HYPERKÄHLER METRICS ON FOCUS-FOCUS FIBRATIONS

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Abstract

In this thesis, we focus on the study of hyperkähler metric in four dimensional cases, and practice GMN's construction of hyperkähler metric on focus-focus fibrations.

We explicitly compute the action-angle coordinates on the local model of focus-focus fibration, and show its semi-global invariant should be harmonic to admit a compatible holomorphic 2-form. Then we study the canonical semi-flat metric on it. After the instanton correction inspired by physics, we get a family of generalized Ooguri-Vafa metric on focus-focus fibrations, which becomes more local examples of explicit hyperkähler metric in four dimensional cases.

In addition, we also make some exploration of the Ooguri-Vafa metric in the thesis. We study the potential function of the Ooguri-Vafa metric, and prove that its nodal set is a cylinder of bounded radius $1 < R < \infty$. As a result, we get that only on a finite neighborhood of the singular fibre the Ooguri-Vafa metric is a hyperkähler metric. Finally, we give some estimates for the diameter of the fibration under the Oogui-Vafa metric, which confirms that the Oogui-Vafa metric is not complete.

The new family of metric constructed in the thesis, we think, will provide more examples to further study of Lagrangian fibrations and mirror symmetry in future.

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Chapter 1

Introduction

1.1 Background

The well-known Calabi conjecture solved by Yau [42] in 1978 promises that given any elliptic K3 surface there always exists a unique hyperkähler metric in each given Kähler class. After this fundamental existence result, it has been an open problem to write down the explicit expression of such metrics.

Motivated by the celebrated Strominger-Yau-Zaslow conjecture [38], nowadays it is a folklore that the hyperkähler metrics near large complex limits are approximated by semi-flat metrics with instanton correction from the holomorphic discs with boundaries on special Lagrangian fibres [12]. The semi-flat metric is written down under the special Lagrangian fibration setting in [17]. Later, Gross and Wilson [20] proved that such hyperkähler metrics indeed can be approximated by the semi-flat metrics glued with generalized Ooguri-Vafa metrics around each singular fibre. However, in this procedure the instanton corrections are not included.

Recently, Gaiotto, Moore and Neitzke make a significant breakthrough and propose a new approach on this problem with the instanton corrctions in their papers [14] [15]. It brings lots of new ingredients into the field, which includes: Kontsevich-Soibelman wall-crossing formula on BPS states or generalized Donaldson-Thomas invariants [26], and construction of twistor spaces of hyperkähler metrics [22] from associated Riemann-Hilbert problems determined by the wall-crossing data.

In this thesis, we try to practice the GMN's construction in one of the important cases of completely integrable systems: focus-focus fibration. Following Vũ Ngọc's classification result on focus-focus fibration [34], here we view the local model in [34] as a total neighborhood of a *type I* or A_1 singular fibre in the special Lagrangian fibration of an elliptic K3 surface (up to some symplectomorphism). We adapt GMN's construction as outlined in [33] on the local model, and study the explicit hyperkähler metric on it.

1.2 Outlines

The thesis is organized as follows:

First, we study our local model and state Vũ Ngọc's classification result on focus-focus fibrations, and then explore its explicit action-angle coordinates. As an important example, we also calculate the action-angle coordinates for Ooguri-Vafa case.

Second, we follow Arnold's integration over vanishing cycle technique [1], show the semi-global invariant S introduced by Vũ Ngọc in [34] on the local model should be a harmonic function to admit a compatible holomorphic 2-form.

Then, we study the canonical semi-flat metric on the regular part of local model. We show that under certain conditions on the semi-global invariant S, such semi-flat metric will become a real hyperkähler metric on the regular part of the fibration. However, such metric generally cannot be extended over the singular fibre.

Finally, we apply the GMN ansatz to modify the semi-flat metric to a global metric with central fibre completion. In stead of making the modification on the metric directly, we consider the associated twistor space and translate the problem into a Riemann-Hilbert problem on the holomorphic Darboux coordinates of holomorphic 2-forms. Then we follow the GMN integral ansatz to solve the Riemann-Hilbert problem and construct the modified twistor space. From the twistor space, we achieve the final modified metric. It turns out to be the generalized Ooguri-Vafa metric with similar extra harmonic term in the potential function as used in Gross-Wilson's work on hyperkähler metric[20].

As a detailed review and also further exploration, we study the geometry of the Ooguri-Vafa metric in the last part. We start with the classical Gibbons-Hawking ansatz, and analyze the periodic Gibbons-Hawking ansatz, and then the Ooguri-Vafa metric. Based on this, we make some geometric estimates of the fibration as Gross-Wilson did in their work.

In the appendix part, we give the Fourier expansion of the Ooguri-Vafa potential and also the curvature calculation of the Gibbons-Hawking ansatz for further interests.

Chapter 2

Local Models

In this chapter, we first state the local model of focus-focus fibrations, which will be the main object for our further study. Then we study the important action-angle coordinates for the local model. Generally for dynamic systems, it is not easy to get the explicit expression of the coordinates. Here we will give an explicit expression in the focus-focus case. As a further example, we will also explore that for Ooguri-Vafa case, and make some interesting observations.

2.1 Focus-focus fibrations

Definition 2.1 *A Lagrangian fibration* $f : (M, \omega) \to B \in \mathbb{R}^2$ *is called a focus-focus fibration if: (1) each fibre is compact; (2) the central fibre* $\pi^{-1}(0)$ *is the unique singular fibre, which has one* A_1 *singularity, i.e. the central fibre is a pinched torus.*

Definition 2.2 *Two focus-focus fibrations* $f_i : (M_i, \omega_i) \to B_i$ *with* $i = 1, 2$ *are called equivalent if there exist subsets* $\widetilde{B}_i \subset B_i$ *such that we have the following bundle symplectomorphism:*

$$\begin{array}{ccc} (M_1|_{\widetilde{B}_1}, \omega_1) & \xrightarrow{F} & (M_2|_{\widetilde{B}_2}, \omega_2) \\ \downarrow f_1 & & \downarrow f_2 \\ \widetilde{B}_1 & \xrightarrow{g} & \widetilde{B}_2 \end{array}$$

For the further construction, we need a quick review of the local model of focus-focus fibration, and also the classification result of focus-focus fibration founded by Vũ Ngọc [34] here.

Focus-focus Singularity. Take the space $W = \mathbb{R}^2 \times \mathbb{R}^2$, with the symplectic structure:

$$\omega_{can} = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$$

We consider the following Lagrangian fibration with isolated singularity, namely the focus-focus singularity:

$$\pi_{can} : W \longrightarrow \mathbb{R}^2$$

$$\pi_{can}(x_1, y_1; x_2, y_2) = (\pi_1, \pi_2) = (x_1 y_1 + x_2 y_2, x_1 y_2 - x_2 y_1)$$

If we take an auxiliary complex structure J_{au} on W with:

$$z_1 = x_1 - iy_1, \quad z_2 = x_2 + iy_2$$

Then the symplectic structure becomes: $\omega_{can} = Re(dz_1 \wedge dz_2)$, and consequently the fibration simply becomes:

$$\pi_{can} : W \longrightarrow \mathbb{C}$$

$$\pi_{can}(z_1, z_2) = z_1 z_2$$

$$with : \pi_1 = Re(z_1 z_2), \quad \pi_2 = Im(z_1 z_2)$$

As a completely integral system, here $\{\pi_i\}$ induce independent hamitonian flows on W. Under the auxiliary complex coordinates, they can be simply written as:

$$\phi_1^t(z_1, z_2) = (e^t \cdot z_1, e^{-t} \cdot z_2)$$

$$\phi_2^t(z_1, z_2) = (e^{-it} \cdot z_1, e^{it} \cdot z_2)$$

Gluing Procedure. We denote the space of smooth functions on $\mathbb{R}^2$ with vanishing value at 0 by $\mathbb{R}[[x,y]]_0$. It will be our classification space. For any $S \in \mathbb{R}[[x,y]]_0$, we denote its partial derivatives by S_1 and S_2. Then we take two Poincare surfaces in W as follows:

$$\Pi_1 = \{(c,1) \mid |c| < \epsilon\}$$
$$\Pi_2 = \{(e^{S_1(c)-iS_2(c)}, c \cdot e^{-S_1(c)+iS_2(c)}) \mid |c| < \epsilon\}$$

Here $\{\Pi_i\}$ are smooth surfaces constructed in such a way that for any $c \neq 0$, Π_2 is the image of Π_1 by the joint flow of (π_1, π_2) at the time $(S_1 - \ln|c|, S_2 + \arg(c))$.

Consider the S^1-orbit of Π_i under the ϕ_2 flow, denoted by $\phi_2(\Pi_i)$. We use the symplectomorphism induced by the joint flow to glue collar neighborhoods of $\phi_2(\Pi_i)$ inside each torus $\pi^{-1}(c)$, that is:

$$\psi : \phi_2(\Pi_1) \longrightarrow \phi_2(\Pi_2)$$
$$\psi(z_1, z_2) = (e^{S_1(c)-iS_2(c)} \cdot z_2^{-1}, e^{-S_1(c)+iS_2(c)} \cdot z_1 z_2^2)$$

Notice that the gluing is carried out on each Lagrangian fibre. After the gluing procedure, each regular fibre $\pi_{can}^{-1}(c)$ with $c \neq 0$ becomes a compact torus, and the central fibre $\pi_{can}^{-1}(0)$ becomes a pinched torus.

Now let us denote the space after the gluing procedure by $(\widetilde{W}, \omega_{can}, S)$. Then we are ready to state the classification result:

Theorem 2.3 ([34]) *The equivalent classes of focus-focus fibration are classified by the local model:*

$$\pi_{can} : (\widetilde{W}, \omega_{can}, S) \longrightarrow B = \{c \mid |c| < \epsilon\}$$

with the classification space $\{S \mid S \in \mathbb{R}[[x,y]]_0\}$.

Remark. Since the classification space $\mathbb{R}[[x,y]]_0$ is path connected, by the standard Moser's trick, we will get all the local models with the same base are symplectomorphic to each other.

Semi-global Invariant. The classification data S above is also called the *semi-global invariant* of focus-focus fibration. It has the following geometric interpretation in each focus-focus fibration.

Given a focus-focus fibration $f : (M, \omega) \to B \in \mathbb{C} \cong \mathbb{R}^2$. Let us take $\{\gamma_e, \gamma_m\}$ as the generators of $H_1(\pi^{-1}(c))$. If we consider the action integral (central charge) along the 1-cycle:

$$z_{\gamma_m}(c) = \frac{1}{2\pi}\int_{\gamma_m} \alpha, \quad z_{\gamma_e}(c) = \frac{1}{2\pi}\int_{\gamma_e} \alpha$$

where α is any 1-form on some neighbourhood of $\pi^{-1}(c)$ in $\widetilde{W}$ such that $d\alpha = \omega$ (which always exists since $\pi^{-1}(c)$ is Lagrangian). Then the semi-global invariant S can be interpreted as a regularised action integral:

$$S(c) = 2\pi \cdot [z_{\gamma_m}(c) - z_{\gamma_m}(0)] + Re(c \ln c - c). \qquad (*)$$

Notice that the classification is purely about the Lagrangian fibration structure. The auxiliary complex structure J_{aux} used above is not necessary compatible with the gluing. In fact, we have the following result:

Lemma 2.4 *The auxiliary complex structure J_{au} is compatible with the gluing if and only if the semi-global invariant S is harmonic.*

Proof 2.5 *Recall that by definition, we have $S_i(c) = S_i(\frac{z_1z_2+\overline{z_1z_2}}{2}, \frac{z_1z_2-\overline{z_1z_2}}{2i})$. We consider* *the Cauchy-Riemann equation for the gluing maps, that is:*

$$\frac{\partial}{\partial \bar{z}_1}(e^{S_1 - iS_2}) = 0, \quad \frac{\partial}{\partial \bar{z}_2}(e^{S_1 - iS_2}) = 0$$

Such equations can be simplified to: $S_{11} + S_{22} = 0$. *Thus we get the proof.*

2.2 Action-Angle coordinates

Now we study the action-angle coordinates of the focus-focus fibration. Let us take a local model $(\widetilde{W}, \omega_{can}, S)$, then denote the punctured disc by B_0, and restricted fibration over punctured disc B_0 by $\widetilde{W}_0$. We will call $\widetilde{W}_0$ the regular part of the fibration in the later discussion.

Following the general strategy, we pick a Lagrangian section of the fibration and then use Hamiltonian flows to construct the coordinates.

Recall that from the gluing construction we have the local model given as:

$$\widetilde{W} = W/(\Pi_1 \sim \Pi_2)$$

Here the two Poincare surfaces are chosen as:

$$\Pi_1 = \{(c, 1) \mid |c| < \epsilon < 1\}$$
$$\Pi_2 = \{(e^{S_1(c) - iS_2(c)}, c \cdot e^{-S_1(c) + iS_2(c)}) \mid |c| < \epsilon < 1\}$$

We take a simple Lagrangian section as another initial data:

$$\Gamma(c) = (c, 1)$$

Then we follow the standard procedure in [9] to construct the action-angel coordinates. Using the Hamiltonian flows $\{\phi_i^t\}$ with $\Gamma(c)$ as the initial level set, we get a parametrization of $\widetilde{W}_0$ as follows:

$$\begin{array}{ccc} B_0 \times \mathbb{R}/L & \xrightarrow{T} & (\widetilde{W}_0, \omega_{can}, S) \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_{can}} \\ B_0 & \xrightarrow{id} & B_0 \end{array}$$

$$T(c; t_1, t_2) = \phi_1^{-t_1} \circ \phi_2^{-t_2}(c, 1) = (c \cdot e^{-t_1 + i \cdot t_2}, e^{t_1 - i \cdot t_2})$$

Here the period lattice is $L =< (S_1 - \ln|c|, S_2 + \arg c), (0, 2\pi) >$ from the above gluing construction, which should be normalized to achieve the angle coordinates.

Remark. Notice here $\arg c$ or $\ln c$ is not globally defined function on B_0. To clarify the affine coordinates on the base, we need at least two affine charts, with different choice of branches of the $\arg c$ or $\ln c$ function. In our calculation, we will skip this part, and formally use $\arg c$ or $\ln c$ directly if no confusion happens.

From the relation between the action integral and the semi-global invariant above, we have the following identity:

Theorem 2.6 *Given the Lagrangian section $\Gamma(c)$ as the initial level set, we have the action-angle coordinates on $\widetilde{W}_0$ as follows:*

$$z_{\gamma_m} = \frac{1}{2\pi}[-\ln|c| \cdot c_1 + \arg c \cdot c_2 + c_1 + S], \quad z_{\gamma_e} = c_2$$

$$\theta_{\gamma_e} = \frac{2\pi \cdot t_1}{S_1 - \ln|c|}, \qquad \theta_{\gamma_m} = t_2 - \frac{S_2 + \arg c}{S_1 - \ln|c|} \cdot t_1$$

i.e. $$T^*(\omega_{can}) = dz_{\gamma_m} \wedge d\theta_{\gamma_e} + dz_{\gamma_e} \wedge d\theta_{\gamma_m}$$

Proof 2.7 *It is a direct calculation to get the identity:*

$$T^*(\omega_{can}) = dc_1 \wedge dt_1 + dc_2 \wedge dt_2$$

Moreover, from the gluing procedure in the local model, we can write down the action integrals directly:

$$z_{\gamma_m} = \frac{1}{2\pi}\left(-\ln|c| \cdot c_1 + \arg c \cdot c_2 + c_1 + S\right), \quad z_{\gamma_e} = c_2$$

Consequently, we are able to figure out the frequency data. Recall we have the implicit relations:

$$c_1 = c_1(z_{\gamma_m}, z_{\gamma_e}), \quad c_2 = c_2(z_{\gamma_m}, z_{\gamma_e})$$

Compute the implicit derivatives, then we will get the frequency data:

$$\omega_{1,1} = \frac{\partial c_1}{\partial z_{\gamma_m}} = \frac{2\pi}{S_1 - \ln|c|}, \quad \omega_{1,2} = \frac{\partial c_1}{\partial z_{\gamma_e}} = -\frac{S_2 + \arg c}{S_1 - \ln|c|}$$

Similarly,

$$\omega_{2,1} = \frac{\partial c_2}{\partial z_{\gamma_m}} = 0, \quad \omega_{2,2} = \frac{\partial c_2}{\partial z_{\gamma_e}} = 1$$

Thus we get the angle coordinates:

$$\theta_{\gamma_e} = \omega_{1,1} \cdot t_1 + \omega_{2,1} \cdot t_2 = \frac{2\pi \cdot t_1}{S_1 - \ln|c|}$$
$$\theta_{\gamma_m} = \omega_{1,2} \cdot t_1 + \omega_{2,2} \cdot t_2 = t_2 - \frac{S_2 + \arg c}{S_1 - \ln|c|} \cdot t_1$$

Finally follow the dynamic identity of the integrable system, we arrive at:

$$dc_1 \wedge dt_1 + dc_2 \wedge dt_2 = dz_{\gamma_m} \wedge d\theta_{\gamma_e} + dz_{\gamma_e} \wedge d\theta_{\gamma_m}$$

That finishes the proof of identities in the lemma.

Notice that$(dz_{\gamma_m} \wedge d\theta_{\gamma_e} + dz_{\gamma_e} \wedge d\theta_{\gamma_m})$ is invariant under the gluing determined by the period lattice L, and also the monodromy transformation of z_{γ_m} and θ_{γ_e}. Thus the action-angle coordinates above is well defined. Moreover, under the angle coordinates

$\{\theta_{\gamma_m}, \theta_{\gamma_e}\}$, the period lattice becomes the standard one: $<(2\pi, 0), (0, 2\pi)>$.

Remark. Notice that generally the action-angle coordinates is not unique, different choice of Lagrangian section as the zero level set of the Hamiltonian flow may give us different angle coordinates.

2.3 Examples

There are lots of interesting examples of focus-focus fibration studied in different fields. However due to the complexity of elliptic integral generally the action-angle coordinates and thus the semi-global invariant is not easy to calculate. We discuss several cases here.

Example.1. Spherical pendulum is a famous example equipped with the focus-focus fibration structure. The action integrals and also the semi-global invariant is recently calculated by Dullin in [10].

Example.2. The Ooguri-Vafa space $M_{O.V.}$ is also an important case of focus-focus fibration. Geometrically it is a S^1 bundle over $\mathbb{R}^2 \times S^1$ with first chern class ± 1 as constructed in [35]. Follow the Gibbons-Hawking ansatz [16], we choose the following sympletic form on $M_{O.V.}$ (which is a rescaling by -2π of the standard one):

$$\omega_0 = -2\pi \cdot [dc_2 \wedge (\frac{d\theta_m}{2\pi} + A_0) + V_0 d(\frac{\theta_e}{2\pi R}) \wedge dc_1]$$

Here the Lagrangian fibration is given by:

$$f : M_{O.V.} \longrightarrow \mathbb{R}^2$$

$$f(c_1, c_2; \theta_e, \theta_m) = (c_1, c_2)$$

Recall that the standard Ooguri-Vafa potential [35] is given as:

$$V_{O.V.} = \frac{R}{4\pi} \cdot \sum_{n \in \mathbb{Z}} [\frac{1}{\sqrt{R^2|c|^2 + (\frac{\theta_e}{2\pi} + n)^2}} - \kappa(n)]$$

with the regularization terms: $\kappa(0) = 0$, and $\kappa(n) = \frac{1}{|n|}$ if $n \neq 0$ for the convergence consideration.

Here we make some generalization and choose the following potential functions for the symplectic structure ω_0:

$$V_0 = \frac{R}{4\pi} \cdot \left(\sum_{n \in \mathbb{Z}} [\frac{1}{\sqrt{R^2|c|^2 + (\frac{\theta_e}{2\pi} + n)^2}} - \kappa(n)] + 2S_1(c_1, c_2) \right)$$

As above, here S is any smooth harmonic function with $S(0) = 0$, and $S_i = \frac{\partial}{\partial c_i} S$. It satisfies the following positivity condition along the θ_e-axis:

$$S_1(0) > -\min_{\theta_e \in [0, 2\pi]} \frac{1}{2} \sum_{n \in \mathbb{Z}} [\frac{1}{|\frac{\theta_e}{2\pi} + n|} - \kappa(n)] \qquad (**)$$

Then V is still a local positive harmonic function with one singularity at origin. The connection 1-form is given from the standard relation: $dA_0 = *dV_0$.

Property 2.8 *The action-angle coordinates on $(M_{O.V.}, \omega_0)$ with respect to the above Lagrangian fibration can be given by:*

$$\omega_0 = dz_m \wedge d\tilde{\theta}_e + dz_e \wedge d\tilde{\theta}_m$$

with:

$$z_m = \frac{1}{2\pi} \cdot [Re(c - c\ln c) + S], \qquad z_e = c_2$$
$$\tilde{\theta}_e = \theta_e + \frac{2\pi \cdot R\sigma}{S_1 - \ln|c|}, \qquad \tilde{\theta}_m = -\theta_m - \frac{S_2 + \arg c}{S_1 - \ln|c|} \cdot R\sigma$$

here when $c \neq 0$, the angle correction term is given by:

$$\sigma = \int V_0^{inst} d\theta_e = \frac{1}{2\pi} \sum_{n\neq 0} \frac{1}{i\cdot n} e^{i\cdot n\theta_e} K_0(2\pi|nc|) + C$$

Proof 2.9 *It mainly comes from the calculation of the action integrals. Notice that ω^{sf} and ω share the same action integrals [5] [14]. It is a special property comes from the Gibbons-Hawking ansatz construction.*

Recall that from Fourier expansion, the semi-flat or zero mode part of the potential is simply given as:

$$V_0^{sf} = -\frac{R}{4\pi}(\ln c + \ln \bar{c} - 2S_1)$$

Consequently, we have the semi-flat or zero mode part of the connection 1-form:

$$A_0^{sf} = \frac{i}{8\pi^2}(\ln c - \ln \bar{c} + 2i \cdot S_2) d\theta_e$$

Then from direct calculation, we have the action-angle coordinates for the semi-flat part:

$$\begin{aligned} \omega_0^{sf} &= -2\pi \cdot [dc_2 \wedge (\frac{d\theta_m}{2\pi} + A_0^{sf}) + V_0^{sf} d(\frac{\theta_e}{2\pi R}) \wedge dc_1] \\ &= d\left[\frac{1}{2\pi} \cdot Re(c - c\ln c) + \frac{1}{2\pi} \cdot S\right] \wedge d\theta_e + dc_2 \wedge d(-\theta_m) \end{aligned}$$

Now let us take the instanton part into account. Recall that the instanton part $\omega_0^{inst} = \omega_0 - \omega_0^{sf}$ is similarly determined by the instanton part of the potential and connection

1-form:

$$
\begin{aligned}
V_0^{inst} &= \frac{R}{2\pi}\sum_{n\neq 0} e^{i\cdot n\theta_e} K_0(2\pi|nc|) \\
A_0^{inst} &= -\frac{R}{4\pi}\left(\frac{dc}{c}-\frac{d\bar{c}}{\bar{c}}\right)\sum_{n\neq 0} sign(n)\cdot e^{i\cdot n\theta_e}|c|K_1(2\pi|nc|) \\
\omega_0^{inst} &= -2\pi\cdot[dc_2\wedge A_0^{inst} + V_0^{inst} d(\frac{\theta_e}{2\pi R})\wedge dc_1]
\end{aligned}
$$

From property of the Bessel function, we get the following action-angle coordinates for the instanton part:

$$
\omega_0^{inst} = d\left[\frac{1}{2\pi}\cdot Re(c - c\ln c) + \frac{1}{2\pi}\cdot S\right]\wedge d\left(\frac{2\pi\cdot R\sigma}{S_1-\ln|c|}\right) + dc_2\wedge d\left(-\frac{S_2+\arg c}{S_1-\ln|c|}\cdot R\sigma\right)
$$

Add ω_0^{sf} and ω_0^{inst} together, we finish the proof of the lemma.

Notice that ω_0^{sf} and ω_0 share the same action coordinates but different angle coordinates. In the Ooguri-Vafa space, θ_e and θ_m are global coordinates away from the singular point, while the angle coordinates $\tilde{\theta}_e$ and $\tilde{\theta}_m$ generally are only defined on the regular part of the fibration or away from the singular fibre. The above formula in the Ooguri-Vafa case indicates a way to deform the angle coordinates to make them extendable over the singular fibre, which might be helpful in other geometry cases.

In addition, the formula also shows that in the Ooguri-Vafa case, the instanton correction only contributes to the deformation of the angle coordinates, which comes in the form of an infinite series labeled by wrapping number n as explained in [35].

Example.3. Another example of focus-focus fibration comes from the famous special Lagrangian fibration model [19] used in the study of mirror symmetry and wall crossing phenomena:

$$
\pi : \mathbb{C}^2 - \{z_1 z_2 + 1 = 0\} \longrightarrow \mathbb{R}^2
$$

$$\pi(z_1, z_2) = (\ln|1 + z_1 z_2|, \frac{|z_1|^2 - |z_2|^2}{2})$$

Here the symplectic structure on $\mathbb{C}^2 - \{z_1 z_2 + 1 = 0\}$ is not the normal one, which is given as:

$$\omega = \frac{i}{2}\frac{1}{|1 + z_1 z_2|}\sum_{j=1}^{2} dz_j \wedge d\bar{z}_j$$

It will be quite interesting to study the dynamic system for this fibration, and find its action integral and moreover the semi-global invariant.

Chapter 3

Construction of Metrics

In this chapter, we construct hyperkähler metrics on focus-focus fibrations. Motivated by the equivalent relation between hyperkähler manifold and its twistor space, we focus on the construction of twistor space. We start with the study of holomorphic 2-form, and then adapt the semi-flat metric inspired by physics, finally use the GMN ansatz to derive the metric which could smoothly extend to the whole fibration. After that, we provide some interesting discussion about the metric and also the whole construction.

3.1 Hyperkähler metric and twistor space

Definition 3.1 *A hyperkähler manifold is a closed smooth Riemannian manifold M, g with a triple of compatible complex structure I, J and K which satisfy the quaternionic relations: $I^2 = J^2 = K^2 = IJK = -Id$.*

Notice that I, J and K give each tangent space the structure of a quaternionic vector space, so the real dimension of a hyperkähler manifold must be divisible by 4. Since I, J and K are covariantly constant, a parallel transport commutes with the quaternionic multiplication and thus the holonomy group is contained in $O(4n) \cap GL_n(\mathbf{H}) \cong Sp(n)$.

In particular, since $Sp(n) \subseteq SU(2n)$ every hyperkähler manifold is Calabi-Yau. In addition, when $n = 1$, we have $Sp(1) \cong SU(2)$, so every Calabi-Yau surface is hyperkähler. The only compact examples are T^4 and K3 surfaces.

From another point of view, if we consider the triple kähler structures induced by I, J and K: $\omega_I, \omega_J, \omega_K$, and their complex linear combination, then we will get into more interesting geometry structures. For example, it is easy to check $\Omega_I = (\omega_J + i \cdot \omega_K)$ will become a holormorphic 2-form with respect to complex structure I. In fact, we have a S^2-family of such structures. From later discussion we will see, they would be essential important data to a hyperkähler manifold.

Notice that for any $u = (u_1, u_2, u_3) \in \mathbf{R}^3$, if we consider $I_u = u_1 I + u_2 J + u_3 K$, then we have:

$$I_u^2 = -(u_1^2 + u_2^2 + u_3^2) \cdot Id.$$

So as long as $u \in S^2$, I_u will become a complex structure on M, which will also induce some complex structure on $M \times S^2$.

Definition 3.2 *The manifold $Z = M \times S^2$ with the complex structure given as:*

$$I_{p,u}(v, w) = (I_u v, I_0 w),$$

where $(p, u) \in Z$, $v \in T_p M$, $w \in T_u S^2$, and I_0 is the natural complex structure on $S^2 \cong \mathbb{C}P^1$, is called the twistor space of the hyperkähler manifold M.

From the above construction, we will get the following structures on the twistor space of a hyperkähler manifold.

Theorem 3.3 *The twistor space Z of a hyperkähler manifold M^{2n} is a complex manifold with the following structures:*

(1) a holomorphic fibration $p : Z \to \mathbb{C}P^1$*,*

(2) a real structure $\sigma : Z \to Z$ *covering the antipodal map of* $\mathbb{C}P^1$*,*

(3) a section ϖ *of* $\bigwedge^2 T^*_{Z/\mathbb{C}P^1} \otimes p^*\mathcal{O}(2)$ *which is real with respect to* σ*, and defines a non-degenerate holomorphic 2-form on each fiber,*

(4) a family of holomorphic sections on $p : Z \to \mathbb{C}P^1$*, which are real with respect to* σ*, and whose normal bundles are all isomorphic to* $\mathbb{C}^{2n} \otimes_{\mathbb{C}} p^*\mathcal{O}(1)$*.*

Moreover, from the twistor space we are able to recover the original hyperkähler manifold in the following sense.

Theorem 3.4 ([22]) *Suppose a complex manifold* Z *of dimension* $2n+1$ *has the structures (1)-(4) given by the previous theorem. Then the parameter space of real sections (4) is a hyperkähler manifold whose twistor space is Z. It's hyperkähler metric can be derived from the section of (3).*

In practice, such theorems provide us with another approach to construct examples of hyperkähler metrics. Instead of the explicit metric, first we construct a twistor space of the underlying manifold, and then use Hithcin's machine to recover its corresponding hyperkähler metric. That is the key idea of GMN ansatz, and of our work here.

3.2 Holomorphic 2-form

Motivated by study of the hyperkähler metric on (local) elliptic K3, here we study holomorphic 2-form or holomorphic symplectic structure on the local model of focus-focus fibration.

Recall that on elliptic K3 surface X with the hyperkähler data (ω, J, Ω), following the standard hyperkähler rotation, we can always transfer the elliptic fibration structure into a Lagrangian fibration structure with respect to the symplectic form $Re(\Omega)$ or $Im(\Omega)$. Generally we will take local elliptic K3, denoted by M, as a total neighborhood of an A_1 singular fibre in such fibration. Then equipped with the restricted geometry data, M acquires a focus-focus fibration structure with respect to the symplectic structure $Re(\Omega|_M)$ or $Im(\Omega|_M)$.

From classification result [34], we know such Lagrangian fibration is also equivalent to certain local model of focus-focus fibration. Therefore we would like to study similar geometric structure directly on the local model of focus-focus fibration, which can be viewed as the pull back of the geometric structure from local elliptic K3 through the corresponding bundle symplectomorphism.

Based on Andreotti's observation [23] about holomorphic 2-form on K3 surface, we make the following definition on the local model:

Definition 3.5 *We call a 2-form* Ω *on the local model* $(\widetilde{W}, \omega_{can}, S)$ *is a compatible holomorphic 2-form, if it has the following specialty properties:*

1) $\omega_{can} = Re(\Omega)$*;*

2) $d\Omega = 0, \quad \Omega \wedge \Omega = 0, \quad \Omega \wedge \overline{\Omega} > 0$*;*

3) the fibration $\pi_{can} : \widetilde{W} \to B$ *becomes an elliptic fibration with respect to the complex structure determined by 2).*

Notice that from Andreotti's argument, in fact any 2-form Ω with property 2) will determine a unique complex structure J_0 such that Ω becomes a holomorphic 2-form with respect to J_0. Thus no ambiguity would happen in our definition.

Now let us consider the condition of existence of such 2-form on the local model. Suppose our local model $(\widetilde{W}, \omega_{can}, S)$ admits such a homomorphic 2-form Ω, and the fibration $\pi_{can} : (\widetilde{W}, J, \Omega) \to B$ becomes an elliptic fibration, then similarly as before, we have the action integral (central charge) along the 1-cycle:

$$Z_{\gamma_m}(c) = \frac{1}{2\pi}\int_{\gamma_m} \kappa, \quad Z_{\gamma_e}(c) = \frac{1}{2\pi}\int_{\gamma_e} \kappa$$

where κ is any 1-form on some neighbourhood of $\pi^{-1}(c)$ in $\widetilde{W}$ such that $d\kappa = \Omega$ (which always exists since $\pi^{-1}(c)$ is Lagrangian). By construction, we have the simple relation:

$$Re(Z_{\gamma_e}) = z_{\gamma_e}, \quad Re(Z_{\gamma_m}) = z_{\gamma_m}$$

Observations. The first observation comes from the result of *integral over vanishing cycles* in singularity theory [1]. Since the local model now is equipped with the structure of elliptic fibration with A_1 singularity, we have the property:

Lemma 3.6 *The action integrals $Z_{\gamma_e}, Z_{\gamma_m}$ are holomorphic functions on B_0. For the integral over vanishing cycle, we have the local expression:*

$$Z_{\gamma_m} = f(c) + g(c)\ln(c), \quad \forall c \in B_0$$

here f, g are local holomorphic functions defined near c. Consequently, the action integrals z_{γ_e} and z_{γ_m} are always harmonic functions on B_0.

Secondly, we have the simple but important identity:

$$c - c \cdot \ln c = (-\ln|c| \cdot c_1 + \arg c \cdot c_2 + c_1) + i \cdot (-\ln|c| \cdot c_2 - \arg c \cdot c_1 + c_2)$$

Thus, we arrive at the following statement about the homomorphic 2-form on the local model:

Corollary 3.7 *If a local model* $(\widetilde{W}, \omega_{can}, S)$ *admits a compatible holomorphic 2-form* Ω *as defined above, then:*

1) the semi-global invariant S *is harmonic,*

2) the action integral has the form:

$$Z_{\gamma_m} = \frac{1}{2\pi} \cdot [c - c \cdot \ln c + (S + i \cdot \widetilde{S})], \quad Z_{\gamma_e} = c_2 - i \cdot c_1 = -i \cdot c$$

here $\widetilde{S}$ *is conjugate harmonic function of* S*,*

3) the holomorphic 2-form Ω *is determined as:*

$$\Omega = dZ_{\gamma_m} \wedge d\theta_{\gamma_e} + dZ_{\gamma_e} \wedge d\theta_{\gamma_m} + i \cdot h(c_1, c_2) dc_1 \wedge dc_2$$

with the positive definite condition: $S_1 > \ln |c|$*. Here* $h(c_1, c_2)$ *is a smooth function.*

Proof 3.8 *The first two properties directly comes from the observations made above. We just check the three characteristic properties of two form* Ω *here.*

1) The closeness of Ω *is given directly.*

2) Notice that we have $(S + i \cdot \tilde{S})$ *as a homomorphic function, by its Cauchy-Riemann equation, we have the identity:*

$$dZ_{\gamma_m} \wedge dZ_{\gamma_e} = 0$$

which implies the second property $\Omega \wedge \Omega = 0$ *through a short calculation.*

3) After arrangement, we get the expression:

$$\begin{aligned} \Omega \wedge \overline{\Omega} &= (d\overline{Z}_{\gamma_e} \wedge dZ_{\gamma_m} + dZ_{\gamma_e} \wedge d\overline{Z}_{\gamma_m}) \wedge d\theta_{\gamma_e} \wedge d\theta_{\gamma_m} \\ &= \frac{2}{\pi} \cdot (S_1 - \ln |c|) dc_1 \wedge dc_2 \wedge d\theta_{\gamma_m} \wedge d\theta_{\gamma_e} \end{aligned}$$

therefore we arrive at the positivity condition: $S_1 > \ln |c|$*. The* h *term appears here since* $\theta_{\gamma_e}, \theta_{\gamma_m}$ *are not necessary angle coordinates for* $Im(\Omega)$ *here.*

Remark. Here the addition term $h(c_1, c_2)$ appears since we just know the fibration is Lagrangian with respect to $Im(\Omega)$. If we further have the section $\Gamma(c)$ is also Lagrangian with respect to $Im(\Omega)$, then the $h(c_1, c_2)$ term will vanish.

Generally it is not easy to write down the explicit expression of the complex structure J_0 determined by Ω. However from the observation in Lemma 2.4, we have the following result in the special case:

Corollary 3.9 *If a local model $(\widetilde{W}, \omega_{can}, S)$ admits a compatible holomorphic 2-form with $h = 0$, then we have the complex structure J_0 and the holomorphic 2-form Ω on $\widetilde{W}$ simply given by:*

$$J_0 = J_{au}, \qquad \Omega = dz_1 \wedge dz_2 = (dc_1 + i \cdot dc_2) \wedge (dt_1 - i \cdot dt_2)$$

Remark. Notice that in this case, J_0 and Ω is well defined on whole $\widetilde{W}$, not just on the regular part $\widetilde{W}_0$. Then the total space $\widetilde{W}$ can be well described as in [45].

3.3 Semi-flat metric

In this part, we study the canonical semi-flat metric [4] [33] on $\widetilde{W}_0$ constructed by the above action-angle coordinates. From now on, we just focus on the simple case with $h = 0$. Notice that in such cases the background complex structure is fixed to be $J_0 = J_{au}$.

Definition 3.10 *Given any $R \in \mathbb{R}_+$, the canonical semi-flat pseudo-metric on $(\widetilde{W}_0, \omega_{can}, S)$ is given by:*

$$\omega^{sf} = \pi R \cdot Re(dZ_{\gamma_m} \wedge d\overline{Z}_{\gamma_e}) + \frac{1}{2\pi R} \cdot d\theta_{\gamma_m} \wedge d\theta_{\gamma_e}$$

It is easy to check this canonical form ω^{sf} is compatible with the gluing, and invariant under the monodromy transformation. Moreover, we have the following properties of the canonical form:

Lemma 3.11

$$\omega^{sf} \wedge \omega^{sf} = \frac{1}{2}\Omega \wedge \overline{\Omega}, \quad \omega^{sf} \wedge \Omega = 0, \quad \omega^{sf} \wedge \overline{\Omega} = 0$$

Thus we get ω^{sf} indeed a non-degenerate (1,1)-form with respect to $J_0 = J_{au}$. As we know in special geometry, generally ω^{sf} is just a pseudo-metric on $\widetilde{W}_0$. We still need to check the positivity condition here. From a direct computation, we have the following:

Proposition 3.12 *The canonical form ω^{sf} gives a hyper-kähler metric on $(\widetilde{W}_0, J_0)$ if and only $S_1 > \ln|c|$.*

Proof 3.13 *Notice that we have the complex structure $J_0 = J_{au}$. Thus the induced complex structure on cooridnates $\{c_1, c_2; t_1, t_2\}$ is given by:*

$$J(dc_1) = dc_2, \ J(dt_1) = -dt_2$$

Consider the canonical form under the $\{c_1, c_2, t_1, t_2\}$ coordinates, which is explicitly given as follows:

$$\begin{aligned}
\omega^{sf} &= R(S_1 - \ln|c|)dc_1 \wedge dc_2 + \frac{1}{2\pi R} d\theta_{\gamma_m} \wedge d\theta_{\gamma_e} \\
&= R(S_1 - \ln|c|)dc_1 \wedge dc_2 \\
&\quad - \frac{t_1}{R(S_1 - \ln|c|)^2}[dS_2 + d\arg(c)] \wedge dt_1 + \frac{t_1}{R(S_1 - \ln|c|)^2}[dS_1 - d\ln|c|] \wedge dt_2 \\
&\quad + \frac{t_1^2}{R(S_1 - \ln|c|)^3}[dS_2 + d\arg(c)] \wedge [dS_1 - d\ln|c|] - \frac{1}{R(S_1 - \ln|c|)} dt_1 \wedge dt_2
\end{aligned}$$

For abbreviation, we take some notations here:

$$m = S_{11} - \frac{c_1}{|c|^2},\ n = S_{12} - \frac{c_2}{|c|^2}$$

Then we continue the calculation and get the simplification:

$$\begin{aligned}\omega^{sf} = & \left[R(S_1 - \ln|c|) + \frac{t_1^2(m^2+n^2)}{R(S_1-\ln|c|)^3}\right] dc_1 \wedge dc_2 \\ & - \frac{t_1}{R(S_1-\ln|c|)^2}[ndc_1 - mdc_2] \wedge dt_1 \\ & + \frac{t_1}{R(S_1-\ln|c|)^2}[mdc_1 + ndc_2] \wedge dt_2 \\ & - \frac{1}{R(S_1-\ln|c|)} dt_1 \wedge dt_2\end{aligned}$$

By Sylvester's criterion, the positivity condition goes to:

$$S_1 - \ln|c| > 0, \quad \left[R(S_1-\ln|c|) + t_1^2 \cdot \frac{m^2+n^2}{R(S_1-\ln|c|)^3}\right] > 0$$

$$\left[R(S_1-\ln|c|) + t_1^2 \cdot \frac{m^2+n^2}{R(S_1-\ln|c|)^3}\right] \cdot \frac{1}{R(S_1-\ln|c|)} > [\frac{t_1 n}{R(S_1-\ln|c|)^2}]^2 + [\frac{t_1 m}{R(S_1-\ln|c|)^2}]^2$$

Previously we already have the condition for Ω *in Corollary 3.7, that is:* $S_1 > \ln|c|$. *Since* $R \in \mathbb{R}_+$, *thus the final condition goes to:* $S_1 > \ln|c|$.

Therefore, for the semi-flat metric, we still need the same condition: $S_1 > \ln|c|$ as for the holomorphic 2-form Ω, which happens to coincide with the positive asymptotic condition for the Ooguri-Vafa potential at infinity.

Furthermore, if we pick complex coordinates as $u_1 = c_1 + ic_2, u_2 = t_1 - it_2$, then from the above calculation, we will get a decomposition of the semi-flat metric. In fact, we have:

$$\begin{aligned}&\pi R \cdot Re(dZ_{\gamma_m} \wedge d\overline{Z}_{\gamma_e}) = R[S_1 - \ln|c|] dc_1 \wedge dc_2 \\ &\frac{1}{2\pi R} \cdot d\theta_{\gamma_m} \wedge d\theta_{\gamma_e} = i\partial\bar{\partial}\left[\frac{t_1^2}{R(S_1-\ln|c|)}\right] = i\partial\bar{\partial}\left[\frac{S_1-\ln|c|}{4\pi^2 R} \cdot \theta_{\gamma_e}^2\right]\end{aligned}$$

Notice that $(S_1 - \ln|c|)$ is a harmonic function on the base B_0, thus the first part generally is not a Ricci flat metric on the base, unless it is a flat one. The second part is a pseudo-metric on $\widetilde{W}_0$ since the metric is positive-semidefinite. This decomposition also indicates that $[\omega^{sf}] \neq 0$ in $H^{1,1}(\widetilde{W}_0, J_0)$.

Remark. It is easy to see that in fact we can generalize the canonical semi-flat metric by admitting the parameter R to be a suitable positive functions which is compatible with the gluing and monodromy condition. Similar results as in Lemma 3.11 is still valid, however ω^{sf} is not Kähler anymore.

Remark. Notice that if the h term in the holomorphic form Ω is non-vanishing, then it is easy to check $\omega \wedge \Omega \neq 0$, and $\omega \wedge \bar{\Omega} \neq 0$, thus the canonical form ω^{sf} cannot be a (1,1) form or a Kähler metric. Moreover, in this case, the complex structure determined by Ω is not J_{au} anymore. It is not easy to write down the explicit expression of the complex structure in general case. Therefore, although we start the construction on the real completely integral system, so far we are just able to carry out further calculation in the complex integrable system, i.e. h term vanishing case. We'd like to explore the general case in further studies.

3.4 Instanton correction

Now we consider the instanton correction of the semi-flat metric.

The main strategy here is that we do not make the modification directly on the semi-flat metric, but instead on its associated twistor space. Then we transfer the correction problem into a Riemann-Hilbert problem of solving the monodromy of the associated holomorphic Darboux coordinates. Finally, we adapt GMN's integral ansatz and read out the metric from the twistor space determined by modified holomorphic Darboux coordinates.

First we consider the twistor space of $\widetilde{W_0}$. By the previous construction of holomorphic 2-form Ω and Kähler form ω^{sf}, we can write the family of holomorphic 2-forms as:

$$\varpi^{sf}(\zeta) = -\frac{1}{2\zeta} \cdot \Omega + \omega^{sf} + \frac{1}{2}\zeta \cdot \overline{\Omega}, \quad \zeta \in \mathbb{C}P^1$$

Here we have an important observation by Gaiotto, Moore and Neitzke. In fact, we can represent the holomorphic 2-forms by holomorphic Darboux coordinates:

Lemma 3.14 ([14]) *The $\mathbb{C}P^1$-family of holomorphic 2-form can be rearranged into the form:*

$$\varpi^{sf}(\zeta) = \frac{1}{2\pi R} \cdot \frac{d\chi^{sf}_{\gamma_m}}{\chi^{sf}_{\gamma_m}} \wedge \frac{d\chi^{sf}_{\gamma_e}}{\chi^{sf}_{\gamma_e}}$$

here the holomorphic Darboux coordinates are given by:

$$\chi^{sf}_{\gamma_m} = \exp[i \cdot \frac{\pi R}{\zeta} \cdot Z_{\gamma_m} - i\theta_{\gamma_m} - i \cdot \pi R \zeta \cdot \overline{Z}_{\gamma_m}]$$

$$\chi^{sf}_{\gamma_e} = \exp[i \cdot \frac{\pi R}{\zeta} \cdot Z_{\gamma_e} + i\theta_{\gamma_e} - i \cdot \pi R \zeta \cdot \overline{Z}_{\gamma_e}]$$

Remark. Recall that we have the action-angle coordinates of Ω given by:

$$Z_{\gamma_m} = \frac{1}{2\pi} \cdot [c - c \cdot \ln c + (S + i \cdot \widetilde{S})], \qquad Z_{\gamma_e} = -i \cdot c$$

$$\theta_{\gamma_e} = \frac{2\pi \cdot t_1}{S_1 - \ln|c|}, \qquad \theta_{\gamma_m} = t_2 - \frac{S_2 + \arg c}{S_1 - \ln|c|} \cdot t_1$$

According to the monodromy transformation:

$$Z_{\gamma_m} \to Z_{\gamma_m} + Z_{\gamma_e}, \quad \theta_{\gamma_m} \to \theta_{\gamma_m} - \theta_{\gamma_e}$$

the pairing in the Darboux coordinates is the unique one which induces the complex structure, although with the global monodromy:

$$\chi^{sf}_{\gamma_m} \to \chi^{sf}_{\gamma_m} \cdot \chi^{sf}_{\gamma_e}$$

Now comes the main idea of GMN's construction. We should carry out instanton correction on the $\mathbb{C}P^1$-family of holomorphic 2-forms by solving the monodromy issue for the whole family of holomorphic Darboux coordinates simultaneously.

The basic idea to achieve this is to produce the inverse monodromy to cancel the original one. The main tool we need is the Cauchy-Plemelj-Sokhotskii formula for Riemann-Hilbert problem, that is:

Theorem 3.15 *Take a smooth simple curve l on $\mathbb{C}$, for every $C^{0,\alpha}(l)$ function φ on l, there exist an unique piecewise holomorphic function on $\mathbb{C}$, which is:*

1) continuously extendable from l_+ to $\bar{l}_+$ as well as from l_- to $\bar{l}_-$,

2) it vanishes for large $|z|$,

3) it has the monodormy: $f_+ - f_- = \varphi$ on l,

Moreover, such function is given by the Cauchy-Plemelj-Sokhotskii formula:

$$f(z) = \frac{1}{2\pi i}\int_l \frac{\varphi}{t-z}dt$$

Taking into account of further compatible conditions for the twistor space [33], we have GMN's integral ansatz for the holomorphic Darboux coordinates:

Theorem 3.16 ([14]) *The instanton corrected holomorphic Darboux coordinates on $\widetilde{W}_0$*

can be given by the following integral formulas:

$$\chi_{\gamma_e} = \chi^{sf}_{\gamma_e}$$
$$\chi_{\gamma_m} = \chi^{sf}_{\gamma_m} \cdot \exp\left[\frac{i}{4\pi}\int_{l_+}\frac{d\zeta'}{\zeta'}\frac{\zeta'+\zeta}{\zeta'-\zeta}\ln(1-X_{\gamma_e}(\zeta')) - \frac{i}{4\pi}\int_{l_-}\frac{d\zeta'}{\zeta'}\frac{\zeta'+\zeta}{\zeta'-\zeta}\ln(1-X^{-1}_{\gamma_e}(\zeta'))\right]$$

Consequently, the instanton corrected holomorphic 2-forms are given by:

$$\varpi(\zeta) = \frac{1}{2\pi R}\cdot\frac{d\chi_{\gamma_m}(\zeta)}{\chi_{\gamma_m}(\zeta)} \wedge \frac{d\chi_{\gamma_e}(\zeta)}{\chi_{\gamma_e}(\zeta)}$$

Here $l_\pm$ *are the rays connecting* 0 *and* ∞*, which are away from the BPS rays:* $\{\zeta \mid Re\frac{c}{\zeta} = 0\}$ *determined by holomorphic discs bounding vanishing cycle, otherwise the above integral will diverge.*

Now we adapt this integral formula and compute the new twistor space after instanton correction on our local model.

Corollary 3.17 *After the instanton correction given by GMN ansatz, we get the modified twistor space given as:*

$$\varpi(\zeta) = \frac{1}{2\pi R}\cdot \xi_m \wedge \xi_e$$

where:

$$\xi_m = -id\theta_{\gamma_m} + 2\pi i\cdot A + \pi i \cdot V \cdot (\frac{1}{\zeta}dc - \zeta d\bar{c})$$
$$\xi_e = id\theta_{\gamma_e} + \pi R\cdot(\frac{1}{\zeta}dc + \zeta d\bar{c})$$

and the potential function here is given by:

$$V = \frac{R}{4\pi}\cdot\left(\sum_{n\in\mathbb{Z}}[\frac{1}{\sqrt{R^2|c|^2 + (\frac{\theta_{\gamma_e}}{2\pi}+n)^2}} - \kappa(n)] + 2S_1\right)$$

Proof 3.18 *The proof of the identity contains the semi-flat part and the instanton part.*

Semi-flat part. For the semi-flat part, we need to check the following identity:

$$\varpi^{sf}(\zeta) = \frac{1}{2\pi R} \cdot \frac{d\chi^{sf}_{\gamma_m}(\zeta)}{\chi^{sf}_{\gamma_m}(\zeta)} \wedge \frac{d\chi_{\gamma_e}(\zeta)}{\chi_{\gamma_e}(\zeta)}$$

Notice that on the left side, we have:

$$\begin{aligned}
\varpi^{sf}(\zeta) &= \frac{1}{2\pi R} \cdot \xi^{sf}_m \wedge \xi_e \\
&= \frac{1}{2\pi R}\left[-id\theta_{\gamma_m} + 2\pi i \cdot A^{sf} + \pi i \cdot V^{sf} \cdot (\frac{1}{\zeta}dc - \zeta d\bar{c})\right] \wedge \xi_e \\
&= \frac{1}{2\pi R}\left[-id\theta_{\gamma_m} + 2\pi i \cdot A^{sf} + \pi i \cdot V^{sf} \cdot (\frac{1}{\zeta}dc - \zeta d\bar{c})\right] \wedge \frac{d\chi_{\gamma_e}(\zeta)}{\chi_{\gamma_e}(\zeta)}
\end{aligned}$$

here

$$\begin{aligned}
V^{sf} &= -\frac{R}{4\pi}(\ln c + \ln \bar{c} - 2S_1) \\
A^{sf} &= \frac{i}{8\pi^2}(\ln c - \ln \bar{c} + 2iS_2)\, d\theta_{\gamma_e}
\end{aligned}$$

Moreover, a direct calculation verifies that:

$$\frac{d\chi^{sf}_{\gamma_m}(\zeta)}{\chi^{sf}_{\gamma_m}(\zeta)} = \left[-id\theta_{\gamma_m} + 2\pi i \cdot A^{sf} + \pi i \cdot V^{sf}(\frac{1}{\zeta}dc - \zeta d\bar{c})\right] - \frac{i}{4\pi}(\ln c - \ln \bar{c} + 2iS_2)\frac{d\chi_{\gamma_e}(\zeta)}{\chi_{\gamma_e}(\zeta)}$$

Thus we get the identity for the semi-flat part.

Instanton part. For the instanton part, we need to check the following identity:

$$\varpi^{inst}(\zeta) = \frac{1}{2\pi R} \cdot \frac{d\chi^{inst}_{\gamma_m}(\zeta)}{\chi^{inst}_{\gamma_m}(\zeta)} \wedge \frac{d\chi_{\gamma_e}(\zeta)}{\chi_{\gamma_e}(\zeta)}$$

The whole calculation is similar as given in [14]. For completeness, we outline the main steps here, which will show how the instanton correction will appear from contour integrals. More details and explanations can be found in [14] and [37].

Notice that on the left side, we have:

$$\begin{aligned}\varpi^{inst}(\zeta) &= \frac{1}{2\pi R}\cdot \xi_m^{inst}\wedge \xi_e \\ &= \frac{1}{2\pi R}\left[2\pi i A^{inst} + \pi i V^{inst}\left(\frac{1}{\zeta}dc - \zeta d\bar{c}\right)\right]\wedge \xi_e\end{aligned}$$

here

$$\begin{aligned}V^{inst} &= \frac{R}{2\pi}\sum_{n\neq 0} e^{in\theta_{\gamma_e}} K_0(2\pi R|nc|) \\ A^{inst} &= -\frac{R}{4\pi}\left(\frac{dc}{c} - \frac{d\bar{c}}{\bar{c}}\right)\sum_{n\neq 0} sign(n)\cdot e^{in\theta_{\gamma_e}}|c|K_1(2\pi R|nc|)\end{aligned}$$

On the right side, first let us take the partial integrals:

$$I_\pm = -\frac{i}{4\pi}\int_{l_\pm}\frac{d\zeta'}{\zeta'}\frac{\zeta'+\zeta}{\zeta'-\zeta}\left[\frac{\chi_{\gamma_e}(\zeta')^{\pm 1}}{1-\chi_{\gamma_e}(\zeta')^{\pm 1}}\frac{d\chi_{\gamma_e}(\zeta')}{\chi_{\gamma_e}(\zeta')}\right]$$

Then the right side of identity goes to:

$$\frac{1}{2\pi R}\cdot (I_+ + I_-)\wedge \frac{d\chi_{\gamma_e}(\zeta)}{\chi_{\gamma_e}(\zeta)}$$

Thus the identity we want to verify can be simplified into:

$$(I_+ + I_-)\wedge \frac{d\chi_{\gamma_e}(\zeta)}{\chi_{\gamma_e}(\zeta)} = \left[2\pi i A^{inst} + \pi i V^{inst}\left(\frac{1}{\zeta}dc - \zeta d\bar{c}\right)\right]\wedge \xi_e$$

Now let us explicitly compute the left side terms. Notice that:

$$\begin{aligned}I_\pm \wedge \frac{d\chi_{\gamma_e}(\zeta)}{\chi_{\gamma_e}(\zeta)} &= \frac{i}{4\pi}\int_{l_\pm}\frac{d\zeta'}{\zeta'}\left(\frac{\zeta'+\zeta}{\zeta'-\zeta}\cdot\frac{d\chi_{\gamma_e}(\zeta)}{\chi_{\gamma_e}(\zeta)}\wedge\frac{d\chi_{\gamma_e}(\zeta')}{\chi_{\gamma_e}(\zeta')}\right)\left[\frac{\chi_{\gamma_e}(\zeta')^{\pm 1}}{1-\chi_{\gamma_e}(\zeta')^{\pm 1}}\right] \\ &= \frac{i}{4\pi}\int_{l_\pm}\frac{d\zeta'}{\zeta'}\left(\frac{\zeta'+\zeta}{\zeta'-\zeta}\cdot\frac{d\chi_{\gamma_e}(\zeta)}{\chi_{\gamma_e}(\zeta)}\wedge\left[\frac{d\chi_{\gamma_e}(\zeta')}{\chi_{\gamma_e}(\zeta')} - \frac{d\chi_{\gamma_e}(\zeta)}{\chi_{\gamma_e}(\zeta)}\right]\right)\left[\frac{\chi_{\gamma_e}(\zeta')^{\pm 1}}{1-\chi_{\gamma_e}(\zeta')^{\pm 1}}\right] \\ &= \frac{i}{4\pi}\int_{l_\pm}\frac{d\zeta'}{\zeta'}\left(-\pi R\cdot\frac{d\chi_{\gamma_e}(\zeta)}{\chi_{\gamma_e}(\zeta)}\wedge\left[\left(\frac{1}{\zeta'}+\frac{1}{\zeta}\right)dc - (\zeta'+\zeta)\,d\bar{c}\right]\right)\left[\frac{\chi_{\gamma_e}(\zeta')^{\pm 1}}{1-\chi_{\gamma_e}(\zeta')^{\pm 1}}\right]\end{aligned}$$

Take the arrangement:

$$
\begin{aligned}
L_1^+ &= \int_{l_+} \frac{d\zeta'}{\zeta'} \left(\frac{1}{\zeta} dc - \zeta d\overline{c} \right) \left[\frac{\chi_{\gamma_e}(\zeta')}{1 - \chi_{\gamma_e}(\zeta')} \right] \\
L_1^- &= \int_{l_-} \frac{d\zeta'}{\zeta'} \left(\frac{1}{\zeta} dc - \zeta d\overline{c} \right) \left[\frac{\chi_{\gamma_e}(\zeta')^{-1}}{1 - \chi_{\gamma_e}(\zeta')^{-1}} \right] \\
L_2^+ &= \int_{l_+} \frac{d\zeta'}{\zeta'} \left(\frac{1}{\zeta'} dc - \zeta' d\overline{c} \right) \left[\frac{\chi_{\gamma_e}(\zeta')}{1 - \chi_{\gamma_e}(\zeta')} \right] \\
L_2^- &= \int_{l_-} \frac{d\zeta'}{\zeta'} \left(\frac{1}{\zeta'} dc - \zeta' d\overline{c} \right) \left[\frac{\chi_{\gamma_e}(\zeta')^{-1}}{1 - \chi_{\gamma_e}(\zeta')^{-1}} \right]
\end{aligned}
$$

Thus we have the simplification:

$$
(I_+ + I_-) \wedge \frac{d\chi_{\gamma_e}(\zeta)}{\chi_{\gamma_e}(\zeta)} = \frac{iR}{4}(L_1^+ + L_1^- + L_2^+ + L_2^-) \wedge \frac{d\chi_{\gamma_e}(\zeta)}{\chi_{\gamma_e}(\zeta)}
$$

Now we are ready to calculate the contour integrals. During the computation, we need some identity for Bessel functions. Notice that after expanding the geometric series, we get the series labeled by n:

$$
\begin{aligned}
\int_{l_+} \frac{d\zeta'}{\zeta'} \frac{\chi_{\gamma_e}(\zeta')}{1 - \chi_{\gamma_e}(\zeta')} &= \sum_{n>0} 2e^{in\theta_{\gamma_e}} K_0(2\pi R|nc|) \\
\int_{l_-} \frac{d\zeta'}{\zeta'} \frac{\chi_{\gamma_e}(\zeta')^{-1}}{1 - \chi_{\gamma_e}(\zeta')^{-1}} &= \sum_{n<0} 2e^{in\theta_{\gamma_e}} K_0(2\pi R|nc|)
\end{aligned}
$$

Thus, we have the first important identity about the V^{inst} part:

$$
\begin{aligned}
\frac{iR}{4}(L_1^+ + L_1^-) \wedge \frac{d\chi_{\gamma_e}(\zeta)}{\chi_{\gamma_e}(\zeta)} &= \frac{iR}{2} \sum_{n\neq 0} e^{in\theta_e} K_0(2\pi R|nc|) \left(\frac{1}{\zeta} dc - \zeta d\overline{c} \right) \wedge \frac{d\chi_{\gamma_e}(\zeta)}{\chi_{\gamma_e}(\zeta)} \\
&= \pi i V^{inst} \left(\frac{1}{\zeta} dc - \zeta d\overline{c} \right) \wedge \frac{d\chi_{\gamma_e}(\zeta)}{\chi_{\gamma_e}(\zeta)} \\
&= \pi i V^{inst} \left(\frac{1}{\zeta} dc - \zeta d\overline{c} \right) \wedge \xi_e
\end{aligned}
$$

Moreover, we have the identity:

$$\int_{l_+} \frac{d\zeta'}{\zeta'}\zeta' \frac{\chi_{\gamma_e}(\zeta')}{1-\chi_{\gamma_e}(\zeta')} = -\sum_{n>0} 2\frac{|c|}{\bar{c}} e^{in\theta_{\gamma_e}} K_1(2\pi R|nc|)$$
$$\int_{l_+} \frac{d\zeta'}{\zeta'}\frac{1}{\zeta'} \frac{\chi_{\gamma_e}(\zeta')}{1-\chi_{\gamma_e}(\zeta')} = -\sum_{n>0} 2\frac{|c|}{c} e^{in\theta_{\gamma_e}} K_1(2\pi R|nc|)$$

Similarly:

$$\int_{l_-} \frac{d\zeta'}{\zeta'}\zeta' \frac{\chi_{\gamma_e}(\zeta')^{-1}}{1-\chi_{\gamma_e}(\zeta')^{-1}} = \sum_{n<0} 2\frac{|c|}{\bar{c}} e^{in\theta_{\gamma_e}} K_1(2\pi R|nc|)$$
$$\int_{l_-} \frac{d\zeta'}{\zeta'}\frac{1}{\zeta'} \frac{\chi_{\gamma_e}(\zeta')^{-1}}{1-\chi_{\gamma_e}(\zeta')^{-1}} = \sum_{n<0} 2\frac{|c|}{c} e^{in\theta_{\gamma_e}} K_1(2\pi R|nc|)$$

Thus $L^{\pm}$ will contribute to the rest part of the identity. We get the second important identity about the A^{inst} part:

$$\begin{aligned}\frac{iR}{4}(L_2^+ + L_2^-)\wedge\frac{d\chi_{\gamma_e}(\zeta)}{\chi_{\gamma_e}(\zeta)} &= \left[-\frac{iR}{2}\left(\frac{dc}{c}-\frac{d\bar{c}}{\bar{c}}\right)\sum_{n\neq 0} sign(n)\cdot e^{in\theta_{\gamma_e}}|c|K_1(2\pi R|nc|)\right]\wedge\frac{d\chi_{\gamma_e}(\zeta)}{\chi_{\gamma_e}(\zeta)} \\ &= 2\pi i A^{inst}\wedge\frac{d\chi_{\gamma_e}(\zeta)}{\chi_{\gamma_e}(\zeta)} \\ &= 2\pi i A^{inst}\wedge\xi_e\end{aligned}$$

Combining the two important identities about the instanton contribution, finally we finish the proof of the instanton part through:

$$\begin{aligned}(I_+ + I_-)\wedge\frac{d\chi_{\gamma_e}(\zeta)}{\chi_{\gamma_e}(\zeta)} &= \frac{iR}{4}(L_1^+ + L_1^- + L_2^+ + L_2^-)\wedge\frac{d\chi_{\gamma_e}(\zeta)}{\chi_{\gamma_e}(\zeta)} \\ &= \left[2\pi i A^{inst} + \pi i V^{inst}\left(\frac{1}{\zeta}dc - \zeta d\bar{c}\right)\right]\wedge\xi_e\end{aligned}$$

Then following the standard procedure from twistor space to the metric, we are able to read out the explicit metric. However, here we adapt an alternative way. In fact, from direct comparison to the twistor space $\varpi_{o.v.}(\zeta)$ of Ooguri-Vafa metric with potential function V, we can figure out the twistor space $\varpi(\zeta)$ we constructed above is just a rescaling by the constant of 2π, that is $\varpi(\zeta) = 2\pi\varpi_{o.v.}(\zeta)$.

Notice that because of the orientation issue of the local model, here we need to take a negative sign for θ_{γ_m}. Up to this orientation adjustment, we end with the following result:

Theorem 3.19 *Given a harmonic semi-global invariant S with $S_1 > \ln|c|$ and positivity condition $(**)$, the twistor structure determined by $\varpi(\zeta)$ gives a construction of metric $2\pi g$ on $\widetilde{W}_0$, here g is the generalized Ooguri-Vafa metric with potential function V given as above.*

Notice that the metric we directly get from the above construction is just defined on the regular part $\widetilde{W}_0$ of the fibration over $B_0 = \{0 < |c| < \epsilon\}$. One natural question is how to carry out certain completion of the metric, by adding the central fibre. Here, we adapt the partial completion through Ooguri-Vafa space as follows.

Partial completion. Since now $\widetilde{W}_0$ is equipped with the metric $2\pi g$, by the property of Ooguri-Vafa space, we have a partial metric completion of $\widetilde{W}_0$ by addition a central fibre, which comes from the following isometric embedding:

$$i : (\widetilde{W}_0, 2\pi g) \longrightarrow (M_{O.V.}, 2\pi g)$$

$$(z_{\gamma_e}, z_{\gamma_m}, \theta_{\gamma_m}, \theta_{\gamma_e}) \longmapsto (z_e, z_m, -\theta_m, \theta_e)$$

Here the space $M_{O.V.}$, and also the coordinates are explicitly given as in the Example 2.

This partial completion is a little tricky here, since it is not directly from $\widetilde{W}_0$ to $\widetilde{W}$, although we know $M_{O.V.}$ is homeomorphism to $\widetilde{W}$ as the total space of same topological torus fibration. The main reason we choose the above approach through embedding i is the following: if originally ω_{can} comes from a hyperkähler metric g_0 on the local model, then from construction we can see $g_0|_{\widetilde{W}_0}$ it is always different from the metric $2\pi g$ determined by the twistor space $\varpi(\zeta)$.

We make more discussion from the point view of Lagrangian fibration here. Recall that twistor strucutre of $(M_{O.V.}, 2\pi g)$ is given by $2\pi\varpi_{o.g.}(\zeta)$. With respect to the symplectic structure: $2\pi\omega_{o.v.} = -4\pi Re\varpi_{o.g.}(0)$, $M_{O.V.}$ has the Lagrangian fibration structure given by:

$$f : M_{O.V.} \longrightarrow B$$

$$f(c_1, c_2; \theta_m, \theta_e) = (c_1, c_2)$$

As for the local model $\widetilde{W}$, we have the global symplectic structure given by: $\omega_{can} = Re(dz_1 \wedge dz_2)$. Its Lagrangian fibration is simply given by:

$$\pi_{can} : \widetilde{W} \longrightarrow B$$

$$\pi_{can}(z_1, z_2) = (c_1, c_2) = (Re(z_1 z_2), Im(z_1 z_2))$$

Notice that on the regular part, we have the important relation from the embedding map:

$$\omega_{can}|_{\widetilde{W}_0} = i^*(2\pi\omega^{sf}_{o.v.})$$

Thus we find $i : (\widetilde{W}_0, \omega_{can}) \longrightarrow (M_{O.V.}, \omega_{o.v.})$ is not a symplectic embedding (or equivalently $\omega_{can}|_{\widetilde{W}_0} \neq -2Re\varpi(0)$), since $\theta_{\gamma_e}, \theta_{\gamma_m}$ are angle coordinates on the left side, however θ_e, θ_m are not angle coordinates on the right side, as showed in Property 2.8. The difference essentially can be read out from similar identity as in Property 2.8. The infinite terms of instanton deformation of the angel coordinates here are directly achieved through GMN's integration formula.

After the above partial completion, finally we get a re-construction of the Ooguri-Vafa metric (up to a rescaling by 2π) with the generalized potential function on a focus-focus fibration through GMN's construction of hyperkähler on completely integrable systems. The type of metrics we get coincides with the one used in Gross and Wilson's work on approximation construction of hyperkähler metric on elliptic K3. Morever, from point of view of focus-focus fibration, we unfold the geometric meaning of the extra harmonic term in the potential function V. In fact, it indicates the semi-global invariant of the integrable system. Such dynamic interpretation also serves as a positive evidence that GMN's project might work finally on the global picture [14] [26].

Chapter 4

Further Discussions

At the end, we make some further discussion about the whole construction. Compared to the Gibbons-Hawking ansatz on S^1 fibration, the GMN ansatz acts as a construction of hyperkähler metric on torus fibration with singular fibres. They require totally different ingredients and technics. However, on the local model, they still have certain relations.

- A_n singularity case:

In our paper, we just study the one singularity case of the Ooguri-Vafa potential, which corresponds to the A_1 singularity case. Generally, it is similar to construct the metric on A_n singularity case. Use the GMN ansatz in the local model, we are also able to construct such metrics. For example, as constructed in [34], we take n copies of the local model with the same semi-global invariant $\{(W_i, \omega, S)\}$, and denote the Poincare surfaces by $\Gamma_{i,1} \equiv \Gamma_1$, and $\Gamma_{i,2} \equiv \Gamma_2$. Then we make the following sequence of gluing:

$$\coprod_{i=1}^{n} (W_i, \omega, S)/\{\Gamma_{i,2} \sim \Gamma_{i+1,1}\}$$

here $\Gamma_{n+1,1} = \Gamma_{1,1}$ and $\Gamma_{n+1,2} = \Gamma_{1,2}$.

Since the compatible property of the final metric $2\pi g$ on $\widetilde{W_0}$, we just need to take the same metric on each copy and glue them together. After the similar partial completion procedure, we thus get a smooth hyperkähler metric in the A_n singularity case.

• S^1-symmetry

As we know, the Ooguri-Vafa metric has the isometric tri-hamiltonian S^1 action which is inherited from the Gibbons-Hawking ansatz [3]. However in the GMN ansatz, there is no *a priori* reason such symmetry will appear in the result. Thus it turns out to be interesting in the local model so far we still cannot get a new metric without such symmetry. It may require a further bundle symplectic automorphism (but not holomorphic automorphism) to break such symmetry.

• Removable singularity of GMN ansatz

In the local model, the extension over singular fibre property of the final metric is borrowed from the good property of Ooguri-Vafa metric. However, in the general case, even for two singular fibres case, we have no such auxiliary metric to make the extension work directly. One possible approach comes from the recent development of geometric analysis on the removable singularity of Kähler-Einstein metrics [6] [8] [44]. GMN ansatz on elliptic K3 may need similar results for hyperkähler metric or even twistor space to justify the extension property of the final metric.

Chapter 5

Gibbons-Hawking Ansatz

In this chapter, we review the Ooguri-Vafa metric, and also make some further exploration, which is important ingredient of our above results.

Back to the story of explicit hyperkähler metric. By Calabi-Yau Theorem, we know such metric always exists, however it is not easy to write down an explicit one. Some newest global approaches are given by gluing of Oaguri-Vafa metric by Gross and Wilson in [20], by gluing Yau-Tian ansatz by Hein in [21] and physically by Kontsevich-Soibelman wall crossing formula by Gaiotto, Moore and Neitzke in [14]. Locally, in a neighborhood of a singular point (fibre), the most important approach is via Gibbons-Hawking ansatz. The first explicit expression is known as the Oogui-Vafa metric [35].

To follow the traditional notation for the Ooguri-Vafa metric, our symbols and parameters adapted here might be different from the ones used above in the main body of the thesis.

5.1 Classical Gibbons-Hawking ansatz

Gibbons-Hawking ansatz is a systematic way to construct equivariant hyperkähler metrics on S^1 bundles over open subsets of $\mathbb{R}^3$ [9].

Classically, let $\pi : M \to U$ be a principal S^1 bundle over some open set $U \subset \mathbb{R}^3$.

Let α be a connection 1-form on M. The curvature of the connection is $d\alpha = \pi^*\omega$ for a 2-form ω on U. And then the first Chern class of the bundle is given by $i\omega/2\pi$. Suppose V is a positive harmonic function on base U satisfying $*dV = \omega/(2\pi i)$. Let

$$\omega_1 = du_1 \wedge \alpha/(2\pi i) + V du_2 \wedge du_3$$

$$\omega_2 = du_2 \wedge \alpha/(2\pi i) + V du_3 \wedge du_1$$

$$\omega_3 = du_3 \wedge \alpha/(2\pi i) + V du_1 \wedge du_2.$$

It follows that ω_1, ω_2, ω_3 defines a hyperkähler metric on M. Let $\alpha_0 = \alpha/(2\pi i)$ denote the real 1-form. Then the metric is explicitly given by:

$$ds^2 = V d\boldsymbol{u} \cdot d\boldsymbol{u} + V^{-1}\alpha_0^2.$$

We are particularly interested in the case when the open set is $\mathbb{R}^3$ with finite discrete points removed. This leads to the following characterization of the Gibbons-Hawking ansatz:

Theorem 5.1 ([16], [29], [31]) *Let $\pi : M \to \mathbb{R}^3 - \{x_1, \ldots, x_k\}$ be an S^1 bundle with first Chern class $-a_i$ ($a_i \in \mathbb{N}$) around each point. Let V be the positive harmonic function:*

$$V(\boldsymbol{x}) = c + \frac{1}{2}\sum_{i=1}^{k} \frac{a_i}{|\boldsymbol{x} - \boldsymbol{x}_i|}, \quad c \in \mathbb{R}^+.$$

Equip the bundle with the connection 1-form α_0 satisfying: $d\alpha_0 = \pi^*(*dV)$, which is unique upto gauge change. Then the metric

$$g = V d\boldsymbol{x} \cdot d\boldsymbol{x} + V^{-1}\alpha_0^2$$

defines a hyperkähler metric on M, which is referred as Gibbons-Hawking ansatz.

Moreover, we have the following results:

If $a_i \equiv 1$, then M has a nonsingular completion $\overline{M} = M \cup \{x_i\}$.

In addition, if $c = 0$, we obtain in this way the multi-Eguchi-Hanson metric; if $c > 0$, we get the multi-Taub-NUT metrics. For example:

1) If $c = 0, k = 1$, then $\overline{M}$ is just the Euclidean space $\mathbb{R}^4$;

2) If $c = 1, k = 1$, then $\overline{M}$ is $\mathbb{R}^4$ with a non-flat hyperkähler metric, which is classified by Lebrun.

In contrast, if $a_i > 1$, then $\overline{M}$ has an orbifold point at x_i of the form $\mathbb{C}^2/\mathbb{Z}_{a_i}$.

In this construction, the positive harmonic potential function $V(\boldsymbol{x})$ plays an important role. Indeed, positive harmonic function on $\mathbb{R}^n$ with isolated singularities can be completely described thanks to Bôcher's Theorem:

Proposition 5.2 *Let $A = \{x_1, x_2, \ldots\}$ be a set of discrete points in $\mathbb{R}^n (n > 2)$. If V is a positive harmonic function on $\mathbb{R}^n - A$, then it has the following form:*

$$V(\boldsymbol{x}) = c + \frac{1}{2}\sum_{i=1}^{\infty} a_i |\boldsymbol{x} - \boldsymbol{x}_i|^{2-n}, \quad c \geq 0$$

for each point x_i, if we take a small sphere with radius r_i centered at x_i which contains only x_i inside, then:

$$a_i = \lim_{\boldsymbol{x} \to \boldsymbol{x}_i} \frac{V(\boldsymbol{x})}{f_i(\boldsymbol{x})}, \text{ here } f_i(\boldsymbol{x}) = \frac{1}{2}|\boldsymbol{x} - \boldsymbol{x}_i|^{2-n}.$$

Different choices of the coefficients will give us different potential functions of the Gibbons-Hawking ansatz. In this way, we can construct a large family of hyerkähler metrics on M.

One interesting problem is that what kind of hyperkähler metric can be reconstructed by the Gibbons-Hawking ansatz. With the help of the above Proposition, we can get the following answer:

Theorem 5.3 ([24]) *Let (M, g) be a complete, simply-connected hyperkähler 4-manifold which admits an isometric, triholomorphic S^1-action. Then the metric g is obtained from the Gibbons-Hawking ansatz.*

5.2 Periodic Gibbons-Hawking ansatz

In this section, we shall discuss the periodic Gibbons-Hawking ansatz, which is closely related to the construction of the Oogui-Vafa metric. In the view of potential function $V(\boldsymbol{x})$ in Theorem 5.1, now we require $V(\boldsymbol{x})$ is periodic in one direction. Therefore such ansatz induces metrics on S^1 bundles over $\mathbb{R}^2 \times S^1$.

Periodic condition is important in the local K3. Since generally a local K3 is an elliptic fibration with one *type I* singularity, which can be regarded as an S^1 bundle over $\mathbb{R}^2 \times S^1$. As pointed by Bernard and Matessi [1], topologically the model of local K3 is the one given by Gross in [9], as follows:

$$M = \mathbb{C}^2 - \{1 + z_1 z_2 = 0\}, \quad \omega = \frac{1}{|1 + z_1 z_2|} \frac{i}{2} \sum_{j=1}^{2} dz_j \wedge d\overline{z}_j$$

$$f_1 : M \to \mathbb{R}^2 \quad \textit{given by} : f_1(z_1, z_2) = (|z_1|^2 - |z_2|^2, \ln|1 + z_1 z_2|)$$

This fibration is a special lagrangian fibration. It induces the S^1 fibration we want to apply the ansatz:

$$f_2 : M - \{(0,0)\} \to \mathbb{R}^2 \times S^1 - \{(0,0,1)\}$$

$$f_2(z_1, z_2) = (|z_1|^2 - |z_2|^2, \ln|1 + z_1 z_2|, \frac{1 + z_1 z_2}{|1 + z_1 z_2|})$$

The fibre S^1 can be observed from the S^1 action on the fibration,which is given as: $T_\theta(z_1, z_2) = (e^{i\theta} z_1, e^{-i\theta} z_2)$. The singular point of this fibration is the origin O in $\mathbb{C}^2$. The first Chern class of the fibration around the singular point is -1.

In this case, the base of the fibration is $\mathbb{R}^2 \times S^1 - \{(0,0,1)\}$. Thus we need to find a potential V on $\mathbb{R}^3$ which is periodic along one direction with period 1. Without loss of generality, we assume the periodic direction is the last component of $\mathbb{R}^3$, so we use the coordinate $(y, z; \phi = x \mod 1)$ on the base, then the potential conditions for V can be stated as follows:

$$\Delta V(y, z; \phi) = \delta(0,0,1), \quad V > 0, \quad \frac{1}{2\pi r^2}\int_{S^2} *dV = -1. \tag{5.1}$$

Here S^2 is a small sphere with radius $r < 1$ centered at $(0,0,1)$.

Nevertheless, we have the striking non-existence result:

Theorem 5.4 *There does not exist a harmonic function V on $\mathbb{R}^3$ satisfying (5.1), which is periodic with period 1 in last component.*

Proof 5.5 *We prove this by contradiction. Suppose there is a periodic function V satisfies (5.1). First, since V is a positive harmonic function on space $\mathbb{R}^3 - \{(0,0,i)|i \in \mathbb{Z}\}$, by Proposition 5.2, we know there are nonnegative constants a_i and c such that:*

$$V(y, z, x) = c + \sum_{i\in\mathbb{Z}} \frac{1}{2}\frac{a_i}{|(y, z, x) - (0,0,i)|}.$$

Second, since V is period in x, we have

$$\sum_{i\in\mathbb{Z}} \frac{a_i}{|(y, z, x) - (0,0,i)|} = \sum_{i\in\mathbb{Z}} \frac{a_{i+1}}{|(y, z, x) - (0,0,i+1)|}$$

By the Heaviside Trick, we obtain: $a_i = a_{i+1}$. Then series converges if and only if $a_i \equiv 0$, in another word: $u = c$, which does not satisfies the integral condition. Thus we get the contradiction.

As a result, we get some statement about the hyperhäler metric on local K3:

Corollary 5.6 *There is no complete hyperkähler metric on local K3 which can be constructed directly from periodic Gibbons-Hawking ansatz.*

Remark. A simple observation here is that: the above analysis also works for general case $\mathbb{R}^{n-1}\times\mathbb{R}$ $(n\geq 3)$. From the argument, we can see the existence of potential function is equivalent to the convergence of the series $\sum(\frac{1}{i})^{\frac{n}{2}}$. Thus in the higher dimensional case, when $n>2$, the potential function always exists, and it will induce function on $\mathbb{R}^{n-1}\times S^1$ which satisfies the potential conditions, and therefore we can construct hyperkähler metrics through the periodic ansatz [3].

5.3 Ooguri-Vafa Metric

From last section, we see that a global positive harmonic function V on $\mathbb{R}^3\setminus\{(0,0,\mathbb{Z})\}$ does not exist. If we relax the positive condition, we will show by explicit construction that such harmonic function exists, and it is positive on a cylinder of the form $D\times\mathbb{R}\setminus\{0,0,\mathbb{Z}\}$, where D is a disk centered at 0 of the $y-z$ plane. Hence it induces an incomplete hyperkähler metric on M, the S^1 bundle over $D\times S^1\setminus(0,0,1)$. By inserting the point back, this metric extends to $\overline{M}$, which can be regarded as a singular torus fibration given as $\pi:\overline{M}\to D$, with the central fibre $\pi^{-1}(0)$ degenerates to a pinched torus. The fist explicit construction is due to Ooguri-Vafa [35], hence referred as Ooguri-Vafa metric.

Now we explain in full detail the explicit construction of the Ooguri-Vafa metric. It is a metric constructed through the periodic Gibbons-Hawking ansatz, with the following amazing potential function on the base $\mathbb{R}^3 - \{(0,0,\mathbb{Z})\}$:

$$V_c(y,z,x) = c + \frac{1}{2}\sum_{n\in\mathbb{Z}}[\frac{1}{|(y,z,x)-(0,0,n)|} - \nu_n]. \tag{5.2}$$

Here the correction term is given as:

$$\nu_0 = 0\ , \quad \nu_n = \frac{1}{|n|}\ (n \neq 0).$$

If we take cylindrical coordinate (r,θ) on $\mathbb{R}^2$, then we get another expression:

$$V_c(r,\theta,x) = c + \frac{1}{2}\frac{1}{\sqrt{r^2+x^2}} + \frac{1}{2}\sum_{n\in\mathbb{Z}-\{0\}}[\frac{1}{\sqrt{r^2+(x-n)^2}} - \frac{1}{|n|}]. \tag{5.3}$$

We state some elementary properties of the potential function by looking at the expression (5.3).

First notice that, for fixed (r,x), we have: $|\frac{1}{\sqrt{r^2+(x-n)^2}} - \nu_n|$ has the order of n^{-2} as $n \to +\infty$, thus the series on the right hand side of (5.3) converges absolutely. As a result, the potential function enjoys some good properties:

(1) Periodic in x: $V_c(r,x) = V_c(r,x+1)$;

(2) Even in x: $V_c(r,x) = V_c(r,-x)$;

(3) $V_c(r,\cdot)$ is a strict decreasing function with respect to r;

(4) The series in (5.3) converges uniformly on any compact subset $K \subset \mathbb{R}^3\backslash\{(0,0,\mathbb{Z})\}$, thus V_c is a singular harmonic function;

(5) $V_c \sim -\ln r$ when $r \to \infty$ (this is a physical requirement set on the Ooguri-Vafa metric).

In order to get a metric by applying the ansatz, we need to figure out the region where V_c is positive. By the property above, it follows that the region of $V_c > 0$ is a cylindrical region around x-axis of the form:

$$\{V_c > 0\} = \{(r,x)\,|\, r \leq R_c(x)\}, \tag{5.4}$$

where $R_c(x)$ is the radius of the slice for each x, which is an even function of x and with period 1. By continuity of V_c, the boundary $N(V_c)$ of $\{V_c > 0\}$ is where V_c vanishes, namely, the nodal set of the harmonic function V_c.

In what follows, we estimate the radius $R_c(x)$ of the cylindrical region $\{V_c > 0\}$. We warm up with the $c = 0$ case, then move to the general case.

Proposition 5.7 *Consider the potential function V_c when $c = 0$, then we have*

a) if $r \leq 1/2$, then $V_0(r, x) > 0$;

b) if $r \geq 2$, then $V_0(r, x) < 0$.

It thus follows: when $c = 0$, $\frac{1}{2} < R_0(x) < 2$.

Proof 5.8 *By property (1) and property (2) of V_0, we only need to check for $x \in [\frac{1}{2}, 1]$.*

Part a). According to the property (3), we just need to show: $V_0(\frac{1}{2}, x) > 0$. We arrange the terms of V_0 into the following expression:

$$2V_0(\frac{1}{2}, x) = I_1 + I_2,$$

where

$$I_1 = \frac{1}{\sqrt{1/4 + x^2}} + (\frac{1}{\sqrt{1/4 + (x-1)^2}} - 1) + (\frac{1}{\sqrt{1/4 + (x+1)^2}} - 1),$$

$$I_2 = \sum_{n \geq 2} [\frac{1}{\sqrt{1/4 + (x-n)^2}} - \frac{1}{n}] + \sum_{n \geq 2} [\frac{1}{\sqrt{1/4 + (x+n)^2}} - \frac{1}{n}].$$

Notice that since $x \in [\frac{1}{2}, 1]$, when $n \geq 2$ we have: $\sqrt{1/4 + (n-x)^2} < n$, thus every term is the first part of I_2 is positive.

Now let us estimate I_1 and I_2. Notice that:

$$\frac{1}{\sqrt{1/4 + (x-n)^2}} - \frac{1}{n} = \frac{2nx - x^2 - 1/4}{n\sqrt{1/4 + (x-n)^2}(n + \sqrt{1/4 + (x-n)^2})},$$

$$\frac{1}{\sqrt{1/4+(x+n)^2}}-\frac{1}{n}=-\frac{2nx+x^2+1/4}{n\sqrt{1/4+(x+n)^2}(n+\sqrt{1/4+(x+n)^2})}.$$

Thus we get

$$\begin{aligned}I_2 &\geq \sum_{n\geq 2}\frac{(2nx-x^2-1/4)-(2nx+x^2+1/4)}{n\sqrt{1/4+(x-n)^2}(n+\sqrt{1/4+(x-n)^2})}\\ &\geq \sum_{n\geq 2}\frac{-2x^2-1/2}{n\sqrt{1/4+(x-n)^2}(n+\sqrt{1/4+(x-n)^2})}\\ &\geq -\sum_{n\geq 2}\frac{5/2}{n\sqrt{1/4+(n-1)^2}(n+\sqrt{1/4+(n-1)^2})}\geq -\frac{3}{5}.\end{aligned}$$

As for the term I_1, it is easy to get

$$I_1 \geq \min I_1 \geq \frac{6}{5}.$$

Therefore, we get $2V_0(\frac{1}{2},x)\geq \frac{6}{5}-\frac{3}{5}>0$, i.e. $V_0(\frac{1}{2},x)>0$.

Part b). According to the property (3), we just need to show: $V_0(2,x)<0$. In this case, we use the following decomposition

$$2V_0(2,x)=\widetilde{I}_1+\widetilde{I}_2,$$

where

$$\widetilde{I}_1=\sum_{-1\leq n\leq 4}[\frac{1}{\sqrt{4+(x-n)^2}}-\nu_n],$$

and

$$\widetilde{I}_2=\sum_{n\geq 5}[\frac{1}{\sqrt{4+(x-n)^2}}-\frac{1}{n}]+\sum_{n\geq 2}[\frac{1}{\sqrt{4+(x+n)^2}}-\frac{1}{n}].$$

We rewrite $\widetilde{I}_2$ as follows:

$$\widetilde{I}_2=\sum_{n\geq 2}[\frac{1}{\sqrt{4+(n+3-x)^2}}-\frac{1}{n+3}]+\sum_{n\geq 2}[\frac{1}{\sqrt{4+(n+x)^2}}-\frac{1}{n}]$$

Notice that

$$\frac{1}{\sqrt{4+(n+3-x)^2}}-\frac{1}{n+3}=\frac{2(n+3)x-x^2-4}{(n+3)\sqrt{4+(n+3-x)^2}(n+3+\sqrt{4+(n+3-x)^2})},$$

and

$$\frac{1}{\sqrt{4+(x+n)^2}}-\frac{1}{n}=-\frac{2nx+x^2+4}{n\sqrt{4+(n+x)^2}(n+\sqrt{4+(n+x)^2})}.$$

Thus we get

$$\begin{aligned}\widetilde{I}_2 &\le \sum_{n\ge 2}\frac{(2(n+3)x-x^2-4)-(2nx+x^2+4)}{n\sqrt{4+(n+x)^2}(n+\sqrt{4+(n+x)^2})}\\ &\le \sum_{n\ge 2}-\frac{2x^2-6x+}{n\sqrt{4+(n+x)^2}(n+\sqrt{4+(n+x)^2})}\\ &\le \sum_{n\ge 2}-\frac{4}{n\sqrt{4+(n+1)^2}(n+\sqrt{4+(n+1)^2})}\le -\frac{1}{10}.\end{aligned}$$

As for the term $\widetilde{I}_1$, it is easy to get

$$\widetilde{I}_1 \le \max I_1 \le \frac{12}{5}-\frac{37}{12}<0.$$

Therefore, we get $2V_0(2,x)<0$, i.e. $V_0(2,x)<0$.

Now let us restrict V_0 on x-axis, denoted the corresponding function by $\varphi_{ov}(x)$, i.e.,

$$\varphi_{ov}(x):=V_0(0,x)=0+\frac{1}{2}\cdot\frac{1}{|x|}+\sum_{n\in\mathbb{N}}\frac{1}{2}\cdot[\frac{1}{n+x}+\frac{1}{n-x}-\frac{2}{n}]. \tag{5.5}$$

It turns out $\varphi_{ov}(x)$ is closely related to the digamma function $\psi(x)$ in number theory and the Euler-Mascheroni constant γ. Digamma function $\psi(x)$ has following expression:

$$\psi(x+1)=-\gamma+\sum_{k=1}^{\infty}(\frac{1}{k}-\frac{1}{x+k}). \tag{5.6}$$

Hence by (5.5) and (5.6), we have

$$\varphi_{ov}(x) = \frac{1}{2}\cdot\frac{1}{|x|} - \sum_{n\in\mathbb{N}}\frac{1}{2}\cdot(\frac{1}{n}-\frac{1}{x+n}) - \frac{1}{2}\cdot\sum_{n\in\mathbb{N}}(\frac{1}{n}-\frac{1}{-x+n}) \tag{5.7}$$
$$= -\gamma + \frac{1}{2}\cdot\frac{1}{|x|} - \frac{1}{2}[\psi(x+1)+\psi(-x+1)]$$
$$= -\gamma + \frac{1}{2|x|} - \frac{1}{2}[\psi(x)+\psi(-x)]$$

From above expression (5.7), we directly get

Lemma 5.9 $\varphi_{ov}(x) \in C^\infty(\mathbb{R}-\mathbb{Z})$, *it is a positive and even function with period 1.*

By the periodic property, we can just consider the function in one period $(0,1)$. As a result of the property of digamma function, we can get

Proposition 5.10 $\varphi_{ov}(x)$ *is strict decreasing on* $(0,\frac{1}{2})$, *and thus strict increasing on* $(\frac{1}{2},1)$, *with:*

$$\min_{0<x<1}\varphi_{ov}(x) = \varphi_{ov}(\frac{1}{2}) = \ln 4.$$

For general $r>0$ case, we can get similar result by the following symmetry of the expression of the potential function for $x\in[0,1]$:

$$V_0(r,x) = \lim_{k\to+\infty}\frac{1}{2}\sum_{n=0}^{k}\{[\frac{1}{\sqrt{r^2+(n+x)^2}} + \frac{1}{\sqrt{r^2+(n+1-x)^2}}] - \frac{2}{n}\}$$

Based on the estimate given in Propostion 5.7, similarly we also have

Proposition 5.11 *For fixed* $\frac{1}{2}\le r\le 2$, *we have the extremal value of* $V_0(r,x)$*:*

$$\min V_0(r,x) = V_0(r,\frac{1}{2}) = \frac{1}{2}\cdot\frac{1}{\sqrt{r^2+\frac{1}{4}}} + \frac{1}{2}\cdot\sum_{n\in\mathbb{Z}-\{0\}}[\frac{1}{\sqrt{r^2+(\frac{1}{2}-n)^2}} - \frac{1}{|n|}].$$

From Proposition 5.11, we obtain a sharp lower bound for $R_0(x)$, which is $R_0(\frac{1}{2})$. By direct calculation of $V_0(r, \frac{1}{2}) > 0$, we can get an estimate:

$$r > 1, \ thus: \ R_0(\frac{1}{2}) > 1.$$

This leads to following result:

Corollary 5.12 *When $c = 0$, the region $\{V_0 > 0\}$ is cylindrical region around the x-axis of the form (5.4) with $1 < R_0(x) < 2$.*

For general $c \geq 0$, notice that for fixed r and x, the function $V_c(r, x)$ is always increasing with respect to c, thus we have:

Corollary 5.13 *the region $\{V_c > 0\}$ is cylindrical region around the x-axis of the form (5.4) with $1 < \varepsilon_1(c) < R_c(x) < \varepsilon_2(c)$. Here ε_1 are smooth increasing functions of c $(c \geq 0)$, moreover when $c \to \infty$, $\varepsilon_2(c)$ has the order of e^c.*

Notice that the radius function $R_c(x)$ automatically descends to the base: $R^2 \times S^1$ of local K3, here we recall ϕ as the coordinate of S^1 component. Now we arrive the locality propery of the Ooguri-Vafa metric:

Theorem 5.14 *The Ooguri-Vafa metric is a hyperkähler metric of the S^1 fibration restricted over an open solid torus, with varying radius $1 < R_c(\phi) < \infty$.*

Remark. From the construction we can see the potential and thus the metric loose uniqueness. Since we just consider positive periodic potential on some solid torus $D \times S^1$, we can construct a lot of other potentials in the form:

$$\widetilde{V} = V_c + h(r, x).$$

For example, in [20] Gross and Wilson make a little correction of the potential function, then consider a new family of potential function:

$$\widetilde{V} = \widetilde{V}_c + f(y, z),$$

where $f(y, z)$ is a harmonic function satisfies some bounded condition.

Apply the Gibbons-Hawking ansatz, then we get a lot of new hyperkähler metrics in a small neighborhood of the singular fibre.

5.4 Geometry of the Ooguri-Vafa Metric

In this part, we follow the calculation in [7] to estimate the diameter of the fibration under the Ooguri-Vafa metric. Finally, we will see the Ooguri-Vafa metric is not a complete metric on the fibration.

We recall the formula of the metric according to the Gibbons-Hawking ansatz, namely:

$$g_{ov} = V_c d\boldsymbol{x} \cdot d\boldsymbol{x} + V_c^{-1} \alpha_0^2$$

Here the potential function V_c is a function on the base: $\mathbb{R}^3 - \{(0, 0, \mathbb{Z})\}$, now we picked the cylindrical coordinate $\{r, \theta; x\}$ on the base. α_0 is the connection 1-form of the S^1 bundle. Since the function is periodic in the x direction, we just need to consider the function in one period, that is: $0 \leq x < 1$.

Generally if we choose a local trivialization of the S^1-firbration so as to have a coordinate t on the fibre. By solving the equation: $d\alpha_0 = *dV_c$ under the cylindrical coordinates $\{r, \theta; x\}$, we can get an explicit formula of the Ooguri-Vafa metric [18]:

Theorem 5.15 *Locally the connection 1-form α is given as:*

$$\alpha_0 = \sum_{n\in\mathbb{Z}}[\frac{x-n}{2\sqrt{r^2+(x-n)^2}} + \frac{sign(n)}{2}]d\theta + \frac{1}{2\pi}dt$$

up to gauge equivalence. Here we set $sign(0) = 0$.

By the Gibbons-Hawking ansatz, we get the local expression of the Ooguri-Vafa metric:

$$g_{ov} = g_1 + g_2,$$

with

$$g_1 = V_c dr^2 + V_c r^2 d\theta^2 + V_c dx^2,$$
$$g_2 = \frac{1}{V_c}\{\sum_{n\in\mathbb{Z}}[\frac{x-n}{2\sqrt{r^2+(x-n)^2}} + \frac{sign(n)}{2}]d\theta + \frac{1}{2\pi}dt\}^2.$$

Remark. The term $sign(n)$ here is added for convergence. The metric above is periodic in the x direction.

Recall that for the potential function, we has the following properties:

1) $\lim_{(r,x)\to(0,0)} V_c(r,x) = +\infty$;

2) $\lim_{r\to R_c(x)} V_c(r,x) = 0$;

By direct calculation, we get the first estimate on one side:

Lemma 5.16 *For fixed x, when $r \to 0$, we have the estimate for the potential function $V_c(r,x)$:*

1) if $x = 0$, then $V_c(r,0) \to \infty$ with the order of r^{-1};

2) if $x \neq 0$, then $V_c(r,x) < \frac{C_1(x)}{\sqrt{r^2+x^2}}$, here $C_1(x)$ is a positive constant depends on x and c.

On the other side, by direct calculation of the derivative we can get a positive constant $C_2(x)$ such that:

$$\partial_r V_c(r,x)|_{r=R_c(x)} = -C_2(x)$$

Therefore, we have the second estimate:

Lemma 5.17 *For fixed x, when $r \to R_c(x)$, we have $V_c(r,x) \to 0$ with the order of: $(R_c(x) - r)$, and thus $V_c^{-1/2}(r,x) \to +\infty$ with the order of: $(R_c(x) - r)^{-1/2}$.*

Now let us start the estimate of diameter of the fibration in each direction.

First, we estimate the diameter of the S^1 fibre, that is the case when r, θ and x are fixed:

Proposition 5.18 *The diameter of S^1 fibre over the base point (r, θ, x), with $0 \le r < R_c(x)$, is given as:*

$$d_1(r,\theta,x) = \int_0^{2\pi} \frac{1}{2\pi} V_c^{-1/2}(r,x)dt = V_c^{-1/2}(r,x).$$

When $(r,x) \to (0,0)$, we get the diameter of the singular fibre: $d_1(0,0) = 0$;

when $r \to R_c(x)$, the diameter $d_1(r,x) \to \infty$, with the order of: $(R_c(x) - r)^{-1/2}$.

Next we consider the diameter of the fibration in the horizontal direction.

We start from the x direction. We consider the central axis of the cylinder, which is the S^1 on the base. In this case, we have fixed $r = 0$. Let us denote the diameter by d_2.

Recall that when $r = 0$, we have the expression of the potential as:

$$V_c(0,x) = c + \varphi_{ov}(x) = c - \gamma + \frac{1}{2x} - \frac{1}{2}[\psi(1+x) + \psi(1-x)].$$

By the formula about digamma function and usual zeta function, we get:

$$-[\psi(1+x) + \psi(1-x)] = 2(1-x^2)^{-1} + 2\gamma - 2 + \sum_{n=1}^{\infty} 2(\zeta(2n+1) - 1)x^{2n}.$$

Thus we have a simple expression of the potential:

$$V_c(0,x) = c + \frac{1}{2x} + \frac{x^2}{1-x^2} + \sum_{n=1}^{\infty}(\zeta(2n+1)-1)x^{2n},$$

which has a radius of convergence at least 2.

Then we get the following estimate:

Proposition 5.19 *The diameter d_2 of the central S^1 in the x direction is given as:*

$$d_2 = \int_0^1 V_c^{1/2}(0,x)dx,$$

which is bounded as:

$$\max\{\sqrt{2},\sqrt{c}\} < d_2 < \sqrt{c} + (1+\sqrt{2}) + \sum_{n=1}^{\infty}\frac{(\zeta(2n+1)-1)^{1/2}}{n+1}.$$

Proof 5.20 *The lower bound follows from the property:*

$$V_c(0,x) > c \ , \ \text{and } V_c(0,x) > \frac{1}{2x},$$

thus,

$$V_c^{1/2}(0,x) > \sqrt{c} \ , \ \text{and } V_c^{1/2}(0,x) > \frac{1}{\sqrt{2x}}.$$

we get the result after integration.

The upper bound follows from the property:

$$V_c^{1/2}(0,x) < \sqrt{c} + \sqrt{\frac{1}{2x}} + \sqrt{\frac{x^2}{1-x^2}} + \sum_{n=1}^{\infty}(\zeta(2n+1)-1)^{1/2}x^n,$$

again, we get the result after integration.

Remark. When the central line moves to the nodal set of the potential, with θ fixed, then it is easy to check: since $V(r,x)$ decreases to 0, the diameter of the S^1 on the base decreases to 0.

At last, we consider the diameter in the radial direction. For $0 \leq x < 1$, we consider a radial curve l_x from $r = 0$ to $r = R_c(x)$ within the slice. There is a horizontal lift $\widetilde{l_x}$ of l_x to the fibration. Let us denote the length of curve $\widetilde{l_x}$ by d_3.

Notice that for $0 \leq x < 1$, we always have: $R_c(x) > 1$. By the property of the potential function, we get the estimate:

Proposition 5.21 *For $0 \leq x < 1$, the length $d_3(x)$ of the radial curve is given as:*

$$d_3(x) = \int_0^{R_c(x)} V_c^{1/2}(r, x)dr,$$

which is bounded as:

$$\sqrt{c + \frac{1}{10}} < d_3(x) \leq C_3(x) + 2C_2^{1/2}(x) \cdot R_c^{3/2}(x) < +\infty.$$

Here C_i are all positive constants which only depend on c and x.

Proof 5.22 *The lower bound comes from the estimate:*

$$d_3(x) > \int_0^1 V_c^{1/2}(r, x)dr > \int_0^1 V_c^{1/2}(1, x)dr > \int_0^1 V_c^{1/2}(1, \frac{1}{2})dr,$$

and here for $V_c^{1/2}(1, \frac{1}{2})$, we have:

$$V_c(1, \frac{1}{2}) = c + \frac{1}{2\sqrt{1 + (\frac{1}{2})^2}} + \frac{1}{2}\sum_{n\in\mathbb{N}}\{[\frac{1}{\sqrt{1 + (n - \frac{1}{2})^2}} + \frac{1}{\sqrt{1 + (n + \frac{1}{2})^2}}] - \frac{2}{n}\},$$

$$V_c(1, \frac{1}{2}) \geq c + \frac{1}{\sqrt{5}} + \frac{1}{2}(-\frac{17}{25}) > c + \frac{1}{10}.$$

We get the result after integration.

Then upper bound comes from the estimate:

$$d_3(x) = \int_0^1 V_c^{1/2}(r, x)dr + \int_1^{R_c(x)} V_c^{1/2}(r, x)dr,$$

by Lemma 5.16 for the first term and Lemma 5.17 for the second term, we get the estimate by integration.

Notice that the Ooguri-Vafa is not defined on the nodal set of the potential function, thus it is just defined on the fibration over an open solid torus, which is not topological closed. Therefore the Proposition 5.21 implies that:

Corollary 5.23 *The Ooguri-Vafa metric is not a complete metric.*

As a result, generally we will get an incomplete hyperkähler metric in a small neighborhood of the singular fibre, but not a complete metric in the entire local K3. It will be an interesting problem to find a complete hyperhähler metric in local K3.

Appendix A

Some calculations

A.1 Fourier expansion of the Ooguri-Vafa potential

In this part, we calculate the Fourier expansion of the potential function V_c first and then discuss its asymptotic expansion when $r \to \infty$.

First, recall that the potential function $V_c(r,x)$ is periodic on the x direction, i.e. $V_c(r,x) = V_c(r,x+1)$, thus it has Fourier expansion with respect to x. Now let us figure out the explicit expansion by the standard calculation.

For the 0-mode term, we apply the trick of [20] and arrange the term in the following way:

$$2V_0(r,x) = \lim_{n\to\infty}\sum_{k=-n}^{n}[\frac{1}{\sqrt{r^2+(x-k)^2}} - \widetilde{\nu}_k] + \lim_{n\to\infty}\sum_{k=-n}^{n}(\widetilde{\nu}_k - \nu_k).$$

Here the correction term is given as: $\nu_k = \frac{1}{|k|}$, $\widetilde{\nu}_k = \ln(|k|+1) - \ln|k|$ if $k \neq 0$, and $\nu_0 = \widetilde{\nu}_0 = 0$. Now let us calculate the 0-mode of each term.

For the first part, we have:

$$\begin{aligned}
&\int_0^1 \sum_{k=-n}^{n} [\frac{1}{\sqrt{r^2+(x-k)^2}} - \widetilde{\nu}_k]dx \\
&= \sum_{k=-n}^{n} \int_0^1 [\frac{1}{\sqrt{r^2+(x-k)^2}} - \widetilde{\nu}_k]dx \\
&= \sum_{k=-n}^{n} \widetilde{\nu}_k + \sum_{k=-n}^{n} \int_k^{k+1} \frac{1}{\sqrt{r^2+x^2}}dx \\
&= \int_1^{n+1} -\frac{1}{x}dx + \int_{-n}^{0} \frac{1}{x-1}dx + \sum_{k=-n}^{n} \int_k^{k+1} \frac{1}{\sqrt{r^2+x^2}}dx \\
&= \int_1^{n+1} (\frac{1}{\sqrt{r^2+x^2}} - \frac{1}{x})dx + \int_{-n}^{0} (\frac{1}{\sqrt{r^2+x^2}} + \frac{1}{x-1})dx + \int_0^1 \frac{1}{\sqrt{r^2+x^2}}dx \\
&= \ln(\frac{x+\sqrt{r^2+x^2}}{x})|_1^{n+1} + \ln[|x-1| \cdot (x+\sqrt{r^2+x^2})]|_{-n}^{0} + \ln(x+\sqrt{r^2+x^2})|_0^1 \\
&= \ln[\frac{(n+1)+\sqrt{r^2+(n+1)^2}}{n+1}] - \ln[(n+1)\cdot(-n+\sqrt{r^2+n^2})] \\
&= \ln[\frac{(n+1)+\sqrt{r^2+(n+1)^2}}{n+1}] - \ln[(n+1)\cdot(-n+\sqrt{r^2+n^2})] \\
&= \ln[\frac{(n+1)+\sqrt{r^2+(n+1)^2}}{n+1}] - \ln[\frac{(n+1)\cdot r^2}{n+\sqrt{r^2+n^2}}].
\end{aligned}$$

Let $n \to \infty$, we get the 0-mode of the first part is:

$$\int_0^1 \lim_{n\to\infty} \sum_{k=-n}^{n} [\frac{1}{\sqrt{r^2+(x-k)^2}} - \widetilde{\nu}_k]dx = \ln 2 - \ln\frac{r^2}{2} = 2\ln 2 - 2\ln r.$$

For the second part, we have:

$$\begin{aligned}
&\int_0^1 \sum_{k=-n}^{n} (\widetilde{\nu}_k - \nu_k)dx \\
&= \sum_{k=-n,k\neq 0}^{n} [\ln(|n|+1) - \ln|n|] - \sum_{k=-n,k\neq 0}^{n} \frac{1}{|n|} \\
&= 2\ln(n+1) - \sum_{k=1}^{n} \frac{2}{n}.
\end{aligned}$$

Let $n \to \infty$, we get the 0-mode of the second part is:

$$\int_0^1 \lim_{n\to\infty} \sum_{k=-n}^{n} (\widetilde{\nu}_k - \nu_k)dx = -2\gamma.$$

Here γ is the Euler-Mascheroni constant as given before.

Thus, we get the 0-mode of the potential function as follows:

$$\int_0^1 V_c(r,x)dx = c + \ln 2 - \gamma - \ln r.$$

It is easy to see here when $r \leq 1$, the 0-mode is always positive.

Now let us calculate the other mode term:

$$\begin{aligned}
&\int_0^1 V_c(r,x)\cdot e^{2\pi imx}dx \\
&= \int_0^1 \sum_{k\in\mathbb{Z}} \frac{1}{2\sqrt{r^2+(x-k)^2}} \cdot e^{2\pi imx}dx \\
&= \sum_{k\in\mathbb{Z}} \int_0^1 \frac{1}{2\sqrt{r^2+(x-k)^2}} \cdot e^{2\pi imx}dx \\
&= \sum_{n\in\mathbb{Z}} \int_{-n}^{-n+1} \frac{1}{2\sqrt{r^2+x^2}} \cdot e^{2\pi imx}dx \\
&= \int_{-\infty}^{+\infty} \frac{1}{2\sqrt{r^2+x^2}} \cdot e^{2\pi imx}dx \\
&= \int_0^{+\infty} \frac{\cos(2\pi mx)}{\sqrt{r^2+x^2}}dx \\
&= \int_0^{+\infty} \frac{\cos(2\pi m\cdot ru)}{\sqrt{1+u^2}}du \quad (here: u = \frac{x}{r}) \\
&= K_0(2\pi|mr|).
\end{aligned}$$

Here $K_0(x)$ is the modified Bessel function of the second kind.

As a result, we get the Fourier expansion of the potential function as follows:

$$V_c(r,x) = (c + \ln 2 - \gamma) + \frac{1}{2}\ln(\frac{1}{r^2}) + \sum_{m\neq 0} e^{2\pi imx}\cdot K_0(2\pi|mr|).$$

Moreover, when $c \gg r \gg 0$, we can get the asymptotic expansion of potential function by the property of Bessel function $K_0(x)$. Notice that when $x \gg 0$, we have the asymptotic expansion:

$$K_0(x) = \sqrt{\frac{\pi}{2x}} \cdot e^{-x} \cdot \sum_{n=0}^{\infty} \frac{(-1)^n[(2n-1)!!]^2}{n!(8x)^n},$$

Thus we have the asymptotic expansion of the potential function when $c \gg r \gg 0$ as follows:

$$V_c(r,x) = (c + \ln 2 - \gamma) + \frac{1}{2}\ln(\frac{1}{r^2}) + V_{instanton}(r,x),$$

here the instanton term is given as follows:

$$\begin{aligned} V_{instanton}(r,x) &= \sum_{m\neq 0}\{\exp[-2\pi(|mr| - imx)] \cdot \sum_{n=0}^{\infty}(-1)^n \frac{2\sqrt{\pi}[(2n-1)!!]^2}{n!} \cdot (\frac{1}{16\pi|mr|})^{n+\frac{1}{2}}\} \\ &= \sum_{m\neq 0}\{\exp[-2\pi(|mr| - imx)] \cdot \sum_{n=0}^{\infty} \frac{\sqrt{\pi}\cdot\Gamma(\frac{1}{2}+n)}{n!\cdot\Gamma(\frac{1}{2}-n)} \cdot (\frac{1}{4\pi|mr|})^{n+\frac{1}{2}}\}. \end{aligned}$$

Physically [35] the term $2\pi|mr|$ in the exponent gives the the Born-Infeld action for the $m-$instanton, and the second term $2\pi imx$ describes the coupling of the D2-brane to the RR field.

A.2 Curvature form for the Gibbons-Hawking Ansatz

In this part, we make some tensor calculation of the Gibbon-Hawking ansatz, which is important ingredient for further construction and application.

Since the Gibbons-Hawking ansatz comes in the good form:

$$g = V((\mathrm{d}u^1)^2 + (\mathrm{d}u^2)^2 + (\mathrm{d}u^3)^2) + V^{-1}(\alpha_0)^2, \quad d\alpha_0 = *dV,$$

we take the orthonormal coframe for the metric, which is just:

$$\omega^1 = V^{1/2}\mathrm{d}u^1,\ \omega^2 = V^{1/2}\mathrm{d}u^2,\ \omega^3 = V^{1/2}\mathrm{d}u^3,\ \omega^4 = V^{-1/2}\alpha_0.$$

Let us solve the Cartan's Structural Equations (torsion free):

$$\mathrm{d}\omega^i = \omega^j \wedge \omega^i_j, \quad \omega^i_j + \omega^j_i = 0, \tag{A.1}$$

$$\mathrm{d}\omega^i_j = \omega^k_j \wedge \omega^i_k + \Omega^i_j. \tag{A.2}$$

We solve the first equation first. A direct computation shows that:

$$\mathrm{d}\omega^1 = \frac{1}{2}V^{-1}\mathrm{d}V \wedge \omega^1,\ \mathrm{d}\omega^2 = \frac{1}{2}V^{-1}\mathrm{d}V \wedge \omega^2,\ \mathrm{d}\omega^3 = \frac{1}{2}V^{-1}\mathrm{d}V \wedge \omega^3,$$

$$\mathrm{d}\omega^4 = -\frac{1}{2}V^{-1}\mathrm{d}V \wedge \omega^4 + V^{-3/2}(V_1\omega^2 \wedge \omega^3 + V_2\omega^3 \wedge \omega^1 + V_3\omega^1 \wedge \omega^2).$$

Thus we can get the expression for $\{\omega^i_j\}$:

$$\omega^1_2 = \frac{1}{2}V^{-3/2}(V_2\omega^1 - V_1\omega^2 - V_3\omega^4),\ \omega^1_3 = \frac{1}{2}V^{-3/2}(V_3\omega^1 - V_1\omega^3 + V_2\omega^4),$$

$$\omega^2_3 = \frac{1}{2}V^{-3/2}(V_3\omega^2 - V_2\omega^3 - V_1\omega^4),\ \omega^1_4 = -\frac{1}{2}V^{-3/2}(V_3\omega^2 - V_2\omega^3 - V_1\omega^4),$$

$$\omega^2_4 = -\frac{1}{2}V^{-3/2}(V_1\omega^3 - V_3\omega^1 - V_2\omega^4),\ \omega^3_4 = -\frac{1}{2}V^{-3/2}(V_2\omega^1 - V_1\omega^2 - V_3\omega^4).$$

Then we solve the second equation and get the curvature form $\{\Omega^i_j\}$. Set the following anti-self dual basis:

$$\omega_a^- = \frac{1}{2}(\omega^1 \wedge \omega^2 - \omega^3 \wedge \omega^4),\ \omega_b^- = \frac{1}{2}(\omega^1 \wedge \omega^3 - \omega^4 \wedge \omega^2),\ \omega_c^- = \frac{1}{2}(\omega^1 \wedge \omega^4 - \omega^2 \wedge \omega^3),$$

For $i < j \leq 3$ cases, we have:

$$\begin{aligned}
\Omega^1_2 &= \omega_a^- \cdot \{-V^{-3}(-V_1^2 - V_2^2 + 2V_3^2) + V^{-2}V_{33}\} + \omega_b^- \cdot \{3V^{-3}V_2V_3 - V^{-2}V_{23}\} \\
&\quad + \omega_c^- \cdot \{3V^{-3}V_1V_3 - V^{-2}V_{13}\} \\
\Omega^1_3 &= \omega_b^- \cdot \{-V^{-3}(-V_1^2 - V_3^2 + 2V_2^2) + V^{-2}V_{22}\} + \omega_a^- \cdot \{3V^{-3}V_2V_3 - V^{-2}V_{23}\} \\
&\quad + \omega_c^- \cdot \{-3V^{-3}V_1V_2 + V^{-2}V_{12}\} \\
\Omega^2_3 &= \omega_c^- \cdot \{-V^{-3}(V_2^2 + V_3^2 - 2V_1^2) - V^{-2}V_{11}\} + \omega_a^- \cdot \{-3V^{-3}V_1V_3 + V^{-2}V_{13}\} \\
&\quad + \omega_b^- \cdot \{3V^{-3}V_1V_2 - V^{-2}V_{12}\}.
\end{aligned}$$

For $j = 4$ cases, we have:

$$\begin{aligned}
\Omega^1_4 &= \omega_c^- \cdot \{-V^{-3}(-V_2^2 - V_3^2 + 2V_1^2) + V^{-2}V_{11}\} + \omega_a^- \cdot \{3V^{-3}V_1V_3 - V^{-2}V_{13}\} \\
&\quad + \omega_b^- \cdot \{-3V^{-3}V_1V_2 + V^{-2}V_{12}\} \\
\Omega^2_4 &= \omega_b^- \cdot \{-V^{-3}(-V_1^2 - V_3^2 + 2V_2^2) + V^{-2}V_{22}\} + \omega_a^- \cdot \{3V^{-3}V_2V_3 - V^{-2}V_{23}\} \\
&\quad + \omega_c^- \cdot \{-3V^{-3}V_1V_2 + V^{-2}V_{12}\} \\
\Omega^3_4 &= \omega_a^- \cdot \{-V^{-3}(V_1^2 + V_2^2 - 2V_3^2) - V^{-2}V_{33}\} + \omega_b^- \cdot \{-3V^{-3}V_2V_3 + V^{-2}V_{23}\} \\
&\quad + \omega_c^- \cdot \{-3V^{-3}V_1V_3 + V^{-2}V_{13}\}.
\end{aligned}$$

Meanwhile, we could observe the curvature forms also have the symmetry relations:

$$\Omega^1_2 = -\Omega^3_4, \quad \Omega^1_3 = \Omega^2_4, \quad \Omega^2_3 = -\Omega^1_4.$$

It is ready to check the Ricci Curvature is 0 here with the above expression.

At last, we consider the point-wise norm square of the curvature form. Notice that the norm square equals to: $\Omega\wedge*\Omega = -\Omega\wedge\Omega$, since the curvature form is anti self dual. It also equals to the square sum of all the coefficients of Ω^i_j under the basis: $\{\frac{\omega_a^-}{\sqrt{2}}, \frac{\omega_b^-}{\sqrt{2}}, \frac{\omega_c^-}{\sqrt{2}}\}$, thus we have:

$$
\begin{aligned}
\frac{1}{2}\|Rm\|^2 &= \sum_{i=1}^{3}\{-V^{-3}|\nabla V|^2 + (3V^{-3}V_i^2 - V^{-2}V_{ii})\}^2 + \sum_{i\neq j}^{3}(3V^{-3}V_iV_j - V^{-2}V_{ij})^2 \\
&= 3V^{-6}|\nabla V|^4 - 2\sum_{i=1}^{3} V^{-3}|\nabla V|^2(3V^{-3}V_i^2 - V^{-2}V_{ii}) + \sum_{i,j}^{3}(3V^{-3}V_iV_j - V^{-2}V_{ij})^2 \\
&= 3V^{-6}|\nabla V|^4 - 6V^{-6}|\nabla V|^4 + 2V^{-5}\Delta V + 9V^{-6}|\nabla V|^4 - 6V^{-5}\sum_{i,j} V_iV_{ij}V_j + V^{-4}\sum_{i,j} V_{ij}^2 \\
&= 6V^{-6}|\nabla V|^4 - 6V^{-5}\sum_{i,j} V_iV_{ij}V_j + V^{-4}\sum_{i,j} V_{ij}^2 \\
&= 6V^{-6}|\nabla V|^4 - 3V^{-5}\nabla V\nabla|\nabla V|^2 + V^{-4}|HessV|^2 \\
&= 6V^{-6}|\nabla V|^4 - 3V^{-5}\nabla V\nabla|\nabla V|^2 + \frac{1}{2}V^{-4}\Delta|\nabla V|^2 - \nabla V\nabla\Delta V \\
&= \frac{1}{4}V^{-1}\Delta\Delta V^{-1}.
\end{aligned}
$$

Therefore norm square of the curvature tensor is given by: $\frac{1}{2}V^{-1}\Delta\Delta V^{-1}$.

Remark. From the above formula we can get the decomposition of the curvature operator on the basis of self-dual 2 forms and anti self-dual 2 forms. It is easy to see that the Gibbons-Hawking ansatz only has the anti self-dual part of the Weyl curvature.

Moreover, we can find that the curvature tensor (W_{ij}) in this orthonormal basis is the trace-free part of the matrix $\left(\frac{V_{ij}}{V^2} - 3\frac{V_iV_j}{V^3}\right)$, which can also be written as $2V$ times the trace-free part of the Hessian of the function V^{-2}, see Donaldson's work for reference.

Remark. During the computation, we always use the fact that V is a harmonic function. In the absence of this condition, the curvature form will have different expression, for instance:

$$\begin{aligned}\Omega_2^1 = \omega_b^-\{-3V^{-3}V_2V_3 + V^{-2}V_{23}\} + \omega_c^-\{-3V^{-3}V_1V_3 + V^{-2}V_{13}\}\\ + \frac{\omega^1\wedge\omega^2}{2}\{V^{-3}(-V_1^2 - V_2^2 + 2V_3^2) + V^{-2}(V_{11} + V_{22})\}\\ - \frac{\omega^3\wedge\omega^4}{2}\{V^{-3}(-V_1^2 - V_2^2 + 2V_3^2) - V^{-2}V_{33}\}.\end{aligned}$$

Then the norm of the curvature tensor goes to:

$$\|Rm\|^2 = 12V^{-6}|\nabla V|^4 + V^{-4}\Delta|\nabla V|^2 - 6\nabla V\nabla|\nabla V|^2 + 4V^{-5}|\nabla V|^2\Delta V - 2V^{-4}(\Delta V)^2.$$

which will be simplified to $1/2V^{-1}\Delta\Delta V^{-1}$ in the presence of harmonicity of V.

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