

Representability in Mixed Integer Quadratic Programming

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Abstract

Representability results play a fundamental role in optimization since they provide characterizations of the feasible sets that arise from optimization problems. In this work we study classes of sets that appear in the feasibility version of mixed integer quadratic optimization problems. We focus on two cases of quadratic inequalities.

In the first half of this work, we assume that the quadratic inequality is an ellipsoidal inequality, a special case of a convex quadratic inequality. We provide complete characterizations of sets that can be obtained as the projection of such feasible regions in spaces extended by mixed binary variables. We provide another characterization allowing for mixed integer extended variables under the assumption that the data defining the feasible region is rational.

In the second half of this work, we assume that the quadratic inequality is convex. We provide a complete characterization of the sets that can be obtained as the projection of such feasible regions in spaces extended by mixed binary variables. In addition, we provide a complete characterization of these sets in the special cases where (i) the feasible region is bounded, (ii) only binary extended variables are present, and (iii) only continuous variables are present.

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Chapter 1

Introduction

1.1 Background

The theory of representability studies one fundamental question: Given a system of algebraic constraints of a specified form, which subsets of $\mathbb{R}^n$ can be represented in this way, possibly using additional variables? A thorough answer to this question would be given by a complete characterization of representable sets. Complete characterizations are useful in that they demonstrate the class of problems which can be modeled using a fixed set of constraints.

Representability is well understood for systems of linear inequalities. It is well known that the projection of a set described by finitely many linear inequalities is again described by finitely many linear inequalities. It follows from the Minkowski-Weyl Theorem that such sets decompose as the Minkowski sum of a polytope and a polyhedral cone.

Integer programming is the study of optimization problems in which some or all of the variables are restricted to be integers, see background references [20, 23, 3]. The feasible regions of mixed integer linear programs are called mixed-integer linear sets. Several researchers have investigated representability questions (see, e.g., [9, 12, 17, 18, 13]), and a systematic study for mixed-integer linear systems is mainly due to Meyer and Jeroslow.

In the case of mixed binary linear systems geometric characterizations have been given by Jeroslow and Lowe [14, 15]. A set $S \subseteq \mathbb{R}^n$ is *mixed binary linear representable* if S is the projected solution set of a mixed binary linear system

$$\begin{aligned} Dw &\leq d \\ w &\in \mathbb{R}^{n+p} \times \{0, 1\}^q. \end{aligned}$$

Then, Jeroslow and Lowe provide the following characterization.

Theorem 1.1 *A set $S \subseteq \mathbb{R}^n$ is mixed binary linear representable if and only if there exist rational polyhedra $P_1, \dots, P_k \subseteq \mathbb{R}^n$ with the same recession cone such that*

$$S = \bigcup_{i=1}^k P_i.$$

In the case of mixed-integer linear systems, Jeroslow and Lowe find a similar characterization. A set $S \subseteq \mathbb{R}^n$ is *mixed-integer linear representable* if S is the projected solution set of a *rational* mixed-integer linear system

$$\begin{aligned} Dw &\leq d \\ w &\in \mathbb{R}^{n+p} \times \mathbb{Z}^q. \end{aligned}$$

Jeroslow and Lowe establish the following result.

Theorem 1.2 *A set $S \subseteq \mathbb{R}^n$ is mixed-integer linear representable if and only if there exist rational polytopes $P_1, \dots, P_k \subseteq \mathbb{R}^n$ and vectors $r^1, \dots, r^t \in \mathbb{Z}^n$ such that*

$$S = \bigcup_{i=1}^k P_i + \text{int.cone}\{r^1, \dots, r^t\}.$$

Representability has also been studied in the case of nonlinear constraints, but few complete characterizations have been established. In second-order cone programming a

linear functional is maximized over a set defined by linear inequalities and constraints of the form $\|Ax + b\|_2 \leq c^\top x + d$. These constraints are quite general and can express a variety of different constraints, including convex quadratic inequalities. There has been a large amount of work [16, 19] that shows different second order cone formulations for a wide range of problems. However, a complete characterization similar to those obtained by Jeroslow and Lowe is missing.

In semidefinite programming, a linear functional is maximized over a set defined by a linear matrix inequality, i.e., a constraint of the form $A_0 + \sum_{i=1}^n x_i A_i \succeq 0$ where the A_i are symmetric matrices. A linear matrix inequality defines a closed, convex, semialgebraic set known as a spectrahedon. In [11], Helton and Vinnikov introduce the notion of rigid convexity and conjecture that a set is a spectrahedron if and only if it is rigidly convex. Another conjecture is stated in [10] where Helton and Nie study which sets can be represented as the projection of a spectrahedon in a higher dimensional space. They conjecture that every convex semialgebraic set can be represented as the projection of a spectrahedron.

There are two main difficulties in establishing these conjectures, as well as forming a characterization in the case of second-order cone programming. One difficulty lies in the complexity of describing the projection of semialgebraic sets. The other lies in finding an appropriate extended formulation of a representable set.

1.2 Our Contributions

In hopes of bridging the gap between characterization results for linear systems and similar results for nonlinear systems, we have considered sets described by linear inequalities

and a single convex quadratic inequality. We say that a region $\mathcal{Q}$ is a *convex quadratic region in $\mathbb{R}^n$* if

$$\mathcal{Q} = \{x \in \mathbb{R}^n \mid x^\top Qx + h^\top x + g \leq 0\}$$

for a positive semidefinite matrix $Q \in \mathbb{R}^{n \times n}$, a vector $h \in \mathbb{R}^n$, and $g \in \mathbb{R}$.

We say that a set $S \subseteq \mathbb{R}^n$ is *mixed binary convex quadratic representable* if it can be obtained as the projection onto $\mathbb{R}^n$ of the solution set of a system of the form

$$\begin{aligned} Dw &\leq d \\ w^\top Qw + h^\top w + g &\leq 0 \\ w &\in \mathbb{R}^{n+p} \times \{0, 1\}^q, \end{aligned} \tag{1.1}$$

where Q is positive semidefinite. Note that if a set S is the projection of the solution set of a system of the form (1.1), but with bounded integer variables in the place of the binary variables, then S is also the projection of the solution set of a system of the form (1.1). We also note that since any convex quadratic region is second-order cone representable, the sets that we characterize can be represented with second-order cone constraints and mixed binary extended variables.

There is a strong connection between mixed binary convex quadratic representable sets and mixed binary convex quadratic programming (MBCQP). This class of problems has applications in many areas, including portfolio optimization and machine learning [2, 1]. Since optimal solutions of MBCQP problems have polynomial size (see [21, 4]), any MBCQP is polynomially equivalent to a polynomial number of MBCQP feasibility problems. In particular, each feasibility problem is over a set of the form (1.1). Moreover, by linearizing the objective, any MBCQP can be transformed to the problem of minimizing a linear function over a set described by (1.1).

We have observed that a characterization of sets representable by more than one convex quadratic inequality seems to be currently out of reach. In fact, the intersection of two convex quadratic inequalities in $\mathbb{R}^3$ may project to a semialgebraic set described by polynomials of degree four in $\mathbb{R}^2$. In this work, we present characterization results for a number of cases of mixed binary convex quadratic representable sets.

In Chapter 2 we characterize mixed binary convex quadratic representable sets under the additional assumption that the convex quadratic inequality has the form $(x - c)^\top Q(x - c) \leq \gamma$. Inequalities of this type are called *ellipsoidal inequalities*. We say that a set $\mathcal{E} \subseteq \mathbb{R}^n$ is an *ellipsoidal region* in $\mathbb{R}^n$ if it is the set of points satisfying an ellipsoidal inequality.

Formally, we say that a set $S \subseteq \mathbb{R}^n$ is *binary ellipsoidal mixed-integer (binary EMI) representable* if it can be obtained as the projection onto $\mathbb{R}^n$ of the solution set of a system of the form

$$\begin{aligned} Dw &\leq d \\ (w - c)^\top Q(w - c) &\leq \gamma \\ w &\in \mathbb{R}^{n+p} \times \{0, 1\}^q, \end{aligned} \tag{1.2}$$

where Q is symmetric and positive semidefinite. In Theorem 2.1 on page 8 we show that a set $S \subseteq \mathbb{R}^n$ is binary EMI-representable if and only if S is a Minkowski sum of a finite union of sets which are the intersection of an ellipsoidal region and a polytope together with a polyhedral cone.

We also provide a characterization in the case where extended variables can be mixed integer in Theorem 2.2 on page 9. In this setting, it is necessary that the data describing system (1.2) is rational. We are able to obtain a similar result and in doing so directly

extend the work of Jeroslow and Lowe in [14, 15].

In Chapter 3 we study the more general case where the quadratic inequality in (1.1) is convex. We present characterization results for a number of cases of mixed binary convex quadratic representable sets. More precisely, we characterize sets when $Dw \leq d$ in (1.1) is bounded in Theorem 3.1 on page 31, when $p = 0$ in (1.1) in Theorem 3.2 on page 33, and when $q = 0$ in (1.1) in Theorem 3.9 on page 54, before proceeding with the general case in Theorem 3.10 on page 56.

We will provide further introductory discussion at the beginning of each chapter.

1.3 Notation

In the remainder of this work we will use the following notation. Given a set $E \subseteq \mathbb{R}^n \times \mathbb{R}^p$ and a vector $\bar{y} \in \mathbb{R}^p$, we define the $\bar{y}$ -restriction of E as

$$E|_{y=\bar{y}} = \{x \in \mathbb{R}^n \mid (x, \bar{y}) \in E\}.$$

Note that $E|_{y=\bar{y}}$ geometrically consists of the intersection of E with coordinate hyperplanes. Sometimes we will need to consider $E|_{y=\bar{y}}$ in the original space $\mathbb{R}^n \times \mathbb{R}^p$, thus we also define

$$\tilde{E}|_{y=\bar{y}} = \{(x, \bar{y}) \in \mathbb{R}^n \times \mathbb{R}^p \mid (x, \bar{y}) \in E\}.$$

We will also need to perform several restrictions $y_1 = \bar{y}_1, \dots, y_k = \bar{y}_k$ at the same time.

In such case we simply write $E|_{y_1=\bar{y}_1, \dots, y_k=\bar{y}_k}$ and $\tilde{E}|_{y_1=\bar{y}_1, \dots, y_k=\bar{y}_k}$.

Given a set $E \subseteq \mathbb{R}^n$, and a positive integer $k \leq n$, we will denote by $\text{proj}_k(E)$ the orthogonal projection of E onto its first k coordinates. Formally,

$$\text{proj}_k(E) = \{x \in \mathbb{R}^k \mid \exists y \in \mathbb{R}^{n-k} \text{ with } (x, y) \in E\}.$$

We note that $\text{proj}_k : \mathbb{R}^n \rightarrow \mathbb{R}^k$ is a linear transformation, and thus respects vector addition, i.e., Minkowski sums.

Given a set $E \subseteq \mathbb{R}^n$ we denote by $\text{rec}(E)$ the recession cone of E , namely the set of vectors $r \in \mathbb{R}^n$ such that for any $\lambda > 0$ and $x \in E$ we have $x + \lambda r \in E$. We note that nearly all of the sets we consider in this paper are closed and convex, in which case $\text{rec}(E)$ coincides with the set of recession directions at *any* point of E . We also denote by $\text{lin}(E)$ the lineality space of E , by $\text{span}(E)$ the linear space generated by E , and by $\text{cone}(E)$ the cone generated by E .

Given a matrix A we denote by $\text{range}(A)$ the range of A and by $\text{ker}(A)$ the kernel of A . If A is positive semidefinite, we write $A \succeq 0$. This implies that A is symmetric. Given a half-space $H^+ = \{x \in \mathbb{R}^n \mid a^\top x \leq b\}$, we write H for the hyperplane $\{x \in \mathbb{R}^n \mid a^\top x = b\}$.

Chapter 2

Ellipsoidal Representability¹

2.1 Introduction

In this chapter we provide characterizations of ellipsoidal mixed integer (EMI) representable sets. Recall that a set $S \subseteq \mathbb{R}^n$ is *binary ellipsoidal mixed-integer (binary EMI) representable* if it can be obtained as the projection onto $\mathbb{R}^n$ of the solution set of a system of the form

$$\begin{aligned} Dw &\leq d \\ (w - c)^\top Q(w - c) &\leq \gamma \\ w &\in \mathbb{R}^{n+p} \times \{0, 1\}^q, \end{aligned} \tag{2.1}$$

where Q is symmetric and positive semidefinite. Our main results are the following geometric characterizations of EMI-representable sets.

Theorem 2.1 *A set $S \subseteq \mathbb{R}^n$ is binary EMI-representable if and only if there exist ellipsoidal regions $\mathcal{E}_i \subseteq \mathbb{R}^n$, $i = 1, \dots, k$, polytopes $\mathcal{P}_i \subseteq \mathbb{R}^n$, $i = 1, \dots, k$, and a polyhedral cone $\mathcal{C} \subseteq \mathbb{R}^n$ such that*

$$S = \bigcup_{i=1}^k (\mathcal{E}_i \cap \mathcal{P}_i) + \mathcal{C}. \tag{2.2}$$

¹This chapter is modified from Del Pia and Poskin [6, 5].

In the rational mixed-integer case, i.e., when the binary variables $\{0, 1\}^q$ are replaced with integer variables and the defining data is rational, we obtain a similar characterization. Formally, we say that a set $S \subseteq \mathbb{R}^n$ is *ellipsoidal mixed-integer (EMI) representable* if it can be obtained as the projection onto $\mathbb{R}^n$ of the solution set of a system of the form

$$\begin{aligned} Dw &\leq d \\ (w - c)^\top Q(w - c) &\leq \gamma \\ w &\in \mathbb{R}^{n+p} \times \mathbb{Z}^q, \end{aligned} \tag{2.3}$$

where Q is positive semidefinite. We say that a set $S \subseteq \mathbb{R}^n$ is *rational EMI-representable* if the data D, Q, d, c, γ in (2.3) can be chosen to be rational.

Theorem 2.2 *A set $S \subseteq \mathbb{R}^n$ is rational EMI-representable if and only if there exist rational ellipsoidal regions $\mathcal{E}_i \subseteq \mathbb{R}^n$, $i = 1, \dots, k$, rational polytopes $\mathcal{P}_i \subseteq \mathbb{R}^n$, $i = 1, \dots, k$, and integral vectors $r^i \in \mathbb{Z}^n$, $i = 1, \dots, t$ such that*

$$S = \bigcup_{i=1}^k (\mathcal{E}_i \cap \mathcal{P}_i) + \text{int.cone}\{r^1, \dots, r^t\}. \tag{2.4}$$

An example of a binary EMI-representable set is given in Figure 1a while an example of an EMI-representable set is given in Figure 1b. Note that the second set is not binary EMI-representable as it is the disjoint union of an infinite number of convex regions.

The presence of rational data in Theorem 2.2 is essential to the development of a meaningful statement. Even in the pure integer linear case, irrational data may cause these kinds of complications. Consider the integer set $S = \{(z_1, z_2) \in \mathbb{Z}_{\geq 0}^2 \mid z_2 \leq \sqrt{2} z_1\}$. It can be shown that S cannot be represented as the Minkowski sum of a finite set and the set of non-negative integer combinations of a finite number of integral vectors.

Both directions of Theorem 2.1 and Theorem 2.2 have geometric implications. Since each set S of the form (2.2) or (2.4) can be obtained as the projection of a set described

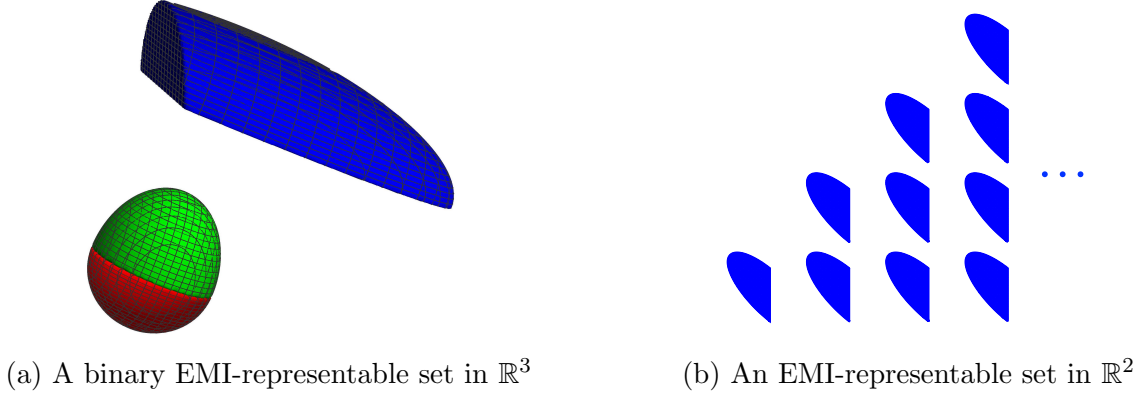


Figure 1: Examples of EMI-representable sets

by a system (2.1) or (2.3) this means that the k ellipsoidal regions $\mathcal{E}_i$ can be expressed with just one ellipsoidal inequality in a higher dimension. We prove this direction of the theorems by explicitly giving extended formulations for the sets S .

The other direction of Theorem 2.1 and Theorem 2.2 states that the projection of each system (2.1) or (2.3) onto $\mathbb{R}^n$ is a set of the form (2.2) or (2.4). An important ingredient of both proofs is showing that the projection of a set $\mathcal{E} \cap \mathcal{P} := \{x \in \mathbb{R}^{n+1} \mid Dx \leq d, (x - c)^\top Q(x - c) \leq \gamma\}$ onto $\mathbb{R}^n$ is a set of the form (2.2). In order to do so, we introduce the key concept of a *shadowing hyperplane*. This hyperplane, that will be formally introduced in Section 2.2, allows us to split the ellipsoidal region into two ‘parts’. In turn, this allows us to compute the projection of $\mathcal{E} \cap \mathcal{P}$ by computing a finite number of projections of $\mathcal{E}$ intersected with a hyperplane. This will show that the projection of $\mathcal{E} \cap \mathcal{P}$ is the union of a finite number of regions that are the intersection of a polyhedron and one nonlinear inequality, which we will prove to be ellipsoidal.

The remainder of this chapter is organized as follows. In Section 2.2, we provide a number of results relating to the intersection of an ellipsoidal region with a polyhedron and the projections of such regions. In Section 2.3, we prove Theorem 2.1. In Section 2.4,

we prove Theorem 2.2.

2.2 Ellipsoidal Regions and Hyperplanes

In this section we formally define ellipsoidal regions. These regions will appear throughout our study of representability. We will prove a few results on the intersection of ellipsoidal regions with half-spaces as well as their projections. These results will be necessary for our proofs of Theorem 2.1 and Theorem 2.2.

Recall, we say that a set $\mathcal{E}$ is an *ellipsoidal region* in $\mathbb{R}^n$ if there exists a real symmetric $n \times n$ matrix $Q \succeq 0$ (i.e., Q is positive semi-definite), a vector $c \in \mathbb{R}^n$, and $\gamma \in \mathbb{R}$, such that

$$\mathcal{E} = \{x \in \mathbb{R}^n \mid (x - c)^\top Q (x - c) \leq \gamma\}.$$

We note that if $Q \succ 0$ (i.e., Q is positive definite) and $\gamma > 0$, then $\mathcal{E}$ is an *ellipsoid*, i.e., the image of the unit ball $\mathcal{B} = \{x \in \mathbb{R}^n \mid \|x\|_2 \leq 1\}$ under an invertible affine transformation.

The following observation is well-known, and we give a proof for completeness.

Observation 1 *Let $q(x) = x^\top Q x + b^\top x$ be a quadratic function on $\mathbb{R}^n$ with Q a symmetric positive definite matrix. Then $q(x)$ has a minimum on $\mathbb{R}^n$ if and only if b is in the range of Q .*

Proof. Assume $b \notin \text{range}(Q)$. Then since Q is symmetric, we can write $b = Qr + c$ with $Qc = 0$ and $c \neq 0$. Consider the line $x(t) = -tc$ for $t \in \mathbb{R}$. Then we have

$$q(x(t)) = b^\top x(t) = -tc^\top c.$$

Since $c \neq 0$, we see that $q(x(t)) \rightarrow -\infty$ as $t \rightarrow +\infty$. Thus, $q(x)$ has no minimum on $\mathbb{R}^n$.

Assume there exists $r \in \mathbb{R}^n$ such that $\frac{1}{2}b = Qr$. Then

$$q(x) = (x + r)^\top Q(x + r) - r^\top Qr$$

and $q(x)$ has a minimum at any $\bar{x}$ such that $\bar{x} + r \in \ker(Q)$. In particular, $-r$ is a minimizer and $q(-r) = -r^\top Qr$ is the optimal value. $\square$

The following lemma shows that ellipsoidal regions are closed under intersections with coordinate hyperplanes. This is equivalent to fixing a number of variables.

Lemma 2.3 *Let $\mathcal{E}$ be an ellipsoidal region in $\mathbb{R}^n \times \mathbb{R}^p$. Then for any $\bar{y} \in \mathbb{R}^p$, the set $\mathcal{E}|_{y=\bar{y}}$ is an ellipsoidal region in $\mathbb{R}^n$.*

Proof. Let $\mathcal{E} = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^p \mid q(x, y) \leq \gamma\}$, where $q(x, y)$ is the quadratic polynomial

$$q(x, y) = \begin{pmatrix} x - c \\ y - c' \end{pmatrix}^\top \begin{pmatrix} Q & R \\ R^\top & Q' \end{pmatrix} \begin{pmatrix} x - c \\ y - c' \end{pmatrix}.$$

For any fixed $\bar{y} \in \mathbb{R}^p$, since Q is symmetric and positive definite it suffices to show there exists $c_{\bar{y}} \in \mathbb{R}^n$ and $\gamma_{\bar{y}} \in \mathbb{R}$ such that

$$\mathcal{E}|_{y=\bar{y}} = \{x \in \mathbb{R}^n \mid (x - c_{\bar{y}})^\top Q(x - c_{\bar{y}}) \leq \gamma_{\bar{y}}\}. \quad (2.5)$$

Let $\bar{y} \in \mathbb{R}^p$. Since $q(x, y)$ has a minimum on $\mathbb{R}^n \times \mathbb{R}^p$ by Observation 4, the quadratic function

$$q(x, \bar{y}) = (x - c)^\top Q(x - c) + 2(\bar{y} - c')^\top R^\top (x - c) + (\bar{y} - c')^\top Q'(\bar{y} - c'),$$

has a minimum on $\mathbb{R}^n$ as it is bounded from below. By Observation 4, $R(\bar{y} - c') \in \text{range}(Q)$, and so there exists $\bar{x} \in \mathbb{R}^n$ such that $Q\bar{x} = R(\bar{y} - c')$. Then (2.5) is satisfied with $c_{\bar{y}} := c - \bar{x}$ and $\gamma_{\bar{y}} := \gamma + \bar{x}^\top Q\bar{x} - (\bar{y} - c')^\top Q'(\bar{y} - c')$. $\square$

We are now ready to provide a geometric description of ellipsoidal regions. A consequence of this description is that any non-empty ellipsoidal region may be decomposed as the Minkowski sum of an ellipsoid and a linear space.

Lemma 2.4 *Let $\mathcal{E}$ be an ellipsoidal region in $\mathbb{R}^n$. Then*

(i) $\mathcal{E} = \emptyset$, or

(ii) $\mathcal{E}$ is an affine space, or

(iii) *There exists an integer $k \in \{0, \dots, n-1\}$, a k -dimensional linear space $L \subseteq \mathbb{R}^n$, and k distinct indices $i_1, \dots, i_k \in \{1, \dots, n\}$ such that the restriction*

$$\mathcal{E}|_{x_{i_1}=\bar{x}_{i_1}, \dots, x_{i_k}=\bar{x}_{i_k}}$$

is an ellipsoid in $\mathbb{R}^{n-k}$, and

$$\mathcal{E} = \tilde{\mathcal{E}}|_{x_{i_1}=\bar{x}_{i_1}, \dots, x_{i_k}=\bar{x}_{i_k}} + L.$$

Proof. Let $\mathcal{E} = \{x \in \mathbb{R}^n \mid (x - c)^\top Q(x - c) \leq \gamma\}$ where Q is a symmetric positive semidefinite matrix. If $\gamma < 0$, then $\mathcal{E} = \emptyset$ since Q is positive semidefinite. Thus, we may assume that $\gamma \geq 0$ and $\mathcal{E}$ is non-empty.

We now show that

$$\text{rec}(\mathcal{E}) = \ker(Q) = \{x \in \mathbb{R}^n \mid x^\top Qx \leq 0\}. \quad (2.6)$$

We first show that $\text{rec}(\mathcal{E}) = \ker(Q)$. Since $\mathcal{E}$ is a closed convex set, $\text{rec}(\mathcal{E})$ is equal to the set of recession directions from any point $x \in \mathcal{E}$. Consider the point $c \in \mathcal{E}$. Then for any $r \in \ker(Q)$ and $\lambda > 0$ we have $c + \lambda r \in \mathcal{E}$ since $\lambda^2 r^\top Qr = 0 \leq \gamma$. Thus, $r \in \text{rec}(\mathcal{E})$. Assume now that $r \in \text{rec}(\mathcal{E})$. Let $Q = L^\top L$ be a Cholesky decomposition of Q . Then

for any $\lambda > 0$ we have $\lambda^2 r^\top Q r = \lambda^2 \|Lr\|^2 \leq \gamma$, which implies $Lr = 0$ and $r \in \ker(Q)$. Next we show that $\ker(Q) = \{x \in \mathbb{R}^n \mid x^\top Q x \leq 0\}$. Clearly, the kernel is contained in the right hand side. Suppose $r \in \mathbb{R}^n$ satisfies $r^\top Q r \leq 0$. Since Q is positive semidefinite, we must have $r^\top Q r = 0$. Replacing Q with its Cholesky decomposition, we see that $\|Lr\|^2 = 0$. This implies $Lr = 0$, and thus $r \in \ker(Q)$.

Now assume $\gamma = 0$. By the above argument, $x \in \mathcal{E}$ if and only if $x \in c + \ker(Q)$. Thus $\mathcal{E} = c + \ker(Q)$ is an affine space.

Assume now $\gamma > 0$. If Q is invertible then $\mathcal{E}$ is an ellipsoid and we are done, in this case $k = 0$. Thus, we may assume $L := \ker(Q)$ is nontrivial. Let $k = \dim(L)$ and $\mathcal{L} = \{l_1, \dots, l_k\}$ be a basis for L . Note if $k = n$ then $\mathcal{E} = \mathbb{R}^n$, an affine space, and we are done. Thus, we may assume that $k < n$. Extend $\mathcal{L}$ to a basis $\mathcal{L}'$ of $\mathbb{R}^n$ by adding a subset of the standard basis vectors $\{e_1, \dots, e_n\}$ of $\mathbb{R}^n$. Let $\mathcal{J} \subseteq \{1, \dots, n\}$ be the set of indices j for which $e_j \in \mathcal{L}' - \mathcal{L}$, and suppose $\{i_1, \dots, i_k\} = \{1, \dots, n\} - \mathcal{J}$. Define

$$\mathcal{E}' := \mathcal{E}|_{x_{i_1}=0, \dots, x_{i_k}=0} \text{ and } \tilde{\mathcal{E}}' := \tilde{\mathcal{E}}|_{x_{i_1}=0, \dots, x_{i_k}=0}.$$

We now show $\mathcal{E} = \tilde{\mathcal{E}}' + L$. Since $\tilde{\mathcal{E}}' \subseteq \mathcal{E}$ and $\text{rec}(\mathcal{E}) = L$, we clearly have $\tilde{\mathcal{E}}' + L \subseteq \mathcal{E}$. Let $v \in \mathcal{E}$. Expanding v in the basis $\mathcal{L}'$, we have for some $l \in L$ and scalars $\alpha_j \in \mathbb{R}$, that $v = l + \sum_{j \in \mathcal{J}} \alpha_j e_j$. Since $L = \text{rec}(\mathcal{E})$ we have $v - l = \sum_{j \in \mathcal{J}} \alpha_j e_j \in \mathcal{E}'$ and $\mathcal{E} \subseteq \tilde{\mathcal{E}}' + L$.

By Lemma 2.3, $\mathcal{E}'$ is an ellipsoidal region in $\mathbb{R}^{n-k}$. Note first that $\mathcal{E}$ is full-dimensional in $\mathbb{R}^n$, i.e., has $n + 1$ affinely independent vectors. This is immediate since $\gamma > 0$ and there exists a vector, namely $c \in \mathbb{R}^n$, for which the continuous function $(x - c)^\top Q(x - c)$ has value 0. If $\mathcal{E}'$ is unbounded, then $\mathcal{E}'$ has some recession direction outside of L which contradicts the fact that $\text{rec}(\mathcal{E}) = L$. Moreover, since $\mathcal{E}'$ is bounded it follows from (2.6) that the matrix defining $\mathcal{E}'$ is invertible, and thus positive definite. Then $\mathcal{E}'$ is either an

ellipsoid or a single point. Since $\mathcal{E} = \tilde{\mathcal{E}}' + L$ is full dimensional, and $\dim(L) = k < n$, $\mathcal{E}'$ cannot be a single point. $\square$

We make the following remark about the proof of (iii) that will be used later. If one of the standard basis vectors of $\mathbb{R}^n$, say e_n , is not contained in L , then we may assume that x_n does not occur among the fixed variables $x_{i_1}, \dots, x_{i_k}$. To see this, note that in completing the basis $\mathcal{L}$ of L to a basis of $\mathbb{R}^n$ we may first add the standard basis vector e_n to the set $\mathcal{L}$.

It can be shown that an appropriate converse of Lemma 2.4 holds. This provides a complete geometric characterization of ellipsoidal regions. We use Lemma 2.4 to make the following observation that distinguishes ellipsoidal regions from general convex quadratic regions.

Observation 2 *Let $\mathcal{E}$ be an ellipsoidal region in $\mathbb{R}^n$. Then there exists a polyhedron $\mathcal{B} \subseteq \mathbb{R}^n$ such that $\mathcal{E} \subseteq \mathcal{B}$ and $\text{rec}(\mathcal{B}) = \text{rec}(\mathcal{E})$.*

Proof. By Lemma 2.4, $\mathcal{E}$ is either empty, an affine space, or the Minkowski sum of an ellipsoid in a possibly lower dimensional space and a linear space. Since affine spaces are polyhedral, and there is nothing to show for the empty case, it suffices to assume that $\mathcal{E} = \tilde{\mathcal{E}}|_{x_{i_1}=\bar{x}_{i_1}, \dots, x_{i_k}=\bar{x}_{i_k}} + L$ for an ellipsoid $\tilde{\mathcal{E}}|_{x_{i_1}=\bar{x}_{i_1}, \dots, x_{i_k}=\bar{x}_{i_k}}$ in $\mathbb{R}^{n-k}$ and a linear space L . Since $\tilde{\mathcal{E}}|_{x_{i_1}=\bar{x}_{i_1}, \dots, x_{i_k}=\bar{x}_{i_k}}$ is a bounded set there exists a polytope $\tilde{\mathcal{B}}$ such that $\tilde{\mathcal{E}}|_{x_{i_1}=\bar{x}_{i_1}, \dots, x_{i_k}=\bar{x}_{i_k}} \subseteq \tilde{\mathcal{B}}$. Then the polyhedron defined by $\mathcal{B} := \tilde{\mathcal{B}} + L$ has the desired properties. $\square$

The next observation gives a description of the recession cones that will be encountered in this chapter.

Observation 3 *Let $\mathcal{P}$ be a polyhedron and $\mathcal{E}$ an ellipsoidal region in $\mathbb{R}^n$. Then $\text{rec}(\mathcal{E} \cap \mathcal{P})$ is a polyhedral cone.*

Proof. Clearly, $\text{rec}(\mathcal{E} \cap \mathcal{P}) = \text{rec}(\mathcal{E}) \cap \text{rec}(\mathcal{P})$. The set $\text{rec}(\mathcal{P})$ is a polyhedral cone (see, e.g., [20]), and $\text{rec}(\mathcal{E})$ is a linear space by (2.6) in Lemma 2.4. As a consequence $\text{rec}(\mathcal{E} \cap \mathcal{P})$ is a polyhedral cone. $\square$

The following lemma shows that to compute the projection of an ellipsoidal region $\mathcal{E}$ in $\mathbb{R}^n$, it suffices to consider the projection of $\mathcal{E} \cap H$ for a specific hyperplane $H \subseteq \mathbb{R}^n$. We will refer to such a hyperplane H as a *shadowing hyperplane*, as it contains enough information to completely describe the projection, or ‘shadow’, of $\mathcal{E}$. See Figure 2 for an illustration.

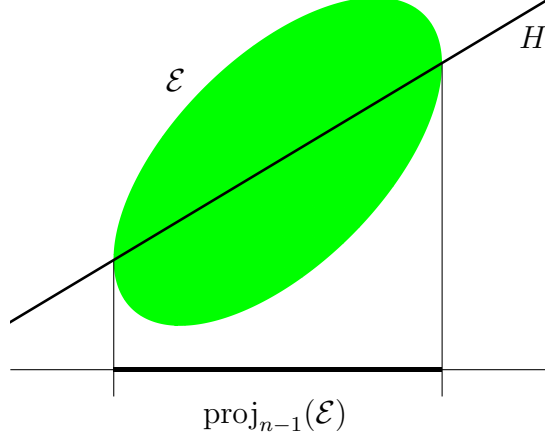


Figure 2: Illustration of a shadowing hyperplane

Lemma 2.5 *Let $\mathcal{E}$ be an ellipsoidal region in $\mathbb{R}^n$. Then there exists a hyperplane $H \subseteq \mathbb{R}^n$ with $e_n \notin \text{lin}(H)$ such that*

$$\text{proj}_{n-1}(\mathcal{E}) = \text{proj}_{n-1}(\mathcal{E} \cap H).$$

Proof. It clearly suffices to show that $\text{proj}_{n-1}(\mathcal{E}) \subseteq \text{proj}_{n-1}(\mathcal{E} \cap H)$. Let $\mathcal{E}$ be described by the ellipsoidal inequality $q(x) = (x - c)^\top Q(x - c) \leq \gamma$. We note that this inequality can be rearranged to $q(x) = x^\top Qx + b^\top x + d \leq 0$ for a specific vector $b \in \text{range}(Q)$ and scalar $d \in \mathbb{R}$. Split the variable x into two pieces $(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ and write

$$Q = \begin{pmatrix} Q' & l \\ l^\top & a \end{pmatrix}$$

for an $(n-1) \times (n-1)$ symmetric matrix $Q' \succeq 0$ and scalar $a \geq 0$. After replacing b with (b', b_n) we can write

$$q(x', x_n) = ax_n^2 + (2l^\top x' + b_n)x_n + x'^\top Q'x' + b'^\top x' + d \leq 0.$$

Assume first that $a = 0$. Then $e_n \in \ker(Q)$ since $Q \succeq 0$. We claim that the hyperplane $H = \{x \in \mathbb{R}^n \mid x_n = 0\}$ has the desired property. For any $\bar{x} \in \text{proj}_{n-1}(\mathcal{E})$, there exists λ such that $(\bar{x}, \lambda) \in \mathcal{E}$. By the proof of Lemma 2.4, namely (2.6), since $e_n \in \ker(Q)$ we have $\pm e_n \in \text{rec}(\mathcal{E})$. Then $(\bar{x}, 0) \in \mathcal{E} \cap H$ which implies $\bar{x} \in \text{proj}_{n-1}(\mathcal{E} \cap H)$.

Assume now that $a \neq 0$. Note that this forces $a > 0$ since $Q \succeq 0$. We claim that the hyperplane $H = \{x \in \mathbb{R}^n \mid 2ax_n + 2l^\top x' = -b_n\}$ has the desired property. Let $\bar{x} \in \text{proj}_{n-1}(\mathcal{E})$. Then the univariate polynomial $q(\bar{x}, x_n)$ has real roots since $a > 0$. It follows from the quadratic formula that the midpoint on the line segment between the two roots say $(\bar{x}, \lambda)$ is in both $\mathcal{E}$ and H . $\square$

The following proposition will be one of the main building blocks of both Theorem 2.1 and Theorem 2.2. It provides a geometric description of the projection of the intersection of an ellipsoidal region and a polyhedron.

Proposition 2.6 *Let $\mathcal{E} \subseteq \mathbb{R}^{n+p}$ be an ellipsoidal region and $\mathcal{P} \subseteq \mathbb{R}^{n+p}$ be a polyhedron.*

Let $S = \text{proj}_n(\mathcal{E} \cap \mathcal{P})$. Then there exist ellipsoidal regions $\mathcal{E}_i \subseteq \mathbb{R}^n$, $i = 1, \dots, k$, polytopes $\mathcal{P}_i \subseteq \mathbb{R}^n$, $i = 1, \dots, k$, and a polyhedral cone $\mathcal{C} \subseteq \mathbb{R}^n$ such that

$$S = \bigcup_{i=1}^k (\mathcal{E}_i \cap \mathcal{P}_i) + \mathcal{C},$$

where $\mathcal{C} = \text{proj}_n(\text{rec}(\mathcal{E}) \cap \text{rec}(\mathcal{P}))$.

Proof. In the first two claims we prove that it suffices to show that S has an equivalent, but simpler, decomposition.

Claim 1. *It suffices to find ellipsoidal regions $\mathcal{E}_i \subseteq \mathbb{R}^n$, polytopes $\mathcal{P}_i \subseteq \mathbb{R}^n$, and polyhedral cones $\mathcal{C}_i \subseteq \mathbb{R}^n$, for $i = 1, \dots, k$, that satisfy*

$$S = \bigcup_{i=1}^k (\mathcal{E}_i \cap \mathcal{P}_i + \mathcal{C}_i). \quad (2.7)$$

Proof of claim. We first show that $\text{rec}(S) = \text{proj}_n(\text{rec}(\mathcal{E}) \cap \text{rec}(\mathcal{P}))$, which by Lemma 2.4 is a polyhedral cone. By definition, $\text{rec}(S) = \text{rec}(\text{proj}_n(\mathcal{E} \cap \mathcal{P}))$. Then since the projection of a ray in $\mathcal{E} \cap \mathcal{P}$ is a ray in S , the containment of $\text{proj}_n(\text{rec}(\mathcal{E}) \cap \text{rec}(\mathcal{P}))$ in $\text{rec}(S)$ is clear. Let $r \in \text{rec}(\text{proj}_n(\mathcal{E} \cap \mathcal{P}))$. Consider a polyhedral approximation $\mathcal{B}$ of $\mathcal{E}$ as in Observation 2 such that $\mathcal{B} \subseteq \mathbb{R}^{n+p}$ is a polyhedron, $\mathcal{E} \subseteq \mathcal{B}$ and $\text{rec}(\mathcal{E}) = \text{rec}(\mathcal{B})$. Then clearly, $r \in \text{rec}(\text{proj}_n(\mathcal{B} \cap \mathcal{P}))$ and since $\mathcal{B} \cap \mathcal{P}$ is a polyhedron we have $\text{rec}(\text{proj}_n(\mathcal{B} \cap \mathcal{P})) = \text{proj}_n(\text{rec}(\mathcal{B} \cap \mathcal{P}))$. Then by construction, $\text{rec}(\mathcal{B} \cap \mathcal{P}) = \text{rec}(\mathcal{E}) \cap \text{rec}(\mathcal{P})$. Henceforth, we denote by $\mathcal{C}$ the polyhedral cone $\text{rec}(S)$.

Assume we have $\mathcal{E}_i, \mathcal{P}_i$, and $\mathcal{C}_i$ that satisfy (2.7). Since $\mathcal{C} = \text{rec}(S)$ for each $i = 1, \dots, k$, we have that $\mathcal{C}_i$ must be contained in $\mathcal{C}$. It follows that

$$S = S + \mathcal{C} = \bigcup_{i=1}^k (\mathcal{E}_i + \mathcal{P}_i + \mathcal{C}_i) + \mathcal{C} = \bigcup_{i=1}^k (\mathcal{E}_i \cap \mathcal{P}_i) + \mathcal{C}. \quad \diamond$$

Claim 2. *It suffices to find ellipsoidal regions $\mathcal{E}_i \subseteq \mathbb{R}^n$, polyhedra $\mathcal{P}_i \subseteq \mathbb{R}^n$, for $i = 1, \dots, k$, that satisfy*

$$S = \bigcup_{i=1}^k (\mathcal{E}_i \cap \mathcal{P}_i). \quad (2.8)$$

Proof of claim. Assume we have ellipsoidal regions $\mathcal{E}_i$ and polyhedra $\mathcal{P}_i$ that satisfy (2.8). Consider a polyhedral approximation $\mathcal{B}_i$ of $\mathcal{E}_i$ as in Observation 2 such that $\mathcal{B}_i \subseteq \mathbb{R}^n$ is a polyhedron, $\mathcal{E}_i \subseteq \mathcal{B}_i$, and $\text{rec}(\mathcal{E}_i) = \text{rec}(\mathcal{B}_i)$. Then $\mathcal{B}_i \cap \mathcal{P}_i$ is a polyhedron and by the Minkowski-Weyl theorem can be decomposed as $\mathcal{R}_i + \mathcal{C}_i$ for a polytope $\mathcal{R}_i$ and a polyhedral cone $\mathcal{C}_i$. We claim that $\mathcal{E}_i \cap \mathcal{R}_i + \mathcal{C}_i = \mathcal{E}_i \cap \mathcal{P}_i$.

Let $x \in \mathcal{E}_i \cap \mathcal{R}_i + \mathcal{C}_i$. Note that $\mathcal{R}_i + \mathcal{C}_i \subseteq \mathcal{P}_i$ so that $x \in \mathcal{P}_i$ and since $\mathcal{C}_i \subseteq \text{rec}(\mathcal{E}_i)$, we have $x \in \mathcal{E}_i$ as well. Thus, $\mathcal{E}_i \cap \mathcal{R}_i + \mathcal{C}_i \subseteq \mathcal{E}_i \cap \mathcal{P}_i$. Let $x \in \mathcal{E}_i \cap \mathcal{P}_i$. Then $x \in \mathcal{B}_i \cap \mathcal{P}_i = \mathcal{R}_i + \mathcal{C}_i$ and we may write $x = r + c$ for some $r \in \mathcal{R}_i$, $c \in \mathcal{C}_i$. Note that $c \in \text{rec}(\mathcal{E}_i)$, and since $\text{rec}(\mathcal{E}_i)$ is a linear space by Lemma 2.4, we obtain $-c \in \text{rec}(\mathcal{E}_i)$ as well. Then $x = (x - c) + c$ and $x - c = r \in \mathcal{E}_i \cap \mathcal{R}_i$, $c \in \mathcal{C}_i$ so $x \in \mathcal{E}_i \cap \mathcal{R}_i + \mathcal{C}_i$. $\diamond$

Claim 3. *We can assume without loss of generality $p = 1$.*

Proof of claim. Let $\mathcal{E} \cap \mathcal{P} \subseteq \mathbb{R}^{n+p}$. We prove that $S = \text{proj}_n(\mathcal{E} \cap \mathcal{P})$ has the desired decomposition (2.8), by induction on p . For this claim, we assume the base case, $p = 1$. Now let $p = m$, and suppose the statement holds for $p < m$. Given $\mathcal{E} \cap \mathcal{P} \subseteq \mathbb{R}^{n+m}$, by the base case $p = 1$ there exist ellipsoidal regions $\mathcal{E}'_i$ and polyhedra $\mathcal{P}'_i$ such that

$$\text{proj}_{n+m-1}(\mathcal{E} \cap \mathcal{P}) = \bigcup_{i=1}^t (\mathcal{E}'_i \cap \mathcal{P}'_i).$$

Since the projection of a union is the union of the projections, we have

$$S = \text{proj}_n(\mathcal{E} \cap \mathcal{P}) = \bigcup_{i=1}^t \text{proj}_n(\mathcal{E}'_i \cap \mathcal{P}'_i).$$

Then by the induction hypothesis there exists ellipsoidal regions $\mathcal{E}'_{i,j}$ and polyhedra $\mathcal{P}'_{i,j}$ such that

$$S = \bigcup_{i=1}^t \left(\bigcup_{j=1}^{t_i} \mathcal{E}'_{i,j} \cap \mathcal{P}'_{i,j} \right),$$

and we are done. $\diamond$

To prove Proposition 2.6 it remains to show the following. Assume we are given $\mathcal{E} \cap \mathcal{P} \subseteq \mathbb{R}^{n+1}$. We must show the existence of ellipsoidal regions $\mathcal{E}_i \subseteq \mathbb{R}^n$, and polyhedra $\mathcal{P}_i \subseteq \mathbb{R}^n$, for $i = 1, \dots, k$, that satisfy (2.8).

Given a half-space $H^+ = \{x \in \mathbb{R}^n \mid a^\top x \geq b\}$, we write H for the hyperplane $\{x \in \mathbb{R}^n \mid a^\top x = b\}$ and H^- for the half-space $\{x \in \mathbb{R}^n \mid a^\top x \leq b\}$. A polyhedron is the intersection of finitely many half-spaces. Thus, there exist half-spaces $H_1^+, \dots, H_s^+ \subseteq \mathbb{R}^{n+1}$ such that $\mathcal{P} = \bigcap_{i=1}^s H_i^+$. By Lemma 2.5, there exists a hyperplane $H_0 \subset \mathbb{R}^{n+1}$ with $e_{n+1} \notin \text{lin}(H_0)$ such that $\text{proj}_n(\mathcal{E}) = \text{proj}_n(\mathcal{E} \cap H_0)$. We arbitrarily pick one closed half-space defined by H_0 to be H_0^+ and the other to be H_0^- . Then

$$\mathcal{E} \cap \mathcal{P} = (\mathcal{E} \cap H_0^+ \cap_{i=1}^s H_i^+) \cup (\mathcal{E} \cap H_0^- \cap_{i=1}^s H_i^+),$$

and it suffices to show the existence of ellipsoidal regions and polyhedra satisfying (2.8) for one of the regions $\mathcal{E} \cap H_0^+ \cap_{i=1}^s H_i^+$ or $\mathcal{E} \cap H_0^- \cap_{i=1}^s H_i^+$. By symmetry, we show this existence for $\mathcal{E} \cap H_0^+ \cap_{i=1}^s H_i^+$.

Claim 4. *Let $\mathcal{H}$ be the collection of hyperplanes H among $H_0, \dots, H_s$ with $e_{n+1} \notin \text{lin}(H)$. Then*

$$\text{proj}_n(\mathcal{E} \cap_{i=0}^s H_i^+) = \bigcup_{H \in \mathcal{H}} \text{proj}_n(\mathcal{E} \cap H \cap_{i=0}^s H_i^+).$$

Proof of claim. The right hand side is clearly contained in the left hand side, so it suffices to show the forward containment. It suffices to show that $\mathcal{E} \cap_{i=0}^s H_i^+$ has the following

property: for any $x \in \mathcal{E} \cap_{i=0}^s H_i^+$ there exists a hyperplane $H \in \mathcal{H}$ and a $\lambda \in \mathbb{R}$ such that $x + \lambda e_{n+1} \in \mathcal{E} \cap H \cap_{i=0}^s H_i^+$.

Let $\bar{x} \in \mathcal{E} \cap_{i=0}^s H_i^+$. To prove the claim, we show that we can translate $\bar{x}$ along $\pm e_{n+1}$, and inside the feasible region, until it meets a half-space in $\mathcal{H}$ at equality. If $e_{n+1} \in \text{lin}(H_i)$ for a half-space H_i , then $\bar{x} + \lambda e_{n+1} \in H_i^+$ for any $\lambda \in \mathbb{R}$. Then, by the existence of the shadowing hyperplane H_0 , there is one direction among $\pm e_{n+1}$ along which $\bar{x}$ may be translated to intersect H_0 while staying inside $\mathcal{E}$. That is, there exists $\bar{\lambda} \in \mathbb{R}$ such that $\bar{x} + \bar{\lambda} e_{n+1} \in \mathcal{E} \cap H_0$. Then, there exists a possibly different $\lambda' \in \mathbb{R}$ with the same sign as $\bar{\lambda}$ and $|\lambda'| \leq |\bar{\lambda}|$ such that $\bar{x} + \lambda' e_{n+1} \in \mathcal{E} \cap_{i=0}^s H_i^+$ and $\bar{x} + \lambda' e_{n+1}$ lies on at least one hyperplane $H \in \mathcal{H}$. $\diamond$

Now it suffices to show that for any $H \in \mathcal{H}$ there exists an ellipsoidal region $\mathcal{E}' \subseteq \mathbb{R}^n$ and a polyhedron $\mathcal{P}' \subseteq \mathbb{R}^n$ such that

$$\text{proj}_n(\mathcal{E} \cap H \cap_{i=0}^s H_i^+) = \mathcal{E}' \cap \mathcal{P}'.$$

Without loss of generality, we may assume that $H_i \cap H \neq \emptyset$ for each $i = 0, \dots, s$. If not, say $H_j \cap H = \emptyset$ for some $0 \leq j \leq s$, i.e., the hyperplanes H_j and H are parallel. Then either $\mathcal{E} \cap H \cap H_j^+ = \emptyset$ and our region is empty, or $\mathcal{E} \cap H \cap H_j^+ = \mathcal{E} \cap H$ and H_j^+ is redundant and may be removed.

We now show that each half-space H_i^+ , with H_i different from H , can be replaced with a different half-space M_i^+ such that $\mathcal{E} \cap H \cap H_i^+ = \mathcal{E} \cap H \cap M_i^+$ and $e_{n+1} \in \text{lin}(M_i^+)$. Fix j such that $0 \leq j \leq s$ and $H_j \neq H$. Consider H_j^+ and the region $\mathcal{E} \cap H \cap H_j^+$. Let $U = H \cap H_j$. Since $U \neq \emptyset$ we have that U is an $(n-1)$ -dimensional affine space in $\mathbb{R}^{n+1}$, say $U = v + V$ for a linear space V of dimension $n-1$. Let $W = V + \text{span}(e_{n+1})$. Since $e_{n+1} \notin \text{lin}(U)$, because $e_{n+1} \notin \text{lin}(H)$, we have that $M_j := v + W$ is a hyperplane in

$\mathbb{R}^{n+1}$ that divides H into the same two regions that H_j does. In particular, upon choice of direction, we have that M_j^+ has the desired properties.

We are now ready to describe the polyhedron $\mathcal{P}'$. First, remove from the intersection $\mathcal{E} \cap H \cap_{i=0}^s H_i^+$ any redundant H_i^+ and the H_i^+ such that $H_i = H$. Then upon relabeling we may rewrite $\mathcal{E} \cap H \cap_{i=0}^s H_i^+$ as $\mathcal{E} \cap H \cap_{i=0}^{s'} H_i^+$. We may now replace each H_i^+ with M_i^+ . By the requirement $e_{n+1} \in \text{lin}(M_i^+)$, we have that each M_i^+ is defined by a linear inequality with the coefficient of x_{n+1} equal to 0. Thus, the projection $\text{proj}_n(M_i^+)$ is a half-space in $\mathbb{R}^n$ which we denote $\bar{H}_i^+$. Further, if each H_i^+ for $i = 0, \dots, s'$ is replaced in this way, we have

$$\text{proj}_n(\mathcal{E} \cap H \cap_{i=0}^{s'} H_i^+) = \text{proj}_n(\mathcal{E} \cap H \cap_{i=0}^{s'} M_i^+) = \text{proj}_n(\mathcal{E} \cap H) \cap_{i=0}^{s'} \bar{H}_i^+,$$

and we have the desired polyhedron $\mathcal{P}' := \cap_{i=0}^{s'} \bar{H}_i^+$.

It remains to show that $\text{proj}_n(\mathcal{E} \cap H)$ is an ellipsoidal region $\mathcal{E}' \subseteq \mathbb{R}^n$. Let $H = \{(x, y) \in \mathbb{R}^n \times \mathbb{R} \mid a^\top(x, y) = b\}$. Consider the linear transformation from $\mathbb{R}^{n+1}$ to itself, defined by the matrix A whose first n rows are the first n standard unit vectors of $\mathbb{R}^{n+1}$ and whose last row is a . Note that A is invertible since e_{n+1} is not in $\text{lin}(H)$, i.e., $a_{n+1} \neq 0$. Then, by the definition of A , for any vector $(x, y) \in \mathbb{R}^{n+1}$ we have $A(x, y) = (x, c)$ where $c = a^\top(x, y)$. It follows that $A(H)$ gets mapped to the hyperplane $\{(x, y) \in \mathbb{R}^{n+1} \mid y = b\}$. Now, since A is invertible we have

$$\begin{aligned} x \in \text{proj}_n(\mathcal{E} \cap H) &\Leftrightarrow \exists y \in \mathbb{R} \text{ such that } (x, y) \in \mathcal{E} \cap H \\ &\Leftrightarrow (x, b) \in A(\mathcal{E} \cap H) \\ &\Leftrightarrow (x, b) \in A(\mathcal{E}). \end{aligned}$$

This shows that $\text{proj}_n(\mathcal{E} \cap H) = A(\mathcal{E})|_{y=b}$. Ellipsoidal regions are clearly preserved under invertible linear transformations, therefore $A(\mathcal{E})$ is an ellipsoidal region. Finally,

by Lemma 2.3, the set $A(\mathcal{E})|_{y=b}$ is an ellipsoidal region. This concludes the proof that $\text{proj}_n(\mathcal{E} \cap H)$ is an ellipsoidal region $\mathcal{E}'$. $\square$

We remark that all of the statements in this section (except Observation 4) behave nicely with respect to rationality. In greater detail, if the given ellipsoidal regions, polyhedra, and vectors are rational, then the resulting objects are also all rational. This observation can be seen directly from the proofs of these results. In particular, the rational version of Proposition 2.6 has the following statement.

Proposition 2.7 *Let $\mathcal{E} \subseteq \mathbb{R}^{n+p}$ be a rational ellipsoidal region and $\mathcal{P} \subseteq \mathbb{R}^{n+p}$ be a rational polyhedron. Let $S = \text{proj}_n(\mathcal{E} \cap \mathcal{P})$. Then there exist rational ellipsoidal regions $\mathcal{E}_i \subseteq \mathbb{R}^n$, $i = 1, \dots, k$, rational polytopes $\mathcal{P}_i \subseteq \mathbb{R}^n$, $i = 1, \dots, k$, and a rational polyhedral cone $\mathcal{C} \subseteq \mathbb{R}^n$ such that*

$$S = \bigcup_{i=1}^k (\mathcal{E}_i \cap \mathcal{P}_i) + \mathcal{C},$$

where $\mathcal{C} = \text{proj}_n(\text{rec}(\mathcal{E}) \cap \text{rec}(\mathcal{P}))$.

2.3 The Binary Case

In this section, we prove Theorem 2.1 which characterizes binary ellipsoidal mixed integer representable sets. We begin with a proposition that establishes the sufficiency of the conditions given in Theorem 2.1.

Proposition 2.8 *Let $\mathcal{E}_i \subseteq \mathbb{R}^n$, $i = 1, \dots, k$ be ellipsoidal regions, $\mathcal{P}_i \subseteq \mathbb{R}^n$, $i = 1, \dots, k$, be polytopes and $\mathcal{C} \subseteq \mathbb{R}^n$ a polyhedral cone. Suppose*

$$S = \bigcup_{i=1}^k (\mathcal{E}_i \cap \mathcal{P}_i) + \mathcal{C}.$$

Then S is binary EMI-representable.

Proof. Assume that we are given a set

$$S = \bigcup_{i=1}^k (\mathcal{E}_i \cap \mathcal{P}_i) + \mathcal{C},$$

where $\mathcal{E}_i = \{x \in \mathbb{R}^n \mid (x - c_i)^\top Q_i (x - c_i) \leq \gamma_i\}$ are ellipsoidal regions, $\mathcal{P}_i = \{x \in \mathbb{R}^n \mid A_i x \leq b_i\}$ are polytopes, and $\mathcal{C} = \text{cone}\{r^1, \dots, r^t\} \subseteq \mathbb{R}^n$ is a polyhedral cone. For each ellipsoidal region $\mathcal{E}_i$, if $\gamma_i > 0$ we can normalize the right hand side of the inequality to 1. Else, $\mathcal{E}_i$ is either empty or an affine space and γ_i can be set to 1 at the cost of adding additional linear inequalities to the system $A_i x \leq b_i$. Thus, we may assume $\gamma_i = 1$ for all $i = 1, \dots, k$.

We introduce new continuous variables $x^i \in \mathbb{R}^n$ and binary variables $\delta_i \in \{0, 1\}$, for $i = 1, \dots, k$, that will model the individual regions $\mathcal{E}_i \cap \mathcal{P}_i + \mathcal{C}$. Then S can be described as the set of $x \in \mathbb{R}^n$ such that

$$\begin{aligned} x &= \sum_{i=1}^k (x^i + \delta_i c_i) + \sum_{j=1}^t \lambda_j r^j \\ A_i x^i &\leq \delta_i (b_i - A_i c_i) & i = 1, \dots, k \\ \sum_{i=1}^k \delta_i &= 1 \\ \begin{pmatrix} x^1 \\ x^2 \\ \vdots \\ x^k \end{pmatrix}^\top \begin{pmatrix} Q_1 & & & \\ & Q_2 & & \\ & & \ddots & \\ & & & Q_k \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ \vdots \\ x^k \end{pmatrix} &\leq 1 \\ x^i &\in \mathbb{R}^n, \delta_i \in \{0, 1\} & i = 1, \dots, k \\ \lambda_j &\in \mathbb{R}_{\geq 0} & j = 1, \dots, t. \end{aligned}$$

Now if $\delta_1 = 1$ the remaining δ_i must be 0. Then for each x^i with $i \neq 1$, we have the

constraint $A_i x^i \leq 0$ which has the single feasible point $x^i = 0$ since $\mathcal{P}_i$ is a polytope.

The remaining constraints reduce to

$$\begin{aligned} x &= x^1 + c_1 + \sum_{j=1}^t \lambda_j r^j \\ A_1(x^1 + c_1) &\leq b_1 \\ (x^1)^\top Q_1 x^1 &\leq 1 \\ x^1 &\in \mathbb{R}^n \\ \lambda_j &\in \mathbb{R}_{\geq 0} \quad j = 1, \dots, t. \end{aligned}$$

By employing a change of variables $x' = x^1 + c_1$, it can be checked that the latter system describes the region $\mathcal{E}_1 \cap \mathcal{P}_1 + \mathcal{C}$. The remaining regions follow symmetrically. Therefore S is binary EMI-representable. $\square$

We are now ready to prove Theorem 2.1.

Proof of Theorem 2.1 Sufficiency of the conditions follows by Proposition 2.8. The remainder of the proof is devoted to proving necessity of the condition. We are given an ellipsoidal region $\mathcal{E}$ and a polyhedron $\mathcal{P}$ in $\mathbb{R}^{n+p+q}$, and we define

$$\begin{aligned} \bar{S} &:= \mathcal{E} \cap \mathcal{P} \cap (\mathbb{R}^{n+p} \times \{0, 1\}^q), \\ S &:= \text{proj}_n(\bar{S}). \end{aligned}$$

We must show the existence of ellipsoidal regions $\mathcal{E}_i \subseteq \mathbb{R}^n$, $i = 1, \dots, k$, polytopes $\mathcal{P}_i \subseteq \mathbb{R}^n$, $i = 1, \dots, k$, and a polyhedral cone $\mathcal{C} \subseteq \mathbb{R}^n$ such that

$$S = \bigcup_{i=1}^k (\mathcal{E}_i \cap \mathcal{P}_i) + \mathcal{C}.$$

Let $\tilde{S} := \mathcal{E} \cap \mathcal{P} \cap (\mathbb{R}^{n+p} \times [0, 1]^q)$. Then for every $\bar{z} \in \mathbb{R}^q$, define $\bar{S}_{\bar{z}} := \mathcal{E} \cap \mathcal{P} \cap (\mathbb{R}^{n+p} \times \{\bar{z}\})$. Clearly, for every $\bar{z} \in \{0, 1\}^q$, we have $\text{rec}(\bar{S}_{\bar{z}}) = \text{rec}(\tilde{S})$, and so $\text{proj}_n(\text{rec}(\bar{S}_{\bar{z}})) = \text{proj}_n(\text{rec}(\tilde{S}))$. Since projections and recession cones operators commute for closed convex sets, we obtain $\text{rec}(\text{proj}_n(\bar{S}_{\bar{z}})) = \text{proj}_n(\text{rec}(\tilde{S}))$. Let $\mathcal{C} := \text{proj}_n(\text{rec}(\tilde{S}))$. By Observation 3, the set $\text{rec}(\tilde{S})$ is a polyhedral cone, thus so is its projection $\mathcal{C}$. Note that $\bar{S} = \cup_{\bar{z} \in \{0, 1\}^q} \bar{S}_{\bar{z}}$ implies $S = \cup_{\bar{z} \in \{0, 1\}^q} \text{proj}_n(\bar{S}_{\bar{z}})$, therefore $\text{rec}(S) = \mathcal{C}$.

Note that, using restrictions, we can write the set S in the form

$$S = \bigcup_{\bar{z} \in \{0, 1\}^q} \text{proj}_n(\bar{S}|_{z=\bar{z}}).$$

It suffices to show that each restriction $\bar{S}|_{z=\bar{z}} = \mathcal{E}' \cap \mathcal{P}'$ for some ellipsoidal region $\mathcal{E}' \subseteq \mathbb{R}^{n+p}$ and polyhedron $\mathcal{P}' \subseteq \mathbb{R}^{n+p}$. Then, by Proposition 2.6, for each $\bar{z} \in \{0, 1\}^q$ we have $\text{proj}_n(\bar{S}|_{z=\bar{z}}) = \cup_{i=1}^k (\mathcal{E}_i \cap \mathcal{P}_i) + \mathcal{C}$. Since S is the finite union of such sets, the result follows.

Let $\bar{z} \in \{0, 1\}^q$. We note $\bar{S}|_{z=\bar{z}} = \mathcal{E}|_{z=\bar{z}} \cap \mathcal{P}|_{z=\bar{z}}$. By Lemma 2.3, $\mathcal{E}' := \mathcal{E}|_{z=\bar{z}}$ is an ellipsoidal region in $\mathbb{R}^{n+p}$. Let $\mathcal{P} = \{(x, y, z) \in \mathbb{R}^{n+p+q} \mid Ax + By + Cz \leq d\}$. Also, $\mathcal{P}' := \mathcal{P}|_{z=\bar{z}} = \{(x, y) \in \mathbb{R}^{n+p} \mid Ax + By \leq d - C\bar{z}\}$ is clearly a polyhedron. This completes the proof of Theorem 2.1.

2.4 The Mixed Integer Case

In this section, we prove Theorem 2.2 which characterizes rational ellipsoidal mixed integer representable sets. A proposition similar to Proposition 2.8 proves sufficiency of the conditions given in Theorem 2.2.

Proposition 2.9 *Let $\mathcal{E}_i \subseteq \mathbb{R}^n$, $i = 1, \dots, k$ be rational ellipsoidal regions, $\mathcal{P}_i \subseteq \mathbb{R}^n$, $i = 1, \dots, k$, be rational polytopes and $r^i \in \mathbb{Z}^n$, $i = 1, \dots, t$ be integral vectors. Suppose*

$$S = \bigcup_{i=1}^k (\mathcal{E}_i \cap \mathcal{P}_i) + \text{int.cone}\{r^1, \dots, r^t\}.$$

Then S is rational EMI-representable.

The proof of Proposition 2.9 is identical to the proof of Proposition 2.8 except that the constraints $\lambda_j \in \mathbb{R}_{\geq 0}$ are replaced with $\lambda_j \in \mathbb{Z}_{\geq 0}$ and the binary constraints $\delta_i \in \{0, 1\}$ are replaced with $0 \leq \delta_i \leq 1$ and $\delta_i \in \mathbb{Z}$.

We now proceed with a proof of Theorem 2.2.

Proof of Theorem 2.2 Sufficiency of the conditions follows by Proposition 2.9. The remainder of the proof is devoted to proving necessity of the condition. We are given a rational ellipsoidal region $\mathcal{E}$ and a rational polyhedron $\mathcal{P}$ in $\mathbb{R}^{n+p+q}$, and we define

$$\bar{S} := \mathcal{E} \cap \mathcal{P} \cap (\mathbb{R}^{n+p} \times \mathbb{Z}^q),$$

$$S := \text{proj}_n(\bar{S}).$$

We must show the existence of rational ellipsoidal regions $\mathcal{E}_i \subseteq \mathbb{R}^n$, $i = 1, \dots, k$, rational polytopes $\mathcal{P}_i \subseteq \mathbb{R}^n$, $i = 1, \dots, k$, and integral vectors $r^1, \dots, r^t \in \mathbb{Z}^n$ such that

$$S = \bigcup_{i=1}^k (\mathcal{E}_i \cap \mathcal{P}_i) + \text{int.cone}\{r^1, \dots, r^t\}.$$

We first show that we can decompose $\bar{S}$ into a bounded region and an integer cone.

Claim 5. *There exists a rational polytope $\mathcal{R} \subseteq \mathbb{R}^{n+p+q}$ and integral vectors $r^1, \dots, r^t \in \mathbb{Z}^{n+p+q}$ such that*

$$\bar{S} = \mathcal{E} \cap \mathcal{R} \cap (\mathbb{R}^{n+p} \times \mathbb{Z}^q) + \text{int.cone}\{r^1, \dots, r^t\}.$$

Proof of claim. Let $\mathcal{B} \subseteq \mathbb{R}^{n+p+q}$ be a rational polyhedral approximation of $\mathcal{E}$ as in Observation 2 such that $\mathcal{B}$ is a rational polyhedron, $\mathcal{E} \subseteq \mathcal{B}$, and $\text{rec}(\mathcal{E}) = \text{rec}(\mathcal{B})$. Then $\mathcal{E} \cap \mathcal{P} = \mathcal{E} \cap (\mathcal{B} \cap \mathcal{P})$. Since $\mathcal{B} \cap \mathcal{P}$ is a rational polyhedron, we can decompose $\mathcal{B} \cap \mathcal{P} = \mathcal{R}' + \mathcal{C}$ for some rational polytope $\mathcal{R}'$ and a rational polyhedral cone $\mathcal{C}$. Since $\mathcal{C}$ is rational, there exist integral vectors $r^1, \dots, r^t \in \mathbb{Z}^{n+p+q}$ such that $\mathcal{C} = \text{cone}\{r^1, \dots, r^t\}$. Note that each $r^i \in \text{rec}(\mathcal{E})$. Let

$$\mathcal{R} = \mathcal{R}' + \left\{ \sum_{i=1}^t \lambda_i r^i \mid 0 \leq \lambda_i \leq 1 \text{ for each } i = 1, \dots, t \right\}.$$

It is well-known that $\mathcal{B} \cap \mathcal{P} = \mathcal{R}' + \mathcal{C} = \mathcal{R} + \text{int.cone}\{r^1, \dots, r^t\}$.

We now show that $\mathcal{R}$ meets the conditions of the claim. Let $p \in \bar{S}$. Then $p \in \mathcal{B} \cap \mathcal{P}$ so $p = q + \sum_{i=1}^t \mu_i r^i$ for some $q \in \mathcal{R}$ and $\mu_i \in \mathbb{Z}_{\geq 0}$. Since $p \in \mathbb{R}^{n+p} \times \mathbb{Z}^q$ and μ_i, r^i are integral, we have $q \in \mathbb{R}^{n+p} \times \mathbb{Z}^q$. Moreover, $q \in \mathcal{E}$ since $p \in \mathcal{E}$ and $\text{rec}(\mathcal{E})$ is a linear space and each $r^i \in \text{rec}(\mathcal{E})$. Thus, $\bar{S} \subseteq \mathcal{E} \cap \mathcal{R} \cap (\mathbb{R}^{n+p} \times \mathbb{Z}^q) + \text{int.cone}\{r^1, \dots, r^t\}$.

For the reverse inclusion, let $q \in \mathcal{E} \cap \mathcal{R} \cap (\mathbb{R}^{n+p} \times \mathbb{Z}^q)$ and $\mu_i \in \mathbb{Z}_{\geq 0}$ for $i = 1, \dots, t$. Let $p = q + \sum_{i=1}^t \mu_i r^i$. Since $q \in \mathbb{R}^{n+p} \times \mathbb{Z}^q$ and μ_i, r^i are integral, we have $p \in \mathbb{R}^{n+p} \times \mathbb{Z}^q$. Also, each $r^i \in \text{rec}(\mathcal{E})$ which implies that $p \in \mathcal{E}$. Finally, $p \in \mathcal{R} + \mathcal{C} = \mathcal{B} \cap \mathcal{P} \subseteq \mathcal{P}$ which implies $p \in \mathcal{P}$. Therefore, $p \in \bar{S}$. $\diamond$

Let $\bar{r}^1, \dots, \bar{r}^t \in \mathbb{Z}^n$ be the vectors consisting of the first n components of $r^1, \dots, r^t$. Then by linearity of the projection operator, we have

$$S = \text{proj}_n(\mathcal{E} \cap \mathcal{R} \cap (\mathbb{R}^{n+p} \times \mathbb{Z}^q)) + \text{int.cone}\{\bar{r}^1, \dots, \bar{r}^t\}.$$

Let $T = \{\bar{z} \in \mathbb{Z}^q \mid \mathcal{E} \cap \mathcal{R} \cap (\mathbb{R}^{n+p} \times \{\bar{z}\}) \neq \emptyset\}$ and note that T is finite since $\mathcal{R}$ is bounded. Let $\bar{S}' = \mathcal{E} \cap \mathcal{R} \cap (\mathbb{R}^{n+p} \times \mathbb{Z}^q)$. Then, by using restriction notation, we can

write the set S in the form

$$S = \bigcup_{\bar{z} \in T} \text{proj}_n(\bar{S}'|_{z=\bar{z}}) + \text{int.cone}\{\bar{r}^1, \dots, \bar{r}^t\}.$$

We now show that for each $\bar{z} \in T$ the restriction $\bar{S}'|_{z=\bar{z}}$ is equal to $\bar{\mathcal{E}} \cap \bar{\mathcal{R}}$ for some ellipsoidal region $\bar{\mathcal{E}} \subseteq \mathbb{R}^{n+p}$ and polytope $\bar{\mathcal{R}} \subseteq \mathbb{R}^{n+p}$. We note $\bar{S}'|_{z=\bar{z}} = \mathcal{E}|_{z=\bar{z}} \cap \mathcal{R}|_{z=\bar{z}}$. By Lemma 2.3, $\bar{\mathcal{E}} = \mathcal{E}|_{z=\bar{z}}$ is an ellipsoidal region in $\mathbb{R}^{n+p}$. Let $\mathcal{R} = \{(x, y, z) \in \mathbb{R}^{n+p+q} \mid Ax + By + Cz \leq d\}$. Then $\bar{\mathcal{R}} = \mathcal{R}|_{z=\bar{z}} = \{(x, y) \in \mathbb{R}^{n+p} \mid Ax + By \leq d - C\bar{z}\}$ is clearly a polyhedron and is bounded since $\mathcal{R}$ is bounded.

Then, by Proposition 2.7, for each $\bar{z} \in T$ we have $\text{proj}_n(\bar{S}'|_{z=\bar{z}}) = \text{proj}_n(\bar{\mathcal{E}} \cap \bar{\mathcal{R}}) = \bigcup_{i=1}^{k_{\bar{z}}} (\mathcal{E}_i \cap \mathcal{P}_i)$ for rational ellipsoidal regions $\mathcal{E}_i$ and rational polytopes $\mathcal{P}_i$. Note that since each $\bar{\mathcal{R}}$ is a polytope, there is no cone $\mathcal{C}$. Since T is a finite set, the theorem follows. $\square$

Chapter 3

Convex Quadratic Representability¹

3.1 Introduction

In this chapter, we consider sets described by linear inequalities and a single convex quadratic inequality. Recall, we say that a region $\mathcal{Q}$ is a *convex quadratic region* in $\mathbb{R}^n$ if

$$\mathcal{Q} = \{x \in \mathbb{R}^n \mid x^\top Qx + h^\top x + g \leq 0\}$$

for a positive semidefinite matrix $Q \in \mathbb{R}^{n \times n}$, a vector $h \in \mathbb{R}^n$, and $g \in \mathbb{R}$. In general, a convex quadratic inequality cannot be factored into an ellipsoidal inequality (see Chapter 2). This implies that the family of ellipsoidal regions is a strict subset of the family of convex quadratic regions.

Recall, we say that a set $S \subseteq \mathbb{R}^n$ is *mixed binary convex quadratic representable* if it can be obtained as the projection onto $\mathbb{R}^n$ of the solution set of a system of the form

$$\begin{aligned} Dw &\leq d \\ w^\top Qw + h^\top w + g &\leq 0 \\ w &\in \mathbb{R}^{n+p} \times \{0, 1\}^q, \end{aligned} \tag{3.1}$$

where Q is positive semidefinite.

¹This chapter is modified from Del Pia and Poskin [7].

In this chapter, we present characterization results for a number of cases of mixed binary convex quadratic representable sets. See Figure 3 and Figure 4 for examples of representable sets. Before proceeding with the proofs, we provide a brief description of the statements and preview the proof techniques.

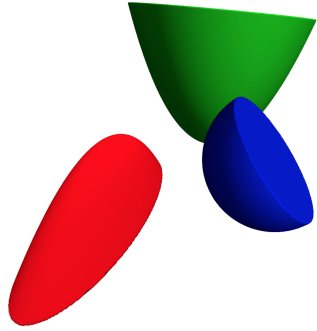


Figure 3: A bounded mixed binary convex quadratic representable set

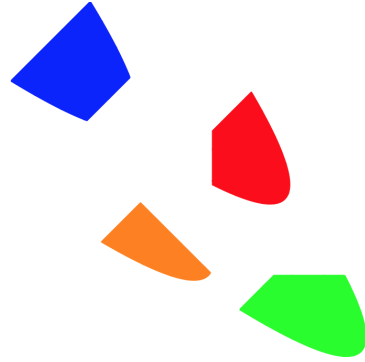


Figure 4: A binary convex quadratic representable set

In Section 3.2, we characterize sets that are *bounded mixed binary convex quadratic representable*, defined as the projection of the solution set of (3.1) where $Dw \leq d$ describes a polytope.

Theorem 3.1 *A set $S \subseteq \mathbb{R}^n$ is bounded mixed binary convex quadratic representable if and only if there exist convex quadratic regions $\mathcal{Q}_i \subseteq \mathbb{R}^n$, $i = 1, \dots, k$, and polytopes $\mathcal{P}_i \subseteq \mathbb{R}^n$, $i = 1, \dots, k$, such that*

$$S = \bigcup_{i=1}^k (\mathcal{Q}_i \cap \mathcal{P}_i). \quad (3.2)$$

The characterization given in Theorem 3.1 is quite general in the sense that there is no restriction on the structure of the convex quadratic regions that may appear in the union (3.2). This is quite similar to what holds for ellipsoidal regions in Chapter 2,

and indeed the fact that each region $\mathcal{Q}_i \cap \mathcal{P}_i$ in (3.2) is bounded allows us to find an extended formulation where each $\mathcal{Q}_i$ appears as a binary slice of a global convex quadratic region $\mathcal{Q}$. In the case of ellipsoidal regions this level of generality still holds even for unbounded regions. We will see in Section 3.5, where the bounded assumption is removed, that although a decomposition of representable sets into a union (3.2) holds, the convex quadratic regions that appear must share common structure.

In Section 3.3, we characterize sets that are *binary convex quadratic representable*, i.e., where $p = 0$ is fixed in (3.1). In order to provide a characterization of such sets, we need to remark on the geometry of convex quadratic sets in more detail. We make the following observation and definition. Let $\mathcal{Q} \subseteq \mathbb{R}^n$ be a convex quadratic region defined by

$$\mathcal{Q} = \{x \in \mathbb{R}^n \mid x^\top Qx + h^\top x + g \leq 0\},$$

where $Q \succeq 0$. Since Q is symmetric, it is a fact of linear algebra that $\mathbb{R}^n = \text{range}(Q) \oplus \ker(Q)$. Thus, we can decompose $h = Qw + v$ where $v \in \ker(Q)$ is uniquely determined. We note that $\mathcal{Q}$ is an ellipsoidal region if and only if $v = 0$.

The pair Q, v defining $\mathcal{Q}$ is essential in understanding the geometry of $\mathcal{Q}$. In this vein, we say that two convex quadratic regions $\mathcal{Q}_1, \mathcal{Q}_2 \subseteq \mathbb{R}^n$ have the *same shape* if there exists a positive semidefinite matrix Q , a vector $v \in \ker(Q)$, vectors w^i , and scalars g_i such that

$$\mathcal{Q}_i = \{x \in \mathbb{R}^n \mid x^\top Qx + (Qw^i + v)^\top x + g_i \leq 0\}, \quad i = 1, 2.$$

Geometrically, this means that $\mathcal{Q}_1$ and $\mathcal{Q}_2$ have the same structure, up to translation and constant term. Note that having the same shape is a transitive property and thus it makes sense for a collection of convex quadratic regions to have the same shape. We

will establish the following result.

Theorem 3.2 *A set $S \subseteq \mathbb{R}^n$ is binary convex quadratic representable if and only if there exist convex quadratic regions $\mathcal{Q}_1, \dots, \mathcal{Q}_k \subseteq \mathbb{R}^n$ with the same shape, and polyhedra $\mathcal{P}_1, \dots, \mathcal{P}_k \subseteq \mathbb{R}^n$ with the same recession cone such that*

$$S = \bigcup_{i=1}^k (\mathcal{Q}_i \cap \mathcal{P}_i). \quad (3.3)$$

In Section 3.4 we obtain an algebraic characterization of *continuous convex quadratic representable sets*, i.e., where $q = 0$ is fixed in (3.1). This sort of algebraic description is quite different from the geometric characterizations obtained prior to this. The combination of extended continuous variables and unbounded regions creates a number of difficulties. Part of this difficulty is due to the complexity of describing the projection of semialgebraic sets. While methods such as Cylindrical Algebraic Decomposition may be used to compute the projection of (3.1), these outputs give little insight into the requirements that must be met for a set to be representable. Another difficulty is that we are not able to use standard disjunctive extended formulations. This is due to the fact that in general a convex quadratic region cannot be decomposed as the Minkowski sum of a bounded region and a polyhedral cone, in contrast to both polyhedra and ellipsoidal regions (see Chapter 2).

In order to overcome these difficulties, we design a method to explicitly compute $S := \text{proj}_n(\mathcal{Q} \cap \mathcal{P})$ for a general convex quadratic region $\mathcal{Q} \subseteq \mathbb{R}^{n+p}$ and a polyhedron $\mathcal{P} \subseteq \mathbb{R}^{n+p}$. A crucial step, stated in Proposition 3.7, is the construction of a ‘shadowing skeleton’ of $\mathcal{Q} \cap \mathcal{P}$, namely a finite set $\mathcal{L}$ of n -dimensional affine spaces that satisfy $\text{proj}_n(\mathcal{Q} \cap \mathcal{P}) = \bigcup_{L \in \mathcal{L}} \text{proj}_n(\mathcal{Q} \cap L) \cap \text{proj}_n(\mathcal{P} \cap L)$. The idea of this skeleton is a general version of the projection method done in Chapter 2. The explicit computation of

$\text{proj}_n(\mathcal{Q} \cap \mathcal{P})$ leaves us with an algebraic description of S which we use to make a technical definition of sets of *Type 1* and *Type 2*. These definitions allow us to characterize continuous convex quadratic representable sets as shown in Theorem 3.9.

In Section 3.5 we present an algebraic characterization for the general case of mixed binary convex quadratic representable sets. Theorem 3.10 follows naturally from the combination of our results in Section 3.2 and Section 3.4. It follows immediately from Theorem 3.10 that mixed binary convex quadratic representable sets can be expressed as a finite union of $\mathcal{Q}_i \cap \mathcal{P}_i$ for convex quadratic regions $\mathcal{Q}_i$ and polyhedron $\mathcal{P}_i$. However, in contrast to Theorem 3.1, the convex quadratic regions $\mathcal{Q}_i$ that appear in a decomposition share a common geometry. This compatibility requirement is captured by our definition of sets with the *same structure* which follows from combining our definition of sets of Type 1 and Type 2 with the notion of convex quadratic regions with the same shape.

In Section 3.6 we work towards a geometric characterization of mixed binary convex quadratic representable sets. We derive obvious necessary conditions for a set to be representable from Theorem 3.9 and Theorem 3.10, and explore whether these necessary conditions are in fact sufficient. We conclude the section, and chapter, with open questions, and an instructive example of a set that is not mixed binary convex quadratic representable.

3.2 The Bounded Case

In this section we give a characterization of bounded mixed binary convex quadratic representable sets, i.e., when the system $Dw \leq d$ in (3.1) describes a polytope.

Proof.[Proof of Theorem 3.1] We prove sufficiency of the condition by giving an explicit extended formulation for a set S described by (3.2). It is similar to the proof of Theorem 2.1. Assume that we are given a set S as in (3.2), where $\mathcal{Q}_i = \{x \in \mathbb{R}^n \mid x^\top Q_i x + (h^i)^\top x + g_i \leq 0\}$ are convex quadratic regions and $\mathcal{P}_i = \{x \in \mathbb{R}^n \mid A^i x \leq b_i\}$ are polytopes. We now introduce new continuous variables $x^i \in \mathbb{R}^n$ and binary variables $\delta_i \in \{0, 1\}$, for $i = 1, \dots, k$, that will model the individual regions $\mathcal{Q}_i \cap \mathcal{P}_i$. Then S can be described as the set of $x \in \mathbb{R}^n$ such that

$$\begin{aligned}
x &= \sum_{i=1}^k x^i \\
A^i x^i &\leq \delta_i b_i & i = 1, \dots, k \\
\sum_{i=1}^k \delta_i &= 1 \\
\sum_{i=1}^k \left((x^i)^\top Q_i x^i + (h^i)^\top x^i + \delta_i g_i \right) &\leq 0 \\
0 \leq \delta_i &\leq 1 & i = 1, \dots, k \\
x^i \in \mathbb{R}^n, \delta_i \in \{0, 1\} & & i = 1, \dots, k.
\end{aligned}$$

Now if $\delta_1 = 1$ the remaining δ_i must be 0. Then for each x^i with $i \neq 1$, we have the constraint $A^i x^i \leq 0$ which has the single feasible point $x^i = 0$ since $\mathcal{P}_i$ is a polytope. The remaining constraints reduce to

$$\begin{aligned}
x &= x^1 \\
A^1 x^1 &\leq b_1 \\
(x^1)^\top Q_1 x^1 + (h^1)^\top x^1 + g_1 &\leq 0 \\
x^1 &\in \mathbb{R}^n.
\end{aligned}$$

This describes the set $\mathcal{Q}_1 \cap \mathcal{P}_1$ exactly. The remaining regions follow symmetrically.

We note that the constraint $\sum_{i=1}^k \left((x^i)^\top Q_i x^i + (h^i)^\top x^i + \delta_i g_i \right) \leq 0$ describes a convex quadratic region since it can be described as a quadratic inequality with defining matrix a block diagonal matrix with blocks either 0 or Q_i , and each $Q_i \succeq 0$.

We show that the linear system is bounded by demonstrating that its set of feasible points is the graph of a linear transformation restricted to a polytope. Each system $A^i x^i \leq \delta_i b_i$, $0 \leq \delta_i \leq 1$ is independent of any other x^j, δ_j . Moreover, each system is bounded in (x^i, δ_i) as it is the convex hull of the polytope $\{x^i \in \mathbb{R}^n \mid A^i x^i \leq b_i\} \times \{1\}$ and the origin. Then the set of feasible points in $x^1, \dots, x^k, \delta_1, \dots, \delta_k$ is just a Cartesian product of bounded sets. Finally, the set of points x satisfying equation $x = \sum_{i=1}^k x^i$ is bounded since it is the image of this Cartesian product under a linear transformation. Thus, S is bounded mixed binary convex quadratic representable.

The remainder of the proof is devoted to proving necessity of the condition. We are given a convex quadratic region $\mathcal{Q}$ and a polytope $\mathcal{P}$ in $\mathbb{R}^{n+p+q}$, and define

$$\bar{S} := \mathcal{Q} \cap \mathcal{P} \cap (\mathbb{R}^{n+p} \times \{0, 1\}^q),$$

$$S := \text{proj}_n(\bar{S}).$$

We must show the existence of convex quadratic regions $\mathcal{Q}_i \subseteq \mathbb{R}^n$, $i = 1, \dots, k$, and polytopes $\mathcal{P}_i \subseteq \mathbb{R}^n$, $i = 1, \dots, k$, such that

$$S = \bigcup_{i=1}^k (\mathcal{Q}_i \cap \mathcal{P}_i).$$

Claim 6. *It is enough to prove the theorem in the case $q = 0$.*

Proof of claim. Note that, using restrictions, we can write the set S in the form

$$S = \bigcup_{\bar{z} \in \{0, 1\}^q} \text{proj}_n(\bar{S}|_{z=\bar{z}}).$$

We first show that each restriction $\bar{S}|_{z=\bar{z}}$ can be written as $\mathcal{Q}' \cap \mathcal{P}'$ for some convex quadratic region $\mathcal{Q}' \subseteq \mathbb{R}^{n+p}$ and polytope $\mathcal{P}' \subseteq \mathbb{R}^{n+p}$. Let $\bar{z} \in \{0, 1\}^q$. We note $\bar{S}|_{z=\bar{z}} = \mathcal{Q}|_{z=\bar{z}} \cap \mathcal{P}|_{z=\bar{z}}$. A short algebraic verification shows that $\mathcal{Q}' := \mathcal{Q}|_{z=\bar{z}}$ is a convex quadratic region and $\mathcal{P}' := \mathcal{P}|_{z=\bar{z}}$ is a polytope.

Now assuming the theorem in the case $q = 0$, for each $\bar{z} \in \{0, 1\}^q$ we have $\text{proj}_n(\bar{S}|_{z=\bar{z}}) = \cup_{i=1}^t (\mathcal{Q}_i \cap \mathcal{P}_i)$. Since S is the finite union of such sets, the theorem follows. $\diamond$

Claim 7. *It is enough to prove the theorem in the case $p = 1$.*

Proof of claim. Let $\mathcal{Q} \cap \mathcal{P} \subseteq \mathbb{R}^{n+p}$. We prove $S = \text{proj}_n(\mathcal{Q} \cap \mathcal{P})$ has the desired decomposition by induction on p . For this claim, we assume the base case, $p = 1$. Now let $p = k$, and suppose the statement holds for $p < k$. Given $\mathcal{Q} \cap \mathcal{P} \subseteq \mathbb{R}^{n+k}$, by the base case $p = 1$ we have

$$\text{proj}_{n+k-1}(\mathcal{Q} \cap \mathcal{P}) = \bigcup_{i=1}^t (\mathcal{Q}_i \cap \mathcal{P}_i)$$

where each $\mathcal{Q}_i$ is a convex quadratic region in $\mathbb{R}^{n+k-1}$ and each $\mathcal{P}_i$ is a polytope in $\mathbb{R}^{n+k-1}$. Since the projection of a union is the union of projections we have

$$S = \text{proj}_n(\mathcal{Q} \cap \mathcal{P}) = \bigcup_{i=1}^t \text{proj}_n(\mathcal{Q}_i \cap \mathcal{P}_i).$$

Then by induction hypothesis, we have

$$S = \bigcup_{i=1}^t \left(\bigcup_{j=1}^{s_i} (\mathcal{Q}_{i,j} \cap \mathcal{P}_{i,j}) \right)$$

where each $\mathcal{Q}_{i,j}$ is a convex quadratic region in $\mathbb{R}^n$ and each $\mathcal{P}_{i,j}$ is a polytope in $\mathbb{R}^n$. $\diamond$

It remains to prove Theorem 3.1 in the case that we have a convex quadratic region $\mathcal{Q} \subseteq \mathbb{R}^{n+1}$ and a polytope $\mathcal{P} \subseteq \mathbb{R}^{n+1}$. The following claim then completes the proof of Theorem 3.1.

Claim 8. *Let $\mathcal{Q} \subseteq \mathbb{R}^{n+1}$ be a convex quadratic region described by*

$$\mathcal{Q} = \left\{ (x, y) \in \mathbb{R}^{n+1} \mid \begin{pmatrix} x \\ y \end{pmatrix}^\top \begin{pmatrix} Q & q \\ q^\top & \gamma \end{pmatrix} + \begin{pmatrix} h_x \\ h_y \end{pmatrix}^\top \begin{pmatrix} x \\ y \end{pmatrix} + g \leq 0 \right\}$$

and $\mathcal{P} \subseteq \mathbb{R}^{n+1}$ be a polytope. Then there exist convex quadratic regions $\mathcal{Q}_i \subseteq \mathbb{R}^n$, $i = 1, \dots, k$, and polytopes $\mathcal{P}_i \subseteq \mathbb{R}^n$, $i = 1, \dots, k$, such that (3.2) holds.

Proof of claim. We first claim that $\text{proj}_n(\mathcal{Q} \cap \mathcal{P}) = \cup_{H \in \mathcal{H}} \text{proj}_n(\mathcal{Q} \cap \mathcal{P} \cap H)$ where $\mathcal{H}$ is a finite set of hyperplanes. Suppose $\mathcal{P}$ is defined as the intersection $H_1^+ \cap \dots \cap H_s^+$ for half-spaces H_i^+ . Let $\mathcal{H}$ be the subset of hyperplanes $H \in \{H_1, \dots, H_s\}$ such that $e_{n+1} \notin \text{lin}(H)$. In the case that $\gamma \neq 0$, define the hyperplane $H_0 := \{(x, y) \in \mathbb{R}^{n+1} \mid q^\top x + \gamma y = -\frac{1}{2}h_y\}$ and include H_0 in the set $\mathcal{H}$. This hyperplane has the property that for any fixed $\bar{x} \in \mathbb{R}^n$, the unique point $(\bar{x}, \bar{y}) \in H_0$ minimizes the univariate quadratic polynomial $q(\bar{x}, y)$ defining $\mathcal{Q}|_{x=\bar{x}}$. Moreover, $e_{n+1} \notin \text{lin}(H_0)$.

We claim that $\text{proj}_n(\mathcal{Q} \cap \mathcal{P}) = \cup_{H \in \mathcal{H}} \text{proj}_n(\mathcal{Q} \cap \mathcal{P} \cap H)$. Let $\bar{x} \in \text{proj}_n(\mathcal{Q} \cap \mathcal{P})$. Define $L_{\bar{x}} = \{(\bar{x}, y) \in \mathbb{R}^{n+1} \mid (\bar{x}, y) \in \mathcal{Q} \cap \mathcal{P}\}$. Since $\mathcal{P}$ is a polytope, $L_{\bar{x}}$ is a non-empty line segment. Consider the endpoints, possibly both the same point, of $L_{\bar{x}}$. If either endpoint lies on the boundary of $\mathcal{P}$ then we are done as this point must lie on some $H \in \mathcal{H}$. Otherwise, both endpoints lie on the boundary of $\mathcal{Q}$ and are thus roots of the quadratic polynomial $q(\bar{x}, y)$. Then the midpoint of $L_{\bar{x}}$ lies on H_0 .

It remains to show that for each $H \in \mathcal{H}$, there exists a convex quadratic region $\mathcal{Q}_H$ and a polytope $\mathcal{P}_H$ such that $\text{proj}_n(\mathcal{Q} \cap \mathcal{P} \cap H) = \mathcal{Q}_H \cap \mathcal{P}_H$. Let $H = \{(x, y) \in \mathbb{R}^{n+1} \mid a^\top x + \alpha y = b\}$ and note that $\alpha \neq 0$ since $e_{n+1} \notin \text{lin}(H)$. It follows that $\text{proj}_n(\mathcal{Q} \cap \mathcal{P} \cap H) = \text{proj}_n(\mathcal{Q} \cap H) \cap \text{proj}_n(\mathcal{P} \cap H)$ as there is a unique point $(x, y) \in H$ lying over any $x \in \mathbb{R}^n$.

We now show that $\mathcal{Q}_H := \text{proj}_n(\mathcal{Q} \cap H)$ is a convex quadratic region and $\mathcal{P}_H := \text{proj}_n(\mathcal{P} \cap H)$ is a polytope. The polyhedron $\mathcal{P}_H$ is clearly a polytope since $\mathcal{P}$ is a polytope. Define the invertible linear transformation $T_A : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$ by multiplication on the left by the matrix

$$A = \begin{pmatrix} I_n & 0 \\ a^\top & \alpha \end{pmatrix}.$$

Then we have that

$$\mathcal{Q}_H = \text{proj}_n(\mathcal{Q} \cap H) = T_A(\mathcal{Q})|_{y=b}.$$

Note that $\mathcal{Q}_H$ is a convex quadratic region as it obtained from $\mathcal{Q}$ by an invertible linear transformation followed by fixing a single variable. ◇

□

3.3 The Binary Case

In this section we characterize binary convex quadratic representable sets, i.e., when $p = 0$ is fixed in (3.1). We refer the reader back to the introduction for the definition of convex quadratic regions with the same shape. Before proving Theorem 3.2 we state a number of lemmas that detail the interaction of binary variables and convex quadratic regions.

Lemma 3.3 *Let $\mathcal{Q} \subseteq \mathbb{R}^{n+q}$ be a convex quadratic region. Then for all $\bar{z} \in \{0, 1\}^q$, the sets $\mathcal{Q}|_{z=\bar{z}}$ are convex quadratic regions with the same shape.*

Proof. Assume that $\mathcal{Q}$ is given by

$$\mathcal{Q} = \left\{ (x, z) \in \mathbb{R}^{n+q} \mid \begin{pmatrix} x \\ z \end{pmatrix}^\top \begin{pmatrix} Q & R \\ R^\top & \bar{Q} \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} + \begin{pmatrix} h_x \\ h_z \end{pmatrix}^\top \begin{pmatrix} x \\ z \end{pmatrix} + g \leq 0 \right\}.$$

Then for any $\bar{z} \in \{0, 1\}^q$ we have

$$\mathcal{Q}|_{z=\bar{z}} = \{x \in \mathbb{R}^n \mid x^\top Qx + (2R\bar{z} + h_x)^\top x + g + h_z^\top \bar{z} + \bar{z}^\top \bar{Q} \bar{z} \leq 0\}.$$

Now since $\mathcal{Q}$ is a convex quadratic region, the matrix Q must be positive semidefinite.

Moreover, the matrix Q is clearly independent of the choice of $\bar{z} \in \{0, 1\}^q$.

It remains to show that the vector $2R\bar{z} + h_x$ decomposes into $Qw + v$ where $v \in \ker(Q)$ is independent of $\bar{z}$. We claim that $2R\bar{z} \in \text{range}(Q)$. Decompose $2R\bar{z} = Qw + v$ for a unique vector $v \in \ker(Q)$. If $v \neq 0$, then for $\lambda < -\frac{\bar{z}^\top \bar{Q} \bar{z}}{v^\top v}$ we have

$$\begin{pmatrix} \lambda v \\ \bar{z} \end{pmatrix}^\top \begin{pmatrix} Q & R \\ R^\top & \bar{Q} \end{pmatrix} \begin{pmatrix} \lambda v \\ \bar{z} \end{pmatrix} = \lambda^2 v^\top Qv + 2\bar{z}^\top R^\top (\lambda v) + \bar{z}^\top \bar{Q} \bar{z} = \lambda v^\top v + \bar{z}^\top \bar{Q} \bar{z} < 0,$$

a contradiction. Since $2R\bar{z} \in \text{range}(Q)$, the vector v depends only on h_x and is thus independent of $\bar{z} \in \{0, 1\}^q$. $\square$

The next lemma can be seen as a converse of Lemma 3.3. We denote by $e_i \in \mathbb{R}^k$ the i th standard basis vector of $\mathbb{R}^k$.

Lemma 3.4 *Let $\mathcal{Q}_1, \dots, \mathcal{Q}_k \subseteq \mathbb{R}^n$ be convex quadratic regions with the same shape. Then there exists a convex quadratic region $\mathcal{Q} \subseteq \mathbb{R}^{n+k}$ such that $\mathcal{Q}|_{z=e_i} = \mathcal{Q}_i$ for each $1 \leq i \leq k$.*

Proof. Suppose that each $\mathcal{Q}_i$ is described by

$$\mathcal{Q}_i = \{x \in \mathbb{R}^n \mid x^\top Qx + (2Qw^i + v)^\top x + g_i \leq 0\},$$

where $Q \succeq 0$ and $v \in \ker(Q)$. Set $\gamma_i \geq k(w^i)^\top Q w^i$ and $h_i = g_i - \gamma_i$, and define $R := \begin{pmatrix} Q w^1 & \cdots & Q w^k \end{pmatrix}$, $\Lambda := \text{diag}(\gamma_1, \dots, \gamma_k)$, and $h^\top := (v^\top, h_1, \dots, h_k)$. We claim that

$$\mathcal{Q} := \left\{ (x, z) \in \mathbb{R}^{n+k} \mid \begin{pmatrix} x \\ z \end{pmatrix}^\top \begin{pmatrix} Q & R \\ R^\top & \Lambda \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} + h^\top \begin{pmatrix} x \\ z \end{pmatrix} \leq 0 \right\}$$

is a convex quadratic region with the desired restriction property. Then

$$\mathcal{Q}|_{z=e_i} = \{x \in \mathbb{R}^n \mid x^\top Q x + (2R e_i + v)^\top x + h_i + \gamma_i \leq 0\}$$

and by construction $2R e_i = 2Q w^i$ and $h_i + \gamma_i = g_i$. Thus, $\mathcal{Q}|_{z=e_i} = \mathcal{Q}_i$.

We now show that $\mathcal{Q}$ is a convex quadratic region by demonstrating that matrix defining $\mathcal{Q}$ is positive semidefinite. Let $(x, z) \in \mathbb{R}^{n+k}$. We have that

$$\begin{aligned} \begin{pmatrix} x \\ z \end{pmatrix}^\top \begin{pmatrix} Q & R \\ R^\top & \Lambda \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} &= x^\top Q x + 2z^\top R^\top x + \sum_{i=1}^k \gamma_i z_i^2 \\ &= x^\top Q x + \sum_{i=1}^k (2(z_i Q w^i)^\top x + \gamma_i z_i^2) \\ &= \frac{1}{k} \sum_{i=1}^k (x^\top Q x + 2(Q k z_i w^i)^\top x + k \gamma_i z_i^2). \end{aligned}$$

We show by choice of γ_i that each summand $x^\top Q x + 2(Q k z_i w^i)^\top x + k \gamma_i z_i^2$ is nonnegative by completing the square. Note

$$x^\top Q x + 2(Q k z_i w^i)^\top x + k \gamma_i z_i^2 = (x + k z_i w^i)^\top Q (x + k z_i w^i) + (k \gamma_i - k^2 (w^i)^\top Q w^i) z_i^2.$$

Now since $\gamma_i \geq k(w^i)^\top Q w^i$ we have expressed each summand as the sum of two non-negative numbers. In particular, each $x^\top Q x + 2(Q k z_i w^i)^\top x + k \gamma_i z_i^2 \geq 0$ and $\mathcal{Q}$ is a convex quadratic region. $\square$

We note that Lemma 3.4 shows that a union of convex quadratic regions with the same shape have a binary lift to a convex quadratic region provided we intersect it with an appropriate polyhedron.

The proof of Theorem 3.2 is now a simple combination of the preceding lemmas. We note that for the construction of the extended formulation, we cannot use a system similar to that which appeared in Theorem 3.1 as it requires additional continuous variables.

Proof.[Proof of Theorem 3.2] We start with sufficiency of the condition. Assume we have convex quadratic regions $\mathcal{Q}_1, \dots, \mathcal{Q}_k \subseteq \mathbb{R}^n$ with the same shape, and polyhedra $\mathcal{P}_1, \dots, \mathcal{P}_k \subseteq \mathbb{R}^n$ with the same recession cone and let S be defined by (3.3). Then by Lemma 3.4, we obtain a convex quadratic region $\mathcal{Q} \subseteq \mathbb{R}^{n+k}$ such that $\mathcal{Q}|_{z=e_i} = \mathcal{Q}_i$ for each $1 \leq i \leq k$. We use a standard technique to obtain a polyhedron $\mathcal{P} \subseteq \mathbb{R}^{n+k}$ such that $\mathcal{P}|_{z=e_i} = \mathcal{P}_i$ for $1 \leq i \leq k$ and $\mathcal{P}|_{z=\bar{z}} = \emptyset$ for $\bar{z} \in \{0, 1\}^k - \{e_1, \dots, e_k\}$. This technique is known as a Big-M formulation, and the existence of such a polyhedron is proved in Proposition 6.1 in [22]. It follows that $S = \text{proj}_n(\mathcal{Q} \cap \mathcal{P} \cap (\mathbb{R}^n \times \{0, 1\}^k))$.

It remains to show necessity. Let $\mathcal{Q} \subseteq \mathbb{R}^{n+q}$ be a convex quadratic region and $\mathcal{P} \subseteq \mathbb{R}^{n+q}$ be a polyhedron. Let $S := \text{proj}_n(\mathcal{Q} \cap \mathcal{P} \cap (\mathbb{R}^n \times \{0, 1\}^q))$. Then

$$S = \bigcup_{\bar{z} \in \{0, 1\}^q} (\mathcal{Q} \cap \mathcal{P})|_{z=\bar{z}}.$$

Choose $\bar{z} \in \{0, 1\}^q$ and note that $(\mathcal{Q} \cap \mathcal{P})|_{z=\bar{z}} = \mathcal{Q}|_{z=\bar{z}} \cap \mathcal{P}|_{z=\bar{z}}$. Then by Lemma 3.3 all $\mathcal{Q}|_{z=\bar{z}}$ are convex quadratic regions with the same shape. Since each polyhedron $\mathcal{P}|_{z=\bar{z}}$ has recession cone independent of $\bar{z}$ the theorem follows. $\square$

3.4 The Continuous Case

In this section we find an algebraic characterization of continuous convex quadratic representable sets, i.e., where $q = 0$ is fixed in (3.1). In the first part of this section we consider a convex quadratic region $\mathcal{Q} \subseteq \mathbb{R}^{n+p}$ and a polyhedron $\mathcal{P} \subseteq \mathbb{R}^{n+p}$. We proceed by computing explicitly the projection $\text{proj}_n(\mathcal{Q} \cap \mathcal{P})$ and in doing so come to the definitions for sets of Type 1 and Type 2. In particular, each set of Type 1 or Type 2 can be written as a finite union of $\mathcal{Q}_i \cap \mathcal{P}_i$ for convex quadratic regions $\mathcal{Q}_i$ and polyhedra $\mathcal{P}_i$. These definitions will be sufficient conditions for a set to be continuous convex quadratic representable. In order to show this, we demonstrate that every set of Type 1 or Type 2 has a lift to $\mathcal{Q} \cap \mathcal{P} \subseteq \mathbb{R}^{n+p}$ for some convex quadratic region $\mathcal{Q}$ and polyhedron $\mathcal{P}$.

Assume now that we are given a convex quadratic region $\mathcal{Q} \subseteq \mathbb{R}^{n+p}$, a polyhedron $\mathcal{P} \subseteq \mathbb{R}^{n+p}$, and wish to compute $S := \text{proj}_n(\mathcal{Q} \cap \mathcal{P})$. We begin by applying an invertible affine transformation to $\mathbb{R}^{n+p}$ that brings $\mathcal{Q}$ to a normalized form.

Lemma 3.5 *Let $\mathcal{Q} \subseteq \mathbb{R}^{n+p}$ be a convex quadratic region defined by*

$$\mathcal{Q} = \left\{ (x, y) \in \mathbb{R}^{n+p} \mid \begin{pmatrix} x \\ y \end{pmatrix}^\top \begin{pmatrix} Q & R \\ R^\top & S \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} h_x \\ h_y \end{pmatrix}^\top \begin{pmatrix} x \\ y \end{pmatrix} + g \leq 0 \right\}.$$

Then there exists an invertible affine transformation $T : \mathbb{R}^{n+p} \rightarrow \mathbb{R}^{n+p}$ that takes $\mathcal{Q}$ to a convex quadratic region $\mathcal{Q}' \subseteq \mathbb{R}^{n+p}$ such that $\text{proj}_n(\mathcal{Q}) = \text{proj}_n(\mathcal{Q}')$ and

$$\mathcal{Q}' = \left\{ (x, y, t) \in \mathbb{R}^{n+k+(p-k)} \mid \begin{pmatrix} x \\ y \\ t \end{pmatrix}^\top \begin{pmatrix} Q' & 0 & 0 \\ 0 & I_k & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ t \end{pmatrix} + \begin{pmatrix} h' \\ 0 \\ l \end{pmatrix}^\top \begin{pmatrix} x \\ y \\ t \end{pmatrix} + g' \leq 0 \right\}, \quad (3.4)$$

where $k = \text{rank}(S)$, I_k is the $k \times k$ identity matrix, and either $l = e_1$ or $l = 0$.

Proof. We will define T as the composition of three invertible affine transformations. Since $S \succeq 0$, there exists an orthogonal $p \times p$ matrix U such that $S = U^\top \Lambda U$ where $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_p)$ and $\lambda_1 \geq \dots \geq \lambda_p \geq 0$. Suppose the first k eigenvalues of S are positive, and define $V = \text{diag}(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_k}, 1, \dots, 1)$. Then $S = U^\top V E V U$ where E is a diagonal matrix whose first k diagonal entries are 1 and the remaining $p - k$ entries are 0.

Define the transformation $T' : \mathbb{R}^{n+p} \rightarrow \mathbb{R}^{n+p}$ by multiplication on the left by the matrix

$$A = \begin{pmatrix} I & 0 \\ 0 & VU \end{pmatrix}.$$

Consider the change of coordinates defined by $(x, u)^\top = A(x, y)^\top$. Then $T'(\mathcal{Q})$ is described by

$$\left\{ (x, u) \in \mathbb{R}^{n+p} \mid \begin{pmatrix} x \\ u \end{pmatrix}^\top \begin{pmatrix} Q & RU^\top V^{-1} \\ V^{-1}UR^\top & E \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} + \begin{pmatrix} h_x \\ V^{-1}Uh_y \end{pmatrix}^\top \begin{pmatrix} x \\ u \end{pmatrix} + g \leq 0 \right\}.$$

Note now that the matrix defining the quadratic region $T'(\mathcal{Q})$ is positive semidefinite. This implies that any diagonal entry being 0 forces the entire corresponding row and column to be 0 as well. Let B denote the first k columns of $RU^\top V^{-1}$. Then

$$\begin{pmatrix} Q & RU^\top V^{-1} \\ V^{-1}UR^\top & E \end{pmatrix} = \begin{pmatrix} Q & B & 0 \\ B^\top & I_k & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Define $T'' : \mathbb{R}^{n+p} \rightarrow \mathbb{R}^{n+p}$ by multiplication on the left by the invertible matrix

$$C = \begin{pmatrix} I_n & 0 & 0 \\ B^\top & I_k & 0 \\ 0 & 0 & I_{p-k} \end{pmatrix}.$$

Consider the change of coordinates defined by $(x, v, w)^\top = C(x, u)^\top$. Then $T''(T'(\mathcal{Q}))$ is described by

$$\left\{ (x, v, w) \in \mathbb{R}^{n+k+(p-k)} \mid \begin{pmatrix} x \\ v \\ w \end{pmatrix}^\top \begin{pmatrix} Q' & 0 & 0 \\ 0 & I_k & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ v \\ w \end{pmatrix} + \begin{pmatrix} h' \\ h_v \\ h_w \end{pmatrix}^\top \begin{pmatrix} x \\ v \\ w \end{pmatrix} + g \leq 0 \right\},$$

where $Q' := Q - BB^\top$, $h' := h_x - B(V^{-1}Uh_y)_+$, and $\begin{pmatrix} h_v \\ h_w \end{pmatrix} := V^{-1}Uh_y$.

Finally, define the affine transformation $L : \mathbb{R}^{n+p} \rightarrow \mathbb{R}^{n+p}$ by $L(x, v, w) = H(x, v, w)^\top + r$ where

$$H = \begin{pmatrix} I_n & 0 & 0 \\ 0 & I_k & 0 \\ 0 & 0 & M \end{pmatrix},$$

and M is either an invertible matrix such that $(M^{-1})^\top h_w = e_1$ if $h_w \neq 0$ or $M = I_{p-k}$ if $h_w = 0$, and $r = (0, -\frac{1}{2}h_v, 0)^\top$. We now change coordinates $(x, y, t)^\top = H(x, v, w)^\top + r$.

Define $T = L \circ T'' \circ T'$. Then T is an invertible affine transformation that takes $\mathcal{Q}$ to $\mathcal{Q}' := T(\mathcal{Q})$ described by (3.4). Note that T is determined by multiplication by a matrix whose first n rows are $\begin{pmatrix} I_n & 0 \end{pmatrix}$ and a vector r whose first n entries are zero. This implies that $\text{proj}_n(\mathcal{Q}) = \text{proj}_n(\mathcal{Q}')$ and the proof is complete. $\square$

Note that by Lemma 3.5, without loss of generality, we may assume that $\mathcal{Q}$ is described by (3.4). We can further simplify the structure of $\mathcal{Q}$ by projecting out all

variables t_i that do not explicitly appear in the description of $\mathcal{Q}$.

Lemma 3.6 *Assume that $\mathcal{Q} \subseteq \mathbb{R}^{n+p}$ is a convex quadratic region described by (3.4) and that $\mathcal{P} \subseteq \mathbb{R}^{n+p}$ is a polyhedron. If $l = 0$ then $\text{proj}_{n+k}(\mathcal{Q} \cap \mathcal{P}) = \mathcal{Q}' \cap \mathcal{P}'$ where $\mathcal{P}' = \text{proj}_{n+k}(\mathcal{P})$ and $\mathcal{Q}'$ is described by*

$$\left\{ (x, y) \in \mathbb{R}^{n+k} \mid \begin{pmatrix} x \\ y \end{pmatrix}^\top \begin{pmatrix} Q & 0 \\ 0 & I_k \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} h \\ 0 \end{pmatrix}^\top \begin{pmatrix} x \\ y \end{pmatrix} + g \leq 0 \right\}. \quad (3.5)$$

If $l = e_1$, then $\text{proj}_{n+k+1}(\mathcal{Q} \cap \mathcal{P}) = \mathcal{Q}' \cap \mathcal{P}'$ where $\mathcal{P}' = \text{proj}_{n+k+1}(\mathcal{P})$ and $\mathcal{Q}'$ is described by

$$\left\{ (x, y, t) \in \mathbb{R}^{n+k+1} \mid \begin{pmatrix} x \\ y \\ t \end{pmatrix}^\top \begin{pmatrix} Q & 0 & 0 \\ 0 & I_k & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ t \end{pmatrix} + \begin{pmatrix} h \\ 0 \\ 1 \end{pmatrix}^\top \begin{pmatrix} x \\ y \\ t \end{pmatrix} + g \leq 0 \right\}. \quad (3.6)$$

Proof. Let $r = k$ in the case of (3.5) and $r = k + 1$ in the case of (3.6). It suffices to show that $\text{proj}_{n+r}(\mathcal{Q} \cap \mathcal{P}) = \text{proj}_{n+r}(\mathcal{Q}) \cap \text{proj}_{n+r}(\mathcal{P})$. Then for any $\bar{x} \in \text{proj}_{n+r}(\mathcal{Q}) \cap \text{proj}_{n+r}(\mathcal{P})$ there exists $s^1, s^2 \in \mathbb{R}^{p-r}$ such that $(\bar{x}, s^1) \in \mathcal{Q}$ and $(\bar{x}, s^2) \in \mathcal{P}$. Since $e_{n+r+j} \in \text{lin}(\mathcal{Q})$ for each $j \geq 1$ we have $(\bar{x}, s^2) \in \mathcal{Q}$ and hence $\bar{x} \in \text{proj}_{n+r}(\mathcal{Q} \cap \mathcal{P})$. The reverse containment is clear. $\square$

Then by Lemma 3.6, without loss of generality, we may assume that $\mathcal{Q}$ is described by either (3.5) or (3.6).

We now construct a family of affine spaces that will simplify the computation of $\text{proj}_n(\mathcal{Q} \cap \mathcal{P})$. These affine spaces will form a sort of skeleton of the region $\mathcal{Q} \cap \mathcal{P}$ that will contain all the essential information of $\text{proj}_n(\mathcal{Q} \cap \mathcal{P})$. We will make use of the following observations whose short proofs we include for completion.

Observation 4 Let $q(x) = x^\top Qx + h^\top x + g$ be a quadratic polynomial in n variables where Q is a positive semidefinite matrix. Then $q(x)$ has a minimum on $\mathbb{R}^n$ if and only if $h \in \text{range}(Q)$. In this case, the set of minimizers of $q(x)$ is $\{x \in \mathbb{R}^n \mid 2Qx + h = 0\}$.

Proof. Assume $h \notin \text{range}(Q)$. Then since Q is symmetric, we can write $h = Qw + v$ with $Qv = 0$ and $v \neq 0$. Consider $x(t) = -tv$ for $t \in \mathbb{R}$. Then we have

$$q(x(t)) = h^\top x(t) + g = -tv^\top v + g.$$

Since $v \neq 0$, we see that $q(x(t)) \rightarrow -\infty$ as $t \rightarrow +\infty$. Thus, $q(x)$ has no minimum on $\mathbb{R}^n$.

We now prove the reverse direction. Since Q is positive semidefinite, the function $q(x)$ attains its minimum at $\bar{x}$ if and only if $\bar{x}$ solves $\nabla q(x) = 2Qx + h = 0$. This set is nonempty since $h \in \text{range}(Q)$. $\square$

Observation 5 Let $\mathcal{Q} = \{x \in \mathbb{R}^n \mid x^\top Qx + h^\top x + g \leq 0\}$ be a non-empty convex quadratic region. Then

$$\text{rec}(\mathcal{Q}) = \{r \in \mathbb{R}^n \mid Qr = 0, h^\top r \leq 0\}.$$

Proof. Let $r \in \mathbb{R}^n$ such that $Qr = 0$ and $h^\top r \leq 0$. Fix $\bar{x} \in \mathcal{Q}$ and $\lambda \geq 0$. Then $(\bar{x} + \lambda r)^\top Q(\bar{x} + \lambda r) + h^\top(\bar{x} + \lambda r) + g = \bar{x}^\top Q\bar{x} + h^\top \bar{x} + g + \lambda h^\top r \leq 0$ and $\bar{x} + \lambda r \in \mathcal{Q}$. It follows that $r \in \text{rec}(\mathcal{Q})$.

Assume now that $r \in \mathbb{R}^n$ either satisfies $Qr \neq 0$ or $Qr = 0$ and $h^\top r > 0$. Fix $\bar{x} \in \mathcal{Q}$. Then for any $\lambda \geq 0$ we have

$$(\bar{x} + \lambda r)^\top Q(\bar{x} + \lambda r) + h^\top(\bar{x} + \lambda r) + g = \lambda^2 r^\top Qr + \lambda(h + 2Q\bar{x})^\top r + \bar{x}^\top Q\bar{x} + h^\top \bar{x} + g,$$

a polynomial in λ . Since $Q \succeq 0$, as $\lambda \rightarrow \infty$, this polynomial increases indefinitely. Thus, $r \notin \text{rec}(\mathcal{Q})$. $\square$

Proposition 3.7 *Assume that $\mathcal{Q} \subseteq \mathbb{R}^{n+p}$ is a convex quadratic region described by (3.5) or (3.6) and that $\mathcal{P} \subseteq \mathbb{R}^{n+p}$ is a polyhedron. Then either $\text{proj}_n(\mathcal{Q} \cap \mathcal{P}) = \text{proj}_n(\mathcal{P})$ or there exists a finite collection $\mathcal{L}$ of affine spaces such that*

$$\text{proj}_n(\mathcal{Q} \cap \mathcal{P}) = \bigcup_{L \in \mathcal{L}} \text{proj}_n(\mathcal{Q} \cap \mathcal{P} \cap L). \quad (3.7)$$

Moreover, each $L \in \mathcal{L}$ has dimension n and can be described by a system $Fx + Gy = d$ where G is an invertible $p \times p$ matrix.

Proof. Assume first that $\mathcal{Q}$ is described by (3.5) and let

$$q(x, y) = \begin{pmatrix} x \\ y \end{pmatrix}^\top \begin{pmatrix} Q & 0 \\ 0 & I_k \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} h \\ 0 \end{pmatrix}^\top \begin{pmatrix} x \\ y \end{pmatrix} + g.$$

Suppose that $\mathcal{P} = \{(x, y) \in \mathbb{R}^{n+p} \mid (a^i)^\top x + (b^i)^\top y \leq c_i \text{ for all } i \in I\}$ where I is some finite index set. Define $\mathcal{J}$ to be the collection of subsets $J \subseteq I$ with $|J| \leq p$ and such that the set $\{b^i\}_{i \in J}$ is linearly independent. For each $J \in \mathcal{J}$ we construct an affine space L_J to include in $\mathcal{L}$. We construct L_J so that for every fixed $\bar{x} \in \mathbb{R}^n$, the unique point $(\bar{x}, y_{\bar{x}}^J) \in L_J$ minimizes $q(\bar{x}, y)$ over the affine space determined by setting to equality the inequalities indexed by J and fixing $x = \bar{x}$.

We first note that $\emptyset \in \mathcal{J}$ trivially. In the case that $J = \emptyset$, we define $L_J = \{(x, y) \in \mathbb{R}^{n+p} \mid y = 0\}$. Then for any fixed $\bar{x} \in \mathbb{R}^n$ the point $y = 0$ minimizes $q(\bar{x}, y)$ on $\mathbb{R}^p$. We will have need to refer to following construction again, so we emphasize it here.

Construction of a Minimizing Affine Space. Consider a nonempty set $J \in \mathcal{J}$, say $|J| = m$, and define R, U , and l to be the matrices with rows $(a^i)^\top$, $(b^i)^\top$, and c_i for $i \in J$, respectively. Consider the affine space K_J defined by $Rx + Uy = l$.

Since $m \leq p$ we know U has rank m and we can partition the y variables into new variables u and v where the v variables correspond to columns of U that define a full

rank submatrix. This division into $(u, v) \in \mathbb{R}^{p-m} \times \mathbb{R}^m$ causes K_J to be described by $Rx + Su + Tv = l$ where T is invertible. Substitute $v = T^{-1}(l - Rx - Su)$ into the polynomial $q(x, u, v)$ and fix a point $\bar{x} \in \mathbb{R}^n$ to obtain the polynomial $q(\bar{x}, u)$ defined by

$$u^\top (I + S^\top (T^{-1})^\top T^{-1} S) u + 2(S^\top (T^{-1})^\top T^{-1} R \bar{x} + 2S^\top (T^{-1})^\top T^{-1} l)^\top u + g(\bar{x}).$$

Now since $I + S^\top (T^{-1})^\top T^{-1} S$ is positive definite, by Observation 4, the unique minimum of $q(\bar{x}, u)$ is the point satisfying

$$2(I + S^\top (T^{-1})^\top T^{-1} S) u + 2(S^\top (T^{-1})^\top T^{-1} R \bar{x} - S^\top (T^{-1})^\top T^{-1} l) = 0.$$

Note that this minimum depends linearly on $\bar{x}$. We thus define L_J to be the affine space determined by

$$\begin{aligned} S^\top (T^{-1})^\top T^{-1} R x + (I + S^\top (T^{-1})^\top T^{-1} S) u &= S^\top (T^{-1})^\top T^{-1} l \\ R x + S u + T v &= l. \end{aligned} \tag{3.8}$$

Since $I + S^\top (T^{-1})^\top T^{-1} S$ and T are invertible matrices, L_J is an affine space of dimension n and is described by a system $Fx + Gy = d$ where G is an invertible $p \times p$ matrix. This marks the end of the construction. †

Set

$$\mathcal{L} = \{L_J \mid J \in \mathcal{J}\}. \tag{3.9}$$

We claim that $\mathcal{L}$ satisfies (3.7). It suffices to show that for any point $\bar{x} \in \text{proj}_n(\mathcal{Q} \cap \mathcal{P})$, there exists $\bar{y} \in \mathbb{R}^p$ and $L \in \mathcal{L}$ such that $(\bar{x}, \bar{y}) \in \mathcal{Q} \cap \mathcal{P} \cap L$.

Let $\bar{x} \in \text{proj}_n(\mathcal{Q} \cap \mathcal{P})$. Then there exists $y^0 \in \mathbb{R}^p$ such that $(\bar{x}, y^0) \in \mathcal{Q} \cap \mathcal{P}$. This implies that $q(\bar{x}, y^0) \leq 0$ and since $q(\bar{x}, 0)$ minimizes $q(\bar{x}, y)$ on $\mathbb{R}^p$ we have $(\bar{x}, 0) \in \mathcal{Q}$ as well. If $y^0 = 0$, we may choose L_J corresponding to $J = \emptyset$ and we are done. Otherwise, the line segment joining $(\bar{x}, y^0)$ and $(\bar{x}, 0)$ is completely contained in $\mathcal{Q}$. Then by moving

along this line segment from $(\bar{x}, y^0)$ toward $(\bar{x}, 0)$ and inside $\mathcal{P}$ we either reach the point $(\bar{x}, 0)$ or stop at a point $(\bar{x}, y^1) \in \mathcal{P}$. Then there exists an inequality $(a^i)^\top x + (b^i)^\top y \leq c_i$ with $b^i \neq 0$ that is satisfied at equality by $(\bar{x}, y^1)$ and is not satisfied by $(\bar{x}, 0)$. We then set $J = \{i\}$ and continue this sliding process recursively.

Assume that we are at the point $(\bar{x}, y^k)$ with current index set J . We now consider the line segment joining $(\bar{x}, y^k)$ and $(\bar{x}, y_x^J)$. Since $(\bar{x}, y_x^J)$ is the minimizer of $q(\bar{x}, y)$ on K_J , this line segment is contained in $\mathcal{Q} \cap K_J$. Again, slide the point $(\bar{x}, y^k)$ toward $(\bar{x}, y_x^J)$ inside $\mathcal{P}$ and we either reach the point $(\bar{x}, y_x^J)$ or stop at a point $(\bar{x}, y^{k+1}) \in \mathcal{P}$. Then there exists an inequality $(a^j)^\top x + (b^j)^\top y \leq c_j$ with $b^j \notin \text{Span}(\{b^i\}_{i \in J})$ that is satisfied at equality by $(\bar{x}, y^{k+1})$ and is not satisfied by $(\bar{x}, y_x^J)$. We update J to include j and repeat this process.

The end result is that we find a point $(\bar{x}, \bar{y}) \in \mathcal{Q} \cap \mathcal{P} \cap L_J$ for some $J \in \mathcal{J}$. In fact, either we hit a point $(\bar{x}, y_x^J)$ at some iteration or after applying the procedure p times we restrict ourselves to an n -dimensional affine space, which by construction must be in $\mathcal{L}$.

Now assume $\mathcal{Q}$ is described by (3.6). There is one degenerate case to consider. Note that for any fixed $\bar{x} \in \mathbb{R}^n$ we have $\text{rec}(\mathcal{Q}|_{x=\bar{x}}) = \{(0, -\lambda) \in \mathbb{R}^{p+1} \mid \lambda \geq 0\}$ by Observation 5. Moreover, for any $(\bar{x}, \bar{y}) \in \mathbb{R}^{n+p}$ there exists $\bar{t} \in \mathbb{R}$ such that $(\bar{x}, \bar{y}, \bar{t}) \in \mathcal{Q}$. To see this, simply take $\bar{t} \leq -(\bar{x}^\top Q \bar{x} + \bar{y}^\top \bar{y} + h^\top \bar{x} + g)$. Suppose that $(0, -1) \in \text{rec}(\mathcal{P}|_{x=\bar{x}})$ for every $\bar{x} \in \mathbb{R}^n$. We claim that $\text{proj}_n(\mathcal{Q} \cap \mathcal{P}) = \text{proj}_n(\mathcal{P})$. Let $\bar{x} \in \text{proj}_n(\mathcal{P})$ so that there exists $(\bar{y}, \bar{t}) \in \mathbb{R}^{p+1}$ such that $(\bar{x}, \bar{y}, \bar{t}) \in \mathcal{P}$. Then, by the note above, for sufficiently large $\lambda \geq 0$ we have $(\bar{x}, \bar{y}, \bar{t} - \lambda) \in \mathcal{Q} \cap \mathcal{P}$ and $\bar{x} \in \text{proj}_n(\mathcal{Q} \cap \mathcal{P})$.

The remaining case when $\mathcal{Q}$ is described by (3.6) and $(0, -1) \notin \text{rec}(\mathcal{P}|_{x=\bar{x}})$ for any $\bar{x} \in \mathbb{R}^n$ follows similarly to the case where $\mathcal{Q}$ is described by (3.5). We make note of

the necessary changes in the proof. Adjust $\mathcal{J}$ to be the set of subsets $J \subseteq I$ such that $|J| \leq p + 1$ and where K_J is described by a system $Rx + Uy + Vt = l$ where $(U \mid V)$ is of full rank and at least one entry of V is negative. This guarantees that for each fixed $\bar{x} \in \mathbb{R}^n$ the polynomial $q(\bar{x}, y)$ has a minimum on $K_J|_{x=\bar{x}}$, the same condition that we needed before. Note that in this case the empty set is not a member of $\mathcal{J}$.

We demonstrate how to modify the first step of the recursive descent described in the previous case. After this initial step the recursion continues exactly as in the general step detailed above. Let $\bar{x} \in \text{proj}_n(\mathcal{Q} \cap \mathcal{P})$. Then there exists $(y^0, t^0) \in \mathbb{R}^p$ such that $(\bar{x}, y^0, t^0) \in \mathcal{Q} \cap \mathcal{P}$. Since $(0, -1) \notin \text{rec}(\mathcal{P}|_{x=\bar{x}})$, the ray based at $(\bar{x}, y^0, t^0)$ and directed along $(0, 0, -1)$ cannot be completely contained in $\mathcal{P}$. In particular, moving in the direction $(0, 0, -1)$ from the point $(\bar{x}, y^0, t^0)$ and inside $\mathcal{P}$ we stop at a point $(\bar{x}, y^1, t^1) \in \mathcal{P}$. Then there exists an inequality $(a^i)^\top x + (b^i)^\top y + v_i t \leq c_i$ with $v_i < 0$ that is satisfied at equality by $(\bar{x}, y^1, t^1)$. We now set $J = \{i\}$ and the recursion process continues identically as before. $\square$

The family $\mathcal{L}$ of affine spaces defined in Proposition 3.7 allows us to explicitly compute the set $\text{proj}_n(\mathcal{Q} \cap \mathcal{P})$. We do so by considering each set $\mathcal{Q} \cap \mathcal{P} \cap L$ in turn. In the next lemma we will see that for each $L \in \mathcal{L}$, the projection $\text{proj}_n(\mathcal{Q} \cap \mathcal{P} \cap L) = \mathcal{Q}_L \cap \mathcal{P}_L$ for some convex quadratic region $\mathcal{Q}_L \subseteq \mathbb{R}^n$ and polyhedron $\mathcal{P}_L \subseteq \mathbb{R}^n$. This implies that the set $\text{proj}_n(\mathcal{Q} \cap \mathcal{P})$ is a finite union of the sets $\mathcal{Q}_L \cap \mathcal{P}_L$. However, in contrast to Theorem 2.1, the $\mathcal{Q}_L$ and $\mathcal{P}_L$ appearing in the projection cannot be arbitrary. We will see that they share a common structure. An understanding of this structure is essential to finding an extended formulation and thus obtaining a full algebraic characterization. This compatibility requirement is captured in our definition of sets of Type 1 and Type

2.

From here on, we compute an algebraic description of $\text{proj}_n(\mathcal{Q} \cap \mathcal{P})$ where $\mathcal{Q}$ is described by (3.5) or (3.6). The region resulting from $\text{proj}_n(\mathcal{Q} \cap \mathcal{P})$ in the case (3.5) will be called a set of *Type 1* and in the case of (3.6) a set of *Type 2*.

We are thus interested in computing $\text{proj}_n(\mathcal{Q} \cap \mathcal{P} \cap L)$ where $L \in \mathcal{L}$. We claim that $\text{proj}_n(\mathcal{Q} \cap \mathcal{P} \cap L) = \text{proj}_n(\mathcal{Q} \cap L) \cap \text{proj}_n(\mathcal{P} \cap L)$. Let $\bar{x} \in \text{proj}_n(\mathcal{Q} \cap \mathcal{P}) \cap \text{proj}_n(\mathcal{P} \cap L)$. Then there exists $y^1, y^2 \in \mathbb{R}^p$ such that $(\bar{x}, y^1) \in \mathcal{Q} \cap L$ and $(\bar{x}, y^2) \in \mathcal{P} \cap L$. Now since L is defined by $Fx + Gy = d$ with G an invertible $p \times p$ matrix, we have $y^i = G^{-1}(d - F\bar{x})$ for $i = 1, 2$. In particular, $y^1 = y^2$ and we have that $\bar{x} \in \text{proj}_n(\mathcal{Q} \cap \mathcal{P} \cap L)$. The reverse containment is obvious.

Therefore, a description of the sets $\text{proj}_n(\mathcal{Q} \cap L)$ and $\text{proj}_n(\mathcal{P} \cap L)$ is of particular interest to us as they serve as the base regions making up continuous convex quadratic representable sets. We define two functions f_1, f_2 that take as input a convex quadratic region $\mathcal{Q}$ of the form (3.5) or (3.6), respectively, and a special n -dimensional affine space L and output a convex quadratic region, which we will show to be the projection onto $\mathbb{R}^n$ of the set $\mathcal{Q} \cap L$.

Let $\mathcal{Q}$ be a convex quadratic region described by (3.5) and L an affine space described by $Fx + Gy = d$ with G an invertible $p \times p$ matrix. We define $f_1[\mathcal{Q}, L]$ to be the set of $x \in \mathbb{R}^n$ satisfying

$$x^\top(Q + F^\top(G^{-1})^\top G^{-1}F)x + (h - 2F^\top(G^{-1})^\top G^{-1}d)^\top x + d^\top(G^{-1})^\top G^{-1}d + g \leq 0.$$

Let $\mathcal{Q}$ be a convex quadratic region described by (3.6) and L an affine space described by $Fx + G(y, t)^\top = d$ where G is an invertible $(p+1) \times (p+1)$ matrix. We define $f_2[\mathcal{Q}, L]$

to be the set of $x \in \mathbb{R}^n$ satisfying

$$\begin{aligned} x^\top (Q + F^\top (G^{-1})^\top E_k G^{-1} F) x + (h - F^\top (G^{-1})^\top e_{k+1} - 2F^\top (G^{-1})^\top E_k G^{-1} d)^\top x \\ + d^\top (G^{-1})^\top E_k G^{-1} d + ((G^{-1})^\top e_{k+1})^\top d + g \leq 0, \end{aligned}$$

where E_k is the $(k+1) \times (k+1)$ matrix with principal $k \times k$ minor the identity matrix and zero elsewhere. We note that $f_1[\mathcal{Q}, L]$ and $f_2[\mathcal{Q}, L]$ are convex quadratic regions in $\mathbb{R}^n$, since the matrices defining them are each the sum of two positive semidefinite matrices.

Assume now that $\mathcal{Q}$ is described by (3.5). We show that $\text{proj}_n(\mathcal{Q} \cap L) = f_1[\mathcal{Q}, L]$. Define the invertible linear transformation $T_A : \mathbb{R}^{n+p} \rightarrow \mathbb{R}^{n+p}$ by multiplication on the left by the matrix

$$A = \begin{pmatrix} I_n & 0 \\ F & G \end{pmatrix}.$$

Then, we have that $\text{proj}_n(\mathcal{Q} \cap L) = T_A(\mathcal{Q})|_{y=d} = f_1[\mathcal{Q}, L]$. A similar proof shows that $\text{proj}_n(\mathcal{Q} \cap L) = f_2[\mathcal{Q}, L]$ when $\mathcal{Q}$ is described by (3.6).

Similarly, we define a function Π that takes as input a polyhedron $\mathcal{P}$ in $\mathbb{R}^{n+r}$ and a special n -dimensional affine space and outputs a polyhedron in $\mathbb{R}^n$. Let $\mathcal{P} = \{x \in \mathbb{R}^{n+r} \mid (a^i)^\top x + (b^i)^\top y \leq c_i \text{ for all } i \in I\}$ be a polyhedron and L an affine space described by $Fx + Gy = d$ with G an invertible $r \times r$ matrix. We define $\Pi[\mathcal{P}, L]$ to be the polyhedron

$$\{x \in \mathbb{R}^n \mid (a^i)^\top x + (b^i)^\top G^{-1}(d - Fx) \leq c_i \text{ for all } i \in I\}.$$

It is immediate from the substitution $y = G^{-1}(d - Fx)$ that $\text{proj}_n(\mathcal{P} \cap L) = \Pi[\mathcal{P}, L]$.

Thus, we have established the following lemma.

Lemma 3.8 *Let $\mathcal{Q} \subseteq \mathbb{R}^{n+p}$ be a convex quadratic region given by (3.5) or (3.6) and $\mathcal{P} \subseteq \mathbb{R}^{n+p}$ a polyhedron. Let $\mathcal{L}$ be the family defined in Proposition 3.7. For each $L \in \mathcal{L}$*

define $\mathcal{P}_L = \Pi[\mathcal{P}, L]$ and either $\mathcal{Q}_L = f_1[\mathcal{Q}, L]$ in the case of (3.5), or $\mathcal{Q}_L = f_2[\mathcal{Q}, L]$ in the case of (3.6). Then

$$\text{proj}_n(\mathcal{Q} \cap \mathcal{P}) = \bigcup_{L \in \mathcal{L}} \mathcal{Q}_L \cap \mathcal{P}_L.$$

We can use the algebraic description from Lemma 3.8 to complete our characterization. We are now ready for our technical definitions of Type 1 and Type 2.

Let $S \subseteq \mathbb{R}^n$. We say that S is a set of *Type 1* if there exists a convex quadratic region $\bar{\mathcal{Q}} = \{x \in \mathbb{R}^n \mid x^\top Qx + h^\top x + g \leq 0\}$, an integer $k \geq 0$, a finite index set I , and vectors $(a^i, b^i, c_i) \in \mathbb{R}^{n+k+1}$ for each $i \in I$ with the following compatibility structure.

Let $\mathcal{J}$ be the collection of subsets $J \subseteq I$ with $|J| \leq k$ such that the set $\{b^i\}_{i \in J}$ is linearly independent. Then for each nonempty $J \in \mathcal{J}$ we define the affine space $L_J \subseteq \mathbb{R}^{n+k}$ to be the output of the construction of a minimizing affine space found in the proof of Proposition 3.7. These objects are required to satisfy

$$S = (\bar{\mathcal{Q}} \cap_{i \in I} \{x \in \mathbb{R}^n \mid (a^i)^\top x \leq c^i\}) \bigcup_{J \in \mathcal{J}} (\mathcal{Q}_{L_J} \cap \mathcal{P}_{L_J})$$

where each $\mathcal{P}_{L_J} = \Pi[\mathcal{P}, L_J]$ and each $\mathcal{Q}_{L_J} = f_1[\mathcal{Q}, L_J]$.

The definition of a set of *Type 2* is exactly as above, except that S is required to satisfy

$$S = \bigcup_{J \in \mathcal{J}} (\mathcal{Q}_{L_J} \cap \mathcal{P}_{L_J})$$

where each $\mathcal{P}_{L_J} = \Pi[\mathcal{P}, L_J]$ and each $\mathcal{Q}_{L_J} = f_2[\mathcal{Q}, L_J]$.

Theorem 3.9 *Let $S \subseteq \mathbb{R}^n$. Then S is continuous convex quadratic representable if and only if S is a set of Type 1 or Type 2.*

Proof. Assume first that $\mathcal{Q} \subseteq \mathbb{R}^{n+p}$ is a convex quadratic region and $\mathcal{P} \subseteq \mathbb{R}^{n+p}$ is a polyhedron. Then Lemma 3.5, Lemma 3.6, Proposition 3.7, and Lemma 3.8 show that S is of Type 1 or Type 2.

Assume now that S is a set of Type 1. Consider the convex quadratic region $\mathcal{Q} \subseteq \mathbb{R}^{n+k}$ described by (3.5), i.e.,

$$\begin{pmatrix} x \\ y \end{pmatrix}^\top \begin{pmatrix} Q & 0 \\ 0 & I_k \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} h \\ 0 \end{pmatrix}^\top \begin{pmatrix} x \\ y \end{pmatrix} + g \leq 0,$$

and the polyhedron $\mathcal{P} \subseteq \mathbb{R}^{n+k}$ described as

$$\{(x, y) \in \mathbb{R}^{n+k} \mid (a^i)^\top x + (u^i)^\top y \leq c^i \text{ for all } i \in I\}.$$

Then Proposition 3.7 and Lemma 3.8 show that $S = \text{proj}_n(\mathcal{Q} \cap \mathcal{P})$.

The case of S being a set of Type 2 is identical to the case of Type 1, save for the construction of $\mathcal{Q}$ satisfying (3.6) instead of (3.5). $\square$

3.5 The Mixed Binary Case

In this section, we combine the results of Sections 3.3 and 3.4 to state a characterization theorem for sets $S \subseteq \mathbb{R}^n$ that are mixed binary convex quadratic representable.

Let $S \subseteq \mathbb{R}^n$ be a set of Type 1 (or 2). Then S is determined by the data of a convex quadratic region $\bar{\mathcal{Q}} \subseteq \mathbb{R}^n$, an integer $k \geq 0$, an index set I , and vectors $(a^i, b^i, c^i) \in \mathbb{R}^{n+k+1}$ for $i \in I$.

Given two sets $S, S' \subseteq \mathbb{R}^n$ both of Type 1 (resp. both of Type 2), we say that S and S' have the *same structure* if the data determining S and S' as sets of Type 1 (resp. Type 2) can be chosen so that

- (i) $k = k'$,
- (ii) $\bar{\mathcal{Q}}$ and $\bar{\mathcal{Q}}'$ have the same shape,

(iii) $I = I'$ and $(a^i, b^i) = (a'^i, b'^i)$ for each $i \in I$.

We can now state and prove our characterization theorem.

Theorem 3.10 *Let $S \subseteq \mathbb{R}^n$. Then S is mixed binary convex quadratic representable if and only if there exist sets $S_1, \dots, S_r \subseteq \mathbb{R}^n$ all of Type 1 (or all of Type 2) with the same structure, such that $S = \cup_{i=1}^r S_i$.*

Proof. Assume first that there exist sets $S_1, \dots, S_r \subseteq \mathbb{R}^n$ of Type 1 all with the same structure such that $S = \cup_{i=1}^r S_i$. Then by Theorem 3.9 there exist convex quadratic regions $\mathcal{Q}_i \subseteq \mathbb{R}^{n+k}$ and polyhedra $\mathcal{P}_i \subseteq \mathbb{R}^{n+k}$ for $i = 1, \dots, r$ such that $S_i = \text{proj}_n(\mathcal{Q}_i \cap \mathcal{P}_i)$. Moreover, it follows from the construction given in the proof of Theorem 3.9 that all the $\mathcal{Q}_i$ have the same shape and all $\mathcal{P}_i$ have the same recession cone. It follows by applying Theorem 3.2 to $\cup_{i=1}^r (\mathcal{Q}_i \cap \mathcal{P}_i)$ that there exists a convex quadratic region $\mathcal{Q} \subseteq \mathbb{R}^{n+k+r}$ and a polyhedron $\mathcal{P} \subseteq \mathbb{R}^{n+k+r}$ such that

$$\bigcup_{i=1}^r (\mathcal{Q}_i \cap \mathcal{P}_i) = \text{proj}_{n+k}(\mathcal{Q} \cap \mathcal{P} \cap (\mathbb{R}^{n+k} \times \{0, 1\}^r)).$$

Now,

$$\text{proj}_n(\bigcup_{i=1}^r (\mathcal{Q}_i \cap \mathcal{P}_i)) = \bigcup_{i=1}^r \text{proj}_n(\mathcal{Q}_i \cap \mathcal{P}_i) = \bigcup_{i=1}^r S_i = S.$$

In particular, S is mixed binary convex quadratic representable. The proof for sets of Type 2 follows similarly.

For the reverse direction, let $\mathcal{Q} \subseteq \mathbb{R}^{n+p+q}$ be a convex quadratic region and $\mathcal{P} \subseteq \mathbb{R}^{n+p+q}$ be a polyhedron and set

$$S := \text{proj}_n(\mathcal{Q} \cap \mathcal{P} \cap (\mathbb{R}^{n+p} \times \{0, 1\}^q)).$$

Then by allowing $\bar{z}$ to vary over $\{0, 1\}^q$ and Lemma 3.3 we have

$$S = \bigcup_{\bar{z} \in \{0, 1\}^q} \text{proj}_n(\mathcal{Q}|_{z=\bar{z}} \cap \mathcal{P}|_{z=\bar{z}}),$$

where each $\mathcal{Q}|_{z=\bar{z}}$ has the same shape and each $\mathcal{P}|_{z=\bar{z}}$ has the same recession cone. We use the following technical claim to complete the proof of the theorem.

Claim 9. *Let $\mathcal{Q}, \mathcal{Q}' \subseteq \mathbb{R}^{n+p}$ be two convex quadratic regions with the same shape and $\mathcal{P}, \mathcal{P}' \subseteq \mathbb{R}^{n+p}$ be polyhedra with the same recession cone. Then $\text{proj}_n(\mathcal{Q} \cap \mathcal{P})$ and $\text{proj}_n(\mathcal{Q}' \cap \mathcal{P}')$ are both sets of Type 1 (or both sets of Type 2) with the same structure.*

Proof of claim. Let $S = \text{proj}_n(\mathcal{Q} \cap \mathcal{P})$ and $S' = \text{proj}_n(\mathcal{Q}' \cap \mathcal{P}')$. We first normalize $\mathcal{Q}$ and $\mathcal{Q}'$ as in Lemma 3.5. Note that an invertible affine transformation takes two convex quadratic regions with the same shape to two convex quadratic regions with the same shape. Similarly, an invertible transformation preserves equality of the recession cones of two polyhedron.

Thus, we can assume that $\mathcal{Q}$ and $\mathcal{Q}'$ have the same shape and are described by (3.4). We can now apply Lemma 3.6 and further assume that $\mathcal{Q}$ and $\mathcal{Q}'$ are both described by (3.5), or both by (3.6), and still having the same shape. Moreover, since $\text{proj}_{n+k}(\text{rec}(\mathcal{P})) = \text{rec}(\text{proj}_{n+k}(\mathcal{P}))$ we may still assume that $\mathcal{P}$ and $\mathcal{P}'$ have the same recession cone.

Assume now that $\mathcal{Q}$ and $\mathcal{Q}'$ are described by (3.5) having the same shape and that $\mathcal{P}$ and $\mathcal{P}'$ have the same recession cone. It is well known that there exists a matrix $M \in \mathbb{R}^{m \times (n+k)}$ and vectors $r, r' \in \mathbb{R}^m$ such that $\mathcal{P} = \{z \in \mathbb{R}^{n+k} \mid Mz \leq r\}$ and $\mathcal{P}' = \{z \in \mathbb{R}^{n+k} \mid Mz \leq r'\}$. It now follows that S and S' are both sets of Type 1 (or both of Type 2) with the same structure. $\diamond$

By Claim 9, it follows that the sets $\text{proj}_n(\mathcal{Q}|_{z=\bar{z}} \cap \mathcal{P}|_{z=\bar{z}})$ are all sets of Type 1 (or all sets of Type 2) with the same structure. $\square$

3.6 Toward a Geometric Characterization

The algebraic characterizations in Section 3.4 of continuous convex quadratic representable sets and in Section 3.5 of mixed binary convex quadratic representable sets lead to a natural question. Are there geometric conditions that characterize continuous and mixed binary convex quadratic representable sets? In this section, we focus on what these algebraic characterizations imply concerning a geometric description of representable sets.

Consider a continuous convex quadratic representable set $S \subseteq \mathbb{R}^n$. As a consequence of Theorem 3.9, there exist convex quadratic regions $\mathcal{Q}_i \subseteq \mathbb{R}^n$ and polyhedra $\mathcal{P}_i \subseteq \mathbb{R}^n$ for $i = 1, \dots, k$ such that $S = \bigcup_{i=1}^k \mathcal{Q}_i \cap \mathcal{P}_i$. Since S is representable it can be realized as the projection of a convex set which implies that S must be convex as well.

It is unclear whether these two obvious necessary conditions are in fact sufficient as well. This leads us to the following question.

Question 3.11 *Let $S \subseteq \mathbb{R}^n$. Is it true that S is continuous convex quadratic representable if and only if S is convex and there exist convex quadratic regions $\mathcal{Q}_i \subseteq \mathbb{R}^n$ and polyhedra $\mathcal{P}_i \subseteq \mathbb{R}^n$ for $i = 1, \dots, k$ such that*

$$S = \bigcup_{i=1}^k \mathcal{Q}_i \cap \mathcal{P}_i \quad ? \quad (3.10)$$

The main difficulty in establishing a positive answer to this question is finding an extended formulation for a set S given by (3.10). As a step in this direction, given a finite collection of convex quadratic regions $\mathcal{Q}_1, \dots, \mathcal{Q}_k$ in $\mathbb{R}^n$ we can show that there exist a convex quadratic region $\mathcal{Q}$ in $\mathbb{R}^{n+k(n+1)}$ and affine spaces $L_1, \dots, L_k$ in $\mathbb{R}^{n+k(n+1)}$, described by $F_i x + G_i y = d^i$ with each G_i an invertible matrix, such that $\mathcal{Q}_i = \text{proj}_n(\mathcal{Q} \cap$

$L_i)$ for $i = 1, \dots, k$. It is unclear whether this construction allows for a polyhedron $\mathcal{P}$ that would complete the extended formulation.

We can make a similar analysis of necessary conditions in the case of a mixed binary convex quadratic representable set $S \subseteq \mathbb{R}^n$. It follows from Theorem 3.10 that S must be the union of convex regions $R_1, \dots, R_k$ where each R_i is a continuous convex quadratic representable set. It can be checked that each of the regions R_i must have the same set of recession directions. However, these necessary conditions are not sufficient.

An example of a set that is not representable. Consider the set $S \subseteq \mathbb{R}^2$ illustrated in Figure 5 below and described by $S = S_1 \cup S_2$ where $S_1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 - y \leq 0, x \geq 1\}$ and $S_2 = \{(x, y) \in \mathbb{R}^2 \mid x^2 - y \leq 0, x \leq -1\}$. Then S is the union of two continuous convex quadratic representable sets with the same recession cone, and thus meets the two obvious necessary conditions described above. We will show however, that S is *not* mixed binary convex quadratic representable. In order to do so, we will derive a stronger necessary condition for mixed binary convex quadratic representable sets. †

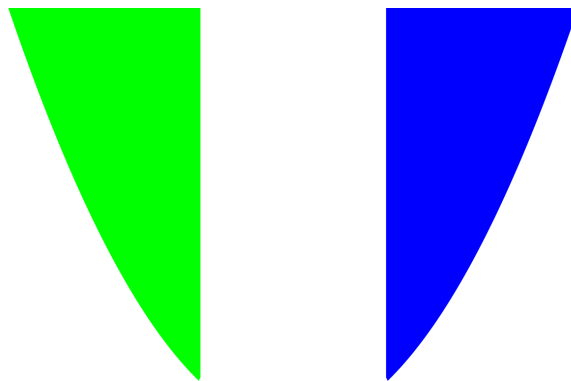


Figure 5: A set that is *not* mixed binary convex quadratic representable

Let $C \subseteq \mathbb{R}^n$ be a nonempty closed convex set. Let $a \in \mathbb{R}^n$ be a nonzero vector. We

say that a is an *unbounded linear objective of C* if $\max\{a^\top x \mid x \in C\} = +\infty$. We can now establish the following proposition.

Proposition 3.12 *Let $\mathcal{Q}$ be a convex quadratic region in $\mathbb{R}^n$ described by*

$$\mathcal{Q} = \{x \in \mathbb{R}^n \mid x^\top Qx + (Qw + v)^\top x + g \leq 0\},$$

where $v \in \ker(Q)$ and $v \neq 0$. Let $\mathcal{P} \subseteq \mathbb{R}^n$ be a polyhedron. Assume that $\mathcal{Q} \cap \mathcal{P}$ is nonempty. Then $a \in \mathbb{R}^n$ is an unbounded linear objective of $\mathcal{Q} \cap \mathcal{P}$ if and only if either

- (a) there exists $r \in \text{rec}(\mathcal{P}) \cap \text{rec}(\mathcal{Q})$ such that $a^\top r > 0$; or*
- (b) there exist both $r \in \text{rec}(\mathcal{P}) \cap \text{relint}(\text{rec}(\mathcal{Q}))$ such that $a^\top r \geq 0$ and $s \in \text{rec}(\mathcal{P})$ such that $a^\top s > 0$.*

Proof. We first note that by Observation 5, $\text{relint}(\text{rec}(\mathcal{Q})) = \{x \in \mathbb{R}^n \mid Qx = 0, v^\top x < 0\}$.

Assume first that there exists $r \in \text{rec}(\mathcal{P}) \cap \text{rec}(\mathcal{Q})$ such that $a^\top r > 0$. Fix $M \geq 0$. We show how to find a point in $\mathcal{Q} \cap \mathcal{P}$ with objective value at least M . Let $\bar{x} \in \mathcal{Q} \cap \mathcal{P}$. There exists $\lambda \geq 0$ such that $\bar{x} + \lambda r \in \mathcal{Q} \cap \mathcal{P}$ and $a^\top(\bar{x} + \lambda r) \geq M$. Thus, a is an unbounded linear direction of $\mathcal{Q} \cap \mathcal{P}$.

Suppose there exist $r \in \text{rec}(\mathcal{P}) \cap \text{relint}(\text{rec}(\mathcal{Q}))$ such that $a^\top r \geq 0$ and $s \in \text{rec}(\mathcal{P})$ such that $a^\top s > 0$. Fix $M \geq 0$. We show how to find a point in $\mathcal{Q} \cap \mathcal{P}$ with objective value at least M . Let $\bar{x} \in \mathcal{Q} \cap \mathcal{P}$. Now, we may assume that $M \geq a^\top \bar{x}$ else we are done. Set $\gamma = \frac{M - a^\top \bar{x}}{a^\top s}$ and $y = \bar{x} + \gamma s$. Then, since $v^\top r < 0$, for any $\lambda \geq \max\{-\frac{y^\top Qy + (Qw + v)^\top y + g}{v^\top r}, 0\}$, we have $a^\top(y + \lambda r) \geq a^\top y = M$. It now suffices to show that $y + \lambda r \in \mathcal{Q} \cap \mathcal{P}$. Since $r, s \in \text{rec}(\mathcal{P})$, clearly $y + \lambda r = \bar{x} + \gamma s + \lambda r \in \mathcal{P}$. Now since $Qr = 0$, we have

$$(y + \lambda r)^\top Q(y + \lambda r) + (Qw + v)^\top(y + \lambda r) + g = \lambda v^\top r + y^\top Qy + (Qw + v)^\top y + g \leq 0,$$

by choice of λ . Thus, $y + \lambda r \in \mathcal{Q} \cap \mathcal{P}$ and a is an unbounded linear objective of $\mathcal{Q} \cap \mathcal{P}$.

We prove the reverse direction by induction on $\dim(\text{lin}(\mathcal{Q}))$. Suppose that a is an unbounded linear objective of $\mathcal{Q} \cap \mathcal{P}$. Then there exists a sequence $\{x^k\}$ in $\mathcal{Q} \cap \mathcal{P}$ such that $a^\top x^k \rightarrow +\infty$. Let $w^k = \frac{x^k}{\|x^k\|}$. Then $\{w^k\}$ is a bounded sequence and therefore must have a convergent subsequence. Suppose $\bar{w}$ is a limit point of this sequence. Then $\bar{w}$ is a unit vector, satisfies $a^\top \bar{w} \geq 0$, and it is a fact of convex analysis that $\bar{w} \in \text{rec}(\mathcal{P}) \cap \text{rec}(\mathcal{Q})$.

If $\bar{w}$ satisfies $a^\top \bar{w} > 0$ then we have met condition (a), and we are done. Thus, we may assume that $a^\top \bar{w} = 0$. Since a is an unbounded linear objective of $\mathcal{P}$, it follows from the Minkowski-Weyl decomposition theorem that there exists $s \in \text{rec}(\mathcal{P})$ such that $a^\top s > 0$. If $\bar{w}$ satisfies $v^\top \bar{w} < 0$ then $\bar{w} \in \text{relint}(\text{rec}(\mathcal{Q}))$ and we are done.

In the base case, $\dim(\text{lin}(\mathcal{Q})) = 0$, we have $\text{rec}(\mathcal{Q}) = \{\lambda v \mid \lambda \leq 0\}$ and since $\bar{w} \neq 0$ it follows that $v^\top \bar{w} < 0$.

In order to prove the inductive step we assume that either condition (a) or (b) holds for an unbounded linear objective provided $\dim(\text{lin}(\mathcal{Q})) < k$. Assume $\dim(\text{lin}(\mathcal{Q})) = k$. By the same construction as before, we either meet condition (a) or (b) or have a vector $\bar{w} \in \text{rec}(\mathcal{P}) \cap \text{rec}(\mathcal{Q})$ satisfying $a^\top \bar{w} = 0$ and $v^\top \bar{w} = 0$.

It remains to find $r \in \text{rec}(\mathcal{P}) \cap \text{relint}(\text{rec}(\mathcal{Q}))$ satisfying $a^\top r \geq 0$. We note that by Observation 5, we have $\bar{w} \in \text{lin}(\mathcal{Q})$. Consider the projection of $\mathcal{Q} \cap \mathcal{P}$ and the vector a onto the orthogonal complement of $\text{Span}(\{\bar{w}\})$. Let $\mathcal{Q}'$ denote the projection of $\mathcal{Q}$, $\mathcal{P}'$ the projection of $\mathcal{P}$, and a' the projection of a . Since $\bar{w} \in \text{lin}(\mathcal{Q})$ and $a^\top \bar{w} = 0$ we have that a' is unbounded linear objective of $\mathcal{Q}' \cap \mathcal{P}'$ and $\dim(\text{lin}(\mathcal{Q}')) = k - 1$. We can now apply the induction hypothesis to obtain either a vector $r' \in \text{rec}(\mathcal{P}') \cap \text{rec}(\mathcal{Q}')$ satisfying $a'^\top r' > 0$ or two vectors $u' \in \text{rec}(\mathcal{P}') \cap \text{relint}(\text{rec}(\mathcal{Q}'))$ satisfying $a'^\top u' \geq 0$ and $s' \in \text{rec}(\mathcal{P}')$ satisfying $a'^\top s' > 0$. We claim that by lifting the vectors r', u' , and s' back

to the original space, we can obtain vectors satisfying either (a) or (b) for the initial region $\mathcal{Q} \cap \mathcal{P}$.

Assume first that there exists $r' \in \text{rec}(\mathcal{P}') \cap \text{rec}(\mathcal{Q}')$ satisfying $a'^\top r' > 0$. Since $r' \in \text{rec}(\mathcal{P}')$ there exists $r \in \text{rec}(\mathcal{P})$ that projects down to r' . In particular, $r = r' + \alpha \bar{w}$ for some $\alpha \in \mathbb{R}$. It follows that $r \in \text{rec}(\mathcal{Q})$ and $a^\top r > 0$ so that condition (a) is met.

Assume now that there exist two vectors $u' \in \text{rec}(\mathcal{P}') \cap \text{relint}(\text{rec}(\mathcal{Q}'))$ satisfying $a'^\top u' \geq 0$ and $s' \in \text{rec}(\mathcal{P}')$ satisfying $a'^\top s' > 0$. Again, there exists $u \in \text{rec}(\mathcal{P})$ such that $u = u' + \alpha \bar{w}$ for some $\alpha \in \mathbb{R}$. Since $\bar{w} \in \text{lin}(\mathcal{Q})$ and $a^\top \bar{w} = 0$, it follows that $u \in \text{rec}(\mathcal{P}) \cap \text{relint}(\text{rec}(\mathcal{Q}))$ and $a^\top u \geq 0$. Similarly, there exists $s \in \text{rec}(\mathcal{P})$ that projects down to s' . Then $a^\top s > 0$ and condition (b) is met. $\square$

A description of unbounded linear objectives for convex quadratic regions, with $v \neq 0$, can be obtained by considering Proposition 3.12 when $\mathcal{P} = \mathbb{R}^n$. In this case, a is an unbounded linear objective of $\mathcal{Q}$ if and only if there exists $r \in \text{relint}(\text{rec}(\mathcal{Q}))$ such that $a^\top r \geq 0$.

We note that a similar characterization of bounded linear objectives holds when $v = 0$, i.e., when $\mathcal{Q}$ is an ellipsoidal region, see Chapter 2 for more details.

Proposition 3.13 *Let $\mathcal{E} \subseteq \mathbb{R}^n$ be an ellipsoidal region and $\mathcal{P} \subseteq \mathbb{R}^n$ be a polyhedron. Assume that $\mathcal{E} \cap \mathcal{P}$ is nonempty. Then $a \in \mathbb{R}^n$ is an unbounded linear objective of $\mathcal{E} \cap \mathcal{P}$ if and only if there exists $r \in \text{rec}(\mathcal{P}) \cap \text{rec}(\mathcal{E})$ such that $a^\top r > 0$.*

Proof. We first note that by the proof of Claim 2, we have $\mathcal{E} \cap \mathcal{P} = B + \mathcal{C}$ for a bounded set $B \subseteq \mathbb{R}^n$ and a polyhedral cone $\mathcal{C}$. Moreover, by Observations 2 and 3, $\mathcal{C}$ is the polyhedral cone $\text{rec}(\mathcal{P}) \cap \text{rec}(\mathcal{E})$. Assume now that a is an unbounded linear objective of $\mathcal{E} \cap \mathcal{P}$. Since B is a bounded set, there exists $r \in \text{rec}(\mathcal{P}) \cap \text{rec}(\mathcal{E})$ such that $a^\top r > 0$.

Assume now that there exists $r \in \text{rec}(\mathcal{P}) \cap \text{rec}(\mathcal{E})$ satisfying $a^\top r > 0$. Fix $\bar{x} \in \mathcal{E} \cap \mathcal{P}$ and $M \geq 0$. Since $a^\top r > 0$, there exists $\lambda \geq 0$ such that $\bar{x} + \lambda r \in \mathcal{E} \cap \mathcal{P}$ and $a^\top(\bar{x} + \lambda r) \geq M$.

□

Again, a description of unbounded linear objectives for ellipsoidal regions can be recovered by considering Proposition 3.13 when $\mathcal{P} = \mathbb{R}^n$. In this case, a is an unbounded linear objective of $\mathcal{E}$ if and only if there exists $r \in \text{rec}(\mathcal{E})$ such that $a^\top r > 0$.

Together Propositions 3.12 and 3.13 describe the set of unbounded linear objectives of sets that are the intersection of a convex quadratic region and a polyhedron. The following corollary to Propositions 3.12 and 3.13 establishes a new necessary condition for mixed binary convex quadratic representable sets.

Corollary 3.14 *Let $S \subseteq \mathbb{R}^n$ be a mixed binary convex quadratic representable set. Then there exist continuous convex quadratic representable sets $R_1, \dots, R_k \subseteq \mathbb{R}^n$ each with the same set of unbounded linear objectives such that $S = \cup_{i=1}^k R_i$.*

Proof. Since S is mixed binary convex quadratic representable, there exists a convex quadratic region $\mathcal{Q} \subseteq \mathbb{R}^{n+p+q}$ and a polyhedron $\mathcal{P} \subseteq \mathbb{R}^{n+p+q}$ such that

$$S = \text{proj}_n(\mathcal{Q} \cap \mathcal{P} \cap (\mathbb{R}^{n+p} \times \{0, 1\}^q)).$$

Take $R_1, \dots, R_k$ to be the nonempty regions among $\text{proj}_n((\mathcal{Q} \cap \mathcal{P})|_{z=\bar{z}})$ for $\bar{z} \in \{0, 1\}^q$. Then clearly, each R_i is continuous convex quadratic representable, $S = \cup_{i=1}^k R_i$, and it remains to show that each R_i has the same set of unbounded linear objectives.

Now, by Propositions 3.12 and 3.13 the set of unbounded linear objectives of a non-empty set $\mathcal{Q}' \cap \mathcal{P}'$ depends only on the recession cone of $\mathcal{Q}'$ and the recession cone of $\mathcal{P}'$. We now apply Lemma 3.3 to observe that each region $(\mathcal{Q} \cap \mathcal{P})|_{z=\bar{z}}$ has shape

of $\mathcal{Q}|_{z=\bar{z}}$ and recession cone of $\mathcal{P}|_{z=\bar{z}}$ independent of choice of $\bar{z} \in \{0,1\}^q$. It then follows by Observation 5, that the recession cone of $\mathcal{Q}|_{z=\bar{z}}$ is independent of choice of $\bar{z} \in \{0,1\}^q$. In particular, each nonempty $(\mathcal{Q} \cap \mathcal{P})|_{z=\bar{z}}$ has the same set of unbounded linear objectives. It follows that their projections, $R_1, \dots, R_k$ have the same set of unbounded linear objectives as well. $\square$

An example of a set that is not representable (cont.). Assume then that S is mixed binary convex quadratic representable. Then, by Corollary 3.14, S decomposes into a union of regions $R_1, \dots, R_k$ each with the same set of unbounded linear objectives. However, the two regions S_1 and S_2 do not have the same set of unbounded linear objectives. In particular, $(1,0)^\top$ must be an unbounded linear objective for at least one R_i contained in S_1 . However, $\max\{x \mid x \in S_2\} = -1$ which implies that $(1,0)^\top$ is not an unbounded linear objective for some R_j , a contradiction. It follows that S is not mixed binary convex quadratic representable. $\dagger$

We note that Corollary 3.14 imposes a stronger necessary condition on mixed binary convex quadratic representable sets than our initial observation provides. It is unclear whether stronger necessary conditions are required. Thus, we are left to consider the following question.

Question 3.15 *Let $S \subseteq \mathbb{R}^n$. Is it true that S is mixed binary convex quadratic representable if and only if there exist continuous convex quadratic representable sets $R_1, \dots, R_k \subseteq \mathbb{R}^n$ each with the same set of unbounded linear objectives such that $S = \cup_{i=1}^k R_i$?*

As for Question 3.11, in order to show that Question 3.15 is true, the main difficulty is in finding a suitable extended formulation for the given set S . This is due to the fact

that the extended formulations pervasive throughout disjunctive programming fail in the presence of nonlinear constraints. While such extended formulations can be altered to behave nicely under certain conditions, e.g., when S is bounded, it seems that entirely different formulations must be found for the general case.

If we were to show that the questions were false, we should search for strictly stronger necessary conditions satisfied by the respective classes of representable sets. The algebraic characterizations found in Sections 3.4 and 3.5 provide a solid foundation for this search. In particular, there is still much to explore in the projection procedure described in Section 3.4. At the current moment however, it is unclear what further sort of geometric conditions are implied by the algebraic characterizations.

An interesting future work would be exploring whether imposing stronger conditions on a given set $S \subseteq \mathbb{R}^n$ would lead to a readily constructible extended formulation. In particular, can we find certain classes of mixed binary convex quadratic representable sets for which we can provide explicit extended formulations?

The notion of unbounded linear objective is quite similar to the notion of *thin convex sets* explored in [8]. Let $K \subseteq \mathbb{R}^n$ be a closed convex set. A set K is *thin* if the following holds for all $a \in \mathbb{R}^n$: $\max\{a^\top x \mid x \in K\} = +\infty$ if and only if there exists $r \in \text{rec}(K)$ such that $a^\top r > 0$. We conclude this section by characterizing which convex quadratic regions are thin.

Proposition 3.16 *Let $\mathcal{Q} \subseteq \mathbb{R}^n$ be a convex quadratic region. Then $\mathcal{Q}$ is thin if and only if $\mathcal{Q}$ is either an ellipsoidal region or a half-space.*

Proof. Suppose that $\mathcal{Q}$ is described by

$$\mathcal{Q} = \{x \in \mathbb{R}^n \mid x^\top Qx + (Qw + v)^\top x + g \leq 0\}$$

where $v \in \ker(Q)$.

Assume first that $\mathcal{Q}$ is either an ellipsoidal region or a half-space. Since a half-space is clearly thin, we may assume that $\mathcal{Q}$ is an ellipsoidal region. Then by the characterization of unbounded linear objectives of $\mathcal{Q}$ following Proposition 3.13 $\mathcal{Q}$ is thin.

Assume now that $\mathcal{Q}$ is neither an ellipsoidal region nor a half-space. This implies that $Q \neq 0$ and $v \neq 0$. Then by the characterization of unbounded linear objectives of $\mathcal{Q}$ following Proposition 3.12, any nonzero vector a in the orthogonal complement of $\ker(Q)$ is an unbounded linear objective of $\mathcal{Q}$. Any such vector a is orthogonal to all vectors in $\text{rec}(\mathcal{Q})$ and thus $\mathcal{Q}$ is not thin. $\square$

In [8], the authors show that if a closed convex set $K \subseteq \mathbb{R}^n$ with $\text{int}(K) \cap \mathbb{Z}^n \neq \emptyset$ is not thin, then the region $\text{conv}(K \cap \mathbb{Z}^n)$ is not a polyhedron. Thus, for a general convex quadratic set $\mathcal{Q} \subseteq \mathbb{R}^n$ the region $\text{conv}(\mathcal{Q} \cap \mathbb{Z}^n)$ is not a polyhedron. The lack of a succinct description of the points in $\mathcal{Q} \cap \mathbb{Z}^n$ is one of the reasons we do not investigate extended integer variables in this work. A possible future work could consider this more general setting.

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