ' in $\mathcal{B}$ if both are of type 0.*

For a vertex lattice Λ, we define

$$n(b, \Lambda) := \max\{n \in \mathbb{Z} \mid \pi^{-n} b \in \Lambda\}. \tag{3.19}$$

It is easy to check that for $\Lambda_2 \cap \Lambda_2' = \Lambda_0$,

$$n(b, \Lambda_0) = \min(n(b, \Lambda_2), n(b, \Lambda_2')). \tag{3.20}$$

We have the following reformulation of Lemma 3.2.6.

Lemma 3.2.9. *If $b = \mathbf{x}(e)$, $n(b, \Lambda) \geq 0$ if and only if $\Lambda \in \mathcal{T}(\mathbf{x})$.*

Lemma 3.2.10. *Assume $q(b) \neq 0$.*

1. *If Λ is a vertex O_F-lattice of type 2, then*

$$n(b, \Lambda) = \mathrm{ord}_{\pi_0}(q(b)) - d(\Lambda, \Lambda_b) + \frac{1}{2}.$$

2. *If Λ is a vertex O_F-lattice of type 0, then*

$$n(b, \Lambda) = \mathrm{ord}_{\pi_0}(q(b)) - d(\Lambda, \Lambda_b).$$

Proof. Claim (2) follows from Claim (1) and (3.20). Now we prove Claim (1). Without lost of generality we can assume $\mathrm{ord}_{\pi_0}(q(b)) = 0$. We prove the lemma by induction on $d(\Lambda, \Lambda_b)$. Let us treat the case $d(\Lambda, \Lambda_b) = \frac{1}{2}$ first, i.e. $\Lambda_b \subset \Lambda$. We have by Lemma 3.2.2

$$\Lambda_b = O_F w_0 + O_F w_1, \quad \Lambda = O_F \pi^{-1} w_0 + O_F w_1.$$

Write $b = xw_0 + yw_1$ with $q(b) = x\bar{y} + y\bar{x} \in O_{F_0}^\times$. Then $x, y \in O_F^\times$, and $b \in \Lambda \setminus \pi\Lambda$. Hence $n(b, \Lambda) = 0$.

Now we assume the assertion holds for all Λ such that $\frac{1}{2} \leq d(\Lambda, \Lambda_b) \leq d + \frac{1}{2}$. For a Λ such that $d(\Lambda, \Lambda_b) = d + \frac{1}{2} \neq \frac{1}{2}$, which satisfies the formula $n := n(b, \Lambda) = -d$, it suffices to show that all its adjacent vertices also satisfy the formula in the lemma.

Choose a basis $\{v_0, v_1\} = \{\pi^{-1}w_0, w_1\}$ of Λ with Gram matrix $\pi^{-1}\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. By Lemma

3.2.2, the $q+1$ neighbors of Λ in the Bruhat-Tits tree are

$$\Lambda_\infty = \mathrm{span}_{\mathcal{O}_F}\{\pi^{-1}v_0, \pi v_1\}, \quad \Lambda_k = \mathrm{span}_{\mathcal{O}_F}\{\pi v_0, \pi^{-1}(kv_0+v_1)\},$$

where k runs through the representatives of $O_{F_0}/(\pi_0)$. We want to show the claim holds for all the neighbors. One of the neighbors will be closer to Λ_b than other neighbors. Without loss of generality, we can assume Λ_∞ is closer to Λ_b than all the other Λ_k. By induction, we know

$$n(b, \Lambda_\infty) = n+1.$$

By the definition of $n(b, \Lambda)$, we can write

$$b = \pi^n(a_0v_0 + a_1v_1) = \pi^{n+1}(\pi^{-1}a_0v_0 + \pi^{-1}a_1v_1),$$

where $\mathrm{ord}_\pi(a_i) \geq 0$ and $\min\{\mathrm{ord}_\pi(a_0), \mathrm{ord}_\pi(a_1)\} = 0$. Since $n(b,\Lambda_\infty) = n+1$, $\pi^{-1}a_0v_0 + \pi^{-1}a_1v_1 \in \Lambda_\infty$ implies $\mathrm{ord}_\pi(a_1) \geq 2$. Hence $\mathrm{ord}_\pi(a_0) = 0$. Claim:

$$\mathrm{ord}_\pi(a_1) \geq 2 \text{ and } \mathrm{ord}_\pi(a_0) = 0 \implies a_0v_0 + a_1v_1 \notin \Lambda_k.$$

To prove this, assume $a_0v_0 + a_1v_1 \in \Lambda_k$ which implies that we can find $a_0', a_1' \in \mathcal{O}_F$ such that

$$a_0v_0 + a_1v_1 = a_0'\pi v_0 + a_1'\pi^{-1}(kv_0 + v_1) = (a_0'\pi + a_1'\pi^{-1}k)v_0 + a_1'\pi^{-1}v_1.$$

As a result,

$$\mathrm{ord}_\pi(a_0'\pi + a_1'\pi^{-1}k) = \mathrm{ord}_\pi(a_0'\pi + a_1k) \geq \min\{\mathrm{ord}_\pi(a_0'\pi), \mathrm{ord}_\pi(a_1)\} \geq 1.$$

But then $a_0'\pi + a_1'\pi^{-1}k$ can not equal to a_0 since we know $\mathrm{ord}_\pi(a_0) = 0$, which is a contradiction.

The claim gives us that $n(b, \Lambda_k) \leq n - 1$. we can easily check $\pi(a_0 v_0 + a_1 v_1) \in \Lambda_k$. Therefore $n(b, \Lambda_k) = n - 1$ as claimed. $\square$

Combining Lemma 3.2.9 with Lemma 3.2.10, we have the following:

Corollary 3.2.11. *Assume $q(b) \neq 0$, $\mathcal{T}(\mathbf{x})$ is a ball centered at the midpoint of Λ_b with radius* $\mathrm{ord}_{\pi_0}(q(b)) + \frac{1}{2}$. *In particular, $\mathcal{T}(\mathbf{x})$ is empty if and only $q(b) \notin O_{F_0}$.*

Lemma 3.2.12. *Assume that $b \neq 0$ and $q(b) = 0$. Let Λ be a vertex lattice of type 2 and assume $n(b, \Lambda) = n$. Then there exists a unique type 2 adjacent lattice Λ_+ of Λ such that $n(b, \Lambda_+) = n + 1$. For any other type 2 adjacent vertex lattice Λ' of Λ, we have $n(b, \Lambda') = n - 1$.*

Proof. By scaling b by a power of π we can assume that $n = 0$. By Lemma 3.2.2, the lattice $\Lambda \approx H$ with Gram matrix $\begin{pmatrix} 0 & \pi^{-1} \\ -\pi^{-1} & 0 \end{pmatrix}$ (as in the introduction). So there is some $b_1 \in \Lambda$ such that $(b, b_1) = \pi^{-1}$. Define $b' = b_1 + \pi a b$ where $a = \frac{1}{2}(b_1, b_1)$. Since $(b_1, b_1) \in \pi^{-1} O_{F_0}$, then $\{b, b'\}$ is a basis of Λ with Gram matrix $\pi^{-1} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

The adjacent type 2 vertex lattices of Λ are

$$\Lambda_\infty = \mathrm{span}_{O_F}\{\pi^{-1} b, \pi b'\},\ \Lambda_k = \mathrm{span}_{O_F}\{\pi b, \pi^{-1}(kb + b')\},$$

where k runs through the representatives of $O_{F_0}/(\pi_0)$. Then it is obvious that $n(b, \Lambda_\infty) = 1$ and $n(b, \Lambda_k) = -1$ for any k. The lemma follows. $\square$

Corollary 3.2.13. *Assume that $b \neq 0$ and $q(b) = 0$. Then $\mathcal{T}(\mathbf{x})$ is a cone with the boundary consisting of vertex lattices Λ of type 2 with $n(b, \Lambda) = 0$. Starting with such a vertex Λ, there is a unique (half) geodesic such that the number $n(b, \Lambda)$ increases along the geodesic. We call such a geodesic an ascending geodesic starting with Λ. Any two ascending geodesics coincide after finitely many steps. An ascending geodesic can be thought of as an 'axis' of the cone $\mathcal{T}(\mathbf{x})$.*

3.2.3 Bruhat-Tits trees of special cycles: rank 2 case

The following is an analogue of Lemma 2.11 from [San17].

Lemma 3.2.14. *Let* $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{V}$, $b_i = \mathbf{x_i}(e)$, $n_i = \mathrm{ord}_{\pi_0}(q(b_i))$ *and assume* $q(b_i) \neq 0$, *for* $i \in \{1, 2\}$. *Assume* $n_1 \leq n_2$. *Suppose that* $\mathcal{T}(\mathbf{x}_1) \cap \mathcal{T}(\mathbf{x}_2) \neq \emptyset$. *Let* Λ_{b_1,b_2} *be the vertex lattice such that*

$$d(\Lambda_{b_1}, \Lambda_{b_1,b_2}) = n_1 + \frac{1}{2} - r \text{ and } d(\Lambda_{b_1}, \Lambda_{b_1,b_2}) + d(\Lambda_{b_2}, \Lambda_{b_1,b_2}) = d(\Lambda_{b_1}, \Lambda_{b_2}).$$

Here

$$r = \min(\frac{n_1 + n_2 + 1 - d(\Lambda_b, \Lambda_{b'})}{2}, n_1 + \frac{1}{2}).$$

Then $\mathcal{T}(\mathbf{x}_1) \cap \mathcal{T}(\mathbf{x}_2) \subset \mathcal{B}$ *is the ball of radius* r *centered at either the vertex* Λ_{b_1,b_2} *or the midpoint of the edge* Λ_{b_1,b_2}, *depending on whether* r *is an integer or not.*

Proof. It follows from Corollary 3.2.11 and the fact that $\mathcal{B}$ is a tree. $\square$

Lemma 3.2.15. *For* $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{V}$, *let* $b_1 = \mathbf{x}_1(e), b_2 = \mathbf{x}_2(e)$. *Assume*

$$T = \begin{pmatrix} (b_1, b_1) & (b_1, b_2) \\ (b_2, b_1) & (b_2, b_2) \end{pmatrix} = \begin{pmatrix} 0 & \pi^n \\ (-\pi)^n & 0 \end{pmatrix}, \, n \geq 0.$$

Then $\mathcal{T}(\mathbf{x}_1) \cap \mathcal{T}(\mathbf{x}_2)$ *is a ball with center* $\Lambda = \mathrm{span}_{O_F}\{\pi^{-r} b_1, \pi^{-r} b_2\}$ *and radius* r, *where* $r = [\frac{n+1}{2}]$ *is the integral part of* $\frac{n+1}{2}$.

Proof. We assume that $n = 2r - 1$ ($r \geq 1$) is odd, the case n is even can be proved similarly. Write $b_i = \pi^r v_i$ for $i = 1, 2$. Define

$$\Lambda_{k,0} = \mathrm{span}_{O_F}\{\pi^{-k} v_1, \pi^{k+1} v_2\}, \qquad \Lambda_{k,2} = \mathrm{span}_{O_F}\{\pi^{-k} v_1, \pi^{k} v_2\}, \; k \in \mathbb{Z}.$$

Then $\Lambda_{k,0}$ are vertex lattices of type 0 and $\Lambda_{k,2}$ are vertex lattices of type 2. It is easy to check that

$$n(b_1, \Lambda_{k,0}) = r + k, \qquad n(b_1, \Lambda_{k,2}) = r + k,$$
$$n(b_2, \Lambda_{k,0}) = r - k - 1, \qquad n(b_2, \Lambda_{k,2}) = r - k.$$

So $\{\Lambda_{k,0}, \Lambda_{k,2} \mid k \geq -r\}$ is the ascending geodesic associated to b_1 starting at $\Lambda_{-r,2}$ and $\{\Lambda_{k,0} \mid k \leq r-1\} \cup \{\Lambda_{k,2} \mid k \leq r\}$ is (the inverse of) the ascending geodesic associated to b_2 ending at $\Lambda_{r,2}$. Let $\mathcal{C}$ be the intersection of the above two half geodesics, namely, the line segment joining $\Lambda_{-r,2}$ and $\Lambda_{r,2}$. By Lemma 3.2.12, both $n(b_1, \Lambda)$ and $n(b_2, \Lambda)$ are decreasing along a half geodesic starting from any vertex on $\mathcal{C}$ other than $\mathcal{C}$ itself. Combine the above facts, it is easy to see that $\mathcal{T}(\mathbf{x}_1) \cap \mathcal{T}(\mathbf{x}_2)$ is a ball centered at $\Lambda_{0,2}$ with radius r. □

The following lemma is an analogue of [San17, Lemma 2.15]. Recall that wo hermitian matrices $T_1, T_2 \in \mathrm{Herm}_2(O_F)$ are said to be equivalent, denoted by $T_1 \approx T_2$ if there is a non-singular matrix $g \in \mathrm{GL}_2(O_F)$ such that $g^t T_1 \bar{g} = T_2$.

Lemma 3.2.16. *Let*

$$T = \begin{pmatrix} (b_1, b_1) & (b_1, b_2) \\ (b_2, b_1) & (b_2, b_2) \end{pmatrix}$$

where $b_i = \mathbf{x}_i(e)$ for $\mathbf{x}_i \in \mathbb{V}$. Assume $q(b_i) \neq 0$, $i \in \{1, 2\}$, and T is non-singular. Let $n_i = \mathrm{ord}_{\pi_0}(q(b_i))$ and assume $n_1 \leq n_2$. Set $d = d(\Lambda_{b_1}, \Lambda_{b_2})$, then

(a) If $\mathcal{T}(\mathbf{x_1}) \subset \mathcal{T}(\mathbf{x_2})$ and $\Lambda_{b_1} \neq \Lambda_{b_2}$, then $T \approx \begin{pmatrix} u_1(-\pi_0)^\alpha & 0 \\ 0 & u_2(-\pi_0)^\beta \end{pmatrix}$ where $u_1, u_2 \in O_{F_0}^\times$ and $-u_1 u_2 \in \mathrm{Nm}(F^\times)$ with

$$\alpha = n_1, \quad \beta = n_2 - d.$$

(b) If $\mathcal{T}(\mathbf{x_1}) \not\subset \mathcal{T}(\mathbf{x_2})$ *, and* $\mathcal{T}(\mathbf{x_1}) \cap \mathcal{T}(\mathbf{x_2}) \neq \emptyset$*, then* $T \approx \begin{pmatrix} 0 & \pi^\alpha \\ (-\pi)^\alpha & 0 \end{pmatrix}$ *with*

$$\alpha = n_1 + n_2 - d.$$

(c) If $\Lambda_{b_1} = \Lambda_{b_2}$*, then* $T \approx \begin{pmatrix} u_1(-\pi_0)^\alpha & 0 \\ 0 & u_2(-\pi_0)^\beta \end{pmatrix}$*. Here* u_1, u_2 *satisfy the same conditions as in* (a) *and*

$$\alpha = n_1, \quad \beta = n_2 + 2\mathrm{ord}_\pi((c_2, c_1')),$$

where $c_i = \pi^{-n_i} b_i$*, and* $c_1' \in \Lambda_{b_1} \setminus \pi\Lambda_{b_1}$ *such that* $(c_1, c_1') = 0$*.*

Proof. If $n_1 < 0$, the lemma holds trivially. So we assume $n_1 \geq 0$.
We treat the case $\Lambda_{b_1} \neq \Lambda_{b_2}$ first. Assume $\Lambda \cap \Lambda' = \Lambda_{b_1}$ where Λ and Λ' are vertex lattices of type 2. We pick a basis $\{v_0, v_1\}$ of Λ such that

$$\begin{pmatrix} (v_0, v_0) & (v_0, v_1) \\ (v_1, v_0) & (v_1, v_1) \end{pmatrix} = \begin{pmatrix} 0 & \pi^{-1} \\ -\pi^{-1} & 0 \end{pmatrix}.$$

Without loss of generality, we can assume

$$\Lambda' = \mathrm{span}_{\mathcal{O}_F}\{\pi^{-1}v_0, \pi v_1\}, \text{ hence } \Lambda_{b_1} = \mathrm{span}_{\mathcal{O}_F}\{v_0, \pi v_1\}.$$

By the symmetry of $\mathcal{B}$, we can also assume

$$\Lambda_{b_2} = \mathrm{span}_{\mathcal{O}_F}\{\pi^{-d}v_0, \pi^{d+1}v_1\} \text{ where } d = d(\Lambda_{b_1}, \Lambda_{b_2}).$$

By Lemma 3.2.10, we can write

$$b_1 = \pi^{n_1}(\alpha_0 v_0 + \alpha_1(\pi v_1)),\ b_2 = \pi^{n_2}(\alpha_0'(\pi^{-d}v_0) + \alpha_1'(\pi^{d+1}v_1)),\ \alpha_i, \alpha_i' \in \mathcal{O}_F.$$

Note that $(b_1, b_1) = (-\pi_0)^{n_1}(-\alpha_0\bar{\alpha}_1 - \alpha_1\bar{\alpha}_0)$. Since $\mathrm{ord}_\pi(-\alpha_0\bar{\alpha}_1 - \alpha_1\bar{\alpha}_0) = 0$, we conclude that $\mathrm{ord}_\pi(\alpha_0) = \mathrm{ord}_\pi(\alpha_1) = 0$. Similarly, $\mathrm{ord}_\pi(\alpha_0') = \mathrm{ord}_\pi(\alpha_1') = 0$. Then a short computation shows:

$$T = \begin{pmatrix} (b_1, b_1) & (b_1, b_2) \\ (b_2, b_1) & (b_2, b_2) \end{pmatrix} = \begin{pmatrix} \pi^{2n_1} \cdot (\text{unit}) & \pi^{n_1+n_2-d} \cdot (\text{unit}) \\ (-\pi)^{n_1+n_2-d} \cdot (\text{unit}) & \pi^{2n_2} \cdot (\text{unit}) \end{pmatrix}.$$

Note that

$$\mathcal{T}(\mathbf{x}_1) \subset \mathcal{T}(\mathbf{x}_2) \iff n_1 \leq n_2 - d \iff n_1 \leq \frac{n_1 + n_2 - d}{2}.$$

Now for the proof of (a), observe that if $\mathcal{T}(\mathbf{x}_1) \subset \mathcal{T}(\mathbf{x}_2)$ then $2n_1 = \min\{\mathrm{ord}_\pi(T_{ij})\}$. This implies

$$T \approx \begin{pmatrix} u_1(-\pi_0)^{n_1} & 0 \\ 0 & u_2(-\pi_0)^{n_2-d} \end{pmatrix}.$$

For (b), the assumption implies that $n_1 > n_2 - d$. Then $n_1 + n_2 - d = \min\{\mathrm{ord}_\pi(T_{ij})\}$, which implies

$$T \approx \begin{pmatrix} 0 & \pi^{n_1+n_2-d} \\ (-\pi)^{n_1+n_2-d} & 0 \end{pmatrix}.$$

The proof for the case $\Lambda_{b_1} = \Lambda_{b_2}$ is essentially the same as the proof in [San17, Lemma 2.15], and is left to the reader. □

Corollary 3.2.17. *Let* $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2) \in \mathbb{V}^2$ *and* $b_i = \mathbf{x}_i(e)$ *for* $i = 1, 2$. *Assume that* $\mathbf{x}_1$ *and* $\mathbf{x}_2$ *are linearly independent. Then the naive intersection* $\mathcal{Z}(\mathbf{x}) = \mathcal{Z}(\mathbf{x}_1) \cap \mathcal{Z}(\mathbf{x}_2)$ *(resp.* $\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x})$ *) is supported in finitely many irreducible components of the special fiber of* $\mathcal{N}_{(1,1)}$ *(resp.* $\mathcal{N}^{\mathrm{Kra}}$*).*

Proof. This follows easily from Lemma 3.2.14 and Lemma 3.2.15. □

3.3 Decomposition of special divisors in the Krämer model

This section is dedicated to the proof of the following theorem.

Theorem 3.3.1. *For a vector* $\mathbf{x} \in \mathbb{V} \setminus \{0\}$*, set* $b = \mathbf{x}(e)$ *as before. Then* $\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}) = 0$ *unless* $q(b) \in O_{F_0}$.

1. *If* $q(b) \neq 0$ *and* $q(b) \in O_{F_0}$*, then we have the following decomposition of special divisor*

$$\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}) = \sum_{\Lambda_2 \in \mathcal{T}(\mathbf{x})} n(b, \Lambda_2)\mathbb{P}_{\bar{\Lambda}_2} + \sum_{\Lambda_0 \in \mathcal{T}(\mathbf{x})} (n(b, \Lambda_0) + 1)\mathrm{Exc}_{\Lambda_0} + \mathcal{Z}^h(\mathbf{x}), \tag{3.21}$$

where the two summations are over type 2 and type 0 *vertex lattices respectively and* $\mathcal{Z}^h(\mathbf{x}) \cong \mathrm{Spf}\mathcal{O}_F$ *is a horizontal divisor meeting the special fiber at* Exc_{Λ_b}*. Recall that* Λ_b *is the unique vertex lattice of type* 0 *such that* $\pi^{-\mathrm{ord}_{\pi_0}(q(b))} b \in \Lambda_b \setminus \pi\Lambda_b$*.*

2. *If* $q(b) = 0$*, then we have the following decomposition of special divisor*

$$\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}) = \sum_{\Lambda_2 \in \mathcal{T}(\mathbf{x})} n(b, \Lambda_2)\mathbb{P}_{\bar{\Lambda}_2} + \sum_{\Lambda_0 \in \mathcal{T}(\mathbf{x})} (n(b, \Lambda_0) + 1)\mathrm{Exc}_{\Lambda_0}, \tag{3.22}$$

where the two summations are over type 2 and type 0 *vertex lattices respectively.*

3.3.1 The Horizontal Component

We say a formal scheme over Spf $\mathcal{O}_{\breve{F}}$ is horizontal if π is not locally nilpotent in its structure sheaf. We say a divisor in $\mathcal{N}^{\mathrm{Kra}}$ is irreducible if it is connected and is an irreducible Cartier divisor in every local ring of $\mathcal{N}^{\mathrm{Kra}}$.

Let $\mathcal{Y}_s$ be the quasi-canonical lifting of $\mathbb{Y}$ of level s over $O_{\breve{F}_0}$ considered by [Gro86] (with $O_s = O_{\breve{F}_0} + \pi^s O_{\breve{F}}$ action). In particular, $\mathcal{Y} = \mathcal{Y}_0$ is the canonical lifting. We show that all horizontal cycles in $\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x})$ comes from canonical lifting.

Theorem 3.3.2. *Let* $\mathcal{Z}$ *be an irreducible horizontal component of* $\mathcal{Z}(\mathbf{x})$*, then* $\mathcal{Z} \cong \mathrm{Spf}\, O_{\breve{F}}$*. Moreover* $\mathcal{Z}$ *intersects with the special fiber of* $\mathcal{N}_{(1,1)}$ *at a superspecial point.*

Proof. By assumption $\mathcal{Z} = \mathrm{Spf}\, R$ where R is a finite extension of $O_{\breve{F}_0}$. Let X be the strict formal O_{F_0}-module over $\mathcal{Z}$ that is the pullback from the universal strict formal O_{F_0}-module over $\mathcal{N}_{(1,1)}$ which by Theorem 3.1.6 carries an O_B action ι_B. By the definition of

$\mathcal{Z}(\mathbf{x})$, $\mathbf{x}$ lifts to a homomorphism $\mathbf{x} : \mathcal{Y} \to X$. We now define a morphism $\phi : \mathcal{Y} \oplus \mathcal{Y} \to X$ by

$$\phi(p_1, p_2) = x(p_1) + \iota_B(\delta) \circ x(p_2),$$

where $p_1, p_2 \in \mathcal{Y}(S)$ for an R-algebra S. We give $\mathcal{Y} \oplus \mathcal{Y}$ an O_B action $\iota : O_B \to \mathrm{End}_{O_{F_0}}(\mathcal{Y} \oplus \mathcal{Y}) \cong \mathrm{M}_{2\times 2}(O_F)$ defined by

$$\iota(\pi) = \begin{pmatrix} \pi & 0 \\ 0 & -\pi \end{pmatrix}, \ \iota(\delta) = \begin{pmatrix} 0 & 1 \\ \delta^2 & 0 \end{pmatrix}. \tag{3.23}$$

Then ϕ becomes an O_B-linear homomorphism.

We claim that ϕ is an isogeny. By [Tat67, Proposition 1], the category of connected O_F modules and the category of divisible commutative formal Lie groups with O_F action are equivalent. Let $A = R[[T_1, T_2]]$ considered as the structure ring of $\mathcal{Y} \oplus \mathcal{Y}$ and B be the structure ring of X. It suffices to show that the induced map $\phi^\sharp : B \to A$ is injective. If $I = \mathrm{Ker}(\phi^\sharp)$ is nontrivial, then ϕ factors through the sub formal group scheme $X' = \mathrm{Spf}\, B/I$ of X. Since A has characteristic zero, so does B/I. By base change to the fraction field $F(R)$ of R we can apply a theorem of Cariter [Car62] and conclude that $X' \otimes_R F(R)$ is a one dimensional formal group law over $F(R)$. But by assumption, $X' \otimes_R F(R)$ has an O_B action which is impossible. Hence $I = \{0\}$. This proves the claim.

Our goal is to show that X is in fact isomorphic to $\mathcal{Y} \oplus \mathcal{Y}$. We define the Tate module for a O_F-module G by

$$T(G) = \varprojlim_n G[\pi^n].$$

We can identify $T = T(\mathcal{Y} \oplus \mathcal{Y})$ with $(O_F)^2$ and further with O_B using the O_B action on $T(\mathcal{Y} \oplus \mathcal{Y})$. To be more specific the element $t = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in T$ is a generator of T under the O_B action given by (3.23). Then the set of O_B-linear isogenies $\phi : \mathcal{Y} \oplus \mathcal{Y} \to X$ has a one-to-one correspondence with O_B stable lattices T' such that

$$T \subseteq T' \subset T^0 = T \otimes_{O_F} F. \tag{3.24}$$

Since T is a free O_B module of rank 1, T' must be of the form $\pi^{-n}\cdot T$ for some $n \geq 0$. This shows that X is in fact isomorphic to $\mathcal{Y}\oplus\mathcal{Y}$. Identifying $X = \mathcal{Y}\oplus\mathcal{Y}$, we see from (3.23) that $j = \mathrm{diag}(\pi, \pi) \in \mathrm{End}_{O_B}(X)$. Since $\mathrm{Tr}_{F/F_0}\mathrm{Tr}j = 0$, this implies that $X \in \mathcal{Z}(j) \subset \mathcal{M}$—the special divisor defined in [KR00, Definition 2.1]. So $\mathcal{Z} \subset Z(j)$ under the isomorphism $\mathcal{N} \cong \mathcal{M}$. Now we conclude from case (iii) of [KR00, Proposition 4.5] that $\mathcal{Z} \cong \mathrm{Spf}\, \mathcal{O}_{\breve{F}}$.

By construction we know that the relative Dieudonné module $M(\bar{X}) = M(\mathbb{Y}) \oplus M(\mathbb{Y})$ is fixed by $\tau = \pi V^{-1}$ where $\bar{X} = X \otimes \bar{\kappa}$. By [KR14a, Lemma 3.2], $\mathcal{Z}$ intersects with the special fiber of $\mathcal{N}_{(1,1)}$ at a superspecial point. $\square$

3.3.2 Hodge filtration and equation of special cycles in $\mathcal{N}_{(1,1)}$

Now we begin to study the equations of special divisors at a superspecial point. We will use Grothendieck-Messing theory to determine the equations, which in turn requires a description of the Hodge filtrations of the relevant relative Dieudonné crystals. In the following we use the Deligne functor to obtain such a description.

In the rest of this section, for $x = (X, \iota, \lambda, \rho) \in \mathcal{N}_{(1,1)}(\bar{\kappa})$, we denote $\rho(M(X)) \subset N$ by $M(x)$. Then identification $\mathcal{N}_{(1,1)}/\bar{\kappa} \cong \mathcal{M}/\bar{\kappa}$ induces an O_B and thus an O_E-action on $M(x)$. This makes $M(x)$ an $O_E \otimes_{O_{F_0}} O_{\breve{F}_0}$-module and induces a $\mathbb{Z}/2\mathbb{Z}$ grading

$$M(x) = M(x)_0 \oplus M(x)_1.$$

Now let x be a superspecial point pt_{Λ_0} of $\mathcal{N}_{(1,1)}$, which implies that $M(x)$ is τ-invariant and that $\Lambda_0 = M(x)^\tau$ is a vertex O_F-lattice of type 0 with $M(x) = \Lambda_0 \otimes_{O_F} O_{\breve{F}}$. As explained in Section 3.1.1 we can choose an O_{F_0}-basis $\{e_0, f_1\}$ for $M_0(x)^\tau$ and $\{f_0, e_1\}$ for $M_1(x)^\tau$ such that

$$\pi e_i = f_i,\ \pi f_i = \pi_0 e_i,\ V e_i = f_i,\ V f_i = \pi_0 e_i,\ i = 0,\ 1. \tag{3.25}$$

and

$$(e_0, e_0) = (e_1, e_1) = 0, (e_0, e_1) = -\delta^2. \tag{3.26}$$

Then $\Lambda_0 = \mathrm{Span}_{O_F}\{e_0, e_1\}$, and the vertex lattices of type 2 containing Λ_0 are

$$\Lambda_2 = \mathrm{Span}_{O_F}\{\pi^{-1}e_0, e_1\}, \qquad \Lambda_2' = \mathrm{Span}_{O_F}\{e_0, \pi^{-1}e_1\}. \tag{3.27}$$

Therefore, $x \in \mathbb{P}_{\bar{\Lambda}_2} \cap \mathbb{P}_{\bar{\Lambda}_2'}$. Since $\mathcal{N} \cong \mathcal{M}$, we have $\mathcal{N}_{(1,1)}(\bar{\kappa}) = \mathcal{M}(\bar{\kappa})$ and there should exist homothety classes of rank 2 O_{F_0}-lattices $[\Lambda]$ and $[\Lambda']$ such that $x \in \mathbb{P}_{[\Lambda]} \cap \mathbb{P}_{[\Lambda']}$. By [San13, Remark 3.4], we can take

$$\Lambda = M_0(x)^\tau = \mathrm{span}_{O_{F_0}}\{e_0, f_1\},$$
$$\Lambda' = \pi^{-1}M_1(x)^\tau = \mathrm{span}_{O_{F_0}}\{\pi^{-1}f_0, \pi^{-1}e_1\} = \mathrm{span}_{O_{F_0}}\{e_0, \pi_0^{-1}f_1\}.$$

Another way to relate these different types of lattices are the following equations

$$\Lambda = ((\Lambda_2 \otimes_{O_{F_0}} O_E)_0)^\tau, \ \Lambda' = ((\Lambda_2' \otimes_{O_{F_0}} O_E)_0)^\tau. \tag{3.28}$$

In particular we can identify $\mathbb{P}_{[\Lambda]}(\bar{\kappa})$ with $\mathbb{P}_{\bar{\Lambda}_2}(\bar{\kappa})$ and $\mathbb{P}_{[\Lambda']}(\bar{\kappa})$ with $\mathbb{P}_{\bar{\Lambda}_2'}(\bar{\kappa})$.

Consider the Deligne functor $F_{[\Lambda,\Lambda']}$ (see [BC91]):

$$\begin{array}{ccccc} \pi_0\Lambda' & \longrightarrow & \Lambda & \longrightarrow & \Lambda' \\ \downarrow{\scriptstyle \alpha'/\pi_0} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha'} \\ \mathcal{L}' & \xrightarrow{c'} & \mathcal{L} & \xrightarrow{c} & \mathcal{L}'. \end{array}$$

For an $O_{\breve{F}_0}$-algebra $R \in \mathrm{Nilp}_{O_{\breve{F}_0}}$, the conditions of α and α' in Proposition 3.1.7 imply that $\alpha(e_0)$ generates $\mathcal{L}$ and $\alpha'(\pi^{-1}e_1)$ generates $\mathcal{L}'$. We identify $\mathcal{L}$ and $\mathcal{L}'$ with R by setting

$$\alpha(e_0) = 1, \quad \text{and} \quad \alpha'(\pi^{-1}e_1) = \alpha'(\pi_0^{-1}f_1) = 1.$$

Let

$$t_0 = \alpha(f_1) \in \mathcal{L} = R, \quad \text{and} \quad t_1 = \alpha'(e_0) = \alpha'(\pi^{-1}f_0) \in \mathcal{L}' = R.$$

Then c is simply the multiplication by t_0 and c' is the multiplication by t_1. So by commutativity of the above diagram, we have

$$t_0 t_1 = \pi_0. \tag{3.29}$$

Consider

$$\alpha \otimes 1 : \Lambda \otimes_{O_{F_0}} R \to \mathcal{L} \quad \text{and} \quad \alpha' \otimes 1 : \Lambda' \otimes_{O_{F_0}} R \to \mathcal{L}'. \tag{3.30}$$

We have

$$\text{kernel of } \alpha \otimes 1 = \mathrm{span}_R\{f_1 \otimes 1 - e_0 \otimes t_0\}, \tag{3.31}$$

$$\text{kernel of } \alpha' \otimes 1 = \mathrm{span}_R\{\pi^{-1} f_0 \otimes 1 - \pi^{-1} e_1 \otimes t_1\}. \tag{3.32}$$

For a strict formal O_{F_0}-module X over R, let $\mathbb{D}(X/R)$ be its relative (to O_{F_0}) Dieudonné crystal with Hodge filtration $\mathrm{Fil}\mathbb{D}(X/R)$, see for example [ACZ16, Section 3]). Then we have the exact sequence:

$$0 \to \mathrm{Fil}\mathbb{D}(X/R) \to \mathbb{D}(X/R) \to \mathrm{Lie}(X/R) \to 0. \tag{3.33}$$

Proposition 3.3.3. *Let $x = pt_{\Lambda_0}$ be a superspecial point. For an $\mathcal{O}_{\mathcal{N}_{(1,1)},x}$-algebra $R \in \mathrm{Nilp}_{O_{\breve{F}_0}}$ where $\mathcal{O}_{\mathcal{N}_{(1,1)},x}$ is the local ring of $\mathcal{N}_{(1,1)}$ at x, let $t_0, t_1 \in R$ be the image of T_0, T_1 under the structure morphism, and X_{t_0,t_1} be the corresponding strict O_{F_0} module over R. Then we have the following identifications*

$$\mathbb{D}(X_{t_0,t_1}/R)_0 = \mathrm{span}_{O_{F_0}}\{e_0, f_1\} \otimes_{O_{F_0}} R, \quad \mathrm{Fil}\mathbb{D}(X_{t_0,t_1}/R)_0 = \mathrm{span}_R\{f_1 \otimes 1 - e_0 \otimes t_0\},$$

and

$$\mathbb{D}(X_{t_0,t_1}/R)_1 = \mathrm{span}_{O_{F_0}}\{e_1, f_0\} \otimes_{O_{F_0}} R, \quad \mathrm{Fil}\mathbb{D}(X_{t_0,t_1}/R)_1 = \mathrm{span}_R\{f_0 \otimes 1 - e_1 \otimes t_1\}.$$

Proof. Since x is a superspecial point, both 0 and 1 are critical indices. Hence $M(x)$ is τ

invariant and $M(x)^\tau = \mathrm{span}_{O_{F_0}}\{e_0, f_0, e_1, f_1\}$. According to the constructions in [BC91], especially how Deligne's functor is related with the moduli funtor of special formal O_B-module, we know that

$$\mathbb{D}(X_{t_0,t_1}/R)_0 = M_0(x)^\tau \otimes_{\mathcal{O}_{F_0}} R = \mathrm{span}_{O_{F_0}}\{e_0, f_1\} \otimes_{\mathcal{O}_{F_0}} R,$$
$$\mathbb{D}(X_{t_0,t_1}/R)_1 = M_1(x)^\tau \otimes_{\mathcal{O}_{F_0}} R = \mathrm{span}_{O_{F_0}}\{e_1, f_0\} \otimes_{\mathcal{O}_{F_0}} R.$$

Under these identifications, the map $\alpha \otimes 1$ in ((3.30)) is the natural quotient map from $\mathbb{D}(X_{t_0,t_1}/R)_0$ to $\mathrm{Lie}(X_{t_0,t_1})_0$, and the maps $\pi\alpha' \otimes 1$ is the natural quotient map from $\mathbb{D}(X_{t_0,t_1}/R)_1$ to $\mathrm{Lie}(X_{t_0,t_1})_1$. Hence $\mathrm{Fil}\mathbb{D}(X_{t_0,t_1}/R)_0$ is the kernel of $\alpha\otimes 1$ and $\mathrm{Fil}\mathbb{D}(X_{t_0,t_1}/R)_1$ is the kernel of $\pi\alpha' \otimes 1$. By (3.31) and (3.32), we obtain the proposition. $\square$

Similarly for the universal object $\mathcal{Y}$ over $\mathcal{N}_{(1,0)}$ and an $O_{\breve{F}}$-algebra $R \in \mathrm{Nilp}_{O_{\breve{F}}}$, it is easy to see that

$$\mathbb{D}(\mathcal{Y}_R) = \mathrm{Span}_R\{e \otimes 1, f \otimes 1\}, \quad \mathrm{Fil}\mathbb{D}(\mathcal{Y}_R) = \mathrm{Span}_R\{f \otimes 1 - e \otimes \pi\}, \tag{3.34}$$

since the O_F-action coincides with the structure action $O_F \to R$ on $\mathrm{Lie}(\mathcal{Y}_R)$: $\pi e\otimes 1 = e\otimes\pi$. Here the tensor is over O_{F_0}.

Proposition 3.3.4. *Let $x \in \mathcal{Z}(\mathbf{x})(\bar{\kappa})$ be a superspecial point pt_{Λ_0}. Choose a basis $\{e_0, e_1\}$ of Λ_0 such that (3.25) and (3.26) are satisfied. Assume $b = \mathbf{x}(e) = \alpha_0 e_0 + \alpha_1 e_1 \in \Lambda_0$ where $\alpha_i \in \mathcal{O}_F$. Recall a neighborhood of x in $\mathcal{N}_{(1,1)}$ is*

$$\breve{\Omega}_{[\Lambda,\Lambda'],\breve{F}} = \mathrm{Spf}(\mathcal{O}_{\breve{F}}[T_0, T_1, (T_0^{q-1} - 1)^{-1}, (T_1^{q-1} - 1)^{-1}]/(T_0 T_1 - \pi_0))^\wedge. \tag{3.35}$$

Then the equations for $\mathcal{Z}(\mathbf{x})$ in the local ring $O_{\mathcal{N}_{(1,1)},x}$ are given by (here $\bar{\alpha}$ stands for the Galois conjugate of α):

$$\bar{\alpha}_1 T_0 - \pi\bar{\alpha}_0 = 0, \ \ \textit{and}\ \bar{\alpha}_0 T_1 - \pi\bar{\alpha}_1 = 0. \tag{3.36}$$

Proof. First, Lemma 3.2.6 implies $b = \mathbf{x}(e) \in \Lambda_0$. Let

$$b' := \mathbf{x}(f) = \pi \mathbf{x}(e) = \alpha_0 f_0 + \alpha_1 f_1.$$

for some $\alpha_i \in O_F$. Consider the local ring $A := \mathcal{O}_{\mathcal{N}_{(1,1)},x}$. Let I denote the ideal corresponding to $\mathcal{Z}(\mathbf{x})$. Set

$$A_n := A/\pi^n A, \quad I_n := (I + \pi^n A)/\pi^n A \subset A_n.$$

Since A is Noetherian, I is π-adic complete. Hence to prove the proposition, we just need to prove I_n are generated by the images of $\bar{\alpha}_1 T_0 - \pi \bar{\alpha}_0$ and $\bar{\alpha}_0 T_1 - \pi \bar{\alpha}_1$ in A_n for all n. Let $\mathfrak{m}_n$ denote the maximal ideal of A_n. Set

$$B := A_n/\mathfrak{m}_n I_n, \quad B' := A_n/I_n.$$

Let $J := I_n/\mathfrak{m}_n I_n$, which is the kernel of of the projection $B \to B'$. Note that $J^2 = 0$, so it has a PD structure. By Nakayama's lemma, it suffices to show J is generated by images of $\bar{\alpha}_1 T_0 - \pi \bar{\alpha}_0$ and $\bar{\alpha}_0 T_1 - \pi \bar{\alpha}_1$ in B.

Let $\mathcal{X}$ be the universal strict formal O_{F_0}-module over $\mathcal{N}_{(1,1)}$. The natural map

$$O_{\breve{\Omega}_{[\Lambda,\Lambda'],\breve{F}}} \to A \to A_n \to B \to B' \to \bar{\kappa}$$

induces the strict formal O_{F_0}-modules $\mathcal{X}_B$, $\mathcal{X}_{B'}$ and $\mathcal{X}_{\bar{\kappa}} = X$ with X being associated to $x \in \mathcal{Z}(\mathbf{x})(\bar{\kappa})$.

Since I is the definition ideal of $\mathcal{Z}(\mathbf{x})$, B' is a quotient of A/I, the quasi-morphism

$$\rho^{-1} \circ \mathbf{x} : \mathbb{Y} \to X$$

lifts to a morphism

$$\mathbf{x}_{B'} : \mathcal{Y}_{B'} \to \mathcal{X}_{B'}.$$

The associated morphism $\mathbb{D}(\mathbf{x}'_B)$ lifts to a morphism

$$\mathbb{D}(\mathbf{x}_R) : \mathbb{D}(\mathcal{Y}_R) \to \mathbb{D}(\mathcal{X}_R)$$

for any ring $R = B/\mathfrak{b}$ with $\mathfrak{b} \subset J$, i.e., $B \twoheadrightarrow R \twoheadrightarrow B'$ as $J^2 = 0$. By Grothebdieck-Messing theory,

$\mathbf{x}$ lifts to a momorphism

$$\mathbf{x}_R : \mathcal{Y}_R \to \mathcal{X}_R \tag{3.37}$$

if and only if

$$\mathbb{D}(\mathbf{x}_R)(f \otimes 1 - e \otimes \pi) = b' \otimes 1 - b \otimes \pi \in \mathrm{Fil}\mathbb{D}(\mathcal{X}_R).$$

Write $\alpha_0 = a_0 + b_0\pi,\ \alpha_1 = a_1 + b_1\pi.$ Then

$$\begin{aligned}
&b' \otimes 1 - b \otimes \pi \\
&= (\alpha_0 f_0 + \alpha_1 f_1) \otimes 1 - (\alpha_0 e_0 + \alpha_1 e_1) \otimes \pi \\
&= ((a_0 + b_0\pi) f_0 + (a_1 + b_1\pi) f_1) \otimes 1 - ((a_0 + b_0\pi) e_0 + (a_1 + b_1\pi) e_1) \otimes \pi \\
&= (a_0 f_0 + b_0\pi_0 e_0 + a_1 f_1 + b_1\pi_0 e_1) \otimes 1 - (a_0 e_0 + b_0 f_0 + a_1 e_1 + b_1 f_1) \otimes \pi \\
&= f_1 \otimes (a_1 - \pi b_1) - e_0 \otimes \pi(a_0 - \pi b_0) + f_0 \otimes (a_0 - \pi b_0) - e_1 \otimes \pi(a_1 - \pi b_1) \\
&= f_1 \otimes \bar{\alpha}_1 - e_0 \otimes \pi\bar{\alpha}_0 + f_0 \otimes \bar{\alpha}_0 - e_1 \otimes \pi\bar{\alpha}_1.
\end{aligned}$$

Combining this with Proposition 3.3.3, we see that the lifting (3.37) exists if and only if

$$\bar{\alpha}_1 T_0 - \pi\bar{\alpha}_0 = 0,\ \text{and}\ \bar{\alpha}_0 T_1 - \pi\bar{\alpha}_1 = 0 \text{ in R},$$

i.e.,

$$\bar{\alpha}_1 T_0 - \pi\bar{\alpha}_0,\ \text{and}\ \bar{\alpha}_0 T_1 - \pi\bar{\alpha}_1 \in \mathfrak{b}.$$

Here we identify T_i with their images in R via $A \to A_n \to B \to R$. Since I is the ideal of $\mathcal{Z}(\mathbf{x})$, the lifting (3.37) exists only when $\mathfrak{b} = J$. So J is generated by $\bar{\alpha}_1 T_0 - \pi\bar{\alpha}_0$ and

$\bar{\alpha}_0 T_1 - \pi\bar{\alpha}_1$ as claimed.

$\square$

3.3.3 Local coordinate charts in the Krämer model

Now we describe the local equation of a special divisor in the Krämer model and use it to give a decomposition of special divisor. Locally around the superspecial point $x \in \mathcal{N}_{(1,1)}(\bar{\kappa})$ corresponding to $\Lambda_0 = \Lambda_2 \cap \Lambda_2'$, we have (3.35) and x corresponds to the maximal ideal $\mathfrak{m}_x = (T_0, T_1, \pi)$. We need to blow it up to get the exceptional divisor of $\mathcal{N}^{\mathrm{Kra}} = \mathcal{N}^{\mathrm{Kra}}_{(1,1)}$.

For simplicity, consider (ignoring the other unimportant restrictions of $\breve{\Omega}_{[\Lambda,\Lambda'],F}$)

$$D \coloneqq \mathrm{Spf}(O_{\breve{F}}[T_0, T_1]/(T_0T_1 - \pi_0))^{\wedge}.$$

Let $\mathrm{Bl}_x D$ denote the blow-up of D at $\mathfrak{m}_x$, which has three charts. Over the first chart D_1, we have

$$\pi S_{\frac{T_0}{\pi}} = T_0,\ \pi S_{\frac{T_1}{\pi}} = T_1, \tag{3.38}$$

where we regard $S_{\frac{x}{\pi}}$ as element in $\mathrm{Frac}(\mathcal{O}_{\breve{F}}[T_0, T_1])$ and

$$\begin{aligned} D_1 &= \mathrm{Spf}\mathcal{O}_{\breve{F}}([T_0, T_1, S_{\frac{T_0}{\pi}}, S_{\frac{T_1}{\pi}}]/(\pi S_{\frac{T_0}{\pi}} - T_0, \pi S_{\frac{T_1}{\pi}} - T_1, S_{\frac{T_0}{\pi}} S_{\frac{T_1}{\pi}} - 1))^{\wedge} \\ &= \mathrm{Spf}\mathcal{O}_{\breve{F}}([S_{\frac{T_0}{\pi}}, S_{\frac{T_1}{\pi}}]/(S_{\frac{T_0}{\pi}} S_{\frac{T_1}{\pi}} - 1))^{\wedge}. \end{aligned} \tag{3.39}$$

Over the second chart D_2, we have:

$$T_0 S_{\frac{T_1}{T_0}} = T_1,\ T_0 S_{\frac{\pi}{T_0}} = \pi, \tag{3.40}$$

and

$$D_2 = \mathrm{Spf}\mathcal{O}_{\breve{F}}([T_0, S_{\frac{\pi}{T_0}}]/(T_0 S_{\frac{\pi}{T_0}} - \pi))^{\wedge}. \tag{3.41}$$

Over the third chart D_3, we have:

$$T_1 S_{\frac{T_0}{T_1}} = T_0,\ T_1 S_{\frac{\pi}{T_1}} = \pi, \tag{3.42}$$

and

$$D_3 = \mathrm{Spf}\mathcal{O}_{\breve{F}}([T_1, S_{\frac{\pi}{T_1}}]/(T_1 S_{\frac{\pi}{T_1}} - \pi))^\wedge \tag{3.43}$$

by symmetry. D_1, D_2 and D_3 are glued in the obvious way, and it is easy to see that D_1, D_2 and D_3 are all regular. Let Exc denote the exceptional divisor of $\mathcal{N}^{\mathrm{Kra}}$, then

$$\begin{aligned}
\mathrm{Exc} \cap D_1 &= \mathrm{Spf}\bar{\kappa}([S_{\frac{T_0}{\pi}}, S_{\frac{T_1}{\pi}}]/(S_{\frac{T_0}{\pi}} S_{\frac{T_1}{\pi}} - 1))^\wedge, \\
\mathrm{Exc} \cap D_2 &= \mathrm{Spf}\mathcal{O}_{\breve{F}}([T_0, S_{\frac{\pi}{T_0}}]/(T_0, T_0 S_{\frac{\pi}{T_0}} - \pi))^\wedge = \mathrm{Spf}\bar{\kappa}([S_{\frac{\pi}{T_0}}])^\wedge, \\
\mathrm{Exc} \cap D_3 &= \mathrm{Spf}\mathcal{O}_{\breve{F}}([T_1, S_{\frac{\pi}{T_1}}]/(T_1, T_1 S_{\frac{\pi}{T_1}} - \pi))^\wedge = \mathrm{Spf}\bar{\kappa}([S_{\frac{\pi}{T_1}}])^\wedge.
\end{aligned} \tag{3.44}$$

$\mathrm{Exc} \cap D_2$ glues with $\mathrm{Exc} \cap D_3$ over $\mathrm{Exc} \cap D_1$ by $S_{\frac{\pi}{T_0}} = \frac{1}{S_{\frac{\pi}{T_1}}}$, so Exc is isomorphic to $\mathbb{P}^1_{\bar{\kappa}}$. The projective line $\mathbb{P}_{\bar{\Lambda}_2}$ only intersects the second chart and is defined by the equation

$$S_{\frac{\pi}{T_0}} = 0. \tag{3.45}$$

Similarly $\mathbb{P}_{\bar{\Lambda}'_2}$ only intersects the third chart and is defined by the equation

$$S_{\frac{\pi}{T_1}} = 0. \tag{3.46}$$

We refer to Example 8.3.53 of [Qin06] for more details about the blow up considered here.

Now we explain how the coordinates of blow up are related with the moduli problem locally around a superspecial point x. Since blow up commutes with flat base change, we have

$$\mathcal{N}_x^{\mathrm{Kra}} \coloneqq \mathrm{Bl}_x D \times_{\mathcal{N}_{(1,1)}} \mathrm{Spf}\, \mathcal{O}_{\mathcal{N}_{(1,1)},x} = \mathrm{Bl}_x \mathrm{Spf}\, \mathcal{O}_{\mathcal{N}_{(1,1)},x}, \tag{3.47}$$

and let $D_{i,x}$, $i = 1, 2, 3$ be the three charts for $\mathcal{N}_x^{\mathrm{Kra}}$ coming from D_i.

Let $R \in \mathrm{Nilp}_{O_{\breve{F}_0}}$ be an $\mathcal{O}_{D_{1,x}}$-algebra, and $s_{\frac{T_0}{\pi}}, s_{\frac{T_1}{\pi}} \in R$ be the image of $S_{\frac{T_0}{\pi}}, S_{\frac{T_1}{\pi}}$ under the structure morphism, and let t_0 and t_1 be given by (3.38). Then R determines a point $(X_{t_0,t_1}, \mathcal{F}) \in D_{1,x}(R)$ where X_{t_0,t_1} is described in Proposition 3.3.3, and $\mathcal{F} = \mathrm{Span}_R\{e_0 \otimes 1 + e_1 \otimes s_{\frac{T_1}{\pi}}\} \subset \mathrm{Lie}(X_{t_0,t_1})$ is the filtration of the Krämer moduli problem.

Let $R \in \mathrm{Nilp}_{O_{\breve{F}_0}}$ be an $\mathcal{O}_{D_{2,x}}$-algebra, and $t_0, s_{\frac{\pi}{T_0}} \in R$ be the image of $T_0, S_{\frac{\pi}{T_0}}$ under the structure morphism, and let t_1 be given by (3.40). Then R determines a point $(X_{t_0,t_1}, \mathcal{F}) \in D_{2,x}(R)$ where X_{t_0,t_1} is as before and $\mathcal{F} = \mathrm{Span}_R\{e_0 \otimes 1 + e_1 \otimes s_{\frac{\pi}{T_0}}\} \subset \mathrm{Lie}(X_{t_0,t_1})$. The corresponding description for an $\mathcal{O}_{D_{3,x}}$-algebra is similar.

3.3.4 Proof of Theorem 3.3.1

Proof of Theorem 3.3.1. In the following, we use $m(\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}), \mathcal{Z})$ to denote the multiplicity of $\mathcal{Z}$ in $\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x})$, where $\mathcal{Z}$ is an irreducible component of $\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x})$. According to Theorem 3.3.2, the horizontal component of $\mathcal{Z}(\mathbf{x})$ intersects with the special fiber of $\mathcal{N}_{(1,1)}$ at some superspecial points. Moreover, each irreducible component of the special fiber of $\mathcal{N}_{(1,1)}$ also passes through some superspecial points. Therefore to determine the multiplicity of each irreducible component, it is enough to consider the equations of special divisors at superspecial points and their pullbacks to the Krämer model.

Write $b = \alpha_0 e_0 + \alpha_1 e_1$ as in Proposition 3.3.4. As before we fix a superspecial point $x = pt_{\Lambda_0} \in \mathcal{N}_{(1,1)}$ for a vertex lattice Λ_0 of type 0. Recall that the equations of $\mathcal{Z}(\mathbf{x})$ in $\mathcal{O}_{\mathcal{N}_{(1,1)},x}$ are:

$$\bar{\alpha}_1 T_0 - \pi\bar{\alpha}_0 = 0, \quad \bar{\alpha}_0 T_1 - \pi\bar{\alpha}_1 = 0, \ \alpha_i \in O_F. \tag{3.48}$$

When $\mathrm{ord}_\pi(\alpha_0) > \mathrm{ord}_\pi(\alpha_1)$, the equations of $\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x})$ in $\mathcal{N}_x^{\mathrm{Kra}}$ are

$$\begin{cases} \bar{\alpha}_1\pi(S_{\frac{T_0}{\pi}} - \frac{\bar{\alpha}_0}{\alpha_1}) = 0, \quad \bar{\alpha}_1\pi(\frac{\bar{\alpha}_0}{\alpha_1}S_{\frac{T_1}{\pi}} - 1) = 0, & \text{in the first chart,} \\ \bar{\alpha}_1 T_0(1 - \frac{\bar{\alpha}_0}{\bar{\alpha}_1}S_{\frac{\pi}{T_0}}) = 0, \quad \bar{\alpha}_1 T_0 S_{\frac{\pi}{T_0}}(\frac{\bar{\alpha}_0}{\bar{\alpha}_1}S_{\frac{\pi}{T_0}} - 1) = 0, & \text{in the second chart,} \\ \bar{\alpha}_1 T_1 S^2_{\frac{\pi}{T_1}}(1 - \frac{\bar{\alpha}_0}{\bar{\alpha}_1}\frac{1}{S_{\frac{\pi}{T_1}}}) = 0, \quad \bar{\alpha}_1 T_1 S_{\frac{\pi}{T_1}}(\frac{\bar{\alpha}_0}{\bar{\alpha}_1}\frac{1}{S_{\frac{\pi}{T_1}}} - 1) = 0, & \text{in the third chart.} \end{cases}$$

Notice that $(\frac{\bar{\alpha}_0}{\bar{\alpha}_1}S_{\frac{T_1}{\pi}} - 1)$ and $(1 - \frac{\bar{\alpha}_0}{\bar{\alpha}_1}S_{\frac{\pi}{T_0}})$ are units in coordinate ring of $D_{1,x}$ and coordinate ring of $D_{2,x}$. In the third chart, we have $T_1 S_{\frac{\pi}{T_1}} = \pi$, which implies that

$$\frac{\bar{\alpha}_0}{\bar{\alpha}_1}\frac{1}{S_{\frac{\pi}{T_1}}} - 1 = \alpha T_1 - 1$$

is a unit in $D_{3,x}$ with $\alpha = \frac{\bar{\alpha}_0}{\bar{\alpha}_1\pi} \in O_F$. So the above equations simplify to

$$\begin{cases} \pi^{\mathrm{ord}_\pi(\alpha_1)+1} = 0, & \text{in the first chart,} \\ \pi^{\mathrm{ord}_\pi(\alpha_1)}T_0 = T_0^{\mathrm{ord}_\pi(\alpha_1)+1}(S_{\frac{\pi}{T_0}})^{\mathrm{ord}_\pi(\alpha_1)} = 0, & \text{in the second chart,} \\ \pi^{\mathrm{ord}_\pi(\alpha_1)+1} = T_1^{\mathrm{ord}_\pi(\alpha_1)+1}(S_{\frac{\pi}{T_1}})^{\mathrm{ord}_\pi(\alpha_1)+1} = 0, & \text{in the third chart.} \end{cases}$$

Therefore, it has no horizonal component, and we have by (3.44), (3.45) and (3.46)

$$\begin{aligned} m(\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}), \mathbb{P}_{\bar{\Lambda}_2}) &= \mathrm{ord}_\pi(\alpha_1), \\ m(\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}), \mathbb{P}_{\bar{\Lambda}'_2}) &= \mathrm{ord}_\pi(\alpha_1) + 1, \\ m(\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}), \mathrm{Exc}_{\Lambda_0}) &= \mathrm{ord}_\pi(\alpha_1) + 1. \end{aligned} \tag{3.49}$$

Similarly if $\mathrm{ord}_\pi(\alpha_0) < \mathrm{ord}_\pi(\alpha_1)$, $\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x})$ has no horizonal component, and

$$\begin{aligned} m(\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}), \mathbb{P}_{\bar{\Lambda}_2}) &= \mathrm{ord}_\pi(\alpha_0) + 1, \\ m(\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}), \mathbb{P}_{\bar{\Lambda}'_2}) &= \mathrm{ord}_\pi(\alpha_0), \\ m(\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}), \mathrm{Exc}_{\Lambda_0}) &= \mathrm{ord}_\pi(\alpha_0) + 1. \end{aligned} \tag{3.50}$$

When $\mathrm{ord}_\pi(\alpha_0) = \mathrm{ord}_\pi(\alpha_1)$(possible only when $q(b) \neq 0$), the equations of $\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x})$ in

$\mathcal{N}_x^{\mathrm{Kra}}$ are

$$\begin{cases} \pi^{\mathrm{ord}_\pi(\alpha_0)+1}(S_{\frac{T_1}{\pi}} - \frac{\bar{\alpha}_1}{\bar{\alpha}_0}) = 0, & \text{in the first chart,} \\ T_0^{\mathrm{ord}_\pi(\alpha_0)+1}(S_{\frac{\pi}{T_0}})^{\mathrm{ord}_\pi(\alpha_0)}(S_{\frac{\pi}{T_0}} - \frac{\bar{\alpha}_1}{\bar{\alpha}_0}) = 0, & \text{in the second chart,} \\ T_1^{\mathrm{ord}_\pi(\alpha_0)+1}(S_{\frac{\pi}{T_1}})^{\mathrm{ord}_\pi(\alpha_0)}(S_{\frac{\pi}{T_1}} - \frac{\bar{\alpha}_0}{\bar{\alpha}_1}) = 0, & \text{in the third chart,} \end{cases}$$

which implies by (3.44), (3.45) and (3.46)

$$\begin{aligned} m(\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}), \mathbb{P}_{\bar{\Lambda}_2}) &= \mathrm{ord}_\pi(\alpha_0), \\ m(\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}), \mathbb{P}_{\bar{\Lambda}_2'}) &= \mathrm{ord}_\pi(\alpha_0), \\ m(\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}), \mathrm{Exc}_{\Lambda_0}) &= \mathrm{ord}_\pi(\alpha_0) + 1. \end{aligned} \tag{3.51}$$

In addition, it has a horizontal component given by

$$\begin{cases} S_{\frac{T_1}{\pi}} - \frac{\bar{\alpha}_1}{\bar{\alpha}_0} = 0, & \text{in the first chart,} \\ S_{\frac{\pi}{T_0}} - \frac{\bar{\alpha}_1}{\bar{\alpha}_0} = 0, & \text{in the second chart,} \\ S_{\frac{\pi}{T_1}} - \frac{\bar{\alpha}_0}{\bar{\alpha}_1} = 0, & \text{in the third chart.} \end{cases} \tag{3.52}$$

This is the local equation of $\mathcal{Z}^h(\mathbf{x})$ along the superspecial point $x = pt_{\Lambda_0}$. In this case, one has to have $\Lambda_0 = \Lambda_b$. From the equation, one can also see that the horizontal component is irreducible when $q(b) \neq 0$, and it intersects with Exc_{Λ_b} at one point (e.g., the image of $S_{\frac{T_1}{\pi}} = \frac{\bar{\alpha}_1}{\bar{\alpha}_0}$ in $\bar{\kappa}$ via the first chart).

Recall $b = \mathbf{x}(e)$. Let $n = n(b, \Lambda_0) = \min\{\mathrm{ord}_\pi(\alpha_0), \mathrm{ord}_\pi(\alpha_1)\}$. Note that

$$n(b, \Lambda_2) = \begin{cases} n+1, & \mathrm{ord}_\pi(\alpha_1) > \mathrm{ord}_\pi(\alpha_0), \\ n, & \text{otherwise,} \end{cases}$$

and

$$n(b, \Lambda_2') = \begin{cases} n+1, & \mathrm{ord}_\pi(\alpha_1) < \mathrm{ord}_\pi(\alpha_0), \\ n, & \text{otherwise.} \end{cases}$$

Hence in all three cases we have

$$\begin{aligned} m(\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}), \mathbb{P}_{\bar{\Lambda}_2}) &= n(b, \Lambda_2), \\ m(\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}), \mathbb{P}_{\bar{\Lambda}_2'}) &= n(b, \Lambda_2'), \\ m(\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}), \mathrm{Exc}_{\Lambda_0}) &= n(b, \Lambda_0) + 1. \end{aligned} \tag{3.53}$$

The above discussion holds for any $\Lambda \in \mathcal{T}(\mathbf{x})$. So we have

$$\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}) = \sum_{\Lambda_2 \in \mathcal{T}(\mathbf{x})} n(b, \Lambda_2)\mathbb{P}_{\bar{\Lambda}_2} + \sum_{\Lambda_0 \in \mathcal{T}(\mathbf{x})} (n(b, \Lambda_0) + 1)\mathrm{Exc}_{\Lambda_0} + \delta_b \mathcal{Z}^h(\mathbf{x}), \tag{3.54}$$

where $\delta_b = 0$ or 1 depends on whether $q(b) = 0$ or not. In the latter case, $\mathcal{Z}^h(\mathbf{x}) \cong \mathrm{Spf}\, \mathcal{O}_{\breve{F}}$ is a horizontal divisor meeting the special fiber at Exc_{Λ_b}.

□

The proof (in particular (3.52)) gives the following corollary, which will be used in next section.

Corollary 3.3.5. *Let $0 \neq \mathbf{x} \in \mathbb{V}$ and $b = \mathbf{x}(e)$. When $q(\mathbf{x}) = 0$, $\mathcal{Z}^h(\mathbf{x})$ is empty. When $q(\mathbf{x}) \neq 0$ with $b \in \Lambda_b$, $\mathcal{Z}^h(\mathbf{x}) = \mathrm{Spf}\, O_{\breve{F}}$ is irreducible, and intersects with the special fiber of $\mathcal{N}^{\mathrm{Kra}}$ at one point on the exceptional divisor Exc_{Λ_b} and is given by image of (3.52) modulo π. More precisely, write*

$$\pi^{-\mathrm{ord}_{\pi_0} q(b)} b = \alpha_0 e_0 + \alpha_1 e_1,$$

where $\{e_0, e_1\}$ is a basis of Λ_b given in Proposition 3.3.4. Then $\alpha_0, \alpha_1 \in O_{\breve{F}}^{\times}$, and the intersection point of $\mathcal{Z}^h(\mathbf{x})$ and $\mathcal{N}^{\mathrm{Kra}}/\bar{\kappa}$ is $S_{\frac{T_1}{\pi}} = \frac{\bar{\alpha}_1}{\bar{\alpha}_0} \pmod{\pi}$ in the first affine chart of the neighborhood of $\mathcal{N}^{\mathrm{Kra}}_x$ where $x = pt_{\Lambda_b}$.

3.4 Intersection between special divisors

In this section we establish a series of lemmas and prove the following theorem.

Theorem 3.4.1. *For a pair of independent vectors $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{V}$, let $T(\mathbf{x}_1, \mathbf{x}_2) = (h(\mathbf{x}_i, \mathbf{x}_j))$ be the associated Gram matrix. Then Int$(L_{\mathbf{x}_1,\mathbf{x}_2})$ depends only on the equivalence class of $T = T(\mathbf{x}_1, \mathbf{x}_2)$. More precisely, Int$(L_{\mathbf{x}_1,\mathbf{x}_2}) = 0$ unless $T(\mathbf{x}_1, \mathbf{x}_2)$ is integral. Assume that $T(\mathbf{x}_1, \mathbf{x}_2)$ is integral.*

1. *When $T \approx diag(u_1(-\pi_0)^\alpha, u_2(-\pi_0)^\beta)$ with $u_i \in \mathcal{O}_{F_0}^\times$ and $0 \leq \alpha \leq \beta$, we have*

$$\mathrm{Int}(L_{\mathbf{x}_1,\mathbf{x}_2}) = \alpha + \beta - \frac{2q(q^\alpha - 1)}{q-1}.$$

2. *When $T \approx \left(\begin{smallmatrix} 0 & \pi^n \\ (-\pi)^n & 0 \end{smallmatrix}\right)$ with n odd (which occurs only when $\mathbb{V}$ is isotropic), we have*

$$\mathrm{Int}(L_{\mathbf{x}_1,\mathbf{x}_2}) = -\frac{(q+1)(q^{\frac{n+1}{2}} - 1)}{q-1} + n + 1.$$

First of all, for two divisors $\mathcal{Z}_1$ and $\mathcal{Z}_2$ of $\mathcal{N}^{\mathrm{Kra}}$ such that $\mathcal{Z}_1 \cap \mathcal{Z}_2$ is supported on finitely many irrreducible components of the special fiber of $\mathcal{N}^{\mathrm{Kra}}$, we define their intersection number to be

$$\mathcal{Z}_1 \cdot \mathcal{Z}_2 := \chi(\mathcal{N}^{\mathrm{Kra}}, O_{\mathcal{Z}_1} \otimes^{\mathbb{L}} O_{\mathcal{Z}_2}), \tag{3.55}$$

where $O_{\mathcal{Z}_i}$ is the structure sheaf of $\mathcal{Z}_i$, $\otimes^{\mathbb{L}}$ is the derived tensor product on the coherent sheaves on $\mathcal{N}^{\mathrm{Kra}}$ and χ is the Euler-Poincaré characteristic.

For a full rank lattice $L_{\mathbf{x}_1,\mathbf{x}_2} \subset \mathbb{V}$ with a basis $\{\mathbf{x}_1, \mathbf{x}_2\}$, let

$$\mathrm{Int}(L_{\mathbf{x}_1,\mathbf{x}_2}) = \mathcal{Z}(\mathbf{x}_1) \cdot \mathcal{Z}(\mathbf{x}_2).$$

According to [How19, Corollary D], this intersection number does not depend on the choice of the basis $\{\mathbf{x}_1, \mathbf{x}_2\}$. First, we recall the following well-known fact.

Proposition 3.4.2. *([Qin06, Proposition 9.1.21]) Assume that X is a regular scheme of dimension 2, S is a Dedekind scheme of dimension 1 and we have a flat proper morphism $X \to S$. Let $s \in S$ be a closed point. The following properties are true:*

(a) For any $E \in \mathrm{Div}_s(X)$, we have $E \cdot X_s$=0. Here $\mathrm{Div}_s(X)$ is the set of Cartier divisors

of X supported on the special fiber X_s.

(b) *Let $\Gamma_1, ...\Gamma_r$ be the irreducible components of X_s of respective multiplicities $d_1, ..., d_r$. Then for any $i \leq r$, we have*

$$\Gamma_i^2 = -\frac{1}{d_i} \sum_{j \neq i} d_j \Gamma_j \cdot \Gamma_i.$$

Lemma 3.4.3. *Let Λ_0 be a fixed vertex O_F-lattice of type 0. Then*

(a) $\mathrm{Exc}_{\Lambda_0} \cdot \mathbb{P}_{\bar{\Lambda}_2} = \begin{cases} 1, & \textit{if } \Lambda_0 \subset \Lambda_2, \\ 0, & \textit{otherwise.} \end{cases}$

(b) $\mathrm{Exc}_{\Lambda_0} \cdot \mathrm{Exc}_{\Lambda_0'} = -2\delta_{\Lambda_0, \Lambda_0'}$ *for any type 0 vertex lattice* Λ_0'.

(c) $\mathrm{Exc}_{\Lambda_0} \cdot \mathcal{Z}^h(\mathbf{x}) = \delta_{\Lambda_0, \Lambda_b} = 1$ *or* 0 *depending on whether* $\Lambda_0 = \Lambda_b$ *or not.*

Proof. By (3.44), (3.45) and (3.46), we can see that

$$\mathrm{Exc}_{\Lambda_0} \cdot \mathbb{P}_{\bar{\Lambda}_2} = \mathrm{Exc}_{\Lambda_0} \cdot \mathbb{P}_{\bar{\Lambda}_2'} = 1 \tag{3.56}$$

if $\Lambda_0 = \Lambda_2 \cap \Lambda_2'$. For a Λ_2'' s.t. $\Lambda_0 \not\subset \Lambda_2''$, then clearly $\mathrm{Exc}_{\Lambda_0} \cap \mathbb{P}_{\bar{\Lambda}_2''} = \emptyset$. Hence (a) is proved. Part (b) follows from Proposition 3.4.2 and part (a). Part (c) follows from Theorem 3.3.2 and (3.52). □

Lemma 3.4.4. *Let Λ_2 be a fixed vertex O_F-lattice of type 2. Then*

(a) $\mathbb{P}_{\bar{\Lambda}_2} \cdot \mathbb{P}_{\bar{\Lambda}_2'} = \begin{cases} -(q+1), & \textit{if } \Lambda_2 = \Lambda_2', \\ 0, & \textit{otherwise.} \end{cases}$

(b) $\mathbb{P}_{\bar{\Lambda}_2} \cdot \mathcal{Z}^h(\mathbf{x}) = 0.$

Proof. Note that $\mathbb{P}_{\bar{\Lambda}_2} \cap \mathbb{P}_{\bar{\Lambda}_2'} = \emptyset$ if $\Lambda_2 \neq \Lambda_2'$. Moreover, there are $q+1$ exceptional divisors intersecting with $\mathbb{P}_{\bar{\Lambda}_2}$. Then part (b) of Proposition 3.4.2 and equation (3.56) imply that $\mathbb{P}_{\bar{\Lambda}_2} \cdot \mathbb{P}_{\bar{\Lambda}_2} + (q+1) = 0$. So part (a) follows. It is clear from (3.44), (3.45) and (3.52) that $\mathbb{P}_{\bar{\Lambda}_2}$ and $\mathcal{Z}^h(\mathbf{x})$ do not intersect. Hence part (b) follows. □

Lemma 3.4.5. $\mathbb{P}_{\bar{\Lambda}_2} \cdot \mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}) = \begin{cases} 1, & \textit{if } \Lambda_2 \in \mathcal{T}(\mathbf{x}), \\ 0, & \textit{otherwise.} \end{cases}$

Proof. When $\Lambda_2 \notin \mathcal{T}(\mathbf{x})$, the intersection number is obviously 0. When $\Lambda_2 \in \mathcal{T}(\mathbf{x})$. We have by Theorem 3.3.1, Lemma 3.4.3 and 3.4.4,

$$\begin{aligned}\mathbb{P}_{\bar{\Lambda}_2} \cdot \mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}) =& \mathbb{P}_{\bar{\Lambda}_2} \cdot [\sum_{\Lambda_2' \in \mathcal{T}(\mathbf{x})} n(b, \Lambda_2') \mathbb{P}_{\bar{\Lambda}_2'} + \sum_{\Lambda_0 \in \mathcal{T}(\mathbf{x})} (n(b, \Lambda_0) + 1) \mathrm{Exc}_{\Lambda_0}] \\ =& -(q+1) n(b, \Lambda_2) + \sum_{\Lambda_0 \subset \Lambda_2} (n(b, \Lambda_0) + 1) \\ =& \sum_{\Lambda_0 \subset \Lambda_2} (n(b, \Lambda_0) + 1 - n(b, \Lambda_2)).\end{aligned}$$

Now combining the information from (3.20) with Lemmas 3.2.10 and 3.2.12 we see that there is exactly one vertex lattice Λ' of type 0 in Λ_2 such that $n(b, \Lambda') = n(b, \Lambda_2)$ and for any other vertex lattice Λ of type 0 in Λ_2 we have $n(b, \Lambda) = n(b, \Lambda_2) - 1$. Hence we have

$$\sum_{\Lambda_0 \subset \Lambda_2} (n(b, \Lambda_0) + 1 - n(b, \Lambda_2)) = 1.$$

The lemma follows. $\square$

Lemma 3.4.6. $\mathrm{Exc}_{\Lambda_0} \cdot \mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}) = \begin{cases} -1, & \Lambda_0 \in \mathcal{T}(\mathbf{x}), \\ 0, & \textit{otherwise.} \end{cases}$

Proof. Assume Λ_2 and Λ_2' are the two vertex lattices of type 2 that contain Λ_0. We treat the cases $\Lambda_0 \neq \Lambda_b$ first. According to (3.20), Lemma 3.2.10 and 3.2.12, we can without loss of generality assume that

$$n(b, \Lambda_2) = n(b, \Lambda_0) + 1, \text{ and } n(b, \Lambda_2') = n(b, \Lambda_0).$$

Then by Theorem 3.3.1 and Lemma 3.4.3, we have

$$\begin{aligned}
\mathrm{Exc}_{\Lambda_0} \cdot \mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}) =& \mathrm{Exc}_{\Lambda_0} \cdot [\sum_{\Lambda_2 \in \mathcal{T}(\mathbf{x})} n(b, \Lambda_2)\mathbb{P}_{\bar{\Lambda}_2} + \sum_{\Lambda_0' \in \mathcal{T}(\mathbf{x})} (n(b, \Lambda_0') + 1)\mathrm{Exc}_{\Lambda_0'}] \\
=& n(b, \Lambda_2)\mathrm{Exc}_{\Lambda_0} \cdot \mathbb{P}_{\bar{\Lambda}_2} + n(b, \Lambda_2')\mathrm{Exc}_{\Lambda_0} \cdot \mathbb{P}_{\bar{\Lambda}_2'} + (n(b, \Lambda_0) + 1)\mathrm{Exc}_{\Lambda_0} \cdot \mathrm{Exc}_{\Lambda_0} \\
=& (n(b, \Lambda_0) + 1) + n(b, \Lambda_0) - 2(n(b, \Lambda_0) + 1) \\
=& -1.
\end{aligned}$$

Now assume $\Lambda_0 = \Lambda_b$, which occurs only when $q(b) \neq 0$. Then notice that $n(b, \Lambda_2) = n(b, \Lambda_2') = n(b, \Lambda_0)$, and $\mathrm{Exc}_{\Lambda_0} \cdot \mathcal{Z}^h(\mathbf{x}) = 1$. The rest of the proof is the same as the previous case. $\square$

Lemma 3.4.7. *Write $b_i = \mathbf{x}_i(e)$, and assume that $(b_1, b_2) = 0$ and $q(b_i) \neq 0$. Then $\mathcal{Z}^h(\mathbf{x}_1) \cdot \mathcal{Z}^h(\mathbf{x}_2) = 0$, and*

$$\mathcal{Z}^h(\mathbf{x}_1) \cdot \mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}_2) = n(b_2, \Lambda_{b_1}) + 1.$$

Proof. Define $\beta_i = \pi^{-\mathrm{ord}_{\pi_0} q(b_i)} b_i$ for $i = 1, 2$. The assumption implies

$$\Lambda_{b_1} = \Lambda_{b_2} = O_F\beta_1 + O_F\beta_2,$$

and the Gram matrix of $\{\beta_1, \beta_2\}$ is diagonal with diagonal entries in $O_{F_0}^{\times}$. Let $\{e_0, e_1\}$ be an O_F-basis of Λ_{b_1} as in Proposition 3.3.4 and write

$$\beta_1 = \alpha_0 e_0 + \alpha_1 e_1, \text{ and } \beta_2 = \alpha_0' e_0 + \alpha_1' e_1.$$

By comparing the determinants of the Gram matrices of $\{e_0, e_1\}$ and $\{\beta_1, \beta_2\}$, we know that the transition matrix $\begin{pmatrix} \alpha_0 & \alpha_1 \\ \alpha_0' & \alpha_1' \end{pmatrix}$ has determinant $\alpha_0\alpha_1' - \alpha_1\alpha_0' = u \in O_F^{\times}$.

On the other hand, if $\mathcal{Z}^h(\mathbf{x}_1)$ and $\mathcal{Z}^h(\mathbf{x}_2)$ intersect, the intersection points would be in the special fiber. By Corollary 3.3.5, they intersect at one point, which is also the intersection between them and the exceptional divisor $\mathrm{Exc}_{\Lambda_{b_i}}$. The same corollary asserts that the point is given by (say in first Chart)

$$\frac{\bar{\alpha}_1}{\bar{\alpha}_0} \equiv \frac{\bar{\alpha}_1'}{\bar{\alpha}_0'} \pmod{\pi}.$$

This implies $\alpha_0\alpha_1' - \alpha_1\alpha_0' \in \pi O_F$, a contradiction. So $\mathcal{Z}^h(\mathbf{x}_1) \cdot \mathcal{Z}^h(\mathbf{x}_2) = 0$. Now we have by Theorem 3.3.1, Lemmas 3.4.3 and 3.4.4,

$$\begin{aligned} \mathcal{Z}^h(\mathbf{x}_1) \cdot \mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}_2) &= \mathcal{Z}^h(\mathbf{x}_1) \cdot (n(b_2, \Lambda_{b_1}) + 1)\mathrm{Exc}_{\Lambda_{b_1}} \\ &= n(b_2, \Lambda_{b_1}) + 1. \end{aligned}$$

□

Theorem 3.4.8. *Let* $\{\mathbf{x}_1, \mathbf{x}_2\}$ *be a basis of a full rank lattice* $L_{\mathbf{x}_1,\mathbf{x}_2} \subset \mathbb{V}$*, and* $b_1 = \mathbf{x}_1(e), b_2 = \mathbf{x}_2(e)$ *as before. Assume*

$$T = \begin{pmatrix} (b_1, b_1) & (b_1, b_2) \\ (b_2, b_1) & (b_2, b_2) \end{pmatrix} \approx \begin{pmatrix} u_1(-\pi_0)^\alpha & 0 \\ 0 & u_2(-\pi_0)^\beta \end{pmatrix}$$

where $\alpha \le \beta$*,* $u_1, u_2 \in O_{F_0}^\times$ *and* $-u_1u_2 \in \mathrm{Nm}(F^\times)$*. When* $\alpha \ge 0$*, we have*

$$\mathrm{Int}(L_{\mathbf{x}_1,\mathbf{x}_2}) = \alpha + \beta - \frac{2q(q^\alpha - 1)}{q - 1}.$$

When $\alpha < 0$*, we have* $\mathrm{Int}(L_{\mathbf{x}_1,\mathbf{x}_2}) = 0$.

Proof. We may assume $T = \mathrm{Diag}(u_1(-\pi_0)^\alpha, u_2(-\pi_0)^\beta)$. In such a case, we see that $\mathrm{span}_{O_F}\{\pi^{-\alpha}\mathbf{x}_1(e), \pi^{-\beta}\mathbf{x}_2(e)\} = \Lambda_{b_1} = \Lambda_{b_2}$. Moreover $\mathcal{T}(\mathbf{x}_1) \subset \mathcal{T}(\mathbf{x}_2)$ and $\mathcal{T}(\mathbf{x}_1)$ is a ball of radius $\alpha + \frac{1}{2}$ centered at Λ_{b_1} by Corollary 3.2.11. We have by Theorem 3.3.1 and

Lemmas 3.4.5, 3.4.6, 3.4.7, and 3.2.10,

$$\begin{aligned}
&\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}_1)\cdot\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}_2)\\
=&\sum_{\Lambda_2\in\mathcal{T}(\mathbf{x}_1)} n(b_1,\Lambda_2)\mathbb{P}_{\bar{\Lambda}_2}\cdot\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}_2)+\sum_{\Lambda_0\in\mathcal{T}(\mathbf{x}_1)}(n(b_1,\Lambda_0)+1)\mathrm{Exc}_{\Lambda_0}\cdot\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}_2)\\
&+\mathcal{Z}^h(\mathbf{x}_1)\cdot\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}_2)\\
=&\sum_{\Lambda_2\in\mathcal{T}(\mathbf{x}_1)} n(b_1,\Lambda_2)-\sum_{\Lambda_0\in\mathcal{T}(\mathbf{x}_1)}(n(b_1,\Lambda_0)+1)+(n(b_2,\Lambda_{b_1})+1)\\
=&2(\sum_{i=0}^{\alpha}(\alpha-i)q^i)-(\alpha+1+2(\sum_{i=1}^{\alpha}(\alpha+1-i)q^i))+(\beta+1)\\
=&\alpha+\beta-2\sum_{i=1}^{\alpha}q^i\\
=&\alpha+\beta-\frac{2q(q^\alpha-1)}{q-1}.
\end{aligned}$$

$\square$

Remark 3.4.9. *As we mentioned before, there is a similar result for intersection product between special divisors on a similarly defined RZ space $\mathcal{N}_E$ obtained by Sankaran in [San17]. Here E is an unramified quadratic field extension of F_0.*

Theorem 3.4.10. *For $\mathbf{x}_1,\mathbf{x}_2\in\mathbb{V}$, let $b_1=\mathbf{x}_1(e), b_2=\mathbf{x}_2(e)$ as before. Assume*

$$T=\begin{pmatrix}(b_1,b_1)&(b_1,b_2)\\(b_2,b_1)&(b_2,b_2)\end{pmatrix}\approx\begin{pmatrix}0&\pi^n\\(-\pi)^n&0\end{pmatrix},$$

with n odd. Write $r=\frac{n+1}{2}$. Then $\mathrm{Int}(L_{\mathbf{x}_1,\mathbf{x}_2})=0$ unless $n\geq 0$. In such a case,

$$\mathrm{Int}(L_{\mathbf{x}_1,\mathbf{x}_2})=-\frac{(q+1)(q^r-1)}{q-1}+2r.$$

Proof. We may assume that $T=\left(\begin{smallmatrix}0&\pi^n\\(-\pi)^n&0\end{smallmatrix}\right)$. By Lemmas 3.4.5, 3.4.6 and Theorem 3.3.1, we have

$$
\begin{aligned}
&\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}_1) \cdot \mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}_2) \\
=& \sum_{\Lambda_2 \in \mathcal{T}(\mathbf{x}_1)} n(b_1, \Lambda_2) \mathbb{P}_{\bar{\Lambda}_2} \cdot \mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}_2) + \sum_{\Lambda_0 \in \mathcal{T}(\mathbf{x}_1)} (n(b_1, \Lambda_0) + 1) \mathrm{Exc}_{\Lambda_0} \cdot \mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}_2) \\
=& \sum_{\Lambda_2 \in \mathcal{T}(\mathbf{x})} n(b_1, \Lambda_2) - \sum_{\Lambda_0 \in \mathcal{T}(\mathbf{x})} (n(b_1, \Lambda_0) + 1),
\end{aligned}
$$

where $\mathcal{T}(\mathbf{x}) = \mathcal{T}(\mathbf{x}_1) \cap \mathcal{T}(\mathbf{x}_2)$.

Write $b_i = \pi^r v_i$ for $i = 1, 2$. Define

$$
\Lambda_{k,2} = \mathrm{span}_{O_F}\{\pi^{-k} v_1, \pi^k v_2\}, \ \Lambda_{k,0} = \mathrm{span}_{O_F}\{\pi^{-k} v_1, \pi^{k+1} v_2\}, \ k \in \mathbb{Z}.
$$

where $\Lambda_{k,2}$ is a vertex lattice of type 2 and $\Lambda_{k,0}$ is a vertex lattice of type 0. It is immediate that

$$
n(b_1, \Lambda_{k,0}) = n(b_1, \Lambda_{k,2}) = r + k.
$$

By Lemma 3.2.15, $\mathcal{T}(\mathbf{x})$ is a ball of radius r centered at $\Lambda_{0,2}$. We define $\mathcal{C} = \mathcal{C}(\Lambda_{0,2}, \Lambda_{r,2})$ to be the geodesic segment joining $\Lambda_{0,2}$ and $\Lambda_{r,2}$.

Now we divide $\mathcal{T}(\mathbf{x})$ into $r+1$ parts $\mathcal{L}_k$ using $\mathbb{V}$ as follows: for $0 \leq k \leq r$, let $\mathcal{L}_k$ be the part of $\mathcal{T}(\mathbf{x})$ such that $\Lambda \in \mathcal{L}_k$ if and only if $\Lambda_{k,2}$ is the first vertex lattice of type 2 that the geodesic from Λ to $\Lambda_{0,2}$ encounters on $\mathcal{C}$. In other words, if we set $\mathcal{L}'_k$ to be the subtree of $\mathcal{T}(\mathbf{x}) \setminus \mathbb{V}$ that starts from $\Lambda_{k,2}$. Then $\mathcal{L}_k = \mathcal{L}'_k \cup \Lambda_{k,0}$ if $\Lambda_{k,0} \in \mathcal{T}(\mathbf{x})$. In particular, $\mathcal{L}_r = \{\Lambda_{r,2}\}$. Then we have

$$
\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}_1) \cdot \mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}_2) = \sum_{k=0}^{r} (S(k) - S'(k))
$$

where

$$
S(k) = \sum_{\Lambda_2 \in \mathcal{L}_k} n(b_1, \Lambda_2), \ \text{and} \ S'(k) = \sum_{\Lambda_0 \in \mathcal{L}_k} (n(b_1, \Lambda_0) + 1)
$$

For $k = 0$ we have

$$S(0) = \sum_{i=0}^{r}(r-i)q^i,\ S'(0) = \sum_{i=0}^{r}(r+1-i)q^i.$$

For $1 \le k \le r-1$ we have

$$S(k) = r + k + \sum_{i=0}^{r-k-1}(r+k-i-1)(q-1)q^i,$$

$$S'(k) = r + k + 1 + \sum_{i=0}^{r-k-1}(r+k-i)(q-1)q^i.$$

For $k = r$ we have

$$S(r) = 2r,\ S'(r) = 0.$$

Summing these terms up, we obtain

$$\begin{aligned}
\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}_1) \cdot \mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}_2) =&(S(0) - S'(0)) + \sum_{k=1}^{r-1}(S(k) - S'(k)) + S(r) \\
=& - \sum_{i=0}^{r} q^i - \sum_{k=1}^{r-1} q^{r-k} + 2r \\
=& - \frac{q^{r+1}-1}{q-1} - \frac{q(q^{r-1}-1)}{q-1} + 2r \\
=& - \frac{(q+1)(q^r-1)}{q-1} + 2r.
\end{aligned}$$

as claimed.

□

By Theorems 3.4.8 and 3.4.10, we see that $\mathrm{Int}(L_{\mathbf{x}_1,\mathbf{x}_2})$ depends only on the Gram matrix $T(b_1, b_2) = -\delta^2 T(\mathbf{x}_1, \mathbf{x}_2)$. Since $-\delta^2 \in O_{F_0}^{\times}$, we see that the formulas in both theorems are not affected when we change $T(b_1, b_2)$ to $T(\mathbf{x}_1, \mathbf{x}_2)$. This proves Theorem 3.4.1.

3.5 Local densities and the Kudla-Rapoport Conjecture

In this section we record basic results on local density and prove Theorem 3.0.1. Let L be an integral O_F-lattice with Gram matrix S (unique up to equivalence), and let T be an $n \times n$ invertible Hermitan matrix over T. Recall the local density polynomial $\alpha(L, T, X) = \alpha(S, T, X)$ defined in the introduction. the following explicit formulas are special cases considered in ([Shi22]).

Theorem 3.5.1. *([Shi22, Theorem 6.2]) Let L be an O_F hermitian lattice with Gram matrix $S = diag(v, 1)$ with $v \in O_{F_0}^{\times}$, and let H be the hermitian hyperbolic O_F-plane defined in the introduction. Set $\epsilon_2 = \chi(-v)$, where χ be the quadratic character of $F_0^{\times}$ associated to the extension of F/F_0.*

1. *Assume that $T \approx \begin{pmatrix} u_1(-\pi_0)^\alpha & 0 \\ 0 & u_2(-\pi_0)^\beta \end{pmatrix}$ with $\alpha \leq \beta$ and $u_i \in O_{F_0}^{\times}$. Set $\epsilon_1 = \chi(-u_1u_2)$. Then $\alpha(L, T, X) = \alpha(H, T, X) = 0$ unless $\alpha \geq 0$. Assume $\alpha \geq 0$, we have*

$$\begin{aligned} \alpha(L, T, X) =&(1 - X)(1 + \epsilon_2 + q\epsilon_2) \sum_{e=0}^{\alpha} (qX)^e - \epsilon_1 q^{\alpha+1} X^{\beta+1}(1 - X) \sum_{e=0}^{\alpha} (q^{-1}X)^e \\ &-\epsilon_1(1 + q)(X^{\alpha+\beta+2} + \epsilon_1\epsilon_2) + (1 + \epsilon_2)q^{\alpha+1}X^{\alpha+1}(1 + \epsilon_1 X^{\beta-\alpha}), \end{aligned}$$

and

$$\begin{aligned} \alpha(H, T, X) =&(1 + \frac{1}{q} + \frac{1}{q^2})(1 - X) \sum_{e=0}^{\alpha} (qX)^e - \epsilon_1 q^{\alpha-1} X^{\beta+1}(1 - X) \sum_{e=0}^{\alpha} (q^{-1}X)^e \\ &- q^{-1}(1 + q^{-1})(1 + \epsilon_1 X^{\alpha+\beta+2}) + q^\alpha(1 + q^{-1})X^{\alpha+1}(1 + \epsilon_1 X^{\beta-\alpha+1}). \end{aligned}$$

In particular,

$$\alpha(L, T) = -\epsilon_1(1 + \epsilon_1\epsilon_2)(1 + q) + (1 + \epsilon_1)(1 + \epsilon_2)q^{\alpha+1},$$

which is zero if and only if $\epsilon_1\epsilon_2 = -1$*. Similarly*

$$\alpha(H,T) = (1+\epsilon_1)(1+q^{-1})(q^{\alpha} - q^{-1}),$$

which is zero if and only if $\epsilon_1 = -1$*.*

2. *Assume that* $T = \begin{pmatrix} 0 & \pi^n \\ (-\pi)^n & 0 \end{pmatrix}$ *where* n *is odd. Then* $\alpha(L,T,X) = \alpha(H,T,X) = 0$ *unless* $n \geq -1$*. When* $n \geq -1$*, we have*

$$\alpha(L,T,X) = -q^{n+2}(1-X)\sum_{e=\frac{n+1}{2}+1}^{n+1}(q^{-1}X)^e + (1-X)(1+\epsilon_2+\epsilon_2 q)\sum_{e=0}^{\frac{n+1}{2}}(qX)^e$$
$$-(q+1)(\epsilon_2 + X^{n+2}) + (1+\epsilon_2)(q+1)q^{\frac{n+1}{2}}X^{\frac{n+1}{2}+1},$$

and

$$\alpha(H,T,X) = -(1-X)q^n\sum_{e=\frac{n+1}{2}+1}^{n+1}(q^{-1}X)^e + (1+\frac{1}{q}+\frac{1}{q^2})(1-X)\sum_{e=0}^{\frac{n+1}{2}}(qX)^e$$
$$-(\frac{1}{q}+\frac{1}{q^2})(1+X^{n+2}) + (1+\frac{1}{q})^2 q^{\frac{n+1}{2}}X^{\frac{n+1}{2}+1}.$$

In particular,

$$\alpha(L,T) = -(1+\epsilon_2)(q+1)(1-q^{\frac{n+1}{2}}),$$

which is zero if and only if $\epsilon_2 = -1$*. Finally,*

$$\alpha(H,T) = \frac{1+q}{q^2}\left[-2+(1+q)q^{\frac{n+1}{2}}\right] \neq 0.$$

Proof of Theorem 3.0.1. First notice that the Gram matrix of L can be chosen to be of the form $S = \mathrm{diag}(v,1)$ with $v \in O_{F_0}^{\times}$ and that the Gram matrix of H is S'. Let $T = T(\mathbf{x}_1,\mathbf{x}_2)$. As L and $L_{\mathbf{x}_1,\mathbf{x}_2}$ represent two different hermitian spaces of dimension 2, we have to have $\epsilon_1\epsilon_2 = -1$ when T can be diagonalized, and $\epsilon_2 = -1$ when T is anti-diagonal. The case that L is isotropic is proved in [Shi22]. Now we assume L is anisotropic, i.e, $\epsilon_2 = -1$. There are two cases.

Case 1: We first assume that $T \approx \mathrm{diag}(u_1(-\pi_0)^{\alpha}, u_2(-\pi_0)^{\beta})$ with $\alpha \leq \beta$. In this case $\epsilon_1 = 1$. When $\alpha < 0$, both sides of the identity are automatically zero. So we assume $\alpha \geq 0$. Theorem 3.5.1 implies that

$$\begin{aligned}&\alpha(L,T,X)\\ =&-q(1-X)\sum_{e=0}^{\alpha}(qX)^e - q^{\alpha+1}X^{\beta+1}(1-X)\sum_{e=0}^{\alpha}(q^{-1}X)^e - (q+1)(X^{\alpha+\beta+2}-1).\end{aligned}$$

So

$$\begin{aligned}-\alpha'(L,T) =&\frac{\partial}{\partial X}\alpha(L,T,X)|_{X=1}\\ =&q\sum_{e=0}^{\alpha}q^e + q^{\alpha+1}\sum_{e=0}^{\alpha}q^{-e} - (q+1)(\alpha+\beta+2)\\ =&2\sum_{e=1}^{\alpha+1}q^e - (q+1)(\alpha+\beta+2).\end{aligned}$$

Theorem 3.5.1(1) also implies

$$\alpha(L,S) = 2(1+q), \tag{3.57}$$

and

$$\alpha(H,T) = 2(1+q^{-1})(q^{\alpha}-q^{-1}).$$

Hence the right hand side of the formula in Theorem 3.0.1 is

$$\begin{aligned}&\frac{2}{\alpha(L,S)}[\alpha'(L,T) - \frac{q^2}{q^2-1}\alpha(H,T)]\\ =&\frac{1}{q+1}[-2\sum_{e=1}^{\alpha+1}q^e + (q+1)(\alpha+\beta+2) + \frac{2}{q-1} - \frac{2q^{\alpha+1}}{q-1}]\\ =&\alpha+\beta+2-2\sum_{e=0}^{\alpha}q^e = \mathrm{Int}(L_{\mathbf{x}_1,\mathbf{x}_2})\end{aligned}$$

as claimed by Theorem 3.0.1.

Case 2: Now we treat the anti-diagonal case

$$T \approx \begin{pmatrix} 0 & \pi^n \\ (-\pi)^n & 0 \end{pmatrix}, \ n \text{ odd.}$$

The case $n < -1$ is trivial as both sides are clearly zero.

When $n \geq -1$, and let $r = \frac{n+1}{2}$. we have by Theorem 3.5.1

$$\begin{aligned} \alpha'(L,T) &= -\frac{\partial}{\partial X}\alpha(L,T,X)|_{X=1} \\ &= 1 - (q+1)\sum_{e=0}^{r} q^e + (q+1)(n+2). \end{aligned}$$

Combining this with Theorem 3.5.1 (2), we see that the right hand side of Theorem 3.0.1 is

$$\begin{aligned} &\frac{2}{\alpha(L,S)}[\alpha'(L,T) - \frac{q^2}{q^2-1}\alpha(H,T)] \\ =&\frac{1}{q+1} - \sum_{e=0}^{r} q^e + 2r + 1 - \frac{q^r}{q-1} + \frac{2}{(q-1)(q+1)} \\ =&\mathrm{Int}(L_{\mathbf{x}_1,\mathbf{x}_2}) \end{aligned}$$

as claimed by Theorem 3.0.1. $\square$

Chapter 4

Kudla-Rapoport conjecture for Krämer models and the case $n = 3$

4.1 Induction formula and primitive local density

In this section, we study various induction formulas of local density polynomials. Let M be a hermitian O_F-lattice of rank m with $\mathrm{v}(M) \coloneqq \min\{\mathrm{v}_\pi(h(v, v')) \mid v, v' \in M\} \geq -1$. and let $M^{[k]} = H^k \obot M$ for an integer $k \geq 0$. Let L be a hermitian O_F-lattice of rank n.

There is a polynomial $\alpha(M, L, X)$ of X—the local density polynomial—such that

$$\alpha(M, L, q^{-2k}) = \int_{\mathrm{Herm}_n(F)} \int_{(M^{[k]})^n} \psi(\langle Y, T(\mathbf{x}) - T \rangle) d\mathbf{x}\, dY, \tag{4.1}$$

where $T(\mathbf{x})$ is the moment matrix of $\mathbf{x}$, $d\mathbf{x}$ is the Haar measure on $(M^{[k]})^n$ with total volume 1, dY is the Haar measures on $\mathrm{Herm}_n(F)$ such that $\mathrm{Herm}_n(O_F)$ has total volume 1, and ψ is an additive character of F_0 with conductor O_{F_0}. Finally, we define $\langle X, Y \rangle = \mathrm{Tr}(XY)$ on Herm_n. We will also denote $\alpha(M, L) = \alpha(M, L, 1)$ and

$$\alpha'(M, L) = -\frac{\partial}{\partial X} \alpha(M, L, X)|_{X=1}. \tag{4.2}$$

There is another way to define $\alpha(M, L, X)$ as follows.

Definition 4.1.1. *Let M and L be two hermitian O_F-lattices of rank m and n respectively. Let a be an integer such that $(x,y) \in \pi_0^{-a}\partial_{F/F_0}^{-1}$ for $x, y \in M$ or $x, y \in L$. Define the local density of M representing L as*

$$\alpha(M,L) := \lim_{d \to \infty} \frac{\left|\mathrm{Herm}_{L,M}(O_{F_0}/(\pi_0^{d+a}))\right|}{q^{2(d+a)nm-dn^2}},$$

which is independent of the choice of a. Here $\mathrm{Herm}_{L,M}(O_{F_0}/(\pi_0^{d+a}))$ *is given by the set*

$$\{\phi \in \mathrm{Hom}_{O_F}(L/\pi_0^{d+a}L, M/\pi_0^{d+a}M) \mid (\phi(x),\phi(y)) \equiv (x,y) \mod (\pi_0^d\partial_{F/F_0}^{-1}),\ x,y \in L\}.$$

in this thesis, we only deal with the case when we can and will choose $a = 0$.Since $\alpha(M,L,X)$ only depends on the Gram matrices of M and L, we may also denote it by $\alpha(S,T,X)$ if S and T are the Gram matrices of M and L.

Now we define primitive local density polynomials. For $1 \leq \ell \leq n$, let

$$(M^{[k]})^{n,(\ell)} = \{\mathbf{x} = (x_1, \cdots, x_n) \in (M^{[k]})^n \mid \dim \mathrm{Span}\{x_1, \cdots, x_\ell\} = \ell \text{ in } M^{[k]}/\pi M^{[k]}\}. \tag{4.3}$$

For $L = L_1 \oplus L_2$, where $L_1 = \mathrm{Span}\{l_1, \cdots, l_\ell\}$ and $L_2 = \mathrm{Span}\{l_{\ell+1}, \cdots, l_n\}$, we define the local ℓ-primitive density to be

$$\beta(M^{[k]}, L_1 \oplus L_2)^{(\ell)} = \int_{\mathrm{Herm}_n(F)} \int_{(M^{[k]})^{n,(\ell)}} \psi(\langle Y, T(\mathbf{x}) - T\rangle) d\mathbf{x}\, dY. \tag{4.4}$$

When $\ell \neq n$, the above definition depends on a choice of $L = L_1 \oplus L_2$. Hence we always fix such a decomposition $L = L_1 \oplus L_2$ in this case. When $L = L_1 \oplus L_2$, and L_i is represented by T_i, we also denote $\beta(M, L_1 \oplus L_2)^{(\ell)}$ as $\beta(S, \mathrm{Diag}(T_1, T_2))^{(\ell)}$. When $\ell = n$, we simply denote $\beta(M, L_1 \oplus L_2)^{(\ell)}$ as $\beta(M, L)$.

Lemma 4.1.2. *Assume* $L = L_1 \oplus L_2$ *where* $\mathrm{rank}(L_1) = n_1$. *Then*

$$\alpha(M, L, X) = \sum_{L_1 \subset L_1' \subset L_{1,F}} (q^{n-m}X)^{\ell(L_1'/L_1)} \beta(M, L_1' \oplus L_2, X)^{(n_1)},$$

where $\ell(L_1'/L_1) = \mathrm{length}_{O_F} L_1'/L_1$.

Proof. This is the analogue of [Kit83, Lemma 3]. Let $G = \mathrm{GL}_{n_1}(F) \cap \mathrm{M}_{n_1,n_1}(O_F)$ and $U = \mathrm{GL}_{n_1}(O_F)$. By choosing a basis $\{l_1, \cdots, l_{n_1}\}$ of L_1, we may identify $U\backslash G$ with $\{L_1' \mid L_1 \subset L_1' \subset L_{1,F}\}$ by sending g to $L_1 \cdot g^{-1}$. Then the identity we want to prove is equivalent to

$$\alpha(M, L, X) = \sum_{g \in U\backslash G} |\det g|^{2k+m-n} \beta(M, L_1 \cdot g^{-1} \oplus L_2, X)^{(n_1)},$$

where $|\pi| = q^{-1}$. By a partition of M_k^n, we have

$$\begin{aligned}\alpha(M, L, X) &= \int_{\mathrm{Herm}_n(F)} dY \int_{(M^{[k]})^n} \psi(\langle Y, T(\mathbf{x}) - T\rangle) d\mathbf{x} \\ &= \sum_{g \in U\backslash G} \int_{\mathrm{Herm}_n(F)} dY \int_{(M^{[k]})^{n,(n_1)} \cdot g_1} \psi(\langle Y, T(\mathbf{x}) - T\rangle) d\mathbf{x},\end{aligned}$$

where $g_1 = \mathrm{Diag}(g, I_{n-n_1})$, and the action of g_1 is simply matrix multiplication on the n components of $M^{n,(n_1)}$. Now

$$\begin{aligned}&\int_{\mathrm{Herm}_n(F)} dY \int_{(M^{[k]})^{n,(n_1)} \cdot g_1} \psi(\langle Y, T(\mathbf{x}) - T\rangle) d\mathbf{x} \\ &= |\det g_1|^{2k+m} \int_{\mathrm{Herm}_n(F)} dY \int_{(M^{[k]})^{n,(n_1)}} \psi(\langle Y, T(\mathbf{x}g_1) - T\rangle) d\mathbf{x} \\ &= |\det g_1|^{2k+m} \int_{\mathrm{Herm}_n(F)} dY \int_{(M^{[k]})^{n,(n_1)}} \psi(\langle Y, (T(\mathbf{x}) - T[g_1^{-1}])[g_1]\rangle) d\mathbf{x} \\ &= |\det g_1|^{2k+m} \int_{\mathrm{Herm}_n(F)} dY \int_{(M^{[k]})^{n,(n_1)}} \psi(\langle Y[g_1^*], T(\mathbf{x}) - T[g_1^{-1}]\rangle) d\mathbf{x} \\ &= |\det g_1|^{2k+m-n} \int_{\mathrm{Herm}_n(F)} dY \int_{(M^{[k]})^{n,(n_1)}} \psi(\langle Y, T(\mathbf{x}) - T[g_1^{-1}]\rangle) d\mathbf{x} \\ &= |\det g_1|^{2k+m-n} \beta(M^{[k]}, L \cdot g_1^{-1})^{(n_1)}.\end{aligned}$$

Here $T[g] := g^*Tg$. Now the lemma is clear. □

Theorem 4.1.3. *Let L be as in Lemma 4.1.2. Then*

$$\alpha(M,L,X) = \sum_{i=1}^{n_1}(-1)^{i-1}q^{i(i-1)/2+i(n-m)}X^i \cdot \sum_{\substack{L_1\subset L_1'\subset \pi^{-1}L_1\\ \dim L_1'/L_1=i}} \alpha(M,L_1'\oplus L_2,X) + \beta(M,L,X)^{(n_1)}.$$

Proof. This is an analogue of [Kat99, Proposition 2.1]. The proof follows from a combination of the argument (in a reverse order) in 4.9.2 and Lemma 4.1.2. □

Motivated by Theorem 4.1.3, we give the following definition.

Definition 4.1.4. *Let $L = L_1 \oplus L_2$ be as in Lemma 4.1.2. We define*

$$\partial\mathrm{Den}(L)^{(n_1)} := \partial\mathrm{Den}(L) - \sum_{i=1}^{n_1}(-1)^{i-1}q^{i(i-1)/2} \sum_{\substack{L_1\subset L_1'\subset L_{1,F}\\ \dim L_1'/L_1=i}} \partial\mathrm{Den}(L_1'\oplus L_2). \tag{4.5}$$

Corollary 4.1.5. *Let $L = L_1 \oplus L_2$ be as in Lemma 4.1.2, and $\epsilon = \chi(L)$. Then*

$$\partial\mathrm{Den}(L)^{(n_1)} = \frac{1}{\alpha(I_n^{-\epsilon}, I_n^{-\epsilon})}\left(2\beta'(I_n^{-\epsilon},L)^{(n_1)} + \sum_i c_\epsilon^{n,i}\beta(H_{n,i}^{\epsilon},L)^{(n_1)}\right).$$

As a corollary of Lemma 4.1.2, we have

Corollary 4.1.6. *Let $L = L_1 \oplus L_2$ be as in Lemma 4.1.2. Then we have the following identity where the summation is finite:*

$$\partial\mathrm{Den}(L) = \sum_{L_1\subset L_1'\subset L_{1,F}} \partial\mathrm{Den}(L_1'\oplus L_2)^{(n_1)}.$$

We may reduce the identity $\mathrm{Int}(L) = \partial\mathrm{Den}(L)$ to a primitive version as the following theorem shows.

Theorem 4.1.7. *Let $L = L_1 \oplus L_2 \subset \mathbb{V}$ be as in Lemma 4.1.2.*

(1) Conjecture 1.0.3 is true for L if for every $L_1 \subset L_1' \subset L_{1,F}$, we have

$$\mathrm{Int}(L_1' \oplus L_2)^{(n_1)} = \partial\mathrm{Den}(L_1' \oplus L_2)^{(n_1)}.$$

(2) If Conjecture 1.0.3 holds for all lattices $L' = L_1' \oplus L_2$ of $\mathbb{V}$ of rank n with $L_1 \subset L_1' \subset L_{1,F}$, then

$$\mathrm{Int}(L_1 \oplus L_2)^{(n_1)} = \partial\mathrm{Den}(L_1 \oplus L_2)^{(n_1)}.$$

(3) For $1 \leq n_1 \leq n$, Conjecture 1.0.3 is true if and only if for every lattice $L = L_1 \oplus L_2 \subset \mathbb{V}$ with $\mathrm{rank}(L_1) = n_1$, one has

$$\mathrm{Int}(L_1 \oplus L_2)^{(n_1)} = \partial\mathrm{Den}(L_1 \oplus L_2)^{(n_1)}.$$

Proof. (1) follows from Lemma 4.9.2 and Corollary 4.1.6. (2) follows from Definitions 4.9.1 and 4.1.4. (3) follows from (1) and (2). □

For the rest of this section, we assume that M is unimodular of rank m with a Gram matrix $\mathrm{Diag}(I_{m-1}, \nu)$. To go further with the calculation of $\alpha(M, L, X)$, we need an induction formula for $\beta(M, L, X)^{(\ell)}$ as follows. The proof is essentially the same as that of Corollary 9.11 of [KR11], and is left to the reader.

Proposition 4.1.8. *Let $L = L_1 \oplus L_2$, where L_j is of rank n_j. Let $C(M^{[k]}, L_1)$ be the $\mathrm{U}(M^{[k]})$-orbits of sublattices $M(i) \subset M^{[k]}$ such that $M(i)$ is isometric to L_1, and write $C(M^{[k]}, L_1) = \sqcup_{i\in J}\{M(i)\}$. Then*

$$\beta(M, L, X)^{(n_1)} = \sum_{i\in J} |M : M(i) \oplus M(i)^{\perp}|^{-n_2} |M(i)^{\vee} : M(i)|^{n_2} \beta_i(M, L_1, X)\alpha(M(i)^{\perp}, L_2),$$

where

$$\beta_i(M, L_1, X) = \lim_{d\to\infty} q^{-dn_1(2m+4k-n_1)} \#\{\phi \in I(M^{[k]}, L_1, d)^{(n_1)} \mid \exists\ \Phi \in \mathrm{U}(M) \text{ with } \phi(L_1) = \Phi(M(i))\},$$

and

$$I(M^{[k]}, L_1, d)^{(n_1)} := \{\phi \in I(M^{[k]}, L_1, d) \mid \mathrm{rank}_{\mathbb{F}_q}\phi(L_1) \otimes_{O_F} \mathbb{F}_q = n_1\}.$$

Recall that $I(M^{[k]}, L_1, d)$ *is defined in* (**??**).

One special case is that $L = H^i \oplus L_2$. Since any sublattice of $M^{[k]} = M \oplus H^k$ isometric to H^i is always a direct summand of $M^{[k]}$ and $\alpha(M, H^i, X) = \beta(M, H^i, X)^{(2i)} = \beta(M, H^i, X)$, the above proposition specializes to

Corollary 4.1.9. *Assume* $L = H^i \oplus L_2$*, then*

$$\alpha(M, L, X) = \beta(M, H^i, X)\alpha(M, L_2, q^{2i}X) = \alpha(M, H^i, X)\alpha(M, L_2, q^{2i}X). \tag{4.6}$$

We end this section with two more special cases of Proposition 4.1.8. Proofs are given in §4.5.

Proposition 4.1.10. *Let the notation be as in Proposition 4.1.8. Assume* $n_1 = 1$ *and* $L_1 = \langle t \rangle$ *where* $t \in O_{F_0}$.

1. *There always exists a primitive vector* $M(1) \in H^k$ *with* $q(M(1)) = t$*, and*

$$M(1)^{\perp} \cong H^{k-1} \oplus I_m^{\chi(M)} \oplus \langle -t \rangle.$$

   Here $\langle t \rangle$ *stands for a lattice* $O_F v$ *of rank one with* $(v, v) = t$.

2. *If* $\mathrm{v}(t) = 0$*, then there exist a primitive vector* $M(0) \in M$ *with* $q(M(0)) = t$*, and*

$$M(0)^{\perp} \cong H^k \oplus I_{m-2}^{\epsilon_{m-2}} \oplus \langle \nu t \rangle.$$

   Here $\epsilon_{m-2} = \chi((-1)^{\frac{(m-2)(m-3)}{2}})$.

3. *If* $\mathrm{v}(t) > 0$*, then there exist a primitive vector* $M(0) \in M$ *with* $q(M(0)) = t$ *only*

when M is isotropic (i.e. $\exists\, v \in M$ with $q(v) = 0$). In this case,

$$M(0)^{\perp} \cong H^k \oplus I_{m-2}^{\chi(M)} \oplus \langle -t \rangle.$$

Assuming the existence of $M(1)$ and $M(0)$, we have

$$|M^{[k]} : M(i) \oplus M(i)^{\perp}|^{-1} |M(i)^{\vee} : M(i)| = \begin{cases} 1 & \text{if } i = 1, \\ q & \text{if } i = 0. \end{cases}$$

4. *Under the action of $\mathrm{U}(M^{[k]})$, v is either in the same orbit of a fixed vector $M(1) \in H^k$ or a fixed vector $M(0) \in M$.*

5. *We have the following induction formula:*

$$\beta(M, L, X)^{(1)} = \beta_1(M, L_1, X)\alpha(M(1)^{\perp}, L_2) + q^{n-1}\beta_0(M, L_1, X)\alpha(M(0)^{\perp}, L_2).$$

Moreover,

   (a) *For any L_1,*

$$\beta_1(M, L_1, X) = 1 - X.$$

   (b) *Assume $\mathrm{v}(t) = 0$, then*

$$\beta_0(M, L_1, X) = \begin{cases} (1 + \chi(M)\chi(L)q^{-\frac{m-1}{2}})X & \text{if } m \text{ is odd,} \\ (1 - \chi(M)q^{-\frac{m}{2}})X & \text{if } m \text{ is even.} \end{cases}$$

   (c) *Assume $\mathrm{v}(t) > 0$, then*

$$\beta_0(M, L_1, X) = \begin{cases} (1 - q^{1-m})X & \text{if } m \text{ is odd,} \\ \left(1 - q^{1-m} + \chi(M)(q-1)q^{-\frac{m}{2}}\right)X & \text{if } m \text{ is even.} \end{cases}$$

Proof. Parts (1)—(4) are proved in subsection 4.5.1. The induction formula for $\beta(M, L, X)^{(1)}$

follows from Proposition 4.1.8. For the formula of $\beta_i(M, L_1, X)$, see Corollaries 4.5.10 and 4.5.12. $\square$

Proposition 4.1.11. *Let the notation be as in Proposition 4.1.8. Assume* $\mathrm{v}(L_1) > 0$ *and* $n_1 = 2$. *Then we have a partition of* $C(M^{[k]}, L_1) = \bigsqcup_{i=0}^{2} C_i(M^{[k]}, L_1)$ *such that for any* $M(i) \in C_i(M^{[k]}, L_1)$, $M(i)^{\perp}$ *is isometric to*

$$(-L_1) \oplus H^{k-i} \oplus M^{(i)}.$$

Here $M^{(i)}$ *is a unimodular* O_F*-lattice of rank* $m-2(2-i)$ *and has determinant* $(-1)^i \det L$.

Moreover, we have

$$\beta(M, L, X)^{(2)} = \sum_{i=0}^{2} q^{(2-i)(n-2)} \beta_i(M, L_1, X) \alpha(M(i)^{\perp}, L_2, X), \tag{4.7}$$

where

$$\beta_2(M, L_1, X) = (1-X)(1-q^2 X),$$
$$\beta_1(M, L_1, X) = q(q+1)\left((1-q^{1-m}) + \delta_e(m)\chi(M)(q-1)q^{-\frac{m}{2}}\right) X(1-X),$$
$$\beta_0(M, L_1, X) = \begin{cases} q(1-q^{1-m})(1-q^{3-m})X^2 & \text{if } m \text{ is odd,} \\ q\left((1-q^{2-m}) + \chi(M)(q^2-1)q^{-\frac{m}{2}}\right)(1-q^{2-m})X^2 & \text{if } m \text{ is even.} \end{cases}$$

Here $\delta_e(m) = 1$ *or* 0 *depending on whether* m *is even or odd.*

Proof. Equation (4.7) follows from Proposition 4.1.8 and Proposition 4.5.5. For the formula of $\beta_i(S, L_1, X)$, see Corollaries 4.5.10, 4.5.13 and Proposition 4.5.14. $\square$

4.2 The modified Kudla-Rapoport conjecture

Recall that the hermitian lattices used to define the correction terms are of the following forms:

$$H_{n,i}^{\epsilon} := H^i \oplus I_{n-2i}^{\epsilon} \quad \text{for } 1 \le i \le \frac{n}{2}, \quad \epsilon = \pm 1 \tag{4.8}$$

where I^{ϵ}_{n-2i} is the unimodular hermitian lattice of rank $n-2i$ with $\chi(I^{\epsilon}_{n-2i}) = \chi(H^{\epsilon}_{n,i}) = \epsilon$. When $n = 2r$ is even, we take $I^{\epsilon}_0 = 0$ and $H^1_{n,r} = H^r$.

Theorem 4.2.1. *Let $r_\epsilon = \frac{n-1}{2}$ when n is odd, and $r_\epsilon = \lfloor \frac{n+\epsilon}{2} \rfloor$ when n is even. In the following we just write r_ϵ as r.*

$$A^{\epsilon} = (A^{\epsilon}_{i,j}) = \begin{pmatrix} \alpha(H^{\epsilon}_{n,1}, H^{\epsilon}_{n,1}) & \alpha(H^{\epsilon}_{n,2}, H^{\epsilon}_{n,1}) & \cdots & \alpha(H^{\epsilon}_{n,r}, H^{\epsilon}_{n,1}) \\ 0 & \alpha(H^{\epsilon}_{n,2}, H^{\epsilon}_{n,2}) & \cdots & \alpha(H^{\epsilon}_{n,r}, H^{\epsilon}_{n,2}) \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \alpha(H^{\epsilon}_{n,r}, H^{\epsilon}_{n,r}) \end{pmatrix},$$

$$B^{\epsilon} = {}^t(\alpha'(I^{-\epsilon}_n, H^{n,1}_{\epsilon}), \cdots, \alpha'(I^{-\epsilon}_n, H^{\epsilon}_{n,r})),$$

and

$$C^{\epsilon} = {}^t(c^{\epsilon}_{n,1}, \cdots, c^{\epsilon}_{n,r}),$$

where $c^{\epsilon}_{n,i}$ is c_{2i} as in Conjecture 1.0.3.

Then C^{ϵ} is the solution of the equation

$$A^{\epsilon} C^{\epsilon} = -2B^{\epsilon}. \tag{4.9}$$

Moreover,

$$A^{\epsilon}_{j,j} = 2q^{\frac{(n-2j)(n-2j-1)}{2}} \prod_{0<s\leq j} (1-q^{-2s}) \prod_{1\leq s\leq \lfloor \frac{n-2j-1}{2} \rfloor} (1-q^{-2s}) \begin{cases} 1 & \text{if } n \text{ is odd,} \\ 1-\epsilon q^{-\frac{n-2j}{2}} & \text{if } n \text{ is even.} \end{cases} \tag{4.10}$$

Finally, for $i < j$,

$$A^{\epsilon}_{i,j} = A^{\epsilon}_{j,j} \cdot \begin{cases} I(n-2i, \frac{n-2i-1}{2}, j-i) & \text{if } n \text{ is odd,} \\ I(n-2i, \frac{n-2i-1+\epsilon}{2}, j-i) & \text{if } n \text{ is even,} \end{cases} \tag{4.11}$$

where

$$I(n,d,k) := \prod_{s=1}^{k} \frac{(q^{d-s+1}-1)(q^{n-d-s}+1)}{q^s-1}.$$

Proof. First notice that $\alpha(H^\epsilon_{n,i}, H^\epsilon_{n,j}) = 0$ if $i < j$. So (1.10) is indeed equivalent to (4.9), and there exists a unique solution C_ϵ.

Now we compute $A^\epsilon_{j,j}$ explicitly. Corollary 4.1.9 and Lemma 4.5.9 imply that

$$\alpha(H^\epsilon_{n,j}, H^\epsilon_{n,j}) = \alpha(H^j, H^j)\alpha(I^\epsilon_{n-2j}, I^\epsilon_{n-2j}).$$

According to Lemma 4.5.8,

$$\alpha(H^j, H^j) = \prod_{0<s\leq j}(1-q^{-2s}).$$

By Lemma 4.5.11,

$$\alpha(I^\epsilon_{n-2j}, I^\epsilon_{n-2j}) = |\mathrm{O}(\overline{I}^\epsilon_{n-2j})(\mathbb{F}_q)|,$$

where $\overline{I}^\epsilon_{n-2j} = I^\epsilon_{n-2j} \otimes_{O_F} O_F/(\pi)$ is the space over $\mathbb{F}_q$ with the naturally induced quadratic form. Now (4.10) follows from the well-known formula:

$$|\mathrm{O}(\overline{I}^\epsilon_{n-2j})(\mathbb{F}_q)| = \begin{cases} 2q^{\frac{(n-2j)(n-2j-1)}{2}} \prod_{s=1}^{\frac{n-2j-1}{2}}(1-q^{-2s}) & \text{if } n \text{ is odd,} \\ 2q^{\frac{(n-2j)(n-2j-1)}{2}}(1-\epsilon q^{-\frac{n-2j}{2}}) \prod_{s=1}^{\frac{n-2j}{2}-1}(1-q^{-2s}) & \text{if } n \text{ is even.} \end{cases}$$

To obtain (4.11), notice that (Corollary 4.1.9)

$$\alpha(H^\epsilon_{n,j}, H^\epsilon_{n,i}) = \alpha(H^\epsilon_{n,j}, H^i)\alpha(H^\epsilon_{n-2i,j-i}, I^\epsilon_{n-2i}),$$

and

$$\alpha(H_{n,j}^{\epsilon}, H_{n,j}^{\epsilon}) = \alpha(H_{n,j}^{\epsilon}, H^{i})\alpha(H_{n-2i,j-i}^{\epsilon}, H_{n-2i,j-i}^{\epsilon}).$$

Hence

$$\frac{A_{i,j}^{\epsilon}}{A_{j,j}^{\epsilon}} = \frac{\alpha(H_{n,j}^{\epsilon}, H_{n,i}^{\epsilon})}{\alpha(H_{n,j}^{\epsilon}, H_{n,j}^{\epsilon})} = \frac{\alpha(H_{n-2i,j-i}^{\epsilon}, I_{n-2i}^{\epsilon})}{\alpha(H_{n-2i,j-i}^{\epsilon}, H_{n-2i,j-i}^{\epsilon})}.$$

Fix an O_F-lattice L that is represented by I_{n-2i}^{ϵ}. According to Lemma 4.2.2, to compute $\frac{A_{i,j}^{\epsilon}}{A_{j,j}^{\epsilon}}$, we need to count the number of lattices L' in L_F such that contain $L \subset L'$ and $L' \cong H_{n-2i,j-i}^{\epsilon}$, which is equivalent to the following condition:

$$\pi L \overset{j-i}{\subset} \pi L' \overset{n-2j}{\subset} (L')^{\sharp} \overset{j-i}{\subset} L \overset{j-i}{\subset} L'.$$

Since L' and $\pi L'$ determine each other, we just need to count $\pi L'$ satisfying the above condition. We regard $\pi L'/\pi L$ as a $(j-i)$-dimensional subspace of $L/\pi L$, where $L/\pi L$ is equipped with quadratic form $(x,y)/\pi$.

Claim: The condition

$$\pi L' \subset (L')^{\sharp}$$

is equivalent to the condition that $\pi L'/\pi L$ is an isotropic subspace of $L/\pi L$.

Indeed, assume $\pi L'/\pi L$ is an isotropic subspace of $L/\pi L$. Then $(\pi x, \pi y) \in \pi O_F$ for any $x, y \in L'$, which is equivalent to $(x, \pi y) \in O_F$ for any $x, y \in L'$. The latter condition is the same as $L' \subset (L')^{\sharp}$. The other direction is clear.

Therefore $\frac{A_{i,j}^{\epsilon}}{A_{j,j}^{\epsilon}}$ is the number of $(j-i)$-dimensional isotropic subspaces of $L/\pi L$. According to [LZ22b, Lemma 3.2.2], it equals to

$$\begin{cases} I(n-2i, \frac{n-2i-1}{2}, j-i) & \text{if } n \text{ is odd,} \\ I(n-2i, \frac{n-2i-1+\epsilon}{2}, j-i) & \text{if } n \text{ is even.} \end{cases}$$

□

According to Theorem 4.2.1, in order to solve C^ϵ, we need to know B^ϵ and A^ϵ. Here, B^ϵ can be calculated by applying Corollary 4.1.9 and Proposition 4.1.10 inductively. The following lemma can be used to compute A^ϵ.

Lemma 4.2.2. *Let F/F_0 be a quadratic p-adic field extension, and let L and M be two O_F-hermitian lattices of rank n. Then $\frac{\alpha(M,L)}{\alpha(M,M)}$ is equal to the number of lattices L' in L_F containing L and isometric to M.*

Proof. The proof is a generalization of that of Proposition 10.2 of [KR14b] and works for both inert and ramified primes.

Let us assume that there is an isometric embedding from L into M, otherwise both sides of the identity in the lemma are zero. In this case, we have a fixed $L_F \cong M_F$. Let α (resp. β) be a top degree translation invariant form on L_F^n (resp. $\mathrm{Herm}_n(F)$). Let $\nu_p = \alpha/h^*(\beta)$ where

$$h: \quad L_F^n \to \mathrm{Herm}_n(F), \quad x \mapsto (x,x).$$

Define X to be the set of F-linear isometric embeddings from L into M. By fixing a basis of L_F and regarding $\phi \in X$ as a linear isometry from L_F to itself, we identify X as a subset of L_F^n. By the argument in Section 3 of [GY00] (in particular Lemma 3.4), we know that

$$\alpha(M,L) = \mathrm{vol}(X, d\nu_p)\frac{\mathrm{vol}(\mathrm{Herm}_n(O_F), d\beta)}{\mathrm{vol}((M)^n, d\alpha)}. \tag{4.12}$$

For any $\phi \in X$ regarded as a linear isometry from L_F to itself, the lattice $L_\phi := \phi^{-1}(M)$ is a lattice containing L. Conversely, for any L' containing L and isometric to M, there is a $\phi \in X$ such that $L_\phi = L'$. Hence we have a partition

$$X = \bigsqcup_{L \subset L'} X_{L'}, \quad X_{L'} := \{\phi \in X \mid L_\phi = L'\}.$$

Since each L' is isomorphic to M, all the $X_{L'}$ have the same volume as that of X_M.

Specializing (4.12) to $L = M$, we see

$$\alpha(M, M) = \mathrm{vol}(X_M, d\nu_p) \frac{\mathrm{vol}(\mathrm{Herm}_n(O_F), d\beta)}{\mathrm{vol}((M)^n, d\alpha)}. \tag{4.13}$$

Dividing equation (4.12) by (4.13), we prove the lemma. □

Remark 4.2.3. *When F/F_0 is unramified and M is unimodular, the lemma was proved by equation (3.6.1.1) of [LZ22a].*

Now we specialize Theorem 4.2.1 to the case $n = 3$.

Lemma 4.2.4. *Assume $n = 3$ and $\epsilon = \chi(L)$. Then $c^{\epsilon}_{3,1} = \frac{q^2}{1+q}$, hence*

$$\partial\mathrm{Den}(L) = 2\frac{\alpha'(I_3^{-\epsilon}, L)}{\alpha(I_3^{-\epsilon}, I_3^{-\epsilon})} + \frac{q^2}{1+q}\frac{\alpha(H^{\epsilon}_{3,1}, L)}{\alpha(I_3^{-\epsilon}, I_3^{-\epsilon})}.$$

Proof. First of all, according to Theorem 4.2.1,

$$\alpha(H^{\epsilon}_{3,1}, H^{\epsilon}_{3,1}) = 2(1 - q^{-2}). \tag{4.14}$$

By Corollary 4.1.9, we have

$$\alpha(I_3^{-\epsilon}, H^{\epsilon}_{3,1}, X) = \alpha(I_3^{-\epsilon}, H, X)\alpha(I_3^{-\epsilon}, I_1^{\epsilon}, q^2X).$$

According to Lemmas 4.5.8 and 4.5.9,

$$\alpha(I_3^{-\epsilon}, H, X) = \beta(H^k, H) = 1 - X.$$

Lemma 4.3.1 gives that

$$\alpha(I_3^{-\epsilon}, I_1^{\epsilon}, q^2X) = 1 - qX.$$

Hence

$$\alpha(I_3^{-\epsilon}, H_{3,1}^{\epsilon}, X) = (1-X)(1-qX),$$

and

$$\alpha'(I_3^{-\epsilon}, H_{3,1}^{\epsilon}) = 1-q.$$

Combining this with (4.14), we solve (4.9) and obtain

$$c_{3,1}^{\epsilon} = \frac{q^2}{1+q}.$$

Now the lemma follows by the definition of $\partial\mathrm{Den}(L)$. □

4.3 Local density formula when $\mathrm{rank}(T) \leq 2$

The main purpose of this section is to give an explicit formula for $\alpha(I,T,X)$ where $I = \mathrm{Diag}(I_{m-1}, \nu)$ with $\nu \in O_{F_0}^{\times}$ and $\mathrm{rank}(T) \leq 2$.

4.3.1 The case $T = (t)$.

In order to apply induction formulas to calculate $\alpha(I,T,X)$ for T with $\mathrm{rank}(T) = 2$, we need to consider the case $T = (t)$ first. Write $t = t_0(-\pi_0)^{\mathrm{v}(t)}$ for $t_0 \in O_{F_0}^{\times}$, and

$$I_{a,b} = \mathrm{Diag}(I, \nu_1(-\pi_0)^a, \nu_2(-\pi_0)^b) = \mathrm{Diag}(s_1, \dots, s_{m+2}) \tag{4.15}$$

for integers $0 \leq a \leq b$.

Lemma 4.3.1. *Assume* $0 \leq a \leq b \leq \mathrm{v}(t)$.

1. *If m is odd, then*

$$\alpha(I_{a,b},(t),X) = 1 + \chi(I)\chi(-\nu_1)(q-1)\sum_{s=a+1}^{b} q^{-ms+a+\frac{m-1}{2}}X^s$$
$$+ \chi(I_{a,b})\chi(t_0)q^{-(m+1)\mathrm{v}(t)+a+b-\frac{m+1}{2}}X^{\mathrm{v}(t)+1}.$$

2. *If m is even, then*

$$\alpha(I_{a,b},(t),X) = 1 + \chi(I)(q-1)\sum_{s=1}^{a} q^{-(m-1)s+\frac{m}{2}-1}X^s$$
$$+ \chi(I_{a,b})q^{a+b}\left((q-1)\sum_{s=b+1}^{\mathrm{v}(t)} q^{-(m+1)s+\frac{m}{2}}X^s - q^{-(m+1)\mathrm{v}(t)-1-\frac{m}{2}}X^{\mathrm{v}(t)+1}\right).$$

Proof. Direct calculation gives

$$\begin{aligned}\alpha(I_{a,b},(t),X) &= \int_{F_0} dY \int_{O_F^{2k+m+2}} \psi(\langle Y, \mathrm{Diag}(H^k, I_{a,b})[\mathbf{x}] - t\rangle)\, d\mathbf{x}\\ &= \int_{F_0} \psi(-tY)\, dY \int_{O_F^{2k}\times O_F^{m+2}} \psi(Y\sum_{i=1}^{k}\mathrm{Tr}(\frac{1}{\pi}x_i\bar{y}_i) + Y\sum_{l=1}^{m+2} s_l z_l \bar{z}_l) \prod_i dx_i dy_i \prod_l dz_l\\ &= 1 + \sum_{s=1}^{\infty}\int_{\mathrm{v}(Y)=-s} I_k(Y) I_{I_{a,b}}(Y)\psi(-tY)\, dY.\end{aligned}$$

Here, according to [Shi22, Lemma 7.6],

$$I_k(Y) = \int_{O_F^{2k}} \psi(Y\sum_{i=1}^{k}\mathrm{Tr}(\frac{1}{\pi}x_i\bar{y}_i))\prod dx_i dy_i = q^{-2ks},$$

and

$$I_{I_{a,b}}(Y) = \int_{O_F^{m+2}} \psi(Y\sum_{l=1}^{m+2} s_l z_l\bar{z}_l)\prod dz_l = \prod_{l=1}^{m+2} J(s_l Y),$$

where

$$J(t) = \int_{O_F}\psi(tz\bar{z})dz = \begin{cases} 1 & \text{if } \mathrm{v}(t)\geq 0,\\ q^{\mathrm{v}(t)}\chi(-t_0)g(\chi,\psi_{\frac{1}{\pi_0}}) & \text{if } \mathrm{v}(t)<0,\end{cases} \tag{4.16}$$

and

$$g(\chi, \psi_{\frac{1}{\pi_0}}) = \sum_{x \in O_{F_0}/\pi_0} \chi(x)\psi(\frac{x}{\pi_0})$$

is the Gauss sum. Write $\psi' = \psi_{\frac{1}{\pi_0}}$. Then

$$\begin{aligned}
\alpha(I_{a,b}, (t), X) = 1 &+ \sum_{s=1}^{a} q^s \int_{O_{F_0}^\times} q^{-2ks} \cdot q^{-ms} \chi(\nu(-Y)^m) g(\chi, \psi')^m \psi(-(-\pi_0)^s Y t) dY \\
&+ \sum_{s=a+1}^{b} \int_{O_{F_0}^\times} q^{-2ks} \cdot q^{-ms+a} \chi(\nu_1 \nu(-Y)^{m+1}) g(\chi, \psi')^{m+1} \psi(-(-\pi_0)^{-s} Y t)\, dY \\
&+ \sum_{s=b+1}^{\infty} \int_{O_{F_0}^\times} q^{-2ks} \cdot q^{-(m+1)s+a+b} \chi(\nu_1 \nu_2 \nu(-Y)^{m+2}) g(\chi, \psi')^{m+2} \psi(-(-\pi_0)^{-s} Y t)\, dY.
\end{aligned}$$

Recall the well-known facts that

$$\begin{aligned}
g(\chi, \psi')^2 &= \chi(-1) \cdot q, \\
\int_{O_{F_0}^\times} \psi((-\pi_0)^{-s} Y t) dY &= \mathrm{Char}(\pi_0^s O_{F_0})(t) - q^{-1} \mathrm{Char}(\pi_0^{s-1} O_{F_0})(t), \\
\int_{O_{F_0}^\times} \chi(Y) \psi((-\pi_0)^{-s} Y t) dY &= \chi(-t_0) q^{-1} g(\chi, \psi') \mathrm{Char}(\pi_0^{s-1} O_{F_0}^\times)(t).
\end{aligned} \tag{4.17}$$

When m is odd, we have

$$\begin{aligned}
\alpha(I_{a,b}, (t), X) = 1 &+ \chi((-1)^{\frac{m+1}{2}} \nu_1 \nu)(q-1) \sum_{s=a+1}^{b} q^{-ms+a+\frac{m-1}{2}} X^s \\
&+ \chi((-1)^{\frac{m+1}{2}} \nu_1 \nu_2 \nu t_0) q^{-(m+1)(\mathrm{v}(t)+1)+a+b+\frac{m+1}{2}} X^{\mathrm{v}(t)+1}.
\end{aligned}$$

When m is even, we have

$$\begin{aligned}
&\alpha(I_{a,b}, (t), X) \\
&= 1 + \chi((-1)^{\frac{m}{2}} \nu)(q-1) \sum_{s=1}^{a} q^{-(m-1)s+\frac{m}{2}-1} X^s + \chi((-1)^{\frac{m+2}{2}} \nu_1 \nu_2 \nu) \\
&\quad \cdot \left((q-1) \sum_{s=b+1}^{\mathrm{v}(t)} q^{-(m+1)s+a+b+\frac{m}{2}} X^s - q^{-(m+1)(\mathrm{v}(t)+1)+a+b+\frac{m}{2}} X^{\mathrm{v}(t)+1} \right).
\end{aligned}$$

Finally, notice that for I of rank m we have

$$\chi(I)=\begin{cases}\chi((-1)^{\frac{m-1}{2}}\nu) & \text{if } m \text{ is odd,}\\ \chi((-1)^{\frac{m}{2}}\nu) & \text{if } m \text{ is even.}\end{cases}$$

Now the lemma is clear. □

Similarly, we have the following lemma.

Lemma 4.3.2. *Let I be unimodular with odd rank m. Then*

$$\alpha(I\oplus H_i,(t),X)=\begin{cases}1+\chi(I)\chi(t_0)q^{-(\mathrm{v}(t)+1)(m+1)+\frac{m+1}{2}+i}X^{\mathrm{v}(t)+1} & \text{if } i\leq 2\mathrm{v}(t),\\ 1+\chi(I)\chi(t_0)q^{-(\mathrm{v}(t)+1)(m-1)+\frac{m-1}{2}}X^{\mathrm{v}(t)+1} & \text{if } i> 2\mathrm{v}(t).\end{cases}$$

4.3.2 The case $T=\mathrm{Diag}(u_1(-\pi_0)^a,u_2(-\pi_0)^b)$

In this subsection, we compute $\alpha(I,T,X)$ for I unimodular of rank $m\geq 2$ and $T=\mathrm{Diag}(u_1(-\pi_0)^a,u_2(-\pi_0)^b)$ with $0\leq a\leq b$. Notice that $\alpha(I,T,X)=0$ when $a<0$.

Proposition 4.3.3. *Assume $T=\mathrm{Diag}(u_1(-\pi_0)^a,u_2(-\pi_0)^b)$ and that I is isotropic of even rank $m\geq 2$, then*

$$\begin{aligned}\alpha(I,T,X)=&(1-X)\left(\sum_{i=0}^{a}(q^{2-m}X)^i+\gamma_e(I,T,X)\right)\\ &+qX(q^{2-m}X)^a(1-\chi(I)q^{-\frac{m}{2}})(1+\chi(I)\chi(T)q^{\frac{m-2}{2}}(q^{2-m}X)^{b+1})\\ &+\left(1-q^{-(m-1)}+(q-1)\chi(I)q^{-\frac{m}{2}}\right)X\\ &\quad\cdot\left(q\sum_{i=0}^{a-1}(q^{2-m}X)^i+\gamma_e(I,T,X)-\chi(I)\chi(T)q^{\frac{m}{2}}(q^{2-m}X)^{a+b+1}\right),\end{aligned}$$

where

$$\gamma_e(I,T,X)=\chi(I)q^{\frac{m}{2}}\left(\sum_{d=1}^{a}(q^d-1)(q^{2-m}X)^d+\chi(T)q^a(q^{2-m}X)^{b+1}\sum_{i=0}^{a}(q^{1-m}X)^i\right).$$

Proof. Since I is of even rank, $u_1 \cdot I^{[k]} \approx I^{[k]}$, and we may assume T is of the form $\mathrm{Diag}((-\pi_0)^a, u(-\pi_0)^b)$ without loss of generality.

According to Theorem 4.1.3 and Proposition 4.1.10, we have

$$\begin{aligned}\alpha(I,T,X) = \beta_1(I,(-\pi_0)^a,X)&\alpha(M(1)^{\perp},u(-\pi_0)^b)\\ &+ q\beta_0(I,(-\pi_0)^a,X)\alpha(M(0)^{\perp},u(-\pi_0)^b)\\ &+ q^{2-m}X\alpha(I,\mathrm{Diag}((-\pi_0)^{a-1},u(-\pi_0)^b),X)\end{aligned}$$

where $M(1)^{\perp} = \mathrm{Diag}(H^{k-1}, -(-\pi_0)^i, I)$ and

$$M(0)^{\perp} = \mathrm{Diag}(H^k, \underbrace{-(-\pi_0)^i, 1, \ldots, 1, -\nu}_{m-1}).$$

Continuing this process, we obtain

$$\begin{aligned}&\alpha(I,T,X)\\ &= \sum_{i=0}^{a}(q^{2-m}X)^{a-i}\cdot\left(\beta_1(I,(-\pi_0)^i,X)\alpha(M(1)^{\perp},u(-\pi_0)^b) + q\beta_0(I,(-\pi_0)^i,X)\alpha(M(0)^{\perp},u(-\pi_0)^b)\right).\end{aligned}$$

By the formulas in Proposition 4.1.10 and Lemma 4.3.1, the above equals to

$$\begin{aligned}&\sum_{i=0}^{a}(q^{2-m}X)^{a-i}(1-X)\\ &\cdot\left(1+(q-1)\chi(I)q^{\frac{m-2}{2}}\sum_{s=-i}^{-1}(q^{(m-1)}(q^2X)^{-1})^s + \chi(I)\chi(T)q^{i+\frac{m}{2}}(q^{2-m}X)^{b+1}\right)\\ &+ q(q^{2-m}X)^aX(1-q^{-\frac{m}{2}}\chi(I))(1+\chi(I)\chi(T)q^{-(b+1)(m-2)+\frac{m-2}{2}}X^{b+1})\\ &+ q\sum_{i=1}^{a}(q^{2-m}X)^{a-i}X\left((1-q^{-(m-1)}) + (q-1)\chi(I)q^{-(m-1)+\frac{m-2}{2}}\right)\\ &\cdot\left(1+(q-1)\chi(I)q^{\frac{m-4}{2}}\sum_{s=-i}^{-1}(q^{(m-3)}X^{-1})^s + \chi(I)\chi(T)q^{i+\frac{m-2}{2}}(q^{2-m}X)^{b+1}\right).\end{aligned}$$

Now the transformation

$$\sum_{i=0}^{a}\sum_{s=1}^{i} q^s (q^{2-m}X)^{a-i+s} = \sum_{d=1}^{a}\sum_{s=1}^{d} q^s (q^{2-m}X)^d$$

and some calculation give us the result we want. □

The case that I is anisotropic (i.e. when $m = 2$ and $\chi(I) = -1$) can be computed similarly and is simpler. We omit the detail here. In particular, we may recover the following formula.

Proposition 4.3.4. *[Shi22, Theorem 6.2(1)] Assume* $I = \mathrm{Diag}(1, \nu)$*, then*

$$\begin{aligned}
&\alpha(I,T,X)\\
&= (1-X)(1+\chi(I)+q\chi(I))\sum_{e=0}^{\alpha}(qX)^e - \chi(T)q^{\alpha+1}X^{\beta+1}(1-X)\sum_{e=0}^{\alpha}(q^{-1}X)^e\\
&\quad - \chi(T)(1+q)(X^{\alpha+\beta+2}+\chi(I)\chi(T)) + (1+\chi(I))q^{\alpha+1}X^{\alpha+1}(1+\chi(T)X^{\beta-\alpha}).
\end{aligned}$$

Moreover, a similar computation yields the following, and we leave the detail to reader.

Proposition 4.3.5. *Assume that* I *is unimodular of odd rank* $m \geq 3$*. Then*

$$\begin{aligned}
&\alpha(I,T,X)\\
&= (1-X)\left(\sum_{i=0}^{a}(q^{2-m}X)^i + \gamma_{o,1}(I,T,X)\right) + (1-q^{-(m-1)})X\left(q\sum_{i=0}^{a-1}(q^{2-m}X)^i + \gamma_{o,0}(S,T,X)\right)\\
&\quad + qX(q^{2-m}X)^a\left(1+\chi(I)\chi(u_1)q^{-\frac{m-1}{2}}\right)\left(1-\chi(I)\chi(u_1)q^{(2-m)b-\frac{m-1}{2}}X^{b+1}\right),
\end{aligned}$$

where $\gamma_{o,1}(I,T,X)$ *equals*

$$\chi(I)\chi(u_1)q^{\frac{m-1}{2}}\left(\sum_{d=a+1}^{a+b}(q^{a+b+1-d}-1)(q^{2-m}X)^d - \sum_{i=b+1}^{a+b+1}(q^{2-m}X)^i\right),$$

and $\gamma_{o,0}(I,T,X)$ *equals*

$$\chi(I)\chi(u_1)q^{\frac{m-1}{2}}\left(\sum_{d=a+1}^{a+b}(q^{a+b+1-d}-q)(q^{2-m}X)^d-\sum_{i=b+1}^{a+b}(q^{2-m}X)^i\right).$$

4.4 Local density formula when $\mathrm{rank}(T)=3$

In this section, we always assume $\mathrm{rank}(T)=3$ and $S=I_3^{-\chi(T)}$. The aim of this section is to compute $\partial\mathrm{Den}(T)$ explicitly. We treat the case $\mathrm{v}(T)\leq -1$ in the first subsection. In the second subsection, we deal with the case when $T=\mathrm{Diag}(1,T_2)$ for T_2 diagonal. In the last subsection, instead of $\partial\mathrm{Den}(T)$, we compute $\partial\mathrm{Den}(T)^{(2)}$ for T of the form not covered by previous subsections.

4.4.1 $\partial\mathrm{Den}(T)$ for T with $\mathrm{v}(T)\leq -1$

Proposition 4.4.1. *If* $\mathrm{v}(T)\leq -1$*, then* $\mathrm{Int}(T)=\partial\mathrm{Den}(T)=0$.

Proof. If $\mathrm{v}(T)<-1$, then $\partial\mathrm{Den}(T)=0$ since $\mathrm{v}(S^{[k]})\geq -1$ where $S\oplus H^k$. If $\mathrm{v}(T)=-1$, then T is of the form $\mathrm{Diag}(H,(u_3(-\pi_0)^c))$ with $\chi(T)=\chi(u_3)$. In this case, according to Corollary 4.1.9, Lemmas 4.5.8 and 4.5.9, we have

$$\alpha(S,T,X)=(1-X)\alpha(S,(u_3(-\pi_0)^c),q^2X).$$

Similarly, we have

$$\begin{aligned}\alpha(H^{3,1}_{\chi(T)},T)&=\beta(H^{3,1}_{\chi(T)},H)\alpha(u_3,(u_3(-\pi_0)^c))\\&=(1-q^{-2})\alpha(u_3,(u_3(-\pi_0)^c)).\end{aligned}$$

Hence, applying Lemma 4.3.1 to $I_{0,0} = S$ where I is of rank 1, we have

$$\begin{aligned}\partial\mathrm{Den}(T) &= 2\alpha(S, (u_3(-\pi_0)^c), q^2) + \frac{q^2}{1+q}(1-q^{-2})\alpha((u_3), (u_3(-\pi_0)^c)) \\ &= 2(1+\chi(S)\chi(u_3)q) + 2(q-1) \\ &= 2(1-q) + 2(q-1) \\ &= 0.\end{aligned}$$

Here we are using the fact $\chi(S)\chi(T) = \chi(S)\chi(u_3) = -1$. □

4.4.2 $\partial\mathrm{Den}(T)$ for $T = \mathrm{Diag}(1, T_2)$ with T_2 diagonal

In this subsection, we assume $T = \mathrm{Diag}(1, T_2)$, where $T_2 = \mathrm{Diag}(u_1(-\pi_0)^a, u_2(-\pi_0)^b)$ with $0 \le a \le b$. Let $u = u_1 u_2$. Also, let $S = \mathrm{Diag}(1, 1, \nu)$ and $S_2 = \mathrm{Diag}(1, \nu)$. We compare $\partial\mathrm{Den}(T)$ and $\partial\mathrm{Den}(T_2)$ in this subsection.

Recall that

$$\partial\mathrm{Den}(T) = 2\frac{\alpha'(S,T)}{\alpha(S,S)} + \frac{q^2}{1+q}\frac{\alpha(H^{3,1}_{\chi(T)}, T)}{\alpha(S,S)}.$$

Moreover, according to [Shi22, Theorem 1.3] and Theorem 3.0.1, the analytic side in the case $n = 2$ is

$$\partial\mathrm{Den}(T_2) = 2\frac{\alpha'(S_2,T_2)}{\alpha(S_2,S_2)} - \frac{2q^2}{q^2-1}\frac{\alpha(H, T_2)}{\alpha(S_2,S_2)}.$$

Proposition 4.4.2.

$$\partial\mathrm{Den}(T) - \partial\mathrm{Den}(T_2) = \begin{cases} 1 + 2\sum_{i=1}^{a} q^i & \text{if } \chi(T) = 1, \\ 1 & \text{if } \chi(T) = -1. \end{cases}$$

Proof. Proposition 4.1.10 implies that $\alpha(S,T,X)$ equals

$$(1-X)\alpha(\mathrm{Diag}(-1,S),T_2,q^2X)+q^2(1+q^{-1}\chi(S))X\alpha(S_2,T_2,X).$$

Hence

$$\alpha'(S,T)=\alpha(\mathrm{Diag}(-1,S),T_2,q^2)+q^2(1+q^{-1}\chi(S))\alpha'(S_2,T_2). \tag{4.18}$$

According to Lemma 4.5.11, one can check that $\alpha(S,S)=\beta(S,S)=2q(q^2-1)$, and $\alpha(S_2,S_2)=2(q-\chi(S_2))$. Then

$$\frac{\alpha'(S,T)}{\alpha(S,S)}-\frac{\alpha'(S_2,T_2)}{\alpha(S_2,S_2)}=\frac{\alpha(\mathrm{Diag}(-1,S),T_2,q^2)}{\alpha(S,S)}. \tag{4.19}$$

Hence we just need to check that

$$\begin{aligned}&2\frac{\alpha(\mathrm{Diag}(-1,S),T_2,q^2)}{\alpha(S,S)}+\frac{q^2}{1+q^2}\frac{\alpha(H^{3,1}_{\chi(T)},T)}{\alpha(S,S)}+\frac{2q^2}{q^2-1}\frac{\alpha(H,T_2)}{\alpha(S_2,S_2)}\\ &=\begin{cases}1+2\sum_{i=1}^a q^i & \text{if } \chi(T)=1,\\ 1 & \text{if } \chi(T)=-1.\end{cases}\end{aligned}$$

By Proposition 4.3.3, we may check that

$$2\alpha(\mathrm{Diag}(-1,S),T_2,q^2)=\begin{cases}2(2q^{a+2}-(q+1)^2)(q-1) & \text{if } \chi(T)=1,\\ 2(q-1)(q^2-1) & \text{if } \chi(T)=-1.\end{cases} \tag{4.20}$$

To compute $\frac{q^2}{1+q}\alpha(H^{3,1}_{\chi(T)},T)$, we may choose $H^{3,1}_{\chi(T)}=\mathrm{Diag}(H,1)$ when $\chi(T)=1$. By Corollary 4.1.9, Proposition 4.3.4, and a direct calculation, we have

$$\frac{q^2}{1+q}\frac{\alpha(H^{3,1}_{\chi(T)},T)}{\alpha(S,S)}=\frac{1}{2q(q^2-1)}\cdot\begin{cases}(q-1)\alpha(\mathrm{Diag}(-1,1),T_2)+\frac{2q^2}{q-1}\alpha(H,T_2) & \text{if } \chi(T)=1\\ (q-1)\alpha(\mathrm{Diag}(-1,-u),T_2) & \text{if } \chi(T)=-1.\end{cases}$$

Combining this with the formulas in Theorem 3.5.1, we have

$$\frac{q^2}{1+q^2}\frac{\alpha(H^{3,1}_{\chi(T)},T)}{\alpha(S,S)}+\frac{2q^2}{q^2-1}\frac{\alpha(H,T_2)}{\alpha(S_2,S_2)}=\frac{1}{q(q^2-1)}\cdot\begin{cases}4q^{a+2}-q^2-2q-1 & \text{if } \chi(T)=1\\ (q^2-1) & \text{if } \chi(T)=-1.\end{cases} \tag{4.21}$$

Now a direct computation combined with (4.20) and (4.21) proves the proposition. □

Corollary 4.4.3. *Assume L is a hermitian lattice with Gram matrix T, then*

$$\partial\mathrm{Den}(T)-\partial\mathrm{Den}(T_2)=|\{\mathcal{V}^0(L)\}|. \tag{4.22}$$

Proof. We can write $L=L^\flat\obot O_F\mathbf{x}$ where $q(\mathbf{x})=1$. If $L^\flat$ is non-split, then $|\{\mathcal{V}^0(L)\}|=1$.

If $L^\flat$ is split, then $|\{\mathcal{V}^0(L)\}|=1+2\sum_{i=1}^a q^i$ since $\mathcal{L}_3(L)$ can be identified with $\mathcal{L}_{2,1}(L^\flat)$, which is a ball in $\mathcal{L}_{2,1}$ centered at a vertex lattice of type 0 with radius a. Here $\mathcal{L}_{2,1}$ is the Bruhat-Tits tree associated with $\mathcal{N}^{\mathrm{Kra}}_{2,1}$ and $\mathcal{L}_{2,1}(L^\flat)$ is the subtree of $\mathcal{L}_{2,1}$ associated with $L^\flat$. □

4.4.3 $\partial\mathrm{Den}(T)^{(2)}$

In this subsection, we assume $T=\mathrm{Diag}(T_2,u_3(-\pi_0)^c)$ with $\mathrm{v}(T_2)>0$, and compute $\partial\mathrm{Den}(T)^{(2)}$. Recall that $\partial\mathrm{Den}(T)^{(2)}=\partial\mathrm{Den}(L^\flat\obot O_F\mathbf{x})^{(2)}$ where the Gram matrix of $L=L^\flat\obot O_F\mathbf{x}$ is T. We consider two cases separately in Propositions 4.4.4 and 4.4.5.

Proposition 4.4.4. *Let $T=\mathrm{Diag}(u_1(-\pi_0)^a,u_2(-\pi_0)^b,u_3(-\pi_0)^c)$ where $0<a\le b\le c$. Then*

$$\partial\mathrm{Den}(T)^{(2)}=1+\chi(-u_2u_3)q^a(q^a-q^b)-q^{a+b}.$$

Proof. Recall that

$$\partial\mathrm{Den}(T)^{(2)}=\frac{1}{2q(q^2-1)}\left(2\beta'(S,T)^{(2)}+\frac{q^2}{1+q}\beta(H^{3,1}_{\chi(T)},T)^{(2)}\right).$$

We compute $\beta'(S,T)^{(2)}$ first. According to Proposition 4.1.11, $\beta_0(S,T_2,X)=0$ and

$$\begin{aligned}\beta(S,T,X)^{(2)} &= \beta_2(S,T_2,X)\alpha(\mathrm{Diag}(S,-T_2),u_3(-\pi_0)^c,q^4X)\\ &\quad + q\beta_1(S,T_2,X)\alpha(\mathrm{Diag}(-\nu,-T_2),u_3(-\pi_0)^c,q^2X)\\ &= (1-X)(1-q^2X)\alpha(\mathrm{Diag}(S,-T_2),u_3(-\pi_0)^c,q^4X)\\ &\quad + (q+1)(q^2-1)X(1-X)\alpha(\mathrm{Diag}(-\nu,-T_2),u_3(-\pi_0)^c,q^2X).\end{aligned}$$

According to Lemma 4.3.1,

$$\alpha(\mathrm{Diag}(S,-T_2),u_3(-\pi_0)^c,q^4X)=1+\chi(S)\chi(u_1)(q-1)\sum_{s=a+1}^{b}q^{a+1}(qX)^s+\chi(u_1u_2u_3\nu)q^{a+b+2}X^{c+1},$$

and

$$\alpha(\mathrm{Diag}(-\nu,-T_2),u_3(-\pi_0)^c,q^2X)=1+\chi(S)\chi(u_1)(q-1)q^a\sum_{s=a+1}^{b}(qX)^s+\chi(u_1u_2u_3\nu)q^{a+b+1}X^{c+1}.$$

The relation $\chi(u_1u_2u_3\nu)=\chi(S)\chi(T)=-1$ and a direct calculation show that

$$\beta'(S,T_2)^{(2)}=1+\chi(-u_2u_3)q^a(q^a-q^b)-q^{a+b}.$$

Finally, $\beta(H^{3,1}_{\chi(T)},T)^{(2)}=0$ by Proposition 4.1.11. The proposition is proved. $\square$

Proposition 4.4.5. *Recall that* $H_a=\begin{pmatrix}0 & \pi^a\\ (-\pi)^a & 0\end{pmatrix}$. *Let* $T=\mathrm{Diag}(H_a,u_3(-\pi_0)^c)$ *where* a *is a positive odd integer and* $c\geq 0$. *Then*

$$\partial\mathrm{Den}(T)^{(2)}=\begin{cases}(1-q^a) & \text{if } a\leq 2c,\\ (1-q^{2c+1}) & \text{if } a>2c.\end{cases}$$

Proof. Recall that

$$\partial\mathrm{Den}(T)^{(2)}=\frac{1}{2q(q^2-1)}\left(2\beta'(S,T)^{(2)}+\frac{q^2}{1+q}\beta(H^{3,1}_{\chi(T)},T)^{(2)}\right).$$

We need to compute $\beta'(S,T)^{(2)}$ and $\beta(H^{3,1}_{\chi(T)},T)^{(2)}$.

According to Proposition 4.1.11, $\beta_0(S,T_2,X)=0$ and

$$\begin{aligned}\beta(S,T,X)^{(2)} &= \beta_2(S,H_a,X)\alpha(\mathrm{Diag}(S,H_a),u_3(-\pi_0)^c,q^4X)\\ &\quad + q\beta_1(S,H_a,X)\alpha(\mathrm{Diag}(-\nu,H_a),u_3(-\pi_0)^c,q^2X)\\ &= (1-X)\Big((1-q^2X)\alpha(\mathrm{Diag}(S,H_a),u_3(-\pi_0)^c,q^4X)\\ &\quad + (q+1)(q^2-1)X\alpha(\mathrm{Diag}(-\nu,H_a),u_3(-\pi_0)^c,q^2X)\Big).\end{aligned}$$

According to Lemma 4.3.2,

$$\alpha(\mathrm{Diag}(S,H_a),u_3(-\pi_0)^c,q^4X)=\begin{cases}1+\chi(S)\chi(u_3)q^{2+a}X^{c+1} & \text{if } a\le 2c,\\ 1+\chi(S)\chi(u_3)q^{2c+3}X^{c+1} & \text{if } a>2c,\end{cases}$$

and

$$\alpha(\mathrm{Diag}(-\nu,H_a),u_3(-\pi_0)^c,q^2X)=\begin{cases}1+\chi(S)\chi(u_3)q^{1+a}X^{c+1} & \text{if } a\le 2c,\\ 1+\chi(S)\chi(u_3)q^{2c+2}X^{c+1} & \text{if } a>2c.\end{cases}$$

A short computation shows that

$$\beta'(S,T)^{(2)}=q(q^2-1)\cdot\begin{cases}1+\chi(S)\chi(u_3)q^{a} & \text{if } a\le 2c,\\ 1+\chi(S)\chi(u_3)q^{2c+1} & \text{if } a>2c.\end{cases}$$

Notice that $\chi(S)\chi(u_3)=\chi(S)\chi(T)=-1$. Finally, $\beta(H^{3,1}_{\chi(T)},T)^{(2)}=0$ by Proposition 4.1.11. The proposition is proved. $\square$

4.5 Calculation of primitive local density

In this section, we provide the proofs of Propositions 4.1.10 and 4.1.11. Throughout this section, M is unimodular of rank $m\ge 2$ unless clearly stated otherwise. Let

$\{v_1, \cdots, v_{2k}, v_{2k+1}, \cdots, v_{2k+m}\}$ be a basis of $M^{[k]} = H^k \oplus M$ with Gram matrix $H^k \oplus \mathrm{Diag}(I_{m-1}, \nu)$. Let L be a hermitian lattice of rank n with Gram matrix T. An isometric embedding $\varphi : L \to M$ is called primitive if its image in $M/\pi M$ has dimension $\mathrm{rank}_{O_F}(L)$. We call a vector v primitive in M if $\pi^{-1}v \notin M$, or equivalently the natural embedding $\varphi : \mathrm{Span}_{\mathrm{O_F}}\{v\} \hookrightarrow M$ is primitive. For a $v \in M^{[k]}$, we let $\mathrm{Pr}_{H^k}(w_i)$ be the projection of w_i to H^k.

4.5.1 Proof of Proposition 4.1.10

The main purpose of this subsection is to prove the first four parts of Proposition 4.1.10. Part (5) of this proposition follows from Proposition 4.1.8 and Corollaries 4.5.10 and 4.5.12.

Proof. For (1), choose $M(1) = \frac{t\pi}{2}v_1 + v_2 \in M^{[k]}$ with $q(M(1)) = t$. Then

$$\begin{aligned} M(1)^{\perp} &= \mathrm{Span}_{\mathrm{O_F}}\{\frac{-t\pi}{2}v_1 + v_2, v_3, ..., v_{2k}, v_{2k+1}, \cdots, v_{2k+m}\} \\ &\cong \langle -t\rangle \oplus H^{k-1} \oplus M, \end{aligned} \tag{4.23}$$

which is represented by $\mathrm{Diag}(-t, H^{k-1}, S)$. It is easy to check

$$|M^{[k]} : M(1) \oplus M(1)^{\perp}|^{-1}|M(1)^{\vee} : M(1)| = |t\pi|_F|t\pi|_F^{-1} = 1.$$

For (2) and (3), assume first that M is isotropic (and unimodular). In this case, we may choose a basis $\{v'_{2k+1}, \dots, v'_{2k+m}\}$ of M with Gram matrix $\mathrm{Diag}(H_0, 1, \dots, 1, -\nu)$. Choose $M(0) = \frac{t}{2}v'_{2k+1} + v'_{2k+2}$ with $q(M(0)) = t$. Then

$$\begin{aligned} M(0)^{\perp} &= \mathrm{Span}\{v_1, \cdots, v_{2k}, -\frac{t}{2}v'_{2k+1} + v'_{2k+2}, v'_{2k+3}, \cdots, v'_{2k+m}\} \\ &\cong H^k \oplus \mathrm{Span}\{v'_{2k+3}, \cdots, v'_{2k+m}\} \oplus \langle -t\rangle. \end{aligned}$$

as claimed. Moreover

$$|M^{[k]} : M(0) \oplus M(0)^{\perp}|^{-1}|M(0)^{\vee} : M(0)| = |t|_F|t\pi|_F^{-1} = q.$$

Next, assume that M is anisotropic. In this case, M has rank 2 and has Gram matrix $\mathrm{Diag}(1,\nu)$ with $\chi(M)=\chi(-\nu)=-1$. In this case, $E=F_0(\sqrt{-\nu})$ is a unramified quadratic field extension of F_0, and $N_{E/F_0}O_E^\times=O_{F_0}^\times$. When $\mathrm{v}(t)=0$, $t\in N_{E/F_0}O_E^\times$, i.e., $t=a\bar{a}+b\bar{b}\nu$. Take $M(0)=av_{2k+1}+bv_{2k+2}$. Then $q(M(0))=t$, and

$$M(0)^\perp=\mathrm{Span}\{v_1,\cdots,v_{2k},-\nu\bar{b}v_{2k+1}+\bar{a}v_{2k+2}\}=H^k\obot\langle t\nu\rangle,$$

and

$$|M^{[k]}:M(0)\obot M(0)^\perp|^{-1}|M(0)^\vee:M(0)|=|\pi|_F^{-1}=q.$$

When $\mathrm{v}(t)>0$, $t\notin N_{E/F_0}O_E^\times$. So there is no primitive $M(0)\in M$ with $q(M(0))=t$. This proves (1)—(3) of Proposition 4.1.10.

The proof of (4) follows from the following 4 lemmas.

Lemma 4.5.1. *For primitive vectors $w_1,\ w_2\in H_i$ with $q(w_1)=q(w_2)$, we can find an element $g\in\mathrm{U}(H_i)$ such that $g(w_1)=w_2$.*

Proof. We treat the case i is odd first. Assume $v=a_1v_1+a_2v_2$. Then v is primitive implies that a_1 or a_2 is a unit. Without loss of generality, we assume a_2 is a unit and we can further assume $a_2=1$ by the action of $\begin{pmatrix}\bar{a}_2 & 0\\ 0 & a_2^{-1}\end{pmatrix}$. Now notice that $q(v)=(v,v)=(a_1-\bar{a}_1)\pi^i$. Hence we can write $a_1=\alpha+\frac{q(v)\pi^{-i}}{2}$, where $\alpha\in O_{F_0}$. Now let $g=\begin{pmatrix}1 & -\alpha\\ 0 & 1\end{pmatrix}$, and it is straightforward to check that $g\in\mathrm{U}(H_i)$ and $g(v)=\frac{q(v)\pi^{-i}}{2}v_1+v_2$.

Now we deal with the case i is even. Again, we can assume $v=a_1v_1+v_2$. Then $q(v)=(a_1+\bar{a}_1)\pi^i$. Hence we can write $a_1=\frac{q(v)}{2}\pi^{-i}+\beta\pi$, where $\beta\in O_{F_0}$. Now let $g=\begin{pmatrix}1 & -\beta\pi\\ 0 & 1\end{pmatrix}$, and it is straightforward to check that $g\in\mathrm{U}(H_i)$ and $g(v)=\frac{q(v)\pi^{-i}}{2}v_1+v_2$. $\square$

Lemma 4.5.2. *Assume M is any lattice such that $\mathrm{v}(M)\geq i$. For $w_1,\ w_2\in H_i^k\obot M$, if $\mathrm{Pr}_{H_i^k}(w_1)$ and $\mathrm{Pr}_{H_i^k}(w_2)$ are primitive and $q(w_1)=q(w_2)$, then there exists $g\in\mathrm{U}(H_i^k\obot M)$*

with $g(w_1) = w_2$.

Proof. Choose a basis $\{v_1, ..., v_{2k}\}$ of H_i^k such that the associated Gram matrix is H_i^k. We also choose a basis $\{v_{2k+1}, ..., v_{2k+m}\}$ of M. Write $w_1 = \sum_{i=1}^{2k+m} a_i v_i$. Since $\mathrm{Pr}_{H_i^k}(w_1)$ is primitive, a_i is a unit for some $i \in \{1, ..., 2k\}$. Without loss of generality, we may assume $a_1 = 1$. Let $w' = w_1 + \frac{(-1)^{i+1} q(w_1)\pi^{-i}}{2} v_2$, then

$$q(w') = q(w_1) + (w_1, \frac{(-1)^{i+1} q(w_1)\pi^{-i}}{2} v_2) + (\frac{(-1)^{i+1} q(w_1)\pi^{-i}}{2} v_2, w_1) = 0,$$

and $(w', v_2) = (v_1, v_2)$. As a result, $M_1 = \mathrm{Span}_{O_F}\{w_1, v_2\} = \mathrm{Span}_{O_F}\{w', v_2\}$ is isometric to H_i. Notice that $\mathrm{val}_\pi(q(w_1)) \geq i$ is guaranteed by the assumption $\mathrm{v}(M) \geq i$.

Similarly, we can show $w_2 \in M_2$ for some M_2 that is isometric to H_i. However, the assumption $\mathrm{v}(M) \geq i$ and [Jac62, Proposition 4.2] imply that there exist $g \in \mathrm{U}(H_i^k \oplus M)$ such that $g(M_1) = M_2$. In particular, $g(w_1) \in M_2$. Since both $g(w_1)$ and w_2 are in M_2, the problem is reduced to Lemma 4.5.1. $\square$

Lemma 4.5.3. *For primitive vectors* $w_1, \ w_2 \in M$ *with* $q(w_1) = q(w_2)$*, we can find an element* $g \in \mathrm{U}(M)$ *such that* $g(w_1) = w_2$.

Proof. Since M is unimodular, we can decompose

$$M = H_0^k \oplus M',$$

where $M' = 0$ or an anisotropic unimodular Hermiatian lattice of rank 1 or 2. If $\mathrm{Pr}_{H_0^k}(w_1)$ and $\mathrm{Pr}_{H_0^k}(w_2)$ are primitive, this is Lemma 4.5.2. If $\mathrm{Pr}_{H_0^k}(w_1)$ is not primitive, then $\mathrm{Pr}_{M'}(w_1)$ is primitive and thus $q(\mathrm{Pr}_{M'}(w_1)) \in O_F^\times$. This implies that $q(w_2) = q(w_1)$ is a unit, and $M = O_F w_i \oplus (O_F w_i)^\perp$. Therefore there is some $g \in \mathrm{U}(M)$ with $g(w_1) = w_2$. $\square$

Lemma 4.5.4. *Assume that* $w_1, \ w_2 \in M^{[k]}$ *are primitive and that* $\mathrm{Pr}_{H^k}(w_1)$ *and* $\mathrm{Pr}_{H^k}(w_2)$ *are not primitive. Then we can find* $g \in \mathrm{U}(M^{[k]})$ *such that* $g(w_1) = w_2$.

Proof. Let $\{v_1, ..., v_{2k+m}\}$ be a basis of $H^k \oplus M$, whose Gram matrix is $H^k \oplus \mathrm{Diag}(1, ..., \nu)$ where ν is a unit. Assume $v \in M^{[k]}$ is primitive and $\mathrm{Pr}_{H^k}(v)$ is not primitive, then we can write $v = \sum_{i=1}^{2k} \pi a_i v_i + \sum_{j=2k+1}^{2k+m} a_j v_j$, where some a_j is a unit for $2k+1 \leq j \leq 2k+m$. Again, without loss of generality, we may assume $a_{2k+m} = 1$. For $i \leq k$, we set

$$v'_{2i-1} = v_{2i-1} + \frac{\bar{a}_{2i}}{\nu} v_{2k+m}, \quad v'_{2i} = v_{2i} + \frac{-\bar{a}_{2i-1}}{\nu} v_{2k+m}.$$

Let $M_v = \mathrm{Span}_{O_F}\{v'_1, ..., v'_{2k}\}$. Then it is easy to check that M_v is perpendicular to v. Moreover, M_v is isometric to H^k since $\mathrm{val}_\pi((v'_{2i-1}, v'_{2i})) = -1$ and $0 \leq \mathrm{val}_\pi((v'_i, v'_j))$ for other $1 \leq i, j \leq 2k$. Hence we can find $g_v \in \mathrm{U}(M^{[k]})$ such that $g_v(M_v) = \mathrm{Span}_{\mathrm{O_F}}\{v_1, ..., v_{2k}\}$, and $g_v(v) \in \mathrm{Span}_{\mathrm{O_F}}\{v_{2k+1}, ..., v_{2k+m}\} = M$.

Applying the above to w_1 and w_2, we can find $g_{w_1}, g_{w_2} \in \mathrm{U}(M^{[k]})$ such that $g_{w_1}(w_1),\ g_{w_2}(w_2) \in M$. Now the problem is reduced to Lemma 4.5.3, and the lemma is proved. □

According to Lemma 4.5.2 and Lemma 4.5.4, a primitive vector $v \in M^{[k]}$ is either in the same orbit of a vector $M(1) \in H^k$ or a vector $M(0) \in M$. Lemma 4.5.1 implies that primitive vectors $M(1), M'(1) \in H^k$ with $q(M(1)) = q(M'(1))$ lie in the same orbit. Lemma 4.5.3 implies the similar result for primitive $M(0), M'(0) \in M$ with $q(M(0)) = q(M'(0))$. A combination of the above proves Part (4) of Proposition 4.1.10. □

4.5.2 Proof of Proposition 4.1.11

In this subsection, we prove the first part of Proposition 4.1.11, which we restate as follows for the convenience of the reader.

Proposition 4.5.5. *Let L be a hermitian O_F-lattice of rank 2 and $\mathrm{v}(L) > 0$. Let $\varphi : L \to M^{[k]}$ be a primitive isometric embedding. Let $d(\varphi)$ be the dimension of the image of the map*

$$\mathrm{Pr}_{H^k} \circ \varphi : L \to H^k$$

in $H^k/\pi H^k$. Then

$$\varphi(L)^{\perp} \cong (-L) \oplus H^{k-d(\varphi)} \oplus M^{(d(\varphi))}$$

where $M^{(d(\varphi))}$ is unimodular of rank equal to $(\mathrm{rank}(M) - 2(2-d(\varphi)))$ *and* $\det M^{(d(\varphi))} = (-1)^{d(\varphi)}\det M$. *In particular, if $d(\varphi) = 1$ then* $\mathrm{rank}(M) \geq 2$, *and if $d(\varphi) = 0$ then* $\mathrm{rank}(M) \geq 4$.

Proof. This proposition follows from Lemmas 4.5.6 and 4.5.7 below. $\square$

Lemma 4.5.6. *Let the notation be as in Proposition 4.5.5. If* $\mathrm{rank}(M^{[k]}) \leq 4$, *then*

$$\varphi(L)^{\perp} \approx -L.$$

In particular, such an φ does not exist if $\chi(M^{[k]}) = -1$ or $\mathrm{rank}(M^{[k]}) < 4$.

Proof. First, assume $M^{[k]} = H^2$ and $L \approx H_i$ where $i > 0$. Let $\varphi(L) = \mathrm{Span}_{\mathrm{O_F}}\{w_1, w_2\}$ such that the Gram matrix of $\{w_1, w_2\}$ is H_i. By Lemma 4.5.1, we may assume $w_1 = v_1$. Then we may write $w_2 = a_1v_1 + \pi^{i+1}v_2 + a_3v_3 + a_4v_4$, and $\min\{\mathrm{v}_\pi(a_3), \mathrm{v}_\pi(a_4)\} = 0$ by assumption. Without loss of generality, we may assume $a_3 = 1$. Now a direct calculation shows that

$$\varphi(L)^{\perp} = \mathrm{Span}_{\mathrm{O_F}}\{v_1 + (-\pi)^{i+1}v_4, v_3 + \bar{a}_4v_4\}.$$

Its Gram matrix is

$$\begin{pmatrix} 0 & (-\pi)^i \\ \pi^i & (a_4 - \bar{a}_4)\pi^{-1} \end{pmatrix} = \begin{pmatrix} 0 & (-\pi)^i \\ \pi^i & -a_1(-\pi)^i - \bar{a}_1\pi^i \end{pmatrix} \approx \begin{pmatrix} 0 & -(-\pi)^i \\ -\pi^i & 0 \end{pmatrix},$$

hence

$$\varphi(L)^{\perp} \approx -L.$$

Now we treat the case $M^{[k]} = H^2$ and $L \approx \mathrm{Diag}(u_1(-\pi_0^a), u_2(-\pi_0)^b)$ where $0 <$

$a \leq b$. Again, let $\varphi(L) = \mathrm{Span}_{\mathrm{O_F}}\{w_1, w_2\}$ such that the Gram matrix of $\{w_1, w_2\}$ is $\mathrm{Diag}(u_1(-\pi_0^a), u_2(-\pi_0)^b)$, and we can assume $w_1 = v_1 - \frac{q(w_1)\pi}{2} v_2$ without loss of generality. Then we may write $w_2 = a_1(v_1 + \frac{q(w_1)\pi}{2} v_2) + a_3 v_3 + a_4 v_4$, hence $\min\{v_\pi(a_3), v_\pi(a_4)\} = 0$ by assumption again. We may assume $a_3 = 1$ and a direct calculation shows that

$$\varphi(L)^\perp = \mathrm{Span}_{\mathrm{O_F}}\{v_1 + \frac{q(w_1)\pi}{2} v_2 - \bar{a}_1 q(w_1)\pi v_4, v_3 + \bar{a}_4 v_4\}.$$

Set $v_3' = v_1 + \frac{q(w_1)\pi}{2} v_2 - \bar{a}_1 q(w_1)\pi v_4$ and $v_4' = a_1 v_3' + v_3 + \bar{a}_4 v_4$. Then $\varphi(L)^\perp = \mathrm{Span}_{\mathrm{O_F}}\{v_3', v_4'\}$ and the Gram matrix of $\{v_3', v_4'\}$ is

$$\begin{pmatrix} -q(w_1) & 0 \\ 0 & a_1\bar{a}_1 q(w_1) - (a_3\bar{a}_4 - \bar{a}_3 a_4)\pi^{-1} \end{pmatrix} = \begin{pmatrix} -q(w_1) & 0 \\ 0 & -q(w_2) \end{pmatrix}.$$

Now assume $M^{[k]} = H \oplus M$, where M is unimodular of rank 2. We only treat the case $L \approx H_i$ in detail, and the argument for L represented by a diagonal matrix is similar. We assume that $M^{[k]}$ has a basis $\{v_1, \dots, v_4\}$ with Gram matrix $H \oplus \mathrm{Diag}(1, \nu)$ where ν is a unit. Let $\varphi(L) = \mathrm{Span}_{\mathrm{O_F}}\{w_1, w_2\}$ where the Gram matrix of $\{w_1, w_2\}$ is H_i. Then one can check that at least one of w_1 and w_2 is primitive in H. By Lemma 4.5.1, we can assume that

$$w_1 \coloneqq \varphi(m_1) = v_1, w_2 \coloneqq \varphi(m_2) = a_1 v_1 + \pi^{i+1} v_2 + a_3 v_3 + a_4 v_4$$

and

$$(w_2, w_2) = a_1 \pi^i - \bar{a}_1 \pi^i + a_3\bar{a}_3 + a_4\bar{a}_4\nu = 0. \tag{4.24}$$

By our assumption we know that $\min\{\mathrm{v}_\pi(a_3), \mathrm{v}_\pi(a_4)\} = 0$. Since we assume $i \geq 1$, (4.24) implies that both a_3 and a_4 are in $O_{F_0}^\times$. This in turn implies that $-\nu \in \mathrm{Nm}_{F/F_0^\times}(O_F^\times) = O_{F_0}^2$. Hence $M^{[k]} \approx H \oplus H_0$ and we can instead assume that $\{v_1, v_2, v_3, v_4\}$ has Gram matrix $H \oplus H_0$. We can furthur assume that

$$w_1 = v_1, w_2 = a_1 v_1 + \pi^{i+1} v_2 + v_3 + a_4 v_4$$

with

$$(w_2, w_2) = a_1 \pi^i - \bar{a}_1 \pi^i + a_4 + \bar{a}_4 = 0.$$

By direct calculation, it is easy to see that

$$\varphi(L)^\perp = \mathrm{Span}_{O_F}\{v_1 - (-\pi)^i v_4, v_3 - \bar{a}_4 v_4\}.$$

Its Gram matrix is

$$\begin{pmatrix} 0 & -(-\pi)^i \\ -\pi^i & -a_4 - \bar{a}_4 \end{pmatrix} = \begin{pmatrix} 0 & -(-\pi)^i \\ -\pi^i & a_1 \pi^i - \bar{a}_1 \pi^i \end{pmatrix} \approx \begin{pmatrix} 0 & -(-\pi)^i \\ -\pi^i & 0 \end{pmatrix}.$$

Finally, assume $M^{[k]}$ is unimodular of rank 4. We treat the case $L \approx H_i$ in detail, and the other cases follow from a similar argument. Let $\varphi(L) = \mathrm{Span}_{\mathrm{O_F}}\{w_1, w_2\}$ such that the Gram matrix of $\{w_1, w_2\}$ is H_i. Apparently $M^{[k]}$ contains a H_0. We can assume that $M^{[k]}$ has a basis $\{v_1, v_2, v_3, v_4\}$ with Gram matrix $H_0 \oplus \mathrm{diag}\{1, \epsilon\}$ where $\epsilon \in O_{F_0}^\times$. By Lemma 4.5.3 we can assume that $w_1 = v_1$. Then we have

$$w_2 = a_1 v_1 + \pi^i v_2 + \sum_{j=3}^{4} a_j v_j,$$

and

$$(w_2, w_2) = a_1(-\pi)^i + \bar{a}_1 \pi^i + a_3 \bar{a}_3 + a_4 \bar{a}_4 \epsilon = 0. \tag{4.25}$$

By our assumption we know that $\min\{\mathrm{v}_\pi(a_3), \mathrm{v}_\pi(a_4)\} = 0$. Since we assume $i \geq 1$, (4.25) implies that both a_3 and a_4 are in $O_{F_0}^\times$. This in turn implies that $-\epsilon \in \mathrm{Nm}_{F/F_0^\times}(O_F^\times) = O_{F_0}^2$. Hence $M^{[k]} = H_0^2$ and we can instead assume that $\{v_1, v_2, v_3, v_4\}$ has Gram matrix

$H_0 \oplus H_0$. We can furthur assume that

$$w_1 = v_1, w_2 = a_1 v_1 + \pi^i v_2 + v_3 + a_4 v_4$$

with

$$(w_2, w_2) = a_1(-\pi)^i + \bar{a}_1 \pi^i + a_4 + \bar{a}_4 = 0.$$

By a direct calculation, it is easy to see that

$$\varphi(L)^\perp = \mathrm{Span}_{O_F}\{v_1 - (-\pi)^i v_4, v_3 - \bar{a}_4 v_4\}.$$

Its Gram matrix is

$$\begin{pmatrix} 0 & -(-\pi)^i \\ -\pi^i & -a_4 - \bar{a}_4 \end{pmatrix} = \begin{pmatrix} 0 & -(-\pi)^i \\ -\pi^i & a_1(-\pi)^i + \bar{a}_1 \pi^i \end{pmatrix} \approx \begin{pmatrix} 0 & -(-\pi)^i \\ -\pi^i & 0 \end{pmatrix}.$$

Notice that, as a byproduct of the above argument, we actually also proved that if $\mathrm{rank}(M^{[k]}) < 4$ or M is not split, then no such φ exists. The lemma is proved.

$\square$

Lemma 4.5.7. *Assume* $\mathrm{v}(L) \geq 0$. *Let* $\varphi : L \to M^{[k]}$ *be a primitive isometric embedding. Let* $d(\varphi)$ *be the dimension of* $\mathrm{Pr}_{H^k}(\varphi(L)) \otimes_{O_F} \mathbb{F}_q$ *in* $H^k/\pi H^k$. *Then there exist a* $g \in \mathrm{U}(M^{[k]})$ *such that*

$$g(\varphi(L)) \subset H^{d(\varphi)} \oplus I_{4-2d(\varphi)} \subset M^{[k]},$$

where $I_{4-2d(\varphi)}$ *is a unimodular sublattice of* $M^{[k]}$ *with rank* $4 - 2d(\varphi)$.

Proof. We prove the case for $L \approx H_i$ in detail, and the other cases are similar. Let $\{v_1, \ldots, v_{2k+m}\}$ be a basis of $M^{[k]}$ whose Gram matrix is $H^k \oplus \mathrm{diag}\{1, \ldots, 1, \nu\}$ where ν is a unit. Set $\varphi(L) = \mathrm{Span}_{\mathrm{O_F}}\{w_1, w_2\}$.

Assume $d(\varphi) = 2$. If $i = -1$, then there is nothing to prove. Therefore, we may assume

$i > -1$. By Lemma 4.5.2, without loss of generality, we can assume that $w_1 = v_1$. Then

$$w_2 = a_1 v_1 + \pi^{i+1} v_2 + \sum_{j=3}^{2k+m} a_j v_j.$$

By the assumption that $d(\varphi) = 2$, we know that

$$\min\{\mathrm{v}_\pi(a_j) \mid 3 \leq j \leq 2k\} = 0.$$

Hence applying Lemma 4.5.2 to $H^{k-1} \oplus M$, we can find a $g \in \mathrm{U}(M^{[k]})$ such that

$$gw_1 = v_1, \quad gw_2 \in H^2$$

where H^2 refers to the first direct summand in the decomposition $H^k \oplus M = H^2 \oplus H^{k-2} \oplus M$.

When $d(\varphi) = 1$, without loss of generality, we can assume $\mathrm{Pr}_{H^k}(w_1)$ is primitive. By Lemma 4.5.2, we can assume that $w_1 = v_1$. Then

$$w_2 = a_1 v_1 + \pi^{i+1} v_2 + \sum_{j=3}^{2k+m} a_j v_j.$$

By the assumption that $d(\varphi) = 1$, we know that

$$\min\{\mathrm{v}_\pi(a_j) \mid 3 \leq j \leq 2k\} \geq 1.$$

Since we assume φ is primitive, we know that

$$\min\{\mathrm{v}_\pi(a_j) \mid 2k+1 \leq j \leq 2k+r\} = 0.$$

Then we are done by applying Lemma 4.5.4 to $H^{k-1} \oplus M$.

When $d(\varphi) = 0$, without loss of generality, we may assume $w_1 = v_{2k+1} + v_{2k+2}$ by Lemma 4.5.4. Here, we pick v_{2k+i} so that the corresponding Gram matrix is $\mathrm{Diag}(1, -1, 1, \ldots, -\nu)$

(this is possible since we assume $m \geq 4$). Since φ is primitive with $d(\varphi) = 0$, then

$$w_2 = \sum_{i=1}^{2k} \pi a_i v_i + \sum_{i=2k+1}^{2k+m} a_i v_i,$$

and

$$\min\{\mathrm{v}_\pi(a_j) \mid 2k+3 \leq j \leq 2k+r\} = 0.$$

We are done by applying Lemma 4.5.4 to $H^k \oplus \mathrm{Span}_{\mathrm{O_F}}\{v_{2k+3}, \ldots, v_{2k+m}\}$. $\square$

4.5.3 Calculation of primitive local density

In this subsection, we compute primitive local density polynomials and prove the formulas in Propositions 4.1.10 and 4.1.11. Assume L is represented by a nonsingular hermitian matrix T of rank $n \leq 2$. We let $\bar{v}$ denote the image of v in $M^{[k]} \otimes_{O_F} \mathbb{F}_q$. Let

$$(M^{[k]})^n(i) := \{(v_j) \in M_k^{n,(n)} \mid \mathrm{Span}_{\mathbb{F}_q}\{\mathrm{Pr}_{H^k}(\bar{v}_j),\ 1 \leq j \leq n\} \text{ has rank } i\}$$

where $M^{n,(n)}$ is as in (4.3), and

$$\beta_i(M, L, X) := \int_{\mathrm{Herm}_n(F)} dY \int_{(M^{[k]})^n(i)} \psi(\langle Y, T(\mathbf{x}) - T \rangle) d\mathbf{x}. \tag{4.26}$$

Notice that

$$\sum_{i=0}^{n} \beta_i(M, L, X) = \beta(M, L, X)^{(n)} \tag{4.27}$$

is the primitive local density defined earlier, and we will shorten it as $\beta(M, L, X)$. Notice that if L is of the form H^j, then $\beta(M, L, X) = \beta_n(M, L, X)$.

First, by a variant of [CY20], Chao Li and Yifeng Liu obtained the following formula of $\beta(H^k, L)$.

Lemma 4.5.8. *[LL22, Lemma 2.16] Let $b_1 \leq \cdots \leq b_n$ be the unique integers such that $L^\vee/L \approx O_F/(\pi^{b_1}) \oplus \cdots \oplus O_F/(\pi^{b_n})$. Let $t_o(L)$ be the number of nonzero entries* *in* $(b_1, \cdots, b_n)$. *Then*

$$\beta(H^k, L) = \prod_{k - \frac{n+t_o(L)}{2} < i \leq k} (1 - q^{-2i}).$$

Lemma 4.5.9. *Assume* L *is of rank* n*, then*

$$\beta_n(M, L, q^{-2k}) = \beta(H^k, L).$$

In particular, if L *is of the form* H^j*, then*

$$\beta(M, L, q^{-2k}) = \beta_n(M, L, q^{-2k}) = \beta(H^k, L).$$

Proof. Recall that $T(\mathbf{x}) = (\mathbf{x}, \mathbf{x})$ is the moment matrix of $\mathbf{x} \in (M^{[k]})^n$. For a $\mathbf{x}_2 \in M^n$, let $T'(\mathbf{x}_2) = T - T(\mathbf{x}_2)$. Then

$$\begin{aligned}
\beta_n(M, L, q^{-2k}) &= \int_{\mathrm{Herm}_n(F)} dY \int_{M^n} \int_{(H^k)^{n,(n)}} \psi(\langle Y, T({}^t(\mathbf{x}_1, \mathbf{x}_2)) - T \rangle) d\mathbf{x}_1 d\mathbf{x}_2 \\
&= \int_{\mathrm{Herm}_n(F)} dY \int_{M^n} \int_{(H^k)^{n,(n)}} \psi(\langle Y, T(\mathbf{x}_1) + T(\mathbf{x}_2) - T \rangle) d\mathbf{x}_1 d\mathbf{x}_2 \\
&= \int_{\mathrm{Herm}_n(F)} dY \int_{M^n} \int_{(H^k)^{n,(n)}} \psi(\langle Y, T(\mathbf{x}_1) - T'(\mathbf{x}_2) \rangle) d\mathbf{x}_1 d\mathbf{x}_2
\end{aligned}$$

Notice that if L and L' are two hermitian O_F-lattices with moment matrix T and T' such that $T - T' \in \mathrm{Herm}_n(O_{F_0})$, then $t_o(L) = t_o(L')$. Hence, for any $\mathbf{x}_2 \in M^n$, we have by Lemma 4.5.8

$$\begin{aligned}
\beta(H^k, T'(\mathbf{x}_2)) &= \int_{\mathrm{Herm}_n(F)} dY \int_{(H^k)^{n,(n)}} \psi(\langle Y, T(\mathbf{x}_1) - T'(\mathbf{x}_2) \rangle) d\mathbf{x}_1 \\
&= \int_{\mathrm{Herm}_n(F)} dY \int_{(H^k)^{n,(n)}} \psi(\langle Y, T(\mathbf{x}_1) - T \rangle) d\mathbf{x}_1 \\
&= \beta(H^k, T).
\end{aligned}$$

Therefore

$$\begin{aligned}\beta_n(M,L,q^{-2k}) &= \mathrm{vol}(M^n, d\mathbf{x}_2)\cdot \int_{\mathrm{Herm}_n(F)} dY \int_{(H^k)^{n,(n)}} \psi(\langle Y, T(\mathbf{x}_1)-T\rangle)d\mathbf{x}_1 \\ &= \int_{\mathrm{Herm}_n(F)} dY \int_{(H^k)^{n,(n)}} \psi(\langle Y, T(\mathbf{x}_1)-T\rangle)d\mathbf{x}_1 \\ &= \beta(H^k, L).\end{aligned}$$

□

Combining the above two lemmas, we have the following.

Corollary 4.5.10.

1. *If L is of rank 1, then we have*

$$\beta_1(M,L,X) = 1-X.$$

2. *If L is of rank 2, then we have*

$$\beta_2(M,L,X) = \begin{cases} (1-X) & \text{if } L = H, \\ (1-X)(1-q^2X) & \text{otherwise.}\end{cases}$$

Lemma 4.5.11. *For an O_F-hermitian lattice, let $\bar{L} = L/\pi L$ be its reduction modulo π with resulting quadratic form. Let $r(\overline{M},\overline{L})$ to be the number of isometries from $\overline{L}$ to $\overline{M}$. Then*

$$\beta_0(M,L,X) = X^n\beta(M,L) = q^{-mn+n^2}r(\overline{M},\overline{L})X^n.$$

Proof. The second identity follows from the same proof of [CY20, Theorem 3.12]. Then a similar argument as in the proof of Lemma 4.5.9 gives the first identity. In this case, we need to replace $M^n \obot (H^k)^{n,(n)}$ in the proof of Lemma 4.5.9 with $M^{n,(n)} \obot (\pi H^k)^n$.

The factor X^n shows up because $\mathrm{vol}((\pi H^k)^n) = (q^{-2k})^n$. We leave the details to the reader. $\square$

Notice that [LZ22b, Lemma 3.2.1] provides a uniform formula for $|r(\overline{M}, \overline{L})|$. As a result, we obtain the following corollaries.

Corollary 4.5.12. *Assume $L = O_F\mathbf{x}$ is of rank 1 (we allow $q(\mathbf{x}) = 0$).*

1. *If $\mathrm{v}(L) = 0$, then*

$$\beta_0(M, L, X) = \begin{cases} (1 + \chi(M)\chi(L)q^{-\frac{m-1}{2}})X & \text{if } m \text{ is odd,} \\ (1 - \chi(M)q^{-\frac{m}{2}})X & \text{if } m \text{ is even.} \end{cases}$$

2. *If $\mathrm{v}(L) > 0$, then*

$$\beta_0(M, L, X) = \begin{cases} (1 - q^{1-m})X & \text{if } m \text{ is odd,} \\ (1 - q^{1-m} + \chi(M)(q-1)q^{-\frac{m}{2}})X & \text{if } m \text{ is even.} \end{cases}$$

Corollary 4.5.13. *Assume L is of rank 2. When $t(L) = 1$, we assume that L has gram matrix $T = \mathrm{Diag}(u_1, u_2(-\pi_0)^b)$ with $b > 0$.*

1. *If m is odd, then*

$$\beta_0(M, L, X) = \begin{cases} q(1 - q^{1-m})X^2 & \text{if } t(L) = 0, \\ q(1 + \chi(M)\chi(u_1)q^{\frac{3-m}{2}})(1 - q^{1-m})X^2 & \text{if } t(L) = 1, \\ q(1 - q^{1-m})(1 - q^{3-m})X^2 & \text{if } t(L) = 2. \end{cases}$$

2. If m is even, then

$$\beta_0(M,L,X) = \begin{cases} q(1-\chi(L)q^{1-m}+\chi(L)\chi(M)(q-\chi(L))q^{-\frac{m}{2}})X^2 & \text{if } t(L)=0, \\ q(1-\chi(M)q^{-\frac{m}{2}})(1-q^{2-m})X^2 & \text{if } t(L)=1, \\ q((1-q^{2-m})+\chi(M)(q^2-1)q^{-\frac{m}{2}})(1-q^{2-m})X^2 & \text{if } t(L)=2. \end{cases}$$

Finally, we calculate $\beta_1(M,L,X)$.

Proposition 4.5.14. *Assume L is as in Corollary 4.5.13. Let $\delta_e(m)=1$ or 0 depending on whether m is even or odd.*

1. If $t(L)=2$, then

$$\beta_1(M,L,X) = q(q+1)((1-q^{1-m})+\delta_e(m)\chi(M)(q-1)q^{-\frac{m}{2}})X(1-X).$$

2. If $t(L)=1$, then

$$\beta_1(M,L,X) = \begin{cases} q(1+q-q^{1-m}+\chi(M)\chi(u_1)q^{\frac{3-m}{2}})X(1-X) & \text{if } m \text{ is odd}, \\ q(1+q-q^{1-m}-\chi(M)q^{-\frac{m}{2}})X(1-X) & \text{if } m \text{ is even}. \end{cases}$$

3. If $t(L)=0$ and $\chi(L)=1$, i.e. $L\cong H_0$, then

$$\beta_1(M,L,X) = q(q+1-2q^{1-m}+\delta_e(m)\chi(M)(q-1)q^{-\frac{m}{2}})X(1-X).$$

4. If $t(L)=0$ and $\chi(L)=-1$, then

$$\beta_1(M,L,X) = q(q+1)(1-\delta_e(m)\chi(M)q^{-\frac{m}{2}})X(1-X).$$

Proof. First we assume $L = H_i$. We claim that

$$\beta_1(M, H_i, X) = \begin{cases} q(1-X)\left(2\beta_0(M,0,X) + \sum_{\alpha \in (O_{F_0}/(\pi_0))^\times} \beta_0(M, \langle -2\alpha \rangle, X)\right) & \text{if } i = 0, \\ q(q+1)(1-X)\beta_0(M,0,X) & \text{if } i \geq 1. \end{cases}$$

Here $\alpha(M,0,X) = \alpha(M, O_F\mathbf{x}, X)$ with $q(\mathbf{x}) = 0$ and $\mathbf{x} \neq 0$. Assuming the claim, the proposition for $L = H_i$ follows from Corollary 4.5.12. To prove the claim, it suffices to show the identity for $X = q^{-2k}$ for sufficiently many $k \geq 0$. Recall

$$\begin{aligned} I(M^{[k]}, L, d) = \{\phi \in &\mathrm{Hom}_{O_F}(L/\pi_0^d L, M^{[k]}/\pi_0^d M^{[k]}) \mid \\ &(\phi(x), \phi(y)) \equiv (x, y) \mod \pi^{2d-1},\ \forall x, y \in L\}. \end{aligned}$$

Let

$$J(M^{[k]}, L, d) \coloneqq \{\phi \in I(M^{[k]}, L, d) \mid \dim_{\mathbb{F}_q} \overline{\mathrm{Pr}_{H^k}(\phi(L))} = \dim_{\mathbb{F}_q} \overline{\mathrm{Pr}_M(\phi(L))} = 1\}.$$

Then

$$\beta_1(M, L, q^{-2k}) = \lim_{d \to \infty} q^{-(4(2k+m)-4)d} |J(M^{[k]}, L, d)|.$$

Let $\{l_1, l_2\}$ be a basis of L with Gram matrix H_i. For $\phi \in J(M^{[k]}, L, d)$, it will be determined by $w_i = \phi(l_i)$. Let $w_{i,H} = \mathrm{Pr}_{H^k}(w_i)$, and $w_{i,M} = \mathrm{Pr}_M(w_i)$. Since $\mathrm{rank}_{\mathbb{F}_q} \overline{\mathrm{Pr}_{H^k}(\phi(L))} = 1$, $\mathrm{rank}_{\mathbb{F}_q} \overline{\mathrm{Pr}_{H^k}(w_i)} = 1$ for $i = 1$ or 2.

Now we define a partition of $J(M^{[k]}, L, d)$. Assume $\alpha \in O_{F_0}$. Let

$$\begin{aligned} J_\alpha(M^{[k]}, L, d) &\coloneqq \{\phi \in I(M^{[k]}, L, d) \mid \mathrm{rank}_{\mathbb{F}_q} \overline{w}_{1,H} = 1,\ \overline{w}_{2,H} = \alpha \overline{w}_{1,H}\}, \text{ and} \\ J_\infty(M^{[k]}, L, d) &\coloneqq \{\phi \in I(M^{[k]}, L, d) \mid \mathrm{rank}_{\mathbb{F}_q} \overline{w}_{2,H} = 1,\ \overline{w}_{1,H} = 0\}. \end{aligned}$$

Then it is easy to verify

$$J(M^{[k]}, L, d) = \bigcup_{\alpha \in O_{F_0}/(\pi_0)} J_\alpha(M^{[k]}, L, d) \cup J_\infty(M^{[k]}, L, d).$$

Now we compute $|J_\alpha(M^{[k]}, L, d)|$. To determine a $\phi \in J_\alpha(M^{[k]}, L, d)$, we choose $w_1 = \phi(l_1)$ first. By definition, we have

$$\lim_{d\to\infty} q^{(2(2k+m)-1)d} \#\{w_1 \in M^{[k]}/\pi_0^d M^{[k]} \mid w_{1,\mathcal{H}} \text{ is primitive, and } q(w_1) \equiv 0 \bmod \pi_0^d\} \tag{4.28}$$

$$= \beta_1(M^{[k]}, 0) = 1 - q^{-2k}.$$

Given such a w_1, now we find the number of $w_2 = \phi(l_2)$ such that ϕ lies in $J_\alpha(M^{[k]}, L, d)$. By Lemma 4.5.2, we may assume $w_{1,S} = 0$. Let $w_2 = w_{2,M} + \alpha w_1 + \pi w_H$, where $w_H \in H^k$. Then the corresponding ϕ lies in $J_\alpha(M^{[k]}, L, d)$ if and only if

$$\pi^i \equiv (w_1, w_2) \equiv (w_1, \pi w_H) \mod \pi^{2d-1}$$

and

$$\begin{aligned} 0 \equiv q(w_2) &\equiv \mathrm{tr}((\alpha w_1, \pi w_H)) - \pi_0 q(w_H) + q(w_{2,M}) \\ &\equiv \alpha \mathrm{tr}(\pi^i) - \pi_0 q(w_H) + q(w_{2,M}) \mod \pi^{2d-1}. \end{aligned}$$

First,

$$\lim_{d\to\infty} q^{-2d(2k-1)} \#\{\pi w_H \in H^k/\pi_0^d H^k \mid (w_1, \pi w_H) \equiv \pi^i \mod \pi^{2d-1}\} = q^{1-2k}. \tag{4.29}$$

Second, for each fixed πw_H we have

$$\lim_{d\to\infty} q^{(-2m+1)d}\#\{w_{2,M} \in M/\pi_0^d M \mid w_{2,M} \text{ primitive, } q(w_{2,M}) \equiv -\alpha \mathrm{tr}(\pi^i) + \pi_0 q(w_H) \mod \pi^{2d-1}\} \tag{4.30}$$

$$= \beta(M, \langle -\alpha\mathrm{tr}(\pi^i) + \pi_0 q(w_H)\rangle)$$

$$= \begin{cases} \beta(M, \langle -2\alpha\rangle) & \text{if } i = 0, \\ \beta(M, 0) & \text{if } i > 0. \end{cases}$$

By symmetry, $|J_\infty(M^{[k]}, L, d)| = |J_0(M^{[k]}, L, d)|$. Now a combination of (4.28), (4.29) and (4.30) implies that

$$\begin{aligned}\beta_1(M, H_i, q^{-2k}) &= \lim_{d\to\infty} q^{(-4(2k+m)+4)d}\left(\sum_{\alpha\in O_{F_0}/(\pi_0)} |J_\alpha(M^{[k]}, L, d)| + |J_\infty(M^{[k]}, L, d)|\right) \\ &= \begin{cases} q(1-X)\left(2\beta_0(M, 0, q^{-2k}) + \sum_{\alpha\in O_{F_0}^\times/(\pi_0)} \beta_0(M, -2\alpha, q^{-2k})\right) & \text{if } i = 0, \\ q(q+1)(1-X)\beta_0(M, 0, q^{-2k}) & \text{if } i \geq 1, \end{cases}\end{aligned}$$

as claimed.

Next, we assume L has a basis $\{l_1, l_2\}$ whose Gram matrix is $\mathrm{Diag}(u_1(-\pi_0)^a, u_2(-\pi_0)^b)$ with $0 \leq a \leq b$. Let $w_i = \phi(l_i)$ as before. Then the number of possible choices for w_1 is given by

$$q^{(2(2k+m)-1)d}\beta_1(M, \langle u_1(-\pi_0)^a\rangle, q^{-2k})$$

for sufficiently large d. We may assume $w_1 = w_{1,H}$ without loss of generality. Let $w_2 = w_{2,M} + \alpha w_1 + \pi w_H$ as before. Then ϕ lies in $J_\alpha(M^{[k]}, L, d)$ if and only if

$$0 \equiv (w_1, w_2) \equiv (w_1, \alpha w_1) + (w_1, \pi w_H) \mod \pi^{2d-1}$$

and

$$\begin{aligned} u_2(-\pi_0)^b &\equiv q(w_2) \equiv (w_{2,M} + \alpha w_1 + \pi w_H, w_2) \\ &\equiv q(w_{2,M}) - \alpha^2 q(w_1) - \pi_0 q(w_H) \mod \pi^{2d-1}. \end{aligned}$$

Now

$$\lim_{d\to\infty} q^{(-4k+2)d} \#\{\pi w_H \in H^k/\pi_0^d H^k \mid (w_1, \pi w_H) \equiv -(w_1, \alpha w_1) \mod \pi^{2d-1}\} = q^{1-2k},$$

and for a fixed πw_H we have

$$\begin{aligned} &\lim_{d\to\infty} q^{(-2m+1)d} \#\{w_{2,M} \in L_S/\pi_0^d L_S \mid w_{2,M} \text{ primitive}, \\ &\qquad\qquad q(w_{2,M}) \equiv u_2(-\pi_0)^b + \alpha^2 q(w_1) + \pi_0 q(w_H) \mod \pi^{2d-1}\} \\ &= \beta(M, \langle u_2(-\pi_0)^b + \alpha^2 q(w_1) + \pi_0 q(w_H)\rangle). \end{aligned}$$

Now this proposition follows from a similar argument as before, and we leave the details to the reader. □

4.6 Special cycles and exceptional divisors

For a formal subscheme $\mathcal{Z}$ of $\mathcal{N}^{\mathrm{Kra}}$, we use the notation $\otimes_{\mathcal{Z}}$ (resp. $\otimes^{\mathbb{L}}_{\mathcal{Z}}$) instead of $\otimes_{\mathcal{O}_{\mathcal{Z}}}$ (resp. $\otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{Z}}}$). We also simply write $\otimes$ (resp. $\otimes^{\mathbb{L}}$) instead of $\otimes_{\mathcal{N}^{\mathrm{Kra}}}$ (resp. $\otimes^{\mathbb{L}}_{\mathcal{N}^{\mathrm{Kra}}}$). Let us first recall the following distribution law of derived tensor product. In this section, we identify $\mathbb{V}$ with C by the isomorphism b defined in (2.5).

Lemma 4.6.1. *Assume that $\mathcal{A}_i$ ($1 \le i \le k$) is in the derived category of bounded coherent sheaves on $\mathcal{N}^{\mathrm{Kra}}$ and $i : \mathcal{Z} \to \mathcal{N}^{\mathrm{Kra}}$ is a closed embedding of formal subscheme. Then the following identity holds in the derived category of bounded coherent sheaves on $\mathcal{Z}$.*

$$i^*(\mathcal{A}_1 \otimes^{\mathbb{L}} \ldots \otimes^{\mathbb{L}} \mathcal{A}_k \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}}) = i^*(\mathcal{A}_1 \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}}) \otimes^{\mathbb{L}}_{\mathcal{Z}} \ldots \otimes^{\mathbb{L}}_{\mathcal{Z}} i^*(\mathcal{A}_k \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}}).$$

Proof. We can take locally free representatives of $A_i^\bullet$ of $\mathcal{A}_i$. Then $A_1^\bullet \otimes \ldots \otimes A_k^\bullet$ is again a complex of locally free sheaves on $\mathcal{N}^{\mathrm{Kra}}$, hence a locally free representatives of $\mathcal{A}_1 \otimes^{\mathbb{L}} \ldots \otimes^{\mathbb{L}} \mathcal{A}_k$. Hence $i^*(\mathcal{A}_1 \otimes^{\mathbb{L}} \ldots \otimes^{\mathbb{L}} \mathcal{A}_k \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}})$ can be represented by $A_1^\bullet \otimes \ldots \otimes A_k^\bullet \otimes \mathcal{O}_{\mathcal{Z}}$. Meanwhile $A_i^\bullet \otimes \mathcal{O}_{\mathcal{Z}}$ is a representative of $\mathcal{A}_i \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}}$ in the derived category of bounded coherent sheaves on $\mathcal{N}^{\mathrm{Kra}}$ and is also a complex of locally free sheaves on $\mathcal{Z}$. Hence $i^*(\mathcal{A}_1 \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}}) \otimes^{\mathbb{L}}_{\mathcal{Z}} \ldots \otimes^{\mathbb{L}}_{\mathcal{Z}} i^*(\mathcal{A}_k \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}})$ can be represented by $(A_1^\bullet \otimes \mathcal{O}_{\mathcal{Z}}) \otimes_{\mathcal{Z}} \ldots \otimes_{\mathcal{Z}} (A_k^\bullet \otimes \mathcal{O}_{\mathcal{Z}})$. Now by the distribution law of tensor products we have

$$A_1^\bullet \otimes \ldots \otimes A_k^\bullet \otimes \mathcal{O}_{\mathcal{Z}} = (A_1^\bullet \otimes \mathcal{O}_{\mathcal{Z}}) \otimes_{\mathcal{Z}} \ldots \otimes_{\mathcal{Z}} (A_k^\bullet \otimes \mathcal{O}_{\mathcal{Z}}).$$

This finishes the proof of the lemma. □

Proposition 4.6.2. *Assume that the dimension of* $\mathbb{V}$ *is* $n \geq 2$*. Then for each* $\mathbf{x} \in \mathbb{V}$*,* $\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x})$ *is a divisor. Moreover, we have the following decomposition of Cartier divisors*

$$\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}) = \tilde{\mathcal{Z}}(\mathbf{x}) + \sum_{\Lambda \in \mathcal{V}^0, \mathbf{x} \in \Lambda} (m_\Lambda(\mathbf{x}) + 1)\mathrm{Exc}_\Lambda \tag{4.31}$$

where $m_\Lambda(\mathbf{x})$ *is the largest integer* m *such that* $\pi^{-m} \cdot \mathbf{x} \in \Lambda$*.*

Proof. The fact that $\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x})$ is a divisor is due to [How19, Proposition 4.3]. By [Shi18, Proposition 3.7], the superspecial point corresponding to a type zero lattice Λ is in $\mathcal{Z}^{\mathrm{Pap}}(\mathbf{x})$ if and only if $\mathbf{x} \in \Lambda$. Hence $\mathrm{Exc}_\Lambda \subset \mathcal{Z}^{\mathrm{Kra}}(\mathbf{x})$ if and only if $\mathbf{x} \in \Lambda$. Since $\mathcal{N}^{\mathrm{Kra}}_{n,\epsilon}$ is regular, we must have a decomposition as in (4.31) and the only job left is to determine the multiplicity of each Exc_Λ.

Fix a type zero lattice Λ and let $m \coloneqq m_\Lambda(\mathbf{x})$. Then $\pi^{-m} \cdot \mathbf{x}$ is a primitive vector in Λ. By Lemma 4.5.3, there exists a decomposition

$$\Lambda = \Lambda_2 \oplus \Lambda'$$

where Λ_2 and Λ' are unimodular lattices of rank 2 and $n-2$ respectively and $\pi^{-m} \cdot \mathbf{x} \in \Lambda_2$. Let $\eta = \chi(\Lambda')$. By applying Proposition 4.7.1, we see that $\tilde{\mathcal{Z}}_{n,\epsilon}(\Lambda') \cong \mathcal{N}^{\mathrm{Kra}}_{2,\epsilon\eta}$. Moreover we have the following proper intersections

$$\mathcal{Z}^{\mathrm{Kra}}_{n,\epsilon}(\mathbf{x}) \cap \tilde{\mathcal{Z}}_{n,\epsilon}(\Lambda') = \mathcal{Z}^{\mathrm{Kra}}_{2,\epsilon\eta}(\mathbf{x}),\ \ \tilde{\mathcal{Z}}_{n,\epsilon}(\mathbf{x}) \cap \tilde{\mathcal{Z}}_{n,\epsilon}(\Lambda') = \tilde{\mathcal{Z}}_{2,\epsilon\eta}(\mathbf{x}),$$

and

$$\mathrm{Exc}_\Lambda \cap \tilde{\mathcal{Z}}_{n,\epsilon}(\Lambda') = \mathrm{Exc}_{\Lambda_2},$$

where Exc_{Λ_2} is the exceptional divisor in $\mathcal{N}^{\mathrm{Kra}}_{2,\epsilon\eta}$ corresponding to the vertex lattice Λ_2. Hence the multiplicity of Exc_Λ in $\mathcal{Z}^{\mathrm{Kra}}_{n,\epsilon}(\mathbf{x})$ is the same as the multiplicity of Exc_{Λ_2} in $\mathcal{Z}^{\mathrm{Kra}}_{2,\epsilon\eta}(\mathbf{x})$. Now the proposition follows from [Shi22, Theorem 4.6] and Theorem 3.3.1. $\square$

The Chow ring $\mathrm{CH}^\bullet(\mathrm{Exc}_\Lambda) \cong \mathrm{Gr}^\bullet K_0(\mathrm{Exc}_\Lambda)$ is isomorphic to $\mathbb{Z}[H_\Lambda]/(H_\Lambda^{n-1} - 1)$ where H_Λ is the hyperplane class of Exc_Λ represented by any $\mathbb{P}^{n-2}_k$ in Exc_Λ.

Proposition 4.6.3. *Assume* $\dim \mathbb{V} = n \geq 2$. *Assume* $\mathbf{x} \in \mathbb{V}$ *such that* $h(\mathbf{x},\mathbf{x}) \neq 0$ *and* Λ *is a type* 0 *vertex lattice containing* $\mathbf{x}$. *Let* $m := m_\Lambda(\mathbf{x})$ *as in Proposition 4.6.2. Then* $\tilde{\mathcal{Z}}(\mathbf{x})$ *and* Exc_Λ *intersect properly and*

$$[\mathcal{O}_{\tilde{\mathcal{Z}}(\mathbf{x}) \cap \mathrm{Exc}_\Lambda}] = (2m+1) H_\Lambda \in \mathrm{CH}^1(\mathrm{Exc}_\Lambda).$$

Proof. First $\tilde{\mathcal{Z}}(\mathbf{x})$ and Exc_Λ are Cartier divisors with no common component, so they intersect properly. Let $m = m_\Lambda(\mathbf{x})$ and $\mathbf{x}' := \pi^{-m} \cdot \mathbf{x}$. By assumption $m \geq 0$. By Proposition 4.1.10, we have

$$\{v \in \Lambda \mid h(\mathbf{x}', v) = 0\} = \mathrm{Span}\{\mathbf{y}\} \obot \Lambda'$$

where $\mathrm{val}(\mathbf{y}) = \mathrm{val}(\mathbf{x}')$ and Λ' is unimodular. Let $\eta = \chi(\Lambda')$ and

$$\Lambda_2 := \{v \in \Lambda \mid v \perp \Lambda'\}.$$

Λ_2 is rank 2 unimodular and contains $\mathbf{x}'$.

By Proposition 4.7.1, we have $\tilde{\mathcal{Z}}(\Lambda') \cong \mathcal{N}^{\mathrm{Kra}}_{2,\epsilon\eta}$. In particular, $\tilde{\mathcal{Z}}(\Lambda')$ is regular. By Corollary 4.7.2, we know that $\tilde{\mathcal{Z}}(\Lambda') \cap \tilde{\mathcal{Z}}(\mathbf{x}) = \tilde{\mathcal{Z}}_{2,\epsilon\eta}(\mathbf{x})$. In particular $\tilde{\mathcal{Z}}(\Lambda')$ and $\tilde{\mathcal{Z}}(\mathbf{x})$ intersect properly as $\tilde{\mathcal{Z}}_{2,\epsilon\eta}(\mathbf{x})$ is a divisor in $\mathcal{N}^{\mathrm{Kra}}_{2,\epsilon\eta}$. On the other hand $\tilde{\mathcal{Z}}(\Lambda') \cap \mathrm{Exc}_\Lambda$ is the exceptional divisor Exc_{Λ_2} in $\mathcal{N}^{\mathrm{Kra}}_{2,\epsilon\eta}$. Since $\mathrm{Exc}_\Lambda \cong \mathbb{P}^{n-1}_k$, it is also regular. Our strategy is to compute the intersection number

$$\chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{O}_{\tilde{\mathcal{Z}}(\mathbf{x})} \otimes^{\mathbb{L}} \mathcal{O}_{\mathrm{Exc}_\Lambda} \otimes^{\mathbb{L}} \mathcal{O}_{\tilde{\mathcal{Z}}(\Lambda')})$$

in two different ways. By Lemma 4.6.1, one way is

$$\chi(\tilde{\mathcal{Z}}(\Lambda'), \mathcal{O}_{\tilde{\mathcal{Z}}(\Lambda') \cap \tilde{\mathcal{Z}}(\mathbf{x})} \otimes^{\mathbb{L}}_{\tilde{\mathcal{Z}}(\Lambda')} \mathcal{O}_{\tilde{\mathcal{Z}}(\Lambda') \cap \mathrm{Exc}_\Lambda}) \tag{4.32}$$

where we use the fact that the intersections $\tilde{\mathcal{Z}}(\Lambda') \cap \tilde{\mathcal{Z}}(\mathbf{x})$ and $\tilde{\mathcal{Z}}(\Lambda') \cap \mathrm{Exc}_\Lambda$ are proper (see for example [Zha21, Lemma B.2]). The other way is, by Lemma 4.6.1,

$$\chi(\mathrm{Exc}_\Lambda, \mathcal{O}_{\tilde{\mathcal{Z}}(\mathbf{x}) \cap \mathrm{Exc}_\Lambda} \otimes^{\mathbb{L}}_{\mathrm{Exc}_\Lambda} \mathcal{O}_{\tilde{\mathcal{Z}}(\Lambda') \cap \mathrm{Exc}_\Lambda}). \tag{4.33}$$

When $\epsilon\eta = -1$, by Proposition 3.11 and Theorem 4.5 of [Shi22], we know that (4.32) is equal to $2m+1$. When $\epsilon\eta = 1$, by Lemma 3.2.9, Theorem 3.3.1 and Lemma 3.4.3, we know that (4.32) is equal to $2m+1$ as well. Since the intersection number of H_Λ with $\mathrm{Exc}_{\Lambda_2} \cong \mathbb{P}^1_k$ in Exc_Λ is 1, the proposition follows. □

4.6.1 Intersection numbers involving the exceptional divisors

Lemma 4.6.4. *The class of* $\underbrace{\mathcal{O}_{\mathrm{Exc}_\Lambda} \otimes^{\mathbb{L}} \cdots \otimes^{\mathbb{L}} \mathcal{O}_{\mathrm{Exc}_\Lambda}}_{m}$ *in* $\mathrm{CH}^{m-1}(\mathrm{Exc}_\Lambda)$ *is* $(-2H_\Lambda)^{m-1}$.

Proof. To study this intersection, it suffices to consider the local model N^{Kra} constructed in [Krä03]. Let N^{Kra}_s be its special fiber. Recall by equation (4.11) loc. cit., we have

$$N^{\mathrm{Kra}}_s = \mathrm{Exc} + Z_2$$

as Cartier divisors where Exc is the exceptional divisor of N^{Kra} and Z_2 is a divisor in N^{Kra} which intersect properly with Exc. Their intersection is $2H$ where H is the hyperplane class of Exc. Since Exc is properly supported on N^{Kra}, we have

$$[\mathcal{O}_{\mathrm{Exc}} \otimes^{\mathbb{L}} \mathcal{O}_{N_s^{\mathrm{Kra}}}] = 0.$$

Hence

$$\begin{aligned} 0 &= [\mathcal{O}_{\mathrm{Exc}} \otimes^{\mathbb{L}}_{N^{\mathrm{Kra}}} \mathcal{O}_{N_s^{\mathrm{Kra}}}] \\ &= [\mathcal{O}_{\mathrm{Exc}} \otimes^{\mathbb{L}}_{N^{\mathrm{Kra}}} \mathcal{O}_{\mathrm{Exc}}] + [\mathcal{O}_{\mathrm{Exc}} \otimes^{\mathbb{L}}_{N^{\mathrm{Kra}}} \mathcal{O}_{Z_2}] \\ &= [\mathcal{O}_{\mathrm{Exc}} \otimes^{\mathbb{L}}_{N^{\mathrm{Kra}}} \mathcal{O}_{\mathrm{Exc}}] + 2H. \end{aligned}$$

This proves the lemma when $m = 2$. The general case now follows from Lemma 4.6.1. $\square$

Corollary 4.6.5. *Let $\Lambda \in \mathcal{V}^0$ and $\mathbf{x} \in \Lambda$. Then we have the following identity in* $\mathrm{CH}^1(\mathrm{Exc}_\Lambda)$*:*

$$[\mathcal{O}_{\mathrm{Exc}_\Lambda} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x})}] = -H_\Lambda.$$

Proof. By Propositions 4.6.2, 4.6.3 and Lemma 4.6.4, we have the following identity in $\mathrm{CH}^1(\mathrm{Exc}_\Lambda)$:

$$[\mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x})} \otimes^{\mathbb{L}} \mathcal{O}_{\mathrm{Exc}_\Lambda}] = [(2m_\Lambda(\mathbf{x}) + 1) - 2(m_\Lambda(\mathbf{x}) + 1)]H_\Lambda = -H_\Lambda.$$

This finishes the proof of the corollary. $\square$

Corollary 4.6.6. *Assume that $n - m \geq 1$ and $\mathrm{Exc}_\Lambda \subset \mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}_1) \cap \ldots \cap \mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}_m)$, then*

$$\chi(\mathcal{N}_n^{\mathrm{Kra}}, \mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}_1)} \otimes^{\mathbb{L}} \ldots \mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}_m)} \otimes^{\mathbb{L}} \underbrace{\mathcal{O}_{\mathrm{Exc}_\Lambda} \otimes^{\mathbb{L}} \cdots \otimes^{\mathbb{L}} \mathcal{O}_{\mathrm{Exc}_\Lambda}}_{n-m}) = (-1)^{n-1} \cdot 2^{n-m-1}.$$

Proof. By Corollary 4.6.5, Lemmas 4.6.1 and 4.6.4, we have

$$\begin{aligned}
&\chi(\mathcal{N}_n^{\mathrm{Kra}}, \mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}_1)} \otimes^{\mathbb{L}} \dots \mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}_m)} \otimes^{\mathbb{L}} \underbrace{\mathcal{O}_{\mathrm{Exc}_\Lambda} \otimes^{\mathbb{L}} \cdots \otimes^{\mathbb{L}} \mathcal{O}_{\mathrm{Exc}_\Lambda}}_{n-m}) \\
&= \chi(\mathrm{Exc}_\Lambda, \underbrace{(-H_\Lambda) \otimes^{\mathbb{L}}_{\mathrm{Exc}_\Lambda} \cdots \otimes^{\mathbb{L}}_{\mathrm{Exc}_\Lambda} (-H_\Lambda)}_{m} \otimes^{\mathbb{L}}_{\mathrm{Exc}_\Lambda} \underbrace{(-2H_\Lambda) \otimes^{\mathbb{L}}_{\mathrm{Exc}_\Lambda} \cdots \otimes^{\mathbb{L}}_{\mathrm{Exc}_\Lambda} (-2H_\Lambda)}_{n-m-1}) \\
&= (-1)^m \cdot (-2)^{n-m-1}.
\end{aligned}$$

□

For $\Lambda \in \mathcal{V}^0$, let $\mathbb{P}^1_\Lambda$ be any $\mathbb{P}^1_k$ in Exc_Λ, and

$$\mathrm{Int}_\Lambda(\mathbf{x}) = \chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x})} \otimes^{\mathbb{L}} \mathcal{O}_{\mathbb{P}^1_\Lambda}). \tag{4.34}$$

Corollary 4.6.7. *For $\Lambda \in \mathcal{V}^0$, we have*

$$\chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{O}_{\mathrm{Exc}_\Lambda} \otimes^{\mathbb{L}} \mathcal{O}_{\mathbb{P}^1_\Lambda}) = -2. \tag{4.35}$$

Proof. By Lemma 4.6.4, we have

$$\begin{aligned}
&\chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{O}_{\mathrm{Exc}_\Lambda} \otimes^{\mathbb{L}} \mathcal{O}_{\mathbb{P}^1_\Lambda}) \\
&= \chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{O}_{\mathrm{Exc}_\Lambda} \otimes^{\mathbb{L}} (\mathcal{O}_{\mathrm{Exc}_\Lambda} \otimes_{\mathcal{O}_{\mathrm{Exc}_\Lambda}} \mathcal{O}_{\mathbb{P}^1_\Lambda})) \\
&= \chi(\mathrm{Exc}_\Lambda, (\mathcal{O}_{\mathrm{Exc}_\Lambda} \otimes^{\mathbb{L}} \mathcal{O}_{\mathrm{Exc}_\Lambda}) \otimes_{\mathcal{O}_{\mathrm{Exc}_\Lambda}} \mathcal{O}_{\mathbb{P}^1_\Lambda}) \\
&= -2\chi(\mathrm{Exc}_\Lambda, H_\Lambda \cdot [\mathcal{O}_{\mathbb{P}^1_\Lambda}]) \\
&= -2.
\end{aligned}$$

□

Corollary 4.6.8. *For $\Lambda \in \mathcal{V}^0$, we have*

$$\mathrm{Int}_\Lambda(\mathbf{x}) = -1_\Lambda(\mathbf{x}).$$

Proof. If $\mathbf{x} \notin \Lambda$, then the intersection number is apparently 0. Otherwise we have by Corollary 4.6.5

$$\begin{aligned}&\chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x})} \otimes^{\mathbb{L}} \mathcal{O}_{\mathbb{P}^1_\Lambda})\\ &= \chi(\mathrm{Exc}_\Lambda, (\mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x})} \otimes^{\mathbb{L}} \mathcal{O}_{\mathrm{Exc}_\Lambda}) \otimes_{\mathcal{O}_{\mathrm{Exc}_\Lambda}} \mathcal{O}_{\mathbb{P}^1_\Lambda})\\ &= -\chi(\mathrm{Exc}_\Lambda, H_\Lambda \cdot [\mathcal{O}_{\mathbb{P}^1_\Lambda}])\\ &= -1.\end{aligned}$$

□

The above results suggest that the difficulty to compute $\mathrm{Int}(L)$ mainly lies in computing

$$\chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{O}_{\tilde{\mathcal{Z}}(x_1)} \otimes^{\mathbb{L}} \cdots \otimes^{\mathbb{L}} \mathcal{O}_{\tilde{\mathcal{Z}}(x_n)}).$$

We end this section by studying the intersection number of difference cycle with exceptional divisors.

Lemma 4.6.9. *If $L^\flat$ has rank $n-1$, then for any $\Lambda \in \mathcal{V}^0(L^\flat)$, we have*

$$\chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{D}(L^\flat) \cdot [\mathcal{O}_{\mathrm{Exc}_\Lambda}]) = \begin{cases}(-1)^{n-1} & \text{if } L^\flat = \Lambda \cap L^\flat_F,\\ 0 & \text{otherwise.}\end{cases}$$

Remark 4.6.10. *$L^\flat = \Lambda \cap L^\flat_F$ if and only if $L^\flat$ is of type (see* (4.42) *and Lemma 4.8.3 below)* 1 *or* 0 *and Λ is at the boundary of the $\mathcal{L}(L^\flat)$.*

Proof. Define

$$M' := \frac{1}{\pi} L^\flat \cap \Lambda \text{ and } m := \dim_{\mathbb{F}_q}(M'/L^\flat).$$

Then for L' such that $L^\flat \subset L' \subset \frac{1}{\pi} L^\flat$, we know that $\mathcal{Z}^{\mathrm{Kra}}(L')$ intersects Exc_Λ if and only if $L' \subset M'$. For such L', by Corollary 4.6.6, we have

$$\chi(\mathcal{N}^{\mathrm{Kra}}, {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L') \cdot [\mathcal{O}_{\mathrm{Exc}_\Lambda}]) = (-1)^{n-1}. \tag{4.36}$$

Hence

$$\chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{D}(L^\flat)\cdot[\mathcal{O}_{\mathrm{Exc}_\Lambda}]) = (-1)^{n-1}[1+\sum_{i=1}^{m}(-1)^i q^{i(i-1)/2} \sum_{\substack{L^\flat\subset L'\subset M' \\ \dim_{\mathbb{F}_q}(L'/L^\flat)=i}} 1].$$

Notice that $m=0$ if and only if $M'=L^\flat$ which is equivalent to the condition $L^\flat=\Lambda\cap L^\flat_F$. In this case the summation in (4.51) is over an empty set hence (4.51) is equal to 1. If $m>0$ we know (4.51) is equal to 0 by (4.48). $\square$

4.7 A geometric cancellation law

Proposition 4.7.1. *Suppose $\chi(\mathbb{V})=\epsilon$. Let L be a self-dual lattice of rank m in $\mathbb{V}$ with $\eta=\chi(L)$. We have*

$$\mathcal{Z}^{\mathrm{Pap}}_{n,\epsilon}(L)\cong\mathcal{N}^{\mathrm{Pap}}_{n-m,\epsilon\eta},\ \textit{and}\ \tilde{\mathcal{Z}}_{n,\epsilon}(L)\cong\mathcal{N}^{\mathrm{Kra}}_{n-m,\epsilon\eta}.$$

Proof. Let us start with the case $L=\mathrm{Span}\{\mathbf{x}_0\}$ where $\mathbf{x}_0\in\mathbb{V}$. Assume that $u=h(\mathbf{x}_0,\mathbf{x}_0)$. Multiplying the hermitian form $(,)_{\mathbb{X}}$ on C by u^{-1} does not affect the various moduli spaces involved. So we can perform this and assume that $h(\mathbf{x}_0,\mathbf{x}_0)=1$. Moreover, the sign of its orthogonal complement in $\mathbb{V}$ becomes

$$\epsilon_1=\epsilon\cdot\chi(u^{-1})\cdot\chi(u^{-(n-1)})\cdot\chi(-1)^{n-1}=\epsilon\chi(u)^n\chi(-1)^{n-1}.$$

Then for $(X,\iota,\lambda,\rho)\in\mathcal{Z}^{\mathrm{Pap}}_{n,\epsilon}(\mathbf{x}_0)(S)$, we define

$$\mathbf{x}_0^*\coloneqq\lambda_{\mathcal{G}}^{-1}\circ\mathbf{x}_0^\vee\circ\lambda,\ e\coloneqq\mathbf{x}_0\circ\mathbf{x}_0^*\in\mathrm{End}(X).$$

By the fact that $h(\mathbf{x}_0,\mathbf{x}_0)=1$ we know that e is an idempotent. It is routine to check that

$$((1-e)X,(1-e)\iota,(1-e^\vee)\lambda(1-e),\rho(1-e))$$

is an object in $\mathcal{N}^{\mathrm{Pap}}_{n-1,\epsilon_1}(S)$. Conversely given $(Y, \iota_Y, \lambda_Y, \rho_Y) \in \mathcal{N}^{\mathrm{Pap}}_{n-1,\epsilon_1}(S)$, the object

$$(Y \times \mathcal{G}_S, \iota_Y \times \iota_{\mathcal{G}_S}, \lambda_Y \times \lambda_{\mathcal{G}_S}, g \circ (\rho_Y \times \rho_{\mathcal{G}_S}))$$

is in $\mathcal{Z}^{\mathrm{Pap}}_{n,\epsilon}(\mathbf{x}_0)(S)$ where $g \in \mathrm{U}(\mathbb{V})$ such that $g^{-1}\mathbf{x}_0$ is the inclusion $0 \times \mathrm{id} : \mathbb{Y} \to \mathbb{X}_{n-1} \times \mathbb{Y}$ where $\mathbb{X}_{n-1}$ is the framing object of $\mathcal{N}^{\mathrm{Pap}}_{n-1,\epsilon_1}$. The above two functors are inverse to each other. This shows that $\mathcal{Z}^{\mathrm{Pap}}_{n,\epsilon}(\mathbf{x}_0) \cong \mathcal{N}^{\mathrm{Pap}}_{n-1,\epsilon_1}$. For general L of rank m and determinant u, find a basis with Gram matrix $\{1, \dots, 1, u\}$ and apply the above result repeatedly. So we have $\mathcal{Z}^{\mathrm{Pap}}_{n,\epsilon}(L) \cong \mathcal{N}^{\mathrm{Pap}}_{n-m,\epsilon_m}$ where

$$\epsilon_m = \epsilon\chi(u)^{n-m+1}\chi(-1)^{(n-m)m+\frac{m(m-1)}{2}}.$$

Notice that by scaling the hermitian form by $(-1)^m u$ again we have $\mathcal{N}^{\mathrm{Pap}}_{n-m,\epsilon_m} = \mathcal{N}^{\mathrm{Pap}}_{n-m,\epsilon\eta}$. It then follows from [Har13, Chapter II, Corollary 7.15] that $\tilde{\mathcal{Z}}_{n,\epsilon}(L)$ is the blow up of $\mathcal{Z}^{\mathrm{Pap}}_{n,\epsilon}(L)$ along its superspecial points, which is $\mathcal{N}^{\mathrm{Kra}}_{n-m,\epsilon\eta}$. □

Corollary 4.7.2. *Let L be as in Proposition 4.7.1 and $\mathbf{y} \in \mathbb{V}$ such that $\mathbf{y} \perp L$. Then*

$$\mathcal{Z}^{\mathrm{Kra}}_{n,\epsilon}(\mathbf{y}) \cap \tilde{\mathcal{Z}}_{n,\epsilon}(L) \cong \mathcal{Z}^{\mathrm{Kra}}_{n-m,\epsilon\eta}(\mathbf{y}).$$

Recall that for two lattices $L, L' \subset \mathbb{V}$ of rank n, we define

$$n(L', L) = \#\{L'' \subset L_F \mid L \subset L'', L'' \cong L'\}.$$

Also recall that $\delta_{\mathrm{odd}}(n) = 1$ or 0 depending on whether n is odd or not.

Proposition 4.7.3. *Let $L = I_\ell \oplus L_2 \subset \mathbb{V}$ where L_2 is of rank r, I_ℓ is unimodular of rank ℓ and $n = \ell + r$. Let I_r be a unimodular lattice that contains L_2. Then*

$$\mathrm{Int}(I_\ell \oplus L_2) - \mathrm{Int}(L_2) = n(I_r, L_2) \cdot (\delta_{\mathrm{odd}}(n) - \delta_{\mathrm{odd}}(r)). \tag{4.37}$$

Moreover,

$$\mathrm{Int}(I_n) = \delta_{\mathrm{odd}}(n). \tag{4.38}$$

Proof. If L_2 is unimodular and $r = 2$, then $\mathrm{Int}(L_2) = 0$ by [Shi22, Theorem 1.3] and Theorem 3.0.1. Combining this with (4.37), we obtain (4.38). In order to prove (4.37), we prove the following equation,

$$\mathrm{Int}(I_1 \oplus L_2) - \mathrm{Int}(L_2) = (-1)^r n(I_r, L_2). \tag{4.39}$$

which is the special case of (4.37) when $\ell = 1$. The general case then follows from an easy induction on n using (4.39) and the fact

$$n(I_n, I_\ell \oplus L_2) = n(I_r, L_2). \tag{4.40}$$

By Proposition 4.6.2, we have the following decomposition of Cartier divisors on $\mathcal{N}_n$

$$\mathcal{Z}(I_1) = \tilde{\mathcal{Z}}(I_1) + \sum_{\Lambda_0 \supset I_1} \mathrm{Exc}_{\Lambda_0},$$

where the summation is over vertex lattices of type 0 in $\mathbb{V}_n$ and $\tilde{\mathcal{Z}}(I_1) \cong \mathcal{N}_{n-1}$ by Corollary 4.7.2. By the same corollary, we know that

$$\chi(\mathcal{N}_n, [\mathcal{O}_{\tilde{\mathcal{Z}}(I_1)}] \cdot^{\mathbb{L}} \mathcal{Z}(L_2)) = \chi(\mathcal{N}_r, {}^{\mathbb{L}}\mathcal{Z}(L_2)) = \mathrm{Int}(L_2).$$

Hence we have

$$\mathrm{Int}(L) - \mathrm{Int}(L_2) = \sum_{\Lambda_0 \supset I_1} \chi(\mathcal{N}_n, [\mathcal{O}_{\mathrm{Exc}_{\Lambda_0}}] \cdot^{\mathbb{L}} \mathcal{Z}(L_2)).$$

If $L_2 \not\subset \Lambda_0$, then $\mathrm{Exc}_{\Lambda_0} \cap \mathcal{Z}(L_2)$ is empty by Proposition 5.1.20 below. If $L_2 \subset \Lambda_0$, then by Corollary 4.6.6, we have

$$\chi(\mathcal{N}_n, [\mathcal{O}_{\mathrm{Exc}_{\Lambda_0}}] \cdot^{\mathbb{L}} \mathcal{Z}(L_2)) = (-1)^r.$$

Hence

$$\mathrm{Int}(L) - \mathrm{Int}(L_2) = \sum_{\Lambda_0 \supset I_1 \obot L_2} (-1)^r.$$

Combining this with (4.40) finishes the proof of (4.39) and the proposition. $\square$

4.8 Horizontal components of special cycles

Given an integral hermitian lattice L we can have its Jordan decomposition:

$$L = \obot_{t \geq 0} L_t \tag{4.41}$$

where L_t is π^t-modular, see [Jac62]. Define the type of L to be

$$t(L) = \sum_{t \geq 1} \mathrm{rank}_{O_F}(L_t). \tag{4.42}$$

4.8.1 Quasi-canonical lifting cycles

Assume that $\dim(\mathbb{V}) = 2$. When $\chi(\mathbb{V}) = -1$, for $\mathbf{y} \in \mathbb{V}$, by [Shi22, Theorem 4.5], we have the following equality of Cartier divisors on $\mathcal{N}_{2,-1}^{\mathrm{Kra}}$.

$$\tilde{\mathcal{Z}}_{2,-1}(\mathbf{y}) = \mathcal{Z}_0 + \sum_{s=1}^{\mathrm{val}(\mathbf{y})} (\mathcal{Z}_s^+ + \mathcal{Z}_s^-).$$

Here $\mathcal{Z}_0$ (resp. $\mathcal{Z}_s^\pm$) is a canonical (resp. quasi-canonical) lifting cycle of level 0 (resp. s), see [Shi22, §3]. Moreover by [Shi22, Proposition 3.12], $\mathcal{Z}_s^+$ and $\mathcal{Z}_s^-$ do not intersect when $s \geq 1$. Let $\mathcal{O}_s := \mathcal{O}_{F_0} + \mathcal{O}_F \cdot \pi_0^s$ and M_s be the finite abelian extension of $\breve{F}$ corresponding to the subgroup $\mathcal{O}_s^\times$ under local class field theory. Let W_s be the integral closure of $\mathcal{O}_{\breve{F}}$ in M_s. Then we have $\mathcal{Z}_0 \cong \mathrm{Spf}\,\mathcal{O}_{\breve{F}}$ and $\mathcal{Z}_s^\pm \cong \mathrm{Spf}\,W_s$. Define the primitive part of $\tilde{\mathcal{Z}}_{2,-1}(\mathbf{y})$ to be

$$\tilde{\mathcal{Z}}_{2,-1}(\mathbf{y})^\circ := \begin{cases} \mathcal{Z}_{\mathrm{val}(\mathbf{y})}^+ + \mathcal{Z}_{\mathrm{val}(\mathbf{y})}^- & \text{if } \mathrm{val}(\mathbf{y}) > 0, \\ \mathcal{Z}_0 & \text{if } \mathrm{val}(\mathbf{y}) = 0. \end{cases}$$

When $\chi(\mathbb{V})=1$, for $\mathbf{y}\in\mathbb{V}$ such that $\mathrm{val}(\mathbf{y})\geq 0$, by Theorem 3.3.1, we have the following equality of Cartier divisors on $\mathcal{N}_{2,1}^{\mathrm{Kra}}$.

$$\tilde{\mathcal{Z}}_{2,1}(\mathbf{y})=\mathcal{Z}_0+\mathcal{Z}_v(\mathbf{y}),$$

where $\mathcal{Z}_0\cong\operatorname{Spf}\mathcal{O}_{\breve{F}}$ is a canonical lifting cycle and $\mathcal{Z}_v(\mathbf{y})$ is a Cartier divisor whose structure sheaf is annihilated by π^N for some $N>0$. Define the primitive horizontal part of $\tilde{\mathcal{Z}}_{2,1}(\mathbf{y})$ to be

$$\tilde{\mathcal{Z}}_{2,1}(\mathbf{y})^{\circ}:=\begin{cases}0 & \text{if } \mathrm{val}(\mathbf{y})>0,\\ \mathcal{Z}_0 & \text{if } \mathrm{val}(\mathbf{y})=0.\end{cases}$$

4.8.2 Horizontal cycles

Definition 4.8.1. *Let $M^\flat$ be a rank $n-1$ integral lattice in $\mathbb{V}$. We say that $M^\flat$ is horizontal if one of the following conditions is satisfied*

1. *$M^\flat$ is unimodular.*
2. *$M^\flat$ is of the form $M^\flat=M\obot\operatorname{Span}\{\mathbf{y}\}$ where M is a unimodular sublattice of rank $n-2$ such that $(M_F)^\perp$ (the perpendicular complement of M_F in $\mathbb{V}$) is nonsplit.*

Notice that condition (2) is independent of the choice of M. We denote the set of horizontal lattices by Hor*.*

For a rank $n-1$ integral lattice $L^\flat$, define

$$\mathrm{Hor}(L^\flat):=\{M^\flat\in\mathrm{Hor}\mid L^\flat\subseteq M^\flat\}. \tag{4.43}$$

Let $M^\flat\subset\mathbb{V}$ be a lattice of rank $n-1$ and type 1 or 0. We can decompose $M^\flat$ as

$$M^\flat=M\obot\operatorname{Span}\{\mathbf{y}\}, \tag{4.44}$$

for some unimodular lattice M of rank $n-2$. Then Proposition 4.7.1 and its corollary imply that

$$\tilde{\mathcal{Z}}(M^\flat) \cong \tilde{\mathcal{Z}}_{2,\chi((M_F)^\perp)}(\mathbf{y}).$$

Under this isomorphism, define $\tilde{\mathcal{Z}}(M^\flat)^\circ$ to be the formal subscheme of $\tilde{\mathcal{Z}}(M^\flat)$ isomorphic to $\tilde{\mathcal{Z}}_{2,\chi((M_F)^\perp)}(\mathbf{y})^\circ$. By the discussion in §4.8.1, $\tilde{\mathcal{Z}}(M^\flat)^\circ$ is nonempty if and only if $M^\flat \in$ Hor, in which case it consists of the union of irreducible components of $\tilde{\mathcal{Z}}(M^\flat)$ isomorphic to $\operatorname{Spf} W_s$. In particular, $\tilde{\mathcal{Z}}(M^\flat)^\circ$ is independent of the choice of M.

Theorem 4.8.2. *Let $L^\flat$ be a rank $n-1$ integral lattice in $\mathbb{V}$, then*

$$\mathcal{Z}^{\mathrm{Kra}}(L^\flat)_h = \bigcup_{M^\flat \in \mathrm{Hor}(L^\flat)} \tilde{\mathcal{Z}}(M^\flat)^\circ. \tag{4.45}$$

In particular, $\mathcal{Z}^{\mathrm{Kra}}(L^\flat)_h$ is of pure dimension 1. Moreover we have the following identity in $\mathrm{Gr}^{n-1} K_0(\mathcal{N}^{\mathrm{Kra}})$:

$$[\mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(L^\flat)_h}] = \sum_{M^\flat \in \mathrm{Hor}(L^\flat)} [\mathcal{O}_{\tilde{\mathcal{Z}}(M^\flat)^\circ}].$$

Proof. The proof largely follows [LZ22a, Section 4.4]. Let K be a finite extension of $\breve{F}$. Assume that z is an irreducible component of $\mathcal{Z}^{\mathrm{Kra}}(L^\flat)(\mathcal{O}_K) = \mathcal{Z}^{\mathrm{Pap}}(L^\flat)(\mathcal{O}_K)$, and let G be the corresponding formal O_F-module over $\mathcal{O}_K$. Define

$$L := \mathrm{Hom}_{O_F}(T_p\mathcal{G}, T_pG)$$

where $\mathcal{G}$ is the canonical lifting and T_p is the integral p-adic Tate module. L is an O_F-module of rank n equipped with the hermitian form

$$\{x, y\} = \lambda_{\mathcal{G}}^\vee \circ y^\vee \circ \lambda_G \circ x,$$

under which it is self-dual. We have two inclusions (preserving hermitian forms)

$$i_K : \mathrm{Hom}_{O_F}(\mathcal{G}, G)_F \to L_F,$$

and

$$i_k : \mathrm{Hom}_{O_F}(\mathcal{G}, G)_F \to \mathbb{V}.$$

By Lemma 4.4.1 of loc.cit., we have

$$\mathrm{Hom}_{O_F}(\mathcal{G}, G) = i_K^{-1}(L). \tag{4.46}$$

Let

$$M^\flat := (L_F^\flat) \cap i_k(i_K^{-1}(L)) \cong \mathrm{Hom}_{O_F}(\mathcal{G}, G).$$

Then $z \subset \mathcal{Z}(M^\flat)(\mathcal{O}_K)$. Lemma 4.8.3 below implies that $t(M^\flat) \leq 1$. Hence we know that z is one of the irreducible component of $\tilde{\mathcal{Z}}(M^\flat)^\circ \cong \tilde{\mathcal{Z}}_{2,\chi((M_F)^\perp)}(\mathbf{y})$ assuming the decomposition of $M^\flat$ as in (4.44). The nonemptiness of $\tilde{\mathcal{Z}}(M^\flat)^\circ$ implies that $M^\flat \in \mathrm{Hor}$. It remains to prove that z has multiplicity 1 in $\mathcal{Z}^{\mathrm{Kra}}(L^\flat)$. Consider R-points of both sides of (4.45), where $R := \mathcal{O}_K[x]/(x^2)$. As in [Krä03] (see [RZ96, Appendix of Chapter 3]) we know

$$\mathbb{D}(\mathcal{G})(R) \cong O_F \otimes_{\mathcal{O}_{F_0}} R, \text{ and } \mathbb{D}(G)(R) \cong (O_F \otimes_{\mathcal{O}_{F_0}} R)^n$$

where $\mathbb{D}$ is the $\mathcal{O}_{F_0}$-relative Dieudonné crystal. Define

$$\tilde{e}_0 = 1 \otimes 1 \in \mathbb{D}(\mathcal{G})(R), \quad \tilde{f}_0 = \pi \otimes 1 \in \mathbb{D}(\mathcal{G})(R).$$

Then the Hodge submodule $\mathcal{F}_0$ of $\mathbb{D}(\mathcal{G})(R)$ is spanned by

$$(1 \otimes \pi)\tilde{e}_0 + \tilde{f}_0.$$

$\mathbb{D}(G)(R)$ is equipped with an O_F-invariant symplectic form $\langle , \rangle$ and we can assume that $\mathbb{D}(G)(R)$ has a basis $\{\tilde{e}_1, \ldots, \tilde{e}_n, \tilde{f}_1, \ldots, \tilde{f}_n\}$ such that

$$(\pi \otimes 1)\tilde{e}_i = \tilde{f}_i, \quad \langle \tilde{e}_i, \tilde{f}_j \rangle = \delta_{ij}.$$

Since any element in $L^\flat$ is O_F-linear, we can arrange a change of basis if necessary and assume that

$$L^\flat((1\otimes\pi)\tilde{e}_0+\tilde{f}_0)=\mathrm{Span}_R\{(1\otimes\pi^{a_1})((1\otimes\pi)\tilde{e}_1+\tilde{f}_1),\dots,(1\otimes\pi^{a_{n-1}})((1\otimes\pi)\tilde{e}_{n-1}+\tilde{f}_{n-1})\}.$$

Now $\mathbb{D}(G)(\mathcal{O}_K)=\mathbb{D}(G)(R)\otimes_R\mathcal{O}_K$. Let $e_i=\tilde{e}_i\otimes 1$ and $f_i=\tilde{f}_i\otimes 1$ respectively. There is an exact sequence of free $\mathcal{O}_F\otimes_{\mathcal{O}_{F_0}}\mathcal{O}_K$-modules (the Hodge filtration)

$$0\to\mathrm{Fil}\to\mathbb{D}(G)(\mathcal{O}_K)\to\mathrm{Lie}G\to 0$$

where Fil is isotropic with respect to $\langle,\rangle$. We must have $L^\flat((1\otimes\pi)e_0+f_0)\subset\mathrm{Fil}$. Hence we have

$$(1\otimes\pi)e_1+f_1,\dots,(1\otimes\pi)e_{n-1}+f_{n-1}\subset\mathrm{Fil}.$$

Since Fil is isotropic and by the signature condition, we have

$$\mathrm{Fil}=\mathrm{Span}_{\mathcal{O}_K}\{(1\otimes\pi)e_1+f_1,\dots,(1\otimes\pi)e_{n-1}+f_{n-1},(1\otimes\pi)e_n-f_n\}.$$

Since $(x)\subset R$ has a nilpotent p.d. structure, by Grothendieck-Messing theory, a lift $\tilde{z}$ of z to $\mathcal{Z}^{\mathrm{Kra}}(L^\flat)(R)$ corresponds to a lift of Fil to an isotropic $O_F\otimes_{\mathcal{O}_{F_0}}R$-module $\widetilde{\mathrm{Fil}}$ in $\mathbb{D}(G)(R)$ containing the image of $L^\flat$. By the same reasonning as above, we must have

$$\widetilde{\mathrm{Fil}}=\mathrm{Span}_R\{(1\otimes\pi)\tilde{e}_1+\tilde{f}_1,\dots,(1\otimes\pi)\tilde{e}_{n-1}+\tilde{f}_{n-1},(1\otimes\pi)\tilde{e}_n-\tilde{f}_n\}.$$

Hence such lift is unique. This implies that the multiplicity of z in $\mathcal{Z}^{\mathrm{Kra}}(L^\flat)$ is one. $\square$

Lemma 4.8.3. *Let L be a self-dual hermitian lattice of rank n and W be a $n-1$ dimensional subspace of L_F. Then $t(M^\flat)\leq 1$ for $M^\flat=L\cap W$.*

Proof. This is exactly the same as the proof of [LZ22a, Lemma 4.5.1]. Notice that in our case we may need some blocks $\begin{pmatrix}0 & \pi^a\\ (-\pi)^a & 0\end{pmatrix}$ in the upper left $(n-1)\times(n-1)$ block of T as in loc.cit. Alternatively, see [LL22, Lemma 2.24(2)]. $\square$

We end this subsection with the following lemma.

Lemma 4.8.4. *Assume $M^\flat \in \mathrm{Hor}$. Then $\tilde{\mathcal{Z}}(M^\flat)^\circ$ intersects the special fiber of $\mathcal{N}_{n,\epsilon}^{\mathrm{Kra}}$ at a unique Exc_Λ for some $\Lambda \in \mathcal{V}^0$. Moreover*

$$\chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{O}_{\tilde{\mathcal{Z}}(M^\flat)^\circ} \otimes^{\mathbb{L}} \mathcal{O}_{\mathrm{Exc}_\Lambda}) = \begin{cases} 1 & \textit{if } M^\flat \textit{ is unimodular,} \\ 2 & \textit{otherwise.} \end{cases}$$

Proof. By the definition of Hor, we can find a decomposition of $M^\flat$

$$M^\flat = M \obot \{\mathbf{x}\}$$

such that M is self dual. Let Λ be any vertex lattice containing $M^\flat$. If $M^\flat$ is unimodular, then Λ has to be of the form $M^\flat \obot L'$ where L' is the unique unimodular lattice in $(M_F^\flat)^\perp$. If $M^\flat$ is of the form $M \obot L'$ such that M is of rank $n-2$ and $(M_F)^\perp$ is nonsplit, then the proof of [Shi18, Theorem 3.10] implies that there is a unique vertex lattice Λ' in $(M_F)^\perp$ which is of unimodular (this fact is the same as the fact that the Bruhat-Tits building of $(M_F)^\perp$ has only one point). Then Λ must be of the form $M \obot \Lambda'$. In both cases, Λ is unique and is of type 0.

Assume $\chi(M) = \eta$. By Proposition 4.7.1, $\tilde{\mathcal{Z}}(M) \cong \mathcal{N}_{2,\epsilon\eta}^{\mathrm{Kra}}$. Moreover $\tilde{\mathcal{Z}}(M) \cap \mathrm{Exc}_\Lambda = \mathbb{P}_k^1$ is an exceptional divisor in $\mathcal{N}_{2,\epsilon\eta}^{\mathrm{Kra}}$. Hence by Lemma 4.6.1, we have

$$\chi(\mathcal{N}_{n,\epsilon}^{\mathrm{Kra}}, \mathcal{O}_{\tilde{\mathcal{Z}}(M^\flat)^\circ} \otimes^{\mathbb{L}} \mathcal{O}_{\mathrm{Exc}_\Lambda}) = \chi(\mathcal{N}_{2,\epsilon\eta}^{\mathrm{Kra}}, \mathcal{O}_{\tilde{\mathcal{Z}}(M^\flat)^\circ} \otimes^{\mathbb{L}}_{\mathcal{N}_{2,\epsilon\eta}^{\mathrm{Kra}}} \mathcal{O}_{\mathbb{P}_k^1}).$$

Now the lemma follows from Lemma 3.4.3 when $\epsilon\eta = 1$, and from [Shi22, Proposition 3.11] when $\epsilon\eta = -1$. $\square$

4.9 Special Difference cycles

Conjecture 1.0.3 and Theorem 4.1.3 motivate us to make the following definition.

Definition 4.9.1. *For $L \subset \mathbb{V}$ a rank ℓ lattice, define the special difference cycle $\mathcal{D}(L) \in K_0^{\mathcal{Z}^{\mathrm{Kra}}(L)}(\mathcal{N}^{\mathrm{Kra}})$ by*

$$\mathcal{D}(L) = {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L) + \sum_{i=1}^{\ell}(-1)^i q^{i(i-1)/2} \sum_{\substack{L \subset L' \subset \frac{1}{\pi}L \\ \dim_{\mathbb{F}_q}(L'/L)=i}} {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L'). \tag{4.47}$$

One interesting observation is the following decomposition of ${}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L)$.

Lemma 4.9.2. *For $L \subset \mathbb{V}$ a lattice of rank ℓ, we have the following identity in $K_0^{\mathcal{Z}^{\mathrm{Kra}}(L)}(\mathcal{N}^{\mathrm{Kra}})$ where the summation is finite.*

$${}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L) = \sum_{\substack{L' \text{integral} \\ L \subset L' \subset L_F}} \mathcal{D}(L').$$

Proof. First of all, if L is not integral, neither is L' if $L \subset L'$. In this case ${}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L) = 0$ and the summation index on the right hand side of the identity in the lemma is empty. This proves the lemma when $\mathrm{val}(L) < 0$. We can now prove the identity by induction on the fundamental invariant of L. Assume that the lemma is proved for all $L' \subset L_F$ with $L \subsetneq L'$.

For L' with $L \subsetneq L' \subset \frac{1}{\pi}L$, we have

$${}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L') = \sum_{L' \subset L'' \subset L'_F} \mathcal{D}(L'')$$

by the induction hypothesis. Combining this with (4.47), we can write

$${}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L) = \sum_{L \subset L'' \subset L_F} m(L'')\mathcal{D}(L'')$$

where $m(L'') \in \mathbb{Z}$. Now it suffices to show $m(L'') = 1$ for any L'' such that $L \subset L'' \subset L_F$.

First, notice that $m(L)=1$. For any L'' such that $L \subset L'' \subsetneq L_F$, let $M' = \frac{1}{\pi}L \cap L''$ and $m = \dim_{\mathbb{F}_q}(M'/L)$. We have

$$m(L'') = -\sum_{i=1}^{m}(-1)^i q^{i(i-1)/2} \sum_{\substack{L \subset L' \subset M' \\ \dim_{\mathbb{F}_q}(L'/L)=i}} 1 = 1 \tag{4.48}$$

by evaluating the identity in the corollary to [Tam63, Lemma 12] at $t=1$. □

Remark 4.9.3. *When $\ell = 1$ and $L = \mathrm{Span}\{\mathbf{x}\}$, the Cartier divisor*

$$\mathcal{D}(L) = \mathcal{Z}(\mathbf{x}) - \mathcal{Z}(\frac{1}{\pi}\mathbf{x})$$

is the difference divisor $\mathcal{D}(\mathbf{x})$ defined in [Ter10, Definition 2.10].

Definition 4.9.4. *Assume $L = L_1 \oplus L_2$, where L_i is of rank n_i and $n_1 + n_2 = n$. We define*

$$\mathrm{Int}(L)^{(n_1)} = \chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{D}(L_1) \cdot {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L_2)). \tag{4.49}$$

Notice that $\mathrm{Int}(L)^{(n_1)}$ depends on the decomposition $L = L_1 \oplus L_2$.

Definition 4.9.1 motivates us to make the following definition.

Definition 4.9.5. *When $L^\flat$ is a rank $n-1$ integral lattice, define $\mathcal{D}(L^\flat)_h \in \mathrm{Gr}^{n-1}K_0(\mathcal{N}^{\mathrm{Kra}})$ by*

$$\mathcal{D}(L^\flat)_h = [\mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(L^\flat)_h}] + \sum_{i=1}^{n-1}(-1)^i q^{i(i-1)/2} \sum_{\substack{L^\flat \subset L' \subset \frac{1}{\pi}L^\flat \\ \dim_{\mathbb{F}_q}(L'/L^\flat)=i}} [\mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(L')_h}]. \tag{4.50}$$

Proposition 4.9.6. *Assume $L^\flat$ is a rank $n-1$ integral lattice, then*

$$\mathcal{D}(L^\flat)_h = \begin{cases} \tilde{\mathcal{Z}}(L^\flat)^\circ & \text{if } L^\flat \in \mathrm{Hor}, \\ 0 & \text{if } L^\flat \notin \mathrm{Hor}. \end{cases}$$

Proof. By Theorem 4.8.2, it suffices to compute the multiplicity of an irreducible component in $\tilde{\mathcal{Z}}(M^\flat)^\circ$ in $\mathcal{D}(L)_h$ for all $M^\flat \in \mathrm{Hor}(L^\flat)$ (see (4.43)). For such a $M^\flat$, define

$$M' := \frac{1}{\pi} L^\flat \cap M^\flat \text{ and } m := \dim_{\mathbb{F}_q}(M'/L^\flat).$$

Then for a lattice L' with $L^\flat \subset L' \subset \frac{1}{\pi} L^\flat$, we know that $\tilde{\mathcal{Z}}(M^\flat)^\circ$ is in $\mathcal{Z}^{\mathrm{Kra}}(L')_h$ if and only if $L' \subset M'$. Hence the multiplicity of an irreducible components in $\tilde{\mathcal{Z}}(M^\flat)^\circ$ in $\mathcal{D}(L)_h$ is

$$1 + \sum_{i=1}^{m} (-1)^i q^{i(i-1)/2} \sum_{\substack{L^\flat \subset L' \subset M' \\ \dim_{\mathbb{F}_q}(L'/L^\flat) = i}} 1. \tag{4.51}$$

Notice that $m = 0$ if and only if $M' = M^\flat = L^\flat$, in this case the summation in (4.51) is over an empty set, hence (4.51) is equal to 1. If $m > 0$, (4.51) is equal to 0 by (4.48). $\square$

4.10 Reduced locus of special cycle

As remarked previously, results of [RTW14] extend to the category of strict formal O_{F_0}-modules using relative Dieudonné theory.

4.10.1 The Bruhat-Tits building for $n = 3$

From now on we assume $n = 3$ and $\mathcal{L} = \mathcal{L}_3$ as in Section 2.1.3.

Lemma 4.10.1.

1. *For every $\Lambda_2 \in \mathcal{V}^2$, $\mathcal{N}_{\Lambda_2}$ is isomorphic to the projective line $\mathbb{P}^1$ over k. Its $q+1$ rational points correspond to all $\Lambda_0 \in \mathcal{V}^0$ contained in Λ_2.*

2. *Every $\Lambda_0 \in \mathcal{V}^0$ is contained in $q+1$ type 2 lattices. In other words, there are $q+1$ projective lines in $(\mathcal{N}_3^{\mathrm{Pap}})_{\mathrm{red}}$ passing through the superspecial point $\mathcal{N}_{\Lambda_0}(k)$. Moreover*

$$\bigcap_{\Lambda_2 \in \mathcal{V}_2,\, \Lambda_0 \subset \Lambda_2} \Lambda_2^\sharp = \pi \Lambda_0. \tag{4.52}$$

Proof. Suppose $z \in \mathcal{N}(k)$ and $M := M(z) \subset N$ is defined as in Proposition 2.1.4. Since $n = 3$, by [RTW14, Proposition 4.1] we have $\Lambda(M) \otimes_{O_F} O_{\breve{F}} = M + \tau(M)$.

Proof of (1): Suppose $z \in \mathcal{N}_{\Lambda_2}(k)$, i.e. $M \subset \Lambda_2$.

If $M = \tau(M)$, then $M = \Lambda_0 \otimes_{O_F} O_{\breve{F}}$ for some $\Lambda_0 \in \mathcal{V}^0$ contained in Λ_2.

If $M \neq \tau(M)$, then by taking the dual of $M \subset \Lambda_2 \otimes_{O_F} O_{\breve{F}}$ we have the following sequence of inclusions

$$(\Lambda_2 \otimes_{O_F} O_{\breve{F}})^\sharp \overset{1}{\subset} M \overset{1}{\subset} M + \tau(M) = \Lambda_2 \otimes_{O_F} O_{\breve{F}}. \tag{4.53}$$

In both cases the class of M in $\Lambda_2 \otimes_{O_F} O_{\breve{F}}/(\Lambda_2 \otimes_{O_F} O_{\breve{F}})^\sharp \cong k^2$ is a line. This finishes the proof of (1).

Proof of (2): For each $\Lambda_0 \in \mathcal{V}^0$ we just need to count the number of lattices $\Lambda_2 \in \mathcal{V}^2$ that contains Λ_0. We have the following sequence of inclusions

$$\pi\Lambda_0 \overset{2}{\subset} \Lambda_2^\sharp \overset{1}{\subset} \Lambda_0 \overset{1}{\subset} \Lambda_2.$$

With respect to the quadratic form $(,)$ (mod π) on $\Lambda_0/\pi\Lambda_0$, the dual lattice $\Lambda_2^\sharp$ corresponds to the 2-dimensional subspaces $U := \Lambda_2^\sharp/\pi\Lambda_0$ in $\Lambda_0/\pi\Lambda_0$ such that $U^\perp \overset{1}{\subset} U$. So we just need to count the number of isotropic lines $U^\perp$. Assume that $\{e_1, e_2, e_3\}$ is a basis of $\Lambda_0/\pi\Lambda_0$ whose Gram matrix with respect to the quadratic form $(,)_{\mathbb{X}}$ (mod π) is

$$\begin{pmatrix} & 1 & \\ 1 & & \\ & & \epsilon \end{pmatrix}.$$

It is easy to see that the isotropic lines are Span$\{e_1\}$, Span$\{e_2\}$ and Span$\{e_1 - \frac{\epsilon a^2}{2} e_2 + a e_3\}$ ($a \in \mathbb{F}_q^\times$). Finally, equation (4.52) can be checked directly using this basis. $\square$

It is well-known that $\mathcal{L}$ is a tree, see for example [Bro89, Section 3 of Chapter VI]. More specifically, the vertices of $\mathcal{L}$ correspond to vertex lattices of type 2 or 0. There is an edge between $\Lambda \in \mathcal{V}^2$ and $\Lambda_0 \in \mathcal{V}^0$ if $\Lambda_0 \subset \Lambda$. We give each edge length $\frac{1}{2}$. This defines a metric $d(,)$ on $\mathcal{L}$. Recall that we have defined $\mathcal{L}(L)$ in (2.8). Then the boundary of $\mathcal{L}(L)$ is the set

$$\mathcal{B}(L) = \{\Lambda \in \mathcal{V}^0(L) \mid \exists\, \Lambda_2 \in \mathcal{V}^2 \text{ such that } \Lambda \subset \Lambda_2, \Lambda_2 \notin \mathcal{L}(L)\}. \tag{4.54}$$

Recall we have the isomorphism $b : \mathbb{V} \to C$ defined in (2.5). Recall that the vertices of $\mathcal{L}_{2,1}$ correspond to vertex lattices of type 2, and an edge corresponds to a vertex lattice of type 0. Each vertex of $\mathcal{L}_{2,1}$ is contained in $q+1$ edges and each edge connects exactly two vertices. For $\mathbf{x} \in \mathbb{V}$ with $\mathrm{val}(\mathbf{x}) = 0$ and $\mathrm{Span}_F\{\mathbf{x}\}^\perp$ split, recall that $\mathcal{L}_{2,1}$ is the Bruhat-Tits tree of $\mathcal{Z}^{\mathrm{Pap}}(\mathbf{x}) \cong \mathcal{N}_{2,1}^{\mathrm{Pap}}$. Then $\mathbf{x}$ determines an embedding $\mathcal{L}_{2,1} \hookrightarrow \mathcal{L}$ defined as follows. First we send each vertex of $\mathcal{L}_{2,1}$ corresponding to a vertex lattice $\Lambda \subset \mathrm{Span}_F\{b(\mathbf{x})\}^\perp$ of type 2 to the vertex of $\mathcal{L}$ corresponding to the type 2 lattice $\Lambda \obot \mathrm{Span}\{b(\mathbf{x})\}$. An edge of $\mathcal{L}_{2,1}$ corresponding to a type zero lattice $\Lambda_0 \subset \mathrm{Span}_F\{b(\mathbf{x})\}^\perp$ is broken into two pieces evenly and sent to the union of the two edges in $\mathcal{L}$ joining the two vertices corresponding to $\Lambda \obot \mathrm{Span}\{b(\mathbf{x})\}$ and $\Lambda' \obot \mathrm{Span}\{b(\mathbf{x})\}$ where Λ and Λ' are the two type 2 lattices containing Λ_0.

4.10.2 Rank 1 case

Lemma 4.10.2. *A point $z \in \mathcal{N}_3^{\mathrm{Pap}}(k)$ is in $\mathcal{Z}^{\mathrm{Pap}}(\mathbf{x})(k)$ if and only if $b(\mathbf{x}) \in M(z)$.*

1. *Assume $\Lambda_0 \in \mathcal{V}^0$, then the superspecial point $\mathcal{N}_{\Lambda_0}(k)$ is in $\mathcal{Z}^{\mathrm{Pap}}(\mathbf{x})(k)$ if and only if $b(\mathbf{x}) \in \Lambda_0$.*

2. *Assume $\Lambda_2 \in \mathcal{V}^2$, then*

$$\mathcal{Z}^{\mathrm{Pap}}(\mathbf{x})(k) \cap \mathcal{N}_{\Lambda_2}(k) = \begin{cases} \mathcal{N}_{\Lambda_2}(k) & \text{if } b(\mathbf{x}) \in \Lambda_2^\sharp, \\ \text{a superspecial point in } \mathcal{N}_{\Lambda_2}(k) & \text{if } b(\mathbf{x}) \in \Lambda_2 \backslash \Lambda_2^\sharp, \\ \emptyset & \text{if } b(\mathbf{x}) \notin \Lambda_2. \end{cases}$$

Proof. By Dieudonné theory, $z \in \mathcal{Z}^{\mathrm{Pap}}(\mathbf{x})(k)$ if and only if $\mathbf{x}(M(\mathbb{Y})) \subset M(z)$ if and only if $b(\mathbf{x}) \in M(z)$ since e is a generator of $M(\mathbb{Y})$. For $z = \mathcal{N}_{\Lambda_0}(k)$ where $\Lambda_0 \in \mathcal{V}^0$, we have $M(z) = \Lambda_0 \otimes_{O_F} O_{\breve{F}}$. Hence (1) immediately follows.

Now we proceed to prove (2). If $b(\mathbf{x}) \in \Lambda^\sharp$, then (4.53) tells us that $z \in \mathcal{Z}^{\mathrm{Pap}}(\mathbf{x})(k)$ for any $z \in \mathcal{N}^\circ_{\Lambda_2}(k)$. The fact that $\Lambda_2^\sharp \subset \Lambda_0$ for any $\Lambda_0 \in \mathcal{L}^0$ contained in Λ_2 implies that $\mathcal{N}_{\Lambda_0}(k) \in \mathcal{Z}^{\mathrm{Pap}}(\mathbf{x})(k)$. So $\mathcal{N}_{\Lambda_2}(k) \subset \mathcal{Z}^{\mathrm{Pap}}(\mathbf{x})(k)$.

If $b(\mathbf{x}) \in \Lambda_2 \backslash \Lambda_2^\sharp$, then $\Lambda_0 \coloneqq \Lambda^\sharp + \mathrm{Span}\{b(\mathbf{x})\}$ is a type 0 lattice contained in Λ_2 and $\mathcal{N}_{\Lambda_0}(k) \in \mathcal{Z}^{\mathrm{Pap}}(\mathbf{x})(k)$. On the other hand, since $\tau(\Lambda_0 \otimes_{O_F} O_{\breve{F}}) = \Lambda_0 \otimes_{O_F} O_{\breve{F}}$, equation (4.53) tells us that $\mathcal{Z}^{\mathrm{Pap}}(\mathbf{x})$ does not contain any point in $\mathcal{N}^\circ_{\Lambda_2}(k)$.

If $b(\mathbf{x}) \notin \Lambda_2$, then $b(\mathbf{x}) \notin M(z)$ for any $z \in \mathcal{N}_{\Lambda_2}(k)$, hence $\mathcal{Z}^{\mathrm{Pap}}(\mathbf{x})(k) \cap \mathcal{N}_{\Lambda_2}(k) = \emptyset$. □

Corollary 4.10.3. *Let $L \subset \mathbb{V}$. Assume $z \in \mathcal{Z}^{\mathrm{Pap}}(L)(k)$ and $z \in \mathcal{N}_\Lambda(k)$ where $\Lambda \in \mathcal{V}^2$. Then $\mathcal{N}_\Lambda \subset \mathcal{Z}^{\mathrm{Pap}}(\pi L)$.*

Corollary 4.10.4. *Assume $\mathbf{x} \in \mathbb{V}$ and $\mathrm{v}(\mathbf{x}) > 0$. Assume $\mathcal{N}_\Lambda \subset \mathcal{Z}^{\mathrm{Pap}}(\mathbf{x})_{\mathrm{red}}$ where $\Lambda \in \mathcal{V}^2$, then either $\mathcal{N}_\Lambda \subset \mathcal{Z}^{\mathrm{Pap}}(\frac{1}{\pi}\mathbf{x})_{\mathrm{red}}$ or $\mathcal{N}_\Lambda \cap \mathcal{Z}^{\mathrm{Pap}}(\frac{1}{\pi}\mathbf{x})_{\mathrm{red}}$ is a unique superspecial point.*

Lemma 4.10.5. *For $L \subset \mathbb{V}$ a lattice of arbitrary rank, $\mathcal{Z}^{\mathrm{Pap}}(L)_{\mathrm{red}}$ is connected.*

Proof. Suppose $\mathcal{Z}^{\mathrm{Pap}}(L)_{\mathrm{red}}$ has two different connected components U_1 and U_2. Since $\mathrm{SU}(\mathbb{V})$ acts transitively on $\mathcal{L}$, we can find a $\mathbf{x} \in \mathbb{V}$ such that $\mathcal{Z}^{\mathrm{Pap}}(\mathbf{x}) \cong \mathcal{N}^{\mathrm{Pap}}_{2,1}$ (i.e. $\{\mathbf{x}\}^\perp$ is split) and $\mathcal{Z}^{\mathrm{Pap}}(\mathbf{x})_{\mathrm{red}} \cap U_i \neq \emptyset$ for $i = 1, 2$. Hence the reduced locus of

$$\mathcal{Z}^{\mathrm{Pap}}(L \oplus \mathrm{Span}\{\mathbf{x}\}) \cong \mathcal{Z}^{\mathrm{Pap}}_{2,1}(L')$$

is not connected where L' is the orthogonal projection of L onto $\{\mathbf{x}\}^\perp$. This contradicts the description of the reduced locus of special cycles in Chapter 3. □

Recall that for a lattice $L \subset \mathbb{V}$ (resp. $\mathbf{x} \in \mathbb{V}$), we have defined $\mathcal{V}(L)$ and $\mathcal{L}(L)$ (resp. $\mathcal{V}(\mathbf{x})$ and $\mathcal{L}(\mathbf{x})$) in Section 2.1.3.

Proposition 4.10.6. *Assume that* $\mathbf{x} \in \mathbb{V}$ *such that* $h(\mathbf{x}, \mathbf{x}) \neq 0$*. Then we have*

$$\mathcal{Z}^{\mathrm{Pap}}(\mathbf{x})_{\mathrm{red}} = \bigcup_{\Lambda \in \mathcal{V}(\mathbf{x})} \mathcal{N}_\Lambda,$$

where $\mathcal{V}(\mathbf{x})$ *is given as follows.*

1. *When* $\mathrm{v}(\mathbf{x}) = 0$ *and* $\mathrm{Span}_F\{\mathbf{x}\}^\perp$ *is non-split, there is a unique vertex lattice* $\Lambda_{\mathbf{x}} \in \mathcal{V}^0$ *containing* $b(\mathbf{x})$*. In this case* $\mathcal{V}(\mathbf{x}) = \{\Lambda_{\mathbf{x}}\}$*.*

2. *When* $\mathrm{v}(\mathbf{x}) = d$ *and* $\mathrm{Span}_F\{\mathbf{x}\}^\perp$ *is non-split, we have*

$$\mathcal{V}(\mathbf{x}) = \{\Lambda \in \mathcal{V} \mid d(\Lambda, \Lambda_{\mathbf{x}/\pi^d}) \leq d\}$$

where $\Lambda_{\mathbf{x}/\pi^d}$ *is as in (1).*

3. *When* $\mathrm{v}(\mathbf{x}) = 0$ *and* $\mathrm{Span}_F\{\mathbf{x}\}^\perp$ *is split,* $\mathcal{L}(\mathbf{x})$ *is the tree* $\mathcal{L}_{2,1}$*.*

4. *When* $\mathrm{v}(\mathbf{x}) = d$ *and* $\mathrm{Span}_F\{\mathbf{x}\}^\perp$ *is split, we have*

$$\mathcal{V}(\mathbf{x}) = \{\Lambda \in \mathcal{V} \mid d(\Lambda, \mathcal{L}(\mathbf{x}/\pi^d)) \leq d\}$$

where $\mathcal{L}(\mathbf{x}/\pi^d)$ *is as in (3).*

5. *When* $h(\mathbf{x}, \mathbf{x}) \notin O_{F_0}$*,* $\mathcal{V}(\mathbf{x})$ *is empty.*

Proof. Proof of (1): This is a direct consequence of Proposition 4.7.1 and the fact that $\mathcal{N}_{2,-1}^{\mathrm{Pap}}$ has only one reduced point, see [Shi22, Section 2] or [RSZ18, Section 8]. Alternatively since $\mathrm{Span}_F\{b(\mathbf{x})\}^\perp$ is non-split of dimension 2, it contains a unique self dual lattice Λ', then $\Lambda_{\mathbf{x}} \coloneqq \mathrm{Span}\{b(\mathbf{x})\} \oplus \Lambda'$ is the unique type 0 lattice containing $b(\mathbf{x})$.

Proof of (3): Applying Proposition 4.7.1,we see that $\mathcal{Z}^{\mathrm{Pap}}(\mathbf{x}) \cong \mathcal{N}_{2,1}^{\mathrm{Pap}}$ is the Drinfeld p-adic half space. The required properties of $\mathcal{L}(\mathbf{x})$ and $\mathcal{V}(\mathbf{x})$ follow.

Proof of (2): We prove this by induction. The case $d = 0$ is just (1). Now we assume

$d > 0$ and that the statement holds for $d-1$, i.e.

$$\mathcal{V}(\mathbf{x}/\pi) = \{\Lambda \in \mathcal{V} \mid d(\Lambda, \Lambda_{\mathbf{x}/\pi^d}) \leq d-1\}.$$

Then applying Corollary 4.10.3 to the lattice $L = \mathrm{Span}\{\mathbf{x}/\pi\}$ we have

$$\bigcup_{\Lambda \in \mathcal{V}^2,\ d(\Lambda, \Lambda_{\mathbf{x}/\pi^d}) \leq d} \mathcal{N}_\Lambda \subset \mathcal{Z}^{\mathrm{Pap}}(\mathbf{x})_{\mathrm{red}}.$$

Corollary 4.10.4 and the induction hypothesis imply that every $\Lambda_2 \in \mathcal{V}^2(\mathbf{x})$ satisfies $d(\Lambda, \Lambda_{\mathbf{x}/\pi^d}) \leq d$. By Lemma 4.10.5 there is no isolated $\Lambda_0 \in \mathcal{V}^0(\mathbf{x})$, i.e. every $\Lambda_0 \in \mathcal{V}^0(\mathbf{x})$ is contained in some $\Lambda_2 \in \mathcal{V}^2(\mathbf{x})$ if $\mathrm{v}(\mathbf{x}) > 0$. This finishes the proof of (2).

Similarly we can prove (4) by an induction on d, the case $d = 0$ is just (3).

(5) follows directly from Lemma 4.10.2. □

4.10.3 Rank 2 case

Proposition 4.10.7. *Assume that $L^\flat = \mathrm{Span}\{\mathbf{x}_1, \mathbf{x}_2\} \subset \mathbb{V}$ is integral of rank 2. Then*

$$\mathcal{Z}^{\mathrm{Pap}}(L^\flat)_{\mathrm{red}} = \bigcup_{\Lambda \in \mathcal{V}(L^\flat)} \mathcal{N}_\Lambda$$

is a finite union, where $\mathcal{V}(L^\flat)$ is the set of vertices of the tree $\mathcal{L}(L^\flat)$ described as follows.

1. *Assume $L^\flat \approx H_{2a+1}$ for some $a \in \mathbb{Z}_{\geq 0}$. Then $\mathcal{L}(L^\flat)$ is a ball centered at a vertex lattice of type 2 with radius $\frac{2a+1}{2}$.*

2. *Assume $L^\flat = \mathrm{Span}\{\pi^a \mathbf{x}_1, \pi^a \mathbf{x}_2\}$ where $\mathrm{v}(\mathbf{x}_1) = 0$, $\mathrm{v}(\mathbf{x}_2) \geq 0$ and $\mathrm{Span}_F\{\mathbf{x}_1\}^\perp$ is nonsplit. Then $\mathcal{L}(L^\flat)$ is a ball centered at a vertex lattice of type 0 with radius a.*

3. *Assume $L^\flat = \mathrm{Span}\{\pi^a \mathbf{x}_1, \pi^{a+r} \mathbf{x}_2\}$ where $\mathbf{x}_1 \perp \mathbf{x}_2$, $\mathrm{v}(\mathbf{x}_1) = \mathrm{v}(\mathbf{x}_2) = 0$, $r \geq 0$ and $\mathrm{Span}_F\{\mathbf{x}_1\}^\perp$ is split. Then*

$$\mathcal{L}(L^\flat) = \{\Lambda \in \mathcal{V} \mid d(\Lambda, \mathcal{L}(\pi^{-a} L^\flat)) \leq a\},$$

where

$$\mathcal{L}(\pi^{-a}L^\flat) = \{\Lambda \in \mathcal{L}(\mathbf{x}_1) \mid d(\Lambda, \Lambda_0) \le r\},$$

$\mathcal{L}(\mathbf{x}_1)$ *is described in (3) of Proposition 4.10.6 and* Λ_0 *is the unique type* 0 *vertex lattice containing* $\{\mathbf{x}_1, \mathbf{x}_2\}$.

Proof. As in the proof of Proposition 4.10.2, for a $\Lambda \in \mathcal{V}$, $\mathcal{N}_\Lambda \subset \mathcal{Z}^{\mathrm{Pap}}(L^\flat)_{\mathrm{red}}$ if and only if $\Lambda^\sharp$ contains $b(\mathbf{x}_1), b(\mathbf{x}_2)$.

We first prove (1) when $a = 0$. Suppose $\Lambda \in \mathcal{V}^2(L^\flat)$. Extend $\{b(\mathbf{x}_1), b(\mathbf{x}_2)\}$ to a basis $\{b(\mathbf{x}_1), b(\mathbf{x}_2), b_3\}$ of $\mathbb{V}$ with Gram matrix $H_1 \oplus \{-\epsilon\}$. Choose a basis $\{v_1, v_2, v_3\}$ of $\Lambda^\sharp$ with the same Gram matrix $H_1 \oplus \{-\epsilon\}$. Then $b(\mathbf{x}_i) \in \Lambda^\sharp$ $(i = 1, 2)$ by Lemma 4.10.2 and

$$b(\mathbf{x}_i) = a_{i1}v_1 + a_{i2}v_2 + a_{i3}v_3$$

where $a_{ij} \in O_F$ $(j = 1, 2, 3)$. The fact that $(b(\mathbf{x}_i), b(\mathbf{x}_j))_{1\le i,j\le 2} = T$ implies $a_{i3} \in \pi O_F$ for $i = 1, 2$ and $(a_{ij})_{1\le i,j\le 2}$ is in $\mathrm{GL}_2(O_F)$. This guarantees that $L^\flat$ is a direct summand of $\Lambda^\sharp$ by Gram-Schmit process. Hence $\Lambda^\sharp$ is in fact the lattice $\mathrm{Span}_{O_F}\{b(\mathbf{x}_1), b(\mathbf{x}_2), b_3\}$. The fact that all $\Lambda_0 \in \mathcal{V}^0(L^\flat)$ are in Λ follows from Lemma 4.10.5.

When $a = 0$, (2) follows from the fact that $\mathcal{Z}^{\mathrm{Pap}}(\mathbf{x}_1) = \mathcal{N}^{\mathrm{Pap}}_{2,-1}$ (by Proposition 4.7.1) and $\mathcal{Z}^{\mathrm{Pap}}(L^\flat)_{\mathrm{red}} = \mathcal{Z}^{\mathrm{Pap}}(\mathbf{x}_1)_{\mathrm{red}}$ is a unique superspecial point. Similarly when $a = 0$, (3) follows from the fact that $\mathcal{Z}^{\mathrm{Pap}}(\mathbf{x}_1) = \mathcal{N}^{\mathrm{Pap}}_{2,1}$ and Corollary 3.2.11.

Now we prove (1), (2) and (3) for general a. First of all, $\mathcal{L}(L^\flat) = \mathcal{L}(\pi^a\mathbf{x}_1) \cap \mathcal{L}(\pi^a\mathbf{x}_2)$ by definition. By Corollary 4.10.3 we have

$$\{\Lambda \in \mathcal{V} \mid d(\Lambda, \mathcal{L}(\pi^{-a}L^\flat)) \le a\} \subset \mathcal{L}(\pi^a\mathbf{x}_1) \cap \mathcal{L}(\pi^a\mathbf{x}_2).$$

Notice that for a sub-tree $\mathcal{L}'$ of a tree $\mathcal{L}$ and a vertex $x \in \mathcal{L} \setminus \mathcal{L}'$, there is a unique geodesic segment joining x with $\mathcal{L}'$. Given $\Lambda \in \mathcal{L}(L^\flat) = \mathcal{L}(\pi^a\mathbf{x}_1) \cap \mathcal{L}(\pi^a\mathbf{x}_2)$, let γ be the unique geodesic segment joining Λ with $\mathcal{L}(\pi^{-a}L^\flat)$. Assume that γ intersects $\mathcal{L}(\pi^{-a}L^\flat)$ at $\Lambda(L^\flat)$. Since $\mathcal{L}(\pi^{-a}L^\flat) = \mathcal{L}(\mathbf{x}_1) \cap \mathcal{L}(\mathbf{x}_2)$, γ necessarily intersects both $\mathcal{L}(\mathbf{x}_1)$ and $\mathcal{L}(\mathbf{x}_2)$.

Without loss of generality we assume that γ intersects $\mathcal{L}(\mathbf{x}_1)$ at $\Lambda(\mathbf{x}_1)$ first. Hence the intersection of γ with $\mathcal{L}(\mathbf{x}_2)$ is $\Lambda(L^\flat)$ and

$$d(\Lambda, \Lambda(\mathbf{x}_1)) = d(\Lambda, \mathcal{L}(\mathbf{x}_1)) \leq d(\Lambda, \Lambda(L^\flat)) = d(\Lambda, \mathcal{L}(\mathbf{x}_2)).$$

Now by Proposition 4.10.6, we have

$$d(\Lambda, \mathcal{L}(\mathbf{x}_1)) \leq a,\ d(\Lambda, \mathcal{L}(\mathbf{x}_2)) \leq a.$$

Hence $d(\Lambda, \mathcal{L}(\pi^{-a}L^\flat)) \leq a$. This shows that

$$\{\Lambda \in \mathcal{V} \mid d(\Lambda, \mathcal{L}(\pi^{-a}L^\flat)) \leq a\} = \mathcal{L}(\pi^a\mathbf{x}_1) \cap \mathcal{L}(\pi^a\mathbf{x}_2).$$

The general case of (1), (2) and (3) follows from the above equation and the case $a = 0$.

Notice that (1), (2) and (3) have covered all possibilities of $L^\flat$ due to the classification of hermitian lattices. Notice that in every case $\mathcal{V}(L^\flat)$ is finite. This finishes the proof of the proposition. $\square$

Definition 4.10.8. *Assume that $L^\flat$ is an integral lattice of rank 2 in $\mathbb{V}$. Define $\mathcal{S}(L^\flat)$, the skeleton of $\mathcal{L}(L^\flat)$, as follows. If the fundamental invariant of $L^\flat$ is $(2a, b)$ ($b \geq 2a$), define $\mathcal{S}(L^\flat) := \mathcal{L}(\pi^{-a}L^\flat)$. If the fundamental invariant of $L^\flat$ is $(2a+1, 2a+1)$, define $\mathcal{S}(L^\flat) := \emptyset$.*

Remark 4.10.9. *The skeleton $\mathcal{S}(L^\flat)$ is isomorphic to a ball in the Bruhat-Tits tree of $\mathcal{N}_{2,\pm1}^{\mathrm{Pap}}$.*

Corollary 4.10.10. *For each $\Lambda_2 \in \mathcal{V}^2(L^\flat)$ not on the skeleton $\mathcal{S}(L^\flat)$, one can find $\Lambda_0 \in \mathcal{V}^0(L^\flat)$ such that Λ_2 has the largest distance to the boundary $\mathcal{B}(L^\flat)$ of $\mathcal{L}(L^\flat)$ among all type 2 lattices in $\mathcal{V}^2(L^\flat)$ containing Λ_0.*

Proof. Assmue the fundamental invariant of $L^\flat$ is $(2a, b)$ or $(2a+1, 2a+1)$. Define $M^\flat := \pi^{-a}L^\flat$. Let b be the unique integer such that $\Lambda_2 \in \mathcal{L}(\pi^b M^\flat) \setminus \mathcal{L}(\pi^{b-1}M^\flat)$. Choose any

$\Lambda_0 \in \mathcal{B}(\pi^b M^\flat)$ such that $\Lambda_0 \subset \Lambda$. Then by Proposition 4.10.7, Λ_0 satisfies the assumption of the corollary. $\square$

4.10.4 The Krämer model

For $\Lambda \in \mathcal{V}^2$, let $\tilde{\mathcal{N}}_\Lambda$ be the strict transform of $\mathcal{N}_\Lambda$ under the blow-up $\mathcal{N}^{\mathrm{Kra}} \to \mathcal{N}^{\mathrm{Pap}}$. Since the strict transform of a regular curve along any of its closed point is an isomorphism, we know $\tilde{\mathcal{N}}_\Lambda \cong \mathbb{P}^1$.

Lemma 4.10.11. *For $\Lambda \neq \Lambda' \in \mathcal{V}^2$, $\tilde{\mathcal{N}}_\Lambda$ and $\tilde{\mathcal{N}}_{\Lambda'}$ do not intersect.*

Proof. If $\mathcal{N}_\Lambda$ and $\mathcal{N}_{\Lambda'}$ do not intersect in $\mathcal{N}^{\mathrm{Pap}}$, then obviously $\tilde{\mathcal{N}}_\Lambda$ and $\tilde{\mathcal{N}}_{\Lambda'}$ do not intersect. Without loss of generality we can assume $\Lambda = \mathrm{Span}\{e_1, e_2, e_3\}$ and $\Lambda' = \mathrm{Span}\{\pi^{-1}e_1, \pi e_2, e_3\}$ where the Gram matrix of $\{e_1, e_2, e_3\}$ is $\mathrm{Diag}(H, \epsilon)$. Take $\mathbf{x}_0 = e_3$. Then by Proposition 4.10.7, both $\tilde{\mathcal{N}}_\Lambda$ and $\tilde{\mathcal{N}}_{\Lambda'}$ are in $\tilde{\mathcal{Z}}(\mathbf{x}_0) \cong \mathcal{N}^{\mathrm{Kra}}_{2,1}$. Now by Lemma 3.4.4, $\tilde{\mathcal{N}}_\Lambda$ and $\tilde{\mathcal{N}}_{\Lambda'}$ do not intersect. $\square$

Lemma 4.10.12. *Let $\Lambda \in \mathcal{V}^2$ and $\Lambda_0 \in \mathcal{V}^0$. When $\Lambda_0 \subset \Lambda$, $\tilde{\mathcal{N}}_\Lambda$ intersects properly with Exc_{Λ_0} and*

$$\chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{O}_{\tilde{\mathcal{N}}_\Lambda} \otimes \mathcal{O}_{\mathrm{Exc}_{\Lambda_0}}) = 1. \tag{4.55}$$

When Λ_0 is not contained in Λ, $\tilde{\mathcal{N}}_\Lambda$ does not intersect with Exc_{Λ_0}.

Proof. First assume $\Lambda_0 \subset \Lambda$. Since $\tilde{\mathcal{N}}_\Lambda$ is a strict transformation of a curve, it intersects the exceptional divisor properly. Let $\mathbf{x}_0$ be as in the proof of Lemma 4.10.11. Then $\tilde{\mathcal{N}}_\Lambda$ is in $\tilde{\mathcal{Z}}(\mathbf{x}_0) \cong \mathcal{N}^{\mathrm{Kra}}_{2,1}$.

$$\begin{aligned}
\chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{O}_{\tilde{\mathcal{N}}_\Lambda} \otimes \mathcal{O}_{\mathrm{Exc}_{\Lambda_0}}) &= \chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{O}_{\tilde{\mathcal{N}}_\Lambda} \otimes_{\mathcal{O}_{\tilde{\mathcal{Z}}(\mathbf{x}_0)}} \mathcal{O}_{\tilde{\mathcal{Z}}(\mathbf{x}_0)} \otimes \mathcal{O}_{\mathrm{Exc}_{\Lambda_0}}) \\
&= \chi(\tilde{\mathcal{Z}}(\mathbf{x}_0), \mathcal{O}_{\tilde{\mathcal{N}}_\Lambda} \otimes_{\mathcal{O}_{\tilde{\mathcal{Z}}(\mathbf{x}_0)}} \mathcal{O}_{\mathrm{Exc}'}).
\end{aligned}$$

Here $\mathrm{Exc}' \cong \mathbb{P}^1_k$ is the exceptional divisor on $\tilde{\mathcal{Z}}(\mathbf{x}_0)$ corresponding to the rank 2 self-dual lattice

$$\Lambda' = \{v \in \Lambda_0 \mid v \perp \mathbf{x}_0\}.$$

By Lemma 3.4.3, we know $\chi(\tilde{\mathcal{Z}}(\mathbf{x}_0), \mathcal{O}_{\tilde{\mathcal{N}_\Lambda}} \otimes_{\mathcal{O}_{\tilde{\mathcal{Z}}(\mathbf{x}_0)}} \mathcal{O}_{\mathrm{Exc}'}) = 1$. When Λ_0 is not contained in Λ, the superspecial point $\mathcal{N}_{\Lambda_0}(k)$ is not contained in $\mathcal{N}_\Lambda$, hence $\tilde{\mathcal{N}_\Lambda}$ does not intersect with Exc_{Λ_0}. □

4.11 Intersection of vertical components and special divisors

In this section we study the intersection of $\tilde{\mathcal{N}}_\Lambda$ and special divisors. The main result is Theorem 4.11.2. To proceed we first study the decomposition of ${}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L^\flat)$ when $\mathrm{v}(L^\flat) = 0$. Since n is odd, we can without loss of generality assume that $\chi(\mathbb{V}) = \chi(C) = 1$. In the rest of the thesis, we identify $\mathbb{V}$ with C by the isomorphism b defined in (2.5).

4.11.1 Decomposition of ${}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L^\flat)$

Let $L^\flat = \mathrm{Span}\{\mathbf{x}_1, \mathbf{x}_2\}$ where $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{V}$ are linearly independent and the hermitian form restricted to L is non-degenerate.

Lemma 4.11.1. ${}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L^\flat) = [\mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}_1)} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}_2)}] \in K_0(\mathcal{N}^{\mathrm{Kra}})$ *is in fact in* $\mathrm{F}^2 K_0(\mathcal{N}^{\mathrm{Kra}})$. *Moreover we have the decomposition in* $\mathrm{Gr}^2 K_0^{\mathcal{Z}^{\mathrm{Kra}}(L^\flat)}(\mathcal{N}^{\mathrm{Kra}})$

$$ {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L^\flat) = \mathcal{Z}^{\mathrm{Kra}}(L^\flat)_h + {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L^\flat)_v. \tag{4.56} $$

where $\mathcal{Z}^{\mathrm{Kra}}(L^\flat)_h$ *is described in Theorem 4.8.2 and* ${}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L)_v \in \mathrm{Gr}^2 K_0^{\mathcal{Z}^{\mathrm{Kra}}(L^\flat)_v}(\mathcal{N}^{\mathrm{Kra}})$.

Proof. By Lemma 2.2.3, $\mathcal{Z}^{\mathrm{Kra}}(L^\flat)$ is Noetherian and has a decomposition

$$ \mathcal{Z}^{\mathrm{Kra}}(L^\flat) = \mathcal{Z}^{\mathrm{Kra}}(L^\flat)_h \cup \mathcal{Z}^{\mathrm{Kra}}(L^\flat)_v. $$

Expressing $\mathcal{Z}(\mathbf{x}_i)$ $(i = 1, 2)$ as in (4.31) and applying Propositions 4.6.2, 4.6.3 and Lemma 4.6.4, ${}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L^\flat)$ equals

$$ [\mathcal{O}_{\tilde{\mathcal{Z}}(\mathbf{x}_1)} \otimes^{\mathbb{L}} \mathcal{O}_{\tilde{\mathcal{Z}}(\mathbf{x}_2)}] + \sum_{\Lambda_0 \in \mathcal{V}^0(L^\flat)} (2m_{\Lambda_0}(\mathbf{x}_1)m_{\Lambda_0}(\mathbf{x}_2) + m_{\Lambda_0}(\mathbf{x}_1) + m_{\Lambda_0}(\mathbf{x}_2))H_{\Lambda_0}. $$

$\mathcal{Z}^{\mathrm{Kra}}(L^\flat)_h$ is contained in $\tilde{\mathcal{Z}}(\mathbf{x}_1)\cap\tilde{\mathcal{Z}}(\mathbf{x}_2)$ and has dimension 1 by Theorem 4.8.2. $\tilde{\mathcal{Z}}(\mathbf{x}_1)\cap\tilde{\mathcal{Z}}(\mathbf{x}_2)_v$ also has dimension 1 as it is supported on the reduced locus of $\mathcal{N}^{\mathrm{Kra}}$ by Lemma 2.2.4 and does not contain any exceptional divisor Exc_{Λ_0}. Hence

$$[\mathcal{O}_{\tilde{\mathcal{Z}}(\mathbf{x}_1)}\otimes^{\mathbb{L}}\mathcal{O}_{\tilde{\mathcal{Z}}(\mathbf{x}_2)}]=[\mathcal{O}_{\tilde{\mathcal{Z}}(\mathbf{x}_1)\cap\tilde{\mathcal{Z}}(\mathbf{x}_2)}]\in \mathrm{F}^2K_0(\mathcal{N}^{\mathrm{Kra}}), \tag{4.57}$$

see for example [Zha21, Lemma B.2]. Hence we know that ${}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L^\flat)\in \mathrm{F}^2K_0(\mathcal{N}^{\mathrm{Kra}})$. The desired decomposition then follows from Theorem 4.8.2. □

By Lemma 4.11.1 and (2.12) we know that ${}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L^\flat)_v\in K_0'(Y)$ where we can take Y to be the reduced locus of $\mathcal{N}^{\mathrm{Kra}}$. By the Bruhat-Tits stratification of $\mathcal{N}^{\mathrm{Kra}}$ and the fact that $\mathrm{Gr}^1K_0^{\mathrm{Exc}_{\Lambda_0}}(\mathcal{N}^{\mathrm{Kra}})\cong \mathrm{CH}^1(\mathrm{Exc}_{\Lambda_0})$ is generated by H_{Λ_0}, we have the following decomposition in $\mathrm{Gr}^2K_0(\mathcal{N}^{\mathrm{Kra}})$:

$${}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L^\flat)_v=\sum_{\Lambda_2\in\mathcal{V}^2(L^\flat)} m(\Lambda_2,L^\flat)[\mathcal{O}_{\tilde{\mathcal{N}}_{\Lambda_2}}]+\sum_{\Lambda_0\in\mathcal{V}^0(L^\flat)} m(\Lambda_0,L^\flat)H_{\Lambda_0}. \tag{4.58}$$

We will determine the multiplicities $m(\Lambda_2,L^\flat)$ and $m(\Lambda_0,L^\flat)$ when $\mathrm{val}(L^\flat)=0$ in this section and deal with the general case in Section 4.12.

Now assume $L^\flat=\mathrm{Span}\{\mathbf{x}_1,\mathbf{x}_2\}$ with Gram matrix $\mathrm{Diag}(u_1,u_2(-\pi_0)^n)$ with $u_1,u_2\in O_{F_0}^\times$. Applying Proposition 4.6.2 to $\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}_1)$, we find

$${}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L^\flat)=[\mathcal{O}_{\tilde{\mathcal{Z}}(\mathbf{x}_1)}\otimes^{\mathbb{L}}\mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}_2)}]+\sum_{\Lambda_0\in\mathcal{V}^0(\mathbf{x}_1)}[\mathcal{O}_{\mathrm{Exc}_{\Lambda_0}}\otimes^{\mathbb{L}}\mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}_2)}].$$

By Proposition 4.7.1, we know the intersection $\tilde{\mathcal{Z}}(\mathbf{x}_1)\cap\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}_2)$ is proper and is isomorphic to $\mathcal{Z}^{\mathrm{Kra}}_{2,\chi(u_1)}(\mathbf{x}_2)$. Combining this with Corollary 4.6.5 we obtain

$${}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L^\flat)=i_*({}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}_{2,\chi(u_1)}(\mathbf{x}_2))-\sum_{\Lambda_0\in\mathcal{V}^0(L^\flat)}H_{\Lambda_0}. \tag{4.59}$$

where i_* is the map $\mathrm{Gr}^1K_0(\mathcal{N}^{\mathrm{Kra}}_{2,\chi(u_1)})\to \mathrm{Gr}^2K_0(\mathcal{N}^{\mathrm{Kra}}_{3,1})$ induced by the closed immersion $i:\mathcal{N}^{\mathrm{Kra}}_{2,\chi(u_1)}\to\mathcal{N}^{\mathrm{Kra}}_{3,1}$. Equation (4.59) reduces the problem of decomposing ${}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L^\flat)$ in this case to [Shi22, Theorem 4.5] and Theorem 3.3.1. We do not make the effort to write the complete result down, but instead look at two basic examples.

Let us begin by the case when $L^\flat$ is unimodular. By (4.59) and either [Shi22, Theorem 4.5] or Theorem 3.3.1, we have

$$^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L^\flat) = [\mathcal{O}_{\tilde{\mathcal{Z}}(L^\flat)^\circ}] \tag{4.60}$$

in the notation of Theorem 4.8.2.

Next consider $L^\flat = \mathrm{Span}\{\mathbf{x}_1, \mathbf{x}_2\}$ with Gram matrix $\mathrm{Diag}(1, -u\pi_0)$ where $u \in O_{F_0}^\times$. Then $\mathrm{Span}\{\mathbf{x}_1\}^\perp$ is split and $\tilde{\mathcal{Z}}(\mathbf{x}_1) \cong \mathcal{N}_{2,1}^{\mathrm{Kra}}$. By Proposition 4.10.7 (3), $\mathcal{V}^2(L^\flat)$ consists of two adjacent lattices Λ and Λ'. Moreover by Theorem 3.3.1 and (4.59), we have

$$^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L^\flat)_v = [\mathcal{O}_{\tilde{\mathcal{N}}_\Lambda}] + [\mathcal{O}_{\tilde{\mathcal{N}}_{\Lambda'}}] + H_{\Lambda\cap\Lambda'}. \tag{4.61}$$

4.11.2 The intersection number

Assume $\Lambda \in \mathcal{V}^2$. For $\mathbf{x} \in \mathbb{V}\backslash\{0\}$, define

$$\mathrm{Int}_\Lambda(\mathbf{x}) \coloneqq \chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{O}_{\tilde{\mathcal{N}}_\Lambda} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x})}). \tag{4.62}$$

In this subsection we prove the following theorem.

Theorem 4.11.2. *Let $\Lambda \in \mathcal{V}^2$ and $\mathbf{x} \in \mathbb{V}\backslash\{0\}$. Then*

$$\mathrm{Int}_\Lambda(\mathbf{x}) = 1_\Lambda(\mathbf{x})$$

where 1_Λ is the characteristic function of $\Lambda \subset \mathbb{V}$.

Corollary 4.11.3. *Assume that $\Lambda_0 \in \mathcal{L}_0$ and $\Lambda \in \mathcal{L}_2$ such that $\Lambda_0 \subset \Lambda$. Then for any $y_0 \in \Lambda_0 \setminus \pi\Lambda_0$ such that $y_0^\perp$ is nonsplit, we have*

$$\chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{O}_{\tilde{\mathcal{N}}_\Lambda} \otimes^{\mathbb{L}} \mathcal{O}_{\tilde{\mathcal{Z}}(y_0)}) = 0.$$

Proof. By Proposition 4.6.2, we know

$$\mathcal{Z}^{\mathrm{Kra}}(y_0) = \tilde{\mathcal{Z}}(y_0) + \mathrm{Exc}_{\Lambda_0}.$$

Now the corollary follows immediately from Theorem 4.11.2 and Lemma 4.10.12. $\square$

Proof of Theorem 4.11.2: We consider three different cases. First if $x \notin \Lambda$ or $\mathrm{v}(\mathbf{x}) < 0$, then by Lemma 4.10.2, $\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}) \cap \tilde{\mathcal{N}}_\Lambda = \emptyset$ hence $\mathrm{Int}_\Lambda(\mathbf{x}) = 0$. From now on we assume $x \in \Lambda$ and $\mathrm{v}(\mathbf{x}) \geq 0$. Write $\mathbf{x} = \mathbf{x}_0 \pi^n$ with $\mathbf{x}_0 \in \Lambda \backslash \pi\Lambda$ and $n \geq 0$.

Case 1: First we assume $\mathbf{x}_0 \in \Lambda \backslash \Lambda^\sharp$. Choose a basis $\{e_1', e_2', e_3'\}$ of Λ with Gram matrix $H_{3,1}^1$ such that

$$\mathbf{x}_0 = x e_1' + y e_2' + z e_3'.$$

Then one of x and y is in $O_F^\times$ as $\Lambda^\sharp = \mathrm{Span}\{\pi e_1', \pi e_2', e_3\}$. Apparently the equation

$$2u - v\bar{v} = h(\mathbf{x}_0, \mathbf{x}_0)$$

has a solution $(u, v) \in O_{F_0}^2$ with $u \in O_{F_0}^\times$. Now according to Lemma 4.5.2, we can find a matrix $g \in \mathrm{U}(H_{3,1}^1)(O_{F_0})$ such that

$$g \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \pi u \\ 1 \\ v \end{pmatrix}.$$

Now replace the basis $\{e_1', e_2', e_3'\}$ by $\{e_1, e_2, e_3\} = \{e_1', e_2', e_3'\} g^{-1}$, we have

$$\mathbf{x}_0 = \pi u e_1 + e_2 + v e_3$$

where $u \in O_{F_0}^\times$, $v \in O_F$.

Define

$$f_1 = \frac{1}{\pi} u^{-1} e_2, \ f_2 = \pi u e_1, \ f_3 = e_3. \tag{4.63}$$

Then $\{f_1, f_2, f_3\}$ has also Gram matrix $H^1_{3,1}$ and $\Lambda' \coloneqq \mathrm{Span}\{f_1, f_2, f_3\}$ is a type 2 lattice adjacent to Λ with $\Lambda_c = \Lambda \cap \Lambda' = \mathrm{Span}\{\pi e_1, e_2, e_3\}$ is a type 0 lattice. Now in terms of the basis $\{f_1, f_2, f_3\}$ we have

$$\mathbf{x} = \pi^n(\pi u f_1 + f_2 + v f_3).$$

Define $\theta \in \mathrm{U}(\mathbb{V})$ by taking the basis $\{e_1, e_2, e_3\}$ to $\{f_1, f_2, f_3\}$. Then

$$\theta(\mathbf{x}) = \mathbf{x}, \quad \theta(\Lambda) = \Lambda'.$$

In particular $\theta(\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x})) = \mathcal{Z}^{\mathrm{Kra}}(\mathbf{x})$ and

$$\mathrm{Int}_{\Lambda'}(\mathbf{x}) = \mathrm{Int}_{\Lambda}(\mathbf{x}). \tag{4.64}$$

Now let

$$\mathbf{y}_0 = e_3, \quad \mathbf{y}_1 = \pi(-\pi u e_1 + e_2),$$

$L^\flat = \mathrm{Span}\{\mathbf{y}_0, \mathbf{y}_1\}$, and $L = \mathrm{Span}\{\mathbf{y}_0, \mathbf{y}_1, \mathbf{x}\}$. Then by (4.61) and Theorem 4.8.2 we have

$$^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L^\flat) = [\mathcal{O}_{\tilde{\mathcal{N}}_\Lambda}] + [\mathcal{O}_{\tilde{\mathcal{N}}_{\Lambda'}}] + H_{\Lambda_c} + [\mathcal{O}_{\tilde{\mathcal{Z}}(M^\flat)}]. \tag{4.65}$$

where $\tilde{\mathcal{Z}}(M^\flat)$ is the quasi canonical-lifting cycle of the lattice

$$M^\flat \coloneqq \mathrm{Span}\{e_3, -\pi u e_1 + e_2\}.$$

Combining with (4.64), we have

$$\mathrm{Int}(L) = 2 \cdot \mathrm{Int}_\Lambda(\mathbf{x}) + \chi(\mathcal{N}^{\mathrm{Kra}}, {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}) \cdot H_{\Lambda_c}) + \chi(\mathcal{N}^{\mathrm{Kra}}, {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}) \cdot [\mathcal{O}_{\tilde{\mathcal{Z}}(M^\flat)}]). \tag{4.66}$$

Let $\mathbf{x}' = \pi^n(\pi u e_1 + e_2) = \mathbf{x} - \pi^n v e_3$. Then we have

$$\begin{aligned}
\mathrm{Int}(L) &= \chi(\mathcal{N}^{\mathrm{Kra}}, {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(\mathbf{y}_0) \cdot {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}) \cdot {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(\mathbf{y}_1)) \\
&= \chi(\mathcal{N}^{\mathrm{Kra}}, {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(\mathbf{y}_0) \cdot {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}') \cdot {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(\mathbf{y}_1)) \\
&= \chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{O}_{\tilde{\mathcal{Z}}(\mathbf{y}_0)} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}')} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{y}_1)}) \\
&\quad + \sum_{\Lambda_0 \in \mathcal{V}^0(L)} \chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{O}_{\mathrm{Exc}_{\Lambda_0}} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}')} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{y}_1)})
\end{aligned}$$

where we have used linear invariance ([How19, Corollary D]) and Proposition 4.6.2. Notice that the Gram matrix of $\{\mathbf{x}', \mathbf{y}_1\}$ is $\mathrm{Diag}(2u(-\pi_0)^n, -2u\pi_0)$. By Proposition 4.7.1 and Theorem 3.4.1,

$$\chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{O}_{\tilde{\mathcal{Z}}(\mathbf{y}_0)} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}')} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{y}_1)}) = \begin{cases} 1 & \text{if } n = 0, \\ 1 + n - 2q & \text{if } n \geq 1. \end{cases}$$

By Corollary 4.6.6 and Lemma 3.2.14, we know that

$$\sum_{\Lambda_0 \in \mathcal{V}^0(L)} \chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{O}_{\mathrm{Exc}_{\Lambda_0}} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}')} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{y}_1)}) = |\mathcal{V}^0(L)| = \begin{cases} 1 & \text{if } n = 0, \\ 2q + 1 & \text{if } n \geq 1. \end{cases}$$

Combining the above two equations we know that

$$\chi(\mathcal{N}^{\mathrm{Kra}}, {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}) \cdot {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L^\flat)) = n + 2.$$

On the other hand, by Corollary 4.6.8,

$$\chi(\mathcal{N}^{\mathrm{Kra}}, H_{\Lambda_0} \cdot {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x})) = -1.$$

By [Gro86, Proposition 3.3]

$$\chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{O}_{\tilde{\mathcal{Z}}(M^\flat)} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x})}) = n + 1.$$

Hence we obtain by (4.66)

$$\mathrm{Int}_\Lambda(\mathbf{x}) = 1. \tag{4.67}$$

Case 2: Now we Assume $\mathbf{x}_0 \in \Lambda^\sharp \backslash \pi\Lambda$. As in the proof of the previous case, we can find a basis $\{e_1, e_2, e_3\}$ of Λ with Gram matrix $H^1_{3,1}$ by Lemma 4.5.4 such that

$$\mathbf{x} = \pi^n(ue_3 + \pi e_1).$$

where $u \in O^\times_{F_0}$. Define

$$\Lambda' = \mathrm{Span}\{\pi e_1, \frac{1}{\pi}e_2, e_3\},\ \Lambda_c = \Lambda \cap \Lambda',$$

then $\mathbf{x}_0 \in \Lambda' \backslash \Lambda'^\sharp$. Also define

$$\mathbf{y}_0 = e_3,\ \mathbf{y}_1 = \pi(-\pi e_1 + e_2),$$

and $L^\flat \coloneqq \mathrm{Span}\{\mathbf{y}_0, \mathbf{y}_1\}$. Then by Theorem 4.8.2 and (4.61) we have

$${}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L^\flat) = [\mathcal{O}_{\tilde{\mathcal{N}}_\Lambda}] + [\mathcal{O}_{\tilde{\mathcal{N}}_{\Lambda'}}] + H_{\Lambda_c} + [\mathcal{O}_{\tilde{\mathcal{Z}}(M^\flat)}], \tag{4.68}$$

where $\tilde{\mathcal{Z}}(M^\flat)$ is the quasi-canonical lifting cycle of the lattice

$$M^\flat \coloneqq \mathrm{Span}\{e_3, -\pi e_1 + e_2\}.$$

Let $\mathbf{x}' \coloneqq \pi^{n+1}e_1 = \mathbf{x} - \pi^n u e_3$, then we have

$$\begin{aligned}\chi(\mathcal{N}^{\mathrm{Kra}}, {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}) \cdot {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L^\flat)) =& \chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{O}_{\tilde{\mathcal{Z}}(\mathbf{y}_0)} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}')} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{y}_1)}) \\ &+ \sum_{\Lambda_0 \in \mathcal{V}^0(L)} \chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{O}_{\mathrm{Exc}_{\Lambda_0}} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}')} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{y}_1)}).\end{aligned}$$

Notice that the Gram matrix of $\{\mathbf{x}', \mathbf{y}_1\}$ is equivalent to H_1 when $n = 0$, and to $\mathrm{Diag}(u_1\pi_0^n, u_2\pi_0)$ for some $u_1, u_2 \in O^\times_{F_0}$ when $n \geq 1$. Hence by Proposition 4.7.1 and Theorem 3.4.1, we know that

$$\chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{O}_{\tilde{\mathcal{Z}}(\mathbf{y}_0)} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}')} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{y}_1)}) = \begin{cases} -(q-1) & \text{if } n = 0, \\ 1 + n - 2q & \text{if } n \geq 1. \end{cases}$$

By Corollary 4.6.6 and Lemmas 3.2.14 and 3.2.15, we know that

$$\sum_{\Lambda_0 \in \mathcal{V}^0(L)} \chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{O}_{\mathrm{Exc}_{\Lambda_0}} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}')} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{y}_1)}) = |\mathcal{V}^0(L)| = \begin{cases} q+1 & \text{if } n = 0, \\ 2q+1 & \text{if } n \geq 1. \end{cases}$$

Hence we know that

$$\chi(\mathcal{N}^{\mathrm{Kra}}, {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}) \cdot {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L^\flat)) = n + 2.$$

On the other hand, by Corollary 4.6.8,

$$\chi(\mathcal{N}^{\mathrm{Kra}}, H_{\Lambda_0} \cdot {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x})) = -1.$$

By [Gro86, Proposition 3.3]

$$\chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{O}_{\tilde{\mathcal{Z}}(M^\flat)} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x})}) = n + 1.$$

Since $\mathbf{x} \in \Lambda' \backslash \Lambda'^{\sharp}$, by the previous case we also have

$$\mathrm{Int}_{\Lambda'}(\mathbf{x}) = 1.$$

Combining all above, we have by (4.68)

$$\mathrm{Int}_{\Lambda}(\mathbf{x}) = 1. \tag{4.69}$$

This finishes the proof of Theorem 4.11.2. $\square$

4.12 Proof of the modified Kudla-Rapoport conjecture for the case $n = 3$

First, we need some preparation.

Proposition 4.12.1. *Assume that $L \subset \mathbb{V}$ has a Gram matrix $T = \mathrm{Diag}(u_1, u_2(-\pi_0)^b, u_3(-\pi_0)^c)$ with $u_i \in O_{F_0}^\times$ and $0 \leq b \leq c$. Then*

$$\mathrm{Int}(L) = \partial\mathrm{Den}(L).$$

Moreover, for every decomposition $L = L^\flat \oplus \mathrm{Span}\{\mathbf{x}\}$, we have

$$\mathrm{Int}(L)^{(2)} = \partial\mathrm{Den}(L)^{(2)}.$$

Proof. Fix a basis $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ of L such that the Gram matrix of $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is $T = \mathrm{Diag}(u_1, T_2)$ where $u_1 \in O_{F_0}^\times$ and $T_2 \in \mathrm{Herm}_2(O_F)$. Let $u_1^{-1} \cdot L$ be a lattice represented by $u_1^{-1} \cdot T$. Since $\mathrm{Int}(u_1^{-1} \cdot L) = \mathrm{Int}(L)$ and $\partial\mathrm{Den}(u_1^{-1} \cdot L) = \partial\mathrm{Den}(L)$, we may assume $u_1 = 1$. Let $L^\flat = \mathrm{Span}\{\mathbf{x}_2, \mathbf{x}_3\}$. According to Propositions 4.7.1, 4.6.2, and Corollary 4.6.6, we have

$$\begin{aligned}\mathrm{Int}(L) - \mathrm{Int}(L^\flat) &= \chi(\mathcal{N}^{\mathrm{Kra}}, {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}_1) \cdot {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L^\flat)) - \chi(\mathcal{N}^{\mathrm{Kra}}, {}^{\mathbb{L}}\tilde{\mathcal{Z}}(\mathbf{x}_1) \cdot {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L^\flat)) \qquad (4.70)\\ &= \sum_{\Lambda_0 \in \mathcal{V}^0(L)} \chi(\mathcal{N}^{\mathrm{Kra}}, [\mathcal{O}_{\mathrm{Exc}_{\Lambda_0}}] \cdot {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L^\flat))\\ &= |\{\mathcal{V}^0(L)\}|.\end{aligned}$$

Now the result we want follows by comparing (4.70) with (4.22), and the identity $\mathrm{Int}(L^\flat) = \partial\mathrm{Den}(L^\flat)$ proved in [Shi22, Theorem 1.3] and Theorem 3.0.1. (2) follows from (1) and Theorem 4.1.7 (2). $\square$

Theorem 4.12.2. *If* $\mathrm{v}(L^\flat) > 0$*, we have the following decomposition of cycles in* $\mathrm{Gr}^2 K_0(\mathcal{N}_3^{\mathrm{Kra}})$

$$\mathcal{D}(L^\flat) = \sum_{\substack{\Lambda_2\in\mathcal{V}^2\\ L^\flat\subset\Lambda_2^\sharp}} (2[\mathcal{O}_{\tilde{\mathcal{N}}_{\Lambda_2}}] + \sum_{\substack{\Lambda_0\in\mathcal{V}^0\\ \Lambda_0\subset\Lambda_2}} H_{\Lambda_0})$$

where $\mathrm{Gr}^\bullet K_0(\mathcal{N}_3^{\mathrm{Kra}})$ *is the associated graded ring of* $K_0(\mathcal{N}_3^{\mathrm{Kra}})$ *with respect to the codimension filtration.*

Proof. Under the assumption $\mathrm{v}(L^\flat) > 0$, we can decompose $\mathcal{D}(L^\flat)$ in $\mathrm{Gr}^2 K_0(\mathcal{N}^{\mathrm{Kra}})$ as

$$\mathcal{D}(L^\flat) = \sum_{\Lambda_2\in\mathcal{V}(L^\flat)} m(\mathcal{D}(L^\flat), \Lambda_2)[\mathcal{O}_{\tilde{\mathcal{N}}_{\Lambda_2}}] + \sum_{\Lambda_0\in\mathcal{V}(L^\flat)} m(\mathcal{D}(L^\flat), \Lambda_0) H_{\Lambda_0}, \tag{4.71}$$

by (4.58) and Proposition 4.9.6.

Claim 1: $m(\mathcal{D}(L^\flat), \Lambda_0) = 0$ unless $L^\flat \subset \Lambda_0$. In such a case,

$$m(\mathcal{D}(L^\flat), \Lambda_0) = \begin{cases} q+1 & \text{if } \Lambda_0 \in \mathcal{V}(L^\flat)\setminus\mathcal{B}(L^\flat), \\ 1 & \text{if } \Lambda_0 \in \mathcal{B}(L^\flat). \end{cases} \tag{4.72}$$

Indeed, since Λ_0 is of type 0, we may choose a $y_0 \in \mathbb{V}\setminus L^\flat_F$ such that $\mathrm{Span}\{y_0\}^\perp$ is non-split and $y_0 \in \Lambda_0 \setminus \pi\Lambda_0$. In this case, Proposition 4.6.2, Corollaries 4.6.7 and 4.6.8 imply that

$$\chi(\mathcal{N}^{\mathrm{Kra}}, H_{\Lambda_0}\cdot[\mathcal{O}_{\tilde{\mathcal{Z}}(y_0)}]) = 1.$$

So by (4.71), Corollaries 4.11.3 and 4.7.2, we have

$$m(\mathcal{D}(L^\flat), \Lambda_0) = \chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{D}(L^\flat)\cdot[\mathcal{O}_{\tilde{\mathcal{Z}}(y_0)}]).$$

Let $(2a, 2b)$ $(b > a)$ be the fundamental invariant of the projection of $L^\flat$ onto $\mathrm{Span}\{y_0\}^\perp$. Let φ be the natural quotient map $\Lambda_0 \to \Lambda_0/\pi\Lambda_0$ and define

$$m := \dim_{\mathbb{F}_q}\varphi(L^\flat) \leq 2.$$

Equation (4.52) implies that $m = 0$ if and only if $\Lambda_0 \in \mathcal{L}(L^\flat) \setminus \mathcal{B}(L^\flat)$. First assume $m = 0$, in other words, $L^\flat \subset \pi\Lambda_0$ so $b \geq a \geq 1$. By the definition of $\mathcal{D}(L^\flat)$ and [Shi22, Theorem 1.2], we have

$$m(\mathcal{D}(L^\flat), \Lambda_0) = \mu(a, b) - q\mu(a-1, b) - \mu(a, b-1) + q\mu(a-1, b-1) = q+1,$$

as claimed where

$$\mu(a, b) = \chi(\mathcal{N}^{\mathrm{Kra}}, {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L^\flat) \cdot [\mathcal{O}_{\tilde{\mathcal{Z}}(y_0)}]) = \begin{cases} 2\sum_{s=0}^{a} q^s(a+b+1-2s) - a - b - 2 & \text{if } a \geq 0 \\ 0 & \text{if } a < 0. \end{cases} \tag{4.73}$$

Now assume $m = 1$, then $\varphi(L^\flat)$ is a line ℓ and $b \geq 1$. By the assumption that $y_0 \notin L^\flat_F$, we know ℓ is not in $\mathrm{Span}\{\varphi(y_0)\}$, hence the projection of ℓ onto $\varphi(y_0)^\perp$ is nonzero. Since $\varphi(y_0)^\perp$ is nonsplit, we must have $a = 0$. Hence by the definition of $\mathcal{D}(L^\flat)$ and (4.73), we have

$$m(\mathcal{D}(L^\flat), \Lambda_0) = \mu(0, b) - q\mu(-1, b) - \mu(0, b-1) + q\mu(-1, b-1) = 1$$

as claimed. Finally, $m = 2$ is impossible since $v(L^\flat) > 0$. This finshes the proof of Claim 1.

Claim 2: $m(\mathcal{D}(L^\flat), \Lambda_2) = 2$ for any $\Lambda_2 \in \mathcal{V}^2(L^\flat)$.

Indeed, according to Lemma 4.6.9, we have $\chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{D}(L^\flat) \cdot [\mathcal{O}_{\mathrm{Exc}_{\Lambda_0}}]) = 0$. On the other hand, Corollary 4.6.7 and Lemma 4.10.12 imply that

$$\chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{D}(L^\flat) \cdot [\mathcal{O}_{\mathrm{Exc}_{\Lambda_0}}]) = \sum_{\Lambda_0 \subset \Lambda_2} m(\mathcal{D}(L^\flat), \Lambda_2) - 2m(\mathcal{D}(L^\flat), \Lambda_0). \tag{4.74}$$

Combining the above with Claim 1, we have

$$0 = \chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{D}(L^\flat) \cdot [\mathcal{O}_{\mathrm{Exc}_{\Lambda_0}}]) = \begin{cases} \sum_{\Lambda_0 \subset \Lambda_2} m(\mathcal{D}(L^\flat), \Lambda_2) - 2(q+1) & \text{if } \Lambda_0 \in \mathcal{L}(L^\flat) \setminus \mathcal{B}(L^\flat), \\ \sum_{\Lambda_0 \subset \Lambda_2} m(\mathcal{D}(L^\flat), \Lambda_2) - 2 & \text{if } \Lambda_0 \in \mathcal{B}(L^\flat). \end{cases} \tag{4.75}$$

Recall $\mathcal{S}(L^\flat)$ in Definition 4.10.8. First assume $\Lambda_2 \in \mathcal{L}(L^\flat) \setminus \mathcal{S}(L^\flat)$. If $d(\Lambda_2, \mathcal{B}(L^\flat))$ equals to $\frac{1}{2}$, choose $\Lambda_0 \in \mathcal{B}(L^\flat)$ such that $\Lambda_0 \subset \Lambda_2$, then Λ_2 is the unique lattice in $\mathcal{V}^2(L^\flat)$ that contains Λ_0, hence (4.75) implies that $m(\mathcal{D}(L^\flat), \Lambda_2) = 2$ in this case. Now Corollary 4.10.10 allows us to show $m(\mathcal{D}(L^\flat), \Lambda_2) = 2$ by induction on the distance $d(\Lambda_2, \mathcal{B}(L^\flat))$ for any $\Lambda_2 \in \mathcal{L}(L^\flat) \setminus \mathcal{S}(L^\flat)$.

Similarly for $\Lambda_2 \in \mathcal{S}(L^\flat)$, we can show $m(\mathcal{D}(L^\flat), \Lambda_2) = 2$ by induction on its distance to $\mathcal{S}(L^\flat) \cap \mathcal{B}(L^\flat)$. This finishes the proof of Claim 2.

Notice that for $\Lambda_0 \in \mathcal{V}(L^\flat)$

$$\sum_{\Lambda_2 \in \mathcal{V}^2(L^\flat)} \sum_{\Lambda_0 \subset \Lambda_2} 1 = \begin{cases} q+1 & \text{if } \Lambda_0 \in \mathcal{V}(L^\flat) \setminus \mathcal{B}(L^\flat), \\ 1 & \text{if } \Lambda_0 \in \mathcal{B}(L^\flat). \end{cases}$$

This finishes the proof of Theorem 4.12.2. □

In the following discussion we freely use Theorem 4.11.2 and Corollary 4.6.8 without explicitly referring to them.

Proposition 4.12.3. *Assume $L = L^\flat \obot \mathrm{Span}\{\mathbf{x}\}$ with Gram matrix*

$$T = \mathrm{Diag}(H_a, u_3(-\pi_0)^c)$$

where a is a positive odd integer, and $c \geq 0$. Then

$$\mathrm{Int}(L)^{(2)} = \partial\mathrm{Den}(L)^{(2)} = \begin{cases} 1 - q^a & \text{if } a \leq 2c, \\ 1 - q^{2c+1} & \text{if } a > 2c. \end{cases} \tag{4.76}$$

Proof. By Proposition 4.4.5, it suffices to prove the identity for $\mathrm{Int}(L)^{(2)}$.

Now we compute $\mathrm{Int}(L)^{(2)}$. We may take $L^\flat = \mathrm{Span}\{\pi^{\frac{a+1}{2}} e_1, \pi^{\frac{a+1}{2}} e_2\}$, where the Gram matrix of $\{e_1, e_2\}$ is H. Let $e_3 = \pi^{-c}\mathbf{x}$. Then $\mathcal{L}(L^\flat)$ is centered at $\mathrm{Span}\{e_1, e_2, e_3\}$ of radius $\frac{a}{2}$ by Proposition 4.10.7.

Assume $a \leq 2c$ first. In this case, $\mathcal{L}(L^\flat) \subset \mathcal{L}(\mathbf{x})$. As a result, we have $\mathrm{Int}_{\Lambda_2}(\mathbf{x}) = 1$ and $\mathrm{Int}_{\Lambda_0}(\mathbf{x}) = -1$ for any $\Lambda_2 \in \mathcal{V}^2(L^\flat)$ and $\Lambda_0 \in \mathcal{V}^0(L^\flat)$. Hence by Theorem 4.12.2, we have

$$\begin{aligned}
\mathrm{Int}(L)^{(2)} &= \sum_{\Lambda_2 \in \mathcal{L}(L^\flat)} \chi(\mathcal{N}^{\mathrm{Kra}}, (2[\mathcal{O}_{\tilde{\mathcal{N}}_{\Lambda_2}}] + \sum_{\Lambda_0 \subset \Lambda_2} H_{\Lambda_0}) \cdot \mathcal{Z}^{\mathrm{Kra}}(\mathbf{x})) \qquad (4.77)\\
&= (1-q)|\{\Lambda_2 \mid \Lambda_2 \in \mathcal{L}(L^\flat)\}| \\
&= (1-q)(1 + (1+q)q + (1+q)q^3 + \cdots + (1+q)q^{a-2}) \\
&= 1 - q^a,
\end{aligned}$$

as claimed.

Now we assume $a > 2c$. We consider the case $c = 0$ first. Recall that $\tilde{\mathcal{Z}}(e_3) \approx \mathcal{N}^{\mathrm{Kra}}_{2,1}$, hence $\mathcal{L}(L^\flat) \cap \mathcal{L}(e_3)$ is a ball of radius $\frac{a}{2}$ in the Bruhat-Tits tree $\mathcal{L}_{2,1}$ of $\mathcal{N}^{\mathrm{Pap}}_{2,1}$ centerd at the vertex lattice corresponding to $\pi^{-\frac{a+1}{2}} \cdot L^\flat$, within which a vertex lattice Λ_0 of type 0 is contained in two vertex lattices of type 2, and a vertex lattice Λ_2 of type 2 contains $q+1$ vertex lattice of type 0. Hence

$$|\{\Lambda_0 \mid \Lambda_0 \in (\mathcal{L}(L^\flat) \setminus \mathcal{B}(L^\flat)) \cap \mathcal{L}(e_3)\}| = 1 + q + (1+q)q + \cdots + (1+q)q^{\frac{a-3}{2}},$$

and

$$|\{\Lambda_0 \mid \Lambda_0 \in \mathcal{B}(L^\flat) \cap \mathcal{L}(e_3)\}| = (1+q)q^{\frac{a-1}{2}}.$$

Moreover, notice that if $e_3 \in \Lambda_0$, then $\mathrm{Int}_{\Lambda_2}(e_3) = 1$ for any Λ_2 such that $\Lambda_0 \subset \Lambda_2$. As a result,

$$\begin{aligned}\chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{D}(L^\flat)\cdot \mathcal{Z}^{\mathrm{Kra}}(e_3)) &= 2(1+q\cdot |\{\Lambda_0 \mid \Lambda_0 \in (\mathcal{L}(L^\flat)\setminus \mathcal{B}(L^\flat))\cap \mathcal{L}(e_3)\}|)\\ &\quad -(q+1)|\{\Lambda_0 \mid \Lambda_0 \in (\mathcal{L}(L^\flat)\setminus \mathcal{B}(L^\flat))\cap \mathcal{L}(e_3)\}|\\ &\quad -|\{\Lambda_0 \mid \Lambda_0 \in \mathcal{B}(L^\flat)\cap \mathcal{L}(e_3)\}|\\ &= 2+(q-1)(1+q+(1+q)q+\cdots+(1+q)q^{\frac{a-3}{2}})-(1+q)q^{\frac{a-1}{2}}\\ &= 1-q,\end{aligned}$$

which is compatible with (4.76).

Next we show

$$\chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{D}(L^\flat)\cdot (\mathcal{Z}^{\mathrm{Kra}}(\pi e_3)-\mathcal{Z}^{\mathrm{Kra}}(e_3))) = q-q^3.$$

According to Proposition 4.10.6, $\mathcal{V}(\pi e_3) = \{\Lambda \mid d(\Lambda, \mathcal{L}(e_3)) \leq 1\}$. Hence, around each $\Lambda_0 \in (\mathcal{L}(L^\flat)\setminus \mathcal{B}(L^\flat))\cap \mathcal{L}(e_3)$, there will be $q(q-1)$ many new vertex lattices of type 0 in $\mathcal{L}(L^\flat)\cap \mathcal{L}(\pi e_3)\setminus \mathcal{L}(L^\flat)\cap\mathcal{L}(e_3)$. Hence,

$$\begin{aligned}&\chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{D}(L^\flat)\cdot (\mathcal{Z}^{\mathrm{Kra}}(\pi e_3)-\mathcal{Z}^{\mathrm{Kra}}(e_3)))\\ &= 2q\cdot q(q-1)(1+q+(1+q)q+(1+q)q^2+\cdots+(1+q)q^{\frac{a-1}{2}-2})\\ &\quad -q(q-1)(q+1)(1+q+(1+q)q+(1+q)q^2+\cdots+(1+q)q^{\frac{a-1}{2}-2})\\ &\quad -q(q-1)(1+q)q^{\frac{a-1}{2}-1}\\ &= q-q^3.\end{aligned}$$

Continuing in this way, we can show

$$\chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{D}(L^\flat)\cdot (\mathcal{Z}^{\mathrm{Kra}}(\pi^i e_3)-\mathcal{Z}^{\mathrm{Kra}}(\pi^{i-1}e_3))) = q^{2i-1}-q^{2i+1}$$

for $2i < a$. So

$$\mathrm{Int}(L)^{(2)} = \mathcal{D}(L^\flat) \cdot \mathcal{Z}^{\mathrm{Kra}}(\pi^c e_3) = 1 - q^{2c+1} = \partial\mathrm{Den}^{(2)}(L)$$

as claimed. □

Proposition 4.12.4. *Assume* $L = L^\flat \obot \mathrm{Span}\{\mathbf{x}\}$ *with Gram matrix*

$$T = \mathrm{Diag}(u_1(-\pi_0)^a, u_2(-\pi_0)^b, u_3(-\pi_0)^c)$$

where $0 < a \leq b \leq c$, *then*

$$\mathrm{Int}(L)^{(2)} = \partial\mathrm{Den}(L)^{(2)} = 1 + \chi(-u_2u_3)q^a(q^a - q^b) - q^{a+b}.$$

Proof. By Proposition 4.4.4, it suffices to show

$$\mathrm{Int}(L)^{(2)} = 1 + \chi(-u_2u_3)q^a(q^a - q^b) - q^{a+b}. \tag{4.78}$$

Notice that since $a \leq b \leq c$, we have $\mathcal{L}(L^\flat) \subset \mathcal{L}(\mathbf{x})$ by Propositions 4.10.6 and 4.10.7.

First, we assume $\chi(-u_2u_3) = -1$, then (4.78) specializes to

$$\partial\mathrm{Den}(T)^{(2)} = 1 - q^{2a}.$$

On the other hand, $\mathcal{L}(L^\flat)$ is a ball centered at a vertex lattice of type 0 with radius a in this case. One can show $\mathrm{Int}(L)^{(2)} = 1 - q^{2a}$ exactly as in (4.77).

Now we assume $\chi(-u_2u_3) = 1$. In this case, (4.78) specializes to

$$\partial\mathrm{Den}(L)^{(2)} = 1 + q^{2a} - 2q^{a+b}.$$

Let $r = b - a$, and $L^\flat = \mathrm{Span}\{x_1, x_2\}$. Then $\mathcal{L}(\pi^{-a}L^\flat)$ is a ball centered at a vertex lattice of type 0 with radius r in the Bruhat-Tits tree $\mathcal{L}_{2,1}$. Hence,

$$\mathcal{L}(L^\flat) = \{\Lambda \mid \Lambda \in \mathcal{L}_3,\ d(\Lambda, \mathcal{L}(\pi^{-a}L^\flat)) \leq a\}.$$

When $a = 1$, $\mathcal{V}^0(\pi^{-1}L^\flat) = \mathcal{V}^0(L^\flat) \setminus \mathcal{B}(L^\flat)$. Then combining with Theorem 4.12.2, it is not hard to see

$$\begin{aligned}\mathrm{Int}(L)^{(2)} &= 2(q+1+q\cdot 2(q+q^2+\cdots+q^r)) - (q+1)|\mathcal{V}^0(L^\flat)\setminus\mathcal{B}(L^\flat)| - |\mathcal{B}(L^\flat)| \\ &= 1+q^2-2q^{b+1},\end{aligned}$$

where we use the fact

$$|\mathcal{V}^0(\pi^{-1}L^\flat)| = 1+2(q+q^2+\cdots+q^r),$$

and

$$|\mathcal{B}(L^\flat)| = (q-1)q(1+2(q+q^2+\cdots+q^{r-1})) + 2q^{r+2}.$$

Now assume $a > 1$. Let T be the hermitian matrix associated with $L^\flat \obot \mathrm{Span}\{\mathbf{x}\}$, then

$$\begin{aligned}&\partial\mathrm{Den}(\pi L^\flat \obot \mathrm{Span}\{\mathbf{x}\})^{(2)} - \partial\mathrm{Den}(L^\flat \obot \mathrm{Span}\{\mathbf{x}\})^{(2)} \\ &= 1+q^{2a+2}-2q^{r+2a+2} - (1+q^{2a}-2q^{r+2a}) \\ &= q^{2a}(q^2-1)(1-q^{2r}) \\ &= q^2\left(\partial\mathrm{Den}(L^\flat \obot \mathrm{Span}\{\mathbf{x}\})^{(2)} - \partial\mathrm{Den}(\pi^{-1}L^\flat \obot \mathrm{Span}\{\mathbf{x}\})^{(2)}\right),\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{Int}(\pi L^\flat \obot \mathrm{Span}\{\mathbf{x}\})^{(2)} - \mathrm{Int}(L^\flat \obot \mathrm{Span}\{\mathbf{x}\})^{(2)}\\
&= 2q|\mathcal{B}(L^\flat)| - q|\mathcal{B}(L^\flat)| - |\mathcal{B}(\pi L^\flat)|\\
&= (2q - q - q^2)|\mathcal{B}(L^\flat)|\\
&= q^2\left(\mathrm{Int}(L^\flat \obot \mathrm{Span}\{\mathbf{x}\})^{(2)} - \mathrm{Int}(\pi^{-1}L^\flat \obot \mathrm{Span}\{\mathbf{x}\})^{(2)}\right),
\end{aligned}$$

where we use the fact $|\mathcal{B}(\pi L^\flat \obot \mathrm{Span}\{\mathbf{x}\})| = q^2|\mathcal{B}(L^\flat)|$. Since r is arbitrary, an induction on a gives the result we want. □

Theorem 4.12.5. *Assume $n = 3$. Then for any non-degenerate lattice $L \subset \mathbb{V}$ of rank 3, we have*

$$\mathrm{Int}(L) = \partial\mathrm{Pden}(L).$$

Proof. The case $\mathrm{v}(L) < 0$ follows from Proposition 4.4.1 and the fact that $\mathrm{Int}(L) = 0$ under this condition. Assume $v(L) \geq 0$. There are three cases.

Case 1: When L has a Gram matrix $\mathrm{Diag}(u_1, u_2(-\pi_0)^b, u_3(-\pi_0)^c)$ as in Proposition 4.12.1, it is proved by Proposition 4.12.1.

Case 2: When L has a basis $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ whose Gram matrix is $T = \mathrm{Diag}(H_a, u_3(-\pi_0)^c)$, take $L^\flat = \mathrm{Span}(\mathbf{x}_1, \mathbf{x}_2)$ and $\mathbf{x} = \mathbf{x}_3$. By Propositions 4.12.1, 4.12.3, and 4.12.4, we have

$$\mathrm{Int}(L^{\flat,\prime} \oplus \mathrm{Span}\{\mathbf{x}\})^{(2)} = \partial\mathrm{Den}(L^{\flat,\prime} \oplus \mathrm{Span}\{\mathbf{x}\})^{(2)}$$

for any $L^\flat \subset L^{\flat,\prime} \subset L^\flat_F$ (direct sums in the above identity are actually orthogonal direct sums). Thus we have by Theorem 4.1.7 (1)

$$\mathrm{Int}(L) = \partial\mathrm{Den}(L).$$

Case 3: When L has a Gram matrix $\mathrm{Diag}(u_1(-\pi_0)^a, u_2(-\pi_0)^b, u_3(-\pi_0)^c)$ with $0 \leq a \leq b \leq c$), the same argument as Case 2 gives $\mathrm{Int}(L) = \partial\mathrm{Den}(L)$. This finishes the proof of the theorem. ☐

Theorem 4.12.5 and Theorem 4.1.7 imply the following corollary.

Corollary 4.12.6. *For any lattice* $L = L^\flat \oplus O_F \mathbf{x} \subset \mathbb{V}$ *of rank* 3*, we have*

$$\mathrm{Int}(L)^{(2)} = \partial\mathrm{Den}(L)^{(2)}.$$

Chapter 5

Proof of the Kudla-Rapoport conjecture for Krämer models

First, for convenience, we take the following conventions in this chapter. In this chapter, we let $\mathrm{Den}(M, L, X) := \alpha(M, L, X)$ and

$$\mathrm{Den}'(M, L) := -2 \cdot \frac{\partial}{\partial X}\mathrm{Den}(M, L, X)|_{X=1}. \tag{5.1}$$

Also, in this chapter, we say a lattice Λ is a vertex lattice of type t if $\pi\Lambda^\sharp \subset \Lambda \subset \Lambda^\sharp$ and $[\Lambda^\sharp : \Lambda] = t$.

5.0.1 Strategy and novelty of the proof of the Main Theorem 1.0.3

First, we review the overall proof strategy in this subsection. Our general strategy is closest to the unramified orthogonal case proved in [LZ22b]. More precisely, fix an O_F-lattice $L^\flat \subset \mathbb{V}$ of rank $n-1$ and denote by $\mathbb{W} = (L^\flat_F)^\perp \subset \mathbb{V}$. Consider functions on $\mathbb{V} \setminus L^\flat_F$,

$$\mathrm{Int}_{L^\flat}(x) := \mathrm{Int}(L^\flat + \langle x\rangle), \quad \partial\mathrm{Den}_{L^\flat}(x) := \partial\mathrm{Den}(L^\flat + \langle x\rangle).$$

Then it remains to show the equality of the two functions $\mathrm{Int}_{L^\flat} = \partial\mathrm{Den}_{L^\flat}$. To show this equality, we find a decomposition

$$\mathrm{Int}_{L^\flat} = \mathrm{Int}_{L^\flat,\mathscr{H}} + \mathrm{Int}_{L^\flat,\mathscr{V}}, \quad \partial\mathrm{Den}_{L^\flat} = \partial\mathrm{Den}_{L^\flat,\mathscr{H}} + \partial\mathrm{Den}_{L^\flat,\mathscr{V}}$$

into "horizontal" and "vertical" parts such that the horizontal identity $\mathrm{Int}_{L^\flat,\mathscr{H}} = \partial\mathrm{Den}_{L^\flat,\mathscr{H}}$ holds and that the vertical parts $\mathrm{Int}_{L^\flat,\mathscr{V}}$ and $\partial\mathrm{Den}_{L^\flat,\mathscr{V}}$ behaves well under Fourier transform along $L^\flat_F$.

The horizontal identity essentially reduces to the horizontal computation for $n = 2$ in [Shi22] and Chapter 3. For the vertical identity, we perform a *partial Fourier transform* along $L^\flat_F$ and consider new functions on $\mathbb{W} \setminus \{0\}$,

$$\mathrm{Int}^\perp_{L^\flat,\mathscr{V}}(x) := \int_{L^\flat_F} \mathrm{Int}_{L^\flat,\mathscr{V}}(y+x)\mathrm{d}y, \quad \partial\mathrm{Den}^\perp_{L^\flat,\mathscr{V}}(x) := \int_{L^\flat_F} \partial\mathrm{Den}_{L^\flat,\mathscr{V}}(y+x)\mathrm{d}y.$$

The key is to show that $\mathrm{Int}^\perp_{L^\flat,\mathscr{V}}$ and $\partial\mathrm{Den}^\perp_{L^\flat,\mathscr{V}}$ are both *constant* on $\mathbb{W}^{\geq 0} \setminus \{0\} := \{x \in \mathbb{W}\setminus\{0\} : \mathrm{val}(x) \geq 0\}$ of $\mathbb{W}$ (see §1.0.6 for notation) as in Theorem 5.2.16 and Theorem 5.6.2. Using an induction on the valuation of $L^\flat$, we show that the difference $\mathrm{Int}^\perp_{L^\flat,\mathscr{V}} - \partial\mathrm{Den}^\perp_{L^\flat,\mathscr{V}}$ vanishes on $\mathbb{W}^{\leq 0} := \{x \in \mathbb{W} : \mathrm{val}(x) \leq 0\}$, and hence it vanishes identically and allows us to conclude that $\mathrm{Int}_{L^\flat,\mathscr{V}} = \partial\mathrm{Den}_{L^\flat,\mathscr{V}}$.

On the geometric side, we prove a *Bruhat–Tits stratification* for the Krämer model (Theorem 5.1.19), analogous to the case of the Pappas model treated in Rapoport–Terstiege–Wilson [RTW14]. We make use of the linear invariance of special cycles [How19] to express $\mathrm{Int}_{L^\flat,\mathscr{V}}$ as a linear combination of functions on $\mathbb{V}$ which are *translation invariant* under vertex lattices. A new observation in our ramified case is that the translation invariance already allows us to control the support of its Fourier transform well enough (Lemma 5.2.13) to conclude the desired key constancy of $\mathrm{Int}^\perp_{L^\flat,\mathscr{V}}$ on $\mathbb{W}^{\geq 0}$. Compared to the unramified case, we *completely avoid* the Tate conjecture of generalized Deligne-Lusztig varieties and explicit computation of their intersections with special divisors. It is not clear that the Deligne-Lusztig cycles span the Tate cycles in this case. §5.1 studies the structure of $\mathcal{N}_{\mathrm{red}}$ and special cycles, and should be of independent interest (in addition to preparation for §5.2).

On the analytic side, we make use of the *primitive decomposition* of the local density polynomial into primitive local density polynomials and obtain a decomposition.

$$\partial\mathrm{Den}(L) = \sum_{L \subset L'} \partial\mathrm{Pden}(L'), \tag{5.2}$$

where L' runs over O_F-lattices in L_F containing L, and the symbol Pden stands for the primitive version of Den (Corollary 5.3.3). Unlike the unramified or exotic smooth case, the primitive local density *polynomial* itself seems rather complicated (see e.g. Corollary 5.4.2). Nevertheless we manage to prove a simple formula for its *modified central derivative* $\partial\mathrm{Pden}(L)$, which we find quite striking.

Theorem 5.0.1 (Theorem 5.5.1)**.** *Let $L \subset \mathbb{V}$ be an O_F-lattice (of full rank n).*

1. *If L is not integral, then $\partial\mathrm{Pden}(L) = 0$.*
2. *If L is unimodular, then*

$$\partial\mathrm{Pden}(L) = \begin{cases} 1, & \text{if } n \text{ is odd,} \\ 0, & \text{if } n \text{ is even.} \end{cases}$$

3. *If L is integral and of type $t > 0$, then*

$$\partial\mathrm{Pden}(L) = \begin{cases} \displaystyle\prod_{\ell=1}^{\frac{t-1}{2}} (1 - q^{2\ell}), & \text{if } t \text{ is odd,} \\ \displaystyle(1 - \chi(L')q^{\frac{t}{2}}) \prod_{\ell=1}^{\frac{t}{2}-1} (1 - q^{2\ell}), & \text{if } t \text{ is even.} \end{cases}$$

Here we write $L \simeq I_{n-t} \obot L'$ with I_{n-t} unimodular of rank $n - t$.

The proof of this theorem occupies the entire §5.4 and §5.5, and is our major technical innovation. One key difference between our case and the unramified or exotic smooth case is that in our case I_n and H (see (1.6) and (1.7)) have different fundamental invariants, hence it is not clear how to reduce the calculation of $\partial\mathrm{Pden}$ into the embedding-counting problems over finite fields in the style of [CY20, §3]. To deal with this difficulty, we first decompose $\partial\mathrm{Pden}(L)$ according to orbits of hermitian embeddings (Theorem 5.4.1). Now a new observation is that the primitive local density polynomial becomes simpler when L is "very integral" (i.e., when its fundamental invariants are all ≥ 1, see Proposition 5.5.6) in which case the decomposition in Theorem 5.4.1 is simple. The primitive local density polynomial vanishes when L is "very non-integral" (e.g., when one of its fundamental invariants is ≤ -2, see the proof of Lemma 5.3.1). When L is the dual of a vertex lattice of positive type, this is just our assumption (1.10). The remaining cases (in particular the unimodular lattice case) are much trickier to handle, whose proof occupies most of §5.5 and is summarized in §5.5.2. The proof relies on a series of non-trivial polynomial identities (e.g., Lemma 5.5.15 and Lemma 5.5.16) involving algebraic combinatorics of quadratic spaces over finite fields, which should be of independent interest.

With the simple formula for $\partial\mathrm{Pden}(L)$ at hand, we finally prove the desired key constancy of $\partial\mathrm{Den}^{\perp}_{L^\flat,\mathscr{V}}$ on $\mathbb{W}^{\geq 0}\setminus\{0\}$ via involved lattice-theoretic computation in §5.6, in a fashion similar to [LZ22b]. The techniques developed here on the analytic side seem to have wide applicability and we hope that they may shed new light on the Kudla–Rapoport conjecture in the context of more general level structures, e.g., for minuscule parahoric levels at unramified places formulated by Cho [Cho22].

We remark again that in this chapter we are using a slightly different notion of vertax lattice for convenience compared with previous chapters. We say a lattice Λ is a vertax lattice of type t if $\pi\Lambda^\sharp\subset\Lambda\subset\Lambda^\sharp$ and $[\Lambda^\sharp:\Lambda]=t$. Basically, a vertex lattice in this chapter is the hermitian dual of a vertex lattice in previous chapters.

5.1 Bruhat–Tits Stratification of Krämer models

We prove a Bruhat–Tits stratification for the Krämer model (Theorem 5.1.19), analogous to the case of the Pappas model (proposed in [Pap00]) treated in [RTW14]. More precisely, we define closed subschemes $\mathcal{N}_\Lambda$ (Definition 5.1.4) and show that the reduced locus of $\mathcal{N}$ is stratified by $\mathcal{N}_\Lambda$ (Theorem 5.1.19). From this stratification we obtain a stratification of the reduced locus of $\mathcal{Z}(L)$ (Proposition 5.1.20). We also show that $\mathcal{N}_\Lambda$ is isomorphic to the (modified) Deligne-Lusztig variety $Y_{V,\bar{\kappa}}$ defined in §5.1.1 (Theorem 5.1.16), and is in particular a smooth projective variety over $\bar{\kappa}$. We remark here that for the purpose of our main result (Theorem 5.7.7), only a weaker version of Proposition 5.1.20 is needed (namely we do not need the reducedness of $\mathcal{N}_\Lambda$). However we believe the rest of this section contributes to the theory of Rapoport-Zink space and is of independent interest.

5.1.1 Deligne-Lusztig varieties

Throughout this subsection we assume $m \geq 1$. Let V be a $2m$-dimensional symplectic space over $\kappa = \mathbb{F}_q$ equipped with the symplectic form $\langle\, ,\, \rangle$. Let $V_{\bar{\kappa}} = V \otimes_\kappa \bar{\kappa}$ and denote the bilinear extension of $\langle\, ,\, \rangle$ to $V_{\bar{\kappa}}$ still by $\langle\, ,\, \rangle$. Let $\mathrm{Gr}(i, V)$ be the Grassmannian variety parametrizing rank i locally direct summands of V_R for any κ-algebra R. Let $\mathrm{SGr}(i, V)$ be the subvariety of $\mathrm{Gr}(i, V)$ whose $\bar{\kappa}$-points are specified by

$$\mathrm{SGr}(i, V)(\bar{\kappa}) = \{z \in \mathrm{Gr}(i, V)(\bar{\kappa}) \mid z \text{ is isotropic with respect to } \langle\, ,\, \rangle\}.$$

Let S_V be the subvariety of $\mathrm{SGr}(m, V)$ as in [RTW14, Equation (5.3)] whose $\bar{\kappa}$-points are specified by

$$S_V(\bar{\kappa}) = \{U \in \mathrm{SGr}(m, V)(\bar{\kappa}) \mid \dim(U \cap \Phi(U)) \geq m - 1\}, \tag{5.3}$$

where Φ is the Frobenius endormophism. By [RTW14, Proposition 5.3] and its remark, S_V has isolated singularities which are exactly the points where $U = \Phi(U)$. We denote by $\mathcal{U}$ the nonsingular locus of S_V. By Proposition 5.5 of loc.cit., $S_{V,\bar{\kappa}}$ is irreducible of dimension m. To resolve the singularities of S_V, define Y_V to be the subvariety of $\mathrm{SGr}(m, V) \times \mathrm{SGr}(m-1, V)$ whose $\bar{\kappa}$-points are specified by

$$Y_V(\bar{\kappa}) = \{(U, U') \in (\mathrm{SGr}(m, V) \times \mathrm{SGr}(m-1, V))(\bar{\kappa}) \mid U' \subset U \cap \Phi(U)\}. \tag{5.4}$$

Then the variety Y_V is a projective subvariety of $\mathrm{Gr}(m, V) \times \mathrm{Gr}(m-1, V)$. The forgetful map $(U, U') \to U$ defines a morphism $\pi_m : Y_V \to S_V$.

Lemma 5.1.1. *The morphism π_m is a projective morphism. It is an isomorphism outside the singular locus of S_V. For a singular point z of S_V, $\pi_m^{-1}(z) \cong \mathbb{P}_{\bar{\kappa}}^{m-1}$.*

Proof. First we know π_m is projective as it is a morphism between projective schemes. Consider a $\bar{\kappa}$-point $z = U$ in $\mathcal{U}(\bar{\kappa})$. Then $U \cap \Phi(U)$ has dimension $m-1$, this entails $U' = U \cap \Phi(U)$. This shows that the morphism has an inverse when restricted on $\pi_m^{-1}(\mathcal{U})$, hence $\pi_m|_{\pi_m^{-1}(\mathcal{U})}$ is an isomorphism of varieties.

If $z = U$ is a singular $\bar{\kappa}$-point, then $U = \Phi(U)$ and U' can be any element in $\mathrm{Gr}(m-1, U) \cong \mathbb{P}_{\bar{\kappa}}^{m-1}$. This finishes the proof of the lemma. □

Proposition 5.1.2. *The projective variety $Y_{V,\bar{\kappa}}$ is irreducible and smooth of dimension m.*

Proof. Define $\mathrm{SGr}(m, m-1, V)$ to be the sub flag variety of $\mathrm{SGr}(m, V) \times \mathrm{SGr}(m-1, V)$ whose $\bar{\kappa}$-points are specified by

$$\mathrm{SGr}(m, m-1, V)(\bar{\kappa}) = \{(U, U') \in (\mathrm{SGr}(m, V) \times \mathrm{SGr}(m-1, V))(\bar{\kappa}) \mid U' \subset U\}.$$

Then $Y_{V,\bar{\kappa}}$ is the intersection of the image of the closed immersion

$$\begin{aligned} (\mathrm{SGr}(m, m-1, V)_{\bar{\kappa}})^2 &\to (\mathrm{SGr}(m, V)_{\bar{\kappa}} \times \mathrm{SGr}(m-1, V)_{\bar{\kappa}})^2 : \\ (U_1, U_1', U_2, U_2') &\mapsto (U_1, U_1', U_2, U_2'), \end{aligned}$$

with the image of the closed immersion

$$\begin{aligned} \mathrm{SGr}(m, V)_{\bar{\kappa}} \times \mathrm{SGr}(m-1, V)_{\bar{\kappa}} &\to (\mathrm{SGr}(m, V)_{\bar{\kappa}} \times \mathrm{SGr}(m-1, V)_{\bar{\kappa}})^2 : \\ (U_3, U_4) &\mapsto (U_3, U_4, \Phi(U_3), U_4). \end{aligned}$$

Since $(\mathrm{SGr}(m, m-1, V)_{\bar{\kappa}})^2$ and $\mathrm{SGr}(m, V)_{\bar{\kappa}} \times \mathrm{SGr}(m-1, V)_{\bar{\kappa}}$ are smooth (as they are homogeneous varieties), and Φ induces the zero map on the tangent space, one can see immediately that the intersection is transversal. Hence $Y_{V,\bar{\kappa}}$ is smooth. Since $S_{V,\bar{\kappa}}$ is irreducible of dimension m, by Lemma 5.1.1, we know $Y_{V,\bar{\kappa}}$ is connected and has an open subvariety of dimension m. Taking into consideration the smoothness, we know $Y_{V,\bar{\kappa}}$ must be irreducible of dimension m. This finishes the proof of the proposition. $\square$

Remark 5.1.3. *One can show that $Y_{V,\bar{\kappa}}$ is in fact the blow-up of $S_{V,\bar{\kappa}}$ along its singular locus.*

5.1.2 Minuscule cycle $\mathcal{N}_\Lambda$ and its tangent space

In this section, we often identify a vertex lattice $\Lambda \subset \mathbb{V}$ with $b^{-1}(\Lambda)$ using the isomorphism of hermitian spaces (2.5) unless otherwise stated.

Definition 5.1.4. *For a vertex lattice $\Lambda \subset \mathbb{V}$ of type $t(\Lambda) = 2m$, define the subfunctor $\mathcal{N}_\Lambda$ to be the subfunctor of $\mathcal{N}$ such that for a $O_{\breve{F}}$-scheme S, $\mathcal{N}_\Lambda(S)$ is the set of isomorphism classes of tuples $(X, \iota, \lambda, \rho, \mathcal{F})$ satisfying the following conditions.*

1. *$(X, \iota, \lambda, \rho, \mathcal{F}) \in \mathcal{Z}(\Lambda)(S)$.*
2. *If $m \geq 1$, we require in addition that $x_*(\mathrm{Lie}(\mathcal{G} \times_{\mathrm{Spf}\, \mathcal{O}_{\breve{F}}} S)) \subset \mathcal{F}$ for any $x \in \Lambda$.*

We first describe the $\bar{\kappa}$-points of $\mathcal{N}$ and $\mathcal{N}_\Lambda$.

Proposition 5.1.5. *There is a bijection between $\mathcal{N}_{\mathrm{red}}(\bar{\kappa})$ and the set of pairs of $O_{\breve{F}}$-lattices (M, M') in N satisfying*

$$M^\sharp = M, \quad \Pi M \subset \tau^{-1}(M) \subset \Pi^{-1} M, \quad \mathbf{V} M \subset M' \subset \tau^{-1}(M) \cap M, \quad \textit{and} \quad \mathrm{length}(M/M') = 1.$$

Proof. Let $(X, \iota, \lambda, \rho, \mathcal{F})$ be a $\bar{\kappa}$-point of $\mathcal{N}$ and $M(X)$ be the O_{F_0}-relative Dieudonné module of X. Define $M = \rho(M(X)) \subset N$ and $M' = \rho(\mathrm{Pr}^{-1}(\mathcal{F})) \subset N$ where $\mathrm{Pr} : M(X) \to \mathrm{Lie} X = M(X)/\mathbf{V} M(X)$ is the natural quotient map. The condition $M^\sharp = M$ is equivalent to the fact that λ is a principal polarization. The condition $\Pi M \subset \tau^{-1}(M) \subset \Pi^{-1} M$

is equivalent to $\pi_0 M \subset \mathbf{V}M \subset M$. The condition $\mathbf{V}M \subset M' \subset \tau^{-1}(M) \cap M$ and $\mathrm{length}(M/M') = 1$ is equivalent to the condition

$$\mathbf{V}M \subset M' \subset M,\ \Pi M' \subset \mathbf{V}M,\ \dim_{\bar{\kappa}}(M/M') = 1,$$

which is in turn equivalent to

$$\mathcal{F} \subset \mathrm{Lie}X, \quad \dim_{\bar{\kappa}}(\mathcal{F}) = n-1, \quad \Pi \cdot \mathcal{F} = \{0\}, \quad \Pi \cdot \mathrm{Lie}X \subset \mathcal{F}.$$

Notice that the condition $\Pi \cdot \mathrm{Lie}X \subset \mathcal{F}$ is automatic once we know $\dim_{\bar{\kappa}}(\mathcal{F}) = n-1$ and $\mathcal{F}$ is stable under the action of Π. Hence the filtration $\mathcal{F} \subset \mathrm{Lie}X$ satisfies Krämer's signature condition and we have translated all conditions in the definition of $\mathcal{N}$ in term of relative Dieudonné modules. The proposition now follows from Dieudonné theory. $\square$

For a vertex lattice Λ in $\mathbb{V}$ or C, define

$$\breve{\Lambda} := \Lambda \otimes_{O_F} O_{\breve{F}},\ \breve{\Lambda}^\sharp := \Lambda^\sharp \otimes_{O_F} O_{\breve{F}}. \tag{5.5}$$

Corollary 5.1.6. *Let Λ be a vertex lattice in C. There is a bijection between $\mathcal{N}_\Lambda(\bar{\kappa})$ and the set of pairs of $O_{\breve{F}}$-lattices (M, M') in N satisfying the conditions in Proposition 5.1.5 and the following condition.*

1. *If $t(\Lambda) = 0$, then $M = \breve{\Lambda}$.*
2. *If $t(\Lambda) \geq 2$, then $\breve{\Lambda} \subset M' \subset M$.*

Proof. By Dieudonné theory, a point $(X, \iota, \lambda, \rho, \mathcal{F}) \in \mathcal{N}(\bar{\kappa})$ is in $\mathcal{Z}(\Lambda)(\bar{\kappa})$ if and only if $\rho^{-1} \circ x(M(\mathbb{Y})) \subset M(X)$ for any $x \in b^{-1}(\Lambda)$. Since $M(\mathbb{Y})$ is generated by e, this is the case if and only if $x(e) \in M = \rho(M(X))$ for any $x \in b^{-1}(\Lambda)$, if and only if $\Lambda \subset M$ (by the definition of b (2.5)), if and only if $\breve{\Lambda} \subset M$. When $t(\Lambda) = 0$, both M and $\breve{\Lambda}$ are unimodular, thus $M = \breve{\Lambda}$. Similarly (2) is equivalent to Condition (2) in Definition 5.1.4 as the Lie algebra of $\mathbb{Y}$ is generated by the image of e under the quotient map

$M(\mathbb{Y}) \to \mathrm{Lie}\mathbb{Y} = M(\mathbb{Y})/\mathbf{V}M(\mathbb{Y})$. $\square$

To study the tangent space of $\mathcal{N}_\Lambda$, we recall the Grothendieck-Messing deformation theory of $\mathcal{N}$ from [How19, §3]. We remark here that although [How19] deals with the case $F_0 = \mathbb{Q}_p$, the argument in fact applies to general F_0 using the relative display theory of [ACZ16]. Let $R \in \mathrm{Nilp}\, O_{\breve{F}}$. For a strict O_{F_0}-module X over $\mathrm{Spec}\, R$, we denote by $D(X)$ the O_{F_0}-relative Dieudonné crystal in the sense of [ACZ16, §3]. A point $z \in \mathcal{N}(R)$ corresponds to a strict O_{F_0}-module (X, ι, λ) over R together with filtration $\mathcal{F} \subset \mathrm{Lie}X$ satisfying Definition 2.1.2. We have the following exact sequence of locally free R-modules

$$0 \to \mathrm{Fil}(X) \to D(X) \to \mathrm{Lie}X \to 0, \tag{5.6}$$

where $\mathrm{Fil}(X)$ and $\mathrm{Lie}X$ are of rank n and $D(X)$ is of rank $2n$. The principal polarization λ induces a symplectic form $\langle\, ,\, \rangle$ on $D(X)$ such that

$$\langle \iota(a)x, y\rangle = \langle x, \iota(\bar{a})y\rangle$$

for all $a \in O_F$ and $x, y \in D(X)$. With respect to $\langle\, ,\, \rangle$ the Hodge filtration $\mathrm{Fil}(X)$ is maximal isotropic. Hence $\langle\, ,\, \rangle$ induces a perfect pairing (still denoted by $\langle\, ,\, \rangle$):

$$\langle\, ,\, \rangle : \mathrm{Fil}(X) \times \mathrm{Lie}X \to R. \tag{5.7}$$

The submodule $\mathcal{F} \subset \mathrm{Lie}X$ and its perpendicular complement $\mathcal{F}^\perp$ (which is locally a direct summand of $\mathrm{Fil}(X)$ of rank one) with respect to (5.7) determine each other. The condition on $\mathcal{F}$ in Definition 2.1.2 is

$$O_F \text{ acts on } \mathcal{F} \text{ by } O_F \xrightarrow{\bar{i}} O_{\breve{F}} \to \mathcal{O}_S \text{ and on } \mathrm{Lie}X/\mathcal{F} \text{ by } O_F \xrightarrow{i} O_{\breve{F}} \to \mathcal{O}_S. \tag{5.8}$$

This is equivalent to the condition that O_F acts on $\mathcal{F}^\perp$ by $O_F \xrightarrow{\bar{i}} O_{\breve{F}} \to \mathcal{O}_S$ and on $\mathrm{Fil}(X)/\mathcal{F}^\perp$ by $O_F \xrightarrow{i} O_{\breve{F}} \to \mathcal{O}_S$. Since O_{F_0} acts on $D(X)$ by i_0 and $O_F = O_{F_0}[\pi]$, (5.8) is further equivalent to

$$(\Pi+\pi)\cdot\mathcal{F}^{\perp}=0,\ (\Pi-\pi)\cdot\mathrm{Fil}(X)\subset\mathcal{F}^{\perp}, \tag{5.9}$$

where we use Π to denote the action $\iota(\pi)$ on $D(X)$.

Definition 5.1.7. *Let $\mathscr{C}$ be the following category. Objects in $\mathscr{C}$ are triples $(\mathcal{O},\mathcal{O}\to\bar{\kappa},d)$ where $\mathcal{O}$ is an Artinian $O_{\breve{F}}$-algebra, $\mathcal{O}\to\bar{\kappa}$ is an $O_{\breve{F}}$-algebra homomorphism, and d is a nilpotent O_{F_0}-pd-structure (see [ACZ16, Definition 1.2.2]) on $\mathrm{Ker}(\mathcal{O}\to\bar{\kappa})$. Morphisms in $\mathscr{C}$ are $O_{\breve{F}}$-algebra homomorphisms compatible with structure maps to $\bar{\kappa}$ and O_{F_0}-pd-structure structures.*

Let $z=(X,\iota,\lambda,\rho,\mathcal{F})\in\mathcal{Z}(\Lambda)(\bar{\kappa})$ and $M=\rho(M(X))\subset N$. Then $\Lambda\subset M$ by Corollary 5.1.6. We can identify (5.6) with

$$0\to\mathbf{V}M/\pi_0M\to M/\pi_0M\to M/\mathbf{V}M\to 0.$$

Let $\mathcal{F}^{\perp}\subset\mathbf{V}M/\pi_0M$ be the perpendicular complement of $\mathcal{F}$ as described above. Denote by $\mathcal{Z}(\Lambda)_z$ (resp. $\mathcal{N}_{\Lambda,z}$) the completion of $\mathcal{Z}(\Lambda)$ (resp. $\mathcal{N}_{\Lambda}$) at z. For any $\mathcal{O}\in\mathscr{C}$ and $\tilde{z}=(\tilde{X},\cdots)\in\mathcal{Z}(\Lambda)_z(\mathcal{O})$, we can identify $D(\tilde{X})$ with $M_{\mathcal{O}}:=M\otimes_{O_{\breve{F}_0}}\mathcal{O}$ and by Grothendieck-Messing theory $\tilde{z}$ corresponds to a filtration of free $\mathcal{O}$-module direct summands

$$\tilde{\mathcal{F}}^{\perp}\subset\widetilde{\mathrm{Fil}}\subset M_{\mathcal{O}},$$

which lifts the filtration $\mathcal{F}^{\perp}\subset\mathrm{Fil}\subset M_{\bar{\kappa}}=M/\pi_0M$. Let $f_{\mathcal{O}}$ be the map

$$f_{\mathcal{O}}:\tilde{z}\mapsto(\tilde{\mathcal{F}}^{\perp},\widetilde{\mathrm{Fil}}). \tag{5.10}$$

Lemma 5.1.8. *Let the notations be as above. Denote by $\Lambda_{M,\mathcal{O}}$ the image of the composition of maps $\breve{\Lambda}\to M\to M_{\mathcal{O}}$, and let $\Lambda_{M,\mathcal{O}}^{\perp}$ be its perpendicular complement in $M_{\mathcal{O}}$ under the alternating form $\langle\,,\rangle$.*

1. *The map $f_{\mathcal{O}}$ defines a bijection from $\mathcal{Z}(\Lambda)_z(\mathcal{O})$ to the set consisting of pairs $(\tilde{\mathcal{F}}^{\perp}, \widetilde{\mathrm{Fil}})$ lifting $(\mathcal{F}^{\perp}, \mathrm{Fil})$ satisfying the following conditions:*
   - *(a) $\tilde{\mathcal{F}}^{\perp}$ and $\widetilde{\mathrm{Fil}}$ are free $\mathcal{O}$-module direct summands of $M_{\mathcal{O}}$ of rank 1 and n respectively and $\tilde{\mathcal{F}}^{\perp} \subset \widetilde{\mathrm{Fil}}$;*
   - *(b) $\widetilde{\mathrm{Fil}}$ is isotropic with respect to $\langle\, , \rangle$;*
   - *(c) $(\Pi + \pi) \cdot \tilde{\mathcal{F}}^{\perp} = 0$ and $(\Pi - \pi) \cdot \widetilde{\mathrm{Fil}} \subset \tilde{\mathcal{F}}^{\perp}$;*
   - *(d) $\widetilde{\mathrm{Fil}}$ contains $\widetilde{\mathrm{Fil}}^{-} := (\Pi + \pi) \cdot \Lambda_{M,\mathcal{O}}$.*
2. *The restriction of $f_{\mathcal{O}}$ to $\mathcal{N}_{\Lambda,z}(\mathcal{O})$ defines a bijection from $\mathcal{N}_{\Lambda,z}(\mathcal{O})$ to the set consisting of pairs $(\tilde{\mathcal{F}}^{\perp}, \widetilde{\mathrm{Fil}})$ satisfying the above conditions together with the extra condition:*
   - *(e) $\tilde{\mathcal{F}}^{\perp} \subset \Lambda_{M,\mathcal{O}}^{\perp}$.*

Proof. Proof of (1): By the previous discussion, $(\tilde{\mathcal{F}}^{\perp}, \widetilde{\mathrm{Fil}})$ satisfies conditions (a), (b) and (c) for any $\tilde{z} \subset \mathcal{N}_z(\mathcal{O})$ ($\mathcal{N}_z$ is the completion of $\mathcal{N}$ at z). Conversely by Grothendieck-Messing theory any pair $(\tilde{\mathcal{F}}^{\perp}, \widetilde{\mathrm{Fil}})$ lifting $(\mathcal{F}^{\perp}, \mathrm{Fil})$ satisfying (a), (b) and (c) gives rise to a unique point $\tilde{z} \in \mathcal{N}_z(\mathcal{O})$. Since the action of O_F on $\mathrm{Lie}\mathcal{G}$ is via the inclusion $\bar{i}$, the Hodge filtration of $\mathcal{G}_{\mathcal{O}}$ is $\mathrm{Span}_{\mathcal{O}}\{(\Pi + \pi) \cdot e \otimes 1\}$ where e is a generator of $M(\mathbb{Y})$ as in §2.1.2. The image of the $\mathrm{Span}_{\mathcal{O}}\{(\Pi + \pi) \cdot e \otimes 1\}$ under elements of $\Lambda \subset \mathbb{V}$ in $M_{\mathcal{O}}$ is exactly $\widetilde{\mathrm{Fil}}^{-}$. By Grothendick-Messing theory again, $\tilde{z} \in \mathcal{Z}(\Lambda)_z(\mathcal{O})$ if and only if condition (d) holds.

(2) is a corollary of (1). For any $\tilde{z} = (\tilde{X}, \ldots, \tilde{\mathcal{F}}) \in \mathcal{Z}(\Lambda)_z(\mathcal{O})$, let $\tilde{\mathcal{F}}'$ be the preimage of $\tilde{\mathcal{F}}$ under the quotient map $M_{\mathcal{O}} \to M_{\mathcal{O}}/\widetilde{\mathrm{Fil}}$. Condition (2) in Definition 5.1.4 is equivalent to $\Lambda_{M,\mathcal{O}} \subset \tilde{\mathcal{F}}'$ by the same reasoning as Corollary 5.1.6. The perpendicular complement of $\tilde{\mathcal{F}}'$ with respect to $\langle\, , \rangle$ is $\tilde{\mathcal{F}}^{\perp}$. Hence condition (2) in Definition 5.1.4 is equivalent to condition (e). Hence $\tilde{z} \in \mathcal{N}_{\Lambda,z}(\mathcal{O})$ if and only if (e) is satisfied. This finishes the proof of the lemma. □

Lemma 5.1.9. *Let Λ be a vertex lattice of type $2m$ in C and $M \subset N$ be an $O_{\breve{F}}$-lattice such that $\Lambda \subset M$ and $M = M^{\sharp}$. Then there is an $O_{\breve{F}}$-basis $\{e_1, \ldots, e_n\}$ of M such that*

$$(e_\alpha, e_{\alpha+m}) = 1, \;\; (e_\mu, e_\mu) \in O_{\breve{F}}^{\times}$$

for $1 \leq \alpha \leq m$, $2m+1 \leq \mu \leq n$, the inner product $(\,,\,)$ between any other basis vectors is zero, and

$$\breve{\Lambda} = \mathrm{Span}_{O_{\breve{F}}}\{\Pi e_1, \dots, \Pi e_m, e_{m+1}, \dots, e_n\}.$$

Proof. By assumption we have $\Pi M \subset \Pi\breve{\Lambda}^\sharp \subset \breve{\Lambda} \subset M$ and $\dim_{\bar{\kappa}}(M/\breve{\Lambda}) = m$. With respect to the $\bar{\kappa}$-valued quadratic form $(\,,\,) \pmod{\pi}$ on $M/\Pi M$, $\breve{\Lambda}/\Pi M$ has a decomposition

$$\breve{\Lambda}/\Pi M = R \ominus W,$$

where R is totally isotropic and W is non-degenerate. Then by the nondegeneracy of $(\,,\,) \pmod{\pi}$ on $M/\Pi M$ we know that there is a totally isotropic subspace R' such that

$$M/\Pi M = (R' \oplus R) \ominus W,$$

and $(\,,\,) \pmod{\pi}$ induces a perfect pairing between R and R'. Hence we can find a basis $\{\bar{e}_1, \bar{e}_n\}$ of $M/\Pi M$ such that $R' = \langle \bar{e}_1, \dots, \bar{e}_m\rangle$, $R = \langle \bar{e}_{m+1}, \dots, \bar{e}_{2m}\rangle$, $W = \langle \bar{e}_{2m+1}, \dots, \bar{e}_n\rangle$, and

$$(\bar{e}_\alpha, \bar{e}_{\alpha+m}) = 1 \pmod{\pi},\ (\bar{e}_\mu, \bar{e}_\mu) \pmod{\pi} \in \bar{\kappa}^\times$$

for $1 \leq \alpha \leq m$ and $2m+1 \leq \mu \leq n$ and the pairing between all other basis vectors are zero. We can lift the above basis to a basis $\{e_1, \dots, e_n\}$ of M which will satisfy the assumptions of the lemma. □

Proposition 5.1.10. *The scheme $\mathcal{Z}(\Lambda)$ has no $O_{\breve{F}}/(\pi^2)$-point.*

Proof. Let $\mathcal{O} = O_{\breve{F}}/(\pi^2)$ with the reduction map $\mathcal{O} \to \bar{\kappa}$ and the natural O_{F_0}-pd structure on $\pi\mathcal{O}$. Then $\mathcal{O} \in \mathscr{C}$. Let $z = (X, \iota, \lambda, \rho, \mathcal{F}) \in \mathcal{Z}(\Lambda)(\bar{\kappa})$ and $M = \rho(M(X)) \subset N$ as in Proposition 5.1.5. Then by Corollary 5.1.6 $\breve{\Lambda} \subset M$, and we can assume there is an $O_{\breve{F}}$-basis $\{e_1, \dots, e_n\}$ of M as in Lemma 5.1.9. Denote the image of e_i in $M_{\mathcal{O}}$ still by e_i. Then $\{e_1, \dots, e_n, \Pi e_1, \dots, \Pi e_n\}$ is an $\mathcal{O}$-basis of $M_{\mathcal{O}}$. With respect to the alternating form $\langle\,,\,\rangle$, we have by (2.2)

$$\langle e_\alpha, \Pi e_{m+\alpha}\rangle = -1/\delta,\ \langle e_{m+\alpha}, \Pi e_\alpha\rangle = -1/\delta,\ \langle e_\mu, \Pi e_\mu\rangle \in O_{\breve{F}_0}^\times, \tag{5.11}$$

for $1 \le \alpha \le m$, $2m+1 \le \mu \le n$, and all other pairings between basis vectors are zero.

Assume that z can be lifted to a point $\tilde{z} \in \mathcal{Z}(\Lambda)_z(\mathcal{O})$, which corresponds to a pair $(\tilde{\mathcal{F}}^\perp, \widetilde{\mathrm{Fil}})$ as in Lemma 5.1.8. First notice that

$$\widetilde{\mathrm{Fil}}^- = (\Pi+\pi)\cdot\Lambda_{M,\mathcal{O}} = \mathrm{Span}_{O_{\breve{F}}\otimes O_{\breve{F}_0}\mathcal{O}}\{\pi\Pi e_1, \dots, \pi\Pi e_m, (\Pi+\pi)e_{m+1}, \dots, (\Pi+\pi)e_n\}. \tag{5.12}$$

With respect to the alternating form $\langle\, ,\, \rangle$, its perpendicular complement $(\widetilde{\mathrm{Fil}}^-)^\perp$ in $\Lambda_{M,\mathcal{O}}$ is generated by

$$\{(\Pi+\pi)e_1, \dots (\Pi+\pi)e_m, \Pi e_{m+1}, \pi e_{m+1}, \dots, \Pi e_{2m}, \pi e_{2m}, (\Pi+\pi)e_{2m+1}, \dots, (\Pi+\pi)e_n\}. \tag{5.13}$$

By Lemma 5.1.8 (c), $\tilde{\mathcal{F}}^\perp$ is annihilated by $\Pi+\pi$, hence it is spanned by a vector

$$v = \sum_{i=1}^n a_i(\Pi-\pi)e_i,$$

where $a_i \in \mathcal{O}^\times$ for some i as $\tilde{\mathcal{F}}^\perp$ is a direct summand of $M_{\mathcal{O}}$. By Lemma 5.1.8, we must have $\widetilde{\mathrm{Fil}}^- \subset \widetilde{\mathrm{Fil}}$, $\tilde{\mathcal{F}}^\perp \subset \widetilde{\mathrm{Fil}}$ and $\widetilde{\mathrm{Fil}}$ is isotropic. Hence $\widetilde{\mathrm{Fil}} \subset (\widetilde{\mathrm{Fil}}^-)^\perp$. Moreover $\langle \widetilde{\mathrm{Fil}}^-, \tilde{\mathcal{F}}^\perp\rangle = 0$, which implies $a_i \in \pi\mathcal{O}$ for $1 \le i \le m$ and $2m+1 \le i \le n$. Hence without loss of generality we can assume that $a_{m+1} = 1$.

Since $\widetilde{\mathrm{Fil}}$ is a direct summand of $M_{\mathcal{O}}$ we have $M_{\mathcal{O}} = \widetilde{\mathrm{Fil}} \oplus S$ where S is an $\mathcal{O}$-module. We can write $\Pi e_1 = w + w'$ where $w \in \widetilde{\mathrm{Fil}}$ and $w' \in S$. Since $\pi\Pi e_1 \in \widetilde{\mathrm{Fil}}^- \subset \widetilde{\mathrm{Fil}}$, we must have $\pi w' = 0$. This implies that $w' \in \pi M_{\mathcal{O}}$ and w is of the form

$$w = (\Pi + b\pi)e_1 + x$$

where $b \in \mathcal{O}$ and $x \in \pi\cdot\mathrm{Span}_{\mathcal{O}}\{e_2, \Pi e_2, \dots, e_n, \Pi e_n\}$. Since $w \in \widetilde{\mathrm{Fil}} \subset (\widetilde{\mathrm{Fil}}^-)^\perp$, by (5.13), we must have $b = 1$ and x is of the form

$$x = \sum_{i=2}^{m} d_i(\Pi + \pi)e_i + \sum_{i=m+1}^{2m} (c_i + d_i\Pi)e_i + \sum_{i=2m+1}^{n} d_i(\Pi + \pi)e_i,$$

where $c_i \in \pi\mathcal{O}$ for $m + 1 \leq i \leq 2m$ and $d_i \in \pi\mathcal{O}$ for $2 \leq i \leq n$. Since $(\Pi + \pi)e_i \in \widetilde{\mathrm{Fil}}$ for $2m + 1 \leq i \leq n$, by changing w and x at the same time if necessary we can assume that $d_i = 0$ for $2m + 1 \leq i \leq n$. By (5.11), we have

$$\langle(\Pi + \pi)e_1, (\Pi - \pi)e_{m+1}\rangle = 2\pi\langle e_1, \Pi e_{m+1}\rangle \neq 0.$$

Moreover

$$\langle x, v\rangle = \sum_{i=2}^{m}\langle d_i(\Pi + \pi)e_i, a_{m+i}(\Pi - \pi)e_{m+i}\rangle + \sum_{i=m+1}^{2m} \langle(c_i + d_i\Pi)e_i, a_{i-m}(\Pi - \pi)e_{i-m}\rangle = 0$$

Here we have used the fact that $a_i \in \pi\mathcal{O}$ for $1 \leq i \leq m$, $c_i \in \pi\mathcal{O}$ for $m + 1 \leq i \leq 2m$ and $d_i \in \pi\mathcal{O}$ for $2 \leq i \leq 2m$. Then $\langle w, v\rangle \neq 0$ which contradicts the fact that $\widetilde{\mathrm{Fil}}$ is isotropic. Hence there is no lift of z into $\mathcal{Z}(\Lambda)(\mathcal{O})$. This proves the lemma. $\square$

As $\mathcal{N}_\Lambda$ is a formal subscheme of $\mathcal{Z}(\Lambda)$, the following corollary is immediate.

Corollary 5.1.11. *$\mathcal{N}_\Lambda$ has no $O_{\breve{F}}/(\pi^2)$-point.*

Proposition 5.1.12. *Let Λ be a vertex lattice of type $2m$ ($m \geq 1$) in $\mathbb{V}$ and $z \in \mathcal{N}_\Lambda(\bar{\kappa})$. Then the tangent space $T_z(\mathcal{N}_{\Lambda,\bar{\kappa}})$ has dimension less or equal to m.*

Proof. Let $z = (X, \iota, \lambda, \rho, \mathcal{F}) \in \mathcal{N}_\Lambda(\bar{\kappa})$ and $M = \rho(M(X)) \subset N$ as in Proposition 5.1.5. Let $\mathcal{O} = \bar{\kappa}[\epsilon]/(\epsilon^2)$, then $\mathcal{O}$ is an $O_{\breve{F}}$-algebra through the map $O_{\breve{F}} \to \bar{\kappa} \to \mathcal{O}$ and the ideal $(\epsilon) \subset \mathcal{O}$ is equipped with a natural O_{F_0}-pd structure. Then $\mathcal{O} \in \mathscr{C}$. Any point $\tilde{z} \in T_z(\mathcal{N}_{\Lambda,\bar{\kappa}}) = \mathcal{N}_{\Lambda,z}(\mathcal{O})$ corresponds to a unique pair $(\tilde{\mathcal{F}}^\perp, \widetilde{\mathrm{Fil}})$ lifting $(\mathcal{F}^\perp, \mathrm{Fil})$ as in Lemma 5.1.8. We prove the lemma in two cases.

Case (a): $\mathrm{Fil} \neq \Pi \cdot M_{\bar{\kappa}}$. Since $M_{\mathcal{O}}$ is a free $O_F \otimes_{O_{F_0}} \mathcal{O}$-modules of rank n, we have the following exact sequence

$$0 \to \Pi \cdot M_{\mathcal{O}} \to M_{\mathcal{O}} \xrightarrow{\Pi} \Pi \cdot M_{\mathcal{O}} \to 0,$$

where $\Pi \cdot M_{\mathcal{O}}$ is a free $\mathcal{O}$-module of rank n and the first arrow is the natural injection. This implies the following sequence is exact.

$$0 \to (\Pi \cdot M_{\mathcal{O}}) \cap \widetilde{\mathrm{Fil}} \to \widetilde{\mathrm{Fil}} \xrightarrow{\Pi} \Pi \cdot \widetilde{\mathrm{Fil}} \to 0. \tag{5.14}$$

Since $\widetilde{\mathrm{Fil}} \neq \Pi \cdot M_{\mathcal{O}}$, by (5.14) we know that $\Pi \cdot \widetilde{\mathrm{Fil}} \neq \{0\}$. By Lemma 5.1.8, $\Pi \cdot \widetilde{\mathrm{Fil}} \subset \tilde{\mathcal{F}}^{\perp}$ and $\tilde{\mathcal{F}}^{\perp}$ has rank 1, we know that $\Pi \cdot \widetilde{\mathrm{Fil}} = \tilde{\mathcal{F}}^{\perp}$ by Nakayama's lemma. In particular $\tilde{\mathcal{F}}^{\perp}$ is determined by $\widetilde{\mathrm{Fil}}$. Moreover $\widetilde{\mathrm{Fil}}$ is determined by its image in the $\mathcal{O}$-module $(\widetilde{\mathrm{Fil}}^{-})^{\perp}/\widetilde{\mathrm{Fil}}^{-}$ where $\widetilde{\mathrm{Fil}}^{-} = \Pi \cdot \Lambda_{M,\mathcal{O}}$ as in Lemma 5.1.8. Equation (5.12) is still true and implies that $\widetilde{\mathrm{Fil}}^{-}$ is an isotropic free $\mathcal{O}$-module direct summand of $M_{\mathcal{O}}$ of rank $n-m$ (notice that $\pi = 0$ in $\mathcal{O}$). Notice that by (5.14) and the fact that $\Pi \cdot \widetilde{\mathrm{Fil}} = \tilde{\mathcal{F}}^{\perp}$ is free (in particular projective), $(\Pi \cdot M_{\mathcal{O}}) \cap \widetilde{\mathrm{Fil}}$ is a free direct summand of $\widetilde{\mathrm{Fil}}$ of corank 1. This implies that $(\Pi \cdot M_{\mathcal{O}}) \cap \widetilde{\mathrm{Fil}}$ is a free direct summand of $\Pi \cdot M_{\mathcal{O}}$ of corank 1 as well. So $(\Pi \cdot M_{\mathcal{O}}) \cap \widetilde{\mathrm{Fil}}/\widetilde{\mathrm{Fil}}^{-}$ is a hyperplane in the $\mathcal{O}$-module $\Pi \cdot M_{\mathcal{O}}/\widetilde{\mathrm{Fil}}^{-}$ of rank m, and is determined by $m-1$ parameters over $\bar{\kappa}$ as the tangent space of $\mathbb{P}^{m-1}_{\bar{\kappa}}$ has dimension $m-1$. Hence $(\Pi \cdot M_{\mathcal{O}}) \cap \widetilde{\mathrm{Fil}}$ is determined by $m-1$ parameters over $\bar{\kappa}$ as well. Since $\widetilde{\mathrm{Fil}}$ is maximal isotropic, it corresponds to a hyperplane in the rank two $\mathcal{O}$-module

$$((\Pi \cdot M_{\mathcal{O}}) \cap \widetilde{\mathrm{Fil}})^{\perp}/(\Pi \cdot M_{\mathcal{O}}) \cap \widetilde{\mathrm{Fil}},$$

hence is further determined by one parameter over $\bar{\kappa}$ as the tangent space of $\mathbb{P}^{1}_{\bar{\kappa}}$ has dimension 1. This proves case (a).

Case (b): $\mathrm{Fil} = \Pi \cdot M_{\bar{\kappa}}$. By Lemma 5.1.9, we know $\Pi \cdot M_{\mathcal{O}} \subset \Lambda_{M,\mathcal{O}}$ and $\Lambda_{M,\mathcal{O}}$ is a free $\mathcal{O}$-module direct summand of $M_{\mathcal{O}}$ of corank m. Hence $(\Lambda_{M,\mathcal{O}})^{\perp}$ is a free $\mathcal{O}$-module of rank m and is in $(\Pi \cdot M_{\mathcal{O}})^{\perp} = \Pi \cdot M_{\mathcal{O}}$. As in [Krä03], we assume that we have a $\mathcal{O} \otimes_{O_{F_0}} O_F$-basis $\{e_1, \ldots, e_n\}$ of $M_{\mathcal{O}}$ such that $\langle e_i, \Pi \cdot e_j \rangle = \delta_{ij}$ for $1 \leq i, j \leq n$ and all other pairings between these basis vectors are zero. The lift $\widetilde{\mathrm{Fil}}$ is spanned by $x_1, \ldots, x_n$ where

$$(x_1, \ldots, x_n) = (e_1, \ldots, e_n, \Pi e_1, \ldots, \Pi e_n) \begin{pmatrix} A\epsilon \\ I_n \end{pmatrix}$$

where $A \in M_n(\bar{\kappa})$ and $A = {}^tA$ since $\widetilde{\mathrm{Fil}}$ is isotropic. Assume $\mathcal{F}^\perp \subset \mathrm{Fil} = \Pi \cdot M_{\bar{\kappa}}$ is spanned by

$$\sum_{i=1}^{n} b_{n+i} \Pi \cdot e_i.$$

Then $b_{n+i} \neq 0$ for some i and we can assume without loss of generality $b_{n+1} = 1$. The lift $\tilde{\mathcal{F}}^\perp$ is spanned by

$$\sum_{i=1}^{n} \tilde{b}_i e_i + \sum_{i=n+1}^{2n} \tilde{b}_i \Pi \cdot e_i,$$

where $\tilde{b}_{n+1} = 1$ and $\tilde{b}_{n+i} = b_{n+i} + \epsilon c_i$ for $2 \leq i \leq n$ and some $c_i \in \bar{\kappa}$. Let

$$\lambda = {}^t(\tilde{b}_{n+1}, \ldots, \tilde{b}_{2n}).$$

Equations (4.7), (4.8) and (4.10) of [Krä03] tell us that

$$A = \gamma_1 \lambda \cdot {}^t\lambda$$

for some $\gamma_1 \in \bar{\kappa}$. Equation (4.5) of loc.cit. tells us

$${}^t(\tilde{b}_1, \ldots, \tilde{b}_n) = A\lambda,$$

which is equal to $\gamma_1 \lambda \cdot {}^t\lambda \cdot \lambda = 0$ as ${}^t\lambda \cdot \lambda = 0$ by (4.9) of loc.cit.. In particular $\tilde{\mathcal{F}}^\perp \subset \Pi M_{\mathcal{O}}$ and a point in $T_z(\mathcal{N}_{\Lambda,\bar{\kappa}})$ is determined by the $n-1$ parameters c_i for $2 \leq i \leq n$ together with the additional parameter γ_1. Now the condition $\tilde{\mathcal{F}}^\perp \subset (\Lambda_{M,\mathcal{O}})^\perp$ (condition (e) of Lemma 5.1.8) imposes further $n - m$ independent linear equations on the parameters c_i for $2 \leq i \leq n$. This shows that the tangent space $T_z(\mathcal{N}_{\Lambda,\bar{\kappa}})$ has dimension less or equal to m. This finishes the proof of the proposition. $\square$

5.1.3 Isomorphism between $\mathcal{N}_\Lambda$ and $Y_{V,\bar{\kappa}}$

By [RTW14, Lemma 6.1], the lattices $\breve{\Lambda}$ and $\breve{\Lambda}^\sharp$ (see (5.5)) are closed under Π, $\mathbf{V}$ and $\mathbf{F}$, hence determine supersingular p-divisible strict O_{F_0}-modules with O_F-action X_- and X_+ (denoted by X_{Λ^-} and X_{Λ^+} resp. in [RTW14, §6]) of dimension n over $\bar{\kappa}$ together with quasi-isogenies $\rho_- : X_- \to \mathbb{X}$ of height m and $\rho_+ : X_+ \to \mathbb{X}$ of height $-m$. The inclusion $\breve{\Lambda} \subset \breve{\Lambda}^\sharp$ also defines an isogeny $\rho_\Lambda : X_- \to X_+$ of height $2m$. Since $X_- \cong \mathbb{Y}^n$ as an O_F-module for any $\bar{\kappa}$-scheme S, on the special fiber condition (1) in Definition 5.1.4 is equivalent to the condition

$$(1)' : \text{ The quasi-isogeny } \rho_{X,-} := \rho^{-1} \circ (\rho_-)_S : (X_-)_S \to X \text{ is an isogeny.} \tag{5.15}$$

This is further equivalent by loc.cit. to the condition

$$(1)'' : \text{ The quasi-isogeny } \rho_{X,+} := (\rho_+)_S^{-1} \circ \rho : X \to (X_+)_S \text{ is an isogeny.} \tag{5.16}$$

Lemma 5.1.13. *The functor $\mathcal{N}_{\Lambda,\bar{\kappa}}$ is representable by a projective scheme over $\bar{\kappa}$. The functor morphism $\mathcal{N}_{\Lambda,\bar{\kappa}} \to \mathcal{N}$ is a closed immersion.*

Proof. $\mathcal{Z}(\Lambda)_{\bar{\kappa}}$ is a closed formal subscheme of $\mathcal{N}$. Since for any $\bar{\kappa}$-scheme S, Condition (1) in Definition 5.1.4 is equivalent to (5.15), the functor $\mathcal{Z}(\Lambda)_{\bar{\kappa}}$ can be represented by a projective scheme over $\bar{\kappa}$ by exact the same argument as that of [VW11, Lemma 3.2]. Condition (2) of Definition 5.1.4 defines $\mathcal{N}_{\Lambda,\bar{\kappa}}$ as a closed subscheme of $\mathcal{Z}(\Lambda)_{\bar{\kappa}}$, hence is itself projective over $\bar{\kappa}$ and a closed formal subscheme of $\mathcal{N}$. This finishes the proof of the lemma. $\square$

In the following discussion we assume that Λ has type greater or equal to 2. Let $V = \Lambda^\sharp/\Lambda$ and define a symplectic form $\langle\, , \rangle_V$ on V as follows. For $\bar{x}, \bar{y} \in V$ with lifts $x, y \in \Lambda^\sharp$, define $\langle \bar{x}, \bar{y}\rangle_V$ by the image of $\pi_0\delta\langle x, y\rangle$ in $\mathbb{F}_q$ (see §2.1.2). Extend this form bilinearly to $V_{\bar{\kappa}}$. Note that τ induces identity on V and the Frobenius Φ on $V_{\bar{\kappa}}$. Let R be a $\bar{\kappa}$-algebra and $(X, \iota, \lambda, \rho, \mathcal{F}) \in \mathcal{N}_\Lambda(R)$. As in the proof of [VW11, Corollary 3.9],

$\mathrm{Image}(D((\rho_\Lambda)_R))$ is a locally free direct summands of $D((X_+)_R)$ of corank $2m$ and

$$D((X_+)_R)/\mathrm{Image}(D((\rho_\Lambda)_R)) \cong \breve{\Lambda}^\sharp/\breve{\Lambda} \otimes_{\bar{\kappa}} R = V_R.$$

As $(\rho_\Lambda)_R = \rho_{X,+} \circ \rho_{X,-}$, we know $\ker(\rho_{X,+}) = \ker((\rho_\Lambda)_R)/\ker(\rho_{X,-})$ as a quotient of finite group schemes over $\mathrm{Spec}\, R$. Since $\Pi\breve{\Lambda}^\sharp \subset \breve{\Lambda}$, by relative Dieudonné theory, we know $\ker(\rho_\Lambda) \subset X_-[\iota_{X_-}(\pi)]$ or equivalently $\iota_{X_-}(\pi) \cdot \ker(\rho_\Lambda) = \{0\}$. Hence $\iota(\pi) \cdot \ker(\rho_{X,+}) = \{0\}$ or equivalently $\ker(\rho_{X,+}) \subset X[\iota(\pi)]$. Thus there exists an isogeny $\tilde{\rho}_{X,+} : X_+ \to X$ such that $\tilde{\rho}_{X,+} \circ \rho_{X,+}$ is the isogeny $\iota(\pi) : X \to X$. Recall Fil in the exact sequence (5.6).

Lemma 5.1.14. $D(\tilde{\rho}_{X,+})^{-1}(\mathrm{Fil})$ *is a locally free direct summand of* $D(X_+)$ *that contains* $D((\rho_\Lambda)_R)$. *Moreover the quotient*

$$U(X) := D(\tilde{\rho}_{X,+})^{-1}(\mathrm{Fil})/\mathrm{Image}(D((\rho_\Lambda)_R)) \tag{5.17}$$

is a locally free direct summand of V_R *of rank* m.

Proof. By universality, it suffices to check the case when $\mathrm{Spec}\, R$ is an affine sub formal scheme of $\mathcal{N}_\Lambda$. In this case, by Nakayama's lemma, it suffices to check the condition on the $\bar{\kappa}$-points of $\mathcal{N}_\Lambda$. A point $z \in \mathcal{N}_\Lambda(\bar{\kappa})$ corresponds to a pair (M, M') as in Corollary 5.1.6. Then the isogeny $\tilde{\rho}_{X,+}$ is induced by the map of relative Dieudonné modules $\breve{\Lambda}^\sharp \to M : x \mapsto \Pi \cdot x$. Recall $\mathrm{Fil} = \mathbf{V}M$. So

$$D(\tilde{\rho}_{X,+})^{-1}(\mathrm{Fil}) = \Pi^{-1}\mathbf{V}M/\pi_0\breve{\Lambda}^\sharp = \tau^{-1}(M)/\pi_0\breve{\Lambda}^\sharp.$$

Since $\breve{\Lambda} \subset M$, we have $\breve{\Lambda} = \tau^{-1}(\breve{\Lambda}) \subset \tau^{-1}(M)$. So

$$\mathrm{Image}(D((\rho_\Lambda)_R)) = \breve{\Lambda}/\pi_0\breve{\Lambda}^\sharp \subset D(\tilde{\rho}_{X,+})^{-1}(\mathrm{Fil}).$$

The condition $M = M^\sharp$ is equivalent to the fact that $\Phi(U(X))$ is Lagrangian in V, which in turn is equivalent to the fact that $U(X)$ is Lagrangian which implies $\dim_{\bar{\kappa}} U(X) = m$. $\square$

By Condition (2) of Definition 5.1.4, we know $\mathrm{Image}(D((\rho_\Lambda)_R)) \subset D(\rho_{X,+})(q_X^{-1}(\mathcal{F}))$ where $q_X : D(X) \to \mathrm{Lie}X$ is the natural quotient homomorphism of R-modules (see the proof of Corollary 5.1.6). Define

$$\mathcal{F}(X) := D(\rho_{X,+})(q_X^{-1}(\mathcal{F}))/\mathrm{Image}(D((\rho_\Lambda)_R)).$$

Then $\mathcal{F}(X)$ is a locally free direct summand of $U(X)$ of rank $m-1$. We define a map $\phi : \mathcal{N}_{\Lambda,\bar{\kappa}} \to \mathrm{Gr}(m, V_{\bar{\kappa}}) \times \mathrm{Gr}(m-1, V_{\bar{\kappa}})$ by

$$\phi : (X, \iota, \lambda, \rho, \mathcal{F}) \mapsto (U(X), \mathcal{F}(X)) \in (\mathrm{Gr}(m, V_{\bar{\kappa}}) \times \mathrm{Gr}(m-1, V_{\bar{\kappa}}))(R).$$

Lemma 5.1.15. *ϕ defines a bijection between $\mathcal{N}_\Lambda(\bar{\kappa})$ and $Y_V(\bar{\kappa})$.*

Proof. A point $z \in \mathcal{N}_\Lambda(\bar{\kappa})$ corresponds to a pair (M, M') as in Corollary 5.1.6. By the definition of ϕ we have $\phi(z) = (U, U')$ where

$$(U, U') = (\Pi^{-1}\mathbf{V}M/\breve{\Lambda}, M'/\breve{\Lambda}) = (\tau^{-1}(M)/\breve{\Lambda}, M'/\breve{\Lambda}) = (\Phi^{-1}(M/\breve{\Lambda}), M'/\breve{\Lambda}).$$

As in the proof of Lemma 5.1.14, the condition $M = M^\sharp$ is equivalent to the condition that U is Lagrangian. The condition $M' \subset M$ is equivalent to $U' \subset \Phi(U)$. The condition $M' \subset \tau^{-1}(M)$ is equivalent to $U' \subset U$. This shows that $\phi(z) \in Y_V(\bar{\kappa})$.

Conversely assume $(U, U') \in Y_V(\bar{\kappa})$ and let $M = \mathrm{Pr}^{-1}(\Phi(U))$ and $M' = \mathrm{Pr}^{-1}(U')$ where $\mathrm{Pr} : \breve{\Lambda}^\sharp \to \breve{\Lambda}^\sharp/\breve{\Lambda}$ is the natural quotient map. Then by definition $\breve{\Lambda} \subset M' \subset M \subset \breve{\Lambda}^\sharp$, and $M = M^\sharp$ as U is Lagrangian. Since $\mathbf{V}M \subset \mathbf{V}\breve{\Lambda}^\sharp = \Pi\breve{\Lambda}^\sharp \subset \breve{\Lambda} \subset M'$, we have $\mathbf{V}M \subset M'$. We also have

$$\Pi M \subset \Pi\breve{\Lambda}^\sharp \subset \breve{\Lambda} = \tau^{-1}\breve{\Lambda} \subset \tau^{-1}(M),$$

and

$$\tau^{-1}(M) \subset \tau^{-1}(\breve{\Lambda}^\sharp) = \breve{\Lambda}^\sharp \subset \Pi^{-1}\breve{\Lambda} \subset \Pi^{-1}M. \tag{5.18}$$

Hence $\Pi M \subset \tau^{-1}(M) \subset \Pi^{-1}M$. This shows that (M, M') satisfies the conditions in Proposition 5.1.5 and Corollary 5.1.6. This defines the inverse of ϕ on the level of $\bar{\kappa}$-points. Hence ϕ defines a bijection between $\mathcal{N}_\Lambda(\bar{\kappa})$ and $Y_V(\bar{\kappa})$. $\square$

Theorem 5.1.16. *Let Λ be a vertex lattice of type $2m$ ($m \geq 1$) in $\mathbb{V}$. Then $\mathcal{N}_\Lambda$ is reduced and the morphism ϕ defines an isomorphism $\mathcal{N}_\Lambda \to Y_{V,\bar{\kappa}}$. In particular $\mathcal{N}_\Lambda$ is smooth of dimension m over $\bar{\kappa}$.*

Proof. Let $\mathcal{N}_\Lambda^{\mathrm{red}}$ be the underlying reduced $\bar{\kappa}$-scheme of $\mathcal{N}_\Lambda$. Lemma 5.1.15 shows that ϕ induces a morphism $\phi^{\mathrm{red}} : \mathcal{N}_\Lambda^{\mathrm{red}} \to Y_{V,\bar{\kappa}}$ which is a bijection on $\bar{\kappa}$-points, in particular quasi-finite. Since ϕ^{red} is a morphism between projective varieties, it is projective. Moreover using the theory of relative displays and windows, working with Cohen rings instead of the Witt ring, we can show that $\mathcal{N}_\Lambda^{\mathrm{red}}(R) = Y_V(R)$ for any field R containing $\bar{\kappa}$ by the same proof as that of Lemma 5.1.15. In particular ϕ^{red} is birational. Being quasi-finite and proper at the same time, it is an isomorphism by Zariski's main theorem since $Y_{V,\bar{\kappa}}$ is normal. Now Proposition 5.1.12 implies that $\mathcal{N}_\Lambda^{\mathrm{red}} = \mathcal{N}_{\Lambda,\bar{\kappa}}$. By [RTZ13, Lemma 10.3] and Corollary 5.1.11, we have $\mathcal{N}_\Lambda = \mathcal{N}_{\Lambda,\bar{\kappa}}$. This finishes the proof of the theorem. $\square$

Proposition 5.1.17. *Let Λ be a vertex lattice of type 0 in $\mathbb{V}$. Then $\mathcal{N}_\Lambda$ is the exceptional divisor Exc_Λ and is isomorphic to $\mathbb{P}^{n-1}_{\bar{\kappa}}$.*

Proof. Let R be any $\bar{\kappa}$-algebra and z be any point in $\mathcal{N}_\Lambda(R)$ and $(X, \iota, \lambda, \rho, \mathcal{F})$ be the pullback of the universal object of $\mathcal{N}$ to z. As Λ is a unimodular lattice, the quasi-isogeny ρ_- has height 0. Thus the isogeny

$$\rho_{X,-} = \rho^{-1} \circ (\rho_-)_R : (X_-)_R \to X$$

has height 0 and is an isomorphism, hence we can identify $(X, \ldots, \rho)$ with $((X_-)_R, \ldots, (\rho_-)_R)$. As $\Pi|_\Lambda = \mathbf{V}|_\Lambda$ for any vertex lattice Λ, and $\mathrm{Lie} X_- = M(X_-)/\mathbf{V}M(X_-)$, the action of $\iota(\pi)$ on $\mathrm{Lie} X_-$ is trivial. The point z is uniquely determined by the filtration $\mathcal{F} \subset \mathrm{Lie} X$. Hence $\mathcal{F}$ can be any rank $n-1$ locally free R-module on $\mathrm{Lie} X$. This shows that $\mathcal{N}_\Lambda$ is isomorphic to $\mathbb{P}_{\bar{\kappa}}^{n-1}$ and is in particular reduced. Moreover if $R = \bar{\kappa}$, then $\rho(M(X)) = \breve{\Lambda}$. This shows that $\mathcal{N}_\Lambda$ is a subscheme of Exc_Λ according Definition 2.1.5. By the proof of Lemma 2.1.6, we know that $\mathcal{N}_\Lambda$ and Exc_Λ have the same $\bar{\kappa}$-points. As they are both reduced subscheme of $\mathcal{N}$, they must be the same. This proves the proposition. □

5.1.4 Bruhat-Tits stratification

Lemma 5.1.18. *For any pair $(M, \mathcal{F})$ satisfying the condition in Proposition 5.1.5, there is a unique vertex lattice $\Lambda(M)$ such that $\Lambda(M) \subset M$ and $\Lambda(M)$ is maximal among all such vertex lattices.*

Proof. This is essentially [RTW14, Proposition 4.1] as such M satisfies the conditions in Proposition 2.4 of loc.cit.. □

Theorem 5.1.19. *There is a stratification of $\mathcal{N}_{\mathrm{red}}$ by closed strata $\mathcal{N}_\Lambda$ given by*

$$\mathcal{N}_{\mathrm{red}} = \bigcup_{\Lambda} \mathcal{N}_\Lambda. \tag{5.19}$$

where the union is over all vertex lattices in $\mathbb{V}$. We call this the Bruhat-Tits stratification of $\mathcal{N}_{\mathrm{red}}$. In the following, assume that Λ and Λ' are vertex lattices of type greater or equal to 2, and Λ_0 and Λ_0' are vertex lattices of type 0.

1. *If $\Lambda \subset \Lambda'$, then $\mathcal{N}_{\Lambda'}$ is a subscheme of $\mathcal{N}_\Lambda$.*
2. *The intersection of $\mathcal{N}_{\Lambda'} \cap \mathcal{N}_\Lambda$ is nonempty if and only if $\Lambda'' = \Lambda + \Lambda'$ is a vertex lattice, in which case we have $\mathcal{N}_{\Lambda'} \cap \mathcal{N}_\Lambda = \mathcal{N}_{\Lambda''}$.*
3. *The intersection of $\mathcal{N}_{\Lambda_0'} \cap \mathcal{N}_{\Lambda_0}$ is always empty if $\Lambda_0 \neq \Lambda_0'$.*
4. *The intersection $\mathcal{N}_\Lambda \cap \mathcal{N}_{\Lambda_0}$ is nonempty if and only if $\Lambda \subset \Lambda_0$ in which case $\mathcal{N}_\Lambda \cap \mathcal{N}_{\Lambda_0}$ is isomorphic to $\mathbb{P}_{\bar{\kappa}}^{m-1}$ where $2m$ is the type of Λ.*

Proof. To prove (5.19), it suffices to check this on $\bar{\kappa}$-points. A point $z \in \mathcal{N}_{\mathrm{red}}(\bar{\kappa})$ corresponds to a pair (M, M') as in Proposition 5.1.5. Take $\Lambda = \Lambda(M)$ as in Lemma 5.1.18.

If Λ has type 0, then both $\breve{\Lambda}$ and M are unimodular and $\breve{\Lambda} \subset M$, so they have to be equal. Hence $z \in \mathcal{N}_\Lambda$ by Corollary 5.1.6. If Λ is not of type 0, then M is not τ-invariant, hence $M' = M \cap \tau^{-1}(M)$ is uniquely determined. Since Λ is τ-invariant, $\breve{\Lambda} \subset M'$. Hence $z \in \mathcal{N}_\Lambda(\bar{\kappa})$ by Corollary 5.1.6. This proves (5.19).

(1) follows immediately from Definition 5.1.4.

(2). If Λ'' is a vertex lattice, then $\mathcal{N}_\Lambda \cap \mathcal{N}_{\Lambda'} = \mathcal{N}_{\Lambda''}$ by Definition 5.1.4. Conversely if $\mathcal{N}_{\Lambda'} \cap \mathcal{N}_\Lambda(\bar{\kappa})$ is nonempty, let $(M, M') \in \mathcal{N}_{\Lambda'} \cap \mathcal{N}_\Lambda(\bar{\kappa})$. Then $\Lambda(M) \supset \Lambda + \Lambda'$ by the maximality of $\Lambda(M)$. Then $\Lambda + \Lambda' \subset \Lambda(M) \subset \Lambda(M)^\sharp \subset \Lambda^\sharp \cap (\Lambda')^\sharp = (\Lambda + \Lambda')^\sharp$. Hence $\Lambda + \Lambda'$ is a vertex lattice.

(3) follows directly from Corollary 5.1.6.

(4). By Corollary 5.1.6, a point $(M, M') \in \mathcal{N}(\bar{\kappa})$ is in $\mathcal{N}_\Lambda \cap \mathcal{N}_{\Lambda_0}$ if and only if $M = \Lambda_0 \otimes_{O_F} O_{\breve{F}}$ and $\Lambda \subset M' \subset M$. This show that $\Lambda \subset \Lambda_0$ and M' corresponds to a point in $\mathbb{P}(\Lambda_0/\Lambda)(\bar{\kappa})$. Hence $\mathcal{N}_\Lambda \cap \mathcal{N}_{\Lambda_0}(\bar{\kappa}) = \mathbb{P}(\Lambda_0/\Lambda)(\bar{\kappa})$. Similarly one can show that

$$\mathcal{N}_\Lambda \cap \mathcal{N}_{\Lambda_0}(R) = \mathbb{P}(\Lambda_0/\Lambda)(R)$$

for any $\bar{\kappa}$-algebra R. This finishes the proof of (4). $\square$

Proposition 5.1.20. *For a rank r lattice $L \subset \mathbb{V}$, the reduced subscheme $\mathcal{Z}(L)_{\mathrm{red}}$ of $\mathcal{Z}(L)$ is a union of Bruhat-Tits strata:*

$$\mathcal{Z}(L)_{\mathrm{red}} = \bigcup_{L \subset \Lambda} \mathcal{N}_\Lambda, \tag{5.20}$$

where the union is taken over all vertex lattices Λ such that $L \subset \Lambda$. Moreover, the intersection of $\mathcal{Z}(L)$ with $\mathcal{N}_\Lambda$ is nonempty if and only if $L \subset \Lambda^\sharp$.

Proof. The proof of (5.20) is the same as that of [Shi18, Proposition 3.8].

If $L \subset \Lambda^\sharp$ and L is integral, define $\Lambda' := L + \Lambda$. Then Λ' is a vertex lattice and $\Lambda \subset \Lambda'$. By Theorem 5.1.19 (1) and the definition of $\mathcal{Z}(L)$, $\mathcal{N}_{\Lambda'}$ is in the intersection of $\mathcal{Z}(L)$ and $\mathcal{N}_\Lambda$.

Conversely if the intersection of $\mathcal{Z}(L)$ and $\mathcal{N}_\Lambda$ is not empty, then by (5.20) and Theorem 5.1.19, there exists a vertex lattice Λ' such that $\Lambda \subset \Lambda'$ and $L \subset \Lambda'$. Since $\Lambda' \subset (\Lambda')^\sharp \subset \Lambda^\sharp$, we know that $L \subset \Lambda^\sharp$. This finishes the proof of the lemma. $\square$

5.2 Fourier transform: the geometric side

5.2.1 Horizontal and vertical part of ${}^{\mathbb{L}}\mathcal{Z}(L^\flat)$

Recall that we have the following definition of horizontal lattices.

Definition 5.2.1. *Let $L^\flat$ be a rank $n-1$ integral lattice in $\mathbb{V}$. We say that $L^\flat$ is horizontal if one of the following conditions is satisfied.*

1. *$L^\flat$ is unimodular.*
2. *$L^\flat$ is of the form $L^\flat = M \obot L'$ where M is a unimodular sublattice of rank $n-2$ such that $(M_F)^\perp$ (the perpendicular complement of M_F in $\mathbb{V}$) is nonsplit.*

We denote the set of horizontal lattices by Hor.

Lemma 5.2.2. *Let $L^\flat$ be a rank $n-1$ lattice in $\mathbb{V}$. Then $L^\flat$ is horizontal if and only if there is a unique vertex lattice Λ which contains $L^\flat$. If this is the case, Λ is of type* 0.

Proof. We first prove the "only if" direction. Let Λ be any vertex lattice containing $L^\flat$. If $L^\flat$ is unimodular, then Λ has to be of the form $L^\flat \obot L'$ where L' is the unique unimodular lattice in $(L^\flat_F)^\perp$. If $L^\flat$ is of the form $M \obot L'$ such that M is of rank $n-2$ and $(M_F)^\perp$ is nonsplit, then the proof of [Shi18, Theorem 3.10] implies that there is a unique vertex lattice Λ' in $(M_F)^\perp$ which is of unimodular (this corresponds to the fact that the Bruhat-Tits building of $(M_F)^\perp$ has only one point). Then Λ must be of the form $M \obot \Lambda'$. In both cases, Λ is unique and is of type 0.

We now prove the "if" direction. If $t(L^\flat) \geq 2$, then there exist a type 2 vertex lattice Λ_2 containing $L^\flat$ and any type 0 vertex lattice containing Λ_2 (there are $q+1$ of them) also contains $L^\flat$. Hence $t(L^\flat) \leq 1$ and $L^\flat$ is of the form $M \obot L'$ such that M is of rank $n-2$.

If $(M_F)^\perp$ is split and $\mathrm{val}(L') > 0$, then there are more than one type 0 vertex lattices Λ' in $(M_F)^\perp$ containing L'. For any such Λ', $M \obot \Lambda'$ is a vertex lattice of type 0 containing $L^\flat$. This shows that in order for such Λ to be unique, $L^\flat$ must satisfies the conditions in Definition 5.2.1. The lemma is proved. $\square$

Recall that for a rank $n-1$ lattice $L^\flat$ in $\mathbb{V}$, we define

$$\mathrm{Hor}(L^\flat) := \{M^\flat \in \mathrm{Hor} \mid L^\flat \subset M^\flat\}. \tag{5.21}$$

When $\dim(\mathbb{V}) = 2$ and $\chi(\mathbb{V}) = -1$, for $y \in \mathbb{V}$, define

$$\tilde{\mathcal{Z}}(y)^\circ := \begin{cases} \mathcal{Z}^+_{\mathrm{val}(y)} \sqcup \mathcal{Z}^-_{\mathrm{val}(y)} & \text{if } \mathrm{val}(y) > 0, \\ \mathcal{Z}_0 & \text{if } \mathrm{val}(y) = 0. \end{cases}$$

Here $\mathcal{Z}_0 \cong \mathrm{Spf}\, \mathcal{O}_{\breve{F}}$ and $\mathcal{Z}^+_s \cong \mathcal{Z}^-_s \cong \mathrm{Spf}\, W_s$ are quasi-canonical lifting cycles defined in [Shi22, §3] where W_s is a totally ramified abelian extension of $O_{\breve{F}}$ of degree q^s. When $\dim(\mathbb{V}) = 2$ and $\chi(\mathbb{V}) = 1$, for $y \in \mathbb{V}^{=0}$, define $\tilde{\mathcal{Z}}(y)^\circ$ to be $\mathcal{Z}^h(y)$, where $\mathcal{Z}^h(y) \cong \mathrm{Spf}\, O_{\breve{F}}$ is as in Theorem 3.3.1. In all cases, $\tilde{\mathcal{Z}}(y)^\circ$ is a closed subscheme of $\mathcal{N}_2$.

For a $M^\flat \in \mathrm{Hor}$, we can decompose $M^\flat$ as $M \obot \mathrm{Span}\{y\}$ where M is unimodular and $\mathrm{val}(y)$ has to be zero if $(M_F)^\perp$ is split. By Proposition 4.7.1, the unimodular lattice M induces a closed embedding $\mathcal{N}_2 \hookrightarrow \mathcal{N}_n$. We define $\tilde{\mathcal{Z}}(M^\flat)^\circ$ to be the image of the composed embedding $\tilde{\mathcal{Z}}(y)^\circ \hookrightarrow \mathcal{N}_2 \hookrightarrow \mathcal{N}_n$ where $\tilde{\mathcal{Z}}(y)^\circ$ is the closed formal subscheme of $\mathcal{N}_2$ defined above. Moreover by loc. cit., the definition of $\tilde{\mathcal{Z}}(M^\flat)^\circ$ is independent of the choice of M.

Recall the following theorem from Chapter 4 (Theorem 4.8.2):

Theorem 5.2.3. *Let $L^\flat$ be a rank $n-1$ non-degenerate integral lattice in $\mathbb{V}$, then*

$$\mathcal{Z}(L^\flat)_{\mathscr{H}} = \bigcup_{M^\flat \in \mathrm{Hor}(L^\flat)} \tilde{\mathcal{Z}}(M^\flat)^\circ. \tag{5.22}$$

In particular, $\mathcal{Z}(L^\flat)_{\mathscr{H}}$ is of pure dimension 1. We have the following identity in $\mathrm{Gr}^{n-1}K_0^{\mathcal{Z}(L^\flat)}(\mathcal{N})$:

$$[\mathcal{O}_{\mathcal{Z}(L^\flat)_{\mathscr{H}}}] = \sum_{M^\flat \in \mathrm{Hor}(L^\flat)} [\mathcal{O}_{\tilde{\mathcal{Z}}(M^\flat)^\circ}]. \tag{5.23}$$

Lemma 5.2.4. *For any formal subscheme Z of $\mathcal{N}$ and $0 \leq i \leq n$, $F^i K_0^Z(\mathcal{N})$ is an ideal in $K_0(\mathcal{N})$.*

Proof. By definition $F^i K_0^Z(\mathcal{N})$ is generated by elements of the form $[\mathcal{F}^\bullet]$ where $\mathcal{F}^\bullet$ is a finite complex of locally free coherent $\mathcal{O}_{\mathcal{N}}$-modules acyclic outside a sub formal scheme Y of Z such that the codimension of Y in $\mathcal{N}$ is greater or equal to i. By Kunneth formula for chain complexes, the product complex $\mathcal{F}^\bullet \otimes_{\mathcal{O}_{\mathcal{N}}} \mathcal{K}^\bullet$ is acyclic outside Y as well for any finite complexes of locally free coherent $\mathcal{O}_{\mathcal{N}}$-modules $\mathcal{K}^\bullet$. This proves the lemma. □

By Lemma 5.2.4, for any formal subscheme Z of $\mathcal{N}$, we can define a quotient ring (not necessary with identity)

$$\mathrm{Gr}' K_0^Z(\mathcal{N}) := K_0^Z(\mathcal{N}) / F^n K_0^Z(\mathcal{N}). \tag{5.24}$$

In particular $\mathrm{Gr}^{n-1} K_0^Z(\mathcal{N}) = F^{n-1} K_0^Z(\mathcal{N}) / F^n K_0^Z(\mathcal{N})$ is a subgroup of $\mathrm{Gr}' K_0^Z(\mathcal{N})$.

Let $L^\flat$ be a rank $n-1$ non-degenerate integral lattice. Since $\mathcal{Z}(L^\flat)_{\mathscr{H}}$ is one-dimensional, the intersection $\mathcal{Z}(L^\flat)_{\mathscr{H}} \cap \mathcal{Z}(L^\flat)_{\mathscr{V}}$ must be 0-dimensional if nonempty. It follows that there is a decomposition

$$\mathrm{Gr}' K_0^{\mathcal{Z}(L^\flat)}(\mathcal{N}) = \mathrm{Gr}' K_0^{\mathcal{Z}(L^\flat)_{\mathscr{H}}}(\mathcal{N}) \oplus \mathrm{Gr}' K_0^{\mathcal{Z}(L^\flat)_{\mathscr{V}}}(\mathcal{N}). \tag{5.25}$$

Under this decomposition, we have

$$^{\mathbb{L}}\mathcal{Z}(L^\flat) = {}^{\mathbb{L}}\mathcal{Z}(L^\flat)_{\mathscr{H}} + {}^{\mathbb{L}}\mathcal{Z}(L^\flat)_{\mathscr{V}} \in \mathrm{Gr}' K_0^{\mathcal{Z}(L^\flat)}(\mathcal{N}), \tag{5.26}$$

where we denote by the same notation the image of $^{\mathbb{L}}\mathcal{Z}(L^\flat)$ under the natural quotient map $K_0^{\mathcal{Z}(L^\flat)}(\mathcal{N}) \to \mathrm{Gr}' K_0^{\mathcal{Z}(L^\flat)}(\mathcal{N})$. It follows that the element $^{\mathbb{L}}\mathcal{Z}(L^\flat)_{\mathscr{V}} \in \mathrm{Gr}' K_0^{\mathcal{Z}(L^\flat)}(\mathcal{N})$

is canonically defined although $\mathcal{Z}(L^\flat)_{\mathscr{V}}$ depends on the choice of a large integer $m \gg 0$.

Since $\mathcal{Z}(L^\flat)_{\mathscr{H}}$ has expected dimension, ${}^{\mathbb{L}}\mathcal{Z}(L^\flat)_{\mathscr{H}}$ is in fact in $\mathrm{Gr}^{n-1}K_0^{\mathcal{Z}(L^\flat)}(\mathcal{N})$ and is represented by the structure sheaf of $\mathcal{Z}(L^\flat)_{\mathscr{H}}$. In order to match the analytic side of our conjecture, we need to slightly modify ${}^{\mathbb{L}}\mathcal{Z}(L^\flat)_{\mathscr{H}}$.

Definition 5.2.5. *Let $L^\flat$ be a horizontal lattice in $\mathbb{V}$. Define ${}^{\mathbb{L}}\mathcal{Z}(L^\flat)^\circ \in \mathrm{Gr}'K_0^{\mathcal{Z}(L^\flat)}(\mathcal{N})$ by*

$$
{}^{\mathbb{L}}\mathcal{Z}(L^\flat)^\circ = \begin{cases} [\mathcal{O}_{\tilde{\mathcal{Z}}(L^\flat)^\circ}] + \frac{1-(-1)^{n-1}}{2}[\mathcal{O}_{\mathbb{P}_\Lambda}] & \text{if } L^\flat \text{ is unimodular,} \\ [\mathcal{O}_{\tilde{\mathcal{Z}}(L^\flat)^\circ}] + [\mathcal{O}_{\mathbb{P}_\Lambda}] & \text{otherwise,} \end{cases}
$$

where Λ is the unique type 0 vertex lattice containing $L^\flat$ as in Lemma 5.2.2 and $\mathbb{P}_\Lambda$ is a projective line over $\bar{\kappa}$ in Exc_Λ.

Remark 5.2.6. *${}^{\mathbb{L}}\mathcal{Z}(L^\flat)^\circ$ is simply the difference cycle $\mathcal{D}(L^\flat)$ defined previously.*

Definition 5.2.7. *Let $L^\flat$ be a rank $n-1$ non-degenerate integral lattice. Define ${}^{\mathbb{L}}\mathcal{Z}(L^\flat)^*_{\mathscr{H}} \in \mathrm{Gr}'K_0^{\mathcal{Z}(L^\flat)}(\mathcal{N})$ by*

$$
{}^{\mathbb{L}}\mathcal{Z}(L^\flat)^*_{\mathscr{H}} := \sum_{M^\flat \in \mathrm{Hor}(L^\flat)} {}^{\mathbb{L}}\mathcal{Z}(M^\flat)^\circ,
$$

where $\mathcal{Z}(M^\flat)^\circ_{\mathscr{H}}$ is as in Definition 5.2.5. Define the modified vertical part of the derived special cycle ${}^{\mathbb{L}}\mathcal{Z}(L^\flat)$ by

$$
{}^{\mathbb{L}}\mathcal{Z}(L^\flat)^*_{\mathscr{V}} := {}^{\mathbb{L}}\mathcal{Z}(L^\flat) - {}^{\mathbb{L}}\mathcal{Z}(L^\flat)^*_{\mathscr{H}} \in \mathrm{Gr}'K_0^{\mathcal{Z}(L^\flat)}(\mathcal{N}).
$$

For any $x \in \mathbb{V} \setminus L^\flat_F$, define

$$
\mathrm{Int}_{L^\flat,\mathscr{H}}(x) := \chi(\mathcal{N}, {}^{\mathbb{L}}\mathcal{Z}(L^\flat)^*_{\mathscr{H}} \cdot [\mathcal{O}_{\mathcal{Z}(x)}]), \text{ and } \mathrm{Int}_{L^\flat,\mathscr{V}}(x) := \chi(\mathcal{N}, {}^{\mathbb{L}}\mathcal{Z}(L^\flat)^*_{\mathscr{V}} \cdot [\mathcal{O}_{\mathcal{Z}(x)}]). \tag{5.27}
$$

Lemma 5.2.8. *For a rank $n-1$ non-degenerate integral lattice $L^\flat$, we have*

$$
{}^{\mathbb{L}}\mathcal{Z}(L^\flat)^*_{\mathscr{V}} \in \mathrm{Gr}'K_0^{\mathcal{Z}(L^\flat)_{\mathscr{V}}}(\mathcal{N}).
$$

Proof. By the definition of ${}^{\mathbb{L}}\mathcal{Z}(L^\flat)^*_{\mathscr{V}}$, the decomposition (5.26) and Theorem 5.2.3, we have

$$\begin{aligned}{}^{\mathbb{L}}\mathcal{Z}(L^\flat)^*_{\mathscr{V}} =& {}^{\mathbb{L}}\mathcal{Z}(L^\flat)_{\mathscr{V}} + {}^{\mathbb{L}}\mathcal{Z}(L^\flat)_{\mathscr{H}} - \sum_{M^\flat \in \mathrm{Hor}(L^\flat)} {}^{\mathbb{L}}\mathcal{Z}(M^\flat)^\circ \\ =& {}^{\mathbb{L}}\mathcal{Z}(L^\flat)_{\mathscr{V}} + \sum_{M^\flat \in \mathrm{Hor}(L^\flat)} ([\mathcal{O}_{\mathcal{Z}(M^\flat)^\circ}] - {}^{\mathbb{L}}\mathcal{Z}(M^\flat)^\circ).\end{aligned}$$

We know all terms in the last expression are in $\mathrm{Gr}' K_0^{\mathcal{Z}(L^\flat)_{\mathscr{V}}}(\mathcal{N})$ by the definition of ${}^{\mathbb{L}}\mathcal{Z}(L^\flat)_{\mathscr{V}}$ and Definition 5.2.5. □

Lemma 5.2.9. *If $L^\flat$ is a horizontal lattice of rank $n-1$ in $\mathbb{V}$, then*

$$ {}^{\mathbb{L}}\mathcal{Z}(L^\flat)^*_{\mathscr{H}} = {}^{\mathbb{L}}\mathcal{Z}(L^\flat). \tag{5.28}$$

In particular for any $x \in \mathbb{V} \setminus L^\flat_F$ we have

$$\mathrm{Int}_{L^\flat, \mathscr{H}}(x) = \mathrm{Int}_{L^\flat}(x).$$

Proof. Let Λ be the unique type 0 vertex lattice containing $L^\flat$ as indicated by Lemma 5.2.2. Then $\Lambda \cap L^\flat_F$ is the unique unimodular lattice in $\mathrm{Hor}(L^\flat)$. By Theorem 5.2.3, we have

$${}^{\mathbb{L}}\mathcal{Z}(L^\flat)^*_{\mathscr{H}} - {}^{\mathbb{L}}\mathcal{Z}(L^\flat) = (m - 1 + \frac{1-(-1)^{n-1}}{2})[\mathcal{O}_{\mathbb{P}_\Lambda}] - {}^{\mathbb{L}}\mathcal{Z}(L^\flat)_{\mathscr{V}},$$

where $m := |\mathrm{Hor}(L^\flat)|$. By Proposition 5.1.20 and Lemma 5.2.2, we know that ${}^{\mathbb{L}}Z(L^\flat)_{\mathscr{V}} \in \mathrm{Gr}' K_0^{\mathcal{N}_\Lambda}(\mathcal{N})$. Corollary 4.6.5 implies that in fact ${}^{\mathbb{L}}Z(L^\flat)_{\mathscr{V}} \in \mathrm{Gr}^{n-1} K_0^{\mathcal{N}_\Lambda}(\mathcal{N})$, hence

$${}^{\mathbb{L}}\mathcal{Z}(L^\flat)_{\mathscr{V}} = m'[\mathcal{O}_{\mathbb{P}_\Lambda}]$$

for some integer m'. In order to prove (5.28), it suffices to show

$$m' = m - \frac{1+(-1)^{n-1}}{2}. \tag{5.29}$$

Now assume $L^\flat = M \oplus L'$ where M is unimodular and of rank $n-2$ and $\mathrm{val}(L') = a$. Then $m = a+1$. By Lemma 4.8.4, we know that

$$\chi(\mathcal{N}, {}^{\mathbb{L}}\mathcal{Z}(L^\flat)_{\mathscr{H}} \cdot [\mathcal{O}_{\mathcal{N}_\Lambda}]) = 2a+1 = 2m-1.$$

By Corollary 4.6.7, we know

$$\chi(\mathcal{N}, {}^{\mathbb{L}}\mathcal{Z}(L^\flat)_{\mathscr{V}} \cdot [\mathcal{O}_{\mathcal{N}_\Lambda}]) = m' \cdot \chi(\mathcal{N}, [\mathcal{O}_{\mathbb{P}_\Lambda}] \cdot [\mathcal{O}_{\mathcal{N}_\Lambda}]) = -2m'.$$

On the other hand, by Corollary 4.6.6,

$$\chi(\mathcal{N}, {}^{\mathbb{L}}\mathcal{Z}(L^\flat)_{\mathscr{H}} \cdot [\mathcal{O}_{\mathcal{N}_\Lambda}]) + \chi(\mathcal{N}, {}^{\mathbb{L}}\mathcal{Z}(L^\flat)_{\mathscr{V}} \cdot [\mathcal{O}_{\mathcal{N}_\Lambda}]) = \chi(\mathcal{N}, {}^{\mathbb{L}}\mathcal{Z}(L^\flat) \cdot [\mathcal{O}_{\mathcal{N}_\Lambda}]) = (-1)^{n-1}.$$

Combine the above equations, we get (5.29). □

5.2.2 hermitian lattices and Fourier transform

We fix an additive character $\psi : F_0 \to \mathbb{C}^\times$ whose conductor is O_{F_0}. Recall that the Fourier transform with respect to ψ is defined by

$$\widehat{\varphi}(x) = \int_{\mathbb{V}} \varphi(y) \cdot \psi(\mathrm{Tr}_{F/F_0}(x,y)) d\mu(y), \tag{5.30}$$

where $d\mu$ is the unique self-dual Haar measure on $\mathbb{V}$ with respect to this transform. For a lattice L in $\mathbb{V}$ we use $L^\vee$ to denote its dual under the quadratic form $\mathrm{Tr}_{F/F_0}((\,,\,))$. The following lemma is well-known and easy to check.

Lemma 5.2.10. *Let $L \subset \mathbb{V}$ be a lattice of rank n and $1_L \in \mathscr{S}(\mathbb{V})$ be its characteristic function. Then*

$$\widehat{1_L} = \mathrm{vol}(L, d\mu) \cdot 1_{L^\vee}.$$

Lemma 5.2.11. *Let L be a rank n lattice in $\mathbb{V}$. A function $\varphi \in \mathscr{S}(\mathbb{V})$ is L-invariant (invariant under the translation of L) if and only if its Fourier transform $\widehat{\varphi}$ is supported* *on* $L^\vee$.

Proof. We first prove the "only if" direction. For any $\mu \in \mathbb{V}$, let $\bar{\mu}$ be its image in the quotient $\mathbb{V}/L$. Define

$$L(\bar{\mu}) := \mu + L.$$

Any L-invariant function $\phi \in \mathscr{S}(\mathbb{V})$ is a linear combination of the characteristic functions $1_{L(\bar{\mu})}$. So it suffices to assume $\phi = 1_{L(\bar{\mu})}$. In this case,

$$\widehat{\phi}(x) = \psi(\mathrm{Tr}_{F/F_0}(x, \mu)) \cdot \widehat{1_L}(x).$$

So $\widehat{\phi}(x)$ is supported on $L^\vee$ by Lemma 5.2.10. This proves the "only if" direction.

For the "if" direction. It suffices to show that if φ is supported on $L^\vee$, then $\widehat{\varphi}$ is L-invariant. For any $z \in L$, we have

$$\widehat{\varphi}(x + z) = \int_{\mathbb{V}} \varphi(y) \cdot \psi(\mathrm{Tr}_{F/F_0}(x, y)) \cdot \psi(\mathrm{Tr}_{F/F_0}(z, y)) d\mu(y).$$

Since $\psi(\mathrm{Tr}_{F/F_0}(z, y)) = 1$ for any $z \in L$ and $y \in L^\vee$ and φ is supported on $L^\vee$, the above is equal to $\widehat{\varphi}(x)$. This finishes the proof of the lemma. $\square$

For an integer m, recall that

$$\mathbb{V}^{\geq m} = \{x \in \mathbb{V} \mid \mathrm{val}(x) \geq m\}.$$

Definition 5.2.12. *Define* $\mathscr{S}(\mathbb{V})^{\geq m}$ *to be the subspace of* $\mathscr{S}(\mathbb{V})$ *consisting of functions* φ *such that* $\widehat{\varphi}$ *is supported on* $\mathbb{V}^{\geq m}$.

Lemma 5.2.13. *Let* Λ *be a vertex lattice in* $\mathbb{V}$. *Any* Λ*-invariant function in* $\mathscr{S}(\mathbb{V})$ *is in* $\mathscr{S}(\mathbb{V})^{\geq -1}$.

Proof. By Lemma 5.2.11, it suffices to show that $\Lambda^\vee \subset \mathbb{V}^{\geq -1}$. Since Λ is a vertex lattice, we have

$$\Lambda^\sharp = H^t \oplus I_{n-2t},$$

for some t. Simple calculation gives then

$$\Lambda^\vee = \frac{1}{\pi}\Lambda^\sharp = \frac{1}{\pi}H^t \oplus \frac{1}{\pi}I_{n-2t} \subset \mathbb{V}^{\geq -1}.$$

$\square$

5.2.3 Fourier transform of $\mathrm{Int}_{L^\flat,\mathscr{V}}$

Theorem 5.2.14. *Let Λ be a vertex lattice and $\mathcal{K} \in K_0^{\mathcal{N}_\Lambda}(\mathcal{N})$. For any $x \in \mathbb{V} \setminus \{0\}$, the function that takes x to $\mathcal{K} \cdot [\mathcal{O}_{\mathcal{Z}(x)}] \in K_0^{\mathcal{N}_\Lambda}(\mathcal{N})$ is Λ-invariant. More precisely, for any $y \in \Lambda$ such that $x + y \neq 0$, we have*

$$\mathcal{K} \cdot [\mathcal{O}_{\mathcal{Z}(x)}] = \mathcal{K} \cdot [\mathcal{O}_{\mathcal{Z}(x+y)}]. \tag{5.31}$$

Moreover, the function

$$\mathrm{Int}_{\mathcal{K}}(x) := \chi(\mathcal{N}, \mathcal{K} \cdot [\mathcal{O}_{\mathcal{Z}(x)}])$$

extends to a Λ-invariant function in $\mathscr{S}(\mathbb{V})^{\geq -1}$.

Proof. Any element $\mathcal{K} \in K_0^{\mathcal{N}_\Lambda}(\mathcal{N}) \cong K_0'(\mathcal{N}_\Lambda)$ is a sum of elements of the form $[\mathcal{F}]$ where $[\mathcal{F}]$ is a coherent sheaf of $\mathcal{O}_{\mathcal{N}_\Lambda}$-module. Hence it suffices to prove the theorem for $\mathcal{K} = [\mathcal{F}]$. By [How19, Corollary C], we know

$$[\mathcal{O}_{\mathcal{Z}(y)} \otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{N}} \mathcal{O}_{\mathcal{Z}(x)}] = [\mathcal{O}_{\mathcal{Z}(y)} \otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{N}} \mathcal{O}_{\mathcal{Z}(x+y)}].$$

For any $y \in \Lambda$ with $x + y \neq 0$, $\mathcal{N}_\Lambda$ is a subscheme of $\mathcal{Z}(y)$ by Proposition 5.1.20. Hence we have

$$\begin{aligned}\mathcal{K}\cdot[\mathcal{O}_{\mathcal{Z}(x)}] =&[\mathcal{F}\otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}}}\mathcal{O}_{\mathcal{Z}(x)}]\\ =&[\mathcal{F}\otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}_\Lambda}}\mathcal{O}_{\mathcal{N}_\Lambda}\otimes_{\mathcal{O}_{\mathcal{Z}(y)}}\mathcal{O}_{\mathcal{Z}(y)}\otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}}}\mathcal{O}_{\mathcal{Z}(x)}]\\ =&[\mathcal{F}\otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}_\Lambda}}\mathcal{O}_{\mathcal{N}_\Lambda}\otimes_{\mathcal{O}_{\mathcal{Z}(y)}}\mathcal{O}_{\mathcal{Z}(y)}\otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}}}\mathcal{O}_{\mathcal{Z}(x+y)}]\\ =&\mathcal{K}\cdot[\mathcal{O}_{\mathcal{Z}(x+y)}]\end{aligned}$$

We have proved the Λ-invariance of $\mathcal{K}\cdot[\mathcal{O}_{\mathcal{Z}(x)}]$. It follows that $\mathrm{Int}_{\mathcal{K}}(x)$ is also Λ-invariant. Hence we can define $\mathrm{Int}_{\mathcal{K}}(0)$ to be $\mathrm{Int}_{\mathcal{K}}(x)$ for any $0\neq x\in\Lambda$ and obtain a (unique) Λ-invariant function (still denoted by $\mathrm{Int}_{\mathcal{K}}(x)$) for all $x\in\mathbb{V}$. In particular $\mathrm{Int}_{\mathcal{K}}(x)$ is locally constant. If $x\notin\Lambda^\sharp$, by Proposition 5.1.20, the intersection of $\mathcal{Z}(x)$ with $\mathcal{N}_\Lambda$ is empty, which implies $\mathrm{Int}_{\mathcal{K}}(x)=0$. This shows that the function $\mathrm{Int}_{\mathcal{K}}(x)$ is compactly supported. Hence it is in $\mathscr{S}(\mathbb{V})$ and is in fact in $\mathscr{S}(\mathbb{V})^{\geq -1}$ by Lemma 5.2.13. This finishes the proof of the theorem. $\square$

Theorem 5.2.15. *For every non-degenerate lattice $L^\flat$ of $\mathbb{V}$ of rank $n-1$, the function $\mathrm{Int}_{L^\flat,\mathscr{V}}$ on $\mathbb{V}\backslash L^\flat_F$ can be extended to an element in $\mathscr{S}(\mathbb{V})^{\geq -1}$ which we denote by the same notation.*

Proof. Lemmas 5.2.8 and 2.2.4 imply that ${}^{\mathbb{L}}\mathcal{Z}(L^\flat)^*_{\mathscr{V}}\in\mathrm{Gr}'K_0^{\mathcal{N}_{\mathrm{red}}}(\mathcal{N})\cap\mathrm{Gr}'K_0^{\mathcal{Z}(L^\flat)_{\mathscr{V}}}(\mathcal{N})$. Lemma 2.2.3 implies that there exist finitely many classes $\mathcal{K}_i\in\mathrm{Gr}'K_0^{\mathcal{N}_{\mathrm{red}}}(\mathcal{N})_{\mathbb{Q}}$ together with $C_i\in\mathbb{Q}$ such that

$$\mathrm{Int}_{L^\flat,\mathscr{V}}(x)=\sum_i C_i\cdot\chi(\mathcal{N},\mathcal{K}_i\cdot[\mathcal{O}_{\mathcal{Z}(x)}]).$$

By Theorem 5.1.19 we may assume that $\mathcal{K}_i$ is supported on some $\mathcal{N}_\Lambda$. Now we can apply Theorem 5.2.14 to conclude the proof. $\square$

5.2.4 Partial Fourier transform

Let $L^\flat$ be a rank $n-1$ non-degenerate lattice in $\mathbb{V}$. Let $\mathbb{W} = (L^\flat_F)^\perp$. For any function φ defined on $\mathbb{V} \setminus L^\flat_F$, we define its partial Fourier transform $\varphi^\perp$ as a function on $\mathbb{W} \setminus \{0\}$ by

$$\varphi^\perp(x) := \int_{L^\flat_F} \varphi(x+y)dy, \quad \forall\, x \in \mathbb{W}. \tag{5.32}$$

Theorem 5.2.16. *The partial Fourier transform* $\mathrm{Int}^\perp_{L^\flat,\mathscr{V}} \in \mathscr{S}(\mathbb{W})^{\geq -1}$ *and is* $\mathbb{W}^{\geq 0}$*-invariant. In particular it is constant on* $\mathbb{W}^{\geq 0}$*.*

Proof. It is easy to see that partial Fourier transform maps $\mathscr{S}(\mathbb{V})$ to $\mathscr{S}(\mathbb{W})$. It remains to show that the Fourier transform of $\mathrm{Int}^\perp_{L^\flat,\mathscr{V}} \in \mathscr{S}(\mathbb{W})$ is supported on $\mathbb{W}^{\geq -1}$. For $x \in \mathbb{W}$, we have

$$\widehat{\mathrm{Int}^\perp_{L^\flat,\mathscr{V}}}(x) = \widehat{\mathrm{Int}_{L^\flat,\mathscr{V}}}(x),$$

where $\widehat{\mathrm{Int}_{L^\flat,\mathscr{V}}}(x)$ is the Fourier transform of $\mathrm{Int}_{L^\flat,\mathscr{V}} \in \mathscr{S}(\mathbb{V})$. Since $\widehat{\mathrm{Int}_{L^\flat,\mathscr{V}}}$ is supported on $\mathbb{V}^{\geq -1}$ by Theorem 5.2.15, we know that $\widehat{\mathrm{Int}^\perp_{L^\flat,\mathscr{V}}}(x)$ is supported on $\mathbb{W}^{\geq -1}$.

Since $\mathbb{W}$ is one-dimensional, $\mathbb{W}^{\geq m}$ is a full rank lattice in $\mathbb{W}$ for any $m \in \mathbb{Z}$. By Lemma 5.2.11 and what we just proved, $\mathrm{Int}^\perp_{L^\flat,\mathscr{V}}$ is invariant under the translation of $(\mathbb{W}^{\geq -1})^\vee = \mathbb{W}^{\geq 0}$. $\square$

5.3 More about local densities and primitive local densities

In this section, we record more results about the basics of local densities and primitive local densities.

5.3.1 Basic properties of local density and primitive local density polynomials

Recall that in this chapter, we let $\mathrm{Den}(M,L,X) := \alpha(M,L,X)$ and

$$\mathrm{Den}'(M,L) := -2 \cdot \frac{\partial}{\partial X}\mathrm{Den}(M,L,X)|_{X=1}. \tag{5.33}$$

Similarly, the primitive local density polynomial $\mathrm{Pden}(M, L, X)$ is defined to be the polynomial in $\mathbb{Q}[X]$ such that

$$\mathrm{Pden}(M, L, q^{-2k}) = \lim_{d\to\infty} q^{-d(2n(m+2k)-n^2)} |\mathrm{Pherm}_{L,M\oplus H^k}(O_{F_0}/(\pi_0^d))|, \tag{5.34}$$

where

$$\mathrm{Pherm}_{L,M\oplus H^k}(O_{F_0}/(\pi_0^d)) := \{\phi \in \mathrm{Herm}_{L,M\oplus H^k}(O_{F_0}/(\pi_0^d)) \mid \ \phi \text{ is primitive}\}.$$

Here we recall that $\phi \in \mathrm{Herm}_{L,M\oplus H^k}(O_{F_0}/(\pi_0^d))$ is primitive if $\dim_{\mathbb{F}_q}((\phi(L) + \pi(M \oplus H^k))/\pi(M \oplus H^k) = n$. In particular, we have $\mathrm{Pden}(M, M) = \mathrm{Den}(M, M)$ since any $\phi \in \mathrm{Herm}_{M,M}(O_{F_0}/(\pi_0^d))$ is primitive for large enough d.

Recall that without explicit mentioning, we assume $\epsilon = \chi(L)$. As an analogue of (1.7) and (1.8), we define

$$\mathrm{Pden}'(L) := -2 \cdot \frac{\frac{\mathrm{d}}{\mathrm{d}X}\big|_{X=1}\mathrm{Pden}(I_n, L, X)}{\mathrm{Den}(I_n, I_n)} \text{ and } \mathrm{Pden}_t(L) := \frac{\mathrm{Pden}(\Lambda_t^\sharp, L)}{\mathrm{Den}(\Lambda_t^\sharp, \Lambda_t^\sharp)}. \tag{5.35}$$

To save notation, we simply denote $\mathrm{Pden}_0(L)$ (resp. $\mathrm{Pden}_0'(L)$) by $\mathrm{Pden}(L)$ (resp. $\mathrm{Pden}'(L)$). We define

$$\partial\mathrm{Pden}(L) := \mathrm{Pden}'(L) + \sum_{j=1}^{\frac{t_{\max}}{2}} c_{2j} \cdot \mathrm{Pden}_{2j}(L), \tag{5.36}$$

where c_{2j} is as in (1.9).

Lemma 5.3.1. *Let L be a lattice. If there exists $x \in L$ such* $\mathrm{val}(x) \le -1$*, then* $\partial\mathrm{Den}(L) = \mathrm{Den}'(L) = \partial\mathrm{Pden}(L) = \mathrm{Pden}'(L) = 0$.

Proof. Assume $M \cong I_n$ or $M \cong \Lambda_{2t}$ for some t. Then $\mathrm{Den}(M \oplus H^k, L) = 0$ and $\mathrm{Pden}(M \oplus H^k, L) = 0$ since there is no vector in M with valuation less than or equal to -1. □

Now we recall several results that describe the relation between local density and primitive local density polynomials discussed in Chapter 4.

Lemma 5.3.2. *Let M and L be lattices of rank m and n. Then we have*

$$\mathrm{Den}(M,L,X)=\sum_{L\subset L'\subset L_F}(q^{n-m}X)^{\ell(L'/L)}\mathrm{Pden}(M,L',X),$$

where $\ell(L'/L)=\mathrm{length}_{O_F}L'/L$. Here $\mathrm{Pden}(M,L',X)=0$ for L' with fundamental invariant less than the smallest fundamental invariant of M. In particular, the summation is finite.

Corollary 5.3.3. *Let L be a lattice. We have the following identity:*

$$\partial\mathrm{Den}(L)=\sum_{L\subset L'\subset L_F}\partial\mathrm{Pden}(L').$$

Proof. Since $\mathrm{Pden}(I_n,L',1)=\mathrm{Pden}(I_n,L')=0$, we have by Lemma 5.3.2

$$-2\frac{\mathrm{d}}{\mathrm{d}X}\bigg|_{X=1}\mathrm{Den}(I_n,L,X)=-2\sum_{L\subset L'\subset L_F}\frac{\mathrm{d}}{\mathrm{d}X}\bigg|_{X=1}\mathrm{Pden}(I_n,L',X)=\sum_{L\subset L'\subset L_F}\mathrm{Pden}'(I_n,L').$$

Similarly, according to Lemma 5.3.2, we have

$$\mathrm{Den}(\Lambda_{2j},L)=\sum_{L\subset L'\subset L_F}\mathrm{Pden}(\Lambda_{2j},L')$$

for $0<j\leq t_{\max}/2$. Now the corollary follows from (1.9) and (5.36). □

Conversely, the primitive local density polynomial is a linear combination of local density polynomials.

Theorem 5.3.4. *Let M and L be lattices of rank m and n. We have*

$$\mathrm{Pden}(M,L,X)=\sum_{i=0}^{n}(-1)^i q^{i(i-1)/2+i(n-m)}X^i\sum_{\substack{L\subset L'\subset\pi^{-1}L\\ \ell(L'/L)=i}}\mathrm{Den}(M,L',X).$$

Corollary 5.3.5. *Let L be a lattice of rank n. Then*

$$\partial\mathrm{Pden}(L) = \sum_{i=0}^{n} (-1)^i q^{i(i-1)/2} \sum_{\substack{L \subset L' \subset \pi^{-1}L \\ \ell(L'/L)=i}} \partial\mathrm{Den}(L').$$

Recall that for two lattices $L, L' \subset \mathbb{V}$ of rank n, $n(L', L) = |\{L'' \subset L_F \mid L \subset L'', L'' \cong L'\}|$.

Lemma 5.3.6. *For two lattices L and M of the same rank n, we have*

$$\mathrm{Pden}(M, L) = \begin{cases} \mathrm{Den}(M, L) & \text{if } M \cong L, \\ 0 & \text{if } M \ncong L. \end{cases} \tag{5.37}$$

Moreover,

$$\mathrm{Den}(M, L) = n(M, L) \cdot \mathrm{Den}(M, M).$$

In particular, if $\chi(M) \neq \chi(L)$, then $\mathrm{Den}(M, L) = 0$.

Proof. First of all, for $M \cong L$, $\mathrm{Pden}(M, L) = \mathrm{Den}(M, L)$ by the definition of primitive local density. Now we show that if $\mathrm{Pden}(M, L) \neq 0$, then $M \cong L$. If $\mathrm{Pden}(M, L) \neq 0$, then for any large enough d we have

$$\mathrm{Pherm}_{L,M}(O_{F_0}/(\pi_0^d)) \neq 0.$$

Let $\phi \in \mathrm{Pherm}_{L,M}(O_{F_0}/(\pi_0^d))$ be a primitive embedding and $L_{(d)} = L \otimes_{O_{F_0}} O_{F_0}/(\pi_0^d)$. Let $\overline{\phi(L_{(d)})}$ be the image of $\phi(L_{(d)})$ in $\overline{M_{(d)}}$. Since ϕ is primitive, we have $\overline{\phi(L_{(d)})} = \overline{M_{(d)}}$. Then by Nakayama's lemma, we know $\phi(L_{(d)}) = M_{(d)}$. Hence ϕ is an isometry between $L_{(d)} \cong M_{(d)}$. Since this holds for any large enough d, we have $L \cong M$.

Now the formula of $\mathrm{Den}(M, L)$ follows from (5.37) and Lemma 5.3.2. $\square$

Corollary 5.3.7. *Let L be a lattice. Then for any even integer t such that* $0 \leq t \leq t_{\max}$, *we have*

$$\mathrm{Den}_t(L) := \frac{\mathrm{Den}(\Lambda_t^\sharp, L)}{\mathrm{Den}(\Lambda_t^\sharp, \Lambda_t^\sharp)} \in \mathbb{Z}.$$

Corollary 5.3.8. *Assume $L \ncong \Lambda_t^\sharp$ for any vertex lattice Λ_t with $t > 0$. Then*

$$\partial \mathrm{Pden}(L) = \mathrm{Pden}'(L).$$

Corollary 5.3.9. *Let c_t be the coefficients in* (1.10) *with even t and $0 < t \le t_{\max}$. Then*

$$c_t = -\mathrm{Pden}'(\Lambda_t^\sharp).$$

Proof. On the one hand, combining Corollary 5.3.5 with (1.10), we obtain

$$\partial \mathrm{Pden}(\Lambda_t^\sharp) = 0.$$

On the other hand, by Lemma 5.3.6 and (1.9),

$$\partial \mathrm{Pden}(\Lambda_t^\sharp) = \mathrm{Pden}'(\Lambda_t^\sharp) + c_t.$$

□

Write $\Lambda_t^\sharp = H^{t/2} \oplus L_1$ where L_1 is unimodular of rank n_1. Then by Lemma 5.3.12, Corollaries 5.4.2 and 5.3.9, we have (see the following subsections for the relevant notations)

$$c_t = -2\frac{\prod_{\ell=1}^{t/2-1}(1-q^{2\ell})}{\mathrm{Den}(I_n, I_n)} \cdot \sum_{i=0}^{n_1} \prod_{\ell=0}^{n_1-i-1}(1-q^{2(\ell+t/2)}) \cdot \sum_{V_1 \in \mathrm{Gr}(i,\overline{L_1})(\mathbb{F}_q)} |\mathrm{O}(V_1, \overline{I_n})|.$$

Combining this formula with Lemma 5.3.15 and Lemma 5.3.17, we can compute c_t explicitly. We give some examples here.

Example 5.3.10. *If n is odd, we have*

$$c_{t_{\max}} = -\frac{\mathrm{Pden}'(I_n, \Lambda^{\sharp}_{t_{\max}})}{\mathrm{Den}(I_n, I_n)} = \frac{(-1)^{\frac{n+1}{2}}}{q^{(\frac{n-1}{2})^2}(q^{\frac{n-1}{2}}+1)}. \tag{5.38}$$

If n is even and $\epsilon = 1$, we have

$$c_{t_{\max}} = -\frac{\mathrm{Pden}'(I_n, \Lambda^{\sharp}_{t_{\max}})}{\mathrm{Den}(I_n, I_n)} = \frac{(-1)^{\frac{n}{2}}}{q^{\frac{n}{2}(\frac{n}{2}-1)}(q^{\frac{n}{2}}+1)}.$$

We also give a list of c_t for small n, t and $\epsilon = 1$ in the following table:

t \ n	2	3	4	5	6
2	$\frac{-1}{q+1}$	$\frac{1}{q(q+1)}$	$\frac{-1}{q^2(q+1)}$	$\frac{1}{q^3(q+1)}$	$\frac{-1}{q^4(q+1)}$
4	0	0	$\frac{1}{q^2(q^2+1)}$	$\frac{-1}{q^4(q^2+1)}$	$\frac{1}{q^6(q^2+1)}$
6	0	0	0	0	$-\frac{1}{q^6(q^3+1)}$

In fact, computation suggests that c_t has the following simple formula:

$$c_t = \begin{cases} \frac{(-1)^{n+t/2}(q^{n/2}+1)}{q^{t/2(n-t-1)}(q^{n/2}-1)(q^{t/2}+1)} & \text{if } n \text{ is even and } \epsilon = -1, \\ \frac{(-1)^{n+t/2}}{q^{t/2(n-t/2-1)}(q^{t/2}+1)} & \text{otherwise.} \end{cases} \tag{5.39}$$

We believe this formula can be proved by similar method as in §5.5. Since this formula is not needed in our proof, we omit the details.

5.3.2 Explicit formulas for some simple primitive local density polynomials

Lemma 5.3.11. *([LL22, Lemma 2.15]) Assume L is an integral lattice of rank n. Then*

$$\mathrm{Pden}(H^k, L) = \prod_{\ell=0}^{n-1}(1 - q^{-2k+2\ell}).$$

Lemma 5.3.12. *Assume $L = H^j \oplus L_1$ where $j > 0$ and L_1 is an integral lattice of rank*

n_1. *Then*

$$\mathrm{Den}(I_m, L, X) = \Big(\prod_{\ell=0}^{j-1}(1-q^{2\ell}X)\Big)\mathrm{Den}(I_m, L_1, q^{2j}X),$$
$$\mathrm{Pden}(I_m, L, X) = \Big(\prod_{\ell=0}^{j-1}(1-q^{2\ell}X)\Big)\mathrm{Pden}(I_m, L_1, q^{2j}X).$$

In particular,

$$\mathrm{Pden}'(I_m, L) = 2\Big(\prod_{\ell=1}^{j-1}(1-q^{2\ell})\Big)\mathrm{Pden}(I_m, L_1, q^{2j}). \tag{5.40}$$

Proof. First, by Corollary 4.1.9 and Lemma 5.3.11,

$$\mathrm{Den}(I_m, L, X) = \Big(\prod_{\ell=0}^{j-1}(1-q^{2\ell}X)\Big)\mathrm{Den}(I_m, L_1, q^{2j}X). \tag{5.41}$$

Notice that if $L \subset L'$ and L' is not of the form $H^j \oplus L_1'$, then there exists $v \in L' \setminus L$ such that $\mathrm{Pr}_{H_F^j}(v) \neq 0$ and $\mathrm{Pr}_{H_F^j}(v) \notin H_j$. Hence some fundamental invariant of L' is smaller or equal to -2. Hence $\mathrm{Den}(I_m, L', X) = 0$ by Lemma 5.3.1. Now Theorem 5.3.4 and (5.41) implies

$$\begin{aligned}
&\mathrm{Pden}(I_m, L, X)\\
&= \sum_{i=0}^{n}(-1)^i q^{i(i-1)/2+i(n-m)}X^i \sum_{\substack{L_1\subset L_1'\subset \pi^{-1}L_1\\ \ell(L_1'/L_1)=i}} \mathrm{Den}(I_m, H^j\oplus L_1', X)\\
&= \Big(\prod_{\ell=0}^{j-1}(1-q^{2\ell}X)\Big)\sum_{i=0}^{n_1}(-1)^i q^{i(i-1)/2+i(n-m-2j)}(q^{2j}X)^i \sum_{\substack{L_1\subset L_1'\subset \pi^{-1}L_1\\ \ell(L'/L)=i}} \mathrm{Den}(I_m, L_1', q^{2j}X)\\
&= \Big(\prod_{\ell=0}^{j-1}(1-q^{2\ell}X)\Big)\mathrm{Pden}(I_m, L_1, q^{2j}X)
\end{aligned}$$

as expected. $\square$

Definition 5.3.13. *Assume U and V are quadratic spaces over $\mathbb{F}_q$. We define* $\mathrm{O}(U, V)$

to be the set of isometries from U into V, and $M(U,V)$ to be the set of subspaces $V_1 \subset V$ such that $V_1 \cong U$. Moreover, we define $m(U,V) = |M(U,V)|$.

Definition 5.3.14. *We define U_i^{ϵ} to be the i-dimensional non-degenerate quadratic space over $\mathbb{F}_q$ with $\chi(U_n^{\epsilon}) = \epsilon$. Moreover, we define 0_i to be the i-dimensional totally isotropic space.*

Recall the following result from Chapter 4.

Lemma 5.3.15. *Assume $L = I_{n-t}^{\epsilon_1} \oplus L_2$ where L_2 is a lattice of full type t and $n \leq m$. Then*

$$\mathrm{Pden}(I_m^{\epsilon_2}, L) = q^{-mn+n^2} \cdot |\mathrm{O}(0_t \oplus U_{n-t}^{\epsilon_1}, U_m^{\epsilon_2})|.$$

Specifically, we have by [LZ22b, Lemma 3.2.1],

$$|\mathrm{O}(0_j \oplus U_k^{\epsilon_1}, U_m^{\epsilon})| = q^{(k+j)(2m-k-j-1)/2} \prod_{\lfloor \frac{m-k}{2} \rfloor + 1 - j \leq l \leq \lfloor \frac{m-1}{2} \rfloor} (1 - q^{-2l}) \cdot \begin{cases} (1 + \epsilon\epsilon_1 q^{-\frac{m-k}{2}+j}) & \text{if } m \equiv k \equiv 1 \pmod 2, \\ 1 & \text{if } m \equiv k-1 \equiv 1 \pmod 2, \\ (1 - \epsilon q^{-\frac{m}{2}}) & \text{if } m \equiv k-1 \equiv 0 \pmod 2, \\ (1 - \epsilon q^{-\frac{m}{2}})(1 + \epsilon\epsilon_1 q^{-\frac{m-k}{2}+j}) & \text{if } m \equiv k \equiv 0 \pmod 2. \end{cases}$$

Corollary 5.3.16. *Let I_n be the unimodular lattice of rank n and sign $-\epsilon$. Then*

$$\mathrm{Den}(I_n, I_n) = \begin{cases} 2q^{\frac{n(n-1)}{2}} \prod_{s=1}^{\frac{n-1}{2}} (1 - q^{-2s}) & \textit{if } n \textit{ is odd}, \\ 2q^{\frac{n(n-1)}{2}} (1 + \epsilon q^{-\frac{n}{2}}) \prod_{s=1}^{\frac{n}{2}-1} (1 - q^{-2s}) & \textit{if } n \textit{ is even}. \end{cases}$$

5.3.3 Counting formulas for subspaces of a quadratic space over $\mathbb{F}_q$

The main results of §5.4 transforms the calculation of primitive local density polynomial into a sum over the subspaces of a quadratic space over $\mathbb{F}_q$. In this subsection, we count the number of such subspaces with a fixed quadratic form.

Lemma 5.3.17. *Given quadratic spaces U and V over $\mathbb{F}_q$, let $M(U,V)$ be the set of subspaces $V_1 \subset V$ such that $V_1 \cong U$, and let $m(U,V) = |M(U,V)|$. Then*

$$m(0_j \oplus U_k^{\epsilon_2}, U_n^{\epsilon}) = \frac{|\mathrm{O}(0_j \oplus U_k^{\epsilon_2}, U_n^{\epsilon})|}{q^{jk}|\mathrm{O}(U_k^{\epsilon_2}, U_k^{\epsilon_2})| \cdot |\mathrm{GL}_j(\mathbb{F}_q)|}.$$

In particular,

$$m(0_j \oplus U_k^{\epsilon_2}, U_n^{\epsilon}) = q^{-jk} m(0_j, U_{n-k}^{\delta(n,k,\epsilon,\epsilon_2)}) m(U_k^{\epsilon_2}, U_n^{\epsilon}), \tag{5.42}$$

where

$$\delta(n,k,\epsilon,\epsilon_2) = \begin{cases} \epsilon & \text{if } k = 0, \\ -\epsilon\epsilon_2 & \text{if both } k \text{ and } n-k \text{ are odd}, \\ \epsilon\epsilon_2 & \text{otherwise.} \end{cases} \tag{5.43}$$

Proof. We prove the first identity first. Fix a subspace V_1 of U_n^{ϵ} such that $V_1 \cong 0_j \oplus U_k^{\epsilon_2}$. Then by Witt's theorem we have a surjection

$$\mathrm{O}(0_j \oplus U_k^{\epsilon_2}, U_n^{\epsilon}) \to M(0_j \oplus U_k^{\epsilon_2}, U_n^{\epsilon}), \quad \phi \to \phi(V_1).$$

Moreover, each fiber of this surjection is in bijection with $\mathrm{O}(0_j \oplus U_k^{\epsilon_2})$. Any $\phi \in \mathrm{O}(0_j \oplus U_k^{\epsilon_2})$ is determined uniquely by $\phi_1 = \phi|_{0_j}$ and $\phi_2 = \phi|_{U_k^{\epsilon_2}}$. The number of different choices of ϕ_1 is $|\mathrm{GL}_j(\mathbb{F}_q)|$. The number of different choices of ϕ_2 is $q^{jk}|\mathrm{O}(U_k^{\epsilon_2}, U_k^{\epsilon_2})|$. □

Lemma 5.3.18. *For any $\epsilon_1, \epsilon_2 \in \{\pm 1\}$, we have*

$$m(0_j \oplus U_k^{\epsilon_2}, 0_t \oplus U_{n-t}^{\epsilon_1}) = \sum_{\ell=0}^{\min\{t,j\}} \binom{t}{\ell}_q \cdot q^{(t-\ell)(j+k-\ell)} m(0_{j-\ell} \oplus U_k^{\epsilon_2}, U_{n-t}^{\epsilon_1}).$$

Proof. Let V and U be quadratic spaces over $\mathbb{F}_q$ such that $V \cong 0_t \oplus U_{n-t}^{\epsilon}$, and $U \cong 0_j \oplus U_k^{\epsilon_2}$.

Let $R \cong 0_t$ be the radical of V. First, we consider a partition of

$$M(U,V) = \bigsqcup_{\ell=0}^{\min\{t,j\}} M_\ell(U,V),$$

where

$$V_1 \in M_\ell(U,V) \text{ if and only if } \dim_{\mathbb{F}_q}(V_1 \cap R) = \ell.$$

The number of choices of ℓ-dimensional subspace of R is $\binom{t}{\ell}_q$. Now we fix an ℓ-dimensional subspace W of R. Let $\overline{R} \cong 0_{t-\ell}$ be the radical of the quotient space of V/W. Then a choice of $V_1 \in M_\ell(U,V)$ such that $V_1 \cap R = W$ corresponds to an element of

$$S = \{\overline{V_1} \subset V/W \mid \overline{V_1} \cap \overline{R} = \{0\} \text{ and } \overline{V_1} \cong 0_{j-\ell} \oplus U_k^{\epsilon_2}\}.$$

Write $V/W = \overline{R} \oplus \overline{V_2} \cong 0_{t-\ell} \oplus U_{n-t}^{\epsilon_1}$. Let $\mathrm{Pr} : V/W \to \bar{V}_2$ be the natural quotient map. For $\overline{V_1} \in S$, the condition $\overline{V_1} \cap \overline{R} = \{0\}$ implies that $\mathrm{Pr}(\overline{V_1}) \cong 0_{j-\ell} \oplus U_k^{\epsilon_2}$ by the rank-nullity theorem. Moreover, the following map

$$S \to M(0_{j-\ell} \oplus U_k^{\epsilon_2}, \overline{V_2}), \quad \overline{V_1} \mapsto \mathrm{Pr}(\overline{V_1})$$

is a surjection with each fiber in a bijection with $\overline{R}^{j+k-\ell}$. $\square$

Corollary 5.3.19.

$$\begin{aligned}|\mathrm{O}(0_j \oplus U_k^{\epsilon_2}, 0_t \oplus U_{n-t}^{\epsilon})| &= q^{jk} |\mathrm{O}(U_k^{\epsilon_2}, U_k^{\epsilon_2})| \cdot |\mathrm{GL}_j(\mathbb{F}_q)| \cdot m(0_j \oplus U_k^{\epsilon_2}, 0_t \oplus U_{n-t}^{\epsilon}) \\ &= |\mathrm{GL}_j(\mathbb{F}_q)| \cdot \sum_{\ell=0}^{\min\{t,j\}} \binom{t}{\ell}_q \cdot q^{(t-\ell)(j+k-\ell)+\ell k} \cdot \frac{|\mathrm{O}(0_{j-\ell} \oplus U_k^{\epsilon_2}, U_{n-t}^{\epsilon})|}{|\mathrm{GL}_{j-\ell}(\mathbb{F}_q)|}.\end{aligned}$$

We will also need the following lemma later (Section 7), which follows from Lemmas 5.3.17 and 5.3.15 .

Lemma 5.3.20. *For integers $0 \le r \le n$ and $\epsilon_1, \epsilon = \pm 1$, we have*

$$\frac{m(U_r^{-\epsilon_1}, U_n^{\epsilon})}{m(U_r^{\epsilon_1}, U_n^{\epsilon})} = \begin{cases} 1 & \text{if } r \equiv n-1 \equiv 1 \pmod 2, \\ \frac{1-\epsilon\epsilon_1 q^{-\frac{n-r}{2}}}{1+\epsilon\epsilon_1 q^{-\frac{n-r}{2}}} & \text{if } r \equiv n \equiv 1 \pmod 2, \\ \frac{1-\epsilon_1 q^{-\frac{r}{2}}}{1+\epsilon_1 q^{-\frac{r}{2}}} & \text{if } r \equiv n-1 \equiv 0 \pmod 2, \\ \frac{1-\epsilon\epsilon_1 q^{-\frac{n-r}{2}}}{1+\epsilon\epsilon_1 q^{-\frac{n-r}{2}}} \cdot \frac{1-\epsilon_1 q^{-\frac{r}{2}}}{1+\epsilon_1 q^{-\frac{r}{2}}} & \text{if } r \equiv n \equiv 0 \pmod 2, \end{cases}$$

and

$$\frac{m(U_{r+1}^{\epsilon_1}, U_n^{\epsilon})}{m(U_r^{\epsilon_1}, U_n^{\epsilon})} = \frac{q^{n-2r-1}(1-(-1)^{n-r}\epsilon\epsilon_1 q^{-\lfloor \frac{n-r}{2} \rfloor})}{1-(-1)^{r+1}\epsilon_1 q^{-\lfloor \frac{r+1}{2} \rfloor}}.$$

Lemma 5.3.21. *Assume $i \le r \le n$ and $\epsilon, \sigma, \delta' \in \{\pm 1\}$. Let $\delta(r, i, \delta', \sigma)$ or $\delta(n, i, \epsilon, \sigma)$ be as in (5.43). Then*

$$m(U_i^{\sigma}, U_n^{\epsilon}) m(U_{r-i}^{\delta(r,i,\delta',\sigma)}, U_{n-i}^{\delta(n,i,\epsilon,\sigma)}) = m(U_r^{\delta'}, U_n^{\epsilon}) m(U_i^{\sigma}, U_r^{\delta'}).$$

Proof. Let $V = U_n^{\epsilon}$ and S be the following set of flags in V,

$$S = \{0 \subset F_1 \subset F_2 \subset V \mid F_1 \cong U_i^{\sigma}, F_2 \cong U_r^{\delta'}\}.$$

We can count the cardinality of S in two ways. One way is to first count the number of $F_1 \in M(U_i^{\sigma}, V)$, then for a fixed F_1 count the number of $F_2' \in M(U_{r-i}^{\delta(r,i,\delta',\sigma)}, (F_1)^{\perp})$ which has a one-to-one correspondence with $F_2 = F_2' \oplus F_1 \in M(U_r^{\delta'}, V)$. This way we get

$$\#|S| = m(U_i^{\sigma}, U_n^{\epsilon}) m(U_{r-i}^{\delta(r,i,\delta',\sigma)}, U_{n-i}^{\delta(n,i,\epsilon,\sigma)}).$$

On the other hand, we can first count the number of $F_2 \in M(U_r^{\delta'}, V)$, then count the number of $F_1 \in M(U_i^{\sigma}, F_2)$ and get

$$\#|S| = m(U_r^{\delta'}, U_n^{\epsilon}) m(U_i^{\sigma}, U_r^{\delta'}).$$

This finishes the proof of the lemma. $\square$

5.3.4 q-binomial theorem

In this subsection we discuss the q-binomial theorem and related results, which are used repeatedly in §5.5 to obtain certain vanishing results and transform complicated linear combinations into simple formulas.

Definition 5.3.22. *The q-analogue of $\binom{n}{i}$ is defined to be*

$$\binom{n}{i}_q := \frac{(q^n-1)\cdots(q^{n-i+1}-1)}{(q^i-1)\cdots(q-1)}.$$

In fact, $\binom{n}{i}_q$ is the number of i-dimensional subspaces of a n-dimensional vector space over $\mathbb{F}_q$. Now we recall the q-binomial theorem.

Lemma 5.3.23 (q-binomial theorem)**.** *The following identity between polynomials of X holds:*

$$\prod_{i=0}^{n-1}(1-q^iX) = \sum_{i=0}^{n}(-1)^i q^{\frac{i(i-1)}{2}}\binom{n}{i}_q X^i. \tag{5.44}$$

Corollary 5.3.24. *Let f be a polynomial of degree $\leq n-1$. Then*

$$\sum_{i=0}^{n}(-1)^i q^{\frac{i(i-1)}{2}}\binom{n}{i}_q \cdot f(q^{-i}) = 0.$$

Proof. Let $f = a_{n-1}x^{n-1} + \cdots + a_0$. For $0 \leq s \leq n-1$, by evaluating (5.44) at $X = q^{-s}$, we have

$$\sum_{i=0}^{n}(-1)^i q^{\frac{i(i-1)}{2}}\binom{n}{i}_q \cdot a_s q^{-si} = 0.$$

Hence

$$\sum_{i=0}^{n}(-1)^i q^{\frac{i(i-1)}{2}}\binom{n}{i}_q \cdot f(q^{-i}) = \sum_{s=0}^{n-1}\sum_{i=0}^{n}(-1)^i q^{\frac{i(i-1)}{2}}\binom{n}{i}_q \cdot a_s q^{-si} = 0.$$

□

The following is in some sense an inverse of q-binomial theorem that will be used in §5.5.

Lemma 5.3.25.

$$\sum_{i=0}^{n}(-1)^i q^{\frac{i(i-1)}{2}} \cdot \binom{n}{i}_q \cdot \prod_{\ell=0}^{i-1}(1+q^{-\ell}X) = (-X)^n.$$

Proof. Let $g_i(X) = \prod_{\ell=0}^{i-1}(q^{-\ell}X+1)$. Since $\{g_i(X)\}$ forms a basis of the vector space of polynomials of degree $\leq n$, there exist $a_{n,i} \in \mathbb{C}$ such that

$$(-X)^n = \sum_{i=0}^{n} a_{n,i} \cdot g_i(X).$$

Notice that $g_{i+1}(X) = (1+q^{-i}X)g_i(X)$, hence $Xg_i(X) = q^i(g_{i+1}(X) - g_i(X))$. Then we have

$$\begin{aligned}\sum_{i=0}^{n+1} a_{n+1,i} \cdot g_i(X) = (-X)^{n+1} = (-X)\cdot(-X)^n &= \sum_{i=0}^{n}(-a_{n,i}) \cdot Xg_i(X) \\ &= \sum_{i=0}^{n}(-a_{n,i})q^i \cdot (g_{i+1}(X) - g_i(X)).\end{aligned}$$

As a result, we have

$$a_{n+1,i} = \begin{cases} a_{n,0} & \text{if } i = 0, \\ q^i a_{n,i} - q^{i-1}a_{n,i-1} & \text{if } 0 < i < n+1, \\ -a_{n,n}q^n & \text{if } i = n+1. \end{cases} \tag{5.45}$$

It is easy to check that $b_{n,i} = (-1)^i q^{\frac{i(i-1)}{2}} \cdot \binom{n}{i}_q$ satisfies (5.45) and that $a_{1,i} = b_{1,i}$. So we have $a_{n,i} = b_{n,i}$. □

5.4 Decomposition of primitive local densities

This section is devoted to prove the following decomposition of the primitive local density polynomial, which is a vast generalization of Proposition 4.5.14 and one of the main tools we use to prove Theorem 5.5.1.

Theorem 5.4.1. *Assume that L is an integral lattice of rank n. For any $m \geq 0$ we have*

$$\mathrm{Pden}(I_m, L, X) = \sum_{i=0}^{n} \mathrm{Pden}^{n-i}(I_m, L, X),$$

where $\mathrm{Pden}^{n-i}(I_m, L, X)$ *is a polynomial characterized by*

$$\mathrm{Pden}^{n-i}(I_m, L, q^{-2k}) = q^{-2ki}\mathrm{Pden}(H^k, 0_{n-i}) \sum_{V_1 \in \mathrm{Gr}(i, \overline{L})(\mathbb{F}_q)} q^{(n-i)i}\mathrm{Pden}(I_m, L_{V_1}), \quad (5.46)$$

where 0_{n-i} is a totally isotropic lattice of rank $n-i$ and $L_{V_1} \subset L$ is a sublattice of rank i such that $\overline{L}_{V_1} = V_1$.

Here, an important special case is when $m = n$. In this case, $\mathrm{Pden}^0(I_n^{-\epsilon}, L, X) = 0$ since $\chi(L) \neq \chi(I_n^{-\epsilon})$. Hence,

$$\mathrm{Pden}(I_n, L, X) = \sum_{i=0}^{n-1} \mathrm{Pden}^{n-i}(I_n, L, X).$$

Applying the formulas of $\mathrm{Pden}(H^k, 0_{n-i})$ and $\mathrm{Pden}(I_m, L_{V_1})$ given in Lemmas 5.3.11 and 5.3.15, we obtain the following corollary.

Corollary 5.4.2. *Let L be an integral lattice of rank n. We have*

$$\mathrm{Pden}(I_m, L, X) = \sum_{i=0}^{n} \left((q^{n-m}X)^i \prod_{\ell=0}^{n-i-1} (1 - q^{2\ell}X) \right) \cdot \sum_{V_1 \in \mathrm{Gr}(i, \overline{L})(\mathbb{F}_q)} |\mathrm{O}(V_1, \overline{I_m})|. \quad (5.47)$$

In particular,

$$\mathrm{Pden}'(I_n, L) = 2\sum_{i=0}^{n}\Big(\prod_{\ell=1}^{n-i-1}(1-q^{2\ell})\Big)\cdot \sum_{V_1\in \mathrm{Gr}(i,\overline{L})(\mathbb{F}_q)} |\mathrm{O}(V_1, \overline{I_n})|.$$

When L is a full type lattice of rank n, $\overline{L}$ is totally isotropic. So we obtain

Corollary 5.4.3. *Let L be a full type lattice of rank n. Then*

$$\mathrm{Pden}(I_m, L, X) = \sum_{i=0}^{n}\Big((q^{n-m}X)^i \prod_{\ell=0}^{n-i-1}(1-q^{2\ell}X)\Big)\cdot \binom{n}{i}_q |\mathrm{O}(0_i, \overline{I_m})|.$$

In particular,

$$\mathrm{Pden}'(I_n, L) = 2\sum_{i=0}^{n}\Big(\prod_{\ell=1}^{n-i-1}(1-q^{2\ell})\Big)\cdot \binom{n}{i}_q |\mathrm{O}(0_i, \overline{I_n})|.$$

Here by Lemma 5.3.15, we have

$$|\mathrm{O}(0_i, \overline{I_m})| = q^{\frac{i(i-1)}{2}} \cdot \begin{cases} \prod_{1\leq \ell\leq i}\left(q^{m+1-2\ell}-1\right) & \textit{if } m \textit{ is odd,} \\ \left(q^{m/2}-\chi(I_m)\right)\left(q^{m/2-i}+\chi(I_m)\right)\cdot \prod_{1\leq \ell< i}\left(q^{m-2\ell}-1\right) & \textit{if } m \textit{ is even.} \end{cases}$$

Proof of Theorem 5.4.1. To save notation, we use M to denote I_m in this proof. Recall that by (5.34),

$$\mathrm{Pden}(M, L, q^{-2k}) = \lim_{d\to\infty} q^{-dn(2(m+2k)-n)}|\mathrm{Herm}_{L,M\oplus H^k}(O_{F_0}/(\pi_0^d))|.$$

First, we define a partition

$$\mathrm{Herm}_{L,M\oplus H^k}(O_{F_0}/(\pi_0^d)) = \bigsqcup_{0\leq i\leq n} \mathrm{Herm}^i_{L,M\oplus H^k}(O_{F_0}/(\pi_0^d)),$$

where

$$\mathrm{Herm}^i_{L,M\oplus H^k}(O_{F_0}/(\pi_0^d)) \coloneqq \{\phi \in \mathrm{Herm}_{L,M\oplus H^k}(O_{F_0}/(\pi_0^d)) \mid \dim_{\mathbb{F}_q} \overline{\mathrm{Pr}_{H^k}(\phi(L))} = i\}. \tag{5.48}$$

Here Pr_{H^k} denote the projection map to H^k, and $\overline{\mathrm{Pr}_{H^k}(\phi(L))}$ denote the image of $\mathrm{Pr}_{H^k}(\phi(L))$ in $\overline{H}^k$. As a result, we have

$$\mathrm{Pden}(M, L, X) = \sum_{i=0}^{n} \mathrm{Pden}^i(M, L, X),$$

where $\mathrm{Pden}^i(M, L, X)$ is the function such that

$$\mathrm{Pden}^i(M, L, q^{-2k}) \coloneqq \lim_{d\to\infty} q^{-(2n(2k+m)-n^2)d} |\mathrm{Pherm}^i_{L,M\oplus H^k}(O_{F_0}/(\pi_0^d))|.$$

We need to count $|\mathrm{Pherm}^i_{L,M\oplus H^k}(O_{F_0}/(\pi_0^d))|$. For $\phi \in \mathrm{Pherm}^i_{L,M\oplus H^k}(O_{F_0}/(\pi_0^d))$, it induces

$$\overline{\phi} : V = \overline{L} \longrightarrow M \oplus H^k/\pi(M \oplus H^k), \text{ and } \overline{\phi}_{H^k} \coloneqq \mathrm{Pr}_{\overline{H}^k} \circ \overline{\phi}.$$

By the definition of $\mathrm{Pherm}^i_{L,M\oplus H^k}(O_{F_0}/(\pi_0^d))$, For a $(n-i)$-dimensional subspace $V_1 \subset \overline{L}$, let

$$\mathrm{Pherm}^{V_1}_{L,M\oplus H^k}(O_{F_0}/(\pi_0^d)) = \{\phi \in \mathrm{Pherm}^i_{L,M\oplus H^k}(O_{F_0}/(\pi_0^d)) \mid \mathrm{Ker}(\overline{\phi}_{H^k}) = V_1 \subset \overline{L}\}.$$

Since $\mathrm{Ker}(\overline{\phi}_{H^k}) \subset \overline{L}$ has dimension $n-i$ for any $\phi \in \mathrm{Pherm}^i_{L,M\oplus H^k}(O_{F_0}/(\pi_0^d))$, we have

$$\mathrm{Pherm}^i_{L,M\oplus H^k}(O_{F_0}/(\pi_0^d)) = \bigsqcup_{V_1 \in \mathrm{Gr}(n-i,V)(\mathbb{F}_q)} \mathrm{Pherm}^{V_1}_{L,M\oplus H^k}(O_{F_0}/(\pi_0^d)). \tag{5.49}$$

We need to show

$$q^{-(2(m+2k)n-n^2)d} |\mathrm{Pherm}^{V_1}_{L,M\oplus H^k}(O_{F_0}/(\pi_0^d))| = q^{(n-i)i} X^{n-i} \mathrm{Pden}(H^k, L_{V_2}) \cdot \mathrm{Pden}(M, L_{V_1}). \tag{5.50}$$

Let V_2 be a subspace of V such that $V = V_1 \oplus V_2$. Let $L_{V_1} \subset L$ be a sublattice of rank $n-i$ such that the image of L_{V_1} in V is V_1. Similarly, let $L_{V_2} \subset L$ be a sublattice of rank i such that the image of L_{V_2} in V is V_2. Let $\phi_i = \phi|_{L_{V_i}}$ for $i \in \{1, 2\}$. According to Lemma 5.4.4, the number of different choices of ϕ_2 is given by

$$|\mathrm{Herm}^i_{L_{V_2}, M \oplus H^k}(O_{F_0}/(\pi_0^d))| = q^{i(2(m+2k)-i)d}\mathrm{Pden}(H^k, L_{V_2}). \tag{5.51}$$

Now for a fixed $\phi_2 \in \mathrm{Herm}^i_{L_{V_2}, M \oplus H^k}(O_{F_0}/(\pi_0^d))$, let

$$\begin{aligned}&\mathrm{Herm}^{\phi_2}_{L_{V_1}, M \oplus H^k}(O_{F_0}/(\pi_0^d))\\ &\coloneqq \{\phi_1 \in \mathrm{Herm}_{L_{V_1}, M \oplus H^k}(O_{F_0}/(\pi_0^d)) \mid (\phi_1, \phi_2) \in \mathrm{Herm}^{V_1}_{L, M \oplus H^k}(O_{F_0}/(\pi_0^d))\}.\end{aligned}$$

Claim: For any $\phi_2 \in \mathrm{Herm}^i_{L_{V_2}, M \oplus H^k}(O_{F_0}/(\pi_0^d))$,

$$\begin{aligned}|\mathrm{Herm}^{\phi_2}_{L_{V_1}, M \oplus H^k}(O_{F_0}/(\pi_0^d))| &= q^{(2d-1)(2k-i)(n-i)}|\mathrm{Herm}_{L_{V_1}, M}(O_{F_0}/(\pi_0^d))| \\ &= q^{(2d-1)(2k-i)(n-i)+(2m(n-i)-(n-i)^2)d}\mathrm{Pden}(M, L_{V_1}).\end{aligned} \tag{5.52}$$

Assuming the claim holds, (5.50) follows from (5.51) and (5.52) since for any fixed ϕ_2 we have

$$|\mathrm{Herm}^{V_1}_{L, M \oplus H^k}(O_{F_0}/(\pi_0^d))| = |\mathrm{Herm}^i_{L_{V_2}, M \oplus H^k}(O_{F_0}/(\pi_0^d))| \cdot |\mathrm{Herm}^{\phi_2}_{L_{V_1}, M \oplus H^k}(O_{F_0}/(\pi_0^d))|.$$

Proof of the claim: For $\phi_1 \in \mathrm{Herm}^{\phi_2}_{L_{V_1}, M \oplus H^k}(O_{F_0}/(\pi_0^d))$, write $\phi_1 = \phi_{1,H^k} + \phi_{1,M}$, where $\phi_{1,H^k} = \mathrm{Pr}_{H^k} \circ \phi_1$ and $\phi_{1,M} = \mathrm{Pr}_M \circ \phi_1$. First, for any $g \in \mathrm{U}(M \oplus H^k)$, one can directly check the map

$$\begin{aligned}\mathrm{Herm}^{\phi_2}_{L_{V_1}, M \oplus H^k}(O_{F_0}/(\pi_0^d)) &\to \mathrm{Herm}^{g \circ \phi_2}_{L_{V_1}, g(M \oplus H^k)}(O_{F_0}/(\pi_0^d)), \\ \phi_1 &\mapsto g \circ \phi_1\end{aligned} \tag{5.53}$$

is well-defined and is in fact a bijection. Then according to Lemma 5.4.5, we may assume $\phi_2(L_{V_2}) \subset H^k$.

Now finding ϕ_1 such that $(\phi_1, \phi_2) \in \mathrm{Perm}^{V_1}_{L,M\oplus H^k}(O_{F_0}/(\pi_0^d))$ is equivalent to finding ϕ_1 such that $\phi_{1,M}$ is primitive, $\phi_{1,H^k} \in \pi H^k$,

$$(\phi_1(v), \phi_2(w)) \equiv (v, w) \mod (\pi^{2d-1}) \text{ for any } v \in L_{V_1}, w \in L_{V_2}, \tag{5.54}$$

and

$$(\phi_1(v), \phi_1(w)) \equiv (v, w) \mod (\pi^{2d-1}) \text{ for any } v \in L_{V_1}, w \in L_{V_1}. \tag{5.55}$$

We consider condition (5.54) first. Since $\phi_2(L_{V_2}) \subset H^k$, we have

$$(\phi_{1,H^k}(v), \phi_2(w)) \equiv (\phi_1(v), \phi_2(w)) \text{ for any } v \in L_{V_1}, w \in L_{V_2}.$$

When k is large enough, we can always find and fix a ϕ'_{1,H^k} that satisfies (5.54). Then finding ϕ_{1,H^k} that satisfies (5.54) is equivalent to find $\Phi_{1,H^k} = \phi_{1,H^k} - \phi'_{1,H^k}$, which satisfies

$$(\Phi_{1,H^k}(v), \phi_2(w)) \equiv 0 \mod (\pi^{2d-1}) \text{ for any } v \in L_{V_1}, w \in L_{V_2},$$

Then according to Lemma 5.4.6, the number of different choices for ϕ_{1,H^k} is $q^{(2d-1)(2k-i)(n-i)}$.

Now we consider condition (5.55). Since $\phi_{1,H^k}(v) \in \pi H^k$ for any $v \in L_{V_1}$, (5.55) is equivalent to

$$(\phi_{1,M}(v), \phi_{1,M}(w)) + \pi\alpha \equiv (\phi_1(v), \phi_1(w)) \equiv (v, w) \mod (\pi^{2d-1}) \text{ for any } v, w \in L_{V_1},$$

for some $\alpha \in \mathcal{O}_F$. By Lemma 5.3.15, for a unimodular lattice M and any integral lattice N, $\mathrm{Pden}(M, N)$ only depends on $\overline{M}$ and $\overline{N}$. Hence, for our purpose, we may replace (5.55) by

$$(\phi_{1,M}(v), \phi_{1,M}(w)) \equiv (\phi_1(v), \phi_1(w)) \equiv (v, w) \mod (\pi^{2d-1}) \text{ for any } v, w \in L_{V_1}.$$

Therefore the number of different choices of primitive $\phi_{1,M}$ is given by

$$q^{(2m(n-i)-(n-i)^2)d}\mathrm{Pden}(M, L_{V_1}).$$

As a result, we have

$$|\mathrm{Perm}^{\phi_2}_{L_{V_1},M\oplus H^k}(O_{F_0}/(\pi_0^d))| = q^{(2d-1)(2k-i)(n-i)} \cdot q^{(2m(n-i)-(n-i)^2)d}\mathrm{Pden}(M, L_{V_1}).$$

This finishes the proof of the claim. □

Lemma 5.4.4. *Assume that L is an integral lattice of rank n and $k \geq n$. Then*

$$|\mathrm{Perm}^n_{L,I_m\oplus H^k}(O_{F_0}/(\pi_0^d))| = q^{2mnd}|\mathrm{Perm}_{L,H^k}(O_{F_0}/(\pi_0^d))|.$$

Proof. For $\phi \in \mathrm{Perm}^n_{L,I_m\oplus H^k}(O_{F_0}/(\pi_0^d))$, we may identify ϕ with (ϕ_{H^k}, ϕ_{I_m}) where $\phi_{H^k} = \mathrm{Pr}_{H^k} \circ \phi$ and $\phi_{I_m} = \mathrm{Pr}_{I_m} \circ \phi$. As

$$|\mathrm{Hom}_{O_F}(L, I_m)(O_{F_0}/(\pi_0^d))| = q^{2mnd},$$

it suffices to show that for any fixed $\varphi \in \mathrm{Hom}_{O_F}(L, I_m)$, we have

$$|\{\phi \in \mathrm{Perm}^n_{L,I_m\oplus H^k}(O_{F_0}/(\pi_0^d)) \mid \phi_{I_m} = \varphi\}| = |\mathrm{Perm}_{L,H^k}(O_{F_0}/(\pi_0^d))|. \tag{5.56}$$

Let L_φ be the lattice L equipped with the hermitian form $(x, y)_{L_\varphi} := (\phi_{H^k}(x), \phi_{H^k}(y))$ where ϕ is any element in $\mathrm{Perm}^n_{L,I_m\oplus H^k}(O_{F_0}/(\pi_0^d))$ such that $\phi_{I_m} = \varphi$. Since each such ϕ is an isometry and ϕ_{I_m} is fixed, $(,)_{L_\varphi}$ is independent of the choice of ϕ. Then we have a bijection

$$\{\phi \in \mathrm{Pherm}^n_{L,I_m\oplus H^k}(O_{F_0}/(\pi_0^d)) \mid \phi_{I_m} = \varphi\} \to \mathrm{Pherm}_{L_\varphi,H^k}(O_{F_0}/(\pi_0^d)), \tag{5.57}$$
$$\phi \mapsto \phi_{H^k}.$$

Since L is integral and ϕ is an isometric embedding, $L_{\phi_{H^k}}$ is also integral. Then according to [LL22, Lemma 2.16],

$$|\mathrm{Pherm}_{L_\varphi,H^k}(O_{F_0}/(\pi_0^d))| = |\mathrm{Pherm}_{L,H^k}(O_{F_0}/(\pi_0^d))|.$$

Combining with the bijection in (5.57), this proves (5.56), hence finishes the proof of the lemma. □

Lemma 5.4.5. *Assume that M is a unimodular lattice, L is an integral lattice of rank n, and $\phi : L \to M \oplus H^k$ is a primitive isometric embedding such that $\mathrm{Pr}_{H^k}(\phi(L))$ is primitive in H^k. Then there exists a $g \in \mathrm{U}(M \oplus H^k)$ such that $g(\phi(L)) \subset H^k$.*

Proof. Consider the non-degenerate symplectic space over $\mathbb{F}_q$: $(\overline{H}^k, \langle\, , \,\rangle) = (\overline{H}^k, \overline{\pi(\, , \,)})$. Let $\overline{v}$ denote the image of $v \in H^k$ in $\overline{H}^k$. Since $\mathrm{Pr}_{H^k}(\phi(L))$ is primitive in H^k, $V(L) := \overline{\mathrm{Pr}_{H^k}(\phi(L))}$ is a n-dimensional subspace. Since M and L are integral, $\mathrm{Pr}_{H^k}(\phi(L))$ is integral. Hence $V(L)$ is an isotropic subspace. Let $\{\ell_1, \cdots, \ell_n\}$ be a basis of L, $\ell_{s,H^k} = \mathrm{Pr}_{H^k}(\ell_s)$ and $e_s = \overline{\ell}_{s,H^k}$. Since $V(L)$ is an n-dimensional isotropic space, we have $k \geq n$ and we can extend $\{e_1, \cdots, e_n\}$ to a standard symplectic basis $\{e_1, f_1, \cdots, e_k, f_k\}$ of $\overline{H}^k$, where $(e_s, f_t) = \delta_{st}$, and $(e_s, e_t) = (f_s, f_t) = 0$ for $1 \leq s, t \leq k$.

Now let $\{\tilde{f}_1, \cdots, \tilde{f}_n\}$ be a lifting of $\{f_1, \cdots, f_n\}$. In particular, for $1 \leq s \leq n$, we have $(\ell_s, \tilde{f}_s) = \pi^{-1} + x$ for some $x \in O_F$. Therefore, $L \oplus \langle \tilde{f}_1 \cdots, \tilde{f}_n\rangle \cong H^n$. Hence, there exists $g \in \mathrm{U}(M \oplus H^k)$ such that $g(L \oplus \langle \tilde{f}_1 \cdots, \tilde{f}_n\rangle) \subset H^k$. □

Lemma 5.4.6. *Let $N \subset H^k$ be a primitive integral lattice of rank i. Then*

$$\#\{w \in \pi H^k/\pi_0^d H^k \mid (N, w) = 0 \mod (\pi^{2d-1})\} = q^{(2d-1)(2k-i)}.$$

Proof. Through this proof, we use $\overline{L}$ to denote the image of L in $\overline{H}^k$ for any sublattice L of H^k. Let $N^\perp$ be the perpendicular lattice of N in H^k. First we show $N^\perp$ is primitive of rank $2k-i$. If $N^\perp$ is not primitive, then there exists $v \in N^\perp$ such that $\pi^{-1}v \in H^k \setminus N^\perp$. However, $(\pi^{-1}v, N) = 0$, hence $\pi^{-1}v \in N^\perp$, which is a contradiction.

We claim that for any $w \in \pi H^k$ and $\pi(N, w) = 0 \mod \pi^a$ with $a \geq 0$, there exists a $x \in \pi^a H^k$ such that $w - x \in N^\perp$. We prove the lemma by assuming the claim, and give the proof of the claim in the last paragraph. Taking $a = 2d$, the claim implies that

$$\begin{aligned}
&\#\{w \in \pi H^k/\pi_0^d H^k \mid (N, w) = 0 \mod (\pi^{2d-1})\} \\
&= \#\{w \in \pi(N^\perp + \pi_0^d H^k)/\pi_0^d H^k \mid (N, w) = 0 \mod (\pi^{2d-1})\}.
\end{aligned}$$

Since $N^\perp$ is primitive of rank $2k-i$, we have

$$\#\{w \in \pi N^\perp/\pi_0^d H^k \mid (N, w) = 0 \mod (\pi^{2d-1})\} = q^{(2d-1)(2k-i)}.$$

This proves the lemma assuming the claim.

Now we prove the claim. Consider the symplectic space over $\mathbb{F}_q$: $(\overline{H}^k, \overline{\pi(\ ,\ )})$. Since N is integral, $\overline{N}$ is isotropic in $\overline{H}^k$. Let $\overline{N}$ be spanned by $\{e_1, \cdots, e_i\}$. Then we may extend $\{e_1, \cdots, e_i\}$ to a standard symplectic basis $\{e_1, f_1, \cdots, e_k, f_k\}$ of $\overline{H}^k$, where $(e_s, f_t) = \delta_{st}$, and $(e_s, e_t) = (f_s, f_t) = 0$ for $1 \leq s, t \leq k$. Let $\{\tilde{e}_1, \tilde{f}_1, \cdots, \tilde{e}_k, \tilde{f}_k\}$ be a lifting of $\{e_1, f_1, \cdots, e_k, f_k\}$. By our choice of $\tilde{e}_s$, we can find a basis $\{w_1, \cdots, w_i\}$ of N such that $w_s - \tilde{e}_s \in \pi H^k$ for any $1 \leq s \leq i$. Consider $x = a_1\tilde{f}_1 + \cdots + a_i\tilde{f}_i \in \langle \tilde{f}_1, \cdots, \tilde{f}_k\rangle$. In order to have $w - x \in N^\perp$, we need to solve the following system of equations:

$$\pi(w_s, x) = \pi(w_s, w) \text{ for } 1 \leq s \leq i. \tag{5.58}$$

Let A denote the $i \times i$ matrix corresponding to this system of linear equations. Since $w_s - \tilde{e}_s \in \pi H^k$, we have $A \equiv \mathrm{Id}_i \mod (\pi)$. Therefore, there exists a unique solution x of (5.58). Moreover, since $\pi(N, w) = 0 \mod \pi^a$, we have $\pi(w_s, w) \in (\pi^a)$ for $1 \leq s \leq i$.

Then (5.58) implies that $a_s \in (\pi^a)$, hence $x \in \pi^a H^k$. The claim is proved. $\square$

5.5 Explicit formulas for $\mathrm{Pden}'(L)$

5.5.1 Explicit formulas and consequences

The goal of this section is to establish the following formulas for

$$\partial\mathrm{Pden}(L) = \mathrm{Pden}'(L) + \sum_{j=0}^{t_{\max}/2} c_{2j}\mathrm{Pden}_{2j}(L). \tag{5.59}$$

Here

$$\mathrm{Pden}'(L) = \frac{\mathrm{Pden}'(I_n, L)}{\mathrm{Den}(I_n, I_n)}$$

is normalized as in (5.35). Recall, from Lemma 5.3.8, that

$$\partial\mathrm{Pden}(L) = \mathrm{Pden}'(L) \tag{5.60}$$

when L is not dual to some vertex lattice Λ_t of positive type $t > 0$.

Theorem 5.5.1. *Let $L \subset \mathbb{V}$ be a lattice of rank n.*

1. *If L is not integral, then $\partial\mathrm{Pden}(L) = 0$.*

2. *If L is unimodular, then*

$$\partial\mathrm{Pden}(L) = \mathrm{Pden}'(L) = \begin{cases} 1, & \text{if } n \text{ is odd,} \\ 0, & \text{if } n \text{ is even.} \end{cases}$$

3. *If $L = I_{n-t} \obot L_2$ where L_2 is of full type t, then*

$$\partial\mathrm{Pden}(L) = \mathrm{Pden}'(L) = \begin{cases} \prod_{\ell=1}^{\frac{t-1}{2}}(1-q^{2\ell}), & \text{if } t \text{ is odd,} \\ (1-\chi(L_2)q^{\frac{t}{2}})\prod_{\ell=1}^{\frac{t}{2}-1}(1-q^{2\ell}), & \text{if } t \text{ is even.} \end{cases}$$

Corollary 5.5.2. *Let L be a lattice. Then $\partial\mathrm{Den}(L) \in \mathbb{Z}$. Moreover, $\partial\mathrm{Den}(L) = 0$ for non-integral L.*

Proof. According to Corollary 5.3.3, we have

$$\partial\mathrm{Den}(L) = \sum_{L \subset L' \subset L_F} \partial\mathrm{Pden}(L').$$

Now Theorem 5.5.1 implies that $\partial\mathrm{Pden}(L') \in \mathbb{Z}$, hence $\partial\mathrm{Den}(L) \in \mathbb{Z}$. If L is non-integral, then $\partial\mathrm{Pden}(L') = 0$ for each L' such that $L \subset L'$ by (1) of Theorem 5.5.1. $\square$

As another corollary, we prove the following cancellation law for $\partial\mathrm{Den}(L)$. Recall that for hermitian lattices L and L' of the same rank, $n(L', L) = \#\{L'' \subset L_F \mid L \subset L'', L'' \cong L'\}$.

Corollary 5.5.3. *Let $L = L_1 \oplus L_2 \subset \mathbb{V}$ be a rank n lattice, with L_1 being unimodular and L_i of rank n_i. Then*

$$\partial\mathrm{Den}(L) - \partial\mathrm{Den}(L_2) = n(I_{n_2}, L_2) \cdot (\delta_{\mathrm{odd}}(n) - \delta_{\mathrm{odd}}(n_2)). \tag{5.61}$$

Proof. By Corollary 5.3.3 and Lemma 5.5.4, we have

$$\partial\mathrm{Den}(L) = \sum_{L \subset L' \subset L_F} \partial\mathrm{Pden}(L') = \sum_{L_2 \subset L_2' \subset L_{2,F}} \partial\mathrm{Pden}(L_1 \oplus L_2').$$

Similarly,

$$\partial\mathrm{Den}(L_2) = \sum_{L_2 \subset L_2' \subset L_{2,F}} \partial\mathrm{Pden}(L_2').$$

Hence

$$\partial\mathrm{Den}(L) - \partial\mathrm{Den}(L_2) = \sum_{L_2 \subset L_2' \subset L_{2,F}} (\partial\mathrm{Pden}(L_1 \oplus L_2') - \partial\mathrm{Pden}(L_2')).$$

If L_2' is not integral, then both $\partial\mathrm{Pden}(L_1 \oplus L_2')$ and $\partial\mathrm{Pden}(L_2')$ vanishes by (1) of Theorem

5.5.1. If L_2' is integral but not unimodular, then (3) of Theorem 5.5.1 implies $\partial\mathrm{Pden}(L_1 \oplus L_2') - \partial\mathrm{Pden}(L_2') = 0$. Hence

$$\partial\mathrm{Den}(L) - \partial\mathrm{Den}(L_2) = \sum_{\substack{L_2 \subset L_2' \subset L_{2,F} \\ L_1 \oplus L_2' \cong \Lambda_0}} (\partial\mathrm{Pden}(L_1 \oplus L_2') - \partial\mathrm{Pden}(L_2')). \tag{5.62}$$

Combining (5.62) with (2) of Theorem 5.5.1, we have

$$\partial\mathrm{Den}(L) - \partial\mathrm{Den}(L_2) = n(I_{n_2}, L_2) \cdot (\delta_{\mathrm{odd}}(n) - \delta_{\mathrm{odd}}(n_2)).$$

This proves the corollary. □

Lemma 5.5.4. *Assume $L = L_1 \oplus L_2$ is a lattice where L_1 is unimodular. If $L \subsetneq L' \subset \pi^{-1}L$ and L' is not of the form $L_1 \oplus L_2'$, then L' is not integral and $\partial\mathrm{Pden}(L') = 0$.*

Proof. Consider the $\mathbb{F}_q$-vector space $\pi^{-1}L/L$. Since we assume L' is not of the form $L_1 \oplus L_2'$, there exists $v \in L' \setminus L$ such that $\mathrm{Pr}_{\pi^{-1}L_1}(v) \neq 0$, which in turn implies that L' is not integral. Hence $\partial\mathrm{Pden}(L') = 0$ by (1) of Theorem 5.5.1. □

5.5.2 Proof strategy

The proof of Theorem 5.5.1 occupies the rest of this section. Since the proof is rather long and technical, we summarize the main idea of the proof first. When there is some $x \in L$ with $\mathrm{val}(x) \leq -1$, $\partial\mathrm{Pden}(L) = 0$ by Lemma 5.3.1. Otherwise, write

$$L = H^j \oplus I_{n_1 - t} \oplus L_2,$$

where L_2 is of full type t. There are four cases.

(a) The case $n_1 - t = 0$ (i.e., $L = H^j \oplus L_2$) is significantly simpler than the general case, and we will deal with it in next subsection although it is part of the general case. For example, when L is of full type, the reduction $\overline{L}$ of L modulo π is a totally isotropic quadratic space over $\mathbb{F}_q$. Hence, the summation in Corollary 5.4.2 is simply:

$$\sum_{V_1 \in \mathrm{Gr}(i,\overline{L})(\mathbb{F}_q)} |\mathrm{O}(V_1, \overline{I_m})| = \binom{n}{i}_q |\mathrm{O}(0_i, \overline{I_m})|.$$

An application of q-binomial theorem settles this case.

(b) The case $j = 0$, i.e., L is integral.

(c) The case $j > 0$ and $t > 0$.

(d) The case that $t = 0$ and $j > 0$ is part of the modification assumption.

In general, the problem becomes harder when $n_1 - t$ is larger. In fact, when $t > 0$, i.e., $n_1 - t < n_1$, (b) and (c) can be proved via Corollary 5.4.2 and an involved application of the induction formulas of $\mathrm{Den}(I_n, L)$ established in Chapter 4. However, when $t = 0$, i.e., L is unimodular, this method fails. To overcome this difficulty and give a uniform proof of (b) and (c), we introduce a new method which is different from the method used in Chapters 3 and 4 even in the case $n \leq 3$.

To illustrate the idea, we stick to case (b) for now. The first key step is to discover a finer structure of $\partial\mathrm{Pden}(L)$ and prove the following formula (Lemma 5.5.11):

$$\partial\mathrm{Pden}(L) = (*) \sum_{r=0}^{\min\{n-t,n-1\}} \sum_{i=0}^{n-1} \binom{n-r}{i-r}_q (-1)^{n-1+i+r} q^{\frac{(i-r)(i-r-1)}{2}} q^{rt} \cdot g(n, n_1, r, q^{-i}), \tag{5.63}$$

where $(*)$ is some constant number and $g(n, n_1, r, X)$ is a linear combination of polynomials of degree $n-1$. The second key observation is that there is a lot of cancellation underlying this linear combination. Indeed, we show for $r < n$ that $g(n, n_1, r, X)$ is actually of degree $\leq n - r - 1$ and is essentially a simple multiple of some simple polynomial (denoted by $h(n, r, X)$) (Lemmas 5.5.13 and 5.5.15). This enables us to apply q-binomial theorem

(Corollary 5.3.24) to the inner sums in (5.63). Consequently, we obtain

$$\partial\mathrm{Pden}(L) = (*) \sum_{r=0}^{\mathrm{m}(n,t)} (-1)^r q^{\frac{(n-r)(n-r-1)}{2}} q^{rt} \cdot g(n, n-t, r, q^{-n}). \tag{5.64}$$

The last step is to evaluate this sum and the result is given by Lemma 5.5.16. It is in this step that the case $n_1 = n$ (L is unimodular) becomes different: the sum above is a sum from $r = 0$ only to $n-1$, not to $n_1 = n$. To make it worse, the 'missing' term $g(n, n, n, X)$ is in fact ill-behaved.

One common strategy in proving Lemmas 5.5.15 and 5.5.16 is to express both sides of the identity as (uniquely) linear combinations of certain basis of polynomials, and prove that the coefficients satisfy the same recursion formulas and the same initial conditions. Here we use crucially the combinatorical properties of $m(U, V)$ (Lemma 5.3.17) for U and V quadratic spaces over $\mathbb{F}_q$.

In Case (c), the derivative becomes the value of some primitive local density polynomials at some non-central point q^j by Lemma 5.3.12. Strikingly, the formula for this value (see (5.97)) is very similar to the formula for the derivative $\mathrm{Pden}(I_{n_1-t} \oplus L_2)$ (see (5.98)). Proof of Cases (b) and (c) will be given in Subsection 5.5.6 after long preparation in Subsections 5.5.4 and 5.5.5.

5.5.3 The case $n_1 - t = 0$

In this subsection we assume that $n_1 - t = 0$ and divide it further into two subcases: $j = 0$ or $j > 0$.

Proposition 5.5.5. *Assume that $L = H^j \oplus L_2$ where $j > 0$ and L_2 is of full type and has rank $n_2 = n - 2j$. Then*

$$\partial\mathrm{Pden}(L) = 0.$$

Proof. By Lemma 5.3.12, we have

$$\mathrm{Pden}'(I_n, L) = 2\Big(\prod_{\ell=1}^{j-1}(1-q^{2\ell})\Big)\mathrm{Pden}(I_n, L_2, q^{2j}).$$

Hence it suffice to show

$$\mathrm{Pden}(I_n, L_2, q^{n-n_2}) = 0.$$

We prove the odd n case and leave the even n case to the reader. According to Corollary 5.4.3, we have

$$\begin{aligned}\mathrm{Pden}(I_n, L_2, q^{n-n_2}) &= \sum_{i=0}^{n_2}(-1)^{n_2-i}q^{\frac{i(i-1)}{2}}\binom{n_2}{i}_q \cdot \prod_{\ell=0}^{n_2-i-1}(q^{2\ell+n-n_2}-1)\cdot\prod_{\ell=1}^{i}(q^{n+1-2\ell}-1)\\ &= \sum_{i=0}^{n_2}(-1)^{n_2-i}q^{\frac{i(i-1)}{2}}\binom{n_2}{i}_q \cdot \prod_{\ell=\frac{n-n_2}{2}}^{\frac{n+n_2}{2}-i-1}(q^{2\ell}-1)\cdot \prod_{\ell=\frac{n+1}{2}-i}^{\frac{n-1}{2}}(q^{2\ell}-1).\end{aligned}$$

We can factor out $\displaystyle\prod_{\ell=\frac{n-n_2}{2}}^{\frac{n-1}{2}}(q^{2\ell}-1)$ so that

$$\mathrm{Pden}(I_n, L_2, q^{n-n_2})\prod_{\ell=\frac{n-1}{2}}^{\frac{n-n_2}{2}}(q^{2\ell}-1)^{-1} = \sum_{i=0}^{n_2}(-1)^{n_2-i}q^{\frac{i(i-1)}{2}}\binom{n_2}{i}_q g(n_2, q^{-i}),$$

where

$$g(n_2, X) = \prod_{\ell=\frac{n+1}{2}}^{\frac{n_2+n}{2}-1}(q^{2\ell}X^2-1) \tag{5.65}$$

is a polynomial of X of degree n_2-1. Now $\mathrm{Pden}(I_n, L_2, q^{n-n_2}) = 0$ by Corollary 5.3.24. $\square$

Proposition 5.5.6. *Assume that L is a full type lattice of rank n. We have*

$$\mathrm{Pden}'(L) = \begin{cases}\prod_{\ell=1}^{\frac{n-1}{2}}(1-q^{2\ell}), & \text{if } n \text{ is odd,}\\ (1-\epsilon q^{\frac{n}{2}})\prod_{\ell=1}^{\frac{n}{2}-1}(1-q^{2\ell}), & \text{if } n \text{ is even.}\end{cases}$$

Proof. First of all, recall that

$$\mathrm{Pden}'(L) = \frac{\mathrm{Pden}'(I_n, L)}{\mathrm{Den}(I_n, I_n)},$$

where

$$\mathrm{Den}(I_n, I_n) = \begin{cases} 2q^{(\frac{n-1}{2})^2} \prod_{\ell=1}^{\frac{n-1}{2}} (q^{2\ell} - 1) & \text{if } n \text{ is odd,} \\ 2q^{(\frac{n}{2})(\frac{n}{2}-1)} (q^{\frac{n}{2}} + \epsilon) \prod_{\ell=1}^{\frac{n}{2}-1} (q^{2\ell} - 1) & \text{if } n \text{ is even.} \end{cases} \tag{5.66}$$

We verify the even n case and leave the odd n case to the reader. Direct calculation using Corollary 5.4.3 gives

$$\begin{aligned} \mathrm{Pden}'(I_n, L)\Big((q^{\frac{n}{2}} + \epsilon) \prod_{\ell=1}^{\frac{n}{2}-1} (q^{2\ell} - 1)\Big)^{-1} &= 2 \sum_{i=0}^{\frac{n}{2}-1} (-1)^{n-i-1} q^{\frac{i(i-1)}{2}} \binom{n}{i}_q (q^{\frac{n}{2}-i} - \epsilon) \prod_{\ell=\frac{n}{2}-i+1}^{n-i-1} (q^{2\ell} - 1) \\ &= 2 \sum_{i=0}^{n-1} (-1)^{n-i-1} q^{\frac{i(i-1)}{2}} \binom{n}{i}_q (q^{\frac{n}{2}-i} - \epsilon) \prod_{\ell=\frac{n}{2}+1}^{n-1} (q^{2\ell-2i} - 1). \end{aligned}$$

According to Corollary 5.3.24,

$$\begin{aligned} \sum_{i=0}^{n-1} (-1)^{n-i-1} q^{\frac{i(i-1)}{2}} \binom{n}{i}_q (q^{\frac{n}{2}-i} - \epsilon) \prod_{\ell=\frac{n}{2}+1}^{n-1} (q^{2\ell-2i} - 1) &= q^{\frac{n(n-1)}{2}} (q^{-\frac{n}{2}} - \epsilon) \prod_{\ell=\frac{n}{2}+1}^{n-1} (q^{2\ell-2n} - 1) \\ &= q^{(\frac{n}{2})(\frac{n}{2}-1)} (1 - \epsilon q^{\frac{n}{2}}) \prod_{\ell=1}^{\frac{n}{2}-1} (1 - q^{2\ell}). \end{aligned}$$

Now by (5.66), we conclude that

$$\mathrm{Pden}'(L) = (1 - \epsilon q^{\frac{n}{2}}) \prod_{\ell=1}^{\frac{n}{2}-1} (1 - q^{2\ell}),$$

as claimed. $\square$

5.5.4 Preparation

In this subsection, we rewrite $\mathrm{Pden}'(I_n, L)$ as a linear combination of special values of certain polynomial $g_{\epsilon_1}(n, m, r, X)$ as in Lemma 5.5.11. We then express general $g_{\epsilon_1}(n, m, r, X)$ as a simple combination of $g_{\epsilon_3}(n, r, r, X)$, see Lemma 5.5.13.

Let $L = I_{n-t}^{\epsilon_1} \oplus L_2$ where L_2 is of full type t. By Corollary 5.4.2, we have

$$\mathrm{Pden}'(I_n, L) = 2\sum_{i=0}^{n-1} \prod_{\ell=1}^{n-i-1} (1 - q^{2\ell}) \sum_{0 \le j \le i} \sum_{\epsilon_2 \in \{\pm 1\}} m\left(0_j \oplus U_{i-j}^{\epsilon_2}, 0_t \oplus U_{n-t}^{\epsilon_1}\right) \left| \mathrm{O}\left(0_j \oplus U_{i-j}^{\epsilon_1}, U_n^{-\epsilon}\right)\right|. \tag{5.67}$$

Here and in the following, we interpret $\sum_{\epsilon_2 \in \{\pm 1\}} f(U_0^{\epsilon_2})$ as $f(U_0^1)$ for a function f with $U_i^{\epsilon_2}$ as input.

Let s and n be integers such that $0 \le s < n$, and let $\epsilon_2 = \pm 1$. For odd n, we define

$$f_{\epsilon_2}(n, s, X) = \begin{cases} \prod_{\ell=\frac{n+1}{2}}^{\frac{n+s-2}{2}} (q^{2\ell} X^2) \cdot q^{\frac{n+s}{2}} X (q^{\frac{n+s}{2}} X - \epsilon\epsilon_2) \cdot \prod_{\ell=\frac{n+s+2}{2}}^{n-1} (q^{2\ell} X^2 - 1) & \text{if } s \text{ is odd,} \\ \prod_{\ell=\frac{n+1}{2}}^{\frac{n-1}{2}+\frac{s}{2}} (q^{2\ell} X^2) \cdot \prod_{\ell=\frac{n+1+s}{2}}^{n-1} (q^{2\ell} X^2 - 1) & \text{if } s \text{ is even.} \end{cases} \tag{5.68}$$

Similarly, for an even integer $n > 0$ and $0 \le s < n$, we define

$$f_{\epsilon_2}(n, s, X) = \begin{cases} \prod_{\ell=\frac{n}{2}}^{\frac{n+s-3}{2}} (q^{2\ell} X^2) q^{\frac{n}{2}+s-1} X \cdot \prod_{\ell=\frac{n+s+1}{2}}^{n-1} (q^{2\ell} X^2 - 1) & \text{if } s \text{ is odd,} \\ \prod_{\ell=\frac{n}{2}}^{\frac{n+s-2}{2}} (q^{2\ell} X^2) \cdot q^{\frac{s}{2}} (q^{\frac{n+s}{2}} X - \epsilon\epsilon_2) \cdot \prod_{\ell=\frac{n+s+2}{2}}^{n-1} (q^{2\ell} X^2 - 1) & \text{if } s \text{ is even.} \end{cases} \tag{5.69}$$

Here when $s = 0$, we always take $\epsilon_2 = 1$. Notice that $f_{\epsilon_2}(n, s, X)$ is a polynomial of degree $n - 1$.

Lemma 5.5.7.

1. *Assume $0 \le i < n$. We have*

$$\prod_{\ell=1}^{n-i-1}(1-q^{2\ell})|\mathrm{O}(0_{i-s}\oplus U_s^{\epsilon_2}, U_n^{-\epsilon})| = (-1)^{n-i-1}q^{\frac{i(i-1)}{2}}I_\epsilon(n)f_{\epsilon_2}(n,s,q^{-i}),$$

where

$$I_\epsilon(n) \coloneqq \begin{cases} \prod_{\ell=1}^{\frac{n-1}{2}}(q^{2\ell}-1) & \text{if } n \text{ is odd,} \\ (q^{\frac{n}{2}}+\epsilon)\prod_{\ell=1}^{\frac{n}{2}-1}(q^{2\ell}-1) & \text{if } n \text{ is even.}\end{cases}$$

2. *Assume $0 \le i \le n$, $s < n$ and that $n' - n > 0$ is even. We have*

$$\prod_{\ell=0}^{n-i-1}(1-q^{2\ell+n'-n})|\mathrm{O}(0_{i-s}\oplus U_s^{\epsilon_2}, U_{n'}^{-\epsilon})| = (-1)^{n-i}q^{\frac{i(i-1)}{2}}I_\epsilon(n',n)f_{\epsilon_2}(n,s,q^{\frac{n'-n}{2}-i}),$$

where

$$I_\epsilon(n',n) \coloneqq \begin{cases} \prod_{\ell=\frac{n'-n}{2}}^{\frac{n'-1}{2}}(q^{2\ell}-1) & \text{if } n \text{ is odd,} \\ (q^{\frac{n'}{2}}+\epsilon)\prod_{\ell=\frac{n'-n}{2}}^{\frac{n'}{2}-1}(q^{2\ell}-1) & \text{if } n \text{ is even.}\end{cases}$$

Proof. This follows from the formula of $|\mathrm{O}(0_{i-s}\oplus U_s^{\epsilon_2}, U_n^{-\epsilon})|$ given in Lemma 5.3.15 and a straightforward computation. □

Lemma 5.5.8. *For integers $0 \le i, t \le n$, we have*

$$\binom{t}{i}_q = \sum_{a=0}^{n-t}(-1)^a \cdot q^{a(t+1-i)}q^{\frac{a(a-1)}{2}} \cdot \binom{n-t}{a}_q \cdot \binom{n-a}{i-a}_q. \tag{5.70}$$

Proof. The identity is automatically true for $i > t$ as both sides are zero. Recall the following analogue of Pascal's identity for q-binomial coefficients:

$$\binom{t}{i}_q = \binom{t+1}{i}_q - q^{t-i+1}\binom{t}{i-1}_q. \tag{5.71}$$

By this identity, we obtain 2 terms, one with the t-index raised, another with the i-index lowered. Applying again (5.71) to $\binom{t+1}{i}_q$ and $\binom{t}{i-1}_q$ respectively, we obtain

$$\binom{t}{i}_q = \binom{t+2}{i}_q - q^{t-i+2}\binom{t+1}{i-1}_q - q^{t-i+1}\left(\binom{t+1}{i-1}_q - q^{t-i+2}\binom{t-1}{i-2}_q\right).$$

We may continue this process and after $n-t$ steps, we obtain 2^{n-t} many terms. Each term corresponds to a lattice path starting from the origin, going to north and east as follows. If the ℓ-th step raises the index of t (resp. reduces the index of i), we define the lattice path goes towards north for the ℓ-th step (resp. east). We use $I = (i_1, \cdots, i_{n-t})$ where $i_\ell \in \{0,1\}$ to denote the path whose ℓ-th step goes towards north (east) if $i_\ell = 0$ (i_ℓ =1) and let $|I| = i_1 + \cdots + i_{n-t}$. We use P_I to denote the term corresponding to I. Now the lemma follows from the following claim.

Claim:

$$\sum_{I,|I|=a} P_I = (-1)^a \cdot q^{a(t+1-i)} q^{\frac{a(a-1)}{2}} \cdot \binom{n-t}{a}_q \cdot \binom{n-a}{i-a}_q.$$

Indeed a direct calculation shows that

$$P_{(1^a,0^{n-t-a})} = (-1)^a \cdot q^{a(t+1-i)} q^{\frac{a(a-1)}{2}} \cdot \binom{n-a}{i-a}_q.$$

Let A_I denote the area bounded by the lattice path I, the horizontal axis, and the vertical line given by $x = |I|$. Then a direct computation shows that

$$P_I = q^{A_I} \cdot P_{(1^a,0^{n-t-a})}.$$

Now the claim follows from the well-known formula of q-binomial coefficient (see [Cam, Theorem 6.9] for example):

$$\sum_{I,|I|=a} q^{A_I} = \binom{n-t}{a}_q.$$

This proves the claim and the lemma. $\square$

Lemma 5.5.9. *For an integer $n \geq 0$ and $\epsilon = \pm 1$, let*

$$\alpha(n) = q^{\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor} = \begin{cases} q^{(\frac{n-1}{2})^2} & \text{if } n \text{ is odd,} \\ q^{\frac{n}{2}(\frac{n}{2}-1)} & \text{if } n \text{ is even,} \end{cases} \quad \text{and} \quad \beta_\epsilon(n) = \begin{cases} (-1)^{\frac{n-1}{2}} & \text{if } n \text{ is odd,} \\ \epsilon(-1)^{\frac{n}{2}} & \text{if } n \text{ is even,} \\ 1 & \text{if } n = 0. \end{cases}$$

Then

$$\sum_{j=0}^{n} (-1)^j q^{\frac{j(j-1)}{2}} m(0_j, U_n^\epsilon) = \alpha(n)\beta_\epsilon(n).$$

Proof. If $n = 0$, the statement both sides are 1 by definition. From now on we assume $n > 0$. By [LZ22b, Lemma 3.2.2.], we have

$$|m(0_j, U_n^\epsilon)| = \binom{d}{j}_q \cdot \prod_{l=1}^{j} (q^{d+e-l} + 1),$$

with

$$d = \begin{cases} \frac{n-1}{2} & \text{if } n \text{ is odd,} \\ \frac{n}{2} - \frac{1-\epsilon}{2} & \text{if } n \text{ is even,} \end{cases} \quad \text{and} \quad e = \begin{cases} 1 & \text{if } n \text{ is odd,} \\ 1-\epsilon & \text{if } n \text{ is even.} \end{cases}$$

So we have

$$\sum_{j=0}^{n} (-1)^j q^{\frac{j(j-1)}{2}} \cdot m(0_j, U_n^\epsilon) = \sum_{j=0}^{d} (-1)^j q^{\frac{j(j-1)}{2}} \binom{d}{j}_q \prod_{l=1}^{j} (q^{d+e-l} + 1),$$

which by Lemma 5.3.25 equals to $(-q^{d+e-1})^d$. A direct calculation checks that $(-q^{d+e-1})^d = \alpha(n)\beta_\epsilon(n)$. ☐

Definition 5.5.10. *For $0 \leq r \leq m \leq n$, we define*

$$g_{\epsilon_1}(n, m, r, X)$$
$$:= \sum_{k=0}^{r} (-1)^k q^{\frac{k(k-1)}{2}} \sum_{\epsilon_2 \in \{\pm 1\}} m(U_k^{\epsilon_2}, U_m^{\epsilon_1}) \sum_{j=0}^{r-k} (-1)^j q^{\frac{j(j-1)}{2}} \cdot \binom{m-j-k}{r-j-k}_q \cdot m(0_j, U_{m-k}^{\delta}) f_{\epsilon_2}(n, k, X)$$

with $\delta = \delta(m, k, \epsilon_1, \epsilon_2)$ as defined in (5.43). In the following, when n is clear in the context, *we simply write $g_{\epsilon_1}(m, r, X)$ for $g_{\epsilon_1}(n, m, r, X)$.*

In particular, $g_{\epsilon_1}(n, m, 0, X) = f_1(n, 0, X)$ and by Lemma 5.5.9,

$$g_{\epsilon_1}(n, r, r, X) = \sum_{k=0}^{r} (-1)^k q^{\frac{k(k-1)}{2}} \cdot \sum_{\epsilon_2 \in \{\pm 1\}} m(U_k^{\epsilon_2}, U_r^{\epsilon_1}) \cdot \alpha(r-k)\beta_\delta(r-k) \cdot f_{\epsilon_2}(n, k, X). \tag{5.72}$$

In the rest of this section, we let $m = n - t$ without explicit mentioning.

Lemma 5.5.11. *Assume L is a lattice of rank n and type t.*

1. *Let $\mathrm{m}(n, t) := \min\{n - t, n - 1\}$. Then for $0 \le i \le n - 1$, we have*

$$(\mathrm{Pden}^{n-i})'(I_n, L) = 2I_\epsilon(n) \cdot \sum_{r=0}^{\mathrm{m}(n,t)} \binom{n-r}{i-r}_q (-1)^{i-r+n-1} q^{\frac{(i-r)(i-r-1)}{2}} q^{r(n-m)} \cdot g_{\epsilon_1}(n, m, r, q^{-i}).$$

2. *Assume that $n' - n$ is a positive even integer and $m < n$. Then for $0 \le i \le n$, we have*

$$\mathrm{Pden}^{n-i}(I_{n'}^{-\epsilon}, L, q^{n'-n}) = I_\epsilon(n', n) \sum_{r=0}^{m} \binom{n-r}{i-r}_q (-1)^{i-r+n} q^{\frac{(i-r)(i-r-1)}{2}} q^{r(n-m)} \cdot g_{\epsilon_1}(n, m, r, q^{\frac{n'-n}{2} - i}).$$

Proof. We prove (1) first. By (5.67), Lemma 5.5.7 and Lemma 5.3.18, we have

$$\begin{aligned}
&(\mathrm{Pden}^{n-i})'(I_n, L) \cdot \Big(2(-1)^{n-i-1} I_\epsilon(n)\Big)^{-1} \\
&= \sum_{\ell=0}^{t} \binom{t}{\ell}_q \sum_{k=0}^{i-\ell} \sum_{\epsilon_2 \in \pm 1} q^{(t-\ell)(i-\ell)} q^{\frac{i(i-1)}{2}} m(0_{i-k-\ell} \oplus U_k^{\epsilon_2}, U_{n-t}^{\epsilon_1}) f_{\epsilon_2}(n, k, q^{-i}) \\
&= \sum_{s=\max\{i-t,0\}}^{i} \binom{t}{i-s}_q \sum_{k=0}^{s} \sum_{\epsilon_2 \in \pm 1} q^{(t-(i-s))s} q^{\frac{i(i-1)}{2}} m(0_{s-k} \oplus U_k^{\epsilon_2}, U_{n-t}^{\epsilon_1}) f_{\epsilon_2}(n, k, q^{-i}),
\end{aligned}$$

where the last identity is obtained by setting $s = i - \ell$. Notice that if $s > n - t$, then $m(0_{s-k} \oplus U_k^{\epsilon_2}, U_{n-t}^{\epsilon_1}) = 0$. Hence we may assume $s \le n - t$, or equivalently $t \le n - s$ in the above summation. Now applying (5.70) to $i - s \le t \le n - s$ and let $m = n - t$, we may rewrite the above summation as

$$\sum_{s=\max\{i-t,0\}}^{i} \sum_{a=0}^{m-s}(-1)^a \binom{n-s-a}{i-s-a}_q \cdot \binom{m-s}{a}_q \cdot$$
$$\sum_{k=0}^{s} \sum_{\epsilon_2 \in \pm 1} q^{a(n-m+1-(i-s))+\frac{a(a-1)}{2}+(n-m+s-i)s} q^{\frac{i(i-1)}{2}} m(0_{s-k} \oplus U_k^{\epsilon_2}, U_m^{\epsilon_1}) f_{\epsilon_2}(n,k,q^{-i})$$
$$= \sum_{s=\max\{i-t,0\}}^{i} \sum_{a=0}^{m-s} \binom{n-s-a}{i-s-a}_q q^{\frac{(i-(s+a))(i-(s+a+1))}{2}} q^{(s+a)(n-m)}$$
$$\cdot (-1)^a \binom{m-s}{a}_q \sum_{k=0}^{s} \sum_{\epsilon_2 \in \pm 1} q^{(s+a)(s-\frac{s+a+1}{2})+\frac{a(a+1)}{2}} m(0_{s-k} \oplus U_k^{\epsilon_2}, U_m^{\epsilon_1}) f_{\epsilon_2}(n,k,q^{-i}).$$

Now let $r = s + a$. Notice that $r \le m$ and $\binom{n-r}{i-r}_q = 0$ for $r > i$. Rearranging the summation index, we obtain

$$(\mathrm{Pden}^{n-i})'(I_n, L) \cdot \Big((-1)^{n-i-1} I_\epsilon(n)\Big)^{-1} = \sum_{r=0}^{\mathrm{m}\{n,t\}} \binom{n-r}{i-r}_q q^{\frac{(i-r)(i-r-1)}{2}} q^{r(n-m)} \cdot g_{\epsilon_1}(n,m,r,q^{-i}),$$

where

$$g_{\epsilon_1}(n,m,r,X)$$
$$= \sum_{s=0}^{r} (-1)^{r-s} \cdot \binom{m-s}{r-s}_q \cdot \sum_{k=0}^{s} \sum_{\epsilon_2 \in \pm 1} q^{\frac{s(s-1)}{2}} m(0_{s-k} \oplus U_k^{\epsilon_2}, U_m^{\epsilon_1}) f_{\epsilon_2}(n,k,X)$$
$$= \sum_{k=0}^{r} (-1)^k q^{\frac{k(k-1)}{2}} \sum_{\epsilon_2 \in \{\pm 1\}} m(U_k^{\epsilon_2}, U_m^{\epsilon_1}) \sum_{j=0}^{r-k} (-1)^j q^{\frac{j(j-1)}{2}} \cdot \binom{m-j-k}{r-j-k}_q \cdot m(0_j, U_{m-k}^{\delta}) f_{\epsilon_2}(n,k,X).$$

Here, we use Lemma 5.3.17 to obtain the last identity.

Using (2) of Lemma 5.5.7, the same proof of (1) proves (2). □

We conclude this subsection by establishing a formula to express $g_{\epsilon_1}(m,r,X)$ in terms of $g_{\epsilon_3}(r,r,X)$, which, as we will see, has a particular simple form (Lemma 5.5.15). First, we need the following identity which might have independent interest.

Lemma 5.5.12. *For any integers $0 \le r \le n$, we have*

$$\sum_{j=0}^{r}(-1)^j q^{j(j-1)/2}\binom{n-j}{r-j}_q m(0_j, U_n^\epsilon) = \sum_{\tau\in\{\pm1\}} m(U_r^\tau, U_n^\epsilon)\alpha(r)\beta_\tau(r), \tag{5.73}$$

where $\alpha(r)$ and $\beta_\tau(r)$ are defined in Lemma 5.5.9.

Proof. We proceed by induction on n. The case $n=1$ is obvious. Now recall the identities

$$\begin{aligned} m(0_j, U_n^\epsilon) &= \binom{n}{j}_q - \sum_{i=1}^{j}\sum_{\sigma\in\{\pm1\}} m(0_{j-i}\oplus U_i^\sigma, U_n^\epsilon) \\ &= \binom{n}{j}_q - \sum_{i=1}^{j}\sum_{\sigma\in\{\pm1\}} q^{-(j-i)i} m(U_i^\sigma, U_n^\epsilon) m(0_{j-i}, U_{n-i}^{\delta(n,i,\epsilon,\sigma)}), \end{aligned}$$

by (5.42), and

$$\sum_{j=0}^{r}(-1)^j q^{j(j-1)/2}\binom{n-j}{r-j}_q\binom{n}{j}_q = \sum_{j=0}^{r}(-1)^j q^{j(j-1)/2}\binom{n}{r}_q\binom{r}{j}_q = 0. \tag{5.74}$$

These imply that

$$\begin{aligned} &\sum_{j=0}^{r}(-1)^j q^{j(j-1)/2}\binom{n-j}{r-j}_q m(0_j, U_n^\epsilon) \\ =& -\sum_{j=0}^{r}(-1)^j q^{j(j-1)/2}\binom{n-j}{r-j}_q \sum_{i=1}^{j}\sum_{\sigma\in\{\pm1\}} q^{-(j-i)i} m(U_i^\sigma, U_n^\epsilon) m(0_{j-i}, U_{n-i}^{\delta(n,i,\epsilon,\sigma)}) \\ =& \sum_{i=1}^{r}\sum_{\sigma\in\{\pm1\}}(-1)^{i+1} q^{i(i-1)/2} m(U_i^\sigma, U_n^\epsilon)\sum_{j=0}^{r-i}(-1)^j q^{j(j-1)/2}\binom{n-i-j}{r-i-j}_q m(0_j, U_{n-i}^{\delta(n,i,\epsilon,\sigma)}). \end{aligned}$$

where in the last step we switch the order of summation and substitute j by $j+i$.

We can now use the induction hypothesis

$$\sum_{j=0}^{r-i}(-1)^j q^{j(j-1)/2}\binom{n-i-j}{r-i-j}_q m(0_j, U_{n-i}^{\delta(n,i,\epsilon,\sigma)}) = \sum_{\tau\in\{\pm1\}} m(U_{r-i}^\tau, U_{n-i}^{\delta(n,i,\epsilon,\sigma)})\alpha(r-i)\beta_\tau(r-i)$$

and

$$m(U_i^\sigma, U_n^\epsilon) m(U_{r-i}^\tau, U_{n-i}^{\delta(n,i,\epsilon,\sigma)}) = m(U_r^{\delta'}, U_n^\epsilon) m(U_i^\sigma, U_r^{\delta'}),$$

where $\delta' \in \{\pm 1\}$ such that $\tau = \delta(r, i, \delta', \sigma)$ (see Lemma 5.3.21) to obtain

$$\begin{aligned}
&\sum_{j=0}^{r}(-1)^j q^{j(j-1)/2}\binom{n-j}{r-j}_q m(0_j, U_n^{\epsilon}) \\
=&\sum_{i=1}^{r}\sum_{\sigma\in\{\pm 1\}}(-1)^{i+1}q^{i(i-1)/2}\sum_{\tau\in\{\pm 1\}} m(U_r^{\delta'}, U_n^{\epsilon})m(U_i^{\sigma}, U_r^{\delta'})\alpha(r-i)\beta_\tau(r-i)
\end{aligned} \tag{5.75}$$

Hence (5.73) is equivalent to

$$\sum_{i=0}^{r}\sum_{\sigma\in\{\pm 1\}}(-1)^{i}q^{i(i-1)/2}\sum_{\tau\in\{\pm 1\}} m(U_r^{\delta'}, U_n^{\epsilon})m(U_i^{\sigma}, U_r^{\delta'})\alpha(r-i)\beta_\tau(r-i) = 0 \tag{5.76}$$

Now applying Lemma 5.5.9, the left hand side of (5.76) is equal to

$$\begin{aligned}
&\sum_{i=0}^{r}\sum_{\sigma\in\{\pm 1\}}(-1)^{i}q^{i(i-1)/2}\sum_{\tau\in\{\pm 1\}} m(U_r^{\delta'}, U_n^{\epsilon})m(U_i^{\sigma}, U_r^{\delta'})\sum_{j=0}^{r-i}(-1)^j q^{j(j-1)/2} m(0_j, U_{r-i}^{\tau}) \\
=&\sum_{\tau\in\{\pm 1\}} m(U_r^{\delta'}, U_n^{\epsilon})\sum_{i=0}^{r}\sum_{j=0}^{r-i}\sum_{\sigma\in\{\pm 1\}}(-1)^{i+j}q^{(i+j)(i+j-1)/2} m(0_j \oplus U_i^{\sigma}, U_r^{\delta'}) \\
=&\sum_{\delta'\in\{\pm 1\}} m(U_r^{\delta'}, U_n^{\epsilon})\sum_{k=0}^{r}(-1)^k q^{k(k-1)/2}\binom{r}{k}_q = 0.
\end{aligned}$$

The lemma is proved. $\square$

Lemma 5.5.13. *For $0 \le r \le m < n$, we have*

$$g_{\epsilon_1}(m, r, X) = \sum_{\epsilon_3\in\{\pm 1\}} m(U_r^{\epsilon_3}, U_m^{\epsilon_1}) g_{\epsilon_3}(r, r, X).$$

Proof. When $r = 0$, we have by definition

$$g(m, 0, X) = f_1(n, 0, X) = \sum_{\epsilon_3\in\{\pm 1\}} m(U_0^{\epsilon_3}, U_m^{\epsilon_1}) g_{\epsilon_3}(0, 0, X).$$

Now we assume $r > 0$, $\delta = \delta(m, k, \epsilon_1, \epsilon_2)$ and $\delta' = \delta(r, k, \epsilon_3, \epsilon_2)$. On the one hand, by definition

$$g_{\epsilon_1}(m,r,X)$$
$$= \sum_{k=0}^{r}(-1)^k q^{\frac{k(k-1)}{2}} \sum_{\epsilon_2\in\{\pm 1\}} m(U_k^{\epsilon_2}, U_m^{\epsilon_1}) \sum_{j=0}^{r-k}(-1)^j q^{\frac{j(j-1)}{2}} \cdot \binom{m-j-k}{r-j-k}_q \cdot m(0_j, U_{m-k}^{\delta}) f_{\epsilon_2}(n,k,X).$$

On the other hand, we have by (5.72),

$$\sum_{\epsilon_3\in\{\pm 1\}} m(U_r^{\epsilon_3}, U_m^{\epsilon_1}) g_{\epsilon_3}(r,r,X)$$
$$= \sum_{k=0}^{r}(-1)^k q^{\frac{k(k-1)}{2}} \cdot \sum_{\epsilon_2\in\{\pm 1\}} \sum_{\epsilon_3\in\{\pm 1\}} m(U_k^{\epsilon_2}, U_r^{\epsilon_3}) m(U_r^{\epsilon_3}, U_m^{\epsilon_1}) \cdot \alpha(r-k)\beta_{\delta'}(r-k) \cdot f_{\epsilon_2}(n,k,X).$$

By Lemma 5.3.21,

$$m(U_k^{\epsilon_2}, U_r^{\epsilon_3}) m(U_r^{\epsilon_3}, U_m^{\epsilon_1}) = m(U_k^{\epsilon_2}, U_m^{\epsilon_1}) m(U_{r-k}^{\delta'}, U_{m-k}^{\delta}).$$

Hence, in order to prove the theorem, it suffices to show for any k and ϵ_2

$$\sum_{j=0}^{r-k}(-1)^j q^{\frac{j(j-1)}{2}} \cdot \binom{m-j-k}{r-j-k}_q \cdot m(0_j, U_{m-k}^{\delta}) = \sum_{\delta'\in\{\pm 1\}} m(U_{r-k}^{\delta'}, U_{m-k}^{\delta}) \cdot \alpha(r-k)\beta_{\delta'}(r-k),$$

which is exactly the content of Lemma 5.5.12. □

5.5.5 Some identities between polynomials

Although $g_{\epsilon_1}(r,r,X)$ by definition is a complicated linear combination of $f_{\epsilon_2}(n,k,X)$. We show in this subsection that in fact $g_{\epsilon_1}(r,r,X)$ has a very simple form (Lemma 5.5.15). Similarly, although $\mathrm{Pden}'(I_n, L)$ is a complicated linear combination of the special values of $g_{\epsilon_1}(n-t,r,X)$, certain linear combination of $g_{\epsilon_1}(n-t,r,X)$ is of a simple form (Lemma 5.5.16). By a direct computation, we can check the following lemma.

Lemma 5.5.14. *For* $0 \leq s \leq n-1$*, let*

$$h_{\epsilon_1}(n,s,X) := \begin{cases} \prod_{l=\frac{n+s+1}{2}}^{n-1}(q^{2l}X^2-1) & \text{if } n-s \text{ is odd,} \\ (q^{\frac{n+s}{2}}X-\epsilon\epsilon_1)\prod_{l=\frac{n+s+2}{2}}^{n-1}(q^{2l}X^2-1) & \text{if } n-s \text{ is even.} \end{cases} \tag{5.77}$$

Then

$$h_{\epsilon_1}(n,j,qX) = h_{\epsilon_1}(n+1,j+1,X), \tag{5.78}$$

$$q^{\lfloor \frac{n+j+2}{2} \rfloor}Xh_{\epsilon_1}(n+1,j+1,X) = h_{\epsilon_1}(n+1,j,X) + (-1)^{n+j+1}\epsilon\epsilon_1 h_{\epsilon_1}(n+1,j+1,X). \tag{5.79}$$

Lemma 5.5.15. *For integers* $0 < r \leq n-1$ *and* $\epsilon, \epsilon_1 = \pm 1$*, we have*

$$g_{\epsilon_1}(r,r,X) = (-1)^{r(n-1)}\epsilon_1^n \epsilon^r \alpha(r) h_{\epsilon_1}(n,r,X).$$

In particular, $g_{\epsilon_1}(r,r,X)$ *is a polynomial of degree* $n-r-1$*.*

Proof. We prove the case when n case is odd and r is even, and leave the other three cases to the reader. The idea is the same (a little bit more complicated). In this case, we need to show

$$\frac{g_{\epsilon_1}(r,r,X)}{h_{\epsilon_1}(n,r,X)} = \sum_{k=0}^{r}(-1)^k q^{\frac{k(k-1)}{2}} \cdot \sum_{\epsilon_2 \in \{\pm 1\}} m(U_k^{\epsilon_2}, U_r^{\epsilon_1}) \cdot \alpha(r-k)\beta_\delta(r-k) \cdot \frac{f_{\epsilon_2}(n,k,X)}{h_{\epsilon_1}(n,r,X)} = \epsilon_1 \alpha(r), \tag{5.80}$$

where $\delta = \delta(r,k,\epsilon_1,\epsilon_2)$. Since n is odd and r is even, we have

$$\frac{f_{\epsilon_2}(n,k,X)}{h_{\epsilon_1}(n,r,X)} = \begin{cases} \prod_{\ell=\frac{n+1}{2}}^{\frac{n+k-2}{2}}(q^{2\ell}X^2) \cdot q^{\frac{n+k}{2}}X(q^{\frac{n+k}{2}}X-\epsilon\epsilon_2)\prod_{\ell=\frac{n+2+k}{2}}^{\frac{n+r-1}{2}}(q^{2\ell}X^2-1) & \text{if } k \text{ is odd,} \\ \prod_{\ell=\frac{n+1}{2}}^{\frac{n-1+k}{2}}(q^{2\ell}X^2) \cdot \prod_{\ell=\frac{n+1+k}{2}}^{\frac{n+r-1}{2}}(q^{2\ell}X^2-1) & \text{if } k \text{ is even.} \end{cases}$$

As a result, dividing (5.80) by $q^{(n+\frac{r-1}{2})(\frac{r-1}{2})+\frac{n+r}{2}} \cdot X^r$ and setting $Y = (q^{\frac{n+1}{2}}X)^{-1}, j = r-k,$

(5.80) is equivalent to

$$\tilde{g}_{\epsilon_1}(r,Y) \coloneqq \sum_{j=0}^{r}(-1)^{r-j}q^{\frac{(r-j)(r-j-1)}{2}} \cdot \alpha(j) \cdot \sum_{\epsilon_2\in\{\pm1\}} m(U_{r-j}^{\epsilon_2}, U_r^{\epsilon_1})\beta_\delta(j) \cdot \tilde{f}_{\epsilon_2}(j,Y) = \epsilon_1 Y^r, \tag{5.81}$$

where $\tilde{f}_{\epsilon_2}(j,Y)$ is a polynomial of degree j defined as follows:

$$\tilde{f}_{\epsilon_2}(j,Y) \coloneqq \begin{cases} \prod_{\ell=\frac{r-j}{2}}^{\frac{r-2}{2}}(1-q^{-2\ell}Y^2) & \text{if } j \text{ is even,} \\ (1-\epsilon\epsilon_2 q^{-\frac{r-j-1}{2}}Y) \cdot \prod_{\ell=\frac{r-j+1}{2}}^{\frac{r-2}{2}}(1-q^{-2\ell}Y^2) & \text{if } j \text{ is odd.} \end{cases}$$

Since $\{\tilde{f}_1(j,Y), 0 \le j \le r\}$ forms a basis of the space of polynomials with degree $\le r$, there exists unique tuples (a_j) and (b_j) such that

$$f(Y) = \epsilon_1 Y^r = \sum_{j=0}^{r} a_j \tilde{f}_1(j,Y), \quad \text{and} \quad \tilde{g}_{\epsilon_1}(r,Y) = \sum_{j=0}^{r} b_j \tilde{f}_1(j,Y).$$

We need to show $(a_j) = (b_j)$. It is easy to check

$$a_r = b_r = (-1)^{\frac{r}{2}}\epsilon_1\alpha(r).$$

Now to prove $a_j = b_j$ for all j, it suffices to prove that both a_j and b_j satisfy the recursion formula for $j < r$

$$a_j = \begin{cases} -q^j \frac{1-q^{-j-2}}{1-q^{j-r}} a_{j+2} & \text{if } j \text{ is even,} \\ 0 & \text{if } j \text{ is odd.} \end{cases} \tag{5.82}$$

We start with a_j. First of all, we have

$$\sum_{j=0}^{r} q^{-r} a_j \tilde{f}_1(j,Y) = q^{-r} f(Y) = f(q^{-1}Y) = \sum_{j=0}^{r} a_j \tilde{f}_1(j, q^{-1}Y). \tag{5.83}$$

Notice that

$$\tilde{f}_1(j, q^{-1}Y) = (1-q^{-r}Y^2)\tilde{f}_1(j-2,Y).$$

Since

$$\tilde{f}_1(j+1,Y)/\tilde{f}_1(j,Y) = \begin{cases} 1-\epsilon q^{-\frac{r-j-2}{2}}Y & \text{if } j \text{ is even,} \\ 1+\epsilon q^{-\frac{r-j-1}{2}}Y & \text{if } j \text{ is odd,} \end{cases}$$

we have

$$Y\tilde{f}_1(j,Y) = \begin{cases} -\epsilon q^{\frac{r-j-2}{2}}(\tilde{f}_1(j+1,Y)-\tilde{f}_1(j,Y)) & \text{if } j \text{ is even,} \\ \epsilon q^{\frac{r-j-1}{2}}(\tilde{f}_1(j+1,Y)-\tilde{f}_1(j,Y)) & \text{if } j \text{ is odd,} \end{cases}$$

$$Y^2\tilde{f}_1(j,Y) = \begin{cases} -q^{r-j-2}(\tilde{f}_1(j+2,Y)-\tilde{f}_1(j,Y)) & \text{if } j \text{ is even,} \\ -q^{r-j-2}(\tilde{f}_1(j+2,Y)+(q-1)\tilde{f}_1(j+1,Y)-q\tilde{f}_1(j,Y)) & \text{if } j \text{ is odd.} \end{cases}$$

Therefore,

$$\tilde{f}_1(j,q^{-1}Y) = \begin{cases} q^{-j}\tilde{f}_1(j,Y)+(1-q^{-j})\tilde{f}_1(j-2,Y) & \text{if } j \text{ is even,} \\ q^{-j}\tilde{f}_1(j,Y)+(q-1)q^{-j}\tilde{f}_1(j-1,Y)+(1-q^{1-j})\tilde{f}_1(j-2,Y) & \text{if } j \text{ is odd.} \end{cases}$$

Plugging this into (5.83), we obtain

$$(q^{-r}-q^{-j})a_j = \begin{cases} (q-1)q^{-j-1}a_{j+1}+(1-q^{-j-2})a_{j+2} & \text{if } j \text{ is even,} \\ (1-q^{-j-1})a_{j+2} & \text{if } j \text{ is odd,} \end{cases}$$

with $a_{r+1}=a_{r+2}=0$. So we have (5.82). We remark that in other cases, we have similar recursion formula as above but could not be simplified like (5.82).

Now we compute b_j for $j<r$. Recall that r is even. First, if $j=0$, we have

$$\sum_{\epsilon_2\in\{\pm 1\}} m(U_{r-j}^{\epsilon_2},U_r^{\epsilon_1})\beta_\delta(j)\cdot\tilde{f}_{\epsilon_2}(j,Y) = \tilde{f}_{\epsilon_1}(0,Y).$$

It is easy to check that

$$\tilde{f}_{\epsilon_2}(j,Y)=\begin{cases}\tilde{f}_1(j,Y) & \text{if } j \text{ is even,}\\ (1-\epsilon\epsilon_2 q^{-\frac{r-j-1}{2}})\tilde{f}_1(j-1,Y) & \text{if } j \text{ is odd,}\end{cases} \tag{5.84}$$

Now Lemmas 5.3.20 and 5.5.9 imply for $j\neq 0$ and $\delta=\delta(r,j,\epsilon_1,\epsilon_2)$, we have

$$\sum_{\epsilon_2\in\{\pm 1\}} m(U^{\epsilon_2}_{r-j},U^{\epsilon_1}_r)\cdot\beta_\delta(j)\cdot\tilde{f}_{\epsilon_2}(j,Y)=\begin{cases}\frac{2\epsilon_1(-1)^{\frac{j}{2}}(\epsilon_1 q^{-\frac{j}{2}}+q^{-\frac{r-j}{2}})}{(1+\epsilon_1 q^{-\frac{j}{2}})(1+q^{-\frac{r-j}{2}})}m(U^1_{r-j},U^{\epsilon_1}_r)\tilde{f}_1(j,Y) & \text{if } j \text{ is even,}\\ 2(-1)^{\frac{j-1}{2}}m(U^1_{r-j},U^{\epsilon_1}_r)\tilde{f}_1(j-1,Y) & \text{if } j \text{ is odd.}\end{cases}$$

Plugging this into the definition of $\tilde{g}_{\epsilon_1}(r,X)$ as in (5.81), we obtain

$$\tilde{g}_{\epsilon_1}(r,X)=\sum_{j=0}^{r} b_j\tilde{f}_1(j,Y)$$

with $b_j=0$ for odd j, $b_0=q^{\frac{r(r-1)}{2}}(1-2q^{-(r-1)}m(U^1_1,U^{\epsilon_1}_r))$, and

$$\begin{aligned}b_j=&2(-1)^{r-\frac{j}{2}}q^{\frac{(r-j)(r-j-1)}{2}}\frac{(q^{-\frac{j}{2}}+\epsilon_1 q^{-\frac{r-j}{2}})}{(1+\epsilon_1 q^{-\frac{j}{2}})(1+q^{-\frac{r-j}{2}})}m(U^1_{r-j},U^{\epsilon_1}_r)\\ &-2(-1)^{r-\frac{j}{2}}q^{\frac{(r-j-1)(r-j-2)}{2}}\alpha(j+1)m(U^1_{r-j-1},U^{\epsilon_1}_r)\end{aligned}$$

for even $j\neq 0$. Applying Lemma 5.3.20, for even $j\neq 0$, we have

$$b_j=2(-1)^{r-\frac{j}{2}}q^{\frac{(r-j)(r-j-1)}{2}}\frac{q^{-\frac{r-j}{2}}(\epsilon_1+q^{-\frac{r}{2}})}{(1+\epsilon_1 q^{-\frac{j}{2}})(1+q^{-\frac{r-j}{2}})}\alpha(j)m(U^1_{r-j},U^{\epsilon_1}_r).$$

Now applying Lemma 5.3.20 twice, we can check that b_j satisfy (5.82). □

Recall that $\mathrm{m}(n,t)\coloneqq\min\{n-t,n-1\}$.

Lemma 5.5.16. *For $0\le t\le n$, and $m=n-t$, we have*

$$\sum_{r=0}^{\mathrm{m}(n,t)}(-1)^r q^{\frac{(n-r)(n-r-1)}{2}}q^{rt}\cdot g_{\epsilon_1}(m,r,X)=F_{\epsilon_1}(n,m,X), \tag{5.85}$$

where

$$F_{\epsilon_1}(n,m,X) = \begin{cases} q^{\frac{n(n-1)}{2}} f_{\epsilon_1}(n,m,X) & \textit{if } t \neq 0, \\ (-1)^{n-1}\alpha(n)\sum_{\ell=0}^{n-1}(-q^n X)^\ell & \textit{if } t = 0. \end{cases}$$

Proof. We treat the case $t=0$ first. In this case, $\epsilon_1 = \epsilon$. Before we give the details of the proof, we summarize the main idea. Since $\{h_\epsilon(n,s,X), 0 \le s \le n-1\}$ forms a basis of the space of polynomials with degree $\le n-1$, there exists unique tuples $(a_{n,j})$ and $(b_{n,j}) \in \mathbb{Q}^n$ such that

$$F_\epsilon(n,n,X) = \sum_{j=0}^{n-1} a_{n,j} h_\epsilon(n,j,X), \quad \text{and} \quad \sum_{r=0}^{n-1}(-1)^r q^{\frac{(n-r)(n-r-1)}{2}} \cdot g_\epsilon(n,r,X) = \sum_{j=0}^{n-1} b_{n,j} h_\epsilon(n,j,X). \tag{5.86}$$

We need to show $a_{n,j} = b_{n,j}$ for all n and j. We first show that $a_{n,j}$ satisfy the recursion relations (5.90), which gives a description of $a_{n+1,j}$ in terms of $a_{n,j}$ and $a_{n,j-1}$. We can directly check $a_{1,0} = b_{1,0} = 1$. Then by an induction on n, it suffices to show $b_{n,j}$ also satisfies (5.90).

Now we derive (5.90). It is easy to check that

$$F_\epsilon(n+1,n+1,X) = q^{\lfloor \frac{n}{2} \rfloor}(q^{n+1}X)F_\epsilon(n,n,qX) + (-1)^n\alpha(n+1). \tag{5.87}$$

Plugging (5.86) into the above formula and applying Lemma 5.5.14, we obtain

$$\sum_{j=0}^{n} a_{n+1,j} h_\epsilon(n+1,j,X) = q^{\lfloor \frac{n}{2} \rfloor}(q^{n+1}X)\sum_{j=0}^{n-1} a_{n,j} h_\epsilon(n+1,j+1,X) + (-1)^n\alpha(n+1)h_\epsilon(n,n-1,X). \tag{5.88}$$

Let

$$\gamma(n,j) = \lfloor \frac{n}{2} \rfloor + n + 1 - \lfloor \frac{n+j+2}{2} \rfloor = \begin{cases} n - \frac{j}{2} & \text{if } j \text{ is even,} \\ n - \lfloor \frac{j-(-1)^n}{2} \rfloor & \text{if } j \text{ is odd.} \end{cases} \tag{5.89}$$

Then Lemma 5.5.14 and (5.88) imply

$$\sum_{j=0}^{n} a_{n+1,j}h_\epsilon(n+1,j,X) = \sum_{j=0}^{n-1} a_{n,j}q^{\gamma(n,j)}h_\epsilon(n+1,j,X) + \sum_{j=1}^{n}(-1)^{n+j}a_{n,j-1}q^{\gamma(n,j-1)}h_\epsilon(n+1,j,X)$$
$$+ (-1)^n\alpha(n+1)h_\epsilon(n+1,n,X).$$

That is

$$\begin{cases} a_{n+1,0} = q^n a_{n,0}, \\ a_{n+1,j} = q^{\gamma(n,j)}a_{n,j} + (-1)^{n+j}q^{\gamma(n,j-1)}a_{n,j-1}, \quad 0 < j < n, \\ a_{n+1,n} = q^{\gamma(n,n-1)}a_{n,n-1} + (-1)^n\alpha(n+1). \end{cases} \tag{5.90}$$

Now we compute $b_{n,j}$. A direct computation shows that $b_{n,0} = q^{\frac{n(n-1)}{2}}$. In the following, we compute b_j for $j \neq 0$.

For $r = 0$, we have

$$g_\epsilon(n,0,X) = f_1(n,0,X) = \begin{cases} h_\epsilon(n,0,X) & \text{if } n \text{ is odd,} \\ h_\epsilon(n,0,X) + (1-\epsilon)h_\epsilon(n,1,X) & \text{if } n \text{ is even.} \end{cases}$$

Now we assume $r \neq 0$. Recall that by Lemma 5.5.13, we have

$$g_\epsilon(n,r,X) = \sum_{\epsilon_3\in\{\pm 1\}} m(U_r^{\epsilon_3}, U_n^\epsilon)g_{\epsilon_3}(r,r,X).$$

Notice that when $n-r$ is odd, $h_{\epsilon_1}(n,r,X)$ is independent of ϵ_1. Then a direct calculation using Lemma 5.5.15 and the formula for $m(U_r^{\epsilon_1}, U_n^\epsilon)$ gives

$$g_\epsilon(n,r,X) = \alpha(r)h_\epsilon(n,r,X)m(U_r^\epsilon, U_n^\epsilon)\begin{cases} \frac{2q^{-\frac{r}{2}}}{1+\epsilon q^{-\frac{r}{2}}} & \text{if } n \equiv r-1 \equiv 1 \pmod 2, \\ -2\epsilon & \text{if } n \equiv r-1 \equiv 0 \pmod 2. \end{cases} \tag{5.91}$$

When $n-r$ is even, we have

$$h_{\epsilon_3}(n,r,X) = (q^{\frac{n+r}{2}}X - \epsilon\epsilon_3)h_\epsilon(n,r+1,X). \tag{5.92}$$

So a direct calculation gives

$$g_\epsilon(n,r,X) = \alpha(r)h_\epsilon(n,r{+}1,X)m(U_r^\epsilon,U_n^\epsilon)\left(q^{\frac{n+r}{2}}X - 1 + (-1)^n(q^{\frac{n+r}{2}}X+1)\frac{m(U_r^{-\epsilon},U_n^\epsilon)}{m(U_r^\epsilon,U_n^\epsilon)}\right). \tag{5.93}$$

We have by Lemma 5.3.17

$$\frac{m(U_r^{-\epsilon},U_n^\epsilon)}{m(U_r^\epsilon,U_n^\epsilon)} = \frac{1-q^{-\frac{n-r}{2}}}{1+q^{-\frac{n-r}{2}}}\begin{cases} 1 & \text{if } n\equiv r\equiv 1 \pmod 2, \\ \frac{1-\epsilon q^{-\frac{r}{2}}}{1+\epsilon q^{-\frac{r}{2}}} & \text{if } n\equiv r\equiv 0 \pmod 2. \end{cases}$$

The equation (5.92) gives

$$Xh_\epsilon(n,r+1,X) = q^{-\frac{n+r}{2}}(h_\epsilon(n,r,X)+h_\epsilon(n,r+1,X)).$$

So when $n\equiv r\equiv 1 \pmod 2$, we have

$$g_\epsilon(n,r,X) = 2\alpha(r)m(U_r^\epsilon,U_n^\epsilon)\left(\frac{q^{-\frac{n-r}{2}}}{1+q^{-\frac{n-r}{2}}}h_\epsilon(n,r,X) + \frac{q^{-\frac{n-r}{2}}-1}{1+q^{-\frac{n-r}{2}}}h_\epsilon(n,r+1,X)\right),$$

and when $n\equiv r\equiv 0 \pmod 2$ and $r\neq 0$, we have

$$\begin{aligned} &g_\epsilon(n,r,X) \\ &= \frac{2\alpha(r)m(U_r^\epsilon,U_n^\epsilon)}{(1+q^{-\frac{n-r}{2}})(1+\epsilon q^{-\frac{r}{2}})}\left((1+\epsilon q^{-\frac{n}{2}})h_\epsilon(n,r,X) + (1-q^{-\frac{n-r}{2}})(1-\epsilon q^{-\frac{r}{2}})h_\epsilon(n,r+1,X)\right). \end{aligned}$$

In summary, we have the numbers $b_{n,j}$ for $j\neq 0$ are given by the following.

If n and j are odd, then

$$b_{n,j} = 2(-1)^j q^{\frac{(n-j)(n-j-1)}{2}}\alpha(j)\frac{m(U_j^\epsilon,U_n^\epsilon)}{1+q^{\frac{n-j}{2}}}.$$

If n is odd and j is even, then

$$b_{n,j} = 2(-1)^j q^{\frac{(n-j)(n-j-1)}{2}}\left(\frac{\alpha(j)q^{-\frac{j}{2}}}{1+\epsilon q^{-\frac{j}{2}}}m(U_j^\epsilon,U_n^\epsilon) - \frac{q^{n-j}\alpha(j-1)(q^{-\frac{n-j+1}{2}}-1)}{1+q^{-\frac{n-j+1}{2}}}m(U_{j-1}^\epsilon,U_n^\epsilon)\right).$$

If n and j are even, then

$$b_{n,j} = \frac{2(-1)^j q^{\frac{(n-j)(n-j-1)}{2}} \alpha(j)(1+\epsilon q^{-\frac{n}{2}})}{(1+q^{-\frac{n-j}{2}})(1+\epsilon q^{-\frac{j}{2}})} m(U_j^\epsilon, U_n^\epsilon).$$

If n is even, then for $j=1$ we have

$$b_{n,1} = q^{\frac{n(n-1)}{2}}(1-\epsilon+2\epsilon q^{-(n-1)}) m(U_1^1, U_n^\epsilon),$$

and for odd $j>1$, $b_{n,j}$ is equal to

$$2(-1)^{j+1} q^{\frac{(n-j)(n-j-1)}{2}} \left(\epsilon\alpha(j) m(U_j^\epsilon, U_n^\epsilon) + \frac{q^{n-j}\alpha(j-1)(1-q^{-\frac{n-j+1}{2}})(1-\epsilon q^{-\frac{j-1}{2}})}{(1+q^{-\frac{n-j+1}{2}})(1+\epsilon q^{-\frac{j-1}{2}})} m(U_{j-1}^\epsilon, U_n^\epsilon) \right).$$

Using the explicit formulas, direct calculation shows that $b_{n,j}$ satisfies (5.90).

From now, we assume $t \neq 0$ and let $m = n-t$. The proof is essentially the same as the proof of Lemma 5.5.15 and we only prove the case that n is odd and m is even in detail. According to Lemma 5.5.13, we have

$$\begin{aligned}
&\sum_{r=0}^{\mathrm{m}(n,t)} (-1)^r q^{\frac{(n-r)(n-r-1)}{2}} q^{rt} \cdot g_{\epsilon_1}(m,r,X) \\
&= \sum_{r=0}^{\mathrm{m}(n,t)} (-1)^r q^{\frac{(n-r)(n-r-1)}{2}} q^{rt} \cdot \sum_{\epsilon_3 \in \{\pm 1\}} m(U_r^{\epsilon_3}, U_m^{\epsilon_1}) g_{\epsilon_3}(r,r,X) \\
&= q^{\frac{(n-m)(n+m-1)}{2}} \sum_{r=0}^{\mathrm{m}(n,t)} (-1)^r q^{\frac{(m-r)(m-r-1)}{2}} \cdot \sum_{\epsilon_3 \in \{\pm 1\}} m(U_r^{\epsilon_3}, U_m^{\epsilon_1}) g_{\epsilon_3}(r,r,X).
\end{aligned}$$

Assume that n is odd and m is even. Factoring out $h_{\epsilon_1}(n,m,X)$, replacing X by $q^{\frac{n-1}{2}}X$, and apply Lemma 5.5.15, we have that (5.85) is equivalent to

$$q^{\frac{(n-m)(n+m-1)}{2}} \sum_{r=0}^{m} (-1)^r q^{\frac{(m-r)(m-r-1)}{2}} \cdot \sum_{\epsilon_3 \in \{\pm 1\}} m(U_r^{\epsilon_3}, U_m^{\epsilon_1}) \alpha(r) g'_{\epsilon_3}(m,r,X) = F'(n,m,X), \tag{5.94}$$

where

$$g'_{\epsilon_3}(m,r,X) = \begin{cases} \epsilon\epsilon_3(q^{\frac{r+1}{2}}X - \epsilon\epsilon_3)\cdot\prod_{\ell=\frac{r+3}{2}}^{\frac{m}{2}}(q^{2\ell}X^2-1) & \text{if } r \text{ is odd,} \\ \epsilon_3\prod_{\ell=\frac{r+2}{2}}^{\frac{m}{2}}(q^{2\ell}X^2-1) & \text{if } r \text{ is even,} \end{cases}$$

and

$$F'(n,m,X) = q^{\frac{n(n-1)}{2}}\prod_{\ell=1}^{\frac{m}{2}}(q^{2\ell}X^2).$$

Since $g'_\epsilon(r,m,X)$ forms a basis of the space of polynomials with degree $\leq m$, there exists unique tuples (a_j) and (b_j) such that

$$\text{LHS of (5.94)} = \sum_{j=0}^{m} a_j g'_\epsilon(m,j,X), \quad \text{RHS of (5.94)} = \sum_{j=0}^{m} b_j g'_\epsilon(m,j,X).$$

It suffices to show that $a_0 = b_0$ and both (a_j) and (b_j) satisfy the following recursive relation for $0 < r \leq m$:

$$a_r = \begin{cases} 0 & \text{if } r \text{ is odd,} \\ \frac{q^{m-r+2}-1}{q^m-q^{m-r}}a_{r-2} & \text{if } r \text{ is even.} \end{cases} \tag{5.95}$$

We derive the recursive relation for a_j first. Notice that

$$g'_\epsilon(m,r,qX) = (q^{m+2}X^2-1)g'_\epsilon(m,r+2,X).$$

Moreover,

$$Xg'_\epsilon(m,r,X) = \begin{cases} q^{-\frac{r+1}{2}}(\epsilon g'_\epsilon(m,r-1,X) - g'_\epsilon(m,r,X)) & \text{if } r \text{ is odd,} \\ q^{-\frac{r}{2}}(\epsilon g'_\epsilon(m,r-1,X) + g'_\epsilon(m,r,X)) & \text{if } r \text{ is even.} \end{cases}$$

$$X^2g'_\epsilon(m,r,X) = \begin{cases} q^{-(r+1)}(qg'_\epsilon(m,r-2,X) + \epsilon(q-1)g'_\epsilon(m,r-1,X) + g'_\epsilon(m,r,X)) & \text{if } r \text{ is odd,} \\ q^{-r}(g'_\epsilon(m,r-2,X) + g'_\epsilon(m,r,X)) & \text{if } r \text{ is even.} \end{cases}$$

Hence if m is even, then

$$g'_\epsilon(m,r,qX)$$
$$= \begin{cases} q^{m-r}g'_\epsilon(m,r,X)+\epsilon(q-1)q^{m-r-1}g'_\epsilon(m,r+1,X)+(q^{m-r-1}-1)g'_\epsilon(m,r+2,X) & \text{if } r \text{ is odd,} \\ q^{m-r}g'_\epsilon(m,r,X)+(q^{m-r}-1)g'_\epsilon(m,r+2,X) & \text{if } r \text{ is even.} \end{cases}$$

Hence for $1 < r \leq m+1$

$$(q^m - q^{m-r})a_r = \begin{cases} (q^{m-r+1}-1)a_{r-2} & \text{if } r \text{ is odd,} \\ \epsilon(q-1)q^{m-r}a_{r-1}+(q^{m-r+2}-1)a_{r-2} & \text{if } r \text{ is even,} \end{cases} \tag{5.96}$$

with $a_{m+1}=0$, which implies (5.95).

Now we compute b_j. First, when $r=0$,

$$\sum_{\epsilon_3\in\{\pm1\}} m(U_r^{\epsilon_3}, U_m^{\epsilon_1})g'_{\epsilon_3}(m,0,X) = \epsilon g'_\epsilon(m,0,X).$$

For $r \neq 0$,

$$\sum_{\epsilon_3\in\{\pm1\}} m(U_r^{\epsilon_3}, U_m^{\epsilon_1})g'_{\epsilon_3}(m,r,X) = \begin{cases} -2\epsilon m(U_r^{\epsilon}, U_m^{\epsilon_1})g'_\epsilon(m,r+1,X) & \text{if } r \text{ is odd,} \\ \epsilon(m(U_r^1, U_m^{\epsilon_1}) - m(U_r^{-1}, U_m^{\epsilon_1}))g'_\epsilon(m,r,X) & \text{if } r \text{ is even.} \end{cases}$$

Then

$$b_r = \begin{cases} 0 & \text{if } r \text{ is odd,} \\ a'_{r-1}+a'_r & \text{if } r \text{ is even.} \end{cases}$$

where

$$a'_r = q^{\frac{(n-m)(n+m-1)}{2}} \begin{cases} -2(-1)^r q^{\frac{(m-r)(m-r-1)}{2}}\alpha(r)\epsilon m(U_{r-1}^{\epsilon}, U_m^{\epsilon_1}) & \text{if } r \text{ is odd,} \\ (-1)^r\alpha(r)q^{\frac{(m-r)(m-r-1)}{2}}\epsilon(m(U_r^1, U_m^{\epsilon_1}) - m(U_r^{-1}, U_m^{\epsilon_1})) & \text{if } r \text{ is even.} \end{cases}$$

Finally, a direct calculation shows that $a_0 = b_0 = \epsilon q^{\frac{n(n-1)}{2}}$ and b_r satisfies (5.96). $\square$

5.5.6 Proof of Theorem 5.5.1

Now we are ready to prove Theorem 5.5.1.

Recall that

$$\mathrm{Pden}'(L) = \frac{\mathrm{Pden}'(I_n, L)}{\mathrm{Pden}(I_n, I_n)},$$

and

$$\mathrm{Pden}(I_n, I_n) = 2\alpha(n) I_\epsilon(n),$$

where $I_\epsilon(n)$ is defined in Lemma 5.5.7.

We first assume that $L = H^j \oplus L_1$ with $j > 0$ where $L_1 = I_{n_1-t} \oplus L_2$ is an integral lattice of rank n_1 and type $t > 0$ (the other non-integral cases were taken care of in the summary of the proof at the beginning of this section). By Lemma 5.3.12, we have

$$\mathrm{Pden}'(I_n, L) = 2\Big(\prod_{\ell=1}^{j-1}(1-q^{2\ell})\Big)\mathrm{Pden}(I_n, L_1, q^{2j}).$$

It suffices to show $\mathrm{Pden}(I_n, L_1, q^{2j}) = 0$. By Theorem 5.4.1 and Lemma 5.5.11, we have

$$\begin{aligned}\mathrm{Pden}(I_n, L, q^{2j}) &= \sum_{i=0}^{n_1} \mathrm{Pden}^{n_1-i}(I_n^{-\epsilon}, L, q^{2j}) \\ &= I_\epsilon(n, n_1) \sum_{r=0}^{n_1-t} \sum_{i=0}^{n_1} \binom{n_1-r}{i-r}_q (-1)^{n_1+i-r} q^{\frac{(i-r)(i-r-1)}{2}} q^{rt} \cdot g_{\epsilon_1}(n_1, n_1-t, r, q^{j-i}).\end{aligned} \tag{5.97}$$

Notice that the assumption $t > 0$ implies that $r \le n_1 - t \le n_1 - 1$. Hence $g_{\epsilon_1}(n_1, n-t, r, X)$ is a polynomial of degree $n_1 - r - 1$ by Lemma 5.5.15. Then we may apply Corollary 5.3.24

to conclude

$$\sum_{i=0}^{n_1} \binom{n_1 - r}{i - r}_q (-1)^{n_1+i-r} q^{\frac{(i-r)(i-r-1)}{2}} q^{rt} \cdot g_{\epsilon_1}(n_1, n_1 - t, r, q^{\frac{n-n_1}{2}-i}) = 0.$$

Hence

$$\mathrm{Pden}'(L) = \frac{2\Big(\prod_{\ell=1}^{j-1}(1 - q^{2\ell})\Big)}{\mathrm{Pden}(I_n, I_n)} \mathrm{Pden}(I_n, L, q^{n-n_1}) = 0.$$

Next, we assume $L = I_{n-t} \obot L_2$ is integral of rank n and type t (Cases (2) and (3) or equivalently Case (b) in the summary of the proof at the beginning of this section). Similarly, by Theorem 5.4.1 and Lemma 5.5.11, we have

$$\begin{aligned} \mathrm{Pden}'(I_n, L) &= \sum_{i=0}^{n-1} (\mathrm{Pden}^{n-i})'(I_n, L) \\ &= 2I_\epsilon(n) \sum_{r=0}^{\mathrm{m}(n,t)} \sum_{i=0}^{n-1} \binom{n - r}{i - r}_q (-1)^{n-1+i+r} q^{\frac{(i-r)(i-r-1)}{2}} q^{rt} \cdot g_{\epsilon_1}(n, n - t, r, q^{-i}). \end{aligned} \tag{5.98}$$

Here, recall that $\mathrm{m}(n,t) := \min\{n - t, n - 1\}$. Applying Corollary 5.3.24 as before, we have

$$\sum_{i=0}^{n} \binom{n - r}{i - r}_q (-1)^r (-1)^{n-1+i} q^{\frac{(i-r)(i-r-1)}{2}} q^{rt} \cdot g_{\epsilon_1}(n, n - t, r, q^{-i}) = 0.$$

Hence,

$$\mathrm{Pden}'(I_n, L) = 2I_\epsilon(n) \sum_{r=0}^{\mathrm{m}(n,t)} (-1)^r q^{\frac{(n-r)(n-r-1)}{2}} q^{rt} \cdot g_{\epsilon_1}(n, n - t, r, q^{-n}).$$

By Lemma 5.5.16, if $t \neq 0$, then

$$\sum_{r=0}^{\mathrm{m}(n,t)} (-1)^r q^{\frac{(n-r)(n-r-1)}{2}} q^{rt} \cdot g_{\epsilon_1}(n, n-t, r, q^{-n}) = F_{\epsilon_1}(n, n-t, q^{-n})$$

$$= \alpha(n) \cdot \begin{cases} \prod_{\ell=1}^{\frac{t-1}{2}} (1-q^{2\ell}) & \text{if } t \text{ is odd,} \\ (1-\epsilon\epsilon_1 q^{\frac{t}{2}}) \prod_{\ell=1}^{\frac{t}{2}-1} (1-q^{2\ell}) & \text{if } t \text{ is even.} \end{cases}$$

Notice that if t is even, then $\chi(I_{n-t}^{\epsilon_1})\chi(L_2) = \epsilon$. Hence $\epsilon\epsilon_1 = \chi(L_2)$.

If $t = 0$, then by Lemma 5.5.16,

$$\sum_{r=0}^{\mathrm{m}(n,t)} (-1)^r q^{\frac{(n-r)(n-r-1)}{2}} q^{rt} \cdot g_{\epsilon_1}(n, n, r, q^{-n}) = F_{\epsilon_1}(n, n, q^{-n}) = \alpha(n) \cdot \begin{cases} 1 & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

This proves the theorem.

5.6 Fourier transform: the analytic side

In this section, we study the partial Fourier transform of the vertical part of the analytic side following Section 8 of [LZ22b]. The main result is Theorem 5.6.2.

Definition 5.6.1. *For a non-degenerate lattice $L^\flat \subset \mathbb{V}$ of rank $n-1$, and $x \in \mathbb{V} \setminus L_F^\flat$, we define*

$$\partial\mathrm{Den}_{L^\flat,\mathscr{V}}(x) = \sum_{\substack{L^\flat \subset L' \subset L'^\sharp \\ L'^\flat \notin \mathrm{Hor}(L^\flat)}} \partial\mathrm{Pden}(L') 1_{L'}(x),$$

where $L'^\flat = L' \cap L_F^\flat$.

Theorem 5.6.2. *Let $L^\flat \subset \mathbb{V}$ be a non-degenerate lattice of rank $n-1$, and let $\mathbb{W} = (L_F^\flat)^\perp$ be the perpendicular space of $L_F^\flat$ in $\mathbb{V}$. Recall the partial Fourier transform*

$$\partial\mathrm{Den}_{L^\flat,\mathscr{V}}^\perp(x) := \int_{L_F^\flat} \partial\mathrm{Den}_{L^\flat,\mathscr{V}}(x+y)\, dy, \quad x \in \mathbb{W} \setminus \{0\}.$$

Then $\partial\mathrm{Den}^{\perp}_{L^\flat,\mathscr{V}}(x)$ *is constant on* $\mathbb{W}^{\geq 0}\setminus\{0\}$ *and is zero for* $x\in\mathbb{W}^{<0}$.

Proof. It suffices to show that if $\mathrm{val}(x)>0$, then

$$\partial\mathrm{Den}^{\perp}_{L^\flat,\mathscr{V}}(x)-\partial\mathrm{Den}^{\perp}_{L^\flat,\mathscr{V}}(\pi^{-1}x)=0.$$

By definition, we have

$$\partial\mathrm{Den}^{\perp}_{L^\flat,\mathscr{V}}(x)=\int_{L^\flat_F}\sum_{\substack{L^\flat\subset L'\subset L'^\sharp\\ L'^\flat\notin\mathrm{Hor}(L^\flat)}}\partial\mathrm{Pden}(L')1_{L'}(x+y)\,dy,$$

where L' runs over lattices of rank n in $L^\flat_F+\langle x\rangle$.

Recall that $\mathrm{Pr}_{L^\flat_F}$ denotes the projection to $L^\flat_F$. We rewrite the summation based on $L'\cap L^\flat_F$ and $\mathrm{Pr}_{L^\flat_F}(L')$. For lattices $L'^\flat\subset\widetilde{L}'^\flat$ in $L^\flat_F$ of rank $n-1$, let

$$\mathrm{Lat}(L'^\flat,\widetilde{L}'^\flat)\coloneqq\{L'\subset\mathbb{V}\mid L'\cap L^\flat_F=L'^\flat,\quad \mathrm{Pr}_{L^\flat_F}(L')=\widetilde{L}'^\flat\}.$$

Then by Lemmas 7.2.1 and 7.2.2 of [LZ22a], we have

$$\partial\mathrm{Den}^{\perp}_{L^\flat,\mathscr{V}}(x)=\sum_{\substack{L^\flat\subset L'^\flat\\ L'^\flat\notin\mathrm{Hor}(L^\flat)}}\sum_{\substack{L'^\flat\subset\widetilde{L}'^\flat\\ \widetilde{L}'^\flat/L'^\flat\text{ cyclic}}}\sum_{L'\in\mathrm{Lat}(L'^\flat,\widetilde{L}'^\flat)}\partial\mathrm{Pden}(L')\int_{L^\flat_F}1_{L'}(x+y)\,dy.$$

Here we can switch the order of the sum and integral because there are only finitely many nonzero terms in the sum for a fixed x. Since $\tilde{L}'^\flat/L'^\flat$ is cyclic, it has a generator $u^\flat\in L^\flat_F$. Moreover, for $L'\in\mathrm{Lat}(L'^\flat,\widetilde{L}'^\flat)$, we can write $L'=L'^\flat+\langle u\rangle$ with $u=u^\flat+u^\perp\in\mathbb{V}$ where $0\neq u^\perp\in\mathbb{W}$. Moreover, write $x=\alpha u^\perp$ with $\alpha\in F^\times$, then

$$x+y=\alpha u+(y-\alpha u^\flat)\in L'$$

if and only if $\alpha \in O_F$ and $y - \alpha u^\flat \in L'^\flat$. As a result, we have

$$\int_{L_F^\flat} 1_{L'}(x+y) - 1_{L'}(\pi^{-1}x+y)dy = \begin{cases} \mathrm{vol}(L'^\flat), & \text{if } \langle x\rangle = \langle u^\perp\rangle, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, we have

$$\partial\mathrm{Den}^\perp_{L^\flat,\mathscr{V}}(x) - \partial\mathrm{Den}^\perp_{L^\flat,\mathscr{V}}(\pi^{-1}x) = \sum_{\substack{L^\flat \subset L'^\flat \\ L'^\flat \notin \mathrm{Hor}(L^\flat)}} \mathrm{vol}(L'^\flat) D(L'^\flat)(x), \tag{5.99}$$

where

$$D(L'^\flat)(x) = \sum_{\substack{L'^\flat \subset \widetilde{L}'^\flat \\ \widetilde{L}'^\flat/L'^\flat \text{ cyclic}}} \sum_{\substack{u^\perp \in \langle x\rangle \text{ generator} \\ L' = L'^\flat + \langle u^\flat + u^\perp\rangle}} \partial\mathrm{Pden}(L') = \sum_{\substack{u^\flat \in (L'^\flat)^\sharp/L'^\flat \\ \mathrm{val}(u^\flat)\geq 0}} \partial\mathrm{Pden}(L'^\flat + \langle u^\flat + x\rangle). \tag{5.100}$$

Here the last step uses the fact that $L' = L'^\flat + \langle u^\flat + x\rangle$ is integral if and only if $u^\flat \in (L'^\flat)^\sharp/L'^\flat$ and $\mathrm{val}(u^\flat) \geq 0$ (since $\mathrm{val}(x) > 0$).

It suffices to show $D(L'^\flat)(x) = 0$ for any $L'^\flat$ such that $L^\flat \subset L'^\flat$ and $L'^\flat \notin \mathrm{Hor}(L^\flat)$. To show this, we write $L'^\flat = I_{n_1}^{\epsilon_1} \oplus L_2$ where L_2 is of full type (of rank $n-1-n_1$). Let u_2 be the projection of $u^\flat$ to L_2. Then

$$(L'^\flat)^\sharp/L'^\flat = (I_{n_1}^{\epsilon_1} \oplus (L_2)^\sharp)/L'^\flat \cong L_2^\sharp/L_2 \quad \text{and} \quad L'^\flat + \langle u^\flat + x\rangle = L'^\flat + \langle u_2 + x\rangle.$$

We consider a partition of

$$(L_2^\sharp)^\circ/L_2 = S^+(L_2) \sqcup S^0(L_2) \sqcup S^-(L_2)$$

with

$$S^+(L_2) = (\pi L_2^\sharp)^{\circ\circ}/L_2, \quad S^0(L_2) = ((\pi L_2^\sharp)^\circ - (\pi L_2^\sharp)^{\circ\circ})/L_2, \quad S^-(L_2) = ((L_2^\sharp)^\circ - (\pi L_2^\sharp)^\circ)/L_2.$$

Here, for a lattice L,

$$L^\circ := \{x \in L \mid \mathrm{val}(x) \geq 0\} \text{ and } L^{\circ\circ} := \{x \in L \mid \mathrm{val}(x) > 0\}.$$

In general, for a full type lattice L_2, we also define

$$\mu^+(L) := |(\pi L^\sharp)^{\circ\circ}/L|, \quad \mu^0(L) := |((\pi L^\sharp)^\circ - (\pi L^\sharp)^{\circ\circ})/L|, \quad \mu^-(L) := |((L^\sharp)^\circ - (\pi L^\sharp)^\circ)/L|. \tag{5.101}$$

For $\nu \in \{\pm 1\}$, let

$$\mu^{0,\nu}(L) := |\{u \in (\pi L^\sharp)^\circ - (\pi L^\sharp)^{\circ\circ} : \chi((u,u)) = \nu\}/L|. \tag{5.102}$$

Since $L' = L'^\flat + \langle u_2 + x\rangle$ with u_2 integral and $\mathrm{val}(x) > 0$, it is not hard to check that

$$t(L') = \begin{cases} t(L'^\flat) + 1 & \text{if } u_2 \in S^+(L_2), \\ t(L'^\flat) & \text{if } u_2 \in S^0(L_2), \\ t(L'^\flat) - 1 & \text{if } u_2 \in S^-(L_2). \end{cases}$$

Set $t = t(L'^\flat)$. There are two cases.

When t is odd, we can write

$$L' = L'^\flat + \langle u_2 + x\rangle = \begin{cases} I_{n_1}^{\epsilon_1} \oplus L_2' & \text{if } u \in S_2^+(L_2), \\ I_{n_1}^{\epsilon_1} \oplus I_2^1 \oplus L_2' & \text{if } u \in S_2^-(L_2). \end{cases}$$

In both cases, a simple calculation gives

$$\chi(L_2') = \epsilon\epsilon_1.$$

For $t > 1$, by Theorem 5.5.1,

$$\partial\mathrm{Pden}(L') = (1 - \epsilon\epsilon_1 q^{\frac{t-1}{2}}) \prod_{\ell=1}^{\frac{t-1}{2}-1} (1 - q^{2\ell}) \cdot \begin{cases} (1 - \epsilon\epsilon_1 q^{\frac{t+1}{2}})(1 + \epsilon\epsilon_1 q^{\frac{t-1}{2}}) & \text{if } u_2 \in S^+(L_2), \\ 1 + \epsilon\epsilon_1 q^{\frac{t-1}{2}} & \text{if } u_2 \in S^0(L_2), \\ 1 & \text{if } u_2 \in S^-(L_2). \end{cases}$$

For $t = 1$, $S^-(L_2)$ is empty and $L'^\flat \notin \mathrm{Hor}(L^\flat)$ implies that $L' = L_{n_1}^{\epsilon_1} \obot L_2'$ with $\chi(L_2') = 1$, i.e. $\epsilon_1 = \epsilon$. In this case, by Theorem 5.5.1,

$$\partial\mathrm{Pden}(L') = \begin{cases} 1 - q & \text{if } u \in S^+(L_2), \\ 1 & \text{if } u \in S^0(L_2). \end{cases}$$

Hence by (5.100), we have $D(L'^\flat) = 0$ if

$$(1 - \epsilon\epsilon_1 q^{\frac{t+1}{2}})(1 + \epsilon\epsilon_1 q^{\frac{t-1}{2}})\mu^+(L_2) + (1 + \epsilon\epsilon_1 q^{\frac{t-1}{2}})\mu^0(L_2) + \mu^-(L_2) = 0. \tag{5.103}$$

When $t = t(L'^\flat)$ is even, $L'^\flat \notin \mathrm{Hor}(L^\flat)$ implies that $t > 0$. Moreover, if $u_2 \in S^0(L_2)$, we have a decomposition (since $u_2 \in L_2$ is perpendicular to $I_{n_1}^{\epsilon_1}$)

$$L' = L'^\flat + \langle u_2 + x \rangle = I_{n_1}^{\epsilon_1} \obot \langle u_2 + x \rangle \obot L_2'$$

for some full type lattice L_2' of rank t. Then a direct calculation gives

$$\chi(L_2') = (-1)^{n_1} \epsilon_1 \epsilon \chi(u_2).$$

So we have by Theorem 5.5.1,

$$\partial\mathrm{Pden}(L') = \prod_{\ell=1}^{\frac{t}{2}-1}(1-q^{2\ell}) \cdot \begin{cases} (1-q^t) & \text{if } u_2 \in S^+(L_2), \\ 1-(-1)^{n_1}\epsilon_1\epsilon\chi(u_2)q^{\frac{t}{2}} & \text{if } u_2 \in S^0(L_2), \\ 1 & \text{if } u_2 \in S^-(L_2). \end{cases}$$

Hence by (5.100), we have $D(L'^{\flat}) = 0$ if

$$(1-q^t)\mu^+(L_2) + (1-(-1)^{n_1}\epsilon_1\epsilon q^{\frac{t}{2}})\mu^{0,1}(L_2) + (1+(-1)^{n_1}\epsilon_1\epsilon q^{\frac{t}{2}})\mu^{0,-1}(L_2) + \mu^-(L_2) = 0. \tag{5.104}$$

Now (5.103) follows from Proposition 5.6.7 and (5.104) follows from Proposition 5.6.9. Hence we have $D(L'^{\flat}) = 0$ for $L'^{\flat}$ such that $L^{\flat} \subset L'^{\flat}$ and $L'^{\flat} \notin \mathrm{Hor}(L^{\flat})$. Now the theorem follows from (5.99). $\square$

To complete the proof of Theorem 5.6.2, we are left to state and prove Propositions 5.6.7 and 5.6.9.

Definition 5.6.3. *Let L and L' be lattices of full type such that $L \subset L' \subset \pi^{-1}L$. For $? \in \{+, 0, -, \{0,+1\}, \{0,-1\}\}$, define*

$$\mu^?(L, L') := \mu^?(L) - [L' : L]\mu^?(L'),$$

where $\mu^?(L)$ is defined in (5.101) and (5.102).

Lemma 5.6.4. *Let L be a full type lattice of rank t. Then*

$$\mu^+(L) + \mu^0(L) + \mu^-(L) = q^t \cdot \mu^+(L).$$

Let L and L' be full type lattices of rank t such that $L \subset L' \subset \pi^{-1}L$. Then

$$\mu^+(L, L') + \mu^0(L, L') + \mu^-(L, L') = q^t \cdot \mu^+(L, L').$$

Proof. It suffices to show that the following map

$$(L^\sharp)^\circ/L \to (\pi L^\sharp)^{\circ\circ}/L, \quad x \mapsto \pi x$$

is surjective and every fiber of this map has size q^t. For $x \in (\pi L^\sharp)^{\circ\circ} = \pi(L^\sharp)^\circ$, the fiber at x is

$$\{\pi^{-1}(y+x) \in (L^\sharp)^\circ : y \in \overline{L}\}.$$

Since $x \in \pi(L^\sharp)^\circ$,

$$\pi^{-1}(y+x) \in (L^\sharp)^\circ \iff (y+x) \in \pi(L^\sharp)^\circ \iff y \in \pi(L^\sharp)^\circ.$$

Moreover, the assumption that L is a full type lattice implies that $L \subset \pi(L^\sharp)^\circ$. Hence

$$|\{\pi^{-1}(y+x) \in (L^\sharp)^\circ : y \in \overline{L}\}| = |\overline{L}| = q^t.$$

This proves the first statement. The second statement follows from the first and the definition of $\mu^?(L, L')$. ☐

Definition 5.6.5. *Let L be a full type lattice of rank t. We call L maximal of type t if $t(L') < t$ for any L' such that $L \subsetneq L' \subset L_F$.*

Lemma 5.6.6. *If L is non-maximal full type lattice of rank t, then there exists a L' such that $L \subset L' \subset \pi^{-1}L$ and*

$$\mu^+(L, L') + \mu^0(L, L') = q \cdot \mu^+(L, L').$$

Proof. We need to find a L' such that $L \subset L' \subset \pi^{-1}L$ and

$$|((\pi L^\sharp)^\circ - (\pi L'^\sharp)^\circ)/L| = q \cdot |((\pi L^\sharp)^{\circ\circ} - (\pi L'^\sharp)^{\circ\circ})/L|.$$

Let $(a_1, \cdots, a_t)$ be the fundamental invariants of L. We consider two cases seperately.

(i) If a_t is even and $a_t \geq 4$, then we may choose a normal basis $\{\ell_1, \cdots, \ell_t\}$ of L such that $\langle \ell_1, \cdots, \ell_{t-1}\rangle \perp \ell_t$. Write $(\ell_t, \ell_t) = u_t(-\pi_0)^{\frac{a_t}{2}}$. In this case, we choose $L' = \langle \ell_1, \cdots, \ell_{t-1}, \pi^{-1}\ell_t\rangle$, with fundamental invariants $(a_1, \cdots, a_{t-1}, a_t - 2)$. Then

$$\pi L^\sharp = \langle \pi^{-a_1+1}\ell_1, \cdots, \pi^{-a_t+1}\ell_t\rangle \text{ and } \pi L'^\sharp = \langle \pi^{-a_1+1}\ell_1, \cdots, \pi^{-a_{t-1}+1}\ell_{t-1}, \pi^{-a_t+2}\ell_t\rangle.$$

For a fixed $x_0 = \sum_{1\leq i<t} s_i\pi^{-a_i+1}\ell_i$ where $s_i \in O_F$, let

$$S^\circ_{x_0} := \{x \in (\pi L^\sharp)^\circ - (\pi L'^\sharp)^\circ : x = x_0 + s_t\pi^{-a_t+1}\ell_t, s_t \in O_F\}/L,$$
$$S^{\circ\circ}_{x_0} := \{x \in (\pi L^\sharp)^{\circ\circ} - (\pi L'^\sharp)^{\circ\circ} : x = x_0 + s_t\pi^{-a_t+1}\ell_t, s_t \in O_F\}/L.$$

It suffices to show $|S^\circ_{x_0}| = q \cdot |S^{\circ\circ}_{x_0}|$. Notice that $x = x_0 + s_t\pi^{-a_t+1}\ell_t \in S^\circ_{x_0}$ if and only if

$$s_t \in O_F^\times, \quad (x, x) = u_t(-\pi_0)^{-\frac{a_t}{2}+1}(u_t^{-1}(-\pi_0)^{\frac{a_t}{2}-1}(x_0, x_0) + s_t\bar{s}_t) \in O_{F_0},$$

and $x \in S^{\circ\circ}_{x_0}$ if and only if

$$s_t \in O_F^\times, \quad (x, x) = u_t(-\pi_0)^{-\frac{a_t}{2}+1}(u_t^{-1}(-\pi_0)^{\frac{a_t}{2}-1}(x_0, x_0) + s_t\bar{s}_t) \in (\pi_0).$$

Consider the π-adic expansions

$$s_t = \sum_{i\geq 0}^{\infty} b_i\pi^i, \quad -u_t^{-1}(-\pi_0)^{\frac{a_t}{2}-1}(x_0, x_0) = \sum_{i\geq 0}^{\infty} c_i\pi^i,$$

where $b_i, c_i \in O_{F_0}/(\pi_0)$. Then $x \in S_{x_0}^{\circ}$ if and only if $s_t \in O_F^{\times}$, and

$$\begin{aligned} c_0 &= b_0^2, \\ c_1 &= b_1 b_0 - b_0 b_1, \\ c_2 &= b_2 b_0 - b_1 b_1 + b_0 b_2, \\ &\cdots \\ c_{a_t-3} &= \sum_{i=0}^{a_t-3} (-1)^i b_{a_t-3-i} b_i. \end{aligned}$$

Similarly, $x \in S_{x_0}^{\circ\circ}$ if and only if $x \in S_{x_0}^{\circ}$ and

$$c_{a_t-2} - \sum_{i=1}^{a_t-3} (-1)^i b_{a_t-2-i} b_i = 2 b_{a_t-2} b_0.$$

Since $s_t \in O_F^{\times}$, $b_0 \neq 0$ and b_{a_t-2} is uniquely determined by the above equation. Hence $|S_{x_0}^{\circ}| = q \cdot |S_{x_0}^{\circ\circ}|$ as a result.

(ii) If a_t is odd (since L is non-maximal, $a_t > 1$ in this case) or $a_t = 2$, then we may choose a normal basis $\{\ell_1, \cdots, \ell_t\}$ of L such that the moment matrix of $\{\ell_{t-1}, \ell_t\}$ is H_{a_t}, where $H_{a_t} := \begin{pmatrix} 0 & \pi^{a_t} \\ (-\pi)^{a_t} & 0 \end{pmatrix}$. We may choose $L' = \langle \ell_1, \cdots, \ell_{t-2}, \pi^{-1}\ell_{t-1}, \ell_t \rangle$ with fundamental invariants $(a_1, \cdots, a_{t-2}, a_t - 1, a_t - 1)$. In this case,

$$\pi L^{\sharp} = \langle \pi^{-a_1+1}\ell_1, \cdots, \pi^{-a_t+1}\ell_t \rangle \text{ and } \pi L'^{\sharp} = \langle \pi^{-a_1+1}\ell_1, \cdots, \pi^{-a_t+1}\ell_{t-1}, \pi^{-a_t+2}\ell_t \rangle.$$

For a fixed $x_0 = \sum_{1 \le i < t-1} s_i \pi^{-a_i+1}\ell_i$ where $s_i \in O_F$, let

$$\begin{aligned} S_{x_0}^{\circ} &:= \{x \in (\pi L^{\sharp})^{\circ} - (\pi L'^{\sharp})^{\circ} : x = x_0 + s_{t-1}\pi^{-a_t+1}\ell_{t-1} + s_t \pi^{-a_t+1}\ell_t, \text{ where } s_{t-1}, s_t \in O_F\}/L, \\ S_{x_0}^{\circ\circ} &:= \{x \in (\pi L^{\sharp})^{\circ\circ} - (\pi L'^{\sharp})^{\circ\circ} : x = x_0 + s_{t-1}\pi^{-a_t+1}\ell_{t-1} + s_t \pi^{-a_t+1}\ell_t, \text{ where } s_{t-1}, s_t \in O_F\}/L. \end{aligned}$$

It suffices to show $|S_{x_0}^{\circ}| = q \cdot |S_{x_0}^{\circ\circ}|$. Notice that $x = x_0 + s_{t-1}\pi^{-a_t+1}\ell_{t-1} + s_t \pi^{-a_t+1}\ell_t \in S_{x_0}^{\circ}$

if and only if

$$s_t \in O_F^\times, \quad (x,x) = (x_0,x_0) + (s_{t-1}\bar{s}_t + (-1)^{a_t}\bar{s}_{t-1}s_t)(-1)^{-a_t+1}\pi^{-a_t+2} \in O_{F_0},$$

and $x \in S_{x_0}^{\circ\circ}$ if and only if

$$s_t \in O_F^\times, \quad (x,x) = (x_0,x_0) + (s_{t-1}\bar{s}_t + (-1)^{a_t}\bar{s}_{t-1}s_t)(-1)^{-a_t+1}\pi^{-a_t+2} \in (\pi_0).$$

Write

$$s_{t-1} = \sum_{i\geq 0}^{\infty} b_i\pi^i, \quad s_t = \sum_{i\geq 0}^{\infty} c_i\pi^i, \quad -(-1)^{a_t-1}\pi^{a_t-2}(x_0,x_0) = \sum_{i\geq 0}^{\infty} d_i\pi^i,$$

where $b_i, c_i, d_i \in O_{F_0}/(\pi_0)$. Then $x \in S_{x_0}^{\circ\circ}$ if and only if $x \in S_{x_0}^{\circ}$ and

$$d_{a_t-2} + S = -2b_{a_t-2}c_0.$$

where S is certain expression involving $b_0, \cdots, b_{a_t-3}$ and $c_1, \cdots, c_{a_t-2}$. Since $s_t \in O_F^\times$, $c_0 \neq 0$. Hence, for any given S, the number of choices of b_{a_t-2} is determined if $x \in S_{x_0}^{\circ\circ}$. As a result, $|S_{x_0}^{\circ}| = q \cdot |S_{x_0}^{\circ\circ}|$. □

Proposition 5.6.7. *Assume that $t \geq 1$ is odd and L is a full type lattice of rank t. Then for any $\chi \in \{\pm 1\}$, we have*

$$(1-\chi q^{\frac{t+1}{2}})(1+\chi q^{\frac{t-1}{2}})\mu^+(L) + (1+\chi q^{\frac{t-1}{2}})\mu^0(L) + \mu^-(L) = 0.$$

Proof. We prove this for maximal L first. We can choose a basis $\{\ell_1, \cdots, \ell_t\}$ of L with moment matrix $\mathrm{Diag}(H_1^{\frac{t-1}{2}}, u_t(-\pi_0))$. Set $L_1 = \langle \ell_1, \cdots, \ell_{t-1}\rangle$ and $L_2 = \langle \ell_t \rangle$. Then we can directly compute that

$$(\pi L^\sharp)^{\circ\circ} = L, \quad (\pi L^\sharp)^{\circ} = L_1 \obot \pi^{-1}L_2, \quad (L^\sharp)^{\circ} = \pi^{-1}L_1 \obot \pi^{-1}L_2.$$

Hence

$$\mu^+(L) = |(\pi L^\sharp)^{\circ\circ}/L| = 1, \quad \mu^0(L) = |((\pi L^\sharp)^\circ - (\pi L^\sharp)^{\circ\circ})/L| = q - 1,$$

and

$$\mu^-(L) = |((L^\sharp)^\circ - (\pi L^\sharp)^\circ)/L| = q^t - q.$$

As a result,

$$\begin{aligned}
&(1 - \chi q^{\frac{t+1}{2}})(1 + \chi q^{\frac{t-1}{2}})\mu^+(L) + (1 + \chi q^{\frac{t-1}{2}})\mu^0(L) + \mu^-(L) \\
&= (1 - \chi q^{\frac{t+1}{2}})(1 + \chi q^{\frac{t-1}{2}}) + (1 + \chi q^{\frac{t-1}{2}})(q - 1) + q^t - q \\
&= 0.
\end{aligned}$$

Now we assume L is not maximal and the proposition holds for L' such that $L \subsetneq L'$ by an induction on $\mathrm{val}(L)$. With this assumption, it suffices to show

$$(1 - \chi q^{\frac{t+1}{2}})(1 + \chi q^{\frac{t-1}{2}})\mu^+(L, L') + (1 + \chi q^{\frac{t-1}{2}})\mu^0(L, L') + \mu^-(L, L') = 0,$$

which follows from a combination of Lemmas 5.6.4 and 5.6.6. □

Lemma 5.6.8. *If L is non-maximal full type lattice of rank t, then there exists a L' such that $L \subset L' \subset \pi^{-1}L$ and*

$$\mu^{0,+}(L, L') = \mu^{0,-}(L, L').$$

Proof. Let

$$S^\nu := \{x \in ((\pi L^\sharp)^\circ - (\pi L^\sharp)^{\circ\circ}) - ((\pi L'^\sharp)^\circ - (\pi L'^\sharp)^{\circ\circ}) : \chi(\langle x \rangle) = \nu\}/L.$$

We need to show $|S^{+1}| = |S^{-1}|$. Let $\{\ell_1, \cdots, \ell_t\}$ be a normal basis of L, and let $\{a_1, \cdots, a_t\}$

be the set of fundamental invariants of L. We consider two cases seperately.

(i) If a_t is even and $a_t \geq 4$, then we may choose a normal basis $\{\ell_1, \cdots, \ell_t\}$ of L such that $\langle \ell_1, \cdots, \ell_{t-1}\rangle \perp \ell_t$. In this case, we choose $L' = \langle \ell_1, \cdots, \ell_{t-1}, \pi^{-1}\ell_t\rangle$, with fundamental invariants $(a_1, \cdots, a_{t-1}, a_t - 2)$. Then

$$\pi L^\sharp = \langle \pi^{-a_1+1}\ell_1, \cdots, \pi^{-a_t+1}\ell_t\rangle \text{ and } \pi L'^\sharp = \langle \pi^{-a_1+1}\ell_1, \cdots, \pi^{-a_{t-1}+1}\ell_{t-1}, \pi^{-a_t+2}\ell_t\rangle.$$

For a fixed $x_0 = \sum_{1\leq i<t} s_i \pi^{-a_i+1}\ell_i$ where $s_i \in O_F$, we set

$$S_{x_0}^\nu := \{x \in ((\pi L^\sharp)^\circ - (\pi L^\sharp)^{\circ\circ}) - ((\pi L'^\sharp)^\circ - (\pi L'^\sharp)^{\circ\circ}) : x = x_0 + s_t\pi^{-a_t+1}\ell_t, s_t \in O_F, \chi(\langle x\rangle) = \nu\}/L.$$

We need to show $|S_{x_0}^{+1}| = |S_{x_0}^{-1}|$. Write $(\ell_t, \ell_t) = u_t(-\pi_0)^{\frac{a_t}{2}}$. Notice that $x = x_0 + s_t\pi^{-a_t+1}\ell_t \in S_{x_0}^\nu$ if and only if

$$s_t \in O_F^\times, \quad (x,x) \in O_{F_0}^\times, \quad \chi((x,x)) = \nu. \tag{5.105}$$

Notice that

$$(x,x) = u_t(-\pi_0)^{-\frac{a_t}{2}+1}(u_t^{-1}(-\pi_0)^{\frac{a_t}{2}-1}(x_0,x_0) + s_t\bar{s}_t).$$

Write

$$s_t = \sum_{i\geq 0}^{\infty} b_i\pi^i, \quad \text{ and } -u_t^{-1}(-\pi_0)^{\frac{a_t}{2}-1}(x_0,x_0) = \sum_{i\geq 0}^{\infty} c_i\pi^i,$$

where $b_i, c_i \in O_{F_0}/(\pi_0)$. Then the conditions in (5.105) are equivalent to the following equations:

$$\begin{aligned}
c_0 &= b_0^2 \neq 0,\\
c_1 &= b_1 b_0 - b_0 b_1,\\
c_2 &= b_2 b_0 - b_1 b_1 + b_0 b_2,\\
&\cdots\\
c_{a_t-3} &= \sum_{i=0}^{a_t-3} (-1)^i b_{a_t-3-i} b_i,\\
\nu &= \chi\Big(u_t(-\pi_0)^{-\frac{a_t}{2}+1}\big(-c_{a_t-2} + \sum_{i=0}^{a_t-2}(-1)^i b_{a_t-2-i} b_i\big)\Big).
\end{aligned}$$

Since a_t is even by assumption, the last equation is the same with

$$\nu = \chi\Big(u_t(-\pi_0)^{-\frac{a_t}{2}+1}\big(-c_{a_t-2} + \sum_{i=1}^{a_t-3}(-1)^i b_{a_t-2-i} b_i + 2b_0 b_{a_t-2}\big)\Big).$$

Notice that the possible choices of $\{b_0, \cdots, b_{a_t-3}\}$ are determined by the first $a_t - 2$ equations. And for a given choice of $\{b_0, \cdots, b_{a_t-3}\}$, the number of choices of b_{a_t-2} that satisfies the last equation is clearly independent of ν since $b_0 \neq 0$.

(ii) If a_t is odd or $a_t = 2$, then we may choose a normal basis $\{\ell_1, \cdots, \ell_t\}$ of L such that the moment matrix of $\{\ell_{t-1}, \ell_t\}$ is H_{a_t}, where $H_{a_t} := \begin{pmatrix} 0 & \pi^{a_t} \\ (-\pi)^{a_t} & 0 \end{pmatrix}$. We may choose $L' = \langle \ell_1, \cdots, \ell_{t-2}, \pi^{-1}\ell_{t-1}, \ell_t\rangle$ with fundamental invariants $(a_1, \cdots, a_{t-2}, a_{t-1} - 1, a_t - 1)$. In this case,

$$\pi L^\sharp = \langle \pi^{-a_1+1}\ell_1, \cdots, \pi^{-a_t+1}\ell_t\rangle \text{ and } \pi L'^\sharp = \langle \pi^{-a_1+1}\ell_1, \cdots, \pi^{-a_{t-1}+1}\ell_{t-1}, \pi^{-a_t+2}\ell_t\rangle.$$

For a fixed $x_0 = \sum_{1\leq i<t-1} s_i \pi^{-a_i+1}\ell_i$, we set

$$\begin{aligned}
S^\nu_{x_0} := \{x \in ((\pi L^\sharp)^\circ - (\pi L^\sharp)^{\circ\circ}) - ((\pi L'^\sharp)^\circ - (\pi L'^\sharp)^{\circ\circ}) : x = x_0 + s_{t-1}\pi^{-a_{t-1}+1}\ell_{t-1} + s_t \pi^{-a_t+1}\ell_t,\\
s_{t-1}, s_t \in O_F, \chi(\langle x\rangle) = \nu\}/L.
\end{aligned}$$

It suffices to show $|S_{x_0}^{+1}| = |S_{x_0}^{-1}|$. Notice that $x = x_0 + s_{t-1}\pi^{-a_{t-1}+1}\ell_{t-1} + s_t\pi^{-a_t+1}\ell_t \in S_{x_0}^{\nu}$ if and only if

$$s_t \in O_F^\times, \quad (x,x) = (x_0,x_0) + (s_{t-1}\bar{s}_t + (-1)^{a_t}\bar{s}_{t-1}s_t)(-1)^{-a_t+1}\pi^{-a_t+2} \in O_{F_0}^\times, \quad \chi((x,x)) = \nu. \tag{5.106}$$

Write

$$s_{t-1} = \sum_{i\geq 0}^{\infty} b_i\pi^i, \quad s_t = \sum_{i\geq 0}^{\infty} c_i\pi^i, \quad -(-1)^{a_t-1}\pi^{a_t-2}(x_0,x_0) = \sum_{i\geq 0}^{\infty} d_i\pi^i,$$

where $b_i, c_i, d_i \in O_{F_0}/(\pi_0)$. Then the condition in (5.106) is equivalent to the following equations:

$$\begin{aligned}
d_0 &= b_0c_0 + (-1)^{a_t}b_0c_0,\\
d_1 &= b_1c_0 - b_0c_1 + (-1)^{a_t}(-b_1c_0 + b_0c_0),\\
&\cdots\\
\nu &= \chi\Big((-1)^{a_t-1}\pi^{-a_t+2}\big(-d_{a_t-2} + S + 2b_{a_t-2}c_0\big)\Big),
\end{aligned}$$

where S is certain expression involving $b_0, \cdots, b_{a_t-3}$ and $c_1, \cdots, c_{a_t-2}$. Since $s_t \in O_F^\times$, $c_0 \neq 0$. Hence, for any given S, the number of choices of b_{a_t-2} that satisfies the last equation is clearly independent of ν. □

Proposition 5.6.9. *Assume that $t \geq 1$ is even and that L is a full type lattice of rank t. Then for any $\chi \in \{\pm 1\}$, we have*

$$(1-q^t)\mu^+(L) + (1-\chi q^{\frac{t}{2}})\mu^{0,+1}(L) + (1+\chi q^{\frac{t}{2}})\mu^{0,-1}(L) + \mu^-(L) = 0.$$

Proof. We prove this for maximal L first. There are two cases we need to consider.
(i) If we can choose a basis $\{\ell_1, \cdots, \ell_t\}$ of L with moment matrix $\mathrm{Diag}(H_1^{\frac{t}{2}-1}, u_{t-1}(-\pi_0), u_t(-\pi_0))$ where $\chi(-u_{t-1}u_t) = -1$, then set $L_1 = \langle \ell_1, \cdots, \ell_{t-2}\rangle$ and $L_2 = \langle \ell_{t-1}, \ell_t\rangle$. In this case, a

direct computation shows that

$$(\pi L^\sharp)^{\circ\circ} = L, \quad (\pi L^\sharp)^\circ = L_1 \oplus \pi^{-1} L_2, \quad (L^\sharp)^\circ = \pi^{-1} L.$$

Hence

$$\mu^+(L) = |(\pi L^\sharp)^{\circ\circ}/L| = 1, \quad \mu^-(L) = |((L^\sharp)^\circ - (\pi L^\sharp)^\circ)/L| = q^t - q^2. \tag{5.107}$$

Moreover,

$$\mu^{0,\nu}(L) = |\{(x,y) \in \mathbb{F}_q^2 - (0,0) \mid \chi(u_{t-1}x^2 + u_t y^2) = \nu\}|.$$

It is well known that

$$|\{(x,y) \in \mathbb{F}_q^2 - (0,0) \mid u_{t-1}x^2 + u_t y^2 = 1\}| = q - \chi(-u_{t-1}u_t) = q + 1.$$

Hence

$$\mu^{0,+1}(L) = \mu^{0,-1}(L) = \frac{q^2 - 1}{2}. \tag{5.108}$$

Combining (5.107) and (5.108), we have

$$\begin{aligned} &(1 - q^t)\mu^+(L) + (1 - \chi q^{\frac{t}{2}})\mu^{0,+}(L) + (1 + \chi q^{\frac{t}{2}})\mu^{0,-}(L) + \mu^-(L) \\ &= (1 - q^t) + (q^2 - 1) + (q^t - q^2) = 0. \end{aligned}$$

(ii) If we can choose a basis $\{\ell_1, \cdots, \ell_t\}$ of L with moment matrix $H_1^{\frac{t}{2}}$, then we can directly compute that

$$(\pi L^\sharp)^{\circ\circ} = L, \quad (\pi L^\sharp)^\circ = L, \quad (L^\sharp)^\circ = \pi^{-1} L.$$

Hence

$$\mu^+(L) = |(\pi L^\sharp)^{\circ\circ}/L| = 1, \quad \mu^0(L) = 0, \quad \mu^-(L) = |((L^\sharp)^\circ - (\pi L^\sharp)^\circ)/L| = q^t - 1.$$

As a result we have

$$(1-q^t)\mu^+(L) + (1-\chi q^{\frac{t}{2}})\mu^{0,+}(L) + (1+\chi q^{\frac{t}{2}})\mu^{0,-}(L) + \mu^-(L) = (1-q^t) + (q^t-1) = 0.$$

Now we assume L is not maximal and the proposition holds for L' such that $L \subsetneq L'$ by an induction on $\mathrm{val}(L)$. With this assumption, it suffices to show

$$(1-q^t)\mu^+(L,L') + (1-\chi q^{\frac{t}{2}})\mu^{0,+1}(L,L') + (1+\chi q^{\frac{t}{2}})\mu^{0,-1}(L,L') + \mu^-(L,L') = 0,$$

which follows from a combination of Lemmas 5.6.4 and 5.6.8. □

5.7 Proof of the main theorem

We prove the main theorem in this section by an induction on $\mathrm{val}(L)$ using the results we obtained about the partial Fourier transform in previous sections.

5.7.1 Comparison of horizontal intersection numbers

Lemma 5.7.1. *Let $L \subset \mathbb{V}$ be a lattice. If $L = L_1 \oplus L_2$ where L_1 is unimodular, then*

$$\mathrm{Int}(L) - \partial\mathrm{Den}(L) = \mathrm{Int}(L_2) - \partial\mathrm{Den}(L_2).$$

Proof. The lemma follows from comparing (5.61) with (4.37). □

Definition 5.7.2. *Let $L^\flat \subset \mathbb{V}$ be a non-degenerate lattice of rank $n-1$, and $x \in \mathbb{V} \setminus L^\flat_F$. Define*

$$\partial\mathrm{Den}_{L^\flat,\mathscr{H}}(x) = \sum_{\substack{L^\flat \subset L' \subset L'^\sharp \\ L'^\flat \in \mathrm{Hor}(L^\flat)}} \partial\mathrm{Pden}(L')1_{L'}(x). \tag{5.109}$$

Lemma 5.7.3. *If $L^\flat \subset \mathbb{V}$ is horizontal, then*

$$\mathrm{Int}_{L^\flat,\mathscr{H}}(x) = \partial\mathrm{Den}_{L^\flat,\mathscr{H}}(x),$$

where $\mathrm{Int}_{L^\flat,\mathscr{H}}$ *is defined in Definition 5.2.7.*

Proof. Let $L = L^\flat \oplus \langle x\rangle$. By Lemma 5.2.9, we know

$$\mathrm{Int}_{L^\flat,\mathscr{H}}(x) = \mathrm{Int}_{L^\flat}(x) = \mathrm{Int}(L). \tag{5.110}$$

On the other hand, since $L^\flat$ is horizontal, by Lemma 5.2.2 any integral lattice of rank $n-1$ containing $L^\flat$ is horizontal, hence we have

$$\partial\mathrm{Den}_{L^\flat,\mathscr{H}}(x) = \partial\mathrm{Den}_{L^\flat}(x) = \partial\mathrm{Den}(L). \tag{5.111}$$

So it suffices to prove $\mathrm{Int}(L) = \partial\mathrm{Den}(L)$.

When $n=2$, by (5.110) and (5.111), the lemma is a consequence of [Shi22, Theorem 1.1] and Theorem 3.4.1. When $n>2$, $L^\flat$ has a unimodular direct summand L_1 of rank $n-2$ such that $L = L_1 \obot L_2$ and $L_2^\flat := L_{2,F} \cap L^\flat$ is a horizontal lattice in $L_{2,F}$. The lemma follows from the the case $n=2$ and Lemma 5.7.1. $\square$

Lemma 5.7.4. *If $M^\flat \subset \mathbb{V}$ is horizontal, then*

$$\chi(\mathcal{N}, {}^{\mathbb{L}}\mathcal{Z}(M^\flat)^\circ \cdot \mathcal{Z}(x)) = \sum_{\substack{M^\flat \subset L' \subset L'^\sharp \\ L'^\flat = M^\flat}} \partial\mathrm{Pden}(L') 1_{L'}(x),$$

where ${}^{\mathbb{L}}\mathcal{Z}(M^\flat)^\circ$ *is as in Definition 5.2.5.*

Proof. By Definition 5.2.7, we have

$$\mathrm{Int}_{M^\flat,\mathscr{H}}(x) = \chi(\mathcal{N}, {}^{\mathbb{L}}\mathcal{Z}(M^\flat)^\circ \cdot \mathcal{Z}(x)) + \sum_{\substack{L'^\flat \in \mathrm{Hor}(M^\flat) \\ L'^\flat \neq M^\flat}} \chi(\mathcal{N}, {}^{\mathbb{L}}\mathcal{Z}(L'^\flat)^\circ \cdot \mathcal{Z}(x)). \tag{5.112}$$

We now prove the lemma by induction on $\mathrm{val}(M^\flat)$. When $M^\flat$ is unimodular, the lemma is the same as Lemma 5.7.3. In general notice that any integral lattice of rank $n-1$ containing $M^\flat$ is horizontal by Lemma 5.2.2. Applying the induction hypothesis to the right hand side of the above formula and applying Lemma 5.7.3 to the left hand had, we obtain

$$\sum_{\substack{M^\flat \subset L' \subset L'^\sharp \\ L'^\flat \in \mathrm{Hor}(M^\flat)}} \partial\mathrm{Pden}(L') 1_{L'}(x) = \chi(\mathcal{N}, {}^{\mathbb{L}}\mathcal{Z}(M^\flat)^\circ \cdot \mathcal{Z}(x)) + \sum_{\substack{M^\flat \subset L' \subset L'^\vee \\ L'^\flat \neq M^\flat}} \partial\mathrm{Pden}(L') 1_{L'}(x). \tag{5.113}$$

Subtract the left hand side by the second term of the right hand side of the equation, the lemma is proved. ☐

Theorem 5.7.5. *For a non-degenerate lattice $L^\flat \subset \mathbb{V}$ of rank $n-1$, and $x \in \mathbb{V} \setminus L^\flat_F$, we have*

$$\mathrm{Int}_{L^\flat, \mathscr{H}}(x) = \partial\mathrm{Den}_{L^\flat, \mathscr{H}}(x).$$

Proof. By the definition of $\mathrm{Int}_{L^\flat, \mathscr{H}}(x)$, we have

$$\mathrm{Int}_{L^\flat, \mathscr{H}}(x) = \sum_{M^\flat \in \mathrm{Hor}(L^\flat)} \chi(\mathcal{N}, {}^{\mathbb{L}}\mathcal{Z}(M^\flat)^\circ \cdot \mathcal{Z}(x)).$$

The theorem now follows from (5.109) and Lemma 5.7.4. ☐

5.7.2 Proof of the main theorem

The following is an analogue of Lemma 9.3.1 of [LZ22b].

Lemma 5.7.6. *Let $L^\flat \subset \mathbb{V}$ be a non-degenerate lattice of rank $n-1$ and $\mathbb{W} = (L^\flat_F)^\perp$. For $x \notin L^\flat \obot \mathbb{W}$, there exists an O_F-lattice $L'^\flat$ of rank $n-1$ and $x' \in \mathbb{V}$ such that*

$$\mathrm{val}(L'^\flat) < \mathrm{val}(L^\flat) \textit{ and } L'^\flat + \langle x' \rangle = L^\flat + \langle x \rangle.$$

Proof. Assume that $L^\flat \subset \mathbb{V}$ has fundamental invariants $(a_1, \cdots, a_{n-1})$. Let $\{\ell_1, \cdots, \ell_{n-1}\}$ be a basis of $L^\flat$ whose moment matrix is

$$\mathrm{Diag}(H_{b_1}, H_{b_3}, \cdots, H_{b_{2s-1}}, u_{2s+1}\pi^{b_{2s+1}}, \cdots, u_{n-1}\pi^{b_{n-1}}),$$

where $b_1, \cdots, b_{2s-1}$ are odd and $H_j = \begin{pmatrix} 0 & \pi^j \\ (-\pi)^j & 0 \end{pmatrix}$. Notice that $\{b_1, \cdots, b_{n-1}\} = \{a_1, \cdots, a_{n-1}\}$. The moment matrix of $\{\ell_1, \cdots, \ell_{n-1}, x\}$ is

$$T = \begin{pmatrix} H_{b_1} & & & (\ell_1, x) \\ & \ddots & & \vdots \\ & & u_{n-1}\pi^{b_{n-1}} & (\ell_{n-1}, x) \\ (x, \ell_1) & \cdots & (x, \ell_{n-1}) & (x, x) \end{pmatrix}.$$

Assume $(a'_1, \cdots, a'_n)$ is the fundamental invariants of $L^\flat + \langle x \rangle$. According to Lemma 2.23 of [LL22], $a'_1 + \cdots + a'_{n-1}$ equals the minimal valuation of the $(n-1) \times (n-1)$ minors of T.

Write $x = x^\flat + x^\perp$ where $x^\flat \in L^\flat_F$ and $x^\perp \in \mathbb{W}$. If $x^\flat \notin L^\flat$, then we can write $x^\flat = \sum_{j=1}^{n-1} \alpha_j \ell_j$ where $\alpha_i \notin O_F$ for some i. First, we assume $\alpha_i \notin O_F$ for some $i \leq 2s$. The valuation of the (n,i)-th minor of T (removing n-th row and i-th column) equals to

$$\begin{cases} \mathrm{val}_\pi((\ell_{i+1}, x)) - b_i + (b_1 + \cdots + b_{n-1}) & \text{if } i \text{ is odd}, \\ \mathrm{val}_\pi((\ell_{i-1}, x)) - b_i + (b_1 + \cdots + b_{n-1}) & \text{if } i \text{ is even}. \end{cases}$$

Since $\alpha_i \notin O_F$, we have $\mathrm{val}_\pi((\ell_{i+1}, x)) < b_i$ if i is odd and $\mathrm{val}_\pi((\ell_{i-1}, x)) < b_i$ if i is even. In particular, $\sum_{j=1}^{n-1} a'_j < \sum_{j=1}^{n-1} a_j$. Now if we choose a normal basis $\{\ell'_1, \cdots, \ell'_n\}$ of $L^\flat + \langle x \rangle$, then $L'^\flat = \langle \ell'_1, \cdots, \ell'_{n-1}\}$ and $x' = \ell'_n$ satisfy the property we want.

Now we assume $\alpha_i \notin O_F$ for some $2s < i \leq n-1$. The valuation of the (n,i)-th minor of T equals to

$$\mathrm{val}_\pi(\ell_i, x) - b_i + (b_1 + \cdots + b_{n-1}).$$

Since $\alpha_i \notin O_F$, $\mathrm{val}_\pi(\ell_i, x) < b_i$, hence $\sum_{j=1}^{n-1} a'_j < \sum_{j=1}^{n-1} a_j$. Now if we choose a normal basis $\{\ell'_1, \cdots, \ell'_n\}$ of $L^\flat + \langle x\rangle$, then $L'^\flat = \langle \ell'_1, \cdots, \ell'_{n-1}\}$ and $x' = \ell'_n$ satisfy the property we want. $\square$

For any L, we can write it as $L^\flat + \langle x\rangle$ where $L^\flat$ is a non-degenerate hermitian O_F-lattice of rank $n-1$, and $x \in \mathbb{V} \setminus L^\flat$. Therefore, in order to show $\mathrm{Int}(L) = \partial\mathrm{Den}(L)$, it suffices to show the following theorem.

Theorem 5.7.7. *Let $L^\flat \subset \mathbb{V}$ be a non-degenerate lattice of rank $n-1$. For any $x \in \mathbb{V}\setminus L^\flat_F$, we have*

$$\mathrm{Int}_{L^\flat}(x) = \partial\mathrm{Den}_{L^\flat}(x).$$

Proof. For $x \in \mathbb{V} \setminus L^\flat_F$, let $\Phi_{L^\flat}(x) = \mathrm{Int}_{L^\flat}(x) - \partial\mathrm{Den}_{L^\flat}(x)$. We need to show $\Phi_{L^\flat}(x) = 0$. We prove the theorem by an induction on $\mathrm{val}(L^\flat)$. If $L^\flat$ is not integral, then $\mathrm{Int}_{L^\flat}(x) = 0$ as $\mathcal{Z}(L)$ is empty by Proposition 5.1.20. Moreover $\partial\mathrm{Den}_{L^\flat}(x) = 0$ by Corollary 5.5.2. Hence the theorem is true in this case.

Now we assume $L^\flat$ is integral. By induction hypothesis and Lemma 5.7.6, we may assume $\mathrm{supp}(\Phi_{L^\flat}) \subset L^\flat \ominus \mathbb{W}$ where $\mathbb{W} = (L^\flat_F)^\perp$. Since $\Phi_{L^\flat}(x)$ is invariant under the translation of $L^\flat$, we may write

$$\Phi_{L^\flat}(x) = 1_{L^\flat} \otimes \phi_{\mathbb{W}}(x), \tag{5.114}$$

where $\phi_{\mathbb{W}}(x)$ is a function on $\mathbb{W} \setminus \{0\}$. Then we have by definition

$$\Phi^\perp_{L^\flat} = \mathrm{vol}(L^\flat)\phi_{\mathbb{W}}.$$

Theorem 5.7.5 implies that

$$\Phi_{L^\flat}(x) = \Phi_{L^\flat,\mathscr{V}}(x) := \mathrm{Int}_{L^\flat,\mathscr{V}}(x) - \partial\mathrm{Den}_{L^\flat,\mathscr{V}}(x).$$

Hence

$$\mathrm{vol}(L^\flat)\phi_{\mathbb{W}} = \Phi^{\perp}_{L^\flat,\mathscr{V}}. \tag{5.115}$$

By induction on the rank of L and Lemma 5.7.1, we may assume $\Phi_{L^\flat}(x) = 0$, hence $\phi_{\mathbb{W}}(x) = 0$ for $x \in \mathbb{W}^{=0}$. Combining this with the non-integral case, we know $\phi_{\mathbb{W}}(x) = 0$ for $x \in \mathbb{W}^{\leq 0}$. As a result, we have $\Phi^{\perp}_{L^\flat,\mathscr{V}}(x) = 0$ for $x \in \mathbb{W}^{\leq 0}$ by (5.115). By Theorem 5.6.2 and Theorem 5.2.16, we have $\Phi^{\perp}_{L^\flat,\mathscr{V}}(x) = 0$ for $x \in \mathbb{W}^{\geq 0} \setminus \{0\}$. Hence $\Phi^{\perp}_{L^\flat,\mathscr{V}}(x) = 0$ for all $x \in \mathbb{W} \setminus \{0\}$. Consequently, $\phi_{\mathbb{W}}(x) = 0$ by (5.115). $\square$

Combining this theorem with Theorem 5.2.15, we have the following corollary.

Corollary 5.7.8. *Let $L^\flat \subset \mathbb{V}$ be a non-degenerate lattice of rank $n-1$. Then $\partial\mathrm{Den}_{L^\flat,\mathscr{V}}(x) \in \mathscr{S}(\mathbb{V})^{\geq -1}$ is a Schwartz function.*

5.8 Global applications

5.8.1 Shimura varieties

In this section, we switch to global situation and will closely follow [RSZ20] and [RSZ21]. Let F be a CM number field with maximal totally real subfield F_0. We fix a CM type $\Phi \subset \mathrm{Hom}(F, \overline{\mathbb{Q}})$ of F and a distinguished element $\phi_0 \in \Phi$. We fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and identify the CM type Φ with the set of archimedean places of F, and also with the set of archimedean places of F_0. Let V be an F/F_0-hermitian space of dimension $n \geq 2$. Let $V_\phi = V \otimes_{F,\phi} \mathbb{C}$ be the associated $\mathbb{C}/\mathbb{R}$-hermitian space for $\phi \in \Phi$. Assume the signature of V_ϕ is given by

$$(r_\phi, r_{\bar\phi}) = \begin{cases} (n-1,1), & \phi = \phi_0, \\ (n,0), & \phi \in \Phi \setminus \{\phi_0\}. \end{cases}$$

Define a variant $G^{\mathbb{Q}}$ of the unitary similitude group $\mathrm{GU}(V)$ by

$$G^{\mathbb{Q}} := \{g \in \mathrm{Res}_{F_0/\mathbb{Q}}\mathrm{GU}(V) : c(g) \in \mathbb{G}_m\},$$

where c denotes the similitude character. Define a cocharacter

$$h_{G^{\mathbb{Q}}} : \mathbb{C}^\times \to G^{\mathbb{Q}}(\mathbb{R}) \subset \prod_{\phi\in\Phi} \mathrm{GU}(V_\phi)(\mathbb{R}) \simeq \prod_{\phi\in\Phi} \mathrm{GU}(r_\phi, r_{\bar\phi})(\mathbb{R}),$$

where its ϕ-component is given by

$$h_{G^{\mathbb{Q}},\phi}(z) = \mathrm{Diag}\{z \cdot 1_{r_\phi}, \bar z \cdot 1_{r_{\bar\phi}}\}$$

under the decomposition of V_ϕ into positive definite and negative definite parts. Then its $G^{\mathbb{Q}}(\mathbb{R})$-conjugacy class defines a Shimura datum $(G^{\mathbb{Q}}, \{h_{G^{\mathbb{Q}}}\})$. Let $E_r = E(G^{\mathbb{Q}}, \{h_{G^{\mathbb{Q}}}\})$ be the reflex field, i.e., the subfield of $\overline{\mathbb{Q}}$ fixed by $\{\sigma \in \mathrm{Aut}(\overline{\mathbb{Q}}/\mathbb{Q}) : \sigma^*(r) = r\}$, where $r : \mathrm{Hom}(F, \overline{\mathbb{Q}}) \to \mathbb{Z}$ is the function defined by $r(\phi) = r_\phi$.

We similarly define the group $Z^{\mathbb{Q}}$ (a torus) associated to a totally positive definite F/F_0-hermitian space of dimension 1 (i.e., of signature $\{(1,0)_{\phi\in\Phi}\}$) and a cocharacter $h_{Z^{\mathbb{Q}}}$ of $Z^{\mathbb{Q}}$. The reflex field $E_\Phi = E(Z^{\mathbb{Q}}, \{h_{Z^{\mathbb{Q}}}\})$ is equal to the reflex field of the CM type Φ, i.e., the subfield of $\overline{\mathbb{Q}}$ fixed by $\{\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) : \sigma \circ \Phi = \Phi\}$.

Now define a Shimura datum $(\widetilde{G}, \{h_{\widetilde{G}}\})$ by

$$\widetilde{G} := Z^{\mathbb{Q}} \times_{\mathbb{G}_m} G^{\mathbb{Q}} = \{(z,g) \in Z^{\mathbb{Q}} \times G^{\mathbb{Q}} \mid \mathrm{Nm}_{F/F_0}(z) = c(g)\}, \quad h_{\widetilde{G}} = (h_{Z^{\mathbb{Q}}}, h_{G^{\mathbb{Q}}}).$$

Then $\widetilde{G} \cong Z^{\mathbb{Q}} \times G$ where $G = \mathrm{Res}_{F_0/\mathbb{Q}}\mathrm{U}(V)$. Its reflex field E is equal to the composite $E_r E_\Phi$, and the CM field F becomes a subfield of E via the embedding ϕ_0. We remark that $E = F$ when $F/\mathbb{Q}$ is Galois, or when $F = F_0\kappa$ for some imaginary quadratic $\kappa/\mathbb{Q}$ and the CM type Φ is induced from a CM type of $\kappa/\mathbb{Q}$ (e.g., when $F_0 = \mathbb{Q}$). Assume that $K_{Z^{\mathbb{Q}}} \subset Z^{\mathbb{Q}}(\mathbb{A}_f)$ is the unique maximal open compact subgroup, and $K_G = \prod_v K_{G,v}$ where v runs over finite places of F_0 is a compact open subgroup of $G(\mathbb{A}_f)$. Let $K = K_{Z^{\mathbb{Q}}} \times K_G \subset \widetilde{G}(\mathbb{A}_f)$. Then the associated Shimura variety $\mathrm{Sh}_K = \mathrm{Sh}_K(\widetilde{G}, \{h_{\widetilde{G}}\})$ is a Deligne-Mumford stack of dimension $n-1$ and has a canonical model over $\mathrm{Spec}\, E$.

5.8.2 Integral model

In this subsection we run through the set-up of [LZ22a, §14]. Let $\mathbf{m} = (m_v)_v$ be a collection of integers $m_v \geq 0$ indexed by finite places of F_0 such that $m_v = 0$ for all but finitely many places and $m_v = 0$ for all places v that are nonsplit in F. Let Λ be an O_F-lattice of V. Assume that for any finite place v of F_0 (with residue characteristic p), the following conditions are satisfied where $\tilde{\nu} : \bar{\mathbb{Q}} \to \bar{\mathbb{Q}}_p$ is an embedding that induces a place ν of E.

(G0) If $p = 2$, then v is unramified in F.

(G1) If v is inert in F and V_v is split, then $\Lambda_v \subset V_v$ is self-dual and $K_{G,v}$ is the stabilizer of Λ_v. If v is further ramified over p and ν is any place of E above v, then the subset $\{\phi \in \Phi : \tilde{\nu} \circ \phi \text{ induces } w\} \subset \mathrm{Hom}(F_w, \overline{\mathbb{Q}}_p)$ is the pullback of a CM type $\Phi^{\mathrm{ur}} \subset \mathrm{Hom}(F_w^{\mathrm{ur}}, \overline{\mathbb{Q}}_p)$ of F_w^{ur}. Here w is the place of F above v and F_w^{ur} is the maximal subfield of F_w unramified over $\mathbb{Q}_p$.

(G2) If v is inert in F and V_v is nonsplit, then v is unramified over p and $\Lambda_v \subset V_v$ is almost self-dual, i.e., $\Lambda_v^{\sharp}/\Lambda_v$ is a 1 dimensional space over the residue field of F_w. Moreover $K_{G,v}$ is the stabilizer of Λ_v.

(G3) If v is ramified in F, then v is unramified over p and $\Lambda_v \subset V_v$ is unimodular.

(G4) If v is split in F and $m_v = 0$, then $\mathrm{U}(V)(F_{0,v}) \cong \mathrm{GL}_n(F_{0,v})$ and we assume $\Lambda_v \subset V_v$ is self-dual. Let $K_{G,v} \cong \mathrm{GL}_n(O_{F_{0,v}})$ be the stabilizer of Λ_v.

(G5) If v is split in F and $m_v > 0$, then again $\mathrm{U}(V)(F_{0,v}) \cong \mathrm{GL}_n(F_{0,v})$ and we assume $\Lambda_v \subset V_v$ is self-dual. Further assume that v splits into w and $\bar{w}$ in F and all places ν of E above v satisfy the following two conditions.

   (a) the place ν of E matches the CM type Φ (in the sense of [RSZ20, §4.3]): if $\phi \in \mathrm{Hom}(F, \overline{\mathbb{Q}})$ induces the p-adic place w of F (via $\tilde{\nu} : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$), then $\phi \in \Phi$.

   (b) the extension $E_\nu/E_{r|_v}$ is unramified, where $E_{r|_v}$ is the local reflex field as defined in [RSZ20, §4.1]. We remark that this condition holds automatically if all p-adic places of F are unramified over p.

   (c) Let v' be any other place of F_0 above p nonsplit in F and ν' be any place of E above v'. Then either w or $\bar{w}$ matches the CM type Φ (let us take w for example): if $\phi \in \mathrm{Hom}(F, \overline{\mathbb{Q}})$ induces the p-adic place w of F (via $\tilde{\nu}' : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$), then $\phi \in \Phi$. This condition is assumption (H5) of [LZ22a, §11.2].

   We remark here that condition (a) and (c) are automatically true when $F = F_0\kappa$ for some imaginary quadratic $\kappa/\mathbb{Q}$ and the CM type Φ is induced from a CM type of $\kappa/\mathbb{Q}$ (e.g., when $F_0 = \mathbb{Q}$), or when v is of degree one over p. Let $K_{G,v}$ be the principal congruence subgroup modulo $\pi_v^{m_v}$ inside the stabilizer of Λ_v where π_v is a uniformizer of $F_{0,v}$.

In the case when the above conditions are satisfied, we denote K by $K^{\mathbf{m}}$. Also denote $K^{\mathbf{m}}$ by K° if $\mathbf{m} = (0)_v$. In other words $K^\circ \subset \tilde{G}(\mathbb{A}_f)$ is the stabilizer of $\Lambda \otimes_{O_F} \widehat{O}_F$. Define the moduli functor $\mathcal{M}_{K^\circ}$ as follows. For a locally noetherian O_E-scheme S, $\mathcal{M}_{K^\circ}(S)$ is the groupoid of tuples $(A_0, \iota_0, \lambda_0, A, \iota, \lambda, \mathcal{F})$ such that

1. A (resp. A_0) is an abelian scheme over S;

2. ι (resp. ι_0) is an action of O_F on A (resp. A_0) satisfying the Kottwitz condition of signature $\{(r_\phi, r_{\bar{\phi}})_{\phi \in \Phi}\}$ (resp. $\{(1,0)_{\phi\in\Phi}\}$):

$$\mathrm{charpol}(\iota(a) \mid \mathrm{Lie}A) = \prod_{\phi\in\Phi} (T - \phi(a))^{r_\phi} \cdot (T - \bar{\phi}(a))^{r_{\bar{\phi}}} \tag{5.116}$$

for any $a \in O_F$;

3. λ (resp. λ_0) is a polarization of A (resp. A_0) whose Rosati involution induces the automorphism given by the nontrivial Galois automorphism of F/F_0 via ι (resp. ι_0);

4. $\mathcal{F}$ is locally a direct summand $\mathcal{O}_S$-submodule of Lie$A$ which is stable under the O_F-action. Moreover O_F acts on $\mathcal{F}$ by the structural morphism and on Lie$A/\mathcal{F}$ by the Galois conjugate of the structural morphism.

We further require the following conditions to be satisfied.

(H1) $(A_0, \iota_0, \lambda_0) \in \mathcal{M}_0^\xi$ where $\mathcal{M}_0^\xi = \mathcal{M}_0^{(1),\xi}$ in the notation of [RSZ21, §4.1] (where (1) is the unit ideal in O_F) is an integral model of $\mathrm{Sh}_{K_{Z^\mathbb{Q}}}(Z^\mathbb{Q}, h_{Z^\mathbb{Q}})$ depending on the choice of a similarity class ξ of 1 dimensional F/F_0-hermitian spaces.

(H2) For each finite place v of F_0, λ induces a polarization λ_v on the p-divisible group $A[v^\infty]$. We require $\ker \lambda_v \subset A[\iota(\varpi_v)]$ and is of rank equal to the size of $\Lambda_v^\sharp/\Lambda_v$, where ϖ_v is a uniformizer of $F_{0,v}$.

(H3) For each place v of F_0, we require the *sign condition* and *Eisenstein condition* as explained in [RSZ20, §4.1]. We remark that the sign condition holds automatically when v is split in F, and the Eisenstein condition holds automatically when the places of F above v are unramified over p.

(H4) We impose the *Krämer condition* on $\mathcal{F}$ as explained in [RSZ21, Definition 6.10].

A morphism $(A_0, \iota_0, \lambda_0, A, \iota, \lambda, \mathcal{F}) \to (A_0', \iota_0', \lambda_0', A', \iota', \lambda', \mathcal{F}')$ is a pair $(f_0 : A_0 \to A_0', f : A \to A')$ of O_F-linear isomorphism of abelian schemes over S such that $f^*(\lambda') = \lambda$, $f_0^*(\lambda_0') = \lambda_0$, $f_*(\mathcal{F}) = \mathcal{F}'$. Let $\mathcal{V}_{\mathrm{ram}}$ (resp. $\mathcal{V}_{\mathrm{asd}}$) be the set of finite places v of F_0 such that v is ramified in F (resp. v is inert in F and Λ_v is almost self-dual). By [RSZ20, Theorem 5.2], the moduli problem $\mathcal{M}_{K^\circ}$ is representable by a Deligne-Mumford stack over O_E which is regular and semistable at all places of E above $\mathcal{V}_{\mathrm{ram}} \cup \mathcal{V}_{\mathrm{asd}}$. The generic fiber of $\mathcal{M}_{K^\circ}$ is the Shimura variety Sh_{K°. For a general $\mathbf{m}$, define $\mathcal{M}_{K^\mathbf{m}}$ to be the normalization of $\mathcal{M}_{K^\circ}$ in $\mathrm{Sh}_{K^\mathbf{m}}$. Then by [RSZ20, Theorem 5.4], $\mathcal{M}_{K^\mathbf{m}}$ is representable by a Deligne-Mumford stack over O_E which is regular and semistable at all places of E above $\mathcal{V}_{\mathrm{ram}} \cup \mathcal{V}_{\mathrm{asd}}$. Its localization at each finite place ν of E agrees with the semi-integral models defined in and [RSZ20, §4] or [LZ22a, §11].

5.8.3 Global main theorems

From now on we assume $K = K^{\mathbf{m}}$ and simply denote $\mathcal{M}_{K^{\mathbf{m}}}$ by $\mathcal{M}$. Let $\mathbb{V}$ be the *incoherent* $\mathbb{A}_F/\mathbb{A}_{F_0}$-hermitian space associated to V, namely $\mathbb{V}$ is totally positive definite and $\mathbb{V}_v \cong V_v$ for all finite places v. Let $\varphi_K \in \mathscr{S}(\mathbb{V}_f^m)$ be a K-invariant (where K acts on $\mathbb{V}_f$ via the second factor K_G) Schwartz function. We say φ_K is admissible if it is K-invariant and $\varphi_{K,v} = \mathbf{1}_{(\Lambda_v)^m}$ at all v nonsplit in F.

First, we consider a special admissible Schwartz function of the form

$$\varphi_K = (\varphi_i) \in \mathscr{S}(\mathbb{V}_f^m), \quad \varphi_i = \mathbf{1}_{\Omega_i}, \quad i = 1, \ldots, m, \tag{5.117}$$

where $\Omega_i \subset \mathbb{V}_f$ is a K-invariant open compact subset such that $\Omega_{i,v} = \Lambda_v$ at all v nonsplit in F. Given $t_i \in F$ and φ_i, there exists a unique special divisor $\mathcal{Z}(t, \varphi_i)$ over $\mathcal{M}_K$ such that for each place ν of E inducing a non-split place of F_0, the base change of $\mathcal{Z}(t, \varphi_i)$ to $\operatorname{Spec} O_{E,(\nu)}$ agrees with the special divisor defined as in [LZ22a, §13.3], and for each ν inducing a split place of F_0, it agrees with the Zariski closure of the special divisor over the generic fiber of $\mathcal{M}_K$. Then we have the following decomposition (cf. [KR14b, (11.2)]),

$$\mathcal{Z}\left(t_1, \varphi_1\right) \cap \cdots \cap \mathcal{Z}\left(t_m, \varphi_m\right) = \bigsqcup_{T \in \mathrm{Herm}_m(F)} \mathcal{Z}\left(T, \varphi_K\right),$$

where $\cap$ denotes taking fiber product over $\mathcal{M}_K$, and the indexes T have diagonal entries $t_1, \ldots, t_m$.

Let $T \in \mathrm{Herm}_n(F)$ be a nonsingular F/F_0-hermitian matrix of size n. Given (T, φ_K), we can define an arithmetic degree as follows. First, analogous to the local situation (1.5), we can define a local arithmetic intersection numbers $\mathrm{Int}_{T,\nu}(\varphi_K)$ for any place ν of E. First we assume ν is finite and let v be the place of F_0 under ν. By the same proof of

[KR14b, Proposition 2.22], it suffices to consider the case when v is nonsplit in F. When φ_K is of the form (5.117), define

$$\mathrm{Int}_{T,\nu}(\varphi_K) \coloneqq \frac{1}{[E:F_0]} \cdot \chi(\mathcal{Z}(T,\varphi_K)_\nu, \mathcal{O}_{\mathcal{Z}(t_1,\varphi_1)_\nu} \otimes^{\mathbb{L}} \cdots \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}(t_n,\varphi_n)_\nu}) \cdot \log q_\nu, \quad (5.118)$$

where q_ν denotes the size of the residue field k_ν of E_ν, $\mathcal{Z}(T,\varphi_K)_\nu$ and $\mathcal{Z}(t,\varphi_i)_\nu$ denote the base change to $O_{E,(\nu)}$, $\mathcal{O}_{\mathcal{Z}(t_i,\varphi_i)_\nu}$ denotes the structure sheaf of the Kudla–Rapoport divisor $\mathcal{Z}(t_i,\varphi_i)$, $\otimes^{\mathbb{L}}$ denotes the derived tensor product of coherent sheaves on $\mathcal{M}$, and χ denotes the Euler–Poincaré characteristic. For a general admissible function φ_K, we can extend the definition $\mathbb{C}$-linearly. Using the star product of Kudla's Green functions, we can also define a local arithmetic intersection number $\mathrm{Int}_{T,\nu}(\mathsf{y},\varphi_K)$ at infinite places ([LZ22a, §15.3]), which depends on a parameter $\mathsf{y} \in \mathrm{Herm}_n(F_\infty)_{>0}$ where $F_\infty = F \otimes_{\mathbb{Q}} \mathbb{R}$. Combining all the local arithmetic numbers together, define the *global arithmetic intersection number*, or the *arithmetic degree* of the special cycle $\mathcal{Z}(T,\varphi_K)$ in the arithmetic Chow group of $\mathcal{M}$,

$$\widehat{\deg}_T(\mathsf{y},\varphi_K) \coloneqq \sum_{\nu \nmid \infty} \mathrm{Int}_{T,\nu}(\varphi_K) + \sum_{\nu \mid \infty} \mathrm{Int}_{T,\nu}(\mathsf{y},\varphi_K).$$

Theorem 5.8.1. *Let* $\mathrm{Diff}(T,\mathbb{V})$ *be the set of places* v *such that* $\mathbb{V}_v$ *does not represent* T. *Let* $T \in \mathrm{Herm}_n(F)$ *be nonsingular such that* $\mathrm{Diff}(T,\mathbb{V}) = \{v\}$ *where* v *is nonsplit in* F *and not above 2. Assume* $\varphi_K \in \mathscr{S}(\mathbb{V}_f^m)$ *is admissible. Then*

$$\widehat{\deg}_T(\mathsf{y},\varphi_K) q^T = c_K \cdot \partial\mathrm{Eis}_T(\mathsf{z},\varphi_K),$$

where $q^T \coloneqq \psi_\infty(\mathrm{Tr} T\mathsf{z})$, $c_K = \frac{(-1)^n}{\mathrm{vol}(K)}$ *is a nonzero constant independent of* T *and* $\mathrm{vol}(K)$ *is the volume of* K *under a suitable Haar measure. Finally,* $\partial\mathrm{Eis}_T(\mathsf{z},\varphi_K)$ *is the* T*-th coefficient of the modified central derivative of Eisenstein series in (1.18)*

Proof. When v is finite and $v \notin \mathcal{V}_{\mathrm{ram}} \cup \mathcal{V}_{\mathrm{asd}}$, the theorem is proved in [LZ22a, Theorem 13.6]. For $v \in \mathcal{V}_{\mathrm{asd}}$, our definition of $\mathrm{Int}_{T,\nu}(\varphi_K)$ differs from that of [LZ22a, (13.5.0.14)]. Correspondingly on the analytic side, our definition of $\partial\mathrm{Eis}_T(\mathsf{z},\varphi_K)$ is also modified, see

(**??**) and (1.17). However using [LZ22a, Theorem 10.5.1] instead of [LZ22a, Theorem 10.3.1], the proof of [LZ22a, Theorem 13.6] works the same way in this case. When v is infinite, the theorem is proved in [Liu11, Theorem 4.17,4.20] and independently in [GS19, Theorem 1.1.2]. When v is finite and $v \in \mathcal{V}_{\mathrm{ram}}$, the theorem is a corollary of Theorem 5.7.7 and can be proved in the same way as [HSY23a, Theorem 12.3]. □

We say $\varphi_v \in \mathscr{S}(\mathbb{V}_v^n)$ is *nonsingular* if its support lies in $\{\mathbf{x} \in \mathbb{V}_v^n : \det T(\mathbf{x}) \neq 0\}$, see [LZ22a, §12.3] or [Liu11, Proposition 2.1].

Theorem 5.8.2. *Assume that F/F_0 is split at all places above 2. Further assume that φ_K is admissible and nonsingular at two places split in F. Then*

$$\widehat{\deg}(\mathsf{z}, \varphi_K) = c_K \cdot \partial \mathrm{Eis}(\mathsf{z}, \varphi_K),$$

where $\widehat{\deg}(\mathsf{z}, \varphi_K)$ is defined in (**??**)*. In particular, $\widehat{\deg}(\mathsf{z}, \varphi_K)$ is a nonholomorphic hermitian modular form of genus n.*

Proof. The theorem can be derived from Theorem 5.8.1 by the same way as [LZ22a, Theorem 15.5.1]. □

Bibliography

[ACZ16] Tobias Ahsendorf, Chuangxun Cheng, and Thomas Zink. $\mathcal{O}$-displays and π-divisible formal $\mathcal{O}$-modules. *Journal of Algebra*, 457:129–193, 2016.

[BC91] J-F Boutot and Henri Carayol. Uniformisation p-adique des courbes de shimura: les théorèmes de čerednik et de drinfeld. *Astérisque*, (196-97):45–158, 1991.

[BHK+20] Jan Bruinier, Benjamin Howard, Stephen S Kudla, Michael Rapoport, and Tonghai Yang. Modularity of generating series of divisors on unitary shimura varieties I. *Astérisque, (421, Diviseurs arithmétiques sur les variétés orthogonales et unitaires de Shimura)*, pages 7–125, 2020.

[Bor99] Richard E. Borcherds. Unique factorization in regular local rings. *Duke Math. J.*, 97(2):219–233, 1999.

[Bro89] Kenneth S Brown. *Buildings*. Springer, 1989.

[BWR15] Jan Bruinier and Martin Westerholt-Raum. Kudla's modularity conjecture and formal fourier-jacobi series. *Forum Math. Pi*, 2015.

[BY20] Jan Bruinier and Tonghai Yang. Arithmetic degrees of special cycles and derivatives of siegel eisenstein series. *Journal of European Math. Soc*, page pp60, 2020.

[Cam] Peter Cameron. Lecture notes of advanced combinatorics mt5821. *http://www-groups.mcs.st-andrews.ac.uk/ pjc/Teaching/MT5821/1/l6.pdf.*

[Car62] Pierre Cartier. Groupes algébriques et groupes formels. In *Colloq. Théorie des Groupes Algébriques (Bruxelles, 1962). Librairie Universitaire, Louvain*, pages 87–111, 1962.

[Cho22] Sungyoon Cho. Special cycles on unitary Shimura varieties with minuscule parahoric level structure. *Mathematische Annalen*, pages 1–67, 2022.

[CY20] Sungmun Cho and Takuya Yamauchi. A reformulation of the Siegel series and intersection numbers. *Mathematische Annalen*, 377(3):1757–1826, 2020.

[DL22] Daniel Disegni and Yifeng Liu. A p-adic arithmetic inner product formula. *arXiv e-prints*, page arXiv:2204.09239, April 2022.

[Dri76] V. G. Drinfeld. Coverings of p-adic symmetric domains. *Funkcional. Anal. i Priložen.*, 10(2):29–40, 1976.

[GQT14] Wee Teck Gan, Yannan Qiu, and Shuichiro Takeda. The regularized siegel–weil formula (the second term identity) and the rallis inner product formula. *Inventiones mathematicae*, 198:739–831, 2014.

[Gro86] Benedict Gross. On canonical and quasi-canonical liftings. *Inventiones mathematicae*, 84(2):321–326, 1986.

[GS19] Luis E. Garcia and Siddarth Sankaran. Green forms and the arithmetic Siegel-Weil formula. *Invent. Math.*, 215(3):863–975, 2019.

[GY00] Wee Teck Gan and Jiu-Kang Yu. Group schemes and local densities. *Duke Mathematical Journal*, 105(3):497–524, 2000.

[Har13] Robin Hartshorne. *Algebraic geometry*, volume 52. Springer Science & Business Media, 2013.

[HLSY22] Qiao He, Chao Li, Yousheng Shi, and Tonghai Yang. A proof of the Kudla–Rapoport conjecture for Krämer models. *preprint*, 2022.

[How15] Benjamin Howard. Complex multiplication cycles and kudla-rapoport divisors, ii. *American Journal of Mathematics*, 137(3):639–698, 2015.

[How19] Benjamin Howard. Linear invariance of intersections on unitary rapoport–zink spaces. In *Forum Mathematicum*, volume 31, pages 1265–1281, 2019.

[HSY23a] Qiao He, Yousheng Shi, and Tonghai Yang. Kudla-Rapoport conjecture for Krämer models. *To appear in Compositio Mathematica*, 2023.

[HSY23b] Qiao He, Yousheng Shi, and Tonghai Yang. The Kudla-Rapoport conjecture at a ramified prime for $U(1,1)$. *Trans. AMS*, 376:2257–2291, 2023.

[Ich04] Atsushi Ichino. A regularized siegel-weil formula for unitary groups. *Mathematische Zeitschrift*, 247:241–277, 2004.

[Ike96] Tamotsu Ikeda. On the residue of the eisenstein series and the siegel-weil formula. *Compositio Mathematica*, 103(2):183–218, 1996.

[Jac62] Ronald Jacobowitz. Hermitian forms over local fields. *American Journal of Mathematics*, 84(3):441–465, 1962.

[Kat99] Hidenori Katsurada. An explicit formula for siegel series. *American Journal of Mathematics*, 121(2):415–452, 1999.

[Kit83] Yoshiyuki Kitaoka. A note on local densities of quadratic forms. *Nagoya Mathematical Journal*, 92:145–152, 1983.

[KM90] Stephen S. Kudla and John J. Millson. Intersection numbers of cycles on locally symmetric spaces and Fourier coefficients of holomorphic modular forms in several complex variables. *Inst. Hautes Études Sci. Publ. Math.*, (71):121–172, 1990.

[KR88a] SS Kudla and S Rallis. On the siegel-weil formula: the isotropic convergent case. *Crelle*, 391:65–84, 1988.

[KR88b] Stephen Kudla and Stephen Rallis. On the weil-siegel formula. *J. reine angew. Math*, 387(1-68):33, 1988.

[KR94] Stephen S Kudla and Stephen Rallis. A regularized siegel-weil formula: the first term identity. *Annals of Mathematics*, 140(1):1–80, 1994.

[KR00] Stephen S. Kudla and Michael Rapoport. Height pairings on Shimura curves and p-adic uniformization. *Invent. Math.*, 142(1):153–223, 2000.

[KR11] Stephen S. Kudla and Michael Rapoport. Special cycles on unitary Shimura varieties I. Unramified local theory. *Invent. Math.*, 184(3):629–682, 2011.

[KR14a] Stephen Kudla and Michael Rapoport. An alternative description of the drinfeld p-adic half-plane. In *Annales de l'Institut Fourier*, volume 64, pages 1203–1228, 2014.

[KR14b] Stephen S. Kudla and Michael Rapoport. Special cycles on unitary Shimura varieties II: Global theory. *J. Reine Angew. Math.*, 697:91–157, 2014.

[Krä03] Nicole Krämer. Local models for ramified unitary groups. In *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, volume 73, pages 67–80. Springer, 2003.

[Kud97] Stephen S. Kudla. Algebraic cycles on Shimura varieties of orthogonal type. *Duke Math. J.*, 86(1):39–78, 1997.

[Kud04] Stephen S. Kudla. Special cycles and derivatives of Eisenstein series. In *Heegner points and Rankin L-series*, volume 49 of *Math. Sci. Res. Inst. Publ.*, pages 243–270. Cambridge Univ. Press, Cambridge, 2004.

[Li23] Chao Li. From sum of two squares to arithmetic siegel-weil formulas. *Bulletin of the American Mathematical Society*, 60(3), 2023.

[Liu11] Yifeng Liu. Arithmetic theta lifting and L-derivatives for unitary groups, I. *Algebra Number Theory*, 5(7):849–921, 2011.

[LL21] Chao Li and Yifeng Liu. Chow groups and L-derivatives of automorphic motives for unitary groups. *Ann. of Math. (2)*, 194(3):817–901, 2021.

[LL22] Chao Li and Yifeng Liu. Chow groups and L-derivatives of automorphic motives for unitary groups, II. In *Forum of Mathematics, Pi*, volume 10. Cambridge University Press, 2022.

[LZ22a] Chao Li and Wei Zhang. Kudla–Rapoport cycles and derivatives of local densities. *Journal of the American Mathematical Society*, 35(3):705–797, 2022.

[LZ22b] Chao Li and Wei Zhang. On the arithmetic Siegel–Weil formula for GSpin Shimura varieties. *Inventiones mathematicae*, 228(3):1353–1460, 2022.

[Mih22] Andreas Mihatsch. Relative unitary RZ-spaces and the arithmetic fundamental lemma. *Journal of the Institute of Mathematics of Jussieu*, 21(1):241–301, 2022.

[Pap00] Georgios Pappas. On the arithmetic moduli schemes of pel shimura varieties. *Journal of Algebraic Geometry*, 9(3):577, 2000.

[Qin06] Liu Qing. *Algebraic geometry and arithmetic curves.* Oxford University Press, 2006.

[RSZ18] Michael Rapoport, Brian Smithling, and Wei Zhang. Regular formal moduli spaces and arithmetic transfer conjectures. *Mathematische Annalen*, 370(3-4):1079–1175, 2018.

[RSZ20] M. Rapoport, B. Smithling, and W. Zhang. Arithmetic diagonal cycles on unitary Shimura varieties. *Compos. Math.*, 156(9):1745–1824, 2020.

[RSZ21] Michael Rapoport, Brian Smithling, and Wei Zhang. On Shimura varieties for unitary groups. *Pure and Applied Mathematics Quarterly*, 17(2):773–837, 2021.

[RTW14] Michael Rapoport, Ulrich Terstiege, and Sean Wilson. The supersingular locus of the Shimura variety for $\mathrm{GU}(1, n-1)$ over a ramified prime. *Mathematische Zeitschrift*, 276(3-4):1165–1188, 2014.

[RTZ13] Michael Rapoport, Ulrich Terstiege, and Wei Zhang. On the arithmetic fundamental lemma in the minuscule case. *Compositio Mathematica*, 149(10):1631–1666, 2013.

[RZ96] M Rapoport and Thomas Zink. *Period spaces for p-divisible groups*. Number 141. Princeton University Press, 1996.

[SABK94] Christophe Soulé, Dan Abramovich, JF Burnol, and JK Kramer. *Lectures on Arakelov geometry*. Number 33 in Cambridge Studies in Advanced Mathematics. Cambridge university press, 1994.

[San13] Siddarth Sankaran. Unitary cycles on shimura curves and the shimura lift i. *Documenta Mathematica*, 18:1403–1464, 2013.

[San17] Siddarth Sankaran. Improper intersections of kudla–rapoport divisors and eisenstein series. *Journal of the Institute of Mathematics of Jussieu*, 16(5):899–945, 2017.

[Shi18] Yousheng Shi. Special cycles on the basic locus of unitary Shimura varieties at ramified primes. *To appear in Algebra & Number Theory*, 2018.

[Shi22] Yousheng Shi. Special cycles on unitary Shimura curves at ramified primes. *Manuscripta Mathematica*, pages 1–70, 2022.

[Sie35] Carl Ludwig Siegel. Über die analytische Theorie der quadratischen Formen. *Ann. of Math. (2)*, 36(3):527–606, 1935.

[Sie51] Carl Ludwig Siegel. Indefinite quadratische Formen und Funktionentheorie. I. *Math. Ann.*, 124:17–54, 1951.

[Tam63] Tsuneo Tamagawa. On the ζ-functions of a division algebra. *Annals of Mathematics*, 77(2):387–405, 1963.

[Tat67] John T Tate. p-divisible groups. In *Proceedings of a conference on Local Fields*, pages 158–183. Springer, 1967.

[Ter10] Ulrich Terstiege. Intersections of special cycles on the shimura variety for $\mathrm{GU}(1,2)$. *Journal fur die Reine und Angewandte Mathematik*, 2013, 06 2010.

[VW11] Inken Vollaard and Torsten Wedhorn. The supersingular locus of the Shimura variety of $\mathrm{GU}(1, n-1)$ II. *Inventiones mathematicae*, 184(3):591–627, 2011.

[Wei65] André Weil. Sur la formule de Siegel dans la théorie des groupes classiques. *Acta Math.*, 113:1–87, 1965.

[Zha09] Wei Zhang. Modularity of generating functions of special cycles on shimura varieties. *Thesis (Ph.D.)-Columbia University*, 2009.

[Zha21] Wei Zhang. Weil representation and arithmetic fundamental lemma. *Annals of Mathematics*, 193(3):863–978, 2021.