ON MIX-NORMS, CORRELATIONS, AND OPTIMAL SPATIALLY DEPENDENT DIFFUSION COEFFICIENTS

by

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To my wife and family.

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CONTENTS

ABSTRACT

The first chapter of this thesis is the contents of the paper *On Mix-norms and the Rate of Decay of Correlations*, which was published June 2021 in *Nonlinearity*. There we explore two quantitative notions of mixing: the decay of correlations and the decay of the mix-norm — a negative Sobolev norm. The intensity of mixing can be measured by the rates of decay of these quantities. We show that the mix-norm and correlations decay at the same rate in the sense that they are Big-O but not Little-O of eachother. This result bridges the connection between two fields: dynamical systems, where correlations are commonly used to study mixing in the context of ergodic theory, and fluid dynamics, where mix-norms are well suited for the partial differential equations context. The second chapter can be seen as a study of mixing rates in the absence of a fluid flow wherein we present a preprint of the paper *Optimal Spatially Dependent Diffusion Coefficients under an L^p Constraint*. We ask what spatially dependent diffusion coefficients will maximize the rate of convergence to equilibrium of solutions to the heat equation in inhomogeneous media. We formulate a variational problem for the optimal spectral gap. We solve a relaxed version of this variational problem which provides an upper bound for the optimal spectral gap. This solution is characterized in terms of the extremals of Sobolev and Poincaré type inequalities.

1 ON MIX-NORMS AND THE RATE OF DECAY OF CORRELATIONS

1.1 Introduction

Consider a spatially-periodic mean-zero function $f^t(\boldsymbol{x}) = f(t, \boldsymbol{x})$ bounded uniformly in $L^2(\mathbb{T}^d)$ for all $t > 0$. For example, $f(t, \boldsymbol{x})$ might be a solution to the advection-diffusion equation

$$\frac{\partial f}{\partial t} + \boldsymbol{u} \cdot \nabla f = D\Delta f, \tag{1.1.1}$$

with $f^0 \in L^2(\mathbb{T}^d)$ and smooth divergence-free velocity field $\boldsymbol{u}(t, \boldsymbol{x})$. We may also consider $D = 0$ in Eq. (1.1.1), in which case it is the transport equation. Another example, in the context of dynamical systems, is when an initial condition $f^0 \in L^2$ is transported by an area-preserving map M via the transfer operator $f^{n+1} = f^n \circ M^{-1}$.

Decay of the correlation function $C_t(g) = |\langle f^t\,, g\rangle|$ as $t \to \infty$ for observables g in $L^2(\mathbb{T}^d)$ corresponds to mixing of f^t as $t \to \infty$ [64]. Mathew et al. [50] introduced the $H^{-1/2}$ norm as another criterion to quantify mixing, and Lin et al. [40] extended this to any negative Sobolev (e.g., H^{-q}) norm and showed that correlations decay to zero if and only if any such "mix-norm" decays to zero. That is,

$$\lim_{t\to\infty} \left\langle f^t\,, g\right\rangle = 0 \quad \forall g \in L^2 \iff \lim_{t\to\infty} \left\| f^t \right\|_{H^{-q}} = 0, \text{ for any } q > 0.$$

Mix-norms are well-suited to quantification of mixing efficiencies [17, 39, 47, 63, 66–68, 70], to lower bounds on the rate of mixing in general [32, 45, 46], and to analyzing mixing [48, 49, 52, 71]. Moreover, such negative Sobolev spaces provide a natural setting for a discussion of enhanced dissipation and relaxation [4, 7, 12–14, 23, 34, 35]. Mathew et al. [50] introduced the mix-norm in the context of spatial averages over strips, and made the connection to weak convergence (see also [73]).

While mix-norms are well-adapted to the PDE context, correlations and weak convergence are more commonly studied in the context of ergodic theory. A central question, then, is the quantitative relationship between the decay rate of correlations and decay of mix-norms. This is the central focus of this chapter where we will work in a setting where the evolution of a function $f^t(\boldsymbol{x})$ is given, arising from the continuous-time solution of a PDE or in discrete times from an iterated map.

When studying a collection of functions converging to zero as $t \to \infty$, such as $|\langle f^t\,,\, g\rangle|$ for $g \in X$ where $X \subset L^2$ is some Banach space, there are several reasonable ways to define a rate of decay:

1. We can consider a *uniform* upper bound [6, 11, 18, 43, 62].

2. We can say that each function is $\mathcal{O}(\varrho)$[1] where $\varrho(t)$ is some rate function [72]. This lifts the tail of the rate function by multiplying by a constant that depends on $g \in X$.

3. We can instead lift the tail of the rate function by translation and say that each function is bounded above by a translation of some rate function [21].

[1]We say that $a(t) = \mathcal{O}(b(t))$ as $t \to \infty$ if there are T, M so that $|a(t)| \leq Mb(t)$ for $t > T$. For $b(t) > 0$, this is equivalent to $\limsup_{t\to\infty} |a(t)|/b(t) = C \in [0, \infty)$. Moreover, we say $a(t) = o(b(t))$ if $C = 0$.

We summarize as follows (for concreteness, fix some $q > 0$ and consider $X = H^q(\mathbb{T}^d)$):

1. Correlations decay at the *uniform* rate $r(t)$ for $g \in H^q$ if

$$\big|\big\langle f^t\,,\, g\big\rangle\big| \le r(t)\|g\|_{H^q}\ \forall g \in H^q\,. \tag{1.1.2}$$

2. Correlations decay at the *asymptotic* rate $\varrho(t)$ for $g \in H^q$ if

$$\big|\big\langle f^t\,,\, g\big\rangle\big| = \mathcal{O}(\varrho), \text{ for each } g \in H^q\,. \tag{1.1.3}$$

   That is,

$$\limsup_{t\to\infty} \frac{|\langle f^t\,,\, g\rangle|}{\varrho(t)} = C_g \in [0,\infty). \tag{1.1.4}$$

3. Correlations decay at the *translational* rate $\lambda(t)$ for $g \in H^q$ if for each $g \in H^q$ there exists $\tau_g \in \mathbb{R}$ such that for all $t > \tau_g$ we have

$$\big|\big\langle f^t\,,\, g\big\rangle\big| \le \lambda(t-\tau_g)\|g\|_{H^q}\,. \tag{1.1.5}$$

From considering duality in Section 1.2, we find that the smallest uniform rate is the mix-norm $\|f^t\|_{H^{-q}}$. Since any uniform rate trivially satisfies the definitions of asymptotic rate and translational rate, the question is whether there is a ϱ (or λ) that decays faster than $\|f^t\|_{H^{-q}}$. We answer this question by showing that we cannot have $\varrho = o(\|f^t\|_{H^{-q}})$. Similarly, given the additional assumption that $\limsup_{t\to\infty} \lambda(t-\tau)/\lambda(t)$ is finite, we cannot have $\lambda = o(\|f^t\|_{H^{-q}})$. We remark that this growth condition on λ is satisfied by power law and exponential functions.[2]

We prove the above facts by constructing an observable $g \in H^q$ such that $|\langle f^t\,,\, g\rangle|$ decays arbitrarily closely to the mix-norm. Namely, for any positive $h(t) = o(\|f^t\|_{H^{-q}})$ there is a $g \in H^q$ such that $|\langle f^t\,,\, g\rangle|$ is Big-O but not Little-O of h. Note that this is not the same as asymptotic equivalence because the correlation may be much smaller than h at certain times.

Let $P_I f^t$ denote the projection of f^t onto the Fourier modes I. We say f^t is *q-recurrent* (otherwise it is *q-transient*) if there is a finite set I where $\|P_I f^t\|_{H^{-q}}$ is Big-O but not Little-O of $\|f^t\|_{H^{-q}}$. Heuristically, in this case, the decay of the mix-norm is characterized by $P_I f^t$. We prove f^t is q-recurrent if and only if there is a $g \in H^q$ such that $|\langle f^t\,,\, g\rangle|$ is Big-O but not Little-O of $\|f^t\|_{H^{-q}}$. Therefore, q-recurrence is the criterion for the existence of a correlation that obtains the decay rate of the mix-norm.

In Section 1.2 we introduce the key definitions and main theorems. Section 1.3 contains examples, and Sections 1.4 and 1.5 contains the full proofs of the theorems.

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[2]The growth condition is not satisfied by functions that decay faster than exponentially, such as $\lambda(t) = \mathrm{e}^{-t^2}$. In this case, λ is not asymptotically equivalent to its own translation: for large t, we see $\mathrm{e}^{-(t-\tau)^2} = \mathrm{e}^{2\tau t - \tau^2}\mathrm{e}^{-t^2} \gg \mathrm{e}^{-t^2}$. For λ not satisfying the growth condition, a translation of λ is much larger than a constant multiple of λ. If there is a correlation bounded by a translation of λ but not a constant multiple of λ, then $\lambda = o(\varrho)$.

1.2 Overview

Throughout, it will be more convenient to work with the homogeneous Sobolev spaces $\dot{H}^\alpha$ for $\alpha \in \mathbb{R}$. Since the torus $\mathbb{T}^d$ is a compact manifold, Poincaré's inequality applies [30] so that the H^α norm and $\dot{H}^\alpha$ norm are equivalent for mean-zero functions. For $\alpha > 0$, the $\dot{H}^{-\alpha}$ norm is typically defined via the duality equation

$$\|f\|_{\dot{H}^{-\alpha}} = \sup_{g \in \dot{H}^\alpha} \frac{|\langle f\,,\,g\rangle|}{\|g\|_{\dot{H}^\alpha}}.$$

However, there is an equivalent definition [24] for all $\alpha \in \mathbb{R}$. Let $f_{\boldsymbol{k}} = \int_{\mathbb{T}^d} f(\boldsymbol{x})\,\mathrm{e}^{-2\pi i \boldsymbol{x}\cdot\boldsymbol{k}}\,d\boldsymbol{x}$ denote the Fourier coefficients of $f(\boldsymbol{x})$. Then

$$\|f\|_{\dot{H}^\alpha} = \left(\sum_{\boldsymbol{k}\neq\boldsymbol{0}} k^{2\alpha}\,|f_{\boldsymbol{k}}|^2\right)^{1/2}$$

where $k^2 = |\boldsymbol{k}|^2 = |k_1|^2 + \cdots + |k_d|^2$. We will typically omit the $\boldsymbol{k} \neq 0$ under the sum since $f_{\boldsymbol{0}} = 0$ for mean-zero functions.

Similarly, correlations have a simple expression. Since the trigonometric functions $\{\mathrm{e}^{2\pi i \boldsymbol{x}\cdot\boldsymbol{k}}\}$ provide an orthonormal basis for $L^2(\mathbb{T}^d)$, the Fourier transform is a unitary map to $\ell^2(\mathbb{Z}^d)$. Therefore the Fourier transform preserves the inner product [24] and we have Plancherel's Theorem:

$$\langle f\,,\,g\rangle = \sum_{\boldsymbol{k}} f_{\boldsymbol{k}}\,\bar{g}_{\boldsymbol{k}} \quad \forall f, g \in L^2(\mathbb{T}^d).$$

Now say $q > 0$ and consider $g \in \dot{H}^q$ for the rest of this paper. For time-

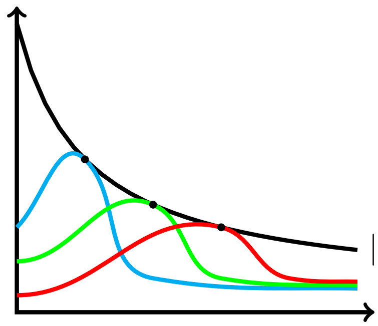

Figure 1.1: Plotted above are the mix-norm and correlations with different choices of g, demonstrating that the mix-norm is the envelope over $\left|\left\langle f^t\,,\,g\right\rangle\right|$ with $\|g\|_{\dot{H}^q} = 1$.

dependent $f^t(\boldsymbol{x})$, the duality equation implies $|\langle f^t\,,\,g\rangle| \leq \|f^t\|_{\dot{H}^{-q}}\|g\|_{\dot{H}^q}$ for all t. Moreover, fix $t = t_0$ and take g with Fourier coefficients

$$g_{\boldsymbol{k}} = f_{\boldsymbol{k}}^{t_0}\, k^{-2q}\left\|f^{t_0}\right\|_{\dot{H}^{-q}}^{-1}. \tag{1.2.1}$$

Then $\|g\|_{\dot{H}^q} = 1$ and Plancherel's Theorem gives $\langle f^{t_0}\,,\,g\rangle = \|f^{t_0}\|_{\dot{H}^{-q}}$. The correlation achieves the mix-norm at the time t_0. Since t_0 is arbitrary, we see that $\|f^t\|_{\dot{H}^{-q}}$ is the envelope of the set of functions $|\langle f^t\,,\,g\rangle|$ with $\|g\|_{\dot{H}^q} = 1$, as in Fig. 1.1. This shows the point-wise smallest uniform rate of decay of correlations, in the sense of Eq. (1.1.2), is the mix-norm $\|f^t\|_{H^{-q}}$.

Using only duality, the most that can be said about the relationship between the rate of decay of a correlation and the rate of decay of the mix-norm is that

$$\left|\left\langle f^t\,,\,g\right\rangle\right| = \mathcal{O}\left(\left\|f^t\right\|_{\dot{H}^{-q}}\right) \text{ for each } g \in \dot{H}^q.$$

However, each correlation could decay strictly faster than the mix-norm as illustrated in Fig. 1.1. We are then led to ask if such a situation is possible.

When is $|\langle f^t\,,\,g\rangle| = o(\|f^t\|_{\dot{H}^{-q}})$ for each $g \in \dot{H}^q$? To answer this question, we must construct functions $g \in \dot{H}^q$ such that the correlations $|\langle f^t\,,\,g\rangle|$ decay as slowly as possible. To do this, we first classify f^t as either q-recurrent or q-transient as follows.

For a set $I \subset \mathbb{Z}^d$ let

$$P_I f^t = \sum_{\boldsymbol{k}\in I} f^t_{\boldsymbol{k}}\, \mathrm{e}^{2\pi i \boldsymbol{x}\cdot\boldsymbol{k}}$$

denote the projection of f^t onto the Fourier modes $\boldsymbol{k} \in I$. Then

$$\left\|P_I f^t\right\|^2_{\dot{H}^{-q}} = \sum_{\boldsymbol{k}\in I} k^{-2q}\, |f^t_{\boldsymbol{k}}|^2$$

measures the amount of mix-norm supported on I. We often refer to this as the *Fourier energy* contained in I. This notion of energy is q-dependent, though the q will usually be clear from the context.

Definition 1. We say f^t is *q-recurrent* if there exists a finite set $I \subset \mathbb{Z}^d$ such that

$$\limsup_{t\to\infty} \frac{\|P_I f^t\|_{\dot{H}^{-q}}}{\|f^t\|_{\dot{H}^{-q}}} > 0. \tag{1.2.2}$$

Functions that are not q-recurrent will be called *q-transient*.

Remark. We emphasize that q-recurrence is a property of f^t which encompasses both the stirring action and the initial condition coupled together. To clarify, in the context of the advection-diffusion equation (1.1.1), q-recurrence is a property of a particular realization of $\boldsymbol{u}$ and f^0 taken together – it is *not* just a property of the vector field $\boldsymbol{u}$. Flows which don't mix, like $\boldsymbol{u} = \text{const.}$, will trivially induce a q-recurrent trajectory f^t for any initial condition and any q. However, this stability does not hold in general. In Example 2 of Section 1.3, we will see that a given f^t may be q-recurrent for some (larger) values of q and q-transient for other (smaller) q. Additionally, we will find that a given flow $\boldsymbol{u}$ may induce a q-recurrent evolution f^t for some initial conditions and q-transient for others.

From inequality (1.2.2) and the trivial bound $\|P_I f^t\|_{\dot{H}^{-q}} \leq \|f^t\|_{\dot{H}^{-q}}$ we see that $\|P_I f^t\|_{\dot{H}^{-q}}$ is Big-O but not Little-O of $\|f^t\|_{\dot{H}^{-q}}$. Unpacking the definition of limit supremum offers another interpretation: there exists $\delta > 0$ and a sequence $t_m \to \infty$ where

$$\left\|P_I f^{t_m}\right\|_{\dot{H}^{-q}} \geq \delta \left\|f^{t_m}\right\|_{\dot{H}^{-q}}. \tag{1.2.3}$$

This means that there is at least a δ fraction of the mix-norm supported on I at arbitrarily large times. As time progresses the Fourier energy could move off of I, but we can always find a future time t_{m+1} where a proportion δ of the mix-norm is again on I. In other words, some Fourier energy always returns to populate the spatial scales in I. In this case, test functions g with coefficients for $\boldsymbol{k} \in I$ similar to that in Eq. (1.2.1) will match well with f^t at times t_m (after possibly taking a subsequence) so that in Section 1.4 we can prove the following theorem, a central result of our paper.

Remark. The case $\|f^{t_0}\|_{\dot{H}^{-q}} = 0$ is degenerate in the context of the advection-diffusion equation and dynamical systems. In those settings, if the mix-norm is zero at any finite time it will remain zero for future times. In such a case where f^t is

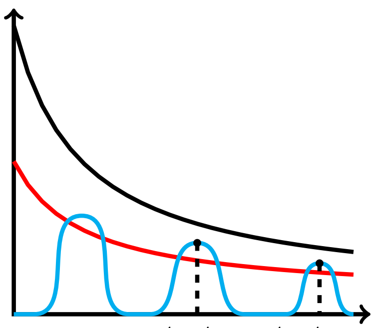

Figure 1.2: There exists $g \in \dot{H}^q$ where $\left|\left\langle f^t , g\right\rangle\right|$ does not decay faster than $c\left\|f^t\right\|_{\dot{H}^{-q}}$.

eventually zero, the mix-norm and correlations will trivially decay at the same rate, the zero function. Otherwise, if the mix-norm is nonzero for a sequence $t_m \to \infty$, we may drop the times where the mix-norm is zero so that $\|f^t\|_{\dot{H}^{-q}} > 0$ for all $t > 0$. Without loss of generality, to simplify the presentation of our results, we will take this as an assumption in the theorems below.

Theorem 1. Let f^t be a mean-zero function in $L^2(\mathbb{T}^d)$ with $\|f^t\|_{\dot{H}^{-q}} > 0$ for all $t > 0$. Then f^t is q-recurrent if and only if there is a function $g \in \dot{H}^q$ such that

$$\limsup_{t\to\infty} \frac{|\langle f^t , g\rangle|}{\|f^t\|_{\dot{H}^{-q}}} > 0.$$

Equivalently, there is a function $g \in \dot{H}^q$, a constant $c > 0$, and a sequence $t_m \to \infty$ where

$$\left|\left\langle f^{t_m} , g\right\rangle\right| \geq c \left\|f^{t_m}\right\|_{\dot{H}^{-q}}. \tag{1.2.4}$$

Remark. As demonstrated in Fig. 1.2, it is possible that $|\langle f^t\,,\,g\rangle|$ is small at times $t \neq t_m$ and so we do not show asymptotic equivalence. We interpret our result as demonstrating that the correlation does not decay asymptotically faster than the mix-norm in the sense that $|\langle f^t\,,\,g\rangle|$ is Big-O but not Little-O of $\|f^t\|_{\dot{H}^{-q}}$. From this theorem, the answer to our previously posed question 'when is $|\langle f^t\,,\,g\rangle| = o(\|f^t\|_{\dot{H}^{-q}})$ for each $g \in \dot{H}^q$?' is exactly when f^t is q-transient.

This naturally prompts us to ask if f^t is q-transient and we carefully choose $g \in \dot{H}^q$, how slowly can we make $|\langle f^t\,,\,g\rangle|$ decay? The following theorem answers this question.

Theorem 2. Let f^t be a mean-zero function in $L^2(\mathbb{T}^d)$ with $\|f^t\|_{\dot{H}^{-q}} > 0$ for all $t > 0$. For any positive function $h(t)$ such that $h(t) = o(\|f^t\|_{\dot{H}^{-q}})$, there is a function $g \in \dot{H}^q$ such that

$$\limsup_{t\to\infty} \frac{|\langle f^t\,,\,g\rangle|}{h(t)} > 0.$$

Remark. Theorems 1 and 2 do not require f^t to be bounded uniformly in $L^2(\mathbb{T}^d)$ in time, nor for the mix-norm to decay to zero. Additionally, Theorem 2 is valid whether f^t is q-recurrent or q-transient.

The proof of Theorem 2 is deferred until Section 1.5, but we present the idea behind the proof now. If f^t is q-recurrent, then the proof is accomplished by a result similar to Theorem 1. For q-transient functions, the proof relies on the construction of sets I_m and times t_m satisfying certain properties, the first being that we want the finite disjoint sets $I_m \in \mathbb{Z}^d$ to capture a large amount of the Fourier energy at time t_m. We can do this since q-transience ensures that we can wait for the next time t_{m+1} where a proportion of the Fourier energy moves off of I_m and never comes back. Then by choosing the Fourier coefficients of g on I_m to agree with f^t at time t_m, we can guarantee that $|\langle f^t\,,\,g\rangle|$ will be large at time t_m. Hence, the function g in Theorem 2 which gives the slowly decaying $|\langle f^t\,,\,g\rangle|$ has Fourier coefficients

$$g_{\boldsymbol{k}} = \begin{cases} f^{t_m}_{\boldsymbol{k}} k^{-2q} \|f^{t_m}\|^{-2}_{\dot{H}^{-q}}\, h(t_m), & \boldsymbol{k} \in I_m; \\ 0, & \text{otherwise}; \end{cases} \tag{1.2.5}$$

where I_m are disjoint and $\|P_{I_m} f^{t_m}\|_{\dot{H}^{-q}}$ captures a nonzero proportion of $h(t_m)$, similar to inequality (1.2.3). These coefficients are similar to those of Eq. (1.2.1) except with an extra factor of $h/\|f\|_{\dot{H}^{-q}}$. This factor is needed so that we can satisfy the second property we require from the sets I_m and times t_m: by taking a subsequence, we can use the fact that $h(t) = o(\|f^t\|_{\dot{H}^{-q}})$ to make the $g_{\boldsymbol{k}}$ decay fast enough as $k \to \infty$ to have $g \in \dot{H}^q$. Hence, although correlations may not achieve the decay rate of the mix-norm, they may achieve the decay rate of h.

These theorems allow us to show the result outlined in the introduction. The following corollary reveals it is not possible to find a ϱ or λ (under a given growth condition) that is Little-O of the mix-norm.

Corollary 1.

1. For any ϱ satisfying Eq. (1.1.3), we have

$$\limsup_{t\to\infty} \frac{\varrho(t)}{\|f^t\|_{H^{-q}}} > 0\,.$$

2. For λ satisfying Eq. (1.1.5) and $\limsup_{t\to\infty} \lambda(t-\tau)/\lambda(t)$ finite for any $\tau \in \mathbb{R}$, we have
$$\limsup_{t\to\infty} \frac{\lambda(t)}{\|f^t\|_{H^{-q}}} > 0\,.$$

Proof of Corollary 1. Seeking contradiction we suppose there is a $\varrho(t)$ satisfying Eq. (1.1.3) such that $\varrho(t) = o(\|f^t\|_{\dot{H}^{-q}})$. Choosing $h(t) = \sqrt{\varrho(t)\|f^t\|_{\dot{H}^{-q}}}$, the geometric mean of ϱ and the mix-norm, we see

$$\limsup_{t\to\infty} \frac{h}{\|f^t\|_{\dot{H}^{-q}}} = \limsup_{t\to\infty} \sqrt{\frac{\varrho}{\|f^t\|_{\dot{H}^{-q}}}} = 0\,. \tag{1.2.6}$$

Then Theorem 2 assures there is a $g \in \dot{H}^q$ with

$$\limsup_{t\to\infty} \frac{|\langle f^t\,,\, g\rangle|}{h} > 0\,.$$

Then we have arrived at a contradiction:

$$\limsup_{t\to\infty} \frac{|\langle f^t\,,\, g\rangle|}{h} \le \limsup_{t\to\infty} \frac{|\langle f^t\,,\, g\rangle|}{\varrho} \cdot \limsup_{t\to\infty} \frac{\varrho}{h} = 0$$

since $\limsup_{t\to\infty} |\langle f^t\,,\, g\rangle|/\varrho$ is finite by Eq. (1.1.3) and $\limsup_{t\to\infty} \varrho/h = 0$ as in Eq. (1.2.6).

A similar argument gives us the second half of the corollary. In this case choose $h(t) = \sqrt{\lambda(t)\|f^t\|_{\dot{H}^{-q}}}$ and apply Theorem 2 to produce a test function g which comes with a τ_g from Eq. (1.1.5). Then we have another contradiction:

$$\limsup_{t\to\infty} \frac{|\langle f^t\,,\, g\rangle|}{h(t)} \le \limsup_{t\to\infty} \frac{|\langle f^t\,,\, g\rangle|}{\lambda(t-\tau_g)} \cdot \limsup_{t\to\infty} \frac{\lambda(t-\tau_g)}{\lambda(t)} \cdot \limsup_{t\to\infty} \frac{\lambda(t)}{h(t)} = 0$$

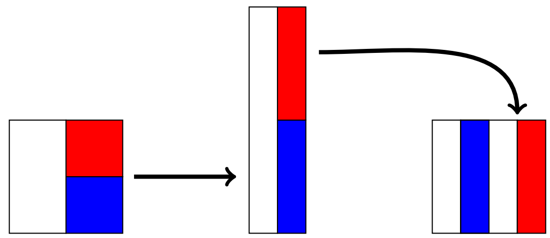

Figure 1.3: The baker's map.

since $\limsup_{t\to\infty} \lambda(t-\tau_g)/\lambda(t)$ is finite by hypothesis. □

In Section 1.3 we present an example of a q-transient function and alter it to send energy down the spectrum *less* efficiently, resulting in q-recurrence for a range of q. We then include diffusion at every time step, demonstrating a transition to q-recurrence for all $q > 0$. We include a numerical example and provide intuition about how to recognize when f^t is q-recurrent. Finally, we prove the theorems in generality in Sections 1.4 and 1.5.

1.3 Examples

Example 1 (baker's map and q-transience)**.** Let B be the baker's map, the area preserving transformation of the domain $[0,1]^2$ as pictured in Fig. 1.3. For the y-independent initial function $f^0(x,y) = 2\cos(2\pi x)$, applying the baker's map simply doubles the frequency in the x direction. After n applications of the baker's map we have $f^n = f^0 \circ B^{-n} = 2\cos(2\pi\ 2^n x)$. As a result, the Fourier coefficients have the simple expression

$$f^n_{\boldsymbol{k}} = \begin{cases} 1 & k_1 = 2^n, k_2 = 0; \\ 0 & \text{otherwise.} \end{cases}$$

This is a one dimensional action on Fourier coefficients $f_k^n = f_{k_1,0}^n$ via an infinite dimensional matrix $A_{k\ell}$ as

$$f_k^{n+1} = \sum_{\ell} A_{k\ell}\, f_\ell^n \tag{1.3.1}$$

where

$$(A_{k\ell}) = \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & \dots \end{array} \\ \left(\begin{array}{ccccc} & & & & \\ 1 & & & & \\ & & & & \\ & 1 & & & \\ & & & & \\ & & 1 & & \\ & & & & \\ & & & 1 & \\ & & & & \ddots \end{array}\right) \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ \vdots \end{array} \end{array}$$

is populated by 1's along a subdiagonal of slope -2 and 0's everywhere else.

The entire mix-norm is supported on just one Fourier mode and given any finite set $I \in \mathbb{Z}^d$, it is clear that, as n increases, the Fourier energy will move off of I and never return. Therefore f^n is q-transient $\forall q > 0$. $\triangle$

Example 2 (baker-like action and q-recurrence)**.** We now alter the previous example so that the energy of f^n is sent down the spectrum *less* effectively, the result being a *q-recurrent* function (if q is large enough). This time, consider the action on the

Fourier coefficients of $f^n(x)$ as in Eq. (1.3.1) via the infinite dimensional matrix

$$
(\widetilde{A}_{k\ell}) = \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & \dots \end{array} \\ \left(\begin{array}{ccccc} a & & & & \\ b & & & & \\ & & & & \\ & 1 & & & \\ & & & & \\ & & 1 & & \\ & & & & \\ & & & 1 & \\ & & & & \ddots \end{array} \right) \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ \vdots \end{array}
$$

where $a, b > 0$ are constants such that $a^2 + b^2 = 1$.

Remark. It is not evident that the current example is still a dynamical systems example. That is, we do not know that there is a map $T : [0,1]^d \to [0,1]^d$ so that $f^{n+1} = f^n \circ T$ and $f_k^{n+1} = \sum_\ell \widetilde{A}_{k\ell} f_\ell^n$. Moreover, such a map might not be injective, surjective, or unique. For example, B^{-1} from the previous example is not injective when thought of as a map on $[0,1]$, but it is when we allow it to move through another dimension ($d = 2$). For the present example, taking $a, b = \sqrt{2}/2$, the initial function $f^0(x) = 2\cos(2\pi x)$ is transformed to $f^1(x) = \sqrt{2}(a\cos(2\pi x) + b\cos(4\pi x))$ after one time step. Since the range of f^0 and f^1 are not the same set, any such T cannot be surjective. In this example, we see the map is also not measure preserving. Lastly, if f^n is constant, then T can be any map. These caveats notwithstanding, we consider the current example as a study of possible ways for energy to move down the spectrum and proceed with an analysis.

We show the coefficients of f^n for initial function $f^0(x) = 2\cos(2\pi x)$ in Table 1.1.

	$k=1$	2	3	4	5	6	7	8	$\dots$
f_k^0	1								
f_k^1	a	b							
f_k^2	a^2	ab		b					
f_k^3	a^3	a^2b		ab				b	
$\vdots$									

Table 1.1: Nonvanishing Fourier coefficients of f^n defined by Eq. (1.3.1), for $f^0(x) = 2\cos(2\pi x)$.

Heuristically, the energy starts concentrated on the $k = 1$ mode and subsequently splits between modes $k = 1, 2$ so that L^2 norm is preserved. After that the $k = 1$ mode continues to donate a proportion b of its energy to $k = 2$ and the energy on $k = 2$ is transported down the spectrum at the same rate as the baker's map ($k = 2^n$).

Notice that f^n is mean-zero because it is the finite sum of cosine functions with full period. We can compute the L^2 norm and find $\|f^n\|_{L^2} = 1\ \forall n$. Hence $A_{k\ell}$ is a unitary map on ℓ^2 by the polarization identity. Therefore f^n is bounded uniformly in L^2 and so f^n satisfies the hypothesis of the theorems in Section 1.2.

Consider the contribution to the mix-norm from mode $k = 1$

$$\mathcal{E}_1^n = \|P_{k=1}f^n\|^2_{\dot{H}^{-q}} = \sum_{k=1} k^{-2q}|f_k^n|^2 = |f_1^n|^2 = a^{2n},$$

and compare it to the contributions from modes $k > 1$ (a geometric sum):

$$\mathcal{E}^n_{k>1} = \|P_{k>1} f^n\|^2_{\dot{H}^{-q}} = \sum_{k>1} k^{-2q} |f^n_k|^2$$
$$= \begin{cases} c_{q,a}(a^{2n} - 2^{-2qn}), & \text{for } a \neq 2^{-q}; \\ b^2 \ a^{2n} n, & \text{for } a = 2^{-q}; \end{cases}$$

where

$$c_{q,a} = \frac{b^2}{a^2 2^{2q} - 1}.$$

For $a > 2^{-q}$, we see that $\mathcal{E}^n_1 \sim \mathcal{E}^n_{k>1}$ as $n \to \infty$ and the mode $k = 1$ captures a non-zero proportion of the mix-norm for arbitrarily large n. Therefore f^n is q-recurrent for $q > \log_2(1/a)$.

For $a \leq 2^{-q}$, $\mathcal{E}^n_1 = o(\mathcal{E}^n_{k>1})$ so the mode $k = 1$ does not capture a non-zero proportion of the mix-norm for arbitrarily large n. This suggests that f^n is q-transient for $q \leq \log_2(1/a)$. To prove q-transience, we need to show the same holds for an arbitrary finite set I. Take $I = \left[0, 2^R\right]$ for some $R \in \mathbb{N}$. For $n > R$,

$$\|P_I f^n\|^2_{\dot{H}^{-q}} = c_{R,q,a} \ a^{2n}$$

where

$$c_{R,q,a} = \begin{cases} 1 + c_{q,a}\left(1 - a^{-2R} 2^{-2qR}\right), & \text{for } a \neq 2^{-q}; \\ 1 + b^2 R, & \text{for } a = 2^{-q} \end{cases}$$

and we conclude that

$$\lim_{n\to\infty} \frac{\|P_I f^n\|^2_{\dot{H}^{-q}}}{\|f^n\|^2_{\dot{H}^{-q}}} = \begin{cases} c_{R,q,a}\big(1+c_{q,a}\big)^{-1}, & \text{for } a > 2^{-q}; \\ 0, & \text{for } a \le 2^{-q}. \end{cases}$$

Therefore f^n is q-transient for $q \le \log_2(1/a)$ and q-recurrent for $q > \log_2(1/a)$. △

Remark. In the above example, an initial function with $f^0_1 = 0$ has a trajectory f^n that is q-transient. This is because the coefficients do not see the alterations we have made to the baker's map and are sent down the spectrum at the same rate as the baker's map ($k = 2^n$). We emphasize that q-recurrence is a property of a particular realization of the flow and initial condition taken together – a given flow may induce a q-recurrent evolution f^t for some initial conditions and q-transient for others.

Example 3 (baker-like action with diffusion)**.** We use the same matrix $\widetilde{A}_{k\ell}$ and initial condition as in Example 2 but now we also include diffusion. Without diffusion the conclusion from the previous section was that f^n is q-recurrent if q was large enough. We now show that with diffusion, f^n is q-recurrent for all q. Along the way, we show the rate of decay of the Sobolev norm $\|f^n\|_{H^\beta}$ is independent of $\beta \in \mathbb{R}$.

Let

$$\gamma_k = \exp\big(-\kappa(2\pi k)^2\big) \tag{1.3.2}$$

and update the Fourier coefficients according to

$$f^{n+1}_k = \sum_\ell \gamma_k \widetilde{A}_{k\ell}\, f^n_\ell \tag{1.3.3}$$

	$k=1$	2	3	4	5	6	7	8	...
f_k^0	1								
f_k^1	$a\gamma_1$	$b\gamma_2$							
f_k^2	$a^2\gamma_1^2$	$a\gamma_1 b\gamma_2$		$b\gamma_2\gamma_4$					
f_k^3	$a^3\gamma_1^3$	$a^2\gamma_1^2 b\gamma_2$		$a\gamma_1 b\gamma_2\gamma_4$				$b\gamma_2\gamma_4\gamma_8$	
$\vdots$									

Table 1.2: Nonvanishing Fourier coefficients of f^n defined by Eq. (1.3.3), for $f^0(x) = 2\cos(2\pi x)$.

where $\widetilde{A}_{k\ell}$ is the matrix defined in Example 2. This matrix multiplication will result in *pulsed diffusion* with diffusion constant κ. We display the coefficients of f^n in Table 1.2.

The amount of mix-norm found on mode $k = 1$,

$$\mathcal{E}_1^n = \|P_{k=1} f^n\|_{\dot{H}^\beta}^2 = \sum_{k=1} k^{2\beta} |f_k^n|^2 = |f_1^n|^2 = (a\gamma_1)^{2n}, \tag{1.3.4}$$

is asymptotically equivalent to the amount found on modes $k > 1$ because

$$\begin{aligned}\mathcal{E}_{k>1}^n = \|P_{k>1} f^n\|_{\dot{H}^\beta}^2 &= \sum_{k>1} k^{2\beta} |f_k^n|^2 \\ &= \sum_{\ell=1}^{n} (2^\ell)^{2\beta} \left((a\gamma_1)^{n-\ell} b \ \Pi_{s=1}^{\ell} \gamma_{2^s} \right)^2 \\ &= (a\gamma_1)^{2n} \sum_{\ell=1}^{n} (2^\ell)^{2\beta} \left((a\gamma_1)^{-\ell} b \ \Pi_{s=1}^{\ell} \gamma_{2^s} \right)^2\end{aligned}$$

where the factor

$$c_{n,\beta,a,\kappa} := \sum_{\ell=1}^{n} (2^\ell)^{2\beta} \left((a\gamma_1)^{-\ell} b \ \Pi_{s=1}^{\ell} \gamma_{2^s} \right)^2 \tag{1.3.5}$$

limits to a finite constant $c_{\beta,a,\kappa}$ as $n \to \infty$. We see that $c_{n,\beta,a,\kappa}$ converges since the factor

$$\gamma_{2^\ell} = \exp\big(-\kappa(2\pi 2^\ell)^2\big) \tag{1.3.6}$$

dominates the terms in the sum to render the sum convergent. Lastly, notice that $c_{\beta,a,\kappa} \to \infty$ as $\beta \to \infty$ or $a \to 0$. We conclude that f^n is q-recurrent for all $q \in \mathbb{R}$ and, moreover, that all of the Sobolev norms decay at the same rate. △

Example 4 (Toral automorphisms and a q-recurrent trajectory)**.** In this example, for any for $q > 0$, we construct an initial condition so that the evolution under a toral automorphism is q-recurrent. A toral automorphism is a map of the form

$$\varphi(\boldsymbol{x}) = A\boldsymbol{x} \pmod{\mathbb{Z}^2}, \tag{1.3.7}$$

where $A \in \mathrm{SL}_2(\mathbb{Z})$ is an integer-valued 2×2 matrix with determinant 1 such as

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}. \tag{1.3.8}$$

The evolution of an initial function $f^0(\boldsymbol{x}) \in L^2(\mathbb{T}^2)$ under this map is given by $f^n(\boldsymbol{x}) = f^0(A^n\boldsymbol{x})$. Toral automorphisms and their mixing properties are well studied, see for example Sturman et al. [64]. The current presentation is inspired by the recent work of Feng and Iyer [23]. Since A is determinant 1,

$$\begin{aligned}
f^1_{\boldsymbol{k}} &= \int_{\mathbb{T}^2} e^{-2\pi i \boldsymbol{x}\cdot\boldsymbol{k}} f^1(\boldsymbol{x})\, d\boldsymbol{x} && (1.3.9)\\
&= \int_{\mathbb{T}^2} e^{-2\pi i \boldsymbol{x}\cdot\boldsymbol{k}} f^0(A\boldsymbol{x})\, d\boldsymbol{x} && (1.3.10)\\
&= \int_{\mathbb{T}^2} e^{-2\pi i A^{-1}\boldsymbol{y}\cdot\boldsymbol{k}} f^0(\boldsymbol{y})\, d\boldsymbol{y} && (1.3.11)\\
&= \int_{\mathbb{T}^2} e^{-2\pi i \boldsymbol{y}\cdot\left(A^{-1}\right)^T\boldsymbol{k}} f^0(\boldsymbol{y})\, d\boldsymbol{y} && (1.3.12)\\
&= f^0_{B\boldsymbol{k}}\,, && (1.3.13)
\end{aligned}$$

where $B = (A^{-1})^T$. We conclude that the Fourier coefficients of f^n transform as

$$f^n_{\boldsymbol{k}} = f^0_{B^n\boldsymbol{k}}\,. \tag{1.3.14}$$

Suppose $B \in \mathrm{SL}_2(\mathbb{Z})$ satisfies the following conditions.

(C1) No eigenvalue of B is a root of unity,

(C2) and the characteristic polynomial of B is irreducible over $\mathbb{Q}$.

Condition (C1) and $\det(B) = 1$ imply $|\lambda_1| > 1 > |\lambda_2| > 0$. Therefore, there is an expanding eigendirection and a shrinking eigendirection in $\mathbb{C}^2$. As a consequence, from Eq. (1.3.14), we see that the Fourier coefficients of the initial condition are shuffled hyperbolically.

Toral automorphisms under conditions (C1) and (C2) are known to be exponentially mixing [23]. We construct an initial condition (depending on $q > 0$) with Fourier coefficients aligned with the shrinking direction so that enough energy is injected to low-frequency modes and the resulting trajectory is q-recurrent. This example demonstrates that we may observe a q-recurrent trajectory even if the flow is an efficient mixer with an exponential mixing rate. In other words, q-recurrence does not a priori imply slow mixing.

We proceed with constructing the initial condition. Feng and Iyer [23] prove the following Lemma which guarantees every nonzero element of $\mathbb{Z}^2$ will have nonzero component in each eigendirection.

Lemma 1. *Suppose $B \in SL_2(\mathbb{Z})$ satisfies assumptions (C1) and (C2). There exists a basis $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ of $\mathbb{C}^2$ such that the following hold:*

1. *Each $\boldsymbol{v}_i$ is an eigenvector of B.*

2. *If $\boldsymbol{k} \in \mathbb{Z}^2 \setminus 0$, and $a, b \in \mathbb{C}$ are such that*

$$\boldsymbol{k} = a\boldsymbol{v}_1 + b\boldsymbol{v}_2\,, \tag{1.3.15}$$

then we must have

$$|ab| \geq 1\,. \tag{1.3.16}$$

We take $\boldsymbol{v}_i$ to be the eigenvector with eigenvalue λ_i, where we have $|\lambda_1| > 1 > |\lambda_2| > 0$. We say $\boldsymbol{v}_1$ is the expanding direction and $\boldsymbol{v}_2$ is the shrinking direction. The $\boldsymbol{v}_i$ are linearly independent since B is determinant 1, and so we may write $\boldsymbol{e}_1 = a\boldsymbol{v}_1 + b\boldsymbol{v}_2$ where $\boldsymbol{e}_1 = (1,0)^T$. Let $\mathbb{Z}_{>0}$ denote the set of positive integers $\{1, 2, 3, \dots\}$. Consider the (forward and backwards) orbit of $\boldsymbol{e}_1$ under B with step size $M \in \mathbb{Z}_{>0}$ given by

$$\boldsymbol{k}_m = B^{mM}\boldsymbol{e}_1 \text{ for } m \in \mathbb{Z}\,. \tag{1.3.17}$$

Lemma 1 guarantees that $\boldsymbol{e}_1$, and therefore $\boldsymbol{k}_m$, will have a nonzero component in the direction of $\boldsymbol{v}_1$. As a result, as $m \to \infty$, $\boldsymbol{k}_m$ will expand in the direction of $\boldsymbol{v}_1$ at a rate of $|\lambda_1|^{mM}$. Similarly, notice that B^{-1} has eigenvectors $\boldsymbol{v}_i$ with eigenvalues $\mu_i = 1/\lambda_i$. Therefore B^{-1} has expanding eigendirection $\boldsymbol{v}_2$. Additionally, B is determinate 1 and so $\lambda_1\lambda_2 = 1$. We see that, as $m \to -\infty$, $\boldsymbol{k}_m$ will grow in the direction of $\boldsymbol{v}_2$ at a rate of $|1/\lambda_2|^{|m|\,M} = |\lambda_1|^{|m|\,M}$. We may therefore approximate $\boldsymbol{k}_m$ in the following way.

Lemma 1 implies $|a| > 0$ and so for $m \geq 1$ we have

$$\boldsymbol{k}_m = a\lambda_1^{mM}\boldsymbol{v}_1 + b\lambda_2^{mM}\boldsymbol{v}_2 \underset{M \gg 1}{\approx} a\lambda_1^{mM}\boldsymbol{v}_1\,. \tag{1.3.18}$$

Let $M_1 \in \mathbb{Z}_{>0}$ be large enough so that for $M \geq M_1$ and $m \geq 1$ we have

$$\frac{|a|}{2}\,|\lambda_1|^{mM}|\boldsymbol{v}_1| \leq |\boldsymbol{k}_m| \leq 2|a|\,|\lambda_1|^{mM}|\boldsymbol{v}_1|\,. \tag{1.3.19}$$

Similarly, $|b| > 0$ and so for $m \leq -1$ we have

$$\boldsymbol{k}_m = a\left(\frac{1}{\lambda_1}\right)^{|m|\,M}\boldsymbol{v}_1 + b\left(\frac{1}{\lambda_2}\right)^{|m|\,M}\boldsymbol{v}_2 \underset{M \gg 1}{\approx} b\left(\frac{1}{\lambda_2}\right)^{|m|\,M}\boldsymbol{v}_2 = b\,\lambda_1^{|m|\,M}\boldsymbol{v}_2\,. \tag{1.3.20}$$

There is $M_2 \in \mathbb{Z}_{>0}$ large enough so that for $M \geq M_2$ and $m \leq -1$ we have

$$\frac{|b|}{2}\,|\lambda_1|^{|m|\,M}|\boldsymbol{v}_2| \leq |\boldsymbol{k}_m| \leq 2|b|\,|\lambda_1|^{|m|\,M}|\boldsymbol{v}_2|\,. \tag{1.3.21}$$

Notice that in the case $m = 0$, the magnitude of $\boldsymbol{k}_m$ is known: $|\boldsymbol{k}_0| = 1$. As a result, we have the following general bound. Let $M = \max(M_1, M_2)$ and take $C = \max(\, 2|a\boldsymbol{v}_1|\,,\, 2|b\boldsymbol{v}_2|\,,\, 2/|a\boldsymbol{v}_1|\,,\, 2/|b\boldsymbol{v}_2|\,,\, 1\,)$. For all $m \in \mathbb{Z}$ we have the bound

$$\frac{1}{C}|\lambda_1|^{|m|\,M} \leq |\boldsymbol{k}_m| \leq C|\lambda_1|^{|m|\,M}\,. \tag{1.3.22}$$

We take the initial condition to be the function with the following Fourier coefficients

$$f^0_{\boldsymbol{k}} = \begin{cases} |\lambda_1|^{-\alpha m} & \text{if } \boldsymbol{k} = \boldsymbol{k}_m \text{ for some } m \geq 0\,, \\ 0 & \text{otherwise}\,, \end{cases} \tag{1.3.23}$$

where the parameter α is some constant greater than zero. Notice that $f^0 \in L^2(\mathbb{T}^2)$ for $\alpha > 0$ since

$$\left\|f^0\right\|^2_{\dot{H}^\beta} = \sum_{\boldsymbol{k}} |\boldsymbol{k}|^0 |f^0_{\boldsymbol{k}}|^2 \tag{1.3.24}$$

$$= \sum_{m\geq 0} |\lambda_1|^{-2\alpha m} \tag{1.3.25}$$

$$< \infty \quad \text{if } \alpha > 0\,. \tag{1.3.26}$$

We will now show that f^n is q-recurrent for q large enough. Using Eq. (1.3.14), we see that the contribution to the mix-norm from mode $\boldsymbol{k}_0 = \boldsymbol{e}_1$ at time $t = nM$ is

$$\begin{aligned}\left\|P_{\boldsymbol{k}=\boldsymbol{e}_1} f^{nM}\right\|_{\dot{H}^{-q}}^2 &= \sum_{\boldsymbol{k}=\boldsymbol{e}_1} |\boldsymbol{k}|^{-2q} \left|f_{\boldsymbol{k}}^{nM}\right|^2 &(1.3.27)\\ &= \left|f_{\boldsymbol{e}_1}^{nM}\right|^2 &(1.3.28)\\ &= \left|f_{B^{nM}\boldsymbol{e}_1}^{0}\right|^2 &(1.3.29)\\ &= \left|f_{\boldsymbol{k}_n}^{0}\right|^2 &(1.3.30)\\ &= |\lambda_1|^{-2\alpha n}. &(1.3.31)\end{aligned}$$

We now compare this to the decay rate of the mix-norm. We proceed by bounding the mix-norm from above at time $t = nM$. Using Eq. (1.3.14) we have

$$\begin{aligned}\left\|f^{nM}\right\|_{\dot{H}^{-q}}^2 &= \sum |\boldsymbol{k}|^{-2q} \left|f_{\boldsymbol{k}}^{nM}\right|^2 &(1.3.32)\\ &= \sum |\boldsymbol{k}|^{-2q} \left|f_{B^{nM}\boldsymbol{k}}^{0}\right|^2. &(1.3.33)\end{aligned}$$

Notice that $B^{nM}\boldsymbol{k} = \boldsymbol{k}_m$ when $\boldsymbol{k} = B^{-nM}\boldsymbol{k}_m = B^{-nM}B^{mM}\boldsymbol{e}_1 = \boldsymbol{k}_{m-n}$. We have

$$\begin{aligned}\left\|f^{nM}\right\|_{\dot{H}^{-q}}^2 &= \sum_{m\geq 0} |\boldsymbol{k}_{m-n}|^{-2q} \left|f_{\boldsymbol{k}_m}^{0}\right|^2\\ &\leq C^{2|q|} \sum_{m\geq 0} \left(|\lambda_1|^{|m-n|\,M}\right)^{-2q} |\lambda_1|^{-2\alpha m}\\ &= C^{2|q|} \sum_{m=0}^{n-1} \left(|\lambda_1|^{(n-m)M}\right)^{-2q} |\lambda_1|^{-2\alpha m} + C^{2|q|} \sum_{m\geq n} \left(|\lambda_1|^{(m-n)M}\right)^{-2q} |\lambda_1|^{-2\alpha m}\\ &= C^{2|q|} \sum_{m=0}^{n-1} |\lambda_1|^{-2q(n-m)M-2\alpha m} + C^{2|q|} \sum_{m\geq n} |\lambda_1|^{-2q(m-n)M-2\alpha m}\end{aligned}$$

$$
\begin{aligned}
&= C^{2|q|}|\lambda_1|^{-2qnM}\sum_{m=0}^{n-1}|\lambda_1|^{2qmM-2\alpha m} + C^{2|q|}|\lambda_1|^{2qnM}\sum_{m\geq n}|\lambda_1|^{-2qmM-2\alpha m} \\
&= C^{2|q|}|\lambda_1|^{-2qnM}\sum_{m=0}^{n-1}\left(|\lambda_1|^{2(qM-\alpha)}\right)^m + C^{2|q|}|\lambda_1|^{2qnM}\sum_{m\geq n}\left(|\lambda_1|^{-2(qM+\alpha)}\right)^m \\
&= C^{2|q|}|\lambda_1|^{-2qnM}\frac{1-\left(|\lambda_1|^{2(qM-\alpha)}\right)^n}{1-|\lambda_1|^{2(qM-\alpha)}} + C^{2|q|}|\lambda_1|^{2qnM}\frac{\left(|\lambda_1|^{-2(qM+\alpha)}\right)^n}{1-|\lambda_1|^{-2(qM+\alpha)}} \\
&= |\lambda_1|^{-2\alpha n}\,c(n)\,,
\end{aligned}
$$

where

$$
c(n) = C^{2|q|}\left(\frac{\left(|\lambda_1|^{-2(qM-\alpha)}\right)^n - 1}{1-|\lambda_1|^{2(qM-\alpha)}} + \frac{1}{1-|\lambda_1|^{-2(qM+\alpha)}}\right). \tag{1.3.34}
$$

Notice that $c(n)$ converges to a finite constant as $n \to \infty$ if and only if $(qM - \alpha) \geq 0$. We conclude that f^n is q-recurrent for $q \geq \alpha/M$. Since α was an arbitrary constant greater than zero and independent of M, we conclude that for any $q > 0$ there is an initial condition f^0 so that the trajectory f^n is q-recurrent.

△

Example 5 (sine flow)**.** Lastly we consider a computational example, the random sine flow, which is a simple model flow that is empirically quite efficient at mixing [61, 68]. The sine flow is a two-dimensional time-periodic flow with a full period consisting of the shear flow

$$
\boldsymbol{u}_1(t,x) = \sqrt{2}\,(0\,,\,\sin(2\pi x + \psi_1)), \qquad 0 \leq t < 1/2, \tag{1.3.35a}
$$

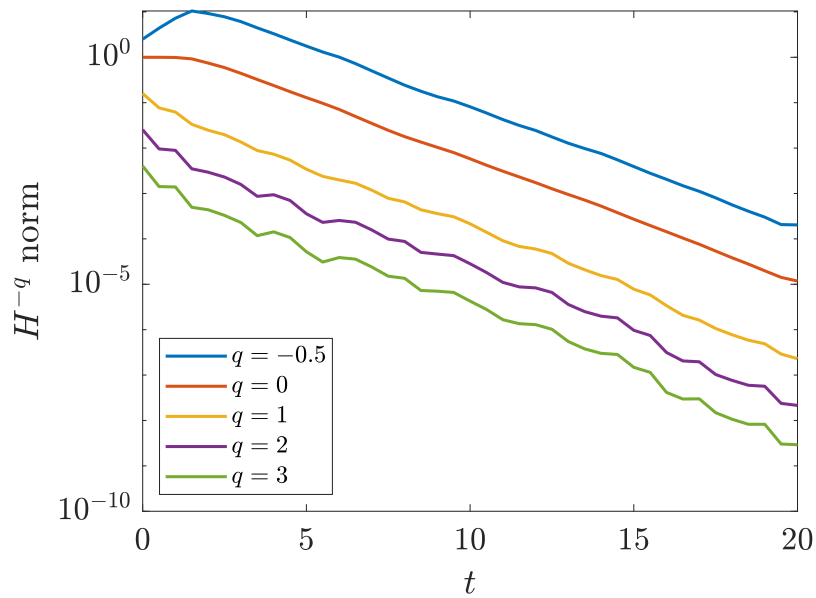

Figure 1.4: For the advection-diffusion equation (1.1.1) with $\boldsymbol{u}$ given by the random sine flow (1.3.35), the rate of decay of the mix-norms is independent of q. The initial condition is $f^0(x) = \sqrt{2}\cos(2\pi x)$, and the diffusivity is $D = 10^{-5}$.

followed by

$$\boldsymbol{u}_2(t, y) = \sqrt{2}\,(\sin(2\pi y + \psi_2)\,,\, 0), \qquad 1/2 \leq t < 1, \tag{1.3.35b}$$

with $(x, y) \in [0, 1]^2$ and periodic spatial boundary conditions. Here ψ_1 and ψ_2 are random phases, uniformly distributed in $[0, 2\pi]$, chosen independently at every period. Unlike the pulsed diffusion in Example 3, diffusion acts continuously by solving the advection–diffusion equation (1.1.1) with diffusivity $D = 10^{-5}$. We display $\|f^t\|_{H^{-q}}$ for various q in Fig. 1.4, for initial condition $f^0(x) = \sqrt{2}\cos(2\pi x)$, and observe that the mix-norms all decay at the same rate, at least within numerical fluctuations. $\triangle$

In general, if f^t is q-recurrent then the decay rate of the mix-norm is independent of q in the following sense:

Theorem 3. If f^t is q-recurrent, then it is also q'-recurrent for any $q' > q$. Moreover, we have

$$\limsup_{t\to\infty} \frac{\|f^t\|_{\dot{H}^{-q'}}}{\|f^t\|_{\dot{H}^{-q}}} > 0\,.$$

Then together with the trivial estimate

$$\left\|f^t\right\|_{\dot{H}^{-q'}} \leq \left\|f^t\right\|_{\dot{H}^{-q}} \tag{1.3.36}$$

we conclude that $\|f^t\|_{\dot{H}^{-q'}}$ is Big-O but not Little-O of $\|f^t\|_{\dot{H}^{-q}}$.

Proof. Since f^t is q-recurrent, there is a finite set I such that

$$0 < \limsup_{t\to\infty} \frac{\|P_I f^t\|_{\dot{H}^{-q}}}{\|f^t\|_{\dot{H}^{-q}}}\,. \tag{1.3.37}$$

Say $I \subset [-R, R]$ for some $R \in \mathbb{N}$. Then

$$\left\|P_I f^t\right\|_{\dot{H}^{-q}} \leq R^{(q'-q)}\left\|P_I f^t\right\|_{\dot{H}^{-q'}}\,. \tag{1.3.38}$$

Putting together Eqs. (1.3.36) to (1.3.38) we obtain

$$0 < R^{(q'-q)} \limsup \frac{\|P_I f^t\|_{\dot{H}^{-q'}}}{\|f^t\|_{\dot{H}^{-q'}}}.$$

We conclude f^t is q'-recurrent. Moreover, the trivial estimate $\|P_I f^t\|_{\dot{H}^{-q'}} \leq \|f^t\|_{\dot{H}^{-q'}}$ together with Eqs. (1.3.37) and (1.3.38) imply

$$0 < R^{(q'-q)} \limsup \frac{\|f^t\|_{\dot{H}^{-q'}}}{\|f^t\|_{\dot{H}^{-q}}}. \tag{1.3.39}$$

□

One question for further investigation is whether a converse to the above theorem exists. That is, can we conclude f^t is q-recurrent for a range of q if the mix-norms decay at the same rate for the those q? Another question concerns the transition from q-transient to q-recurrent when including pulsed diffusion. Does q-recurrence imply an introduction of the Batchelor scale and anomalous dissipation [5, 52, 53]?

1.4 Proof of Theorem 1

We begin by generalizing the definition of q-recurrent functions to the notion of '(q, h)-recurrent' functions.

Definition 2. For positive functions $h(t)$, we say f^t is (q, h)-recurrent if there exists a finite set $I \subset \mathbb{Z}^d$ such that

$$\limsup_{t\to\infty} \frac{\|P_I f^t\|_{\dot{H}^{-q}}}{h} > 0. \tag{1.4.1}$$

Functions that are not (q, h)-recurrent are called (q, h)-transient.

Lemma 2. If f^t is (q, h)-recurrent, then there is a function $g \in \dot{H}^q$ such that

$$\limsup_{t\to\infty} \frac{|\langle f^t\, ,\, g\rangle|}{h} > 0.$$

Moreover, $g \in \dot{H}^\beta$ for any $\beta \in \mathbb{R}$ with

$$\|g\|_{\dot{H}^\beta}^2 = 2 \sum_{\boldsymbol{k} \in I} k^{2(\beta - q)}. \tag{1.4.2}$$

Proof. There exists a constant $c > 0$ and a sequence of times $t_m \to \infty$ where

$$\sum_{\boldsymbol{k} \in I} |f_{\boldsymbol{k}}^{t_m}|^2 \, k^{-2q} \geq c^2 h^2(t_m). \tag{1.4.3}$$

Recall the signum function

$$\operatorname{sgn} x = \begin{cases} 1, & x \geq 0; \\ -1, & x < 0. \end{cases} \tag{1.4.4}$$

Now notice that for each fixed time t_m, $\{f_{\boldsymbol{k}}^{t_m}\}_{\boldsymbol{k} \in I}$ is a list of $|I|$ numbers in $\mathbb{C}$. Write $f_{\boldsymbol{k}}^{t_m} = a_{\boldsymbol{k}}^{t_m} + i b_{\boldsymbol{k}}^{t_m}$ where $a_{\boldsymbol{k}}^{t_m}$ and $b_{\boldsymbol{k}}^{t_m}$ are real. Then $f_{\boldsymbol{k}}^{t_m}$ is found in one of the four quadrants of the complex plane, depending on the two possibilities for $\operatorname{sgn} a_{\boldsymbol{k}}^{t_m}$ and two possibilities for $\operatorname{sgn} b_{\boldsymbol{k}}^{t_m}$. Thus, $\{f_{\boldsymbol{k}}^{t_m}\}_{\boldsymbol{k} \in I}$ has $4^{|I|}$ possible states. Since we have an infinite sequence of times t_m, one of these states must occur infinitely many times. By taking a subsequence t_{m_ℓ}, we can ensure $\{f_{\boldsymbol{k}}^{t_{m_\ell}}\}_{\boldsymbol{k} \in I}$ is the same state for all ℓ. Let $\{(c_{\boldsymbol{k}}, d_{\boldsymbol{k}})\}_{\boldsymbol{k} \in I}$ encode this state, meaning that $c_{\boldsymbol{k}} = \operatorname{sgn} a_{\boldsymbol{k}}^{t_{m_\ell}}$ and $d_{\boldsymbol{k}} = \operatorname{sgn} b_{\boldsymbol{k}}^{t_{m_\ell}}$ for all ℓ. We see that $a_{\boldsymbol{k}}^{t_{m_\ell}} c_{\boldsymbol{k}} = \left|a_{\boldsymbol{k}}^{t_{m_\ell}}\right|$ and $b_{\boldsymbol{k}}^{t_{m_\ell}} d_{\boldsymbol{k}} = \left|b_{\boldsymbol{k}}^{t_{m_\ell}}\right|$ for all ℓ. Let

$$g_{\boldsymbol{k}} = \begin{cases} (c_{\boldsymbol{k}} + i d_{\boldsymbol{k}}) k^{-q}, & \boldsymbol{k} \in I; \\ 0, & \text{otherwise}. \end{cases} \tag{1.4.5}$$

Notice that $g \in \dot{H}^\beta$ because

$$\|g\|_{\dot{H}^\beta}^2 = \sum |g_{\boldsymbol{k}}|^2 \, k^{2\beta} = \sum_{\boldsymbol{k}\in I} (|c_{\boldsymbol{k}}|^2 + |d_{\boldsymbol{k}}|^2) k^{-2q} k^{2\beta} = 2 \sum_{\boldsymbol{k}\in I} k^{2(\beta - q)} < \infty \tag{1.4.6}$$

since I is a finite set. We have

$$\begin{aligned}
\sum f_{\boldsymbol{k}}^{t_{m_\ell}} \, \bar{g}_{\boldsymbol{k}} &= \sum_{\boldsymbol{k}\in I} \left(a_{\boldsymbol{k}}^{t_{m_\ell}} + i b_{\boldsymbol{k}}^{t_{m_\ell}} \right) (c_{\boldsymbol{k}} - i d_{\boldsymbol{k}}) \; k^{-q} \\
&= \sum_{\boldsymbol{k}\in I} \left(a_{\boldsymbol{k}}^{t_{m_\ell}} c_{\boldsymbol{k}} + b_{\boldsymbol{k}}^{t_{m_\ell}} d_{\boldsymbol{k}} + i \left(b_{\boldsymbol{k}}^{t_{m_\ell}} c_{\boldsymbol{k}} - a_{\boldsymbol{k}}^{t_{m_\ell}} d_{\boldsymbol{k}} \right) \right) \; k^{-q} \\
&= \sum_{\boldsymbol{k}\in I} \left(\left| a_{\boldsymbol{k}}^{t_{m_\ell}} \right| + \left| b_{\boldsymbol{k}}^{t_{m_\ell}} \right| + i \left(b_{\boldsymbol{k}}^{t_{m_\ell}} c_{\boldsymbol{k}} - a_{\boldsymbol{k}}^{t_{m_\ell}} d_{\boldsymbol{k}} \right) \right) \; k^{-q}.
\end{aligned}$$

We conclude that

$$\begin{aligned}
\left| \left\langle f^{t_{m_\ell}} \, , \, g \right\rangle \right| &\geq \operatorname{Re} \sum f_{\boldsymbol{k}}^{t_{m_\ell}} \, \bar{g}_{\boldsymbol{k}} \\
&= \sum_{\boldsymbol{k}\in I} \left(\left| a_{\boldsymbol{k}}^{t_{m_\ell}} \right| + \left| b_{\boldsymbol{k}}^{t_{m_\ell}} \right| \right) \; k^{-q} \\
&\geq \sum_{\boldsymbol{k}\in I} \sqrt{ \left| a_{\boldsymbol{k}}^{t_{m_\ell}} \right|^2 + \left| b_{\boldsymbol{k}}^{t_{m_\ell}} \right|^2 } \; k^{-q} \\
&= \sum_{\boldsymbol{k}\in I} \left| f_{\boldsymbol{k}}^{t_{m_\ell}} \right| \; k^{-q},
\end{aligned}$$

as desired. Lastly, we use dominance of ℓ^2 by ℓ^1 together with Eq. (1.4.3) to conclude $|\langle f^{t_{m_\ell}} \, , \, g\rangle| \geq c \, h(t_{m_\ell})$, $\forall m_\ell$. □

Lemma 2 characterizes the behavior of (q, h)-recurrent functions. We now develop the tool we need to further analyze (q, h)-transient functions.

Lemma 3. Let f^t be (q,h)-transient for some positive $h = \mathcal{O}(\|f^t\|_{\dot{H}^{-q}})$. For any δ with $0 < \delta < 1$, there exist sets $I_i = \{\boldsymbol{k} \mid J_{i-1} < |\boldsymbol{k}| \leq J_i\}$ with $J_0 = -1$ and a sequence of times $T_i \to \infty$ satisfying the following.

(i). The set I_i captures a significant proportion of the Fourier energy at time T_i, so that

$$\sum_{\boldsymbol{k}\in I_i} \left|f_{\boldsymbol{k}}^{T_i}\right|^2 k^{-2q} \geq (1-\delta)\left\|f^{T_i}\right\|_{\dot{H}^{-q}}^2 . \tag{1.4.7}$$

(ii). Enough of the Fourier energy does not return to lower frequency modes, so that

$$\sum_{|\boldsymbol{k}|\leq J_{i-1}} \left|f_{\boldsymbol{k}}^{t}\right|^2 k^{-2q} \leq \delta\, h^2(t) \text{ for all } t \geq T_i . \tag{1.4.8}$$

Proof. Base case: $i = 1$. Since (by definition of absolute convergence)

$$\lim_{J\to\infty} \sum_{|\boldsymbol{k}|\leq J} \left|f_{\boldsymbol{k}}^{0}\right|^2 k^{-2q} = \left\|f^{0}\right\|_{\dot{H}^{-q}}^2 \tag{1.4.9}$$

there is a J_1 such that

$$\sum_{|\boldsymbol{k}|\leq J_1} \left|f_{\boldsymbol{k}}^{0}\right|^2 k^{-2q} \geq (1-\delta)\left\|f^{0}\right\|_{\dot{H}^{-q}}^2 . \tag{1.4.10}$$

We see $I_1 = \{\boldsymbol{k} \mid J_0 < |\boldsymbol{k}| \leq J_1\}$ where $J_0 = -1$ and $T_1 = 0$ therefore trivially satisfy Eqs. (1.4.7) and (1.4.8).

Induction step: Suppose that we are given J_{i-2}, J_{i-1} and T_{i-1} that satisfy Eqs. (1.4.7) and (1.4.8). Since $h = \mathcal{O}(\|f^t\|_{\dot{H}^{-q}})$, there is a $c > 0$ and a time T so that $h(t) \leq$

$c\|f^t\|_{\dot{H}^{-q}}\ \forall t \geq T$.

Recall that f^t is (q,h)-transient and so, for any finite set I,

$$\limsup_{t\to\infty} \frac{\|P_I f^t\|_{\dot{H}^{-q}}}{h} = 0. \tag{1.4.11}$$

That is, given finite set I, then for any constant ϵ there is T_ϵ so that

$$\sum_{\boldsymbol{k}\in I} \left|f^t_{\boldsymbol{k}}\right|^2 k^{-2q} \leq \epsilon\, h^2(t) \text{ for all } t \geq T_\epsilon\,. \tag{1.4.12}$$

Take $I = \{\boldsymbol{k} \text{ s.t. } |\boldsymbol{k}| \leq J_{i-1}\}$ and $\epsilon = \min(\delta, \delta/c^2)$. We conclude that there exists T_i with $T_i \geq T_{i-1}+1$ and $T_i \geq T$ such that Eq. (1.4.8) is satisfied. Moreover, we have

$$\sum_{|\boldsymbol{k}|\leq J_{i-1}} \left|f^{T_i}_{\boldsymbol{k}}\right|^2 k^{-2q} < \left(\delta/c^2\right)\, h^2(T_i) \leq \delta\left\|f^{T_i}\right\|^2_{\dot{H}^{-q}}\,. \tag{1.4.13}$$

Hence,

$$\sum_{|\boldsymbol{k}|> J_{i-1}} \left|f^{T_i}_{\boldsymbol{k}}\right|^2 k^{-2q} > (1-\delta)\left\|f^{T_i}\right\|^2_{\dot{H}^{-q}}. \tag{1.4.14}$$

That is,

$$\lim_{J\to\infty} \sum_{J_{i-1}<|\boldsymbol{k}|\leq J} \left|f^{T_i}_{\boldsymbol{k}}\right|^2 k^{-2q} = C\left\|f^{T_i}\right\|^2_{\dot{H}^{-q}}. \tag{1.4.15}$$

where $C > (1-\delta)$. From the definition of the limit, it follows that there is a J_i large enough so that for $I_i = \{\boldsymbol{k} \,|\, J_{i-1} < |\boldsymbol{k}| < J_i\}$ we have

$$\sum_{\boldsymbol{k}\in I_i} |f^{T_i}_{\boldsymbol{k}}|^2\, k^{-2q} \geq (1-\delta)\left\|f^{T_i}\right\|^2_{\dot{H}^{-q}} \tag{1.4.16}$$

which is Eq. (1.4.7). □

Having developed all of the tools we will need, we now prove Theorem 1 and, in the next section, Theorem 2.

Theorem 1. Let f^t be a mean-zero function in $L^2(\mathbb{T}^d)$ with $\|f^t\|_{\dot{H}^{-q}} > 0$ for all $t > 0$. Then f^t is q-recurrent if and only if there is a function $g \in \dot{H}^q$ such that

$$\limsup_{t\to\infty} \frac{|\langle f^t\,,\,g\rangle|}{\|f^t\|_{\dot{H}^{-q}}} > 0.$$

Proof. The forward direction is a special case of Lemma 2 with $h(t) = \|f^t\|_{\dot{H}^{-q}}$. We assume f^t is q-transient and show $\langle f^t\,,\,g\rangle = o(\|f^t\|_{\dot{H}^{-q}})$ for all $g \in \dot{H}^q$. We already know $\langle f^t\,,\,g\rangle = \mathcal{O}(\|f^t\|_{\dot{H}^{-q}})$ for all $g \in \dot{H}^q$, so

$$\limsup_{t\to\infty} \frac{|\langle f^t\,,\,g\rangle|}{\|f^t\|_{\dot{H}^{-q}}} < \infty, \quad \forall g \in \dot{H}^q. \tag{1.4.17}$$

Seeking a contradiction, we suppose there exists a $g \in \dot{H}^q$ such that

$$\limsup_{t\to\infty} \frac{|\langle f^t\,,\,g\rangle|}{\|f^t\|_{\dot{H}^{-q}}} = C > 0. \tag{1.4.18}$$

There is a sequence $t_n \to \infty$ such that

$$\big|\langle f^{t_n}\,,\,g\rangle\big| \geq \frac{C}{2}\big\|f^{t_n}\big\|_{\dot{H}^{-q}}, \quad \forall n. \tag{1.4.19}$$

We will show that Eq. (1.4.19) implies that the Fourier coefficients of g decay too slowly for g to be in $\dot{H}^q$. Since $g \in \dot{H}^q$, we can choose δ small enough that

$\frac{C}{2} - 2\sqrt{\delta}\|g\|_{\dot{H}^q} = C_0 > 0$.

Applying Lemma 3 with $h(t) = \|f^t\|_{\dot{H}^{-q}}$, there exist sets $I_i = \{\boldsymbol{k} \mid J_{i-1} < |\boldsymbol{k}| \leq J_i\}$ and a sequence of times $T_i \to \infty$ (without loss of generality, say that $\{T_i\}$ is a subsequence of $\{t_n\}$ above) such that we have Eqs. (1.4.7) and (1.4.8). Equation (1.4.7) implies

$$\sum_{|\boldsymbol{k}|>J_i} \left|f_{\boldsymbol{k}}^{T_i}\right|^2 k^{-2q} \leq \delta \left\|f^{T_i}\right\|_{\dot{H}^{-q}}^2 . \tag{1.4.20}$$

Note that

$$\begin{aligned}\left\langle f^{T_i}\,,\, g\right\rangle &= \sum_{j\geq 1}\sum_{\boldsymbol{k}\in I_j} f_{\boldsymbol{k}}^{T_i}\;\bar{g}_{\boldsymbol{k}} \\ &= \sum_{\boldsymbol{k}\in I_i} f_{\boldsymbol{k}}^{T_i}\;\bar{g}_{\boldsymbol{k}} + E \end{aligned} \tag{1.4.21}$$

and we can bound the error using the Cauchy–Schwarz inequality

$$E = \sum_{\boldsymbol{k}\notin I_i} f_{\boldsymbol{k}}^{T_i}\;\bar{g}_{\boldsymbol{k}} \leq \left(\sum_{\boldsymbol{k}\notin I_i}\left|f_{\boldsymbol{k}}^{T_i}\right|^2 k^{-2q}\right)^{1/2}\left(\sum_{\boldsymbol{k}\notin I_i}|g_{\boldsymbol{k}}|^2\, k^{2q}\right)^{1/2} . \tag{1.4.22}$$

Applying Eq. (1.4.7) of Lemma 3, we have

$$|E| \leq \sqrt{\delta}\left\|f^{T_i}\right\|_{\dot{H}^{-q}}\|g\|_{\dot{H}^q} . \tag{1.4.23}$$

and therefore

$$\left|\left\langle f^{T_i}\,,\, g\right\rangle\right| \leq \left|\sum_{\boldsymbol{k}\in I_i} f_{\boldsymbol{k}}^{T_i}\;\bar{g}_{\boldsymbol{k}}\right| + \sqrt{\delta}\left\|f^{T_i}\right\|_{\dot{H}^{-q}}\|g\|_{\dot{H}^q} . \tag{1.4.24}$$

Putting together Eqs. (1.4.19) and (1.4.24), we have

$$\left(\frac{C}{2} - \sqrt{\delta}\|g\|_{\dot{H}^q}\right)\left\|f^{T_i}\right\|_{\dot{H}^{-q}} \leq \left|\sum_{\boldsymbol{k}\in I_i} f_{\boldsymbol{k}}^{T_i}\ \bar{g}_{\boldsymbol{k}}\right|. \tag{1.4.25}$$

Since $g \in \dot{H}^q$, we can choose δ small enough that

$$C_0 := \left(\frac{C}{2} - \sqrt{\delta}\|g\|_{\dot{H}^q}\right) > 0\,. \tag{1.4.26}$$

Applying Cauchy–Schwarz to the right-hand side of Eq. (1.4.25) we have

$$C_0\left\|f^{T_i}\right\|_{\dot{H}^{-q}} \leq \left(\sum_{\boldsymbol{k}\in I_i}\left|f_{\boldsymbol{k}}^{T_i}\right|^2 k^{-2q}\right)^{1/2}\left(\sum_{\boldsymbol{k}\in I_i}|g_{\boldsymbol{k}}|^2\, k^{2q}\right)^{1/2} \tag{1.4.27}$$

$$\leq \left\|f^{T_i}\right\|_{\dot{H}^{-q}}\left(\sum_{\boldsymbol{k}\in I_i}|g_{\boldsymbol{k}}|^2\, k^{2q}\right)^{1/2}. \tag{1.4.28}$$

Therefore

$$\sum_{\boldsymbol{k}\in I_i}|g_{\boldsymbol{k}}|^2\, k^{2q} \geq C_0^2, \quad \forall i\,. \tag{1.4.29}$$

This shows that the coefficients of g are large on sets I_i and we have

$$\|g\|_{\dot{H}^q}^2 = \sum_i \sum_{\boldsymbol{k}\in I_i}|g_{\boldsymbol{k}}|^2\, k^{2q} \geq \sum_i C_0^2 = \infty. \tag{1.4.30}$$

We conclude that g is not in $\dot{H}^q$ — a contradiction. □

1.5 Proof of Theorem 2

Theorem 2. Let f^t be a mean-zero function in $L^2(\mathbb{T}^d)$ with $\|f^t\|_{\dot{H}^{-q}} > 0$ for all $t > 0$. For any positive function $h(t)$ such that $h(t) = o(\|f^t\|_{\dot{H}^{-q}})$, there is a function $g \in \dot{H}^q$ such that

$$\limsup_{t\to\infty} \frac{|\langle f^t\,,\, g\rangle|}{h(t)} > 0.$$

Proof of Theorem 2. If f^t is (q,h)-recurrent, then we are done by Lemma 2, so say f^t is (q,h)-transient. Take some $\delta < 1/3$ and apply Lemma 3 to construct sets $\{I_i\}_{i=1}^\infty$ and a sequence $\{T_i\}_{i=1}^\infty$. Let $\{T_{i_\ell}\}_{\ell=1}^\infty$ be a subsequence of $\{T_i\}_{i=1}^\infty$ satisfying

$$\sum_{\ell>L}\left(\frac{h(T_{i_\ell})}{\left\|f^{T_{i_\ell}}\right\|_{\dot{H}^{-q}}}\right)^2 \le \delta^2\left(\frac{h(T_{i_L})}{\left\|f^{T_{i_L}}\right\|_{\dot{H}^{-q}}}\right)^2. \tag{1.5.1}$$

and

$$\sum\left(\frac{h(T_{i_\ell})}{\left\|f^{T_{i_\ell}}\right\|_{\dot{H}^{-q}}}\right)^2 \le \delta. \tag{1.5.2}$$

This can be done since $h(t) = o(\|f^t\|_{\dot{H}^{-q}})$. Let g be the function with Fourier coefficients given by

$$g_{\boldsymbol{k}} = \begin{cases} f_{\boldsymbol{k}}^{T_{i_\ell}} k^{-2q}\left\|f^{T_{i_\ell}}\right\|_{\dot{H}^{-q}}^{-2} h(T_{i_\ell}) & \boldsymbol{k} \in I_{i_\ell}\,; \\ 0 & \text{otherwise.} \end{cases} \tag{1.5.3}$$

Eq. (1.5.2) allows us to conclude that $g \in \dot{H}^q$:

$$\begin{aligned}
\|g\|^2_{\dot{H}^q} &= \sum |g_{\boldsymbol{k}}|^2 \, k^{2q} \\
&= \sum_{i_\ell} \sum_{\boldsymbol{k} \in I_{i_\ell}} \left| f^{T_{i_\ell}}_{\boldsymbol{k}} \right|^2 k^{-4q} \left\| f^{T_{i_\ell}} \right\|^{-4}_{\dot{H}^{-q}} h^2(T_{i_\ell}) \; k^{2q} \\
&= \sum_{i_\ell} \left\| f^{T_{i_\ell}} \right\|^{-4}_{\dot{H}^{-q}} h^2(T_{i_\ell}) \left(\sum_{\boldsymbol{k} \in I_{i_\ell}} \left| f^{T_{i_\ell}}_{\boldsymbol{k}} \right|^2 k^{-2q} \right) \\
&\leq \sum_{i_\ell} \left\| f^{T_{i_\ell}} \right\|^{-2}_{\dot{H}^{-q}} h^2(T_{i_\ell}).
\end{aligned}$$

We will now finish the proof by showing $\left| \left\langle f^{T_{i_\ell}} \, , \, g \right\rangle \right| \geq (1 - 3\delta) \; h(T_{i_\ell})$. We begin with some notation. Split the following sum into two parts:

$$\begin{aligned}
\left\langle f^{T_{i_\ell}} \, , \, g \right\rangle &= \sum_{j_\ell} \sum_{\boldsymbol{k} \in I_{j_\ell}} f^{T_{i_\ell}}_{\boldsymbol{k}} \; \overline{f^{T_{j_\ell}}_{\boldsymbol{k}}} k^{-2q} \left\| f^{T_{j_\ell}} \right\|^{-2}_{\dot{H}^{-q}} h(T_{j_\ell}) \\
&= S^{T_{i_\ell}} + E^{T_{i_\ell}},
\end{aligned}$$

where $S^{T_{i_\ell}}$ is the sum when $j_\ell = i_\ell$:

$$\begin{aligned}
S^{T_{i_\ell}} &= \sum_{j_\ell = i_\ell} \sum_{\boldsymbol{k} \in I_{i_\ell}} f^{T_{i_\ell}}_{\boldsymbol{k}} \; \overline{f^{T_{j_\ell}}_{\boldsymbol{k}}} k^{-2q} \left\| f^{T_{j_\ell}} \right\|^{-2}_{\dot{H}^{-q}} h(T_{j_\ell}) \\
&= \sum_{\boldsymbol{k} \in I_{i_\ell}} \left| f^{T_{i_\ell}}_{\boldsymbol{k}} \right|^2 k^{-2q} \; \left\| f^{T_{i_\ell}} \right\|^{-2}_{\dot{H}^{-q}} \; h(T_{i_\ell}),
\end{aligned}$$

and $E^{T_{i_\ell}}$ is the sum over $j_\ell \neq i_\ell$:

$$E^{T_{i_\ell}} = \sum_{j_\ell \neq i_\ell} \sum_{\boldsymbol{k} \in I_{j_\ell}} f_{\boldsymbol{k}}^{T_{i_\ell}} \, \overline{f_{\boldsymbol{k}}^{T_{j_\ell}}} k^{-2q} \big\| f^{T_{j_\ell}} \big\|_{\dot{H}^{-q}}^{-2} h(T_{j_\ell}) \,. \tag{1.5.4}$$

The idea is that $g_{\boldsymbol{k}}$ is constructed to agree well with $f_{\boldsymbol{k}}^{T_{i_\ell}}$ when $\boldsymbol{k} \in I_{i_\ell}$. We will show that $S^{T_{i_\ell}}$ dominates the error $E^{T_{i_\ell}}$. Consider $j_\ell < i_\ell$ and $j_\ell > i_\ell$ separately in Eq. (1.5.4); taking absolute value, we have

$$\left| E^{T_{i_\ell}} \right| \leq \sum_{j_\ell \neq i_\ell} \sum_{\boldsymbol{k} \in I_{j_\ell}} \left| f_{\boldsymbol{k}}^{T_{i_\ell}} \right| \left| f_{\boldsymbol{k}}^{T_{j_\ell}} \right| \; k^{-2q} \; \big\| f^{T_{j_\ell}} \big\|_{\dot{H}^{-q}}^{-2} \; h(T_{j_\ell}) \tag{1.5.5}$$

and let $E_1^{T_{i_\ell}}$ be the sum over $j_\ell < i_\ell$:

$$E_1^{T_{i_\ell}} := \sum_{j_\ell < i_\ell} \sum_{\boldsymbol{k} \in I_{j_\ell}} \left| f_{\boldsymbol{k}}^{T_{i_\ell}} \right| \left| f_{\boldsymbol{k}}^{T_{j_\ell}} \right| \; k^{-2q} \; \big\| f^{T_{j_\ell}} \big\|_{\dot{H}^{-q}}^{-2} \; h(T_{j_\ell}) \,. \tag{1.5.6}$$

Similarly define $E_2^{T_{i_\ell}}$ to be the sum over $j_\ell > i_\ell$:

$$E_2^{T_{i_\ell}} := \sum_{j_\ell > i_\ell} \sum_{\boldsymbol{k} \in I_{j_\ell}} \left| f_{\boldsymbol{k}}^{T_{i_\ell}} \right| \left| f_{\boldsymbol{k}}^{T_{j_\ell}} \right| \; k^{-2q} \; \big\| f^{T_{j_\ell}} \big\|_{\dot{H}^{-q}}^{-2} \; h(T_{j_\ell}) \,. \tag{1.5.7}$$

We now bound $E_1^{T_{i_\ell}}$ using the Cauchy–Schwarz inequality:

$$\begin{aligned}E_1^{T_{i_\ell}} &\le \left(\sum_{j_\ell<i_\ell}\sum_{\boldsymbol{k}\in I_{j_\ell}}\left|f_{\boldsymbol{k}}^{T_{i_\ell}}\right|^2 k^{-2q}\right)^{1/2}\left(\sum_{j_\ell<i_\ell}\sum_{\boldsymbol{k}\in I_{j_\ell}}\left|f_{\boldsymbol{k}}^{T_{j_\ell}}\right|^2 k^{-2q}\left\|f^{T_{j_\ell}}\right\|_{\dot{H}^{-q}}^{-4} h^2(T_{j_\ell})\right)^{1/2}\\ &\le \left(\sum_{j_\ell<i_\ell}\sum_{\boldsymbol{k}\in I_{j_\ell}}\left|f_{\boldsymbol{k}}^{T_{i_\ell}}\right|^2 k^{-2q}\right)^{1/2}\left(\sum_{j_\ell<i_\ell}\left\|f^{T_{j_\ell}}\right\|_{\dot{H}^{-q}}^{-2} h^2(T_{j_\ell})\right)^{1/2}.\end{aligned}$$

We use Eq. (1.4.8) from Lemma 3 to bound the first factor and Eq. (1.5.2) to bound the second factor:

$$E_1^{T_{i_\ell}} \le \left(\delta h^2(T_{i_\ell})\right)^{1/2}(\delta)^{1/2} = \delta\, h(T_{i_\ell}). \tag{1.5.8}$$

We similarly bound $E_2^{T_{i_\ell}}$:

$$\begin{aligned}E_2^{T_{i_\ell}} &\le \left(\sum_{j_\ell>i_\ell}\sum_{\boldsymbol{k}\in I_{j_\ell}}\left|f_{\boldsymbol{k}}^{T_{i_\ell}}\right|^2 k^{-2q}\right)^{1/2}\left(\sum_{j_\ell>i_\ell}\sum_{\boldsymbol{k}\in I_{j_\ell}}\left|f_{\boldsymbol{k}}^{T_{j_\ell}}\right|^2 k^{-2q}\left\|f^{T_{j_\ell}}\right\|_{\dot{H}^{-q}}^{-4} h^2(T_{j_\ell})\right)^{1/2}\\ &\le \left\|f^{T_{i_\ell}}\right\|_{\dot{H}^{-q}}\left(\sum_{j_\ell>i_\ell}\left\|f^{T_{j_\ell}}\right\|_{\dot{H}^{-q}}^{-2} h^2(T_{j_\ell})\right)^{1/2}.\end{aligned}$$

Using Eq. (1.5.1), we find

$$E_2^{T_{i_\ell}} \le \left\|f^{T_{i_\ell}}\right\|_{\dot{H}^{-q}}\left(\delta^2\left(\frac{h(T_{i_\ell})}{\left\|f^{T_{i_\ell}}\right\|_{\dot{H}^{-q}}}\right)^2\right)^{1/2} = \delta\ h(T_{i_\ell}) \tag{1.5.9}$$

and therefore

$$\begin{aligned}\left\langle f^{T_{i_\ell}}\,,\,g\right\rangle &= S^{T_{i_\ell}} + E^{T_{i_\ell}} \\ &\geq S^{T_{i_\ell}} - E_1^{T_{i_\ell}} - E_2^{T_{i_\ell}} \\ &\geq S^{T_{i_\ell}} - \delta\ h(T_{i_\ell}) - \delta\ h(T_{i_\ell}).\end{aligned}$$

Again using Eq. (1.4.7) from Lemma 3 that the set I_{i_ℓ} captures a large proportion of the Sobolev norm, we conclude

$$\begin{aligned}\left\langle f^{T_{i_\ell}}\,,\,g\right\rangle &\geq (1-\delta)\left\|f^{T_{i_\ell}}\right\|_{\dot{H}^{-q}}^{2}\left\|f^{T_{i_\ell}}\right\|_{\dot{H}^{-q}}^{-2}\ h(T_{i_\ell}) - 2\delta\ h(T_{i_\ell}) \\ &= (1-3\delta)\ h(T_{i_\ell}).\end{aligned}$$

□

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2 OPTIMAL SPATIALLY DEPENDENT DIFFUSION COEFFICIENTS UNDER AN L^p CONSTRAINT

2.1 Introduction

In this chapter, we consider the problem of choosing spatially dependent coefficients $D_{i,j}(\boldsymbol{x})$ that maximize the rate of convergence to equilibrium of solutions to the heat equation in inhomogeneous media [16, 69]

$$\frac{\partial}{\partial t}\theta - \nabla \cdot (\mathbb{D}\nabla\theta) = 0 \quad \text{in } \Omega \,. \tag{2.1.1}$$

This requires us to maximize the spectral gap of the corresponding elliptic operator. Indeed, optimal control of coefficients of elliptic operators is well-studied, see [8, 31, 33, 36, 42, 44] and the references there. Applications include optimal design in conductivity of composite materials [1, 15, 55]. The problem of optimal diffusion processes in stochastic differential equations has been studied by Jafarizadeh [33] in the $n = 1$ case with L^1 norm constraint on $D(x)$ and Biswal et al. [8] in the $n = 2$ case with L^∞ constraint on $\mathbb{D}(\boldsymbol{x})$. The present work may be considered an extension of this work to the arbitrary dimension case with L^p constraint on $\mathbb{D}$. The success of the method presented in this chapter is that the nonlinear, nonlocal L^p constraint couples $\mathbb{D}$ and the scalar θ when we take the functional derivative with respect to the components $D_{i,j}$, allowing for a nontrivial solution using calculus of variations. Other commonly used constraints like L^∞ do not see this coupling.

In Section 2.2, we formulate the problem for the optimal spectral gap using the

Rayleigh quotient. We do not solve this optimization problem. Instead, we solve a relaxed version which produces an upper bound for the optimal spectral gap. It is common to assume the matrix of coefficients $\mathbb{D}(\boldsymbol{x})$ is uniformly elliptic. That is, there exists a constant $\sigma > 0$ such that $\boldsymbol{\xi} \cdot \mathbb{D}(\boldsymbol{x})\boldsymbol{\xi} \geq \sigma|\boldsymbol{\xi}|^2$ for all $\boldsymbol{\xi} \in \mathbb{R}^n, \boldsymbol{x} \in \Omega$. We find that, relaxing the above restriction to $\sigma \geq 0$, the relaxed optimization problem admits a solution $\mathbb{D}_*$. Let $1 < p < \infty$ and take $r = 2p'$ where p' is the conjugate exponent of p such that $1/p + 1/p' = 1$. The optimal matrix $\mathbb{D}_* = \mathbb{D}_{\varphi_*}$ is

$$\mathbb{D}_{\varphi_*}(\boldsymbol{x}) = \left(\int_\Omega |\nabla\varphi_*|^r \, d\boldsymbol{x}\right)^{-1/p} |\nabla\varphi_*|^{r-4}\nabla\varphi_* \, \nabla\varphi_*^\top . \tag{2.1.2}$$

Assuming Dirichlet boundary conditions on Eq. (2.1.1), $\varphi_* = \varphi_{*,\mathrm{Dir}}$ is the minimizer (if it exists) of the Gagliardo-Nirenberg-Sobolev type inequality [22]

$$\inf_{\varphi \in W_0^{1,r},\, \varphi \neq 0} \frac{\|\nabla\varphi\|_{L^r}}{\|\varphi\|_{L^2}} . \tag{2.1.3}$$

If we have Neumann boundary conditions on Eq. (2.1.1), then $\varphi_* = \varphi_{*,\mathrm{Neu}}$ is the minimizer (if it exists) of the generalized $(2, r)-$Poincaré inequality

$$\inf_{\varphi \in W^{1,r},\, \varphi \neq 0,\, \int \varphi = 0} \frac{\|\nabla\varphi\|_{L^r}}{\|\varphi\|_{L^2}} . \tag{2.1.4}$$

We remark that these $\mathbb{D}_{*,\cdot}$ are not necessarily uniformly elliptic. Indeed, as demonstrated in Section 2.5 in the $n = 1$ case, $\varphi_{*,\mathrm{Dir}}$ has a point of $\varphi'_{*,\mathrm{Dir}}(\boldsymbol{x}) = 0$ in the interior of Ω. Therefore, so does $\mathbb{D}_{*,\mathrm{Dir}}$. Similarly, $\varphi'_{*,\mathrm{Neu}}$ and $\mathbb{D}_{*,\mathrm{Neu}}$ are zero on the boundary $\partial\Omega$. Allowing points of degeneracy is natural. In the context of anisotropic continuous media, a point where $\mathbb{D}(\boldsymbol{x}) = 0$ corresponds to a "perfect" insulator [10, 15].

Having recognized the connection between our problem and the Sobolev and Poincaré type inequalities, we may search the literature to find bounds on the relevant constants and understand more of the properties (existence, regularity, sign, points where $\nabla\varphi_*$ vanish) of the extremal functions $\varphi_{*,\mathrm{Dir}}$ and $\varphi_{*,\mathrm{Neu}}$. Indeed there are many researchers who study generalized Poincaré inequalities, $(q,p)-$Poincaré inequalities, Poincaré type inequalities, Sobolev-Poincaré inequalities, the related $p-$Laplacian eigenvalue equation, and sharp constants thereof [3, 9, 25, 27–29, 38, 41, 51, 54, 57, 60, 65]. We now present an overview where we derive the relevant optimization problems, state our results, and outline the rest of the chapter.

2.2 Formulation of the optimization problem and an overview

We begin by considering the parabolic partial differential equation

$$\begin{cases} \dfrac{\partial}{\partial t}\theta - \nabla \cdot (\mathbb{D}\nabla\theta) = 0 & \text{in } \Omega \\ \theta = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{2.2.1}$$

with initial condition $\theta(\boldsymbol{x}, 0) = \theta_0(\boldsymbol{x}) \in L^2(\Omega)$ and open, bounded, connected domain Ω. Suppose that $D_{i,j}(\boldsymbol{x}) \in C^\infty(\bar{\Omega})$ and $\mathbb{D}$ is symmetric ($D_{i,j} = D_{j,i}$). Let the operator $\mathscr{L}$ be such that $\mathscr{L}\theta = -\nabla \cdot (\mathbb{D}\nabla\theta)$ and suppose $\mathscr{L}$ is uniformly elliptic. That is, there exist a $\sigma > 0$ such that $\boldsymbol{\xi} \cdot \mathbb{D}(\boldsymbol{x})\boldsymbol{\xi} \geq \sigma|\boldsymbol{\xi}|^2$ for all $\boldsymbol{\xi} \in \mathbb{R}^n, \boldsymbol{x} \in \Omega$. Then it is well known (see Evans [22], section 6.5) that there is an orthonormal basis $\{w_k\}_{k=1}^{\infty}$ of $L^2(\Omega)$, where the $w_k \in H_0^1(\Omega)$ are Dirichlet eigenfunctions of $\mathscr{L}$ satisfying

$$\begin{cases} \mathscr{L} w_k = \lambda_k w_k & \text{in } \Omega \\ w_k = 0 & \text{on } \partial\Omega \,, \end{cases} \tag{2.2.2}$$

with corresponding real eigenvalues

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \,. \tag{2.2.3}$$

Expand the initial condition in this basis

$$\theta_0 = \sum_{k=1}^{\infty} a_k w_k \,, \tag{2.2.4}$$

and notice that the solution to the heat equation Eq. (2.2.1) is then [58]

$$\theta = \sum_{k=1}^{\infty} e^{-\lambda_k t} a_k w_k \,. \tag{2.2.5}$$

We have

$$\|\theta\|_{L^2} = \sqrt{\sum_{k=1}^{\infty} e^{-2\lambda_k t} \left|a_k\right|^2} \,. \tag{2.2.6}$$

A generic initial condition θ_0 will have $a_1 \neq 0$, and so *generically solutions decay to 0 exponentially with rate* λ_1 [58]. The problem we investigate in this chapter is that of maximizing the decay rate λ_1 over choices of diffusion tensors $\mathbb{D}(\boldsymbol{x})$, where $\mathbb{D}$ is constrained by its L^p norm.

We now formulate the appropriate optimization problem. The principal eigenvalue λ_1 is characterized by the Rayleigh quotient [22, 58],

$$\lambda_1 = \min_{\varphi \in H_0^1(\Omega),\, \varphi \neq 0} \frac{\int_\Omega \nabla\varphi \cdot \mathbb{D}\nabla\varphi \, d\boldsymbol{x}}{\int_\Omega \varphi^2 \, d\boldsymbol{x}}\,, \tag{2.2.7}$$

and the minimum is attained at $\varphi = w_1$, the leading eigenvector. Suppose $1 \leq p \leq \infty$. Taking the supremum of λ_1 over $\mathbb{D}$ described above, we define $\lambda_{*,\sigma,\mathrm{Dir}}$ to be the optimal decay rate given Dirichlet boundary conditions and uniform ellipticity constant $\sigma > 0$,

$$\lambda_{*,\sigma,\mathrm{Dir}} := \sup_{\mathbb{D} \in \mathscr{A}_\sigma} \min_{\varphi \in H_0^1(\Omega),\, \varphi \neq 0} \frac{\int_\Omega \nabla\varphi \cdot \mathbb{D}\nabla\varphi \, d\boldsymbol{x}}{\int_\Omega \varphi^2 \, d\boldsymbol{x}}\,, \tag{2.2.8}$$

where

$$\mathscr{A}_\sigma = \begin{cases} \mathbb{D}(\boldsymbol{x}) \in L^p(\Omega) \quad \text{s.t.} & D_{i,j}(\boldsymbol{x}) \in C^\infty(\bar{\Omega})\,, \\ & D_{i,j} = D_{j,i}\,, \\ & \|\mathbb{D}\|_{L^p} = 1\,, \ \text{and} \\ & \boldsymbol{\xi} \cdot \mathbb{D}(\boldsymbol{x})\boldsymbol{\xi} \geq \sigma|\boldsymbol{\xi}|^2 \quad \forall \boldsymbol{\xi} \in \mathbb{R}^n, \boldsymbol{x} \in \Omega\,. \end{cases} \tag{2.2.9}$$

We define the L^p norm of a matrix $\mathbb{D}(\boldsymbol{x})$ to be the L^p norm of its magnitude, considering $\mathbb{D}$ as a vector in $\mathbb{R}^{(n^2)}$. That is

$$\|\mathbb{D}\|_{L^p}^p = \int_\Omega |\mathbb{D}|^p \, d\boldsymbol{x} \tag{2.2.10}$$

$$= \int_\Omega \left(|\mathbb{D}|^2\right)^{p/2} d\boldsymbol{x} \tag{2.2.11}$$

$$= \int_\Omega \left(\sum_{1\leq i\leq n} \sum_{1\leq j\leq n} D_{i,j}^2 \right)^{p/2} d\boldsymbol{x}\,. \tag{2.2.12}$$

Therefore, the optimal decay rate $\lambda_{*,\mathrm{Dir}}$ given Dirichlet boundary conditions is found by taking the supremum over choices of $\sigma > 0$. We have

$$\lambda_{*,\mathrm{Dir}} := \sup_{\sigma>0} \sup_{\mathbb{D}\in\mathscr{A}_\sigma} \min_{\varphi\in H_0^1(\Omega),\, \varphi\neq 0} \frac{\int_\Omega \nabla\varphi\cdot\mathbb{D}\nabla\varphi \, d\boldsymbol{x}}{\int_\Omega \varphi^2 \, d\boldsymbol{x}}\,. \tag{2.2.13}$$

Instead of solving the variational problem Eq. (2.2.13), we solve a relaxed version where we allow a larger class of $\mathbb{D}$. The result is that we arrive at an upper bound for $\lambda_{*,\mathrm{Dir}}$.

Theorem 4. *Let $1 < p < \infty$ and take $r = 2p'$ where p' is the conjugate exponent of p such that $1/p + 1/p' = 1$.*

1.) *[Dirichlet Boundary Conditions] Define*

$$\widetilde{\lambda}_{*,\mathrm{Dir}} := \mathop{inf}_{\varphi\in\mathscr{B}_{\mathrm{Dir}}} \mathop{sup}_{\mathbb{D}\in\mathscr{A}} \int_\Omega \nabla\varphi\cdot\mathbb{D}\nabla\varphi \, d\boldsymbol{x}\,, \tag{2.2.14}$$

where

$$\mathscr{A} = \{\mathbb{D} \in L^p(\Omega) \quad s.t. \quad \|\mathbb{D}\|_{L^p} = 1\} \;\; and \tag{2.2.15}$$

$$\mathscr{B}_{\mathrm{Dir}} = \{\varphi \in W_0^{1,r}(\Omega) \quad s.t. \quad \|\varphi\|_{L^2} = 1\}\,. \tag{2.2.16}$$

We have that $\lambda_{*,\mathrm{Dir}} \leq \widetilde{\lambda}_{*,\mathrm{Dir}}$.

2.) *[Neumann Boundary Conditions] Define*

$$\lambda_{*,\mathrm{Neu}} := \sup_{\sigma>0} \; \sup_{\mathbb{D}\in\mathscr{A}_\sigma} \; \min_{\varphi\in H^1(\Omega),\, \varphi\neq 0\,, \int \varphi=0} \; \frac{\int_\Omega \nabla\varphi\cdot\mathbb{D}\nabla\varphi\, d\boldsymbol{x}}{\int_\Omega \varphi^2\, d\boldsymbol{x}}\,. \tag{2.2.17}$$

Also define

$$\widetilde{\lambda}_{*,\mathrm{Neu}} := \inf_{\varphi\in\mathscr{B}_{\mathrm{Neu}}} \; \sup_{\mathbb{D}\in\mathscr{A}} \int_\Omega \nabla\varphi\cdot\mathbb{D}\nabla\varphi\, d\boldsymbol{x}\,, \tag{2.2.18}$$

where

$$\mathscr{B}_{\mathrm{Neu}} = \{\varphi \in W^{1,r}(\Omega) \quad s.t. \quad \int_\Omega \varphi\, d\boldsymbol{x} = 0,\;\|\varphi\|_{L^2} = 1\}\,. \tag{2.2.19}$$

We have that $\lambda_{*,\mathrm{Neu}} \leq \widetilde{\lambda}_{*,\mathrm{Neu}}$.

With the introduction of the relaxed problems Eqs. (2.2.14) and (2.2.18), we present our characterizations of their solutions. We begin with

Theorem 5. *Given* $\varphi(\boldsymbol{x}) \in W^{1,r}(\Omega)$, *the spatially dependent matrix*

$$\mathbb{D}_\varphi(\boldsymbol{x}) := \left(\int_\Omega |\nabla\varphi|^r\, d\boldsymbol{x}\right)^{-1/p} |\nabla\varphi|^{r-4}\nabla\varphi\,\nabla\varphi^\top \tag{2.2.20}$$

maximizes the variation

$$\underset{\mathbb{D}\in L^p,\, \|\mathbb{D}\|_{L^p}=1}{sup} \int_\Omega \nabla\varphi \cdot \mathbb{D}\nabla\varphi \, d\boldsymbol{x} \,. \tag{2.2.21}$$

With this, the variational problems (2.2.14 and 2.2.18) simplify to

$$\inf_{\varphi\in\mathscr{B}} \sup_{\mathbb{D}\in\mathscr{A}} \int_\Omega \nabla\varphi \cdot \mathbb{D}\nabla\varphi \, d\boldsymbol{x} = \inf_{\varphi\in\mathscr{B}} \left(\int_\Omega |\nabla\varphi|^r \, d\boldsymbol{x}\right)^{1/p'} \tag{2.2.22}$$

$$= \left(\inf_{\varphi\in\mathscr{B}} \|\nabla\varphi\|_{L^r}\right)^2 . \tag{2.2.23}$$

For $\mathscr{B} = \mathscr{B}_{\mathrm{Dir}}$, notice that

$$\inf_{\varphi\in\mathscr{B}_{\mathrm{Dir}}} \|\nabla\varphi\|_{L^r} = \inf_{\varphi\in W_0^{1,r},\, \varphi\neq 0} \frac{\|\nabla\varphi\|_{L^r}}{\|\varphi\|_{L^2}} = C_{\mathrm{Dir}}^{-1} \,, \tag{2.2.24}$$

where C_{Dir} is the optimal constant in the Gagliardo-Nirenberg-Sobolev type inequality [22]

$$\|\varphi\|_{L^2} \leq C_{\mathrm{Dir}} \|\nabla\varphi\|_{L^r} \quad \forall\varphi \in W_0^{1,r} \,. \tag{2.2.25}$$

Similarly, for $\mathscr{B} = \mathscr{B}_{\mathrm{Neu}}$, notice that

$$\inf_{\varphi\in\mathscr{B}_{\mathrm{Neu}}} \|\nabla\varphi\|_{L^r} = \inf_{\varphi\in W^{1,r},\, \varphi\neq 0,\, \int\varphi=0} \frac{\|\nabla\varphi\|_{L^r}}{\|\varphi\|_{L^2}} = C_{\mathrm{Neu}}^{-1} \,, \tag{2.2.26}$$

where C_{Neu} is the optimal constant in the generalized $(2, r)-$Poincaré inequality

$$\|\varphi\|_{L^2} \leq C_{\mathrm{Neu}} \|\nabla\varphi\|_{L^r} \quad \forall\varphi \in W^{1,r} \ \text{ s.t. } \int_\Omega \varphi \, d\boldsymbol{x} = 0 \,. \tag{2.2.27}$$

Here we have to assume Ω has a C^1 boundary $\partial\Omega$ so that Eq. (2.2.27) is valid [22]. We conclude that

Corollary 2. *We have* $\widetilde{\lambda}_{*,\mathrm{Dir}} = C_{\mathrm{Dir}}^{-2}$ *and* $\widetilde{\lambda}_{*,\mathrm{Neu}} = C_{\mathrm{Neu}}^{-2}$. *Moreover, if* $\varphi_{*,\mathrm{Dir}}$ *achieves equality in Eq.* (2.2.25)*, then the optimal* $\mathbb{D}_{*,\mathrm{Dir}}$ *in Eq.* (2.2.14) *is of the form* $\mathbb{D}_{*,\mathrm{Dir}} = \mathbb{D}_{\varphi_{*,\mathrm{Dir}}}$*, using the notation from Theorem 5. Similarly, if* $\varphi_{*,\mathrm{Neu}}$ *achieves equality in Eq.* (2.2.27)*, then the optimal* $\mathbb{D}_{*,\mathrm{Neu}}$ *in Eq.* (2.2.18) *is of the form* $\mathbb{D}_{*,\mathrm{Neu}} = \mathbb{D}_{\varphi_{*,\mathrm{Neu}}}$.

In Section 2.3, we recall that the nonhomogeneous Dirichlet eigenvalues of the $r-$Laplacian correspond to the optimal constant in the Gagliardo-Nirenberg-Sobolev type inequality. The $r-$Laplacian is defined as $\Delta_r\varphi = \nabla \cdot \left(|\nabla\varphi|^{r-2}\nabla\varphi\right)$. Similarly, the nonhomogeneous Neumann eigenvalues of the $r-$Laplacian correspond to the optimal constant in the generalized Poincaré inequality. This will prepare us for Sections 2.4 to 2.7 which concern the case $n = 1$. This is the setting where we can give explicit examples.

In Section 2.4, we look at the closed solution provided by Drábek and Manásevich [19] for the nonhomogeneous $r-$Laplacian eigenvalue problem. In Section 2.5, we use this closed solution to express the Dirichlet and Neumann eigenvalues and the extremals $\varphi_{*,\mathrm{Dir}}$ and $\varphi_{*,\mathrm{Neu}}$. This result is not new [37], but it prepares us for the example that follows. In Section 2.6, we present an explicit example where we show the heat equation with Dirichlet boundary conditions (resp. Neumann) for various choices of $D(x)$ does not decay faster than rate $\widetilde{\lambda}_{*,\mathrm{Dir}}$ (resp. $\widetilde{\lambda}_{*,\mathrm{Neu}}$). In Section 2.7, we compute a closed form expression for $\widetilde{\lambda}_{*,\mathrm{Neu}}$. Again, this expression is not new [37], but we are able to verify analytically that $\widetilde{\lambda}_{*,\mathrm{Neu}}$ is larger than π^2, the eigenvalue in the constant diffusion case ($D = 1$). From this, we know that spatially dependent diffusion functions can improve decay rates in the heat equation over the constant diffusion case.

In Section 2.8, we return to the arbitrary dimension case. We derive the optimal $\mathbb{D}_\varphi$ for use in Theorem 5. In Section 2.9, we use another choice of matrix norm and arrive at a related eigenvalue equation [59]. In Section 2.10, we prove Theorems 4 and 5. In Section 2.11, we prove a general lower bound for $\widetilde{\lambda}_{*,\text{Neu}}$ if Ω is convex and bounded. This lower bound is an improvement over the constant diffusion case by a factor of $n^{1/2}$.

2.3 The nonhomogeneous $r-$Laplacian eigenvalue equation

Notice that Corollary 2 and Eq. (2.2.24) imply $\widetilde{\lambda}_{*,\text{Dir}} = C_{\text{Dir}}^{-2}$ where

$$C_{\text{Dir}}^{-r} = \inf_{\varphi \in W_0^{1,r},\, \int_\Omega \varphi^2 = 1} \int |\nabla \varphi|^r \, d\boldsymbol{x} \,. \tag{2.3.1}$$

Similarly, Corollary 2 and Eq. (2.2.26) imply $\widetilde{\lambda}_{*,\text{Neu}} = C_{\text{Neu}}^{-2}$ where

$$C_{\text{Neu}}^{-r} = \inf_{\varphi \in W^{1,r},\, \int_\Omega \varphi^2 = 1,\, \int \varphi = 0} \int |\nabla \varphi|^r \, d\boldsymbol{x} \,. \tag{2.3.2}$$

Dividing by r and introducing Lagrange multipliers, we have

$$\frac{1}{r} C_{\text{Dir}}^{-r} = \inf_{\varphi \in W_0^{1,r}} \int_\Omega \frac{1}{r} |\nabla \varphi|^r \, d\boldsymbol{x} + \frac{\mu_{\text{Dir}}}{2} \left(1 - \int_\Omega \varphi^2 \, d\boldsymbol{x} \right), \tag{2.3.3}$$

and

$$\frac{1}{r}C_{\text{Neu}}^{-r} = \inf_{\varphi \in W^{1,r}} \int_\Omega \frac{1}{r}|\nabla\varphi|^r \, d\boldsymbol{x} + \frac{\mu_{\text{Neu}}}{2}\left(1 - \int_\Omega \varphi^2 \, d\boldsymbol{x}\right) - \gamma \int_\Omega \varphi \, d\boldsymbol{x}\,. \tag{2.3.4}$$

The corresponding Euler-Lagrange equation for Eq. (2.3.3) is

$$\begin{cases} \nabla \cdot \left(|\nabla\varphi|^{r-2}\nabla\varphi\right) + \mu_{\text{Dir}}\,\varphi = 0 & \forall \boldsymbol{x} \in \Omega \\ \varphi = 0 & \forall \boldsymbol{x} \in \partial\Omega\,. \end{cases} \tag{2.3.5}$$

This is a nonhomogeneous $r-$Laplacian Dirichlet eigenvalue equation. Recall that the $r-$Laplacian is defined as $\Delta_r\varphi = \nabla\cdot\left(|\nabla\varphi|^{r-2}\nabla\varphi\right)$. The corresponding Euler-Lagrange equation for Eq. (2.3.4) is

$$\begin{cases} \nabla \cdot \left(|\nabla\varphi|^{r-2}\nabla\varphi\right) + \mu_{\text{Neu}}\,\varphi + \gamma = 0 & \forall \boldsymbol{x} \in \Omega \\ \nabla\varphi \cdot \boldsymbol{n} = 0 & \forall \boldsymbol{x} \in \partial\Omega\,. \end{cases} \tag{2.3.6}$$

Neumann boundary conditions appear here as the result of the natural boundary conditions [26]. Taking the integral of the first equation of Eq. (2.3.6), applying the divergence theorem, the Neumann boundary conditions, and the constraint $\int \varphi = 0$, we notice that $\gamma = 0$. Eq. (2.3.6) becomes

$$\begin{cases} \nabla \cdot \left(|\nabla\varphi|^{r-2}\nabla\varphi\right) + \mu_{\text{Neu}}\,\varphi = 0 & \forall \boldsymbol{x} \in \Omega \\ \nabla\varphi \cdot \boldsymbol{n} = 0 & \forall \boldsymbol{x} \in \partial\Omega\,. \end{cases} \tag{2.3.7}$$

This is a nonhomogeneous $r-$Laplacian Neumann eigenvalue equation.

Say $\mu_{\text{Dir},1}$ is the smallest eigenvalue in Eq. (2.3.5) such that $\int \varphi^2\, d\boldsymbol{x} = 1$. We conclude this section by noticing that $\mu_{\text{Dir},1} = C_{\text{Dir}}^{-r} = \widetilde{\lambda}_{*,\text{Dir}}^{p'}$. This can be seen by multiplying both sides of Eq. (2.3.5) by φ, integrating over the domain, integrating by parts, applying the Dirichlet boundary condition, using the constraint $\int \varphi^2\, d\boldsymbol{x} = 1$, and comparing to Eq. (2.3.1). Similarly, the smallest eigenvalue $\mu_{\text{Neu},1}$ in Eq. (2.3.7) such that $\int \varphi^2\, d\boldsymbol{x} = 1$ and $\int \varphi = 0$ is such that $\mu_{\text{Neu},1} = C_{\text{Neu}}^{-r} = \widetilde{\lambda}_{*,\text{Neu}}^{p'}$.

2.4 $n = 1$: A closed solution to a nonhomogeneous eigenvalue problem

Drábek and Manásevich [19] solve the initial value problem

$$(IV) \quad \begin{cases} \left(|u'|^{r-2}u'\right)' + \lambda |u|^{q-2}u = 0 \quad \text{on } (0,\infty)\,, \\ u(0) = 0, \quad u'(0) = \alpha\,. \end{cases} \tag{2.4.1}$$

This is a precursor to the eigenvalue problem discussed in the previous section, Section 2.3. We remark that, in this section, we do not assume $\int u^2 = 1$. Drábek and Manásevich [19] find that for each α, there is exactly one λ (denoted λ_α) for which (IV) has a solution. Moreover, the solution is unique for this choice of α. The solution is given as follows. Let

$$\arcsin_{r,q}(\sigma) = \sigma\ {}_2F_1\left(\frac{1}{q}, \frac{1}{r}; 1 + \frac{1}{q}, \left(\frac{2\sigma}{q}\right)^q\right) \tag{2.4.2}$$

where ${}_2F_1$ is the hypergeometric function. Define

$$\pi_{r,q} = 2\,\mathrm{arcsin}_{r,q}(\frac{q}{2})\,. \tag{2.4.3}$$

We have that $\mathrm{arcsin}_{r,q} : [0, \frac{q}{2}] \mapsto [0, \frac{\pi_{r,q}}{2}]$ is strictly increasing. Denote its inverse by $\sin_{r,q}$ and notice that $\sin_{r,q} : [0, \frac{\pi_{r,q}}{2}] \mapsto [0, \frac{q}{2}]$ is strictly increasing. This is the generalized sine function due to Lindqvist [41]. $\sin_{r,q}$ is the unique solution to the initial value problem

$$\begin{cases} \left(|u'|^{r-2}u'\right)' + \lambda_1|u|^{q-2}u = 0 \quad \text{on } (0, \frac{\pi_{rq}}{2})\,, \\ u(0) = 0, \quad u'(0) = 1\,, \end{cases} \tag{2.4.4}$$

where

$$\lambda_1 = \frac{2^q}{r'q^{q-1}}\,. \tag{2.4.5}$$

r' is the conjugate exponent of r such that $1/r + 1/r' = 1$. Rescaling $\sin_{r,q}$ allows us to conclude that $u = \dfrac{\alpha}{c}\sin_{r,q}(c\,t)$, where $c > 0$, is the unique solution to

$$\begin{cases} \left(|u'|^{r-2}u'\right)' + \lambda_\alpha|u|^{q-2}u = 0 \quad \text{on } (0, \frac{\pi_{rq}}{2c})\,, \\ u(0) = 0, \quad u'(0) = \alpha\,, \end{cases} \tag{2.4.6}$$

where

$$\lambda_\alpha = \lambda_1 c^q |\alpha|^{r-q}\,. \tag{2.4.7}$$

The unique solution to (IV) is then the periodic extension of u to $(0, \infty)$. From here, the authors rescale and shift the above solution to find the eigenvalues on $(0, T)$ under Dirichlet, Neumann and periodic boundary conditions. The conclusion is that in each case, given $r \neq q$, every $\lambda \in (0, \infty)$ is an eigenvalue ($\lambda = 0$ is also an eigenvalue if $q < r$). This is evident from Eq. (2.4.7) and noting that changing α does not change the domain $(0, \frac{\pi_{r,q}}{2c})$ of the solution. The nonhomogeneity of the equation is what causes the spectrum to have this range; scaling results in new eigenvalues.

2.5 $n = 1$: The extremal functions $\varphi_{*,\cdot}$ and $\mathbb{D}_{*,\cdot}$

Now that we have an explicit expression for the eigenfunctions and eigenvalues of the $r-$Laplacian, we may write an expression for the extremal functions $\varphi_{*,\cdot}$ and $D_{*,\cdot}$. Unlike Drábek and Manásevich [19] in the previous section, our setting requires us to normalize the solution to the $r-$Laplace eigenvalue equation in the L^2 norm (see Section 2.3). With this extra restriction, there is now only one choice of α so that $\|u\|_{L^2} = 1$ and u has largest period.

Take $q = 2$, $1 < p < \infty$, and $r = 2p'$. In this setting, we may express the Neumann and Dirichlet eigenvectors in terms of $\sin_{r,q}$ as in [37]. In the construction from the previous section, take $c = \frac{\pi_{r,2}}{2L}$ so that $u_{\text{Neu}} = \frac{\alpha}{c} \sin_{r,2}(c\, t)$ is defined on $[0, L]$. Extend u_{Neu} to $[-L, L]$ via $u_{\text{Neu}}(-x) = -u_{\text{Neu}}(x)$. The α that gives $\|u\|_{L^2} = 1$ is

$$\alpha = \sqrt{\frac{c^3}{2c_0}}, \tag{2.5.1}$$

where

$$c_0 = \int_0^{\frac{\pi_{r,2}}{2}} (\sin_{r,2}(t))^2 dt\,. \tag{2.5.2}$$

From Eq. (2.4.7), the corresponding smallest eigenvalue is

$$\mu_{\mathrm{Neu},1} = \lambda_1 c^2 |\alpha|^{r-2} = \lambda_1 \frac{c^{\frac{3}{2}r-1}}{(2c_0)^{\frac{r}{2}-1}}\,. \tag{2.5.3}$$

We have $\varphi_{*,\mathrm{Neu}} = u_{\mathrm{Neu}}$ and can find $\varphi_{*,\mathrm{Dir}}$ similarly by shifting $\frac{\alpha}{c}\sin_{r,2}(c\,t)$ horizontally and doing an even extension. As a result, $\mu_{\mathrm{Neu},1} = \mu_{\mathrm{Dir},1}$. We have

$$D_{*,\cdot} = \mu_{\mathrm{Neu},1}^{-1/p}\, |\varphi'_{*,\cdot}|^{r-2}\,. \tag{2.5.4}$$

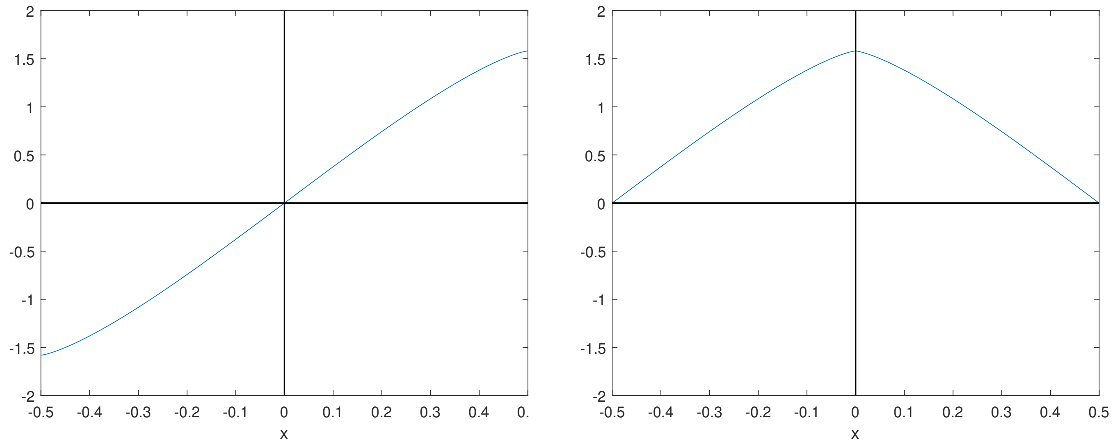
We plot $\varphi_{*,\cdot}$ in Fig. 2.1. In Fig. 2.2, we plot $D_{*,\cdot}$ against the constant function $D_c = 1$ and the parabolas $D_{n,\cdot}$ where

$$D_{n,\mathrm{Neu}} = a_n \left(\left(\frac{1}{2} - x\right)\left(\frac{1}{2} + x\right) + \frac{1}{n} \right), \tag{2.5.5}$$

and

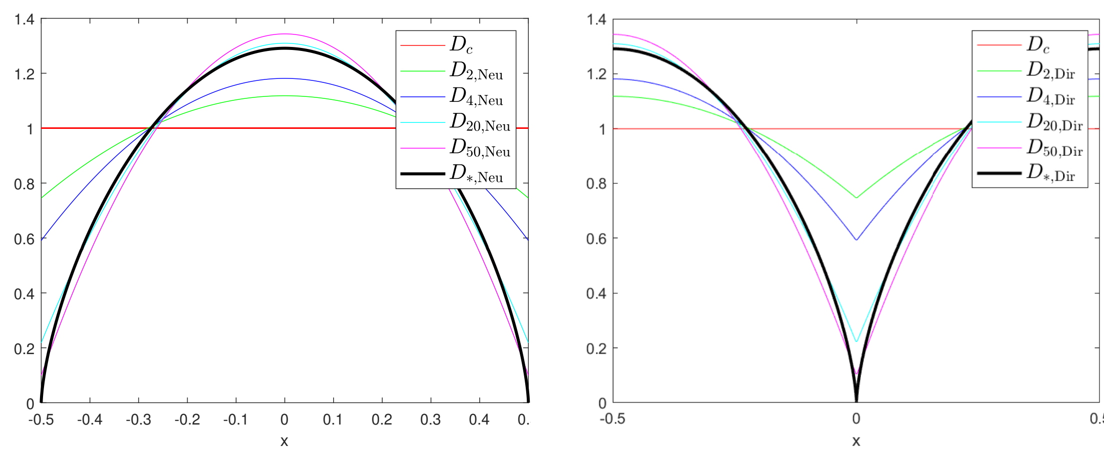
$$D_{n,\mathrm{Dir}} = a_n \left((1 - |x|)|x| + \frac{1}{n} \right). \tag{2.5.6}$$

a_n is chosen such that $\int_{[-.5,.5]} |D_{n,\cdot}|^p dx = 1$.



(a) $\varphi_{*,\text{Neu}}$ (b) $\varphi_{*,\text{Dir}}$

Figure 2.1: $\varphi_{*,\cdot}$ with $L = .5$ and $p = 2$.



(a) $D_{*,\text{Neu}}$ and $D_{n,\text{Neu}}$ (b) $D_{*,\text{Dir}}$ and $D_{n,\text{Dir}}$

Figure 2.2: $D_{*,\cdot}$ plotted against the constant function $D_c = 1$ and the parabolas $D_{n,\cdot}$ with $n = 2, 4, 20, 50$ from Eqs. (2.5.5) and (2.5.6). We take $L = .5$ and $p = 2$.

2.6 $n = 1$: Example, the rate of decay of solutions to the heat equation

In this section, we simulate the solution to the heat equation

$$\frac{\partial}{\partial t}\theta = (D(x)\theta')' \tag{2.6.1}$$

using the Matlab package pdepe. In Fig. 2.3, we take mean-zero initial condition $\theta(x,0) = .1\sin(\pi x) + \sin(3\pi x)$, no flux boundary conditions $D(-.5)\theta(-.5,t) = D(.5)\theta(.5,t) = 0$, and the various choices of $D(x)$ given in Fig. 2.2 from Eq. (2.5.5). Let $\theta_D(x,t)$ denote the solution to the heat equation given that choice of D. In Fig. 2.3a, we plot the log of the variance of θ_D as a function of time. After a phase of superexponential decay where the high frequencies are quickly diffused, the solutions settle into an exponential decay. The slopes are difficult to distinguish, therefore, in Fig. 2.3b, we translate each curve by a constant c_D so that they all pass through the point $(.15, 0)$. We note that in both subfigures, the dashed line represents the decay rate $\widetilde{\lambda}_{*,\mathrm{Neu}}$, and it is shown for comparison.

We have a similar picture for Dirichlet boundary conditions in Fig. 2.4. We take initial condition $\theta(x,0) = .1\cos(\pi x) + \cos(3\pi x)$ and Dirichlet boundary conditions $\theta(-.5,t) = \theta(.5,t) = 0$. In Fig. 2.4a, we plot the L^2 norm of θ_D as a function of time. In Fig. 2.4b, we translate each curve by a constant c_D so that they all pass through the point $(.15, 0)$.

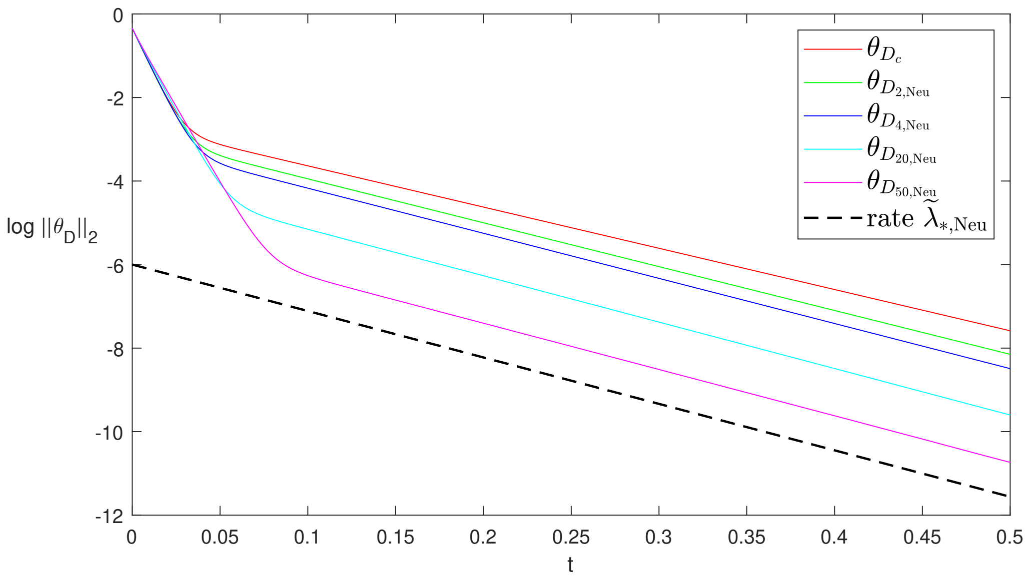
From this example, we remark that $\widetilde{\lambda}_{*,\cdot}$ seems to be the fastest decay rate for the heat equation. Recall that, for Dirichlet boundary conditions, the actual optimal decay rate is $\lambda_{*,\mathrm{Dir}}$. This is evidence that the bound in Theorem 4 is tight.

2.7 $n = 1$: An expression for the optimal eigenvalue.

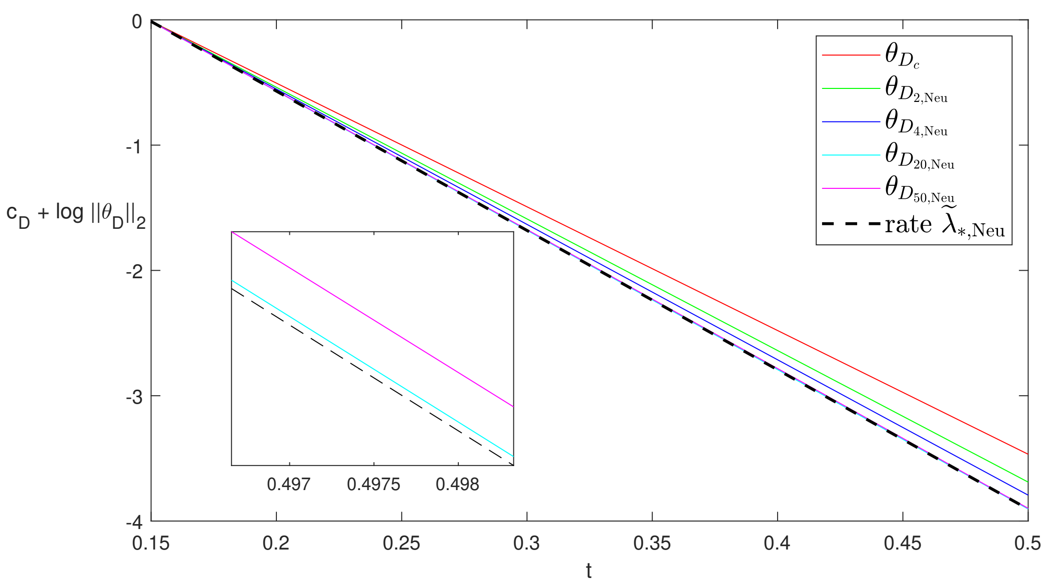
In this section, we solve the variational problem for $n = 1$ with Neumann boundary conditions. This has dual benefits: one can see the computations in a less notionally heavy setting, and we arrive at an analytic expression for $\widetilde{\lambda}_{*,\mathrm{Neu}}$. The same expression may be found by considering the optimal constant in the generalized $(2, r)-$Poincaré inequality [37], but this example demonstrates a simple approach.

We take the constraint $\|D\|_p = 1$ for some $p > 1$. We solve the following variational problem

$$\begin{aligned} \inf_{\varphi(x)} \quad \sup_{D(x)} & \int_\Omega D(x) \left(\frac{d}{dx}\varphi(x)\right)^2 dx \\ s.t. \quad & \int_\Omega \varphi(x)^2\, dx = 1 \\ & \int_\Omega |D(x)|^p\, dx = 1 \\ & \int_\Omega \varphi(x)\, dx = 0\ . \end{aligned} \tag{2.7.1}$$

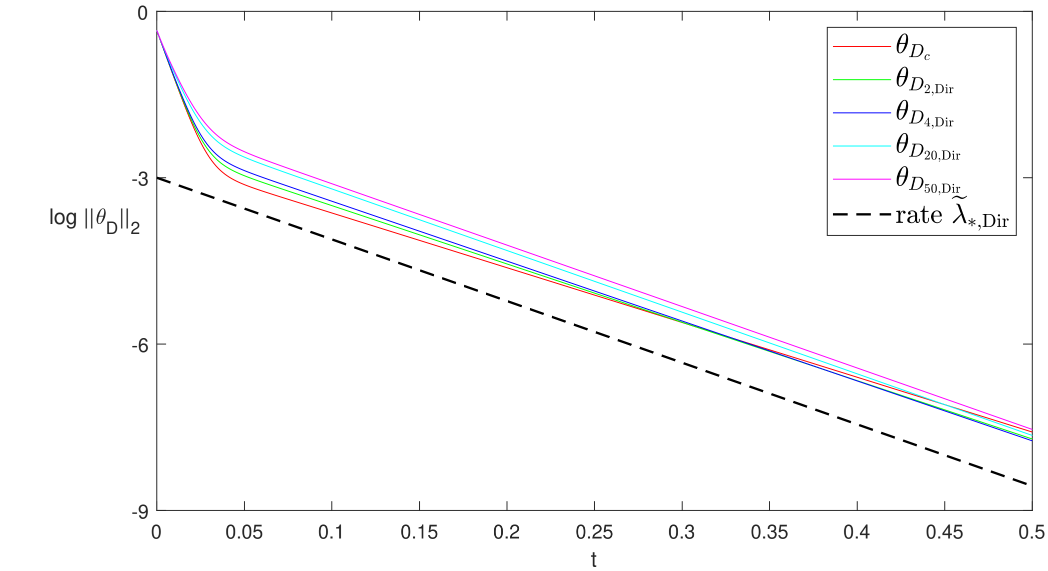


(a) Log of the variance of θ_D and decay rate $\widetilde{\lambda}_{*,\mathrm{Neu}}$ (dashed line)

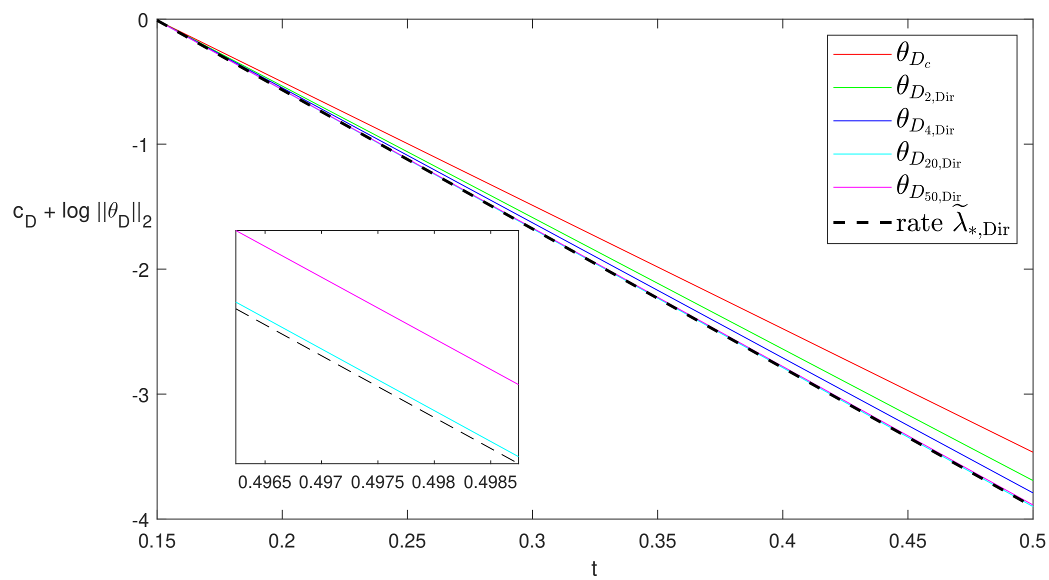


(b) Translated curves from Fig. 2.3a so each pass through $(.15, 0)$

Figure 2.3: Plotted are the log of the variance of θ_D as a function of time for D from Fig. 2.2a. The dashed line corresponds to the decay rate $\widetilde{\lambda}_{*,\mathrm{Neu}}$.



(a) Log of the L^2 norm of θ_D and decay rate $\widetilde{\lambda}_{*,\mathrm{Dir}}$ (dashed line)



(b) Translated curves from Fig. 2.4a so each pass through $(.15, 0)$

Figure 2.4: Plotted are the log of the L^2 norm of θ_D as a function of time for D from Fig. 2.2b. The dashed line corresponds to the decay rate $\widetilde{\lambda}_{*,\mathrm{Dir}}$.

Introducing Lagrange multipliers (λ, ν, γ), the problem is equivalent to

$$\begin{aligned} \inf_{\varphi(x)} \quad \sup_{D(x)} \int_\Omega D(x) \left(\frac{d}{dx}\varphi(x) \right)^2 dx \\ + \lambda \Big(1 - \int_\Omega \varphi(x)^2 \, dx \Big) \\ + \nu \Big(1 - \int_\Omega |D(x)|^p \, dx \Big) \\ + \gamma \int_\Omega \varphi(x) \, dx \, . \end{aligned} \tag{2.7.2}$$

Without loss of generality, we may assume $D \geq 0$. We now compute the Euler-Lagrange equation. Notice that we will have the Natural boundary conditions

$$D(x) \, \frac{d}{dx}\varphi(x) \Big|_{\delta\Omega} = 0 \, . \tag{2.7.3}$$

The variation with respect to D is

$$\int_X \delta D(x) \left(\left(\frac{d}{dx}\varphi(x) \right)^2 - p\nu \, D(x)^{p-1} \right) dx \, . \tag{2.7.4}$$

The variation with respect to φ is

$$- 2 \int_X \delta\varphi(x) \left(\frac{d}{dx} \left(D(x) \frac{d}{dx}\varphi(x) \right) + \lambda\varphi(x) - \frac{1}{2}\gamma \right) dx \, . \tag{2.7.5}$$

We arrive at the system

$$\frac{d}{dx} \left(D(x) \frac{d}{dx}\varphi(x) \right) + \lambda\varphi(x) - \frac{1}{2}\gamma = 0 \tag{2.7.6}$$

$$\left(\frac{d}{dx}\varphi(x)\right)^2 = p\nu D(x)^{p-1} \tag{2.7.7}$$

with boundary condition Eq. (2.7.3).

Remark 1: Notice that $\gamma = 0$. This follows from integrating Eq. (2.7.6) in x, using the boundary condition, and $\int \varphi = 0$.

Remark 2: Notice that $\nu \neq 0$. This follows from a contradiction argument. If $\nu = 0$, then Eq. (2.7.7) implies $\varphi =$ constant. The constraint $\int \varphi = 0$ implies $\varphi = 0$. Then constraint $\int \varphi^2 = 1$ cannot be satisfied. Without loss of generality, we assume $\nu > 0$.

We proceed solving the system of equations. Solve Eq. (2.7.7) for D:

$$D = (p\nu)^{\frac{-1}{p-1}} |\varphi'|^{\frac{2}{p-1}} . \tag{2.7.8}$$

Substitute into Eq. (2.7.6) to get

$$(p\nu)^{\frac{-1}{p-1}} \left(|\varphi'|^{\frac{2}{p-1}} \varphi'\right)' + \lambda\pi\varphi = 0 . \tag{2.7.9}$$

For simplicity, take $\Omega = [0, 1]$. We arrive at the nonhomogeneous eigenvalue problem

$$\left(|\varphi'|^{\frac{2}{p-1}} \varphi'\right)' + \widetilde{\lambda}\varphi = 0 \tag{2.7.10}$$

where $\widetilde{\lambda} = \lambda(p\nu)^{\frac{1}{p-1}}$. We seek solutions where $\varphi' \geq 0$. The equation simplifies to the second-order nonlinear ODE

$$(\varphi')^{\frac{2}{p-1}} \varphi'' = c_p \varphi \tag{2.7.11}$$

where

$$c_p = \frac{-\lambda (p\nu)^{\frac{1}{p-1}}}{\frac{2}{p-1} + 1} \,. \tag{2.7.12}$$

Multiply Eq. (2.7.6) by φ, integrate in x, apply integration by parts, and the boundary condition to arrive at

$$-\int D(\varphi')^2 dx + \int \lambda \varphi^2 = 0 \,. \tag{2.7.13}$$

Substitute Eq. (2.7.7) to get

$$-p\nu \int D^p = -\lambda \int \varphi^2 \,. \tag{2.7.14}$$

Recall the constraints $\|D\|_p = 1$ and $\|\varphi\|_2 = 1$. We conclude

$$p\nu = \lambda \,. \tag{2.7.15}$$

The coefficient c_p simplifies to

$$c_p = \frac{-\lambda^{\frac{1}{p-1}+1}}{\frac{2}{p-1} + 1} \tag{2.7.16}$$

$$= -\frac{p-1}{p+1} \lambda^{\frac{p}{p-1}} \,. \tag{2.7.17}$$

We now solve the ODE

$$(\varphi')^{\frac{2}{p-1}} \varphi'' = c_p \varphi \tag{2.7.18}$$

with zero flux boundary conditions

$$\varphi'(0) = \varphi'(1) = 0 \,. \tag{2.7.19}$$

Suppose $\varphi(x)$ is invertible on the domain $U \subset [0,1]$. On this domain, we may write x as a function of φ, that is $x(\varphi)$. Let $v(x) = \varphi'(x)$ and $w(\varphi) = v(x(\varphi))$. Write the ODE as

$$\varphi'' = f(\varphi, \varphi') \tag{2.7.20}$$

where

$$f(\varphi, \varphi') = c_p \, \varphi \, (\varphi')^{\frac{-2}{p-1}} \tag{2.7.21}$$

From the chain rule

$$\frac{dw}{d\varphi} = \frac{dv}{dx}\bigg|_{x(\varphi)} \frac{dx}{d\varphi}\bigg|_{\varphi} \tag{2.7.22}$$

$$= f(\varphi, w)\, \frac{1}{w} \tag{2.7.23}$$

$$= c_p \, \varphi \, w^{\frac{-2}{p-1}} \, \frac{1}{w} \,. \tag{2.7.24}$$

The result is a separable ODE

$$w'(\varphi) = c_p \, \varphi \, w^{\frac{-2}{p-1} - 1}(\varphi) \,. \tag{2.7.25}$$

We proceed by separation of variables

$$\int w^{\frac{2}{p-1}+1} dw = \int c_p \, \varphi \, d\varphi \,. \tag{2.7.26}$$

$$\left(\frac{2}{p-1} + 2 \right)^{-1} w^{\frac{2}{p-1}+2} = c_p \, \varphi^2 + a_1 \,. \tag{2.7.27}$$

Or simply, after redefining a_1,

$$w^{\frac{2}{p-1}+2} = \widetilde{c}_p\, \varphi^2 + a_1\, . \tag{2.7.28}$$

Recall that $w(\varphi) = v(x(\varphi))$, and so $w(\varphi(x)) = v(x)$. We have

$$\varphi'(x)^{\frac{2}{p-1}+2} = \widetilde{c}_p\, \varphi(x)^2 + a_1 \tag{2.7.29}$$

which is again a separable ODE. Let q be the conjugate exponent of p such that $1/q + 1/p = 1$.

Observations:

(1.) $\widetilde{c}_p = -\frac{p}{p+1}\lambda^q$

This follows from Eq. (2.7.17) and the above arithmetic.

(2.) $a_1 = \frac{2p+1}{p+1}\lambda^q$

This follows from integrating Eq. (2.7.29) over $[0, 1]$ and applying the constraints $\|D\|_p = 1$ and $\|\varphi\|_2 = 1$.

(3.) $\varphi^2(0) = \varphi^2(1) = 2 + \frac{1}{p}$

This follows from evaluating Eq. (2.7.29) at the boundary and applying the boundary conditions $\varphi'(0) = \varphi'(1) = 0$.

From these observations, the ODE Eq. (2.7.29) reduces to

$$\varphi' = \lambda^{1/2}\left(\frac{p}{p+1}\right)^{\frac{1}{2q}}\left(2+\frac{1}{p}-\varphi^2\right)^{\frac{1}{2q}}. \tag{2.7.30}$$

We proceed by separating variables.

$$\int\left(2+\frac{1}{p}-\varphi^2\right)^{-\frac{1}{2q}} d\varphi = \int \lambda^{1/2}\left(\frac{p}{p+1}\right)^{\frac{1}{2q}} dx\,. \tag{2.7.31}$$

The integral on the left hand side is expressed in terms of the hypergeometric function ${}_2F_1(a, b, c, z)$.

$$\frac{\varphi}{\left(2+\frac{1}{p}\right)^{\frac{1}{2q}}}\quad {}_2F_1\left(\frac{1}{2q}, \frac{1}{2}, \frac{3}{2}, \left(2+\frac{1}{p}\right)^{-1}\varphi^2\right) = \lambda^{1/2}\left(\frac{p}{p+1}\right)^{\frac{1}{2q}} x + a_2\,. \tag{2.7.32}$$

For $\mathbb{R}(c-a-b) > 0$, Gauss's summation Theorem gives

$${}_2F_1(a, b, c, 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\,. \tag{2.7.33}$$

Using $\Gamma(1) = 1$, $\Gamma(\frac{1}{2}) = \pi^{1/2}$, and the relation $\Gamma(x+1) = x\Gamma(x)$, we have

$${}_2F_1\left(\frac{1}{2q}, \frac{1}{2}, \frac{3}{2}, 1\right) = \frac{\pi^{1/2}}{2}\frac{\Gamma(1-\frac{1}{2q})}{\Gamma(\frac{3}{2}-\frac{1}{2q})} \tag{2.7.34}$$

$$= \frac{\pi^{1/2}}{2}\frac{\Gamma(\frac{1}{2}+\frac{1}{2p})}{\Gamma(1+\frac{1}{2p})} \tag{2.7.35}$$

$$\tag{2.7.36}$$

Assume that $\varphi' \geq 0$, $\varphi(0) = -\left(2+\frac{1}{p}\right)^{1/2}$, and $\varphi(1) = \left(2+\frac{1}{p}\right)^{1/2}$. Sending $x \to 0$ and $\varphi \to -\left(2+\frac{1}{p}\right)^{1/2}$ in Eq. (2.7.32) gives

$$a_2 = -\left(2+\frac{1}{p}\right)^{\frac{1}{2}-\frac{1}{2q}} \frac{\pi^{1/2}}{2} \frac{\Gamma(\frac{1}{2}+\frac{1}{2p})}{\Gamma(1+\frac{1}{2p})}. \tag{2.7.37}$$

Then sending $x \to 1$ and $\varphi \to \left(2+\frac{1}{p}\right)^{1/2}$ in Eq. (2.7.32), we conclude that

$$\lambda = \widetilde{\lambda}_{*,\mathrm{Neu}} = \pi \, \frac{p+1}{p}\left(2-\frac{1}{p+1}\right)^{1/p}\left(\frac{\Gamma(\frac{1}{2}+\frac{1}{2p})}{\Gamma(1+\frac{1}{2p})}\right)^2. \tag{2.7.38}$$

Observations:

(1.) $\lim_{p\to 1} \lambda = 12$

This follows from $\Gamma(1) = 1$ and $\Gamma(\frac{3}{2}) = \dfrac{\pi^{1/2}}{2}$.

(2.) $\lim_{p\to\infty} \lambda = \pi^2$

This follows from $\Gamma(1) = 1$ and $\Gamma(\frac{1}{2}) = \pi^{1/2}$.

These limits correspond to known cases. In Jafarizadeh [33], the $p = 1$ case on the domain $\Omega = [0,1]$ gives $\lambda = 12$. The $p = \infty$ case is trivial. The integrand in Eq. (2.7.1) being non-negative, it is clear that the maximum of the integral is obtained when D is pointwise largest (which in this case, is allowed). Taking $D = constant$, the problem reduces to finding the second largest eigenvalue for the Laplacian on the interval $\Omega = [0,1]$, which is $\lambda = \pi^2$.

2.8 Derivation of the solution to the variational problem.

In this section, we formally derive the solution to the Neumann variational problem (2.2.18) for $\widetilde{\lambda}_{*,\mathrm{Neu}}$. The computations are later justified by Theorem 5. That is, we consider the variation

$$\inf_{\varphi\in\mathscr{B}} \sup_{\mathbb{D}\in\mathscr{A}} \int_\Omega \nabla\varphi(\boldsymbol{x})\cdot\mathbb{D}(\boldsymbol{x})\nabla\varphi(\boldsymbol{x})\,d\boldsymbol{x}\,. \tag{2.8.1}$$

The admissibility sets are

$$\mathscr{A} = \{\mathbb{D}(\boldsymbol{x}) \quad \text{s.t.} \quad \|\mathbb{D}\|_p = 1\}\,, \tag{2.8.2}$$

$$\mathscr{B} = \{\varphi\in W^{1,r}(\Omega) \quad \text{s.t.} \quad \int_\Omega \varphi\,d\boldsymbol{x} = 0,\ \|\varphi\|_2 = 1\}\,. \tag{2.8.3}$$

Introducing Lagrange multipliers (λ,ν,γ), the problem is equivalent to

$$\begin{aligned}\inf_{\varphi(x)} \sup_{\mathbb{D}(x)} &\int_\Omega \sum_{1\le i\le d}\sum_{1\le j\le d} D_{i,j}\,\partial_{x_i}\varphi\,\partial_{x_j}\varphi\,d\boldsymbol{x} \\ &+\lambda\left(1-\int_\Omega \varphi(\boldsymbol{x})^2\,d\boldsymbol{x}\right) \\ &+\nu\left(1-\int_\Omega |\mathbb{D}|^p\,d\boldsymbol{x}\right) \\ &+\gamma\int_\Omega \varphi(\boldsymbol{x})\,d\boldsymbol{x}\,.\end{aligned} \tag{2.8.4}$$

Notice that the derivative of $|\mathbb{D}|^p$ with respect to $D_{i,j}$ is $p|\mathbb{D}|^{p-2}D_{i,j}$. This may be seen as an application of the generalized binomial theorem: if $|x| > |y|$ then

$$(x+y)^R = \sum_{k=0}^{\infty} \binom{R}{k} x^{R-k} y^k \,. \tag{2.8.5}$$

Recall that $e_i \otimes e_j$ is a matrix with 1 in the $(i,j)^{\text{th}}$ component and 0 everywhere else. In what follows, repeated index does not imply summation over that index. We have

$$|\mathbb{D} + \partial D_{i,j}\, e_i \otimes e_j|^p = \big(|\mathbb{D} + \partial D_{i,j}\, e_i \otimes e_j|^2\big)^{p/2} \tag{2.8.6}$$

$$= \big(|\mathbb{D}|^2 + \big(2D_{i,j}\, \partial D_{i,j} + (\partial D_{i,j})^2\big)\big)^{p/2} \,. \tag{2.8.7}$$

Apply the generalized binomial theorem with $x = |\mathbb{D}|^2$ and $y = 2D_{i,j}\, \partial D_{i,j} + (\partial D_{i,j})^2$ to arrive at

$$|\mathbb{D} + \partial D_{i,j}\, e_i \otimes e_j|^p \tag{2.8.8}$$

$$= (|\mathbb{D}|^2)^{\frac{p}{2}} + \frac{p}{2}(|\mathbb{D}|^2)^{\frac{p}{2}-1}(2D_{i,j}\, \partial D_{i,j} + (\partial D_{i,j})^2) + O((\partial D_{i,j})^2) \tag{2.8.9}$$

$$= |\mathbb{D}|^p + p|\mathbb{D}|^{p-2} D_{i,j}\, \partial D_{i,j} + O((\partial D_{i,j})^2) \,. \tag{2.8.10}$$

Thus, the derivative of $|\mathbb{D}|^p$ with respect to $D_{i,j}$ is $p|\mathbb{D}|^{p-2}D_{i,j}$. We now compute the variation of $\sum_{i,j} D_{i,j}\, \partial_{x_i}\varphi\, \partial_{x_j}\varphi$ with respect to φ:

$$\sum_{i,j} D_{i,j}\, \partial_{x_i}(\varphi + \partial\varphi)\, \partial_{x_j}(\varphi + \partial\varphi) - \sum_{i,j} D_{i,j}\, \partial_{x_i}\varphi\, \partial_{x_j}\varphi \tag{2.8.11}$$

$$= \sum_{i,j} D_{i,j}\big(\partial_{x_i}\varphi\, \partial_{x_j}\partial\varphi + \partial_{x_j}\varphi\, \partial_{x_i}\partial\varphi\big) + O((\partial\varphi)^2) \,. \tag{2.8.12}$$

And so the derivative of $\int_\Omega \sum_{i,j} D_{i,j}\,\partial_{x_i}\varphi\,\partial_{x_j}\varphi$ with respect to φ is

$$-\sum_{i,j} \partial_{x_j}(D_{i,j}\partial_{x_i}\varphi) + \partial_{x_i}\left(D_{i,j}\partial_{x_j}\varphi\right). \tag{2.8.13}$$

Using the notation $\bar{\mathbb{D}} = (\mathbb{D} + \mathbb{D}^T)/2$, this expression is equivalent to

$$-\,2\,\nabla\cdot(\bar{\mathbb{D}}\nabla\varphi)\,. \tag{2.8.14}$$

The Euler-Lagrange system for the variational problem 2.8.4 is then

$$\begin{cases} \partial_{x_i}\varphi\,\partial_{x_j}\varphi - \nu p|\mathbb{D}|^{p-2}D_{i,j} = 0 \\ \nabla\cdot(\bar{\mathbb{D}}\nabla\varphi) + \lambda\,\varphi - \gamma/2 = 0\,, \end{cases} \tag{2.8.15}$$

with the natural boundary conditions

$$\mathbb{D}\nabla\varphi\cdot\boldsymbol{n} = 0 \quad \forall\boldsymbol{x}\in\partial\Omega\,, \tag{2.8.16}$$

where $\boldsymbol{n}$ is the outward pointing normal. Solving for $D_{i,j}$ in the first equations of Eq. (2.8.15), we have

$$D_{i,j} = (\nu p)^{-1}|\mathbb{D}|^{2-p}\,\partial_{x_i}\varphi\,\partial_{x_j}\varphi\,. \tag{2.8.17}$$

From this, we see that the optimal $\mathbb{D}$ is symmetric. Integrating the final equation of Eq. (2.8.15) in space, applying the boundary conditions, and applying that φ is mean zero, we see that $\gamma = 0$. From Eq. (2.8.17), we have

$$|\mathbb{D}|^2 = \sum_{i,j} D_{i,j}^2 \tag{2.8.18}$$

$$= |\nu p|^{-2} |\mathbb{D}|^{2(2-p)} \sum_{i,j} (\partial_{x_i}\varphi)^2 \, (\partial_{x_j}\varphi)^2 \tag{2.8.19}$$

$$= |\nu p|^{-2} |\mathbb{D}|^{2(2-p)} \left(\sum_i (\partial_{x_i}\varphi)^2 \right)^2 \tag{2.8.20}$$

$$= |\nu p|^{-2} |\mathbb{D}|^{2(2-p)} |\nabla\varphi|^4 \, . \tag{2.8.21}$$

And so

$$|\mathbb{D}| = |\nu p|^{\frac{-1}{p-1}} |\nabla\varphi|^{\frac{2}{p-1}} \, . \tag{2.8.22}$$

Therefore Eq. (2.8.17) becomes

$$D_{i,j} = sgn(\nu) \, |\nu p|^{\frac{-1}{p-1}} |\nabla\varphi|^{\frac{2(2-p)}{p-1}} \, \partial_{x_i}\varphi \, \partial_{x_j}\varphi \, , \tag{2.8.23}$$

where $sgn(\nu)$ is 1 if $\nu > 0$ and -1 if $\nu < 0$. Using this, we see that the j^{th} component of $\mathbb{D}\nabla\varphi$ is

$$\sum_i D_{i,j} \, \partial_{x_i}\varphi = sgn(\nu) \, |\nu p|^{\frac{-1}{p-1}} |\nabla\varphi|^{\frac{2(2-p)}{p-1}} \sum_i (\partial_{x_i}\varphi)^2 \partial_{x_j}\varphi \tag{2.8.24}$$

$$= sgn(\nu) \, |\nu p|^{\frac{-1}{p-1}} |\nabla\varphi|^{\frac{2(2-p)}{p-1}+2} \partial_{x_j}\varphi \, . \tag{2.8.25}$$

Therefore

$$\mathbb{D}\nabla\varphi = sgn(\nu) \, |\nu p|^{\frac{-1}{p-1}} |\nabla\varphi|^{\frac{2}{p-1}} \nabla\varphi \, , \tag{2.8.26}$$

and the last equation of Eq. (2.8.15) becomes

$$sgn(\nu)\,|\nu p|^{\frac{-1}{p-1}}\nabla\cdot\left(|\nabla\varphi|^{\frac{2}{p-1}}\nabla\varphi\right)+\lambda\,\varphi=0\,. \tag{2.8.27}$$

Let $\tilde{\lambda}=\lambda\,sgn(\nu)\,|\nu p|^{\frac{1}{p-1}}$ and $r=2p^*$ where the conjugate exponent p^* is given by $\frac{1}{p}+\frac{1}{p*}=1$. Define the $r-$Laplacian operator by $\Delta_r\varphi=\nabla\cdot(|\nabla\varphi|^{r-2}\nabla\varphi)$. Then Eq. (2.8.27) becomes

$$\Delta_r\varphi=-\tilde{\lambda}\,\varphi\,. \tag{2.8.28}$$

Multiply Eq. (2.8.27) by φ, integrate over Ω, integrate by parts, and apply the boundary condition to arrive at

$$\int_\Omega sgn(\nu)\,|\nu p|^{\frac{-1}{p-1}}|\nabla\varphi|^{2+\frac{2}{p-1}}d\boldsymbol{x}=\lambda\int_\Omega\varphi^2 d\boldsymbol{x}\,. \tag{2.8.29}$$

Apply Eq. (2.8.22) to get

$$sgn(\nu)\,|\nu p|^{\frac{-1}{p-1}+\frac{p}{p-1}}\int_\Omega|\mathbb{D}|^p d\boldsymbol{x}=\lambda\int_\Omega\varphi^2 d\boldsymbol{x}\,. \tag{2.8.30}$$

Recall the constraints $\int_\Omega|\mathbb{D}|^p\,d\boldsymbol{x}=1$ and $\|\varphi\|_2=1$. Therefore, $sgn(\nu)\,|\nu p|=\lambda$. Or, more simply, $\nu p=\lambda$. We conclude that

$$\tilde{\lambda}=|\lambda|^{p'}\,. \tag{2.8.31}$$

2.9 Derivation with $L^p(\ell^p)$ norm constraint

This time, we consider a different norm constraint on the matrix $\mathbb{D}$. Take $\mathbb{D}$ with constraint $\|\mathbb{D}\|_{L^p(\ell^p)} = 1$, for some $p > 1$, defined by

$$\|\mathbb{D}\|_{L^p(\ell^p)}^p = \int_\Omega \sum_{1\le i\le d} \sum_{1\le j\le d} |D_{i,j}|^p \, d\boldsymbol{x} \,. \tag{2.9.1}$$

The above definition is consistent with first taking the ℓ^p norm of $\mathbb{D}$, considering $\mathbb{D}$ as a vector in $\mathbb{R}^{(d^2)}$, then taking the L^p norm of the resulting scalar function. This norm is consistent with the definition of the norm of a vector that one sees, for example, in the definition of the $W^{1,p}$ norm.

Since ℓ^p norms on finite dimensional spaces are equivalent, this norm is equivalent to that used in the previous section. Though it is an equivalent norm, we will derive a different Euler-Lagrange equation, a "diagonal" $p-$Laplacian eigenvalue equation [59].

Notice that the derivative of $\|\mathbb{D}\|_{L^p(\ell^p)}^p$ with respect to $D_{i,j}$ is $\int p|D_{i,j}|^{p-2}D_{i,j}$. This may be seen as an application of the generalized binomial theorem: if $|x| > |y|$ then

$$(x+y)^R = \sum_{k=0}^{\infty} \binom{R}{k} x^{R-k} y^k \,. \tag{2.9.2}$$

Expanding, we have

$$|D_{i,j} + \partial D_{i,j}|^p = \big(|D_{i,j} + \partial D_{i,j}|^2\big)^{p/2} \tag{2.9.3}$$

$$= \big(|D_{i,j}|^2 + \big(2D_{i,j}\,\partial D_{i,j} + (\partial D_{i,j})^2\big)\big)^{p/2} . \tag{2.9.4}$$

Apply the generalized binomial theorem with $x = |D_{i,j}|^2$ and $y = 2D_{i,j}\,\partial D_{i,j} + (\partial D_{i,j})^2$ to arrive at

$$|D_{i,j} + \partial D_{i,j}|^p \tag{2.9.5}$$

$$= (|D_{i,j}|^2)^{\frac{p}{2}} + \frac{p}{2}(|D_{i,j}|^2)^{\frac{p}{2}-1}(2D_{i,j}\,\partial D_{i,j} + (\partial D_{i,j})^2) + O((\partial D_{i,j})^2) \tag{2.9.6}$$

$$= |D_{i,j}|^p + p|D_{i,j}|^{p-2}D_{i,j}\,\partial D_{i,j} + O((\partial D_{i,j})^2) . \tag{2.9.7}$$

Thus, the derivative of $\|\mathbb{D}\|^p_{L^p(\ell^p)}$ with respect to $D_{i,j}$ is $\int p|D_{i,j}|^{p-2}D_{i,j}$.

The Euler-Lagrange system for the variational problem is then

$$\begin{cases} \partial_{x_i}\varphi\,\partial_{x_j}\varphi - \nu p|D_{i,j}|^{p-2}D_{i,j} = 0 \\ \nabla\cdot(\bar{\mathbb{D}}\nabla\varphi) + \lambda_{p,p}\,\varphi - \gamma/2 = 0 , \end{cases} \tag{2.9.8}$$

with the natural boundary conditions

$$\mathbb{D}\nabla\varphi\cdot\boldsymbol{n} = 0 \quad \forall\boldsymbol{x}\in\partial\Omega , \tag{2.9.9}$$

where $\boldsymbol{n}$ is the outward pointing normal. Rearranging the first equations of Eq. (2.9.8), we have

$$|D_{i,j}|^{p-2} D_{i,j} = (\nu p)^{-1}\, \partial_{x_i}\varphi\, \partial_{x_j}\varphi\,. \tag{2.9.10}$$

Taking absolute values we see that

$$|D_{i,j}|^{p-1} = \left|(\nu p)^{-1}\, \partial_{x_i}\varphi\, \partial_{x_j}\varphi\right|. \tag{2.9.11}$$

Rearranging Eq. (2.9.10) and using Eq. (2.9.11) we have

$$\begin{aligned} D_{i,j} &= (\nu p)^{-1}\, |D_{i,j}|^{2-p}\, \partial_{x_i}\varphi\, \partial_{x_j}\varphi && (2.9.12)\\ &= (\nu p)^{-1}\, |(\nu p)^{-1}\, \partial_{x_i}\varphi\, \partial_{x_j}\varphi|^{\frac{2-p}{p-1}}\, \partial_{x_i}\varphi\, \partial_{x_j}\varphi && (2.9.13)\\ &= sgn(\nu)\, |\nu p|^{\frac{-1}{p-1}} |\partial_{x_i}\varphi\, \partial_{x_j}\varphi|^{\frac{2-p}{p-1}}\, \partial_{x_i}\varphi\, \partial_{x_j}\varphi\,. && (2.9.14) \end{aligned}$$

From this, we see that the optimal $\mathbb{D}$ is symmetric. We now notice that the j^{th} component of $\mathbb{D}\nabla\varphi$ is

$$\begin{aligned} \sum_i D_{i,j}\, \partial_{x_i}\varphi &= sgn(\nu)\, |\nu p|^{\frac{-1}{p-1}} \left(\sum_i |\partial_{x_i}\varphi|^{\frac{2-p}{p-1}} (\partial_{x_i}\varphi)^2\right) |\partial_{x_j}\varphi|^{\frac{2-p}{p-1}}\, \partial_{x_j}\varphi && (2.9.15)\\ &= sgn(\nu)\, |\nu p|^{\frac{-1}{p-1}} \left(\sum_i |\partial_{x_i}\varphi|^{\frac{p}{p-1}}\right) |\partial_{x_j}\varphi|^{\frac{2-p}{p-1}}\, \partial_{x_j}\varphi && (2.9.16)\\ &= sgn(\nu)\, |\nu p|^{\frac{-1}{p-1}}\, \|\nabla\varphi\|_{\ell^{\frac{p}{p-1}}}^{\frac{p}{p-1}}\, |\partial_{x_j}\varphi|^{\frac{2-p}{p-1}}\, \partial_{x_j}\varphi\,. && (2.9.17) \end{aligned}$$

Let p^* be the conjugate exponent given by $\frac{1}{p} + \frac{1}{p*} = 1$. Notice that $p' = \frac{p}{p-1}$. We have

$$\sum_i D_{i,j}\, \partial_{x_i}\varphi = sgn(\nu)\, |\nu p|^{\frac{-1}{p-1}}\, \|\nabla\varphi\|_{\ell^{p^*}}^{p^*}\, |\partial_{x_j}\varphi|^{p'-2}\, \partial_{x_j}\varphi\,. \tag{2.9.18}$$

Integrating the final equation of Eq. (2.9.8) over Ω, applying the boundary conditions, and using that φ is mean zero, we see that $\gamma = 0$. And the last equation of system 2.9.8 becomes

$$\sum_i \partial_{x_i}\left(\|\nabla\varphi\|_{\ell^{p^*}}^{p^*} \, |\partial_{x_i}\varphi|^{p'-2}\, \partial_{x_i}\varphi \right) = -\tilde{\lambda}_{p,p}\, \varphi\,, \tag{2.9.19}$$

where $\tilde{\lambda}_{p,p} = \lambda_{p,p}\, sgn(\nu)\, |\nu p|^{\frac{1}{p-1}}$. This equation is related to the second order ordinary differential equation investigated by [59].

Multiply Eq. (2.9.19) by φ, integrate over Ω, integrate by parts, and apply the boundary condition to arrive at

$$\int_\Omega \|\nabla\varphi\|_{\ell^{p^*}}^{2p^*}\, d\boldsymbol{x} = \tilde{\lambda}_{p,p} \int_\Omega \varphi^2 d\boldsymbol{x}\,. \tag{2.9.20}$$

Notice that Eq. (2.9.11) implies

$$\|\nabla\varphi\|_{\ell^{p^*}}^{2p^*} = |\nu p|^{p'} \|\mathbb{D}\|_{\ell^p}^{p}\,. \tag{2.9.21}$$

Recall the constraints $\|\mathbb{D}\|_{L^p(\ell^p)}^p = 1$ and $\|\varphi\|_2 = 1$. From Eq. (2.9.20) we have $|\nu p|^{p'} = \tilde{\lambda}_{p,p}$. Therefore, $\nu p = \lambda_{p,p}$, and we conclude that

$$\tilde{\lambda}_{p,p} = |\lambda_{p,p}|^{p'}\,. \tag{2.9.22}$$

2.10 Proof of Theorems 4 and 5

Before proving Theorem 4, we need a Lemma due to Ekeland and Témam [20], Ch. 6. For completion, we repeat the proof here.

Lemma 4. Let $\mathscr{A}$ and $\mathscr{B}$ be arbitrary sets. A function $\mathcal{I}(\mathbb{D},\varphi) : \mathscr{A} \times \mathscr{B} \to \mathbb{R}$ has the property that

$$\sup_{\mathbb{D}\in\mathscr{A}} \inf_{\varphi\in\mathscr{B}} \mathcal{I}(\mathbb{D},\varphi) \leq \inf_{\varphi\in\mathscr{B}} \sup_{\mathbb{D}\in\mathscr{A}} \mathcal{I}(\mathbb{D},\varphi)\,. \tag{2.10.1}$$

Proof. From the definition of infimum,

$$\inf_{\varphi\in\mathscr{B}} \mathcal{I}(\mathbb{D},\varphi) \leq \mathcal{I}(\mathbb{D},v) \quad \forall \mathbb{D} \in \mathscr{A}\,, v \in \mathscr{B}\,. \tag{2.10.2}$$

Take the supremum over $\mathbb{D}$ of both sides to find

$$\sup_{\mathbb{D}\in\mathscr{A}} \inf_{\varphi\in\mathscr{B}} \mathcal{I}(\mathbb{D},\varphi) \leq \sup_{\mathbb{D}\in\mathscr{A}} \mathcal{I}(\mathbb{D},v) \quad \forall v \in \mathscr{B}\,. \tag{2.10.3}$$

The right side is uniformly (in v) bounded below by the left side. Therefore, the left side is still a lower bound for the infimum (over v) of the right side. We conclude that

$$\sup_{\mathbb{D}\in\mathscr{A}} \inf_{\varphi\in\mathscr{B}} \mathcal{I}(\mathbb{D},\varphi) \leq \inf_{\varphi\in\mathscr{B}} \sup_{\mathbb{D}\in\mathscr{A}} \mathcal{I}(\mathbb{D},\varphi)\,. \tag{2.10.4}$$

□

We now prove Theorem 4. Recall its statement:

Theorem 4. *Let* $1 < p < \infty$ *and take* $r = 2p'$ *where* p' *is the conjugate exponent of* p *such that* $1/p + 1/p' = 1$.

1.) *[Dirichlet Boundary Conditions] Define*

$$\widetilde{\lambda}_{*,\mathrm{Dir}} := \mathop{inf}_{\varphi \in \mathscr{B}_{\mathrm{Dir}}} \mathop{sup}_{\mathbb{D} \in \mathscr{A}} \int_\Omega \nabla\varphi \cdot \mathbb{D}\nabla\varphi \, d\boldsymbol{x} \,, \tag{2.2.14}$$

where

$$\mathscr{A} = \{\mathbb{D} \in L^p(\Omega) \quad s.t. \quad \|\mathbb{D}\|_{L^p} = 1\} \;\; and \tag{2.2.15}$$

$$\mathscr{B}_{\mathrm{Dir}} = \{\varphi \in W_0^{1,r}(\Omega) \quad s.t. \quad \|\varphi\|_{L^2} = 1\} \,. \tag{2.2.16}$$

We have that $\lambda_{*,\mathrm{Dir}} \leq \widetilde{\lambda}_{*,\mathrm{Dir}}$.

2.) *[Neumann Boundary Conditions] Define*

$$\lambda_{*,\mathrm{Neu}} := \mathop{sup}_{\sigma > 0} \mathop{sup}_{\mathbb{D} \in \mathscr{A}_\sigma} \mathop{min}_{\varphi \in H^1(\Omega),\, \varphi \neq 0\,,\int \varphi = 0} \frac{\int_\Omega \nabla\varphi \cdot \mathbb{D}\nabla\varphi \, d\boldsymbol{x}}{\int_\Omega \varphi^2 \, d\boldsymbol{x}} \,. \tag{2.2.17}$$

Also define

$$\widetilde{\lambda}_{*,\mathrm{Neu}} := \mathop{inf}_{\varphi \in \mathscr{B}_{\mathrm{Neu}}} \mathop{sup}_{\mathbb{D} \in \mathscr{A}} \int_\Omega \nabla\varphi \cdot \mathbb{D}\nabla\varphi \, d\boldsymbol{x} \,, \tag{2.2.18}$$

where

$$\mathscr{B}_{\mathrm{Neu}} = \{\varphi \in W^{1,r}(\Omega) \quad s.t. \quad \int_\Omega \varphi \, d\boldsymbol{x} = 0, \; \|\varphi\|_{L^2} = 1\} \,. \tag{2.2.19}$$

We have that $\lambda_{*,\mathrm{Neu}} \leq \widetilde{\lambda}_{*,\mathrm{Neu}}$.

Proof. Recall the original variational problem for $\lambda_{*,\mathrm{Dir}}$, that is Eq. (2.2.13):

$$\lambda_{*,\mathrm{Dir}} := \sup_{\sigma>0} \sup_{\mathbb{D}\in\mathscr{A}_\sigma} \min_{\varphi\in H_0^1(\Omega),\,\varphi\neq 0} \frac{\int_\Omega \nabla\varphi\cdot\mathbb{D}\nabla\varphi\,d\boldsymbol{x}}{\int_\Omega \varphi^2\,d\boldsymbol{x}}\,, \tag{2.10.5}$$

where

$$\mathscr{A}_\sigma = \begin{cases} \mathbb{D}(\boldsymbol{x})\in L^p(\Omega) \quad \text{s.t.} \quad D_{i,j}(\boldsymbol{x})\in C^\infty(\bar{\Omega})\,, \\ \qquad\qquad\qquad\qquad\quad D_{i,j} = D_{j,i}\,, \\ \qquad\qquad\qquad\qquad\quad \|\mathbb{D}\|_{L^p} = 1\,,\ \text{and} \\ \qquad\qquad\qquad\qquad\quad \boldsymbol{\xi}\cdot\mathbb{D}(\boldsymbol{x})\boldsymbol{\xi} \geq \sigma|\boldsymbol{\xi}|^2 \quad \forall\boldsymbol{\xi}\in\mathbb{R}^n, \boldsymbol{x}\in\Omega\,. \end{cases} \tag{2.10.6}$$

Since $\bigcup_{\sigma>0}\mathscr{A}_\sigma \subset \mathscr{A} = \{\mathbb{D}\in L^p(\Omega) \quad \text{s.t.} \quad \|\mathbb{D}\|_{L^p} = 1\}$, it is clear that

$$\lambda_{*,\mathrm{Dir}} \leq \sup_{\mathbb{D}\in\mathscr{A}} \min_{\varphi\in H_0^1(\Omega),\,\varphi\neq 0} \frac{\int_\Omega \nabla\varphi\cdot\mathbb{D}\nabla\varphi\,d\boldsymbol{x}}{\int_\Omega \varphi^2\,d\boldsymbol{x}}\,. \tag{2.10.7}$$

Similarly, recall that $r > 2$ and the L^p spaces are nested on the bounded domain Ω. We see that $W_0^{1,r} \subset H_0^1$ and so we have

$$\sup_{\mathbb{D}\in\mathscr{A}} \min_{\varphi\in H_0^1(\Omega),\,\varphi\neq 0} \frac{\int_\Omega \nabla\varphi\cdot\mathbb{D}\nabla\varphi\,d\boldsymbol{x}}{\int_\Omega \varphi^2\,d\boldsymbol{x}} \leq \sup_{\mathbb{D}\in\mathscr{A}} \inf_{\varphi\in W_0^{1,r}(\Omega),\,\varphi\neq 0} \frac{\int_\Omega \nabla\varphi\cdot\mathbb{D}\nabla\varphi\,d\boldsymbol{x}}{\int_\Omega \varphi^2\,d\boldsymbol{x}}\,. \tag{2.10.8}$$

Normalize φ in L^2 to see that

$$\sup_{\mathbb{D}\in\mathscr{A}} \min_{\varphi\in H_0^1(\Omega),\,\varphi\neq 0} \frac{\int_\Omega \nabla\varphi\cdot\mathbb{D}\nabla\varphi\,d\boldsymbol{x}}{\int_\Omega \varphi^2\,d\boldsymbol{x}} = \sup_{\mathbb{D}\in\mathscr{A}} \inf_{\varphi\in\mathscr{B}_{\mathrm{Dir}}} \int_\Omega \nabla\varphi\cdot\mathbb{D}\nabla\varphi\,d\boldsymbol{x}\,, \tag{2.10.9}$$

where

$$\mathscr{B}_{\mathrm{Dir}} = \{\varphi \in W_0^{1,r}(\Omega) \quad \text{s.t.} \quad \|\varphi\|_{L^2} = 1\}\,. \tag{2.10.10}$$

Finally, we apply Lemma 4 and see that

$$\lambda_{*,\mathrm{Dir}} \leq \widetilde{\lambda}_{*,\mathrm{Dir}} := \inf_{\varphi \in \mathscr{B}_{\mathrm{Dir}}} \sup_{\mathbb{D} \in \mathscr{A}} \int_\Omega \nabla\varphi \cdot \mathbb{D}\nabla\varphi \, d\boldsymbol{x}\,, \tag{2.10.11}$$

where we have recalled the related variational problem for $\widetilde{\lambda}_{*,\mathrm{Dir}}$ from Eq. (2.2.14). The proof of $\lambda_{*,\mathrm{Neu}} \leq \widetilde{\lambda}_{*,Nue}$ is similar.

□

We finish this section by proving

Theorem 5. *Given $\varphi(\boldsymbol{x}) \in W^{1,r}(\Omega)$, the spatially dependent matrix*

$$\mathbb{D}_\varphi(\boldsymbol{x}) := \left(\int_\Omega |\nabla\varphi|^r \, d\boldsymbol{x}\right)^{-1/p} |\nabla\varphi|^{r-4} \nabla\varphi \, \nabla\varphi^\top \tag{2.2.20}$$

maximizes the variation

$$\mathop{sup}_{\mathbb{D} \in L^p,\, \|\mathbb{D}\|_{L^p} = 1} \int_\Omega \nabla\varphi \cdot \mathbb{D}\nabla\varphi \, d\boldsymbol{x}\,. \tag{2.2.21}$$

Proof of Theorem 5. First, we check that $\mathbb{D}_\varphi$ satisfies the condition $\|\mathbb{D}_\varphi\|_p = 1$.

$$\int_\Omega |\mathbb{D}_\varphi|^p \, d\boldsymbol{x} = \left(\int_\Omega |\nabla\varphi|^{2p'} \, d\boldsymbol{x}\right)^{-1} \int_\Omega |\nabla\varphi|^{\frac{2p(2-p)}{p-1}} \left|\nabla\varphi \, \nabla\varphi^\top\right|^p d\boldsymbol{x} \tag{2.10.12}$$

$$= \left(\int_\Omega |\nabla\varphi|^{2p'} \, d\boldsymbol{x}\right)^{-1} \int_\Omega |\nabla\varphi|^{\frac{2p(2-p)}{p-1}+2p} \, d\boldsymbol{x} \tag{2.10.13}$$

$$= 1\,. \tag{2.10.14}$$

Now, we apply the Cauchy-Schwarz inequality for vectors and Hölder's inequality for functions to arrive at the bound

$$\int_\Omega \nabla\varphi \cdot \mathbb{D}\, \nabla\varphi \, d\boldsymbol{x} \tag{2.10.15}$$

$$= \int_\Omega \sum_{i,j=1}^n D_{i,j}\, \varphi_{x_i}\varphi_{x_j} \, d\boldsymbol{x} \tag{2.10.16}$$

$$\leq \int_\Omega \left(\sum_{i,j=1}^n D_{i,j}^2\right)^{1/2} \left(\sum_{i,j=1}^n \varphi_{x_i}^2\varphi_{x_j}^2\right)^{1/2} d\boldsymbol{x} \tag{2.10.17}$$

$$= \int_\Omega \left(\sum_{i,j=1}^n D_{i,j}^2\right)^{1/2} |\nabla\varphi|^2 \, d\boldsymbol{x} \tag{2.10.18}$$

$$\leq \|\mathbb{D}\|_p \left(\int_\Omega |\nabla\varphi|^{2p'} \, d\boldsymbol{x}\right)^{1/p'} \tag{2.10.19}$$

$$\leq \left(\int_\Omega |\nabla\varphi|^{2p'} \, d\boldsymbol{x}\right)^{1/p'} . \tag{2.10.20}$$

We finish the proof by showing that $\mathbb{D}_\varphi$ achieves this upper bound.

$$\int_\Omega \nabla\varphi \cdot \mathbb{D}_\varphi \nabla\varphi \, d\boldsymbol{x} \tag{2.10.21}$$

$$= \left(\int_\Omega |\nabla\varphi|^{2p'} \, d\boldsymbol{x}\right)^{-1/p} \int_\Omega |\nabla\varphi|^{\frac{2(2-p)}{p-1}} \nabla\varphi \cdot \nabla\varphi \, \nabla\varphi^\top \nabla\varphi \, d\boldsymbol{x} \tag{2.10.22}$$

$$= \left(\int_\Omega |\nabla\varphi|^{2p'} \, d\boldsymbol{x}\right)^{-1/p} \int_\Omega |\nabla\varphi|^{\frac{2(2-p)}{p-1}+4} \, d\boldsymbol{x} \tag{2.10.23}$$

$$= \left(\int_\Omega |\nabla\varphi|^{2p'} \, d\boldsymbol{x}\right)^{-1/p} \int_\Omega |\nabla\varphi|^{2p'} \, d\boldsymbol{x} \tag{2.10.24}$$

$$= \left(\int_\Omega |\nabla\varphi|^{2p'} \, d\boldsymbol{x}\right)^{1/p'} . \tag{2.10.25}$$

□

2.11 Bounding $\widetilde{\lambda}_{*,\mathrm{Neu}}$ from below

It is well known that the Poincaré inequality holds given certain conditions on the domain Ω. The following Theorem is from Evans [22].

Theorem 6. (Poincaré's inequality) *Let Ω be a bounded, connected, open subset of $\mathbb{R}^n$, with a C^1 boundary $\partial\Omega$. Assume $1 \leq r \leq \infty$. Then there exists a constant C_1 depending on n, r and Ω such that*

$$\left\| \varphi - \frac{1}{|\Omega|} \int_\Omega \varphi \, d\boldsymbol{x} \right\|_{L^r(\Omega)} \leq C_1 \left\| \nabla\varphi \right\|_{L^r(\Omega)} \tag{2.11.1}$$

for each function $\varphi \in W^{1,r}(\Omega)$.

We have used the notation that $|\Omega|$ is the volume of the domain Ω. If we apply

Hölder's inequality, we immediately see that for $1 \le q \le r$ we have a "Poincaré type inequality"

$$\left\| \varphi - \frac{1}{|\Omega|} \int_\Omega \varphi \, d\boldsymbol{x} \right\|_{L^q(\Omega)} \le C_2 \left\| \nabla \varphi \right\|_{L^r(\Omega)}, \tag{2.11.2}$$

where $C_2 \le |\Omega|^{\frac{1}{q}-\frac{1}{r}} C_1$. We are fortunate that, for us, $q = 2$ and $1 < p < \infty$ implies $r = 2p' > 2$, and so Eq. (2.11.2) holds. Unfortunately for us, the exact constant (and the extremal function $\varphi_{*,\mathrm{Neu}}$) in Eq. (2.11.2) is not known for this choice of q and r. Exact constants in Eq. (2.11.2) are known in only 4 cases (see the recent 2015 survey Kuznetsov and Nazarov [37]):

- $r = q = 2$ (the quadratic case);
- $\Omega = (0, \ell)$ (the one-dimensional case);
- $r < n$, $q = \dfrac{nr}{n-r}$ (the critical case);
- $r = q = 1$ (the "geometric" case).

We considered the $n = 1$ case in Sections 2.4 to 2.7, where much can be made explicit. However, we now wish to consider the arbitrary dimension n.

Instead of exact constants, we turn to the literature for upper bounds on the constants. For example, we may apply a result due to Payne and Weinberger [60]: when the domain is convex and $r = 2$, we have $C_1 \le \mathrm{diam}(\Omega)/\pi$ in Eq. (2.11.1). Applying Hölder's inequality, we have the following bound on the constant C_{Neu} from the generalized $(2, r)-$Poincaré inequality Eq. (2.2.26):

$$C_{\mathrm{Neu}} \le \frac{\mathrm{diam}(\Omega)\, |\Omega|^{\frac{1}{2p}}}{\pi}. \tag{2.11.3}$$

Recall from Corollary 2 that $\widetilde{\lambda}_{*,\mathrm{Neu}} = C_{\mathrm{Neu}}^{-2}$. We conclude that

$$\widetilde{\lambda}_{*,\mathrm{Neu}} \geq \frac{\pi^2}{\mathrm{diam}(\Omega)^2\,|\Omega|^{1/p}}\,. \tag{2.11.4}$$

We remark that the bound Eq. (2.11.3) is sharp in the power of volume V [9]. Consider, for example, the domain $\Omega = [0,1] \times [0,1/k]^{n-1}$ and the test function $\varphi(\boldsymbol{x}) = x_1$. We see that

$$\frac{\|\varphi\|_{L^2}}{\|\nabla\varphi\|_{L^r}} = \sqrt{3}\,V^{\frac{1}{2}-\frac{1}{r}} = \sqrt{3}\,V^{\frac{1}{2p}}\,. \tag{2.11.5}$$

Comparison to spatially independent diffusivity

We wish to compare this result to the spatially independent diffusivity case. Take the diffusion matrix to be a constant times the identity, $\mathbb{D}_{const.} = c\,\mathbb{I}$. Then the constraint $\|\mathbb{D}_{const.}\|_p = 1$ implies $c = n^{-1/2}\,V^{-1/p}$. The variational problem (Eq. (2.2.14)) becomes

$$c\,\Delta\varphi = -\lambda_{const.}\,\varphi\,. \tag{2.11.6}$$

Therefore, $C_1 \leq \mathrm{diam}(\Omega)/\pi$ from Payne and Weinberger [60] implies

$$\frac{\lambda_{const.}}{c} \geq \frac{\pi^2}{\mathrm{diam}(\Omega)^2}\,, \tag{2.11.7}$$

and so

$$\lambda_{const.} \geq n^{-1/2}\,\frac{\pi^2}{\mathrm{diam}(\Omega)^2\,|\Omega|^{1/p}}\,. \tag{2.11.8}$$

We notice that the lower bound Eq. (2.11.4) is an improvement over the lower bound Eq. (2.11.8) by a factor of $n^{1/2}$.

2.12 Future Work

Recall that we are interested in accelerating the rate of decay of the heat equation by choosing optimal coefficients. We formulated the corresponding optimization problem for $\lambda_{*,\cdot}$. Instead of solving this optimization problem, we studied the corresponding relaxed problem for $\widetilde{\lambda}_{*,\cdot}$. We were able to characterize the solution to this relaxed problem using the extremals of well known Sobolev and Poincaré type inequalities. Moreover, we have proved $\lambda_{*,\cdot} \leq \widetilde{\lambda}_{*,\cdot}$.

For future work, we plan to study the $\Gamma-$convergence of the original variational problem to the relaxed problem with the hope of showing $\lambda_{*,\cdot} = \widetilde{\lambda}_{*,\cdot}$, see Allaire [1]. This requires us to bound $\int \nabla\varphi \cdot \mathbb{D}_{*,\cdot}\nabla\varphi \, d\boldsymbol{x}$ from below in terms of $\|\nabla\varphi\|_{L^2}^2$ without the use of the uniform ellipticity assumption. To show this, we need that $\boldsymbol{\xi} \cdot \mathbb{D}_{*,\cdot}\boldsymbol{\xi}$ is not zero "too often".

By this, we mean there is a $\sigma(\boldsymbol{x}) \geq 0$ such that $\boldsymbol{\xi} \cdot \mathbb{D}(\boldsymbol{x})\boldsymbol{\xi} \geq \sigma(\boldsymbol{x})|\boldsymbol{\xi}|^2$ for all $\boldsymbol{\xi} \in \mathbb{R}^n, \boldsymbol{x} \in \Omega$, and $\sigma(\boldsymbol{x})$ is not equal to zero "too often" in the sense that $\sigma^{-1} \in L^q$ for some q sufficiently large. In that case, we may prove results about the existence/regularity of solutions to and spectrum of the heat equation, see [2, 56]. Therefore, we must study the properties of the extremal functions (if they exist) in the Sobolev and Poincaré type inequalities. This is an open problem; in general, the extremal functions for the Sobolev and Poincaré type inequalities that we consider in this chapter are not known [37]. Perhaps finding the extremal functions is too difficult, but studying the order of the zeros of their gradients may be tractable.

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