LEBESGUE INEQUALITIES FOR CONVOLUTION WITH SURFACE MEASURE ON PROTOTYPICAL HYPERSURFACES IN THREE DIMENSIONS

By

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Abstract

We find the precise range of Lebesgue space inequalities satisfied by convolution with surface measure on compact subsets of certain prototypical polynomial hypersurfaces in three dimensions. We derive these results using non-oscillatory, geometric methods, for a model class of polynomials bearing a strong connection to the general real-analytic case.

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Chapter 1

Preliminaries

1.1 Introduction

Curvature plays a dominant role in many questions in harmonic analysis, from maximal operators to restriction and averaging operators. When manifolds have nonvanishing "curvature", much more is known, and theorems tend to be more concise, with simpler proofs (though even those "simpler" proofs can be very challenging). Vanishing curvature often makes these operators worse, and must be carefully accounted for in proofs of results that can vary greatly from one manifold to another.

Two primary ways of accounting for the curvature information are as follows: First, the measure supported on the manifold can be chosen in such a way to incorporate the effects of curvature. Such a measure is often referred to as the affine surface measure. One advantage with the affine surface measure is that results tend to have a high level of uniformity over large classes of manifolds. A disadvantage is that such measures are distinctly non-Euclidean, so recovering results for measures comparable to the Euclidean surface measure is nontrivial.

Second, one can attain results with the Euclidean surface measure, or a comparable measure, using precise curvature information of the surfaces, and possibly making use of known affine measure results. As a downside, such Euclidean results often lack uniformity and a conciseness in the statement of results.

One such longstanding question has been the effects of curvature on the behavior of general averaging operators

$$\tilde{\mathcal{T}}f(x) := \int_{B\subset\mathbb{R}^d} f(x-(t,\varphi(t)))d\mu(t),$$

for various measures μ on $\mathbb{R}^n$, and with $x \in \mathbb{R}^n$, $n > d$, and with B a ball containing the origin. The two simplest cases have been averages on hypersurfaces and curves. Cases where $n = 1$ (averages along curves) are largely understood due in part to Tao and Wright [14]. For the cases of hypersurfaces ($n = d - 1$), much is now known when $d\mu(t)$ is equipped with the Affine surface measure $d\mu(t) = |\det(H\varphi(t))|^{\frac{1}{n+1}}dt$, with H being the Hessian, due especially to the work of Oberlin [12] and Gressman [7]. For the Euclidean surface measure or equivalently the unweighted push-forward measure ($d\mu(t) = dt$), on the other hand, only partial results exist for any $d \geq 2$. In the case of a hypersurface with the Euclidean surface measure, Ferreyra, Godoy, and Urciuolo completed the homogeneous polynomial case for $n = 3$ [6][15], and the additive case $\varphi(t) = \sum |t_i|^{a_i}$ for every n [5]. In addition, for $n = 3$, Iosevich, Sawyer, and Seeger [10] include results for convex hypersurfaces, based partially on multi-type. More recently, Dendrinos and Zimmerman [4] investigated this question in the case that φ is a polynomial of mixed-homogeneous type, that is, for φ satisfying $\varphi(\sigma^{\kappa_1}t_1, \sigma^{\kappa_2}t_2) = \sigma\varphi(t)$ for all $\sigma > 0$ and some fixed $\kappa_1, \kappa_2 \in (0,\infty)$. Such polynomials form a natural model class, because their members contain all anisotropic scaling limits of polynomials of two variables, and because mixed-homogeneous polynomials are related to the faces of the Newton diagram of all real-analytic functions. Thus, techniques for these polynomials should give insight on how to proceed in a more general case. In many cases, the results in [4] were nearly optimal, missing only the boundary, while in other cases, like $\varphi(t) = t_1^4 + t_1^2 t_2 + \frac{1}{6} t_2^2$, there remained an additional region between the conjecture and the theorem.

In this article we complete the mixed-homogeneous picture, establish the sharp range of restricted weak type (p, q) inequalities for averaging operators of the form

$$\mathcal{T} f(x) := \int_{[-1,1]^2} f(x' - t, x_3 - \varphi(t)) dt, \qquad x =: (x', x_3) \in \mathbb{R}^3,$$

with φ a mixed homogeneous polynomial. We use an alternate approach to [4], which allows us to obtain the full range of restricted weak type estimates for all such polynomials and which provides some clarification of the relationship between the Euclidean versions of these results and the affine versions. More precisely, we completely avoid using oscillatory methods, instead using methods such as Christ's method of refinements in [1], often complimented by additional techniques such as orthogonality arguments as in [3]. In several places, we modify the method of refinements to find the influence of lower-dimensional curvature information (in the simpler cases, this is instead accomplished with Minkowski's inequality). Finally, for the cases where regions remain between the Dendrinos-Zimmermann [4] results and conjecture, we alter the method of refinement to exploit the geometric information of the surface.

In Sections 1.1 and 1.2, we present two versions of our main theorem, the former written fully in terms of the Newton polytope, and the latter written more in terms of the factorization of φ, with notation similar to the results in [4]. Then Section 1.3 and 1.4 will give further detail into our techniques, along with an outline of our proof. One additional key purpose of the techniques here will be to clarify the complex relationship between the affine measure results of Oberlin [12] and Gressman [7], restated in Theorem

1.2.1, and the Euclidean or push-forward measure results proven here.

One hope is that the relationship between mixed-homogeneous polynomials and general real analytic functions will allow for these methods to be greatly generalized. Additionally, some of the concepts demonstrated here can be applied to analyzing the effects of curvature in a much broader class of problems. For example, in [13], which is a joint work of the author and Stovall, these concepts are applied to analyzing the effects of curvature in Fourier restriction.

In this article, we consider the averaging operator

$$\mathcal{T}f(x) = \int_{[-1,1]^2} f(x' - t, x_3 - \varphi(t))dt \qquad \text{where } x = (x', x_3) \in \mathbb{R}^3.$$

where φ are polynomials satisfying the following definition:

Definition 1.1.1. A polynomial $\varphi : \mathbb{R}^2 \to \mathbb{R}$ is mixed homogeneous if there exists $\kappa_1, \kappa_2 \in (0, \infty)$ such that $\varphi(\sigma^{\kappa_1} t_1, \sigma^{\kappa_2} t_2) = \sigma\varphi(t)$ for all $\sigma > 0$.

To state our main theorem, we need some additional definitions.

Definition 1.1.2. : Let $\varphi(z_1, z_2) = \sum_{\alpha_1,\alpha_2=0}^{M} c_{\alpha_1,\alpha_2} z_1^{\alpha_1} z_2^{\alpha_2}$ be a mixed homogeneous polynomial mapping of $\mathbb{R}^2$ into $\mathbb{R}$. We will denote the Taylor support of φ at $(0, 0)$, the Newton polyhedron of φ, and the Newton distance, respectively, to be

$$\mathcal{S}(\varphi) := \{(\alpha_1, \alpha_2) \in \mathbb{N}_0^2 : c_{\alpha_1,\alpha_2} \neq 0\};$$

$$\mathcal{N}(\varphi) := \overline{\text{Conv}}(\{(\alpha_1, \alpha_2) + \mathbb{R}_+^2 : (\alpha_1, \alpha_2) \in \mathcal{S}(\varphi)\});$$

$$d(\varphi) := \inf\{c : (c, c) \in \mathcal{N}(\varphi)\};$$

where $\overline{\text{Conv}}$ denotes the closed convex hull, and where $(d(\varphi), d(\varphi))$ is the intersection of the bisetrix $x_1 = x_2$ with the Newton polyhedron $\mathcal{N}(\varphi)$.

Next, define $\varphi_{R1} = \varphi - \{\text{all terms for which } \alpha_1 = 0\}$, and denote the z_1-reduced Taylor support of φ as

$$\mathcal{S}(\varphi_{R1}) = \{(\alpha_1, \alpha_2) \in \mathcal{S}(\varphi) : \alpha_1 \neq 0\},$$

which removes all z_2-axis terms from the Taylor support of φ. Under this definition, the Newton polytope $\mathcal{N}(\varphi_{R1})$ of φ_{R1} is a simple translate of the Newton polytope of $\partial_{z_1}\varphi$. Let $\mathcal{S}(\varphi_{R2})$ be defined similarly, and denote the reduced Newton distance as $d(\varphi_R) = \max(d(\varphi_{R1}), d(\varphi_{R2}))$, with φ_R defined accordingly. If $d(\varphi_{R1}) = d(\varphi_{R2})$, then set φ_R to be φ_{R1}.

A mixed-homogeneous function φ will be said to be *linearly adapted* if every factor of the form $(z_2 - \lambda z_1)$, some $\lambda \in \mathbb{R}/\{0\}$, has multiplicity less that or equal to $d(\varphi)$. By this definition, if φ is mixed homogeneous but not homogeneous, then φ is already linearly adapted. This ends up being equivalent to the definition given in [9]. For brevity the proof of this equivalence is omitted.

Finally, let $o(\varphi)$ denote the maximal multiplicity of any real irreducible (over $\mathbb{C}[z_1, z_2]$) factor of φ, and denote the height of φ by $h(\varphi) = \max(d(\varphi), o(\varphi))$.

Theorem 1.1.3. *Let $\varphi(z_1, z_2) : \mathbb{R}^2 \to \mathbb{R}$ be a mixed homogeneous polynomial, with $\varphi(0) = 0$, $\nabla\varphi(0) = 0$. If φ is homogeneous, we additionally assume that φ is linearly adapted and that $\varphi(z_1, z_2) \neq C(\lambda_1 z_1 + \lambda_2 z_2)^J$, for any $C \in \mathbb{R}$, $\lambda_i \in \mathbb{C}$, $J \in \mathbb{N}$. Then $\mathcal{T}$ is bounded from $L^p(\mathbb{R}^3)$ to $L^q(\mathbb{R}^3)$ in the restricted-weak sense iff p, q satisfy each of the following conditions:*

$$\tfrac{1}{q} \leq \tfrac{1}{p} \tag{1.1.1}$$

$$\tfrac{1}{q} \geq \tfrac{1}{3p} \qquad\qquad \tfrac{1}{q} \geq \tfrac{3}{p} - 2 \tag{1.1.2}$$

$$\tfrac{1}{q} \geq \tfrac{1}{p} - \tfrac{1}{d(\varphi)+1} \tag{1.1.3}$$

$$\tfrac{1}{q} \geq \tfrac{d(\varphi_R)+1}{2d(\varphi_R)+1}\tfrac{1}{p} - \tfrac{1}{2d(\varphi_R)+1} \qquad \tfrac{1}{q} \geq \tfrac{2d(\varphi_R)+1}{d(\varphi_R)+1}\tfrac{1}{p} - 1 \tag{1.1.4}$$

$$\tfrac{1}{q} \geq \tfrac{h(\varphi)+1}{h(\varphi)+2}\tfrac{1}{p} - \tfrac{1}{h(\varphi)+2} \qquad \tfrac{1}{q} \geq \tfrac{h(\varphi)+2}{h(\varphi)+1}\tfrac{1}{p} - \tfrac{2}{h(\varphi)+1} \tag{1.1.5}$$

$$\tfrac{1}{q} \geq \tfrac{1}{p} - \tfrac{1}{h(\varphi)}. \tag{1.1.6}$$

Remark 1.1.4. Since non-degenerate linear transformations preserve $L^p \to L^q$ norms, up to a constant, the assumptions that φ is linearly adapted and that $\varphi(0) = 0$, $\nabla\varphi(0) = 0$ do not weaken the generality of the theorem.

Remark 1.1.5. Equations (1.1.4), (1.1.5), and (1.1.6) only become relevant if $d(\varphi_R) > 2d(\varphi)$, $h(\varphi) > d(\varphi) + \frac{1}{2}$, or $h(\varphi) > d(\varphi) + 1$, respectively.

Remark 1.1.6. The two cases not covered by our theorem are when, after a linear transformation, $\varphi \equiv 0$ or $\varphi = z_1^J$, $J \geq 2$. In the former case, it is well known that $\mathcal{T}$ is bounded if and only if $p = q$. In the latter case, it was proved in [6] that $\mathcal{T}$ maps L^p boundedly into L^q precisely when p, q satisfy all of the following:

$$\tfrac{1}{q} \leq \tfrac{1}{p} \qquad \tfrac{1}{q} \geq \tfrac{1}{2p} \qquad \tfrac{1}{q} \geq \tfrac{2}{p} - 1 \qquad \tfrac{1}{q} \geq \tfrac{1}{p} - \tfrac{1}{J+1}.$$

This can also be shown with a proof similar to that in Subsection 4.2.1, namely, by interpolating $L^{\frac{3}{2}} \to L^3$ and $L^\infty \to L^\infty$ bounds after a dyadic decomposition. For brevity this proof is omitted.

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1.2 Reformulation

In this section, we will state (Theorem 1.2.4) an alternate formulation of Theorem 1.1.3, and prove in Proposition 1.2.5 that these theorems are equivalent. By way of background, we restate a result of Gressman, Theorem 3 of [7]. For $\varphi : \mathbb{R}^{d-1} \to \mathbb{R}$, and $x \in \mathbb{R}^d$, define the affine averaging operator $\mathcal{A}$ as

$$\mathcal{A}f(x) := \int_{\mathbb{R}^{d-1}} f(x' - t, x_d - \varphi(t))|\det(D^2\varphi)|^{\frac{1}{d+1}}\, dt,$$

where $D^2\varphi$ is the Hessian.

Theorem 1.2.1 ([7])**.** *Let $\varphi : \mathbb{R}^{d-1} \to \mathbb{R}$ be a polynomial. Then $\mathcal{A}$ extends as a bounded linear operator from $L^{\frac{d+1}{d}}(\mathbb{R}^d)$ to $L^{d+1}(\mathbb{R}^d)$, with operator norm bounded by a constant depending only on the dimension d and the degree of the polynomial φ.*

In fact, the restricted strong type version of this theorem, due to Oberlin ([12]), suffices for most of our applications. We will use this result extensively in proving Theorem 1.1.3.

Definition 1.2.2. Let $\boldsymbol{\kappa} = (\kappa_1, \kappa_2) \in (0,\infty)^2$, and let $\psi : \mathbb{R}^2 \to \mathbb{R}$. We say ψ is $\boldsymbol{\kappa}$*-mixed homogeneous* if, for every $\sigma > 0$, $\psi(\sigma^{\kappa_1}\cdot, \sigma^{\kappa_2}\cdot) = \sigma\psi(\cdot,\cdot)$.

If φ is mixed-homogeneous, there exists a $\boldsymbol{\kappa} = (\kappa_1, \kappa_2) \in (0,\infty)^2$ such that φ is $\boldsymbol{\kappa}$-mixed homogeneous. If φ is a not a monomial, this $\boldsymbol{\kappa}$ is unique, while for monomials, uniqueness holds under the additional assumption $\kappa_1 = \kappa_2$. We denote this unique $\boldsymbol{\kappa}$ as $\boldsymbol{\kappa_\varphi}$. We define homogeneous distance as

$$d_h := d_h(\varphi) := \tfrac{1}{\kappa_1+\kappa_2}$$

Finally, there exist unique r, s, m satisfying $\gcd\{r, s\} = 1$ such that $\boldsymbol{\kappa_\varphi} = (\frac{s}{m}, \frac{r}{m})$.

The mixed-homogeneous polynomial φ may be expanded as $\varphi = \sum_{l=0}^{l_f} c_l z_1^{J-rl} z_2^{K+sl}$, for some J, K, c_l. We observe that

$$d_h(\varphi) = \tfrac{Js+Kr}{r+s}.$$

By the Fundamental Theorem of Algebra, we can factor φ as

$$\varphi(z_1, z_2) = C z_1^{\tilde{\nu}_1} z_2^{\tilde{\nu}_2} \prod_{j=3}^{\tilde{m}_2} (z_2^s - \lambda_j z_1^r)^{\tilde{n}_j}, \text{ with } \lambda_j \text{ real iff } j \le \tilde{m}_1 \le \tilde{m}_2, \tag{1.2.1}$$

For brevity, we refer to the irreducible factors $z_1, z_2, z_2^s - \lambda_j z_1^r$, simply as factors. The homogeneous distance d_h can be rewritten as

$$d_h = \tfrac{s\tilde{\nu}_1 + r\tilde{\nu}_2 + rs\sum \tilde{n}_j}{r+s}.$$

Consider $\omega := \det(D^2\varphi)$. As we will show in Lemma 2.1.5 and (2.1.2), if $d_h \neq 1$, ω is $\frac{\boldsymbol{\kappa_\varphi}}{2-2/d_h}$-mixed homogeneous, so ω is mixed homogeneous with the same r, s, and its mixed homogeneity satisfies

$$d_\omega := d_h(\omega) = 2d_h - 2.$$

If $d_h = 1$, ω is constant (see (2.1.1)), and we can say $d_\omega = 0$. Thus, in either case, we can factor ω as

$$\omega(z_1, z_2) = C z_1^{\nu_1} z_2^{\nu_2} \prod_{j=3}^{m_2} (z_2^s - \lambda_j z_1^r)^{n_j}, \text{ with } \lambda_j \text{ real iff } j \le m_1 \le m_2. \tag{1.2.2}$$

We will use the following proposition from [8], which follows from simple algebra:

Proposition 1.2.3 ([8]). *Let ψ be a mixed homogeneous polynomial as in* (1.2.1) *or* (1.2.2), *and denote d_ψ as the homogeneous distance of ψ.*

1. *If $\min\{r,s\} > 1$ then any irreducible factor of ψ the form $(z_2^s - \lambda z_1^r)$, $\lambda \neq 0$ has multiplicity strictly less than d_ψ.*

2. *If ψ has a factor of multiplicity greater than d_ψ, then every other factor of ψ has multiplicity less than d_ψ.*

Let T denote the maximum multiplicity of any real factor of ω, i.e. $T = \max\{\nu_1, \nu_2\} \cup \{n_j : j \leq m_1\}$. As we will see in Proposition 2.1.6, our assumptions imply that $\omega \not\equiv 0$, so T is well-defined.

Suppose $T > d_\omega$. By Proposition 1.2.3, there exists a unique factor of ω, denoted f_T, having multiplicity T. If f_T is linear, let ν denote its multiplicity in φ, and if f_T is nonlinear, let N denote its multiplicity in φ. If f_T is linear and $\nu = 0$, interchanging indices allows for the assumption $f_T(z) \neq z_1$, so we may expand

$$\varphi(z) = z_1^J + z_1^{J-lr} f_T^{ls} + \mathcal{O}(f_T^{ls+s}), \text{ some } l \geq 1,$$

and in any such case where $T > d_\omega$, f_T is linear, and $\nu = 0$, we set $A := ls$.

If we are in any case such that ν is not defined in the preceding (likewise A, N) we set it to be 0. We use the convention that $\frac{1}{0} = \infty$.

Theorem 1.2.4. *Let $\varphi(z_1, z_2)$ be a mixed homogeneous polynomial, with vanishing gradient at the origin, and not of the form $(\lambda_1 z_1 + \lambda_2 z_2)^J$, $\lambda_i \in \mathbb{C}$, $J \in \mathbb{N}$. Then for $1 \leq p, q \leq \infty$, $\mathcal{T}$ is of restricted-weak type (p,q) iff the following hold:*

$$\tfrac{1}{q} \leq \tfrac{1}{p} \qquad \tfrac{1}{q} \geq \tfrac{1}{3p} \qquad \tfrac{1}{q} \geq \tfrac{3}{p} - 2 \tag{1.2.3}$$

$$\frac{1}{q} \geq \frac{1}{p} - \frac{1}{d_h+1} \tag{1.2.4}$$

$$\frac{1}{q} \geq \frac{1}{p} - \frac{1}{\nu+1} \tag{1.2.5}$$

$$\frac{1}{q} \geq \frac{A+1}{2A+1}\frac{1}{p} - \frac{1}{2A+1} \qquad \frac{1}{q} \geq \frac{2A+1}{A+1}\frac{1}{p} - 1 \tag{1.2.6}$$

$$\frac{1}{q} \geq \frac{N+1}{N+2}\frac{1}{p} - \frac{1}{N+2} \qquad \frac{1}{q} \geq \frac{N+2}{N+1}\frac{1}{p} - \frac{2}{N+1} \tag{1.2.7}$$

$$\frac{1}{q} \geq \frac{1}{p} - \frac{1}{N} \tag{1.2.8}$$

Proposition 1.2.5. *Theorems 1.1.3 and 1.2.4 are equivalent.*

Proposition 1.2.5 will be proved in Proposition 2.1.11.

1.3 Overview

Our strategy will be as follows: we will use the mixed homogeneity of φ (and by implication that of the Gaussian curvature, which is comparable to $\omega := \det(D^2\varphi)$) to decompose $[-1,1]^2$ into dyadic level-set "strips" on which $|\omega| \approx 2^{-j}$, and further decompose those strips into dyadic rectangles (after a possible coordinate change). On any such strip or rectangle, we will have three useful restricted weak type $L^p \to L^q$ estimates: For $p = q$, Young's inequality implies $L^p \to L^q$ bounds with constant equal to the measure of the strip or rectangle. Since ω is nearly constant on each strip or rectangle, Theorem 1.2.1 gives us $L^{\frac{4}{3}} \to L^4$ bounds on each dyadic strip. And finally, at $(p,q) = (\frac{3}{2}, 3)$, we will use bounds for averages on curves in $\mathbb{R}^2$ ([11] and later [7]), and Minkowski's Inequality with Young's Inequality or a variant of Christ's method of refinements to achieve either strong-type or restricted-weak-type versions of $L^{\frac{3}{2}} \to L^3$ bounds. In some cases, by summing over the dyadic regions, summing the minimum of the $L^{\frac{4}{3}} \to L^4$, the $L^\infty \to L^\infty$, and the $L^{\frac{3}{2}} \to L^3$ bounds, we can achieve restricted-weak-type bounds for $\mathcal{T}$ in the optimal range. We sometimes also utilize the scaling symmetry of mixed-homogeneous polynomials in this argument, as in Prop 2.3.4. However, these methods are not always sufficient. Two types of obstructions can arise.

One issue is that degeneracies sometimes lead to a logarithmic factor in the $L^p \to L^p$ bounds on the relevant strips. To remove the logarithmic factor, we use an orthogonality argument incorporated into Christ's method of refinements, found in [3].

A more delicate issue arises when $\omega = \det(D^2\varphi)$ possesses a factor f_T of high multiplicity $T > d_\omega$, but all components of $\nabla\varphi$ lack f_T as a factor. This causes the portion of the surface over our dyadic "rectangles" to have a structure more closely resembling a helix than a parabola or hyperbola. As a consequence, the image of any such dyadic "rectangle" under $\nabla\varphi$ is highly non-convex. We can beat the above mentioned interpolation argument by quantifying this non-convexity. We use a variant of Christ's method of refinements [1], to show that non-convexity makes it impossible for $L^\infty \to L^\infty$ quasi-extremals to also be $L^{\frac{4}{3}} \to L^4$ quasi-extremals, and we interpolate after quantifying this trade-off. Further detail will be provided in the next section.

1.4 Outline

We first divide into the cases $T \leq d_\omega$ and $T > d_\omega$, and we further break the case $T > d_\omega$ into the following cases:

<table>
<tr><td>Case(ν):</td><td>f_T linear and $\nu \geq 1$</td><td rowspan="3">} Rectangular Cases</td></tr>
<tr><td>Case(A):</td><td>f_T linear and $\nu = 0,\ A \geq 2$</td></tr>
<tr><td>Case(N):</td><td>f_T nonlinear and $N \geq 2$</td></tr>
</table>

Case(i):	f_T linear and $\nu = 0$, $A = 1$	Twisted Cases
Case(iia):	f_T nonlinear and $N = 0$	
Case(iib):	f_T nonlinear and $N = 1$	

When $T > d_\omega$ and f_T is linear with $\nu = 0$, A cannot be 0, so this decomposition covers all cases where $T > d_\omega$. As we will see in Corollary 2.1.4, in cases (ν), (A), and (N), T satisfies $T = 2\nu - 2$, $T = A - 2$, and $T = 2N - 3$, respectively. In these cases, on a local scale, the graph of φ will resemble a rectangular piece of a paraboloid or a saddle. We classify these three cases as Rectangular Cases. In cases (i), (iia), and (iib), on the other hand, T has no such restrictions, and on a local scale the graph of φ has a helix-like structure. These cases will be denoted as Twisted Cases.

In Section 2.1, we perform prove algebraic lemmas relating φ and its derivatives, show that the hypotheses of Theorem 1.2.4 imply that $\omega \not\equiv 0$, show that if φ is homogeneous and $T > d_\omega$, it can only belong to case(ν), and prove that the two versions of our main theorem, Theorems 1.1.3 and 1.2.4, are equivalent. Section 2.2 is dedicated to proving that every condition stated in Theorem 1.2.4 is necessary. In Section 8 we exploit the mixed-homogeneous scaling symmetry of φ to shorten the interpolation arguments in numerous later sections.

In Section 2.4, we make a finite decomposition of the region $[-1, 1]$ into neighborhoods of each irreducible subvariety of $\{\omega := detD^2\varphi = 0\}$. This induces a finite decomposition of the operator. The most troublesome of these neighborhoods, which we denote by R_T, is the neighborhood of the subvariety $\{f_T = 0\}$. When $T \leq d_\omega$, we set $R_T = \emptyset$.

We begin Section 3.1 by showing in Lemmas 3.1.2 and 3.1.3 that to prove Theorem

1.2.4, it suffices to prove that $\mathcal{T}$ is of rwt (p_{v_1}, q_{v_1}), for a specific (p_{v_1}, q_{v_1}) satisfying $q_{v_1} = 3p_{v_1}$, and in cases (N) and (A), to additionally prove that $\mathcal{T}$ is of rwt (p_{v_2}, q_{v_2}), for a specific (p_{v_2}, q_{v_2}) satisfying $p'_{v_2} \leq q_{v_2} < 3p_{v_2}$.

Under these decompositions, our proof of Theorem 1.2.4 reduces to proving each of the following:

(I) $\mathcal{T}_{[-1,1]^2 \setminus R_T}$ is of restricted-weak-type $(\frac{2d_h+2}{3}, 2d_h + 2)$.

(II) In the Rectangular Cases, $\mathcal{T}_{R_T}$ is of restricted-weak-type $(\frac{T+4}{3}, T + 4)$.

(III) In the Twisted Cases, $\mathcal{T}_{R_T}$ is of restricted-weak-type $(\frac{2d_h+2}{3}, 2d_h + 2)$.

(IV) In Cases (N) and (A), $\mathcal{T}_{R_T}$ is of restricted-weak-type (p_{v_2}, q_{v_2}).

In Section 3.1, we prove (II) and, when $T \neq d_\omega$, we prove (I). In Sections 3.2 and 3.3 we finish the proof of (I) when $T = d_\omega$ by incorporating an orthogonality argument into Christ's method of refinements, as in [3]. We then prove (III) in Section 3.4, by analyzing the "twisting" of the graph of φ by quantifying the nonconvexity of the image of local regions under $\nabla\varphi$, and using those calculations with a variant of the method of refinements.

Finally, in Section 4.1, we turn our attention to proving (IV). We first begin by decomposing cases (A) and (N) into subcases, based on the location of $(\frac{1}{p_{v_2}}, \frac{1}{q_{v_2}})$.

In Section 4.2, we complete the proof of (IV), by going through each subcase and proving that $\mathcal{T}_{R_T}$ is of restricted-weak-type (p_{v_2}, q_{v_2}). In each subcase, our argument begins by computing restricted-weak-type $(\frac{3}{2}, 3)$ bounds over small regions. In several of the subcases of Case (N), this computation also involves a variant of the method of refinements.

We then proceed to interpolate the restricted-weak-type $(\frac{3}{2}, 3)$ bound with one or both of the restricted-weak-type $(0, 0)$ and restricted-weak-type $(\frac{4}{3}, 4)$ bounds to achieve our result. However, one of the subcases, denoted $(N^{scal}_{q=2p})$, relies on a more complex argument that involves incorporating an orthogonality argument into the method of refinements, similar to the argument in Section 3.2.

1.5 Notation

For the operator $\mathcal{T}$, most of our results will be bounds of restricted-weak-type. $\mathcal{T}$ is of rwt (p, q), with bound $C_{p,q,w}$, iff for all measurable sets E of finite measure,

$$\|\mathcal{T}\chi_E\|_{L^{q,w}} \leq C_{p,q,w}\|\chi_E\|_{L^p} = C_{p,q,w}|E|^{\frac{1}{p}},$$

or, equivalently, if for all E, F finite,

$$\langle \mathcal{T}\chi_E, \chi_F\rangle \leq C_{p,q,w}\|\chi_E\|_{L^p}\|\chi_F\|_{L^{q'}} = C_{p,q,w}|E|^{\frac{1}{p}}|F|^{1-\frac{1}{q}}.$$

We will sometimes use the notation $\mathcal{T}(E, F) := \langle \mathcal{T}\chi_E, \chi_F\rangle$.

We define π_1 and π_2 to be the projection onto the first and second coordinate in $\mathbb{R}^2$, respectively, so that $\pi_1(z_1, z_2) = z_1$. And for $G \in \mathbb{R}^2$, we denote $\langle G\rangle$ to be the span of G, which would be a line, whenever $G \neq 0$.

The notation $\overline{Conv}(S)$ will denote the closed convex hull of S, while $\mu(\overline{Conv}(S))$ will refer to the Lebesgue measure of the convex hull.

To represent mixed Lebesgue norm spaces, we will use the notation

$$\|F(u,t)\|_{L^a_{u_2}L^b_{u_1,u_3}L^c_t(\{t\in S\})} := \|\|\|F(u,t)\|_{L^c(t\in S)}\|_{L^b((u_1,u_3)\in\mathbb{R}^2)}\|_{L^a(u_2\in\mathbb{R})}.$$

We will sometimes need to decompose the operator $\mathcal{T}$. If $R \subset \mathbb{R}^2$ is a measurable set, we define

$$\mathcal{T}_R f(x) := \int_R f(x' - t, x_3 - \varphi(t))dt.$$

Also, unless otherwise stated, we will use $A \lesssim B$ to imply that there exists a constant C, uniform except for a possible φ dependence, such that $A \leq CB$, and similarly for $\gtrsim$. If we are incorporating extra dependence, for instance ϵ-dependence, we will use $\lesssim_\epsilon$. Finally, $A \approx B$ is equivalent to $A \lesssim B$ and $A \gtrsim B$. By $(A, B) \approx (D, E)$, we mean $A \approx D$ and $B \approx E$. Also, C will be a constant that can change from line to line, which will depend only on φ unless otherwise stated.

Chapter 2

Setup

2.1 Algebraic Lemmas

Our first few lemmas will connect the form that φ and ω take.

Lemma 2.1.1. *If* $\varphi = z_1^J(z_2 - z_1^r)^{N'} + \mathcal{O}((z_2 - z_1^r)^{N'+1})$*, some* $J \geq 0$*,* $N' \geq 2$*, and* $r \geq 2$*, then* $\omega = Cz_1^{2J+r-2}(z_2 - z_1^r)^{2N'-3} + \mathcal{O}((z_2 - z_1^r)^{2N'-2})$*, some* $C \neq 0$*.*

Lemma 2.1.2. *If* $\varphi = z_1^J + cz_1^{J-lr}z_2^{ls} + \mathcal{O}(z_2^{ls+1})$*, some* $c \neq 0$ *and* $l \geq 1$*, with* $ls \geq 2$ *and* $J \geq 1$*, then* $\omega = Cz_1^{2J-lr-2}z_2^{ls-2} + \mathcal{O}(z_2^{ls-1})$*, some* $C \neq 0$*.*

Lemma 2.1.3. *If* $\varphi = z_1^J z_2^{\nu'} + \mathcal{O}(z_2^{\nu'+1})$*, some* $J, \nu \geq 1$*, then* $\omega = Cz_1^{2J-2}z_2^{2\nu-2} + \mathcal{O}(z_2^{2\nu-1})$*, some* $C \neq 0$*.*

Corollary 2.1.4. *: In Case(N),* $T = 2N - 3$*, in Case(A),* $T = A - 2$*, and in Case(*ν*),* $T = 2\nu - 2$*.*

Next, we will prove each of these lemmas.

Proof of Lemma 2.1.1. Making a change of variables, we set $x = z_1$ and $y = z_2 - z_1^r$.

Then $\partial_{z_1}^2 = \partial_x^2 + r^2x^{2r-2}\partial_y^2 - 2rx^{r-1}\partial_{xy} - r(r-1)x^{r-2}\partial_y$, $\partial_{z_2}^2 = \partial_y^2$, and $\partial_{z_1z_2} = \partial_{xy} - rx^{r-1}\partial_y^2$, so

$$\omega = [\partial_x^2\varphi\partial_y^2\varphi - (\partial_{xy}\varphi)^2] - r(r-1)x^{r-2}\partial_y\varphi\partial_y^2\varphi.$$

If $\varphi = x^J y^N + \mathcal{O}(y^{N+1})$, with $N' \geq 2$, then the terms $\partial_x^2\varphi\partial_y^2\varphi$ and $(\partial_{xy}\varphi)^2$ are both $\mathcal{O}(y^{2N'-2})$, while the final term

$$x^{r-2}\partial_y\varphi\partial_y^2\varphi = N'^2(N'-1)x^{2J+r-2}y^{2N'-3} + \mathcal{O}(y^{2N'-2}).$$

Thus,

$$\omega = Cx^{2J+r-2}y^{2N'-3} + \mathcal{O}(y^{2N'-2}), \text{ some } C \neq 0. \qquad \square$$

Proof of Lemma 2.1.2. For $\varphi = z_1^J + cz_1^{J-lr}z_2^l s + \mathcal{O}(z_2^{ls+1})$,

$$\omega = \partial_{z_1}^2\varphi\partial_{z_2}^2\varphi - (\partial_{z_1z_2}\varphi)^2.$$

The term $(\partial_{z_1z_2}\varphi)^2$ is $\mathcal{O}(z_2^{2ls-2})$, while

$$\partial_{z_1}^2\varphi\partial_{z_2}^2\varphi = cJ(J-1)ls(ls-1)z_1^{2J-lr-2}z_2^{ls-2} + \mathcal{O}(z_2^{ls-1}).$$

Since $ls \geq 2$, and since $\nabla\varphi(0) = 0$ implies that $J \geq 2$,

$$\omega = Cz_1^{2J-lr-2}z_2^{ls-2} + \mathcal{O}(z_2^{ls-1}), \text{ some } C \neq 0. \qquad \square$$

Proof of Lemma 2.1.3. For $\varphi = z_1^J z_2^{\nu'} + \mathcal{O}(z_2^{\nu'+1})$, with $J, \nu' \geq 1$,

$$\omega = \partial_{z_1}^2\varphi\partial_{z_2}^2\varphi - (\partial_{z_1z_2}\varphi)^2.$$

Looking at each term separately,

$$\begin{aligned}(\partial_{z_1z_2}\varphi)^2 &= [J\nu' z_1^{J-1}z_2^{\nu'-1}]^2 + \mathcal{O}(z_2^{2\nu'-1}),\\ \partial_{z_1}^2\partial_{z_2}^2\varphi &= [J(J-1)z_1^{J-2}z_2^{\nu'}][\nu'(\nu'-1)z_1^J z_2^{\nu'}] + \mathcal{O}(z_2^{2\nu'-1}).\end{aligned}$$

Thus,

$$\begin{aligned}\omega &= [1-\nu'-J]J\nu' z_1^{2J-2}z_2^{2\nu'-2} + \mathcal{O}(z_2^{2\nu'-1})\\ &= Cz_1^{2J-2}z_2^{2\nu'-2} + \mathcal{O}(z_2^{2\nu'-1}), \text{ some } C \neq 0. \square\end{aligned}$$

Proof of Corollary 2.1.4. In Case(N), $\min\{r, s\} = 1$ by Prop 1.2.3, so after rescaling φ can be written as in Lemma 2.1.1 with $N = N' \geq 2$, implying that $T = 2N - 3$. In Case(A), after rescaling φ can be written as in Lemma 2.1.2, with $A = ls \geq 2$. Additionally, if J were less that 2, then the hypothesis $\nabla\varphi(0) = 0$ of Theorem 1.2.4 wouldn't be satisfied. Thus, by Lemma 2.1.2, $T = A - 2$. In Case(ν), after rescaling, φ can be written as in Lemma 2.1.3 with $\nu = \nu' \geq 1$. Additionally, if $J = 0$, then $\varphi = z_2^\nu$, violating the hypothesis $\nabla\varphi(0) = 0$ of Theorem 1.2.4. Thus, by Lemma 2.1.3, $T = 2\nu - 2$. □

Next, we look at the relationship between the homogeneous distances for φ and its derivatives.

Lemma 2.1.5. *The homogeneous distances of φ and its derivatives obey the following relations: $d_\omega = 2d_h - 2$, $d_h(\partial_{z_1}\varphi) = d_h - \frac{s}{r+s}$, and $d_h(\partial_{z_2}\varphi) = d_h - \frac{r}{r+s}$.*

Proof. There exists some m such that $\sigma\varphi(z_1, z_2) = \varphi(\sigma^{\frac{s}{m}} z_1, \sigma^{\frac{r}{m}} z_2)$. Taking derivatives,

$$\sigma^2 \partial_{z_1}^2 \varphi(z_1, z_2) \partial_{z_2}^2 \varphi(z_1, z_2) = \sigma^{2\frac{s+r}{m}} \partial_{z_1}^2 \varphi(\sigma^{\frac{s}{m}} z_1, \sigma^{\frac{r}{m}} z_2) \partial_{z_2}^2 \varphi(\sigma^{\frac{s}{m}} z_1, \sigma^{\frac{r}{m}} z_2)$$

$$\sigma^2 (\partial_{z_1 z_2} \varphi(z_1, z_2))^2 = \sigma^{2\frac{s+r}{m}} (\partial_{z_1 z_2} \varphi(\sigma^{\frac{s}{m}} z_1, \sigma^{\frac{r}{m}} z_2))^2.$$

Putting these together, and using $d_h = \frac{m}{r+s}$:

$$\sigma^2 \omega(z_1, z_2) = \sigma^{\frac{2}{d_h}} \omega(\sigma^{\frac{s}{m}} z_1, \sigma^{\frac{r}{m}} z_2). \tag{2.1.1}$$

When $d_h \neq 1$, we can change variables using $\gamma = \sigma^{2 - \frac{2}{d_h}}$ to get

$$\gamma \omega(z_1, z_2) = \omega(\gamma^{\frac{s}{m(2-2/d_h)}} z_1, \gamma^{\frac{r}{m(2-2/d_h)}} z_2). \tag{2.1.2}$$

Thus, $d_\omega = \frac{m(2-2/d_h)}{r+s} = d_h(2 - \frac{2}{d_h}) = 2d_h - 2$. And when $d_h = 1$, ω is invariant under scaling, implying that ω is constant and $d_\omega = 0$. □

Next, by symmetry, it suffices to show the claim for $d_h(\partial_{z_1}\varphi) = d_h - \frac{s}{r+s}$. Taking a derivative, we get

$$\sigma\partial_{z_1}\varphi(z_1, z_2) = \sigma^{\frac{s}{m}}\partial_{z_1}\varphi(\sigma^{\frac{s}{m}}z_1, \sigma^{\frac{r}{m}}z_2),$$

which becomes, under the change of variables $\gamma = \sigma^{1-\frac{s}{m}} = \sigma^{\frac{m-s}{m}}$,

$$\gamma\partial_{z_1}\varphi(z_1, z_2) = \partial_{z_1}\varphi(\gamma^{\frac{s}{m-s}}z_1, \gamma^{\frac{r}{m-s}}z_2).$$

Thus, $d_h(\partial_{z_1}\varphi) = \frac{m-s}{s+r} = d_h - \frac{s}{s+r}$, and therefore by symmetry, $d_h(\partial_{z_2}\varphi) = d_h - \frac{r}{s+r}$. □

Next, we want to show that if $\omega \equiv 0$, then φ does not satisfy the assumptions of Theorem 1.2.4.

Proposition 2.1.6. *If φ is a mixed homogeneous polynomial with $\nabla\varphi(0) = 0$ and $\omega \equiv 0$, then φ is a constant multiple of z_1^J, z_2^J, or $(z_1 + \lambda z_2)^J$, with $\lambda \in \mathbb{R}$ and $J \in \mathbb{N}$. Consequently, in Theorem 1.2.4, ω is never identically* 0.

Proof. Exclude the aforementioned cases. Then after possibly swapping z_1 and z_2 and rescaling φ, either

(1) $\varphi = z_1^J z_2^\nu + \mathcal{O}(z_2^{\nu'+1})$, some $J, \nu' \geq 1$, or

(2) $\varphi = z_1^J + cz_1^{J-lr}z_2^{ls} + \mathcal{O}(z_2^{ls+s})$

. $= Cz_2^K + \tilde{c}z_2^{K-ks}z_1^{kr} + \mathcal{O}(z_1^{kr+r})$, for some k, l, with $C, c, \tilde{c} \neq 0$, and $J, K \geq 2$.

In case 1, by Lemma 2.1.3,

$$\omega = Cz_1^{2J-2}z_2^{2\nu-2} + \mathcal{O}(z_2^{2\nu-1}) \not\equiv 0$$

In case 2, if $\max\{ls, kr\} > 1$, then we can use Lemma 2.1.2. By symmetry, we need only consider $ls > 1$. Then for some $C \neq 0$,

$$\omega = Cz_1^{2J-lr-2}z_2^{ls-2} + \mathcal{O}(z_2^{ls-1}) \not\equiv 0.$$

Lastly, if $\max\{ls, kr\} = 1$, then φ is homogeneous and case 2 simplifies to

$$\varphi = z_1^J + c_1 z_1^{J-1} z_2 + ... + c_{J-1} z_1 z_2^{J-1} + c_J z_2^J, \text{ with } c_1 \neq 0.$$

We want to show that if $\omega \equiv 0$, then $\varphi = (z_1 - \lambda z_2)^J$, for some λ.

For later convenience, we will prove an even stronger statement here:

Lemma 2.1.7. *Suppose that* $\varphi = z_1^J + c_1 z_1^{J-1} z_2 + ... + c_{J-1} z_1 z_2^{J-1} + c_J z_2^J$, *with* $c_1 \neq 0$, *and suppose that* $\omega = \mathcal{O}(z_2^{J-1})$. *Then* $\varphi = (z_1 - \lambda z_2)^J$, *with* $\lambda = \frac{c_1}{J}$, *and consequently* $\omega \equiv 0$.

Proof. Replacing c_1 with $J\lambda$, we can write φ as

$$\varphi = z_1^J + J\lambda z_1^{J-1} z_2 + c_2 z_1^{J-2} z_2^2 + ... + c_{J-1} z_1 z_2^{J-1} + c_J z_2^J, \text{ with } \lambda \neq 0.$$

Since $\tilde{\varphi} = (z_1 + \lambda z_2)^J$ produces such a φ, with c_2,...,c_J fixed in terms of λ and J, and since $detD^2\tilde{\varphi} \equiv 0$, it suffices to show that:

If $\omega = \mathcal{O}(z_2^{J-1})$, then c_2,...,c_J are uniquely determined by λ and J.

Expanding, there exist functions $f_{1,K}$ and $f_{2,K}$, for $2 \leq K \leq J$, such that

$$(\partial_{z_1}^2 \varphi \partial_{z_2}^2 \varphi) = \sum_{K=2}^{J} [JK(J-1)(K-1)c_K + f_{2,K}(J, \lambda, c_2, ..., c_{K-1})] z_1^{2J-K-2} z_2^{K-2} + \mathcal{O}(z_2^{J-1})$$

and

$$(\partial_{z_1 z_2} \varphi)^2 = \sum_{K=2}^{J} f_{1,K}(J, \lambda, c_2, ..., c_{K-1}) z_1^{2J-K-2} z_2^{K-2} + \mathcal{O}(z_2^{J-1}),$$

so there exist functions $f_{3,K}$, $2 \leq K \leq J$, such that

$$\omega = \sum_{K=2}^{J} [JK(J-1)(K-1)c_K - f_{3,K}(J, \lambda, c_2, ..., c_{K-1})] z_1^{2J-K-2} z_2^{K-2} + \mathcal{O}(z_2^{J-1}).$$

Therefore, $\omega = \mathcal{O}(z_2^{J-1})$ implies that, for $K = 2, 3, ..., J$,

$$c_K = \frac{f_{3,K}(J,\lambda,c_2,...,c_{K-1})}{JK(J-1)(K-1)},$$

and therefore inductively each c_K is uniquely determined by J and λ, and the Lemma proof is complete. □

Then due to Lemma 2.1.7, the proof of Proposition 2.1.6 is complete. □

Corollary 2.1.8. *Theorem 1.2.4 holds in the case $d_h = 1$, and in the case $d_h < 1$, Theorem 1.2.4 holds vacuously.*

Proof. If $d_h = 1$, then ω is constant by Lemma 2.1.5. If that constant is nonzero, then Theorem 1.2.1 applies, while if $\omega \equiv 0$, then Theorem 1.2.4 holds vacuously by Proposition 2.1.6. If $d_h < 1$, then $d_\omega < 0$ by Lemma 2.1.5, implying $\omega \equiv 0$, so again Theorem 1.2.4 holds vacuously. □

Next, we show that homogeneous φ can only belong to to a couple cases of Theorem 1.2.4.

Proposition 2.1.9. *If φ is homogeneous, and φ satisfies the hypotheses of Theorem 1.2.4, then either $T \leq d_\omega$ or φ belongs to case(ν). Consequently, in case(A) and twisted case(i), we have* $\max\{r, s\} \geq 2$.

Proof. If φ is homogeneous, then every factor of ω is linear. Thus, when $T > d_\omega$, φ can only belong to case(ν), case(A), or twisted case(i).

If φ belongs to case(A), then after a linear change of coordinates and rescaling, $\varphi = z_1^J + cz_1^{J-A}z_2^A + \mathcal{O}(z_2^{A+1})$, $A \geq 2, c \neq 0$, and by Lemma 2.1.2 or Corollary 2.1.4,

$T = A - 2$. Also, $d_\omega = 2d_h - 2 = 2\frac{J}{2} - 2 = J - 2$. But since $c \neq 0$ implies that $A \leq J$, then $T > d_\omega$ is impossible, resulting in a contradiction.

If φ belongs to case(i), then after a linear change of coordinates and rescaling, $\varphi = z_1^J + cz_1^{J-1}z_2 + \mathcal{O}(z_2^2)$, $c \neq 0$, and $\omega = Cz_1^M z_2^T + \mathcal{O}(z_2^{T+1})$, some $M, C \neq 0$. Then $T > d_\omega = 2d_h - 2 = 2\frac{J}{2} - 2 = J - 2$, so $T \geq J - 1$. Thus, $\omega = \mathcal{O}(z_2^{J-1})$, and by Lemma 2.1.7, $\omega \equiv 0$ and $\varphi = (z_1 + \frac{c}{J}z_2)^J$, so φ violates the hypotheses of Theorem 1.2.4. □

Remark 2.1.10. The argument for case(A) in Proposition 2.1.9 also implies that if $\varphi = z_1^J + cz_1^{J-ls}z_2^{ls} + \mathcal{O}(z_2^{ls+1})$, with $ls \geq 2$, and z_2 has multiplicity $T = d_\omega$ in ω, then $J = ls$ and $\varphi = z_1^J + cz_2^J$, some $c \neq 0$.

We will now complete the proof of Proposition 1.2.5.

Proposition 2.1.11. *Theorems 1.1.3 and 1.2.4 are equivalent.*

We begin with three technical lemmas.

Lemma 2.1.12. *The Newton distance, $d(\varphi)$, equals* $\max\{\nu, d_h\}$.

Proof of Lemma 2.1.12. First, the Newton diagram of a mixed homogeneous polynomial will have at most 3 edges: a horizontal edge $y = \tilde{\nu}_2$ corresponding to the multiplicity of z_2 in φ, a vertical edge $x = \tilde{\nu}_1$ corresponding to the multiplicity of z_1 in φ, and an edge corresponding to the mixed homogeneity of φ, which if extended into an infinite line, would intersect the bisectrix at (d_h, d_h). (One can check that $\varphi(z_1, z_2) + z_1^{d_h} z_2^{d_h}$ is still mixed homogeneous to verify this.) Thus, since the bisectrix intersects the Newton diagram at $(d(\varphi), d(\varphi))$, then $d(\varphi) = \max\{\tilde{\nu}_1, \tilde{\nu}_2, d_h\}$.

If $\tilde{\nu}_1 > d_h$, then by Lemmas 2.1.3 and 2.1.5, z_1 has multiplicity greater than d_ω in ω, so by our definitions, $\nu = \tilde{\nu}_1$. Similarly for $\tilde{\nu}_2$. Likewise, if $\nu > d_h$, then after linearly adapting φ as is assumed by Theorem 1.1.3, ν will be the multiplicity of either z_1 or z_2 in φ, so $\nu = \tilde{\nu}_1$ or $\nu = \tilde{\nu}_2$. Thus, $d(\varphi) = \max\{\tilde{\nu}_1, \tilde{\nu}_2, d_h\} = \max\{\nu, d_h\}$. □

Lemma 2.1.13. *If* $\max\{A, d(\varphi_R)\} > 2d(\varphi)$*, then* $A = d(\varphi_R)$*.*

Proof of Lemma 2.1.13. Suppose $d(\varphi_R) > 2d(\varphi)$. Then, since $d(\varphi_R) > d(\varphi)$, and therefore by Lemma 2.1.12 $d(\varphi_R) > d_h$, the bisectrix must intersect the Newton diagram of φ_R on a vertical or horizontal edge, and so we have that $\varphi = c_1 z_1^J + c_2 z_1^K z_2^{d(\varphi_R)} + o(z_2^{d(\varphi_R)})$, some $J \geq 1, K \geq 0$, $c_1, c_2 \neq 0$, up to a swapping of z_1 and z_2. Then, by Lemma 2.1.2 and Lemma 2.1.5, since $d(\varphi_R) > 2d_h$, z_2 will have multiplicity greater than d_ω as a factor of ω, so by the definition of A, $d(\varphi_R) = A$.

Similarly, suppose $A > 2d(\varphi)$. Then by Lemma 2.1.12, $A > 2d_h$, so by Lemma 2.1.5, $A - 2 > d_\omega$. Since $A \neq 0$, then after a possible swap of z_1 and z_2 and rescaling, $\varphi(z) = z_1^J + c z_1^K f_T^A + o(f_T^A)$, some $J \geq 1, K \geq 0, c \neq 0$ by the definition of A, where $f_T \neq z_2$ is linear. Then, after a linear transformation, we can use Lemma 2.1.2 to find that f_T has multiplicity greater than d_ω in ω. Then φ belongs to Case(A), so by Lemma 2.1.9, φ cannot be homogeneous, so $f_T = z_2$ up to a constant. Then, $\varphi_{R2} = z_1^K z_2^A + o(z_2^A)$, so since $A > d_h$, the bisectrix will intersect the Newton diagram of φ_{R2} on a vertical or horizontal edge, at (A, A), implying that $d(\varphi_{R2}) = A$. Thus, $d(\varphi_R) \geq A > 2d(\varphi)$, so by the first part of the proof, $d(\varphi_R) = A$. □

Lemma 2.1.14. *If* $\max\{N, h(\varphi)\} > d(\varphi) + \frac{1}{2}$*, then* $N = d(\varphi)$*.*

Proof of Lemma 2.1.14. Suppose $h(\varphi) > d(\varphi) + \frac{1}{2}$. Since $h(\varphi) = \max\{d(\varphi), o(\varphi)\}$, this implies that $h(\varphi) = o(\varphi)$, the maximal multiplicity of the real irreducible factors of φ. Since all the linear factors of φ have multiplicity not exceeding ν, and therefore not exceeding $d(\varphi)$ by Lemma 2.1.12, any factor associated with $o(\varphi)$ must be nonlinear.

Since $d(\varphi) \geq d_h$ by Lemma 2.1.12, then by Lemma 2.1.5, $h(\varphi) > \frac{d_\omega+3}{2}$. Since $\nabla\varphi(0) = 0$ and $\varphi(0) = 0$ imply that $d(\varphi) > \frac{1}{2}$, then $o(\varphi) = h(\varphi) > 1$. Additionally, since $d(\varphi) \geq d_h$, then $o(\varphi) > d_h$, so by Proposition 1.2.3, $\min(r, s) = 1$. Therefore, Lemma 2.1.1 can be applied to φ with $h(\varphi) = N'$, implying that the factor associated with $h(\varphi)$ will have multiplicity greater than d_ω in ω, so by the definition of N, $h(\varphi) = o(\varphi) = N$.

Similarly, suppose $N > d(\varphi)+\frac{1}{2}$. Since $N > d_h$ by Lemma 2.1.12, then by Proposition 1.2.3, N is the highest multiplicity of any real irreducible factor of φ, so $N = o(\varphi) = h(\varphi)$. □

Now we turn to the proof of Proposition 2.1.11 by proving Proposition 1.2.5:

Proof of Proposition 1.2.5. Inequality (1.2.3) is equivalent to the validity of (1.1.1) and (1.1.2). By Lemma 2.1.12, the validity of (1.2.4) and (1.2.5) is equivalent to (1.1.3).

If $\max\{A, d(\varphi_R)\} \leq 2d(\varphi)$, then by Lemma 2.1.12, (1.2.6) follows from (1.2.3), (1.2.4), and (1.2.5), while (1.1.4) follows from (1.1.1), (1.1.2), and (1.1.3). If $\max\{A, d(\varphi_R)\} > 2d(\varphi)$, then by Lemma 2.1.13, $A = d(\varphi_R)$, in which case (1.2.6) is equivalent to (1.1.4).

If $\max\{N, h(\varphi)\} \leq d(\varphi) + \frac{1}{2}$, then inequalities (1.2.3), (1.2.4), and (1.2.5) imply (1.2.7) and (1.2.8), while (1.1.1), (1.1.2), and (1.1.3) imply (1.1.5) and (1.1.6). By Lemma 2.1.14, if $\max\{N, h(\varphi)\} > d(\varphi) + \frac{1}{2}$, (1.2.7) is equivalent to (1.1.5) and (1.2.8) is equivalent to (1.1.6).

This completes the proof of Proposition 2.1.11, and likewise Proposition 1.2.5. □

2.2 Necessary Conditions

The necessity of each bound in Theorems 1.2.4 and 1.1.3 was proven in [4] for the strictly mixed homogeneous cases, with proofs that can also be applied in the homogeneous cases, and by [5] in the homogeneous cases. For completeness, we will include similar proofs here.

Necessity of $q \geq p$: Let $E = [-3K, 3K]^3$ and $F = [-K, K]^3$, where $K >> 1$. Then $|E| \approx |F| \approx K^3$, and on χ_F, $\mathcal{T}\chi_E \approx 1$. Then $\mathcal{T}$ being of rwt (p, q) requires that

$$1 \approx \text{avg}_F \mathcal{T}\chi_E \lesssim |E|^{\frac{1}{p}}|F|^{-\frac{1}{q}} \approx K^{3(\frac{1}{p}-\frac{1}{q})},$$

implying $1 \lesssim K^{3(\frac{1}{p}-\frac{1}{q})}$. Since K can be taken to be arbitrarily large, this implies that $0 \leq \frac{1}{p} - \frac{1}{q}$, which simplifies to

$$q \geq p.$$

Necessity of $q \leq 3p$: (Note: By duality this will also prove optimality of the line $\frac{1}{q} = \frac{3}{p} - 2$.)

Let $E = [-\frac{1}{4}, \frac{1}{4}]^2 \times [\varphi - C\epsilon, \varphi + C\epsilon]$ and $F = [-\epsilon, \epsilon]^3$, where $\epsilon << 1$. Then $|E| \approx \epsilon$, $|F| \approx \epsilon^3$, and since $|\nabla\varphi|$ is bounded on $[-1, 1]^2$, there exists a C large enough, independent of ϵ, so that on χ_F, $\mathcal{T}\chi_E \approx 1$. Then $\mathcal{T}$ being of rwt (p, q) requires that

$$1 \approx \text{avg}_F \mathcal{T}\chi_E \lesssim |E|^{\frac{1}{p}}|F|^{-\frac{1}{q}} \approx \epsilon^{\frac{1}{p}-\frac{3}{q}},$$

implying $1 \lesssim \epsilon^{\frac{1}{p}-\frac{3}{q}}$. Since ϵ can be taken to be arbitrarily small, this implies that $0 \geq \frac{1}{p} - \frac{3}{q}$, which after simplifying becomes

$$q \leq 3p.$$

Necessity of the Scaling line $\frac{1}{q} \geq \frac{1}{p} - \frac{1}{d_h+1}$: Let S be a hypersurface in $\mathbb{R}^3$, and S_R be the portion of the hypersurface over region $R \in \mathbb{R}^2$. Denote $\mathcal{T}_R$ as the averaging with hypersurface S_R. By the structure of $\mathcal{T}$, if $R' \supset R$, then $\mathcal{T}_{R'}\chi_E \geq \mathcal{T}_R\chi_E$, so the (strong or restricted-weak-type) bound of $\mathcal{T}$ cannot increase as R increases.

Denote σI as $[-\sigma^{\kappa_1}, \sigma^{\kappa_1}] \times [-\sigma^{\kappa_2}, \sigma^{\kappa_2}]$. Then by the upcoming Lemma 2.3.3 (letting $\|\mathcal{T}\|_{p,q}$ denote either the strong or restricted-weak-type bound at $(\frac{1}{p}, \frac{1}{q})$),

$$\|\mathcal{T}_{\sigma I}\|_{p,q} = \sigma^{\frac{d_h+1}{d_h}(\frac{1}{q} - \frac{1}{p} + \frac{1}{d_h+1})} \|\mathcal{T}_{[-1,1]^2}\|_{p,q}.$$

Then, since $\sigma I \supset [-1,1]^2$ for $\sigma > 1$, we need $\sigma^{\frac{d_h+1}{d_h}(\frac{1}{q} - \frac{1}{p} + \frac{1}{d_h+1})} \geq 1$ for $\sigma > 1$, implying that boundedness at $(\frac{1}{p}, \frac{1}{q})$ requires

$$\frac{1}{q} \geq \frac{1}{p} - \frac{1}{d_h+1}.$$

Necessity of $\frac{1}{q} \geq \frac{1}{p} - \frac{1}{\nu+1}$ *in Case*(ν): In this case, it suffices to consider $\varphi = z_1^J z_2^\nu + \mathcal{O}(z_2^{\nu+1})$. Then, choose $E = [-3,3] \times [-3\epsilon, 3\epsilon] \times [-3\epsilon^\nu, 3\epsilon^\nu]$ and $F = [-1,1] \times [-\epsilon, \epsilon] \times [-\epsilon^\nu, \epsilon^\nu]$, where $\epsilon << 1$. Then $|E| \approx |F| \approx \epsilon^{\nu+1}$, and on χ_F, $\mathcal{T}\chi_E \approx \epsilon$. Then $\mathcal{T}$ being bounded for $(\frac{1}{p}, \frac{1}{q})$ requires that

$$\epsilon \approx \text{avg}_F \mathcal{T}\chi_E \lesssim |E|^{\frac{1}{p}} |F|^{-\frac{1}{q}} \approx \epsilon^{(\nu+1)(\frac{1}{p} - \frac{1}{q})},$$

implying $\epsilon \lesssim \epsilon^{(\nu+1)(\frac{1}{p} - \frac{1}{q})}$. Since ϵ can be taken to be arbitrarily small, this implies that $1 \geq (\nu+1)(\frac{1}{p} - \frac{1}{q})$, or after simplifying,

$$\frac{1}{q} \geq \frac{1}{p} - \frac{1}{\nu+1}.$$

Necessity of $\frac{1}{q} \geq \frac{1}{p} - \frac{1}{N}$ *in Case(N)*: In this case, by Prop 1.2.3, we can assume $s = 1$ and write $\varphi = z_1^J (z_2 - \lambda z_1^r)^N + \mathcal{O}((z_2 - \lambda z_1^r)^{N+1})$. Then, choose $E = [-3,3] \times [-3\lambda, 3\lambda] \times$

$[-3\epsilon^N, 3\epsilon^N]$ and $F = [-1,1] \times [-\lambda, \lambda] \times [-\epsilon^N, \epsilon^N]$, where $\epsilon << 1$. Then $|E| \approx |F| \approx \epsilon^N$, and on χ_F, $\mathcal{T}\chi_E \approx \epsilon$. Then $\mathcal{T}$ being bounded for $(\frac{1}{p}, \frac{1}{q})$ requires that

$$\epsilon \approx \text{avg}_F \mathcal{T}\chi_E \lesssim |E|^{\frac{1}{p}}|F|^{-\frac{1}{q}} \approx \epsilon^{N(\frac{1}{p}-\frac{1}{q})},$$

implying $\epsilon \lesssim \epsilon^{N(\frac{1}{p}-\frac{1}{q})}$. Since ϵ can be taken to be arbitrarily small, this implies that $1 \geq N(\frac{1}{p} - \frac{1}{q})$, or after simplifying,

$$\tfrac{1}{q} \geq \tfrac{1}{p} - \tfrac{1}{N}.$$

Necessity of $\frac{1}{q} \geq \frac{N+1}{N+2}\frac{1}{p} - \frac{1}{N+2}$ *in Case(N)*: (By duality the necessity of $\frac{1}{q} \geq \frac{N+2}{N+1}\frac{1}{p} - \frac{2}{N+1}$ follows.) In this case, by Prop 1.2.3, we can assume $s = 1$ and write $\varphi = z_1^J(z_2 - \lambda z_1^r)^N + \mathcal{O}((z_2 - \lambda z_1^r)^{N+1})$. Then, choose $E = [\frac{1}{2}, 1] \times [\lambda z_1^r - 3\epsilon, \lambda z_1^r + 3\epsilon] \times [-3\epsilon^N, 3\epsilon^N]$ and $F = [-\epsilon, \epsilon]^2 \times [-\epsilon^N, \epsilon^N]$, where $\epsilon << 1$. Then $|E| \approx \epsilon^{N+1}$, $|F| \approx \epsilon^{N+2}$, and on χ_F, $\mathcal{T}\chi_E \approx \epsilon$. Then $\mathcal{T}$ being bounded for $(\frac{1}{p}, \frac{1}{q})$ requires that

$$\epsilon \approx \text{avg}_F \mathcal{T}\chi_E \lesssim |E|^{\frac{1}{p}}|F|^{-\frac{1}{q}} \approx \epsilon^{(N+1)\frac{1}{p}-(N+2)\frac{1}{q}},$$

implying $\epsilon \lesssim \epsilon^{(N+1)\frac{1}{p}-(N+2)\frac{1}{q}}$. Since ϵ can be taken to be arbitrarily small, this implies that $1 \geq (N+1)\frac{1}{p} - (N+2)\frac{1}{q}$, or

$$\tfrac{1}{q} \geq \tfrac{N+1}{N+2}\tfrac{1}{p} - \tfrac{1}{N+2}.$$

Necessity of $\frac{1}{q} \geq \frac{A+1}{2A+1}\frac{1}{p} - \frac{1}{2A+1}$ *in Case(A)*: (By duality the necessity of $\frac{1}{q} \geq \frac{2A+1}{A+1}\frac{1}{p} - 1$ follows.) In this case, it suffices to consider $\varphi = z_1^J + z_1^{J-lr}z_2^A + \mathcal{O}(z_2^{A+1})$, $A \geq 2$. Then, choose $E = [\frac{1}{2}, 1] \times [-3\epsilon, 3\epsilon] \times [z_1^J + z_1^{J-lr}z_2^A - 3\epsilon^A, z_1^J + z_1^{J-lr}z_2^A + 3\epsilon^A]$ and $F = [-\epsilon^A, \epsilon^A] \times [-\epsilon, \epsilon] \times [-\epsilon^A, \epsilon^A]$, where $\epsilon << 1$. Then $|E| \approx \epsilon^{A+1}$, $|F| \approx \epsilon^{2A+1}$, and on χ_F, $\mathcal{T}\chi_E \approx \epsilon$. Then $\mathcal{T}$ being bounded for $(\frac{1}{p}, \frac{1}{q})$ requires that

$$\epsilon \approx \text{avg}_F \mathcal{T}\chi_E \lesssim |E|^{\frac{1}{p}}|F|^{-\frac{1}{q}} \approx \epsilon^{(A+1)\frac{1}{p}-(2A+1)\frac{1}{q}},$$

implying $\epsilon \lesssim \epsilon^{(A+1)\frac{1}{p}-(2A+1)\frac{1}{q}}$. Since ϵ can be taken to be arbitrarily small, this implies that $1 \geq (A+1)\frac{1}{p} - (2A+1)\frac{1}{q}$, or

$$\tfrac{1}{q} \geq \tfrac{A+1}{2A+1}\tfrac{1}{p} - \tfrac{1}{2A+1}. \qquad \square$$

2.3 Scaling Symmetries

When a vertex of the polygon arising in Theorem 1.2.4 occurs on the scaling line $\frac{1}{q} = \frac{1}{p} - \frac{1}{d_h+1}$, the mixed-homogeneity of φ and scale-invariance of $\mathcal{T}$ can be exploited as follows.

Let $R \subset \mathbb{R}^2$, $f : \mathbb{R}^3 \to \mathbb{R}$, $\boldsymbol{\kappa} = \boldsymbol{\kappa_\varphi}$, and $\sigma > 0$. We will use notation

$$R_{\boldsymbol{\kappa},\sigma} := \{(\sigma^{\kappa_1}z_1, \sigma^{\kappa_2}z_2)|(z_1, z_2) \in R\}, \qquad f_\sigma(\cdot,\cdot,\cdot) := f(\sigma^{\kappa_1}\cdot, \sigma^{\kappa_2}\cdot, \sigma\cdot). \tag{2.3.1}$$

Definition 2.3.1. Let $\boldsymbol{\kappa} = (\kappa_1, \kappa_2) \in (0,\infty)^2$, and let $\psi : \mathbb{R}^2 \to \mathbb{R}$. We say ψ is $\boldsymbol{\kappa}$-*mixed homogeneous* if, for every $\sigma > 0$, $\psi(\sigma^{\kappa_1}\cdot, \sigma^{\kappa_2}\cdot) = \sigma\psi(\cdot,\cdot)$.

Definition 2.3.2. $\mathcal{R} \subset \mathbb{R}^2$ is $\boldsymbol{\kappa}$-*scale invariant* if $\mathcal{R} = \mathcal{R}_{\kappa,\sigma}$ for all $\sigma > 0$.

Lemma 2.3.3. *Let $R \subset \mathbb{R}^2$, let $\boldsymbol{\kappa} = \boldsymbol{\kappa_\varphi}$, and let $\sigma > 0$. Let $\|\cdot\|_{p,q}$ refer to either the strong-type bound $\|\cdot\|_{L^p\to L^q}$ or the restricted-weak-type bound $\|\cdot\|_{L^{p,1}\to L^{q,\infty}}$. If $\|\mathcal{T}_R\|_{p,q} < \infty$, then $\|\mathcal{T}_{R_{\boldsymbol{\kappa},\sigma}}\|_{p,q} = \sigma^{(\kappa_1+\kappa_2)[1+\frac{1}{q}-\frac{1}{p}]+[\frac{1}{q}-\frac{1}{p}]}\|\mathcal{T}_R\|_{p,q}$.*

Proof.

$$\begin{aligned}
\mathcal{T}_R f_\sigma &:= T_R[f(\sigma^{\kappa_1}\cdot, \sigma^{\kappa_2}\cdot, \sigma\cdot)] \\
&= \int_R f(\sigma^{\kappa_1}x_1 - (\sigma^{\kappa_1}t_1), \sigma^{\kappa_2}x_2 - (\sigma^{\kappa_2}t_2), \sigma x_3 - \varphi(\sigma^{\kappa_1}t_1, \sigma^{\kappa_2}t_2))dt_1dt_2 \\
&= \sigma^{-(\kappa_1+\kappa_2)}\int_{R_{\boldsymbol{\kappa},\sigma}} f(\sigma^{\kappa_1}x_1 - u_1, \sigma^{\kappa_2}x_2 - u_2, \sigma x_3 - \varphi(u_1,u_2))du_1du_2
\end{aligned}$$

$$= \sigma^{-(\kappa_1+\kappa_2)}(\mathcal{T}_{R_{\boldsymbol{\kappa},\sigma}} f)(\sigma^{\kappa_1}x_1, \sigma^{\kappa_2}x_2, \sigma x_3) = \sigma^{-(\kappa_1+\kappa_2)}(\mathcal{T}_{R_{\boldsymbol{\kappa},\sigma}} f)_\sigma.$$

Then, by scaling,

$$\begin{aligned}
\sigma^{-(\kappa_1+\kappa_2)}\sigma^{-\frac{(\kappa_1+\kappa_2+1)}{q}}\|\mathcal{T}_{R_{\boldsymbol{\kappa},\sigma}} f\|_{L^q(\mathbb{R}^3)} &= \sigma^{-(\kappa_1+\kappa_2)}\|(T_{R_{\boldsymbol{\kappa},\sigma}} f)_\sigma\|_{L^q(\mathbb{R}^3)} \\
&= \|T_R(f_\sigma)\|_{L^q(\mathbb{R}^3)} \\
&\leq \|\mathcal{T}_R\|_{L^p\to L^q}\|f_\sigma\|_{L^p(\mathbb{R}^3)} \\
&= \sigma^{-\frac{(\kappa_1+\kappa_2+1)}{p}}\|\mathcal{T}_R\|_{L^p\to L^q}\|f\|_{L^p(\mathbb{R}^3)},
\end{aligned}$$

and by looking at f that are near-extremal, we get, for $\|\mathcal{T}_R\|_{L^p\to L^q} < \infty$,

$$\begin{aligned}
\|\mathcal{T}_{R_{\boldsymbol{\kappa},\sigma}}\|_{L^p\to L^q} &= \sigma^{(\kappa_1+\kappa_2)[1+\frac{1}{q}-\frac{1}{p}]+[\frac{1}{q}-\frac{1}{p}]}\|\mathcal{T}_R\|_{L^p\to L^q} \\
&= \sigma^{\frac{1}{d_h}[1+\frac{1}{q}-\frac{1}{p}]+[\frac{1}{q}-\frac{1}{p}]}\|\mathcal{T}_R\|_{L^p\to L^q} \\
&= \sigma^{\frac{d_h+1}{d_h}(\frac{1}{q}-\frac{1}{p}+\frac{1}{d_h+1})}\|\mathcal{T}_R\|_{L^p\to L^q}.
\end{aligned}$$

This proves Lemma 2.3.3 for strong-type bounds. Replacing $\|\mathcal{T}f\|_{L^q(\mathbb{R}^3)}$ with $\|\mathcal{T}f\|_{L^q_w(\mathbb{R}^3)}$, and f with χ_E, we get an identical result for restricted-weak-type bounds. □

Proposition 2.3.4. *Let* $\boldsymbol{\kappa} = \boldsymbol{\kappa}_{\boldsymbol{\varphi}}$. *Let* $\mathcal{R}$ *be* $\boldsymbol{\kappa}$*-scale invariant, and let* ψ *be* $\frac{\boldsymbol{\kappa}}{D}$*-mixed homogeneous, some* $D > 0$. *Suppose that* $\frac{1}{q_S} = \frac{1}{p_S} - \frac{1}{d_h+1}$ *and that* $(\frac{1}{p_S}, \frac{1}{q_S}) = (1-\theta)(\frac{1}{p_0}, \frac{1}{q_0}) + \theta(\frac{1}{p_I}, \frac{1}{q_I})$, *for some* $\theta \in (0,1)$, *for some* (p_0, q_0) *and* (p_I, q_I) *where* $\frac{1}{q_0} \neq \frac{1}{p_0} - \frac{1}{d_h+1}$ *and* $\frac{1}{q_I} \neq \frac{1}{p_I} - \frac{1}{d_h+1}$. *If* $\mathcal{T}_{\mathcal{R}\bigcap\{|\psi|\approx 1\}}$ *is of rwt* (p_0, q_0) *and* (p_I, q_I), *then* $\mathcal{T}_{\mathcal{R}}$ *is of rwt* (p_S, q_S).

Proof. It suffices to consider $\frac{1}{q_0} < \frac{1}{p_0} - \frac{1}{d_h+1}$ and $\frac{1}{q_I} > \frac{1}{p_I} - \frac{1}{d_h+1}$. Let $\mathcal{T}_j := \mathcal{T}_{\mathcal{R}\bigcap\{|\psi|^{-1}([2^{jD}, 2^{(j+1)D}))\}}$. Then $\mathcal{T}_j = \mathcal{T}_{\mathcal{R}\bigcap\{|\psi|^{-1}([1,2^D))\}_{\boldsymbol{\kappa},2^j}}$, by the scale-invariance of $\mathcal{R}$ and the mixed-homogeneity of ψ. Thus, by Lemma 2.3.3,

$$\mathcal{T}_j(E,F) \lesssim \min\{2^{j\frac{d_h+1}{d_h}(\frac{1}{q_0}-\frac{1}{p_0}+\frac{1}{d_h+1})}|E|^{\frac{1}{p_0}}|F|^{1-\frac{1}{q_0}}, 2^{j\frac{d_h+1}{d_h}(\frac{1}{q_I}-\frac{1}{p_I}+\frac{1}{d_h+1})}|E|^{\frac{1}{p_I}}|F|^{1-\frac{1}{q_I}}\}.$$

Then, interpolating,

$$\mathcal{T}_{\mathcal{R}}(E,F) \lesssim \sum_j \min\{2^{j\frac{d_h+1}{d_h}(\frac{1}{q_0}-\frac{1}{p_0}+\frac{1}{d_h+1})}|E|^{\frac{1}{p_0}}|F|^{1-\frac{1}{q_0}}, 2^{j\frac{d_h+1}{d_h}(\frac{1}{q_I}-\frac{1}{p_I}+\frac{1}{d_h+1})}|E|^{\frac{1}{p_I}}|F|^{1-\frac{1}{q_I}}\}.$$

$$\approx |E|^{\frac{(\frac{1}{p_0}-\frac{1}{p_I})\frac{1}{d_h+1}+\frac{1}{p_0q_I}-\frac{1}{q_0p_I}}{(\frac{1}{q_I}-\frac{1}{p_I})-(\frac{1}{q_0}-\frac{1}{p_0})}}|F|^{1-\frac{(\frac{1}{q_0}-\frac{1}{q_I})\frac{1}{d_h+1}+\frac{1}{q_Ip_0}-\frac{1}{q_0p_I}}{(\frac{1}{q_I}-\frac{1}{p_I})-(\frac{1}{q_0}-\frac{1}{p_0})}}.$$

Thus, $\mathcal{T}_{\mathcal{R}}$ is of rwt (p_S, q_S) for

$$(\frac{1}{p_S},\frac{1}{q_S}) = (\frac{(\frac{1}{p_0}-\frac{1}{p_I})\frac{1}{d_h+1}+\frac{1}{p_0q_I}-\frac{1}{q_0p_I}}{(\frac{1}{q_I}-\frac{1}{p_I})-(\frac{1}{q_0}-\frac{1}{p_0})}, \frac{(\frac{1}{q_0}-\frac{1}{q_I})\frac{1}{d_h+1}+\frac{1}{q_Ip_0}-\frac{1}{q_0p_I}}{(\frac{1}{q_I}-\frac{1}{p_I})-(\frac{1}{q_0}-\frac{1}{p_0})}), \tag{2.3.2}$$

the point on the scaling line $\frac{1}{q}=\frac{1}{p}-\frac{1}{d_h+1}$ directly between $(\frac{1}{p_0},\frac{1}{q_0})$ and $(\frac{1}{p_I},\frac{1}{q_I})$. □

There are three cases where we will use this result:

<u>Case 1</u>: $(\frac{1}{p_0},\frac{1}{q_0}) = (\frac{3}{4},\frac{1}{4})$, and $(\frac{1}{p_I},\frac{1}{q_I})$ satisfies $\frac{1}{p_I}=\frac{3}{q_I}$.

In this case, the point $(\frac{1}{p_S},\frac{1}{q_S})$ is the intersection of the lines $\frac{1}{p}=\frac{3}{q}$ and scaling line $\frac{1}{q}=\frac{1}{p}-\frac{1}{d_h+1}$, namely $(\frac{1}{p_S},\frac{1}{q_S}) = (\frac{3}{2d_h+2},\frac{1}{2d_h+2}) = (\frac{3}{d_\omega+4},\frac{1}{d_\omega+4})$, using the fact that $d_\omega = 2d_h-2$.

<u>Case 2</u>: $(\frac{1}{p_0},\frac{1}{q_0}) = (\frac{2}{3},\frac{1}{3})$, and $(\frac{1}{p_I},\frac{1}{q_I})$ satisfies $\frac{1}{p_I}=\frac{2}{q_I}$.

In this case, the point $(\frac{1}{p_S},\frac{1}{q_S})$ is the intersection of the lines $\frac{1}{p}=\frac{2}{q}$ and scaling line $\frac{1}{q}=\frac{1}{p}-\frac{1}{d_h+1}$, namely $(\frac{1}{p_S},\frac{1}{q_S}) = (\frac{2}{d_h+1},\frac{1}{d_h+1})$.

<u>Case 3</u>: $(\frac{1}{p_0},\frac{1}{q_0}) = (\frac{3}{4},\frac{1}{4})$, and $(\frac{1}{p_I},\frac{1}{q_I})$ satisfies $\frac{1}{p_I}=\frac{2}{q_I}$.

Here, by simplifying (2.3.2), $(\frac{1}{p_S},\frac{1}{q_S})$ simplifies to

$$(\frac{2}{p_S},\frac{2}{q_S}) = (\frac{3-\frac{8}{q_I}+\frac{1}{q_I}(d_h+1)}{(1-\frac{2}{q_I})(d_h+1)}, \frac{1-\frac{4}{q_I}+\frac{1}{q_I}(d_h+1)}{(1-\frac{2}{q_I})(d_h+1)}).$$

Additionally, using identity $d_\omega = 2d_h - 2$, we can rewrite $\frac{1}{p_S}$ as

$$\frac{1}{p_S} = \frac{3 - \frac{8}{q_I} + \frac{1}{2}\frac{1}{q_I}(d_\omega + 4)}{(1 - \frac{2}{q_I})(d_\omega + 4)}. \tag{2.3.3}$$

2.4 Decomposition

Our goal in this section will be to decompose $[-1,1]^2$, and hence our operator $\mathcal{T}_{[-1,1]^2}$, around the factors of the determinant Hessian ω. Since ω is mixed homogeneous, with homogeneous distance $d_\omega = 2d_h - 2$,

$$\omega(z_1, z_2) = Cz_1^{\nu_1} z_2^{\nu_2} \prod_{j=3}^{M_2} (z_2^s - \lambda_j z_1^r)^{n_j}, \text{ with } \lambda_j \text{ real iff } j \le M_1, \text{ some } M_1 \le M_2.$$

Let $\tilde{\epsilon} > 0$ be sufficiently small. We use the following covering of $[-1,1]^2$:

$R_j := [-1,1]^2 \cap \{|z_2^s - \lambda_j z_1^r| < \tilde{\epsilon}|z_1|^r\}, j = 3, ..., M_1$ (each real λ_j);

$R_2 := [-1,1]^2 \cap \{|z_2|^s < \tilde{\epsilon}|z_1|^r\}$ if $\nu_2 \neq 0$, and $R_2 := \emptyset$ otherwise;

$R_1 := [-1,1]^2 \cap \{|z_1|^r < \tilde{\epsilon}|z_2|^s\}$ if $\nu_1 \neq 0$, and $R_1 := \emptyset$ otherwise;

$R_0 := [-1,1]^2 \backslash \{\bigcup_{j=1}^{j=M_1} R_j\}$.

Sometimes, especially when we have a vertex on the scaling line $\frac{1}{q} = \frac{1}{p} - \frac{1}{d_h+1}$, it will be more useful to have this decomposition extended to all of $\mathbb{R}^2$:

$R_j^e := \{|z_2^s - \lambda_j z_1^r| < \tilde{\epsilon}|z_1|^r\}, j = 3, ..., M_1$ (each real λ_j);

$R_2^e := \{|z_2|^s < \tilde{\epsilon}|z_1|^r\}$ if $\nu_2 \neq 0$, and $R_2^e := \emptyset$ otherwise;

$R_1^e := \{|z_1|^r < \tilde{\epsilon}|z_2|^s\}$ if $\nu_1 \neq 0$, and $R_1^e := \emptyset$ otherwise.

These R_j^e are $\boldsymbol{\kappa_\varphi}$-scale invariant, which will allow us to apply Proposition 2.3.4. By Proposition 1.2.3, when $T > d_\omega$ there exists a unique index j_0 such that either $j_0 \in \{1, 2\}$

and $\nu_{j_0} > d_\omega$ or $j_0 \geq 3$ and $n_{j_0} > d_\omega$. We set

$$R_T := R_{j_0} \quad \text{and} \quad R_T^e := R_{j_0}^e.$$

when $T > d_\omega$. When $T \leq d_\omega$, we set $R_T := \emptyset$ and $R_T^e := \emptyset$.

Chapter 3

First Relevant Vertex

3.1 First relevant vertex, Initial observations

Definition 3.1.1. A *relevant vertex* is a vertex $(\frac{1}{p}, \frac{1}{q})$ of the polygon described in Theorem 1.2.4 that satisfies $q' \leq p < q$.

To prove Theorem 1.2.4, by Young's Inequality, duality, and real interpolation, it suffices to prove that $\mathcal{T}(E, F) \lesssim |E|^{\frac{1}{p}} |F|^{\frac{1}{q'}}$ for all relevant vertices.

We recall from Section 1.4 that we can divide our problem into three broad cases: the case $T \leq d_\omega$, the rectangular cases, and the twisted cases, and further decompose the rectangular cases into Cases (ν), (A), and (N).

Lemma 3.1.2. *The polygon described in Theorem* 1.2.4 *has exactly one relevant vertex, denoted* $(\frac{1}{p_{v_1}}, \frac{1}{q_{v_1}})$*, that lies on the line* $q = 3p$*. If we are in the twisted cases, case* (ν)*, or if* $T \leq d_\omega$*, this is the only relevant vertex. In Cases* (N) *and* (A)*, there exists exactly one additional relevant vertex* $(\frac{1}{p_{v_2}}, \frac{1}{q_{v_2}})$*, and this additional vertex satisfies* $(\frac{1}{p_{v_2}}, \frac{1}{q_{v_2}}) \in \overline{Conv}\{(0,0), (\frac{3}{4}, \frac{1}{4}), (\frac{2}{3}, \frac{1}{3})\}$.

This lemma follows by simple algebra with the boundaries in Theorem 1.2.4. For the rest of Sections 3.1-3.4, we will focus on the first relevant vertex.

Lemma 3.1.3. *For the polygon described in Theorem 1.2.4,* $(\frac{1}{p_{v_1}}, \frac{1}{q_{v_1}}) = (\frac{3}{d_\omega+4}, \frac{1}{d_\omega+4})$ *in the twisted cases and case* $T \le d_\omega$*. In the rectangular cases,* $(\frac{1}{p_{v_1}}, \frac{1}{q_{v_1}}) = (\frac{3}{T+4}, \frac{1}{T+4})$*.*

Proof. In the case $T \le d_\omega$ and the twisted cases, (p_{v_1}, q_{v_1}) lies on the intersection of the curves $q = 3p$ and $\frac{1}{q} = \frac{1}{p} - \frac{1}{d_h+1}$, and since $d_\omega = 2d_h - 2$,

$$(\frac{1}{p_{v_1}}, \frac{1}{q_{v_1}}) = (\frac{3}{2d_h+2}, \frac{1}{2d_h+2}) = (\frac{3}{d_\omega+4}, \frac{1}{d_\omega+4}),$$

In Case(ν), (p_{v_1}, q_{v_1}) lies on the intersection of the curves $q = 3p$ and $\frac{1}{q} = \frac{1}{p} - \frac{1}{\nu+1}$, and since $T = 2\nu - 2$,

$$(\frac{1}{p_{v_1}}, \frac{1}{q_{v_1}}) = (\frac{3}{2\nu+2}, \frac{1}{2\nu+2}) = (\frac{3}{T+4}, \frac{1}{T+4}),$$

In Case(A), (p_{v_1}, q_{v_1}) lies on the intersection of the curves $q = 3p$ and $\frac{1}{q} = \frac{A+1}{2A+1}\frac{1}{p} - \frac{1}{2A+1}$, and since $T = A - 2$,

$$(\frac{1}{p_{v_1}}, \frac{1}{q_{v_1}}) = (\frac{3}{A+2}, \frac{1}{A+2}) = (\frac{3}{T+4}, \frac{1}{T+4}),$$

In Case(N), (p_{v_1}, q_{v_1}) lies on the intersection of the curves $q = 3p$ and $\frac{1}{q} = \frac{N+1}{N+2}\frac{1}{p} - \frac{1}{N+2}$, and since $T = 2N - 3$,

$$(\frac{1}{p_{v_1}}, \frac{1}{q_{v_1}}) = (\frac{3}{2N+1}, \frac{1}{2N+1}) = (\frac{3}{T+4}, \frac{1}{T+4}).$$ □

This lemma leads to the following proposition, whose proof will occupy the next 3 sections:

Proposition 3.1.4. *The operator* $\mathcal{T}_{R_j}$ *is of rwt* $(\frac{d_\omega+4}{3}, d_\omega + 4)$ *when* $j = 0$ *or when* n_j *(likewise* ν_j*) is less than or equal to* d_ω*. When* n_j *(likewise* ν_j*) is greater than* d_ω*,* $\mathcal{T}_{R_j}$ *is of rwt* $(\frac{T+4}{3}, T + 4)$*.*

Together, Proposition 3.1.4 and Lemma 3.1.3 imply the following corollary:

Corollary 3.1.5. *The operator $\mathcal{T}_{[-1,1]^2\setminus R_T}$ is of rwt (p,q) for $(\frac{1}{p},\frac{1}{q})$ lying in the polygon of Theorem 1.2.4. Additionally, in the rectangular cases, $\mathcal{T}_{R_T}$ is of rwt (p_{v_1},q_{v_1}).*

We will spend the remainder of this section proving Proposition 3.1.4 in all cases except when n_j or ν_j equals d_ω. The remaining cases will be handled in Sections 3.2 and 3.3.

The following lemma will allow us to use Hölder's Inequality to compute $L^\infty \to L^\infty$ bounds, which will be useful in interpolation.

Lemma 3.1.6. *Let μ be the standard Lebesgue measure on $\mathbb{R}^2$. The regions in the dyadic decomposition of ω satisfy the following inequalities:*

- $\mu(R_0 \cap \{|\omega| \approx 2^{-m}\}) \lesssim_{\tilde{\epsilon}} 2^{-\frac{m}{d_\omega}}$.
- $\mu(R_i \cap \{|\omega| \approx 2^{-m}\}) \lesssim_{\tilde{\epsilon}} 2^{-\frac{m}{\max(\nu_i,d_\omega)}}$ *for $i = 1,2$, whenever $\nu_i \neq d_\omega$.*
- $\mu(R_i \cap \{|\omega| \approx 2^{-m}\}) \lesssim_{\tilde{\epsilon}} 2^{-\frac{m}{\max(n_i,d_\omega)}}$ *for $3 \leq i \leq M_1$, whenever $n_i \neq d_\omega$.*

Proof. On R_0, $|z_2 - \lambda_j z_1^r| \sim |z_2|^s \sim |z_1|^r \sim |z_1|^r + |z_2|^s$ for all j, so $|\omega(z_1,z_2)| \gtrsim (|z_1|^r + |z_2|^s)^{\frac{(r+s)d_\omega}{rs}}$. Therefore,

$$\mu(R_0 \cap \{|\omega| \approx 2^{-m}\}) \lesssim_{\tilde{\epsilon}} \mu([-1,1]^2 \cap \{(|z_1|^r + |z_2|^s) \leq 2^{-\frac{m}{d_\omega}\frac{rs}{r+s}}\}) \lesssim_{\tilde{\epsilon}} 2^{-\frac{m}{d_\omega}}.$$

Next, on R_2, we have $|z_2|^s < \tilde{\epsilon}|z_1|^r$, so for $\tilde{\epsilon}$ sufficiently small, $|z_2^s - \lambda_i z_1^r| \sim |z_1|^r$ for all i. Thus, $|\omega| \gtrsim |z_1|^Q |z_2|^{\nu_2}$, for some Q satisfying $\frac{Qs+\nu_2 r}{r+s} = d_\omega$. Therefore, after a few elementary calculus calculations,

$$\mu(R_2 \cap \{|\omega| \approx 2^{-m}) \lesssim_{\tilde{\epsilon}} \mu(R_2 \cap \{|z_1|^Q |z_2|^{\nu_2} \leq 2^{-m}\})$$

$$\lesssim_{\tilde{\epsilon}} \mu(\{|z_2| \leq \min(2^{-\frac{m}{\nu_2}}|z_1|^{-\frac{Q}{\nu_2}}, \tilde{\epsilon}|z_1|^{\frac{r}{s}}), |z_1| \leq 1\})$$
$$\lesssim_{\tilde{\epsilon}} \max(2^{-\frac{m}{\nu_2}}, 2^{-\frac{m(s+r)}{Qs+\nu_2 r}}) = 2^{-\frac{m}{\max(\nu_2, d_\omega)}}$$

as long as $\nu_2 \neq d_\omega$. The case for R_1 follows similarly.

Finally, on R_j, for $j \geq 3$, we have $|z_2^s - \lambda_j z_1^r| < \tilde{\epsilon}|z_1|^r$, so for $\tilde{\epsilon}$ sufficiently small, $|z_2^s - \lambda_i z_1^r| \sim |z_2^s| \sim |z_1^r|$ for all $i \neq j$. Thus, $|\omega| \gtrsim |z_1|^Q|z_2^s - \lambda_j z_1^r|^{n_j}$, for some Q satisfying $\frac{Qs+n_j rs}{r+s} = d_\omega$. Therefore, after a few elementary calculus calculations,

$$\mu(R_j \cap \{|\omega| \approx 2^{-m}\}) \lesssim_{\tilde{\epsilon}} \mu(R_j \cap \{|z_1|^Q|z_2^s - \lambda_j z_1^r|^{n_j} \leq 2^{-m}\})$$
$$\lesssim_{\tilde{\epsilon}} \mu(\{|z_2^s - \lambda_j z_1^r| \leq \min(2^{-\frac{m}{n_j}}|z_1|^{-\frac{Q}{n_j}}, \tilde{\epsilon}|z_1|^r), |z_1| \leq 1\})$$
$$\lesssim_{\tilde{\epsilon}} \max(2^{-\frac{m}{n_j}}, 2^{-\frac{m(s+r)}{Qs+n_j rs}}) = 2^{-\frac{m}{\max(n_j, d_\omega)}}$$

as long as $n_j \neq d_\omega$. □

Proposition 3.1.7. *For $\tilde{\epsilon}$ sufficiently small, the operator $\mathcal{T}_{R_j}$ is of rwt (p,q) when $(\frac{1}{p}, \frac{1}{q})$ equals:*

- $(\frac{3}{d_\omega+4}, \frac{1}{d_\omega+4})$ *for $j = 0$.*
- $(\frac{3}{\max(\nu_j,d_\omega)+4}, \frac{1}{\max(\nu_j,d_\omega)+4})$ *for $j = 1, 2$, and $\nu_j \neq d_\omega$.*
- $(\frac{3}{\max(n_j,d_\omega)+4}, \frac{1}{\max(n_j,d_\omega)+4})$ *for $3 \leq j \leq M_1$, and $n_j \neq d_\omega$.*

Proof. Define $\mathcal{T}_{j,m} := \mathcal{T}_{R_j \bigcap \{|\omega| \approx 2^{-m}\}}$, and denote $h_j := \max(\nu_j, d_\omega)$ for $j \in \{1,2\}$, $h_j := \max(n_j, d_\omega)$ for $j \geq 3$, and $h_0 := d_\omega$. We assume that n_j (likewise ν_j) is not equal to d_ω. By Hölder's Inequality and Lemma 3.1.6, $\|\mathcal{T}_{j,m}\|_{\infty\to\infty} \lesssim 2^{-\frac{m}{h_j}}$. In addition, by Theorem 1.2.1, $\|\mathcal{T}_{j,m}\|_{\frac{4}{3}\to 4} \lesssim 2^{\frac{m}{4}}$. Combining these,

$$\mathcal{T}_{R_j}(E,F) \lesssim \sum_m \min(2^{-\frac{m}{h_j}}|F|, 2^{\frac{m}{4}}|E|^{\frac{3}{4}}|F|^{\frac{3}{4}}) \lesssim |E|^{\frac{3}{h_j+4}}|F|^{1-\frac{1}{h_j+4}}.$$

Thus, $\mathcal{T}_{R_j}$ is of rwt $(\frac{h_j+4}{3}, \frac{h_j+4}{1})$. □

Proposition 3.1.7 implies Proposition 3.1.4 in all cases except when n_j or ν_j equals d_ω. However, if we extend Lemma 3.1.6 to the remaining cases, the measures and thereby the $L^\infty \to L^\infty$ bounds will contain logarithmic terms, which are undesirable for the interpolation performed in Proposition 3.1.7. To avoid this, we will show in Sections 3.2 and 3.3 that for q arbitrarily large, the rwt $(\frac{q}{3}, q)$ bounds lack the extra logarithmic term, by proving the following lemma:

Lemma 3.1.8. *If n_j or ν_j equals d_ω, then $\mathcal{T}_{R_j^e \cap \{|\omega| \approx 1\}}$ is of rwt $(\frac{q}{3}, q)$ for all q satisfying $4 \leq q < \infty$.*

By Proposition 2.3.4 (we are in Case 1, as defined after the proof of that proposition), this lemma implies the following corollary:

Corollary 3.1.9. *If n_j or ν_j equals d_ω, then $\mathcal{T}_{R_j^e}$ is of rwt $(\frac{d_\omega+4}{3}, d_\omega + 4)$.*

Together, Proposition 3.1.7 and Corollary 3.1.9 imply Proposition 3.1.4. In Sections 3.2 and 3.3, we will complete our argument by proving Lemma 3.1.8, with an argument that will have further applications in later sections.

3.2 Application of Method of Refinements

We begin our argument by looking at a general subset S of $\mathbb{R}^2$, which is then subdivided into smaller subsets τ_n. In light of Section 3.1, for now consider S to be the set $R_j^e \cap \{|\omega| \approx 1\}$, where n_j (likewise ν_j) equals d_ω, and consider the sets τ_n to be some sort of dyadic decomposition of S.

Notation 3.2.1. Let τ_n be subsets of $\mathbb{R}^2$, let $S := \bigcup_{n\in\mathbb{N}} \tau_n$, and denote $\mathcal{T}_n := \mathcal{T}_{\tau_n}$. For any $E, F \subset \mathbb{R}^3$, to capture the primary contributors to $\mathcal{T}_n(E,F)$, we will define the following refinements:

$$F_n := \{u \in F : \mathcal{T}_n\chi_E(u) \geq \tfrac{1}{4}\tfrac{\mathcal{T}_n(E,F)}{|F|} = \tfrac{1}{4}\text{avg}_F \mathcal{T}_n\chi_E\};$$

$$E_n := \{w \in E : \mathcal{T}_n^*\chi_{F_n}(w) \geq \tfrac{1}{4}\tfrac{\mathcal{T}_n(E,F_n)}{|E|} = \tfrac{1}{4}\text{avg}_E \mathcal{T}_n^*\chi_{F_n} =: \alpha_{E_n}\};$$

$$\tilde{F}_n := \{u \in F_n : \mathcal{T}_n\chi_{E_n}(u) \geq \tfrac{1}{4}\tfrac{\mathcal{T}_n(E_n,F_n)}{|F_n|} = \tfrac{1}{4}\text{avg}_{F_n} \mathcal{T}_n\chi_{E_n} =: \alpha_{F_n}\};$$

$$F_{kl} := \{u \in F_k : \mathcal{T}_k\chi_{E_k\cap E_l}(u) \geq \tfrac{1}{4}\tfrac{\mathcal{T}_k(E_k\cap E_l,F_k)}{|F_k|} = \tfrac{1}{4}\text{avg}_{F_k} \mathcal{T}_k\chi_{E_k\cap E_l} =: \beta_{F_{kl}}\};$$

$$E_{kl} := \{w \in E_k : \mathcal{T}_k^*\chi_{\tilde{F}_k\cap\tilde{F}_l}(w) \geq \tfrac{1}{4}\tfrac{\mathcal{T}_k(E_k,\tilde{F}_k\cap\tilde{F}_l)}{|E_k|} = \tfrac{1}{4}\text{avg}_{E_k} \mathcal{T}_k^*\chi_{\tilde{F}_k\cap\tilde{F}_l} =: \beta_{E_{kl}}\}.$$

With these refinements,

$$\begin{aligned}\mathcal{T}_n(E,F) &= \int_{F_n} \mathcal{T}_n\chi_E + \int_{F\setminus F_n} \mathcal{T}_n\chi_E \;\leq\; \mathcal{T}_n(E,F_n) + \int_{F\setminus F_n} \tfrac{1}{4}\tfrac{\mathcal{T}_n(E,F)}{|F|} \\ &\leq \mathcal{T}_n(E,F_n) + \tfrac{1}{4}\mathcal{T}_n(E,F),\end{aligned}$$

and consequently $\mathcal{T}_n(E,F_n) \geq \frac{3}{4}\mathcal{T}_n(E,F)$. Likewise, $\mathcal{T}_n(E_n,F_n) \geq \frac{3}{4}\mathcal{T}_n(E,F_n)$ and $\mathcal{T}_n(E_n,\tilde{F}_n) \geq \frac{3}{4}\mathcal{T}_n(E_n,F_n)$, so

$$\mathcal{T}_n(E_n,\tilde{F}_n) \approx \mathcal{T}_n(E,F). \tag{3.2.1}$$

Therefore, E_n and $\tilde{F}_n$ capture the primary contributions of E and F pertaining to $\mathcal{T}_n$. Additionally, let $b \in \{2,3\}$ be fixed, and define $\eta(\epsilon) := \{n \in \mathbb{N} : \mathcal{T}_n(E,F) \approx \epsilon|E|^{\frac{b}{b+1}}|F|^{\frac{b}{b+1}}\}$, for $\epsilon \in 2^{\mathbb{Z}}$. (Note that η depends implicitly on b, E, F.) Pertaining to Section 3.1, we will use $b = 3$, but in a later section, we will reuse these arguments with $b = 2$.

When $b = 3$ and when $S = R_j \cap \{|\omega| \approx 1\}$, the following lemma implies Lemma 3.1.8 if all the hypotheses can be shown to be satisfied.

Lemma 3.2.2. *Let* $\|\mathcal{T}_n\|_{L^{\frac{b}{b+1},1}\to L^{\frac{1}{b+1},\infty}} \leq \Lambda_\varphi \lesssim 1$*, some* Λ_φ *depending only on* φ*, and let* $|\tau_n| \lesssim 1$ *uniformly in* n*. If*

$$(1) \sum_{n\in\eta(\epsilon)} |E_n| \lesssim_\delta \epsilon^{-\delta}|E| \quad \textit{and} \quad (2) \sum_{n\in\eta(\epsilon)} |\tilde{F}_n| \lesssim_\delta \epsilon^{-\delta}|F|$$

for all $\delta > 0$*, uniformly over* E, F, ϵ*, then* $\|\mathcal{T}_S\|_{L^{\frac{\theta b}{b+1},1}\to L^{\frac{\theta}{b+1},\infty}} < \infty$ *for all* $\theta \in (0,1]$*.*

Proof. In what follows, we will introduce an $a \in [0,1]$ to be chosen near the end. In the first line, we use $\|\mathcal{T}_n\|_{\frac{b}{b+1},\frac{1}{b+1}} \lesssim 1$ to imply $\epsilon \lesssim 1$; the second line is due to (3.2.1); the definition of $\eta(\epsilon)$, Young's Inequality using $|\tau_n| \lesssim 1$, and the hypothesis that $\|\mathcal{T}_n\|_{\frac{b}{b+1},\frac{1}{b+1}} \lesssim 1$ give the third line; the fourth line is simple rearrangement; in the fifth line a is chosen so that $1 - a = \frac{b+1}{b+1+(b-1)\theta}$, and Hölder is applied; for the final line, conditions (1) and (2) give the first inequality, and a geometric sum with δ sufficiently small completes the argument:

$$\begin{aligned}
\langle \mathcal{T}_S\chi_E, \chi_F\rangle &= \sum_{\epsilon\in 2^{-\mathbb{N}}} \sum_{n\in\eta(\epsilon)} \langle \mathcal{T}_n\chi_E, \chi_F\rangle \\
&\approx \sum_{\epsilon\lesssim 1} \sum_{n\in\eta(\epsilon)} \langle \mathcal{T}_n\chi_E, \chi_F\rangle^{a\theta} \langle \mathcal{T}_n\chi_E, \chi_F\rangle^{a(1-\theta)} \langle \mathcal{T}_n\chi_{E_n}, \chi_{\tilde{F}_n}\rangle^{1-a} \\
&\lesssim \sum_{\epsilon\lesssim 1} \sum_{n\in\eta(\epsilon)} (\epsilon|E|^{\frac{b}{b+1}}|F|^{\frac{b}{b+1}})^{a\theta} |F|^{a(1-\theta)} (|E_n|^{\frac{b\theta}{b+1}}|\tilde{F}_n|^{1-\frac{\theta}{b+1}})^{1-a} \\
&= \sum_{\epsilon\lesssim 1} \epsilon^{a\theta} (|E|^{\frac{b\theta}{b+1}}|F|^{1-\frac{\theta}{b+1}})^a \sum_{n\in\eta(\epsilon)} (|E_n|^{\frac{b\theta}{b+1}}|\tilde{F}_n|^{(1-\frac{\theta}{b+1})})^{(1-a)} \\
&\leq \sum_{\epsilon\lesssim 1} \epsilon^{a\theta} (|E|^{\frac{b\theta}{b+1}}|F|^{1-\frac{\theta}{b+1}})^a [(\sum_{n\in\eta(\epsilon)} |E_n|)^{\frac{b\theta}{b+1}} (\sum_{n\in\eta(\epsilon)} |\tilde{F}_n|)^{1-\frac{\theta}{b+1}}]^{(1-a)} \\
&\lesssim \sum_{\epsilon\lesssim 1} \epsilon^{a\theta-\delta(1+\theta\frac{b-1}{b+1})(1-a)} |E|^{\frac{b\theta}{b+1}}|F|^{1-\frac{\theta}{b+1}} \lesssim |E|^{\frac{b\theta}{b+1}}|F|^{1-\frac{\theta}{b+1}}
\end{aligned}$$

□

To use Lemma 3.2.2, we will need to show that hypotheses (1) and (2) are satisfied. In Lemma 3.2.4, we will prove that the following condition will suffice:

Condition 3.2.3. *There exist $\tilde{M}, B > 0$, $\xi \in \{-1, 1\}$ independent of E, F such that, for all $k, l \in \mathbb{N}$ satisfying $\max\{k, l, |k-l|\} > \tilde{M}$, with $sgn(k-l) = \xi$, we have $|F_l|^{b-1} \gtrsim 2^{|k-l|B} \alpha_{E_l}^b \beta_{F_{kl}}$ and $|E_l|^{b-1} \gtrsim 2^{|k-l|B} \alpha_{F_l}^b \beta_{E_{kl}}$, with implicit constants independent of l, k, E, F.*

Lemma 3.2.4. *Let all conditions of Lemma 3.2.2 be satisfied save for (1) and (2). If Condition 3.2.3 is satisfied, then (1) and (2) are also satisfied.*

Proof. (Previously done in [3].) Since $\|\mathcal{T}_n\|_{L^{\frac{b}{b+1},1} \to L^{\frac{1}{b+1},\infty}} \leq \Lambda_\varphi \lesssim 1$ uniformly, then by (3.2.1),

$$|E_n|^{\frac{b}{b+1}} |F|^{\frac{b}{b+1}} \gtrsim \mathcal{T}_n(E_n, F) \approx \mathcal{T}_n(E, F) \approx \epsilon |E|^{\frac{b}{b+1}} |F|^{\frac{b}{b+1}},$$

so $|E_n| \gtrsim \epsilon^{\frac{b+1}{b}} |E|$ uniformly. By a similar argument, $|\tilde{F}_n| \gtrsim \epsilon^{\frac{b+1}{b}} |F|$.

Suppose condition (1) fails. Then, for $M > \tilde{M}$ arbitrarily large, there exist $E, F, \epsilon \leq \Lambda_\varphi$ such that $\sum_{n \in \eta(\epsilon) \cap (\tilde{M}, \infty)} |E_n| > 10M \log(10\Lambda_\varphi \epsilon^{-1})|E|$. By the pigeonhole principal, there exists a finite, $M \log(10\Lambda_\varphi \epsilon^{-1})$-separated set $\eta' \subset \eta(\epsilon) \cap (\tilde{M}, \infty)$ such that $\sum_{n \in \eta'} |E_n| > 10|E|$. Then by Hölder's inequality,

$$\sum_{n \in \eta'} |E_n| = \int_E \sum_{n \in \eta'} \chi_{E_n} \leq |E|^{\frac{1}{2}} \Big(\int_E \big(\sum_{n \in \eta'} \chi_{E_n}\big)^2\Big)^{\frac{1}{2}}$$
$$= |E|^{\frac{1}{2}} \Big(\sum_{n \in \eta'} |E_n| + \sum_{n \neq m \in \eta'} 2|E_n \cap E_m|\Big)^{\frac{1}{2}}.$$

Since $\sum_{n \in \eta'} |E_n| \geq \sum_{n \neq m \in \eta'} |E_n \cap E_m|$ would imply $\sum_{n \in \eta'} |E_n| \leq 3^{\frac{1}{2}} |E|^{\frac{1}{2}} (\sum_{n \in \eta'} |E_n|)^{\frac{1}{2}}$ and contradict $\sum_{n \in \eta'} |E_n| > 10|E|$, then $\sum_{n \in \eta'} |E_n| < \sum_{n \neq m \in \eta'} |E_n \cap E_m|$, and therefore $\sum_{n \in \eta'} |E_n| \leq 3^{\frac{1}{2}} |E|^{\frac{1}{2}} (\sum_{n \neq m \in \eta'} |E_n \cap E_m|)^{\frac{1}{2}}$.

Then, choosing $n_1, k, l \in \eta'$, $k \neq l$ so that $|E_{n_1}| = \min_{n \in \eta'} |E_n|$ and $|E_l \cap E_l| =$

$\max_{n\neq m\in\eta'}|E_n\cap E_m|$, we have $(\#\eta')|E_{n_1}|\leq 3^{\frac{1}{2}}|E|^{\frac{1}{2}}(\#\eta')^{2\frac{1}{2}}|E_k\cap E_l|^{\frac{1}{2}}$, implying

$$|E_k\cap E_l|\geq\frac{|E_{n_1}|^2}{3|E|}\gtrsim(\epsilon^{\frac{b+1}{b}})^2\frac{|E|^2}{|E|}=\epsilon^{\frac{2b+2}{b}}|E|. \tag{3.2.2}$$

Since $|k-l|>10M\log(10\Lambda_\varphi\epsilon^{-1})>M$, then by Condition 3.2.3,

$$|F_l|^{b-1}\gtrsim(10\Lambda_\varphi\epsilon^{-1})^{MB}\alpha_{E_l}^b\beta_{F_{kl}}.$$

To simplify this, by the definitions of $\mathcal{T}(\cdot,\cdot)$ and E_k, (3.2.1), and (3.2.2),

$$\begin{aligned}\mathcal{T}_k(E_k\cap E_l,F_k)&=\int_{E_k\cap E_l}\mathcal{T}_k^*\chi_{F_k}(w)dw\gtrsim\int_{E_k\cap E_l}\frac{\mathcal{T}_k(E,F_k)}{|E|}\\&\approx\mathcal{T}_k(E,F)\frac{|E_k\cap E_l|}{|E|}\gtrsim\epsilon^{\frac{2b+2}{b}}\mathcal{T}_k(E,F),\end{aligned}$$

so $\beta_{F_{kl}}=\frac{1}{4}\frac{\mathcal{T}_k(E_k\cap E_l,F_k)}{|F_k|}\gtrsim\epsilon^{\frac{2b+2}{b}}\frac{\mathcal{T}_k(E,F)}{|F_k|}$ and $\alpha_{E_l}=\frac{1}{4}\frac{\mathcal{T}_l(E,F_l)}{|E|}\approx\frac{\mathcal{T}_l(E,F)}{|E|}$.

Putting these together,

$$|F_l|^{b-1}\gtrsim(10\Lambda_\varphi\epsilon^{-1})^{MB-\frac{2b+2}{b}}\frac{\mathcal{T}_k(E,F)\mathcal{T}_l(E,F)^b}{|E|^b|F_k|}\gtrsim(10\Lambda_\varphi\epsilon^{-1})^{MB-\frac{2b+2}{b}-b-1}\frac{|E|^b|F|^b}{|E|^b|F_k|}.$$

Consequently, $|F_l|^{b-1}|F_m|\gtrsim(10\Lambda_\varphi\epsilon^{-1})^{MB-\frac{2b+2}{b}-b-1}|F|^b$, which for sufficiently large M implies a contradiction, since $10\Lambda_\varphi\epsilon^{-1}\geq 10$ and $F_n\subset F$ for all n.

And if (2) were false, by replacing each F_n with E_n, E_n with $\tilde{F}_n$, F_{kl} with E_{kl}, and swapping E and F, the proof plays out identically. □

Now, we can finally move on to the main proposition.

Proposition 3.2.5. *The operator* $\mathcal{T}_{R_j^e\cap\{|\omega|\in[1,2]\}}$ *is of rwt* $(\frac{q}{3},q)$ *for all* $q\in[4,\infty)$ *if the following hold:*

0) We can decompose $R_j^e\cap\{|\omega|\in[1,2]\}$ *into a finite number of subsets* $S\in\mathcal{S}$ *such that for each* S*,* S *is either bounded or the following hold:*

1) S can be broken into regions τ_n, where n belongs to some subset of $\mathbb{N}$, and where $|\tau_n| \lesssim 1$ for each n.

2) On each τ_n, $|\partial_{z_{i_1}}\varphi| \approx 2^{nD}$, and $|\partial_{z_{i_2}}\varphi| \approx 2^{nK}$, for some fixed $D, K \in \mathbb{R}\backslash\{0\}$, $K \geq D$, that may depend on S, where $\{i_1, i_2\}$ is some permutation of $\{1, 2\}$.

Proof. If S is bounded, then Young's Inequality, Theorem 1.2.1 (since $|\omega| \approx 1$), and interpolation give us the result directly. For S not bounded, we will use the notation of Notation 3.2.1, with $b = 3$. By our hypothesis, $|\tau_n| \lesssim 1$, and since $|\omega| \approx 1$ on S, then by Theorem 1.2.1, $\|\mathcal{T}_n\|_{L^{\frac{4}{3}} \to L^4} \lesssim 1$ uniformly in n. Thus, by Lemma 3.2.2 and Lemma 3.2.4, it suffices to prove the following version of Condition 3.2.3:

Lemma 3.2.6. *Let $k, l \in \mathbb{N}$, and let k, l, and $|k-l|$ be sufficiently large, independent of E and F. Furthermore, let $sgn(k-l) = sgn(D)$. Then hypotheses (1) and (2) of Proposition 3.2.5 imply that $|F_l|^2 \gtrsim 2^{|k-l||D|}\alpha_{E_l}^3\beta_{F_{kl}}$ and $|E_l|^2 \gtrsim 2^{|k-l||D|}\alpha_{F_l}^3\beta_{E_{kl}}$, with implicit constants independent of l, k, E, F.*

Proof. By symmetry, it will suffice to prove the F_l inequality. Fix $u_0 \in F_{kl}$. Define

$$\Omega_1 := \{t \in \tau_k : u_0 - (t, \varphi(t)) =: w(t) \in E_k \cap E_l\}.$$

Then $|\Omega_1| = \mathcal{T}_k \chi_{E_k \cap E_l}(u_0) \geq \beta_{F_{kl}}$. For $t \in \Omega_1$, define

$$\Omega_2(t) := \{s \in \tau_l : w(t) + (s, \varphi(s)) \in F_l\}.$$

Then $|\Omega_2(t)| = T_l^* \chi_{F_l}(w(t)) \geq \alpha_{E_l}$. Finally, we define $\Omega \subset \mathbb{R}^6$ and $\Psi : \Omega \to \mathbb{R}^6$:

$$\Omega := \{(t, s_1, s_2) \in \mathbb{R}^6 : t \in \Omega_1, s_i \in \Omega_2(t), i = 1, 2\};$$

$$\Psi(t, s_1, s_2) := u_0 - (t, \varphi(t)) + (s_i, \varphi(s_i)) : i = 1, 2.$$

Since Ψ is a polynomial mapping $\Omega \subset \mathbb{R}^6$ into $\mathbb{R}^6$ with $\det D\Psi \not\equiv 0$, this map is $\mathcal{O}(1)$-to-one off a set of measure zero. Since $\Psi(\Omega) \subset F_l \times F_l$, then $|F_l|^2 \gtrsim \int_\Omega |\det D\Psi(t_1, s_1, s_2)| dt_1 ds_1 ds_2$. Defining $G(t, s_i) := \nabla\varphi(t) - \nabla\varphi(s_i)$, we can expand the Jacobian as follows:

$$\begin{aligned} |\det D\Psi(t; s_1, s_2)| &= |\det(G(t, s_1), G(t, s_2))| \\ &\approx |G(t, s_1)| \operatorname{dist}(G(t, s_2), \langle G(t, s_1)\rangle), \end{aligned}$$

which implies that $|F_l|^2 \gtrsim \int_\Omega |G(t, s_1)| dist(G(t, s_2), \langle G(t, s_1)\rangle) dt ds_1 ds_2$.

By hypothesis (2) of Proposition 3.2.5, after a possible reordering we can assume that that $|\partial_{z_1}\varphi| \approx 2^{mK}$ and $|\partial_{z_2}\varphi| \approx 2^{mD}$ on τ_m, for each m. Additionally, by hypothesis $sgn(k-l) = sgn(D)$, so $2^{kD} \geq 2^{lD}$ and $(k-l)D = |k-l||D|$.

Since $K \neq 0$ and $K \geq D$, and since $k, l, |k-l|$ are all sufficiently large, $|G(t, s_1)| \approx |\pi_1(G(t, s_1))| \approx \max(2^{kK}, 2^{lK})$. Here we recall that $t \in \tau_k$ and $s_i \in \tau_l$ for $i = 1, 2$.

Fix t, s_1, and denote $\overline{G}_{1,t,s_1} := G_1(t, s_1)$, which will be fixed, and $\overline{G}_{2,t}(s_2) := G_2(t, s_2)$. Then the map $s_2 \mapsto \overline{G}_{2,t}(s_2)$ has a 1^{st} derivative comparable to $|\omega| \approx 1$, so that $|\overline{G}_2(\Omega_2(t))| \gtrsim \alpha_{E_l}$ and

$$\int_{\Omega_2(t)} \operatorname{dist}(G(t, s_2), \langle G(t, s_1)\rangle) ds_2 \approx \int_{\overline{G}_{2,t}(\Omega_2(t))} dist(\overline{G}_{2,t}, \langle \overline{G}_{1,t,s_1}\rangle) d\overline{G}_{2,t}.$$

Since $\pi_2(|\overline{G}_{1,t,s_1} - \overline{G}_{2,t}|) = \pi_2(|\nabla\varphi(s_2) - \nabla\varphi(s_1)|) \lesssim 2^{lD}$, then $\overline{G}_2(\Omega(t)) \subset \mathbb{R} \times [G_1 - C2^{lD}, G_1 + C2^{lD}]$ for some C sufficiently large. However, $slope(\langle \overline{G}_1\rangle) \approx \frac{2^{kD}}{\max(2^{kK}, 2^{lK})}$, which also implies that $slope(\langle \overline{G}_1\rangle) < 1$, so if we define $\gamma := \alpha_{E_l} \frac{2^{|k-l||D|}}{\max(2^{kK}, 2^{lK})}$, then $|\mathcal{N}_\gamma(\langle \overline{G}_1\rangle) \cap \mathbb{R} \times [G_1 - C2^{lD}, G_1 + C2^{lD}]| \lesssim \gamma \frac{\max(2^{kK}, 2^{lK})}{2^{kD}} 2^{lD} = \alpha_{E_l}$. Since $|\overline{G}_2(\Omega_2(t))| \gtrsim \alpha_{E_l}$, then for some sufficiently small c, $|\overline{G}_2(\Omega(t)) \backslash \mathcal{N}_{c\gamma}(\langle \overline{G}_1\rangle)| \geq \frac{1}{2}|\overline{G}_2(\Omega(t))|$. Therefore,

$$\int_{\overline{G}_2(\Omega_2(t))} \operatorname{dist}(\langle \overline{G}_1\rangle, \overline{G}_2) d\overline{G}_2 \gtrsim \gamma |\overline{G}_2(\Omega(t))| \gtrsim \frac{\alpha_{E_l}^2 2^{|k-l||D|}}{\max\{2^{kK}, 2^{lK}\}}.$$

Thus, $|F_l|^2 \gtrsim \int_{\Omega_1} \int_{\Omega_2(t)} \max\{2^{kK}, 2^{lK}\} \frac{\alpha_{E_l}^2 2^{|k-l||D|}}{\max\{2^{kK}, 2^{lK}\}} ds_1 dt \gtrsim 2^{|k-l||D|} \alpha_{E_l}^3 \beta_{F_{kl}}$.

A near-identical argument gives $|E_l|^2 \gtrsim 2^{|k-l||D|} \alpha_{F_l}^3 \beta_{E_k}$, by swapping E_n and F_n, and using $\Psi(t, s_1, s_2) = y_0 + (t, \varphi(t)) - (s_i, \varphi(s_i)) : i = 1, 2$, with $y_0 \in E_{kl}$. This completes the proof of Lemma 3.2.6. □

Hence, by Lemmas 3.2.2 and 3.2.4, with $b = 3$, the proof of Prop 3.2.5 is complete. □

3.3 Conclusion of case $T = d_\omega$

Proposition 3.3.1. *If ν_j (likewise n_j) equals d_ω, then φ satisfies the hypotheses of Proposition 3.2.5.*

Proof. First, consider the case $j \geq 3$. If $n_j \geq d_\omega$, part (a) of Proposition 1.2.3 implies that $\min(r, s) = 1$. By symmetry, we may then choose $s = 1$ and perform the following change of variables over R_j^e: $x = z_1$ and $y = z_2 - \lambda_j z_1^r$. Then

$$\partial_{z_1} = \partial_x - \lambda_j r x^{r-1} \partial_y \quad \text{and} \quad \partial_{z_2} = \partial_y \tag{3.3.1}$$

Then $\partial_{z_2}^2 = \partial_y^2$, $\partial_{z_1 z_2} = \partial_{xy} - \lambda_j r x^{r-1} \partial_y^2$,

and $\partial_{z_1}^2 = \partial_x^2 + \lambda_j^2 r^2 x^{2r-2} \partial_y^2 - 2\lambda_j r x^{r-1} \partial_{xy} - \lambda r(r-1) x^{r-2} \partial_y$.

And finally the determinant Hessian $\omega = det D^2 \varphi$ satisfies

$$\omega = [\partial_x^2 \varphi \partial_y^2 \varphi - (\partial_{xy} \varphi)^2] - \lambda_j r(r-1) x^{r-2} \partial_y \varphi \partial_y^2 \varphi. \tag{3.3.2}$$

The relation $d_\omega = T$, with y being a factor of ω of multiplicity T, requires that $\omega = C x^T y^T + \mathcal{O}(y^{T+1})$, so in the region $R_j^e \cap \{|\omega| \approx 1\}$, $|x| \approx |y|^{-1}$. Similarly, when $j = 1, 2$, $d_\omega = T$ implies $\omega = C z_1^T z_2^T + \mathcal{O}(z_2^{T+1})$ after possibly swapping z_1 and z_2, so with the choice $x = z_1$, $y = z_2$, the last sentence also holds for $j = 1, 2$.

By symmetry, it suffices to consider cases with $\lambda_j \geq 0$ and regions with $z_1, z_2 \geq 0$. Additionally, we will focus on $y > 0$, as the proof in the region $y < 0$ is essentially identical.

Next, we decompose $R_j^e \cap \{|\omega| \approx 1\} \cap \{z_1, z_2, z_2 - \lambda_j z_1 > 0\}$ into $\tau_n = \{x \approx 2^n, y \approx 2^{-n}\}$, with $n = 0, 1, 2, ...$ By hypothesis (0) of Proposition 3.2.5, it suffices to consider only τ_n where n is sufficiently large, which will allow us to make the asymptotics in the next paragraph work. Each τ_n satisfies $|\tau_n| \approx 1$, satisfying the first condition of Proposition 3.2.5.

The partial derivative $\partial_{z_1}\varphi$ satisfies $\partial_{z_1}\varphi = Cx^J y^L + \mathcal{O}(y^{L+1}) \approx 2^{(J-L)n}$ for some $J, L \in \mathbb{N}_0$. Thus $\partial_{z_1}\varphi$ satisfies the second condition of Proposition 3.2.5 as long as $J \neq L$. Suppose $J = L$. Then $d_h(\partial_{z_1}\varphi) = \frac{Js+Jr}{s+r} = J$, and since $T = d_\omega$, then by Lemma 2.1.5, $d_h(\partial_{z_1}\varphi) = d_h - \frac{s}{r+s} = \frac{d_\omega+2}{2} - \frac{s}{r+s} = \frac{T+2}{2} - \frac{s}{r+s}$. Combining these equations, $\frac{2s}{r+s} = T + 2 - 2J \in \mathbb{N}$, so for some $m \in \mathbb{N}$, $2s = m(r+s)$, so $(2-m)s = mr$. This can only be satisfied if $m = 1$ and $s = r$. A similar argument leads to an identical result for $\partial_{z_2}\varphi$. Hence, except for the homogeneous case where $s = r = 1$, we have $J \neq L$, and the final condition of the Proposition 3.2.5 is satisfied.

Next, consider the homogeneous case. All factors are linear, so up to a linear transformation, z_2^T is a factor of ω. We can decompose our options for φ in the following way:

1. $\varphi = z_1^M z_2^{\tilde{\nu}} + \mathcal{O}(z_2^{\tilde{\nu}+1})$, with $\tilde{\nu} \geq 2$;

2. $\varphi = z_1^M z_2 + \mathcal{O}(z_2^2)$;

3. $\varphi = z_1^M + \mathcal{O}(z_2^2)$;

4. $\varphi = z_1^M + c z_1^{M-1} z_2 + \mathcal{O}(z_2^2)$.

By Lemma 2.1.3, option 2 would imply $T = 0$, so this case is trivial.

In option 1, $\omega = Cz_1^{2M+2}z_2^{2\tilde{\nu}+2}+\mathcal{O}(z_2^{2\tilde{\nu}+3})$ by Lemma 2.1.3, and since $d_\omega = T$, it follows that $2M + 2 = 2\nu + 2 = T$, implying that $M = \nu$. Then, $\nabla\varphi \approx (z_1^{\tilde{\nu}-1}z_2^{\tilde{\nu}}, z_1^{\tilde{\nu}}z_2^{\tilde{\nu}-1}) \approx (2^{-n}, 2^n)$ on τ_n, so the lemma conditions are satisfied.

In option 3, by Remark 2.1.10 following Proposition 2.1.9, $\varphi = z_1^M + cz_2^M$, some $c \neq 0$. Therefore, $\omega = Cz_1^{M-2}z_2^{M-2}$. Then, $\nabla\varphi \approx (z_1^{M-1}, z_2^{M-1}) = (2^{(M-1)n}, 2^{-(M-1)n})$, and since $\nabla\varphi(0,0) = 0$ and $T > 0$ each require $M \neq 1$, we are done.

Finally, in option 4, $\nabla\varphi \approx (z_1^{M-1}, z_1^{M-1})$, and since $\nabla\varphi(0,0) = 0$ requires $M \neq 1$, we are done.

Thus, in each case where $d_\omega = T$, the conditions of Proposition 3.2.5 hold. □

Hence, by Proposition 3.2.5, the proof of Lemma 3.1.8 and thereby Proposition 3.1.4 is complete. □

3.4 Twisted Cases

In this section, we will prove that in the twisted cases, the high order vanishing of ω is has no effect on the L^p bounds, by proving the following:

Proposition 3.4.1. *In the twisted cases (i), (iia), and (iib), $\mathcal{T}_{R_T}$ is of rwt (p_{v_1}, q_{v_1}).*

A few concrete examples of functions φ falling into the twisted cases are:

$$\varphi = z_1^4 + z_1^2 z_2 + \tfrac{1}{6}z_2^2; \qquad \varphi = z_1^5 + z_1^3 z_2 + \tfrac{9}{40}z_1 z_2^2 \qquad (Case(i));$$

$$\varphi = z_1^4 + z_1^2 z_2 + z_2^2 \qquad (Case(iia));$$

$$\varphi = (z_2 - z_1^2)(z_2 - 2z_1^2) \qquad (Case(iib)).$$

First, we will discuss the idea of the proof. As before, we will decompose $R_T \subset [-1,1]^2$ into suitable dyadic rectangles $\tau_{j,k}$. In rectangular cases, the dominant factor f_T of ω is also a factor of one or both components of $\nabla\varphi$, which causes $\nabla\varphi(\tau_{j,k})$ to be essentially convex. In the twisted cases, on the other hand, the dominant factor f_T of ω arises from a "fortuitous" cancellation, and not in an obvious way from the structure of φ and $\nabla\varphi$. This leads to $\nabla\varphi$ "twisting" near the associated curve $f_T = 0$. Because of this, $\nabla\varphi(\tau_{j,k}) \subset \mathbb{R}^2$ is highly non-convex, and similar to the neighborhood of a parabola.

In the method of refinements, the Jacobian determinant, which is used to find $L^{4/3} \to L^4$ bounds, relies primarily on the convex hull of $\nabla\varphi(\tau_{j,k})$, while the $L^\infty \to L^\infty$ bounds are connected to the measure of $\nabla\varphi(\tau_{j,k})$. Hence, $\nabla\varphi(\tau_{j,k})$ being highly non-convex causes the classes of $L^\infty \to L^\infty$ and $L^{4/3} \to L^4$ near-extremizers to be disjoint, and quantifying that tradeoff leads to a much better bounds, implying that $\mathcal{T}_{R_T}$ is of rwt (p_{v_1}, q_{v_1}), with (p_{v_1}, q_{v_1}) lying on the scaling line.

We start our arguments with two lemmas that will be necessary to show that $\nabla\varphi(\tau_{j,k})$ is always highly non-convex in the twisted cases.

Lemma 3.4.2. *Set $y := z_2 - z_1^r$, where $r \geq 2$. If y is a factor of ω with multiplicity $T > d_\omega$, and y is not a factor of φ, then y is not a factor of $\partial_{z_2}\varphi$.*

Proof. We express φ as a polynomial in $x := z_1$ and $y := z_2 - z_1^r$. By the hypotheses of Theorem 1.2.4 and the lemma, $\varphi \neq z_1^J$ and y does not divide φ. Thus, by the mixed homogeneity, after rescaling,

$$\varphi = x^J + c_l x^{J-lr} y^l + \mathcal{O}(y^{l+1}),$$

for some $J \neq 0$, $c_l \neq 0$. Suppose $l \geq 2$. From (3.3.2),

$$\omega = J(J-1)c_l l(l-1)x^{J-lr-2}y^{l-2} + \mathcal{O}(y^{l-1}).$$

Thus, y is a factor of ω with multiplicity $T = l - 2$. However, since we require $T > d_\omega$, and we know $d_\omega = 2d_h - 2 = \frac{2J}{r+1} - 2$, then $2J < l(r+1)$. From the c_l term of φ, we know that $J \geq lr \geq l$, contradicting $2J < l(r+1)$. Thus, $l = 1$, in which case y is not a factor of $\partial_y \varphi = \partial_{z_2} \varphi$. □

Lemma 3.4.3. *If $y := z_2 - z_1^r$, $r \geq 2$, is a factor of $\partial_{z_1}\varphi$, and a factor of ω with multiplicity $T > d_\omega$, but not a factor of φ, then $\partial_{z_1}\varphi \equiv 0$.*

Proof. Assume $\partial_{z_1}\varphi \not\equiv 0$ and let $M \neq 0$ denote the multiplicity of y in $\partial_{z_1}\varphi$. Defining $x := z_1$, by Lemma 3.4.2 we can write φ as follows, for some $c_1 \neq 0$, after rescaling:

$$\varphi = x^J + c_1 x^{J-r} y + \mathcal{O}(y^2).$$

<u>Claim:</u> Let M be the multiplicity of y in $\partial_{z_1}\varphi$, and suppose that $M \neq 0$ and $\partial_{z_1}\varphi \not\equiv 0$. Then y has multiplicity $M - 1$ in ω.

<u>Proof:</u> First, we rewrite ω in a way that preserves its relationship with $\partial_{z_1}\varphi$, while still taking advantage of the coordinates x and y:

$$\omega := \partial_x(\partial_{z_1}\varphi)\partial_{yy}\varphi - \partial_y(\partial_{z_1}\varphi)\partial_{xy}\varphi.$$

Since $M \neq 0$, y is a factor of $\partial_y(\partial_{z_1}\varphi)$ of multiplicity $M - 1$, and since $c_1 \neq 0$ (and $J > r$), y is not a factor of $\partial_{xy}\varphi$. Additionally, y is a factor of $\partial_x(\partial_{z_1}\varphi)$ with multiplicity at least M. Thus, y is factor of ω of multiplicity $M - 1$. ■

Since $d(\partial_{z_1}\varphi) = \frac{Mr+K}{r+1}$ for some r, and by Lemma 2.1.5, Lemma 2.1.5, the assumption $d_\omega < T$, and our above claim,

$$\begin{aligned} 2\tfrac{Mr}{1+r} \leq 2d(\partial_{z_1}\varphi) &= 2d_h - \tfrac{2}{1+r} = d_\omega + 2 - \tfrac{2}{1+r} \\ &< T + 2 - \tfrac{2}{1+r} = M + 1 - \tfrac{2}{1+r}. \end{aligned}$$

This simplifies to $(M-1)(r-1) < 0$, which is impossible by assumptions $M \geq 1$ and $r \geq 1$. Hence $\partial_{z_1}\varphi \equiv 0$. $\square$

Next, to shorten the remainder of the argument, we will unify the twisted cases. In Case(i), we can assume that $f_T = z_2$, so that z_2 has multiplicity T in ω and multiplicity 0 in φ. Thus, after rescaling, because of mixed homogeneity φ can take the following form:

$$(i) \quad \varphi = z_1^J + c_1 z_1^{J-r} z_2 (1 + \mathcal{O}(\tfrac{z_2}{z_1^r})) = x^J + c_1 x^{J-r} y (1 + \mathcal{O}(\tfrac{y}{x^r})),$$

for some $J \in \mathbb{N}$, where we used the change of coordinates $x := z_1$, $y = z_2$ for (i). Similarly, in Cases(iia) and (iib), we can assume after rescaling that $f_T = z_2 - z_1^r$, so that $z_2 - z_1^r$ has multiplicity T in ω and multiplicity 0 or 1 respectively in φ. Thus, after rescaling, because of mixed homogeneity, φ can take the following form:

$$(iia) \quad \varphi = z_1^J (1 + \mathcal{O}(\tfrac{z_2 - z_1^r}{z_1^r})) = x^J (1 + \mathcal{O}(\tfrac{y}{x^r}));$$

$$(iib) \quad \varphi = z_1^{J-r} (z_2 - z_1^r)(1 + \mathcal{O}(\tfrac{z_2 - z_1^r}{z_1^r})) = x^{J-r} y (1 + \mathcal{O}(\tfrac{y}{x^r})),$$

for some J, where we used the change of coordinates $x := z_1$, $y := z_2 - z_1^r$ for (iia-b). Also, we can assume that $r \geq 2$ by Lemma 2.1.9. Observe that in each case,

$$d_h = \tfrac{J}{r+1}. \tag{3.4.1}$$

The following lemma will unify the form of φ:

Lemma 3.4.4. *In each twisted case, for some $a_0, b_0 \neq 0$, $\nabla\varphi$ can be written as*

$$\nabla\varphi = (\partial_{z_1}\varphi, \partial_{z_2}\varphi) = (x^{J-1}(a_0 + \mathcal{O}(\tfrac{y}{x^r})), x^{J-r}(b_0 + \mathcal{O}(\tfrac{y}{x^r})).$$

Proof. In case(i), $x = z_1$ and $y = z_2$, and

$$\nabla\varphi = (J z_1^{J-1} + \mathcal{O}(z_2), c_1(J-r) z_1^{J-r} + \mathcal{O}(z_2))$$

where by our definition of case(i), $c_1 \neq 0$. Thus $a_0 = J$ and $b_0 = c_1(J - r)$ are both nonzero, since if $J = r$, φ would violate the assumption $\nabla\varphi(0) = 0$.

For cases (iia) and (iib), we will need to rewrite ∂_{z_1} and ∂_{z_2} in terms of ∂_x and ∂_y. Doing so, we have

$$\partial_{z_1} = \partial_x - rx^{r-1}\partial_y \qquad \partial_{z_2} = \partial_y.$$

In case (iia), $\varphi = x^J + c_1 x^{J-r} y + \mathcal{O}(y^2)$, and so

$$\nabla\varphi = (\partial_x\varphi - rx^{r-1}\partial_y\varphi, \partial_y\varphi) = (J - rc_1)x^{J-1} + \mathcal{O}(y), c_1 x^{J-r} + \mathcal{O}(y)).$$

By Lemma 2.1.6, $\omega \not\equiv 0$, so by Lemmas 3.4.3 and 3.4.2, respectively, $a_0, b_0 \neq 0$.

In this case (iib), $\varphi = x^{J-r}y + \mathcal{O}(y^2)$, and so

$$\nabla\varphi = (\partial_x\varphi - rx^{r-1}\partial_y\varphi, \partial_y\varphi) = (-rx^{J-1} + \mathcal{O}(y), x^{J-r} + \mathcal{O}(y)),$$

resulting in $a_0 = -r$ and $b_0 = 1$ being both nonzero. □

Additionally, we want to consider ω itself. There exists some Q such that

$$\omega \approx x^Q y^T + \mathcal{O}(y^{T+1}) = x^{Q+Tr}(\tfrac{y}{x^r})^T + \mathcal{O}(y^{T+1}),$$

where $d_\omega = \frac{Qr+T}{r+1} = 2d_h - 2 = \frac{2J-2r-2}{r+1}$ by (3.4.1). Thus, $Qr + T = 2J - 2r - 2$, so

$$\omega = x^{Q+Tr}(\tfrac{y}{x^r})^T + \mathcal{O}(y^{T+1}) = x^{2J-2r-2}(\tfrac{y}{x^r})^T + \mathcal{O}(y^{T+1}). \tag{3.4.2}$$

Next, we decompose $R_T = \{|\frac{y}{x^r}| < \epsilon, 0 < x < 1\}$. Rescaling, it suffices to only consider $x < \tilde{c}$, for some small constant $\tilde{c}$ independent of x and y. For simplicity, we give details when $y > 0$, the case $y < 0$ being similar.

We first define , in cases (iia-b), function $z(t_1, t_2) := (t_1, t_2 + t_1^r)$, and in case (i), we define $z(t)$ to be the identity. Thus, in each case, $z(x, y) = (z_1, z_2)$. Dyadically decomposing R_T, we define $\mathbf{m} := (m_1, m_2) \in \mathbb{N}^2$, $x_{\mathbf{m}} := 2^{-m_1}$, $\frac{y_{\mathbf{m}}}{x_{\mathbf{m}}^r} := 2^{-m_2}$, and define curved rectangles $\tau_{\mathbf{m}} := z([x_{\mathbf{m}}, 2x_{\mathbf{m}}] \times [y_{\mathbf{m}}, 2y_{\mathbf{m}}])$, along with extended rectangles $\tau_{\mathbf{m}}^e := z([\frac{1}{2}x_{\mathbf{m}}, 3x_{\mathbf{m}}] \times [y_{\mathbf{m}}, 2y_{\mathbf{m}}])$. Denote $\mathcal{T}_{\mathbf{m}} := \mathcal{T}_{\tau_{\mathbf{m}}}$.

To begin, we want to refine $E \rightsquigarrow E_{\mathbf{m}}$ and $F \rightsquigarrow F_{\mathbf{m}}$ so that $\mathcal{T}_{\mathbf{m}}(E, F) \approx \mathcal{T}_{\mathbf{m}}(E_{\mathbf{m}}, F_{\mathbf{m}})$. Define

$$E_{\mathbf{m}} := \{w \in E : \mathcal{T}_{\mathbf{m}}^* \chi_F(w) \geq \tfrac{1}{4}\tfrac{\mathcal{T}_{\mathbf{m}}(E,F)}{|E|}\}$$

$$F_{\mathbf{m}} := \{u \in F : \mathcal{T}_{\mathbf{m}} \chi_{E_{\mathbf{m}}}(u) \geq \tfrac{1}{4}\tfrac{\mathcal{T}_{\mathbf{m}}(E_{\mathbf{m}},F)}{|F|} =: \alpha_{\mathbf{m}}\}$$

$$E'_{\mathbf{m}} := \{w \in E_{\mathbf{m}} : \mathcal{T}_{\mathbf{m}}^* \chi_{E_{\mathbf{m}}}(w) \geq \tfrac{1}{4}\tfrac{\mathcal{T}_{\mathbf{m}}(E_{\mathbf{m}},F_{\mathbf{m}})}{|E_{\mathbf{m}}|} =: \beta_{\mathbf{m}}\}$$

Then $\mathcal{T}_{\mathbf{m}}(E_{\mathbf{m}}, F_{\mathbf{m}}) \approx \mathcal{T}_{\mathbf{m}}(E, F)$, by an argument similar to the argument leading to (3.2.1).

To proceed, we will create a map, and use the size of the Jacobian to acquire the desired bound. Fix $e_0 \in E'_{\mathbf{m}}$, and define:

$$\Omega_1 := \{t \in \mathbb{R}^2 : z(t) \in \tau_{\mathbf{m}} \text{ and } e_0 + (t, \varphi(z(t))) =: e_1(t) \in F\}.$$

Then $|\Omega_1| = \mathcal{T}_{\mathbf{m}}^* \chi_{F_{\mathbf{m}}}(e_0) \geq \beta_{\mathbf{m}}$. Next, define, for $t \in \Omega_1$,

$$\Omega_2(t) := \{s \in \mathbb{R}^2 : z(s) \in \tau_{\mathbf{m}} \text{ and } e_1(t) - (s, \varphi(z(s))) \in E_{\mathbf{m}}\}.$$

Then $|\Omega_2(t)| = T_{\mathbf{m}} \chi_{E_{\mathbf{m}}}(e_2(t)) \geq \alpha_{\mathbf{m}}$. Define the following:

$$\Omega := \{(t; s^{(1)}, s^{(2)}) \in \mathbb{R}^6 : t \in \Omega_1, s^{(i)} \in \Omega_2(t), i = 1, 2\}$$

$$\Psi(t; s^{(1)}, s^{(2)}) = (e_0 + (t, \varphi(z(t))) - (s^{(i)}, \varphi(z(s^{(i)})))) : i = 1, 2$$

Since Ψ is a polynomial mapping $\Omega \subset \mathbb{R}^6$ into $\mathbb{R}^6$, with $\det D\Psi \not\equiv 0$, it is $\mathcal{O}(1)$-to-one off a set of measure zero. Since $\Psi(\Omega) \subset E_{\mathbf{m}}^2$, we have $|E_{\mathbf{m}}|^2 \gtrsim \int_\Omega |\det D\Psi(t; s^{(1)}, s^{(2)})| dt ds^{(1)} ds^{(2)}$.

Expanding,

$$|\det D\Psi(t;s^{(1)},s^{(2)})| = |\det(\nabla\varphi(z(s^{(1)})) - \nabla\varphi(z(t)), \nabla\varphi(z(s^{(2)})) - \nabla\varphi(z(t)))|$$
$$\approx \mu(\overline{\text{Conv}}\{\nabla\varphi(z(s^{(1)})), \nabla\varphi(z(s^{(2)})), \nabla\varphi(z(t))\}),$$

where $\mu(\overline{\text{Conv}}(S))$, $S \subset \mathbb{R}^2$, is the area of the convex hull of S, implying that

$$|E_{\mathbf{m}}|^2 \gtrsim \int_\Omega \mu(\overline{\text{Conv}}\{\nabla\varphi(z(s^{(1)})), \nabla\varphi(z(s^{(2)})), \nabla\varphi(z(t))\}) dt ds^{(1)} ds^{(2)}. \qquad (3.4.3)$$

To proceed, we will consider two regimes: $\alpha_{\mathbf{m}} \leq C\frac{y_{\mathbf{m}}}{x_{\mathbf{m}}^r} x_{\mathbf{m}} y_{\mathbf{m}}$ and $C\frac{y_{\mathbf{m}}}{x_{\mathbf{m}}^r} x_{\mathbf{m}} y_{\mathbf{m}} \leq \alpha_{\mathbf{m}} \lesssim x_{\mathbf{m}} y_{\mathbf{m}}$, for some sufficiently large constant C to be decided later. For small $\alpha_{\mathbf{m}}$, we will directly calculate the $L^{\frac{4}{3}} \to L^4$ bound using Theorem 1.2.1 and interpolate. For the larger $\alpha_{\mathbf{m}}$, we will instead quantify how far the set $\nabla\varphi(z(\Omega_2(t)))$ violates convexity and use (3.4.3) as follows:

Lemma 3.4.5. *For $\alpha_{\mathbf{m}} \geq C\frac{y_{\mathbf{m}}}{x_{\mathbf{m}}^r} x_{\mathbf{m}} y_{\mathbf{m}}$,*

$$\mu(\overline{Conv}\{\nabla\varphi(z(t)), \nabla\varphi(z(s^{(1)})), \nabla\varphi(z(s^{(2)}))\}) \gtrsim x_{\mathbf{m}}^{-[J-2r+1]} \left(\frac{\alpha_{\mathbf{m}}}{x_{\mathbf{m}} y_{\mathbf{m}}} x_{\mathbf{m}}^{J-r}\right)^3$$

on some subset $\Omega' \subset \Omega$, of size $|\Omega'| \geq \frac{1}{100}|\Omega|$.

We will use Lemma 3.4.6, together with Lemmas 3.4.7-3.4.14, to prove Lemma 3.4.5.

Lemma 3.4.6. *Let γ be a C^2 curve in $\mathbb{R}^2$, with curvature $\kappa \approx \Theta$, and such that the set of all unit tangent vectors of γ belongs to the same half quadrant, or its mirror image. Let $p_1, p_2, p_3 \in \mathbb{R}^2$ lie in a $\delta^2\Theta$ neighborhood of γ, and satisfy $\|p_i - p_j\| > a$ for all $i \neq j$. If $a > 40\max\{\delta^2\Theta, \delta(\frac{\Theta}{\inf\kappa})^{\frac{1}{2}}\}$, then $\mu(\overline{Conv}\{p_1, p_2, p_3\}) \gtrsim \Theta a^3$.*

Proof. There exist points $\tilde{p}_1, \tilde{p}_2, \tilde{p}_3$ on γ closest to p_1, p_2, p_3, respectively, and since $a > 40\delta^2\Theta$, the points $\tilde{p}_i$ have separation at least $\frac{a}{2}$. Since γ cannot be a closed curve by our hypothesis on tangent lines, we can reorder indices such that $\tilde{p}_2$ lies between $\tilde{p}_1$ and $\tilde{p}_3$ on γ.

We translate and rotate $\mathbb{R}^2$ so that $\tilde{p}_2 = 0$ and the tangent line of γ at 0 becomes horizontal. We then reflect $\mathbb{R}^2$ if needed so that γ lies in the upper half plain and so that $\tilde{p}_1$ lies in the left plain, and $\tilde{p}_3$ in the right. We can then write γ as the graph $(x, g(x))$ of some function g, where $g(0) = g'(0) = 0$, $|g'| \leq 1$, and $g'' \in (\frac{1}{2} \inf \kappa, 4 \sup \kappa)$. Therefore, $g(x) \leq \frac{1}{4}(\inf \kappa)x^2$.

We recall that π_i projects points in $\mathbb{R}^2$ onto their i-th coordinate. The separation of the $\tilde{p}_i$ must be at least $\frac{a}{2}$, and since $|g'| \leq 1$, then $|\pi_1(\tilde{p}_1)|$, $|\pi_1(\tilde{p}_3)| \geq \frac{a}{2\sqrt{2}}$. Therefore, $|\pi_2(\tilde{p}_1)|, |\pi_2(\tilde{p}_3)| \geq \frac{1}{16}(\inf \kappa)a^2$.

Therefore, $\tilde{p}_1, \tilde{p}_2, \tilde{p}_3$ form a triangle of triangle with base and height greater than $\frac{a}{\sqrt{2}}$ and $\frac{1}{16}(\inf \kappa)a^2$, respectively. Since $\tilde{p}_i$ is within $\delta^2\Theta$ of p_i for each i, and since $a > 40 \max\{\delta^2\Theta, \delta(\frac{\Theta}{\inf \kappa})^{\frac{1}{2}}\}$, then p_1, p_2, p_3 form a triangle with base and height greater than $\frac{a}{2\sqrt{2}}$ and $\frac{1}{32}(\inf \kappa)a^2$, respectively. Hence, the triangle formed by p_1, p_2, p_3 has area $\gtrsim \Theta a^3$. $\square$

Using Lemma 3.4.6 to prove Lemma 3.4.5 will require us to identify a curve, bound its curvature, restrict the possible unit tangent vectors, establish a minimal separation of points, and show points remain in a neighborhood of the curve. A suitable curve γ_y, or γ_y^*, will be defined in Lemmas 3.4.8-3.4.10. The curvature will be bounded in Lemma 3.4.8. Lemma 3.4.11 will restrict the possible unit tangent vectors. The minimal separation comes from Lemma 3.4.7 and our lower bound on $\alpha_{\mathbf{m}}$. And finally, the neighborhood size $\mathcal{H}$ will be bounded in Lemmas 3.4.11-3.4.15.

Our goal will be to look at the structure of $\nabla\varphi(z(x,y))$, and then use Lemma 3.4.6

on the points $\nabla\varphi(z(s_1))$, $\nabla\varphi(z(s_2))$, and $\nabla\varphi(z(t))$. To proceed, we will consider $(x, y) \in z^{-1}(\tau_{\mathbf{m}})$, and analyze $\nabla\varphi(\tau_{\mathbf{m}})$. Define

$$\Phi \equiv (\Phi_1, \Phi_2) := (\mathrm{sgn}(a_0)(\partial_{z_1}\varphi) \circ z, \mathrm{sgn}(b_0)(\partial_{z_2}\varphi) \circ z),$$

where by Lemma 3.4.4, $a_0, b_0 \neq 0$ and

$$\Phi(x, y) = (x^{J-1}[|a_0| + \tilde{a_1}\tfrac{y}{x^r} + \mathcal{O}((\tfrac{y}{x^r})^2)], x^{J-r}[|b_0| + \tilde{b_1}\tfrac{y}{x^r} + \mathcal{O}((\tfrac{y}{x^r})^2)]) \tag{3.4.4}$$

where $\tilde{a_1} := \mathrm{sgn}(a_0)a_1$, and $\tilde{b_1} := \mathrm{sgn}(b_0)b_1$ may be zero. Because Φ and $\nabla\varphi \circ z$ are equal up to coordinate sign changes, we can work purely with Φ, since

$$\mu(\overline{\mathrm{Conv}}\{\Phi(t), \Phi(s^{(1)}), \Phi(s^{(2)})\}) = \mu(\overline{\mathrm{Conv}}\{\nabla\varphi(z(t)), \nabla\varphi(z(s^{(1)})), \nabla\varphi(z(s^{(2)}))\}). \tag{3.4.5}$$

First, we will show that x and $\xi_2 = \Phi_2(x, y)$ are strongly related.

Lemma 3.4.7. *Let $(x, y), (x + \Delta x, y + \Delta y) \in z^{-1}(\tau^e_{\mathbf{m}})$, and define $\delta_\flat := x_{\mathbf{m}}\frac{y_{\mathbf{m}}}{x^r_{\mathbf{m}}}$ and $\delta_\Phi := x^{J-r}_{\mathbf{m}}\frac{y_{\mathbf{m}}}{x^r_{\mathbf{m}}}$ as the x, Φ_2 uncertainty scales, respectively. Then if either $|\Phi_2(x+\Delta x, y+\Delta y) - \Phi_2(x, y)| \geq C\delta_\Phi$, or $|\Delta x| \geq C\delta_\flat$, for a sufficiently large C, then*

$$sgn(\Delta x)\big(\Phi_2(x + \Delta x, y + \Delta y) - \Phi_2(x, y)\big) \approx x^{J-r-1}_{\mathbf{m}}|\Delta x|.$$

Proof. By (3.4.4), recalling that $b_0 \neq 0$, we have

$$\partial_x\Phi_2 = x^{J-r-1}[|b_0|(J - r) + \mathcal{O}(\tfrac{y}{x^r})] \approx x^{J-r-1}_{\mathbf{m}}$$
$$|\partial_y\Phi_2| = |x^{J-2r}[\tilde{b_1} + \mathcal{O}(\tfrac{y}{x^r})]| \lesssim x^{J-2r}_{\mathbf{m}}.$$

Therefore, since $|\Delta y| \leq y_{\mathbf{m}}$ in $z^{-1}(\tau^e_{\mathbf{m}})$, a change in y will only change Φ_2 by at most $x^{J-2r}_{\mathbf{m}}y_{\mathbf{m}} = \delta_\Phi$. Next, consider $x_2 > x_1$. Since $\partial_x\Phi_2 \approx x^{J-r-1}_{\mathbf{m}}$, then $\Phi_2(x_2, y_1) - \Phi_2(x_1, y_1) \approx (x_2-x_1)x^{J-r-1}_{\mathbf{m}}$. Thus, by the triangle inequality, if $|\Phi_2(x_2, y_2)-\Phi_2(x_1, y_1)| >$

$C\delta_\Phi$, or if $|x_2 - x_1| > C\delta_\flat$, for some large $C > 0$, then $\Phi_2(x_2, y_2) - \Phi_2(x_1, y_1) \approx (x_2 - x_1)x_{\mathbf{m}}^{J-r-1}$. This concludes the proof of Lemma 3.4.7. □

The next lemmas define and analyze the curves that will be used with Lemma 3.4.6.

Lemma 3.4.8. *For $y \in [y_{\mathbf{m}}, 2y_{\mathbf{m}}]$, $\gamma_y : x \mapsto \Phi(x, y)$ has curvature $\kappa_y \approx x_{\mathbf{m}}^{-[J-2r+1]} := \tilde{\Theta}$.*

Proof of Lemma 3.4.8. By direct computation using (3.4.4),

$$\kappa_y(x) = x^{-[J-2r+1]} \frac{|a_0 b_0|(J-1)(J-r)(r-1) + \mathcal{O}(\frac{y}{x^r})}{[x^{2r-2}a_0^2 + b_0^2 + \mathcal{O}(\frac{y}{x^r})]^{\frac{3}{2}}}.$$

Then, since $0 < x \leq 1$ and $a_0, b_0 \neq 0$,

$$\kappa_y(x) \approx x^{-[J-2r+1]} \approx x_{\mathbf{m}}^{-[J-2r+1]} =: \tilde{\Theta},$$

concluding the proof of Lemma 3.4.8. □

Lemma 3.4.9. *Let $y \in [y_{\mathbf{m}}, 2y_{\mathbf{m}}]$. Then there exists a function $\gamma_y^* : \Phi_2(z^{-1}(\tau_{\mathbf{m}})) \to \mathbb{R}$ such that*

$$\gamma_y([\tfrac{1}{2}x_{\mathbf{m}}, 3x_{\mathbf{m}}]) \cap \{\mathbb{R} \times \Phi_2(z^{-1}(\tau_{\mathbf{m}}))\} = \{(\xi_1, \xi_2) : \xi_1 = \gamma_y^*(\xi_2), \xi_2 \in \Phi_2(z^{-1}(\tau_{\mathbf{m}}))\}.$$

This lemma is equivalent to the following lemma:

Lemma 3.4.10. *Let $y \in [y_{\mathbf{m}}, 2y_{\mathbf{m}}]$. Then, for each $\xi_2 \in \Phi_2(z^{-1}(\tau_{\mathbf{m}}))$, there exists a unique ξ_1 such that $(\xi_1, \xi_2) \in \gamma_y([\frac{1}{2}x_{\mathbf{m}}, 3x_{\mathbf{m}}])$.*

Proof of Lemma 3.4.10. .

1. *Existence*: If $\xi_2 \in \Phi_2(z^{-1}(\tau_{\mathbf{m}}))$, there exists $(\tilde{x}, \tilde{y}) \in z^{-1}(\tau_{\mathbf{m}})$ such that $\Phi_2(\tilde{x}, \tilde{y}) = \xi_2$. Now consider points in $z^{-1}(\tau_{\mathbf{m}}^e)$ with minimal and maximal $x-$values: $(\frac{x_{\mathbf{m}}}{2}, y)$ and $(3x_{\mathbf{m}}, y)$. Since $\frac{x_{\mathbf{m}}}{2} \leq \tilde{x} - \frac{x_{\mathbf{m}}}{2} \leq \tilde{x} - c\delta_\flat$ and $3x_{\mathbf{m}} \geq \tilde{x} + x_{\mathbf{m}} \geq \tilde{x} + c\delta_\flat$, then

$$\Phi_2(\tfrac{x_{\mathbf{m}}}{2}, y) < \xi_2 < \Phi_2(3x_{\mathbf{m}}, y)$$

by Lemma 3.4.7. Then, by continuity of $\gamma_y(x)$ and the Intermediate Value Theorem, there exists $x \in [\frac{1}{2}x_{\mathbf{m}}, 3x_{\mathbf{m}}]$ such that $\Phi_2(x,y) \equiv \tilde{\pi}_2(\gamma_y(x)) = \xi_2$.

2. *Uniqueness*: By (3.4.4), $\tilde{\pi}_1(\frac{\mathrm{d}}{\mathrm{d}x}\gamma_y(x)) \equiv \partial_x \Phi_1(x,y) = x^{J-2}(|a_0|(J-1)+\mathcal{O}(\frac{y}{x^r})) > 0$ for each $(x,y) \in z^{-1}(\tau^e_{\mathbf{m}})$, so for fixed $y \in [y_{\mathbf{m}}, 2y_{\mathbf{m}}]$, the map $x \mapsto \tilde{\pi}_1(\gamma_y(x)) \equiv \Phi_1(x,y)$ is one-to-one. Hence, for fixed y, ξ_1 is mapped to by at most a single unique $\overline{x}$, and hence a unique $\overline{\xi_2} = \Phi_2(\overline{x}, y)$.

This concludes the proof of Lemma 3.4.10. □

Lemma 3.4.11. *For every* $y \in [y_{\mathbf{m}}, 2y_{\mathbf{m}}]$, $|\frac{d}{d\xi_2}\gamma^*_y(\xi_2)| < \frac{1}{10}$ *and* $|\frac{d^2}{d\xi_2^2}\gamma^*_y(\xi_2)| \approx x_{\mathbf{m}}^{-[J-2r+1]} =: \tilde{\Theta}$ *for all* $\xi_2 \in \Phi_2(\tau_{\mathbf{m}})$.

Proof. Since $|\partial_x \Phi_1| \approx x_{\mathbf{m}}^{J-2}$, and $|\partial_x \Phi_2| \approx x_{\mathbf{m}}^{J-r-1}$, then $|\frac{\mathrm{d}}{\mathrm{d}\xi_2}\gamma^*_y(\xi_2)| \approx x_{\mathbf{m}}^{r-1} < \frac{1}{10}$, so the curvature of $\xi_2 \mapsto (\gamma^*_y(\xi_2), \xi_2)$ is comparable to $|\frac{\mathrm{d}^2}{\mathrm{d}\xi_2^2}\gamma^*_y(\xi_2)|$. Finally, by Lemma 3.4.8, the curve $\xi_2 \to (\gamma^*_y(\xi_2), \xi_2)$ has curvature $\approx x_{\mathbf{m}}^{-[J-2r+1]}$, concluding the proof of Lemma 3.4.11. □

Next, define

$$\Phi(z^{-1}(\tau_{\mathbf{m}}))^e := \Phi(z^{-1}(\tau^e_{\mathbf{m}})) \cap \{\mathbb{R} \times \Phi_2(z^{-1}(\tau_{\mathbf{m}}))\},$$

and observe that $\Phi(z^{-1}(\tau_{\mathbf{m}}))^e$ satisfies

$$\Phi(z^{-1}(\tau_{\mathbf{m}})) \subset \Phi(z^{-1}(\tau_{\mathbf{m}}))^e \subset \Phi(z^{-1}(\tau^e_{\mathbf{m}})). \tag{3.4.6}$$

The expanded image $\Phi(z^{-1}(\tau_{\mathbf{m}}))^e$ essentially rectangularizes our image $\Phi(z^{-1}(\tau_{\mathbf{m}}))$, making it more practical for proving results.

Lemma 3.4.12. *The set* $\Phi(z^{-1}(\tau_{\mathbf{m}}))^e$ *satisfies*

$$\Phi(z^{-1}(\tau_{\mathbf{m}}))^e = \cup_{\xi_2 \in \Phi_2(z^{-1}(\tau_{\mathbf{m}}))}[\min(\gamma^*_{y_{\mathbf{m}}}(\xi_2), \gamma^*_{2y_{\mathbf{m}}}(\xi_2)), \max(\gamma^*_{y_{\mathbf{m}}}(\xi_2), \gamma^*_{2y_{\mathbf{m}}}(\xi_2))] \times \{\xi_2\}.$$

Proof. Since $|Jac\Phi| \approx |\omega| > 0$ on $\tau^e_{\mathbf{m}}$, Φ is an open map by the inverse function theorem, so $\Phi^{-1}(\mathrm{Bdry}\Phi(z^{-1}(\tau^e_{\mathbf{m}}))) \subset \mathrm{Bdry}z^{-1}(\tau^e_{\mathbf{m}})$. By Lemma 3.4.7, for all $y \in [y_{\mathbf{m}}, 2y_{\mathbf{m}}]$, $\Phi_2(\frac{1}{2}x_{\mathbf{m}}, y)$, $\Phi_2(3x_{\mathbf{m}}, y) \notin \Phi_2(z^{-1}(\tau_{\mathbf{m}}))$, so the boundary of $\Phi(z^{-1}(\tau_{\mathbf{m}}))^e$ is a union of two horizontal line segments and subsets of the curves $(\gamma^*_{y_{\mathbf{m}}}(\cdot), \cdot)$ and $(\gamma^*_{2y_{\mathbf{m}}}(\cdot), \cdot)$ over $\Phi_2(z^{-1}(\tau_{\mathbf{m}})))$. □

Lemma 3.4.13. *For $\xi_2 \in \Phi_2(z^{-1}(\tau_{\mathbf{m}}))$, μ the Lebesgue measure on $\mathbb{R}$, $\mu(\{\xi_1 : (\xi_1, \xi_2) \in \Phi(z^{-1}(\tau_{\mathbf{m}}))^e\}) = diam\{\xi_1 : (\xi_1, \xi_2) \in \Phi(z^{-1}(\tau_{\mathbf{m}}))^e\} = |\gamma^*_{2y_{\mathbf{m}}}(\xi_2) - \gamma^*_{y_{\mathbf{m}}}(\xi_2)|$.*

Proof. Follows directly from Lemma 3.4.12. □

To proceed, we will use the following definitions:

$$\mathcal{W}_{[\xi_2^{(1)}, \xi_2^{(2)}]} := \tfrac{1}{|\xi_2^{(2)} - \xi_2^{(1)}|} \int_{\xi_2 \in [\xi_2^{(1)}, \xi_2^{(2)}]} \mathrm{diam}\{\xi_1 : (\xi_1, \xi_2) \in \Phi(z^{-1}(\tau_{\mathbf{m}}))^e\} \mathrm{d}\xi_2; \tag{3.4.7}$$

$$\mathcal{H} := \max_{\xi_2 \in \Phi_2(z^{-1}(\tau_{\mathbf{m}}))} \mathrm{diam}\{\xi_1 : (\xi_1, \xi_2) \in \Phi(z^{-1}(\tau_{\mathbf{m}}))\}. \tag{3.4.8}$$

Lemma 3.4.14. *If $\xi_2^{(1)}, \xi_2^{(2)} \in \Phi_2(z^{-1}(\tau_{\mathbf{m}}))$, and $\xi_2^{(2)} - \xi_2^{(1)} > C\delta_\Phi$, then*

$$\mathcal{W}_{[\xi_2^{(1)}, \xi_2^{(2)}]} \approx x_{\mathbf{m}}^{J-1} (\tfrac{y_{\mathbf{m}}}{x^r_{\mathbf{m}}})^{T+1}.$$

Proof. By Lemma 3.4.7, $\Phi([x_1, x_2] \times [y_{\mathbf{m}}, 2y_{\mathbf{m}}]) \subset \Phi(z^{-1}(\tau_{\mathbf{m}}))^e \cap (\mathbb{R} \times [\xi_2^{(1)}, \xi_2^{(2)}]) \subset \Phi([x'_1, x'_2] \times [y_{\mathbf{m}}, 2y_{\mathbf{m}}])$, some $x'_1, x'_2, x_1, x_2 \in [\frac{1}{2}x_{\mathbf{m}}, 3x_{\mathbf{m}}]$, where $x_2 - x_1 \approx x'_2 - x'_1 \approx (\frac{1}{x_{\mathbf{m}}})^{J-r-1}(\xi_2^{(2)} - \xi_2^{(1)})$. Then

$$\begin{aligned} \mathcal{W}_{[\xi_2^{(1)}, \xi_2^{(2)}]} &\approx \tfrac{|\Phi([x_1,x_2] \times [y_{\mathbf{m}}, 2y_{\mathbf{m}}])|}{\xi_2^{(2)} - \xi_2^{(1)}} \approx \tfrac{|Jac\Phi| y_{\mathbf{m}} (x_2 - x_1)}{\xi_2^{(2)} - \xi_2^{(1)}} \\ &\approx x_{\mathbf{m}}^{2J-2r-2} (\tfrac{y_{\mathbf{m}}}{x^r_{\mathbf{m}}})^T y_{\mathbf{m}} \tfrac{1}{x_{\mathbf{m}}^{J-r-1}} \approx x_{\mathbf{m}}^{J-1} (\tfrac{y_{\mathbf{m}}}{x^r_{\mathbf{m}}})^{T+1} \end{aligned}$$

since $|Jac\Phi| \approx |\omega| \approx x_{\mathbf{m}}^{2J-2r-2} (\frac{y_{\mathbf{m}}}{x^r_{\mathbf{m}}})^T$ by (3.4.2). □

Lemma 3.4.15. *The following inequality holds: $\mathcal{H} \lesssim \tilde{\Theta}\delta_\Phi^2$.*

Proof. We can consider two cases: $\mathcal{H} >> \mathcal{W}_{\Phi_2(z^{-1}(\tau_{\mathbf{m}}))}$ or $\mathcal{H} \lesssim \mathcal{W}_{\Phi_2(z^{-1}(\tau_{\mathbf{m}}))}$.

If $\mathcal{H} \lesssim \mathcal{W}_{\Phi_2(z^{-1}(\tau_{\mathbf{m}}))}$, then

$$\mathcal{H} \lesssim x_{\mathbf{m}}^{J-1}(\tfrac{y_{\mathbf{m}}}{x_{\mathbf{m}}^r})^{T+1} \leq x_{\mathbf{m}}^{J-1}(\tfrac{y_{\mathbf{m}}}{x_{\mathbf{m}}^r})^2 = x_{\mathbf{m}}^{-[J-2r+1]}(x_{\mathbf{m}}^{J-r}\tfrac{y_{\mathbf{m}}}{x_{\mathbf{m}}^r})^2 = \tilde{\Theta}\delta_\Phi^2$$

by Lemma 3.4.14, $T \geq 1$, simple reordering, and the definitions of $\tilde{\Theta}$ and δ_Φ.

If $\mathcal{H} >> \mathcal{W}_{\Phi_2(z^{-1}(\tau_{\mathbf{m}}))}$, then by Lemma 3.4.13, there exist c_1, ξ_2' such that $c_1 \in [\xi_2', \xi_2' + 3C\delta_\Phi] \subset \Phi_2(z^{-1}(\tau_m))$ and

$$|\gamma_{2y_{\mathbf{m}}}^*(c_1) - \gamma_{y_{\mathbf{m}}}^*(c_1)| \gtrsim \mathcal{H}.$$

Additionally, by Lemma 3.4.14,

$$\text{avg}_{[\xi_2',\xi_2'+C\delta_\Phi]}|\gamma_{2y_{\mathbf{m}}}^* - \gamma_{y_{\mathbf{m}}}^*| \approx \mathcal{W}_{\Phi_2(z^{-1}(\tau_{\mathbf{m}}))} \approx \text{avg}_{[\xi_2'+2C,\xi_2'+3C\delta_\Phi]}|\gamma_{2y_{\mathbf{m}}}^* - \gamma_{y_{\mathbf{m}}}^*|$$

and so, by the Intermediate Value Theorem, there exist $c_2 \in [\xi_2', \xi_2' + C\delta_\Phi]$ and $c_3 \in [\xi_2', \xi_2' + C\delta_\Phi]$ such that

$$|\gamma_{2y_{\mathbf{m}}}^*(c_2) - \gamma_{y_{\mathbf{m}}}^*(c_2)|, |\gamma_{2y_{\mathbf{m}}}^*(c_3) - \gamma_{y_{\mathbf{m}}}^*(c_3)| \lesssim \mathcal{W}_{\Phi_2(z^{-1}(\tau_{\mathbf{m}}))}$$

Then, since $|c_2 - c_3| \approx \delta_\Phi$, $|c_1 - c_3| \lesssim \delta_\Phi$, and $\mathcal{H} >> \mathcal{W}_{\Phi_2(z^{-1}(\tau_{\mathbf{m}}))}$, by the Mean Value Theorem there exist $c_4, c_5 \in [\xi_2', \xi_2' + 3C\delta_\Phi]$ such that

$$|\gamma_{2y_{\mathbf{m}}}^{*'}(c_4) - \gamma_{y_{\mathbf{m}}}^{*'}(c_4)| \lesssim \tfrac{\mathcal{W}_{\Phi_2(z^{-1}(\tau_{\mathbf{m}}))}}{\delta_\Phi} \text{ and } |\gamma_{2y_{\mathbf{m}}}^{*'}(c_5) - \gamma_{y_{\mathbf{m}}}^{*'}(c_5)| \gtrsim \tfrac{\mathcal{H}}{\delta_\Phi}$$

Since $\mathcal{H} >> \mathcal{W}_{\Phi_2(z^{-1}(\tau_{\mathbf{m}}))}$ and $|c_4 - c_5| \lesssim \delta_\Phi$, by the Mean Value Theorem there exists $c_6 \in [\xi_2', \xi_2' + 3C\delta_\Phi]$ such that

$$|\gamma_{2y_{\mathbf{m}}}^{*''}(c_6) - \gamma_{y_{\mathbf{m}}}^{*''}(c_6)| \gtrsim \tfrac{\mathcal{H}}{\delta_\Phi^2}.$$

Since Lemma 3.4.11 implies that $|\gamma_{2y_{\mathbf{m}}}^{*''}(c_6) - \gamma_{y_{\mathbf{m}}}^{*''}(c_6)| \lesssim \tilde{\Theta}$, then $\mathcal{H} \lesssim \tilde{\Theta}\delta_\Phi^2$. $\square$

Now, we will proceed with the proof of Proposition 3.4.5:

Lemma 3.4.5 proof. There exists some $\Omega' \subset \Omega$, where $|\Omega'| \geq \frac{1}{100}|\Omega|$, and $t_1, s_1^{(1)}$, and $s_1^{(2)}$ are separated by at least $\frac{1}{4}\frac{\alpha_{\mathbf{m}}}{y_{\mathbf{m}}} \geq C\frac{y_{\mathbf{m}}}{x_{\mathbf{m}}^r}x_{\mathbf{m}} = C\delta_\flat$ on Ω'. Then, by Lemma 3.4.7, $\Phi(t), \Phi(s^{(1)})$, and $\Phi(s^{(2)})$ have mutual $\xi_2 -$ *separation* of at least $\tilde{C}\delta_\Phi$, where we can make $\tilde{C}$ arbitrarily large my making the C on the lower bound of $\alpha_{\mathbf{m}}$ sufficiently large. Furthermore, by Lemmas 3.4.15, these points lie in an $\mathcal{H} \approx \tilde{\Theta}\delta_\Phi^2$ neighborhood of a curve, and by Lemmas 3.4.8 and 3.4.11, respectively, that curve has curvature $\approx \tilde{\Theta} = x_{\mathbf{m}}^{-[J-2r+1]}$, and derivative bounded by $\frac{1}{10}$, which bounds the possible unit tangent vectors.

Thus, by choosing the constant on the lower bound of $\alpha_{\mathbf{m}}$ to be large enough, we can separate $p_1 = \Phi(t), p_2 = \Phi(s^{(1)})$, and $p_3 = \Phi(s^{(2)})$ by $a = \tilde{C}\delta_\Phi = \tilde{C}x_{\mathbf{m}}^{J-r}\frac{y_{\mathbf{m}}}{x_{\mathbf{m}}^r}$, for a sufficiently large $\tilde{C}$. By choosing Θ and δ so that $\Theta = \tilde{\Theta}$ and $\Theta\delta^2 = \mathcal{H} \approx \tilde{\Theta}\delta_\Phi^2$, the neighborhood $\Theta\delta^2$ will equal $\mathcal{H}$ so that $\delta \approx \delta_\Phi$. Since x_m, $\frac{y_{\mathbf{m}}}{x_{\mathbf{m}}^r} < 1$, then $a = C\delta_\Phi$ will satisfy $a > 40\max\{\delta^2\Theta, \delta(\frac{\Theta}{\inf\kappa})^{\frac{1}{2}}\}$ if C is sufficiently large (note that $\inf\kappa \approx \Theta$), completing the proof of Lemma 3.4.6, and implying that on Ω',

$$\mu(\overline{\text{Conv}}\{\nabla\varphi(z(t)), \nabla\varphi(z(s^{(1)})), \nabla\varphi(z(s^{(2)}))\}) \gtrsim x_{\mathbf{m}}^{-[J-2r+1]}(\frac{\alpha_{\mathbf{m}}}{x_{\mathbf{m}}y_{\mathbf{m}}}x_{\mathbf{m}}^{J-r})^3. \qquad \square$$

Next, recall our decomposition of R_T into $\tau_{\mathbf{m}}$. This induces a decomposition $\mathcal{T} = \sum_{m_2}\sum_{m_1}\mathcal{T}_{\mathbf{m}}$. Define $\mathcal{T}_{m_2} := \sum_{m_1}\mathcal{T}_{(m_1,m_2)}$.

Proposition 3.4.16. *There exists $\sigma(T, J, r) > 0$ such that*

$$\mathcal{T}_{m_2}(E, F) \lesssim 2^{-\sigma m_2}|E|^{\frac{3}{d_\omega+4}}|F|^{1-\frac{1}{d_\omega+4}}.$$

Proof. By Young's Inequality, since $|\tau_{\mathbf{m}}| = x_{\mathbf{m}}y_{\mathbf{m}}$, then $\alpha_{\mathbf{m}} \leq x_{\mathbf{m}}y_{\mathbf{m}}$. First, consider $\alpha_{\mathbf{m}} \geq C\frac{y_{\mathbf{m}}}{x_{\mathbf{m}}^r}x_{\mathbf{m}}y_{\mathbf{m}}$. By Lemmas 3.4.3 and 3.4.5,

$$|E_{\mathbf{m}}|^2 \gtrsim \int_{\Omega'} x_{\mathbf{m}}^{-[J-2r+1]}(\frac{\alpha_{\mathbf{m}}}{x_{\mathbf{m}}y_{\mathbf{m}}}x_{\mathbf{m}}^{J-r})^3 dt ds_1 ds_2$$

$$\gtrsim x_{\mathbf{m}}^{2[J-r-1]}(\tfrac{y_{\mathbf{m}}}{x_{\mathbf{m}}^r})^{-1}(\tfrac{\alpha_{\mathbf{m}}}{x_{\mathbf{m}}y_{\mathbf{m}}})^2\alpha_{\mathbf{m}}\min_t(|\Omega_2(t)|)^2|\Omega_1|$$
$$= x_{\mathbf{m}}^{2[J-r-1]}(\tfrac{y_{\mathbf{m}}}{x_{\mathbf{m}}^r})^{-1}(\tfrac{\alpha_{\mathbf{m}}}{x_{\mathbf{m}}y_{\mathbf{m}}})^2\alpha_{\mathbf{m}}^3\beta_{\mathbf{m}}.$$

We recall that $\beta_{\mathbf{m}} = \frac{1}{4}\frac{\mathcal{T}_{\mathbf{m}}(E,F)}{|E|}$ and $\alpha_{\mathbf{m}} = \frac{1}{4}\frac{\mathcal{T}_{\mathbf{m}}(E_{\mathbf{m}},F)}{|F|} \approx \frac{\mathcal{T}_{\mathbf{m}}(E,F)}{|F|}$, so that

$$|E|^2 \geq |E_{\mathbf{m}}|^2 \gtrsim x_{\mathbf{m}}^{2[J-r-1]}(\tfrac{y_{\mathbf{m}}}{x_{\mathbf{m}}^r})^{-1}(\tfrac{\alpha_{\mathbf{m}}}{x_{\mathbf{m}}y_{\mathbf{m}}})^2\alpha_{\mathbf{m}}^3\beta_{\mathbf{m}} \gtrsim x_{\mathbf{m}}^{2[J-r-1]}(\tfrac{y_{\mathbf{m}}}{x_{\mathbf{m}}^r})^{-1}(\tfrac{\alpha_{\mathbf{m}}}{x_{\mathbf{m}}y_{\mathbf{m}}})^2\frac{\mathcal{T}_{\mathbf{m}}(E,F)^4}{|E|^3|F|}.$$

Then

$$\mathcal{T}_{\mathbf{m}}(E,F) \lesssim (\tfrac{\alpha}{x_{\mathbf{m}}y_{\mathbf{m}}})^{-\frac{1}{2}}x_{\mathbf{m}}^{-\frac{1}{2}[J-r-1]}(\tfrac{y_{\mathbf{m}}}{x_{\mathbf{m}}^r})^{\frac{1}{4}}|E|^{\frac{3}{4}}|F|^{\frac{3}{4}}. \tag{3.4.9}$$

Also, by the definition of $\alpha_{\mathbf{m}}$, we can write the following $L^\infty \to L^\infty$ bound:

$$\mathcal{T}_{\mathbf{m}}(E,F) \lesssim \alpha_{\mathbf{m}}|F| = x_{\mathbf{m}}y_{\mathbf{m}}(\tfrac{\alpha_{\mathbf{m}}}{x_{\mathbf{m}}y_{\mathbf{m}}})|F| = x_{\mathbf{m}}^{r+1}(\tfrac{y_{\mathbf{m}}}{x_{\mathbf{m}}^r})(\tfrac{\alpha_{\mathbf{m}}}{x_{\mathbf{m}}y_{\mathbf{m}}})|F|. \tag{3.4.10}$$

Since $\frac{y_{\mathbf{m}}}{x_{\mathbf{m}}^r} := 2^{-m_2} \lesssim \frac{\alpha_{\mathbf{m}}}{x_{\mathbf{m}}y_{\mathbf{m}}} \lesssim 1$, with $x_{\mathbf{m}} := 2^{m_1}$, define

$$\eta(m_3) := \{\mathbf{m} : \tfrac{\alpha_{\mathbf{m}}}{x_{\mathbf{m}}y_{\mathbf{m}}} \approx 2^{-m_3}\},$$

which induces a function $m_3 = m_3(m_1, m_2)$. Define $\mathcal{T}_{m_2,2} := \sum_{m_1:m_3(m_1,m_2)<m_2-C}\mathcal{T}_{\mathbf{m}}$ and $\mathcal{T}_{m_2,1} := \sum_{m_1:m_3(m_1,m_2)\geq m_2-C}\mathcal{T}_{\mathbf{m}}$, for a suitable $C \geq 0$. Then, combining (3.4.9) and (3.4.10),

$$\begin{aligned}
\mathcal{T}_{m_2,2}(E,F) &\lesssim \sum_{m_3=0}^{m_2-C}\sum_{m_1=0}^{\infty}\min\{2^{\frac{-m_3}{2}}2^{\frac{m_1}{2}[J-r-1]}2^{\frac{-m_2}{4}}|E|^{\frac{3}{4}}|F|^{\frac{3}{4}}, 2^{-(r+1)m_1}2^{-m_2}2^{-m_3}|F|\}\\
&\lesssim \sum_{m_3=0}^{m_2-C}2^{-m_2[\frac{2J-r-1}{2(J+r+1)}]}2^{-m_3[\frac{J-2r-2}{J+r+1}]}|E|^{\frac{3}{2}\frac{r+1}{J+r+1}}|F|^{1-\frac{1}{2}\frac{r+1}{J+r+1}}\\
&\lesssim 2^{-m_2[\frac{\min(2J-r-1,4J-5r-5)}{2(J+r+1)}]}|E|^{\frac{3}{d_\omega+4}}|F|^{1-\frac{1}{d_\omega+4}},
\end{aligned}$$

where we used $d_h = \frac{J}{r+1}$ from (3.4.1) and $d_\omega = 2d_h - 2$ from Lemma 2.1.5 in the last line. Next, since $\omega = Cx^Qy^T + \mathcal{O}(y^{T+1})$, then $d_\omega = \frac{Q+Tr}{r+1} > \frac{Tr}{r+1} \geq \frac{2}{3}T \geq \frac{2}{3}$, and the relationship $d_\omega = 2d_h - 2$ implies that $\frac{J}{r+1} = d_h > \frac{4}{3}$. Thus, $\frac{\min(2J-r-1, 4J-5r-5)}{2(J+r+1)} > 0$, so $\mathcal{T}_{m_2,2}$ satisfies Proposition 3.4.16.

Finally, consider the case $\frac{\alpha_{\mathbf{m}}}{x_{\mathbf{m}} y_{\mathbf{m}}} \lesssim \frac{y_{\mathbf{m}}}{x_{\mathbf{m}}^r}$. By the definition of $\alpha_{\mathbf{m}}$,

$$\mathcal{T}_{\mathbf{m}}(E,F) \lesssim x_{\mathbf{m}}^{r+1} (\tfrac{y_{\mathbf{m}}}{x_{\mathbf{m}}^r})^2 |F|, \tag{3.4.11}$$

and by equation (3.4.2) and Theorem 1.2.1,

$$\mathcal{T}_{\mathbf{m}}(E,F) \lesssim x_{\mathbf{m}}^{-\frac{1}{2}(J-r-1)} (\tfrac{y_{\mathbf{m}}}{x_{\mathbf{m}}^r})^{-\frac{T}{4}} |E|^{\frac{3}{4}} |F|^{\frac{3}{4}}. \tag{3.4.12}$$

Then, combining (3.4.11) and (3.4.12),

$$\begin{aligned}
\mathcal{T}_{m_2,1}(E,F) &\lesssim \sum_{m_1=0}^{\infty} \min\{2^{-(r+1)m_1} 2^{-2m_2} |F|, 2^{\frac{m_1}{2}[J-r-1]} 2^{\frac{m_2 T}{4}} |E|^{\frac{3}{4}} |F|^{\frac{3}{4}}\} \\
&\lesssim 2^{-m_2 \frac{4J-4r-4-T(r+1)}{2[J+r+1]}} |E|^{\frac{3}{2}\frac{r+1}{J+r+1}} |F|^{1-\frac{1}{2}\frac{r+1}{J+r+1}} \\
&= 2^{-m_2 \frac{2Q+T(r-1)}{2[J+r+1]}} |E|^{\frac{3}{d_\omega+4}} |F|^{1-\frac{1}{d_\omega+4}}.
\end{aligned}$$

where we used the relation $2(J-r-1) = Q + Tr$ shown in (3.4.2), as well as $d_h = \frac{J}{r+1}$ from (3.4.1) and $d_\omega = 2d_h - 2$ from Lemma 2.1.5 in the last line. Thus, $\mathcal{T}_{m_2,1}$ satisfies Proposition 3.4.16, so the proof of Proposition 3.4.16 is complete. □

Then, since $m_2 \geq 0$ on R_T, we can sum over all m_2 to get

$$\mathcal{T}_{R_T}(E,F) \leq \sum_{m_2=0}^{\infty} \mathcal{T}_{m_2}(E,F) \lesssim |E|^{\frac{3}{d_\omega+4}} |F|^{1-\frac{1}{d_\omega+4}}.$$

Thus, $\mathcal{T}_{R_T}$ is of rwt $(\frac{d_\omega+4}{3}, d_\omega+4)$, concluding the proof of Proposition 3.4.1. □

Chapter 4

Second Relevant Vertex

4.1 Second Relevant Vertex: Case breakdown

In Lemma 3.1.2, we found that the polygon arising in Theorem 1.2.4 has at most two relevant vertices. With Theorem 1.2.4 now proven for $\mathcal{T}_{[-1,1]^2\setminus\mathcal{R}_T}$, and proven at the first relevant vertex for $\mathcal{T}_{R_T}$, we now turn our attention to proving Theorem 1.2.4 for $\mathcal{T}_{R_T}$ at the second relevant vertex. We recall that the second vertex, $(\frac{1}{p_{v_2}}, \frac{1}{q_{v_2}})$, exists only in cases (N) and (A), and belongs in the set $\overline{Conv}\{(0,0),(\frac{3}{4},\frac{1}{4}),(\frac{2}{3},\frac{1}{3})\}$, either in the interior or on the boundary. Based on the location of $(\frac{1}{p_{v_2}}, \frac{1}{q_{v_2}})$, we further decompose cases (N) and (A).

Definition 4.1.1. Define $\mathcal{D} := \overline{Conv}\{(0,0),(\frac{3}{4},\frac{1}{4}),(\frac{2}{3},\frac{1}{3})\}$. Depending on the location of $(\frac{1}{p_{v_2}}, \frac{1}{q_{v_2}})$, we break Cases (N) and (A) into the following cases:

$(N_{Int}),(A_{Int})$: $(\frac{1}{p_{v_2}}, \frac{1}{q_{v_2}}) \in Int(\mathcal{D})$;

$(N_{q=p'}),(A_{q=p'})$: $q_{v_2} = p'_{v_2}, (\frac{1}{p_{v_2}}, \frac{1}{q_{v_2}}) \neq (\frac{2}{3},\frac{1}{3})$;

$(N_{(\frac{2}{3},\frac{1}{3})})$: $(\frac{1}{p_{v_2}}, \frac{1}{q_{v_2}}) = (\frac{2}{3},\frac{1}{3})$;

$(N^{scal}_{q=2p})$: $q_{v_2} = 2p_{v_2} \neq 3$, $\frac{1}{q_{v_2}} = \frac{1}{p_{v_2}} - \frac{1}{d_h+1}$;

$(N^{\neq scal}_{q=2p})$: $q_{v_2} = 2p_{v_2} \neq 3$, $\frac{1}{q_{v_2}} \neq \frac{1}{p_{v_2}} - \frac{1}{d_h+1}$.

We will show in the following lemma that the above subcases completely cover cases (N) and (A), and we will furthermore identify qualities of the functions φ that belong to each subcase.

Lemma 4.1.2. *Every φ belonging to Cases (N) or (A) belongs to one of the subcases of Definition 4.1.1. Additionally, for each subcase, the following hold, where we refer to the line $\frac{1}{q} = \frac{1}{p} - \frac{1}{d_h+1}$ as the scaling line:*

(A_{Int}): *$(\frac{1}{p_{v_2}}, \frac{1}{q_{v_2}})$ lies on the scaling line, when $\frac{1}{p_{v_2}} = \frac{(2T+5)-(d_h+1)}{(d_h+1)(T+2)}$.*

(N_{Int}): *$(\frac{1}{p_{v_2}}, \frac{1}{q_{v_2}})$ lies on the scaling line, when $\frac{1}{p_{v_2}} = \frac{N+1-d_h}{d_h+1}$. Additionally, in this case $N < d_h + 1$.*

$(N^{scal}_{q=2p}) : (\frac{1}{p_{v_2}}, \frac{1}{q_{v_2}}) = (\frac{2}{N}, \frac{1}{N})$*, with $N = d_h + 1$, and $N > 3$.*

$(N^{\neq scal}_{q=2p}) : (\frac{1}{p_{v_2}}, \frac{1}{q_{v_2}}) = (\frac{2}{N}, \frac{1}{N})$*, with $N > d_h + 1$, and $N > 3$*

$(N_{(\frac{2}{3},\frac{1}{3})}) : (\frac{1}{p_{v_2}}, \frac{1}{q_{v_2}}) = (\frac{2}{3}, \frac{1}{3})$*, and up to a rescaling and a reordering of z_1 and z_2, $\varphi = (z_2 - z_1^2)^3$.*

$(N_{q=p'}) : (\frac{1}{p_{v_2}}, \frac{1}{q_{v_2}})$ *lies where the scaling line intersects the line $q = p'$, and up to a rescaling and a reordering of z_1 and z_2, $\varphi = (z_2 - z_1^2)^2$.*

$(A_{q=p'}) : (\frac{1}{p_{v_2}}, \frac{1}{q_{v_2}})$ *lies where the scaling line intersects the line $q = p'$, and up to a rescaling and a reordering of z_1 and z_2, $\varphi = z_1^2 \pm z_2^S$, for some $S \geq 3$.*

Proof. In case (A), by Corollary 2.1.4, $T = A - 2$. Thus, lines $\frac{1}{q} = \frac{A+1}{2A+1}\frac{1}{p} - \frac{1}{2A+1} = \frac{T+3}{2T+5}\frac{1}{p} - \frac{1}{2T+5}$ and $\frac{1}{q} = \frac{1}{p} - \frac{1}{d_h+1}$ intersect when

$$(\frac{1}{p_{v_2}}, \frac{1}{q_{v_2}}) = (\frac{(2T+5)-(d_h+1)}{(d_h+1)(T+2)}, \frac{(T+3)-(d_h+1)}{(d_h+1)(T+2)}),$$

which proves the statement for subcase (A_{Int}). Since $d_h > 0$ and $T > d_\omega = 2d_h - 2$, $2 < \frac{q_{v_2}}{p_{v_2}} < 3$ is always satisfied. Additionally, the inequality $\frac{1}{p_{v_2}} + \frac{1}{q_{v_2}} < 1$ is equivalent to

$$d_h > \frac{2(T+2)}{T+4}, \tag{4.1.1}$$

which always holds when $d_h \geq 2$. To find when it fails, since φ is not homogeneous in Case (A) by Proposition 2.1.9, then by the definition of A, up to a rescaling and an interchanging of z_1 and z_2,

$$\varphi = z_1^J + c_1 z_2^{ls} z_1^{J-lr} + o(z_2^{ls}), \text{ some } J, l \geq 1, c_1 \neq 0,$$

where $A = ls \geq 2$, and by Corollary 2.1.4, $T = ls - 2$. Then, (4.1.1) is equivalent to $d_h > \frac{2ls}{ls+2}$. Also, from the structure of φ, $d_h = \frac{Js}{s+r}$ and $J \geq \min(lr, 2)$ (J cannot be 1 since $\nabla\varphi(0,0) = 0$). When $lr = 1$, we have $l = r = 1$, and

$$d_h = \tfrac{Js}{s+r} \geq \tfrac{2s}{s+1} > \tfrac{2s}{s+2} = \tfrac{2ls}{ls+2},$$

When $lr > 2$, then $\frac{r}{s+r} > \frac{2}{ls+2}$, and

$$d_h = \tfrac{Js}{s+r} \geq \tfrac{lsr}{s+r} > \tfrac{2ls}{ls+2},$$

When $lr = 2$ and $J > 2$, then $\frac{r}{s+r} = \frac{2}{ls+2}$, so

$$d_h = \tfrac{Js}{s+r} > \tfrac{lsr}{s+r} = \tfrac{2ls}{ls+2}.$$

Thus, q_{v_2} is always greater than p'_{v_2} when either $lr \neq 2$ or when $lr = 2$ and $J \neq 2$. However, if $J = lr = 2$, then $d_h = \frac{lsr}{s+r} = \frac{2ls}{ls+2}$, implying that $q_{v_2} = p'_{v_2}$. And since $J = lr = 2$, after rescaling, φ can be written as

$$\varphi = z_1^2 \pm z_2^S$$

for some $S \geq 3$, as was claimed for subcase ($A_{q=p'}$). This completes the proof for the subcases of Case(A).

For case (N), when $N < d_h + 1$, lines $\frac{1}{q} = \frac{N+1}{N+2}\frac{1}{p} - \frac{1}{N+2}$ and $\frac{1}{q} = \frac{1}{p} - \frac{1}{d_h+1}$ intersect when

$$\left(\tfrac{1}{p_{v_2}}, \tfrac{1}{q_{v_2}}\right) = \left(\tfrac{(N+2)-(d_h+1)}{d_h+1}, \tfrac{(N+1)-(d_h+1)}{d_h+1}\right),$$

as stated for subcase (N_{Int}). Then

$$\frac{q_{v_2}}{p_{v_2}} = \frac{(N+2)-(d_h+1)}{(N+1)-(d_h+1)}$$

satisfies $\frac{q_{v_2}}{p_{v_2}} < 3$ since $T > 2d_h - 2$, and $\frac{q_{v_2}}{p_{v_2}} > 2$ is satisfied since $N < d_h + 1$.

Furthermore, since $\frac{1}{p_{v_2}} + \frac{1}{q_{v_2}} = \frac{(2N+3)-(2d_h+2)}{d_h+1}$, then the inequality $\frac{1}{p_{v_2}} + \frac{1}{q_{v_2}} < 1$ is equivalent to

$$d_h > \tfrac{2}{3}N.$$

First, if $N > 2$, then $N < d_h + 1$ implies that $d_h > \frac{2}{3}N$. Now, for some $k \geq 0$ in $\mathbb{N} \cup \{0\}$, depending on φ, $d_h = \frac{rN+k}{r+1}$. When $r > 2$ or $k > 0$, then

$$d_h = \frac{rN+k}{r+1} > \tfrac{2}{3}N.$$

Therefore, when $N < d_h + 1$, then if $N \neq 2$, or $k \neq 0$, or $r \neq 2$, then $q_{v_2} > p'_{v_2}$.

However, when $r = 2$, $k = 0$, and $N = 2$, then $N < d_h + 1$ and $d_h = \frac{2}{3}N$, implying that $q_{v_2} = p'_{v_2}$ and that up to rescaling and a swapping of z_1 and z_2,

$$\varphi = (z_2 - z_1^2)^2,$$

as was claimed for subcase ($N_{q=p'}$).

Finally, for case (N), with $N \geq d_h + 1$, the lines $\frac{1}{q} = \frac{N+1}{N+2}\frac{1}{p} - \frac{1}{N+2}$ and $\frac{1}{q} = \frac{1}{p} - \frac{1}{N}$ intersect when $(\frac{1}{p_{v_2}}, \frac{1}{q_{v_2}}) = (\frac{2}{N}, \frac{1}{N})$, as stated for subcases ($N^{scal}_{q=2p}$) and ($N^{\neq scal}_{q=2p}$). Thus, when $N \geq d_h + 1$, $\frac{q_{v_2}}{p_{v_2}} = 2$ is always satisfied.

If $N > 3$, then $\frac{1}{p_{v_2}} + \frac{1}{q_{v_2}} < 1$. Now, for some $k \geq 0$ in $\mathbb{N} \cup \{0\}$, depending on φ, $d_h = \frac{rN+k}{r+1}$. Thus, when either $N > d_h + 1$, or when $N = d_h + 1$ and $l \neq 0$ or $r \neq 2$, N must be larger than 3 implying that $\frac{1}{p_{v_2}} + \frac{1}{q_{v_2}} < 1$.

Alternatively, when $N = d_h + 1$, $r = 2$, $l = 0$, and $N = 3$, then $(\frac{1}{p_{v_2}}, \frac{1}{q_{v_2}}) = (\frac{2}{3}, \frac{1}{3})$, and up to a rescaling and a swapping of z_1 and z_2,

$$\varphi = (z_2 - z_1^2)^3,$$

as was claimed for subcase $(N_{(\frac{2}{3},\frac{1}{3})})$. This completes the proof for the subcases of Case(N). $\square$

4.2 The rwt $(\frac{3}{2}, 3)$ bound and the second relevant vertex.

In this section, we will go through each subcase from Definition 4.1.1, proving that $\mathcal{T}_{R_T}$ is of rwt (p_{v_2}, q_{v_2}) and completing the proof of Theorem 1.2.4.

4.2.1 Case $(A_{q=p'})$

By Lemma 4.1.2, after rescaling, we may assume

$$\varphi = t_1^2 \pm t_2^S,$$

where $S \geq 3$. By symmetry, it suffices to consider $\mathcal{T}$ over the set $\{t_1, t_2 \geq 0\}$. We decompose $(\mathbb{R}_+)^2$ into strips $S_j = \{t_2 \approx 2^{-j}\}$. This induces a decomposition $\mathcal{T} = \sum \mathcal{T}_j$. We now bound $\|T_j\|_{L^{\frac{3}{2}} \to L^3}$ using Minkowski's Inequality, Theorem 1.2.1 on $\mathbb{R}^2$, and Young's Inequality as follows:

$$\begin{aligned}\|\mathcal{T}_j f\|_{L^3(\mathbb{R}^3)} &= \|f(x - (t, \varphi(t)))\|_{L^3_x L^1_t(\{t \in S_j\})} \\ &\lesssim \|f(x - (t, \varphi(t)))|\partial_{t_1}^2 \varphi|^{\frac{1}{3}}\|_{L^3_{x_2} L^1_{t_2} L^3_{x_1 x_3} L^1_{t_1}(\{t \in S_j\})}\end{aligned}$$

$$\lesssim \|f(y_1, x_2 - t_2, y_3)\|_{L^3_{x_2} L^1_{t_2} L^{\frac{3}{2}}_{y_1 y_3}(\{t_2 \sim 2^{-j}\})}$$

$$= \|(\|f(y_1, \cdot, y_3)\|_{L^{\frac{3}{2}}_{y_1 y_3}} * \chi_{\{\cdot \approx 2^{-j}\}})\|_{L^3}$$

$$\leq \|\chi_{\{\cdot \approx 2^{-j}\}}\|_{L^{\frac{3}{2}}} \|f\|_{L^{\frac{3}{2}}(\mathbb{R}^3)} = 2^{-\frac{2}{3}j} \|f\|_{L^{\frac{3}{2}}(\mathbb{R}^3)}. \tag{4.2.1}$$

Since $\omega := detD^2\varphi \approx t_2^{S-2} \approx 2^{-(S-2)j}$ on S_j, Theorem 1.2.1 implies

$$\|\mathcal{T}_j f\|_{L^4(\mathbb{R}^3)} \lesssim 2^{\frac{(S-2)j}{4}} \|f\|_{L^{\frac{4}{3}}(\mathbb{R}^3)}. \tag{4.2.2}$$

Combining (4.2.1) and (4.2.2),

$$\mathcal{T}_{(\mathbb{R}_+)^2}(E, F) \lesssim \sum_{j=-\infty}^{\infty} \min(2^{-\frac{2}{3}j} |E|^{\frac{2}{3}} |F|^{\frac{2}{3}}, 2^{\frac{(S-2)j}{4}} |E|^{\frac{3}{4}} |F|^{\frac{3}{4}}) \approx |E|^{\frac{2S+2}{3S+2}} |F|^{\frac{2S+2}{3S+2}}.$$

Thus, $\mathcal{T}_{\mathbb{R}^2}$ is of rwt (p_S, q_S) for $(\frac{1}{p_S}, \frac{1}{q_S}) = (\frac{2S+2}{3S+2}, \frac{S}{3S+2})$. As $\frac{1}{p_S} + \frac{1}{q_S} = 1$ and (p_S, q_S) lies on the scaling line of $\mathcal{T}$, then $(p_S, q_S) = (p_{v_2}, q_{v_2})$ by Lemma 4.1.2. □

4.2.2 Case $(N_{q=p'})$

By Lemma 4.1.2, after rescaling, we may assume

$$\varphi = (t_2 - t_1^2)^2.$$

We decompose $\mathbb{R}^2$ into strips $S_j = \{|t_2 - t_1^2| \approx 2^{-j}\}$, which induces a decomposition $\mathcal{T} = \sum \mathcal{T}_j$. By the change of coordinates $u_1 := t_1$, $u_2 := t_2 - t_1^2$, (setting $\tilde{\varphi}(u_1, u_2) := u_2 + u_1^2$), Minkowski's Inequality, Theorem 1.2.1 (on $\mathbb{R}^2$), the change of variables $v = u_2^2$, and Young's Inequality,

$$\|\mathcal{T}_j f\|_{L^3(\mathbb{R}^3)} := \|f(x - (t, \varphi(t))\|_{L^3_x L^1_t(\{t \in S_j\})}$$

$$\approx \|f(x - (u_1, \tilde{\varphi}(u), u_2^2))|\partial^2_{u_1} \tilde{\varphi}|^{\frac{1}{3}}\|_{L^3_x L^1_u(\{|u_2| \sim 2^{-j}\})}$$

$$\leq \|f(x-(u_1,\tilde{\varphi}(u),u_2^2))|\partial_{u_1}^2\tilde{\varphi}|^{\frac{1}{3}}\|_{L^3_{x_3}L^1_{u_2}L^3_{x_1x_2}L^1_{u_1}(\{|u_2|\sim 2^{-j}\})}$$

$$\lesssim \|f(y_1,y_2,x_3-u_2^2)\|_{L^3_{x_3}L^1_{u_2}L^{\frac{3}{2}}_{y_1y_2}(\{|u_2|\sim 2^{-j}\})}$$

$$\approx 2^j\|f(y_1,y_2,x_3-v)\|_{L^3_{x_3}L^1_{v}L^{\frac{3}{2}}_{y_1y_2}(\{v\sim 2^{-2j}\})}$$

$$= 2^j\|\|f(y_1,y_2,\cdot)\|_{L^{\frac{3}{2}}_{y_1y_2}} * \chi_{\{|\cdot|\approx 2^{-2j}\}}\|_{L^3}$$

$$\leq 2^j\|\chi_{\{|\cdot|\approx 2^{-2j}\}}\|_{L^{\frac{3}{2}}}\|f\|_{L^{\frac{3}{2}}(\mathbb{R}^3)} = 2^j2^{-\frac{4j}{3}}\|f\|_{L^{\frac{3}{2}}(\mathbb{R}^3)} = 2^{-\frac{j}{3}}\|f\|_{L^{\frac{3}{2}}(\mathbb{R}^3)}. \quad (4.2.3)$$

Since $|\omega| := |det(D^2\varphi)| \approx |t_2-t_1^2| \approx 2^{-j}$ on S_j, Theorem 1.2.1 implies

$$\|\mathcal{T}_jf\|_{L^4(\mathbb{R}^3)} \lesssim 2^{\frac{j}{4}}\|f\|_{L^{\frac{4}{3}}(\mathbb{R}^3)}. \quad (4.2.4)$$

Combining 4.2.3 and 4.2.4,

$$\mathcal{T}_{\mathbb{R}^2}(E,F) \lesssim \sum_{j=-\infty}^{\infty}\min(2^{-\frac{1}{3}j}|E|^{\frac{2}{3}}|F|^{\frac{2}{3}}, 2^{\frac{j}{4}}|E|^{\frac{3}{4}}|F|^{\frac{3}{4}}) \approx |E|^{\frac{5}{7}}|F|^{\frac{5}{7}}.$$

Thus, $\mathcal{T}_{\mathbb{R}^2}$ is of rwt $(\frac{7}{5},\frac{7}{2})$. As $\frac{5}{7}+\frac{2}{7}=1$, and as $(\frac{7}{5},\frac{7}{2})$ lies on the scaling line of $\mathcal{T}$, then $(\frac{7}{5},\frac{7}{2}) = (p_{v_2},q_{v_2})$ by Theorem 4.1.2.

4.2.3 Case $(N_{(\frac{2}{3},\frac{1}{3})})$

By Lemma 4.1.2, after rescaling, we may assume

$$\varphi = (t_2-t_1^2)^3.$$

By the change of variable $u_1 := t_1$, $u_2 := t_2 - t_1^2$ (setting $\tilde{\varphi}(u_1,u_2) := u_2+u_1^2$), Minkowski's inequality, Theorem 1.2.1 (on $\mathbb{R}^2$), the change of variables $v = u_2^3$, and Young's Inequality,

$$\|\mathcal{T}f\|_{L^3(\mathbb{R}^3)} = \|f(x-(t,\varphi(t)))\|_{L^3_xL^1_t}$$

$$
\begin{aligned}
&\approx \|f(x-(u_1,\tilde{\varphi}(u),u_2^3))|\partial_{u_1}^2\tilde{\varphi}|^{\frac{1}{3}}\|_{L_x^3L_u^1}\\
&\leq \|f(x-(u_1,\tilde{\varphi}(u),u_2^3))|\partial_{u_1}^2\tilde{\varphi}|^{\frac{1}{3}}\|_{L_{x_3}^3L_{u_2}^1L_{x_1x_2}^3L_{u_1}^1}\\
&\lesssim \|f(y_1,y_2,x_3-u_2^3)\|_{L_{x_3}^3L_{u_2}^1L_{y_1y_2}^{\frac{3}{2}}}\\
&\approx \|v^{-\frac{2}{3}}\|f(y_1,y_2,x_3-v)\|_{L_{y_1y_2}^{\frac{3}{2}}}\|_{L_{x_3}^3L_v^1}\\
&= \|\|f(y_1,y_2,\cdot)\|_{L_{y_1y_2}^{\frac{3}{2}}}*(\cdot)^{-\frac{2}{3}}\|_{L^3}\leq\|(\cdot)^{-\frac{2}{3}}\|_{L^{\frac{3}{2},w}}\|f\|_{L^{\frac{3}{2}}(\mathbb{R}^3)}=\|f\|_{L^{\frac{3}{2}}(\mathbb{R}^3)}.
\end{aligned}
$$

Thus, $\mathcal{T}_{\mathbb{R}^2}$ has a strong type bound at $(p,q)=(\frac{3}{2},3)$, which equals (p_{v_2},q_{v_2}) by Lemma 4.1.2. $\square$

4.2.4 Case (A_{Int})

In Case(A), we can assume that $f_T=t_2$, so that t_2 has multiplicity T in ω and multiplicity 0 in φ. Thus, after rescaling, because of mixed homogeneity, φ will take the following form:

$$\varphi=t_1^J\pm t_2^{ls}t_1^{J-lr}(1+\mathcal{O}(\tfrac{t_2^s}{t_1^r}))\quad\text{where}\quad ls\geq 2, r\geq 1, J\geq 1,$$

for some l,J. We recall from the definition of case(A) that $A=ls\geq 2$. Then, by Lemma 2.1.2, ω will take the form

$$\omega=Ct_1^{2J-lr-2}t_2^{ls-2}(1+\mathcal{O}(\tfrac{t_2^s}{t_1^r})),\quad C\neq 0.$$

Thus, $T=ls-2=A-2$, and we define $Q:=2J-lr-2$, so that $|\omega|\approx|t_1|^Q|t_2|^T$ on R_T^e, where $|t_2|^s\leq\tilde{\epsilon}|t_1|^r$. By symmetry, it will suffice to consider only the region where $t_1,t_2\geq 0$. We define $S:=R_T^e\cap\{|\omega|\approx 1\}\cap\{t_1,t_2\geq 0\}$, which decomposes into regions $\tau_j:=\{t_1\approx 2^{\frac{j}{Q}},t_2\approx 2^{-\frac{j}{T}}\}$ and induces the decomposition $\mathcal{T}_S=\sum_j T_j$. Then, by

Minkowski's Inequality, Theorem 1.2.1 (on $\mathbb{R}^2$), and Young's Inequality,

$$\begin{aligned}
\|\mathcal{T}_j f\|_{L_3(\mathbb{R}^3)} &= \|f(x-(t,\varphi(t))\|_{L^3_x L^1_t(\{t\in\tau_j\})} \\
&\approx 2^{-\frac{1}{3}\frac{j}{Q}(J-2)}\|f(x-(t,\varphi(t))|\partial^2_{t_1}\varphi|^{\frac{1}{3}}\|_{L^3_x L^1_t(\{t\in\tau_j\})} \\
&\leq 2^{-\frac{1}{3}\frac{j}{Q}(J-2)}\|f(x-(t,\varphi(t))|\partial^2_{t_1}\varphi|^{\frac{1}{3}}\|_{L^3_{x_2}L^1_{t_2}L^3_{x_1x_3}L^1_{t_1}(\{t\in\tau_j\})} \\
&\lesssim 2^{-\frac{1}{3}\frac{j}{Q}(J-2)}\|f(y_1,x_2-t_2,y_3)\|_{L^3_{x_2}L^1_{t_2}L^{\frac{3}{2}}_{y_1y_3}(\{t_2\sim 2^{-j/T}\})} \\
&= 2^{-\frac{1}{3}\frac{j}{Q}(J-2)}\|\|f(y_1,\cdot,y_3)\|_{L^{\frac{3}{2}}_{y_1y_3}} * \chi_{\{\cdot\approx 2^{-\frac{j}{T}}\}}\|_{L^3} \\
&\leq 2^{-\frac{1}{3}\frac{j}{Q}(J-2)}\|\chi_{\{\cdot\approx 2^{-\frac{j}{T}}\}}\|_{L^{\frac{3}{2}}}\|f\|_{L^{\frac{3}{2}}(\mathbb{R}^3)} = 2^{-\frac{1}{3}\frac{j}{Q}(J-2)}2^{-\frac{2}{3}\frac{j}{T}}\|f\|_{L^{\frac{3}{2}}(\mathbb{R}^3)} \\
&= 2^{-\frac{j}{3}[\frac{(J-2)}{Q}+\frac{2}{T}]}\|f\|_{L^{\frac{3}{2}}(\mathbb{R}^3)}. \qquad (4.2.5)
\end{aligned}$$

Additionally, since $|\tau_j| \approx 2^{(\frac{1}{Q}-\frac{1}{T})j}$, we have

$$||\mathcal{T}_j||_{\infty\to\infty} \leq 2^{(\frac{1}{Q}-\frac{1}{T})j}. \qquad (4.2.6)$$

Combining (4.2.5) and (4.2.6),

$$\mathcal{T}_j(E,F) \lesssim \min(2^{(\frac{1}{Q}-\frac{1}{T})j}|F|, 2^{-\frac{j}{3}[\frac{(J-2)}{Q}+\frac{2}{T}]}|E|^{\frac{2}{3}}|F|^{\frac{2}{3}}).$$

Noting that $T-Q>0$ since $T>d_\omega = \frac{Qs+Tr}{r+s}$, we interpolate as follows:

$$\begin{aligned}
T_S(E,F) &\lesssim \sum_{j=-\infty}^{\infty} \min(2^{(\frac{1}{Q}-\frac{1}{T})j}|F|, 2^{-\frac{j}{3}[\frac{(J-2)}{Q}+\frac{2}{T}]}|E|^{\frac{2}{3}}|F|^{\frac{2}{3}}) \\
&= |E|^{\frac{2(T-Q)}{(J+1)T-Q}}|F|^{1-\frac{T-Q}{(J+1)T-Q}}.
\end{aligned}$$

Thus, $\mathcal{T}_{R^e_T\cap\{|\omega|\approx 1\}}$ is of rwt $(\frac{q_I}{2}, q_I)$ for $(\frac{2}{q_I}, \frac{1}{q_I}) = (\frac{2(T-Q)}{(J+1)T-Q}, \frac{T-Q}{(J+1)T-Q})$. Using equalities $Qs+Tr = d_\omega(r+s)$, $d_h = \frac{Js}{r+s}$, and $d_\omega = 2d_h-2$, and some messy arithmetic, we can rewrite $\frac{1}{q_I}$ as

$$\frac{1}{q_I} = \frac{2(T-d_\omega)}{4(T-d_\omega)+(T+2)d_\omega}. \qquad (4.2.7)$$

Since $\mathcal{T}_{R_T^e \cap \{|\omega|\approx 1\}}$ is of stong type $(\frac{4}{3}, 4)$ by Theorem 1.2.1, and of rwt (p_I, q_I), where $q_I = 2p_I$, and since R_T^e is κ_φ-scale invariant, then by Case 3 of Proposition 2.3.4, $\mathcal{T}_{R_T^e}$ is of rwt (p_S, q_S), when (p_S, q_S) lies on the scaling line of $\mathcal{T}$ and

$$\frac{1}{p_S} = \frac{3 - \frac{8}{q_I} + \frac{1}{2}\frac{1}{q_I}(d_\omega + 4)}{(1 - \frac{2}{q_I})(d_\omega + 4)}.$$

Since (p_S, q_S) lies on the scaling line, then by Lemma 4.1.2 it suffices to verify that $p_S = p_{v_2}$. Using (4.2.7) and the identity $d_\omega = 2d_h - 2$, after some messy arithmetic, we find that

$$\frac{1}{p_S} = \frac{2T + 4 - d_h}{(T+2)(d_h+1)}.$$

Thus, by Lemma 4.1.2, $p_{v_2} = p_S$, and so $\mathcal{T}_{R_T^e}$ is of rwt (p_{v_2}, q_{v_2}). □

4.2.5 Cases (N_{Int}), $(N_{q=2p}^{\neq scal})$, $(N_{q=2p}^{scal})$

In Cases(N), we can assume after rescaling that $f_T = z_2 - z_1^r$, so that $z_2 - z_1^r$ has multiplicity T in ω and multiplicity N in φ. Thus, after rescaling, because of mixed homogeneity, φ can take the following form:

$$\varphi = z_1^J (z_2 - z_1^r)^N (1 + \mathcal{O}(\tfrac{z_2 - z_1^r}{z_1^r})), \qquad \text{with } d_h = \tfrac{J + rN}{r+1},$$

for some J. Then, by Lemma 2.1.1,

$$\omega = C z_1^{2J+r-2} (z_2 - z_1^r)^{2N-3} (1 + \mathcal{O}(\tfrac{z_2 - z_1^r}{z_1^r})), \quad C \neq 0. \tag{4.2.8}$$

Thus, $T = 2N - 3$. We define $Q := 2J + r - 2$, so that $|\omega| \approx |z_1|^Q |z_2 - z_1^r|^T$ on $R_T^e = \{|z_2 - z_1^r| < \tilde{\epsilon}|z_1|^r\}$. To simplify the argument, we will demonstrate the proof when $z_1, z_2 - z_1^r \geq 0$. All other regions follow similarly. We decompose $R_T^e \cap \{z_1, z_2 - z_1^r \geq 0\}$ into sets $\tau_{j,k} := \{z_1 \approx 2^{-j} := \mathbb{x}_j, |z_2 - z_1^r| \approx 2^{-k} := \mathbb{y}_k\}$, which induces a decomposition

$\mathcal{T}_{R_T^e \cap \{z_1, z_2 - \lambda z_1^r \geq 0\}} = \sum_{j,k} \mathcal{T}_{j,k}$. For ease, we perform the change of variables $x = z_1, y = z_2 - z_1^r$. To find the rwt $(\frac{3}{2}, 3)$ bound in these cases, our previous methods will not work, so we instead use a variant of the method of refinements.

Lemma 4.2.1. *The operator $\mathcal{T}_{j,k}$ satisfies the following rwt $(\frac{3}{2}, 3)$ bound:*

$$\mathcal{T}_{j,k}(E, F) \lesssim 2^{\frac{j}{3}(J+r-2)} 2^{\frac{k}{3}(N-3)} |E|^{\frac{2}{3}} |F|^{\frac{2}{3}} = \left(\mathrm{x}_j^{J+r-2} \mathrm{y}_k^{N-3}\right)^{-\frac{1}{3}} |E|^{\frac{2}{3}} |F|^{\frac{2}{3}}$$
$$= 2^{\frac{j}{3}[(r+1)(1+d_h) - (rN+3)] + \frac{k}{3}(N-3)} |E|^{\frac{2}{3}} |F|^{\frac{2}{3}}.$$

Proof. First, we will refine $E \rightsquigarrow E_{j,k}$ and $F \rightsquigarrow F_{j,k}$ as follows. Define

$$F_{j,k} := \{u \in F : \mathcal{T}_{j,k}\chi_E(u) \geq \tfrac{1}{4} \tfrac{\mathcal{T}_{j,k}(E,F)}{|F|}\}$$
$$E_{j,k} := \{w \in E : \mathcal{T}_{j,k}^* \chi_{F_{j,k}}(w) \geq \tfrac{1}{4} \tfrac{\mathcal{T}_{j,k}(E, F_{j,k})}{|E|} =: \alpha\}$$
$$F'_{j,k} := \{u \in F_{j,k} : \mathcal{T}_{j,k}\chi_{E_{j,k}}(u) \geq \tfrac{1}{4} \tfrac{\mathcal{T}_{j,k}(E_{j,k}, F_{j,k})}{|F_{j,k}|} =: \beta\}.$$

Then $\mathcal{T}_{j,k}(E_{j,k}, F_{j,k}) \approx \mathcal{T}_{j,k}(E, F)$ by the short argument leading to (3.2.1). To proceed, we will construct a map, and use the size of the Jacobian to prove the lemma. First, we define $\psi(x) := x^r$ and $z(t) := (t_1, t_2 + \psi(t_1))$. Then, the surface equation has the following equivalent forms: $(z_1, z_2, \varphi(z_1, z_2)) = (x, y + \psi(x), \varphi(x, y + \psi(x)) = (z(x, y), \varphi(z(x, y))$.

Fix $u_0 \in F'_{j,k}$ and define

$$\Omega_1 := \{t \in \mathbb{R}^2 : z(t) \in \tau_{j,k} \text{ and } u_0 - (t_1, t_2 + \psi(t_1), \varphi(z(t))) =: w(t) \in E_{j,k}\}.$$

Then $|\Omega_1| = \mathcal{T}_{j,k}\chi_{E_{j,k}}(u_0) \geq \beta$. To achieve a lower-dimensional result with the method of refinements, we fix one variable. Rewriting $|\Omega_1|$,

$$\frac{|\Omega_1|}{2^{-k}} = \fint_{t_2 \approx 2^{-k}} \int_{t_1 \approx 2^{-j}} \chi_{\Omega_1}(t_1, t_2) dt_1 dt_2.$$

Therefore, by Hölder's inequality, there exists a fixed $t_2 \approx 2^{-k}$ such that

$$|\Omega_{1,t_2}| := |\{t_1 : t \in \Omega_1\}| = \int_{t_1 \approx 2^{-j}} \chi_{\Omega_1}(t_1, t_2) dt_1 \geq \tfrac{\beta}{2^{-k}} = \tfrac{\beta}{\mathrm{y}_k}.$$

For $t_1 \in \Omega_{1,t_2}$, define

$$\Omega_{2,t_2}(t_1) := \{s \in \mathbb{R}^2 : s \in \tau_{j,k} \text{ and } w(t) + (s_1, s_2 + \psi(s_1), \varphi(s)) \in F_{j,k}\}.$$

Then $|\Omega_{2,t_2}(t_1)| = T^*_{j,k}\chi_{F_{j,k}}(y(t)) \geq \alpha$.

Define $\Omega_{t_2} := \{(t_1, s_1, s_2) \in \mathbb{R}^3 : t_1 \in \Omega_{1,t_2}, s = (s_1, s_2) \in \Omega_{2,t_2}(t_1)\}$, and

$$\Psi_{t_2}(t_1, s_2, s_2) := x_0 - (t_1, t_2 + \psi(t_1), \varphi(z(t))) + (s_1, s_2 + \psi(s_1), \varphi(z(s))).$$

Since Ψ_{t_2} is a polynomial mapping $\Omega \subset \mathbb{R}^3$ into $\mathbb{R}^3$, it is $\mathcal{O}(1)$-to-one off a set of measure zero. Since $\Psi_{t_2}(\Omega_{t_2}) \subset F_{j,k}$, we have $|F_{j,k}| \gtrsim \int_{\Omega_{t_2}} |\det D\Psi_{t_2}(t_1, s_1, s_2)| dt_1 ds_1 ds_2$. Expanding,

$$\begin{aligned}|\det D\Psi_{t_2}(t_1, s_1, s_2)| &= |[\psi'(t_1) - \psi'(s_1)]\partial_{s_2}\varphi(z(s)) - [\partial_{t_1}\varphi(t) - \partial_{s_1}\varphi(z(s))]| \\ &= |\psi''(\tilde{x})\partial_{s_2}\varphi(z(s))(t_1 - s_1) - (\partial_{v_1}, \partial_{v_2})\partial_{v_1}\varphi(z(v)) \cdot (t - s)| \\ &= |[\psi''(\tilde{x})\partial_{s_2}\varphi(z(s)) - \partial_{v_1v_1}\varphi(z(v))](t_1 - s_1) - \partial_{v_1v_2}\varphi(z(v))(t_2 - s_2)|,\end{aligned}$$

for some $\tilde{x}$ between t_1 and s_1, and some v lying on the line between t and s, by the Mean Value Theorem.

Comparing each term, and recalling definition $\mathbb{x}_j := 2^{-j}$, $\mathbb{y}_k := 2^{-k}$:

$$\begin{aligned}|\psi''(\tilde{x})\partial_{s_2}\varphi(z(s))| &\approx \mathbb{x}_j^{r-2}\mathbb{x}_j^J\mathbb{y}_k^{N-1} = \mathbb{x}_j^{J+r-2}\mathbb{y}_k^{N-1}; \\ |\partial_{v_1v_1}\varphi(z(v))| &\approx \mathbb{x}_j^{J-2}\mathbb{y}_k^N; \\ |\partial_{v_1v_2}\varphi(z(v))| &\approx \mathbb{x}_j^{J-1}\mathbb{y}_k^{N-1}.\end{aligned}$$

Since $\mathbb{y}_k < \epsilon\mathbb{x}_j^r$ on R_T^e, we have $|\psi''(\tilde{x})\partial_{s_2}\varphi(z(s))| >> |\partial_{v_1v_1}\varphi(v)|$, so we can disregard the $\partial_{v_1v_1}\varphi(z(v))$ term. Comparing the remaining terms, as long as

$$|s_1 - t_1|\mathbb{x}_j^{r-1} >> |s_2 - t_2| \tag{4.2.9}$$

the $\psi''(\tilde{x})\partial_{s_2}\varphi(z(s))(t_1 - s_1)$ term dominates, and we get

$$|\det D\Psi_{t_2}(t_1, s_1, s_2)| \approx |s_1 - t_1|\mathbb{x}_j^{J+r-2}\mathbb{y}_k^{N-1}. \tag{4.2.10}$$

Let t_1 be fixed. Since $|\Omega_{2,t_2}(t_1)| \geq \alpha$ and since $s \in \Omega_{2,t_2}(t_1)$ implies $s_2 \approx \mathbb{y}_k$, then on over half of $\Omega_{2,t_2}(t_1)$, we have $|s_1 - t_1| \gtrsim \frac{\alpha}{\mathbb{y}_k}$. Also, since $s_2, t_2 \approx 2^{-k} = \mathbb{y}_k$, we get $|s_2 - t_2| \lesssim \mathbb{y}_k$.

Thus, based on (4.2.9), we have two cases: $\frac{\alpha}{\mathbb{y}_k}\mathbb{x}_j^{r-1} >> |s_2 - t_2|$ and $\frac{\alpha}{\mathbb{y}_k}\mathbb{x}_j^{r-1} \lesssim \mathbb{y}_k$. For the first case, if $\frac{\alpha}{\mathbb{y}_k}\mathbb{x}_j^{r-1} >> \mathbb{y}_k$, then (4.2.9) holds on over half of $\Omega_{2,t_2}(t_1)$, and

$$|\det D\Psi_{t_2}(t_1, s_1, s_2)| \approx |s_1 - t_1|\mathbb{x}_j^{J+r-2}\mathbb{y}_k^{N-1} \gtrsim \frac{\alpha}{\mathbb{y}_k}\mathbb{x}_j^{J+r-2}\mathbb{y}_k^{N-1}$$

on over half of $\Omega_{2,t_2}(t_1)$, implying that

$$\begin{aligned} |F_{j,k}| &\gtrsim \int_\Omega \alpha \mathbb{y}_k^{-1}\mathbb{x}_j^{J+r-2}\mathbb{y}_k^{N-1} dt_1 ds_1 ds_2 \gtrsim \alpha \mathbb{x}_j^{J+r-2}\mathbb{y}_k^{N-2} \min_{t_1}(|\Omega_2(t_1)|)|\Omega_{1,t_2}| \\ &\gtrsim \alpha \mathbb{x}_j^{J+r-2}\mathbb{y}_k^{N-2}\alpha\frac{\beta}{\mathbb{y}_k} = \mathbb{x}_j^{J+r-2}\mathbb{y}_k^{N-3}\alpha^2\beta. \end{aligned}$$

By definition, $\beta = \frac{1}{4}\frac{\langle \mathcal{T}_{j,k}\chi_E, \chi_F\rangle}{|F|}$ and $\alpha = \frac{1}{4}\frac{\langle \mathcal{T}_{j,k}\chi_E, \chi_{F_0}\rangle}{|E|} \approx \frac{\mathcal{T}_{j,k}(E,F)}{|E|}$, and so

$$|F| \geq |F_{j,k}| \gtrsim \mathbb{x}_j^{J+r-2}\mathbb{y}_k^{N-3}\alpha^2\beta \gtrsim \mathbb{x}_j^{J+r-2}\mathbb{y}_k^{N-3}\frac{\mathcal{T}_{j,k}(E,F)^3}{|E|^2|F|}.$$

Then, using $d_h = \frac{J+rN}{r+1}$, with $\mathbb{x}_j := 2^{-j}$ and $\mathbb{y}_k := 2^{-k}$, we conclude

$$\begin{aligned} \mathcal{T}_{j,k}(E,F) &\lesssim (\mathbb{x}_j^{J+r-2}\mathbb{y}_k^{N-3})^{-\frac{1}{3}}|E|^{\frac{2}{3}}|F|^{\frac{2}{3}} \\ &= 2^{\frac{j}{3}[(r+1)(1+d_h)-(rN+3)]+\frac{k}{3}(N-3)}|E|^{\frac{2}{3}}|F|^{\frac{2}{3}} \\ &= 2^{-j}2^{-k}2^{-\frac{j}{3}[(r+1)(d_h+1)-rN]}2^{\frac{kN}{3}}|E|^{\frac{2}{3}}|F|^{\frac{2}{3}}, \end{aligned}$$

concluding the first case where $\frac{\alpha}{\mathbb{y}_k}\mathbb{x}_j^{r-1} >> |s_2 - t_2|$.

For the second case, we combine $\frac{\alpha}{\mathbb{y}_k}\mathbb{x}_j^{r-1} \lesssim \mathbb{y}_k$ and $\mathcal{T}_{j,k}(E,F) \lesssim \alpha|E|$ to get the $L^1 \to L^1$ bound

$$\mathcal{T}_{j,k}(E,F) \lesssim \mathbb{y}_k^2 \mathbb{x}_j^{-(r-1)}|E|. \tag{4.2.11}$$

Since $\tau_{j,k}$ has measure $\mathbb{x}_j\mathbb{y}_k$, Young's Inequality gives an $L^\infty \to L^\infty$ bound of

$$\mathcal{T}_{j,k}(E,F) \lesssim \mathbb{x}_j\mathbb{y}_k|F|. \tag{4.2.12}$$

Interpolating (4.2.11) and (4.2.12), we get the $L^2 \to L^2$ bound

$$\mathcal{T}_{j,k}(E,F) \lesssim \mathbb{y}_k^{\frac{3}{2}} \mathbb{x}_j^{-\frac{1}{2}(r-2)}|E|^{\frac{1}{2}}|F|^{\frac{1}{2}}. \tag{4.2.13}$$

Next, by (4.2.8), we know that on $\tau_{j,k} \subset R_T^e$,

$$|\omega| \approx \mathbb{x}_j^{2J+r-2}\mathbb{y}_k^{2N-3},$$

so by Theorem 1.2.1, we have the $L^{\frac{4}{3}} \to L^4$ bound

$$\mathcal{T}_{j,k}(E,F) \lesssim (\mathbb{x}_j^{2J+r-2}\mathbb{y}_k^{2N-3})^{-\frac{1}{4}}|E|^{\frac{3}{4}}|F|^{\frac{3}{4}}. \tag{4.2.14}$$

Finally, interpolating (4.2.13) and (4.2.14),

$$\begin{aligned}\mathcal{T}_{j,k}(E,F) &\lesssim \left(\mathbb{y}_k^{\frac{3}{2}} \mathbb{x}_j^{-\frac{1}{2}(r-2)}|E|^{\frac{1}{2}}|F|^{\frac{1}{2}}\right)^{\frac{1}{3}} \left((\mathbb{x}_j^{2J+r-2}\mathbb{y}_k^{2N-3})^{-\frac{1}{4}}|E|^{\frac{3}{4}}|F|^{\frac{3}{4}}\right)^{\frac{2}{3}} \\ &= \left(\mathbb{x}_j^{J+r-2}\mathbb{y}_k^{N-3}\right)^{-\frac{1}{3}}|E|^{\frac{2}{3}}|F|^{\frac{2}{3}},\end{aligned}$$

as in the first case. Thus, the proof of Lemma 4.2.1 is complete. □

Case (N_{Int})

We start by defining $\tilde{S} := R_T^e \cap \{|\omega| \approx 1\} \cap \{z_1, z_2 - \lambda z_1^r \geq 0\}$, which we can decompose into regions $\tau_{\frac{-j}{Q},\frac{j}{T}} := \{z_1 =: x \sim 2^{\frac{j}{Q}}, z_2 - z_1^r =: y \sim 2^{-\frac{j}{T}}\}$, recalling that $T = 2N - 3$ and

$Q = 2J + r - 2$ from (4.2.8). This induces the decomposition $\mathcal{T}_{\tilde{S}} = \sum \mathcal{T}_j$. From Lemma 4.2.1, replacing j with $\frac{-j}{Q}$ and k with $\frac{j}{T}$,

$$\begin{aligned} \mathcal{T}_j(E,F) &\lesssim 2^{-\frac{j}{3Q}[(r+1)(1+d_h)-(rN+3)]+\frac{j}{3T}(N-3)}|E|^{\frac{2}{3}}|F|^{\frac{2}{3}} \\ &= 2^{-\frac{j}{3}[\frac{B}{Q}-\frac{N-3}{T}]}|E|^{\frac{2}{3}}|F|^{\frac{2}{3}}, \end{aligned} \tag{4.2.15}$$

where $B = (r+1)[1+d_h] - (rN+3)$. Since $|\tau_j| \approx 2^{(\frac{1}{Q}-\frac{1}{T})j}$, we also have

$$\mathcal{T}_j(E,F) \lesssim 2^{(\frac{1}{Q}-\frac{1}{T})j}|F|. \tag{4.2.16}$$

Combining (4.2.15) and (4.2.16),

$$\mathcal{T}_j(E,F) \lesssim \min(2^{(\frac{1}{Q}-\frac{1}{T})j}|F|, 2^{-\frac{j}{3}(\frac{B}{Q}-\frac{N-3}{T})}|E|^{\frac{2}{3}}|F|^{\frac{2}{3}}).$$

Noting that $T - Q > 0$ since $T > d_\omega = \frac{Q+Tr}{r+1}$, we interpolate as follows:

$$\begin{aligned} T_{\tilde{S}}(E,F) &\lesssim \sum_j \min(2^{(\frac{1}{Q}-\frac{1}{T})j}|F|, 2^{-\frac{j}{3}(\frac{B}{Q}-\frac{N-3}{T})}|E|^{\frac{2}{3}}|F|^{\frac{2}{3}}) \\ &\approx |E|^{\frac{2(T-Q)}{(3+B)T-NQ}}|F|^{1-\frac{T-Q}{(3+B)T-NQ}}. \end{aligned}$$

Thus, $T_{R^e_T \cap \{|\omega|\approx 1\}}$ is of rwt $(\frac{q_I}{2}, q_I)$ for $(\frac{2}{q_I}, \frac{1}{q_I}) = (\frac{2(T-Q)}{(3+B)T-NQ}, \frac{T-Q}{(3+B)T-NQ})$.

Since $T_{R^e_T \cap \{|\omega|\approx 1\}}$ is of strong type $(\frac{4}{3}, 4)$ by Theorem 1.2.1, and of rwt $(\frac{q_I}{2}, q_I)$, and since R^e_T is κ_φ-scale invariant, then by Case 3 of Proposition 2.3.4, $\mathcal{T}_{R^e_T}$ is of rwt (p_S, q_S), where (p_S, q_S) lies on the scaling line of $\mathcal{T}$ and

$$\left(\frac{2}{p_S}, \frac{2}{q_S}\right) = \left(\frac{3-\frac{8}{q_I}+\frac{1}{q_I}(d_h+1)}{(1-\frac{2}{q_I})(d_h+1)}, \frac{1-\frac{4}{q_I}+\frac{1}{q_I}(d_h+1)}{\left(1-\frac{2}{q_I}\right)(d_h+1)}\right).$$

Since (p_S, q_S) lies on the scaling line, then by Lemma 4.1.2 it suffices to verify that $p_S = p_{v_2}$. Using the identities $Q = 2J + r - 2$, $B = (r+1)[1+d_h] - (rN+3)$,

$T = 2N-3$, $d_h = \frac{J+rN}{r+1}$, and our equation for $\frac{1}{q_I}$, after some messy arithmetic we find that

$$\tfrac{1}{p_S} = \tfrac{N+1-d_h}{d_h+1}.$$

Thus, by Lemma 4.1.2, $p_{v_2} = p_S$, and so $\mathcal{T}_{R_T^e}$ is of rwt (p_{v_2}, q_{v_2}). □

Case $(N_{q=2p}^{\neq scal})$

In this case, $(\frac{1}{p_{v_2}}, \frac{1}{q_{v_2}}) = (\frac{2}{N}, \frac{1}{N})$ lies off the scaling line, so we will be using R_T instead of R_T^e. We decompose $R_T \cap \{z_1, z_2 - z_1^r \geq 0\}$ into regions $\tau_{j,k} = \{x = z_1 \sim 2^{-j}, y = z_2 - z_1^r \sim 2^{-k}\} \cap [-1,1]^2$, where $-\infty \leq k \leq \infty$ and $0 \leq j \leq \infty$. This induces the decomposition $\mathcal{T}_{R_T \cap \{z_1, z_2 - z_1^r \geq 0\}} = \sum_{j,k} \mathcal{T}_{j,k}$. Combining the result of Lemma 4.2.1 with the implications of $|\tau_{j,k}| \lesssim 2^{-j}2^{-k}$,

$$\mathcal{T}_{j,k}(E,F) \lesssim \min\{2^{-j}2^{-k}|F|, 2^{-j}2^{-k}2^{\frac{j}{3}[(r+1)(d_h+1)-rN]}2^{\frac{kN}{3}}|E|^{\frac{2}{3}}|F|^{\frac{2}{3}}\}.$$

Defining $\mathcal{T}_j := \sum_k \mathcal{T}_{j,k}$, and interpolating over k,

$$\begin{aligned}\mathcal{T}_j(E,F) &\lesssim \sum_k \min\{2^{-j}2^{-k}|F|, 2^{-j}2^{-k}2^{\frac{j}{3}[(r+1)(d_h+1)-rN]}2^{\frac{kN}{3}}|E|^{\frac{2}{3}}|F|^{\frac{2}{3}}\} \\ &\approx 2^{-\frac{j}{N}(r+1)[1-\frac{d_h+1}{N}]}|E|^{\frac{2}{N}}|F|^{1-\frac{1}{N}}.\end{aligned}$$

Since $N > d_h + 1$,

$$\mathcal{T}_{R_T \cap \{z_1, z_2 - z_1^r \geq 0\}}(E,F) \lesssim \sum_{j=0}^{\infty} 2^{-\frac{j}{N}(r+1)[1-\frac{d+1}{N}]}|E|^{\frac{2}{N}}|F|^{1-\frac{1}{N}} \lesssim |E|^{\frac{2}{N}}|F|^{1-\frac{1}{N}},$$

so $\mathcal{T}_{R_T}$ is of rwt $(\frac{N}{2}, N)$, which is (p_{v_2}, q_{v_2}) by Lemma 4.1.2. □

Case $(N_{q=2p}^{scal})$

By Lemma 4.1.2, it suffices to prove the following proposition:

Proposition 4.2.2. *In Case* $(N^{scal}_{q=2p})$, $\mathcal{T}_{R^e_T}(E,F) \lesssim |E|^{\frac{2}{N}}|F|^{1-\frac{1}{N}}$.

Proof. Our result will quickly follow from the following lemma:

Lemma 4.2.3. *In Case* $(N^{scal}_{q=2p})$, $\|\mathcal{T}_{R^e_T\cap\{|xy|\approx 1\}}\|_{L^{\frac{q}{2},1}\to L^{q,\infty}} < \infty$ *for all* $q \in [3,\infty)$.

Proof. Recall the change of coordinates $x = z_1, y = z_2 - z_1^r$. It suffices to consider the region with $x, y \geq 0$. We decompose $R^e_T \cap \{|xy| \approx 1\} \cap \{x, y \geq 0\}$ into regions $\tau_{-n,n} := \{z_1 \approx 2^n, z_2 - z_1^r \approx 2^{-n}\}$ for $n \in \mathbb{N}$, and denote $\mathcal{T}_n := \mathcal{T}_{\tau_{-n,n}}$. By Lemma 4.2.1, using $N = d_h + 1$,

$$\mathcal{T}_n(E,F) \lesssim 2^{-\frac{n}{3}[(r+1)N-(rN+3)]+\frac{n}{3}(N-3)}|E|^{\frac{2}{3}}|F|^{\frac{2}{3}} = |E|^{\frac{2}{3}}|F|^{\frac{2}{3}}.$$

We will use the notation of Notation 3.2.1, with $b = 2$, to refine E and F. Since $|\tau_{-n,n}| \lesssim 1$, and since $\mathcal{T}_n(E,F) \lesssim |E|^{\frac{2}{3}}|F|^{\frac{2}{3}}$, by Lemma 3.2.2 and Lemma 3.2.4, it suffices to prove the following version of Condition 3.2.3:

Lemma 4.2.4. *Let* $k, l \in \mathbb{N}$, *and let* k, l, *and* $k - l$ *be sufficiently large, independent of* E *and* F. *Then* $|F_l| \gtrsim 2^{|k-l|r}\alpha^2_{E_l}\beta_{F_{kl}}$ *and* $|E_l| \gtrsim 2^{|k-l|r}\alpha^2_{F_l}\beta_{E_{kl}}$, *with implicit constants independent of* l, k, E, F.

Proof. By symmetry, it will suffice to prove the F_l inequality. Fix $u_0 \in F_{kl}$. Define

$$\Omega_1 := \{t \in \mathbb{R}^2 : z(t) \in \tau_{-k,k} \text{ and } u_0 - (t_1, t_2 + t_1^r, \varphi(z(t))) =: w(t) \in E_k \cap E_l\},$$

recalling that $z(t) := (t_1, t_2 + t_1^r)$. Then $|\Omega_1| = \mathcal{T}_k\chi_{E_k\cap E_l}(u_0) \geq \beta_{F_{kl}}$. To achieve a lower-dimensional result, we fix one variable. Rewriting $|\Omega_1|$,

$$\frac{|\Omega_1|}{2^{-k}} = \fint_{t_2\approx 2^{-k}} \int_{t_1\approx 2^k} \chi_{\Omega_1}(t_1,t_2)dt_1dt_2.$$

Therefore, by Hölder's inequality, there exists a fixed $t_2 \approx 2^{-k}$ such that

$$|\Omega_{1,t_2}| := |\{t_1 : (t_1, t_2) \in \Omega_1\}| = \int_{t_1 \approx 2^k} \chi_{\Omega_1}(t_1, t_2) dt_1 \geq \frac{\beta_{F_k}}{2^{-k}}.$$

For $t_1 \in \Omega_{1,t_2}$, define

$$\Omega_{2,t_2}(t_1) := \{s \in \mathbb{R}^2 : z(s) \in \tau_{-l,l} \text{ and } w(t) + (s_1, s_2 + s_1^r, \varphi(z(s))) \in F_l\}.$$

Then $|\Omega_{2,t_2}(t_1)| = T_l^* \chi_{F_l}(y(t)) \geq \alpha_{E_l}$. Finally, define the following:

$$\Omega_{t_2} := \{(t_1, s_1, s_2) \in \mathbb{R}^3 : t_1 \in \Omega_{1,t_2}, s = (s_1, s_2) \in \Omega_2(t_1)\};$$

$$\Psi_{t_2}(t_1, s_2, s_2) := x_0 - (t_1, t_2 + t_1^r, \varphi(t)) + (s_1, s_2 + s_1^r, \varphi(s_i)).$$

Since Ψ_{t_2} is a polynomial mapping $\Omega_{t_2} \subset \mathbb{R}^3 \to \mathbb{R}^3$, it is $\mathcal{O}(1)$-to-one off a set of measure zero. Since $\Psi_{t_2}(\Omega_{t_2}) \subset F_l$, then $|F_l| \gtrsim \int_{\Omega_{t_2}} |\det D\Psi_{t_2}(t_1, s_1, s_2)| dt_1 ds_1 ds_2$. Then

$$|\det D\Psi_{t_2}(t_1, s_1, s_2)| = r[s_1^{r-1} - t_1^{r-1}]\partial_{s_2}\varphi(z(s)) - [\partial_{s_1}(z(s)) - \partial_{t_1}(z(t))].$$

Then, for $k, l, k-l$ sufficiently large, $t_1 \approx 2^k >> 2^l \approx s_1 >> 1$.

Claim: : $|t_1^{r-1}\partial_{s_2}\varphi(z(s))| >> |s_1^{r-1}\partial_{s_2}\varphi(z(s))| >> |\partial_{s_1}\varphi(z(s))| \gtrsim |\partial_{t_1}\varphi(z(t))|$ holds for $k, l, k-l$ sufficiently large, independent of E, F.

Proof:

$|t_1^{r-1}\partial_{s_2}\varphi(z(s))| >> |s_1^{r-1}\partial_{s_2}\varphi(z(s))|$: Since $t_1 >> s_1$, this is clear.

Next, recall that $\varphi(z(x, y)) = x^J y^N + o(y^N)$. If $J = 0$, then $\partial_{s_1}\varphi(z(s)) = \partial_{t_1}\varphi(z(t)) = 0$, and $|s_1^{r-1}\partial_{s_2}\varphi(z(s))| > 0$, so the rest of the claim holds. Thus, it suffices to consider $J \geq 1$.

$\underline{|s_1^{r-1}\partial_{s_2}\varphi(s)| >> |\partial_{s_1}\varphi(s)|}$: First, $s_1 >> 1$. Also, $\varphi(z(s)) = s_1^J s_2^N + \mathcal{O}(s_2^{N+1})$, with $J \geq 1$, so $|\partial_{s_2}\varphi(z(s))| \approx s_1^J |s_2|^{N-1} >> s_1^{J-1}|s_2|^N \approx |\partial_{s_1}\varphi(z(s))|$, since $s_1 >> 1$ and $|s_2| << 1$.

$\underline{|\partial_{s_1}\varphi(z(s))| \gtrsim |\partial_{t_1}\varphi(z(t))|}$: Since $\varphi(z(x,y)) \approx x^J y^N$, with $N = d_h + 1$, we have the relation $N - 1 = d_h = \frac{J+Nr}{r+1}$, implying $J = N - r - 1$. Since $|s_1 s_2| \approx |t_1 t_2| \approx 1$ and $|s_2| >> |t_2|$, then $|\partial_{s_1}\varphi(z(s))| \approx s_1^{J-1}|s_2|^N = s_1^{N-r-2}|s_2|^N \approx |s_2|^{r-2} \geq |t_2|^{r-2} \approx t_1^{J-1}|t_2|^N \approx |\partial_{t_1}\varphi(z(t))|$. ■

$$\text{Therefore} \quad |\det D\Psi_{t_2}(t_1, s_1, s_2)| \approx |t_1^{r-1}\partial_{s_2}\varphi(z(s))| \approx t_1^{r-1} s_1^J |s_2|^{N-1}$$
$$= t_1^{r-1} s_1^{N-r-1}|s_2|^{N-1} \approx t_1^{r-1}|s_2|^r$$
$$\approx 2^{k(r-1)}2^{-lr} = 2^{|k-l|r}2^{-k}.$$

Then, since $1 \geq \alpha_{E_l}$ since $1 = |\tau_l| \geq |\Omega_2(t_1)| \geq \alpha_{E_l}$,

$$|F_l| \gtrsim \int_{\Omega_{t_2}} 2^{|k-l|r}2^{-k} dt_1 ds_1 ds_2 \gtrsim 2^{|k-l|r}2^{-k} \min_{t_1}(|\Omega_{2,t_2}(t_1)|)|\Omega_{1,t_2}|.$$
$$\gtrsim 2^{|k-l|r}2^{-k}\frac{\alpha_{E_l}}{2^{-k}}\beta_{F_{kl}} \geq 2^{|k-l|r}\alpha_{E_l}^2\beta_{F_k}.$$

A near-identical argument gives $|E_l| \gtrsim 2^{|k-l|r}\alpha_{F_l}^2\beta_{E_k}$, by swapping E_n and F_n, and using $\Psi_{t_2}(t_1, s_2, s_2) = w_0 + (t_1, t_2 + t_1^r, \varphi(z(t))) - (s_1, s_2 + s_1^r, \varphi(z(s)))$, for $w_0 \in E_{kl}$. This completes the proof of Lemma 4.2.4. □

Hence, by Lemmas 3.2.2 and 3.2.4, with $b = 2$, the proof of Lemma 4.2.3 is complete.

□

Finally, since $|xy|$ is $\frac{\boldsymbol{\kappa}_{\boldsymbol{\varphi}}}{D}$-mixed homogeneous for some $D > 0$, and since R_T^e is $\boldsymbol{\kappa}_{\boldsymbol{\varphi}}$-scale-invariant, then by Case 2 of Proposition 2.3.4, we conclude (using $d_h + 1 = N$)

$$\mathcal{T}_{R_T^e}(E, F) \lesssim |E|^{\frac{2}{N}}|F|^{1-\frac{1}{N}},$$

so $\mathcal{T}_{R_T^e}$ is of rwt $(\frac{N}{2}, N)$, which is (p_{v_2}, q_{v_2}) by Lemma 4.1.2. $\square$

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