

Spaces of Pants Decompositions for Surfaces of Infinite Type

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Abstract

We study the pants complex of surfaces of infinite type. When S is a surface of infinite type, the usual definition of the pants graph $\mathcal{P}(S)$ yields a graph with infinitely many connected-components. In the first part of our paper, we study this disconnected graph. In particular, we show that the extended mapping class group $\text{Mod}(S)$ is isomorphic to a proper subgroup of $\text{Aut}(\mathcal{P})$, in contrast to the finite-type case where $\text{Mod}(S) \cong \text{Aut}(\mathcal{P}(S))$.

In the second part of the paper, motivated by the Metaconjecture of Ivanov [9], we seek to endow $\mathcal{P}(S)$ with additional structure. To this end, we define a coarser topology on $\mathcal{P}(S)$ than the topology inherited from the graph structure. We show that our new space is path-connected, and that its automorphism group is isomorphic to $\text{Mod}(S)$.

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1 Introduction

Let S be an orientable surface of infinite type. The *graph of pants decompositions* $\mathcal{P}(S)$ contains one vertex for each isotopy class of decompositions of S into pairs of pants. Vertices of $\mathcal{P}(S)$ are connected by an edge if they differ by an “elementary move” (see section 2.)

Unlike in the finite-type case, the pants graph of an infinite-type surface is always disconnected. As an example, consider a pants decomposition X of S containing only nonseparating curves, and a pants decomposition Y of S containing infinitely many separating curves. An elementary move replaces only one curve, hence no finite sequence of elementary moves can take X to Y .

The (extended) mapping class group $\text{Mod}(S)$ acts on $\mathcal{P}(S)$ by graph isomorphisms. This action is well understood for finite-type surfaces. In particular, this action is faithful and $\text{Aut}(\mathcal{P}(S)) \cong \text{Mod}(S)$ for all but finitely many surfaces [12].

The author is aware of only two prior papers which consider pants graphs for infinite-type surfaces. Fossas and Parlier [6] add additional edges to the pants graph to make it connected. They are concerned primarily with the coarse geometry of the graphs, and do not consider the automorphism group. Later, Aroca showed [2] that some of the connected pants graphs of Fossas and Parlier have automorphism group $\text{Mod}^\pm(S)$.

The outline of the rest of this paper is as follows. In section 2, we recall some background information. In section 3, we study the automorphism group of the pants graph of infinite-type surfaces. In particular we show that it is not isomorphic to the mapping class group. In section 4, we define a family of topologies on the set of isotopy classes of pants decompositions of a surface (that is, the 0-skeleton of the pants graph). This topology is analogous to the Gromov-Hausdorff topology on a set of metric spaces. In section 5, we explicitly construct a metric on the vertex space, prove it is complete, and use it to prove that the vertex space is homeomorphic to the Baire space $\mathbb{N}^{\mathbb{N}}$.

In section 6, we extend the topology on the vertices to the entire pants graph, creating the *pants space* $\mathcal{PS}(S)$. In section 7, we prove that the automorphism group of the pants space is the extended mapping class group, generalizing Margalit’s result for finite-type surfaces. In section 8, we prove that the natural action of the mapping class group on $\mathcal{PS}(S)$ is continuous.

In section 9, we briefly discuss some open questions about the pants space.

2 Preliminaries

In this section we recall the terminology and basic results that we will use in this paper.

If S is an orientable surface, a *pair of pants decomposition* X for S is a set of disjoint, simple-closed curves in S such that $S \setminus X$ is homeomorphic to a disjoint union of thrice-punctured spheres. By abuse of notation, we will sometimes refer to an isotopy class of pants decompositions as a single pants decomposition.

The *complexity* $\kappa(S)$ of a surface S is the number of curves in a pants decomposition for S . If $S = S_{g,n}^b$ has finite type, then $\kappa(S) = 3g - 3 + b + n$, otherwise $\kappa(S) = \infty$.

We recall the *Alexander Method* for finite-type surfaces [5, Prop 2.8].

Theorem 2.1. *Let S have finite-type and let $\kappa(S) \geq 6$, and let $f : S \rightarrow S$ be a homeomorphism. Then f fixes each simple closed curve in S up to isotopy if and only if f is isotopic to the identity on S .*

The main result of [7] extends the Alexander Method to infinite-type surfaces.

Theorem 2.2 ([7]). *Let S be a surface of infinite type, and let $f : S \rightarrow S$ be a homeomorphism. Then f fixes each simple closed curve in S up to isotopy if and only if f is isotopic to the identity on S .*

Let S be an infinite-type surface and let $S_0 \subset S_1 \subset \dots$ be an exhaustion of S by finite-type surfaces. For notational convenience, we let $S_{-1} = \emptyset$. Following the convention of [7], we say that the S_j form a *principal exhaustion* for S if

1. For all $n \geq 0$, $\overline{S_n} \subseteq S_{n+1}^\circ$. (This condition ensures that any compact set in S is contained in S_n for some n .)
2. For all $n \geq 0$, $\partial S_n \setminus \partial S$ is a union of finitely many pairwise-disjoint essential simple closed curves.
3. For all $n \geq 0$, each connected-component of $S_n \setminus S_{n-1}$ has complexity at least 6.

The *curve complex* $C(S)$ of a surface whose vertex set is the set of isotopy classes of simple closed curves on S , and where k vertices form a $k - 1$ -simplex if and only if the corresponding curves have mutually disjoint representatives.

The mapping class group acts on $C(S)$ in the obvious way. The Alexander Method implies that this action is faithful. 1997, Ivanov proved the following:

Theorem 2.3 ([8]). *Let S be a finite-type surface of genus at least 2. Then $\text{Aut}(C(S)) \cong \text{Mod}^\pm(S)$.*

In [11], F Luo extended the result to finite-type surfaces of complexity at least 2. Two decades later, Bavard, Dowdell, and Rafi [3] extended the result to infinite-type surfaces.

There are many related results which state that some simplicial complex associated with a surface has automorphism group $\text{Mod}^\pm(S)$ (see the references in [4] for examples). In 2006, Ivanov [9] proposed the *Metaconjecture*

Conjecture 2.4 ([9]). *Every object naturally associated to a surface S and having sufficiently rich structure has $\text{Mod}^\pm(S)$ as its automorphism group. Moreover, this can be proven using a reduction to the theorem about the automorphisms of $C(S)$.*

The phrases "naturally associated" and "sufficiently rich structure" are not given rigorous definitions.

One case of the metaconjecture which was shown by Margalit in [12] was that of the *graph of pants decompositions*. Two pants decompositions X and Y are said to *differ by an elementary move* if

1. $X \setminus Y$ is a single curve α .
2. $Y \setminus X$ is a single curve α'
3. The intersection number $i(\alpha, \alpha')$ is positive minimal for the complexity 1 subsurface containing α and α' .

The curves α, α' must lie in a copy of either $S_{1,2}$ or $S_{0,4}$. Any two nonisotopic curves in $S_{1,2}$ intersect at least once, so for an elementary move we require that they intersect exactly once. Similarly, any two nonisotopic curves in $S_{0,4}$ intersect at least twice, so for an elementary move we require that they intersect exactly twice.

The pants graph $\mathcal{P}(S)$ is the graph whose vertices are isotopy classes of pants decompositions of S , and where two pants decompositions are joined by an edge if they differ by an elementary move. There is also a related notion of the *pants complex*, which is a 2-complex whose 1-skeleton is $\mathcal{P}(S)$, but we will not use the pants complex in this paper.

Margalit proved the following theorem.

Theorem 2.5 ([12]). *Let S be a finite-type surface with $\kappa(S) \geq 2$. Then $\text{Aut}(\mathcal{P}(S)) \cong \text{Mod}^\pm(S)$.*

If S is an infinite-type surface, then $\mathcal{P}(S)$ has uncountably many connected components. The reason is that an elementary move can only replace one curve at a time. In an infinite-type surface, each pants decomposition contains infinitely many curves. For two pair of pants decompositions X and Y such that $X \setminus Y$ is infinite, no finite sequence of elementary moves can turn X into Y .

For any surface S , $\text{Mod}(S)$ can be given the structure of a topological group. We endow $\text{Homeo}(S)$ with the compact-open topology, and quotient by isotopy to obtain a topology on $\text{Mod}(S)$. When S has finite type, the resulting topology is always discrete [5, Section 2.1]. By contrast, when S has infinite type, $\text{Mod}(S)$ is always homeomorphic to the Baire space $\mathbb{N}^{\mathbb{N}}$ [15, Corollary 9].

3 Automorphisms of $\mathcal{P}(S)$

The goal of this section is to understand the automorphism group of $\mathcal{P}(S)$.

3.1

The goal of this subsection is to establish the following result.

Theorem 3.1. *Let S be an infinite-type surface. Then the extended mapping class group of S is isomorphic to a proper subgroup of $\text{Aut}(\mathcal{P}(S))$.*

This theorem at first glance seems to contradict the Ivanov Metaconjecture. However, $\mathcal{P}(S)$ is a disconnected graph with uncountably many connected components. Thus, it can be said to lack the "sufficiently rich structure" required by the Metaconjecture.

We will need several lemmas to prove Theorem 3.1.

Lemma 3.2. *Let S be an infinite-type surface, and let X and Y be pants decompositions of S . Then X and Y lie in the same connected-component of $\mathcal{P}(S)$ if and only if $|X \setminus Y|$ is finite.*

Proof. The "only if" direction follows from the fact that an elementary move can only replace one curve at a time. Conversely, if $|X \setminus Y|$ is finite, then $|Y \setminus X| = |X \setminus Y|$ is also finite. Hence the union of all curves in $(X \setminus Y) \cup (Y \setminus X)$ is contained in some finite-type subsurface of S . The "if" direction now follows from the fact that finite-type pants graphs are connected. $\square$

Lemma 3.3. *Let S be a surface of infinite type and let the support of $f \in \text{Mod}^{\pm}(S)$ have finite type. Then f induces automorphisms on every connected component of $\mathcal{P}(S)$.*

Proof. If f has finite type support and X is a pants decomposition of S , then f fixes all but finitely-many curves of X up to isotopy. Hence, $f(X)$ and X are in the same connected-component of $\mathcal{P}(S)$. $\square$

Lemma 3.4. *Let Γ be a connected-component of $\mathcal{P}(S)$ and let $f \in \text{Mod}^\pm(S)$ be a mapping class such that the restriction of f to Γ is the identity. Then f is isotopic to the identity.*

Proof. Let a and b be distinct curves in S . Then there is some pants decomposition $X \in \Gamma$ such that $a \in X$ but $b \notin X$. By construction, $f(X) = X$, hence $f(a) \neq b$. Since the only assumption we needed to make about b was that $b \neq a$, we can conclude that $f(a) = a$. Hence, by the Alexander method, $f \cong \text{Id}_S$ $\square$

Lemma 3.5. *Let Γ be a connected-component of $\mathcal{P}(S)$ and let $f, g \in \text{Mod}^\pm(S)$ be mapping classes such that $f|_\Gamma = g|_\Gamma$. Then f and g are isotopic.*

Proof. Since the restrictions of f and g to Γ are equal, the restriction of $f^{-1} \circ g$ to Γ is the identity. Hence $f^{-1} \circ g$ is isotopic to the identity by the previous lemma. $\square$

Let $\text{Mod}_{\text{Fin}}^\pm(S)$ denote the group of mapping classes which contain a homeomorphism whose support is a finite-type subsurface.

Lemma 3.6. *The automorphism group $\text{Aut}(\mathcal{P})$ contains a subgroup isomorphic to the direct product*

$$\prod_{\Gamma \in \Pi_0(\mathcal{P}(S))} \text{Mod}_{\text{Fin}}^\pm(S)$$

Proof. $\mathcal{P}$ is a disconnected graph, and hence its automorphism group contains a copy of

$$\prod_{\Gamma \in \Pi_0(\mathcal{P}(S))} \text{Aut}(\Gamma)$$

The finite-type-supported mapping class group acts faithfully on each connected component of S , and hence $\text{Aut}(\Gamma)$ contains a subgroup isomorphic to $\text{Mod}_{\text{Fin}}^\pm(S)$. The result follows. $\square$

We are now ready to prove Theorem 3.1.

Proof. Choose some component $\Gamma \in \Pi_0(\mathcal{P}(S))$ and some nontrivial mapping class $f \in \text{Mod}_{\text{Fin}}^\pm(S)$. Then f induces a nontrivial automorphism on Γ . Let $\bar{f}$ be the automorphism of $\mathcal{P}(S)$ given by

$$\bar{f}(X) = \begin{cases} f(X), & X \in \Gamma \\ X, & X \notin \Gamma \end{cases}$$

Then $\bar{f}$ gives an automorphism on each component of $\mathcal{P}(S)$ and hence is an automorphism of $\mathcal{P}(S)$. On the other hand, $\bar{f}$ acts trivially on all but one component, and hence by Lemma 3.4 cannot be induced by a nontrivial mapping class. $\square$

3.2

The goal of this section is to prove a converse to Lemma 3.3. Namely, we will show

Theorem 3.7. *Let $f \in \text{Mod}^\pm(S)$. Then f induces automorphisms on every connected component of $\mathcal{P}(S)$ if and only if f is finite-type-supported.*

We will need several lemmas to prove this theorem.

Lemma 3.8. *Let S be an infinite-type surface, X be a pants decomposition of S , and C be an infinite subset of the curves of X . Then there exists an infinite subset $A = \{\alpha_1, \dots\} \subseteq C$ such that all curves in A lie on pairwise-disjoint complexity 1 subsurfaces.*

Proof. We will construct A inductively. We can choose any curve in C to be α_1 . Now suppose we have already chosen $\alpha_1, \dots, \alpha_{n-1} \in C$ which all lie on disjoint complexity 1 subsurfaces. Each α_i shares a pair of pants with at most four curves in $C \setminus \{\alpha_1, \dots, \alpha_{n-1}\}$. Hence, all but finitely many curves in C lie on disjoint complexity 1 subsurfaces from each α_i for $1 \leq i \leq n-1$, and hence there are infinitely many choices in C for α_n . $\square$

Lemma 3.9. *Let X be a pants decomposition of S and let $A = \{\alpha_1, \dots\} \subseteq X$ be an infinite family of curves in X which all lie on mutually disjoint complexity 1 subsurfaces. Let $f \in \text{Mod}^\pm(S)$ be a mapping class such that $f(\alpha_i) \neq \alpha_i$ for all $i \in \mathbb{N}$. Then there exists a family of curves $B = \{\beta_1, \dots\}$ in S such that*

1. *The curve β_i is disjoint from $X \setminus \{\alpha_i\}$.*
2. *The curves α_i and β_i are distinct.*
3. *For all $i \in \mathbb{N}$, $f(\beta_i) \notin B$*
4. *For all $i \in \mathbb{N}$, $f^{-1}(\beta_i) \notin B$.*

Proof. We can construct B inductively. As a base case, $B_0 = \emptyset$ vacuously satisfies conditions (1)-(4). As our induction hypothesis, suppose that we have chosen a sequence of curves $B_{n-1} = \{\beta_1, \dots, \beta_{n-1}\}$ such that for each $1 \leq i \leq n-1$, β_i satisfies the conditions

1. The curve β_i is disjoint from $X \setminus \{\alpha_i\}$.
2. The curves α_i and β_i are distinct.

3. For all $1 \leq i \leq n-1$, $f(\beta_i) \notin B_{n-1}$

4. For all $1 \leq i \leq n-1$, $f^{-1}(\beta_i) \notin B_{n-1}$.

We want to choose a curve β_n such that

(i) The curve β_n is disjoint from $X \setminus \{\alpha_n\}$.

(ii) The curves α_n and β_n are distinct

(iii) $f(\beta_n) \notin B_{n-1}$

(iv) $f^{-1}(\beta_n) \notin B_{n-1}$

(v) $f(\beta_n) \neq \beta_n$

Conditions (i)-(iv) can be easily satisfied: there are infinitely many isotopy classes of curves in the complexity 1 subsurface containing α_n , and conditions (ii)-(iv) only require us to avoid a finite number of curves. Now recall that $f(\alpha_n) \neq \alpha_n$. Hence, f is not isotopic to the identity on the complexity 1 subsurface containing α_n , and hence there are infinitely many curves on this subsurface which are not fixed by f . Thus, we can find a β_n satisfying conditions (i)-(v).

Conditions (iii) and (v) ensure that $f^{-1}(\beta_i) \neq \beta_n$ for $1 \leq i \leq n$. Conditions (iv) and (v) ensure that $f(\beta_i) \neq \beta_n$ for $1 \leq i \leq n$. We have thus shown that $B_n := B_{n-1} \cup \{\beta_n\}$ satisfies conditions (1)-(4) of the induction hypothesis. Hence, the desired set of curves B exists by induction. $\square$

Lemma 3.10. *Let $f \in \text{Mod}^\pm(S)$ and let X be a pants decomposition of S . Suppose that there are infinitely many curves $c \in X$ such that $f(c) \neq c$. Then there exists a pants decomposition Y such that Y and $f(Y)$ are in different components of $\mathcal{P}(S)$.*

Proof. If X and $f(X)$ are in different components, then we are done.

Now suppose that X and $f(X)$ are in the same connected component. Our goal is to modify X in such a way that we will obtain the desired pants decomposition Y .

Because X and $f(X)$ are in the same component, the set of curves $X \setminus f^{-1}(X)$ is finite. Hence, there are infinitely many curves $c \in X$ such that $f(c) \neq c$ but $f(c) \in X$. Thus, we can find an infinite subset $A = \{\alpha_1, \dots\} \subseteq X$ such that $f(A) \subseteq X$, $f(\alpha_i) \neq (\alpha_i)$ for all $i \geq 1$, and all the curves in A lie on mutually disjoint subsurfaces.

Invoking the previous lemma, we obtain a family of curves $B = \{\beta_1, \dots\}$. We obtain Y from X by replacing α_i with β_i for all $i \geq 1$. This process still yields a pants decomposition because the α_i all lie on disjoint subsurfaces.

Now, we want to show that $f(\beta_i) \notin Y$ for all $i \geq 1$. First, we consider the case where $f(\alpha_i) \notin A$. Then $f(\alpha_i) \in Y$. Also, α_i and β_i are distinct and intersect, so $f(\alpha_i)$ and $f(\beta_i)$ are also distinct and intersect. Thus, $f(\beta_i)$ cannot be contained in a pants decomposition with $f(\alpha_i)$, so $f(\beta_i) \notin Y$.

On the other hand, suppose $f(\alpha_i) \in A$. Then $f(\alpha_i) = \alpha_j$ for some $j \in \mathbb{N}$, and $\beta_j \in Y$. Since α_i and β_i lie on the same complexity 1 subsurface, so do α_j , β_j , and $f(\beta_i)$. Recall that β_j was chosen so that $f(\beta_i) \neq \beta_j$, hence $f(\beta_i) \notin Y$.

Thus, we have infinitely many curves that are in $f(Y)$ but not Y , so $f(Y)$ and Y are in different components. □

We are now ready to prove Theorem 3.7

Proof. We have already proven the “if” direction in Lemma 3.3. Now, suppose that f induces automorphisms on every connected component of $\mathcal{P}(S)$. Choose a finite set of pants decompositions $P_1, \dots, P_n$ which fill S . By the previous lemma, f fixes all but finitely many curves in each pants decomposition. The mapping class group of a pair of pants is generated by Dehn twists around the boundary components. Hence, outside of a finite type subsurface, f is isotopic to a product of Dehn twists along curves in P_1 . But the P_i fill S , hence each curve in P_1 intersects with at least one curve in P_j for some $2 \leq j \leq n$. Since f also fixes all but finitely many curves in P_j for all $2 \leq j \leq n$, it follows that f is isotopic to the identity on all but finitely many pairs of pants in P_1 , and hence f is finite-type-supported. □

3.3

The goal of this subsection is to prove the following two theorems. Let $\phi : S \rightarrow S'$ be a π_1 -injective simplicial embedding between surfaces. Let Γ and Γ' respectively be connected components of S and S' , and let $f : \Gamma \rightarrow \Gamma'$ be a simplicial injective embedding. We say that ϕ *induces* f if there exists a multicurve Q of S' such that $f(X) = \phi(X) \cup Q$ for all vertices X of Γ .

Theorem 3.11. *Let Γ and Γ' be connected components of $\mathcal{P}(S)$ and $\mathcal{P}(S')$, respectively. let $f : \Gamma \rightarrow \Gamma'$ be an isomorphism. Then there is a homeomorphism $\phi : S \rightarrow S'$, unique up to isotopy, which induces f .*

This result will be important in section seven when we determine the automorphism group of the pants space.

We will need the following theorem, which is an extension of the main result of [1].

Theorem 3.12 (Pants Graph Embedding Theorem). *Let S and S' be (possibly finite-type) surfaces such that each connected-component has complexity at least 2. Let Γ and Γ' respectively be connected components of $\mathcal{P}(S)$ and $\mathcal{P}(S')$. Let $f : \Gamma \rightarrow \Gamma'$ be an injective simplicial map. Then there exists a π_1 -injective embedding $\phi : S \rightarrow S'$ which induces f .*

The case where both S and S' are of finite-type is Theorem A in [1].

We start by proving the case of the Pants Graph Embedding Theorem where S has finite-type and S' has infinite type.

Lemma 3.13. *Let S be a compact orientable surface of negative Euler characteristic and such that each connected-component has complexity at least 2. Let S' be an infinite-type orientable surface, and let $f : \mathcal{P}(S) \rightarrow \mathcal{P}(S')$ be an injective simplicial map. Then there exists a π_1 -injective embedding $\phi : S \rightarrow S'$ which induces f .*

Definition 3.14. Let Q be a multicurve of S . The *deficiency* of Q is the cardinality of $P \setminus Q$, where P is a pants decomposition containing Q . Note that if S is of finite type, then the deficiency of Q is $\kappa(S) - |Q|$.

We note that in the case where S has finite type, our definition is equivalent to the one given in [1].

With our altered definition of deficiency, the proof Lemma 3.13 is exactly the same as the case where both surfaces are of finite-type. Writing out the full proof would simply involve copying Aramayona's paper word for word, except that our surface S' has infinite type.

In particular, Aramayona's proof gives us a lemma similar to Theorem C in [1]. Let $Q \subset S$ be a multicurve. We denote by $\mathcal{P}(S)_Q$ the full subgraph of $\mathcal{P}(S)$ spanned by vertices which contain Q . Note that if Q has finite deficiency, then $\mathcal{P}(S)_Q$ is connected by Lemma 3.2. A connected-component of $S \setminus Q$ is said to be *nontrivial* if it has nonzero complexity.

Lemma 3.15. *Let S be a finite-type surface such that each connected-component has complexity at least 2, and let S' be an infinite-type surface. Let Γ' be a connected component of $\mathcal{P}(S')$. Let $f : \mathcal{P}(S) \rightarrow \Gamma'$ be an injective simplicial map. Then*

- *There exists an infinite multicurve $Q \subset S'$, of deficiency $\kappa(S)$, such that $f(\mathcal{P}(S)) = \mathcal{P}(S')_Q$. In particular, $\mathcal{P}(S) \cong \mathcal{P}(S')_Q$.*

- S and $S' \setminus Q$ have the same number of nontrivial components. Moreover, if $S_1, \dots, S_r$ and $S'_1, \dots, S'_r$ are respectively nontrivial components of S and $S' \setminus Q$, then, up to reordering of the indices, f induces isomorphisms $f_i : \mathcal{P}(S_i) \rightarrow \mathcal{P}(S'_i)$. In particular $\kappa(S_i) = \kappa(S'_i)$.

We now present a proof of Theorems 3.12. This argument is similar to the one used in [3] to prove Theorem 1.3.

Proof of Theorem 3.12. Choose a pants decomposition $X \in \Gamma$. Choose an exhaustion $S_0 \subset S_1 \subset \dots \subset S$ of S by finite-type surfaces such that for all $n \geq 0$, ∂S_n is a disjoint union of essential curves in X . This exhaustion induces natural embeddings $\iota_n : \mathcal{P}(S_n) \rightarrow \Gamma$. Composing with f gives embeddings $f \circ \iota_n : \mathcal{P}(S_n) \rightarrow \Gamma'$.

By Lemma 3.15, each $f \circ \iota_n$ is an isomorphism between $\mathcal{P}(S_n)$ and $\mathcal{P}(S')_{Q_n}$ for some deficiency $\kappa(S_n)$ multi-curve Q_n . Since $f \circ \iota_n$ is a restriction of $f \circ \iota_{n+1}$, $Q_{n+1} \subset Q_n$ for all natural numbers n .

By Lemma 3.13, each $f \circ \iota_n$ is induced by a π_1 -injective embedding $\phi_n : S_n \rightarrow S'$. Moreover, by the second part of Lemma 3.15, the image of ϕ_n is the union of the nontrivial components of $S' \setminus Q_n$. Since $Q_{n+1} \subset Q_n$, the image of ϕ_n is a subsurface of the image of ϕ_{n+1} , and moreover ϕ_n is isotopic to a restriction of ϕ_{n+1} . Hence we can take the colimits of the ϕ_n to get a π_1 -injective embedding $\phi : S \rightarrow S'$. $\square$

Now we are ready to prove Theorem 3.11.

Proof of Theorem 3.11. Our argument here is almost the same as the proof of Theorem 3.12. The only difference is that we need to show the embedding $\phi : S \rightarrow S'$ is actually a homeomorphism. Suppose $S \setminus S'$ is nonempty. Since ϕ is π_1 -injective, there is an essential curve $\alpha \subset (S \setminus S')$. Since ϕ induces f , α is either a curve in every curve of $f(\Gamma)$ or none of the curves of $f(\Gamma)$. But this is impossible as f is surjective. $\square$

3.4

The goal of this section is to prove the following theorem.

Theorem 3.16. *Let $\Gamma \in \Pi_0(\mathcal{P}(S))$ be a connected-component let $X \in \Gamma$ be a pants decomposition. Then there is an infinite-support mapping class $f \in \overline{\text{Mod}_{\text{Fin}}^{\pm}(S)} \setminus \text{Mod}_{\text{Fin}}^{\pm}(S)$ such that $f(X) = X$.*

Proof. Choose an orientation for each curve X . Let f be the mapping class of the homeomorphism obtained by performing a Dehn twist simultaneously along each curve in X . Then f fixes X and hence induces an automorphism on Γ . If we let $\alpha_1, \alpha_2, \dots$ denote the curves of X , then f is a limit of the sequence $T_{\alpha_1}, T_{\alpha_1} \cdot T_{\alpha_2}, T_{\alpha_1} \cdot T_{\alpha_2} \cdot$

$T_{\alpha_3}, \dots$ of compactly supported mapping classes. Hence f lies in the closure of $\text{Mod}_{\text{Fin}}^{\pm}(S)$. $\square$

3.5

The goal of this section is to prove the following theorem.

Theorem 3.17. *Let $\Gamma \in \Pi_0(\mathcal{P}(S))$ be a connected-component. Then there is a mapping class $f \in \overline{\text{Mod}_{\text{Fin}}^{\pm}(S)} \subset \text{Mod}^{\pm}(S)$ such that $f(\Gamma) \neq \Gamma$.*

First, we recall a theorem of Taylor and Zupan[14].

Theorem 3.18. *Let S be a finite surface, let G be a Cartesian product of Farey graphs, and let $i : G \rightarrow \mathcal{P}(S)$ be a simplicial embedding. Then $i(G)$ is totally geodesic in $\mathcal{P}(S)$.*

To prove Theorem 3.17, we will need an infinite-type version of Taylor and Zupan's theorem. We will need the following lemma.

Lemma 3.19. *Let S have infinite type, let $\Gamma \in \Pi_0(\mathcal{P}(S))$ be a connected-component, let G be a finite graph, and let $\iota : G \rightarrow \Gamma$ be a simplicial embedding. Then there is a finite-type subsurface $S' \subset S$ such that $\iota(G) \subset i(S')$, where $i : \mathcal{P}(S') \rightarrow \mathcal{P}(S)$ is the natural inclusion.*

Proof. Let $X_1, \dots, X_m$ be the vertices of $\iota(G)$. Since Γ is connected, Lemma 3.2 implies that

$$X_j \setminus \left(\bigcap_{k=1}^m X_k \right)$$

is finite for each $1 \leq j \leq m$. Hence,

$$A = \bigcup_{j=1}^m (X_j \setminus \left(\bigcap_{k=1}^m X_k \right))$$

is also finite. Let S' be a finite-type surface containing every curve in A . Then S' satisfies the conclusion of the lemma. $\square$

Lemma 3.20. *Let S have infinite type, let $\Gamma \in \Pi_0(\mathcal{P}(S))$ be a connected-component, let G be a Cartesian product of finitely many Farey graphs, and let $i : G \rightarrow \Gamma$ be a simplicial embedding. Then $i(G)$ is totally geodesic in Γ .*

Proof. Let X and Y be vertices of G . Let P be a geodesic in Γ from $i(X)$ to $i(Y)$. Then $P \cup i(G)$ is a finite graph, and hence by Lemma 3.19 there is a finite-type subsurface $S' \subset S$ such that $(P \cup i(G)) \subset \mathcal{P}(S')$. By Taylor and Zupan's theorem for finite-type surfaces, $i(G)$ is totally geodesic in $\mathcal{P}(S')$, and hence $P \subset i(G)$, as desired. $\square$

Before proving Theorem 3.17, we note that Lemma 3.20 provides us with an interesting corollary about the pants graph. Recall that the *geometric rank* of a metric space G is the greatest n such that there is a quasi-isometric embedding of $\mathbb{Z}^n$ in G .

Corollary 3.21. *Let S have infinite type, and let $\Gamma \in \Pi_0(\mathcal{P}(S))$ be a connected-component. Then Γ has infinite geometric rank.*

Proof. For any $n \in \mathbb{N}$, S contains a subsurface S' consisting of n disjoint complexity 1 subsurfaces. The pants graph of S' is the product of n Farey graphs. By Lemma 3.20, this product embeds totally geodesically in Γ , hence the geometric rank of Γ is at least n . $\square$

Now we are ready to prove Theorem 3.17.

Proof. Pick a pants decomposition $X \in \Gamma$. Pick a multicurve $Y \subset X$ such that $S \setminus Y$ is a disjoint union of complexity 1 surfaces. Call these complexity 1 surfaces $S_1, \dots$. Let $X_j = X \cap S_j$ for $j \in \mathbb{N}$. For each $j \in \mathbb{N}$, choose a pseudo-Anosov map ϕ_j on S_j such that $d_{\mathcal{P}(S_j)}(X_j, \phi_j(X_j)) > j$. Let $\tilde{\phi}_j \in \text{Homeo}(S)$ be an extension of ϕ_j to S which is the identity outside of $\overline{S_j}$. By Lemma 3.20, the image of the natural inclusion of $\mathcal{P}(S_j)$ is totally geodesic in Γ , hence $d_\Gamma(X, \tilde{\phi}_j(X)) > j$. Consider the homeomorphism $\tilde{\phi}$ obtained as the limit of the sequence $\tilde{\phi}_1, \tilde{\phi}_2 \circ \tilde{\phi}_1, \tilde{\phi}_3 \circ \tilde{\phi}_2 \circ \tilde{\phi}_1, \dots$. Then $\tilde{\phi}(X)$ cannot be any finite distance from X , and hence $\tilde{\phi}(X) \notin \Gamma$. Thus the mapping class of $\tilde{\phi}$ satisfies the conclusion of Theorem 3.17. $\square$

We now state a corollary which ties together several results from this section.

Corollary 3.22. *Let S have infinite type, and let $\Gamma \in \Pi_0(\mathcal{P}(S))$ be a connected-component. There are natural monomorphisms $\text{Mod}_{\text{Fin}}^\pm(S) \hookrightarrow \text{Aut}(\Gamma) \hookrightarrow \text{Mod}^\pm(S)$. Neither map is a surjection. Moreover, the complement of the image of both maps contain elements of $\overline{\text{Mod}_{\text{Fin}}^\pm(S)}$.*

Proof. The first embedding comes from Lemma 3.3. The fact that the first embedding is not surjective, and that the complement of the image contains an element of $\overline{\text{Mod}_{\text{Fin}}^\pm(S)}$, comes from Theorem 3.16.

The second embedding comes from Theorem 3.11. The fact that the second embedding is not surjective, and that the complement of the image contains an element of $\overline{\text{Mod}_{\text{Fin}}^\pm(S)}$, comes from Theorem 3.17. $\square$

4 The Vertex Spaces $V_i(S)$

In this section, we endow the vertex set of $\mathcal{P}(S)$ with a family of topologies. We will prove that the resulting vertex spaces are metrizable, second-countable, and totally disconnected. We will use the vertex spaces again in the next section.

We fix a principal exhaustion $S_0 \subseteq S_1 \subseteq \dots$ of S by finite-type subsurfaces. Also fix a hyperbolic metric on S . For pants decompositions X and Y , we wish to define some sense in which X and Y "agree" on a compact subsurface $Z \subset S$. We wish for this definition to be preserved by isotopy. We will use this notion of agreement to define the limit of a sequence of pants decompositions, and use this definition of a limit to get a topological structure on the set of pants decompositions.

For the chosen hyperbolic metric on S , each pants decomposition has a unique geodesic representative. Thus, we can define agreement for the geodesic representatives, and consider two pants decompositions to agree if their geodesic representatives agree.

Definition 4.1. Two pants decompositions X and Y *0-agree* if they are isotopic.

Definition 4.2. Let X and Y be two geodesic pants decompositions, viewed as sets of (geodesic) curves, and let $S' \subset S$ be a compact subsurface. X and Y are said to *1-agree on S'* if

$$\{x \in X : x \cap S' \neq \emptyset\} = \{y \in Y : y \cap S' \neq \emptyset\}$$

There is another way we could have defined 1-agreement.

Lemma 4.3. *Let X and Y be two geodesic pants decompositions, viewed as subsets of S . Then X and Y 1-agree on S' if and only if*

$$X \cap S' = Y \cap S'$$

Proof. The "only if" direction is trivial. Suppose that $X \cap S' = Y \cap S'$ and that γ is a curve in X which intersects S' . Then Y must contain a curve γ' such that $\gamma \cap S' = \gamma' \cap S'$. Any two tangent geodesics are equal, so $\gamma' = \gamma$. Thus X and Y 1-agree on S' . $\square$

Definition 4.4. Let X and Y be two geodesic pants decompositions, and let $S' \subset S$ be a compact subsurface. View X and Y as subsets of S . X and Y are said to *2-agree on Z* if the subsets $X \cap S'$ and $Y \cap S'$ are properly isotopic, counting multiplicities of proper arcs.

If we ignore the multiplicities of arcs in $X \cap S'$ and $X \cap Y'$, we get a weaker notion of agreement.

Definition 4.5. Let X and Y be two geodesic pants decompositions, and let $S' \subset S$ be a compact subsurface. X and Y are said to *3-agree* on S' if every curve and proper arc in $X \cap S'$ is isotopic to a curve or proper arc in $Y \cap S'$, and vice versa. In other words, X and Y 3-agree on S' if the multicurves $X \cap S'$ and $Y \cap S'$ correspond to the same simplex in the arc-and-curve complex of S' .

Definition 4.6. Let X and Y be two geodesic pants decompositions, viewed as sets of curves, and let $S' \subset S$ be a compact subsurface. X and Y are said to *4-agree* on Z if

$$\{x \in X : x \subseteq S'\} = \{y \in Y : y \subseteq S'\}$$

Lemma 4.7. Let X and Y be two pants decompositions of S , and let S' be a compact subsurface, and let $0 \leq i \leq j \leq 4$. If X and Y i -agree on S' , then they also j -agree on S' .

Proof. This result follows immediately from the definitions. □

Lemma 4.8. For $0 \leq i \leq 4$, and for a compact subsurface $S' \subset S$, i -agreement on S' is an equivalence relation.

Proof. The result follows from the uniqueness of geodesics. □

Definition 4.9. Let $0 \leq i \leq 4$ and let $X_1, X_2, \dots$ be a sequence of pants decompositions for S . The sequence is said to *i -converge* to a pants decomposition Y if, for all $n \in \mathbb{N}$, there exists a $m \in \mathbb{N}$ such that for all $k \geq m$, X_k and Y i -agree on S_n .

Note that a sequence of pants decompositions 0-converges if and only if it is eventually constant.

Definition 4.10. Let $0 \leq i \leq 4$. Define $V_i(S)$ to be the set of pants decompositions of S , endowed with the finest topology such that, for all sequences $\{X_j\}_{j \geq 1}$ in $V_i(S)$, $\{X_j\}_{j \geq 1}$ converges to Y in $V_i(S)$ if and only if $\{X_j\}_{j \geq 1}$ i -converges to Y .

The space $V_0(S)$ is just the discrete topology on the set of isotopy classes of pants decompositions of S . Our goal for this section is to understand the structure of $V_i(S)$ for $1 \leq i \leq 4$.

Lemma 4.11. For $1 \leq i \leq 4$, the topology on $V_i(S)$ does not depend on the choice of exhaustion for S .

Proof. Let $S'_0 \subset \dots$ be another exhaustion for S . Let $V'_i(S)$ be the topological space obtained by replacing S_i with S'_i in the above definition. Suppose $X_1, X_2, \dots$ is a sequence of pants decompositions which converge to Y in $V_i(S)$. We want to show that it also converges to Y in $V'_i(S)$. Choose some $n' \in \mathbb{N}$. Then there is some $n \in \mathbb{N}$ such that $S'_{n'} \subseteq S_n$. By the definition of $V_i(S)$, there exists a $m \in \mathbb{N}$ such that for all $k \geq m$, X_k and Y i -agree on S_n , and hence X_k and Y i -agree on $S'_{n'}$. Hence the sequence $X_1, \dots$ converges to Y in $V'_i(S)$, so $V_i(S)$ and $V'_i(S)$ have the same topology. $\square$

Definition 4.12. Suppose $X \in V_i(S)$ for some $1 \leq i \leq 4$. Let

$$B_n^i(X) = \{Y \in V_i(S) : X \text{ and } Y \text{ } i\text{-agree on } S_n\}$$

Lemma 4.13. For $1 \leq i \leq 4$,

$$\{B_n^i(X) : X \in V_i(S), n \in \mathbb{N}\}$$

is a basis for $V_i(S)$

Proof. First, we must show that each $B_n^i(X)$ is open in $V_i(S)$. Suppose $Y \in B_n^i(X)$ and $X_1, \dots$ converge to Y in $V_i(S)$. Then $\{X_j\}_{j \geq 1}$ i -converges to Y , hence all but finitely many of the X_j are in $B_n^i(X)$. The openness of $B_n^i(X)$ thus follows from the fact that $V_i(S)$ carries the finest possible topology such that all i -convergent sequences converge in $V_i(S)$.

Now suppose $X \in V_i(S)$ and that $U \subseteq V_i(S)$ is a neighborhood of X . We want to show that there exists some $n \in \mathbb{N}$ such that $B_n^i(X) \subseteq U$. Suppose not. Then for all $n \in \mathbb{N}$, there exists a pants decomposition $X_n \in B_n^i(X) \setminus U$. But this means the sequence $\{X_n\}_{n \geq 1}$ converges to X , but none of the X_n are in U , a contradiction because U is open. Hence there is some $B_n^i(X) \subseteq U$. $\square$

Lemma 4.14 (Disjointness Lemma). Let $X, Y \in V_i(S)$. Then $B_n^i(X)$ and $B_n^i(Y)$ are either disjoint or equal.

Proof. $B_n^i(X)$ and $B_n^i(Y)$ are equivalence classes of vertices with i -agreement on S_n as the equivalence relation. Any two equivalence classes are either disjoint or equal. $\square$

Definition 4.15. For $X \in V_i(S)$, let

$$V_i(S, X) = \{Y \in \mathcal{P}(S) : X \text{ and } Y \text{ are in the same connected-component of } \mathcal{P}\}$$

Lemma 4.16. *For $1 \leq i \leq 4$, $V_i(S)$ is second-countable. In particular, for any $X \in V_i(S)$, the space has a countable basis given by*

$$\{B_n^i(Y) : Y \in V_i(S, X), n \in \mathbb{N}\}$$

Proof. Let $Y \in V_i(S)$ and $n \in \mathbb{N}$. Then $B_n^i(Y)$ intersects nontrivially with $V_i(S, X)$, hence there is some $\hat{Y} \in (B_n^i(Y) \cap V_i(S, X))$. By the Disjointness Lemma, $B_n^i(Y) = B_n^i(\hat{Y})$. Thus,

$$\{B_n^i(Y) : Y \in V_i(S), n \in \mathbb{N}\} = \{B_n^i(Y) : Y \in V_i(S, X), n \in \mathbb{N}\}$$

is a basis for $V_i(S)$. □

Lemma 4.17. *The space $V_i(S)$ is zero-dimensional: i.e., it has a basis of clopen sets.*

Proof. For each $n \geq 0$, the set

$$\{B_n^i(Y) : Y \in V_i(S)\}$$

gives a partition of $V_i(S)$ into disjoint open sets, hence each set in the basis from the previous lemma is clopen. □

Lemma 4.18. *For $1 \leq i \leq 4$, $V_i(S)$ is Hausdorff.*

Proof. Let $X, Y \in V_i(S)$ with $X \neq Y$. Choose an $n \in \mathbb{N}$ such that X and Y do not i -agree on S_n . Then by the Disjointness Lemma, $B_n^i(X)$ and $B_n^i(Y)$ are disjoint open sets that respectively contain X and Y . □

Lemma 4.19. *For $1 \leq i \leq 4$, $V_i(S)$ is totally disconnected.*

Proof. Let $X, Y \in V_i(S)$ with $X \neq Y$. Choose an $n \in \mathbb{N}$ such that X and Y do not i -agree on S_n . Then by Lemma 4.17, $B_n^i(X)$ and $B_n^i(Y)$ are disjoint clopen sets, so X and Y are not in the same connected-component. □

Recall that a space M is said to be T_3 if for any nonempty closed set $C \subseteq M$ and any point $m \in M \setminus C$, there exist open sets $U, V \subset M$ such that $C \subseteq U$ and $m \in V$.

Lemma 4.20. *For $1 \leq i \leq 4$, $V_i(S)$ is T_3 .*

Proof. Let $X \in V_i(S)$ be a pants decomposition and $C \subset V_i(S)$ be a closed set with $X \notin C$. Note that $V_i(S) \setminus C$ is an open neighborhood of X . We claim that there is an $n \in \mathbb{N}$ such that $B_n^i(X) \cap C = \emptyset$. Suppose not. Then for all $m \in \mathbb{N}$, there is some $Y_m \in C$ such that $Y_m \in B_m^i(X)$. But this implies that $Y_1, Y_2, \dots$ is a sequence of

pants decompositions which converge to X , and hence this sequence must eventually enter $V_i(S) \setminus C$, which is a contradiction. Thus, we have some n such that $B_n^i(X) \cap C = \emptyset$.

Now let

$$U = \bigcup_{Y \in C} B_n^i(Y)$$

Then U is an open set containing C , and by the Disjointness Lemma, $B_n^i(X) \cap U = \emptyset$. □

Corollary 4.21. *For $1 \leq i \leq 4$, $V_i(S)$ is metrizable.*

Proof. Combining the previous lemmas, $V_i(S)$ is a Hausdorff, T_3 , second-countable topological space. Hence, by the Urysohn Metrization Theorem[16], $V_i(S)$ is metrizable. □

Lemma 4.22. *For all $X \in V_i(S)$ and all $n \geq 1$, $B_n^i(X)$ is not compact.*

Proof. The set $B_n^i(X)$ has an open cover by a disjoint union of clopen sets:

$$B_n^i(X) = \bigcup_{X' \in B_n^i(X)} B_{n+1}^i(X')$$

By the assumption that $\kappa(S_{n+1} \setminus S_n) \geq 6$, this cover is infinite. Since the sets in the cover are disjoint, they clearly cannot have a finite subcover. □

The following Lemma will be useful in the next section. We state and prove it here because it is a statement about the topology of $V_i(S)$.

Lemma 4.23. *Every compact subset of $V_i(S)$ has empty interior.*

Proof. Suppose $C \subseteq V_i(S)$ has nonempty interior. Then there is a point $X \in C$ and an $n \geq 1$ such that $B_n^i(X) \subset C$. But since $B_n^i(X)$ is closed, and closed subsets of compact sets are compact, then $B_n^i(X)$ is also compact, which is a contradiction. □

The following Lemma will be useful in the section six. We state and prove it here because it is a statement about the topology of $V_i(S)$.

Lemma 4.24. *Let $X, Y \in V_i(S)$. Then there exists a sequence of pants decompositions $X = X_1, X_2, \dots$ such that X_j and X_{j+1} differ by an elementary move and such that the X_j converge to Y .*

Proof. Recall the assumption that the exhaustion $S_0 \subset \cdots$ is *principal*. Also recall that the topology of $V_i(S)$ does not depend on the choice of principal exhaustion. Thus, we may assume that $(\partial S_j \setminus \partial S) \subset Y$ for all $j \geq 0$.

Since the exhaustion is principal, S_0 has connected pants graph. Thus, there is a path $X_1, \dots, X_{m_0}$ in $\mathcal{P}(S)$ such that X_{m_0} agrees with Y on S_0 and such that $(\partial S_0 \setminus \partial S) \subset X_{m_0}$.

We will inductively build a path satisfying the conclusion of this lemma. Suppose by way of induction that there exist integers $1 \leq m_0 < m_1 < \cdots < m_k$ and a path $X = X_1, X_2, \dots, X_{m_k}$ in $\mathcal{P}(S)$ such that

- For all $m_j \leq \ell \leq m_k$, the pants decompositions X_ℓ and Y agree on S_j .
- For all $m_j \leq \ell \leq m_k$, the pants decomposition X_ℓ contains $\partial S_j \setminus \partial S$.

Starting from X_{m_k} , we make additional elementary moves to extend the sequence. The exhaustion is principal, and hence each component of $S_{k+1} \setminus S_k$ has a connected pants graph. Thus, there is a pants decomposition $X_{m_{k+1}}$ in the same connected-component as X_{m_k} such that

- $(\partial S_{k+1} \setminus \partial S) \subset X_{m_{k+1}}$.
- The pants decompositions $X_{m_{k+1}}$ and Y agree on S_{k+1} .

Moreover, by the induction hypothesis, X_{m_k} contains $\partial S_k \setminus \partial S$, and so we can find a path from X_{m_k} to $X_{m_{k+1}}$ such that all added and removed curves are entirely contained in $S \setminus S_k$. The result follows. $\square$

5 $V_i(S)$ as a Metric Space

In this section, we construct a metric on the set of pants decompositions of S , which induces the topology of $V_i(S)$ from section 4. We will show that this metric is complete, and we will use it to prove that $V_i(S)$, as a topological space, is homeomorphic to the Baire space $\mathbb{Z}^{\mathbb{Z}}$.

Definition 5.1. For $1 \leq i \leq 4$, denote by $\hat{d}_i$ the *partial function* on $V_i(S) \times V_i(S)$ with the following values.

$$\hat{d}_i(X, Y) = \begin{cases} 0 & X = Y \\ \frac{1}{n} & Y \in B_{n-1}^i(X) \setminus B_n^i(X) \\ 1 & Y \notin B_1^i(X) \text{ and } d_0(X, Y) = 1 \end{cases}$$

We will show that $\hat{d}_i$ can be extended to a distance function d_i on $V_i(S)$.

Lemma 5.2. *For $X, Y \in V_i(S)$, $\hat{d}_i(X, Y) = 0$ if and only if $X = Y$.*

Proof. This lemma follows immediately from the definition. $\square$

Lemma 5.3. *Suppose $\hat{d}_i(X, Y)$ is defined. Then $\hat{d}_i(X, Y) = \hat{d}_i(Y, X)$.*

Proof. If $Y \in B_n^i(X)$ for some $n \geq 0$, then the Disjointness Lemma implies that $X \in B_n^i(Y)$, hence $\hat{d}_i(X, Y) = \hat{d}_i(Y, X)$. If $Y \notin B_0^i(X)$ but $d_0(X, Y) = 1$, then $d_0(Y, X) = 1$ because d_0 is symmetric, and hence $\hat{d}_i(X, Y) = \hat{d}_i(Y, X) = 1$. $\square$

Lemma 5.4. *Suppose X, Y , and Z are distinct vertices such that $\hat{d}_i(X, Y)$, $\hat{d}_i(X, Z)$, and $\hat{d}_i(Y, Z)$ are all defined. Then $\hat{d}_i(X, Z) < \max(\{\hat{d}_i(X, Y), \hat{d}_i(Y, Z)\})$*

Proof. First, suppose $\hat{d}_i(X, Y) = 1$. Since $\hat{d}_i$ never takes values greater than 1, we have

$$\hat{d}_i(X, Z) \leq 1 = \hat{d}_i(X, Y) = \max(\{\hat{d}_i(X, Y), \hat{d}_i(Y, Z)\})$$

By a similar argument, if $\hat{d}_i(Y, Z) = 1$, then

$$\hat{d}_i(X, Z) \leq 1 = \hat{d}_i(Y, Z) = \max(\{\hat{d}_i(X, Y), \hat{d}_i(Y, Z)\})$$

Now suppose that $\hat{d}_i(X, Y)$ and $\hat{d}_i(Y, Z)$ are both less than 1. Then there exist integers $m, n \geq 1$ such that $B_m^i(X) = B_m^i(Y)$ and $B_n^i(Y) = B_n^i(Z)$. Without loss of generality, we will assume $m \leq n$. Then $Z \in B_n^i(Y) \subseteq B_m^i(Y) = B_m^i(X)$, hence $\hat{d}_i(X, Z) \leq \hat{d}_i(X, Y) \leq \max(\{\hat{d}_i(X, Y), \hat{d}_i(Y, Z)\})$. $\square$

Theorem 5.5. *There is a distance function d_i on the set of pants decompositions of S such that*

1. *Whenever $\hat{d}_i(X, Y)$ is defined, $\hat{d}_i(X, Y) = d_i(X, Y)$.*
2. *If $\hat{d}_i(X, Y)$ is undefined, then $d_i(X, Y) \geq 1$.*
3. *The topology induced by d_i is the same as the topology on $V_i(S)$ defined in section 4.*

Proof. To construct d_i , consider the set $\mathbb{S}_i(X, Y)$ of all finite sequences of vertices $X = X_0, X_1, \dots, X_\ell = Y$ from X to Y such that $\hat{d}_i(X_j, X_{j+1})$ is defined for all $0 \leq j < \ell$. Let

$$d_i(X, Y) = \inf_{\mathbb{S}_i(X, Y)} (\hat{d}_i(X_0, X_1) + \hat{d}_i(X_1, X_2) + \dots + \hat{d}_i(X_{\ell-1}, X_\ell))$$

To show that $d_i(X, Y)$ is well-defined, we must show that $\mathbb{S}_i(X, Y)$ is nonempty for all $X, Y \in V_i(S)$. For $n \geq 0$, $B_n^i(X)$ has nonempty intersection with every connected-component of $\mathcal{P}(S)$. Thus, there is some $X_1 \in B_0^i(X)$ such that X_1 and Y are in the same connected component. Hence, there exists a sequence $(X = X_0, X_1, \dots, X_\ell = Y)$ such that X_j is adjacent to X_{j+1} for $1 \leq j < \ell$, and hence $\hat{d}_i(X_j, X_{j+1})$ is defined. Thus $\mathbb{S}_i(X, Y)$ is nonempty, so we can take the infimum of a nonnegative function over it.

The fact that d_i is symmetric follows from the symmetry of $\hat{d}_i$ and the construction of d_i . Similarly, d_i satisfies the triangle inequality because $\hat{d}_i$ does when it is defined.

If $\hat{d}_i(X, Y)$ is defined, then $\mathbb{S}_i(X, Y)$ contains the two vertex sequence (X, Y) . No other sequence can have a shorter length by Lemma 5.4. Thus $\hat{d}_i(X, Y) = d_i(X, Y)$.

Next suppose $\hat{d}_i(X, Y)$ is undefined, and consider a sequence $(X, X_1, \dots, X_\ell = Y) \in \mathbb{S}_i(X, Y)$. Since $\hat{d}_i(X, Y)$ is undefined, we must have $\ell \geq 2$. We claim that there exists a $0 \leq j < \ell$ such that $\hat{d}_i(X_j, X_{j+1}) = 1$, and hence $\sum_{j=0}^{\ell-1} \hat{d}_i(X_j, X_{j+1}) > 1$. Suppose not. Then $X_{j+1} \in B_1^i(X_j)$ for all j , so by the Disjointness Lemma $Y \in B_1^i(X)$, so $\hat{d}_i(X, Y)$ is defined, which is a contradiction. Thus, $d_i(X, Y) \geq 1$ whenever $\hat{d}_i(X, Y)$ is undefined. In particular, $d_i(X, Y) = 0$ if and only if $X = Y$, so d_i is actually a distance function.

Finally, we must show that the metric given by d_i is compatible with the topology on $V_i(S)$ defined in section 4. Note that for $0 < \epsilon < 1$, the open ϵ -ball around a point X is $B_{\lfloor \frac{1}{\epsilon} \rfloor}^i(X)$. Hence, the topology induced by d_i has a basis

$$\{B_n^i(X) : X \in V_i(S), n \geq 1\}$$

which is also a basis for the previously defined topology on $V_i(S)$. □

We also have the following.

Lemma 5.6. *$V_i(S)$ is a complete metric space.*

Proof. Let $X_1, \dots$ be a Cauchy sequence in $V_i(S)$. Let $n \geq 2$. Then by the definition of a Cauchy sequence, there exists an $N \in \mathbb{N}$ such that $d_i(X_j, X_k) < \frac{1}{n}$ whenever $j, k \geq N$. But by construction, this means that X_j and X_k agree on S_n . Hence, all but a finite number of terms in the sequence agree on any finite-type subsurface of S , and hence the sequence converges. □

The preceding lemma implies that, as a topological space, $V_i(S)$ is completely metrizable. We now recall the *Baire Space*, which is the countably infinite Cartesian product of a countable infinite discrete space. The Baire set

is also homeomorphic to $\mathbb{R} \setminus \mathbb{Q}$ with the usual topology. The Alexandrov-Urysohn Theorem characterizes when a space is homeomorphic to the Baire Space.

Theorem 5.7 ([10], Theorem 7.7). *A space is homeomorphic to the Baire space if and only if it is non-empty, zero-dimensional, separable, completely metrizable, and if every compact subset has empty interior.*

Corollary 5.8. *For any infinite-type surface S , $V_i(S)$ is homeomorphic to the Baire space.*

Proof. Any infinite-type surface has a pants decomposition, hence $V_i(S)$ is nonempty. In section 4, we have shown that $V_i(S)$ is separable, zero-dimensional, and that all compact subsets of $V_i(S)$ have empty interior. The previous lemma shows that $V_i(S)$ is completely metrizable. $\square$

Corollary 5.9. *For a pants decomposition $X \in V_i(S)$ and an integer $n \geq 0$, the subspace $B_n^i(X) \subset V_i(S)$ is homeomorphic to the Baire space.*

Proof. The subspace $B_n^i(X)$ is a closed subset of a completely metrizable space, and hence is completely metrizable. The space $B_n^i(X)$ has a basis

$$\{B_m^i(Y) : Y \in B_n^i(X), m \geq n\}$$

We can show that $B_n^i(X)$ is zero-dimensional and separable and such that all compact subsets have empty interior using the same arguments as for $V_i(X)$. $\square$

Remark 5.10. By Corollary 9 of [15], $\text{Mod}(S)$ with the compact-open topology is homeomorphic to a Baire space, and hence to $V_i(S)$. This phenomenon also occurs in the finite-type setting: for a finite-type hyperbolic surface S , the zero skeleton of $\mathcal{P}(S)$ and $\text{Mod}(S)$ with the compact-open topology are both countable discrete spaces.

The goal of the next two lemmas is to sharpen part (2) of Theorem 5.5.

Lemma 5.11. *Let $n \geq 3$, and let X and Y be pants decompositions such that $1 < d_i(X, Y) < 1 + \frac{1}{n}$. Then there exist pants decompositions X' and Y' such that*

- $X' \in B_n^i(X)$.
- $Y' \in B_n^i(Y)$.
- X' and Y' disagree on S_0 .

- X' and Y' differ by an elementary move.

Proof. By the definition of d_i , there exists a sequence of pants decompositions

$$(X_0, \dots, X_\ell) \in \mathbb{S}_i(X, Y)$$

such that

$$\sum_{j=1}^{\ell} (\hat{d}_i(X_{j-1}, X_j)) < 1 + \frac{1}{n}$$

Our goal is to modify this sequence to obtain a sequence $(X, X', Y', Y) \in \mathbb{S}_i(X, Y)$ such that X' and Y' have the desired properties.

Call an element of $\mathbb{S}_i(X, Y)$ *reduced* if, for all $0 \leq j_1 < j_2 \leq \ell$, $\hat{d}_i(X_{j_1}, X_{j_2})$ is defined if and only if $j_2 - j_1 = 1$.

Suppose that $(X_0, \dots, X_\ell)$ is not reduced: i.e., there exist indices $0 \leq j_1 < j_1 + 2 \leq j_2 \leq \ell$ such that $\hat{d}_i(X_{j_1}, X_{j_2})$ is defined. Then we can remove all pants decompositions in between X_{j_1} and X_{j_2} to obtain a shorter sequence,

$$(X_0, \dots, X_{j_1}, X_{j_2}, \dots, X_\ell) \in \mathbb{S}_i(X, Y).$$

By nonnegativity of $\hat{d}_i$, the sum of the distances of this sequence is less than that of the original sequence. Hence, we can repeat this process up to ℓ times until we get a reduced element of $\mathbb{S}_i(X, Y)$. Thus, we may assume $(X_0, \dots, X_\ell)$ is reduced.

If X_j and X_{j+1} disagree on S_0 , then $\hat{d}_i(X_j, X_{j+1}) = 1$. But $\sum_{j=1}^{\ell} (\hat{d}_i(X_{j-1}, X_j)) < 2$. Hence there can be at most one index j such that X_j and X_{j+1} disagree on S_0 .

Now consider an index $0 \leq j \leq \ell - 2$. Since our sequence is reduced, X_j and X_{j+2} disagree on S_0 . Since i -agreement on S_0 is an equivalence relation, the previous sentence implies that either X_j and X_{j+1} disagree on S_0 or X_{j+1} and X_{j+2} disagree on S_0 .

Combining the results of the previous two paragraphs, we deduce that $\ell \leq 3$. We tackle each possible value of ℓ .

If $\ell = 3$, then X_1 and X_2 must disagree on S_0 in order for the sequence to be reduced (otherwise three consecutive X_j would agree on S_0 .) Thus X_1 and X_2 differ by an elementary move. We now claim that $X' = X_1$ and $Y' = X_2$ satisfy the conclusion of the lemma. We have already seen that X' and Y' disagree on S_0 and differ by an elementary move. Also, by hypothesis,

$$1 + \frac{1}{n} > \hat{d}_i(X_0, X_1) + \hat{d}_i(X_1, X_2) + \hat{d}_i(X_2, X_3) = 1 + \hat{d}_i(X_0, X_1) + \hat{d}_i(X_2, X_3)$$

Hence, $\hat{d}_i(X_0, X_1) < \frac{1}{n}$ and $\hat{d}_i(X_2, X_3) < \frac{1}{n}$, so by definition $X_1 \in B_n^i(X_0) = B_n^i(X)$ and $X_2 \in B_n^i(X_3) = B_n^i(Y)$, as desired.

Now we turn our attention to the case $\ell = 2$. Either X_0 and X_1 disagree on S_0 , or X_1 and X_2 disagree on S_0 . If X_0 and X_1 disagree on S_0 , then we must have $X_1 \in B_n^i(X_2) = B_n^i(Y)$ by the same argument as the $\ell = 3$ case. Thus, we can let $X_1 = Y'$ and $X = X_0 = X'$, and we are done. Similarly, if X_1 and X_2 disagree on S_0 , then $X_1 \in B_n^i(X)$, so we can set $X_1 = X'$ and $X_2 = Y = Y'$, and we are done.

Finally, we note that since $\hat{d}_i(X, Y)$ is undefined, we must have $\ell \geq 2$, so we don't need to consider $\ell = 1$ or $\ell = 0$. $\square$

Lemma 5.12. *If $X, Y \in V_i(S)$, then $\hat{d}_i(X, Y)$ is undefined if and only if $d_i(X, Y) > 1$.*

Proof. If $\hat{d}_i(X, Y)$ is defined, then Theorem 5.5 (1) immediately tells us that $d_i(X, Y) = \hat{d}_i(X, Y) \leq 1$.

Now suppose $\hat{d}_i(X, Y)$ is undefined. We will tackle this lemma in two cases, depending on the cardinality of $X \setminus Y$.

First, consider the case where $|X \setminus Y| = 1$, i.e., there is a unique curve α in X but not in Y . Let S_α denote the complexity 1 subsurface contained in the complement of the union of $X \setminus \{\alpha\}$.

Note that $|Y \setminus X| = 1$, and let α' the unique curve in $Y \setminus X$. Observe that $\alpha' \subset S_\alpha$, because $X \setminus \alpha = Y \setminus \alpha'$ has only one component of positive complexity.

Since S_α has finite type, there exists a natural number n such that $\overline{S_\alpha} \subset S_n$. We claim that $d_i(X, Y) \geq 1 + \frac{1}{n}$. Suppose, by way of contradiction, that $d_i(X, Y) < 1 + \frac{1}{n}$. Then by the previous lemma, there exist pants decompositions X' and Y' such that

- $X' \in B_n^i(X)$.
- $Y' \in B_n^i(Y)$.
- X' and Y' differ by an elementary move.

Since X and X' agree on S_n , and $\alpha \subset S_\alpha \subset S_n$, we must have $\alpha \in X'$. By the same argument, $\alpha' \in Y'$. Thus, the elementary move that takes X' to Y' must replace α by α' . Hence, α and α' intersect minimally. But this statement is impossible, because it implies that X and Y differ by an elementary move, and hence $\hat{d}_i(X, Y)$ is defined. Thus, we can conclude $d_i(X, Y) \geq 1 + \frac{1}{n} > 1$.

Now, consider the case where $|X \setminus Y| > 1$, and choose two distinct curves $\alpha, \beta \in X \setminus Y$. As in the previous case, we let S_α denote the complexity 1 subsurface contained in the complement of $X \setminus \alpha$. Similarly, we let S_β denote the complexity 1 subsurface contained in the complement of $X \setminus \beta$. Note that since α and β are disjoint and nonisotopic, $S_\alpha \neq S_\beta$, although the subsurfaces are not necessarily disjoint.

We may pass to the geodesic representatives of X and Y , and thus we can assume that S_α and S_β have geodesic boundaries. Hence $S_\alpha \cup S_\beta$ is of finite type, so there exists a natural number n such that $S_\alpha \cup S_\beta \subset S_n$. As in the previous case, suppose that $d_i(X, Y) < 1 + \frac{1}{n}$. Then by the previous lemma, there exist pants decompositions X' and Y' such that

- $X' \in B_n^i(X)$.
- $Y' \in B_n^i(Y)$.
- X' and Y' differ by an elementary move.

Since α and β are contained in $S_\alpha \cup S_\beta \subset S_n$, and since X and X' agree on S_n , we must have $\alpha, \beta \in X'$. Similarly, Y and Y' agree on S_n , so $\alpha, \beta \notin Y'$. Thus $X' \setminus Y'$ contains at least two curves. But this is impossible since X' and Y' differ by an elementary move. Thus we can conclude that $d_i(X, Y) \geq 1 + \frac{1}{n} > 1$.

□

Recall that a metric space (M, d) is said to be an *ultrametric space* if for all $x, y, z \in M$, $d(x, z) \leq \max(\{d(x, y), d(y, z)\})$. With the usual Euclidean metric, $\mathbb{R}^n$ is not an ultrametric space, because any three distinct colinear points violate the ultrametric inequality.

Corollary 5.13. *If $U \subset V_i(X)$ is a subspace of diameter ≤ 1 , then U with the restriction of d_i is an ultrametric space.*

Proof. By the previous lemma, $\hat{d}_i$ is defined on all of $U \times U$, and hence $d_i(x, y) = \hat{d}_i(x, y)$ for all $x, y \in U$. The result now follows from Lemma 5.4. □

Corollary 5.14. *The metric d_i is not an ultrametric on $V_i(S)$.*

Proof. Let $X, Y \in V_i(S)$ such that X and Y disagree on S_0 and such that X and Y do not differ by an elementary move. Then $\hat{d}_i(X, Y)$ is undefined, and hence $d(X, Y) > 1$ by the previous lemma. But by Theorem 5.5, there

exists a sequence of points $X = X_0, X_1, \dots, X_\ell = Y$ such that $d_i(X_j, X_{j+1}) = \hat{d}_i(X_j, X_{j+1}) \leq 1$ for all $0 \leq j < \ell$. If d_i were an ultrametric, then we would have $d_i(X, Y) \leq 1$, a contradiction. $\square$

6 The spaces $\mathcal{PS}_i(S)$

The family of spaces $V_i(S)$ defined in the previous section are totally disconnected. This fact is not very surprising, because $V_i(S)$ contains only the vertex set of $\mathcal{P}(S)$, without the edges. In this section, using the definitions of $V_i(S)$ as a template, we will define a topology on the entire pants graph (including the edges), and call this space $\mathcal{PS}_i(S)$. The topology on $\mathcal{PS}_i(S)$ will be weaker than the usual topology on $\mathcal{P}(S)$, but it will still encode all of the structure of the pants graph. In this section, we will construct a basis of open sets for $\mathcal{PS}_i(S)$, and we will prove it is second-countable.

$\mathcal{PS}_i(S)$ will be equal as a set to $\mathcal{P}(S)$. We think of $\mathcal{P}(S)$ as the set of vertices with copies of the unit interval glued to each pair of adjacent vertices. We can therefore represent a point in $\mathcal{PS}_i(S)$ by a three-tuple (X, a, Y) , where X and Y are adjacent vertices in $\mathcal{P}(S)$ and $a \in [0, 1]$ is the distance between the point and X . Each edge point has two such representations, since $(X, a, Y) = (Y, 1 - a, X)$. Also, each vertex is represented by infinitely many three-tuples:

$$X = (X, 0, Y) = (Y, 1, X)$$

for any vertex X and any vertex Y adjacent to X .

Definition 6.1 states that a sequence converges in $\mathcal{PS}_i(S)$ if it satisfies any of three conditions. Intuitively, the first two conditions are rough analogs of the fact that a sequence converges in a product space converges if and only if the projections of the sequence to each coordinate converge.

If we think of edges as analogous to real intervals bounded by their incident vertices, then the third condition is an analog of the Squeeze Theorem.

We are now ready to define i -converges for $\mathcal{PS}_i(S)$.

Definition 6.1. Let $\{P_k\}_{k \geq 1}$ be a sequence of points (either vertex or edge points) in $\mathcal{P}(S)$. For $0 \leq i \leq 4$, the sequence is said to i -converge to a point P if there exist representatives $P_k = (X_k, a_k, Y_k)$ for $k \in \mathbb{N}$ and $P = (X, a, Y)$ such that *at least one* of the following conditions hold:

1. $\{X_k\} \rightarrow X$ and $\{Y_k\} \rightarrow Y$ in $V_i(S)$, and $\{a_k\} \rightarrow a$ in $[0, 1]$.

2. P is a vertex, $\{X_k\} \rightarrow P$ in $V_i(S)$, and $\{a_k\} \rightarrow 0$ in $[0, 1]$.
3. P is a vertex, and $\{X_k\}$ and $\{Y_k\}$ both converge to P in $V_i(S)$

Definition 6.2. For $0 \leq i \leq 4$, let $\mathcal{PS}_i(S)$ be the set $\mathcal{P}(S)$ equipped with the finest topology such that any i -convergent sequence converges in $\mathcal{PS}_i(S)$.

Let $\Gamma \subset \mathcal{P}(S)$ be a connected component. Recall that Γ can be made into a metric space by letting each edge have length 1.

Lemma 6.3. For any surface S , $\mathcal{PS}_0(S) \cong \mathcal{P}(S)$ with topology induced by the graph distance metric.

Proof. We will show that a sequence converges in $\mathcal{PS}_0(S)$ if and only if it converges in $\mathcal{P}(S)$. By construction, any convergent sequence in $\mathcal{P}(S)$ also converges in $\mathcal{PS}_0(S)$. Now choose a 0-convergent sequence $\{(X_k, a_k, Y_k) \rightarrow (X, a, Y)$ in $\mathcal{PS}_0(S)$. Recall that $V_0(S)$ carries the discrete topology. Thus, condition (1) in definition 6.1 can occur only if $\{X_k\}$ and $\{Y_k\}$ are eventually constant. In this case it follows that $\{(X_k, a_k, Y_k)\}$ eventually agrees with $\{X, a_k, Y\}$, which obviously converges to $\{X, a, Y\}$ in $\mathcal{P}(S)$.

Now suppose that $\{(X_k, a_k, Y_k) \rightarrow P$ in $\mathcal{PS}_0(S)$ by condition 2 in definition 6.1. Then $\{X_k\}$ is eventually constant and converges to P . Also, $\{a_k\} \rightarrow 0$, so $\{P, a_k, Y_k\}$ converges to P in $\mathcal{P}(S)$.

Condition 3 in definition 6.1 cannot occur in the case $i = 0$. This is because $V_0(S)$ is discrete, so condition 3 says that $\{X_k\}$ and $\{Y_k\}$ are eventually equal. But $C_P(S)$ is a simple graph, so $X_k = Y_k$ cannot be adjacent to itself, which is a contradiction.

Thus, in all three cases, a convergent sequence in $\mathcal{PS}_0(S)$ also converges to the same limit in $\mathcal{P}(S)$, so $\mathcal{PS}_0(S) \cong \mathcal{P}(S)$. □

Lemma 6.4. For $0 \leq i \leq j \leq 4$, the topology on $\mathcal{PS}_i(S)$ is finer than the topology on $\mathcal{PS}_j(S)$.

Proof. This lemma follows immediately from the fact that $V_i(S)$ has a finer topology than $V_j(S)$ for $0 \leq i \leq j \leq 4$. □

Lemma 6.5. For $0 \leq i \leq 4$, the restriction of the topology on $\mathcal{PS}_i(S)$ to the vertex set agrees with that of $V_i(S)$.

Proof. Immediate from the definition. □

Our next goal is to construct a basis for $\mathcal{PS}_i(S)$ that is relatively easy to work with. Intuitively, we can think of points in $\mathcal{PS}_i(S)$ as close if they are either close in the underlying graph, or if they agree on a large finite-type subsurface. Thus far, we lack a definition of two points "agreeing on S_n " if one or both of the points is an edge point. To this end, we will define sets $A_n^i(P)$ for each $P \in \mathcal{PS}_i(S)$. The $A_n^i(P)$ will not be open, but we will use them to construct open neighborhoods of each point in $\mathcal{PS}_i(S)$. The definition of $A_n^i(P)$ will be different depending on whether P is a vertex point or an edge point.

Definition 6.6. If $X \in \mathcal{PS}_i(S)$ is a vertex, let $A_n^i(X)$ be the induced subgraph on $B_n^i(X)$.

If $P = (X, a, Y)$ is an edge point, let

$$A_n^i(P) = \{(X', a, Y') : X' \in B_n^i(X) \text{ and } Y' \in B_n^i(Y)\}$$

If P is a vertex, then $A_n^i(X)$ can be thought of as the "cloud" of vertices and edges near X . If P is an edge point, then $A_n^i(X)$ can be thought of as a "horizontal slice" of the edges near P .

Note that $A_n^i(P)$ need not be open in $\mathcal{PS}_i(S)$. Also note that for the case $i = 0$, $A_n^0(P)$ is just a single point P .

Lemma 6.7 (Vertex Disjointness Lemma). *Let X and Y be two vertices in $\mathcal{PS}_i(S)$. Then $A_n^i(X)$ and $A_n^i(Y)$ are either disjoint or equal.*

Proof. Follows immediately from the disjointness lemma for $V_i(S)$. □

Lemma 6.8 (Edge Disjointness Lemma). *Let P and Q be two edge points in $\mathcal{PS}_i(S)$. Then $A_n^i(P)$ and $A_n^i(Q)$ are either disjoint or equal.*

Proof. Follows immediately from the disjointness lemma for $V_i(S)$. □

Note that if X is a vertex and P is a point on an edge incident to X , then disjointness between $A_n^i(X)$ and $A_n^i(P)$ can fail. If $P = (X, a, Y)$ is an edge point, and if X i -agrees on S_n with Y , then $A_n^i(P)$ is a proper subset of $A_n^i(X) = A_n^i(Y)$.

We need one more family of auxillary sets, $D_{\epsilon,n}^i(P)$. Starting from P , we move a distance of up to ϵ in the graph metric on $\mathcal{P}(S)$ until we reach another point P' . From there, we jump to another point which agrees with P' on S_n : i.e., a point in $A_n^i(P')$. The set of all possible ending points of these two steps is $D_{\epsilon,n}^i(P)$.

If P_1 and P_2 are points in a connected-component of $\mathcal{P}(S)$, we denote by $d_0(P_1, P_2)$ their distance in the graph metric. If P_1 and P_2 are in different components, then we let $d_0(P_1, P_2) = \infty$. We can now restate the previous two paragraphs as a formal definition.

Definition 6.9. Let $P \in \mathcal{PS}_i(S)$, $n \in \mathbb{N}$, and $\epsilon > 0$. Then let

$$D_{\epsilon,n}^i(P) = \bigcup_{P': d_0(P, P') < \epsilon} A_n^i(P')$$

Now we are ready to define the basis of open sets for $\mathcal{PS}_i(S)$.

Definition 6.10. Let $X \in \mathcal{PS}_i(S)$ be a vertex. Let $0 \leq i \leq 4$ and $n \geq 0$ be integers and let $\epsilon > 0$ be a real number.

Let

$$\mathbb{D}_{\epsilon,n}^i(X) = \bigcup_{X': X' \text{ and } X \text{ } i\text{-agree on } S_n} D_{\epsilon,n}^i(X')$$

.

Definition 6.11. If $P = (X, a, Y)$ and $Q = (X', a', Y')$ are edge points, we say that P and Q *i-agree on S_n* if X and X' *i-agree on S_n* , Y and Y' *i-agree on S_n* , and $a = a'$. As in the case of vertices, we will sometimes say that P and Q *agree on S_n* if i is understood.

Definition 6.12. Let P be an edge point. Let $0 \leq i \leq 4$ and $n \geq 0$ be integers and let $\epsilon > 0$ be a real number. Let

$$\mathbb{D}_{\epsilon,n}^i(P) = \bigcup_{P \text{ and } P' \text{ agree on } S_n} D_{\epsilon,n}^i(P')$$

.

It will sometimes be convenient to have a basis indexed only by a single positive real number. Thus, we define

$$\mathbb{D}_{\epsilon}^i(P) = \mathbb{D}_{\epsilon, \lfloor \frac{1}{\epsilon} \rfloor}^i(P)$$

Note that for $i = 0$, $\mathbb{D}_{\epsilon,n}^0(P)$ is the open ball of radius ϵ around P in the graph-distance metric on $\mathcal{P}(S)$.

The goal for the rest of this section is to prove the following theorem.

Theorem 6.13 (Basis Theorem). *For $1 \leq i \leq 4$, the set*

$$\{\mathbb{D}_{\epsilon}^i(P) : \epsilon > 0, P \in \mathcal{PS}_i(S)\}$$

is a basis for $\mathcal{PS}_i(S)$.

We will prove this theorem by breaking it up into several lemmas.

Lemma 6.14. *Let $P = (X, a, Y)$ be an edge point, $n \in \mathbb{N}$, and $0 < \epsilon < \max(a, 1 - a)$. Then $\mathbb{D}_{\epsilon, n}^i(P)$ is an open set.*

Proof. Let $Q \in \mathbb{D}_{\epsilon, n}^i(P)$. By definition, there are points $P', P'' \in \mathcal{PS}_i(S)$ such that P' and P agree on S_n , $d_0(P', P'') < \epsilon$, and $Q \in A_n^i(P'')$. Hence, P' is an edge point with a representation (X', a, Y') . Since $d_0(P', P'') < \epsilon < \max(a, 1 - a)$, P'' is also an edge point on the same edge as P' : it has a representation $P'' = (X', a', Y')$ with $|a - a'| < \epsilon$. Hence, Q is also an edge point with a representation (X'', a', Y'') where $X'' \in B_n^i(X') = B_n^i(X)$ and $Y'' \in B_n^i(Y') = B_n^i(Y)$.

Now suppose $\{Q_j\}_{j \geq 1} \rightarrow Q$. Then each Q_j has a representation (X_j, a_j, Y_j) with $X_j \rightarrow X''$, $Y_j \rightarrow Y''$, and $a_j \rightarrow a' \in (a - \epsilon, a + \epsilon)$. Hence, for all sufficiently large j , we have $X_j \in B_n^i(X'') = B_n^i(X)$, $Y_j \in B_n^i(Y'')$, and $a_j \in (a - \epsilon, a + \epsilon)$. Hence $Q_j \in \mathbb{D}_{\epsilon, n}^i(P)$ for all sufficiently large j . $\square$

Lemma 6.15. *Let $X \in V_i(S)$, $n \in \mathbb{N}$, $0 < \epsilon < 1$, and let $Q \in \mathbb{D}_{\epsilon, n}^i(X)$ be an edge point. Let $\{Q_j\}_{j \geq 1} \rightarrow Q$. Then all but finitely many of the Q_j are in $\mathbb{D}_{\epsilon, n}^i(X)$*

Proof. Since $Q \in \mathbb{D}_{\epsilon, n}^i(X)$, there exists a vertex $X' \in B_n^i(X)$ and a point $P \in \mathcal{PS}_i(S)$ such that $d_0(X', P) < \epsilon$ and $Q \in A_n^i(P)$. Since $d_0(X', P) < \epsilon < 1$, P is either an edge point on an edge incident to X' , or else $P = X'$. We tackle these two cases separately.

First, suppose $P = (X', a, Y)$ with $0 < a < \epsilon$. Then Q is also an edge point with a representation (X'', a, Y') where $X'' \in B_n^i(X') = B_n^i(X)$ and $Y' \in B_n^i(Y)$. Hence, for all sufficiently large j , $Q_j = (X_j, a_j, Y_j)$ with $X_j \in B_n^i(X'') = B_n^i(X)$, $Y_j \in B_n^i(Y') = B_n^i(Y)$, and $0 < a_j < \epsilon$. Thus

$$Q_j \in A_n^i((X', a_j, Y)) \subseteq \mathbb{D}_{\epsilon, n}^i(X).$$

Next suppose $P = X'$. Then $Q = (X'', a, Y)$ where $X'', Y \in B_n^i(X') = B_n^i(X)$. Thus, for all sufficiently large j , Q_j has a representation (X_j, a_j, Y_j) with $X_j, Y_j \in B_n^i(X') = B_n^i(X)$, hence $Q_j \in \mathbb{D}_{\epsilon, n}^i(X)$. $\square$

Lemma 6.16. *Let $X \in V_i(S)$, $n \in \mathbb{N}$, $0 < \epsilon < 1$, and $Q \in \mathbb{D}_{\epsilon, n}^i(X)$ is a vertex. Let $\{Q_j\}_{j \geq 1} \rightarrow Q$. Then all but finitely many of the Q_j are in $\mathbb{D}_{\epsilon, n}^i(X)$*

Proof. By definition, there exists points $P, P' \in \mathcal{PS}_i(S)$ such that $P \in B_n^i(X)$, $d_0(P, P') < \epsilon$ and $Q \in A_n^i(P')$. Since Q is a vertex, P' must also be a vertex and $Q \in B_n^i(P')$. Since $d_0(P, P') < 1$, and both P and P' are vertices, $P = P'$, hence the vertices X, P , and Q all agree on S_n .

We must consider two subcases, depending on which of the conditions from Definition 6.1 the Q_j satisfy.

In the first subcase, corresponding to Definition 6.1 (2), $Q_j = (X_j, a_j, Y_j)$ where $X_j \rightarrow Q$ and $a_j \rightarrow 0$. For all sufficiently large j , $X_j \in B_n^i(Q) = B_n^i(X)$ and $0 \leq a_j < \epsilon$. Thus $d_0(X_j, Q_j) = a_j < \epsilon$ and $X_j \in B_n^i(X)$, so $Q_j \in \mathbb{D}_{\epsilon,n}^i(X)$.

In the second subcase, corresponding to Definition 6.1 (3), $Q_j = (X_j, a_j, Y_j)$ with $X_j \rightarrow Q$ and $Y_j \rightarrow Q$. Thus, for all sufficiently large j , $X_j, Y_j \in B_n^i(Q) = B_n^i(X)$, so $Q_j \in A_n^i(X) \subseteq \mathbb{D}_{\epsilon,n}^i(X)$. $\square$

Corollary 6.17. *For all $P \in \mathcal{PS}_i(S)$, $\epsilon > 0$, and $n \in \mathbb{N}$, the set $\mathbb{D}_{\epsilon,n}^i(P)$ is open.*

Proof. Lemmas 6.14, 6.15, and 6.16 show that $\mathbb{D}_{\epsilon,n}^i(P)$ is open for all sufficiently small values of ϵ . If $Q \in \mathbb{D}_{\epsilon,n}^i(P)$, then there is a $0 < \psi < \epsilon$ such that $\mathbb{D}_{\psi,n}^i(Q) \subset \mathbb{D}_{\epsilon,n}^i(P)$. Hence $\mathbb{D}_{\epsilon,n}^i(P)$ is a union of open sets. $\square$

Lemma 6.18. *Let $U \subset \mathcal{PS}_i(S)$ be an open set, and let $P \in U$. Then there exists an $\epsilon > 0$ such that $D_\epsilon^i(P) \subset U$.*

Proof. First, assume P is an edge point with a representation $P = (X, a, Y)$ (we will tackle the case where P is a vertex separately). For sufficiently small $\epsilon > 0$, note that $D_\epsilon^0(P)$ is contained entirely in the edge between X and Y , and in particular it contains no vertices. It follows from the definitions that $D_\epsilon^i(P)$ also contains no vertices.

Consider the descending sequence of open sets $D_{\frac{1}{n}}^i(P)$ where $n \in \mathbb{N}$. Suppose that for each n , there exists a point $P_n \in D_{\frac{1}{n}}^i(P) \setminus U$. For all but finitely many n , P_n is an edge point. Hence, we can find representatives $P_n = (X_n, a_n, Y_n)$ where $X_n \in B_n^i(X)$, $Y_n \in B_n^i(Y)$, and $|a_n - a| < \frac{1}{n}$. By construction, the sequence of points $\{P_n\}$ converges to P , and hence all but finitely many must be contained in U , which is a contradiction. Thus, there is some n such that $D_{\frac{1}{n}}^i(P) \subset U$.

For P a vertex, we can use a similar argument. Consider the descending sequence of open sets $D_{\frac{1}{n}}^i(P)$ where $n \in \mathbb{N}$. Suppose that for each n , there exists a point $P_n \in D_{\frac{1}{n}}^i(P) \setminus U$. Each P_n is either a vertex with $P_n \in B_n^i(P)$, or an edge point with a representation $P_n = (X_n, a_n, Y_n)$ such that $X_n \in B_n^i(P)$ and $0 < a < \epsilon$. But by definition, this is a sequence which converges to P , so all but finitely many of the P_j are in U , a contradiction. $\square$

Lemma 6.19. *Let $U \subseteq \mathcal{PS}_i(S)$ be an open set and let $X \in U$ be a vertex. Then there exists an $n \geq 0$ such that*

$$\mathbb{D}_{\frac{1}{n}}^i(X) \subseteq U$$

.

Proof. By Lemma 6.18, there exists an $m \geq 2$ such that $D_{\frac{1}{m}}^i(X) \subseteq U$. Thus, it suffices to find an $n \geq 2$ such that

$$\mathbb{D}_{\frac{1}{n}}^i(X) \subseteq D_{\frac{1}{m}}^i(X)$$

Suppose no such n exists. Then for all $n \geq 2$, there exists a point P_n such that

$$P_n \in \left(\mathbb{D}_{\frac{1}{n}}^i(X) \right) \setminus \left(D_{\frac{1}{m}}^i(X) \right)$$

By definition, that inclusion means that there exists a vertex X_n such that X_n and X agree on S_n and $P_n \in D_{\frac{1}{n}}^i(X_n)$ but $P_n \notin D_{\frac{1}{m}}^i(X)$.

Suppose that P_n is a vertex. Then P_n must agree with X_n on S_n but disagree with X on S_m . When $n \geq m$, this is a contradiction, since X and X_n agree on S_n .

Hence, $P_n = (X'_n, a_n, Y'_n)$ is an edge point for all $n \geq m$. But this implies that $a_n < \frac{1}{n}$ and that X'_n agrees with X on S_n . Hence, the sequence $\{P_n\}_{n \geq m}$ converges to X , so all but finitely many of the P_n are in $D_{\frac{1}{m}}^i(X)$, which is a contradiction. $\square$

Lemma 6.20. *Let P and Q be points in $\mathcal{PS}_i(S)$ that are either both vertices or both edge points. Let $0 \leq i \leq 4$ and $n \geq 0$ be integers and let $\epsilon > 0$ be a real number. If P and Q i -agree on S_n , then*

$$\mathbb{D}_{\epsilon,n}^i(P) = \mathbb{D}_{\epsilon,n}^i(Q)$$

.

Proof. Immediate from the definitions. $\square$

Definition 6.21. An edge point $P \in \mathcal{PS}_i(S)$ is called *rational* if it has a representation $P = (X, a, Y)$ where a is rational.

Lemma 6.22. *Let $U \subset \mathcal{PS}_i(S)$ be an open set containing no vertices. Then U is the union of sets of the form $\mathbb{D}_{\epsilon,n}^i((X, a, Y))$ where ϵ is a rational number and P is a rational edge point.*

Proof. Let $P = (X, a, Y) \in U$. We will try to find values of n such that $\mathbb{D}_{\frac{1}{n}}^i(P) \subset U$.

Suppose that no such n exists. Then for all $n \geq 0$, there exists a point

$$P_n \in \left(\mathbb{D}_{\frac{1}{n}}^i(P) \setminus U \right)$$

. For all sufficiently large n , X and Y disagree on S_n . Also, for all sufficiently large n , $0 < a - \frac{1}{n} < a + \frac{1}{n} < 1$.

Tracing through the definition of $\mathbb{D}_{\frac{1}{n}}^i(P)$, the statement $P_n \in \mathbb{D}_{\frac{1}{n}}^i(P)$ says that there are points $Q_n = (X'_n, a, Y'_n)$ and $R_n = (X'_n, a'_n, Y'_n)$ such that

1.

$$|a - a'_n| < \frac{1}{n}$$

.

2. P_n has a representation (X''_n, a'_n, Y''_n) .

3. X, X'_n , and X''_n all agree on S_n .

4. Y, Y'_n , and Y''_n all agree on S_n .

In particular, the sequence of edge points $\{P_n\}$ have representations $\{(X''_n, a'_n, Y''_n)\}$ such that $X''_n \rightarrow X, Y''_n \rightarrow Y$, and $a'_n \rightarrow a$. Thus $P_n \rightarrow P$, and hence all but finitely many of the P_n are in U , a contradiction. Thus, there is some n such that $\mathbb{D}_{\frac{1}{n}}^i(P) \subset U$.

If a is rational, then we are done. If a is irrational, then we can find a rational edge point P' very close to P such that $\mathbb{D}_{\frac{1}{n}}^i(P') \subset U$. □

We can now easily prove the Basis Theorem (Theorem 6.13).

Proof. Corollary 6.17 tells us that the $\mathbb{D}_{\epsilon, n}^i$ sets are all open, while Lemmas 6.19 and 6.22 says they generate all open sets. □

Corollary 6.23. $\mathcal{PS}_i(S)$ is a second-countable space.

Proof. Our goal is to find a countable subset of

$$\{\mathbb{D}_{\epsilon}^i(P) : \epsilon > 0, P \in \mathcal{PS}_i(S)\}$$

which is still a basis. For $m \in \mathbb{N}$, let V_m be a set containing one vertex in each equivalence class for the equivalence relation " i -agrees on S_m ." By Lemma 4.15, V_m is countable. Likewise, let E_m be the union of one edge in each equivalence class. E_m contains a countable number of edges (but an uncountable number of points.) Let

$$\tau = \bigcup_{m=0}^{\infty} (V_m \cup E_m) \subset \mathcal{PS}_i(S).$$

By Lemma 6.20,

$$\{\mathbb{D}_{\epsilon}^i(P) : \epsilon > 0, P \in \mathcal{PS}_i(S)\} = \{\mathbb{D}_{\epsilon}^i(P) : \epsilon > 0, P \in \tau\}.$$

By Lemma 6.22, it suffices to consider basic open sets with rational ϵ values based around vertices and rational edge points. Hence, if we let τ' be the subset of τ containing only vertices and rational edge points, then

$$\{\mathbb{D}_\epsilon^i(P) : \epsilon \in \mathbb{Q}^+, P \in \tau'\}$$

is also a basis. τ' is countable, hence the above basis is also countable. $\square$

7 Properties of $\mathcal{PS}_i(S)$

In this section we study the basic properties of $\mathcal{PS}_i(S)$, as well as its homeomorphism group. In particular, we show that it is metrizable, path-connected, not locally path-connected, and that every automorphism of $\mathcal{PS}_i(S)$ is isotopic to a unique graph automorphism of $\mathcal{P}(S)$. We then show that the mapping class group of $\mathcal{PS}_i(S)$ is naturally isomorphic to $\text{Mod}^\pm(S)$, generalizing Margalit's result for finite-type pants graphs and confirming this case of the metaconjecture.

Lemma 7.1. *The space $\mathcal{PS}_i(S)$ is first-countable.*

Proof. By the Basis Theorem, each point $P \in \mathcal{PS}_i(S)$ has a countable neighborhood basis $\{\mathbb{D}_{\frac{1}{n}}^i(P) : n \in \mathbb{N}\}$. Thus $\mathcal{PS}_i(S)$ is first-countable. $\square$

Lemma 7.2. *The vertex set $V_i(S)$ is closed in $\mathcal{PS}_i(S)$.*

Proof. For an edge (X, Y) in the pants graph, consider the $\frac{1}{2}$ -ball around the midpoint:

$$\begin{aligned} & \mathbb{D}_{\frac{1}{2}}^i((X, \frac{1}{2}, Y)) \\ &= \bigcup_{(X', \frac{1}{2}, Y') \text{ agrees with } (X, \frac{1}{2}, Y) \text{ on } S_2} \bigcup_{P': d_0(P', (X, \frac{1}{2}, Y)) < \frac{1}{2}} A_2^i(P') \\ &= \bigcup_{(X', \frac{1}{2}, Y') \text{ agrees with } (X, \frac{1}{2}, Y) \text{ on } S_2} \bigcup_{(X', a', Y'): 0 < a' < 1} A_2^i((X', a', Y')) \end{aligned}$$

We note that this open set is a disjoint union of edges, and that it contains no vertices. Moreover, every edge of $\mathcal{PS}_i(S)$ is contained in a similar neighborhood, so the union of all edges is open in $\mathcal{PS}_i(S)$, and hence $V_i(S)$ is closed. $\square$

Lemma 7.3. *$\mathcal{PS}_i(S)$ is Hausdorff.*

Proof. First suppose $P_1 = (X_1, a_1, Y_1)$ and $P_2 = (X_2, a_2, Y_2)$ are distinct edge points, and that $a_1 \neq a_2$. Choose an $n \in \mathbb{N}$ such that $B_n^i(X_1) \cap B_n^i(Y_1) = B_n^i(X_2) \cap B_n^i(Y_2) = \emptyset$, and choose an $\epsilon > 0$ such that $|a_1 - a_2| < \frac{\epsilon}{2}$. Then $\mathbb{D}_{\epsilon,n}^i(P_1)$ and $\mathbb{D}_{\epsilon,n}^i(P_2)$ are disjoint open neighborhoods.

On the other hand, if $a_1 = a_2$, then P_1 and P_2 necessarily lie on distinct edges, since $P_1 \neq P_2$. Without loss of generality, we may assume that $X_1 \neq X_2$. Choose an $n \geq 1$ such that $B_n^i(X_1) \cap B_n^i(Y_1) = B_n^i(X_2) \cap B_n^i(Y_2) = B_n^i(X_1) \cap B_n^i(X_2) = \emptyset$. Also choose an $\epsilon > 0$ such that $\epsilon < \min(\{a, 1 - a\})$. Then $\mathbb{D}_{\epsilon,n}^i(P_1)$ and $\mathbb{D}_{\epsilon,n}^i(P_2)$ are disjoint open neighborhoods.

A similar argument works when one or both points are vertices. $\square$

The next two lemmas tell us the closure of the basis sets. Intuitively, they say that we can get the closure of a basis set by replacing " $d_0(P', P'') < \epsilon$ " in the definition of $D_{\epsilon,n}^i$ with $d_0(P', P'') \leq \epsilon$. They will be used to show that $\mathcal{PS}_i(S)$ is regular.

Lemma 7.4. *Let $X \in \mathcal{PS}_i(S)$ be a vertex, and let $0 < \epsilon \leq \frac{1}{2}$. Then the closure of $\mathbb{D}_{\epsilon,n}^i(X)$ is*

$$C_{\epsilon,n}^i = \bigcup_{X' \in B_n^i(X)} \left(\bigcup_{P: d_0(P, X') \leq \epsilon} A_n^i(P) \right)$$

Proof. It is clear from the definition of $C_{\epsilon,n}^i$ that each point in $C_{\epsilon,n}^i$ is a limit point of a sequence in $\mathbb{D}_{\epsilon,n}^i(X)$, and hence $C_{\epsilon,n}^i \subseteq \overline{\mathbb{D}_{\epsilon,n}^i(X)}$. Our goal now is to show that $C_{\epsilon,n}^i$ is closed.

Let $P_1, \dots$ be a sequence in $\mathbb{D}_{\epsilon,n}^i$ which converges to some point $Q \in \mathcal{PS}_i(S)$. We will show that $Q \in C_{\epsilon,n}^i$.

First suppose that Q is an edge point. Then Q has a representation $Q = (Y, a, Z)$, $0 < a < 1$, and each P_i has a representation $P_j = (Y_j, a_j, Z_j)$ such that $\{Y_j\} \rightarrow Y$, $\{Z_j\} \rightarrow Z$, and $\{a_j\} \rightarrow a$. Note that if Y and Z are both in $B_n^i(X)$, then $Q \in D_{\epsilon,n}^i(X) \subset \mathbb{D}_{\epsilon,n}^i(X)$ by definition and we are done. On the other hand, At least one of Y_j or Z_j must be in $B_n^i(X)$, since $P_j \in \mathbb{D}_{\epsilon,n}^i(X)$. Thus, we may assume that exactly one of X_j or Y_j is in $B_n^i(X)$.

Due this assumption, the definition of $\mathbb{D}_{\epsilon,n}^i(X)$ implies that either $a_j < \epsilon \leq \frac{1}{2}$ or $a_j > 1 - \epsilon \geq \frac{1}{2}$. Since the a_j converge as a sequence of real numbers, then one of those two cases must occur only a finite number of times. Assume, without loss of generality, that $a_j < \epsilon \leq \frac{1}{2}$ for all but finitely many $j \in \mathbb{N}$. But that means $Y_j \in B_n^i(X)$ for all but finitely many $j \in \mathbb{N}$, and hence $Y \in B_n^i(X)$. We also have $a \leq \epsilon$. Thus $Q \in C_{\epsilon,n}^i$, as desired.

We now turn our attention to the separate case where Q is a vertex. As usual, there are two separate subcases, depending on whether the P_j satisfy conditions (2) or (3) of definition 6.1. In the first subcase, we have representations $P_j = (Y_j, a_j, Z_j)$ such that $Y_j \rightarrow Q$, $a_j \rightarrow 0$, and there are no restrictions on Z_j . Since $(Y_j, a_j, Z_j) \in \mathbb{D}_{\epsilon,n}^i(X)$,

either $Y_j \in B_n^i(X)$ and $a_j < \epsilon$, or $Z_j \in B_n^i(X)$ and $a_j > 1 - \epsilon$. But the a_j approach 0, so we must have $Y_j \in B_n^i(X)$ for all but finitely many $j \geq 1$. Hence

$$Q \in B_n^i(X) \subset \mathbb{D}_{\epsilon,n}^i(X) \subset C_{\epsilon,n}^i.$$

In the second subcase, we have representations $P_j = (Y_j, a_j, Z_j)$ such that $Y_j \rightarrow Q$ and $Z_j \rightarrow Q$. Hence, for large j , the vertices Y_j, Z_j , and Q all agree on S_n . Since $P_j \in \mathbb{D}_{\epsilon,n}^i(X)$ for all j , at least one of Y_j or Z_j agrees with X on S_n . Hence, by transitivity, for large j the vertices X, Y_j, Z_j , and Q all agree on S_n , thus

$$Q \in B_n^i(X) \subset \mathbb{D}_{\epsilon,n}^i(X) \subset C_{\epsilon,n}^i.$$

□

Lemma 7.5. *Let $P = (X, a, Y) \in \mathcal{PS}_i(S)$ be an edge point, let $0 < \epsilon < \max a, 1 - a$, and let $n \geq 1$ be large enough so that X and Y disagree on S_n . Then the closure of $\mathbb{D}_{\epsilon,n}^i(P)$ is*

$$\begin{aligned} C_{\epsilon,n}^i &= \bigcup_{P'=(X',a,Y'):X' \in B_n^i(X), Y' \in B_n^i(Y)} \left(\bigcup_{P'':d_0(P',P'') \leq \epsilon} A_n^i(P'') \right) \\ &= \{(X', a', Y') : X' \in B_n^i(X), Y' \in B_n^i(Y), a - \epsilon \leq a' \leq a + \epsilon\}. \end{aligned}$$

Proof. The equality in the lemma follows immediately from the definitions. It is clear from the definition of $C_{\epsilon,n}^i$ that each point in $C_{\epsilon,n}^i$ is a limit point of a sequence in $\mathbb{D}_{\epsilon,n}^i(P)$, and hence $C_{\epsilon,n}^i \subseteq \overline{\mathbb{D}_{\epsilon,n}^i(P)}$. Our goal now is to show that $C_{\epsilon,n}^i$ is closed.

Suppose $P_1, \dots$ is a sequence of points in $\mathbb{D}_{\epsilon,n}^i(P)$ which converge to some point Q . Note that $\mathbb{D}_{\epsilon,n}^i(P)$ contains only edge points. Since the P_j converge, we can choose representations $P_j = (W_j, a_j, Z_j)$ such that the W_j converge as a sequence of points in $V_i(S)$ to some vertex W . Each W_j lies in either $B_n^i(X)$ or $B_n^i(Y)$. Since we chose n large enough such that $B_n^i(X) \cap B_n^i(Y) = \emptyset$, the only way for the W_j to converge is if either $W_j \in B_n^i(X)$ for all but finitely many j , or $W_j \in B_n^i(Y)$ for all but finitely many j . Without loss of generality, we assume that $W_j \in B_n^i(X)$ for all j . Hence, $W \in B_n^i(X)$, and $Z_j \in B_n^i(Y)$ for all j .

Since Z_j and W disagree on S_n for all j , the sequence $Z_1, \dots$ does not converge to W . Also, since $a - \epsilon \leq a_j \leq \epsilon$, the a_j do not converge to either 0 or 1. Hence, the only way for the P_j to converge at all is if the Z_j converge to a vertex $Z \in B_n^i(Y)$ and the a_j converge to a point b with $a - \epsilon \leq b \leq a + \epsilon$. Thus $Q = (W, b, Z) \in C_{\epsilon,n}^i$. □

We recall that a space $\mathbb{X}$ is said to be T_3 if, for any closed set $C \subset \mathbb{X}$ and any point $x \in \mathbb{X} \setminus C$, there are disjoint open sets U and V with $x \in U$ and $C \subseteq V$. A space is said to be *regular* if it is both T_3 and Hausdorff.

We also recall the well known theorem [13, Chapter 4 Lemma 31.1 a] that a Hausdorff space is regular if for every point x and for every open set U containing x , there is an open set V such that $x \in V$ and $\overline{V} \subset U$.

Lemma 7.6. $\mathcal{PS}_i(S)$ is a regular space.

Proof. Recall $\mathcal{PS}_i(S)$ is Hausdorff by Lemma 7.3. the previous two lemmas, each point $P \in \mathcal{PS}_i(S)$ has a neighborhood basis of open sets $\mathbb{D}_{\frac{1}{n}}^i(P)$ where $n \in \mathbb{N}$. Moreover, for all sufficiently large n , we have $\overline{\mathbb{D}_{\frac{1}{n}}^i(P)} \subset \mathbb{D}_{\frac{1}{n-1}}^i(P)$, so $\mathcal{PS}_i(S)$ is regular. $\square$

Corollary 7.7. $\mathcal{PS}_i(S)$ is metrizable.

Proof. The pants space is regular by the previous lemma, and second-countable by Corollary 6.23. Thus, by the Urysohn Metrization Theorem[16], $\mathcal{PS}_i(S)$ is metrizable. $\square$

Lemma 7.8. Let $\Gamma \subset \mathcal{P}(S)$ be a connected-component. Then for $1 \leq i \leq 4$, Γ is dense in $\mathcal{PS}_i(S)$.

Proof. First, we note that by Lemma 4.23, every vertex in $\mathcal{PS}_i(S)$ is a limit point of Γ . Now choose an edge point $P = (X, a, Y)$. We want to show that $P \in \overline{\Gamma}$.

Since X and Y differ by an elementary move, there exists curves α and β such that $Y = (X \setminus \{\alpha\}) \cup \{\beta\}$. Let S' be the complexity one subsurface in the complement of $X \setminus \{\alpha\} = Y \setminus \{\beta\}$. Let M be the least integer such that $S' \subset S_M$.

For each $n \geq 1$, choose a vertex $X_n \in \Gamma$ such that X_n and X agree on S_n (such a vertex must exist by Lemma 4.23). For $n \geq M$, there is an elementary move on X_n which replaces α by β . Call the result of this elementary move Y_n : i.e. $Y_n = (X_n \setminus \{\alpha\}) \cup \{\beta\}$.

Then there is an edge point $(X_n, a, Y_n) \in \Gamma$. By construction, $\{X_n\}_{n \geq M} \rightarrow X$ and $\{Y_n\}_{n \geq M} \rightarrow Y$, thus $\{(X_n, a, Y_n)\}_{n \geq M}$ is a sequence of points in Γ which converges to P . $\square$

Lemma 7.9. For $1 \leq i \leq 4$, $\mathcal{PS}_i(S)$ is a separable space.

Proof. Let $\Gamma \subset \mathcal{P}(S)$ be a connected-component. Then Γ has countably many vertices and edges. The subset of Γ consisting of vertices and rational edge points is a countable dense subset of $\mathcal{PS}_i(S)$. $\square$

Lemma 7.10. For $1 \leq i \leq 4$, $\mathcal{PS}_i(S)$ is path-connected.

Proof. First, note that because $\mathcal{P}(S) = \mathcal{PS}_0(S)$ carries a finer topology than $\mathcal{PS}_i(S)$, every path-component of $\mathcal{P}(S)$ is also path-connected in $\mathcal{PS}_i(S)$. Thus, it is sufficient to find a path between an arbitrary pair of vertices X and Y in different $\mathcal{P}(S)$ -components.

By Lemma 4.23, there exists a sequence of pants decompositions $X_1, X_2, \dots$ such that $X = X_1$, X_j and X_{j+1} differ by an elementary move, and such that the sequence converges to Y in $\mathcal{PS}_i(S)$. Consider the function $f : [0, 1] \rightarrow \mathcal{PS}_i(S)$ with

$$f(t) = \begin{cases} X_n & t = 1 - 1/n \\ Y & t = 1 \\ (X_n, t - (1 - 1/n), X_n + 1) & 1 - 1/n < t < 1 - 1/(n+1) \end{cases}$$

We first claim that f is continuous everywhere except possibly at $t = 1$. To see this, note that the restriction of f to $[0, 1)$ is a continuous path in $\mathcal{P}(S)$, and $\mathcal{P}(S)$ has a coarser topology than $\mathcal{PS}_i(S)$.

Let $t_1, t_2, \dots$ be a sequence of real numbers in $[0, 1]$ which converges to 1. Then for any $n \geq 1$, and for all but finitely many of the t_j , $f(t_j)$ lies on the edge between X_M and X_{M+1} for some $M \geq n$. Thus, by the definition of pants convergence, $f(t_j) \rightarrow Y = f(1)$. Thus f is continuous at 1, and hence X and Y are in the same path-component of $\mathcal{PS}_i(S)$. $\square$

Lemma 7.11. $\mathcal{PS}_i(S)$ is locally path-connected at a point P if and only if P is a vertex point.

Proof. First, suppose P is a vertex. Then $\mathbb{D}_{1/2}^i(X)$ is path-connected, by the same argument used in the previous lemma.

Now suppose $P = (X, a, Y)$ is an edge point. Recall that P has a neighborhood $E_i(S) = \mathcal{PS}_i(S) \setminus V_i(S)$ consisting of all edge points. We will show that each edge is a connected component of $E_i(S)$.

A single edge is path-connected and hence connected. To show that $E_i(S)$ is disconnected, we consider the quotient space

$$\hat{E} = E_i(S) / ((X, a, Y) \sim (X, a', Y))$$

obtained by identifying all points on an edge. We will show that $\hat{E}$ is totally disconnected. Each point in $\hat{E}$ can be represented by an unordered pair $\{X, Y\}$ of adjacent vertices in $\mathcal{P}(S)$.

For a point $\{X, Y\} \in \hat{E}$ and an integer $n \geq 1$, there is a neighborhood

$$\hat{A}_n(\{X, Y\}) = \{\{X', Y'\} : X' \in B_n^i(X) \text{ and } Y' \in B_n^i(Y)\}$$

This set is open because its preimage in $E_i(S)$ is open. It follows from the disjointness lemma that for two points $\{X, Y\}$ and $\{X', Y'\}$, the neighborhoods $\hat{A}_n(\{X, Y\})$ and $\hat{A}_n(\{X', Y'\})$ are either disjoint or equal. Thus, for sufficiently large n , the $\hat{A}_n$ partition $\hat{E}$ into disjoint clopen sets such that $\{X, Y\}$ and $\{X', Y'\}$ are in different parts of the partition. Hence, $\hat{E}$ is totally disconnected. Hence, each edge of $E_i(S)$ is a connected-component.

Note also that any smaller neighborhood U such that $P \in U \subseteq E_i(S)$ must intersect infinitely many edges, and hence is also disconnected. Thus, $\mathcal{PS}_i(S)$ is not locally connected at P . $\square$

Corollary 7.12. $\mathcal{PS}_i(S)$ is not locally path-connected.

Proof. Immediate from the previous lemma. $\square$

Corollary 7.13. $\mathcal{PS}_i(S)$ is not homeomorphic to a CW complex.

Proof. This follows from the fact that $\mathcal{PS}_i(S)$ not locally path-connected. $\square$

In particular, the preceeding corollary implies that $\mathcal{PS}_i(S)$ is different from the generalization of the pants graph in [6].

Corollary 7.14. Let $f : \mathcal{PS}_i(S) \rightarrow \mathcal{PS}_i(S)$ be a homeomorphism. Then f takes vertices to vertices and edge points to edge points.

Proof. Homeomorphisms must preserve local-path-connectedness, so the result follows from the previous lemma. $\square$

Lemma 7.15. Let $f : \mathcal{PS}_i(S) \rightarrow \mathcal{PS}_i(S)$ be a homeomorphism and let $P, Q \in \mathcal{PS}_i(S)$ be edge points. Then P and Q lie in the same edge if and only if $f(P)$ and $f(Q)$ lie in the same edge.

Proof. By the previous corollary, f restricts to a homeomorphism of $E_i(S)$. Recall from the proof of Lemma 7.5 that P and Q lie on the same edge if and only if they are in the same connected-component of $E_i(S)$. $\square$

Lemma 7.16. Let $f : \mathcal{PS}_i(S) \rightarrow \mathcal{PS}_i(S)$ be a homeomorphism. Let X be a vertex and P be an edge point. Then P lies on an edge incident to X if and only if $f(P)$ lies on an edge incident to X .

Proof. Suppose P is on an edge incident to X . Then P has a representative triple $P = (X, a, Y)$. Consider the sequence of points $\{(X, \frac{a}{n}, Y)\}_{n \geq 1}$. This sequence converges to X in $\mathcal{PS}_i(S)$. Hence, the sequence $\{f((X, \frac{a}{n}, Y))\}_{n \geq 1}$ converges to $f(X)$. By the previous lemma, all the points in $\{f((X, \frac{a}{n}, Y))\}_{n \geq 1}$ lie in a single edge, and hence their

limit $f(X)$ must be one of the end points of that edge. Thus $f(P)$ is on an edge incident to $f(X)$. The converse follows from replacing f by f^{-1} . $\square$

Theorem 7.17. *Let S and S' be infinite-type surfaces and let $f, f' : \mathcal{PS}_i(S) \rightarrow \mathcal{PS}_i(S')$ be two homeomorphisms. Then the following are equivalent:*

1. *The homeomorphisms f and f' are isotopic.*
2. *The restrictions of f and f' to $V_i(S)$ are equal.*
3. *For all edge points $P \in \mathcal{PS}_i(S)$, $f(P)$ and $f'(P)$ lie on the same edge in $\mathcal{PS}_i(S')$.*

Proof. (1) $\Rightarrow$ (2): Let f and f' be isotopic homeomorphisms. By Corollary 7.14, f and f' restrict to homeomorphisms from $V_i(S)$ to $V_i(S')$, and any isotopy between f and f' restricts to an isotopy between $f|_{V_i(S)}$ and $f'|_{V_i(S)}$. By Lemma 4.19, $V_i(S)$ is totally disconnected, so isotopic homeomorphisms on $V_i(S)$ must be equal. Thus $f|_{V_i(S)} = f'|_{V_i(S)}$.

(2) $\Leftrightarrow$ (3): This implication is immediate from Lemma 7.16.

(2) and (3) $\Rightarrow$ (1): Consider the edge between two vertices $X, Y \in V_i(S)$. By hypothesis (2), $f(X) = f'(X)$ and $f(Y) = f'(Y)$. By hypothesis (3), f and f' restrict to homeomorphisms from the edge between X and Y to the edge between $f(X)$ and $f(Y)$. An edge in the pants space is homeomorphic to the real unit interval, which has trivial (orientation-preserving) mapping class group, hence the restrictions of f and f' to this edge are isotopic via the straight line isotopy. We can apply the straight line isotopy to every edge simultaneously to get an isotopy between f and f' . $\square$

Definition 7.18. Let S and S' be infinite-type surfaces. A (not necessarily continuous) map $f : \mathcal{PS}_i(S)$ to $\mathcal{PS}_i(S')$ is a *graph isomorphism* if

1. The restriction of f to $V_i(S)$ is a bijection between $V_i(S)$ and $V_i(S')$.
2. For all edge points $P = (X, a, Y) \in \mathcal{PS}_i(S)$, $f(P) = (f(X), a, f(Y))$.

Note: In an earlier version of this paper, Theorem 7.18 was stated only for the special case $S = S'$. The author is grateful to J. Aramayona for suggesting that theorem holds in the cases when $S \neq S'$.

Theorem 7.19. *Let S and S' be infinite-type surfaces. Then every homeomorphism from $\mathcal{PS}_i(S)$ to $\mathcal{PS}_i(S')$ is isotopic to a unique graph isomorphism from $\mathcal{P}(S)$ to $\mathcal{P}(S')$.*

Proof. Let $f : \mathcal{PS}_i(S) \rightarrow \mathcal{PS}_i(S')$ be a homeomorphism. By combining Corollary 7.14, Lemma 7.15, and Lemma 7.16, f takes vertices to vertices, and for any edge point $P = (X, a_P, Y)$, $f(P)$ has a representative of the form $(f(X), a'_P, f(Y))$ for some $0 < a' < 1$. By isotoping f along the edges, we can make it so that $a_P = a'_P$ for all edge points. Hence, f is isotopic to a simplicial isomorphism.

To show that this isomorphism is unique, suppose that $g, h : \mathcal{P}_i(S) \rightarrow \mathcal{P}_i(S')$ are two distinct graph isomorphisms. Then there exists some $X \in V_i(S)$ such that $g(X) \neq h(X)$. Hence, by Theorem 7.17, g and h are not isotopic. $\square$

Theorem 7.18 tells us that each isotopy class of homeomorphisms between pants spaces has a "canonical" representative: namely, the one which is a graph isomorphism of the underlying pants graphs. In the particular case of $S = S'$, this means that the mapping class group of $\mathcal{PS}_i(S)$ is naturally isomorphic to the subgroup of $\text{Aut}(\mathcal{P}(S))$ consisting of the automorphisms which are continuous in the topology of $\mathcal{PS}_i(S)$. Our goal for the rest of this section is to determine which graph automorphisms of $\mathcal{P}(S)$ are homeomorphisms of $\mathcal{PS}_i(S)$.

Lemma 7.20. *Let Γ be a connected-component of $\mathcal{P}(S)$. Let f and g be homeomorphisms of $\mathcal{PS}_i(S)$ which are also graph automorphisms. Then $f = g$ if and only if $f|_\Gamma = g|_\Gamma$.*

Proof. The result follows from the fact that Γ is dense in $\mathcal{PS}_i(S)$. $\square$

Theorem 7.21. *Let S and S' be infinite-type surfaces. A graph automorphism $\phi : \mathcal{PS}_i(S) \rightarrow \mathcal{PS}_i(S')$ is a homeomorphism of pants spaces if and only if it is induced by a homeomorphism from S to S' .*

Proof. The "if" direction follows immediately from the definition of the pants space topology. Now suppose ϕ is a homeomorphism of pants spaces. Choose a connected-component $\Gamma \in \Pi_0(\mathcal{P}(S))$. Since ϕ is a graph isomorphism, the restriction of ϕ to Γ is an isomorphism between Γ and some $\Gamma' \in \Pi_0(\mathcal{P}(S'))$. By Theorem 3.11, there is a homeomorphism $f : S \rightarrow S'$ which induces an isomorphism from Γ to Γ' . Then f also induces a homeomorphism from $\mathcal{PS}_i(S)$ to $\mathcal{PS}_i(S')$, and by Lemma 7.20, this homeomorphism must be ϕ . $\square$

This theorem immediately provides us with the following corollary.

Corollary 7.22. *The mapping class groups $\text{Mod}^\pm(\mathcal{PS}_i(S))$ and $\text{Mod}^\pm(S)$ are naturally isomorphic.*

We thus have a generalization to infinite-type surfaces of Margalit's result from [12]

8 A continuous group action on $\mathcal{PS}_i(S)$

The goal of this section is to prove the following theorem.

Theorem 8.1. *Let S be an infinite-type surface with no planar ends. Then the natural action of $\text{Mod}^\pm(S)$ on $\mathcal{PS}_i(S)$ is continuous.*

First, we will need the following lemma.

Lemma 8.2. *Let S be an infinite-type surface. Let $(f_1, X_1), (f_2, X_2), \dots$ be a sequence in $\text{Mod}^\pm(S) \times V_i(S)$ which converges to (f, X) . Then the sequence $f_1(X_1), f_2(X_2), \dots$ converges to $f(X)$.*

Proof. Fix some hyperbolic metric on S . For each finite-type surface $S_n \subset S$, there is a compact subsurface $S'_n \subseteq S_n$ such that any geodesic simple closed curve is disjoint from $S_n \setminus S'_n$. Since the f_j converge to f in the compact-open topology on $\text{Mod}^\pm(S)$, the restrictions of the f_j to S'_k uniformly converge to $f|_{S'_k}$ for all $k \geq 0$.

Fix an $n \geq 1$. Let $\hat{n}$ be the least natural number such that every curve in $f(X)$ which essentially intersects S_n is contained in $S'_{\hat{n}} \subseteq S_{\hat{n}}$. By uniform convergence, for all sufficiently large m , $f_{m|S'_{\hat{n}}} = f|_{S'_{\hat{n}}}$, and X_m agrees with X on $S_{\hat{n}}$. Thus, $f_m(X_m)$ agrees with $f(X)$ on S_n . The result follows now from the definition of convergence in $V_i(S)$. $\square$

Now we are ready to prove Theorem 9.1.

Proof. Suppose that $(f_1, P_1), \dots$ is a sequence in $\text{Mod}^\pm(S) \times \mathcal{PS}_i(S)$ which converges to (f, P) . We will show that $\lim_{n \rightarrow \infty} f_n(P_n) = f(P)$, and therefore conclude that the action of $\text{Mod}^\pm(S)$ is continuous. As usual, we must consider three cases, depending on which of the conditions of definition 6.1 the P_j satisfy.

First, we consider the case where $P = (X, a, Y)$ is an edge point. Then there are representations $P_j = (X_j, a_j, Y_j)$ such that $\{X_j\} \rightarrow X$, $\{Y_j\} \rightarrow Y$, and $a_j \rightarrow a$. By the definition of the action of $\text{Mod}^\pm(S)$ on $\mathcal{PS}_i(S)$, $f_j(P_j) = (f(X_j), a_j, f(Y_j))$. By Lemma 8.2, $f_j(X_j) \rightarrow f(X)$ and $f_j(Y_j) \rightarrow f(Y)$, so $f_j(P_j) \rightarrow f(P)$, as desired.

Next, suppose that P is a vertex, and that for $j \geq 1$, we have representations $P_j = (X_j, a_j, Y_j)$ such that $\{X_j\} \rightarrow P$ and $\{a_j\} \rightarrow 0$. Once again, we have $f_j(P_j) = (f_j(X_j), a_j, f(Y_j))$. Again by Lemma 8.2, $f_j(X_j) \rightarrow f(X)$, and hence $f_j(P_j) \rightarrow f(P)$.

Finally, suppose P is a vertex, and that there are representations $P_j = (X_j, a_j, Y_j)$ where the sequences $\{X_j\}$ and $\{Y_j\}$ both converge to P . Then $f_j(P_j) = (f_j(X_j), a_j, f(Y_j))$, so invoking Lemma 8.2 again shows that $\{f_j(X_j)\}$ and $\{f_j(Y_j)\}$ both converge to $f(P)$, and hence $f_j(P_j) \rightarrow f(P)$. $\square$

9 Open Questions

There are two main open questions concerning $\mathcal{PS}_i(S)$ and $V_i(S)$ which we intend to tackle in a sequel paper. We briefly discuss them in this section.

9.1 Coarse Geometry of $V_i(S)$

In section 5, we constructed a metric on $V_i(S)$ and showed that it has diameter strictly greater than one. The definition of the metric allows us to "walk around" the vertex space using two kinds of steps: jumping to another pants decomposition which agrees on a finite-type subsurface, and making elementary moves. These two types of moves do not necessarily commute. Hence, it is nontrivial to compute distances greater than one in the vertex space.

Ultimately, we would like to know the diameter of $V_i(S)$. If it is infinite diameter, we would like to understand its coarse geometry.

9.2 Naturality of $V_i(S)$

The topologies on $V_i(S)$ and $\mathcal{PS}_i(S)$ does not depend on the choice of the exhaustion of S . However, the metric on $V_i(S)$ does depend on the exhaustion. We do not know if there is some equivalence relation (such as quasi-isometry) under which the metric on $V_i(S)$ is independent of the exhaustion.

9.3 Natural Metrics on $\mathcal{PS}_i(S)$

In section seven, we showed that the pants space is metrizable using the Urysohn Metrization Theorem. The proof of the Urysohn Metrization Theorem involves embedding any second-countable regular space in the Hilbert cube. While this embedding gives a metric on $\mathcal{PS}_i(S)$, there is no guarantee that this metric has anything to do with the graph structure underlying the pants space, or with the family of metrics d_i on the vertex space we constructed in section five.

We would like to know the answer to the following questions.

Question 9.1. Is there a metric on $\mathcal{PS}_i(S)$ such that the edges are geodesics and the length of an edge is the distance between its endpoints?

Question 9.2. Is there a metric on the pants space such that the inclusion of $(V_i(S), d_i) \hookrightarrow \mathcal{PS}_i(S)$ is an isometric embedding?

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