Supporting Pre-Service Teachers' Skill of Crafting a Pedagogical Response That Builds on Children's Mathematical Thinking

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Abstract

Teaching that attends to and builds on students' resources and treats students as capable mathematical sense makers has been referred to as responsive teaching. A body of research reports teachers' tendency to overlook students' mathematical thinking when their answer is correct and to correct it procedurally when students make a mistake. This practice prevents teachers from gaining insights about students' underlying mathematical understanding regardless of the correctness of the answer and noticing the potential for conceptual understanding. Teachers need support to use students' mathematical thinking as a source for mathematically responsive pedagogical actions.

In many studies, pre-service teachers' responsive pedagogical practice is studied in the later stages of the teacher education program in relation to field placements, which may cause a separation between learning to teach and doing the work of teaching. In this study, I investigate the potential in the mathematics content course to support PSTs in developing their ability to respond to students' mathematical thinking to deepen their mathematical understanding. To that end, I designed a semester-long intervention in the content course which is typically the first course PSTs take in relation to learning to teach mathematics. I used Jacobs et al.'s (2010) framework of professional noticing of children's mathematical thinking to design the course materials.

The key finding of this research uncovered the potential in content courses, where PSTs are beginning to develop their mathematical knowledge for teaching, to address the skills required to notice children's mathematical thinking. In addition, the results confirmed previous research findings by showing that a high level of attending to student ideas is necessary to develop a response that leverages students' mathematical thinking; solely developing this skill is insufficient to craft sophisticated responses. Pre-service teachers in this study began to probe each other's thinking by asking for evidence and to challenge each other about how the proposed response aligned with the instructional purpose. The findings indicate that PSTs could broaden their vision of better possibilities to leverage children's mathematical understanding. Implications for future research call for aligning noticing with asset-based mathematical teaching.

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Introduction

Throughout history, discussions about what it means to learn mathematics have been motivated by societal demands and changing perspectives on learning. For example, from a cognitive perspective of learning mathematics, scholars argue that students are expected to develop mathematical proficiency that has been outlined in five strands: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition (Kilpatrick et al., 2001). Scholars from other perspectives argue that learning mathematics also entails cultural, social, and critical dimensions of learning. Culture, social interaction, participation structure, and collaboration mediate mathematical sense-making (Lerman, 2000; Nasir & Hand, 2006). While both cultural and cognitive perspectives have contributed significant inroads into students' mathematical sensemaking (Nasir & de Royston, 2013), learning and teaching mathematics remains predominantly structured around procedural fluency with little focus on students' sense-making and higher-order thinking, and it is characterized by an initiate-response-feedback participation structure. These practices contribute, from a sociopolitical perspective, to privileging certain ways of doing mathematics and positioning students who meet the expectation of this form of mathematics as capable and others as incapable (Louie, 2018). Scholars argue that creating a learning environment that is responsive to diverse learners, where they can engage in authentic mathematical work and demonstrate conceptual understanding is imperative for equitable mathematics teaching.

Teaching that attends to and builds on students' productive and diverse resources and treats all students as capable mathematical sense makers has been referred to as responsive teaching. This view of teaching requires teachers to gauge and elevate the mathematical value of students' thinking and use current student understanding to foster deeper mathematical understanding (Jacobs et al., 2010). Responsiveness to students' mathematical thinking is important particularly in light of what studies show about teachers' tendency to deemphasize investigating and extending students' mathematical thinking when students reach the correct result in given mathematics tasks and relying on procedural correction when they notice student mistakes (Munson, 2019; Son, 2013). This pervasive teaching practice prevents teachers from gaining insights about students' underlying mathematical understanding. Students who reach the correct solution need to be given an opportunity to reflect on their thinking, which may help teachers to gain insights about their mathematical conceptualization and to create opportunities for further connections within mathematical ideas. Teachers should also be able to see the reflected understanding in students thinking regardless of the correctness of the answer, as well as identifying the conceptual ideas that needs to be supported. Student errors create potential for conceptual understanding, and they should be seen as windows to gain insights about their mathematical reasoning instead of dead ends (Larrain & Kaiser, 2022; Son, 2013). Regardless of the correctness of the answer, teachers should inquire about student reasoning, identify the reflected understandings, and adapt their pedagogical responses based on the students' learning needs. This would create opportunities to deepen students' mathematical understanding and position them as mathematical sense-makers.

In order to effectively position students as mathematical sense makers and have them develop mathematical proficiency; teachers are the gatekeepers of this vision of teaching. Teachers are required to not only attend to and interpret students' mathematical thinking, but also craft a pedagogical response building on what they recognize in students' thinking to create opportunities for new mathematical connections. Teachers' response is the key point to expanding, maintaining, or shutting down students' sense-making (Schwarz et al., 2021). Researchers claim that only well-equipped teachers can recognize, value, and draw on students' knowledge. Studies show that beginning teachers, in particular, struggle with attending to students' ideas (Franke et al., 2009; Jacobs et al., 2010; Munson, 2019). Although teachers may develop expertise in attending to students' ideas as they gain experience, this same relation is not the case for building on students' ideas (Jacobs et al., 2010). Considering the highlighted struggle in the literature about responding to student's mathematical thinking based on their learning needs, the overarching question motivated me for this study is "How do PSTs develop in their capacity to craft evidence-based responses to children's mathematical thinking?"

Researchers argue that teachers need to not only hold deep knowledge of content to be able to craft a response but also use this knowledge strategically by using what students know (e.g., Philipp et al., 2019; Shulman, 1986). Crafting a pedagogical response is complex and requires teachers to engage with pedagogical reasoning. This includes teachers weighing the alternative course of reasonable action and facing pedagogical dilemmas (Kennedy, 2006; Lampert, 1985). Teachers have to develop the pedagogical judgment to navigate mathematically-based dilemmas about what mathematical idea to pursue and how to pursue it while they craft a mathematically-based pedagogical response to leverage students' mathematical understanding (Kavanagh et al., 2020). Considering the complexity of crafting a mathematically-based pedagogical response that builds on students' mathematical thinking and its significance in expanding students' mathematical sense-making, teachers need support to use students' mathematical thinking as a source for mathematically responsive pedagogical actions (Leatham et al., 2015).

In the view of responsive teaching, teachers constantly make instructional decisions about what to pursue and how to pursue based on emergent student thinking (Jacobs & Empson, 2016). In order to craft a response to deepening children's mathematical understanding, teachers need to engage in pedagogical reasoning drawing on their Mathematical Knowledge for Teaching (MKT), specifically specialized content knowledge (SCK) and knowledge of content and students (KCS). Typically, pre-service teachers' (PSTs) pedagogical reasoning is mostly studied in the later stage of their teacher preparation, in connection with the field experience. Given that PSTs start developing MKT in the early years of the teacher education program, this results in a separation between the learning of mathematical knowledge and the work of teaching (Steele & Hillen, 2012). PSTs' engagement with pedagogical judgment needs to be supported while they are making sense of MKT. Considering the difficulty teachers face, especially in building on students' mathematical understanding and extending it, this study focused on initiating PSTs' development of pedagogical judgment to craft a mathematically-based pedagogical response by drawing on their SCK and KCS skillfully. I hypothesize that creating a space for PSTs to attend to, interpret, and craft a pedagogical response to build on students' mathematical thinking in mathematics content classes contributes to developing judgment of responsive practice. In this study, the invisible work of decision-making of how to respond and PSTs' judgment drawing on their SCK and KCS was unpacked in the learning community meanwhile PSTs were supported to develop a vision of possible ways of leveraging children's sense-making of mathematics.

Overall, my study aims to understand how to support PSTs' development in their capacity to craft evidence-based responses to children's mathematical thinking? To that end, the research questions I aim to answer in this study as follows:

1. How does PSTs' noticing students' mathematical thinking change from the beginning of the semester to the end of the semester?
2. What relationships emerge among facets in PSTs' noticing of students' mathematical thinking?
3. What opportunities to learn were available for PSTs in the content course to support their capacity to develop noticing?

Theoretical Framework

In this chapter, I lay the theoretical foundation of my study. First of all, I summarize different views about noticing and responding, then discuss how I conceptualize the responding to children's mathematical thinking. Following, I summarize theoretical foundations of how teachers learn.

Noticing and Responding

In everyday language, noticing signifies the act of recognizing noteworthy events in the complexities of the world. However, in the professional contexts it signifies to define strategic ways of noticing. Mason's (2011) idea of the discipline of noticing lays the foundation for the definition of noticing in the professional context. He defines the discipline of noticing as "a collection of practices designed to sensitize oneself so as to notice opportunities in the future in which to act freshly rather than automatically out of habit." (p. 35) The concept of mathematics teachers' noticing has been an area of inquiry among researchers and how researchers conceptualize of what constitutes teacher noticing differs.

Researchers agree on noticing involves *attending* to particular elements and *making sense* of these elements, whereas how researcher defines making sense varies (Sherin et al., 2011). For example, van Es and Sherin (2002) conceptualize making sense as solely interpreting, so they conceptualize teacher noticing as consisting of *attending* and *interpreting*. Conversely, some researchers define sense making as more than *interpreting*. Jacobs et al. (2010) claim that *attending* to the details of children's strategies and *interpreting* the children's understanding reflected in these strategies are "not ends in themselves but are instead starting points for making effective instructional responses" (pp. 99–100). They highlight that a critical aspect of professional noticing is teachers' decision about how to respond based on students' understanding.

Although some researchers consider noticing without the aspect of responding, they still emphasize that teachers' responses are shaped by what they attend to and how they interpret students' strategies (Barnhart et al., 2024; Richards et al., 2020). Further insights from Barnhart and van Es (2015) suggest that responding plays a significant role in determining the sophistication of noticing. Their findings indicate that sophisticated responses necessitate sophisticated attending to students' ideas. Although these researchers agree that teacher noticing constitutes *attending* to and *interpreting* students' mathematical thinking, they argue that responding is a separate but closely related aspect of noticing. This indicates that researchers conceptualize noticing students' disciplinary thinking and responding to it as two interrelated aspects of ambitious teaching and learning regardless of terminological differences.

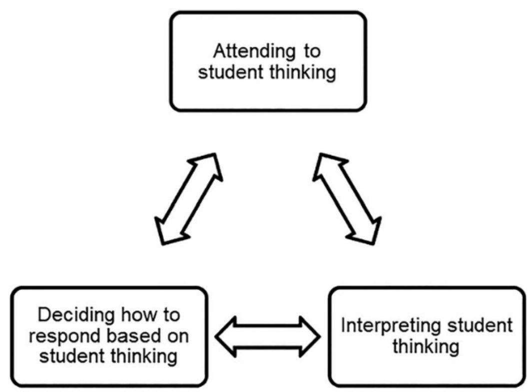
In my view, I conceptualize the act of responding separate from act of noticing. However, considering Mason's (2011) definition of noticing that stresses on informing future practices and considering the documented connections by researchers between what teachers attend and how they interpret it informs their responses, I used a framework that includes the aspect of decision making. That aspect takes into account proposing an approach that shapes the future act. In my study, I used Jacobs et al.'s (2010) framework of professional noticing of children's mathematical thinking. They define this construct in set of skills consisting of *attending, interpreting,* and *deciding how to respond.*

Attending is defined as recognizing the mathematical elements in children's thinking. Identifying the reflected mathematical understandings in these noteworthy mathematical details is called *interpreting. Deciding how to respond* to students' thinking characterized by crafting a response that builds on substance of students thinking by drawing on professional knowledge of learning and teaching mathematics. This skill focuses on not executing the response but generating the intended responses by focusing on teacher's decision making. In addition, Jacobs et al. (2010) take the complexity of pedagogical judgment into consideration and acknowledge that this skill, *deciding how to respond*, does not claim that there is one best response to leverage student's mathematical thinking. Instead, it highlights the connection with what is being noticed and the intended responses. Given my goal was supporting PST's skill of crafting response that is responsive to students' mathematical thinking, whenever I use the word "responsiveness" for the rest of this study, I will refer the third skill of professional noticing of children's thinking that is *deciding of how to respond* based on students' mathematical understanding.

To support PSTs' responsiveness to students' mathematical understanding, I focuse on developing the set of skills of professional noticing of children's mathematical thinking which are *attending, interpreting*, and *deciding how to respond*. Jacobs et al. (2010) argue that these skills should not be interpreted as a linear trajectory, and they highlight that they are an interrelated set of skills. They are often cyclical in nature as shown in Figure 1, so the development of the three skills needs to be addressed in relation to each other (Jacobs et al., 2010; Lam & Chan, 2020).

Figure 1

Interrelated Skills for Professional Noticing of Children's Mathematical Thinking



Note: Adapted from "Characterizing pre-service secondary science teachers' noticing of different forms of evidence of student thinking" by D. S. H., Lam, and K. K. H., Chan, 2020, *International Journal of Science Education*, *42*(4), p. 578. Copyright by Routledge.

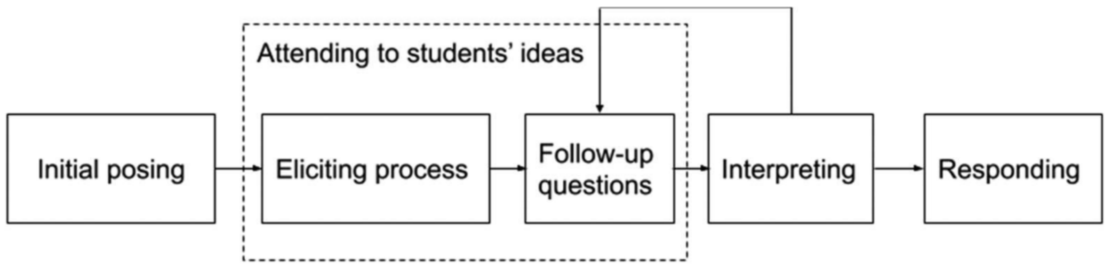
The skill of attending and interpreting requires teachers to notice the conceptual underpinnings of students' mathematical thinking and demands pedagogical judgment to decide on a response that leverages students' mathematical understanding. These skills are informed by various subdomains of their mathematical knowledge for teaching (MKT) (Ball et al., 2008). One of the main domains is specialized content knowledge (SCK) which is required to be able to do the work of teaching. SCK requires unpacking mathematical conceptions and developing mathematical reasoning and understanding (Ball et al., 2008). The other domain that affects teachers' understanding and interpreting is knowledge of content and students (KCS). Teachers must know what common student conceptions and misconceptions are and anticipate how students would think or what will make them confused (Ball et al., 2008).

To build on and leverage children's mathematical thinking, teachers need to uncover students' mathematical reasoning. Eliciting is a process of uncovering further details of students' thinking processes. Gaining more information about students' understanding informs teachers' instructional decision. In the eliciting process, teachers could ask follow-up questions that invites students for further explanation. The eliciting questions could focus on either understanding the actions taken by students or uncovering the rationale behind their mathematical choices (Munson, 2019). Although both question type are necessary to understand students' thinking, the latter one probes conceptual understanding (Reinke et al., 2022).

As an example, Figure 2 below shows the process of eliciting takes place between the facet of attending and interpreting student thinking. In this demonstration, following the eliciting questions and interpreting the reflected understandings, the teacher proceeds to elicit further to understand the student's comprehension. However, eliciting is not restricted with the presented example. Eliciting is an ongoing process, and this cycle could repeat until the instructor uncovers students thinking and makes an instructional decision. Furthermore, new cycle of eliciting could take place following simultaneous interaction between students and teachers. Eliciting is an essential skill for teachers to notice student thinking.

Figure 2

A Synthesis of Decomposition of Eliciting and Interpreting Student Thinking



Note: Adapted from "Developing Student Teachers' Skills at Eliciting Students' Mathematical Thinking Using the Coaching Cycle." by L. T., Reinke, L. W., Schmidt, and D., Polly, 2022, *The Teacher Educator*, *57*(2), p. 218. Copyright 2021 by the Routledge.

While attending to and interpreting students' mathematical thinking are foundational to understanding student thinking, the teacher's use of these understandings as a resource for instruction is central to deepening students' conceptual understanding (Bishop et al., 2022; Jacobs & Empson, 2016; Monson et al., 2020; Robertson et al., 2016; Stockero et al., 2020). A teacher's response is key leverage for children's higher-order mathematical thinking because the response should give students an opportunity to reason by drawing on their intellectual resources, meanwhile, it needs to set higher-order thinking goals (Lampert et al., 2013). There is an intertwined relationship between attending, interpreting, and deciding how to respond. Furthermore, responding is the significance factor to position students as sense-makers and afford them higher-order mathematical thinking, my study focuses on supporting PSTs development in these nested skills of noticing in the context of a math content course for elementary teachers. This study focuses on developing these three nested skills which are required to leverage students' mathematical sense-making.

Teacher Learning

Scholars of teacher education have identified the types of knowledge necessary for teaching and learning mathematics. Ball et al. (2008) built on Shulman's (1986) framework to conceptualize mathematical knowledge for teaching (MKT), specifying domains that mathematics teachers need to grasp for their professional development to do the work of teaching mathematics. This framework consists of two main domains: subject matter knowledge and pedagogical content knowledge. Subject Matter Knowledge consists of Common Content Knowledge (CCK) along with Specialized Content Knowledge (SCK). Pedagogical Content Knowledge consists of Knowledge of Content and Students (KCS), and Knowledge of Content and Teaching (Hill et al., 2008). CCK is defined as the general mathematical knowledge held by an adult while, SCK is defined as unique mathematical content knowledge to do the work of teaching (Copur-Gencturk et al., 2019). SCK requires unpacking mathematical conceptions and developing mathematical reasoning and understanding. Teachers draw on their SCK when they analyze various solutions to see the mathematical ideas, make connections between mathematical ideas and representation, and ask questions to press students' thinking. Another domain that affects teachers' understanding and interpreting is knowledge of content and students (KCS). Teachers must know what common student conceptions and misconceptions are and anticipate how students would think or what will make them confused (Ball et al., 2008). Teachers draw especially on their SCK and KCS while attending, interpreting, and deciding how to build on students' mathematical thinking. SCK and KCS, the subset of MKT, are the focus of the study.

Scholars agree that teachers need to have a deep understanding of MKT to do the work of teaching. Shulman (1986) argues teachers need to develop *strategic pedagogical knowledge* which means developing judgment of how to use these knowledge bases strategically when confronted with dilemmas (p. 12). Teaching is a complex endeavor, and it requires simultaneous consideration of various concerns and competent goals, and the alternative course of action should be weighted in the moments of decision-making (Kennedy, 2006). It requires teachers to learn to make decisions by using their professional judgment to navigate dilemmas that arise in the complexity of the moment (Kavanagh et al., 2020; Lefstein et al., 2020). The invisible intellectual work and unpacking of the unseen aspect of the act of teaching are generally referred to as pedagogical reasoning (Loughran, 2019; Shulman, 1987). Making pedagogical reasoning explicit is a powerful way of helping PSTs to comprehend the complex nature of teaching and initiating the development of a vision for their future teaching (Loughran, 2019). Schön (1983) defines two types of reflection, reflection in action and reflection on action, to make sense of the critical moments and to make visible the key teaching events. In this study, I used reflection to have PSTs make their decision-making process of crafting a response visible while they are making sense of MKT.

My analyses of PSTs' discussions around the designed tasks are guided by Rogoff's (1997) notion of *transformation of participation* view of learning. Determining the occurrence of learning is one of the fundamental inquiries of educational research. Historically, learning is conceptualized in two ways: the *transmission* and *acquisition* of knowledge. The former conceptualization is defined as transferring information from the outside world to the brain. In this conceptualization the learner is framed as a passive recipient. The latter conceptualization is defined by the acquisition of information and ideas. In this conceptualization, the learner is positioned as an active participant in the cognitive process. Nonetheless, Rogoff (1997) introduces an alternative viewpoint by highlighting that both conceptions define learning as one-sided development. She defines a third concept centered around the *transformation of participation* that surpasses the one-sided limitation of *transmission* and *acquisition of* *knowledge* and it accounts for multidimensionality. In this viewpoint, people engage in shared and ongoing endeavors. The change occurs in individual's understanding during this shared engagement, while the shared endeavors evolve and change.

My approach involves examining evidence of change in PSTs' noticing students' thinking to understand their development of skills of *attending*, *interpreting*, and *deciding how to respond* by drawing on professional knowledge of teaching. Rogoff highlights three layers of analysis: personal, interpersonal, and community. The personal plane of analysis focuses on changes in individuals in terms of their roles and understandings in the context that they are part of. The interpersonal plane of analysis moves beyond the individual level. It considers the dynamic relationship and exchanges occur between people. The community plane of analysis expands within a larger social context and accounts for interaction of broader settings.

Among those three planes of analysis, I focus on interpersonal analysis by focusing on how the PSTs' group discussions around the given tasks evolved. Analyzing PSTs' group discussions revealed opportunities to learn that become available in the interpersonal plane of analysis. By opportunities to learn I mean "the types of interpersonal engagements and involvement with particular resources that are available to novice teachers in a particular context" (Ghousseini et al., 2015, p. 463). I focus on opportunities to learn that become available through the discussion in the small group setting, around the tasks, to develop the skills that are required to do the work of noticing in tandem with using MKT strategically.

Literature Review

This chapter summarizes the body of literature concerning the preparation of PSTs to do work of noticing children's mathematical thinking. Identifying the current state of knowledge about what is known and what needs to be done to support PSTs development of skill of taking up and building on students' mathematical shapes the structure of my study.

The Cognitively Guided Instruction research group (Carpenter et al., 2014) has documented the multiple ways of children's mathematics, the benefits of knowing more about children' various mathematical thinking and using it for instruction. However, researchers continually report that pre-service teachers, novice teachers, and even experienced teachers have difficulty in using students' understandings to leverage their mathematical understanding. The literature suggests increased attention to change this situation in ways that relate to the three prongs of Jacobs et al.'s (2010) framework of attending to, interpreting, and deciding how to respond to students' mathematical thinking. Researchers also highlight that the components of the framework are interrelated, claiming that attending to, interpreting, and deciding how to respond to children's mathematical thinking develop simultaneously (Jacobs et al., 2010; Krupa, 2017; Monson et al., 2020). In my study, I focus on the three intertwined skills to develop mathematically responsive teachers.

I elaborated on these three interrelated skills in detail in the theoretical framework. These skills could be summarized as attending to students' thinking refers to recognizing mathematical details in students' strategies, interpreting refers to what understanding the mathematical elements reflect, and deciding how to respond refers to crafting a mathematically-based response to extend students' mathematical thinking. Despite the cyclical relationship between these skills, researchers have tended to study their development separately, suggesting the affordances of pre-service teachers developing these skills and what remains missing. In reviewing literature, I elaborate on what researchers do to develop necessary skills to develop mathematically responsive teachers. I summarize the literature in three parts. The first part summarizes the studies that show the general characteristics of teacher response, the second part involves the studies showing the relationship between MKT and being mathematically responsive, and the third one involves the studies focusing on developing pedagogical judgment to develop mathematical responsiveness.

The first body of research addresses the general characteristics of teacher responses and what stands in their way of leveraging children's mathematical understanding. Teachers effectively guide their instruction based on children's mathematical needs if they demonstrate the ability to attend to children's strategies, interpret their understanding, and decide how to respond to leverage children's understanding (van den Kieboom et al., 2017). Conversely, when teachers struggle to craft a response based on students' emergent mathematical ideas, the instruction provide little benefit to students (Monson et al., 2020). A body of research shows that PSTs struggle with crafting a response that leverages children's mathematical thinking and creates opportunities for extended mathematical understanding. Studies report that PSTs' responses are mostly related to procedural actions (Sánchez-Matamoros et al., 2019). Additionally, other studies demonstrate that PSTs encounter difficulties in how to respond especially when students make mistakes, and tend to ask funneling questions that lead them to the intended answer (Webel & Yeo, 2021; Weiland et al., 2014).

Moreover, literature portrays the way teachers struggle with asking questions that broaden children's mathematical thinking. For instance, a case study conducted by Munson

(2019) reports on teachers' pathways of responding to children's thinking. The study reveals that teachers are less likely to probe or leverage children's thinking when children demonstrate understanding of the task, while teachers most likely funnel students to answer when the children have confusion. The findings confirm the teachers' concern with the correctness of students' ideas (Louie, 2017). These studies suggest that PSTs need to be supported to shift their focus towards to mathematical elements in children's strategies to see the mathematical understanding it reflects regardless of the correctness of the result.

Further research by Philipp et al. (2002) highlights that incorporating children's thinking into the courses allows PSTs to understand the depth in children's thinking and how they make sense of mathematical ideas as well as changing PSTs' belief about learning and teaching mathematics. Additionally, studies demonstrate that when PSTs' engagement with students' mathematical thinking is guided by scaffolding questions focusing on mathematical elements, it helps PSTs to identify the mathematics in children's various strategies and the understanding it reflects (e.g.; Ivars & Fernández, 2018; Krupa, 2017; Lee & Choy, 2017; Webel & Yeo, 2021; Weiland et al., 2014). These findings underscore the importance of providing PSTs with opportunities to engage deeply with children's various mathematical thinking, regardless of their error, might foster meaningful learning experiences.

The second body of research shows the relationship between the skills of mathematical responsiveness and MKT. A study conducted by Sanche-Matamoros et al. (2019) with secondary prospective mathematics teachers found that SCK is necessary to attend to mathematical elements in children's strategies, but not enough to interpret what understanding it reflects. The researchers argue that KCS is also necessary to interpret children's understanding. Another study conducted by Namakshi et al. (2022), investigated the relationship between PSTs' development of SCK and KCS and their ability to attend to and interpret what students know. The study found that as SCK and KCS increase, PSTs' ability to attend to students' various strategies also improves. Similarly, Tyminski et al. (2014) showed that PSTs demonstrate a strong foundation in attending to and interpreting students' mathematics thinking as PSTs increase their SCK and KCS. In addition, they argue that making sense of MKT by exploring children's thinking and interpreting the mathematical understanding help PSTs to focus on the conceptual understanding of children. Consistent with these findings, a study conducted by Webel and Yeo (2021) shows that with the support of probing questions from the instructor directed at uncovering the mathematical elements of various student strategies, PSTs stop taking over students' thinking and start to ask more probing questions to understand it. However, a contribution to extending students' thinking is not found.

In summary, the literature cited above on pre-service teacher learning to be responsive to student thinking mainly shows attention to supporting PSTs in attending to and interpreting students' mathematical thinking, with less attention to teachers' ability to craft a pedagogical response to build on students' mathematical knowledge. Learning to respond to students' thinking requires scaffolds (Leatham et al., 2015) and should be informed by pedagogical judgment. Researchers highlight that the crucial part is the judgment of when to and how to use the knowledge bases when teachers are faced with the complex nature of decision making (Philip et al., 2019; Shulman, 1986).

The last body of research includes the studies that positioned teachers as instructional decision makers and focused on the need to develop pedagogical judgment to develop responsiveness. These studies for the most part is conducted in the later stage of teacher preparation, particularly connected to the field experience. There is a body of research that did not embed Jacobs et al.’s (2010) framework, but they inform us that PSTs learn the work of teaching through actively engaging in pedagogical judgment and the decision-making process (Ghousseini et al., 2015; Hallman-Thrasher, 2017; Kazemi & Wæge, 2015; Lampert et al., 2013). This body of research shows that making the decision-making process visible, reflecting, and discussing the affordances of possible pedagogical moves help PSTs to develop judgment related to students’ mathematical needs in the complexity of teaching. In addition, Steele and Hillen (2012) demonstrate that engaging MKT and pedagogical reasoning in relation to each other enhance both mathematical knowledge and learning the complex work of teaching. Pedagogical dilemmas that PSTs face when they need to make a decision, facilitate re-examining and further development of content knowledge. They also inform the relationship between knowledge bases (Philipp et al., 2007; Speer & Wagner, 2009) and the judgment of when and how to use them (Shulman, 1986). These arguments support Jacobs et al.’s (2010) claim that the components of their framework need to be addressed in relation to each other as these skills are drawn on MKT, and the learning and the work of teaching should be addressed in relation to each other (Steele & Hillen, 2012).

In light of the need to initiate the development of judgment in using MKT strategically, researchers identify characteristics of crafting a good response to guide the PSTs’ sense-making. Jacobs et al. (2010) highlight that a good response cannot be defined, and the productivity of the response could be differ based on contextual conditions. However, they articulate some characteristics of a good response that can guide researchers, teacher educators, and teachers. They posit that a response that deepens mathematical thinking should 1) draw on and is consistent with the student thinking presented, 2) draw on and is consistent with research on students’ mathematical development, and 3) leave space for students’ future thinking. Since research shows that keeping their attention on the math goal is also difficult for PSTs and teachers, McDonald et al. (2013) and Leatham et al. (2015) argue that working toward a math goal is also an important characteristic of a good response. Monson et al. (2020) used these four characteristics in their tasks, assignments, and classroom discussions to scaffold PST's ability to craft a good response to students' mathematical thinking. They report that giving PSTs the opportunity to examine, improve, and reflect on their proposed responses based on children's thinking in the light of the characteristics of a good response is a powerful practice to support PSTs' development of pedagogical judgment of using the knowledge bases skillfully. Other researchers also suggest that reflecting on missing opportunities and discussing possible different directions is significant for PSTs to create a vision of possible responses to leverage students' mathematical thinking (Teuscher et al., 2017; van den Kieboom et al., 2017).

Van Es et al. (2014) conducted a study that focused on making decision-making visible within collaborative contexts to assist teachers in learning to recognize students' mathematical thinking. They highlight some principles to facilitate teacher learning in a collaborative context including sustaining an inquiry stance, maintaining the focus on mathematics, and supporting group collaboration. They define sustaining inquiry stance as highlighting the key mathematical concepts for deeper exploration and prompting participants to elaborate their reasoning. Maintaining the focus on mathematics is characterized by redirecting teachers' attention to mathematical elements, prompting them about identifying the evidence, and connecting the ideas participants raised. The final principle they highlight for teacher learning is supporting group collaboration that allows participants to share alternative ideas and discuss and expand upon each other's ideas. These principles have important implication on cultivating teacher learning in the collaborative context by highlighting the importance of given opportunities for teachers to unpack the mathematical concepts and explore the reasoning of each other.

The summarized research highlights the critical role of teacher education programs to develop noticing children's mathematical thinking. Focusing on PSTs' ability to craft a pedagogical response that deepens students conceptual understanding is foundational to improve PSTs' noticing skill. To improve PSTs' ability to craft pedagogical responses that create opportunities for children to deepen their mathematical sense-making, teacher education programs need to initiate PSTs to develop pedagogical judgment in using their MKT. Furthermore, PSTs benefit from opportunities to examine, improve, and reflect on their examination of students' mathematical understandings. Discussions in collaborative contexts enhance PSTs' learning to be responsive by creating affordances for PSTs to unpacking mathematical concepts, discovering missed opportunities, and discussing the affordances of alternative ideas.

In the present study, I implement the implication of the summarized literature in the content course design by aiming to cultivate PSTs' responsiveness to children's mathematical thinking. In this study, I aim to address how to support PSTs development in their capacity to craft evidence-based responses to children's mathematical thinking, specifically in the content course by addressing the work of teaching and learning in relation to each other. I aim to answer my overarching question by answering the following research questions:

1. How does PSTs' noticing students' mathematical thinking change from the beginning of the semester to the end of the semester?
2. What relationships emerge among facets in PSTs' noticing of students' mathematical thinking?

3. What opportunities to learn were available for PSTs in the content course to support their capacity to develop noticing?

Methodology

In this study, I investigate how PSTs' ability to notice children's mathematical thinking develops in a content course aimed at developing their noticing and responding in tandem with developing their MKT. I particularly focus on development in their skill in formulating mathematically grounded pedagogical responses that deepen children's mathematical understanding. This qualitative case study (Creswell, 2007) involved designing and implementing a classroom teaching intervention in a semester-long mathematics content course for elementary teachers.

The PST participants in this study were enrolled in a two-year undergraduate elementary teacher education program at a public university located in Midwest within the United States. The program comprises a sequence of two content courses and a methods course. The content courses are typically taken during the first year of the program. This study was conducted during the Fall 2022 semester in the first content course, Pedagogical Content Knowledge for Teaching Elementary Mathematics-I. During the course PSTs did not have a field teaching component. The course met once a week in a 150-minute session, in total 14 sessions throughout the semester and covered counting and cardinality, arithmetic operations and place-value understanding, number theory, and integers.

In this study, I served a dual role of both course instructor and researcher. Accordingly, I differentiate this duality in my roles through the use of two different voices: As the researcher, I use first person to refer to my role in designing, conducting, and analyzing the data. As the course instructor and teacher educator, I use the word instructor to refer myself as the facilitator of the content course in which the study was conducted. Although I try to differentiate my roles using two different voices, their influence on each other in practice is inevitable. Before I conducted the present study, I had a history of teaching the sequence of two content courses and the method course that are offered in the elementary teacher education program. My experience of teaching these courses over three years allowed me to gain some insights about PSTs' tendency to use standard algorithms, their resistance about using invented strategies strategically, and their rigid belief about what it means to do mathematics and how it needs to be taught. This experience as an instructor allowed me to do several intentional choices in designing the course materials, accordingly, designing my data collection tools as a researcher. In addition, my knowledge of the work of noticing children's mathematical thinking influenced my facilitation skills as an instructor. I will delve into how my researcher and instructor role informed each other in the section related to task design. In the following part, I summarize the details about the course design, data collection, participants, data collection instruments, and data analysis.

Course Design

Pedagogical Content Knowledge for Teaching Elementary Mathematics-I is the first mathematics course that PSTs take to develop the knowledge specific to mathematics teaching. There were four sections of the course and four distinct instructors who collaborated to create the syllabus and the course structure, as well as the design of assignments so they are aligned with the goals of the course. Despite this joint effort, each instructor had autonomy in adapting and modifying their instructional materials and course structure for their section. There were 25 PSTs in the section the study was conducted. Table 1 below summarizes the covered topics week by week over the semester.

Table 1

Weekly Content Covered Throughout the Semester in Pedagogical Content Knowledge for Teaching Elementary Mathematics-I in the Fall of 2022

Week 1	Introducing the course & syllabus Discussing what pedagogical content knowledge for teaching is Getting the know each other Discussing the participation expectation and rubric to understand what is considered "full participation"
Week 2	Foundations of Arithmetic • Counting and cardinality • Stages of early arithmetic learning • Number word sequences (one more/one less) • Children's early difficulties with numbers
Week 3	Number Sense • Number sense and number relationships Basic Math Facts (1-20) Part-whole Thinking Problem posing, story problems, and their role in developing basic number facts
Week 4	Addition and Subtraction Problem Types • Join, separate, part-part-whole, and compare problem types • Properties of Addition Children's Solution Strategies for Addition and Subtraction Problems
Week 5	The Base Ten Place Value System Addition and Subtraction Strategies (1-1000)
Week 6	Multiplicative Thinking • Examining the difference between multiplicative and additive thinking • Principles of Multiplicative (Composite) Thinking Development Multiplication and Division Problem Types • Grouping and Partitioning Problems • Multiplication, Partitive Division, Measurement Division, Multiplicative Comparison Problems
Week 7	Multiplication and Division Strategies Array and Area Representations Properties of multiplication
Week 8	Standard algorithms Multiplying and dividing by 0 and 1 Historical multiplication and division methods around the world

Week 9	Number Theory • Factors, multiples, divisors • Prime numbers
Week 10	The fundamental theorem of arithmetic Factorization, Prime factorization, and its applications Greatest Common Factors (GCF) and Lowes Common Multiple (LCM) Divisibility Rules
Week 11	Integers • Number sets • Integer Addition and Subtraction • Using number lines and chip models to make sense of integers
Week 12	Thanksgiving Recess
Week 13	Integers • Integer Multiplication and Division
Week 14	Post-Baseline Assessment

The instructor engaged PSTs in targeted discussions through questions that promoted the development of pedagogical judgement by drawing on their MKT. These inquiries aimed to facilitate the development of their pedagogical understanding. In addition, course materials centered around PSTs' engagement with children's mathematical reasoning and informing their pedagogical strategies by using and developing their MKT.

Course Guiding Principles

This course is based on some guiding principles to develop the conception of mathematics as a sense-making activity and position students as sense-makers. First, this course positions PSTs themselves as sense-makers. PSTs were provided with the opportunity to examine key mathematical concepts in elementary school mathematics and engage in meaningful ways of doing mathematics through discussing and making sense of each other's thinking.

Second, children can engage with mathematics and bring informal mathematical knowledge to a class that the teacher can use to create room for children to build on their authentic mathematical thinking. This guiding principle relates to the principle that mathematics is not neutral, learning and teaching mathematics is social and cultural, and social interaction, participation, and collaboration mediate mathematical sensemaking (Nasir & Hand, 2006). This principle also highlights that learning is interactive and individuals interact and learn from each other (Russ et al., 2016).

Third, learning to teach requires careful reflection and using mathematical knowledge to teach strategically (Schön, 1993; Shulman 1986). In this class, PSTs were situated in groups to contribute to a shared understanding as they proposed alternative ways of explanation and negotiated and developed meaning and actions as it defined in Greeno and Engestrom's (2006) perspective of learning. PSTs engaged with crafting their ideas, discussing them, and working on recrafting them in their small groups and the whole class discussions. The following excerpt from the course syllabus illustrates the guiding principles of the course:

> We will engage in the following mathematical practices that are vital for your as well as your future students' mathematical learning:
>
> - noticing, wondering, and asking questions;
> - focusing on the hows and whys of the mathematical actions instead of thoughtless procedures;
> - explaining, justifying, and representing mathematical thinking clearly and concisely;
> - revising and re(revising) ideas and solutions based on feedback and discussions;
> - persevering in problem-solving by using collective support structures designed as part of the course;
> - making connections across different concepts to extend the mathematical inquiry. (Pedagogical Content Knowledge for Teaching Elementary Mathematics I, Fall 2022)

To meet the course objectives and integrate the fundamental course principles in the course design, the instructor's primary focus was fostering a learning community where PSTs could comfortably articulate their ideas, share their confusions and developing ideas, and engage with each other's mathematical ideas. To initiate this process, the instructor conducted a survey to get to know their students at the beginning of the semester. The survey aimed to gather information about the PSTs' previous mathematics experiences as learners, the strengths they bring to the classroom community, and the kind of support they need from the classroom community. Based on the survey findings, the instructor created semester-long groups. I will call these groups as home groups for the rest of the dissertation to indicate that the group members stayed with the same group during the semester. The grouping consisted of seven home groups including four groups of four people and three groups of three people. These groupings aimed to be heterogeneous based on the diversity of PSTs' previous mathematical exposures and the support they need. The overarching objective behind these home groups was to cultivate peer support, create a sense of community, and establish a safe space for PSTs to exchange thoughts and experiences as they create a trustworthy setting.

Participation was one of the foundational principles of the course to foster a learning community and move the class forward. Table 2 shows these norms and participation domains that were shared on the syllabus and discussed on the first day of the class.

Table 2

Participation Rubric

Dimension	What "full participation" looks like:

Find a way to contribute and move the class forward.	• Share your own ideas, observations, and thinking in whole class and/or small group settings. • Elevate the ideas of others. • Actively listen to the ideas, observations, and thinking of others and elaborate, connect, or respond. • During group activities, keep focused on the goal and encourage group members to do the same. • Be on time and prepared and notify the instructor if you will be late/absent.
Take on a stance of inquiry rather than assumption.	• Assume we all have more to learn about most topics, content, and experiences. • Appreciate nuance and avoid blanket statements. • Ask for clarification when possible. “Can you say more about that?” can be powerful. • Stay curious about why things are the way they are. • Ask questions like “Why does this seem difficult?” rather than using terms like “can’t or won’t” - especially with regards to children.
Practice patience & empathy for all (including yourself).	• When working in groups or with partners on mathematics, ask questions rather than show answers. • Appreciate and learn from struggles - your own and others’. • Ask questions if you don’t understand or see something. • Find help when you need it - from the instructor, the internet, other students, or student services.

Note: Adapted from “Gwyneth, H. (Spring, 2020). Teaching Mathematics in Inclusive Settings [Syllabus]. University of Wisconsin, Madison.”

The instructor integrated Featherstone et al.'s (2011) recommended roles to foster collaboration and equity in elementary mathematics classrooms to cultivate active participation and establishing group norms. These roles were names as task manager, reasoning monitor, recorder/reporter, and timekeeper/resource manager. The roles were adapted in small group settings and were rotated weekly among group members.

The task manager actively managed the group's progress. They had the responsibility for maintaining group work norms such as facilitating discussions where all members can engage and ensuring questions about the task by group members were addressed. The reasoning monitor's role was posing clarifying questions to ensure explanations were detailed and evidence of reasonings was shown. Furthermore, they prompted group members to explore alternate perspectives by questions like "Will that always be true?" or "What would happen if...?" The main goal of reporter/recorder was creating representations and explanations that reflects the groups' work. Additionally, they shared these representations and explanations during whole-class discussions and contributed to collective learning. Lastly, the timekeeper/resource manager aimed to manage the group's pace by being sure the group has enough time to review all the questions and communicated with the instructor for additional time if required. In addition, they ensured the availability of necessary materials and requested any needed resources for the representations. As a summary, these randomly rotating roles within the small group setting aimed to encourage active engagement and equitable contributions.

The course was structured around a variety of instructional activities to meet the previously outlined overarching goals. These activities included watching video clips of students engaging in math, watching videos of diagnostic interviews conducted between a teacher and a student, analyzing children's written work, and developing and using Cognitively Guided

Instruction (GCI) problem types. In addition, Children's Mathematics: Cognitively Guided Instruction by Carpenter et al., (2014) was used as a resource for videos of children engaging in mathematics to watch in class and analyze children's mathematical thinking collectively. The instructor aimed to create room to unpack the invisible pedagogy through these various materials. Unpacking the mathematics knowledge and examining the underlying mathematical ideas and cognitive demands is related to the development of SCK (Ball et al., 2008). SCK is the knowledge that only teachers use to do the work of teaching which is beyond what students are being taught. For example, PSTs are required to understand the different meanings of operations, and different understandings inherited under each meaning. Comprehending the difference between "take away" and "distance" meaning of subtraction requires teachers to see the different cognitive demands of these questions although both are seen as subtraction (Ball et al., 2008). In addition, making connections across the mathematical concepts and representing them was part of the class. For example, examining the standard algorithm for different operations and making connections with the base 10 system or making connections with a standard algorithm of multiplication, partial product, distributive property, and area model are part of supporting SCK development (Max & Amstutz, 2019).

Understanding children's conceptions and misconceptions of mathematics, what children are likely to think, and what they might find difficult is related to developing KCS (Ball et al., 2008). Specifically, children's written student solution and video clips children engaging with mathematics were discussed to unpack students though process and to identify the reflected understandings. For example, watching various videos children engage in counting and reflecting on it collectively with what principles of early arithmetic learning they seemed to attend to. In addition, the connection between culture and mathematics were discussed throughout the semester. For example, difficulty in learning numbers based on the language structure, how the operations are being performed based on different cultures, and their underlying mathematical ideas were discussed. These discussions aimed to contribute to the learning mathematics is not neutral, it is integrated with the informal knowledge children bring into the class, and it is important to see the mathematical elements in the different ways of doing mathematics.

Please see Table 3 for more examples of how SCK and KCS were addressed in this course. In addition, more detailed information on how each data collection item addresses MKT domains and the components of professional noticing of children's mathematical thinking will be addressed in the section on data collection instruments.

Table 3

MKT Domains and Sample Activities

Content Domain	MKT Domains	Sample Activities
Counting Cardinality	SCK, KCS	Understanding the early stages of arithmetic learning (emergent counting, perceptual counting, figurative counting, initial number sequence). Watching various videos analyzing children's counting and reflecting on it with what principles of early arithmetic learning they seem to attend to. Reflecting on the affordances of material use, such as "Why do you think the teacher used the ten frameworks in the given video?"
	SCK, KCS	Children's early difficulty with counting. Examining the connection between the structure of the numbers, and how the numbers are named in different languages and discussing why it might be difficult for children to count in some languages.

Arithmetic operations & Place-value understanding	SCK, KCS	Showing various story problems with different problem types (Join, Separate, Part-Part-Whole, Compare) having PSTs to have them order from least difficult to most difficult for children, and unpacking why they think one could be more difficult than the others for children.
	SCK	Examining the example of children's solution of basic facts, doubling strategy, using 10 as a benchmark, decomposing numbers to use with friendly numbers, and making connections with number lines and ten frames to a scaffold of using these strategies.
	SCK	Examining some problems to understand the difference between the distance meaning and the takeaway meaning of subtraction. Understanding these two leads to different strategies. Discussing how the way of posing a problem may lead to a certain strategy and working on revising the problem structure based on cultivating different strategies.
	SCK, KCS	Examining various student strategies and naming them such as think addition, take from 10 strategies, down over 10 strategies, and discuss the similarities and differences between what these strategies reflect in terms of student understanding.

Note: This table briefly summarizes how SCK and KCS were addressed in the first two units of the course to give a better description of the course.

In addition to developing MKT, the course materials are designed to support PSTs' pedagogical judgment. For example, when PSTs were given various children solutions to analyze in their small groups, they were also asked to modify the task based on the given purpose. Furthermore, they were asked which student solution they would discuss on the board to have them think about the affordances of sharing each student's strategy in terms of the mathematical elements in it and make PSTs' thinking visible by discussing them in the small group and whole class settings. The most important scaffoldings to support PSTs' judgment were the task design and the instructor's questions to guide the discussions. Table 4 shows the guiding questions the instructor used to facilitate the whole class and small group discussions. The guiding questions are synthesized by the instructor drawing from the literature on characteristics of good response (Jacobs et al., 2010; Leatham et al., 2015; McDonald et al., 2013).

Table 4

Characteristics of a Good Response and Related Guiding Questions to Facilitate the Discussion

<table>
<tr><th>Characteristics of a good response</th><th>Guiding questions to facilitate the discussion</th></tr>
<tr><td>The response is directed to the student learning objective.</td><td rowspan="4">How are the proposed responses consistent with what you identified about what the student knows?

How does the proposed response create room for students toward the mathematical idea that needs to be scaffolded?

How can we revise this question to open room for students to make progress toward the mathematical objective of the task or mathematical idea that needs to be scaffolded?

Does this response leave room for students to build on what they know?</td></tr>
<tr><td>The response draws on and is consistent with the student thinking that is presented.</td></tr>
<tr><td>The response draws on and is consistent with research on students' mathematical development</td></tr>
<tr><td>The proposed interaction with students leaves space for students' future thinking.</td></tr>
</table>

Note: Characteristics of a good response are synthesized based on three researchers' characterization of response that builds on and leverages children's mathematical understanding (Jacobs et al., 2010; Leatham et al., 2015; McDonald et al., 2013).

Participants

At the beginning of the semester, in the first class, the PSTs were informed about the study and received clear explanations from a faculty member other than the course instructor. They were assured of their ability to withdraw their consent for participation at any time during the study. Ultimately, 24 PSTs (N=24) consented to the use of their coursework for research purposes. As both the instructor and researcher conducting the study, I remained unaware of the identities of consenting participants until after the semester was over and final grades were submitted to protect the participants' confidentiality and the study from potential bias.

Data Collection

In this study, data were collected through (1) a baseline assessment of PSTs' knowledge of ability to respond to children's mathematical thinking, (2) students' written work related to five in-class small group tasks including audio recordings of small group and whole class discussion, (3) PSTs' written reflections after completing the tasks. Baseline assessments and the reflections on the small group tasks were conducted individually to understand change over time and what PSTs grapple with over time, while the PSTs worked on the designed tasks as a small group in their home group. The tasks were designed to engage PSTs with various facets of noticing based on Jacobs et al.'s (2010) framework. Their discussions around each task were used to understand opportunities where PSTs make sense of various MKT domains and worked on their skill of noticing. Before I gave you a detailed explanation of each data collection item, I presented the data collection timeline below in Table 5 to give a better sense of my design.

Table 5

Timeline of Data Collection

Week 2	Week 4	Week 5	Week 7	Week 8	Week 11	Week 13
Pre-Baseline Assessment	Task 1	Task 2	Task 3	Task 4	Task 5	Post-Baseline Assessment
	Multi-Digit Subtraction		Multi-Digit Multiplication		Subtraction of Integers	

Baseline Assessments

Baseline assessments were conducted at two different times during the semester as pre- and post-assessments for diagnostic assessment. PSTs' answers to these assessments were considered as evidence of their developing MKT and of their skills in attending to children's mathematical thinking, interpreting the understandings that are reflected in children's mathematical thinking, and in deciding how to respond based on children's mathematical understanding.

Pre-assessment was conducted at the beginning of the semester to diagnose PSTs' knowledge about MKT related to the subtraction of multidigit numbers and their ability to notice children's mathematical knowledge in the same topic. The PSTs completed this assessment individually during the second week of the course. The assessment consisted of two parts (See Appendix A). In the first part, PSTs were presented with a multidigit subtraction question, and they were tasked to solve it in as many ways as possible. This part is specifically connected with common content knowledge and specialized content knowledge (Ball et al., 2008) as the task addressed PSTs' knowledge of non-standard ways of solving the problem. In the second part of the assessment, a corresponding student solution to the same question was provided to assess PSTs' capacity to notice children's mathematical thinking. The given student's solution intentionally contained an error since existing research indicates that PSTs tend to focus on guiding children who made an error toward the correct answer rather than understanding their thought processes (e.g., Munson et al., 2019; Webel & Yeo, 2021; Weiland et al., 2014).

To facilitate PSTs' engagement with children's mathematical reasoning and inform their pedagogical strategies in each part, the task guided them through a structured process that drew on the three facets of noticing children's mathematical thinking framework by Jacobs et al. (2010): Initially, PSTs were directed to focus on the first facet of the framework (attending to what the child did) by analyzing the student's actions within their solution. Second, they were prompted to interpret the mathematical understandings reflected in the student's solution. This approach aimed to shift PSTs' attention toward underlying mathematical ideas inherent in the student's thought process, without focusing on the correctness of the answer. In addition, a follow-up question prompted the PSTs to gather more information about the student's thought process by eliciting their understanding. PSTs were guided to identify areas where the student may require support to develop conceptual understanding of mathematical ideas. This question aimed to scaffold PSTs' response that creates opportunity to leverage student's mathematical thinking. Final question oriented PSTs to the third facet of the noticing framework which is deciding on a response that builds on children's mathematical thinking. Specifically, PSTs were prompted to articulate their pedagogical strategies aimed at extending the student's mathematical thinking by using the substance of their mathematical knowledge and by aiming to address the missing mathematical concept in their understanding.

The same assessment was conducted at the end of the semester as a post-baseline assessment to understand if there is a change in PSTs' MKT in the topic of subtraction with multi-digit numbers and if their nuanced understanding of content and developing MKT strategically improved their noticing children's mathematical thinking in the same topic. The post-baseline assessment was conducted with an additional reflection part. When the PSTs completed the post-assessment at the end of the semester, they were given their pre-assessment and were asked to compare their answers from between the pre- and post-assessments and reflect on how their answers had changed. The reflection questions included 1) What was the most challenging part for you among the questions at the beginning of the semester, 2) Considering the progress you made, what is still challenging for you? 3) If you believe your answers changed from the beginning of the semester to now, please share the changes/progress you made, 4) If you believe your answers changed from the beginning of the semester to now, what helped you to make this progress? 5) Comparing your answers from the beginning of the semester to now, is there anything else you noticed and want to reflect on? These inquiries aimed to gain insight information about PSTs' reflection on their learning.

Small Group Task Designs

The discussions of five tasks that PSTs completed in their small group setting across the semester were audio-recorded for research purposes. Those tasks were around multi-digit subtraction, multi-digit multiplication, and integers. I intend to underscore that this course aims to uncover the mathematical ideas behind both standard and non-standard approaches and make a connection between underlying mathematical ideas across different approaches. This goal shaped how I designed the tasks in this study. For example, Task 3 (see Appendix D) includes analyzing three student strategies that includes various invented strategies in multiplication. The goal of the task is unpacking the underlying mathematical ideas in these strategies and connecting these strategies with standard algorithm. This task allows us to discover the foundation of conceptual understanding behind the standard approach. Followingly, Task 4 (see Appendix E) exposes PSTs to student solution with an error in the multiplication in standard algorithm. This sequence aims developing PSTs to target the conceptual understanding rather than procedural corrections even in standard algorithm. The course materials such as videos students engaging in mathematics, weekly assignments, and classroom tasks involved using and unpacking both standard and non-standard approaches. When the tasks are examined, although some of them include invented strategies, it could be seen that most of them rely on standard algorithm as well as the baseline assessments. However, intentional decision lays down behind the choice of using standard algorithm in the tasks especially in given student solutions with error such as in Task 2 and Task 4, as PSTs often show tendency to direct students to the intended answer without leaving room for mathematical sensemaking and conceptual understanding. The goal is moving beyond the procedural corrections and developing the skill of fostering the mathematical ideas behind standard algorithm.

These recordings of the small group discussions served as exemplars of the activities and inquiries PSTs encountered during the course. They also provided evidence of the opportunities to learn provided by the course design for them to strategically use their pedagogical knowledge, develop nuanced understandings across various domains of Mathematics Knowledge for Teaching (MKT), and actively engage in components of noticing children's mathematical thinking. Table 5 above shows the five tasks that were audio recorded during the course of the study and the content they addressed. I aimed to use two consecutive tasks to collect data in each content area. I was only able to achieve this in the content areas of multi-digit subtraction and multi-digit multiplication, as Task 6 had to be canceled due to an unexpected event. The content of subtraction of integers had only one task, Task 5.

The two consecutive tasks in the same content were structured with slightly different purposes. The first of the two tasks aimed to expose PSTs to various student strategies and to create opportunities to unpack the knowledge behind the standard and non-standard approaches. This addressed one of the main goals of the course, developing SCK by developing nuanced understanding between different strategies and mathematical ideas. Furthermore, examining students' multiple strategies aimed to develop PSTs' KCS by increasing their awareness about what students are likely to think and to learn more about children's mathematics.

The second of the two tasks had a similar structure with the pre and post assessments. In the first part, PSTs were presented with a question, and they were asked to solve it in as many ways as possible. This question directed PSTs to use not only the standard algorithm to solve the question, but also to utilize various non-standard ways of solving the questions. In addition, PSTs were asked to identify the underlying mathematical ideas in each solution to facilitate comprehending the nuanced differences. After PSTs solved individually the question in as many ways as possible, they discussed their various solution with the members of their home team, brainstormed about the similarities and differences, and the mathematical ideas behind those solution. Following small groups discussion, the whole class got back together to discuss and unpack the knowledge behind the proposed solution strategies.

After the whole class discussion, the second part of the task were given to the PSTs that included analyzing a corresponding student solution to the question. The student solution intentionally contained an error to co-opt PSTs' focus on making sense of the student's work despite the mistake. In the rest of the task, PSTs were guided through a structured process that drew on the three facets of Jacobs et al. (2010) noticing framework. This process aimed to facilitate PSTs' engagement with student's mathematical thinking and inform their pedagogical decision making to leverage student's mathematical understanding: The questions started with the attending facet by asking PSTs to identify the student' possible thought process based on their given solution. The following questions focused on interpreting facet. Firstly, PSTs were asked to focus on the substance of student's thinking by identifying the reflected understanding. Secondly, they were asked to identify areas the student might need support to deepen their conceptual understanding. The questions focused on facilitating the interpreting facet, especially the latter one, aimed to set the stage for the responding facet of the noticing framework, "deciding how to respond to students' thinking" by providing a base for PSTs to craft a response that leverages student's mathematical understanding.

In the last question, PSTs were asked to articulate their pedagogical strategies aimed at extending the student's mathematical thinking by using the substance of the students' mathematical thinking and aiming the scaffold the identified missing mathematical ideas. After PSTs had the opportunity think about and answer these questions individually, they engaged in small group discussions. In these small group discussions they shared their reasoning, discussed, and negotiated about the mathematical details in the student's solution, and then examined the reflected understandings that might need to be scaffolded. Furthermore, they crafted a response to deepen student's understanding. Following that, the whole class discussion was held to examine various group's answers.

During the whole class discussion, PSTs were encouraged to share how they unpacked the underlying reasoning of student, and the peers were invited to ask follow-up question and to seek evidence for how the proposed responses were working toward the mathematical ideas need to be scaffolded. The overarching goal of the whole class discussion was unpacking the invisible work of decision making and illustrating the affordances of various responses that could extend students' thinking. This allowed whole class to reflect on the better and possible ways of deepening student's mathematical thinking.

Given that crafting a mathematically-based response is a difficult skill, after the whole class discussion, PSTs were asked to revise their thinking for interpreting the mathematical ideas needs to be supported. As this is the base for their response to leverage student's mathematical thinking, in addition they were tasked to refine their mathematically based response that will create room for deepening mathematical understanding.

As an example, I described how Task 1 creates base for developing the skill of noticing and drawing on their MKT. Task 1 (see Appendix B) and Task 2 (see Appendix C), were conducted in consecutive weeks after covering topics such as concepts of numbers, place value notation, and addition and during the weeks of covering subtraction. Task 1 focused on having PSTs examine two student solutions as a response to a subtraction word problem, separate result unknown; "A teacher asked her students the question "Paul had 83 strawberries in his basket. He gave 38 strawberries to his friend. How many strawberries did Paul have left?" (Carpenter et al., 2014). Following, the PSTs were presented two different student solutions. The first student solved the question "83 take away 30 is 53 and take away 3 is 50. Then take away 5 more. That's 45." The other student solved the question "83 take away is 38 is the same as 85 take away 40. That's 45." Then, PSTs were tasked to answer the prompted questions that aimed to have them make judgments, weigh the affordances, and use their knowledge strategically. In table (6) below, I identified how the questions prompted in Task 1 aimed to prepare PSTs to attend to the components of noticing children's mathematical thinking and utilizing their MKT. I wanted to stress that SCK and KCS are not always mutually exclusive, and researchers might interpret them differently. I shared my own interpretation in the table below.

Table 6

Nature of Questions Provided in Task 1

Questions Asked	Addressed Domain of MKT and Facet of Noticing
1) What did the student do to solve this problem?	Attending (Jacobs et al., 2010) • Finding out what students think and identifying the mathematical details in their solution MKT- Knowledge of Content and Students (KCS) (Ball et al., 2008) • Student A broke the number 38 down to decade numbers and subtracted. The student first subtracted the 30 from 83. Then subtracted the larger amount 30 from 83. Then incremented down the first 3, then 5. • Student B used the distance meaning of subtraction, and they know the distance between 83 and 38 is the same as 85 and 40.
2) Considering what the student did, what do you think they know? What mathematical understandings are reflected in their solution?	Interpreting (Jacobs et al., 2010) • Identifying what mathematical understandings reflected in children's solutions. MKT- Specialized Content Knowledge (SCK) (Ball et al., 2008) • Student A knows that they can decompose 38 into various addends by subtracting the first 30, then 3, and the last 5. • Student A uses the numbers flexibly, and they use the decade numbers as a benchmark. They took 30 away from 83, that is 53. Instead of taking away 8 from 53, again they used benchmark numbers and they took 3 away and got a decade number 50. Then, they took 5 away easily and got 45.

	• Student B knows that when they take away 38 from 83, they distance between them. They know that when we add 2 to both numbers that would not change the distance between them. They added 2 to both numbers to make both numbers benchmark numbers and ended up with 40 and 85.
3) What do you think is the major difference between the way of interpreting the subtraction between two students?	MKT- Specialized Content Knowledge (SCK) (Ball et al., 2008) • These questions focus on the ability to unpack the mathematical ideas behind two different solutions responding to a separate result unknown word problem to a multidigit subtraction question. • These questions lead PSTs to discuss the affordances of mathematical ideas reflected in different strategies.
4) Which of these students would you judge to be using a method that could be used to subtract any two whole numbers?	
5) Which of these student solutions would you use to discuss on the board?	
6) How would you revise the question to have students solve it as a "think-addition"?	MKT- Knowledge of Content and Students (KCS) & Specialized Content Knowledge (SCK) (Ball et al., 2008) • This question focuses on strategically using KCS and SCK

As it is seen in the Table 6 above, Task 1 aimed to prompt PSTs to engage in attending to students' mathematical thinking by identifying what students did and interpreting the reflected understanding in the given students' mathematical thinking by drawing on their SCK and KCS in the content of subtraction. During the small group work, an audio recorder captured the small group discussion while PSTs were working on Task 1. As the instructor, I interacted with different home groups, providing support by addressing questions and offering clarification as needed. My goal was to engage with these groups, listen to their discussions, and scaffold their sense-making processes through targeted questions. Especially considering the early stage of the semester and it being the initial pedagogy course, my questions aimed to prompt PSTs to provide more evidence and elaborate on their thinking, emphasizing evidence-based reasoning. Please see Table 4 for the question structures.

A whole-group discussion was conducted upon completion of the small group tasks. This discussion aimed to encourage PSTs to articulate their reasoning, unpack their thought processes and foster a collaborative exploration of KCS and SCK. This collective discussion aimed to refine and negotiate a more nuanced understanding of mathematical concepts by sharing and critiquing reasoning the groups shared within the larger class context.

Please see the appendixes, for the following tasks, Task 2, Task 3, Task 4, and Task 5 aim to prepare PSTs to attend to the components of noticing children's mathematical thinking and utilizing their MKT in the related content.

Reflections

After completing each group task, PSTs were requested to engage in individual reflections on their learning process. These reflections aimed to gather information on the challenges PSTs encountered, their moments of sensemaking, what helped them to make sense, and what they were still grappling with. The first reflection question focused on challenges to thinking. Participants were asked to articulate aspects of the task that posed challenges to their thinking processes and to elucidate the reasons behind these challenges. The second question focused on PSTs' sensemaking process. In this question, PSTs were prompted to identify instances during their group work where they experienced personal 'aha moments' meaning moments of sudden clarity. Additionally, they were asked to elaborate on the factors contributing to these moments of insight. The last question of the reflection focused on ongoing questions and confusion. This question aimed to explore lingering uncertainties, wonderings, and questions that remained even after task completion. Participants were encouraged to articulate these areas of confusion to gain more insight into what they were grappling with. The following guiding questions were used in the reflections:

1) What challenged your thinking in the task? Please explain why.

2) Identify a moment(s) where you personally had an aha moment when you were working on the task with your group. What helped you to have this aha moment? Please explain.

3) What are you still wondering about? What are some questions or confusions you still have?

These reflective inquiries aimed to capture the nuanced dimensions of the PSTs' learning experiences by offering valuable insights into what they grappled with and ongoing areas of development.

Data Analysis

In this section, I described analytical procedures employed to answer the following research questions,

RQ1: How does PSTs' noticing students' mathematical thinking change from the beginning of the semester to the end of the semester?

RQ2: What relationships emerge among facets in PSTs' noticing of students' mathematical thinking?

RQ3: What opportunities to learn were available for PSTs in the content course to support their capacity to develop noticing?

Analysis Related to the First Research Question

In order to craft a response that deepens students' mathematical understanding, it is essential for teachers to engage in pedagogical reasoning drawing on their Mathematical

Knowledge for Teaching (MKT), specifically specialized content knowledge (SCK) and knowledge of content and students (KCS). The content course the study took place in aimed to initiate PSTs' development of pedagogical judgment related to noticing children's mathematical thinking and crafting a mathematically-based pedagogical response by drawing on their professional knowledge bases. As a researcher, I hypothesized that facilitating PSTs' professional noticing of children's mathematical thinking will contribute to the development of judgment related to mathematically-based responsive practice by unpacking the invisible work of teaching and utilizing their MKT.

My analysis related to the first research question relied on the baseline assessments (see Appendix A) to identify the change in PSTs' skill of noticing children's mathematical thinking in multi-digit subtraction between the beginning and end of the semester. In my analysis, I drew on two theoretical frameworks: noticing children's mathematical thinking by Jacobs et al. (2010) and mathematical knowledge domains for teaching by Ball et al. (2008) to create analytical rubrics for the baseline assessments.

Coding Baseline Assessments. In coding of baseline assessments, I used a modified version of the coding scheme outlined by Jacobs et al. (2010). In their analytic framework, Jacobs et al. created a coding rubric that structured each component of their framework (attending to, interpreting, and deciding how to respond to children's mathematical thinking) along three levels: lacking evidence, limited evidence, and robust evidence of mathematically significant details. Starting with their framework as a foundational guide to distinguish the components of noticing—attending, interpreting, and deciding how to respond, I made modifications to their coding rubric, since I found their criteria for these three levels are limiting, especially the parts relying on correctness of PSTs' answers. For example, they defined their limited evidence in attending as "Mentions some INCORRECT specifics of mathematics they notice", while their robust evidence explanation is as follows "Mentions CORRECT specific of mathematics they notice" (Dick, 2017, p. 345). In my perspective these definitions were based on dichotomy of right or wrong answers and does not demonstrate utilization of knowledge of content and teaching. Given noticing children's mathematical thinking requires drawing on knowledge of content and teaching, the coding scheme should rely on the level of utilization of MKT.

Initially, I conducted preliminary coding (Charmaz, 2006) for each question prompted in the assessments in relation to examining given student's mathematical thinking. PSTs' responses to the questions related to attending to, interpreting, and responding to the student's mathematical thinking, as well as eliciting the student's thinking, revealed three broad categories. Those categories were broadly identified as general comments without incorporating noteworthy mathematical details, identifying some of the mathematical elements or emphasizing procedural actions without conceptual connections, and identifying all the mathematically significant details and making conceptual connections. Three categories emerged from the preliminary analysis, laid the foundation of my codes: low, medium, and high evidence of drawing on and utilizing MKT. The precise definition of these codes varies based on the facet of noticing (Please see the Appendix G for the coding rubric). Detailed examples and descriptions of each level for attending to, interpreting, and responding to student's mathematical thinking, as well as eliciting student's thinking, can be found in the results section.

In order to enhance the robustness of the data coding scheme, I met with a fellow researcher two times to discuss the codes, and accordingly I revised the descriptions of the codes. Next, I met with the same researcher to code a sample dataset to test the reliability of my coding.

After coding the sample dataset, we achieved a level of agreement at and above 90%. Following, any disparities were reconciled. Building upon this consensus, I proceeded to code the rest of the dataset.

To begin, pre- and post-baseline assessments were coded based on using the provided coding rubric in Appendix G. Out of the 24 PSTs, 18 were included in the coding process for both pre-and post-assessments since six PSTs did not attend the class on the day the post-assessment was conducted and were not able to take the post-assessment. After completing the coding of pre- and post- assessments, I focused on identifying PSTs' development throughout the semester in each facet related to the professional noticing of children's mathematical thinking, as well as the eliciting. I identified the development as a shift in the codes from the lower level to the higher level which demonstrates the sophistication of utilizing MKT and their level of attending to the corresponding skills required for developing noticing (attending, interpreting, responding, and eliciting).

First of all, I focused on assessing the overall improvement in each facet of noticing by identifying the number of PSTs fell into the corresponding levels (low, medium, and high evidence). To achieve that, I compared the number of PSTs categorized within each level in the pre- and post-assessments aiming to identify the change throughout the study in each facet of noticing. After identifying the overall change in each facet of noticing, I conducted a second stage of analysis to unpack the observed changes.

In the second stage of analysis, I focused on tracking the specific changes in levels for each PSTs to describe their progress to the higher evidence level, their regression to the lower evidence level, or their stability if they stayed in the same level. The changes observed in the facets of the professional noticing are categorized according to the transition from low to medium evidence, medium to high evidence, and low to high evidence levels. Additionally, a category labeled "stayed the same" was included to identify PSTs who demonstrated no development and remained in the same level between the pre-and post-assessments such as from low to low, medium to medium, and high to high evidence levels. Notably, the findings do not include categories that showed transition from medium to low, high to medium, or high to low evidence categories, as no instances of such transitions were observed in the results.

After identifying the possible transitions as from low to medium, medium to high, low to high, and stayed the same, I reported the number of the PSTs who fell into each transition category. Furthermore, I created subcategorized under the level of "stayed the same". These categories are low to low, medium to medium, and high to high indicating no level change between pre- and post-assessments. This subcategorization allowed me to interpret the change across the study. In addition, it allowed me to see that the part of the stability between pre- and post-assessments might stem from retaining the high level of evidence in the facets of noticing across the study. Identifying the possible transitions and the number of PSTs in each transition category allowed me to understand whether and to what extent PSTs demonstrated progress in noticing to students' mathematical thinking between pre- and post-assessments.

Analysis Related to Second Research Question

To address the second research question, I utilized the first stage of coding which was already conducted to answer the first research question described above. After coding the pre- and post-assessments based on the low, medium, and high levels across the facets of noticing, I combined the datasets of pre- and post-assessments. That resulted in 36 assessments for examination. My goal was to identify the emerging relationships among the facets of noticing to gain insights into how the skills required for noticing are related to each other. By identifying these relationships, I aimed to contribute to the knowledge base of how to cultivate PSTs' responsiveness to leverage students' mathematical understanding.

Analysis Related to Third Research Question

In this phase of analysis, I focused on PSTs' learning in their small group to investigate the opportunities that supported the development of the various domains of MKT. I do not make any claims here about how individual PSTs' participation in small group work contributed to their development of MKT, but I focus on how small group discussions centered around tasks focused on students' thinking provided access to the development of MKT domains for the group participants and utilizing this knowledge in noticing students' mathematical thinking. Such analysis affords gathering more information about the kinds of pedagogical opportunities that might support developing various domains of MKT and how it relates to the development of their noticing of children's mathematical thinking.

I drew on instrumental case study (Creswell, 2006) and I employed "embedded analysis" which is defined as particular aspects of the case (p. 75). All audio-recorded small group discussions were transcribed and underwent multiple readings while notes were taken to document major themes. Throughout the initial readings of transcripts, my focus was guided by inquiries into the nature of mathematical and pedagogical questions raised, and how PST' talk about student work to understand what might help PSTs deepen their understanding of MKT. The units of analysis referred to as episodes of pedagogical reasoning (EPRs) (Horn, 2005). I adapted the definition of EPR as episodes revolve around the particular idea and can be as a single turn or an exchange involving various participants.

Findings

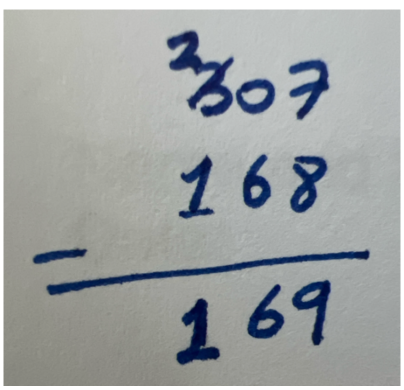
The presentation of the results is organized by the three research questions aimed at addressing the overarching question of "How do PSTs develop in their capacity to craft evidence-based responses to children's mathematical thinking?" First, I presented the results from the baseline assessments to identify the change in PSTs' noticing and responding to children's mathematical thinking to answer the question "How does PSTs' noticing students' mathematical thinking change from the beginning of the semester to the end of the semester?" Next, I presented the findings related to the changes that were most significant throughout the facets of noticing children's mathematical thinking. This addresses the research question "What relationships emerge among facets in PSTs' noticing of students' mathematical thinking?" Lastly, I discussed the results from PSTs' small group discussions centered around student work to identify the opportunities to learn become available in that setting to answer the question of "What opportunities to learn were available for PSTs in the content course to support their capacity to develop noticing?"

Change in PSTs' Noticing Students' Mathematical Thinking

To answer the first research question "How does PSTs' noticing students' mathematical thinking change from the beginning of the semester to the end of the semester?", pre- and post-baseline assessments were examined. The student solution corresponding to the question of subtracting 168 from 307 shown in Figure 3 below was presented to the PSTs in the assessments. PSTs were guided through a structured process by drawing on the three facets of noticing children's mathematical thinking framework by Jacobs et al. (2010), as well as eliciting. To recap from what I described in the method section, the first question of the assessment concentrated on the first aspect of the professional noticing framework, attending that is defined by recognizing the mathematical elements in the student's solution. The next question focused on the second aspect of professional noticing, interpreting. In this question, PSTs were guided to interpret the mathematical understanding reflected in the given student solution. In addition, a question was posed about identifying the mathematical ideas that may require support. Furthermore, a follow-up question was asked to surface how PSTs elicit the student's understanding to gather more information about the students' mathematical thinking. Lastly, a question related to the facet of responding was asked to elicit the pedagogical approaches PSTs would use to extend the student's mathematical understanding. I presented the findings of each of these questions in the below sections.

Figure 3

Presented Student Solution in The Baseline Assessments



The coding involved an examination of 18 PSTs' both pre- and post-assessments (See Appendix G for coding rubric). During the coding process, each facet related to the professional noticing of children's mathematical thinking, as well as the eliciting process, organized into three levels. These levels were low evidence, medium evidence, and high evidence based on the extent

PSTs used mathematical elements to describe the student's mathematical thinking presented in the assessments. Table 7 outlines number of PSTs that fall into the corresponding level in each facet of noticing. In the following sections, I presented each facet individually by delving into the specific changes observed.

Table 7

Number of PSTs Fall into the Relative Level in Each Facets of Noticing

Level	Attending		Interpreting the reflected understanding		Interpreting the math ideas that need to be scaffolded		Responding		Eliciting	
	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post
Low	11/18	1/18	12/18	2/18	12/18	0/18	13/18	0/18	4/18	2/18
Medium	4/18	3/18	5/18	9/18	6/18	14/18	5/18	13/18	13/18	3/18
High	3/18	14/18	1/18	7/18	0/18	4/18	0/18	5/18	1/18	13/18

Change in Attending to Children's Mathematical Thinking

Attending is the first facet of the three intertwined facets of the professional noticing of children's mathematical thinking. Attending is defined as identifying the noteworthy mathematical details in children's thinking. In this section, I described to what extent PSTs recognize the mathematical elements in the given student's solution in the context of subtracting multi-digit numbers in the assessments. The development of PSTs' skills in identifying noteworthy steps taken by the student in their problem solution showed promising progress. As summarized in Table 8, the majority of PSTs (11/18) fell into the low evidence level in the pre-assessment. This level is primarily characterized by missing the mathematical elements to describe the student's answer and focusing on providing general descriptions or assessing the correctness of the student's solution. For example, a PST's answer "They used the borrowing technique but made a mistake along the way" was coded as low evidence since there is no specific identification of mathematical elements. In this level, PSTs missed the opportunity to recognize the student might have skipped the tens place and borrowed from the hundreds place, and the student might have thought 6-0 instead of 0-6 in the tens place. In the post-assessment, it was evident that only 1 out of 18 PST made a general description or missed drawing their explanation on the mathematical elements in the student's solution.

Table 8

Number of PSTs in Different Levels of Attending to Children's Mathematical Thinking

Level of attending	Assessment		Description of levels
	Pre	Post	
Low	11/18	1/18	Using the specifics of KCS and SCK is missed. Only a general description of students' answers was provided. Attended only to the correctness of answers
Medium	4/18	3/18	Identification of some of the noteworthy KCS and SCK, but not all of them or giving vague explanations without proper evidence
High	3/18	14/18	Full identification of the mathematical elements related to KCS and SCK

In the pre-assessment, 4 out of 18 PSTs were categorized in the medium evidence category, compared to 3 out of 18 in the post-assessment. In this category, PSTs explained the mathematical elements of the student's solution partially and did not attend to all of them. For example, a PST said, "The child may have subtracted 6-0 in the 10s place value to get the 6 in 169. However, they got the 1 and 9 correct, so they have some understanding of place value". In this example, the PST was able to identify that the student might have gotten 6 in the tens place because they might have done 6-0 instead of 0-6. However, they interpreted the student has some place value understanding, since they got 1 and 9 correct without specifying how the student got 1 in the hundreds place and 9 at the ones place if the student did not do any borrowing to subtract 0-6 in the tens place. In this situation, the PSTs only explained the student's solution partially. However, the PST identified that the students might have done 6-0, instead of 0-6, since the tens place was 0. This is related with KCS which is about anticipating how students would think.

In the high evidence category, PSTs were able to recognize all noteworthy mathematical elements in the student's solution and draw on their professional knowledge. When the high evidence category was examined in the pre-assessment, 3 out of 18 PSTs identified all the mathematical elements in the student's solution while in the post-assessment the number of PSTs drastically increased into 14 out of 18. An example of a PST answer for high level is "The kid borrowed 1 from hundreds since there is zero in the tens place and made it 10 in the ones place". In this example, it is evident that the PST recognizes why the student skipped the tens-place and they borrowed "1" from a hundred-place and transferred it to the ones place as one ten without acknowledging the tens place. In addition, they showed evidence of knowledge of reference units between place values by identifying one hundred was transferred as "1" ten by the student which is evidence of SCK. In this example, the PST was able demonstrate knowledge behind mathematical approaches and anticipated what students likely to think. This showed more nuanced understanding of MKT compared to the medium level.

Although these results show us the overall improvement in the attending component of noticing during the study, they do not give us insights into whether the PSTs who were in the specific categories had progressed to higher evidence categories or stayed in the same category. For example, PSTs primarily categorized in the lower evidence of attending could either improve to medium evidence or high evidence category or not improve at all and could stay in the same category across the semester. Accordingly, in the following part, I will present findings from a detailed analysis that allowed me to track PSTs progress (or the lack of) in various categories.

Changes in Participation Between Category Levels. The changes observed in the facet of attending to children's mathematical thinking are outlined in Table 9 according to the transition from low to medium evidence, medium to high evidence, and low to high evidence categories. Additionally, a category labeled "stayed the same" is included to identify PSTs who demonstrated no development and remained in the same category between the pre-and post-assessments such as from low to low, medium to medium, and high to high evidence category. Notably, the findings do not include categories that show transition from medium to low, high to medium, or high to low evidence categories, as no instances of such transitions were observed in the results.

Table 9 below shows the number of PSTs in different change categories relative to the facet of attending to children's mathematical thinking. Table 9, it becomes evident that the majority of PSTs demonstrated improvement in their ability to identifying the noteworthy mathematical details in children's thinking. Specifically, 8 out of 18 PSTs who initially failed to use the mathematical elements in describing the student's answer in the pre-assessment and fell into the low evidence category. These PSTs were able to recognize all noteworthy mathematical elements in the student's solution by drawing on their MKT in the post-assessment and fell into the high evidence category. It is worth emphasizing that these PSTs, who previously offered only general descriptions or solely focused on the correctness of the answers, demonstrated a significant evolution in their capacity to attend to the mathematical details of children's thinking.

Table 9

Level Changes in the Facet of Attending

Change in levels of attending	Number of PSTs
Low-Low	1/18
Medium-Medium	1/18
High-High	3/18
Low to Medium	2/18
Medium to High	3/18
Low to High	8/18

A subset of the PSTs (3/18) progressed from the medium evidence level, which involved identifying some mathematical details in the student's thinking, to the high evidence level. This latter level involves recognizing *all* noteworthy mathematical elements in the student's solution by drawing on their MKT. In addition, 2 out of 18 PSTs improved their skills from the low evidence level where no evidence of identifying any mathematical details in the student's thinking was observed, to the medium evidence level, where they explained mathematical elements partially but were not able to address all of them.

Interestingly, 5 out of 18 PSTs showed no progress between the pre- and post-assessments. Nevertheless, within this group, 3 out of 18 of them were already in the high evidence level, showcasing their proficiency in identifying the mathematical details in the student's thinking across both assessments. Meanwhile, 1 out of 18 PST remained in the medium evidence level, and 1 out of 18 PST retained their position in the low evidence level. Although these findings showed diverse trajectories of skill development among PSTs throughout the study, the majority of the PSTs (8 out of 18) improved their skills in attending to the student's mathematical thinking from low evidence to the high evidence level by drastically enhancing their ability to identify the mathematical details in the student's thinking.

Change in Interpreting Children's Mathematical Thinking

Interpreting is one of the three facets of the professional noticing of children's mathematical thinking framework. Interpreting is defined as identifying the underlying mathematical understandings present in children's solutions or mathematical thinking. In this study, I added a unique dimension to the interpreting facet. Not only does it involve identifying the student's understanding, but it also incorporates an additional aspect—spotting mathematical ideas that may require support. This dual focus aimed to provide scaffolding for PSTs in making informed decisions to enhance children's mathematical thinking. Specifically, I chose to have PSTs analyze a student's solution that includes an error to direct PSTs' attention to the reflected understanding in the student's thinking and the mathematical ideas that require scaffolding, rather than solely focusing on correctness. Through this reporting, I aimed to present the findings addressing both aspects of interpreting the student's mathematical reasoning respectively.

Change in Interpreting - The Mathematical Understanding Evident in the Student's Thinking

PSTs' interpretation of the student's mathematical understanding is reported in Table 10 below based on the levels of low, medium, and high evidence. In the pre-assessment, majority of the PSTs (12 out of 18) fell into the low evidence level which is characterized by making general claims without relying on mathematical elements or drawing on the knowledge domains of MKT. For instance, a PTS stated, "They know subtraction means to take away. They know that they need to manipulate the numbers to be able to subtract, but they are still working on the proper way to do so". This example falls into the low evidence level as the PST asserts the student's understanding without providing supporting evidence. Furthermore, the language used, such as “manipulating the numbers” and “working on the proper way” of doing subtraction lacked specificity and failed to demonstrate specialized content knowledge related to subtraction, such as place value or borrowing. Upon examining the post-assessment results, a notable improvement was observed, with only 2 out of 18 PSTs falling into the low evidence level. This positive shift could be an indicator of proficiency in articulating evidence-based claims and deeper integration of specialized content knowledge in their assessments of students’ underlying mathematical understanding.

Table 10

Number of PSTs in Different Levels of Interpreting Children’s Understanding

Level of interpreting the student’s understanding	Assessment		Description of the levels
	Pre	Post	
Low	12/18	2/18	Interpretations are not evidence-based or general comments are provided without mentioning mathematical elements.
Medium	5/18	9/18	Drawing on some evidence, however, interpretation is still vague or limited and the interpretation explains only part of the student's work
High	1/18	7/18	Making sense of the details of a student strategy and noting how these details reflected what the student understood in specific situations

The medium level was characterized by the inclusion of some evidence or focusing on a portion of the student’s work; however, the interpretation remained vague or limited. A notable improvement was observed in the medium evidence level in the interpreting facet between the pre- and post-assessments. While 5 out of 18 PSTs were categorized into the medium evidence level in the pre-assessment, this number increased to 9 PSTs in the post-assessment. An example from this level is, “They understand the place value and subtracting from positive numbers, they don’t know what to do with 0." In this example, PST inferred that the student struggled with subtracting from a zero as in the problem given to the student zero was used as a placeholder in tens place. However, although they mentioned that the student knows about place value, the explanation lacked evidence, and the connection between having difficulty with zero in subtraction and its relation to place value was not articulated.

The high evidence level was characterized by the ability to make sense of the enlisted mathematical details within the student's strategy and to articulate how these details reflect the student's understanding and explain the student's solution completely. Given the higher cognitive demand associated with this level, one PST fell into this level in the classification of pre-assessment. In the post-assessment, this number increased 7 PSTs. A PST who fell into this level stated that "They know that when the value in the ones place on top is smaller than the values in the ones place on the bottom, they have to borrow. They knew they couldn't borrow from a place value that had a zero". In this instance, the PST's answer indicated an ability to identify that the student can borrow to facilitate the subtraction of a larger number from a smaller one and also identified the part that the student does not have a clear understanding of when there is a zero in the number. In the following paragraphs, I unpacked the presented results to demonstrate whether and how PSTs who were in the specific level of interpreting were improved into higher evidence levels or stayed at the same level without showing progress.

Change in Participation Between Category Levels. As presented in Table 11, the most noteworthy change was observed from the low to the high evidence level in 8 out of 18 PSTs. In the pre-assessments, these PSTs were observed making general claims without relying on mathematical elements in the student's thinking or without drawing on the knowledge domains of MKT in the pre-assessment. However, they progressed into the inclusion of some evidence focusing on mathematical details of the student's thinking in the post-assessment.

Table 11

Level Changes in the Facet of Interpreting-Identifying Reflected Understandings in Students' Solution

Change	Number of the PSTs
Low-Low	2/18
Medium-Medium	1/18
High-High	1/18
Low to Medium	8/18
Medium to High	4/18
Low to High	2/18

In total, 6 PSTs advanced their skill of interpreting to the high evidence level. They demonstrated their ability to make sense of the involved mathematical details within the student's strategy to articulate how these details reflect the student's understanding and explain the student's solution completely. Within this level, 2 PSTs showed improvement from the low evidence level, where they only made general claims without relying on mathematical elements in the student's thinking or without drawing on the knowledge domains of MKT in the pre-assessments. Similarly, another 4 PSTs showed improvement from the medium evidence level where they utilized some mathematical elements to make an inference about the student's understanding, to the high evidence level.

Although the majority of the PSTs showed a significant improvement in their interpreting skill by identifying the reflected understanding of the student's solution, a minority of the PSTs showed no progress between the pre-and post-assessments. In total, 4 out of 18 PSTs showed no progress across the assessments. However, when Table 11 was examined, it was seen that one of these PSTs already showed proficiency in both assessments and they remained in the high evidence level. Contrastingly, one of the PSTs remained in the medium evidence level, and two PSTs remained in the low evidence level by indicating limited or no development in their interpreting skills.

Change in Interpreting- The Mathematical Ideas That Need to be Scaffolded

Table 12 below presented the number of the PSTs across the three levels in the interpreting phase that requires identification of the mathematical concepts the student might need to have support with. In the pre-assessment, the majority of the PSTs fell within the low and medium evidence levels, with none in the high evidence level. Specifically, 12 out of 18 PSTs were categorized as low evidence level, indicating responses characterized by the general comment lacking evidence of drawing on SCK and KCS. For instance, a PST's response that is coded in this level states "Might need help with how to subtract bigger numbers." As can be seen from this instance, the PST's response does not rely on any mathematical elements and professional knowledge of teaching. However, when the post-assessment result was examined, it was evident that no PSTs provided responses lacking the identification of a mathematical idea.

Table 12

Number of PSTs in Different Levels of Interpreting the Math Ideas Children Seed Support With

Level of interpreting the math ideas that need to be scaffolded	Assessment		Description of the level
	Pre	Post	

Low	12/18	0/18	Interpretations are not evidence-based or general comments are provided without mentioning mathematical elements
Medium	6/18	14/18	Identifying place value as a math idea that needs to be scaffolded, but missing the making a connection with other mathematical ideas such as borrowing
High	0/18	4/18	In addition to focusing on scaffolding the idea of place value, they also focus on reference units that show the relation between different place values and suggest a connection with borrowing

Within the medium evidence level, PSTs demonstrated the ability to identify a single mathematical idea but often missed opportunities to recognize multiple interconnected mathematical concepts. In the pre-assessment, 6 out of 18 PSTs were in this level. An example response from this level is "More understanding with the zero, and what it means in different positions at ones, tens, 100s, 1000s." This example showcases the PST's focus on place value comprehension and the significance of zero as a placeholder in various positions. In this example, the student would benefit from the place value understanding, however, making a connection with borrowing would be more meaningful to leverage the student's understanding of subtraction in relation to students' error in the context of subtraction (See Figure 3). By establishing connections between reference units for each digit and relating these units to one another, the student's understanding of subtraction can be enhanced significantly. In the post-assessment, the number of PSTs capable of identifying at least one mathematical idea to broaden the student's thinking increased to 14. This could be an example of how PSTs increased their utilization of SCK in the post-assessment compared to the pre-assessment.

The high evidence level is the one that requires more cognitive demand and using the SCK knowledge. In this level, PSTs did not only identify an isolated mathematical idea. They identified multiple mathematical ideas, and they were able to address those mathematical ideas in relation to each other. For example, a PST stated that "Place value understanding, what each place value means. In addition, its relation with borrowing which is decomposing the numbers to make subtraction possible." Despite no PSTs demonstrating proficiency in this level in the pre-assessment phase, subsequent numbers of PSTs showed improvement and 4 out of 18 PSTs were identified as high evidence level. In the following paragraphs, I investigated specific changes in interpreting the math ideas the students need support with across the levels of low, medium, and high evidence that will allow me to demonstrate specific transitions of PSTs.

Change in Participation Between Category Levels. The transitions observed between the evidence level in the interpreting facet, where PSTs were expected to demonstrate an ability to identify the mathematical understandings in need of scaffolding for a student, are reported in Table 13. The most notable transition observed was from the low to the medium evidence level. Specifically, 11 out of 18 PSTs who were initially categorized in the low evidence level in the pre-assessment, characterized by general comments lacking specific knowledge related to teaching and learning mathematics and interpretations lacking evidence-based support, progressed to the medium level evidence level in the post-assessment. In the medium evidence level, while PSTs' interpretation may not fully elucidate a student's conceptual comprehension need, demonstrated evidence of their ability to draw on their mathematical knowledge for teaching to identify conceptual ideas. For example, a PST mentioned "More understanding with the zero, and what it means in different positions at ones, tens, 100s, 1000s." In this example, it is evident that PST was able to identify the need for support in understanding place value. However, there was no further explanation regarding the connection between place value and subtraction, nor how place value would be utilized in borrowing, grouping, and regrouping referent units.

Table 13

Level Changes in the Facet of Interpreting Math Ideas Need to Be Scaffolded

Change	Number of the PSTs
Medium-Medium	3/18
Low to Medium	11/18
Medium to High	3/18
Low to High	1/18

As summarized in Table 13, only one of PSTs progressed from the low to high evidence level in interpreting the mathematical ideas that the student might need support with, while the majority of the PSTs transitioned from the low to medium evidence level. This result could be expected as a shift from overgeneralizing children's understanding to carefully linking interpretations to the specific details of children's understanding is too challenging to develop and robust development might not be expected in a semester. Jacobs et al. (2010) suggested that developing expertise in the professional noticing of children's mathematical thinking is complex and their study showed that developing this expertise may require years. Nevertheless, a shift from the low to the medium level was observed in 11 out of 18 PSTs, and from the medium to the high evidence level was observed in 3 out of 18 PSTs is a promising result as, in total, the majority of the PSTs (15 out of 18) showed growth and shifted to their interpretations to the connecting to missing mathematical concepts that needs to be scaffolded. While 3 out of 18 PSTs did not show growth in terms of interpreting the math ideas the student might need support with, they still demonstrated an understanding of linking interpretations to specific mathematical elements. This finding suggested that while progress varied among individuals, there was an overall improvement in the PSTs' ability to identify and articulate the mathematical understanding needs of children.

Change in Deciding How to Respond to Children's Mathematical Thinking

The change in the number of PSTs across the three levels in the facet of deciding on how to respond to the student's understanding is outlined in Table 14 below. The change in this phase exhibited a parallel trend with the previously discussed facet of identifying the mathematical ideas the student needs support with. Similar to the facet reported above, the majority of the PSTs fell within the low and medium evidence levels, with none in the high evidence level in the pre-assessment. Specifically, 13 out of 18 PSTs were classified in the low evidence level, characterized by actions or questions primarily focused on procedures, lacking connections to broader mathematical concepts, or merely seeking correct answers. For instance, "I could show a large, detailed example of how to solve the problem, using different colors to show the different steps" or "I would show them how to cross out the numbers so they could see it visually." In these instances, PSTs described methods like demonstrating a detailed step-by-step process using different colors or instructing students to visually cross out numbers. Their focus was on the procedural action of borrowing without focusing on conceptual understanding and lack mentioning of any mathematical elements that the student might need support with. In these approaches, they overlooked crucial mathematical concepts that students may struggle with such as the meaning of borrowing and underlying concepts like place value understanding and reference unit understanding. However, when the post-assessment result is examined, it is evident that no PSTs provided responses solely centered on procedural actions without connection with the mathematical concepts.

Table 14

Number of PSTs in Different Levels of Responding to Children's Mathematical Thinking

Level of responding	Assessment		Description of the level
	Pre	Post	
Low	13/18	0/18	Questions or actions focused on procedures without connections or producing correct answers.
Medium	5/18	13/18	Responses attend to one of the missing mathematical ideas identified in the student's mathematical thinking and/or solution strategy.
High	0/18	5/18	Questions or actions related to conceptual understanding and focusing on interrelated mathematical ideas.

The medium evidence level was distinguished by responses that aimed to cultivate an understanding of a missing mathematical concept inferred from the student's solution. When this level was examined, 5 out of 18 PSTs were categorized in the medium evidence group while this number increased to 13 out of 18 in the post-assessment. An example response in this level was "I would split the problem into 300+0+7 to have him see each value individually to show that the 0 isn't nothing, but it is the tens place" This response focuses on the place value understanding by breaking down the numbers based on a digit as 3 means three 300 and 0 means 0 tens, 7 means 7 ones. Unlike the examples in the low evidence level, this response was focused on grasping a mathematical concept, specifically place value. As it was seen from the example above, PSTs in this level drew on their SCK to decide their pedagogical actions.

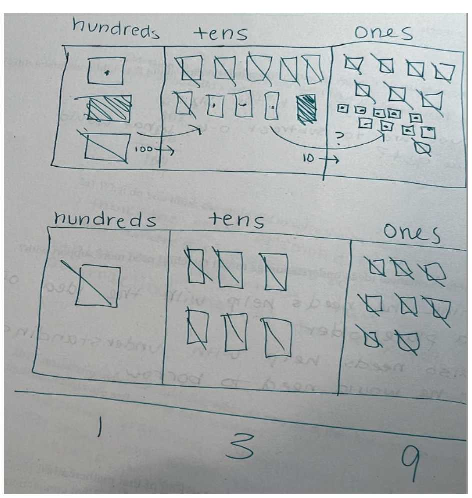
The high evidence level requires the highest level of engagement with MKT among the three levels. Actions or proposed follow-up questions falling into this level require not only understanding a single mathematical idea but also addressing the interconnectedness of multiple mathematical ideas. The distribution of number of the PSTs within this level yields a promising finding. During the pre-assessment phase, none of the PSTs' responses met the criteria for the high evidence level. However, during the post-assessment, 5 out of 18 PSTs demonstrated proficient responding skills by falling into the high evidence level. A PST's response from the post-assessment demonstrates their improved skill of responding:

> "I would first ask them what each number represents with the base ten blocks. This would help them to see borrowing involves taking a place value over and breaking it up. You cannot take 6 tens away from 0 tens, so you have to break one of the hundreds into 10 tens and move one ten over to the ones because you cannot take 8 ones away from 7."

Similarly, another PST also focused on using base ten blocks to represent each number to endorse place value understanding as shown in Figure 4. In addition, they represented decomposing place value to show what is being borrowed.

Figure 4

A PST's Example of the High Evidence Response



In these examples, the PSTs emphasized place value understanding by prompting the student to represent numbers with base ten blocks. Moreover, the PSTs focused on establishing reference unit understanding for each digit and elucidating their interrelation. This suggests that there was an improvement in their ability to analyze and synthesize mathematical concepts to decide how to respond to leverage the student's existing mathematical knowledge. Now, I will explore specific transition PSTs demonstrated across the low, medium, and high evidence levels of responding.

Change in Participation Between Category Levels. The most noteworthy shift observed in Table 15 from the low to the medium evidence level in the responding phase with 11 out of 18 PSTs. In the pre-assessment, these PSTs crafted questions or proposed next moves centered around drilling the procedures of subtraction by aiming to produce correct answers without connecting them to conceptual understanding. However, in the post-assessment, these answers evolved into aiming to address at least one missing mathematical concept. For example, focusing on enhancing place value understanding that was identified as missing in the student's solution.

Table 15

Level Changes in the Facet of Responding-Deciding How to Respond to the Student's Mathematical Thinking

Change	Number of the PSTs
Medium-Medium	2/18
Low to Medium	11/18
Medium to High	3/18
Low to High	2/18

Table 15 shows, 3 out of 18 PSTs demonstrated progress into the high evidence level from the medium evidence level. These PSTs initially focused on a mathematical concept of place value understanding to develop with their responses to the student's mathematical thinking in the pre-assessment. In the post-assessment, in addition to focusing on place value understanding, they also focused on developing connections with other mathematical ideas such as making a connection between borrowing and decomposing place values to perform multi-digit subtraction.

In addition, 2 out of 18 PSTs demonstrated progress from the low to the high evidence level between pre- and post-assessments. This shift is particularly significant as initially, these PSTs focused on drilling the procedures of subtraction without emphasizing conceptual comprehension. By the end of the semester, their responses evolved into a response that incorporates the foundational mathematical ideas in relation to each other that are required to do the subtraction. For instance, these PSTs focused on place value understanding and its connection with borrowing by focusing on understanding the decomposition and re-composition of place values.

Although the majority of the PSTs (16 out of 18) showed progress in the level of responses they proposed across the assessments, a minority of the PSTs remained at the same level. Specifically, 2 out of 18 PSTs showed no improvement and proposed a response that fell within the medium evidence level and continued emphasis on scaffolding the student's place value understanding between both pre- and post-assessments.

Change in Eliciting Children's Mathematical Thinking

Eliciting aims to gather more information about students' mathematical understanding through follow-up questions. Unlike the cyclical relationship between the facets of professional noticing that are attending, interpreting, and deciding how to respond, eliciting does not follow a specific order. Teachers have the flexibility to elicit information at any stage when they require further insight into children's thought processes. Despite its non-linear nature, I have presented findings related to eliciting after the others. This decision is not based on its chronological occurrence following attending, interpreting, and responding, but rather on its inherent flexibility, which allows it to manifest at any point within the cycle. Consequently, I presented the findings related to eliciting in the low, medium, and high evidence levels to provide an understanding of how the change occurred as a result of the intervention in the course.

Follow-up questions that primarily focus on the correctness or guide the student toward the correct or intended answer, without demonstrating an intention to understand the student's thought process, were categorized into the low evidence level. For example, one PST's response exemplifies this level: "If we check our work and add 169 back to 168, what number do we get?" This question directs the student's attention solely to the accuracy of the solution and prompts them to recognize their mistake without delving into their reasoning or understanding of the problem-solving process. Such questions may limit opportunities for deep comprehension of mathematical concepts as they simply lead the student toward the desired outcome without exploring their cognitive processes. It is instrumental for teachers to pose open-ended questions that encourage reflection and analysis, enabling students to articulate their reasoning and providing valuable insights for teachers to comprehend any misconceptions or underlying thought processes. Table 16 shows that while 4 out of 18 PSTs employed this approach in the pre-assessment phase, this number decreased to 2 out of 18 in the post-assessment, suggesting a positive shift towards more effective questioning strategies.

Table 16

Number of PSTs in Different Levels of Eliciting Children's Mathematical Thinking

Level of eliciting	Assessment		Description of the Levels
	Pre	Post	
Low	4/18	2/18	Questions focusing on the correctness or funneling students to the right or intended answers
Medium	13/18	3/18	Focusing on "what" students did instead of "why"
High	1/18	13/18	Focusing on understanding why students did what they did

The medium evidence level of eliciting was characterized by the follow-up questions that primarily focused on what students did. Understanding what students did to solve the problem is a crucial step for teachers to comprehend children's thought processes. However, solely concentrating on the actions taken by students might not be sufficient to inquire deeply their understanding of mathematical concepts. Responses categorized into the medium evidence level succeeded in inquiring about what the student did but lacked in inquiring about why the student took those particular actions. For instance, PSTs' questions such as "I would ask them to show me the steps they took" or "When doing so where did you borrow from?" exemplified this approach. Although this question allows teachers to gather more information about students' thought processes, it might not be sufficient to reveal the underlying reasoning behind what the student did.

Examining the number of the PSTs shifts in Table 16, it becomes apparent that while the majority of PSTs (13 out of 18) focused on understanding what the student did in the pre-assessment, this figure decreased to 3 out of 18 in the post-assessment. At first glance, this shift may appear as a regression. However, a closer examination of the post-assessment data revealed a noteworthy trend: In the post-assessment, PSTs not only concentrated on understanding what the student did but also probed deeper to comprehend why the student approached the problem in a certain way, thereby gathering more insights into their reasoning. This shift contributed to the majority of PSTs moving into the high evidence level in the post-assessment phase. I explained the findings of the high evidence level in the next paragraph.

As previously discussed, the high evidence level comprised follow-up questions that aim to unpack the actions of children to understand the underlying reasoning behind these actions. For instance, a PST's questions such as "What does the 0 represent in 307? What do the three represent? How many tens are in one hundred? How many are you borrowing for the ones place? Why did you borrow from the 3 (hundreds place)?" exemplified the types of inquiries categorized within this high evidence level. These types of questions aim to delve deeper into the student's thought processes and reasoning behind their actions, providing the PSTs with valuable insights into the student's comprehension of place value and borrowing.

By posing these follow-up questions, the PST can more effectively assess the student's conceptual understanding and identify any underlying misconceptions. Examining the change in number of PSTs categorized in the low level in Table 16, it became apparent that while only 1 out of 18 of PSTs asked follow-up questions focused on evaluating conceptual understanding in the pre-assessment, this number increased to 13 out of 18 in the post-assessment. This notable increase demonstrates a marked improvement in the PSTs' ability to elicit responses that target conceptual understanding.

When considering the overall transition across the levels of low, medium, and high evidence, in the post-assessment, it is evident that the majority of PSTs were successful in enhancing their elicitation techniques to prioritize conceptual understanding. This improvement infers that the PSTs have developed a deeper understanding of how to effectively assess students' conceptual understanding. Now, I will investigate specific transitions PSTs demonstrated among the low, medium, and high evidence levels.

Change in Participation Between Category Levels. A noteworthy transition was observed in the eliciting level, where PSTs demonstrated improvement in their ability to ask follow-up questions to gather more information about a student's thought process. Specifically, 10 out of 18 PSTs, who were initially categorized in the medium evidence level during the pre-assessment, progressed to the high evidence level. Interpreting these levels, this result suggests that while these PSTs initially focused on gathering more information about what steps students took, later they expanded their inquire about on why the student took certain actions and asked questions to understand the underlying reasoning behind the student's thought process. Although focusing on what students did is necessary to gather information about the student's thinking, in addition, focusing on why the student solved the question in a certain way shows evidence of attention of underlying mathematical ideas and conceptual understanding in children's mathematical thinking.

Table 17

Level Changes in Eliciting

Change	Eliciting
Low-Low	2/18
Medium-Medium	3/18
High-High	1/18
Low to Medium	-
Medium to High	10/18
Low to High	2/18

However, as it reported in Table 17, a subset of PSTs did not demonstrate progress throughout the study. Specifically, 2 out of 18 PSTs remained in the low evidence level in both pre-and post-assessments and continued to pose general questions that did not delve into understanding the student's actions or the reasoning behind them. 3 out of 18 PSTs remained in the medium evidence level in both assessments. The PSTs managed to focus on what the student did but did not explore the underlying mathematical thought processes. Although this indicates a partial proficiency in asking eliciting questions, these PSTs did not show the development of their eliciting skill throughout the study.

Despite these findings, it is encouraging to note that, in total, the majority of the PSTs (12 out of 18) exhibited progress in their ability to elicit mathematical thinking. Among them, 10 out of 18 PSTs advanced from the medium to high-level evidence level, while 2 out of 18 PSTs made a significant leap from the low to the high evidence level. This distinction is noteworthy, as it suggests that a substantial portion of PSTs improved their questioning techniques, moving from general inquiries to those that probe deeper into the student's mathematical reasoning and understanding. This shift implies that PSTs are increasingly focusing on grasping the mathematical concepts understood by children and identifying gaps in children's mathematical conceptual knowledge.

These results underscore the complexity of developing effective questioning skills among PSTs, a critical component of teaching that facilitates a deeper understanding of students' mathematical thinking. The significant progress observed in a majority of PSTs highlights the effectiveness of targeted interventions and training programs to enhance these skills. However, the persistence of general questioning among a minority of PSTs suggests the need for continued emphasis on this area within teacher education programs. By fostering PSTs' ability to elicit and interpret students' mathematical thinking more effectively, we can better prepare them to address the diverse learning needs of their future students.

The findings regarding PSTs' noticing and eliciting abilities evolving over the semester are promising. It appears that overall PSTs demonstrated improvement in every component of noticing children's mathematical thinking. This outcome might be expected, given that the pre-assessment was conducted at the beginning of the semester, and they were not exposed to specific knowledge about teaching and learning mathematics before this course. Consequently, observing a development in each component could be interpreted as typical. However, considering the literature reports that the responding component poses the greatest challenge for development not only for PSTs but also for in-service teachers, observed improvement across all components remains encouraging. I will delve into the possible reasons for the promising improvement in the responding component in the discussion section.

Observed Trends Among the Facets of Noticing

In this section, I focused on the observed trends in the data to gain insights into how the skills required for noticing are related to each other to cultivate PSTs' responsiveness to leverage students' mathematical understanding. The examination of these trends could contribute to the knowledge base of professional noticing of children's mathematical thinking and the development of teachers who take up and build on students' mathematical thinking.

Relationship Between Attending and Responding

Upon examining the pre-and post-assessments, 15 out of 36 assessments were identified as following a trend where each facet of noticing fell into the same evidence level (see Table 18). Among these, 7 out of 36 assessments were identified as low evidence of attending, interpreting, and responding. A similar pattern was observed in the medium and high evidence levels. 4 out of

36 assessments were identified as medium evidence in all three facets of noticing, and another 4 out of 36 assessments were identified as high level of evidence in all three facets of noticing.

Table 18

Level of Attending and Its Relationship with the Interpreting and Deciding How to Respond

Number of the Assessments	Level of attending	Level of interpreting the math ideas needs to be scaffolded	Level of deciding how to respond
7/36	Low	Low	Low
4/36	Medium	Medium	Medium
4/36	High	High	High

However, the rest of the data revealed that a high level of attending did not necessarily lead to a corresponding level of interpreting and responding. Analysis of the pre- and post-assessment data showed various examples where PSTs successfully pinpointed mathematical elements in students' thinking, but this skill did not translate into the same level of interpreting and responding. Table 19 below illustrated the trajectories identified in the data. For example, the first row showed that these PSTs successfully identified all the mathematical elements in a student's solution. Although they recognized the need for scaffolding in understanding place value, they were not able to make a connection between decomposing place values and how borrowing works. This caused these PSTs to generate a response at a medium level. Similarly, the second and third row highlighted the instances where PSTs identified the mathematical details in students' solutions, but they failed to identify the mathematical concepts missing in the students' thinking. Consequently, their proposed next steps fell into the low evidence level since they were centered on procedural actions, such as borrowing, without addressing conceptual comprehension.

Table 19

High Level of Attending Does not Guarantee High Level of Interpreting and Responding

Number of the Assessments	Level of attending	Level of interpreting the math ideas needs to be scaffolded	Level of deciding how to respond
9/36	High	Medium	Medium
2/36	Medium	Low	Low
2/36	High	Low	Low

While a high level of attending did not necessarily ensure a high level of interpretation and high level of response, an examination of the high level of responses demonstrated that it required a high level of attending. This showed that to be able to respond that leverages students' thinking, teachers need to recognize the mathematical details of students' work. Although recognizing the mathematical understanding in the students' thinking was a crucial step, it did not necessarily mean that it equips teachers to provide responses that further elevate student's understanding. This argument was supported in the change throughout pre and post assessments. There were incidents where although PSTs showed remarkable progress in their level of attending, the level of change in interpreting and responding were not aligned.

For example, in the pre-assessment, a PST whom I referred to as Jessie showed a low level attending to students thinking by giving an explanation that doesn't include any mathematical elements, "they must have done subtraction, but I think they got confused with what numbers to use". Consequently, her identification of the support the student might need and her proposed follow up response relied on general comment without identification and comprehension of mathematical ideas. Jessie's response was, "The child might need help with how to subtract bigger numbers. I would make it a fun game with a rhyme for them to learn how to subtract." As it seen from her proposed response, Jessie's proposed approach relied on learning the procedurals of the subtraction without any emphasis on the conceptual understanding. When Jessie's post-assessment was examined, it was evident that she made a remarkable progress in attending to student's thinking and she shifted from low to high level of attending. Her recognition of noteworthy mathematical thinking of student was:

> I think that the child knew that they needed to borrow because they didn't have enough in the ones place. My guess is they decided they couldn't borrow from zero and so they borrowed from the 3, made it a 2. Then they transferred it as 10 to the ones place, so changed 7 to 17 and subtracted 8 to get 9. They knew you can't take 6 away from 0 and just left it at 6, because 0 has "no effect" then they took 2-1 to get 1 and ended up 169.

Consequently, her level of interpreting and responding shifted in the post assessment, however this shift was less salient compared to the shift in level in attending. In the post assessment, Jessie's interpreting and responding level were categorized into the medium level evidence as she focused on place-value understanding. Her response was as follows:

> The child needs support with how to subtract with zero. Understanding a zero has a value in the numbers and is just a placeholder. I would start with using a smaller number. I would show 2 and ask what that means. Then, I would place a zero next to the 2 and ask wat number that is now, which is 20. Then, I will ask what place value zero is in to highlight it in tens place and how many 10s I have.

Relationship Between Interpreting and Responding

In addition, the analysis revealed that there is a strong relationship between the level of interpreting the math ideas that need to be scaffolded and the level of responding. Upon examining pre- and post-assessments, the instances observed where PSTs had the same level of interpreting and responding while the level of attending varies. Table 20 below reported that in 11 out of 36 assessments, PSTs' interpretation of the math ideas students needed support with was categorized in the low evidence level and showed a consistent, low evidence level of responding. These PSTs in responding were characterized by a lack of evidence-based explanations and general comments that did not rely on mathematical elements and professional knowledge of teaching mathematics, MKT. Correspondingly, these PSTs' proposed follow-up actions or questions focused on the correctness of the answer or focused on procedures without making connections to broader mathematical concepts. For instance, a PST identified a student's needs as "Taking their time and not rushing," offering an overgeneralized response that lacked any mathematical elements that are specific to the mathematical content. Subsequently, this led PSTs to propose follow-up steps that solely focus on procedural actions rather than conceptual understanding. The PST's proposed follow-up response was "Highlighting the new numbers above crossed out ones." This example showed that the PST focused on procedural actions by focusing on crossing out the numbers to get to the right answer by failing to address conceptual understanding.

Table 20

Relation Between Interpreting and Responding

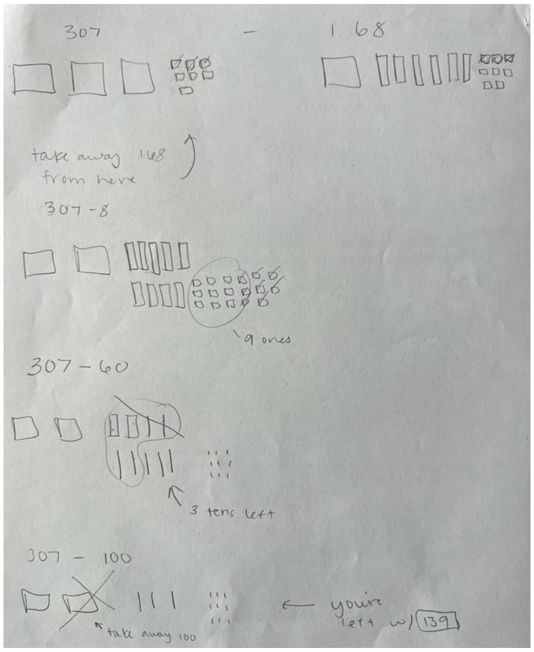
Number of the Assessments	Level of interpreting the math ideas needs to be scaffolded	Level of deciding how to respond
11/36	Low	Low
17/36	Medium	Medium
4/36	High	High

I found a similar pattern in 17 out of 36 assessments, indicating that PSTs who fell into the medium evidence level for the interpreting facet also fell into the medium evidence level in the responding facet. PSTs who fell into the medium evidence level in the interpreting identified place value as a missing concept in the student's solution, as a follow-up their suggested response focused on enhancing understanding of place value. For example, one PST identified the student's need for support as follows: "The child may need support with what 0 means in a number. He also may need to be retaught that when borrowing, you always take the value close to the left, and if that does not work, you borrow from the next". In this example, although the PST aimed to enhance understanding of place value, their approach focused on the procedural aspects of borrowing rather than on developing a comprehension of the reference units, which would enable the students to connect the concept of place value with the borrowing process. Consequently, this PST proposed a follow-up action "I would split the numbers into 300+0+7 and 100+60+8 to have him see each place value, to show him 0 isn't nothing, but it is the tens place" that targets place value understanding by decomposing the numbers, specifically 0 signifies a place value, in this case ten zeros, rather than nothing.

In the high evidence level, a similar trend was observed in 4 out of 36 of the assessments. In these assessments, PSTs who fell into the high evidence level in the interpreting facet also fell into the high evidence level in the responding facet. The high evidence level of interpreting is characterized by not only scaffolding the idea of place value but also focusing on reference units that show the relation between different place values and suggest a connection with borrowing. Subsequently, it was observed that the suggested action embedded this interconnectedness of borrowing and place value, and it was categorized as high-level evidence in the facet of responding. For example, one PST identified the missing mathematical concept in the student solution as “The child may need support with what it means to regroup, and how you regroup and adjust, especially when there is a zero involved”. Consequently, the PST built her response when she identified and crafted her proposed approach. She stated, “I would ask them how they subtract a smaller number from a larger number in a place value when there is another place value to borrow from. I would ask “How can you borrow from another place value and adjust? Why does that work? I would use the blocks.” Please see the figure below for what the PSTs drew.

Figure 5

Above PST's Drawing on the Given Task



Relationship Between Eliciting Questions and The Focus on Conceptual Understanding

The eliciting questions aim to gather more information about student's thinking processes and uncover the underlying reasoning behind their actions. When the pre- and post-assessments were examined, it was revealed that when PSTs started to inquire about the student's underlying reasoning behind their actions, they were better at recognizing the missing conceptual elements in the student's solutions. Subsequently, they proposed responses centered on conceptual actions.

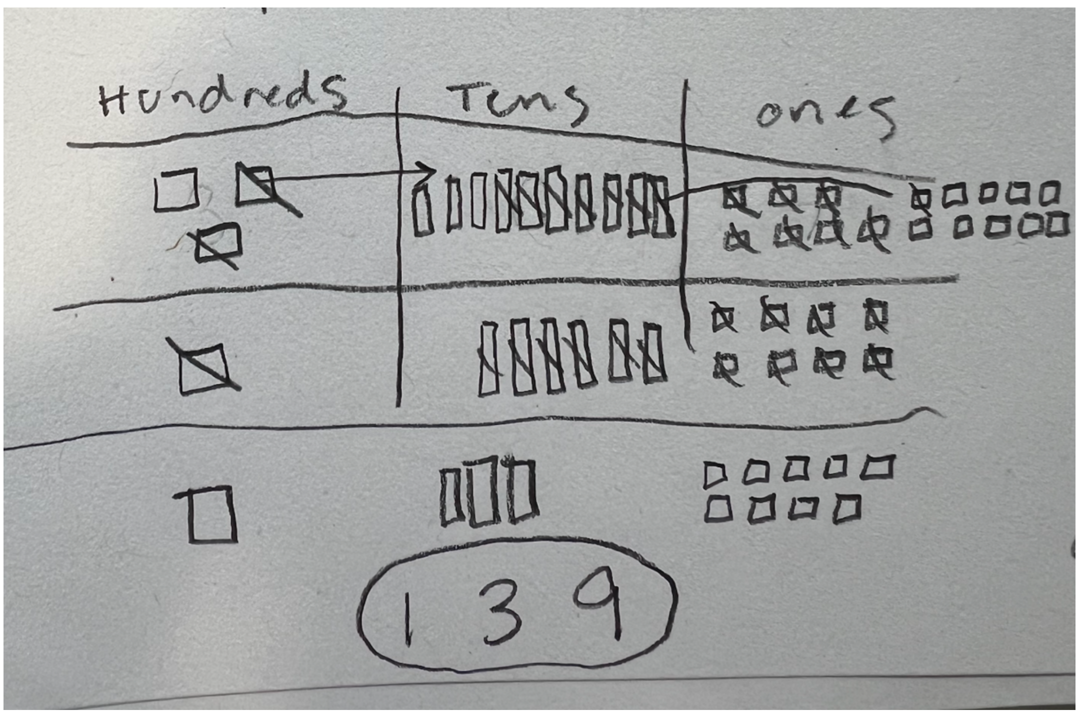
When the eliciting questions were examined, it was revealed that PSTs who were categorized into the high evidence level, characterized by questions that focused on not only what students did but also uncovering the reasoning behind their actions—prioritized conceptual understanding rather than procedural actions in the responding facet. This led PSTs to fall into either the medium or high evidence level in the facets of interpreting and responding. The observed relationship is reported in the table below (Table 21). The first row shows that in 9 out of 36 assessments, PSTs who posed eliciting questions focused on understanding the underlying reasoning of students achieved identifying a mathematical concept of place value as needing to be scaffolded and subsequently focused on developing place value comprehension in their follow-up moves. In 9 out of 36 of the assessments, PSTs who were categorized into high evidence level of eliciting were also categorized into the high evidence level of interpreting and responding. In the third row, 1 out of 36 showed a different trajectory. In trajectory, a PST whom I will refer to Jaclyn proposed a high level of eliciting questions, while she was categorized into the medium level interpreting and high level of responding.

Table 21

Eliciting and Its Relationship with Interpreting and Responding

	Eliciting	Interpreting	Responding
9/36	High	Medium	Medium
4/36	High	High	High
1/36	High	Medium	High

Jaclyn, shared her eliciting questions, “How did you know that you need to use borrowing with this problem? Where did the 6 in your solution come from? What does each number represent?” As is seen in this example, Jaclyn’s focus was on understanding the student’s reasoning rather than leading them to the correct answer. This intention was further evident in their interpreting phase, where they identified the missing mathematical concept that required support based on the student’s solution. Jaclyn’s answer for that was, “I think the child might need support with the idea of what place values represent in a number since they said 0 tens minus 6 tens was 6. They likely do not know that each number represents a place value." Jaclyn's response reflected her emphasis on conceptual understanding rather than procedural actions, as she recognized the need to scaffolding the student's understanding of place value. In her response to scaffold the student's understanding of place value, Jaclyn proposed a move that not only focuses on place value understanding but also stressed the connection between borrowing and decomposing place values;

> "I would first ask them what each number individually represents in the whole. I would then use the following representation:
>
> 
>
>
> This would help them to see that borrowing involves taking a place value over and breaking it up. You cannot take 6 tens away from 0 tens, so you have to break one of the hundreds into 10 tens, and make one ten over the ones because you cannot take 8 ones away from 7."

These responses demonstrate Jaclyn's commitment to fostering conceptual understanding in their students.

These results demonstrated a strong relationship between the high level of eliciting questions and a conceptual focus within the interpreting and responding facets of professional noticing of children's mathematical thinking. Whenever PSTs posed a high level of eliciting questions that focused on uncovering the student's reasoning, they consistently formulated their subsequent move centered around conceptual understanding, specifically addressing either solely place values understanding or its interconnectedness with borrowing and decomposition of place values.

However, such a relationship was not consistently observed among PSTs at the low and medium levels of eliciting. PSTs who posed low-level eliciting questions, characterized by focusing on correctness or funneling students towards the intended answers, were either categorized into medium or low levels of interpreting and responding. A similar pattern was observed for PSTs at the medium level of eliciting questions. PSTs who proposed eliciting questions at the medium level, characterized by focusing on what students did and what actions they took, were categorized into either medium or low levels of interpreting and responding. Notably, it was evident that none of the PSTs who suggested low or medium-level eliciting questions were categorized at the high level of interpreting and responding.

Opportunities to Learn Noticing Students' Mathematical Thinking

In the previous sections, I presented results that focus on the changes observed throughout the study, and the trends observed in these findings to contribute to understanding the dynamics of developing professional noticing of children's mathematical thinking. In this section, the purpose of my analysis was to elaborate on the nature of the opportunities related to noticing children's mathematical thinking that became available to the PSTs during the participation in the tasks designed around students' mathematical thinking. My analysis of "opportunities to learn" is guided by the opportunities of developing mathematical knowledge for teaching and specifically drawing on Specialized Content Knowledge (SCK) and Knowledge of Content and Students (KCS) to attend to, interpret, and decide how to respond to students' mathematical thinking.

The opportunities to learn that were available to the PSTs could be different in each home group based on their discussions of the tasks. In this section, I focused on two home groups to understand how small group discussions around targeted tasks created opportunities for PSTs to enhance their noticing skills in relation to developing their MKT. I identified these two home groups based on their progress across pre- and post-assessments to picture the different transitions across the semester and to identify the possible opportunities that contributed to the PSTs' progress. As I reported in the above sections, identifying the missing mathematical idea created a ground for crafting responses that target the conceptual understanding of the identified missing mathematical ideas in students' thinking. I identified the focus groups based on this finding, and my criteria to choose these two groups was their progress in the interpreting the missing mathematical ideas and deciding a response that would aim the missing mathematical ideas. The members of the first home group showed transition from low evidence level to medium evidence level in both interpreting and responding facets of noticing children's mathematical thinking. In contrast, the members of the second group progressed into high level from the medium level in interpreting and responding facets of noticing.

The first home group consisted of three PSTs. For clarity, I will refer to these PSTs using the pseudonyms Cindy, Carol, and Mary. This home group was chosen as a focus group to zoom in on, because in the majority of the facets of the professional noticing, Cindy and Mary demonstrated a transition from the low to the medium evidence levels. This transition included the interpreting facet and notably included the responding facet, which was identified in the literature as the most challenging facet to make progress for both in-service and pre-service teachers. However, since Carol was absent on the last day of the class, she was not able to take the post-assessment. This resulted in no information on her progress throughout the semester.

Table 22 reported Cindy and Mary's transformation of participation in the facet of noticing throughout the semester. In the pre-assessment, Mary fell into the low evidence level in each facet, while in the post-assessment she progressed through the medium evidence level in attending, interpreting, and deciding how to respond. In addition, she transitioned to a high level of eliciting in the post assessment. Although Cindy demonstrated a similar transition, hers is slightly different from Mary's. In the pre-assessment, Cindy demonstrated medium evidence level attending and proposed medium level eliciting questions. While she progressed into a high level of attending, she did not show a progress in proposing eliciting questions and remained at the medium level of eliciting. Cindy demonstrated a similar transition in the interpreting and responding facet, and she progressed from low to medium evidence level like Mary.

Table 22

Cindy and Mary's Transition Throughout the Semester

Group Member	Assessment	Attending	Interpreting the math ideas that need to be scaffolded	Deciding how to respond	Eliciting
Mary	Pre	Low	Low	Low	Low
	Post	Medium	Medium	Medium	High
Cindy	Pre	Medium	Low	Low	Medium
	Post	High	Medium	Medium	Medium

Table 23, documented Mary, Cindy, and Carol's development of noticing that is evident in shifts in participation in small group discussion centered around student work overtime. The table showed the shifts in Task 2, Task 4, and Task 5, which were conducted at different times during the course and the groups were given opportunities to revise their thinking following the whole class discussions.

Table 23

Shift in Participation in Small Group Discussion Centered on Student Work Overtime

	Attending	Interpreting the math ideas that need to be scaffolded	Deciding how to respond
Task 2	Low	Low	Low
Task 2 Revised	-	Medium	Medium
Task 4	High	Low	Low
Task 4 Revised	-	Medium	Medium
Task 5	High	High	High

Task 5 Revised	-	High	High

Now, I will share some examples of learning opportunities that became available through small group discussions focused on tasks that centered around student work, which allowed them to improve their noticing in relation to developing their MKT.

Excerpt 1

My analysis revealed that revision of their answers after whole class discussions offered PSTs opportunities to draw on their knowledge of content in noticing Antonio's mathematical understanding. The episode below is part of their small group discussion during Task 2 (see Appendix C), where they were answering the questions "What understanding(s) might the child need to be scaffolded?" and "How might you further support their understanding of that mathematical idea?"

Cindy: Probably he needs to understand subtracting from a zero.

Carol: He obviously knows if you have a smaller number you have to make it into bigger number and borrow. He just doesn't know how to deal with zero. Subtracting from zero, once you show them how to do it and when they practice a couple of times, they will be good at it.

Cindy: Yeah, I feel like talking through the problem, and giving him another similar problem is a good idea.

Mary: Honestly, just to show, look you have to take away to proceed for the next step.

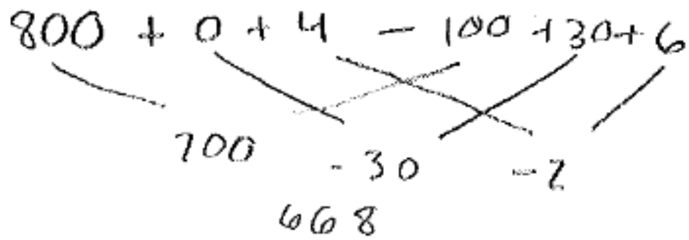
Cindy: I think another way is, you could use number line, but I think you could use multiple number lines and break up each place value to show them, instead of doing it on one number line.

In the above episode, Mary, Cindy, and Carol were categorized into the low level of interpreting and responding since their interpretation did not give enough evidence of utilizing and drawing on mathematical elements. They identified the student's need for support as how to deal with zero without connecting it to underlying mathematical ideas, specifically place value understanding. Consequently, they suggested a follow-up move focusing on drilling the procedural actions of subtracting from zero. In addition, they talked about using a number line, but they did not make a specific connection as to why they think it would help Antonio understand what they identified as subtraction from zero. After PSTs completed Task 2, they were invited to share their reasoning with the whole class, where peers and instructors asked for justification and evidence in their reasoning.

During the whole class discussion, the class worked on the identification of what Antonio did to solve the problem and what understandings were reflected in his solution. When one group shared that "Antonio knows borrowing", however another group built on that by stating "he borrowed from the hundreds straight to the ones, omitting the step in between at the ones". This created an opportunity for PSTs to discuss decomposing place value and making a connection with what is being borrowed. PSTs started to discuss whether Antonio knew what "1" represents when he borrowed. The class started to discuss that Antonio borrowed "1" from the hundreds place and moved the borrowed 100 as 10 ones to the ones place. They concluded that Antonio borrowed "1" from the hundreds place and moved it as "1" tens to the ones place. The class started to focus on place value understanding was lacking and began to discuss the ways of supporting place value understanding as their next step. One group shared breaking down the numbers might be helpful for Antonio to see what each number represents:

Figure 6

Place Value Strategy Shared During Whole Class Discussion



However, one PST pointed out that it might not be enough for Antonio to understand that 0 represents tens place value. Building on this idea, another PST suggested using representation with the base ten blocks to scaffold Antonio's place value understanding. Before wrapping up the discussion, the instructor prompted them to think in their small groups, whether and how these suggested strategies address the identified misconception of Antonio which was "always borrow 1". The PSTs returned to their home groups to think about the prompt provided by the instructor and revised their answers if their reasoning changed after discussing various possibilities and focusing on evidence during the whole class discussion. The following episode belongs to Carol, Mary, and Cindy's discussion during their revision after the whole class discussion.

Carol: They were talking how it is difficult to represent the absence-like 0 with the blocks, I feel like we can revise that.

Mary: I think it is more about units-like place value. Like in part one we broke the numbers up to show the place values, like in that one there is nothing in tens place.

Cindy: Maybe we can do something like, remember the ten frames? Instead of a ten frame, it is like a place value. Like you can show hundreds place- tens place in different columns, and there is two of them, so putting 0 shows they have 0 tens, so it is physically showing the place value.

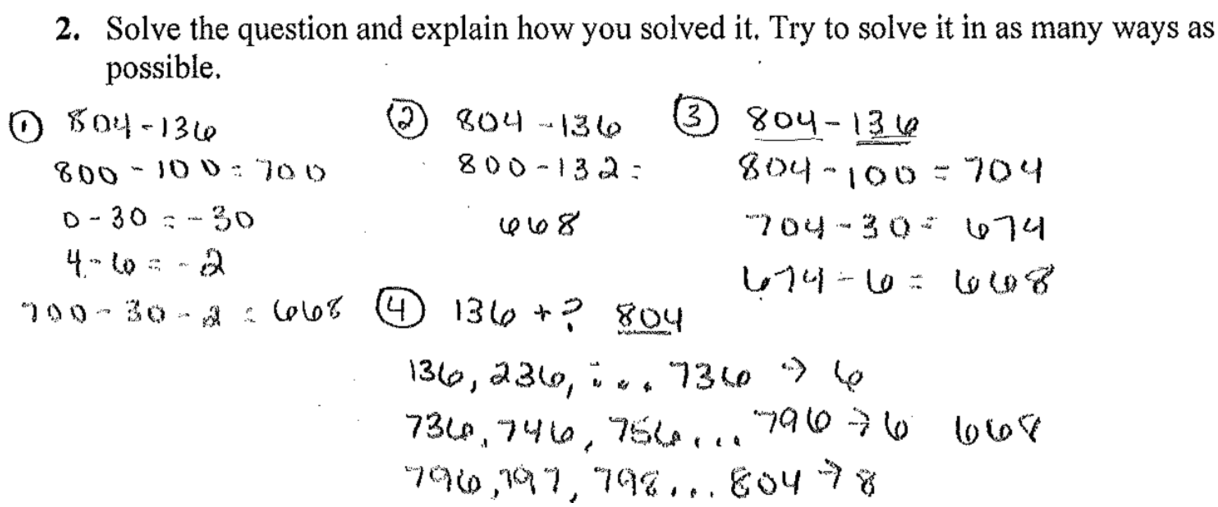
Mary: Yes, then maybe a question focusing on place value of zero, what zero means in there.

Carol: Maybe we can ask them to break down the 804 and try to have them see tens place is 0.

The whole class discussion directed Mary, Carol, and Cindy to draw on specific mathematical ideas rather than making general suggestions that do not rely on mathematical elements and evidence. While initially they focused on subtracting from zero in their response, after the whole class discussion, they started to delve into more details and discussed about place value of zero in the given problem. The revision initiated an opportunity for them to negotiate missing mathematical ideas and craft a response addressing these mathematical ideas. In addition, the revision created an opportunity for PSTs to connect with their SCK they already utilized in the first part of the task. In the first part of the task, before they examined Antonio's solution, when PSTs were asked to solve the question as many ways they can, one of the strategies they discussed was a place value strategy. See the solution 1, in their group work in the figure below.

Figure 7

Mary, Carol, and Cindy's Multiple Ways of Solving 804-136



In this solution, they broke two numbers down based on place value, showing that 0 represents 0 tens. However, while they were analyzing Antonio's error and crafting a response to leverage Antonio's math understanding, they were not able to connect with place value strategy that they used themselves to solve the question, and their next step was showing Antonio more drills to master him in the procedures rather than focusing on scaffolding the lacking place value understanding. In her comment Mary pointed out that connection "I think it is more about units-like place value. Like in part one, we broke the numbers up to show the place values, like in that one there is nothing in tens place." That allowed PSTs to focus on the missing mathematical concept by drawing on their mathematical knowledge for teaching. They decided to focus on fostering the place value understanding rather than drilling subtraction questions. Although, their suggestion only focused on place value understanding without making connection with how borrowing works though decomposing the place values, their revised suggested move focuses on conceptual understanding rather than memorizing the procedural way of borrowing in subtraction.

When the PSTs reflection was examined, it was evident that they were still thinking about the possible and better ways to have students understand the place value especially if the digit is 0. For example, Carol shared in her reflection that:

> Something that challenged my thinking in this task was when I had to find a way to explain to Antonio what the absence of a number or the 0 in 804 meant. It was difficult because I found that it was hard to create a visual representation, which often helps children understand concepts better. In Antonio's work, it seemed like he skipped over the 0…Something that I am still wondering about is the part that was challenging for me. I am still not 100% sure what the best strategy is to be able to explain what the 0 in 804 represents, and how I can explain it in a way to show that it is in the tens place. When I thought about using blocks of 100's, 10's, and 1's, it was helpful because it visually represented the problem, but it still did not put emphasis on the 0 being in the tens place in the subtraction.

In her reflection, Carol reflected that she was still wondering how to scaffold the student's understanding of place value especially when 0 is the place holder and decomposition needs to be done to do borrowing in the subtraction. This reflection showed that, although initially this group of PSTs focused on teaching Antonio the procedural way of subtraction without connection with conceptual understanding, the task created an opportunity to focus on comprehension of conceptual understanding. In addition, it created PSTs to draw on their content knowledge to leverage Antonio's thinking.

Furthermore, this reflection showed that completing the task, and revising their thinking by discussing in the home groups is not an end for the PSTs' comprehension, but it created opportunity for them to think about better and possible ways to support Antonio's understanding. As it seen on Carols self-reflection, she still grappled and wondered the possible ways of fostering conceptual understanding of the lacking mathematical ideas.

Following excerpts belonged to a group of four consisting Madison, Claire, Emily, and Kamila (all names are pseudonyms). Table 24 showed that Madison, Claire, and Emily's individual transformation of the participation to each skill of noticing. The table did not present Kamila's shift in participation to these skills since she was absent on the last day of the class and was not able to take the post-assessment. The table showed that Madison and Claire shifted their participation from medium level to the high level in interpreting the math ideas that need to be supported and in deciding how to respond, while Emily shifted her participation from low level to the medium level throughout the semester.

Table 24

Madison, Claire, And Emily's Transition Throughout the Semester

Home Group Member	Assessment	Attending	Interpreting the math ideas that need to be scaffolded	Deciding how to respond	Eliciting
Madison	Pre	High	Medium	Medium	Medium
	Post	High	High	High	High
Claire	Pre	Medium	Medium	Low	Low
	Post	High	High	High	High
Emily	Pre	Low	Low	Low	Medium
	Post	Medium	Medium	Medium	Medium

Before I delved into opportunities to develop noticing in tandem with developing MKT, I reported Madison, Claire, Emily, and Kamila's development of noticing that was evident in shifts in small group discussion centered around student works. Table 25 reported the shift in their noticing in Task 2, Task 4, and Task 5 where PSTs had opportunity to examine a student solution with error and worked on leveraging the student's mathematical understanding regardless of error in their solution. In addition, those tasks created opportunities to revise their thinking after a discussion where multiple responses and their affordances were discussed in the whole class setting.

Table 25

Shifts in Participation in Small Group Discussion Centered on Student Work (Madison, Claire, Emily, and Kamila)

	Attending	Interpreting the math ideas that need to be scaffolded	Deciding how to respond
Task 2	Medium	Medium	Medium
Task 2 Revised		High	High
Task 4	High	Medium	Medium
Task 4 Revised		High	High
Task 5	High	High	High
Task 5 Revised		High	High

Excerpt 2

This excerpt comes from a 40-minute-long, small group discussion. The excerpt did not include Kamila since she was absent on that day. The PSTs were given a task aimed developing understanding of underlying mathematical ideas in different strategies. My analysis revealed that, discussion around the task created various opportunities for PSTs to attend the SCK, specifically pinpointing the underlying mathematical understanding behind the students' solutions and developing nuanced understanding of the content. The students' solutions presented in the task are shown in the figure below. Please see Appendix D to see the questions prompted in Task 3.

Figure 8

Student Solutions Presented in Task 3

Student A	Student B	Student C
35 x 25 125 +75 875	35 x25 175 +700 875	35 x 25 25 150 100 +600 875

Note: Adapted from "Hill, H.C., Schilling, S.G., & Ball, D.L. (2004) Developing measures of teachers' mathematics knowledge for teaching. *Elementary School Journal 105*, 11-30."

The following episode is a part of the discussion where group members focused on understanding what Student A did in Task 3.

10 **Madison:** I think they multiplied 25 times 5 first. And then 25 times 3 because it was 75, but not 750. That's what I was thinking. I don't know. They either knows that it is times 30 or they just know to put the 75 like in standard algorithm.

13 **Claire:** I think it shows an understanding that, instead of putting a zero, 25 times 3 is and adding a 0 is really like you are doing 25x30. Maybe he just shifted over, the same thing. But, actually, I have never thought what and why I was doing when I do standard algorithm.

16 **Emily:** But I wonder, how they knew not to put a zero here?

17 **Madison:** Yeah, because they knew there was a zero there.

18 **Emily:** I wonder how they knew that it would be like a three-digit number instead of like a four digit. I wonder why they put 75 right there instead of like, under the two and the five.

20 **Claire:** I think it's like they know that they're actually multiplying 25 times 30 like they already know the zero is there.

22 **Madison:** I just find it interesting that they didn't put it in.

23 **Emily**: So, did they do 25 times 30?

24 **Claire:** It's just like, it's the same as when you do standard algorithm and when you do, you just add zero to whatever you get, so that when you're multiplying, you're getting 70. But really, you're getting 700.

27 **Emily:** Ohhh. Yeah. Okay. I got it.

28 **Madison:** I just don't know if they were doing that because they think it's standard algorithm. Should we say they multiplied by 30 or three?

30 **Emily:** I don't know. I said 3, because they got 75 because they wrote down 75 and didn't add a zero. But now I'm confused.

In this episode, group members focused on attending Students A’s thinking and grappled with identifying the reflected understanding from their solution. PSTs identified that 75 represents 75 tens in the student’s solution. However, they wondered if the student new that 75 represents 75 tens or if the student just “shifted it one place” like in standard algorithm without knowing it represents 75 tens. As the exchange showed above, the PSTs were questioning the reflected understandings from Student A’s solution based on the evidence. They discussed that shifting over might not be an indication of that multiplying with 20. Claire brought up (line 13) although she shifted a digit while she was doing double digit multiplication, she had never thought or knew why she did it. This discussion created an opportunity for PSTs to focus on students' underlying reasoning behind the actions rather than solely focusing on how the student solved the question and developing their SCK since this knowledge domain related with grasping nuanced differences across multiple strategies.

PSTs individual reflection upon completing the task, revealed the benefits gained from engaging collaborative discussion in unpacking of the knowledge and understandings embedded within the students' strategies. Claire's reflection upon completing the task supported this argument:

> My group mates helped me to see how the student A didn't use a 0 in 750 and pushed my thinking about what that meant about the student's knowledge and understanding. I overlooked this part when I was first looking at the problem but saw more significance in it when we looked as a group.

Another group member's, Madison's, reflection further illustrated the depth of inquiry stimulated by the task about the possible underlying knowledge embedded in the students' solution. In her reflection Madison shared that:

> I was challenged to see what student A did when they put the 75 over a place value without adding a zero. I could not tell if they simply knew that there should be a zero or if it was just how they learned how to do this in standard algorithm…I had an aha moment when Claire pointed out that student B used standard algorithm. Even though I grew up learning standard algorithm, it was still hard for me to see that the child [student B] used that [standard algorithm].

In addition, Madison reflected how the collaborative inquiry helped her to make connection with her knowledge of standard algorithm and its connection with Student B's solution strategy. As this helped Madison to comprehend the mathematics behind the standard algorithm and demonstrating the similar mathematical ideas in various ways created opportunity for her to attend the knowledge of content.

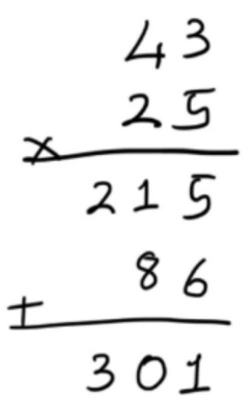
In the shared excerpt and in both reflections, it was evident that opportunities to interpreting the nuances of student thinking were made available to PSTs which is related with developing MKT, specifically SCK. In addition, collaborative analysis helped PSTs to unpack the invisible and embedded mathematical ideas in the various student strategies by negotiating the meanings and evidence. Furthermore, it created opportunities to grapple with nuanced understandings and enhance their pedagogical comprehension of MKT.

Excerpt 3

The following excerpt is taken from Task 4 that focused on analyzing a student solution with an error. Task 4 was conducted the following week of Task 3 where the part of the discussion shared in the excerpt above. This excerpt is a part of a 40-minute-long, small group discussion around Charlie's solution presented in figure below. Please see Appendix E for the prompted questions. All participants of the group, Emily, Claire, Madison, and Kamila were present on that they, and the following excerpt includes all the participants.

Figure 9

Presented Student, Charlie's, Solution in Task 4



487 **Kamila:** I think they did standard algorithm.

488 **Madison:** That's standard algorithm but they didn't account for what the 2 represents. Instead of doing like 20 or like two times 43 and then just putting a zero there, they just did not.

491 **Emily:** Yes, I see what you mean. But "considering what Charlie did, what do you think they know. What mathematical understandings are reflected in their solution?" (Reads the question)

492 **Kamila:** I think they have a decent grasp on knowing how to use place value.

493 **Emily:** I would disagree with that.

494 **Kamila:** Is it because they were carrying the one and they did not represent it (referring the 43x5).

495 **Madison:** That is borrowing.

496 **Emily:** Because they didn't, they didn't understand that the two in this represents 20, so that he was supposed to like to add a zero like he put 86, so the eight under the two. So that's like not representing the place value.

499 **Claire:** I think they have some knowledge of place value with the way that they did five times three and five times 40 but that they haven't mastered it because then they didn't do it with 20.

502 **Madison:** I also said, he knows how to borrow, even though he did not show it with the 21.

504 **Claire:** Is that borrowing?

505 **Madison:** I think so, wait, it is not borrowing, you are right. What is it then?

506 **Emily:** Is it decomposing?

507 **Claire:** It is carrying**.**

508 **Madison:** Yes, yes, it is carrying.

509 **Claire:** I feel like that carrying could also show place value understanding knowing that, that one 10 goes into tens place. That could show place value understanding. But it also could just be like when I used to it as a kid, I didn't know what it meant. It was just to carry without knowing what it meant.

In the exchange above, it was shown that how PSTs pressed for each other to elaborate their thoughts which created opportunity to drawing on evidence. For example, in line 494 Kamila asked others why they think Charlie doesn't have a decent grasp of place value understanding. This gave opportunity for Emily to respond and unpack Charlie's solution that he multiplied by 2 instead of 20. Claire built on it (line 499) and pointed that Charlie have some understanding of place value connected on the evidence that he did 5 x (40+3). However, Claire (line 509) built on her own experience of using standard algorithm that she had never knew what she was doing while she was shifting the digits or carrying while she was multiplying. Based on that Claire claimed that Charlie might not know the meaning of carrying, and they cannot be sure if he has a decent place value understanding. Carlie's actions might be because of memorization of the steps without knowing what they meant.

When the group started to discuss the question "what understanding might the child still need to be scaffolded?" the instructor visited them and asked some prompting question.

641 **Instructor:** What did you discuss?

642 **Madison:** We said, if you broke up the it like this (40+3)x(20+5). They can see what is being multiplied.

644 **Claire:** Yes, if we do partial product and see if he recognizes his mistake, because they are the same number.

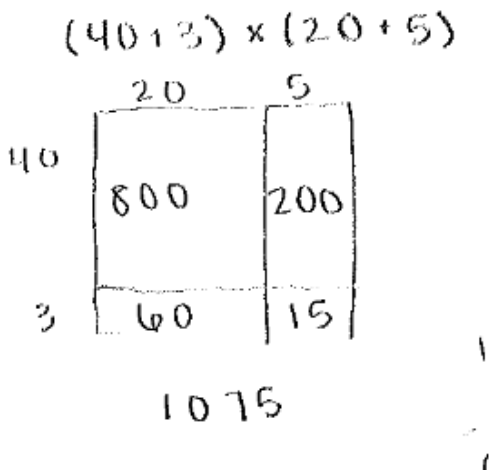
646 **Instructor:** Okay, I have one question though. Now he is going to multiply 20 by 40, and 40 by 5, then 3 by 5 and 3 by 20. After adding all of them up, he might find the right answer. But my question is, how would you make connection with what he missed in their original solution. My main point is, I don't want Charlie to think that we are just teaching him a different method, like you made a mistake and let me teach you another strategy. But how can we make a connection with the missed idea then he can see the connection in your suggestion with what he originally did?

In this excerpt, the instructor pressed for more explicit connection upon the group's answer of "we do partial product and see if he recognizes his mistake" and invited them to think for more deliberate pedagogical move rather than hoping Charlie to recognize their own mistake after solving the question by using partial product. The instructor opened the floor for PSTs to pursue more explicit and intentional connections across the mathematical ideas and the strategies. After the instructor left, the group continued brainstorming and finalized their answer as "Charlie needs to understand place value. Asking him: why is the first line of work 215 when you did 3x5 and 5x4? What does the 4 represent? What does the second line of your work mean? When does the 2 in 25 represent? Ask him these so that he may find a way a connection between place values." In these answers, it was evident that PSTs focused on place value understanding which is a fundamental mathematical idea that needs to be understood in order to grasp the multiplication in a meaningful way. However, PSTs were not able make explicit connection with the Charlie's method which was standard algorithm.

Later, the whole class came together, and various groups started to share their thinking. One group suggested that using an open array could help Charlie to understand the place value better. In addition, other groups claimed that open array could help Charlie to see what is being multiplied and that could be a way of making connection with partial product and standard algorithm. This group's work was as it followed:

Figure 10

Open Array Representation During the Whole Class Discussion

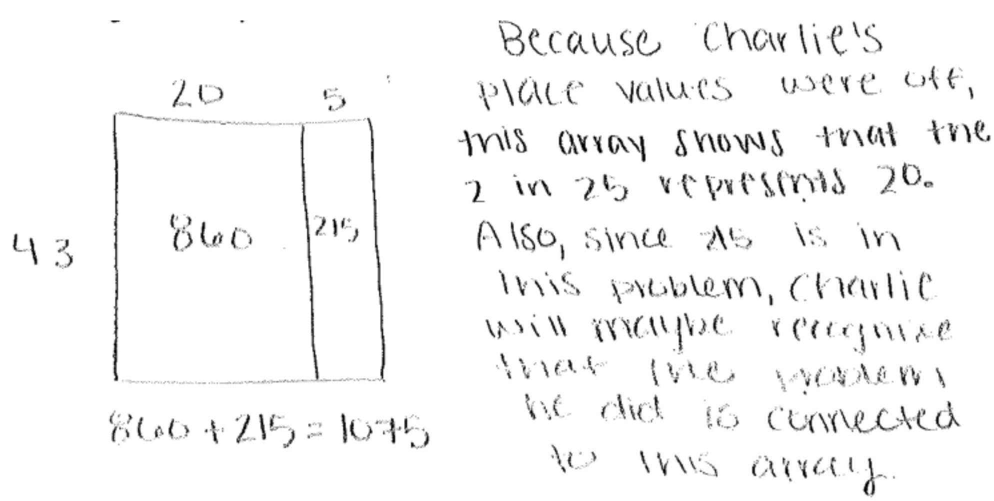


Some of the other groups used the same approach to scaffold Charlie's understanding that 4 in 43 represent 40, when it is being multiplied by 5, indeed it is 5x40 instead 5x4. However, there were some PSTs claimed that although this approach might help Charlie to understand that 4 in 43 represents 40, they suggested to finding ways of making explicit connection with standard algorithm to support Charlie's understanding to see the connection between the two strategies. Some PSTs claimed that although open array is a good way of explaining the multiplication, they claimed that the proposed representation would be better representation for partial product, and they referred Student C's solution in Task 3 they engaged in the previous class. Another PST chimed in to build on the Task 3 reference, and she claimed that Charlie's way of thinking is closer to Student B’s solution. Then, the PSTs started to wonder how to create a representation that would align with Charlie’s solution, claiming that would help Charlie to make better sense of standard algorithm. As the class time was running out, the instructor asked the PSTs to go back to their small groups and revise their thinking upon the whole class discussion.

Emily, Claire, Madison, and Kamila started their discussion by building on the whole class discussion. They decided to use a representation that would directly connect with Carlie’s strategy. They discussed that representing breaking into addends strategy on the array would make direct connection with Charlie’s standard algorithm and his mistake. They specifically highlighted that it would help Charlie to see 43 is multiplied by (20+5). In addition, they addressed that the areas in the representation would help Charlie to comprehend where 215 and 860 comes from that he used in the standard algorithm. Their revised proposed move was as follows:

Figure 11

Emily, Claire, Madison, And Kamila’s Representation After the Revising Their Thinking



As it seen, the task structure provided space for PSTs to gain deeper understanding of MKT and use the knowledge strategically to craft a respond that aim to builds on and extends Charlie's understanding. Madison's reflection after completing the task also identified the opportunities to learn to build on students' thinking by using the substance of their thinking. Madison's reflection that includes self-reported opportunities to learn:

> I had an aha moment when I realized that the best way to explain to a child is not through a different method, but through going through the current method and having the child walk you through. Then, you can see what the child did, as well as make connections to a representation based off of what they did. The whole class discussion made me have this aha moment when Erin brought up how the array could be connected with the child's solution.

In summary, this section of the findings presented instances of opportunities to learn to develop noticing children's mathematical thinking in tandem with developing their MKT. These instances included several affordances. First, these instances presented opportunities of pedagogical reasoning by bringing up alternative ideas, pressing for evidence, and negotiating and validating each other's ideas. Second, these instances presented affordances of developing nuanced understanding of SCK and KCS by examining mathematical ideas enlisted in various strategies. In addition, they included affordances of drawing on evidence when they identify the reflected and missing understanding in students' understanding. Additionally, it created opportunity for PSTs to leverage representation to craft pedagogical response that deepen students' understanding by engaging in students' existing reasoning.

Discussion

In this study, I focus on PSTs' development of noticing children's mathematical thinking, with specific interest in their development of the responding facet in the context of an intervention situated in a mathematics content course. Existing research typically addresses the challenge of developing the skill of responding to children's mathematical thinking in pre-and in-service teachers as an end in and of itself. In contrast, my study focuses on addressing the skills required to notice children's understanding in relation to each other, within a content course where PSTs are beginning to develop their mathematical knowledge for teaching. In addition to embedding the facets of professional noticing in the course structure, the main goal of the intervention is to engage PSTs in pedagogical reasoning by unpacking what to pursue and how to pursue in students' emergent understandings.

This section discusses the findings of my overarching research question: "How do PSTs develop in their capacity to craft evidence-based responses to children's mathematical thinking?" To answer this overarching research question, I first investigated the changes that PSTs demonstrated individually throughout the semester. Second, I examined the relationship among various facets of noticing to understand whether and how they affect the development of each other and contribute to the development of responsiveness. Lastly, I investigated the nature of the opportunities related to noticing around the tasks through small and whole class discussion. In this process, I identified the opportunities of developing MKT and using it strategically to craft a response that leverages children's mathematical understanding. I also identified the opportunities to learn become available for PSTs in the content course to cultivate their capacity to develop noticing. The findings summarized above chapter, demonstrate that the implemented design provided opportunities to PSTs to utilize their developing MKT strategically and allowed them to refine their capacity of developing professional noticing of children's mathematical thinking. I will discuss the results in relation to the three research questions below.

Change Throughout the Study

When the findings are examined, it is clear that PSTs made progress in the skills of attending, interpreting, and deciding how to respond, as well as in eliciting. This result confirms the previous research findings in relation to necessity of robust MKT to develop the skills required for noticing children's mathematical thinking. The existing research indicates that as PSTs enhance their understanding of MKT, their skill of noticing children's mathematical thinking improves. Given that the overarching goal of the content course is to develop PSTs' MKT, an increase in noticing skills could be expected. Prior studies suggest that the attending component could be readily developed as the teachers are exposed to a variety of children's thinking by leading to a more nuanced understanding of SCK (Jacobs et al., 2010). A study conducted by Sánchez-Matamoro et al. (2019), for example, showed that the skill of interpreting students' mathematical understanding requires KCS in addition to SCK.

Throughout the course, PSTs were exposed to various solution strategies employed by children and unpacked the enlisted mathematical ideas in the invented strategies. They actively engaged with written examples of children's diverse approaches and watched videos showcasing children engaging with mathematical thinking. These opportunities could help them develop nuanced SCK and KCS, thereby enhancing their MKT and allowing them to notice the mathematical details in students thinking and identifying the reflected understanding. Furthermore, in this study, the interconnectedness of the three skills of noticing (Jacobs et al.,

2010) was emphasized as addressing these nested skills together might cause greater gain. This approach contrasts with the tendency of studies conducted with PSTs to focus solely on developing the skill of attending and interpreting while leaving the development of responding in relation to field placements. Such fragmented focus on the facets of noticing may lead to a separation between learning to teach and doing the work of teaching (Steele & Hillen, 2012). While my overarching aim in this study was investigating the development of PSTs' responding skill, I approached this aim through a focus on the nested and interrelated skills of noticing.

Considering that the extant literature addresses the challenge of crafting a response that takes up and builds on students' current understanding, the most promising result from this study is the development in the skill of responding. Considering previous research suggests that developing MKT is not enough to develop the skill of responding, the main inquiry is how PSTs showed progress in the skill of responding as it reported in Table 14. Discussion of the second and third research questions helps me to answer this question further.

Relationships Among Noticing Skills

In this section, I discuss the observed relationships among noticing skills to understand how to support teachers' responsiveness to children's mathematical thinking. One of the connections identified from the data is the necessity of a high level of attending for a high level of analyzing and crafting responses that would extend student understanding. However, the data shows that a high level of attending does not guarantee the sophisticated level of interpreting and responding. This finding aligned with the previous research conducted by Barnhart and van Es (2015) and Sánchez-Matamoro et al. (2019). Barnhart and van Es (2015) found that crafting a pedagogical response demanded sophistication in attending to student ideas. However, sophistication in attending to students' ideas did not result in sophisticated responding to student thinking. In the study conducted by Sánchez-Matamoros et al. (2019), they found that although identifying the mathematical elements in students' solution was necessary, it was not sufficient to identify the reflected understandings. My findings align with both studies by indicating the necessity of developing the skill of attending to be able to develop the skill of interpreting and responding, while the high level of attending does not guarantee the high-level interpreting and responding. These findings show that addressing solely attending by supporting the development of SCK would not yield to sophistication in the skill of responding.

A study conducted by Monson et al. (2020) showed that addressing the skill of responding created higher gain in the skill of attending and interpreting compared to the addressing solely the skill of attending and interpreting. This finding aligns with my study and might explain the remarkable improvement in all facets of noticing throughout the semester. These findings show that the importance of addressing the responding skill which demands pedagogical judgment and decision making. Addressing the skill of responding not only cultivates crafting a mathematically-based pedagogical response, but also facilitates MKT development and improves the skill of attending and interpreting. In addition, it confirms that the development of these nested and interrelated skills needs to be addressed in relation to each other.

My results highlight that although a high level of attending to student ideas is necessary to develop a response that leverages students' mathematical thinking, merely developing this skill is insufficient to craft sophisticated responses. The work of responsive teaching is complex. The inability to use the knowledge of what teachers recognize as noteworthy in students' mathematical thinking prevents leveraging students' mathematical thinking (Robertson et al., 2016). How could we support teachers to identify the reflected understanding and use this knowledge skillfully to respond to students' thinking? The other identified relationships between noticing skills in my study could contribute to understanding how teachers enhance their responsiveness to children's mathematical thinking by drawing on noteworthy mathematical elements.

In the assessment data, whether pre- or post-assessment, it is found that PSTs' level of interpreting is more related to their level of responding. The data shows that when PSTs identified a missing mathematical idea in students' solutions, they based their response on the identified missing idea and targeted the comprehension of this idea in the following pedagogical approach. When PSTs are able to identify more than one missing mathematical idea in relation to each other, their responses aim at the comprehension of these mathematical ideas by focusing on interconnectedness of these ideas to each other. These results prove the necessity of identifying the missing mathematical idea during the interpretation facets in crafting a response that extends students' mathematical thinking.

Taking this finding into consideration, the identification of the mathematical ideas that need to be supported in students' thinking could be a pivotal aspect of scaffolding the PSTs' response to build on children's thinking. This outcome holds promise for the efficacy of pedagogical interventions aimed at enhancing PSTs' responses to build upon children's mathematical reasoning. Now, how we can support PSTs' identification of missing mathematical ideas will be discussed in the following part based on the findings of the third research question.

Opportunities to Learn in the Content Course to Support PSTs Capacity to Develop Noticing

The findings show that the employed design in this study yielded multiple opportunities for PSTs to develop a nuanced understanding of MKT, learn to use their developing MKT strategically, and develop professional noticing of children's mathematical thinking.

First, incorporating the facets of the professional noticing of children's mathematical thinking framework through as a sequence of questions that invited PSTs to examine given student solutions creates multiple opportunities for them to develop noticing in tandem with developing a nuanced understanding of MKT. Tasks that required examining written student solutions, communicating to validate one's interpretation with other PSTs, bringing up alternative ideas, and convincing each other help PSTs focus on uncovering the reasoning behind students' solutions and identifying the mathematical ideas that need to be supported. Fernandez et al., (2012) highlighted the value of the written text in facilitating the establishment of focus points around the negotiated communication. In my study, although PSTs communicated alternative ideas and weighed multiple approaches through collaborative talk in small group and whole class settings, they had to make a decision and come up with the written plan to demonstrate the response they proposed aiming to support the students' deeper understanding.

The existing literature indicates that opportunities for debriefing and reflecting play a pivotal role in developing skills to analyze student thinking (Lampert et al., 2013). Debriefing about alternative ideas in the whole class setting as well as in their small group setting, allows PSTs to brainstorm about affordances of the different approaches, especially in proposing a response as a follow-up move. Discussing the affordances of various approaches and focusing on decision making based on instructional purposes rather than a didactic "best possible approaches" is central to responsive teaching. While it might be challenging for PSTs who typically follow the press of covering the content (Horn & Kane, 2015; Parks, 2008), PSTs needs to learn that the eventual goal is learning to center the teaching on emergent and productive students thinking (Parks, 2008). The present study suggests that this work of responsive teaching can be learned by PSTs: Compared to the beginning of the semester where the instructor was the one who initiated the examination of whether the proposed responses aligned with the missing idea or whether the responses fulfilled the instructional purpose, over time, PSTs began to probe each other's thinking by asking for evidence and to challenge the peers how the proposed response aligned with the instructional purpose. This indicates that PSTs could broaden their vision on better possibilities to leverage children's mathematical understanding. In addition, collaborative discussion and open-ended nature of the task allow PSTs to begin to work on collectively improving and revising the proposed responses to align with missing mathematical ideas, or to use representations to scaffold students' understanding by utilizing their nuanced knowledge of content and teaching.

Furthermore, giving PSTs room to revise their thinking after the whole class discussion allows them to refine their thinking and take up and build on what is available for them to comprehend knowledge of content and teaching based on their level of readiness and to draw on it to develop noticing student's mathematical understanding.

Implications and Future Work

My dissertation has several implications. First, the findings confirm that noticing is a learnable skill, and three facets of noticing are nested as claimed by Jacobs et al. (2010). Considering the cognitively demanding structure and the complexity of the facet of responding, this skill has been addressed in the later stages of the teacher education program in relation to PSTs' field experience, resulting in the separation between the learning of mathematical knowledge for teaching and the work of teaching (Steele & Hillen, 2012). My study brings attention to the possible opportunities in the content courses to address attending, interpreting, and responding in relation to each other through unpacking the invisible pedagogical reasoning and drawing on the pedagogical knowledge strategically to make decisions.

My findings suggest that PSTs would benefit from unpacking explicitly what it means to respond to students' thinking and deepen students' mathematical understanding by drawing on their MKT. This finding aligns with Barnhart et al.'s (2024) study that shows that PSTs benefited from examining thoroughly the concept of responding to students' thinking. The authors argue that breaking apart the key components of professional noticing and engaging with the decision-making process could benefit PSTs in what needs to be done to build on the substance of students' mathematical ideas. These studies align in viewing responding as navigating the decision-making about what to pursue and how to pursue it by creating the opportunity to unpack the pedagogical reasoning and focusing on the instructional purpose of the response. These findings also show that there is an opportunity in content courses to address the skill of responding to students' mathematical ideas by unpacking the invisible complexity of pedagogical reasoning and navigating the decision-making. However, I am cautious to claim that this promising result is enough to affect PSTs' in-the-moment instruction in the classroom context. Further studies are needed to explore to what extent PSTs' skill of noticing is reflected in their instruction.

Additionally, although PSTs could benefit from breaking apart the key components of noticing for the reasons highlighted above, this might cause oversimplification of teaching. Teaching and learning are situated in sociocultural contexts and focusing solely on cognition attends only partially to the complexity of teaching and learning: Researchers argue that teacher noticing is situated in cultural, historical, political, and social contexts that are shaped by teacher and student identities (Louie, 2018; van Es et al., 2022). Students' funds of knowledge, their lived experience, and their sociocultural background influence how students make sense and how they position themselves in the learning environment. Moreover, teachers see the students and their mathematical reasoning through some lenses, that are shaped by their own lived experiences although those lenses might be implicit (Hand, 2012; Shah & Coles, 2020). These implicit lenses influence what teachers notice and do not notice. Researchers underscore that noticing is multidimensional that entails teachers' and students' sociocultural selves and sociocultural aspect of learning environments (van Es et al., 2022). Focusing solely on cognition might cause reproduction of the inequities and contribute the dominant narratives such as attributing mathematical achievement to specific racial groups (Louie, 2018; Shah & Coles, 2020). In this regard, my study has limitations in its focus on sociocultural factors that influence teacher noticing. Further research needs to focus on how to better prepare PSTs for multidimensional noticing that would not only focus on cognition but also foregrounds language, gender, and race that are part of students' identity and shape who they are, not only in the learning environment but also in the society. Furthermore, multidimensional noticing might allow PSTs to be aware of their own lenses that they see students through and might allow them to notice to disrupt the historical inequities.

Although this study shows promising results in terms of opportunities to support the development of responsiveness of pre-service teachers within the content course, considerable effort is required from teacher educators to support the development of this skill throughout the teacher education program. First, simply using the tasks used in this study and the guiding questions (see Table 4) to facilitate discussion might not yield to the same results for other teacher educators. I intend to highlight that, my researcher identity and my motivation for supporting PSTs' noticing students' mathematical thinking might have affected my facilitation skills as the instructor. This study's professional noticing of children's mathematical thinking framework (Jacobs et al., 2010) informed my prompts to PSTs during the discussion. I recognized that I began to shape my prompts around the skills of attending, interpreting, and responding especially when we worked on students' solutions. For example, asking PSTs how their proposed action is connected with what they already identified as conceptual ideas that need to be scaffolded or asking them to make connection with the proposed representation and present student's strategy are examples of how I embedded the framework in my facilitation skills. In addition, I may have unintentionally begun to use this framework to notice my PSTs' understanding. My follow up questions relied on uncovering the reflected understanding of my PSTs, and I crafted my responses to give them opportunity for sensemaking by building on what they already did. It is evident that the framework I used to scaffolded PSTs' noticing also scaffolded my facilitation skills and allowed me to have a critical role in facilitating the discussion.

Second, this study relied mainly on written student work with less emphasis on non-verbal student thinking or gestures. Walkoe et al. (2023) highlighted the importance of non-verbal student thinking in developing PSTs' noticing. Their study showed that when PSTs attended to non-verbal student thinking, their discussion of student thinking took more asset-based approach where they focused on productive student thinking rather than the correctness of an answer. More research needs to be conducted to understand the opportunities to support PSTs' development of responsiveness in the content courses not only by focusing on written student work, but also verbal and non-verbal student thinking.

Third, my study does not account for the interaction with the students. Interacting with students requires in-the-moment decision-making. However, taking up and building on students thinking at the moment is demanding and complex. As the ultimate objective is to develop PSTs' responsiveness to students' ideas during the instructional moments, I suggest continually creating opportunities to address three nested skills of noticing in the following content and methods courses throughout the teacher education program. To support the development of these intertwined skills, PSTs should be gradually introduced to the complexity of in-the-moment noticing and responding. This could be achieved by creating similar opportunities for one-on-one interactions with students first, where PSTs are given an opportunity to take up and build on students' thinking (Webel & Yeo, 2021).

Then, opportunities should be gradually created for PSTs to lead a discussion with a group of students, where they can develop their skill of building on students' understanding (e.g; Leatham et al., 2015). Although field experiences allow teacher educators to work on attending, interpreting, and responding skills, I believe improving technology makes other productive tools available. Simulation use has been increased in recent years to create a safe space for PSTs to develop their various skills before they interact with real students (e.g.; Mikeska et al., 2022). The use of simulation will also allow teacher educators to use these simulated interactions for reflection. The recorded interactions could allow PSTs to spend time unpacking and reflecting on their own experience in a collaborative context with their peers and their instructors. As it was used in my study, having an opportunity to reflect and unpack the pedagogical reasonings and bringing up and validating alternative ideas would bring opportunities to revise and broaden possible pedagogical approaches. That would allow PSTs revise their thinking and decision-making and discuss the possible opportunities collectively to learn what it means to take up and build on children's' thinking. My argument is not simply to take up one practice or design to embed in a teacher education program, but to plan and implement these opportunities deliberately and intentionally throughout the teacher education program gradually without waiting for field placements to address the skill of responding.

Promising results from this study demonstrated that, by the end of the semester, PSTs began to generate responses focused on conceptual understanding rather than their initial responses at the beginning of the course, which focused on operational understanding without connecting to conceptual understanding. In addition, their eliciting questions shifted from what students did to why they did it by focusing on understanding students' underlying thinking rather than focusing on only students' actions and results. Their procedure-oriented responses at the beginning of the semester and their questions focusing solely on what students did, demonstrated that PSTs entered the program with their belief about math and their conceptualization of what it means to do math. Researchers have emphasized that most of the PSTs start their program with a focus on the correctness of the work rather than focusing on the details of the students' thinking, and their goal is to have students reach the right answer (Louie, 2017). The findings demonstrated that the younger generation of teacher candidates still holds a similar conceptualization of what it means to do math when they enter the teacher education program.

The shift in PSTs' procedure-oriented responses to the conceptual understanding-oriented responses and the shift in their eliciting questions throughout the semester might be an indicator of PSTs' orientation to asset-based mathematics where they value the mathematical thinking in students' thinking regardless of the correct answer. However, this study did not discover whether identifying the mathematical elements, reflected and missing understandings in students' thinking, and working on crafting a response that aims to deepen student thinking affected PSTs' belief about mathematics teaching and learning. Considering PSTs' rigid beliefs about math, which stem from their apprenticeship experiences (Lortie, 1975), future research could examine whether engaging in noticing helps PSTs to change their rigid belief about learning and teaching mathematics.

In this study, I aimed to scaffold PSTs' skill of extending students mathematical thinking by foregrounding students' current understanding. I targeted this scaffolding by focusing on both the mathematical ideas reflected in students' thinking and math ideas that need to be scaffolded. This focus on the mathematical ideas that need to be scaffolded, however, runs the risk of unintentionally contributing to the reproduction of deficit perspectives about focusing on what students' do not know, instead of seeing their strengths which contradicts with the asset-based mathematical teaching. Complementary studies that not only focus on students' mathematical knowledge but also entailing students' sociocultural selves might align with the asses-based perspective as students bring their social, cultural, and linguistic strengths that form the basis to deepen their mathematical thinking.

Another limitation comes from the design of the data collection tools. I conjecture that variables such as given student solutions in the assessments might be limited to capturing PSTs' skills of attending, interpreting, and deciding how to respond. Their responses could differ when presented with different student solutions in the same content. Another set of limitations might pertain to the analysis itself. In this research, especially in answering the second research question "What relationships emerge among facets in PSTs' noticing of students' mathematical thinking?" and third research question "What are the opportunities to learn that become available to develop through the discussion around the designed tasks?" findings reflect what I "see" interesting in the data.

In my future research, I aim to focus on the connection with noticing and how it plays a role in disrupting inequities. With this purpose, I aim to support PSTs' multidimensional noticing that foregrounds on who the students and who the teachers are. I intend to find ways of supporting PSTs to see the noticing patterns in the classroom environment that would allow them to understand the predominant ideologies play a role in learning environments. I wonder how to make PSTs' noticing visible to increase their awareness about their and students' sociocultural selves that influence their noticing of students' mathematical thinking in the learning context.

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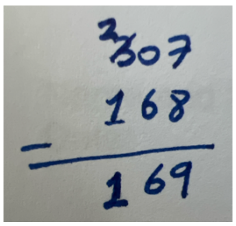
Appendices

Appendix A - Baseline Assessment (Pre- and Post-Assessment)

Question 1) Solve the following question and explain how you solved it. Try to solve it in as many ways as you can.

307-168

Question 2) A child solved the question in the following way. Answer the following questions based on the child's solution.



2a) What do you think the child did to solve this problem?

2b) Considering what the child did, what do you think they know? What mathematical understandings are reflected in their solution?

2c) What questions would you ask to gather more information about the child's understanding?

2d) What mathematical ideas/understandings might the child need more support with?

2e) How might you further support their understanding of that mathematical idea? (What questions would you ask? What representations would you use? What connections would you make?) Please explain and show it.

Appendix B - Task 1

A teacher asked her students the question "Paul had 83 strawberries in his basket. He gave 38 strawberries to his friend. How many strawberries did Paul have left?" Two students answered the question in the following ways.

Student A	Student B
83 take away 30 is 53 and take away 3 is 50. Then take away 5 more. That's 45.	83 take away is 38 is the same as 85 take away 40. That's 45.

1A. What did Student A do to solve this problem? **2A.** Considering what Student A did, what do you think they know? What mathematical understandings are reflected in their solution?	**1B**. What did Student B do to solve this problem? **2B.** Considering what Student B did, what do you think they know? What mathematical understandings are reflected in their solution?

3) What do you think is the major difference between the way of interpreting the subtraction between two students?
4) Which of these students would you judge to be using a method that could be used to subtract any two whole numbers?
5) Which of these student solutions would you use to discuss on the board?
6) How would you revise the question to have students solve it as a "think-addition"?

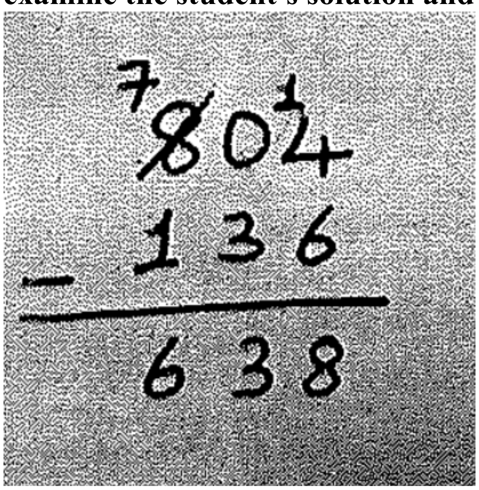
Appendix C - Task 2

This task consists of two parts, Part 1 & Part 2. First, you will complete Part 1 in your small groups. Then, a whole class discussion will be held to discuss your answers to the questions, and various solutions across the groups will be compared to discuss the connections across the strategies. After the whole class discussion, Part 2 will be distributed.

Part 1: 804-136

1. Identify the mathematical idea of the task.
2. Solve the question and explain how you solved it. Try to solve it in as many ways as possible.
3. Identify the mathematical knowledge in various solutions you found in your small group.

This part includes a student's answer to the same question you worked on in Part 1. Please examine the student's solution and answer the following questions.



Part 2: Examine Antonio's solution below and answer the following questions in your small group

1. What do you think Antonio did to solve this problem?
2. Considering what Antonio did, what do you think they know? What mathematical understandings are reflected in their solution?
   3. What understanding might the child need to be scaffolded?
   4. We examined as a whole class various strategies and how to represent them in different ways while we examined the mathematical thinking behind them after completing Part 1. Considering what you identified as what the child knows and considering the various ways of thinking we discussed as a whole class after Part 1, what kind of strategy will be helpful for Antonio to develop a better understanding of regrouping? You can use representations and materials. Please explain your reasoning.
   5. What would you ask as a next step question to Antonio, to have him develop the strategy you chose?
   6. Leave this question blank for now. Your instruction will ask you to revise your questions 4 and 5 after the whole class discussion.

   Revise Question 4: If you change/revise your question, explain why you have decided to this change/revision.

   Revise Question 5: If you change/revise your question, explain why you have decided to this change/revision.

Appendix D - Task 3

Imagine that you are working with your class on multiplying large numbers. Among your students' papers, you notice that some have displayed their work in the following ways:

Student A	Student B	Student C
35 x 25 125 +75 875	35 x25 175 +700 875	35 x 25 25 150 100 +600 875

1A. What do you think student A did to solve this problem? **2A.** Considering what student A did, what do you think they know? What mathematical understandings about multiplications are reflected in their solution?	**1B**. What do you think student B did to solve this problem? **2B.** Considering what student B did, what do you think they know? What mathematical understandings about multiplications are reflected in their solution?	**1C**. What do you think student C did to solve this problem? **2C.** Considering what student C did, what do you think they know? What mathematical understandings about multiplications are reflected in their solution?

3) Which of these students would you judge to be using a method that could be used to multiply any two whole numbers? Please explain your reasoning clearly.
4A) Which solution might you focus on first during a whole class discussion to highlight its connection to meaning of the multiplication? Explain your reasoning clearly.
4B) Which one would you focus on last and why?
4C) How would you highlight the connection between the two strategies?

Appendix E - Task 4

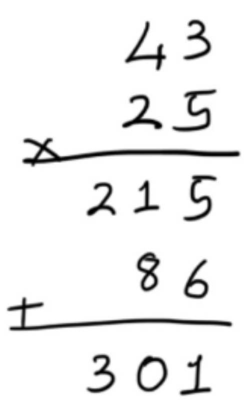
This task consists of two parts, Part 1 & Part 2. First, you will complete Part 1 in your small groups. Then, a whole class discussion will be held to discuss your answers to the questions, and various solutions across the groups will be compared to discuss the connections across the strategies. After the whole class discussion, Part 2 will be distributed.

Part 1: 43x25

1. Identify the mathematical idea of the task.
2. Solve the question and explain how you solved it. Try to solve it in as many ways as possible.
3. Identify the mathematical knowledge in various solutions you found in your small group.

This part includes a student's answer to the same question you worked on in Part 1. Please examine the student's solution and answer the following questions.

Part 2: Examine Charlie's solution below and answer the following questions in your small group



1. What do you think Charlie did to solve the problem?

2. Considering what Charlie did, what do you think they know? What mathematical understandings are reflected in their solution?

3. What understandings might the child still need to be scaffolded? How would you support them? Please explain your reasoning and provide examples of next step questions you will ask Charlie to support their learning.

4. Leave this question blank for now. Your instructor will ask you to revise your question 3 after the whole class discussion.

Revise Question 3: If you change/revise your answer, explain why you have decided to change or revise your answer.

Appendix F - Task 5

This task consists of two parts, Part 1 & Part 2. First, you will complete Part 1 in your small groups. Then, a whole class discussion will be held to discuss your answers to the questions, and various solutions across the groups will be compared to discuss the connections across the strategies. After the whole class discussion, Part 2 will be distributed.

Part 1: -9 – 4

1.What math understandings/ideas does this task target? (What are the deep mathematical understandings that a student develops from this task)

2.Solve the question and explain how you solved it. Try to solve it in as many ways as possible and explain how you solved it by using modeling and real-world situation problems.

3.Identify the mathematical knowledge in various solutions you found in your small group.

This part includes a student's answer to the same question you worked on in Part 1. Please examine the student's solution and answer the following questions.

Part 2: Examine Sam's solution below and answer the following questions in your small group.

-9 – 4 = -5

1. What do you think Sam did to solve the problem? What could be possible explanation(s) why the students might have gotten -5 as an answer?

2. Considering what Sam did, what do you think they know? What mathematical understandings are reflected in their solution?

3.What understandings might the child still need to be scaffolded? How would you support them? Please explain your reasoning and provide examples of next step questions you will ask Sam or provide examples of next-step moves you will take to support their learning? (Support your explanations by using modeling, real-world situations, etc.)

4.Leave this question blank for now. Your instructor will ask you to revise your question 3 after the whole class discussion.

Revise Question 3: If you change/revise your answer, explain why you have decided to change or revise your answer.

Appendix G - Coding Rubric

Noticing Components	Code Name	Explanation of the Code	Examples from Data
Attending	Low evidence	Attending the specifics of KCS and SCK is missed Only a general description of students' answers was provided Attended only to the correctness of answers	"They used the borrowing technique but made a mistake along the way" "Instead of 9-6, they did 0-6, the rest was correct" "Child forgot to take away from the tens" "The child made the 0 13 instead of 10"
	Medium Evidence	Identification of some of the noteworthy KCS and SCK, but not all of them or giving vague explanations without proper evidence	"The child may have subtracted 6-0 in the 10s place value, in order to get the "6" in 169. However, they got the 1 and 9 correct, so they have some understanding of place value."
	High evidence	Full identification of the mathematical elements related to KCS and SCK *the evidence is more nuanced compared to the medium evidence level as it is seen in the example. In this example, PST identifies that borrowed 1 from hundred should have transferred to ones place as ten 10s which is a evidence of more nuanced knowledge.	"The kid borrowed 1 from hundreds since there is zero in the tens place, and make it 10 in the ones place"
Interpreting	Low evidence	Interpretations are not evidence-based or general comments are provided	"Knows how to subtract but not correctly", "They just made a

		without mentioning mathematical elements.	simple mistake by forgetting to look at the new number.”, “They know subtraction means take away. They know they need to manipulate the numbers to be able to subtract, but they are still working on the proper way to do so.”, “they know how to move over and replace numbers but I think they got confused what number to use”
	Medium evidence	Drawing on some evidence, however, interpretation is still vague or limited and the interpretation explains only part of the student's work	“They understand the place value and subtracting from positive numbers, they don't know what to do with 0.” *In this example it is not clear how PSTs know that the child has an understanding of place value
	High evidence	Making sense of the details of a student strategy and noting how these details reflected what the student understood in specific situations	“I think they know you have to borrow when the number below is higher than the number above. This can be seen because they most likely borrowed from the hundreds place to make 7 into 17, then took 8 away from 17.”
Interpreting the math ideas that need to be scaffolded	Low evidence	Interpretations are not evidence-based or general comments are provided without mentioning mathematical elements	“Might need help with how to subtract bigger numbers” “Support with how to borrow” “Taking their time and not rushing”
	Medium Evidence	Identifying place value as a math idea that needs to be scaffolded, but missing the making a connection with	”More understanding with the zero, and what it means in different positions at ones, tens, 100s, 1000s ”

		other mathematical ideas such as borrowing	
	High Evidence	In addition to focusing on scaffolding the idea of place value, they also focus on reference units that show the relation between different place values and suggest a connection with borrowing	What it means to regroup, and how you regroup and adjust when there is a 0”
Deciding how to respond	Low Evidence	Questions or actions focused on procedures without connections or producing correct answers.	“I could show a detailed example of how to solve the problem, using different colors to show the different steps” “Teach them a song, so they will know the principles” “ would show them how to cross out the numbers so they could see it visually”
	Medium Evidence	Responses attend to one of the missing mathematical ideas identified in the student’s mathematical thinking and/or solution strategy.	“Connecting the borrowing with place value, 0 represents 0 tens, 3 represents 3 hundreds” “I would split the problem into 300+0+7 to have him see each value individually to show that the 0 isn’t nothing, but it is the tens place”
	High Evidence	Questions or actions related to conceptual understanding and focusing on interrelated mathematical ideas.	“I would first ask them what each number represents with the base ten blocks. (She draws represent both numbers with the base then block and separates the place values in different column to show the subtraction). This would help them to see borrowing involves taking a place value over and breaking it up. You cannot take 6 tens away from 0 tens, so you have

			to break one of the hundreds into 10 tens, and move one ten over to the ones because you cannot take 8 ones away from 7.”
Eliciting	Low Evidence	Questions focusing on the correctness or funneling students to the right or intended answers	“If we check our work and add 169 back to 168, what number do we get?”
	Medium Evidence	Focusing on procedures not the process of the child’s thought process Focusing on “what” student did instead of “why”	“I would ask them to show me the steps they took”
	High Evidence	Focusing on understanding why the student did what they did	Asking them to use the block, asking them for a representation “What does the 0 represent in 307? What does the 3 represent? How many tens are in one hundred? How many are you borrowing for the ones place? Why did you borrow from the 3/hundreds place?”