THE CONTACT TRIAD CONNECTION AND CONTACT INSTANTIONS

By

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Abstract

Assume (Q, ξ) is a contact manifold with a contact one-form λ. We assign an almost complex structure J compatible with (Q, λ) and call (Q, λ, J) a *contact triad.*

In this thesis, we define a *contact instanton* as a map from a Riemannan surface into a contact manifold, satisfying the following generalized Cauchy-Riemann typed equations

$$\overline{\partial}^\pi_J w = 0, \quad d(w^*\lambda \circ j) = 0.$$

We derive a tensorial way for the analysis of contact instantons. The thesis mainly contains the following three contents.

First, we define the so called *contact triad connection* for each triad (Q, λ, J) and prove the existence and uniqueness of such connection. This connection shows several advantages in the study the pseudo-holomorphic curves (contact instantons) in contact manifolds.

Second, we use the contact triad connection to study the analytic properties of contact instantons. In particular, we derive the energy density equality in a coordinate free form. Some new exploration of the tensorial calculations for the vector space valued forms are given for this context. Then we apply the Weitzenböck formula to the density equality, and derive coercive estimates for contact instantons from closed Riemann surfaces.

For a punctured Riemann surface, we study the asymptotical behavior of contact instantons defined on it. The subsequence convergence theorem is proven. We also apply the energy density equality under cylindrical coordinates and derive the exponential decay of the charge zero contact instantons to a limiting Reeb orbit under nondegenerate situation. An alternating boot-strapping method is presented in particular.

Third, we study the exponentially asymptotical behavior of contact instantons near a clean submanifold foliated by Reeb orbits which is of Morse-Bott setting (under some technical assumption for the clean submanifold). We give a normal form theorem to describe the tubular neighborhood of such clean submanifold. We give the splitting of contact instanton equations into vertical and horizontal parts with respect to the normal form. We prove the exponential decay of a contact instanton with vanishing charge to some Reeb orbit living in the clean submanifold.

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Contents

Chapter 1

Introduction

1.1 Motivation and main results

A contact manifold (Q, ξ) is a $2n + 1$ dimensional manifold equipped with a completely non-integrable distribution of rank $2n$, called a contact structure. Complete non-integrability of ξ can be expressed by the non-vanishing property

$$\lambda \wedge (d\lambda)^n \neq 0$$

for a one-form λ which defines the distribution, i.e., $\ker \lambda = \xi$. Such a one-form λ is called a contact form associated to ξ. Associated to the given contact form λ, we have the unique vector field X_λ determined by

$$X_\lambda \rfloor \lambda \equiv 1, \quad X_\lambda \rfloor d\lambda \equiv 0.$$

In relation to the study of pseudo-holomorphic curves, people consider an endomorphism $J : \xi \to \xi$ with $J^2 = -id$ and regard (ξ, J) as a complex vector bundle. In the presence of the contact form λ, one usually considers the set of J that is compatible to $d\lambda$ in the sense that the bilinear form $g_\xi = d\lambda(\cdot, J\cdot)$ defines a Hermitian vector bundle (ξ, J, g_ξ) on Q. We call the triple (Q, λ, J) a *contact triad.*

Motivated Gromov's pseudo-holomorphic curves in symplectic manifolds, Hofer [H1] started the study of pseudo-holomorphic curves in the symplectization $(\mathbb{R} \times M, d(e^t\alpha))$

of contact manifolds (M, α) and successfully used it to prove the Weinstein conjecture in some cases.

However, since there exists already a natural symplectic structure for every contact manifold (Q, ξ) which is the contact distribution ξ with symplectic form $d\lambda|_\xi$, we find it is more natural to look at the pseudo-holomorphic curves directly lives in the contact manifold without involving symplectization. This is the main object studied in this thesis. We would like to mention that such generalization (not exactly the same) was firstly considered by Hofer his survey paper [H2], and then studied by Abbas in [Ab] and further by Abbas-Cieliebak-Hofer in [ACH]. From the progress so far, it seems to the author that one can expect that the study of such contact instantons possible to lead to a better understanding of the Weinstein conjecture, which we would like to choose as a further study.

Now we start with the map w satisfying just $\overline{\partial}^\pi_J w = 0$, which is a nonlinear elliptic equation.

Definition 1.1.1 (Contact Cauchy-Riemann map)**.** Let (Q, λ, J) be a contact triad and let (Σ, j) be a Riemann surface. We call any map $w : \Sigma \to Q$ a *contact Cauchy-Riemann map* if it satisfies $\overline{\partial}^\pi_J w = 0$.

We first introduce a canonical connection, called the contact triad connection, on each contact triad (Q, λ, J) (see [OW1]).

Theorem 1.1.2 (Contact triad connection [OW1])**.** *Let (Q, λ, J) be any contact triad of the contact manifold (Q, ξ). Denote by*

$$g_\xi + \lambda \otimes \lambda =: g$$

the natural Riemannian metric on Q induced by (λ, J), which we call a contact triad metric. Then there exists a unique affine connection ∇ that has the following properties:

1. *∇ is a Riemannian connection of the triad metric.*

2. *The torsion tensor of ∇ satisfies $T(X_\lambda, Y) = 0$ for all $Y \in TQ$.*

3. *$\nabla_{X_\lambda} X_\lambda = 0$ and $\nabla_Y X_\lambda \in \xi$, for $Y \in \xi$.*

4. *$\nabla^\pi := \pi \nabla|_\xi$ defines a Hermitian connection of the vector bundle $\xi \to Q$ with Hermitian structure $(d\lambda|_\xi, J)$.*

5. *The ξ projection of the torsion T, denoted by $T^\pi := \pi T$ satisfies the following properties:*

$$T^\pi(JY, Y) = 0$$

   for all Y tangent to ξ.

6. *For $Y \in \xi$,*

$$\partial^\nabla_Y X_\lambda = \frac{1}{2}(\nabla_Y X_\lambda - J\nabla_{JY} X_\lambda) = 0.$$

We call ∇ the contact triad connection.

We denote by ∇ the contact triad connection on Q and ∇^π the contact Hermitian connection on the Hermitian vector bundle $(\xi, d\lambda|_\xi, J)$. Various symmetry properties carried by the connections ∇ and ∇^π enable us to derive the following precise formulae concerning the second covariant differential of w and the Laplacian of the π-harmonic energy density function for any contact Cauchy-Riemann map w.

Theorem 1.1.3 (Fundamental equation [OW2])**.** *Let w be a contact Cauchy-Riemann map. Then*

$$\begin{aligned} d^{\nabla^\pi}(d^\pi w) &= d^{\nabla^\pi}(\partial^\pi w) \\ &= -w^*\lambda \circ j \wedge \left(\frac{1}{2}(\mathcal{L}_{X_\lambda} J)\, \partial^\pi w\right). \end{aligned}$$

We define the ξ-component of the standard harmonic energy density function by

$$e^\pi := |d^\pi w|^2 = |\pi dw|^2.$$

Definition 1.1.4. For any smooth map $w : \Sigma \to Q$,

$$E^\pi_{(\lambda,J)}(w,j) = \frac{1}{2}\int_\Sigma |d^\pi w|^2$$

and call it the π-harmonic energy of a smooth map.

Theorem 1.1.5 (π-harmonic energy identity [OW2])**.** *Let w be a contact Cauchy-Riemann map. Then*

$$\begin{aligned} \frac{1}{2}\Delta e^\pi &= -|\nabla^\pi(\partial^\pi w)|^2 - 2\delta\langle *d^{\nabla^\pi}\partial^\pi w, *\partial^\pi w\rangle + 2|\delta^{\nabla^\pi}\partial^\pi w|^2 \\ & \quad -\langle K^\pi(dw,dw)\partial^\pi w, \partial^\pi w\rangle - R\,|\partial^\pi w|^2 \end{aligned}$$

where R is the Gaussian curvature of the given Kähler metric h on (Σ, j) and K^π is the curvature tensor of the contact Hermitian connection ∇^π.

It turns out that to establish the geometric analysis necessary for the study of associated moduli space, one needs to augment the equation $\overline{\partial}^\pi_J w = 0$ by

$$d(w^*\lambda \circ j) = 0,$$

which is a natural elliptic twisting of the contact Cauchy-Riemann map equation.

Definition 1.1.6 (Contact instanton)**.** Let Σ be as above. We call a pair (j, w) of j a complex structure on Σ and a map $w : \Sigma \to Q$ a *contact instanton* if they satisfy

$$\overline{\partial}^{\pi}_{J} w = 0, \quad d(w^*\lambda \circ j) = 0. \tag{1.1}$$

The very fact that the twisting (1.1) is a natural one is evidenced by the a priori elliptic estimates and certain asymptotical convergence results (up to this thesis, we are able to get exponential convergence for the case of vanishing charge, see Hypothesis 5.3.1).

Then we first establish the following a priori $W^{2,2}$-estimates for such a map.

Theorem 1.1.7 ([OW2])**.** *Let (Σ, j) be a closed Riemann surface. Suppose w satisfies* (1.1) *on Σ. Then there exist uniform constants C_1, C_2 depending only on $\|K^\pi\|_{C^0}$, $\|R\|_{C^0}$ and $\|\mathcal{L}_{X_\lambda} J\|_{C^0}$ but independent of w such that*

$$\|dw\|^2_{W^{1,2}} \leq C_1 \|dw\|^4_{L^4} + C_2 \|dw\|^2_{L^2}.$$

We also need to have the following local version of the main estimates which will be an important ingredient for the local regularity and the bubbling-off analysis.

Theorem 1.1.8 ([OW2])**.** *Let $D = D^2(1)$ be the unit disc and let $D' \subset \overline{D}' \subset D$ be another smaller disc. Then there exists $\epsilon > 0$ such that if $w : D \to Q$ is a smooth map satisfying* (1.1)*, and $E_{(\lambda,J)}(w) < \epsilon$,*

$$\|dw\|^2_{1,2;D'} \leq C_3 \|dw\|^4_{4;D} + C_4 \|dw\|^2_{2;D}$$

for some constants C_3, C_4 which depend only on $\|K^\pi\|_{C^0}$, $\|R\|_{C^0}$ and $\|\mathcal{L}_{X_\lambda} J\|_{C^0}$.

For the punctured Riemann surface $\dot{\Sigma}$, one needs to put suitable asymptotic conditions in terms of the cylindrical metric and its associated isothermal coordinates denoted by (τ, t). For this purpose, we need to impose asymptotic convergence behavior of the following L^2-integral function

$$f(\tau) = \frac{1}{2}\int_{S^1} |d^\pi w|^2(\tau,t)\, dt$$

and the one-form $w^*\lambda = a_1\, d\tau + a_2\, dt$, besides requiring $|d^\pi w| \in L^2 \cap L^4$.

Theorem 1.1.9 ([OW2])**.** *Let (Σ, j) be a closed Riemann surface with a finite number of marked points $\{r_1, \cdots, r_k\}$. Denote by $\dot\Sigma$ the associated punctured Riemann surface with a Kähler metric h on (Σ, j) which is cylindrical near the punctures. Let f_ℓ be the function defined as above associated to the ℓ-th puncture r_ℓ. Suppose w satisfies* (1.1) *on $\dot\Sigma$ and $|d^\pi w| \in L^2 \cap L^4$ and $\|w^*\lambda\|_{C^0} < \infty$ on $\dot\Sigma$*

$$\begin{aligned} \lim_{\tau\to\infty} a_1 &= a, & \lim_{\tau\to\infty} a_2 &= T \\ \lim_{\tau\to\infty} f_\ell(\tau) &= 0, & \lim_{\tau\to\infty} f_\ell'(\tau) &= 0 \end{aligned} \tag{1.2}$$

for all $\ell = 1, \cdots, k$. Then

$$\|dw\|^2_{W^{1,2}} \leq C_4\|dw\|^4_{L^4} + C_5\|dw\|^2_{L^2}.$$

We remark that the asymptotic boundary conditions (1.2) imposed in this theorem will be also established in Chapter 5 under the Hypothesis 4.1.2 together with Morse-Bott assumption. In particular, they are obtained from subsequence convergence theorem, Theorem 4.1.3.

Once this $W^{2,2}$-estimate is proved, further differentiations of ∇dw and inductive alternating bootstrapping give rise to the all the higher regularity estimates too. We just state the more nontrivial punctured case here.

Theorem 1.1.10 ([OW2]). *Let* (Σ, j) *and* w *be as above Then if* w *satisfies* (1.1) *on* $\dot\Sigma$ *and* $|d^\pi w| \in L^2 \cap L^4$ *and* $\|w^*\lambda\|_{C^0} < \infty$ *on* $\dot\Sigma$, *and* (1.2), *then*

$$\int_{\dot\Sigma} |(\nabla)^{k+1}(dw)|^2 \leq \int_{\dot\Sigma} J'_k(d^\pi w, w^*\lambda).$$

Here J'_{k+1} *a polynomial function of the norms of the covariant derivatives of* $d^\pi w$, $w^*\lambda$ *up to* $0, \ldots, k$ *with degree at most* $2k+4$ *whose coefficients depend on*

$$\|R\|_{C^k}, \|K^\pi\|_{C^k}, \|\mathcal{L}_{X_\lambda} J\|_{C^k}, \ \|w^*\lambda\|_{C^0}.$$

In particular,

$$\|dw\|_{W^{k+1,2}} \leq C_k(\|dw\|_{L^2}, \|dw\|_{L^4})$$

for a similar polynomial function $C_k = C_k(s,t)$.

The local version of the estimate also holds.

Theorem 1.1.11 ([OW2]). *Let* $D = D^2(1)$ *be the unit disc. There exist* $C_{5;k}$, $C_{6;k} > 0$ *depending only on* $D' \subset \overline{D}' \subset D$ *and on* $\|K^\pi\|_{C^k}$, $\|\mathcal{L}_{X_\lambda} J\|_{C^k}$ *and* $\|R\|_{C^k;D}$ *but independent of* w *such that for any smooth map* $w : D \to Q$ *satisfying* (1.1), *then*

$$\|dw\|_{k+1,2;D'} \leq C_{k;D,D'}(\|dw\|_{2;D}, \|dw\|_{4;D})$$

for a polynomial function $C_{k;D,D'}(s,t)$ *of* s, t *up to* $0, \ldots, k$ *of degree at most* $2k+4$ *depending also on* D', D. *In particular, any weak solution of* (1.1) *in* $W^{1,4}$ *automatically becomes a classical solution of* (3.2).

We refer to Theorem 4.3.4 and 3.2.6 and discussions around them for further expounding of these estimates.

Next, we state the results of the behavior of contact instantons near each puncture of a punctured Riemann surface. In this regard, it is crucial to prove some asymptotic convergence result to a closed Reeb orbit under a suitable finite energy hypothesis. We prove the following convergence result and refer Theorem 4.1.3 for more precise statements.

Theorem 1.1.12. *Let w be any contact instanton on $[0,\infty)\times S^1$ with finite π-harmonic energy and gradient bound*

$$E^\pi(w) := \frac{1}{2}\int_{[0,\infty)\times S^1} |d^\pi w|^2 < \infty, \quad \|dw\|_{C^0;[0,\infty)\times S^1} < \infty. \tag{1.3}$$

Then for any sequence $\tau_k \to \infty$, there exists a subsequence, still denoted by τ_k, and a massless instanton $w_\infty(\tau,t) = \gamma(a\,\tau + T\,t)$ (i.e., $E^\pi(w_\infty) = 0$) on the cylinder $\mathbb{R}\times S^1$ along a closed Reeb orbit γ with period T such that

$$\lim_{k\to\infty} w(\tau_k+\tau,t) = w_\infty(\tau,t)$$

uniformly on $[-K,K]\times S^1$ for any given $K \geq 0$, where γ is a T-periodic orbit of X_λ. Here T and a are determined by

$$\begin{aligned} T &= \int_{[0,\infty)\times S^1} |d^\pi w|^2 + \int_{S^1} w(0,\cdot)^*\lambda \\ a &:= -\int_{S^1} w(0,\cdot)^*\lambda\circ j = \int_{S^1} \lambda\left(\frac{\partial w}{\partial \tau}(0,t)\right)\, dt. \end{aligned}$$

We call a the charge. In particular, when $a=0$, the limiting instanton w_∞ is translation invariant and so $w_\infty(\tau,t) \equiv \gamma(Tt)$, and the convergence is uniform and exponentially fast.

In the context of symplectization at ends, which roughly corresponds to the vanishing charge $a = 0$, this subsequence convergence theorem can be derived from Hofer's subsequence convergence result proved in [H1].

When (γ, T) is a nondegenerate Reeb orbit, the limit z does not depend on the choice of subsequence and the convergence is exponentially fast, whose proof we give in this thesis is essentially different from that of [HWZ1, HWZ2], even for the exact context. Our tensor calculations also clarify the geometry behind Hofer-Wysocki-Zehnder's coordinate calculations involved in their study of exponential decay estimates for the pseudoholomorphic curves on the symplectization of contact manifolds (or on the symplectic manifolds with cylindrical ends). We would like to recall that this on-shell exponential decay estimate is one of the crucial analytical ingredients in setting up the off-shell functional analytic framework for the study of moduli space of contact instantons on the punctured Riemann surfaces residing in the contact manifold Q.

The nondegeneracy condition of Reeb orbits enables us to look at the globally defined Cauchy-Riemann equation directly and apply the third-order method to derive the exponential decay estimate, which is different from the second-order method we apply for the Morse-Bott case. To be specific, the estimate for nondegenerate case relies on the study of the second derivative of the canonical integral

$$\int_{S^1} |d^\pi w|^2 \, dt$$

through the invariant tensorial calculation using the special connections. We are able to to make our C^k-exponential estimates (for $k \geq 1$) use only C^{k-1}-exponential estimates and the standard bootstrapping argument. For this purpose, we need to derive a precise geometric formula of the Laplacian of the energy density function and the second covariant differential of w. We also perform several times of integration by parts to remove all the second order derivative terms of the map w after integrating over $t \in S^1$.

While for the Morse-Bott case, it is still unclear to the author that if we can expect the C^k-exponential estimates (for $k \geq 1$) use only C^{k-1}-exponential estimates and give a similar alternating bootstrapping as for the nondegenerate case. The main difficulty causes this seems to be that the Morse-Bott condition is a condition involving submanifold itself which contains some information that will lose if we only identify tangent spaces. Hence we have to derive some normal form theorem to keep the information based on submanifold.

For the Morse-Bott case, we look at a special type of Morse-Bott clean submanifold which can be considered as the symplectic normal bundle of some prequantization manifold. We remark that this is only a technical requirement which can be removed (see [OW3]). We assign a normal form (U_E, λ_E) to it and relate the given contact manifold (U_E, λ) with the normal form by the following normal form theorem.

Theorem 1.1.13 (Normal form [OW3])**.** *There exists a diffeomorphism ϕ from U_E to itself and a function $f > 0$ defined on U_E, such that*

$$\phi^*\lambda = f \cdot \lambda_E.$$

Moreover,

$$d\phi|_N = id|_{TM|_N}, \quad f|_{o_E} \equiv 1, \quad df|_{o_E} \equiv 0$$

and

$$\phi^* d\lambda|_{TE|_{o_E}} = (d\lambda_E)_{TE|_{o_E}}$$

Then we study the contact instanton under such normal form and give the splitting of the Cauchy-Riemann equations under the vertical and horizontal decomposition, by which we prove the exponential decay to a Reeb orbit under Morse-Bott settings ([OW4]).

We would like to summarize that since we do not take symplectization of the contact triad (Q, λ, J) but directly work on the contact manifold Q, it enables us to get ready to construct the compactification of the smooth moduli space of contact instantons with either closed or punctured Riemann surfaces as their domains and contact manifolds Q as their targets, and so to define a genuinely contact topological invariant *without taking the symplectization of* Q. Indeed the question if two contact manifolds having symplectomorphic symplectization are contactomorphic or not was addressed in the book by Cieliebak and Eliashberg. (See p. 239 [CE].) In this regard, S. Courte [Co] recently provided a construction of two contact manifolds that have symplectomorphic symplectization which are not contactomorphic (actually, even not diffeomorphic). It would be interesting to see in the future whether our study leads to a construction of genuinely contact topological quantum invariants of the Gromov-Witten or Floer theoretic type that can be used to investigate the following kind of question. (See [Co] where a similar question was explicitly stated.)

Another point we would like to mention is that the equation (1.1) was first mentioned by Hofer in p.698 of [H2] in a little different setting,

$$\begin{cases} \overline{\partial}^\pi w = 0 \\ w^*\lambda \circ j = da + \gamma \end{cases}$$

for a given *smooth* harmonic one-form γ defined on the *closed* Riemann surface Σ smoothly extending across the punctures. (See also [ACH]). In the language of this thesis, the second part of the equation forces *any* w *arising from a finite* π*-energy solution thereof will have vanishing 'charge' at the puncture where the charge is given by*

$$\int_{S^1} w(\tau, \cdot)^*\lambda \circ j$$

in the notation of the thesis: any solution to their equation hence does not have spiral along the limiting Reeb orbit. The same form of the equation was also used by Abbas [Ab] to prove some existence result in relation to the open book decomposition in 3 dimension. It seems to be a useful restriction for the purpose of exponential convergence result near the punctures. At least in this thesis, we are not able to prove general exponential decay result for nonvanishing charge case and it is a very interesting problem to find some appropriate way to study it in the future.

1.2 Organization of the thesis

This thesis is organized as follows.

In Chapter 2, we study the differential geometry of contact triads. It consists of three parts. First, we introduce the contact triad connection for each contact triad and prove its uniqueness and existence. Second, we give a characterization of the linearized operator along Reeb orbits and interpret the Morse-Bott condition by using this operator. Third, we study the local geometric structure near a clean Morse-Bott submanifold of a special type - the contact thickening of prequantization and prove the normal form theorem.

In Chapter 3, we study the contact instantons *without* finite π-harmonic energy assumption. We derive the energy identity and use it to get the coercive estimates for contact instantons defined on closed domains.

In Chapter 4, we consider *finite π-harmonic energy* contact instantons and study the asymptotic properties *without* the assumption on the types of contact one-forms. We also derive coercive estimates for the punctured case under some asymptotic assumptions.

In Chapter 5, we continue to consider the asymptotic properties of *finite energy* contact instantons for *nondegenerate* contact one-forms. C^∞ exponential convergence and C^0 convergence are derived *using by the contact triad connection*. The alternating bootstrapping argument is also presented there.

In Chapter 6, we consider the asymptotic properties of *finite energy* contact instantons for *Morse-Bott* type contact one-forms *by using the normal form theorem.*

Chapter 2

The geometry of contact triads

Assume Q is a contact manifold, ξ is the contact distribution. We choose a globally defined contact one-form λ such that $\ker \lambda = \xi$, and an almost complex structure J compatible with λ. We call the triad (Q, λ, J) the contact triad. In this Chapter, we study the differential geometry of the contact triad.

This chapter consists of three parts. In the first part, we introduce the contact triad connection for each contact triad, which is the analogue of the canonical connection on almost Hermitian manifold. We give the proof of its uniqueness and existence. In the second part, we look at the dynamics of the Reeb vector field, and study the Morse-Bott condition on contact one-forms. In particular, we give a characterization of the linearized operator along Reeb orbits and interpret the Morse-Bott condition by using this operator. In the third part, we study the local geometric structure near a clean orbit submanifold of a special Morse-Bott type - the contact thickening of prequantization.

2.1 The canonical connection on contact triad

2.1.1 The definition of contact triad

A contact manifold (Q, ξ) is a $2n+1$ dimensional C^∞ manifold Q equipped with a completely non-integrable distribution ξ of rank $2n$, called a contact structure. The

Complete non-integrability of ξ can be expressed by the non-vanishing property

$$\lambda \wedge (d\lambda)^n \neq 0$$

for a one-form λ which defines the distribution locally, i.e., $\ker \lambda = \xi$. Such a one-form λ is called a contact form associated to ξ. We assume the contact distribution ξ is co-oriented throughout the thesis, which indicates the contact one-form λ is globally defined. Associated to the given contact form λ, we have the unique vector field X_λ called the Reeb vector field determined by

$$X_\lambda \rfloor \lambda \equiv 1, \quad X_\lambda \rfloor d\lambda \equiv 0. \tag{2.1}$$

Lemma 2.1.1.

$$\mathcal{L}_{X_\lambda} \lambda = 0 = \mathcal{L}_{X_\lambda} d\lambda.$$

Theorem 2.1.2 (Darboux's theorem)**.** *Every contact structure (Q, ξ) is locally diffeomorphic to the standard contact structure on $\mathbb{R}^{2n+1}$, where $\dim Q = 2n+1$.*

Theorem 2.1.3 (Gray's stability theorem)**.** *For a family of contact forms λ_t on a closed manifold Q there exists a family of diffeomorphisms ϕ_t and a family of functions $f_t > 0$, such that*

$$\phi_t^* \lambda_t = f_t \lambda_0.$$

The proofs of both theorems can be given by Moser's trick which we are going to give, but to a more general form, in Section 2.3.4. More preliminaries of contact forms will be given in Section 2.3.1.

In the study of pseudoholomorphic curves, one considers an endomorphism $J : \xi \to \xi$ with $J^2 = -id|_\xi$ and regards (ξ, J) as a complex vector bundle over Q. In the presence of the contact form λ, one usually considers the set of J that are compatible with $d\lambda$ in the sense that the bilinear form $g_\xi = d\lambda(\cdot, J\cdot)$ defines a Hermitian vector bundle (ξ, J, g_ξ) on Q. We call the triple (Q, λ, J) a *contact triad*.

For each given contact triad, we equip Q with the triad metric

$$g = d\lambda(\cdot, J\cdot) + \lambda \otimes \lambda.$$

2.1.2 Review of the canonical connection of almost Kähler manifold

Before we introduce the contact triad connection for the contact triad (Q, λ, J), we first recall the construction of the canonical connection for the case of almost Kähler manifolds (M, ω, J).

Recall that an almost Hermitian manifold is a triple (M, J, g) of an almost complex structure J and a metric that satisfies

$$g(J\cdot, J\cdot) = g(\cdot, \cdot).$$

An affine connection ∇ is called J-linear if $\nabla J = 0$. There always exists a J-linear connection for a given almost complex manifold. We denote by T the torsion tensor of ∇.

Definition 2.1.4. Let (M, J, g) be an almost Hermitian manifold. A J-linear connection is called a canonical connection (or the Chern connection) if it satisfies

$$T(JY, Y) = 0, \tag{2.2}$$

for any vector field Y on M.

Recall that any J-linear connection extended to the complexification $T_{\mathbb{C}}M = TM \otimes_{\mathbb{R}} \mathbb{C}$ complex linearly preserves the splitting

$$T_{\mathbb{C}}M = T^{(1,0)}M \oplus T^{(0,1)}M.$$

Similarly we can extend the torsion tensor T complex linearly which we denote by $T_{\mathbb{C}}$.

For the later purpose, we will need to derive general properties of the torsion tensor. See [Ko], [Oh1, Section 7.1] for some basic properties of the canonical connection. A nice exhaustive discussion on the general almost Hermitian connection is given by Gauduchon in [Ga].

Following the notation of [Ko], we denote

$$\Theta = \Pi' T_{\mathbb{C}}$$

which is a $T^{(1,0)}M$-valued two-form on M. Here $T_{\mathbb{C}} = T \otimes \mathbb{C}$ is the complex linear extension of T and Π' is the projection to $T^{(1,0)}M$. We have the decomposition

$$\Theta = \Theta^{(2,0)} + \Theta^{(1,1)} + \Theta^{(0,2)}.$$

We can define the canonical connection in terms of the induced connection on the complex vector bundle $T^{(1,0)}M \to M$. The following lemma is easy to check by definition.

Lemma 2.1.5. *An affine connection ∇ on M is a canonical connection if and only if the complex torsion form $\Theta = \Pi' T_{\mathbb{C}}$ of the induced connection ∇ on the complex vector bundle $T^{(1,0)}M$ satisfies $\Theta^{(1,1)} = 0$.*

We particularly quote two theorems from Gauduchon [Ga], Kobayashi [Ko].

Theorem 2.1.6. *On any almost Hermitian manifold (M, J, g), there exists a unique Hermitian connection ∇ on TM leading to the canonical connection on $T^{(1,0)}M$. We call this connection the canonical Hermitian connection of (M, J, g).*

We recall that (M, J, g) almost-Kähler if the fundamental two-form $\Phi = g(J\cdot, \cdot)$ is closed [KN].

Theorem 2.1.7. *Let (M, J, g) be almost Kähler and ∇ be the canonical connection of $T^{(1,0)}M$. Then $\Theta^{(2,0)} = 0$ in addition, and hence Θ is of type $(0, 2)$.*

The following properties can be derived from this latter theorem easily. (See [Ga], [Ko] for details.)

Proposition 2.1. Let (M, J, g) be an almost Kähler manifold and ∇ be the canonical connection. Denote by T its torsion tensor. Then

$$T(JY, Z) = T(Y, JZ) \tag{2.3}$$

and

$$JT(JY, Z) = T(Y, Z) \tag{2.4}$$

for all vector fields Y, Z on M.

Now we describe one way of constructing the canonical connection on an almost complex manifold described in [KN, Theorem 3.4] which will be useful for our purpose of constructing the contact analogue thereof later. This connection has its torsion which satisfies

$$N = 4T$$

where N is the Nijenhuis tensor of the almost complex structure J defined as

$$N(X, Y) = [JX, JY] - [X, Y] - J[X, JY] - J[JX, Y].$$

In particular, the complexification $\Theta = \Pi' T_{\mathbb{C}}$ is of $(0, 2)$-type.

We now describe the construction of this canonical connection by following [KN]. Let ∇^{LC} be the Levi-Civita connection (in fact, we can do it for any torsion free affine connection) of an almost Hermitian manifold (M, g, J) and consider the tensor field Q defined by

$$4Q(X,Y) = (\nabla^{LC}_{JY} J)X + J((\nabla^{LC}_{Y} J)X) + 2J((\nabla^{LC}_{X} J)Y) \tag{2.5}$$

for vector fields X, Y on M. It turns out that when (M, g, J) is almost Kähler, i.e., the two-form $g(J\cdot, \cdot)$ is closed, the sum of the first two terms vanish.

Lemma 2.1.8 ((2.2.10)[Ga])**.** *Assume (M, g, J) is almost Kähler. Then*

$$\nabla^{LC}_{JY} J + J(\nabla^{LC}_{Y} J) = 0 \tag{2.6}$$

and so $Q(X,Y) = \frac{1}{2} J(\nabla^{LC}_X J)Y$.

We now consider the standard averaged connection ∇^{av} of multiplication $J : TM \to TM$

$$\begin{aligned} \nabla^{av}_X Y &:= \frac{\nabla^{LC}_X Y + J^{-1}\nabla^{LC}_X (JY)}{2} \\ &= \nabla^{LC}_X Y - \frac{1}{2} J(\nabla^{LC}_X J)Y \\ &= \nabla^{LC}_X Y - Q(X,Y). \end{aligned}$$

We then have the following Proposition stating that this connection becomes the canonical connection. Its proof can be found in [KN, Theorem 3.4] or from section 2 [Ga] with a little more strengthened argument by using (2.6) for the metric property.

Proposition 2.2. Assume that (M, g, J) is almost Kähler, i.e, the two-form $\omega = g(J\cdot, \cdot)$ is closed. Then the average connection ∇^{av} defines the canonical connection of (M, g, J), i.e., the connection is J-linear, preserves the metric and its complexified torsion is of $(0,2)$-type.

In later sections, we will need a contact analogue to this proposition.

2.1.3 Definition of the contact triad connection and its consequences

Let (Q, ξ) be a contact manifold and a contact form λ be given. On Q, the Reeb vector field X_λ associated to the contact form λ is the unique vector field satisfying (2.1).

Therefore the tangent bundle TQ has the splitting $TQ = \mathbb{R}\{X_\lambda\} \oplus \xi$. We denote by

$$\pi : TQ \to \xi$$

the corresponding projection. J is a complex structure on ξ, and we extend it to TQ by defining $J(X_\lambda) = 0$. We will use such $J : TQ \to TQ$ throughout the thesis. If there is no danger of confusion, we do not distinguish J and $J|_\xi$.

Definition 2.1.9 (Contact triad metric)**.** Let (Q, λ, J) be a contact triad. We call the metric defined by

$$g(h, k) = g_{(\lambda,J)}(h, k) := \lambda(h)\lambda(k) + d\lambda(\pi h, J\pi k). \tag{2.7}$$

for any $h,\ k \in TQ$ the contact triad metric associated to the triad (Q, λ, J).

In this section, we associate a particular type of affine connection on Q to the given triad (Q, λ, J) which we call *the contact triad connection* of the triple.

To construct this contact analogue to the canonical connection of the case of almost Kähler manifolds, we note that the space of Reeb foliations of (Q, λ) becomes naturally a (non-Hausdorff) almost Kähler manifold $(\widehat{Q}, \widehat{d\lambda}, \widehat{J}_\xi)$, which characterize this connection in the contact distribution ξ. At the same time, we expect this connection behaves in the Reeb vector field direction as the Levi-Civita connection, i.e., it is torsion free and metric. This intuition motivates the following definition of canonical connection of the contact triads (Q, λ, J).

We recall

$$TQ = \mathbb{R}\{X_\lambda\} \oplus \xi,$$

and we denote by $\pi : TQ \to \xi$ the projection. Denote $\Pi : TQ \to TQ$ the corresponding endomorphism of π, with $\Pi^2 = \Pi$. Under this splitting, we may regard a section Y of $\xi \to Q$ as a vector field $Y \oplus 0$. We will just denote the latter by Y with slight abuse of notation.

For an affine connection ∇, define ∇^π the connection of the bundle $\xi \to Q$ by

$$\nabla^\pi Y = \pi \nabla Y.$$

If there is no danger of confusion, we also use $\nabla^\pi = \Pi\nabla$ by abuse of notation.

Definition 2.1.10 (Contact triad connection)**.** We call an affine connection ∇ on Q a *contact triad connection* of the contact triad (Q, λ, J), if it satisfies the following properties:

1. ∇^π is a Hermitian connection of the Hermitian bundle ξ over the contact manifold Q with the Hermitian structure $(d\lambda, J)$.

2. The ξ projection, denoted by $T^\pi := \pi T$, of the torsion T satisfies the following properties:

$$T^\pi(JY, Y) = 0 \tag{2.8}$$

for all Y tangent to ξ.

3. $T(X_\lambda, Y) = 0$ for all $Y \in TQ$.

4. $\nabla_{X_\lambda} X_\lambda = 0$ and $\nabla_Y X_\lambda \in \xi$, for $Y \in \xi$.

5. For $Y \in \xi$,
$$\nabla_{JY} X_\lambda + J \nabla_Y X_\lambda = 0.$$

6. For any $Y,\ Z \in \xi$,
$$\langle \nabla_Y X_\lambda, Z\rangle + \langle X_\lambda, \nabla_Y Z\rangle = 0.$$

It follows from the definition that the contact triad connection is a Riemannian connection of the triad metric.

Note that Axioms (1) and (2) in the direction of ξ are nothing but the properties of canonical connection on the tangent bundle of the (non-Hausdorff) almost Kähler manifold $(\widehat{Q}, \widehat{d\lambda}, \widehat{J}_\xi)$ lifted to ξ. In addition to them, Axiom (1) also requires the property that ∇_{X_λ} preserve the Hermitian structure $(d\lambda, J)$. On the other hand, Axioms (3), (4), (6) indicate that this connection behaves like the Levi-Civita connection when the Reeb direction X_λ gets involved. Axiom (5) is an extra requirement to connect the information in ξ part and X_λ part, which is used to dramatically simplify our calculations of contact Cauchy-Riemann maps in later sections. It turns out that this condition has the following elegant interpretation in terms of CR-geometry.

By the second part of Axiom (4), the covariant derivative ∇X_λ restricted to ξ can be decomposed into
$$\nabla X_\lambda = \partial^\nabla X_\lambda + \overline{\partial}^\nabla X_\lambda$$
where $\partial^\nabla X_\lambda$ (respectively, $\overline{\partial}^\nabla X_\lambda$) is J-linear (respectively, J-anti-linear part). Axiom

(6) then is nothing but the requirement that $\partial^\nabla X_\lambda = 0$, i.e., X_λ is anti J-holomorphic in the CR-sense.

One can also consider similar decompositions of one-form λ. For this, we need some digression. Define $J\alpha$ for a k-form α by the formula

$$J\alpha(Y_1, \cdots, Y_k) = \alpha(JY_1, \cdots, JY_k).$$

Definition 2.1.11. Let (Q, λ, J) be a contact triad. We call a k-form α is CR-holomorphic if it satisfies

$$\nabla_{X_\lambda} \alpha = 0, \tag{2.9}$$

$$\nabla_Y \alpha + J \nabla_{JY} \alpha = 0 \quad \text{for } Y \in \xi. \tag{2.10}$$

Proposition 2.3. In the CR-sense in the presence of other defining properties of contact triad connection, Axiom (5) is equivalent to the statement that λ is holomorphic .

Proof. We first prove $\nabla_{X_\lambda} \lambda = 0$ by evaluating it against vector fields on Q. For X_λ, the first half of Axiom (4) gives rise to

$$\nabla_{X_\lambda} \lambda(X_\lambda) = -\lambda(\nabla_{X_\lambda} X_\lambda) = 0.$$

For the vector field $Y \in \xi$, we compute

$$\begin{aligned} \nabla_{X_\lambda} \lambda(Y) &= -\lambda(\nabla_{X_\lambda} Y) \\ &= -\lambda(\nabla_Y X_\lambda + [X_\lambda, Y] + T(X_\lambda, Y)) \\ &= -\lambda(\nabla_Y X_\lambda) - \lambda([X_\lambda, Y]) - \lambda(T(X_\lambda, Y)). \end{aligned}$$

Here the third term vanishes by Axiom (3), the first term by the second part of Axiom (4) and the second term vanishes since

$$\lambda([X_\lambda, Y]) = \lambda(\mathcal{L}_{X_\lambda} Y) = X_\lambda[\lambda(Y)] - \mathcal{L}_{X_\lambda} \lambda(Y) = 0 - 0 = 0.$$

Here the first term vanishes since $Y \in \xi$ and the second because $\mathcal{L}_{X_\lambda}\lambda = 0$ by the definition of the Reeb vector field. This proves

$$\nabla_{X_\lambda}\lambda = 0. \tag{2.11}$$

We next compute $J\nabla_Y\lambda$ for $Y \in \xi$. For a vector field $Z \in \xi$,

$$(J\nabla_Y\lambda)(Z) \;=\; (\nabla_Y\lambda)(JZ) = \nabla_Y(\lambda(JZ)) - \lambda(\nabla_Y(JZ)) = -\lambda(\nabla_Y(JZ))$$

since $\lambda(JZ) = 0$ for the last equality. Then by the definitions of the Reeb vector field and the triad metric and the skew-symmetry of J, we derive

$$-\lambda(\nabla_Y(JZ)) = -\langle \nabla_Y(JZ), X_\lambda\rangle = \langle JZ, \nabla_Y X_\lambda\rangle = -\langle Z, J\nabla_Y X_\lambda\rangle.$$

Finally, applying (6), we obtain

$$-\langle Z, J\nabla_Y X_\lambda\rangle = \langle Z, \nabla_{JY} X_\lambda\rangle = -\langle \nabla_{JY} Z, X_\lambda\rangle = -\lambda(\nabla_{JY} Z) = (\nabla_{JY}\lambda)(Z).$$

Combining the above, we have derived

$$J(\nabla_Y\lambda)(Z) = \nabla_{JY}\lambda(Z)$$

for all $Z \in \xi$. On the other hand, for X_λ, we evaluate

$$J(\nabla_Y\lambda)(X_\lambda) = \nabla_Y\lambda(JX_\lambda) = \nabla_Y\lambda(0) = 0.$$

We also compute

$$\nabla_{JY}\lambda(X_\lambda) = \mathcal{L}_{JY}(\lambda(X_\lambda)) - \lambda(\nabla_{JY}X_\lambda)).$$

The first term vanishes since $\lambda(X_\lambda) \equiv 1$ and the second vanishes since $\nabla_{JY}X_\lambda \in \xi$ by the second part of Axiom (4). Therefore we have derived

$$J(\nabla_Y\lambda) = \nabla_{JY}\lambda \tag{2.12}$$

which is equivalent to (2.10). Combining (2.11) and (2.12), we have proved that Axiom (6) implies λ is holomorphic in the CR-sense. The converse can be proved by reading the above proof backwards. □

One very interesting consequence of this uniqueness is the following naturality result of the contact-triad connection.

Theorem 2.1.12 (Naturality). *Let ∇ be the contact triad connection of the triad (Q, λ, J) associated to any given constant $c \in \mathbb{R}$. Consider any strict contact diffeomorphism $\phi : Q \to Q$, i.e., a diffeomorphism ϕ satisfying $\phi^*\lambda = \lambda$. Then the pull-back connection $\phi^*\nabla$ is the contact triad connection associated to the triad $(Q, \lambda, \phi^* J)$. In particular, this applies to any diffeomorphism arising from the Reeb flow ϕ^t of $\dot{x} = X_\lambda(x)$.*

Proof. A straightforward computation shows that the pull-back connection $\phi^*\nabla$ satisfies all Axioms (1) – (6). Therefore by the uniqueness, $\phi^*\nabla$ is the canonical connection. □

Remark 2.1.13. Axiom (1) includes the property

$$\nabla_{X_\lambda} d\lambda = 0$$

as a part of Hermitian property of the connection ∇^π on the Hermitian bundle $(\xi, d\lambda, J)$ over Q. However this is not a part of algebraic properties lifted from those of the canonical connection on $(\widehat{Q}, \widehat{d\lambda}, \widehat{J})$ because the lifted properties do not say anything about the X_λ-direction. Because of this, it is not obvious whether the vanishing $\nabla_{X_\lambda} d\lambda$ is consistent with other part of axioms. An examination of the uniqueness proof given in the section 2.1.4, though, shows that we never used the condition $\nabla_{X_\lambda} d\lambda = 0$ and so we can drop this requirement from Axiom (1) just by requiring $\nabla_Y d\lambda = 0$ for $Y \in \xi$. Furthermore the vanishing $\nabla_{X_\lambda} d\lambda = 0$ will automatically follow from Axioms (3) and

(5;c) (which is a generalization of Axiom (5), see (2.14) in next section) and does not lead to any contradiction with other axioms: the existence proof given in section 2.1.6 will ensure that the connection satisfying all these axioms of the contract triad connection exist. Therefore Axiom (1) in the presence of other axioms is equivalent to the latter. We prefer to state Axiom (1) as above because the statement is simpler and more natural to state.

The fact that the automatic vanishing of $\nabla_{X_\lambda} d\lambda$ derived from other parts of the axioms reflects the nice interplay between the geometric structures of contact form λ and of the endomorphism $J : \xi \to \xi$ via the compatibility requirement $g|_\xi = d\lambda(\cdot, J\cdot)$. The vanishing is a consequence of the symmetry of the operator $\mathcal{L}_{X_\lambda} J : \xi \to \xi$ as stated in Lemma 2.1.16 in section 2.1.5, whose proof in turn follows from the invariance property $\mathcal{L}_{X_\lambda} d\lambda = 0$ of $d\lambda$ under the Reeb flow.

2.1.4 Proof of the uniqueness of the contact triad connection

In this section, we give the uniqueness proof by analyzing the first structure equation and showing how every axiom determines the connection one-forms. In the next two sections, we explicitly construct a connection by carefully examining properties of the Levi-Civita connection and modifying the constructions in Section 2.1.2 as [KN], [Ko] for the canonical connection, and then show it satisfies all the requirements and thus the unique contact triad connection.

We are going to prove the existence and uniqueness for a more general family of connections. First, we generalize the Axiom (5) to the following Axiom: For $Y \in \xi$,

$$\nabla_{JY} X_\lambda + J\nabla_Y X_\lambda \in \mathbb{R} \cdot Y, \tag{2.13}$$

and we denote Axiom by (5; c)

(5; c) For a given $c \in \mathbb{R}$,

$$\nabla_{JY} X_\lambda + J \nabla_Y X_\lambda = c\, Y, \quad Y \in \xi. \tag{2.14}$$

In particular, Axiom (5) corresponds to Axiom (5; 0).

Theorem 2.1.14. *For any $c \in \mathbb{R}$, there exists a unique connection satisfying Axiom (1)-(4), (6) and (5; c).*

In particular, there exists a unique contact triad connection ∇ for the triad (Q, λ, J).

Proof. (Uniqueness)

Choose a moving frame of $TQ = \mathbb{R}\{X_\lambda\} \oplus \xi$ given by

$$X_\lambda, E_1, \cdots, E_n, JE_1, \cdots, JE_n$$

and denote its dual co-frame by

$$\lambda, \alpha^1, \cdots, \alpha^n, \beta^1, \cdots, \beta^n.$$

We use the Einstein summation convention to denote the sum of upper indices and lower indices in this paper.

Assume the connection matrix is (Ω^i_j), $i, j = 0, 1, ..., 2n$, and we write the first structure equations as follows.

$$\begin{aligned}
d\lambda &= -\Omega^0_0 \wedge \lambda - \Omega^0_k \wedge \alpha^k - \Omega^0_{n+k} \wedge \beta^k + T^0 \\
d\alpha^j &= -\Omega^j_0 \wedge \lambda - \Omega^j_k \wedge \alpha^k - \Omega^j_{n+k} \wedge \beta^k + T^j \\
d\beta^j &= -\Omega^{n+j}_0 \wedge \lambda - \Omega^{n+j}_k \wedge \alpha^k - \Omega^{n+j}_{n+k} \wedge \beta^k + T^{n+j}
\end{aligned}$$

Denote

$$\begin{aligned}
\Omega_j^i &= \Gamma_{0,j}^i\lambda + \Gamma_{k,j}^i\alpha^k + \Gamma_{n+k,j}^i\beta^k \\
\Omega_{n+j}^{n+i} &= \Gamma_{0,n+j}^{n+i}\lambda + \Gamma_{k,n+j}^{n+i}\alpha^k + \Gamma_{n+k,n+j}^{n+i}\beta^k \\
\Omega_j^{n+i} &= \Gamma_{0,j}^{n+i}\lambda + \Gamma_{k,j}^{n+i}\alpha^k + \Gamma_{n+k,j}^{n+i}\beta^k \\
\Omega_{n+j}^i &= \Gamma_{0,n+j}^i\lambda + \Gamma_{k,n+j}^i\alpha^k + \Gamma_{n+k,n+j}^i\beta^k
\end{aligned} \tag{2.15}$$

Throughout the section, if not stated otherwise, we let i, j and k take values from 1 to n.

We will analyze each axiom in Definition 2.1.10 and show how they set down the matrix of connection one-forms.

We first state that Axioms (1) and (2) uniquely determine $(\Omega_j^i|_\xi)_{i,j=1,\cdots,2n}$. This is exactly the same as Kobayashi's proof for the uniqueness of Hermitian connection given in [Ko]. To be more specific, we can restrict the first structure equation to ξ and get the following equations for α and β since ξ is the kernel of λ.

$$\begin{aligned}
d\alpha^j &= -\Omega_k^j|_\xi \wedge \alpha^k - \Omega_{n+k}^j|_\xi \wedge \beta^k + T^j|_\xi \\
d\beta^j &= -\Omega_k^{n+j}|_\xi \wedge \alpha^k - \Omega_{n+k}^{n+j}|_\xi \wedge \beta^k + T^{n+j}|_\xi
\end{aligned}$$

We can see $(\Omega_j^i|_\xi)_{i,j=1,\cdots,2n}$ is skew-Hermitian from Axiom (1). We also notice that from the proof of Proposition 2.1 that Axiom (2) is equivalent to say that $\Theta^{(1,1)} = 0$, where $\Theta = \Pi' T_{\mathbb{C}}$. Then one can strictly follow Kobayashi's proof of Theorem 2.1.6 in [Ko] and get $(\Omega_j^i|_\xi)_{i,j=1,\cdots,2n}$ are uniquely determined. For this part, we refer the proofs of [Ko, Theorem 1.1 and 2.1] for details.

In the rest of the proof, we will clarify how the Axioms (3), (4), (5;c), (6) uniquely determine Ω^0, Ω_0 and $(\Omega^i_j(X_\lambda))_{i,j=1,\cdots,2n}$. Compute the first equality in Axiom (4) and we get

$$\nabla_{X_\lambda} X_\lambda = \Gamma^0_{0,0} X_\lambda + \Gamma^k_{0,0} E_k + \Gamma^{n+k}_{0,0} J E_k = 0.$$

Hence

$$\Gamma^0_{0,0} = 0 \tag{2.16}$$

$$\Gamma^k_{0,0} = 0 \tag{2.17}$$

$$\Gamma^{n+k}_{0,0} = 0. \tag{2.18}$$

The second claim in Axiom (4) is equal to say

$$\nabla_{E_k} X_\lambda \in \xi, \quad \nabla_{JE_k} X_\lambda \in \xi. \tag{2.19}$$

Similar calculation shows that

$$\Gamma^0_{k,0} = 0 \tag{2.20}$$

$$\Gamma^0_{n+k,0} = 0. \tag{2.21}$$

Now (2.16), (2.20) and (2.21) uniquely settle down

$$\Omega^0_0 = \Gamma^0_{0,0} \lambda + \Gamma^0_{k,0} \alpha^k + \Gamma^0_{n+k,0} \beta^k = 0.$$

The vanishing of (2.17) and (2.18) will be used to determine Ω_0 in the later part. From Axiom (3), we can get

$$\Gamma^k_{j,0} - \Gamma^k_{0,j} = \langle [E_j, X_\lambda], E_k \rangle = -\langle \mathcal{L}_{X_\lambda} E_j, E_k \rangle \tag{2.22}$$

$$\Gamma^k_{n+j,0} - \Gamma^k_{0,n+j} = \langle [JE_j, X_\lambda], E_k \rangle = -\langle \mathcal{L}_{X_\lambda} (JE_j), E_k \rangle \tag{2.23}$$

and

$$\begin{aligned}\Gamma^{n+k}_{j,0} - \Gamma^{n+k}_{0,j} &= \langle [E_j, X_\lambda], JE_k\rangle = -\langle \mathcal{L}_{X_\lambda} E_j, JE_k\rangle &(2.24)\\ \Gamma^{n+k}_{n+j,0} - \Gamma^{n+k}_{0,n+j} &= \langle [E_j, X_\lambda], JE_k\rangle = -\langle \mathcal{L}_{X_\lambda}(JE_j), JE_k\rangle. &(2.25)\end{aligned}$$

From Axiom (5; c), we have

$$\begin{aligned}\Gamma^{k}_{j,0} + \Gamma^{n+k}_{n+j,0} &= 0 &(2.26)\\ \Gamma^{n+k}_{j,0} - \Gamma^{k}_{n+j,0} &= -c\,\delta_{j,k}. &(2.27)\end{aligned}$$

Now we show how to determine Ω^j_0 for $j = 1, \ldots, 2n$. For this purpose, we calculate $\Gamma^k_{j,0}$. First, by using (2.26), we write

$$\Gamma^k_{j,0} = \frac{1}{2}\Gamma^k_{j,0} - \frac{1}{2}\Gamma^{n+k}_{n+j,0}.$$

Furthermore, using (2.22) and (2.25) , we have

$$\begin{aligned}\Gamma^k_{j,0} &= \frac{1}{2}\Gamma^k_{j,0} - \frac{1}{2}\Gamma^{n+k}_{n+j,0}\\ &= \frac{1}{2}(\Gamma^k_{0,j} - \langle \mathcal{L}_{X_\lambda} E_j, E_k\rangle) - \frac{1}{2}(\Gamma^{n+k}_{0,n+j} - \langle \mathcal{L}_{X_\lambda}(JE_j), JE_k\rangle)\\ &= \frac{1}{2}(\Gamma^k_{0,j} - \Gamma^{n+k}_{0,n+j}) - \frac{1}{2}(\langle \mathcal{L}_{X_\lambda} E_j, E_k\rangle - \langle \mathcal{L}_{X_\lambda}(JE_j), JE_k\rangle)\\ &= \frac{1}{2}(\Gamma^k_{0,j} - \Gamma^{n+k}_{0,n+j}) - \frac{1}{2}\langle \mathcal{L}_{X_\lambda} E_j + J\mathcal{L}_{X_\lambda}(JE_j), E_k\rangle\\ &= \frac{1}{2}(\Gamma^k_{0,j} - \Gamma^{n+k}_{0,n+j}) - \frac{1}{2}\langle J(\mathcal{L}_{X_\lambda} J)E_j, E_k\rangle\\ &= \frac{1}{2}(\Gamma^k_{0,j} - \Gamma^{n+k}_{0,n+j}) + \frac{1}{2}\langle (\mathcal{L}_{X_\lambda} J)JE_j, E_k\rangle\end{aligned}$$

Notice the first term vanishes by Axiom (2). In particular, that is from $\nabla_{X_\lambda} J = 0$. Hence we get

$$\Gamma^k_{j,0} = \frac{1}{2}\langle (\mathcal{L}_{X_\lambda} J)JE_j, E_k\rangle. \tag{2.28}$$

Following the same idea, we use (2.27) and calculate

$$\begin{aligned}
\Gamma^{n+k}_{j,0} &= -\frac{1}{2}c\delta_{jk} + \frac{1}{2}\Gamma^{n+k}_{j,0} + \frac{1}{2}\Gamma^{k}_{n+j,0} \\
&= -\frac{1}{2}c\delta_{jk} + \frac{1}{2}(\Gamma^{n+k}_{0,j} - \langle \mathcal{L}_{X_\lambda} E_j, JE_k\rangle) + \frac{1}{2}(\Gamma^{k}_{0,n+j} - \langle \mathcal{L}_{X_\lambda}(JE_j), E_k\rangle) \\
&= -\frac{1}{2}c\delta_{jk} + \frac{1}{2}(\Gamma^{n+k}_{0,j} + \Gamma^{k}_{0,n+j}) - \frac{1}{2}(\langle \mathcal{L}_{X_\lambda} E_j, JE_k\rangle + \langle \mathcal{L}_{X_\lambda}(JE_j), E_k\rangle) \\
&= -\frac{1}{2}c\delta_{jk} - \frac{1}{2}\langle \mathcal{L}_{X_\lambda} E_j + J\mathcal{L}_{X_\lambda}(JE_j), JE_k\rangle \\
&= -\frac{1}{2}c\delta_{jk} - \frac{1}{2}\langle J(\mathcal{L}_{X_\lambda} J)E_j, JE_k\rangle \\
&= -\frac{1}{2}c\delta_{jk} + \frac{1}{2}\langle (\mathcal{L}_{X_\lambda} J)JE_j, JE_k\rangle.
\end{aligned}$$

Here the forth equality is due to $\nabla_{X_\lambda} J = 0$ as before. Then substituting this into (2.26) and (2.27), we get

$$\begin{aligned}
\Gamma^{k}_{n+j,0} &= \frac{1}{2}c\delta_{jk} + \frac{1}{2}\langle (\mathcal{L}_{X_\lambda} J)JE_j, JE_k\rangle \\
&= \frac{1}{2}c\delta_{jk} - \frac{1}{2}\langle (\mathcal{L}_{X_\lambda} J)E_j, E_k\rangle.
\end{aligned}$$

and

$$\begin{aligned}
\Gamma^{n+k}_{n+j,0} &= -\frac{1}{2}\langle (\mathcal{L}_{X_\lambda} J)JE_j, E_k\rangle \\
&= \frac{1}{2}\langle (\mathcal{L}_{X_\lambda} J)E_j, JE_k\rangle.
\end{aligned}$$

Together with (2.16), (2.17) and (2.18), Ω_0 is uniquely determined by this way.

Furthermore (2.22),(2.23),(2.24) and (2.25), uniquely determine $\Omega^i_j(X_\lambda)$ for $i, j = 1, \ldots, 2n$.

Notice that for any $Y \in \xi$, we derive

$$\nabla_{X_\lambda} Y \in \xi$$

from Axiom (3). This is because the axiom implies $\nabla_{X_\lambda} Y = \nabla_Y X_\lambda + \mathcal{L}_{X_\lambda} Y$ and the latter is contained in ξ: the second part of Axiom (4) implies $\nabla_Y X_\lambda \in \xi$ and the Lie derivative along the Reeb vector field preserves the contact structure ξ. It then follows that $\Gamma^0_{0,l} = 0$ for $l = 1, \ldots, 2n$. At the same time, Axiom (6) implies

$$\Gamma^0_{j,k} = -\Gamma^k_{j,0}.$$

for $j,\ k = 1, \ldots, 2n$. Hence together with (2.20) and (2.21), Ω^0 is uniquely determined.

We are done with the proof of uniqueness.

□

We end this section by giving a summary of the procedure we took in the proof of uniqueness which actually indicates a way how to construct this connection in later sections.

First, we used the Hermitian connection property, i.e., Axiom (1) and the torsion property Axiom (2), i.e., $T^\pi|_\xi$ has vanishing $(1,1)$ part, to uniquely fix the connection on ξ projection of ∇ when taking values on ξ.

Then we used the metric property

$$\langle X_\lambda, \nabla_Y Z\rangle + \langle \nabla_Y X_\lambda, Z\rangle = 0,$$

for any $Y, Z \in \xi$, to determine the X_λ component of ∇ when taking values in ξ.

To do this, we needed information about $\nabla_Y X_\lambda$. As mentioned before the second part of Axiom (4) enabled us to decompose

$$\nabla X_\lambda = \partial^\nabla X_\lambda + \overline{\partial}^\nabla X_\lambda$$

using

$$\partial^\nabla X_\lambda = \frac{\nabla X_\lambda - J\nabla_{J(\cdot)}X_\lambda}{2}, \quad \overline{\partial}^\nabla X_\lambda = \frac{\nabla X_\lambda + J\nabla_{J(\cdot)}X_\lambda}{2}.$$

The axiom $\nabla_{X_\lambda} J = 0$ embedded into the Hermitian property of (ξ, g, J) is nothing but

$$\nabla_{X_\lambda}(JY) - J\nabla_{X_\lambda} Y = 0$$

Axiom (3), the torsion property $T(X_\lambda, Y) = 0$, then became

$$\nabla_{JY} X_\lambda - J\nabla_Y X_\lambda = -(\mathcal{L}_{X_\lambda} J)Y$$

which is also equivalent to saying

$$J\overline{\partial}^\nabla_Y X_\lambda = \frac{1}{2}(\mathcal{L}_{X_\lambda} J)Y \quad \text{or} \quad \overline{\partial}^\nabla_Y X_\lambda = \frac{1}{2}(\mathcal{L}_{X_\lambda} J)JY. \tag{2.29}$$

It turned out that we can vary Axiom (5) by replacing it to (5;c)

$$\nabla_{JY} X_\lambda + J\nabla_Y X_\lambda = cY, \quad \text{or equivalently } \partial^\nabla_Y X_Y = \frac{c}{2}Y \tag{2.30}$$

for any given real number c. This way gives a one-parameter family of affine connections parameterized by $\mathbb{R}$ each of which satisfies Axioms (1) - (4) and (6) with (5) replaced by (5;c).

When c is fixed, i.e., under Axiom (5; c), we can uniquely determine $\nabla_Y X_\lambda$ to be

$$\nabla_Y X_\lambda = -\frac{1}{2}cJY + \frac{1}{2}(\mathcal{L}_{X_\lambda} J)JY.$$

Therefore, ∇_Y, $Y \in \xi$ is uniquely determined in this process by getting the formula for $\nabla_Y X_\lambda$ and the torsion property. Then the remaining property $\nabla_{X_\lambda} X_\lambda = 0$ now completely determines the connection.

2.1.5 Properties of the Levi-Civita connection on contact manifolds

From the discussion in previous sections, the only thing left to do for the existence of the contact triad connection is to globally define a connection such that it can patch the

ξ part of $\nabla|_\xi$ and the X_λ part of it. In particular, we seek a connection that satisfies the following properties:

1. it satisfies all the algebraic properties of the canonical connection of an almost Kähler manifold [Ko] when restricted to ξ.

2. it satisfies the metric property and has vanishing torsion in X_λ direction.

We construct such a connection by examining the properties of the Levi-Civita connection of the triad metric associated to the contact triad (Q, λ, J). Our interest in the special connection of contact triad is motivated by our attempt in later chapters to optimally organize the complicated tensorial expressions in the tensor calculations that appear in the analytical study of the maps of pseudoholomorphic curves in a contact manifold (or in its symplectization). We will give more details of this part in Chapter 3. Without this guidance, it would not have been possible for us to pinpoint the geometric properties laid out in our definition of the canonical connection in the way that the uniqueness and existence can be established. The presence of such a construction is a manifestation of delicate interplay between the geometric structures ξ, λ, and J in the geometry of contact triads (Q, λ, J). In this regard, the closedness of $d\lambda$ and the definition of Reeb vector field X_λ play important roles. In particular $d\lambda$ plays the role similar to that of the fundamental two-form Φ in the case of almost Kähler manifold [KN] (in a non-strict sense) in that it is closed.

This interplay turns out to be reflected already in several basic properties of the Levi-Civita connection of the contact triad metric exposed in this section. We would like to mention that similar results had been previously derived in Blair's book [Bl, Section 6.1, 6.2] in a much more general context of contact metric manifolds. However, our sign convention of the compatible metric is different from the one in [Bl] and also our derivation may be simpler for this particular case. To make our exposition self-contained, we include full derivation of these properties in this section.

Before we study the Levi-Civita connection, we would like to remind that we extended J to TQ by defining $J(X_\lambda) = 0$. We denote by $\Pi : TQ \to TQ$ the idempotent associated to the projection $\pi : TQ \to \xi$, i.e., the endomorphism satisfying

$$\Pi^2 = \Pi, \quad \operatorname{Im}\Pi = \xi, \quad \ker\Pi = \mathbb{R}\{X_\lambda\}.$$

We have now $J^2 = -\Pi$. Moreover, for any connection ∇ on Q,

$$(\nabla J)J = -(\nabla\Pi) - J(\nabla J). \tag{2.31}$$

Notice for $Y \in \xi$, we have

$$\Pi(\nabla\Pi)Y = 0 \tag{2.32}$$

$$(\nabla\Pi)X_\lambda = -\Pi\nabla X_\lambda. \tag{2.33}$$

We denote the triad metric $g = g_{(\lambda,J)}$ as

$$\langle X, Y\rangle := d\lambda(\Pi X, J\Pi Y) + \lambda(X)\lambda(Y)$$

for any $X, Y \in TQ$ for our computations hereafter.

We state the following obvious properties of $(g =: \langle\cdot,\cdot\rangle, \lambda, J)$.

Lemma 2.1.15.

$$\langle X, Y\rangle = d\lambda(X, JY) + \lambda(X)\lambda(Y)$$

$$d\lambda(X, Y) = d\lambda(JX, JY).$$

Therefore,

$$\begin{aligned}\langle JX, JY\rangle &= d\lambda(X, JY)\\ \langle X, JY\rangle &= -d\lambda(X, Y)\\ \langle JX, Y\rangle &= -\langle X, JY\rangle.\end{aligned}$$

However, we remark

$$\langle JX, JY\rangle \neq \langle X, Y\rangle$$

in general now.

The following preparation lemma says that the linear operator $\mathcal{L}_{X_\lambda}J$ is symmetric with respect to the metric $g = \langle \cdot, \cdot \rangle$.

Lemma 2.1.16 (Lemma 6.2 [Bl])**.** *For* $Y, Z \in \xi$*,*

$$\langle (\mathcal{L}_{X_\lambda}J)Y, Z\rangle = \langle Y, (\mathcal{L}_{X_\lambda}J)Z\rangle.$$

Proof.

$$
\begin{aligned}
& \langle(\mathcal{L}_{X_\lambda}J)Y, Z\rangle \\
= \;& \langle\mathcal{L}_{X_\lambda}(JY), Z\rangle - \langle J\mathcal{L}_{X_\lambda}Y, Z\rangle \\
= \;& d\lambda(\mathcal{L}_{X_\lambda}(JY), JZ) - d\lambda(J\mathcal{L}_{X_\lambda}Y, JZ) \\
= \;& d\lambda(\mathcal{L}_{X_\lambda}(JY), JZ) - d\lambda(\mathcal{L}_{X_\lambda}Y, Z) \\
= \;& X_\lambda(d\lambda(JY, JZ)) - (\mathcal{L}_{X_\lambda}d\lambda)(JY, JZ) - d\lambda(JY, \mathcal{L}_{X_\lambda}(JZ)) \\
& - d\lambda(\mathcal{L}_{X_\lambda}Y, Z) \\
= \;& X_\lambda(d\lambda(JY, JZ)) - d\lambda(JY, \mathcal{L}_{X_\lambda}(JZ)) - d\lambda(\mathcal{L}_{X_\lambda}Y, Z) \\
= \;& X_\lambda(d\lambda(JY, JZ)) - d\lambda(JY, (\mathcal{L}_{X_\lambda}J)Z) - d\lambda(JY, J\mathcal{L}_{X_\lambda}Z) - d\lambda(\mathcal{L}_{X_\lambda}Y, Z) \\
= \;& X_\lambda(d\lambda(Y, Z)) - d\lambda(JY, (\mathcal{L}_{X_\lambda}J)Z) - d\lambda(Y, \mathcal{L}_{X_\lambda}Z) - d\lambda(\mathcal{L}_{X_\lambda}Y, Z) \\
= \;& (X_\lambda(d\lambda(Y, Z)) - d\lambda(Y, \mathcal{L}_{X_\lambda}Z) - d\lambda(\mathcal{L}_{X_\lambda}Y, Z)) - d\lambda(JY, (\mathcal{L}_{X_\lambda}J)Z) \\
= \;& (\mathcal{L}_{X_\lambda}d\lambda)(Y, Z) + \langle Y, (\mathcal{L}_{X_\lambda}J)Z\rangle \\
= \;& \langle Y, (\mathcal{L}_{X_\lambda}J)Z\rangle.
\end{aligned}
$$

Here we use $\mathcal{L}_{X_\lambda}d\lambda = 0$ for the fifth and the last equalities. □

The following properties of the Levi-Civita connection on contact manifolds can be also found in [D, Lemma 3] as well as in [Bl]. One amusing consequence of this lemma is that the Reeb foliation is a geodesic foliation for the Levi-Civita connection (and so for the contact triad connection) of the contact triad metric.

Lemma 2.1.17. *For any vector field Z on Q,*

$$\nabla^{LC}_Z X_\lambda \in \xi, \tag{2.34}$$

and

$$\nabla^{LC}_{X_\lambda} X_\lambda = 0. \tag{2.35}$$

Proof. We first note

$$\left\langle \nabla^{LC}_Z X_\lambda, X_\lambda \right\rangle = \frac{1}{2} Z \langle X_\lambda, X_\lambda \rangle = 0 \tag{2.36}$$

for all vector fields Z. This already finishes the proof of $\nabla^{LC}_Z X_\lambda \in \xi$, since $\xi = \ker \lambda$ and $\langle \nabla^{LC}_Z X_\lambda, X_\lambda \rangle = \lambda(\nabla^{LC}_Z X_\lambda)$ by definition of the triad metric (2.7).

Next we compute

$$\begin{aligned}
\left\langle \nabla^{LC}_{X_\lambda} X_\lambda, Y \right\rangle &= -\left\langle X_\lambda, \nabla^{LC}_{X_\lambda} Y \right\rangle \\
&= -\left\langle X_\lambda, \nabla^{LC}_Y X_\lambda + [X_\lambda, Y] \right\rangle \\
&= 0 + \lambda[X_\lambda, Y]) = \lambda(\mathcal{L}_{X_\lambda} Y) \\
&= (\mathcal{L}_{X_\lambda} \lambda)(Y) - X_\lambda(\lambda(Y)).
\end{aligned}$$

The second term vanishes since $\lambda(Y) \equiv 0$ and the first vanishes by Cartan's formula

$$(\mathcal{L}_{X_\lambda} \lambda) = d(X_\lambda \rfloor \lambda) + X_\lambda \rfloor d\lambda = 0$$

and the definition of Reeb vector field X_λ. Together with (2.34), this implies (2.35). This finishes the proof. □

Next we derive the following lemma which is the contact analogue to the Prop 4.2 in [KN] for the almost Hermitian case. This lemma can be also extracted from [Bl, Corollary 6.1].

Lemma 2.1.18.

$$\begin{aligned}
2\langle (\nabla^{LC}_X J) Y, Z \rangle &= \langle N(Y, Z), JX \rangle \\
&\quad -\langle JX, JY \rangle \lambda(Z) + \langle JX, JZ \rangle \lambda(Y)
\end{aligned}$$

for $X,\ Y,\ Z \in TQ$*, where* N *is the Nijenhuis tensor defined as*

$$N(X,Y) = [JX,JY] - [X,Y] - J[X,JY] - J[JX,Y].$$

Proof. The left hand side

$$\begin{aligned}\langle(\nabla^{LC}_X J)Y,Z\rangle &= \langle\nabla^{LC}_X(JY),Z\rangle - \langle J\nabla^{LC}_X Y,Z\rangle \\ &= \langle\nabla^{LC}_X(JY),Z\rangle + \langle\nabla^{LC}_X Y,JZ\rangle\end{aligned}$$

Since ∇^{LC} is the Levi-Civita connection with respect to the Riemannian manifold $(Q, g = \langle\cdot,\cdot\rangle)$, we have the following formula

$$\begin{aligned}2\langle\nabla^{LC}_X Y,Z\rangle &= X\langle Y,Z\rangle + Y\langle X,Z\rangle - Z\langle X,Y\rangle \\ &\quad +\langle[X,Y],Z\rangle + \langle[Z,X],Y\rangle + \langle X,[Z,Y]\rangle.\end{aligned}$$

Using Lemma 2.1.15, we derive

$$\begin{aligned}2\langle(\nabla^{LC}_X J)Y,Z\rangle &= 2\langle\nabla^{LC}_X(JY),Z\rangle + 2\langle\nabla^{LC}_X Y,JZ\rangle \\ &= R(X,Y,Z) + S(X,Y,Z),\end{aligned}$$

From it, where

$$\begin{aligned}R(X,Y,Z) &:= (JY)d\lambda(X,JZ) - Yd\lambda(X,Z) + Zd\lambda(X,Y) - (JZ)d\lambda(X,JY) \\ &\quad +d\lambda([X,JY],JZ) - d\lambda([X,Y],Z) \\ &\quad -d\lambda([Z,X],Y) + d\lambda([JZ,X],JY) \\ &\quad +d\lambda([Z,JY],JX) + d\lambda([JZ,Y],JX), \qquad (2.37)\end{aligned}$$

and

$$\begin{aligned}S(X,Y,Z) &:= (JY)\,(\lambda(X)\lambda(Z)) - (JZ)\,(\lambda(X)\lambda(Y)) \\ &\quad +\lambda([X,JY])\lambda(Z) + \lambda([JZ,X])\lambda(Y) \\ &\quad +\lambda([Z,JY])\lambda(X) + \lambda([JZ,Y])\lambda(X).\end{aligned}$$

The way to deal with $R(X,Y,Z)$ is exactly the same as in the proof of [KN, Proposition 4.2] *if we replace* $d\lambda$ *by the fundamental two-form* Φ *therein.*

Lemma 2.1.19.

$$R(X,Y,Z) = \langle N(Y,Z), JX\rangle.$$

Proof. The proof follows by a straightforward calculation using $d(d\lambda) = 0$ and organizing $R(X,Y,Z)$ into

$$R(X,Y,Z) = (d(d\lambda))(X,JY,JZ) - (d(d\lambda))(X,Y,Z) + \langle N(Y,Z), JX\rangle.$$

Indeed, we can rewrite (2.37) into

$$\begin{aligned} R(X,Y,Z) \;\; = \;\; & -(JY)d\lambda(JZ,X) - (JZ)d\lambda(X,JY) - Xd\lambda(JY,JZ) \\ & +d\lambda([X,JY],JZ) + d\lambda([JZ,X],JY) + d\lambda([JY,JZ],X) \\ & +X(d\lambda(Y,Z)) + Yd\lambda(Z,X) + Zd\lambda(X,Y) \\ & -d\lambda([X,Y],Z) - d\lambda([Z,X],Y) - d\lambda([Y,Z],X) \\ & +d\lambda([Z,JY],JX) + d\lambda([JZ,Y],JX) \\ & -d\lambda([JY,JZ],X) + d\lambda([Y,Z],X). \end{aligned}$$

The difference between this formula for R and (2.37) is as follows: Here beside rearranging the terms, we subtracted $Xd\lambda(JY,JZ)$ in the first line and add back $X(d\lambda(Y,Z)) = Xd\lambda(JY,JZ)$ to the third line, and add $d\lambda([JY,JZ],X)$ to the second line and subtract it back in the fifth line.

Then the first two lines of this formula become $-(d(d\lambda))(X,JY,JZ)$ and the second two lines become $(d(d\lambda))(X,Y,Z)$ and final two lines can be re-written into

$$\begin{aligned} & d\lambda([Z,JY],JX) + d\lambda([JZ,Y],JX) - d\lambda([JY,JZ],X) + d\lambda([Y,Z],X) \\ = & -\langle J[JY,Z],JX\rangle - \langle J[Y,JZ],JX\rangle - \langle([Y,Z],JX) + \langle[JY,JZ],JX) \\ = & \langle N(Y,Z),JX\rangle. \end{aligned}$$

This finishes the proof. □

For $S(X,Y,Z)$, we use the formula

$$d\lambda(X,Y) = X(\lambda(Y) - Y(\lambda(X)) - \lambda([X,Y]),$$

to simplify it and get

$$\begin{aligned} S(X,Y,Z) &= -d\lambda(X,JY)\lambda(Z) + d\lambda(X,JZ)\lambda(Y) \\ &= -\langle JX,JY\rangle\lambda(Z) + \langle JX,JZ\rangle\lambda(Y). \end{aligned}$$

Combining all these, we have derived the formula

$$\begin{aligned} 2\langle(\nabla_X^{LC}J)Y,Z\rangle &= \langle N(Y,Z),JX\rangle \\ &\quad -\langle JX,JY\rangle\lambda(Z) + \langle JX,JZ\rangle\lambda(Y). \end{aligned}$$

□

In particular, we obtain the following corollary.

Corollary 2.1.20. *For* $Y, Z \in \xi$,

$$\begin{aligned} 2\langle(\nabla^{LC}_Y J)X_\lambda, Z\rangle &= \langle N(X_\lambda, Z), JY\rangle + \langle Y, Z\rangle \\ &= -\langle(\mathcal{L}_{X_\lambda} J)Z, Y\rangle + \langle Y, Z\rangle \\ 2\langle(\nabla^{LC}_Y J)Z, X_\lambda\rangle &= \langle N(Z, X_\lambda), JY\rangle - \langle Y, Z\rangle \\ &= \langle(\mathcal{L}_{X_\lambda} J)Z, Y\rangle - \langle Y, Z\rangle \\ 2\langle(\nabla^{LC}_X J)Y, Z\rangle &= \langle N(Y, Z), JX\rangle. \end{aligned}$$

Proof. This is a direct corollary from Lemma 2.1.18 except that we also use

$$N(X_\lambda, Z) = -J(\mathcal{L}_{X_\lambda} J)Z \tag{2.38}$$

$$N(Z, X_\lambda) = J(\mathcal{L}_{X_\lambda} J)Z. \tag{2.39}$$

for the first two conclusions. □

Next we prove the following lemma.

Lemma 2.1.21. *For* $Y, Z \in \xi$,

$$\begin{aligned} JN(Y, JZ) - \Pi N(Y, Z) &= 0 \\ \Pi N(Y, JZ) + \Pi N(Z, JY) &= 0. \end{aligned}$$

Proof. We compute for $Y, Z \in \xi$,

$$\begin{aligned}
& JN(Y, JZ) - \Pi N(Y, Z) \\
= \; & J\left([JY, JJZ] - [Y, JZ] - J[Y, JJZ] - J[JY, JZ]\right) \\
& -\Pi\left([JY, JZ] - [Y, Z] - J[Y, JZ] - J[JY, Z]\right) \\
= \; & J\left(-[JY, Z] - [Y, JZ] + J[Y, Z] - J[JY, JZ]\right) \\
& -\Pi\left([JY, JZ] - [Y, Z] - J[Y, JZ] - J[JY, Z]\right) \\
= \; & -J[JY, Z] - J[Y, JZ] - \Pi[Y, Z] + \Pi[JY, JZ] \\
& -\Pi[JY, JZ] + \Pi[Y, Z] + J[Y, JZ] + J[JY, Z] \\
= \; & 0.
\end{aligned}$$

For the second one, similarly, we compute

$$\begin{aligned}
& \Pi N(Y, JZ) + \Pi N(Z, JY) \\
= \; & \Pi\left([JY, JJZ] - [Y, JZ] - J[Y, JJZ] - J[JY, JZ]\right) \\
& +\Pi\left([JZ, JJY] - [Z, JY] - J[Z, JJY] - J[JZ, JY]\right) \\
= \; & -\Pi[JY, Z] - \Pi[Y, JZ] + J[Y, Z] - J[JY, JZ] \\
& -\Pi[JZ, Y] - \Pi[Z, JY] + J[Z, Y] - J[JZ, JY] \\
= \; & 0.
\end{aligned}$$

□

Together with the last equality in Corollary 2.1.20 and Lemma 2.1.21, we obtain the following lemma, which is the contact analogue to Lemma 2.1.8.

Lemma 2.1.22.

$$\Pi(\nabla^{LC}_{JY} J)X + J(\nabla^{LC}_{Y} J)X = 0. \tag{2.40}$$

Proof. We look at for any $Z \in \xi$,

$$
\begin{aligned}
& \langle \Pi(\nabla^{LC}_{JY} J)X + J(\nabla^{LC}_{Y} J)X, Z\rangle \\
= \;& \langle (\nabla^{LC}_{JY} J)X, Z\rangle - \langle (\nabla^{LC}_{Y} J)X, JZ\rangle \\
= \;& \frac{1}{2}\langle N(X,Z), JJY\rangle - \frac{1}{2}\langle N(X,JZ), JY\rangle \\
= \;& \frac{1}{2}\langle JN(X,JZ) - \Pi N(X,Z), Y\rangle = 0,
\end{aligned}
$$

and then (2.40) follows. □

The following result stated in [Bl, Corollary 6.1] is an immediate but important consequence of Corollary 2.1.20 and the property $\nabla_{X_\lambda} X_\lambda = 0$ of X_λ.

Proposition 2.4 (Corollary 6.1 [Bl])**.**

$$\nabla^{LC}_{X_\lambda} J = 0.$$

Proof. We will prove $(\nabla^{LC}_{X_\lambda} J)Z = 0$ for all vector field Z on Q.

For $(\nabla^{LC}_{X_\lambda} J)X_\lambda$, we have

$$(\nabla^{LC}_{X_\lambda} J)X_\lambda = \nabla^{LC}_{X_\lambda}(JX_\lambda) - J\nabla^{LC}_{X_\lambda} X_\lambda = 0 - 0 = 0.$$

Notice that

$$(\nabla^{LC}_{X_\lambda} J)Y = \nabla^{LC}_{X_\lambda}(JY) - J\nabla^{LC}_{X_\lambda} Y \in \xi$$

since $\nabla^{LC}_{X_\lambda}(JY) \in \xi$ for any vector field Y. Therefore we have

$$\langle (\nabla^{LC}_{X_\lambda} J)Y, X_\lambda\rangle = 0.$$

From Corollary 2.1.20, we also derive

$$\langle (\nabla^{LC}_{X_\lambda} J)Y, Z\rangle = 0$$

for any $Z \in \xi$. Altogether, we have proved

$$(\nabla^{LC}_{X_\lambda} J)Y = 0$$

for any $Y \in TQ$ and this completes the proof of the proposition. □

The following is equivalent to the second part of Lemma 6.2 [Bl] after taking into consideration of different sign convention of the definition of compatibility of J and $d\lambda$.

Lemma 2.1.23 (Lemma 6.2 [Bl])**.** *For any $Y \in \xi$, we have*

$$\nabla^{LC}_Y X_\lambda = \frac{1}{2} JY + \frac{1}{2}(\mathcal{L}_{X_\lambda} J)JY.$$

Proof. Since the Levi-Civita connection is Riemannian, for any $Y,\ Z \in \xi$, we have

$$\langle \nabla^{LC}_Y X_\lambda, Z\rangle = -\langle X_\lambda, \nabla^{LC}_Y Z\rangle.$$

Next we write

$$-\langle X_\lambda, \nabla^{LC}_Y Z\rangle \ = \ \langle X_\lambda, (\nabla^{LC}_Y J)(JZ)\rangle,$$

and then further by using the second equality in the Corollary 2.1.20, we have

$$\begin{aligned}
\langle \nabla^{LC}_Y X_\lambda, Z\rangle &= -\langle X_\lambda, \nabla^{LC}_Y Z\rangle \\
&= \langle X_\lambda, (\nabla^{LC}_Y J)(JZ)\rangle \\
&= \frac{1}{2}\langle(\mathcal{L}_{X_\lambda} J)(JZ), Y\rangle - \frac{1}{2}\langle Y, JZ\rangle \\
&= \frac{1}{2}\langle(\mathcal{L}_{X_\lambda} J)Y, JZ\rangle - \frac{1}{2}\langle Y, JZ\rangle \\
&= -\frac{1}{2}\langle J(\mathcal{L}_{X_\lambda} J)Y, Z\rangle + \frac{1}{2}\langle JY, Z\rangle \\
&= \langle \frac{1}{2} JY + \frac{1}{2}(\mathcal{L}_{X_\lambda} J)JY, Z\rangle
\end{aligned}$$

for any $Z \in \xi$. Here we use Lemma 2.1.16 for the forth equality. Since $\nabla^{LC}_Y X_\lambda \in \xi$ for $Y \in \xi$, we are done with the proof. □

2.1.6 Existence of the contact triad connection

In this section, we establish the existence theorem of the contact triad connection in two stages. Recall that the space of affine connections on a given smooth manifold M is an affine space and so for any given affine connection ∇ the sum $\nabla^B := \nabla + B$ defines a new affine connection for any given tensor B of type $\binom{1}{2}$ by the formula

$$\nabla^B_{Z_1} Z_2 = \nabla_{Z_1} Z_2 + B(Z_1, Z_2).$$

Now consider the endomorphism $\widetilde{B}(Z_1)$ of TM defined by

$$\langle \widetilde{B}(Z_1)(Z_2), Z_3\rangle := \langle B(Z_1, Z_2), Z_3\rangle.$$

When ∇ is Riemannian with respect to the given metric, the connection ∇^B remains Riemannian if $\widetilde{B}$ is skew-symmetric with respect to the associated inner product, i.e., it satisfies

$$\langle \widetilde{B}(Z_1)Z_2, Z_3\rangle = -\langle \widetilde{B}(Z_1)Z_3, Z_2\rangle. \tag{2.41}$$

First, we examine the relationship between the connections of two different c's. Denote by $\nabla^{\lambda;c}$ the unique connection associated to the constant c. The following proposition shows that $\nabla^{\lambda;c}$ and $\nabla^{\lambda;c'}$ for two different nonzero constants with the same parity are essentially the same in that it arises from the scale change of the contact form.

Proposition 2.5. Let (Q, λ, J) be a contact triad and consider the triad $(Q, a\lambda, J)$ for a constant $a > 0$. Then

$$\nabla^{a\lambda;1} = \nabla^{\lambda;a}.$$

Proof. By definition, $\nabla^{a\lambda;1}$ is characterized by Axioms (1) - (4), (6) and (5;-1) for the triad $(Q, a\lambda, J)$. Since Axioms (1) - (4) and (6) are obviously scale-invariant of the contact form λ, this connection satisfies all axioms for the triad (Q, λ, J). The only thing left to check is Axiom (5;a). But this immediately follows from the relationship of the Reeb vector fields under the multiplication by a positive constant, which is

$$X_{a\lambda} = \frac{1}{a} X_\lambda.$$

Therefore $\nabla^{a\lambda;1}$ satisfies

$$\nabla^{a\lambda;1}_{JY} X_{a\lambda} + J\nabla^{a\lambda;1}_{Y} X_{a\lambda} = Y$$

which is equivalent to

$$\nabla^{a\lambda;1}_{JY} X_{\lambda} + J\nabla^{a\lambda;1}_{Y} X_{\lambda} = a\, Y.$$

By the defining Axiom (5;a) for $\nabla^{\lambda;a}$, and the uniqueness thereof, we have proved $\nabla^{a\lambda;1} = \nabla^{\lambda;a}$. This finishes the proof. □

In regard to this proposition, one could say that for each given contact structure (Q, ξ), there are essentially two inequivalent ∇^0, ∇^1 (respectively three, ∇^0, ∇^1 and ∇^{-1}, if one fixes the orientation) choice of triad connections for each given projective equivalence class of the contact triad (Q, λ, J). In this regard, the connection ∇^0 is essentially different from others in that this argument of scaling procedure of contact form λ does not apply to the case $a = 0$ since it would lead to the zero form $0 \cdot \lambda$. This proposition also reduces the construction essentially two connections of $\nabla^{\lambda;0}$ and $\nabla^{\lambda;1}$ (or $\nabla^{\lambda;-1}$).

In the rest of this section, we will explicitly construct $\nabla^{\lambda;-1}$ and $\nabla^{\lambda;c}$ in two stages, by modifying the Levi-Civita connection by adding suitable tensors B as described above.

In the first stage, partially motivated by the construction of the Hermitian connection on almost Kähler manifold and exploiting the properties of the Levi-Civita connection we extracted in the previous section, we construct a connection named $\nabla^{tmp;1}$ and calculate its metric and torsion properties. This part itself provides a contact analogue of Kobayashi's construction of the Hermitian connection and the resulting connection $\nabla^{tmp;1}$ satisfies Axioms (1)-(4), (5;−1), (6).

In the second stage, we modify $\nabla^{tmp;1}$ to the final connection $\nabla^{tmp;2}$ mainly to deforming the property (5;−1) thereof to (5;c) leaving other properties of $\nabla^{tmp;1}$ intact for given constant c. This $\nabla^{tmp;2}$ then satisfies all the axioms in Definition 2.1.10. The construction of $\nabla^{tmp;2}$ exhibits a way of combining the Levi-Civita connection and the canonical connection, so that the resulting connection behaves like the canonical connection on ξ and like the Levi-Civita connection on X_λ. Indeed this is the original idea behind the choice of the axioms laid out in Definition 2.1.10.

Remark 2.1.24. We find that it is rather mysterious to see that the second stage of modification does not differentiate the case of $c = 0$ from that of $c \neq 0$, although the above mentioned scaling procedure of contact form clearly single out this case from others. It seems to say that even though the contact form degenerates as $c \to 0$, the associated contact triad connection itself converges to a smooth well-defined connection in the space of affine connections.

Modification 1; $\nabla^{tmp;1}$

We begin the first stage of constructing our first connection $\nabla^{tmp;1}$ motivated by the construction of canonical connection on almost Kähler manifold. This is the contact analogue to this construction.

Define an affine connection $\nabla^{tmp;1}$ by the formula

$$\nabla^{tmp;1}_{Z_1} Z_2 = \nabla^{LC}_{Z_1} Z_2 - \Pi P(\Pi Z_1, \Pi Z_2)$$

where the bilinear map $P : \Gamma(TQ) \times \Gamma(TQ) \to \Gamma(TQ)$ over $C^\infty(Q)$ is defined by

$$4P(X, Y) = (\nabla^{LC}_{JY} J)X + J((\nabla^{LC}_{Y} J)X) + 2J((\nabla^{LC}_{X} J)Y) \tag{2.42}$$

for vector fields X, Y in Q. (To avoid confusion with our notation Q for the contact manifold and to highlight that P is not the same tensor field as Q but is the contact analogue thereof, we use P instead for its notation.) From (2.40), we have now

$$\Pi P(\Pi Z_1, \Pi Z_2) = \frac{1}{2} J(\nabla^{LC}_{\Pi Z_1} J)\Pi Z_2.$$

According to the remark made in the beginning of the section, we choose B to be

$$B_1(Z_1, Z_2) = -\Pi P(\Pi Z_1, \Pi Z_2) = -\frac{1}{2} J(\nabla^{LC}_{\Pi Z_1} J)\Pi Z_2. \tag{2.43}$$

First we consider the induced vector bundle connection on the Hermitian bundle $\xi \to Q$, which we denote by $\nabla^{tmp;1,\pi}$: it is defined by

$$\nabla^{tmp;1,\pi}_X Y := \pi \nabla^{tmp;1}_X Y \tag{2.44}$$

for a vector field Y tangent to ξ, i.e., a section of ξ for arbitrary vector field X on Q. We now prove the J linearity of $\nabla^{tmp;1,\pi}$.

Lemma 2.1.25. *Let $\pi : TQ \to \xi$ be the projection. Then*

$$\nabla^{tmp;1,\pi}_X (JY) = J\nabla^{tmp;1,\pi}_X Y$$

for $Y \in \xi$ and all $X \in TQ$.

Proof. For $X \in \xi$,

$$\begin{aligned}
& \nabla_X^{tmp;1}(JY) \\
= \;& \nabla_X^{LC}(JY) - \Pi P(X, JY) \\
= \;& (J\nabla_X^{LC}Y + (\nabla_X^{LC}J)Y) - \frac{1}{2}J((\nabla_X^{LC}J)JY) \\
= \;& J\nabla_X^{LC}Y + (\nabla_X^{LC}J)Y - \frac{1}{2}\Pi((\nabla_X^{LC}J)Y) + \frac{1}{2}J((\nabla_X^{LC}\Pi)Y) \qquad (2.45) \\
= \;& J\nabla_X^{LC}Y + (\nabla_X^{LC}J)Y - \frac{1}{2}\Pi((\nabla_X^{LC}J)Y)
\end{aligned}$$

where we use (2.31) to get the last two terms in the third equality and use (2.32) to see that the last term in (2.45) vanishes. Hence,

$$\begin{aligned}
\pi\nabla_X^{tmp;1}(JY) &= \pi\nabla_X^{tmp;1}(JY) \\
&= J\nabla_X^{LC}Y + \frac{1}{2}\pi((\nabla_X^{LC}J)Y).
\end{aligned}$$

On the other hand, we compute

$$\begin{aligned}
J\pi\nabla_X^{tmp;1}Y &= J\left(\nabla_X^{LC}Y - \frac{1}{2}J((\nabla_X^{LC}J)Y)\right) \\
&= J\nabla_X^{LC}Y + \frac{1}{2}\pi((\nabla_X^{LC}J)Y).
\end{aligned}$$

Hence we have now

$$\pi\nabla_X^{tmp;1}(JY) = J\pi\nabla_X^{tmp;1}Y$$

for $X,\ Y \in \xi$.

We notice that $\nabla_{X_\lambda}^{tmp;1}Y = \nabla_{X_\lambda}^{LC}Y$. By using Proposition 2.4, the equality

$$\pi\nabla_X^{tmp;1}(JY) = J\pi\nabla_X^{tmp;1}Y$$

also holds for $X = X_\lambda$, and we are done with the proof. $\square$

Next we study the metric property of $\nabla^{tmp;1}$ by computing $\langle \nabla^{tmp;1}_X Y, Z\rangle + \langle Y, \nabla^{tmp;1}_X Z\rangle$ for arbitrary $X, Y, Z \in TQ$.

Using the metric property of the Levi-Civita connection, we derive

$$\begin{aligned} & \langle \nabla^{tmp;1}_X Y, Z\rangle + \langle Y, \nabla^{tmp;1}_X Z\rangle - X\langle Y, Z\rangle \\ = \; & \langle \nabla^{LC}_X Y, Z\rangle + \langle Y, \nabla^{LC}_X Z\rangle - X\langle Y, Z\rangle - \langle \Pi P(\Pi X, \Pi Y), Z\rangle - \langle Y, \Pi P(\Pi X, \Pi Z)\rangle \\ = \; & -\langle \Pi P(\Pi X, \Pi Y), Z\rangle - \langle Y, \Pi P(\Pi X, \Pi Z)\rangle, \end{aligned} \tag{2.46}$$

The following lemma shows that when $X, Y, Z \in \xi$ this last line vanishes. This is the contact analogue to Proposition 2.2 whose proof is also similar thereto this time based on Lemma 2.1.21.

Lemma 2.1.26. *For $X, Y, Z \in \xi$,*

$$\langle P(X, Y), Z\rangle + \langle Y, P(X, Z)\rangle = 0.$$

Therefore,

$$\langle \nabla^{tmp;1}_X Y, Z\rangle + \langle Y, \nabla^{tmp;1}_X Z\rangle = X\langle Y, Z\rangle.$$

Proof. We compute for $X, Y, Z \in \xi$,

$$\begin{aligned} & \langle P(X, Y), Z\rangle + \langle Y, P(X, Z)\rangle \\ = \; & \frac{1}{2}\langle J((\nabla^{LC}_X J)Y), Z\rangle + \frac{1}{2}\langle Y, J((\nabla^{LC}_X J)Z)\rangle \\ = \; & -\frac{1}{2}\langle (\nabla^{LC}_X J)Y, JZ\rangle - \frac{1}{2}\langle JY, (\nabla^{LC}_X J)Z\rangle \\ = \; & -\frac{1}{4}\langle N(Y, JZ), JX\rangle - \frac{1}{4}\langle N(Z, JY), JX\rangle \qquad (2.47) \\ = \; & -\frac{1}{4}\langle \Pi N(Y, JZ) + \Pi N(Z, JY), JX\rangle = 0, \qquad (2.48) \end{aligned}$$

where we use the third equality of Corollary 2.1.20 for (2.47) and use the second equality of Lemma 2.1.21 for the vanishing of (2.48). □

Now, we are ready to state the following proposition.

Proposition 2.6. The vector bundle connection $\nabla^{tmp;1,\pi} := \pi\nabla^{tmp;1}$ is an Hermitian connection of the Hermitian bundle $\xi \to Q$.

Proof. What is now left to show is that for any $Y,\ Z \in \xi$,

$$\langle \nabla^{tmp;1}_{X_\lambda} Y, Z\rangle + \langle Y, \nabla^{tmp;1}_{X_\lambda} Z\rangle = X_\lambda \langle Y, Z\rangle,$$

which immediately follows from our construction of $\nabla^{tmp;1}$ since

$$\begin{aligned} \nabla^{tmp;1}_{X_\lambda} Y &= \nabla^{LC}_{X_\lambda} Y \\ \nabla^{tmp;1}_{X_\lambda} Z &= \nabla^{LC}_{X_\lambda} Z. \end{aligned}$$

□

Next, we look at the metric property when the Reeb direction gets involved.

Lemma 2.1.27. *For $Y,\ Z \in \xi$,*

$$\langle \nabla^{tmp;1}_{Y} X_\lambda, Z\rangle + \langle X_\lambda, \nabla^{tmp;1}_{Y} Z\rangle = 0.$$

Proof. We compute for $Y,\ Z \in \xi$,

$$\begin{aligned} & \langle \nabla^{tmp;1}_{Y} X_\lambda, Z\rangle + \langle X_\lambda, \nabla^{tmp;1}_{Y} Z\rangle \\ = \ & \langle \nabla^{LC}_{Y} X_\lambda, Z\rangle + \langle X_\lambda, \nabla^{LC}_{Y} Z\rangle - \langle X_\lambda, \Pi P(Y,Z)\rangle = 0. \end{aligned}$$

This finishes the proof. □

Now we study the torsion property of $\nabla^{tmp;1}$. Denote the torsion of $\nabla^{tmp;1}$ by $T^{tmp;1}$. We complexify the contact structure ξ and denote it by $\xi_{\mathbb{C}} = \xi \otimes \mathbb{C}$. Then $\xi_{\mathbb{C}}$ has the decomposition

$$\xi_{\mathbb{C}} = \xi^{(1,0)} \oplus \xi^{(0,1)}.$$

Denote Π' the projection to $\xi^{(1,0)}$ and $T^{tmp;1}_{\mathbb{C}}$ is the complexification of $T^{tmp;1}$. Define

$$\Theta^{\pi} = \Pi' T^{tmp;1,\pi}_{\mathbb{C}}.$$

The proof of the following lemma follows essentially the same strategy as that of the proof of [KN, Theorem 3.4]. But we would like to highlight two conventions we are using which are different therefrom:

1. We use the definition

$$N(Y,Z) = [JY, JZ] - [Y,Z] - J[Y,JZ] - J[JY,Z]$$

   without the factor of 2 differently from [KN].

2. Our definition of the wedge product is the one from [S] but not the one from [KN]. More specifically, in our convention, we have

$$d\lambda(X,Y) = X[\lambda(Y)] - Y[\lambda(X)] - \lambda([X,Y])$$

   while the one from [KN] gives rise to

$$2\, d\lambda(X,Y) = X[\lambda(Y)] - Y[\lambda(X)] - \lambda([X,Y].$$

Besides these differences of the convention, since the current case deals with the contact case, whose statements are significantly different from the almost Hermitian case.

Lemma 2.1.28. *For $Y \in \xi$,*

$$T^{tmp;1}(X_\lambda, Y) = 0.$$

If we decompose

$$T^{tmp;1}|_\xi = \pi T^{tmp;1}|_\xi + \lambda(T^{tmp;1,\pi}|_\xi)\, X_\lambda$$

and denote $T^{tmp;1,\pi}|_\xi := \pi T^{tmp;1,\pi}|_\xi$*, then*

$$\begin{aligned} T^{tmp;1,\pi}|_\xi &= \frac{1}{4}N^\pi|_\xi \\ \lambda(T^{tmp;1}|_\xi) &= 0. \end{aligned}$$

In particular, $\Theta^\pi|_\xi$ *is of* $(0,2)$ *form or equivalently* $JT^{tmp;1}(JY,Z) = T^{tmp;1}(Y,Z)$ *for all* $Y,\ Z \in \xi$.

Proof. Since $\nabla^{tmp;1} = \nabla^{LC} - \Pi P(\Pi,\Pi)$ and ∇^{LC} is torsion free, we derive for $Y,\ Z \in \xi$,

$$\begin{aligned} T^{tmp;1}(Y,Z) &= T^{LC}(Y,Z) - \Pi P(Y,Z) + \Pi P(Z,Y) \\ &= \frac{1}{2}J\nabla^{LC}_Y JZ - \frac{1}{2}J\nabla^{LC}_Z JY. \end{aligned}$$

from the general torsion formula.

Next we calculate $-\Pi P(\Pi Y,\Pi Z) + \Pi P(\Pi Z,\Pi Y)$ using the formula

$$\begin{aligned} \frac{1}{2}J\nabla^{LC}_Y JZ - \frac{1}{2}J\nabla^{LC}_Z JY &= \frac{1}{4}\pi([JY,JZ] - \pi[Y,Z] - J[JY,Z] - J[Y,JZ]) \\ &= \frac{1}{4}\pi N(Y,Z). \end{aligned}$$

This follows from the general formula

$$-P(Y,Z) + P(Z,Y) = \frac{1}{4}([JY,JZ] - \Pi[Y,Z] - J[JY,Z] - J[Y,JZ]), \tag{2.49}$$

whose derivation we postpone till Appendix A.

On the other hand, since the added terms to ∇^{LC} only involves ξ-directions, the X_λ-component of the torsion does not change and so

$$\lambda(T^{tmp;1}|_\xi) = \lambda(T^{LC}|_\xi) = 0.$$

This finishes the proof. □

From the definition of $\nabla^{tmp;1}$, we have the following lemma from the properties of the Levi-Civita connection in Proposition 2.1.17.

Lemma 2.1.29. $\nabla^{tmp;1}_{X_\lambda} X_\lambda = 0$ *and* $\nabla^{tmp;1}_Y X_\lambda \in \xi$ *for any* $Y \in \xi$.

We also get the following property by using Lemma 2.1.23 for Levi-Civita connection.

Lemma 2.1.30. *For any* $Y \in \xi$, *we have*

$$\nabla^{tmp;1}_Y X_\lambda = \frac{1}{2} JY + \frac{1}{2}(\mathcal{L}_{X_\lambda} J)JY.$$

We end the construction of $\nabla^{tmp;1}$ by summarizing that $\nabla^{tmp;1}$ satisfies Axioms (1)-(4),(6) and (5;−1).

Modification 2; $\nabla^{tmp;2}$

Now we introduce another modification $\nabla^{tmp;2}$ starting from $\nabla^{tmp;1}$ to make it satisfy Axiom (5;c) and preserve other axioms for any given constant $c \in \mathbb{R}$. Recall that $\nabla^{\lambda;0}$ is our definition of the contact triad connection and that $\nabla^{tmp;1}$ satisfies (5;−1) and so $\nabla^{tmp;1} = \nabla^{\lambda;-1}$.

We define

$$\begin{aligned} \nabla^{tmp;2}_{Z_1} Z_2 &= \nabla^{tmp;1}_{Z_1} Z_2 - \frac{1}{2}(1+c)\left\langle Z_2, X_\lambda \right\rangle JZ_1 - \frac{1}{2}(1+c)\left\langle Z_1, X_\lambda \right\rangle JZ_2 \\ &\quad + \frac{1}{2}(1+c)\left\langle JZ_1, Z_2 \right\rangle X_\lambda. \end{aligned}$$

In other words, we define $\nabla^{tmp;2} = \nabla^{tmp;1} + B_2$ for the tensor B_2 defined by

$$B_2(Z_1, Z_2) = \frac{1}{2}(1+c)\left(-\left\langle Z_2, X_\lambda \right\rangle JZ_1 - \left\langle Z_1, X_\lambda \right\rangle JZ_2 + \left\langle JZ_1, Z_2 \right\rangle X_\lambda\right). \tag{2.50}$$

From its expression, it follows

$$
\begin{aligned}
B_2(X_\lambda, X_\lambda) &= 0 && (2.51)\\
B_2(Y_1, Y_2) &= \frac{1}{2}(1+c)\langle JY_1, Y_2\rangle X_\lambda, && (2.52)\\
B_2(Y, X_\lambda) &= -\frac{1}{2}(1+c)JY = B_2(X_\lambda, Y), && (2.53)
\end{aligned}
$$

for any Y_1, Y_2, $Y \in \xi$.

Now let's check all the properties required for $\nabla^{tmp;2}$.

Proposition 2.7. The connection $\nabla^{tmp;2}$ satisfies all the properties of the canonical connection with constant c. In particular $\nabla := \nabla^{tmp;2}$ with $c = 0$ is the contact triad connection.

Proof. For $Y,\ Z \in \xi$, (2.52) gives $\nabla_Y^{tmp;2,\pi} Z = \nabla_Y^{tmp;1,\pi} Z$. Hence we have

$$
\begin{aligned}
\nabla_Y^{tmp;2,\pi}(JZ) &= J\nabla_Y^{tmp;2,\pi} Z\\
\left\langle \nabla_X^{tmp;2} Y, Z\right\rangle + \left\langle Y, \nabla_X^{tmp;2} Z\right\rangle &= X\left\langle Y, Z\right\rangle
\end{aligned}
$$

for $X,\ Y,\ Z \in \xi$ from the properties of $\nabla^{tmp;1}$.

For $Z \in \xi$, $\nabla_{X_\lambda}^{tmp;2,\pi}(JZ) = J\nabla_{X_\lambda}^{tmp;2,\pi} Z$ follows from (2.53) and $\nabla_{X_\lambda}^{tmp;1,\pi}(JZ) = J\nabla_{X_\lambda}^{tmp;1,\pi} Z$.

The metric property,

$$
\left\langle \nabla_{X_\lambda}^{tmp;2} Y_1, Y_2\right\rangle + \left\langle Y_1, \nabla_{X_\lambda}^{tmp;2} Y_2\right\rangle = 0
$$

for Y_1, $Y_2 \in \xi$ immediately follows from that of $\nabla^{tmp;1}$ and (2.53). Hence we have checked that Axiom (1) is satisfied.

We also have $T^{tmp;2,\pi}(Y_1, Y_2) = T^{tmp;1,\pi}(Y_1, Y_2)$ again by (2.52) and (2.53). Therefore, Axiom (2) is satisfied.

For Axiom (3), we calculate for $Y \in \xi$,

$$\begin{aligned} T^{tmp;2}(X_\lambda, Y) &= \nabla^{tmp;2}_{X_\lambda} Y - \nabla^{tmp;2}_Y X_\lambda - [X_\lambda, Y] \\ &= \nabla^{tmp;1}_{X_\lambda} Y - \frac{1}{2}(1+c)\, JY - \nabla^{tmp;1}_Y X_\lambda + \frac{1}{2}(1+c)\, JY - [X_\lambda, Y] \\ &= T^{tmp;1}(X_\lambda, Y) = 0. \end{aligned}$$

Axiom (4) immediately follows from the definition.

For Axiom (5;c), we compute

$$\begin{aligned} & \nabla^{tmp;2}_{JY} X_\lambda + J\nabla^{tmp;2}_Y X_\lambda \\ = \; & \nabla^{tmp;1}_{JY} X_\lambda - \frac{1}{2}(1+c)\, JJY + J\nabla^{tmp;1}_Y X_\lambda - J\frac{1}{2}(1+c)\, JY \\ = \; & -Y + (1+c)Y = cY. \end{aligned}$$

Then we can uniquely get the following property

$$\nabla^{tmp;2}_Y X_\lambda = -\frac{1}{2} cJY + \frac{1}{2}(\mathcal{L}_{X_\lambda} J)JY$$

as explained at the end of Section 2.1.3, and Axiom (6) is preserved by $\nabla^{tmp;2}$ after a short calculation by recalling Lemma 2.1.30 together with $\nabla^{tmp;2} X_\lambda = \nabla^{tmp;1} X_\lambda$. □

This proposition finally completes our construction of the contact triad connection, which is $\nabla^{\lambda;0}$.

We now summarize the modifications that we have performed in the previous section. Our connection $\nabla^{\lambda;-1}$ is nothing but

$$\nabla^{\lambda;-1} = \nabla^{LC} + B_1$$

with the tensor B_1 of the type $\binom{1}{2}$ given by $B_1(Z_1, Z_2) = -\Pi P(\Pi Z_1, \Pi Z_2)$, and $\nabla = \nabla^{\nabla;0}$ is

$$\nabla = \nabla^{\lambda;-1} + B_2 = \nabla^{LC} + B_1 + B_2 \tag{2.54}$$

where

$$B_2(Z_1, Z_2) = \frac{1}{2}\left(-\langle Z_2, X_\lambda\rangle JZ_1 - \langle Z_1, X_\lambda\rangle JZ_2 + \langle JZ_1, Z_2\rangle X_\lambda\right). \tag{2.55}$$

Before ending this section, we restate the following properties which will be useful for calculations involving contact Cauchy-Riemann maps performed in Sections 3, 4, 5.

Proposition 2.8. Let ∇ be the connection satisfying Axiom (1)-(4),(6) and (5; c), then

$$\nabla_Y X_\lambda = -\frac{1}{2}cJY + \frac{1}{2}(\mathcal{L}_{X_\lambda}J)JY.$$

In particular, for the contact triad connection,

$$\nabla_Y X_\lambda = \frac{1}{2}(\mathcal{L}_{X_\lambda}J)JY.$$

Proof. We already gave its proof in the last part of Section 2.1.3. □

Proposition 2.9. Decompose the torsion of ∇ into $T = \pi T + \lambda(T)\, X_\lambda$. The triad connection ∇ has its torsion given by $T(X_\lambda, Z) = 0$ for all $Z \in TQ$, and

$$\begin{aligned} \pi T(Y,Z) &= \frac{1}{4}\pi N(Y,Z) = \frac{1}{4}\left((\mathcal{L}_{JY}J)Z + (\mathcal{L}_Y J)JZ\right) \\ \lambda(T(Y,Z)) &= d\lambda(Y,Z) \end{aligned}$$

for all Y, $Z \in \xi$.

Proof. We have seen

$$\pi T^{tmp;2}|_\xi = \pi T^{tmp;1}|_\xi = \frac{1}{4}N^\pi|_\xi.$$

On the other hand, a simple computation shows

$$N^\pi(Y,Z) = (\mathcal{L}_{JY}J)Z - J(\mathcal{L}_Y J)Z = (\mathcal{L}_{JY}J)Z + (\mathcal{L}_Y J)JZ.$$

This proves the first equality.

For the second, a straightforward computation shows

$$\begin{aligned}\lambda(T^{tmp;2}(Y,Z)) &= \lambda(T^{tmp;1}(Y,Z)) + (1+c)\,\langle JY, Z\rangle \\ &= (1+c)\,d\lambda(Y,Z)\end{aligned}$$

for general c. Substituting $c = 0$, we obtain the second equality. This finishes the proof. □

2.2 Morse-Bott contact form and the linearization of Reeb orbits

Let (Q, ξ) be a contact manifold and λ be a contact form of ξ. Recall λ defines the Reeb vector field X_λ by (2.1)

$$X_\lambda \rfloor \lambda \equiv 1, \quad X_\lambda \rfloor d\lambda \equiv 0.$$

The existence of closed integral orbits of X_λ is phrased as the famous Weinstein conjecture which is partially proved for some cases, especially for dimension three, but still unknown for the general case. Assume there exists a closed Reeb orbit γ of period $T > 0$. That is to say, $\gamma : \mathbb{R} \to Q$ is a solution of $\dot\gamma = X_\lambda(\gamma)$ satisfying $\gamma(T) = \gamma(0)$. We would like to study the linearization of the equation $\dot x = X_\lambda(x)$ along the closed Reeb orbit γ.

By definition, we can write $\gamma(t) = \phi^t(\gamma(0))$ for the Reeb flow $\phi^t := \phi^t_{X_\lambda}$ of the Reeb vector field X_λ. In particular $p = \gamma(0)$ is a fixed point of the diffeomorphism ϕ^T when γ is a closed Reeb orbit of period T. We will call the pair (T, z) a Reeb orbit of period T instead for such closed orbit γ of period T by writing $z(t) = \gamma(Tt)$ for a loop parameterized over the unit interval $S^1 = [0,1]/\sim$. It is obvious that $\dot z = T X_\lambda(z)$.

Since $L_{X_\lambda}\lambda = 0$, the contact diffeomorphism ϕ^T canonically induces the isomorphism on ξ,

$$\Psi_z := d\phi^T(p)|_{\xi_p} : \xi_p \to \xi_p$$

which is the linearized Poincaré return map ϕ^T restricted to ξ_p via the splitting

$$T_pM = \mathbb{R}\cdot\{X_\lambda(p)\} \oplus \xi_p.$$

Definition 2.2.1. We say a T-closed Reeb orbit (T, z) is *nondegenerate* if the linearized return map $\Psi_z(p) : \xi_p \to \xi_p$ with $p = \gamma(0)$ has no eigenvalue 1.

It is well known that there exists a generic family of contact one-forms with respect to a contact manifold (Q, ξ), such that every closed Reeb orbit (of any period) is non-degenerate (e.g., the appendix in [ABW] provides a proof which is closest to the setting of this thesis).

Denote $\mathrm{Cont}(Q, \xi)$ the set of contact one-forms with respect to the contact structure ξ and $\mathcal{L}(Q) = C^\infty(S^1, M)$ the space of loops $z : S^1 = \mathbb{R}/\mathbb{Z} \to M$. Let $\mathcal{L}^{1,2}(Q)$ be the $W^{1,2}$-completion of $\mathcal{L}(Q)$. We would like to consider some Banach bundle $\mathcal{L}$ over the Banach manifold $(0,\infty) \times \mathcal{L}^{1,2}(Q) \times \mathrm{Cont}(Q, \xi)$ whose fiber at (T, z, λ) is given by $L^{1,2}(z^*TQ)$. We consider the assignment

$$\Upsilon : (T, z, \lambda) \mapsto \dot{z} - T\,X_\lambda(z)$$

which is a section. Then $(T, z, \lambda) \in \Upsilon^{-1}(0) := \mathfrak{Reeb}(Q, \xi)$ if and only if there exists some Reeb orbit $\gamma : \mathbb{R} \to Q$ with period T, such that $z(\cdot) = \gamma(T\cdot)$.

From the formula of a T-periodic orbit (T, γ),

$$T = \int_\gamma \lambda.$$

it follows that the period varies smoothly on γ.

Definition 2.2.2. We say that the contact form λ is of Morse-Bott type if every connected component of $\mathfrak{Reeb}(Q, \xi)$ is a smooth submanifold of $(0, \infty) \times \mathcal{L}^{1,2}(Q)$ with its tangent space at every Reeb orbit (T, z) therein coincides with $\ker d_{(T,z)}\Upsilon$.

Lemma 2.2.3. *Suppose λ is of Morse-Bott type, then on each connected component of $\mathfrak{Reeb}(Q, \xi)$, the period remains constant.*

Proof. Let γ_0, γ_1 be two elements in the same connected component. We connect them by a smooth one-parameter family γ_s for $0 \leq s \leq 1$ and denote by $T = T(s)$ the corresponding period function. It is enough to prove

$$\frac{d}{ds}\int_{\gamma_s} \lambda = \frac{d}{ds}\int_{S^1} \gamma_s^* \lambda \equiv 0. \tag{2.56}$$

We compute

$$\frac{d}{ds}\gamma_s^*\lambda = \gamma_s^*(d(\gamma_s' \rfloor \lambda) + \gamma_s' \rfloor d\lambda).$$

Therefore we obtain

$$\frac{d}{ds}\int_{\gamma_s} \lambda = \int_{S^1} \gamma_s' \rfloor d\lambda = \int_{S^1} d\lambda(\gamma_s', X_\lambda)\, dt = 0$$

where we use the equation $\dot\gamma_s = X_\lambda(\gamma_s)$. This finishes the proof. □

This lemma enables us to have the following finitely dimensional characterization of Morse-Bott type.

Proposition 2.10. The contact form λ is Morse-Bott if and only if the subset $N_T \subset Q$ formed by the closed Reeb trajectories of period T is a smooth closed submanifold of M, such that the rank of $d\lambda|_{N_T}$ is locally constant and $T_p N_T = \ker(\Psi_z(p) - id|_{\xi_p})$.

To prove this proposition, we would like to study the linearization operator carefully. Recall

$$\Upsilon : (T, z, \lambda) \mapsto \dot{z} - T\, X_\lambda(z)$$

which is a section of $\mathcal{L}$. Then $(T, z, \lambda) \in \Upsilon^{-1}(0)$ if and only if there exists some Reeb orbit $\gamma : \mathbb{R} \to Q$ with period T, such that $z(\cdot) = \gamma(T\cdot)$.

We also denote $DX_\lambda : \Omega^0(\xi) \to \Omega^0(\xi)$ the covariant derivative of X_λ induced from the contact triad connection ∇ to highlight its aspect as a linear operator, whenever we feel convenient. The following derivation of the linearization of Υ is a routine exercise. (See [ABW, Appendix] for a formula that is close to the current form.)

Lemma 2.2.4. *For any torsion free connection,*

$$d(T, z, \lambda)\Upsilon(a, Y, B) = \frac{DY}{dt} - TDX_\lambda(z)(Y) - aX_\lambda - T\delta_\lambda X_\lambda(B),$$

where $a \in \mathbb{R}$, $Y \in T_z\mathcal{L}^{1,2}(Q) = W^{1,2}(z^*TQ)$, $B \in T_\lambda \mathrm{Cont}(Q, \xi)$ and the last term $\delta_\lambda X_\lambda$ is some linear operator.

We remark that the contact triad connection we use here is *not* torsion-free. However, when $(T, z, \lambda) \in \Upsilon^{-1}(0)$, i.e., $z(\cdot) = \gamma(T\cdot)$ for some γ which is a Reeb orbit with period T with respect to contact one-form λ, the torsion Axiom (3) in Definition 2.1.10 is already enough to derive Lemma 2.2.4. From now on, we use the contact triad connection through out this section, but we remark the linearization at $(T, z, \lambda) \in \Upsilon^{-1}(0)$ actually doesn't depend of the choice of connections.

In our study here, we only need to look at the linearization restricted to subspace $W^{1,2}(z^*\xi)$ for fixed (T, λ). Denote the corresponding operator by

$$\Upsilon_{T,\lambda} = \Upsilon(T, \cdot, \lambda).$$

The rest of the section will be occupied by the proof of Proposition 2.10.

From Lemma 2.2.4 and Proposition 2.8, we compute

$$\begin{aligned} d_z^\pi \Upsilon_{T,\lambda}|_{W^{1,2}(z^*\xi)}(\zeta) &= \pi \frac{D\zeta}{dt} - T \cdot \pi D X_\lambda(z)(\zeta) \\ &= \frac{D^\pi \zeta}{dt} - \frac{T}{2}(\mathcal{L}_{X_\lambda} J) J \zeta. \end{aligned}$$

Since from Axiom (3) of Definition 2.1.10, the image of $d_z \Upsilon_{T,\lambda}|_{W^{1,2}(z^*\xi)}$ automatically in ξ.

Lemma 2.2.5. *Let $DX_\lambda(z) = \nabla_{(\cdot)} X_\lambda : z^*TQ \to z^*TQ$ be the covariant derivative of X_λ with respect to the pull-back connection $z^*\nabla$ of the contact triad connection. Consider a Reeb orbit (T, z) i.e., a map $z : S^1 \to Q$ satisfying $\dot{z} = T X_\lambda(z)$ with $z(1) = z(0)$. Then*

$$DX_\lambda(z)(Y) = \frac{1}{2}(\mathcal{L}_{X_\lambda} J(z)) J(z) Y$$

for any section $Y \in \Omega^0(z^*TQ)$.

Proof. By definition, we have

$$DX_\lambda(z)(Y) = \nabla_Y X_\lambda$$

and then apply Proposition 2.8, which proves the equality. □

Recall by Lemma 2.1.16, both $(\mathcal{L}_{X_\lambda} J)J$ and $\mathcal{L}_{X_\lambda} J$ are pointwise symmetric with respect to the triad metric of (Q, λ, J).

Combining the above discussion, we have derived

Proposition 2.11. The linear operator

$$J d_z^\pi \Upsilon_{T,\lambda} = J\pi \left(\frac{D}{dt} - DX_\lambda(z) \right) = J \frac{D^\pi}{dt} - \frac{T}{2}(\mathcal{L}_{X_\lambda} J) : L^2(z^*\xi) \to L^2(z^*\xi)$$

is a self-adjoint operator. In particular, we obtain

$$\text{Index}\, d^\pi_z \Upsilon_{T,\lambda} = \text{Index}\, J d^\pi_z \Upsilon_{T,\lambda} = 0. \tag{2.57}$$

By Proposition 2.11, the surjectivity of $d^\pi_z \Upsilon_{T,\lambda}$ is equivalent to the injectivity of the operator. In fact, we prove the following characterization of kernel elements of the linearization map $d^\pi_z \Upsilon_{T,\lambda}$ in terms of the eigenvectors of the linear map $\Psi_p : \xi_p \to \xi_p$ where $\Psi_p = d\phi^T(p)|_{\xi_p}$.

Proposition 2.12. Let $p = z(0)$ be a fixed point of $\phi^T : Q \to Q$ lying in the given Reeb orbit (T, z). Then there exists a one-one correspondence

$$v \in \xi_p \mapsto \eta; \quad \eta(t) := d\phi^{tT}(v), \quad t \in [0,1]$$

between the set of eigenvectors v of $\Psi_\gamma = d\phi^T|_{\xi_p} : \xi_p \to \xi_p$ with eigenvalue 1 and the set of solutions η to $\frac{D\eta}{dt} - \frac{T}{2}(\mathcal{L}_{X_\lambda} J)\eta = 0$.

Proof. Recall that any closed Reeb orbit of period T has the form $z(t) = \phi^{tT}(p)$ for a fixed point p of ϕ^{tT}.

Suppose η is a solution to $0 = d^\pi_z \Upsilon_{T,\lambda}(\eta) = \frac{D\eta}{dt} - \frac{T}{2}(\mathcal{L}_{X_\lambda} J)J\eta$. We consider the one-parameter family

$$v(t) = (d\phi^{tT})^{-1}(\eta(t))$$

of tangent vectors at $p \in M$, and so $\eta(t) = d\phi^{tT}(v(t))$. We compute $\nabla_t \eta(t)$ by considering the map $\Gamma(s,t) = \phi^{tT}(\alpha(s,t))$ such that $\alpha(0,t) \equiv p$ and $\frac{\partial}{\partial s}\Big|_{s=0} \alpha(s,t) = v(t)$. Then we compute

$$\frac{\partial \Gamma}{\partial s} = d\phi^{tT}\left(\frac{\partial \alpha}{\partial s}\right), \quad \frac{\partial \Gamma}{\partial t}(s,t) = TX_\lambda(\Gamma(s,t)) + d\phi^{tT}\left(\frac{\partial \alpha}{\partial t}(s,t)\right)$$

and so

$$\begin{aligned}
\nabla_t \eta &= \frac{D}{dt}\frac{\partial \Gamma}{\partial s}\Big|_{s=0} = \frac{D}{ds}\frac{\partial \Gamma}{\partial t}\Big|_{s=0} \\
&= T\frac{D}{ds}(X_\lambda(\Gamma(s,t))\Big|_{s=0} + d\phi^{tT}\left(\frac{D}{\partial s}\Big|_{s=0}\frac{\partial \alpha}{\partial t}(s,t)\right) \\
&= T\frac{D}{ds}(X_\lambda(\Gamma(s,t))\Big|_{s=0} + d\phi^{tT}\left(\frac{D}{\partial t}\frac{\partial \alpha}{\partial s}\Big|_{s=0}(s,t)\right) \\
&= T\frac{D}{ds}(X_\lambda(\Gamma(s,t))\Big|_{s=0} + d\phi^{tT}\left(v'(t)\right).
\end{aligned}$$

Here the second and the fourth equalities follow from the torsion property of the triad connection

$$T\left(\frac{\partial \Gamma}{\partial t}\Big|_{s=0}, \frac{\partial \Gamma}{\partial s}\Big|_{s=0}\right) = T(d\phi^{tT}v(t), X_\lambda(\phi^{tT}(p)) = 0.$$

The first term of the farthest right becomes

$$T\frac{D}{ds}(X_\lambda(\Gamma(s,t))\Big|_{s=0} = \frac{T}{2}(\mathcal{L}_{X_\lambda}J)J\eta.$$

Therefore we have derived

$$v'(t) = (d\phi^{tT})^{-1}\left(\nabla_t^\pi \eta - \frac{T}{2}(\mathcal{L}_{X_\lambda}J)J\eta\right) = 0.$$

by the hypothesis that η satisfies the equation $\frac{D\eta}{dt} - \frac{T}{2}(\mathcal{L}_{X_\lambda}J)J\eta = 0$. Therefore we have

$$v(1) = v(0), i.e., (d\phi^T)^{-1}(\eta(1)) = \eta(0).$$

Since $\eta(0) = \eta(1)$, it implies that $J\eta(0)$ is an eigenvector of eigenvalue 1 if $\eta(0) \neq 0$.

Conversely suppose that v is an eigenvector of $\phi^{tT} : \xi_p \to \xi_p$. Then the above computation of v' applied to constant function $v(t) \equiv v$ proves that the vector field $t \mapsto d\phi^{tT}(v)$ satisfies $\nabla_t^\pi \eta - \frac{T}{2}(\mathcal{L}_{X_\lambda}J)J\eta = 0$. This finishes the proof. □

This proposition in particular characterizes the nondegeneracy of Reeb orbits, i.e., $d_z^\pi \Upsilon_{T,\lambda}|_{W^{1,2}(z^*\xi)}$ is surjective if and only if $\Psi_\gamma = d\phi^T|_{\xi_p}$ has an eigenvalue 1.

Proposition 2.13. A closed Reeb orbit γ with period T is nondegenerate if and only if the ξ projection of the linearization restricted to $W^{1,2}(z^*\xi)$, i.e.,

$$d_z^\pi \Upsilon_{T,\lambda}|_{W^{1,2}(z^*\xi)} := \pi d_z \Upsilon_{T,\lambda}|_{W^{1,2}(z^*\xi)} : W^{1,2}(z^*\xi) \to L^2(z^*\xi)$$

is surjective, where $z(\cdot) := \gamma(T\cdot) : S^1 \to Q$.

2.3 The geometry of prequantization and its contact thickening

2.3.1 Some preliminaries of contact forms

Let (Q, ξ) be a contact manifold. We start with stating the following lemma.

Lemma 2.3.1. *Let λ be a contact form of the contact manifold (Q, ξ). Then for any given one-form α, there exists a unique $Y \in \xi$ such that*

$$\alpha = Y \rfloor d\lambda + \alpha(X_\lambda)\lambda.$$

We write such uniquely determined Y as Y_α^λ.

Proof. From the pointwise decomposition of $T_xQ = \mathbb{R} \cdot \{X_\lambda(x)\} \oplus \xi_x$, we have

$$T_x^*Q = \operatorname{span}\{X_\lambda(x)\}^\perp \oplus \xi_x^\perp$$

where the annihilators are given by

$$\operatorname{span}\{X_\lambda(x)\}^\perp = \{v \rfloor d\lambda(x) \mid v \in \xi_x\} \quad \xi_x^\perp = \operatorname{span}\{\lambda_x\},$$

where the former comes from the nondegeneracy of $d\lambda$. The conclusion follows immediately. $\square$

This lemma gives a characterization of contact forms. A contact form λ provides a frame to $\Omega^1(Q)$, with coordinates $(Y^\lambda_\cdot, X_\lambda \rfloor \cdot) \in \xi \times \Omega^0(Q)$. In particular, any contact one-form of (Q,ξ) lives in $\{0\}\times\Omega^0(Q)^+ \cup \{0\}\times\Omega^0(Q)^-$, which is the following well-known fact.

Corollary 2.3.2. *Assume α is another contact one-form of (Q,ξ), then $\alpha = f\lambda$ for some nowhere vanishing function $f : Q \to \mathbb{R}$.*

We remark that such family of contact one-forms $f\lambda$ has the same conformal class of symplectic forms $d\lambda$ on ξ. Hence when we impose the extra structure of compatible almost complex structures to ξ, such complex structure is independent of the choice of contact one-forms.

Next, we look at the relation between the Reeb vector fields of $X_{f\lambda}$ and X_λ, where f is a nowhere vanishing function. We assume f is positive from now on.

Proposition 2.14. Assume $X_{f\lambda}$ and X_λ are the Reeb vector fields of $f\lambda$ and λ respectively. Denote by $\pi_{f\lambda}$ and π_λ their projections to the contact distribution. Then we have

$$X_{f\lambda} = \frac{1}{f}(Y^\lambda_{\log f} + X_\lambda) \tag{2.58}$$

$$\pi_{f\lambda}(\cdot) = \pi_\lambda(\cdot) - \lambda(\cdot) Y^\lambda_{\log f}, \tag{2.59}$$

where $Y^\lambda_{\log f} := Y^\lambda_{d\log f}$ as given in Lemma 2.3.1.

Proof. It turns out to be easier to consider $f\, X_{f\lambda}$ and so we consider the decomposition

$$f\, X_{f\lambda} = c \cdot X_\lambda + \eta$$

with respect to the splitting $TM = \mathbb{R}\{X_\lambda\} \oplus \xi_\lambda$. We evaluate

$$c = \lambda(f X_{f\lambda}) = (f\lambda)(X_{f\lambda}) = 1.$$

Notice that

$$d(f\lambda) = fd\lambda + df \wedge \lambda,$$

so

$$\begin{aligned}
\eta \rfloor d\lambda &= (fX_{f\lambda})\rfloor d\lambda \\
&= X_{f\lambda}\rfloor d(f\lambda) - X_{f\lambda}\rfloor(df \wedge \lambda) \\
&= -X_{f\lambda}\rfloor(df \wedge \lambda) \\
&= -X_{f\lambda}(f)\lambda + \lambda(X_{f\lambda})df \\
&= -\frac{1}{f}(X_\lambda + \eta)(f)\lambda + \frac{1}{f}\lambda(X_\lambda + \eta)df \\
&= -\frac{1}{f}X_\lambda(f)\lambda - \frac{1}{f}\eta(f)\lambda + \frac{1}{f}df.
\end{aligned}$$

Take value of X_E for both sides, we get

$$\eta(f) = 0,$$

and hence

$$\begin{aligned}
\eta \rfloor d\lambda &= -\frac{1}{f}X_\lambda(f)\lambda + \frac{1}{f}df \\
&= -\frac{1}{f}X_\lambda(f)\lambda + d\log f.
\end{aligned}$$

This gives the proof of (2.58).

To see (2.59), we compute for any vector field Z,

$$\begin{aligned}
\pi_{f\lambda}(Z) &= Z - f\lambda(Z)X_{f\lambda} \\
&= Z - \lambda(Z)(fX_{f\lambda}) \\
&= Z - \lambda(Z)X_\lambda + (\lambda(Z)X_\lambda - \lambda(Z)(fX_{f\lambda})) \\
&= \pi_\lambda Z + \lambda(Z)(X_\lambda - fX_{f\lambda}) \\
&= \pi_\lambda Z - \lambda(Z)Y^\lambda_{\log f}.
\end{aligned}$$

This finishes the proof. □

2.3.2 Prequantization and its contact thickening

We recall the definition of Morse-Bott condition and the symplectomorphism $\Psi_z(q) : \xi_q \to \xi_q$, $q \in N_T$ as given in Section 2.2. Without the danger of confusion, we omit T and use N to denote the clean submanifold from now on. The proof of this following lemma can be found in [Kl, Proposition 3.2.1].

Lemma 2.3.3. *Assume $\Psi : V \to V$ is a symplectomorphism from the symplectic space V to itself. Then the generalized eigenspace of eigenvalue 1 is a symplectic subspace of V.*

In this thesis, we restrict ourselves to *a special case that T is the minimal period and the generalized eigenspace of eigenvalue 1 of Ψ_z is the eigenspace.* Thus from the above lemma, the contact structure restricted to the Morse-Bott clean submanifold $\xi|_{TN}$ forms a symplectic subspace with respect to the symplectic form $d\lambda$ at every point $q \in N$. By the Morse-Bott condition, we require also the dimension of this space is constant for $q \in N$ and is the same as the multiplicity of the eigenvalue 1, which is always even.

Next, we give the prequantization picture of such clean Morse-Bott submanifold N and also its contact thickening.

Proposition 2.15. The submanifold $N \subset Q$ required as above is a contact submanifold. The restriction of λ to N defines a contact form for $(N, \xi|_{TN})$ such that the followings hold:

1. We have the splitting

$$TQ = TN \oplus (TN)^{d\lambda},$$

where $(TN)^{d\lambda} := \{e \in TQ \big| d\lambda(\tilde{e}, e) = 0, \text{ for any } \tilde{e} \in TN\}$ is the symplectic normal bundle of N in Q with respect to $d\lambda$.

2. The two-form $d\lambda$ restricts to a nondegenerate skew-symmetric two-form on the symplectic normal bundle $(TN)^{d\lambda}$.

3. The quotient $P := N/\sim$, the set of closed Reeb orbits, is a smooth manifold and it carries a canonical symplectic form ω_P so that the circle bundle with connection form $\theta := \lambda|_N$ has its curvature become $\pi^*\omega$.

Proof. We have shown statement (1) and (2). For the proof of (3), we note that $d\lambda$ defines a presymplectic form on N such that $\ker d\lambda|_{TN} = \mathbb{R} \cdot \{X_\lambda\}$. By the general symplectic reduction theorem, the statement follows. □

Denote by $E := (TN)^{d\lambda}$ the symplectic normal bundle over N, which has the symplectic vector bundle structure with each fiber over $q \in N$ a symplectic vector space $(E_q, \Omega := d\lambda_{(TN)^{d\lambda}}(q))$.

Proposition 2.16. The S^1-action on N canonically induces the S^1-equivariant vector bundle structure on E such that the form Ω is equivariant under the S^1-action on E.

Proof. The action of S^1 on N by $t \cdot q = \phi^t(q)$ canonically induces a S^1 action on $T_N Q$ by $t \cdot v = (d\phi^t)(v)$, for $v \in T_N Q$. Hence it follows the following identity since the Reeb flow preserves λ,

$$t^* d\lambda = d\lambda. \tag{2.60}$$

We first show it is well-defined on $E \to N$, i.e., if $v \in (T_q N)^{d\lambda}$, then $t \cdot v \in (T_{t\cdot q} N)^{d\lambda}$. In fact, by using (2.60), for $w \in T_{t\cdot q} N$,

$$d\lambda(t \cdot v, w) = \left((\phi^t)^* d\lambda\right)\left(v, (d\phi^t)^{-1}(w)\right) = d\lambda\left(v, (d\phi^t)^{-1}(w)\right).$$

This vanishes, since N consists of Reeb orbits and thus $d\phi^t$ preserves TN.

Secondly, the same identity (2.60) further indicates that this S^1 action preserves Ω on fibers, i.e., $t^*\Omega = \Omega$, and we are done with the proof of this Proposition. □

Remark 2.3.4. By this proposition, it follows that $E \to Q$ further induces a symplectic vector bundle $E \to P$ with Ω as symplectic form on each fiber.

Motivated by this, we will examine a natural contact structure on a neighborhood of the zero section of the S^1-equivariant symplectic vector bundle (E, Ω) on the base N equipped with the prequantization circle bundle, and prove a normal form theorem.

Firstly, we recall the definition of prequantization circle bundle. Let (P, ω) be an integral compact symplectic manifold, i.e., ω be an integral symplectic form. The prequantization circle bundle $\pi : N \to P$ of (P, ω) is one whose connection form θ satisfies

$$d\theta = \pi^*\omega. \tag{2.61}$$

Then θ defines a contact form whose associated contact structure $\ker\theta \subset TN$ is given by the horizontal distribution of the Ehresman connection assiciated to the connection form θ. Its associated Reeb vector field is given by $X_\theta = \frac{\partial}{\partial\theta}$ which is nothing but the generator of the circle action on N which is a fiberwise rotation of the constant period. We note that θ defines a canonical contact form that is invariant under the S^1-action.

Next, we consider an S^1-invariant symplectic vector bundle (E, Ω). We denote by $\vec{R}$ the radial vector field which generates the family of radial multiplication

$$(c, e) \mapsto c\,e.$$

This vector field is invariant under the given S^1-action on E, and vanishes on the zero section. By its definition, $d\pi(\vec{R}) = 0$, i.e., $\vec{R}$ is in the vertical distribution, denoted by VTE, of TE.

Denote the canonical isomorphism $V_eTE \cong E_{\pi(e)}$ by $I_{e;\pi(e)}$. It obviously intertwines the scalar multiplication, i.e.,

$$I_{e;\pi(e)}(\mu\,\xi) = \mu I_{e;\pi(e)}(\xi)$$

for a scalar μ. It also satisfies the following identity (2.62) with respect to the derivative of the fiberwise scalar multiplication map $R_c : E \to E$.

Lemma 2.3.5. *Let $\xi \in V_eTE$. Then*

$$I_{c\,e;\pi(c\,e)}(dR_c(\xi)) = c\, I_{e;\pi(e)}(\xi) \tag{2.62}$$

on $E_{\pi(c\,e)} = E_{\pi(e)}$ for any constant c.

Proof. We compute

$$\begin{aligned} I_{c\,e;\pi(c\,e)}(dR_c(\xi)) &= I_{c\,e;\pi(c\,e)}\left(\frac{d}{ds}\Big|_{s=0} c(e+s\xi)\right) \\ &= I_{c\,e;\pi(c\,e)}(R_c(\xi)) = c\, I_{e;\pi(e)}(\xi) \end{aligned}$$

which finishes the proof. □

We then define the fiberwise two-form Ω^v on $VTE \to E$ by

$$\Omega^v_e(\xi_1, \xi_2) = \Omega_{\pi_E(e)}(I_{e;\pi(e)}(\xi_1), I_{e;\pi(e)}(\xi_2))$$

for $\xi_1, \xi_2 \in V_eTE$, and one-form $\vec{R}\rfloor\Omega^v$ respectively.

Now we introduce an S^1-invariant symplectic (vector bundle) connection on (E, Ω) and denote by

$$TE = HTE \oplus VTE$$

the associated splitting. Existence of such an invariant connection follows, e.g., by averaging over the compact group S^1. Using the splitting, we extend the fiberwise two-form Ω^v on $VTE \to E$ to a genuine differential two-form on E by setting $\Omega^v|_{HTE} \equiv 0$. We denote this two form by $\widetilde{\Omega}$. We define a one-form on E by

$$\lambda_E = \pi_E^*\theta + \frac{1}{2}\vec{R}\rfloor\widetilde{\Omega}. \tag{2.63}$$

Remark 2.3.6. Suppose $d\lambda_E(\cdot, J_E\cdot) =: g_{E;J_E}$ defines a Hermitian vector bundle $(\xi_E, g_{E,J}, J_E)$. Then we can write the radial vector field considered in the previous section as

$$\vec{R} = r\,\frac{\partial}{\partial r}$$

where $r^2(p,v) = g(v,v) = |v|_g^2$. Let (E, Ω, J_E) be a Hermitian vector bundle. Motivated by the terminology used in [BT], we call the one-form

$$\psi = \psi_\Omega = \frac{1}{r}\frac{\partial}{\partial r}\rfloor\Omega^v \tag{2.64}$$

the *global angular form* for the Hermitian vector bundle (E, Ω, J_E). Note that ψ is defined only on $E \setminus o_E$ although Ω is globally defined.

Lemma 2.3.7. *Let $\Omega = d\lambda|_{(TN)^{d\lambda}}$ be as in the previous section. Then,*

1. $\vec{R}\rfloor d\widetilde{\Omega} = 0$.

2. *For any non-zero constant $c > 0$, we have*

$$R_c^*\widetilde{\Omega} = c^2\,\widetilde{\Omega}.$$

Proof. Notice that $\widetilde{\Omega}$ is compatible with Ω in the sense of symplectic fiberation and the symplectic vector bundle connection is nothing but the Ehresmann connection induced by $\widetilde{\Omega}$, which is a symplectic connection now. Since $\vec{R}$ is vertical, the statement (1) immediately follows from the fact that the symplectic connection is vertical closed.

It remains to prove statement (2). Let $e \in E$ and $\xi_1, \xi_2 \in T_eE$. By definition, we derive

$$\begin{aligned}
(R_c^*\widetilde{\Omega})_e(\xi_1, \xi_2) &= \widetilde{\Omega}_{c\,e}(dR_c(\xi_1), dR_c(\xi_2)) \\
&= \Omega^v_{c\,e}(dR_c(\xi_1), dR_c(\xi_2))) \\
&= \Omega_{\pi_E(c\,e)}(I_{c\,e;\pi(e)}(dR_c(\xi_1)), I_{c\,e;\pi(e)}(dR_c(\xi_2))) \\
&= \Omega_{\pi_E(e)}(c\,I_{e;\pi(e)}(\xi_1), c\,I_{e;\pi(e)}(\xi_2)) \\
&= c^2\Omega_{\pi_E(e)}(I_{e;\pi(e)}(\xi_1), I_{e;\pi(e)}(\xi_2)) = c^2\Omega^v_e(\xi_1, \xi_2) \\
&= c^2\widetilde{\Omega}_e(\xi_1, \xi_2)
\end{aligned}$$

where we use the equality (2.62) and $\pi_E(c\,e) = \pi(e)$ for the fourth equality.

This proves $R_c^*\widetilde{\Omega} = c^2\widetilde{\Omega}$. □

It follows from this lemma that $\mathcal{L}_{\vec{R}}\widetilde{\Omega} = 2\widetilde{\Omega}$. By Cartan's formula, we get

$$d(\vec{R}\rfloor\widetilde{\Omega}) = 2\widetilde{\Omega}.$$

Therefore we have derived

$$d\lambda_E = \pi_E^*(d\theta) + \widetilde{\Omega}. \tag{2.65}$$

Proposition 2.17. There exists some $\delta > 0$ such that the one-form λ_E is a contact form on the disc bundle $D^\delta(E)$, where

$$D^\delta(E) = \{(q, v) \in E \mid \|v\| < \delta\}.$$

Proof. This immediately from (2.63) and (2.65) and the compactness of N. □

Definition 2.3.8. We define the *contact width* of (N, λ) in $(E, \widetilde{\Omega})$ to be the supremum of the δ's that satisfies the property stated in Lemma 2.17. Denote the contact width by $\delta_{(N,\lambda)}(E)$ and by $U_E := D^\delta(E)$ for $\delta = \delta_{(N,\lambda)}(E)$ the *(maximal) contact thickening* of the contact embedding $N \hookrightarrow E$.

2.3.3 The contact manifold (U_E, λ_E)

Now we look at the contact structure ξ_E and the Reeb vector field X_{λ_E} of U_E given by the contact one-form λ_E.

We first note that the two-form $d\lambda_E$ is a pre-symplectic form with one dimensional kernel such that

$$d\lambda_E|_{VTE} = \Omega^v|_{VTE}.$$

Denote by $\widetilde{X} := (d\pi_{E;H})^{-1}(X)$ the horizontal lifting of the vector field X on N, where

$$d\pi_{E;H} := d\pi_E|_H : HTE \to TN$$

is the bijection of the horizontal distribution and TN.

Recall X_θ is the Reeb vector field of (N, θ), then we have the following proposition state that the Reeb vector field of (E, λ_E) is nothing but the horizontal lifting of X_θ.

Proposition 2.18 (Reeb vector field)**.** The vector field $\widetilde{X}_\theta$ is the Reeb vector field of λ_E, i.e., $X_E = \widetilde{X}_\theta$. In particular, X_E is horizontal.

Proof. Recall X_E is characterized by

$$X_E \rfloor \lambda_E = 1, \quad X_E \rfloor d\lambda_E = 0. \tag{2.66}$$

It is enough to show that $\widetilde{X}_\theta$ satisfies (2.66).

For the first, we evaluate

$$\lambda_E(\widetilde{X}_\theta) = \theta(d\pi_E(\widetilde{X}_\theta)) + (\vec{R}\rfloor\widetilde{\Omega})(\widetilde{X}_\theta) = \theta(X_\theta) + 0 = 1,$$

where $(\vec{R}\rfloor\widetilde{\Omega})(\widetilde{X}_\theta) = \widetilde{\Omega}(\vec{R}, \widetilde{X}_\theta) = 0$ since $\widetilde{X}_\theta$ is horizontal.

We then compute

$$\begin{aligned}\widetilde{X}_\theta\rfloor d\lambda_E &= \widetilde{X}_\theta\rfloor(\pi_E^* d\theta + \widetilde{\Omega}) = \widetilde{X}_\theta\rfloor\pi_E^* d\theta + \widetilde{X}_\theta\rfloor\widetilde{\Omega} \\ &= d\theta(d\pi_E(\widetilde{X}_\theta), d\pi_E(\cdot)) + 0 = d\theta(X_\theta, d\pi_E(\cdot)) = 0.\end{aligned}$$

This finishes the proof. □

Next, we look at the contact structure of (U_E, λ_E). Recall the contact form λ_E gives the decomposition of TE by

$$TE = \mathbb{R}\cdot\{X_E\} \oplus \xi_E.$$

At the same time, we have the horizontal and vertical decomposition given by the symplectic connection

$$TE = HTE \oplus VTE,$$

and moreover, X_E is horizontal. Hence for any $\zeta \in \xi_E$, we can assume

$$\zeta = \zeta^h + \zeta^v$$

with respect to this decomposition, and further $\zeta^h = b\cdot\widetilde{X_\theta} + \widetilde{\eta}$, where $\eta \in \xi_\theta$. Then we calculate

$$\begin{aligned}0 = \lambda_E(\zeta) &= \lambda_E(b\cdot\widetilde{X_\theta} + \widetilde{\eta} + \zeta^v) \\ &= \pi_E^*\theta(b\cdot\widetilde{X_\theta} + \widetilde{\eta} + \zeta^v) + \frac{1}{2}\vec{R}\rfloor\Omega(b\cdot\widetilde{X_\theta} + \widetilde{\eta} + \zeta^v) \\ &= \theta(b\cdot X_\theta + \eta) + \frac{1}{2}\Omega^v(\vec{R}, \zeta^v) \\ &= b + \frac{1}{2}\Omega^v(\vec{R}, \zeta^v),\end{aligned}$$

and enlightened by this, we denote a subspace in TE by

$$W := \{-\frac{1}{2}\Omega^v(\vec{R}, v) \cdot X_E + v \mid v \in VTE\},$$

then we have the following characterization of the contact distribution.

Proposition 2.19 (Contact distribution)**.** The associated contact distribution of λ_E is given by the direct sum

$$\xi_E = (d\pi_{E;H})^{-1}(\xi_\theta) \oplus W. \tag{2.67}$$

Proof. By the calculation above, we already see that

$$\xi_E = (d\pi_{E;H})^{-1}(\xi_\theta) + W,$$

we just need to show the right hand side is a direct sum, which immediately follows from the fact that X_E is horizontal. In fact, if we can write

$$0 = \widetilde{\zeta} + b \cdot X_E + \eta,$$

where $\zeta \in \xi_\theta$, $\eta \in VTE$ and $b \in \mathbb{R}$, then after applying $d\pi_E$ to both sides,

$$0 = \zeta + b \cdot X_\theta.$$

Hence $\zeta = 0$, $b = 0$ and further it indicates $\eta = 0$, and we are done with the proof. □

We remark that on the zero section o_E we have $b = 0$ since $\vec{R}|_{o_E} \equiv 0$. Therefore the contact distribution on the zero section coincides with the direct sum

$$(d\pi_{E;H})^{-1}(\xi_\theta|_{o_q}) \oplus V_{o_q}E,$$

and so the vertical subbundle $V_{o_E}E$ is completely decoupled from $X_E|_{o_E} = di(X_\theta)$, where $i : N \hookrightarrow E$ is the embedding $q \mapsto o_q$ as the zero section.

We summarize the decompositions of TE by

$$\begin{aligned} TE &= (d\pi_{E;H})^{-1}(\mathbb{R}\cdot\{X_\theta\}) \oplus (d\pi_{E;H})^{-1}(\xi_\theta) \oplus W \\ \mathbb{R}\cdot\{X_E\} &= (d\pi_{E;H})^{-1}(\mathbb{R}\cdot\{X_\theta\}) \\ \xi_E &= (d\pi_{E;H})^{-1}(\xi_\theta) \oplus W, \end{aligned}$$

and note that W is vertical only at the zero section.

2.3.4 Canonical neighborhoods of the clean manifold of Reeb orbits

Now let N be the clean submanifold of Q that is foliated by the closed Reeb orbits of λ with constant period T. We regard $(N, \theta := \lambda|_N)$ as the prequantization bundle of the integral symplectic manifold (P, ω) consisting of Reeb orbits. In Section 2.3.2, we introduce a local model (U_E, λ_E) and study the contact data in Section 2.3.3.

On the other hand, we can identify a tubular neighborhood of N in Q with U_E with N identified with the zero section o_E, e.g., by considering a contact triad (Q, λ, J) and require that J preserves TN and using exponential map induced from the triad metric (that is the structure we are going to use for the study of contact instantons in Chapter 6 for Morse-Bott case). U_E then posses another contact one-form induced from (Q, λ) and we still use λ to denote it by abusing of notation.

In this section, we give the normal form theorem which relate the two structures (U_E, λ) and (U_E, λ_E).

We first give the following general submanifold version of Gray's theorem, by which we can get the Darboux theorem 2.1.2 and the Gray's stability theorem 2.1.3, and also the normal form theorem we use to study the asymptotic behavior of contact intantons in the Morse-Bott case in Chapter 6.

Theorem 2.3.9. (M, ξ) *is a contact manifold, and* λ_0 *and* λ_1 *are two contact one-forms on it. Let* N *be a closed manifold in* M *and*

$$\lambda_0(q) = \lambda_1(q), \quad \text{for any } q \in N.$$

Then there exists a diffeomorphism ϕ *from a neighborhood* $\mathcal{U}$ *of* N *to a neighborhood* $\mathcal{V}$ *such that*

$$\phi|_N = id|_N, \tag{2.68}$$

and a function $f > 0$ *such that*

$$\phi^* \lambda_1 = f \cdot \lambda_0,$$

and

$$f|_N \equiv 1, \quad df|_{TN} \equiv 0. \tag{2.69}$$

Proof. Since λ_0 and λ_1 coincide on N, there exists a small tubular neighborhood of N in M, denote by $\mathcal{U}$, such that the isotopy $\lambda_t = (1-t)\lambda_0 + t\lambda_1$, $t \in [0,1]$, are contact forms in $\mathcal{U}$. Moreover, we have

$$\lambda_t(q) \equiv \lambda_0(q)(= \lambda_1(q)), \quad \text{for any } q \in N, \quad t \in [0,1].$$

Next we apply the Moser's trick. We are looking for a family of diffeomorphisms onto its image $\phi_t : \mathcal{U}' \to \mathcal{U}$ for some smaller open subset $\mathcal{U}' \subset \overline{\mathcal{U}}' \subset \mathcal{U}$ such that

$$\phi_t|_N = id|_N, \quad d\phi_t|_{TM|_N} = id|_{TM|_N}$$

for all $t \in [0,1]$, together with a family of functions $f_t > 0$ defined on $\phi_t(\overline{\mathcal{U}}')$ such that

$$\begin{aligned} \phi_t^* \lambda_t &= f_t \cdot \lambda_0 \quad \text{on } \phi_t(\mathcal{U}') \\ \phi_t|_N &\equiv id|_N \end{aligned}$$

for $0 \le t \le 1$. We will further require $f_t \equiv 1$ on N and $df_t|_{TN} \equiv 0$.

Since N is a closed manifold, it is enough to look for the vector fields Y_t generated by ϕ_t via

$$\frac{d}{dt}\phi_t = Y_t \circ \phi_t, \quad \phi_0 = id, \tag{2.70}$$

satisfying

$$\begin{cases} \phi_t^* \left(\frac{d}{dt}\lambda_t + \mathcal{L}_{Y_t}\lambda_t \right) = \frac{f_t'}{f_t}\phi_t^* \lambda_t \\ Y_t|_N \equiv 0. \end{cases}$$

By Cartan's formula, the first equation gives rise to

$$d(Y_t \rfloor \lambda_t) + Y_t \rfloor d\lambda_t = \mathcal{L}_{Y_t}\lambda_t = (\frac{f_t'}{f_t} \circ \phi_t^{-1})\lambda_t - \alpha_t, \tag{2.71}$$

where

$$\alpha_t = \frac{d\lambda_t}{dt} = \lambda_1 - \lambda_0.$$

Now, we need to show that there exists Y_t such that $\frac{d}{dt}\lambda_t + \mathcal{L}_{Y_t}\lambda_t$ is proportional to λ_t. Actually, we can make our choice of Y_t unique if we restrict ourselves to those tangent to ξ_t by Lemma 2.3.1.

We require $Y_t \in \xi_t$ and then (2.71) becomes

$$\alpha_t = -Y_t \rfloor d\lambda_t + (\frac{f_t'}{f_t} \circ \phi_t^{-1})\lambda_t. \tag{2.72}$$

This in turn determines ϕ_t by integration. Since we have $\alpha_t|_N = \frac{d}{dt}\lambda_t|_N = (\lambda_1 - \lambda_0)|_N = 0$, hence $Y_t|_N = 0$. Therefore by compactness of $[0,1] \times N$, the domain of existence of the

ODE $\dot{x} = Y_t(x)$ includes an open neighborhood of $[0,1] \times N \subset \mathbb{R} \times M$ which we may assume is of the form $(-\epsilon, 1+\epsilon) \times \mathcal{V}$.

Now going back to (2.72), it $(\frac{f_t'}{f_t} \circ \phi_t^{-1})$ is uniquely determined. We evaluate $\lambda_1 - \lambda_0$ against the vector fields $X_t = (\phi_t)_* X_0$, and get

$$\frac{d}{dt} \log f_t = \frac{f_t'}{f_t} = (\lambda_1(X_t) - \lambda_0(X_t)) \circ \phi_t, \tag{2.73}$$

which determines f_t by integration with the initial condition $f_0 \equiv 1$. □

Corollary 2.3.10. *Under the assumption of Theorem 2.3.9 and assume for any $q \in N$, $X_1(q) = X_0(q) \in T_q N$ where X_i is the Reeb vector field of λ_i, $i = 0, 1$. Then we have $df|_{TM|_N} = 0$. As a consequence, we have $d\phi^* \lambda_1(q) = d\lambda_0(q)$ for any $q \in N$.*

Proof. In Theorem 2.3.9, we have derived

$$\phi^* \lambda_1 = f \cdot \lambda_0, \tag{2.74}$$

Take differential to both sides and we get

$$\phi^* d\lambda_1 = df \wedge \lambda_0 + f d\lambda_0.$$

Now take the interior product with X_0,

$$X_0 \rfloor (\phi^* d\lambda_1 - f d\lambda_0) = X_0 \rfloor (df \wedge \lambda_0).$$

The LHS

$$X_0 \rfloor (\phi^* d\lambda_1 - f d\lambda_0) = X_0 \rfloor \phi^* d\lambda_1,$$

Since $X_0 \in TN$, we have $d\phi(X_0) = X_0$ on N which is also the Reeb vector field of λ_1 by assumption, hence the LHS vanishes.

The RHS,

$$\begin{aligned} X_0 \rfloor (df \wedge \lambda_0) &= df(X_0)\lambda_0 - df \lambda_0(X_0) \\ &= df(X_0)\lambda_0 - df. \end{aligned}$$

The first term vanishes since $X_0 \in TN$.

Hence $df|_{TM|_N} = 0$. □

We remark that actually for the isotopy of contact one-forms λ_t we constructed in the proof of Theorem 2.3.9, we have $df_t|_{TM|_N} = 0$ with little modification of the proof.

Applying this theorem to λ and λ_E on E with N as the zero section o_E, and notice that $\lambda = \lambda_E$, $d\lambda = d\lambda_E$ on the zero section, we immediately get the following corollary which proves the normal form near the clean submanifold of Reeb orbits.

Corollary 2.3.11. *There exists a diffeomorphism ϕ from U_E to itself and a function $f > 0$ defined on U_E, such that*

$$\phi_* \lambda = f \cdot \lambda_E.$$

Moreover,

$$d\phi|_N = id|_{TM|_N}, \quad f|_{o_E} \equiv 1, \quad df|_{o_E} \equiv 0$$

and

$$\phi_* d\lambda|_{TE|_{o_E}} = (d\lambda_E)_{TE|_{o_E}}$$

From now on, we fix the target that we would like to study as $(U_E, o_E, \lambda = f\lambda_E)$, where (U_E, ξ) is the contact manifold with two contact one-forms λ and λ_E with the relation $\lambda = f\lambda_E$. o_E as the zero section is foliated by Reeb orbits of λ (and also of

λ_E). $f|_{o_E} \equiv 1$, $df|_{TE|_{o_E}} = 0$. Denote by X_E the Reeb vector field of λ_E. Recall formulae (2.58) and (2.59), we rewrite them as

$$X_\lambda = X_{f\lambda_E} = \frac{1}{f}(Y^{\lambda_E}_{\log f} + X_E) \tag{2.75}$$

$$\pi_\lambda(\cdot) = \pi_{f\lambda_E}(\cdot) = \pi_{\lambda_E}(\cdot) - \lambda_E(\cdot)Y^{\lambda_E}_{\log f}. \tag{2.76}$$

Now we give the explicit vertical and horizontal decomposition of $Y^{\lambda_E}_{\log f}$ by using the definition of λ_E. To simplify notation, we just use $Y := Y^{\lambda_E}_{\log f}$ if there is no danger of confusion.

Decompose $Y = Y^h + Y^v$, where we denote by Y^v the vertical part of Y, and by Y^h the horizontal part thereof. Denote $g := \log f$. Such function g gives a hamiltonian vector field $X_g \in \xi$ with respect to $d\lambda_E$ by

$$X_g \rfloor d\lambda_E = dg.$$

Since $d\lambda_E = \pi_E^* d\theta + \widetilde{\Omega}$, the vertical and horizontal decomposition of $X_g = X_g^v + X_g^h$ and

$$dg = d^v g + d^h g$$

where $d^v g = dg|_{VTE}$ and $d^h g = dg|_{HTE}$ have the relations

$$\begin{aligned} X_g^v \rfloor \Omega^v &= d^v g \\ X_g^h \rfloor \pi_E^* d\theta &= d^h g. \end{aligned}$$

Proposition 2.20.

$$\begin{aligned} Y^v &= -\frac{1}{2} X_E(g)\vec{R} + X_g^v \\ Y^h &= X_g^h - \frac{1}{2}\Omega^v(\vec{R}, X_g^v) X_E. \end{aligned}$$

Proof. Recalling

$$Y \rfloor d\lambda_E = -X_E(g)\lambda_E + dg, \quad g := \log f.$$

and

$$\begin{aligned}\lambda_E &= \pi_E^*\theta + \frac{1}{2}\vec{R}\rfloor\widetilde{\Omega} \\ d\lambda_E &= \pi_E^* d\theta + \widetilde{\Omega},\end{aligned}$$

we can immediately get

$$\begin{aligned} Y^v \rfloor \widetilde{\Omega}\big|_{VTE} &= -\frac{1}{2}dg(X_E)\vec{R}\rfloor\Omega^v + d^v g = -\frac{1}{2}dg(X_E)\vec{R}\rfloor\Omega^v + X_g^v\rfloor\Omega^v \\ Y^h\rfloor\pi_E^* d\theta\big|_{HTE} &= -dg(X_E)\pi_E^*\theta + d^h g = -dg(X_E)\pi_E^*\theta + X_g^h\rfloor\pi_E^* d\theta. \end{aligned}$$

By the nondegeneracy of Ω^v in VTE, we get from the first identity that

$$Y^v = -\frac{1}{2}dg(X_E)\vec{R} + X_g^v.$$

By the nondegeneracy of $d\theta$ on the contact distribution of the submanifold (o_E, ξ_θ), we get

$$\pi_\theta(d\pi_E(Y^h)) = d\pi_E(X_g^h).$$

Recall that $Y \in \xi_E$, by (2.67), we have

$$Y^h = X_g^h - \frac{1}{2}\Omega^v(\vec{R}, X_g^v)X_E.$$

□

2.3.5 Linearization of Reeb orbits on the normal form

In this section, we derive the linearization of Reeb orbits in the clean submanifold o_E in the normal form picture $(E, o_E, \lambda = f\lambda_E)$. This formula will be used in Chapter 6 for the study of the asymptotic behavior of contact instantons under Morse-Bott situation. Since we only look at the neighborhood along a Reeb orbit, we simplify the notation U_E to E in this section.

Denote by $z : S^1 \to Q$ a Reeb orbit with period T which lives in the zero section o_E. We are going to derive the linearization of the operator

$$\Upsilon_{(T,\lambda_E)} : z \mapsto \dot{z} - T\, X_{f\lambda_E}(z)$$

along a Reeb orbit. We look at the curve $w_\epsilon(t) := (z(t), \epsilon\, e(t)) \in E$, where $e(t) \in E_{z(t)}$, and calculate

$$d^\pi_z \Upsilon_{(T,\lambda_E),z}(e) = \pi \frac{D}{d\epsilon}\Big|_{\epsilon=0} (\dot{w} - T\, X_{f\lambda_E}(w))$$

by introducing an affine connection D on the contact manifold E from the symplectic connection ∇ of $E \to N$ and some connection on N (for example, we can impose an almost complex structure J_N on N and choose the triad connection ∇^{can} of the triad (N, θ, J_N)).

Assume $r(t)$ is a curve in E and $\zeta(t)$ is a vector field along $r(t)$. Define covariant derivative of ζ along r as

$$\frac{D}{dt}\zeta \;\; := \;\; \nabla_t \zeta^v + (\pi_E^* \nabla^{can})_{\dot{r}} \zeta^h.$$

Since the linearization along a Reeb orbit is independent of the choice of connections, we are going to derive the formula by using this connection.

Proposition 2.21. Assume J_E is a complex structure compatible with Ω^v in VTE. Then we can express $d_z \Upsilon_{(T,\lambda_E),z}(e)$ as

$$d_z \Upsilon_{(T,\lambda_E),z}(e) = \nabla_t e + T\, J_E Hess(g)(z)(e),$$

for z as a Reeb orbit living in o_E.

Proof.

$$\begin{aligned} d_z\Upsilon_{(T,\lambda_E),z}(e) &= \frac{D}{d\epsilon}\Big|_{\epsilon=0}(\dot{w} - T\,X_{f\lambda_E}(w)) \\ &= \frac{D}{d\epsilon}\Big|_{\epsilon=0}\left(\dot{w} - \frac{T}{f}(Y + X_E)\right) \\ &= \frac{d}{d\epsilon}\Big|_{\epsilon=0}\nabla_t(\epsilon\, e) - \frac{d}{d\epsilon}\Big|_{\epsilon=0}(\frac{T}{f}\,Y^v) \qquad (2.77) \\ &= \nabla_t e - T\frac{d}{d\epsilon}\Big|_{\epsilon=0}Y^v \end{aligned}$$

For the calculation of the last term in (2.77), we use $dg = 0$ on zero section and the expression of Y in Proposition 2.20.

We then compute $\frac{d}{d\epsilon}\big|_{\epsilon=0}Y^v$. By the expression of Y^v given in Proposition 2.20, we have

$$\begin{aligned} \frac{d}{d\epsilon}\Big|_{\epsilon=0}Y^v &= \frac{d}{d\epsilon}\Big|_{\epsilon=0}(-\frac{1}{2}X_E(g)\vec{R} + X^v g) \\ &= \frac{d}{d\epsilon}\Big|_{\epsilon=0}X^v g \\ &= (\bar{\nabla}_e X^v g)(z) \\ &= -J_E Hess(g)(z)(e). \qquad (2.78) \end{aligned}$$

where $\bar{\nabla}$ here denote the Hermitian connection in the linear space (VTE, Ω^v, J_E).

We are done with the proof. □

Corollary 2.3.12. *There exists some $\delta > 0$ such that*

$$\|\nabla_t e + T\,J_E Hess(g)(z)(e)\|^2_{L^2} \geq \delta^2\,\|e\|^2_{L_2},$$

for any Reeb orbit z on the zero section o_E.

Proof. This follows from the definition of Morse-Bott condition and Proposition 2.12. To get the uniform constant δ for any Reeb orbit z on the zero section o_E, we also use the compactness of the clean submanifold N. □

2.3.6 Adapted CR-almost complex structures

Let (P,ω), (N,θ) and $(E \to N, \lambda = f\,\lambda_E)$ be as in previous sections, we assign some almost complex structure in this section which satisfies some adapted conditions.

We start with the discussion of contact triad connection on the prequantization (N,θ). We equip an almost Kähler structure (P,ω,J_P) with (P,ω) and denote by $\mathcal{J}_\omega(P)$ the set of compatible almost complex structure thereon. Since the contact distribution $\xi_\theta := \ker\theta \subset TN$ of (N,θ) is horizontal, each such J_P naturally induces a CR-almost complex structure $\widetilde{J_P}$ on ξ_θ by

$$\widetilde{J_P}\widetilde{v} = \widetilde{J_P v},$$

where $v \in TP$. We extend $\widetilde{J_P}$ to TN by defining $\widetilde{J_P}X_\theta = 0$, then we get a contact triad $(N,\theta,\widetilde{J_P})$. The contact distribution ξ_θ is S^1-equivariant, which induces the invariant splitting

$$TN = \mathcal{L} \oplus \xi_\theta$$

where $\mathcal{L}$ is the trivial line bundle $\mathcal{L} = \mathbb{R}\{X_\lambda\}$.

In particular, it carries the canonical affine connection formed by the direct sum

$$\nabla^{\mathcal{L}} \oplus \pi^*\nabla^{can}$$

where ∇^{can} is the canonical connection (or Ehresman-Limbermann connection) on the almost Kähler manifold (P,ω,J_P) and $\nabla^{\mathcal{L}}$ is the trivial connection on the trivial line bundle $\mathcal{L}$.

Denote by $\widetilde{J}_P : \ker\theta \to \ker\theta$ the associated CR-almost complex structure on N. The following immediately follows from the definition of the contact triad connection, see Definition 2.1.10.

Proposition 2.22. The contact triad connection for the triad $(N, \theta, \widetilde{J}_P)$ is given by

$$\nabla^{\mathcal{L}} \oplus \pi^* \nabla^{can}$$

We now look at the contact manifold (Q, ξ) and λ is a special type of Morse-Bott contact one-form as given in Section 2.3.2. N is a clean submanifold foliated by Reeb orbits as before. In particular, (N, θ) is the prequantization of (P, ω).

Definition 2.3.13. Let (Q, N, λ) be given as above. Suppose J defines a contact triad (Q, λ, J). We say a CR-almost complex structure J is adapted to the prequantization N if $J_N := J|_{TN} = \widetilde{J_P}$, where (P, ω, J_P) is an almost Kähler manifold.

The following automatically holds from this definition.

Lemma 2.3.14. *Suppose J is adapted to N. Then*

$$\begin{aligned} J(TN) &\subset TN \\ J(TN^{d\lambda}) &\subset TN^{d\lambda}. \end{aligned}$$

In fact, by this lemma, we would like to generalize the definition adaptedness to the following one, which turns to be enough for the asymptotic study of Morse-Bott case in Chapter 6.

Definition 2.3.15. Say J is adapted to N if $J(TN) \subset TN$.

As a consequence, we have $J(TN^{d\lambda}) \subset TN^{d\lambda}$ too.

Remark 2.3.16. The nondegenerate case automatically satisfies Definition 2.3.15, but not Definition 2.3.13, while the latter additionally requires that $\mathcal{L}_{X_\lambda} J$ vanishes on Reeb orbits.

Now we briefly summarize how this adapted almost CR-structure enters into the normal form. First we identify the tubular neighborhood of N in Q with the symplectic normal bundle (E, Ω), by using the exponential map induced from the contact triad metric. Each fiber over $q \in N$, $(E_q, \Omega) := (E_q, d\lambda_q)$ is a symplectic space with a compatible almost complex structure $J_E := J|_{E_q}$.

Second, we construct λ_E by choosing a symplectic connection. We can of course take the Hermitian connection of the Hermitian bundle $(E \to N, \Omega, J_E)$ at this moment. Such connection gives a vertical and horizontal splitting of TE.

As a consequence, regard this splitting, there is a natural adapted almost complex structure J_0 defined as

$$J_0 = \widetilde{J_N} \oplus J_E.$$

where $\widetilde{J_N}$ is the horizontal lifting of J_N given by

$$\widetilde{J_N}\widetilde{\eta} = \widetilde{J_N \eta}, \qquad \text{for } \eta \in TN.$$

By following the definition, it is easy to check that J_0 is compatible to λ_E. We have add CR-structure to the normal form now and get the triad (E, λ_E, J_0).

Recall Corollary 2.3.11, there exists a diffeomorphism ϕ such that $\phi^*\lambda = f\,\lambda_E$, and in particular, such ϕ preserves the structure of J_N, i.e., $(\phi^*J)_N = J_N$. Moreover, we use ϕ^*J to constructed $(\phi^*J)_0$ as above, and then the last identity in Corollary 2.3.11 leads to the fact that $(E, \lambda_E, (\phi^*J)_0)$ is still a contact triad.

However, we remark that it seems there is no way to make $(\phi^*J)_E = J_E$, which causes a problem that we have to give up the Hermitian property of the symplectic connection we have chosen before for later use, but it won't cause big trouble in the study of exponential decay of contact instantons (see Chapter 6 for details).

With the discussion above, now it is convenient to omit the diffeomorphism ϕ, and focus ourselves on two contact triads $(U_E, \lambda = f\,\lambda_E, J)$ and (U_E, λ_E, J_0). This is the setting we are going to look at in Chapter 6.

Chapter 3

Contact instantons

In this chapter, we conduct geometric analysis to contact instantons by using the contact triad connection introduced in section 2.1. This chapter consists of two parts.

In the first part, we derive the energy density equality which will be used to get the coercive estimates and the asymptotic properties of contact instantons in later sections.

In the second part, we focus on the coercive estimates after bubbling, i.e., with the assumption of bounded gradient of w. Without assuming finite π-harmonic energy, the coercive estimates here are for any contact instanton from closed domains. We will study the case of punctured domain in the next chapter, since there we need to require contact instantons have finite π-harmonic energy.

3.1 Tensorial calculations for geometric energy density function

Let (Q, λ, J) be a contact triad which is the target manifold and is fixed. In the chapter, we don't impose any requirement to the contact form λ, i.e., we don't require it to be nondegenerate or of Morse-Bott type.

Definition 3.1.1 (Contact Cauchy-Riemann map)**.** Let (Q, λ, J) be a contact triad and let (Σ, j) be a Riemann surface. We call any map $w : \Sigma \to Q$ a *contact Cauchy-Riemann* *map* if it satisfies $\overline{\partial}^\pi_J w = 0$.

It turns out that to establish the geometric analysis necessary for the study of associated moduli space, one needs to augment the equation $\overline{\partial}^\pi_J w = 0$ by

$$d(w^*\lambda \circ j) = 0. \tag{3.1}$$

Definition 3.1.2 (Contact instanton)**.** Let Σ be as above. We call a pair (j, w) of j a complex structure on Σ and a map $w : \dot\Sigma \to Q$ a *contact instanton* if they satisfy

$$\overline{\partial}^\pi_J w = 0, \quad d(w^*\lambda \circ j) = 0. \tag{3.2}$$

We would like to point out that the system (3.2) (for a fixed j) forms an elliptic system, which is a natural elliptic twisting of the Cauchy-Riemann equation $\overline{\partial}^\pi_J w = 0$.

In hindsight, the more common twisting of $\overline{\partial}^\pi w = 0$ initiated by Hofer [H1] has been the twisting to the pseudoholomorphic curve system

$$\begin{cases} \pi_\lambda\left(\frac{\partial w}{\partial \tau}\right) + J(w)\pi_\lambda\left(\frac{\partial w}{\partial t}\right) = 0 \\ \\ w^*\lambda \circ j = da. \end{cases} \tag{3.3}$$

of the pair (a, w) through the symplectization, *when* $w^*\lambda\circ j$ *is assumed to be exact*, where $a : \Sigma \to \mathbb{R}$ is an auxiliary potential function satisfying $w^*\lambda \circ j = da$. In this regard, the twisting (3.2) may be more natural in some respect in that it does not introduce additional auxiliary variable a.

From this section, we will use the contact Hermitian connection ∇^π for the hermitian bundle ξ over Q and the triad connection ∇ on Q introduced in 2.1 to do the calculation. First, we combine the pull-back connection on $w^*\xi$, again denoted by ∇^π, and the Hermitian connection of the Riemann surface (Σ, j, h). We get a connection on $T^*\Sigma \otimes w^*\xi$ which is still denoted by ∇^π.

Fix a Kähler metric h on (Σ, j). The norm $|dw|$ of the map

$$dw : (T\Sigma, h) \to (TQ, g)$$

with respect to the metric g is defined by

$$|dw|_g^2 := \sum_{i=1}^{2} |dw(e_i)|_g^2,$$

where $\{e_1, e_2\}$ is an orthonormal frame of $T\Sigma$ with respect to h.

The following off-shell formulae are immediate conseuences of the compatibility of J to $d\lambda$ on ξ.

Proposition 3.1. Denote $g_J = d\lambda(\cdot, J\cdot)|_\xi$ and the associated norm by $|\cdot| = |\cdot|_J$. Fix a Hermitian metric h of (Σ, j), and consider a smooth map $u : \Sigma \to M$. Then we have

(1) $|d^\pi w|^2 = |\partial^\pi w|^2 + |\overline{\partial}^\pi w|^2$,

(2) $2\, w^* d\lambda = (-|\overline{\partial}^\pi w|^2 + |\partial^\pi w|^2)\, dA$ where dA is the area form of the metric h on Σ.

(3) $w^*\lambda \wedge w^*\lambda \circ j = -|w^*\lambda|^2\, dA$

(4) $|\nabla w^*\lambda|^2 = |dw^*\lambda|^2 + |\delta w^*\lambda|^2$.

In particular, if $\overline{\partial}^\pi w = 0$, then

$$|d^\pi w|^2 = |\partial^\pi w|^2, \quad w^* d\lambda = \frac{1}{2}|d^\pi w|^2\, dA \tag{3.4}$$

Proof. The proofs of (1), (2) are exactly the same as the case of pseudoholomorphic maps in symplectic manifolds with replacement of dw by $d^\pi w$ and the symplectic form by $d\lambda$ and so omitted. (See e.g., Proposition 7.19 [Oh1] for the statements and their proofs in the symplectic case corresponding the statements (1), (2) here.) Statement (3) follows from the identity $w^*\lambda \circ j = - * w^*\lambda$ and by definition of the Hodge star operator, and then (4) is nothing but the Gärding's equality. □

We denote by d^{∇^π} the skew-symmetrization of covariant derivative ∇^π given by

$$d^{\nabla^\pi}(\alpha)(\xi_1, \xi_2) = (\nabla^\pi_{\xi_1}\alpha)(\xi_2) - (\nabla^\pi_{\xi_2}\alpha)(\xi_1)$$

where $\xi_1,\ \xi_2 \in T\Sigma$.

It defines an operator

$$d^{\nabla^\pi} : \Omega^1(w^*\xi) \to \Omega^2(w^*\xi).$$

We denote by

$$\Delta^\pi = \delta^{\nabla^\pi} d^{\nabla^\pi} + d^{\nabla^\pi}\delta^{\nabla^\pi}$$

the Hodge Laplacian acting on one-forms, where δ^{∇^π} is the formal adjoint of d^{∇^π}. Then we have the following Weitzenböck formula

$$\Delta^\pi \beta = (\nabla^\pi)^* \nabla^\pi \beta + R dw + K^\pi(dw, dw)\beta \tag{3.5}$$

in general where R is the Gaussian curvature of the surface (Σ, j) and K^π is the curvature of the connection ∇^π on the vector bundle $\xi \to Q$.

On the flat cylinder, we have $R \equiv 0$, and so the equation further simplifies to

$$\Delta^\pi \beta = (\nabla^\pi)^* \nabla^\pi \beta + K^\pi(dw, dw)\beta.$$

In the remaining section, considering $\beta = \partial^\pi w := \pi \partial w$ as a one-form in $\Omega^1(w^*TQ)$, we will compute $\Delta^\pi(\partial^\pi w)$ for a contact Cauchy-Riemann map, i.e., a map satisfying $\overline{\partial}^\pi w = 0$.

We start with the following lemma. Recall we denote by $\Pi : TQ \to TQ$ the idempotent associated to the projection $\pi : TQ \to \xi$, i.e., the endomorphism satisfying

$$\Pi^2 = \Pi, \quad \operatorname{Im}\Pi = \xi, \quad \ker \Pi = \mathbb{R}\{X_\lambda\}.$$

Lemma 3.1.3. *Let $w : \Sigma \to Q$ be any smooth map. Denote $d^\pi w = \pi dw \in \Omega^1(w^*\xi)$. As a two-form with value in $w^*\xi$, $d^{\nabla^\pi}(d^\pi w)$ has the expression*

$$d^{\nabla^\pi}(d^\pi w) = T^\pi(\Pi dw, \Pi dw) + w^*\lambda \wedge \left(\frac{1}{2}(\mathcal{L}_{X_\lambda} J)\, J d^\pi w\right) \tag{3.6}$$

where T is the torsion tensor of ∇.

Proof. For given ξ_1, $\xi_2 \in \Gamma(T\Sigma)$, we evaluate

$$\begin{aligned}
& d^{\nabla^\pi}(d^\pi w)(\xi_1, \xi_2) = d^{\nabla^\pi}(\pi dw)(\xi_1, \xi_2) \\
= \;& (\nabla^\pi_{\xi_1}(\pi dw))(\xi_2) - (\nabla^\pi_{\xi_2}(\pi dw))(\xi_1) \\
= \;& \big(\nabla^\pi_{\xi_1}(\pi dw(\xi_2)) - \pi dw(\nabla_{\xi_1}\xi_2)\big) - \big(\nabla^\pi_{\xi_2}(\pi dw(\xi_1)) - \pi dw\,(\nabla_{\xi_2}\xi_1)\big) \\
= \;& \pi\Big(\,(\nabla_{\xi_1}(dw(\xi_2)) - \nabla_{\xi_1}(\lambda(dw(\xi_2))X_\lambda)) - (\nabla_{\xi_2}(dw(\xi_1)) - \nabla_{\xi_2}(\lambda(dw(\xi_1))X_\lambda)) \\
& -dw\,(\nabla_{\xi_1}\xi_2 - \nabla_{\xi_1}\xi_2)\,\Big) \\
= \;& \pi\,(\nabla_{\xi_1}(dw(\xi_2)) - \nabla_{\xi_2}(dw(\xi_1)) - [dw(\xi_1), dw(\xi_2)]) \\
& -\nabla_{\xi_1}(\lambda(dw(\xi_2))X_\lambda) + \nabla_{\xi_2}(\lambda(dw(\xi_1))X_\lambda)\big) \\
= \;& \pi(T(dw(\xi_1), dw(\xi_2)) - \lambda(dw(\xi_2))\nabla_{\xi_1}X_\lambda - \xi_1[\lambda(dw(\xi_2))]X_\lambda \\
& +\lambda(dw(\xi_1))\nabla_{\xi_2}X_\lambda + \xi_2[\lambda(dw(\xi_1))]X_\lambda\big) \\
= \;& \pi(T(dw(\xi_1), dw(\xi_2))) - \lambda(dw(\xi_2))\nabla_{\xi_1}X_\lambda + \lambda(dw(\xi_1))\nabla_{\xi_2}X_\lambda \\
= \;& T^\pi(\Pi dw(\xi_1), \Pi dw(\xi_2)) \\
& +\frac{1}{2}\lambda(dw(\xi_2))J(\mathcal{L}_{X_\lambda}J)\pi dw(\xi_1) - \frac{1}{2}\lambda(dw(\xi_1))J(\mathcal{L}_{X_\lambda}J)\pi dw(\xi_2) \\
= \;& T^\pi(\Pi dw(\xi_1), \Pi dw(\xi_2)) \\
& -\frac{1}{2}\lambda(dw(\xi_2))(\mathcal{L}_{X_\lambda}J)J\pi dw(\xi_1) + \frac{1}{2}\lambda(dw(\xi_1))(\mathcal{L}_{X_\lambda}J)J\pi dw(\xi_2)
\end{aligned}$$

Here for the last second equality, we use Proposition 2.8 and Axiom (3) for this connection. We rewrite the above result as

$$d^{\nabla^\pi}(d^\pi w) = T^\pi(\Pi dw, \Pi dw) + w^*\lambda \wedge \left(\frac{1}{2}(\mathcal{L}_{X_\lambda} J)\, J d^\pi w\right)$$

for any w. We have finished the proof. □

Now let w be a solution to

$$\overline{\partial}^\pi w = 0. \tag{3.7}$$

Then we have

$$\begin{aligned} d^\pi w &= \partial^\pi w \\ J\partial^\pi w &= \partial^\pi w \circ j. \end{aligned} \tag{3.8}$$

As an immediate corollary of the previous lemma applied to the solution w, we derive the following theorem of the fundamental equation. This is the contact analogue to [Oh1, Proposition 7.27].

Theorem 3.1.4 (Fundamental equation)**.** *Let w be a contact Cauchy-Riemann map, i.e., a solution of* (3.7)*. Then*

$$\begin{aligned} d^{\nabla^\pi}(d^\pi w) &= d^{\nabla^\pi}(\partial^\pi w) \\ &= -w^*\lambda \circ j \wedge \left(\frac{1}{2}(\mathcal{L}_{X_\lambda} J)\, \partial^\pi w\right). \end{aligned} \tag{3.9}$$

Proof. The first equality follows since $d^\pi w = \partial^\pi w$ for the solution w. Also, it follows

$$T^\pi(\Pi dw, \Pi dw) = T^\pi(\partial^\pi w, \partial^\pi w) = 0$$

since the torsion $T^\pi|_\xi$ is of $(0,2)$-type, in particular, has vanishing $(1,1)$-component. Further we write (3.6) as

$$\begin{aligned} d^{\nabla^\pi}(d^\pi w) &= w^*\lambda \wedge \left(\frac{1}{2}(\mathcal{L}_{X_\lambda} J)\, J\partial^\pi w\right) \\ &= w^*\lambda \wedge \left(\frac{1}{2}(\mathcal{L}_{X_\lambda} J)\, \partial^\pi w\right) \circ j \\ &= -w^*\lambda \circ j \wedge \left(\frac{1}{2}(\mathcal{L}_{X_\lambda} J)\, \partial^\pi w\right), \end{aligned}$$

where we use the identity $J\partial^\pi w = \partial^\pi w \circ j$. This finishes the proof. □

Now we compute

$$\Delta^\pi(\partial^\pi w) = \delta^{\nabla^\pi} d^{\nabla^\pi}(\partial^\pi w) + d^{\nabla^\pi}\delta^{\nabla^\pi}(\partial^\pi w).$$

Recall the Hodge dual formula for dimension 2 spaces

$$\delta^{\nabla^\pi} = - * d^{\nabla^\pi} * .$$

We have the following lemma which can greatly simplify our calculation of deriving the energy density formulae. This lemma is an interpretation of the metric property of the connection for forms. We postpone its proof till the appendix, Section B.

Lemma 3.1.5. *Assume α is a zero-form in $\Omega^0(w^*\xi)$ and β is a one-form in $\Omega^1(w^*\xi)$. $\langle\cdot,\cdot\rangle$ is the inner production on $w^*\xi$ introduced from the metric of Q. Then we have*

$$\langle d^{\nabla^\pi}\alpha, \beta\rangle - \langle \alpha, \delta^{\nabla^\pi}\beta\rangle = -\delta\langle\alpha,\beta\rangle.$$

The following lemma is another useful formula whose proof is a straightforward calculation and will be given in the appendix.

Lemma 3.1.6. *For any connection ∇ and vector-valued one-form α,*

$$|\nabla\alpha|^2 = |d^\nabla\alpha|^2 + |\delta^\nabla\alpha|^2 - *(\nabla\alpha \wedge \nabla_{j(\cdot)}\alpha).$$

In the cylindrical coordinates, it has the expression

$$|\nabla\alpha|^2 \quad = \quad |d^\nabla\alpha|^2 + |\delta^\nabla\alpha|^2 + 2\langle\nabla_\tau\eta, \nabla_t\zeta\rangle - 2\langle\nabla_t\eta, \nabla_\tau\zeta\rangle.$$

We would like to point out that the last term $*(\nabla\alpha \wedge \nabla_{j(\cdot)}\alpha)$ vanishes for real valued (or any line bundle valued) forms but does not for general higher rank vector bundle valued forms.

Before we use these formulae to compute $\langle\Delta^\pi\partial^\pi w, \partial^\pi w\rangle$, we first state the following lemma which is due to (3.8) and J-compatibility of the metric. We also remark that this lemma holds only for the connection ∇^π which is J-linear.

Lemma 3.1.7. *For any smooth map w, we have*

$$\langle d^{\nabla^\pi}\delta^{\nabla^\pi}\partial^\pi w, \partial^\pi w\rangle = \langle\delta^{\nabla^\pi}d^{\nabla^\pi}\partial^\pi w, \partial^\pi w\rangle.$$

As a consequence,

$$\langle\Delta^\pi\partial^\pi w, \partial^\pi w\rangle = 2\langle\delta^{\nabla^\pi}d^{\nabla^\pi}\partial^\pi w, \partial^\pi w\rangle. \tag{3.10}$$

Proof.

$$
\begin{aligned}
\langle \delta^{\nabla^\pi} d^{\nabla^\pi} \partial^\pi w, \partial^\pi w\rangle &= -\langle * d^{\nabla^\pi} * d^{\nabla^\pi} \partial^\pi w, \partial^\pi w\rangle \\
&= -\langle d^{\nabla^\pi} * d^{\nabla^\pi} \partial^\pi w, * \partial^\pi w\rangle \\
&= -\langle d^{\nabla^\pi} * d^{\nabla^\pi} \partial^\pi w, -\partial^\pi w \circ j\rangle \qquad (3.11) \\
&= \langle d^{\nabla^\pi} * d^{\nabla^\pi} \partial^\pi w, J\partial^\pi w\rangle \\
&= -\langle J d^{\nabla^\pi} * d^{\nabla^\pi} \partial^\pi w, \partial^\pi w\rangle \\
&= -\langle d^{\nabla^\pi} * d^{\nabla^\pi} J\partial^\pi w, \partial^\pi w\rangle \qquad (3.12) \\
&= -\langle d^{\nabla^\pi} * d^{\nabla^\pi} \partial^\pi w \circ j, \partial^\pi w\rangle \qquad (3.13) \\
&= \langle d^{\nabla^\pi} * d^{\nabla^\pi} * \partial^\pi w, \partial^\pi w\rangle \qquad (3.14) \\
&= \langle d^{\nabla^\pi} \delta^{\nabla^\pi} \partial^\pi w, \partial^\pi w\rangle.
\end{aligned}
$$

Here for (3.11) and (3.14), we use $*\alpha = -\alpha \circ j$ for any one-form α. For (3.12), we use the connection is J-linear. □

Now we are ready to state the following lemma.

Lemma 3.1.8.

$$\langle \Delta^\pi \partial^\pi w, \partial^\pi w\rangle = -2\delta \langle * d^{\nabla^\pi} \partial^\pi w, * \partial^\pi w\rangle + 2|\delta^{\nabla^\pi} \partial^\pi w|^2.$$

Proof. Using (3.10) and Lemma 3.1.5 we compute

$$\begin{aligned}
\langle \Delta^\pi \partial^\pi w, \partial^\pi w\rangle &= 2\langle \delta^{\nabla^\pi} d^{\nabla^\pi} \partial^\pi w, \partial^\pi w\rangle \\
&= 2\langle - * d^{\nabla^\pi} * d^{\nabla^\pi} \partial^\pi w, \partial^\pi w\rangle \\
&= 2\langle *(- * d^{\nabla^\pi} * d^{\nabla^\pi} \partial^\pi w), *\partial^\pi w\rangle \\
&= 2\langle d^{\nabla^\pi} * d^{\nabla^\pi} \partial^\pi w, *\partial^\pi w\rangle \\
&= -2\delta\langle *d^{\nabla^\pi} \partial^\pi w, *\partial^\pi w\rangle + 2\langle *d^{\nabla^\pi} \partial^\pi w, \delta^{\nabla^\pi} * \partial^\pi w\rangle \\
&= -2\delta\langle *d^{\nabla^\pi} \partial^\pi w, *\partial^\pi w\rangle - 2\langle *d^{\nabla^\pi} \partial^\pi w, (*d^{\nabla^\pi} *) * \partial^\pi w\rangle \\
&= -2\delta\langle *d^{\nabla^\pi} \partial^\pi w, *\partial^\pi w\rangle + 2\langle *d^{\nabla^\pi} \partial^\pi w, *d^{\nabla^\pi} \partial^\pi w\rangle \\
&= -2\delta\langle *d^{\nabla^\pi} \partial^\pi w, *\partial^\pi w\rangle + 2|d^{\nabla^\pi} \partial^\pi w|^2 \\
&= -2\delta\langle *d^{\nabla^\pi} \partial^\pi w, *\partial^\pi w\rangle + 2|\delta^{\nabla^\pi} \partial^\pi w|^2.
\end{aligned}$$

For the last equality we use

$$|d^{\nabla^\pi} \partial^\pi w| = |\delta^{\nabla^\pi} \partial^\pi w| \tag{3.15}$$

which comes from the fact that

$$\begin{aligned}
\delta^{\nabla^\pi} \partial^\pi w &= - * d^{\nabla^\pi} * \partial^\pi w = *d^{\nabla^\pi} \partial^\pi w \circ j \\
&= *d^{\nabla^\pi} J \partial^\pi w = J * d^{\nabla^\pi} \partial^\pi w.
\end{aligned}$$

□

Now we consider the energy density function for a contact Cauchy-Riemann map w defined as

$$e^\pi := |d^\pi w|^2 = \left(|\pi dw(e_1)|^2 + |\pi dw(e_2)|^2\right)$$

for a local orthonormal frame $\{e_1, e_2\}$ of $T\Sigma$. This becomes $|\partial^\pi w|^2$ for a contact Cauchy-Riemann map w since $d^\pi w = \partial^\pi w + \overline{\partial}^\pi w$.

We now compute the Hodge Laplacian of e^π,

$$\begin{aligned}\frac{1}{2}\Delta e^\pi &= \frac{1}{2}\Delta|\partial^\pi w|^2\\ &= -|\nabla^\pi(\partial^\pi w)|^2 - \langle \mathbf{tr}(\nabla^\pi)^2\partial^\pi w, \partial^\pi w\rangle\\ &= -|\nabla^\pi(\partial^\pi w)|^2 + \langle(\nabla^\pi)^*\nabla^\pi\partial^\pi w, \partial^\pi w\rangle.\end{aligned}$$

Using the Weitzenböck formula (3.5), we have derived the following general formula

$$\begin{aligned}\frac{1}{2}\Delta e^\pi &= \frac{1}{2}\Delta|\partial^\pi w|^2 = -|\nabla^\pi(\partial^\pi w)|^2 + \langle\nabla^\pi)^*\nabla^\pi(\partial^\pi w), \partial^\pi w\rangle\\ &= -|\nabla^\pi(\partial^\pi w)|^2 + \langle\Delta^\pi\partial^\pi w, \partial^\pi w\rangle - \langle K^\pi(dw,dw)\partial^\pi w, \partial^\pi w\rangle - R\langle dw, \partial^\pi w\rangle\\ &= -|\nabla^\pi(\partial^\pi w)|^2 + \langle\Delta^\pi\partial^\pi w, \partial^\pi w\rangle - \langle K^\pi(dw,dw)\partial^\pi w, \partial^\pi w\rangle - R|\partial^\pi w|^2\end{aligned}$$

where the last equality holds since $\partial^\pi w$ and $\overline{\partial}^\pi w$ are orthogonal to each other. Finally, applying Lemma 3.1.8, we have derived the following important formula for the Hodge Laplacian Δe^π

Theorem 3.1.9. *Let w be a contact Cauchy-Riemann map. Then*

$$\begin{aligned}\frac{1}{2}\Delta e^\pi &= \frac{1}{2}\Delta|\partial^\pi w|^2 = -|\nabla^\pi(\partial^\pi w)|^2 + \langle(\nabla^\pi)^*\nabla^\pi(\partial^\pi w), \partial^\pi w\rangle\\ &= -|\nabla^\pi(\partial^\pi w)|^2 + \langle\Delta^\pi\partial^\pi w, \partial^\pi w\rangle - \langle K^\pi(dw,dw)\partial^\pi w, \partial^\pi w\rangle - R\langle dw, \partial^\pi w\rangle\\ &= -|\nabla^\pi(\partial^\pi w)|^2 + 2|\delta^{\nabla^\pi}\partial^\pi w|^2 - 2\delta\langle *d^{\nabla^\pi}\partial^\pi w, *\partial^\pi w\rangle\\ &\quad -\langle K^\pi(dw,dw)\partial^\pi w, \partial^\pi w\rangle - R|\partial^\pi w|^2. \end{aligned}\tag{3.16}$$

3.2 The coercive estimates: closed domain case and local case

In this section, we first derive the $W^{2,2}$-coercive estimates for the equation (3.2) on the closed Riemann surface Σ, following the spirit of [Oh1, Chapter 7], which is the contact analogue to [Oh1, Proposition 7.33]. Then by further taking higher derivatives, we derive general $W^{k,2}$ estimates.

3.2.1 $W^{2,2}$ estimates

Using the formula of Δe^π, we derive the following $W^{2,2}$ a priori estimates of the map w satisfying (3.2).

Theorem 3.2.1. *Assume Σ is a closed Riemann surface with finite number of marked points. Let $w : \Sigma \to Q$ be any smooth solution to* (3.2)*. Then*

$$\begin{aligned} \int_\Sigma |\nabla dw|^2 &\leq \frac{1}{2}(C_1^2+1)\cdot \int_\Sigma |\partial^\pi w|^4 + \frac{3}{2}C_1^2 \int_\Sigma |w^*\lambda|^2 |\partial^\pi w|^2 \\ &\quad -2\min R \cdot \int_{\operatorname{supp} R} |\partial^\pi w|^2 \qquad (3.17) \\ &\quad +2\max|K^\pi| \int_\Sigma \left(|w^*\lambda|^2|\partial^\pi w|^2 + |\partial^\pi w|^4\right) \qquad (3.18) \end{aligned}$$

where C_1 is the constant given by

$$C_1 = \|\mathcal{L}_{X_\lambda} J\|_{C^0}.$$

Proof. We start with (3.16)

$$\begin{aligned} \frac{1}{2}\Delta e^\pi &= -|\nabla^\pi(\partial^\pi w)|^2 - 2\delta\langle *d^{\nabla^\pi}\partial^\pi w, *\partial^\pi w\rangle + 2|\delta^{\nabla^\pi}\partial^\pi w|^2 \\ &\quad -\langle K^\pi(dw,dw)\partial^\pi w, \partial^\pi w\rangle - R|\partial^\pi w|^2. \end{aligned}$$

computed before in Theorem 3.1.9. We re-write this into

$$\begin{aligned} |\nabla^\pi(\partial^\pi w)|^2 &= -\frac{1}{2}\Delta e^\pi - 2\delta\langle *d^{\nabla^\pi}\partial^\pi w, *\partial^\pi w\rangle + 2|\delta^{\nabla^\pi}\partial^\pi w|^2 \\ &\quad -\langle K^\pi(dw,dw)\partial^\pi w, \partial^\pi w\rangle - R|\partial^\pi w|^2. \qquad (3.19) \end{aligned}$$

On the other hand, using Gärding's identity (3.4), we obtain

$$|\nabla w^*\lambda|^2 = |dw^*\lambda|^2 + |\delta w^*\lambda|^2 = |dw^*\lambda|^2 = \frac{1}{4}|\partial^\pi w|^4 \tag{3.20}$$

for which we use $d(w^*\lambda \circ j) = 0$ for the second equality and Proposition 3.1 (2) for the last.

By adding up the two, we obtain

$$\begin{aligned} & |\nabla^\pi(\partial^\pi w)|^2 + |\nabla(w^*\lambda)|^2 \\ = & -\frac{1}{2}\Delta e^\pi - 2\delta\langle * d^{\nabla^\pi}\partial^\pi w, *\partial^\pi w\rangle + 2|\delta^{\nabla^\pi}\partial^\pi w|^2 \\ & -\langle K^\pi(dw, dw)\partial^\pi w, \partial^\pi w\rangle - R|\partial^\pi w|^2 + \frac{1}{4}|\partial^\pi w|^4. \end{aligned} \tag{3.21}$$

Substituting the formula in Theorem 3.1.4, we derive

$$d^{\nabla^\pi}\partial^\pi w = -(w^*\lambda \circ j) \wedge \left(\frac{1}{2}(\mathcal{L}_{X_\lambda} J)\partial^\pi w\right).$$

Finally using (3.15), we obtain

$$\begin{aligned} |\delta^{\nabla^\pi}\partial^\pi w|^2 &= |d^{\nabla^\pi}\partial^\pi w|^2 \\ &= \left| w^*\lambda \circ j \wedge \left(\frac{1}{2}(\mathcal{L}_{X_\lambda} J)\partial^\pi w\right)\right|^2 \leq \frac{C_1^2}{4}|w^*\lambda|^2|\partial^\pi w|^2 \end{aligned} \tag{3.22}$$

where

$$C_1 = \|\mathcal{L}_{X_\lambda} J\|_{C^0}.$$

We then note that

$$
\begin{aligned}
|\nabla dw|^2 &= |\nabla(\partial^\pi w) + \nabla(w^*\lambda\, X_\lambda)|^2 \\
&\leq 2|\nabla(\partial^\pi w)|^2 + 2|\nabla(w^*\lambda\, X_\lambda)|^2 \\
&= 2|\nabla(\partial^\pi w)|^2 + 2|\nabla(w^*\lambda)X_\lambda|^2 + 2|w^*\lambda|^2|\nabla_{d^\pi w}X_\lambda|^2 \\
&\leq 2|\nabla(\partial^\pi w)|^2 + 2|\nabla(w^*\lambda)|^2 + 2|w^*\lambda|^2|\nabla X_\lambda|^2|d^\pi w|^2 \\
&\leq 2|\nabla(\partial^\pi w)|^2 + 2|\nabla(w^*\lambda)|^2 + \frac{1}{2}C_1^2|w^*\lambda|^2|d^\pi w|^2 \\
&= 2(|\nabla(\partial^\pi w)|^2 + |\nabla(w^*\lambda)|^2) + \frac{1}{2}C_1^2|w^*\lambda|^2|\partial^\pi w|^2 \qquad (3.23)
\end{aligned}
$$

where we use $dw = d^\pi w + w^*\lambda\, X_\lambda$, $d^\pi w = \partial^\pi w + \overline{\partial}^\pi w$ and $\overline{\partial}^\pi w = 0$ for the first equality, $\nabla_{X_\lambda} X_\lambda = 0$ and $\langle X_\lambda, \nabla X_\lambda\rangle = 0$ for the second equality and

$$\nabla_{d^\pi w} X_\lambda = \frac{1}{2}(\mathcal{L}_{X_\lambda} J) J d^\pi w \qquad (3.24)$$

for the third inequality. And we have the decomposition

$$
\begin{aligned}
|\nabla(\partial^\pi w)|^2 &= |\nabla^\pi(\partial^\pi w) + \langle X_\lambda, \nabla(\partial^\pi w)\rangle X_\lambda|^2 \\
&= |\nabla^\pi(\partial^\pi w)|^2 + |\langle X_\lambda, \nabla(\partial^\pi w)\rangle|^2 \\
&= |\nabla^\pi(\partial^\pi w)|^2 + |\langle \nabla_{d^\pi w} X_\lambda, \partial^\pi w\rangle|^2 \\
&\leq |\nabla^\pi(\partial^\pi w)|^2 + \frac{1}{4}C_1^2|\partial^\pi w|^4 \qquad (3.25)
\end{aligned}
$$

where we use $\langle \nabla^\pi(\partial^\pi w), X_\lambda\rangle = 0$ for the second equality. Substituting this into (3.23), we obtain

$$
\begin{aligned}
|\nabla dw|^2 &\leq 2|\nabla^\pi(\partial^\pi w)|^2 + \frac{1}{2}C_1^2|\partial^\pi w|^4 + 2|\nabla(w^*\lambda)|^2 + \frac{1}{2}C_1^2|w^*\lambda|^2|\partial^\pi w|^2 \\
&= 2\left(|\nabla^\pi(\partial^\pi w)|^2 + |\nabla(w^*\lambda)|^2\right) + \frac{1}{2}C_1^2|\partial^\pi w|^4 + \frac{1}{2}C_1^2|w^*\lambda|^2|\partial^\pi w|^2.
\end{aligned}
$$

Now substituting (3.21) and (3.22) into this, we derive

$$\begin{aligned} |\nabla dw|^2 &\leq -\Delta e^\pi - 4\delta\langle *d^{\nabla^\pi}\partial^\pi w, *\partial^\pi w\rangle \\ &\quad -2\langle K^\pi(dw,dw)\partial^\pi w, \partial^\pi w\rangle - 2R|\partial^\pi w|^2 \\ &\quad +\frac{1}{2}(C_1^2+1)|\partial^\pi w|^4 + \frac{3}{2}C_1^2|w^*\lambda|^2|\partial^\pi w|^2. \end{aligned} \tag{3.26}$$

Integrating (3.26) over Σ, we have finished the proof of Theorem 3.2.1. □

An immediate corollary of this theorem, when combined with the standard Hölder's inequality and interpolation inequality between L^p-norms, is the following $W^{2,2}$-coercive estimates

Corollary 3.2.2. *Let Σ and w be as above. Suppose w satisfies* (3.2) *on Σ. Then there exist uniform constants C_3, C_4 depending only on $\|K^\pi\|_{C^0}$, $\|R\|_{C^0}$ and $\|\mathcal{L}_{X_\lambda}J\|_{C^0}$ but independent of w such that*

$$\|dw\|^2_{W^{1,2}} \leq C_3\|dw\|^4_{L^4} + C_4\|dw\|^2_{L^2}.$$

The last statement follows from the standard bootstrapping argument by differentiating the equation (3.2).

We also need to have the following local version of the main estimates which is an important ingredient for the local regularity and the bubbling-off analysis.

Theorem 3.2.3. *Let $D = D^2(1)$ be the unit disc. There exists C_5, $C_6 > 0$ depending only on $D' \subset \overline{D}' \subset D$ and on $\|K^\pi\|_{C^0}$, $\|\mathcal{L}_{X_\lambda}J\|_{C^0}$ and $\|R\|_{C^0;D}$ but independent of w such that*

$$\|dw\|^2_{1,2,D'} \leq C_5\|dw\|^4_{4,D} + C_6\|dw\|^2_{2,D}$$

for any smooth map $w : D \to Q$ satisfying

$$\overline{\partial}^\pi w = 0, \quad d(w^*\lambda \circ j) = 0.$$

Proof. The proof is a standard practice in geometric analysis by multiplying a cut-off function ρ to dw such that $\rho \equiv 1$ on D' while $\rho \equiv 0$ outside $D \subset \Sigma$. We refer to [SU] for details in the context of harmonic maps and omit the details. $\square$

3.2.2 $W^{k,2}$ estimates for $k \geq 3$

Starting from the above $W^{2,2}$-estimate, we proceed the higher $W^{k,2}$-estimate inductively. For this purpose, consider the decomposition

$$dw = d^\pi w + w^*\lambda\, X_\lambda$$

and estimate $|\nabla^{k+1} d^\pi w|$ and $|\nabla^k(w^*\lambda X_\lambda)|$ inductively by alternatively bootstrapping staring from $k = 0$ as for the case of $|\nabla dw|$ in the previous subsection.

We start with estimating

$$\frac{1}{2}\Delta|(\nabla^\pi)^k d^\pi w|^2 = -|\nabla^\pi((\nabla^\pi)^k d^\pi w)|^2 + \langle (\nabla^\pi)^*\nabla^\pi((\nabla^\pi)^k d^\pi w), \nabla^k(d^\pi w)\rangle$$

similarly as for $\frac{1}{2}\Delta e^\pi = \frac{1}{2}\Delta|d^\pi w|^2$. Rewriting this and then combining the Weitzenböck formula applied to $(\nabla^\pi)^k d^\pi w$, we obtain

$$\begin{aligned} |\nabla^\pi((\nabla^\pi)^k d^\pi w)|^2 &= -\frac{1}{2}\Delta|(\nabla^\pi)^k d^\pi w|^2 + \langle \Delta^\pi((\nabla^\pi)^k d^\pi w), (\nabla^\pi)^k d^\pi w\rangle \\ &\quad -\langle K^\pi(dw, dw)(\nabla^\pi)^k d^\pi w, (\nabla^\pi)^k d^\pi w\rangle \\ &\quad -\langle R\,(\nabla^\pi)^k d^\pi w, (\nabla^\pi)^k d^\pi w\rangle. \end{aligned} \tag{3.27}$$

Therefore we have derived

$$\begin{aligned}\int_\Sigma |\nabla^\pi((\nabla^\pi)^k d^\pi w)|^2 &= \int_\Sigma \langle \Delta^\pi((\nabla^\pi)^k d^\pi w), (\nabla^\pi)^k d^\pi w\rangle \\ &\quad - \int_\Sigma \langle K^\pi(dw, dw)(\nabla^\pi)^k d^\pi w, (\nabla^\pi)^k d^\pi w\rangle \\ &\quad - \int_\Sigma \langle R\,(\nabla^\pi)^k d^\pi w, (\nabla^\pi)^k d^\pi w\rangle.\end{aligned}$$

Obviously the last two terms are bounded by the norm $\|dw\|^2_{k,2}$. It remains to examine the integral

$$\int_\Sigma \langle \Delta^\pi((\nabla^\pi)^k d^\pi w), (\nabla^\pi)^k d^\pi w\rangle.$$

Recalling $\Delta^\pi = d^{\nabla^\pi}\delta^{\nabla^\pi} + \delta^{\nabla^\pi} d^{\nabla^\pi}$, we rewrite

$$\int_\Sigma \langle \Delta^\pi((\nabla^\pi)^k d^\pi w), (\nabla^\pi)^k d^\pi w\rangle = \int_\Sigma |d^{\nabla^\pi}((\nabla^\pi)^k d^\pi w))|^2 + \int_\Sigma |\delta^{\nabla^\pi}((\nabla^\pi)^k d^\pi w))|^2.$$

On the other hand, we compute

$$d^{\nabla^\pi}((\nabla^\pi)^k d^\pi w)). \tag{3.28}$$

For this purpose, we quote the following lemma

Lemma 3.2.4. *For any ξ-valued one-form α,*

$$d^{\nabla^\pi}(\nabla^\pi \alpha) = \nabla^\pi(d^{\nabla^\pi}\alpha) + w^* K^\pi\, \alpha \tag{3.29}$$

or equivalently

$$\left[d^{\nabla^\pi}, \nabla^\pi\right] \alpha = w^* K^\pi\, \alpha \tag{3.30}$$

for the commutator $[\cdot,\cdot]$.

Applying this to $d^{\nabla^\pi}((\nabla^\pi)^k d^\pi w))$ iteratively, we derive

$$|d^{\nabla^\pi}((\nabla^\pi)^k d^\pi w))|^2 \leq |(\nabla^\pi)^k (d^{\nabla^\pi} d^\pi w)|^2 + G_k(|d^\pi w|, |w^*\lambda|) \tag{3.31}$$

where G_k is a polynomial function of $|d^\pi w|$, $|w^*\lambda|$ and their covariant derivatives up to order k. And applying the fundamental equation (3.9) to $d^{\nabla^\pi} d^\pi w$, the term itself has the bound

$$|(\nabla^\pi)^k (d^{\nabla^\pi} d^\pi w)|^2 \leq H_k(|d^\pi w|, |w^*\lambda|)$$

for a polynomial function H_k of the form G_k.

Similarly, we obtain

$$|\delta^{\nabla^\pi}((\nabla^\pi)^k d^\pi w))|^2 \leq I_k(|d^\pi w|, |w^*\lambda|)$$

for similar polynomial function I_k since $|\delta^{\nabla^\pi}((\nabla^\pi)^k d^\pi w))|^2 = |d^{\nabla^\pi}((\nabla^\pi)^k d^\pi w))|^2$.

We now summarize the above computations into

Proposition 3.2. Let $w : \Sigma \to Q$ satisfy $\overline{\partial}^\pi w = 0, \quad d(w^*\lambda \circ j) = 0$, on Σ. Then

$$\int_{\dot\Sigma} |(\nabla^\pi)^{k+1}(d^\pi w)|^2 \leq \int_{\dot\Sigma} J_k(|\partial^\pi w|, |w^*\lambda|) \tag{3.32}$$

for a polynomial function J_k of $|d^\pi w|$, $|w^*\lambda|$ its covariant derivatives up to $0, \ldots, k$ of degree at most $2k+4$.

Next we compute

$$\nabla^{k+1}(w^*\lambda X_\lambda) = \nabla^{k+1}(w^*\lambda)\, X_\lambda + \sum_{l=1}^{k+1} \begin{pmatrix} k+1 \\ l \end{pmatrix} \nabla^l(w^*\lambda)\, \nabla^{k+1-l} X_\lambda$$

Here we recall the formula ∇X_λ in (3.24). Therefore it follows that

$$\left| \sum_{l=1}^{k+1} \nabla^l(w^*\lambda) \nabla^{k+1-l} X_\lambda \right| \leq L_k(|d^\pi w|, |w^*\lambda|)$$

for a polynomial function L_k similar to J_k. We write

$$\begin{aligned} |\nabla^{k+1}(w^*\lambda)|^2 &= |\nabla(\nabla^k(w^*\lambda))|^2 \\ &= -\frac{1}{2}\Delta|\nabla^k(w^*\lambda))|^2 + \langle \nabla^*\nabla((\nabla^k(w^*\lambda)), \nabla^k(w^*\lambda)\rangle. \end{aligned}$$

Applying the Weitzenböck formula, we obtain

$$\nabla^*\nabla((\nabla^k(w^*\lambda)) = -\Delta(\nabla^k(w^*\lambda)) - R\,\nabla^k(w^*\lambda).$$

Therefore we have obtained

$$\begin{aligned} & \int_\Sigma \langle \nabla^*\nabla((\nabla^k(w^*\lambda)), \nabla^k(w^*\lambda)\rangle \\ = & \ -\int_\Sigma |d(\nabla^k(w^*\lambda))|^2 + |\delta(\nabla^k(w^*\lambda))|^2 - \int_\Sigma R\,|\nabla^k(w^*\lambda)|^2. \end{aligned}$$

By applying similar arguments considering the commutators $[d, \nabla^k]$, $[\delta, \nabla^k]$ and the equations $dw^*\lambda = \frac{1}{2}|d^\pi w|^2\, dA$ from Proposition 3.1 and $\delta w^*\lambda = 0$, we have derived

Proposition 3.3. Let $w : \Sigma \to Q$ satisfy $\overline{\partial}^\pi w = 0$, $d(w^*\lambda \circ j) = 0$. Then

$$\int_\Sigma |\nabla^{k+1} w^*\lambda|^2 \leq L_k(|\partial^\pi w|, |w^*\lambda|)$$

for a polynomial function L_k of $|d^\pi w|$, $|w^*\lambda|$ its covariant derivatives up to $0, \ldots, k$ of degree at most $2k+3$.

Now combining Propositions 3.2, 3.3, we derive

Theorem 3.2.5. *Let (Σ, j) be a closed Riemann surface. Let $w : \Sigma \to Q$ satisfy $\overline{\partial}^\pi w = 0, \quad d(w^*\lambda \circ j) = 0$, on Σ. Then*

$$\int_\Sigma |(\nabla)^{k+1}(dw)|^2 \leq \int_\Sigma J'_k(|\partial^\pi w|, |w^*\lambda|). \tag{3.33}$$

Here J'_{k+1} a polynomial function of covariant derivatives of $|d^\pi w|$, $|w^*\lambda|$ up to $0, \ldots, k$ with degree at most $2k+4$ whose coefficients are bounded by

$$\|R\|_{C^k;\operatorname{supp} R}, \|K^\pi\|_{C^k}, \|\mathcal{L}_{X_\lambda} J\|_{C^k}.$$

In particular,

$$\|dw\|_{k+1,2} \leq C_k(\|dw\|_{L^2}, \|dw\|_{L^4}) \tag{3.34}$$

for a similar polynomial function $C_k = C_k(s,t)$.

Proof. It remains to check the second statement, which itself follows expressing the bound of $\|dw\|^2_{k,2}$ inductively staring from $k = 1$, i.e., Corollary 3.2.2. This finishes the proof. $\square$

Similar inductive computation also leads to the following local higher regularity.

Theorem 3.2.6. *Let $D = D^2(1)$ be the unit disc. There exists $C_{5;k}$, $C_{6;k} > 0$ depending only on $D' \subset \overline{D}' \subset D$ and on $\|K^\pi\|_{C^k}$, $\|\mathcal{L}_{X_\lambda} J\|_{C^k}$ and $\|R\|_{C^k;D}$ but independent of w such that for any smooth map $w : D \to Q$ satisfying*

$$\overline{\partial}^\pi w = 0, \quad d(w^*\lambda \circ j) = 0,$$

$$\|dw\|_{k+1,2;D'} \leq C_{k;D,D'}(\|dw\|_{2;D}, \|dw\|_{4;D})$$

for a similar polynomial function $C_{k;D,D'}(s,t)$ of s, t as C_k above depending also on D', D.

In particular, any weak solution of (3.2) in $W^{1,4}$ automatically lies in $W^{3,2}$ and becomes the classical solution to (3.2) and so becomes smooth.

Chapter 4

Finite energy contact instantons I

In this chapter, we study the behavior of contact instantons *with finite π-harmonic energy at the punctures* (Definition 4.1.1). To be specific, we study the limiting behavior of contact instantons and derive the coercive estimates for punctured domain case. However, we don't need any condition on contact one-forms, i.e., we don't require it is of nondegenerate or Morse-Bott type in the chapter.

This chapter consists of three sections. In the first section, we study the limiting behavior of finite energy contact instantons and get the subsequence convergence to limiting Reeb orbits.

In the second section, we derive the fundamental equation under cylindrical coordinates.

In the third section, we study the coercive estimates for punctured case.

4.1 Asymptotic convergence to Reeb orbits at puncture

We introduce the relevant energy in the asymptotic study of contact instanton map near the punctures of $\dot{\Sigma}$.

Let (Σ, j) be a compact Riemann surface and let $\{r_1, \cdots, r_k\} \subset \Sigma$ be given marked points (including $k = 0$). Denote by $\dot\Sigma$ the associated punctured Riemann surface equipped with a Kähler metric h that is cylindrical on disjoint punctured discs $D_i \setminus \{0\}$ near each punctures r_i. We fix an associated isothermal coordinates $z_i = e^{-2\pi(\tau+it)}$ on each $D_i \setminus \{0\} \cong [0,\infty) \times S^1$.

The following is the relevant off-shell energy that controls the asymptotic behavior of the map near the punctures.

Definition 4.1.1. Let $w : \dot\Sigma \to Q$ be any smooth map and let r be a given puncture with isothermal coordinates z centered at r. We define

$$E^\pi_{(\lambda,J;D\setminus\{0\})}(w) = \frac{1}{2}\int_{[0,\infty)\times S^1} |d^\pi w|^2. \tag{4.1}$$

We put the following hypotheses in our asymptotic study of the finite energy contact instanton maps w:

Hypothesis 4.1.2. Let $r \in \{r_1, \cdots, r_k\}$ be one of the given marked points and consider the map w restricted to the associated punctured disc $D \setminus \{r\}$.

1. Assume $w : [0,+\infty) \times S^1 \to Q$ is a contact instanton map, i.e., satisfies (3.2).
2. $E_{(\lambda,J;D\setminus\{0\})}(w) < \infty$,
3. $|\nabla w| < C$.
4. $\frac{1}{2}\int_{[0,\infty)\times S^1} |d^\pi w|^2 + \int_{S^1} w(0,\cdot)^*\lambda \neq 0$.

The following asymptotic convergence result of the finite energy contact instanton maps will be the fundamental ingredient for the applications thereof to the contact topology. The result essentially relies on our (local) $W^{3,2}$ a priori estimates which was already established in Theorem 3.2.6, and the finiteness of the above introduced end energy. This is an analogue to Theorem 31 [H1] and its proof somewhat resembles that of [H1], [HWZ1, HWZ2] with the usage of bounded harmonic functions therein replaced by that of bounded harmonic one-forms. However, *we would like to emphasize that our proof does not involve symplectization.*

We denote

$$w^*\lambda = a_1\, d\tau + a_2\, dt, \qquad \text{i.e., } a_1 = \lambda\left(\frac{\partial w}{\partial \tau}\right)\ a_2 = \lambda\left(\frac{\partial w}{\partial t}\right).$$

Then

$$w^*\lambda \circ j = a_2\, d\tau - a_1\, dt.$$

The equation

$$d(w^*\lambda \circ j) = 0$$

from (3.2) implies that the divergence

$$\nabla \cdot w^*\lambda := -\delta(w^*\lambda \circ j) = \frac{\partial a_1}{\partial \tau} + \frac{\partial a_2}{\partial t} = 0.$$

Let w be as in Hypothesis 4.1.2. Then we can associate two natural asymptotic invariants of w defined by

$$T := \int_{[0,\infty)\times S^1} |d^\pi w|^2 + \int_{S^1} w(0,\cdot)^*\lambda \neq 0 \tag{4.2}$$

$$a := -\int_{S^1} w(0,\cdot)^*\lambda \circ j = \int_{S^1} \lambda\left(\frac{\partial w}{\partial \tau}(0,t)\right) dt. \tag{4.3}$$

Due to the closedness of $w(0,\cdot)^*\lambda \circ j$ the integral

$$\int_{S^1} w(\tau,\cdot)^*\lambda \circ j = -a$$

for all $\tau \geq 0$. We call T the *asymptotic contact action* and a the *asymptotic contact charge* of the instanton w.

Theorem 4.1.3 (Subsequence convergence)**.** *Let w be as in Hypothesis 4.1.2. Then for any sequence $\tau_k \to \infty$, there exists a subsequence, still denote by τ_k, and a Reeb orbit γ with period, the action T, and with charge a such that*

$$\lim_{k\to\infty} w(\tau_k + \tau, t) = \gamma(a\,\tau + T\,t)$$

uniformly on compact set $[-K, K] \times S^1$ for any given $K \geq 0$.

Proof. For any sequence $\tau_k \to \infty$, we can always choose a subsequence, still denoted by τ_k for $k = 1, 2, \cdots$, so that

$$\lim_{k\to\infty} \int_{[\frac{1}{2}\tau_k,\infty)\times S^1} |d^\pi w|^2 = 0$$

by Hypothesis (2). We define a sequence of translated maps

$$w_k(\tau, t) = w(\tau + \tau_k, t) : [-\tau_k, \infty) \times S^1 \to Q$$

which gives rise to

$$\lim_{k\to\infty} \int_{[-\frac{1}{2}\tau_k,\infty)\times S^1} |d^\pi w_k|^2 = 0$$

We also have $|\nabla w_k| < C$ from Hypothesis (3) because the translations preserves the norm on the cylindrical metric near the puncture, and each w_k satisfies Hypothesis (1).

Let $K > 0$ be any given number and consider $[-K, K] \times S^1$ and note that eventually $[-K, K] \subset [-\frac{1}{2}\tau_k, \infty)$. Then the bound $\|w_k\|_{W^{3,2}([-K,K]\times S^1)} < C_K$ follows from compactness of Q and Theorem 3.2.6 for $k = 1$. Using the compactness of the embedding $W^{3,2} \hookrightarrow C^1$ on $[-K, K] \times S^1$, we get a subsequence w_k and $w_{\infty;K} : [-K, K] \times S^1 \to Q$ such that $w_k \to w_{\infty;K}$ in C^1 topology.

By letting $K \to \infty$ and taking the diagonal sequence argument, we obtain a map $w_\infty : \mathbb{R} \times S^1 \to Q$ such that $w_k \to w_\infty$ in compact C^1 topology on $\mathbb{R} \times S^1 \to Q$. We have

$$\|w_\infty\|_{C^1([-K,K]\times S^1)} \leq \sup_k \|w_k\|_{C^1([-K,K]\times S^1)}$$

which is uniformly bounded in the meaning that the upper bound is independent of K. In particular, we get $\|\nabla w_\infty\|_{C^0(\mathbb{R}\times S^1)} \leq C$.

Furthermore C^1 convergence gives that $w_{\infty,K} : [-K,K] \to Q$ satisfies

$$\begin{aligned} \int_{[-K,K]\times S^1} |d^\pi w_{\infty;K}|^2 &= \lim_{k\to\infty} \int_{[-K,K]\times S^1} |d^\pi w_k|^2 \\ &\leq \lim_{k\to\infty} \int_{[-\tau_k,\infty)\times S^1} |d^\pi w_k|^2 \\ &= 0 \end{aligned} \tag{4.4}$$

Hence $d^\pi w_\infty = 0$. Also since w_∞ also satisfies Hypothesis (1), we have $w_\infty^* d\lambda = \frac{1}{2}|d^\pi w_\infty|^2\, dA$, which in turn implies

$$w_\infty^* d\lambda = 0. \tag{4.5}$$

The equation (4.5) indicates that $w_\infty^*\lambda$ is closed, i.e., $d(w_\infty^*\lambda) = 0$. Hence together with

$$\delta w_\infty^* \lambda = *d(w^*\lambda \circ j) = 0,$$

we derive

$$\Delta w_\infty^* \lambda = 0,$$

i.e. $w_\infty^*\lambda$ is a harmonic one-form on $\mathbb{R}\times S^1$ (with respect to the standard flat metric). It is also bounded by Hypothesis (2) on $\mathbb{R}\times S^1$. Therefore it follows that

$$w_\infty^*\lambda \circ j = a_{2,\infty}\, d\tau - a_{1,\infty}\, dt$$

for some constants $a_{1,\infty}\, a_{2,\infty}$. By the connectedness of $[0,\infty)\times S^1$, the image of w is contained in a single leaf of the Reeb foliation. Let $\gamma : \mathbb{R} \to Q$ be a parameterization of the leaf so that $\dot\gamma = X_\lambda(\gamma)$. This parameterization is unique modulo a time-shift.

These imply that

$$w_\infty(\tau, t) = \gamma(a_{1,\infty}\tau + a_{2,\infty}t + c)$$

for a Reeb orbit γ as a map a priori defined on the universal covering space $\mathbb{R} \times \mathbb{R}$ of $\mathbb{R} \times S^1$ on the right hand side, where c can be chosen arbitrarily. But from $w(0, t+1) = w(0, t+1)$, we also obtain $\gamma(b(t+1)) = \gamma(bt)$ and hence γ is a closed period of period b.

By the C^1-convergence of $w(\tau_k, t) \to w_\infty(0, t)$ it follows

$$\int_{S^1} w_\infty(0, \cdot)^*\lambda = \lim_{k\to\infty} \int_{S^1} w(\tau_k, \cdot)^*\lambda \tag{4.6}$$

$$\int_{S^1} w_\infty(0, \cdot)^*\lambda \circ j = \lim_{k\to\infty} \int_{S^1} w(\tau_k, \cdot)^*\lambda \circ j = -a. \tag{4.7}$$

From (4.7), we have shown $a_{1,\infty} = a$. For (4.6), the identity $\frac{1}{2}|d^\pi w|^2\, dA = d(w^*\lambda)$ and Stokes' formula provides

$$\int_{S^1} w(\tau_k, \cdot)^*\lambda = \int_{S^1} w(0, \cdot)^*\lambda + \int_{[0,\tau_k]\times S^1} d(w^*\lambda) = \int_{S^1} w(0, \cdot)^*\lambda + \frac{1}{2}\int_{[0,\tau_k]\times S^1} |d^\pi w|^2.$$

Then using finiteness hypothesis of the integral $\int_{[0,\infty)\times S^1} |d^\pi w|^2 < \infty$ and Fatou's lemma, we derive

$$\begin{aligned} a_{2,\infty} &= \int_{S^1} w_\infty(0, \cdot)^*\lambda \\ &= \lim_{k\to\infty} \int_{S^1} w(\tau_k, \cdot)^*\lambda \\ &= \lim_{k\to\infty} \left(\int_{S^1} w(0, \cdot)^*\lambda + \frac{1}{2}\int_{[0,\tau_k]\times S^1} |d^\pi w|^2 \right) \\ &= \int_{S^1} w(0, \cdot)^*\lambda + \frac{1}{2}\int_{[0,\infty)\times S^1} |d^\pi w|^2 = T. \end{aligned}$$

Therefore $w_\infty(0, \cdot)$ is a Reeb orbit of period $T \neq 0$ with its charge a. □

By specializing to the case $a = 0$, we have derived the following

Corollary 4.1.4. *Let w be as in Hypothesis 4.1.2 and assume $a = 0$. Then for any sequence $\tau_k \to \infty$, there exists a subsequence, still denote by τ_k, and a Reeb orbit γ with period, the action T such that*

$$\lim_{k\to\infty} w(\tau_k + \tau, \cdot) = \gamma(t)$$

uniformly on compact set $[-K, K] \times S^1$ for any given $K \geq 0$.

In section 5.3, we will improve this subsequence convergence result to the full exponential convergence result for the vanishing charge case, i.e., $a = 0$. We hope to come back to the corresponding convergence result for $a \neq 0$ elsewhere.

Remark 4.1.5. This corollary includes Hofer's subsequence convergence result in the standard case of symplectization of contact manifold in [H1]. It also covers the case of Hofer's generalized equation

$$\begin{cases} \overline{\partial}^\pi_J w = 0 \\ w^*\lambda \circ j = da + \gamma \end{cases} \tag{4.8}$$

introduced in [H2] in which γ is a smooth harmonic one-form on the closed Riemann surface Σ pulled back to $\dot\Sigma$ via the natural inclusion map $\dot\Sigma \hookrightarrow \Sigma$: This is because by definition, the charge vanishes, i.e., we have

$$\int_\alpha w^*\lambda \circ j = \int_\alpha da + \gamma = 0$$

for any local loop α around the given puncture.

4.2 Fundamental equation in cylindrical coordinates

Consider the punctured Riemann surface $(\dot\Sigma, j)$ and fix a puncture. Equip a punctured neighborhood thereof with a cylindrical metric and let (τ, t) be the associated isothermal coordinates. We recall the decomposition of the w^*TQ-valued one-form dw on Σ,

$$dw = d^\pi w + w^*\lambda\, X_\lambda$$

associated the splitting $TQ = \mathbb{R}\{X_\lambda\} \oplus \xi$ and the vector bundle connection ∇^π of $\xi \to Q$ is defined by

$$\nabla^\pi_X Z := \pi(\nabla_X Z) \tag{4.9}$$

for any vector fields Z tangent to ξ and X on Q. Denote

$$\begin{aligned} \zeta = \pi\frac{\partial w}{\partial \tau}, \quad \eta = \pi\frac{\partial w}{\partial t} \\ w^*\lambda \circ j = a_2\, d\tau - a_1\, dt, \end{aligned} \tag{4.10}$$

then we have

$$d^\pi w = \partial^\pi w = \zeta\, d\tau + \eta\, dt,$$

and

$$\nabla \cdot w^*\lambda = \frac{\partial a_2}{\partial t} + \frac{\partial a_1}{\partial \tau} = 0. \tag{4.11}$$

from (3.2).

We apply (3.9) to the pair $(\frac{\partial}{\partial\tau}, \frac{\partial}{\partial t})$. Then the left hand side becomes

$$J\nabla^\pi_\tau \zeta - \nabla^\pi_t \zeta,$$

while the right hand side becomes

$$\begin{aligned} & -\frac{1}{2}\lambda\left(\frac{\partial w}{\partial \tau}\right)(\mathcal{L}_{X_\lambda}J)\zeta - \frac{1}{2}\lambda\left(\frac{\partial w}{\partial t}\right)(\mathcal{L}_{X_\lambda}J)J\zeta \\ = \quad & -\frac{1}{2}a_1(\mathcal{L}_{X_\lambda}J)\zeta - \frac{1}{2}a_2(\mathcal{L}_{X_\lambda}J)J\zeta, \end{aligned}$$

where we use (4.10) to get the second line. This immediately gives rise to the following form of fundamental equation in cylindrical coordinates, which is nothing but the linearization of $\overline{\partial}^\pi_J w = 0$ in the direction of $\zeta = \pi\frac{\partial w}{\partial \tau}$.

Proposition 4.1 (Fundamental equation in cylindrical coordinates)**.** Let $\zeta = \pi\frac{\partial w}{\partial \tau}$ as a section of $\xi \to Q$. Then

$$\nabla^\pi_\tau \zeta + J\nabla^\pi_t \zeta - \frac{1}{2}a_2(\mathcal{L}_{X_\lambda}J)\zeta + \frac{1}{2}a_1(\mathcal{L}_{X_\lambda}J)J\zeta = 0. \tag{4.12}$$

Now we define an $\mathbb{R}$-family of operators

$$A_\tau : C^\infty(w_\tau^*\xi) \to C^\infty(w_\tau^*\xi)$$

with w_τ being the loop defined by $w_\tau(t) := w(\tau, t)$ defined by

$$A_\tau(Y) = J\nabla^\pi_t Y - \frac{1}{2}a_2(\mathcal{L}_{X_\lambda}J)Y + \frac{1}{2}a_1(\mathcal{L}_{X_\lambda}J)JY \tag{4.13}$$

for $Y \in C^\infty(w_\tau^*\xi)$. This family of operators will enter in the study of perturbation results of the eigenvalues of the asymptotic operators at z which is the linearization operator

$$d\Upsilon_{(T,\lambda)} : C^\infty(z^*\xi) \to C^\infty(z^*\xi); \quad Y \mapsto \frac{DY}{dt} - T\,DX_\lambda(z)(Y)$$

of the map $\Upsilon_{(T,\lambda)}$ derived in Section 2.2 along the limit orbit z. Here z is determined by

$$z(t) = \lim_{\tau\to\infty} w(\tau, t).$$

By identifying $C^\infty(z^*\xi)$ with $C^\infty(S^1, w_\tau^*\xi)$ by suitable parallel transport for sufficiently large τ, it follows that the operator A_τ converges to $d\Upsilon_{(\lambda,T)}$ as $\tau \to \infty$. We refer readers to next chapter for the precise meaning of this convergence.

When λ is nondegenerate and $w^*\lambda \circ j$ is exact, we will see that $a_1 \to 0$ uniformly. However when $w^*\lambda \circ j \neq 0$, a_1 will converge to the constant determined by

$$\int_{S^1} (w(\tau, \cdot))^* \lambda \circ j$$

which does not depend on τ by the closedness condition $d(w^*\lambda \circ j) = 0$.

4.3 $W^{k,2}$-coercive estimates: the punctured case

In this section, we derive the $W^{k,2}$ estimates on punctured Riemann surfaces $\dot\Sigma$ equipped with cylindrical metric near the punctures.

We recall

$$\begin{aligned}
|\nabla dw|^2 \;\leq\; & -\Delta e^\pi - 4\delta\langle *d^{\nabla^\pi}\partial^\pi w, *\partial^\pi w\rangle \\
& -2\langle K^\pi(dw, dw)\partial^\pi w, \partial^\pi w\rangle - 2R|\partial^\pi w|^2 \\
& +\frac{1}{2}(C_1^2 + 1)|\partial^\pi w|^4 + \frac{3}{2}C_1^2|w^*\lambda|^2|\partial^\pi w|^2.
\end{aligned}$$

from (3.26). Unlike the close case, the first two terms of the right hand side will give rise to some 'asymptotic boundary terms' after integration by parts. Therefore we need to impose some asymptotic boundary condition at the punctures.

For this purpose, we re-write the two terms as

$$-\Delta e^\pi \, dA \;=\; d(*de^\pi) \tag{4.14}$$

$$-\delta\langle *d^{\nabla^\pi}\partial^\pi w, *\partial^\pi w\rangle \, dA \;=\; d\langle *d^{\nabla^\pi}\partial^\pi w, *\partial^\pi w\rangle. \tag{4.15}$$

We denote by $\Sigma(\rho)$ the Riemann surface obtained by excising the discs $|z| \leq \rho$ around the punctures for the given analytic coordinates $z = e^{-2\pi(\tau+it)}$ centered at the punctures where (τ, t) is the corresponding cylindrical coordinates. Then we have its boundary

$$\partial\Sigma(\rho) = \bigcup_{\ell=1}^{k} \partial_\ell \Sigma(\rho) \tag{4.16}$$

where $\partial_\ell\Sigma(\rho)$ is the component of $\Sigma(\rho)$ associated to the ℓ-th puncture r_ℓ equipped with the boundary orientation of $\Sigma(\rho)$. In terms of the cylindrical coordinates (τ, t), $\frac{\partial}{\partial\tau}$ corresponds to the outward normal vector of $\partial\Sigma(\rho)$.

By Stokes' formula, we obtain

$$\begin{aligned} \int_{\Sigma(\rho)} -\Delta e^\pi \, dA &= \int_{\partial\Sigma(\rho)} *de^\pi = \sum_{\ell=1}^{k} \int_{\partial_\ell\Sigma(\rho)} *de^\pi \\ &= \sum_{\ell=1}^{k} \int_{\partial_\ell\Sigma(\rho)} \frac{\partial e^\pi}{\partial\tau} \, dt \\ &= \sum_{\ell=1}^{k} \frac{\partial}{\partial\tau} \int_{\partial_\ell\Sigma(\rho)} e^\pi(\tau, t) \, dt \\ &= 2\sum_{\ell=1}^{k} \frac{\partial}{\partial\tau} \int_{\partial_\ell\Sigma(\rho)} |\zeta|^2 \, dt. \end{aligned} \tag{4.17}$$

For the term $-\delta\langle *d^{\nabla^\pi}\partial^\pi w, *\partial^\pi w\rangle \, dA$, we need some digression. In cylindrical coordinates (τ, t) (or in any isothermal coordinates on Σ), we recall $\partial^\pi w = \zeta \, d\tau + \eta \, dt$. where

$$\zeta = \pi\left(\frac{\partial w}{\partial\tau}\right), \quad \eta = \pi\left(\frac{\partial w}{\partial t}\right).$$

A straightforward calculation leads to the following general formulae.

Lemma 4.3.1. *Let w be any smooth map. Then*

$$\begin{aligned} \delta^{\nabla^\pi}\partial^\pi w &= -(\nabla^\pi_\tau \zeta + \nabla^\pi_t \eta) \\ *d^{\nabla^\pi}\partial^\pi w &= \nabla^\pi_\tau \eta - \nabla^\pi_t \zeta. \end{aligned}$$

In particular when w satisfies $\overline{\partial}^\pi_J w = 0$, we have $\eta = J\zeta$ and so

$$\partial^\pi w = \zeta \, d\tau + J\zeta \, dt, \quad *\partial^\pi w = -J\zeta \, d\tau + \zeta \, dt. \tag{4.18}$$

Then

$$*d^{\nabla^\pi}\partial^\pi w = d^{\nabla^\pi}\partial^\pi w\left(\frac{\partial}{\partial\tau},\frac{\partial}{\partial t}\right) = J(\nabla^\pi_\tau\zeta + J\nabla^\pi_t\zeta) = J\bar{D}\zeta,$$

where we define

$$\bar{D}\zeta := \nabla^\pi_\tau\zeta + J\nabla^\pi_t\zeta.$$

Then we obtain

$$\begin{aligned}
\int_{\Sigma(\rho)} -\delta\langle *d^{\nabla^\pi}\partial^\pi w, *\partial^\pi w\rangle\, dA &= \int_{\Sigma(\rho)} d*\langle *d^{\nabla^\pi}\partial^\pi w, *\partial^\pi w\rangle \\
&= \sum_{\ell=1}^{k}\int_{\partial_\ell\Sigma(\rho)} *\langle *d^{\nabla^\pi}\partial^\pi w, *\partial^\pi w\rangle\, dt \\
&= \sum_{\ell=1}^{k}\int_{\partial_\ell\Sigma(\rho)} *\langle J\bar{D}\zeta, (-J\zeta\, d\tau - \zeta\, dt)\rangle \\
&= \sum_{\ell=1}^{k}\int_{\partial_\ell\Sigma(\rho)} -\langle \bar{D}\zeta, \zeta\rangle\, dt.
\end{aligned}$$

Using the fundamental equation in the cylindrical coordinate derived in Proposition 4.1, we obtain

$$\bar{D}\zeta = \frac{1}{2}a_2(\mathcal{L}_{X_\lambda}J)\zeta - \frac{1}{2}a_1(\mathcal{L}_{X_\lambda}J)J\zeta \tag{4.19}$$

and hence

$$\int_{\partial_\ell\Sigma(\rho)} -\left\langle \bar{D}\zeta, \zeta\right\rangle\, dt = \int_{\partial\ell\Sigma(\rho)} \left\langle \frac{1}{2}a_1(\mathcal{L}_{X_\lambda}J)J\zeta - \frac{1}{2}a_2(\mathcal{L}_{X_\lambda}J)\zeta, \zeta\right\rangle\, dt. \tag{4.20}$$

Motivated by these explicit formulae, we introduce the following function

$$f(\tau) = \frac{1}{2}\int_{S^1} e^\pi(\tau,t)\, dt = \int_{S^1} |\zeta|^2(\tau,t)\, dt \tag{4.21}$$

which will be also important for the later study of exponential convergence in part 5.

We then obtain

$$\begin{aligned}
\int_{\Sigma(\rho)} -\Delta e^\pi \, dA &= \sum_{\ell=1}^k f_\ell'(\tau) \\
\int_{\partial_\ell \Sigma(\rho)} |\langle \bar D \zeta, \zeta\rangle| \, dt &\leq C_1(\|a_1\|_{C^0} + \|a_2\|_{C^0}) \sum_{\ell=1}^k \int_{\partial_\ell\Sigma(\rho)} |\zeta|^2 \, dt \\
&\leq 2C_1 \|w^*\lambda\|_{C^0} \sum_{\ell=1}^k f_\ell(\tau) \qquad (4.22)
\end{aligned}$$

from (4.17) and (4.20) respectively, where $C_1 = \|\mathcal{L}_{X_\lambda} J\|_{C^0}$, and f_ℓ is the L^2-integral function f above defined on the puncture disc around r_ℓ.

The following is the $W^{2,2}$ a priori estimates on the punctured case which is the counterpart of Theorem 3.2.1 of the closed case. The proof is the same as that of Theorem 3.2.1 incorporating the asymptotic boundary condition.

Theorem 4.3.2. *Let (Σ, j) be a closed Riemann surface with a finite number of marked points $\{r_1, \cdots, r_k\}$. Denote by $\dot\Sigma$ the associated punctured Riemann surface with a Kähler metric h on (Σ, j) which is cylindrical near the puncture. Let f_ℓ be the function defined as above associated to the ℓ-th puncture r_ℓ. Suppose w satisfies* (3.2) *on $\dot\Sigma$ and $|d^\pi w| \in L^2 \cap L^4$ and $\|w^*\lambda\|_{C^0} < \infty$ on $\dot\Sigma$*

$$\begin{aligned}
\lim_{\tau\to\infty} a_1 = a, &\qquad \lim_{\tau\to\infty} a_2 = T \\
\lim_{\tau\to\infty} f_\ell(\tau) = 0, &\qquad \lim_{\tau\to\infty} f_\ell'(\tau) = 0 \qquad (4.23)
\end{aligned}$$

for all $\ell = 1, \cdots, k$. Then

$$\begin{aligned}
\int_{\dot\Sigma} |\nabla dw|^2 \leq\ & \frac{1}{2}(C_1^2+1)\|\partial^\pi w\|_{L^4}^4 - 2\min R \|\partial^\pi w\|^2_{L^2;\operatorname{supp} R} + \frac{3}{2} C_1^2 \|w^*\lambda\|^2_{C^0} \|\partial^\pi w\|^2_{L^4} \\
& + 2\max|K^\pi| \left(\|w^*\lambda\|^2_{C^0} \|\partial^\pi w\|^2_{L^2} + \|\partial^\pi w\|^4_{L^4}\right).
\end{aligned}$$

Here K^π is the curvature of ∇^π, R the Gaussian curvature of the metric h on Σ and

$$C_1 = \|\mathcal{L}_{X_\lambda} J\|_{C^0}.$$

We would like to remark that the asymptotic boundary conditions imposed in this theorem will be automatically satisfied under the Hypothesis 4.1.2 together with nondegeneracy of the asymptotic Reeb orbits obtained from subsequence convergence theorem, Theorem 4.1.3. These will be established in Chapter 5.

An immediate corollary of this theorem, when combined with the standard Hölder's inequality, is the following $W^{2,2}$-coercive estimates.

Corollary 4.3.3. *Let $\dot\Sigma$ and w be as above. Suppose w satisfies* (3.2) *on $\dot\Sigma$ and $|d^\pi w| \in L^2 \cap L^4$ and $\|w^*\lambda\|_{C^0} < \infty$ on $\dot\Sigma$ and assume* (4.23)*. Then there exists uniform constants C_3', C_4' depending only on*

$$\|K^\pi\|_{C^0},\ \|R\|_{C^0},\ \|\mathcal{L}_{X_\lambda} J\|_{C^0}\ \|w^*\lambda\|_{C^0}$$

but independent of w such that

$$\|dw\|^2_{W^{1,2}} \leq C_3' \|\partial^\pi w\|^4_{L^4} + C_4' \|\partial^\pi w\|^2_{L^2}.$$

Once we establish the above $W^{2,2}$-estimate, the alternating inductive bootstrapping arguments will establish the following higher $W^{k,2}$-estimate.

Theorem 4.3.4. *Let (Σ, j) be a closed Riemann surface. Let $w : \Sigma \to Q$ satisfy $\overline{\partial}^\pi_J w = 0, \quad d(w^*\lambda \circ j) = 0$, on $\dot\Sigma$ and* (4.23)*. Then if $|d^\pi w| \in L^2 \cap L^4$ and $\|w^*\lambda\|_{C^0} < \infty$ on $\dot\Sigma$,*

$$\int_{\dot\Sigma} |(\nabla)^{k+1}(dw)|^2 \leq \int_{\dot\Sigma} J_k'(|d^\pi w|, |w^*\lambda|). \tag{4.24}$$

Here J_{k+1}' a polynomial function of covariant derivatives of $|d^\pi w|$, $|w^*\lambda|$ up to $0, \ldots, k$ with degree at most $2k+4$ whose coefficients are bounded by

$$\|R\|_{C^k;\operatorname{supp} R}, \|K^\pi\|_{C^k}, \|\mathcal{L}_{X_\lambda} J\|_{C^k}.$$

In particular,

$$\|dw\|_{k+1,2} \leq C_k(\|dw\|_{L^2}, \|dw\|_{L^4}) \tag{4.25}$$

for a similar polynomial function $C_k = C_k(s,t)$.

Remark 4.3.5. We would like to note that starting from the asymptotic decay condition (4.23) of the functions f_ℓ, i.e., the L^2-integral of e^π at the punctures r_ℓ, we also need to inductively establish the asymptotic exponential decay for the k-th derivatives of f_ℓ themselves as a part of the alternating bootstrapping process. In summary, on the punctured Riemann surface $\dot\Sigma$, the proofs of the above $W^{k,2}$-estimates and of the exponential estimates should be performed together simultaneously in the alternating bootstrapping process performed in here and in Chapter 5.

Chapter 5

Finite energy contact instantons II - nondegenerate case

5.1 Overview and hypothesis

In this part, we continue to use the contact triad connection ∇ on Q and on the Hermitian vector bundle $\xi \to Q$ in tensorial calculations of the sections of ξ in cylindrical coordinates near the given puncture. More specifically we use this connection to differentiate $\zeta = \pi\left(\frac{\partial w}{\partial \tau}\right)$ and prove the exponential convergence property of the contact Cauchy-Riemann map w satisfying

$$\overline{\partial}^\pi w_J = 0, \quad d(w^*\lambda \circ j) = 0$$

to a Reeb orbit z as $\tau \to \infty$ when z is nondegenerate and $a = 0$. The convergence is uniform *when the uniform gradient bound*

$$|\nabla w| < C$$

is assumed, which will be always achieved after bubbling-off analysis which is by now standard. Of course, this derivation covers the exact case, i.e, the case of pseudoholomorphic maps $(a, w) : \dot{\Sigma} \to \mathbb{R} \times Q$, for which $w^*\lambda \circ j = da$.

To give an overview, in this chapter, we directly study the derivative dw as a vector valued one-form on $\mathbb{R} \times S^1$ by exploiting the presence of splitting $TQ = \mathbb{R}\{X_\lambda\} \oplus \xi$ and so

$$dw = \partial^\pi w + w^*\lambda \, X_\lambda.$$

In cylindrical coordinates, we have

$$\partial^\pi w = \zeta \, d\tau + J\zeta \, dt, \quad w^*\lambda \circ j = a_2 \, d\tau - a_1 \, dt$$

where we denote

$$\pi \frac{\partial w}{\partial \tau} = \zeta, \, \pi \frac{\partial w}{\partial t} = J\zeta, \quad a_1 = \lambda \left(\frac{\partial w}{\partial \tau} \right), \, a_2 = \lambda \left(\frac{\partial w}{\partial t} \right).$$

As sections on the circle S^1, we have

$$\begin{aligned} \frac{\partial w}{\partial \tau} &= \pi \frac{\partial w}{\partial \tau} + a_1 \, X_\lambda(w) \\ \frac{\partial w}{\partial t} &= \pi \frac{\partial w}{\partial t} + a_2 \, X_\lambda(w). \end{aligned}$$

We first perform the tensorial calculation of $\zeta = \pi \frac{\partial w}{\partial \tau}$ as a section of $w(\tau, \cdot)^*\xi$ in terms of the contact Hermitian connection ∇^π (and the canonical contact connection ∇ on Q). As a consequence, after this step, we obtain a stronger $W^{1,2}$-exponential estimate for ζ, rather than just L^2.

Our proof of C^∞ exponential convergence is based on the purely inductive bootstrapping arguments, i.e., the C^1-exponential estimate and its proof does not depend on the estimates of the second (or higher) derivatives of the contact Cauchy-Riemann map w (or the function a associated to the pseudoholomorphic map (a, w) in the symplectization.) It turns out that this task involves highly non-trivial combinatorics of tensor calculations and depends on our choice of special connections and precise tensorial computations.

In this part, we add the following additional nondegeneracy hypothesis of the relevant Reeb orbits.

Hypothesis 5.1.1 (Nondegeneracy)**.** Assume that the T-periodic orbit in Hypothesis 4.1.2 is nondegenerate: If $z = \gamma(T\cdot)$ for a nondegenerate T-periodic Reeb orbit γ on Q, we denote the S^1-family of rotations of the loop $z : S^1 \to Q$ by

$$Z = \{z_\theta \in C^1(S^1, Q) \mid z_\theta(t) := z(t-\theta),\, \theta \in S^1\}.$$

5.2 Asymptotic perturbation results of eigenvalues

We recall from Section 2.2 that the linearization operator of the Reeb orbit z

$$A_z : W^{1,2}(z^*\xi) \to L^2(z^*\xi),$$

has the form

$$\begin{aligned} A_z(\eta) &= Jd(\Upsilon_T)_z(\eta) = J\frac{D\eta}{dt} - TJDX_\lambda(\eta) \\ &= J\frac{D\eta}{dt} - \frac{1}{2}T(\mathcal{L}_{X_\lambda}J)\eta. \end{aligned} \tag{5.1}$$

Nondegeneracy hypothesis of z implies $\ker A_z = \{0\}$ and then since the Fredholm index of A_z is zero its cokernel is also trivial. We note that the operator $A_z : L^2 \to L^2$ is a self-adjoint unbounded operator and so has real eigenvalues. It follows from the open mapping theorem that there exists $\delta = \delta(z) > 0$ such that

$$\|A_z\eta\|_{L^2} \geq \delta\|\eta\|_{L^2}$$

for all $\eta \in W^{1,2}(S^1, z^*\xi)$. Since the rotation $R_\theta : z \to z_\theta$ induces an isometry between the relevant Sobolev spaces of $W^{1,2}$, L^2 associated to z and z_θ, we have derived the following lemma.

Lemma 5.2.1. *There exists $\lambda_1 > 0$ such that for any $z \in Z$,*

$$\|A_z \eta\|_{L^2} \geq \lambda_1 \|\eta\|_{L^2} \tag{5.2}$$

for all $\eta \in W^{1,2}(z^*\xi)$.

We will also need the following technical lemma. (We need only the version with $W^{1,2}$ replaced by C^1 in this lemma but would like to give this stronger version since the proofs will not make much difference.)

Lemma 5.2.2. *Consider the exponential map*

$$\widetilde{exp}_z : W^{1,2}(z^*TQ) \to W^{1,2}(S^1, Q)$$

defined by $\widetilde{\exp}_z(v)(t) = \exp_{z(t)}(v(t))$ for $v \in W^{1,2}(z^*TQ)$ and define the parameterized map

$$\widetilde{\exp}_Z : \bigcup_{z \in Z} W^{1,2}(z^*TQ) \to W^{1,2}(S^1, Q)$$

by $\widetilde{\exp}_Z(z, v) = \exp_z(v)$. For any given $\epsilon > 0$, consider the image

$$N_\epsilon(Z) := \bigcup_{z \in Z} \widetilde{\exp}_z(W^{1,2}(z, \epsilon)) \subset W^{1,2}(S^1, Q)$$

where

$$W^{1,2}(z, \epsilon) = \{v \in W^{1,2}(z^*TQ) \mid \|v\|_{1,2} \leq \epsilon\}.$$

Then the image is compact in C^δ-topology with $0 \leq \delta < \frac{1}{2}$.

Proof. This is an immediate consequence of compactness of Z and the compactness of the embedding $W^{1,2}(S^1, Q) \hookrightarrow C^\delta(S^1, Q)$. □

Using this lemma and Theorem 4.1.3 and Hypothesis 4.1.2 (4), we now prove

Proposition 5.1. Assume the contact charge vanishes, i.e., $a = 0$. Let $\lambda_1 > 0$ be the constant given in Lemma 5.2.1. Consider the completion of the operator (4.13)

$$A_\tau : W^{1,2}(w(\tau,\cdot)^*\xi) \to L^2(w(\tau,\cdot)^*\xi).$$

Then there exists some $\tau_1 \in \mathbb{R}$ such that for all $\tau \geq \tau_1$,

$$\|A_\tau(\eta)\|_{L^2} \geq \frac{3}{4}\lambda_1\|\eta\|_{L^2} \tag{5.3}$$

for all $\eta \in W^{1,2}(w(\tau,\cdot)^*\xi)$.

Proof. First of all, Hypothesis 4.1.2 and Theorem 4.1.3 imply that for any given $\epsilon > 0$, there exists some $\tau_1 > 0$ such that

$$w(\tau,\cdot) \in N_\epsilon(Z) \tag{5.4}$$

for all $\tau \geq \tau_1$.

Again by Hypothesis 4.1.2 and Theorem 4.1.3, it follows that the union

$$\{w(\tau,\cdot)\}_{\tau\geq\tau_1} \cup Z$$

is a (fiberwise) closed bounded subset of $N_\epsilon(Z) \subset W^{1,2}(S^1, Q)$. Therefore it is compact in C^ϵ-topology. Now for each given τ, we consider $z_\tau \in Z$ that is the shortest distance point of Z from $w(\tau,\cdot)$, i.e.,

$$d_{C^\epsilon}(w(\tau,\cdot), z) = d_{C^\epsilon}(w(\tau,\cdot), Z) = \min_{z\in Z} d_{C^\epsilon}(w(\tau,\cdot), z).$$

Such z_τ exists by the compactness of the subset $Z \subset C^0$. There could be more than one such z_τ but any one choice will do our purpose. We denote by $\Pi = \Pi_{z_\tau}^{w(\tau,\cdot)}$ the parallel transport map of the vector bundle ξ with respect to ∇^π from z_τ to $w(\tau, \cdot)$ along the short geodesic between $z_\tau(t)$ and $w(\tau, t)$ at each $t \in S^1$. Then we consider the operator

$$\Pi^{-1} A_\tau \Pi : W^{1,2}(z_\tau^* \xi) \to W^{1,2}(z_\tau^* \xi)$$

The following lemma is the key ingredient in our optimal exponential decay estimate under the main hypothesis put in this beginning of the section, but without assuming any assumption on the higher derivatives of w. (The latter will be an automatic consequence by our bootstrap arguments.)

Lemma 5.2.3. *There exists some τ_2 such that for all $\tau \geq \tau_2$*

$$|(\Pi^{-1} A_\tau \Pi - A_{z_\tau})\eta| \leq \frac{1}{4}\lambda_1 |\eta|$$

for any $\eta \in C^1(z_\tau^* \xi)$.

Proof. We recall

$$A_\tau = J \nabla_t^\pi + B$$

where B is the zero-order operator on $w(\tau, \cdot)^* \xi$ by

$$B\eta = -\frac{1}{2} a_2 (\mathcal{L}_{X_\lambda} J)\eta + \frac{1}{2} a_1 (\mathcal{L}_{X_\lambda} J) J\eta \tag{5.5}$$

for $\eta \in w(\tau, \cdot)^* \xi$. Then using the J-linearity of the Hermitian connection ∇^π, we compute

$$\begin{aligned} (\Pi^{-1} A_\tau \Pi)\eta &= \Pi^{-1}(J \nabla_t^\pi + B)(\Pi \eta) \\ &= J \Pi^{-1} \nabla_t^\pi (\Pi \eta) + \Pi^{-1} B \Pi(\eta). \end{aligned} \tag{5.6}$$

Now for given τ, we consider the map

$$\Gamma(s, t) = \exp_{z_\tau(t)}(s E(z_\tau(t), w(\tau, t)))$$

where we recall the definition $E(x,y) = \exp_x^{-1} y$. Then by definition of the parallel transport, we have

$$\Pi\eta(t) = \Xi(1,t)$$

where $\Xi = \Xi(s,t)$ is the solution to the ordinary differential equation

$$\nabla_s^\pi \Xi = 0, \quad \Xi(0,t) = \eta(t)$$

which defines the parallel transport of $\eta(t)$ along the geodesic

$$s \mapsto \Gamma(s,t) := \exp_{z_\tau(t)}(sE(z_\tau(t), w(\tau,t))).$$

Now we write

$$\Pi^{-1}\nabla_t^\pi(\Pi\eta) - \nabla_{\dot{z}_\tau}^\pi \eta = \int_0^1 \frac{d}{ds}\left(\Pi_s^{-1}(\nabla_t^\pi(\Pi_s\eta))\right) ds \tag{5.7}$$

where Π_s is the parallel transport from z_τ to $\exp_{z_\tau}(sE(z_\tau, w(\tau,\cdot)))$. Then we compute

$$\begin{aligned}
\frac{d}{ds}\left(\Pi_s^{-1}\nabla_t^\pi(\Pi_s\eta)\right) &= \Pi_s^{-1}\nabla_s^\pi(\nabla_t^\pi(\Pi_s\eta))) \\
&= \Pi_s^{-1}\left(\nabla_t^\pi\nabla_s^\pi(\Pi_s\eta)\right) + \Pi_s^{-1}K^\pi\left(\frac{\partial\Gamma}{\partial s}, \frac{\partial\Gamma}{\partial t}\right)(\Pi_s\eta) \\
&= \Pi_s^{-1}K^\pi\left(\frac{\partial\Gamma}{\partial s}, \frac{\partial\Gamma}{\partial t}\right)(\Pi_s\eta).
\end{aligned}$$

But we note

$$\begin{aligned}
\frac{\partial\Gamma}{\partial s} &= d_2\exp_{z_\tau}(sE(z_\tau, w(\tau,\cdot)))(E(z_\tau, w(\tau,\cdot)) \\
\frac{\partial\Gamma}{\partial t} &= D_1\exp_{z_\tau}(sE(z_\tau, w(\tau,\cdot)))\left(\frac{\partial z_\tau}{\partial t}\right) + d_2\exp_{z_\tau}(sE(z_\tau, w(\tau,\cdot)))\left(s\frac{\partial E(z_\tau, w(\tau,\cdot))}{\partial t}\right).
\end{aligned}$$

The constants C, C' appearing in the computations below may vary place by place but always depend only on the triad (Q,λ,J) and the $W^{1,2}$-bound of w.

Using the equality $|E(x,y)| = d(x,y)$ when $d(x,y)$ is less than injective radius, it follows that

$$\begin{aligned} |d_2 \exp_{z_\tau}(sE(z_\tau, w(\tau,\cdot)))(E(z_\tau, w(\tau,\cdot))| &\leq C|E(z_\tau, w(\tau,\cdot))| \leq C\|d(z_\tau, w(\tau,\cdot))\|_{C^0} \\ \left|D_1 \exp_{z_\tau}(sE(z_\tau, w(\tau,\cdot)))\left(\frac{\partial z_\tau}{\partial t}\right)\right| &\leq C\left|\frac{\partial z_\tau}{\partial t}\right| \leq CT\|X_\lambda\|_{C^0} = CT. \end{aligned}$$

On the other hand,

$$\begin{aligned} &\left|D_1 \exp_{z_\tau}(sE(z_\tau, w(\tau,\cdot)))\left(s\frac{\partial E(z_\tau, w(\tau,\cdot))}{\partial t}\right)\right| = \\ &\left|D_1 \exp_{z_\tau}(sE(z_\tau, w(\tau,\cdot)))s\left(D_1(E(z_\tau, w(\tau,\cdot))\frac{\partial z_\tau}{\partial t} + d_2(E(z_\tau, w(\tau,\cdot))\frac{\partial w}{\partial t}\right)\right|. \end{aligned}$$

It follows from the standard Jacobi field estimate [Ka]

$$\begin{aligned} \left|D_1(E(z_\tau, w(\tau,\cdot))\frac{\partial z_\tau}{\partial t} + d_2(E(z_\tau, w(\tau,\cdot))\frac{\partial w}{\partial t}\right| &\leq C\left(\left|\frac{\partial z_\tau}{\partial t}\right| + \left|\frac{\partial w}{\partial t}\right|\right) \\ &\leq C'T \end{aligned}$$

where the second inequality comes since $\frac{\partial z_\tau}{\partial t} = TX_\lambda(z_\tau)$ and $\left|\frac{\partial w}{\partial t}\right| \to T|X_\lambda| = T$ uniformly as $\tau \to \infty$. In summary, we have derived

$$\left|\frac{\partial \Gamma}{\partial s}\right| \leq C\|d(z_\tau, w(\tau,\cdot))\|_{C^0}, \quad \left|\frac{\partial \Gamma}{\partial t}\right| \leq C'T.$$

Therefore we have established

$$\left|\Pi_s^{-1} K^\pi\left(\frac{\partial \Gamma}{\partial s}, \frac{\partial \Gamma}{\partial t}\right)(\Pi_s \eta)\right| \leq CC'T\|d(z_\tau, w(\tau,\cdot))\|_{C^0}|\eta|.$$

We also recall $\|d(z_\tau, w(\tau,\cdot))\|_{C^0} \leq C\,\|d(z_\tau, w(\tau,\cdot))\|_{W^{1,2}}$. Substituting these into (5.7), there exists some τ_1 such that for all $\tau \geq \tau_1$, we obtain

$$|\Pi^{-1}\nabla_t^\pi(\Pi\eta) - \nabla_{\dot{z}_\tau}^\pi \eta| \leq \frac{1}{8}|\eta|.$$

Substituting this into (5.6), we have obtained

$$\begin{aligned}
|(\Pi^{-1}A_\tau\Pi\eta) - A_{z_\tau}\eta| &\leq |\Pi^{-1}\nabla_t^\pi(\Pi\eta) - \nabla_{\dot z_\tau}^\pi\eta| + |\Pi^{-1}B\Pi(\eta) - DX_\lambda(z_\tau)\eta| \\
&\leq \frac{1}{8}|\eta| + \frac{1}{8}|\eta| = \frac{1}{4}|\eta|.
\end{aligned}$$

Here we use the convergence $|\Pi^{-1}B\Pi(\eta) - DX_\lambda(z_\tau)\eta| = o(\tau)|\eta|$ from the expression of B (5.5) and the convergence

$$a_1 \to 0, \quad a_2 \to T$$

where the first convergence to zero follows from the assumption $a = 0$.

This finishes the proof of the lemma. □

To prove

$$\|A_\tau(\eta)\|_{L^2} \geq \frac{3}{4}\lambda_1\|\eta\|_{L^2} \tag{5.8}$$

it is enough to prove

$$|\langle A_\tau(\eta), \eta\rangle_{L^2}| \geq \frac{3}{4}\lambda_1\langle\eta, \eta\rangle_{L^2}$$

for all $\eta \in W^{1,2}(w(\tau,\cdot)^*\xi)$. We write

$$\begin{aligned}
|\langle A_\tau(\eta), \eta\rangle_{L^2}| &= |\langle \Pi^{-1}A_\tau\Pi\Pi^{-1}(\eta), \Pi^{-1}\eta\rangle_{L^2}| \\
&\geq |\langle A_{z_\tau}(\Pi^{-1}\eta), \Pi^{-1}\eta\rangle_{L^2}| - |\langle(\Pi^{-1}A_\tau\Pi - A_{z_\tau}(\Pi^{-1}\eta), \Pi^{-1}\eta\rangle_{L^2}| \\
&\geq \lambda_1\|\Pi^{-1}\eta\|_{L^2}^2 - \frac{1}{4}\lambda_1\|\Pi^{-1}\eta\|_{L^2}^2 = \frac{3}{4}\lambda_1\|\Pi^{-1}\eta\|_{L^2}^2 = \frac{3}{4}\lambda_1\|\eta\|_{L^2}^2
\end{aligned}$$

where we use the fact that Π is an isometry. This now finishes the proof the proposition.

□

Remark 5.2.4. The subtlety of the above proof of the lemma (and hence that of Proposition 9.5) lies in the fact that we would like to prove the uniform lower bound of the eigenvalues of the family A_τ for $\tau \geq \tau_2$ for all τ_2. Because we only assume $w(\tau, \cdot)$ is $W^{1,2}$-close to z_τ (even under the C^1-closeness hypothesis), it is a nontrivial task for the a priori noncompact family of the operators A_τ whose domain and target, which are $W^{1,2}(w(\tau, \cdot)^*\xi)$ and $L^2(w(\tau, \cdot)^*\xi)$, have uniform bound below away from zero. Here enters the compactness of the embedding $W^{1,2} \hookrightarrow C^\epsilon$ and our careful usage of exponential map.

5.3 L^2 exponential convergence of ξ-component of dw

Under the hypothesis, Hypothesis 5.1.1, in addition to Hypothesis 4.1.2, we prove the following stronger convergence result in this section.

Proposition 5.2. Under the Hypothesis 4.1.2 and 5.1.1, the T-periodic orbit given in Theorem 4.1.3 satisfies the following additional properties:

$$\lim_{\tau \to +\infty} \left| \pi \frac{\partial w}{\partial \tau} \right| = 0 \tag{5.9}$$

$$\lim_{\tau \to +\infty} \left| \pi \frac{\partial w}{\partial t} \right| = 0, \tag{5.10}$$

$$\lim_{\tau \to +\infty} a_2(\tau, t) = \lim_{\tau \to +\infty} \lambda \left(\frac{\partial w}{\partial t} \right) = T \tag{5.11}$$

$$\lim_{\tau \to +\infty} a_1(\tau, t) = \lim_{\tau \to +\infty} \lambda \left(\frac{\partial w}{\partial \tau} \right) = a \tag{5.12}$$

uniformly in t, where a is determined by

$$a = -\int_{S^1} (w(0, \cdot)^* \lambda \circ j = \int_{S^1} \lambda \left(\frac{\partial w}{\partial \tau}(0, t) \right) dt.$$

Proof. By the closedness of $w^*\lambda \circ j$, it follows

$$\int_{S^1} w(\tau,\cdot)^*\lambda \circ j = \int_{S^1} w(0,\cdot)^*\lambda \circ j$$

for all $\tau \geq 0$. In addition, by recalling from (4.7), we have derived

$$w_\infty^*\lambda\left(\frac{\partial}{\partial\tau}\right) = a.$$

By the C^1-convergence of $w(\tau_k,\cdot) \to w_\infty(0,\cdot)$, it follows

$$\int_{S^1} w_\infty(0,\cdot)^*\lambda \circ j - a\, d\tau = -\int_{S^1} a\, dt = -a$$

We also have

$$w_\infty^*\lambda \circ j = T\, d\tau - a\, dt, \quad w_\infty^*\lambda = T\, dt + a\, d\tau. \tag{5.13}$$

Next, we derive the derivative convergence. From the subsequence convergence proved above, it follows that for any sequence $\tau_k \to +\infty$, there exists a subsequence, still denoted by τ_k, such that $\lim_{k\to+\infty} w(\tau_k,\cdot) = z_\theta(\cdot)$ in C^1 topology on S^1 under Hypothesis (4) (The argument is exactly the same [HWZ1, HWZ2, Proposition 2.1], so we omit the details), where $z_\theta \in Z$ is some rotation of z whose associated rotation constant θ may depend on the choice of subsequence. Hence

$$w(\tau_k,\cdot) \to \dot{z}_\theta,$$

in $C^1(S^1)$, where $z_\theta \in Z$. Then we get

$$\lim_{k\to+\infty}\left|\pi\frac{\partial w}{\partial t}(\tau_k,t)\right| = 0, \tag{5.14}$$

$$\lim_{k\to+\infty}\lambda\left(\frac{\partial w}{\partial t}\right) = T. \tag{5.15}$$

Notice here, the period T does not depend on the choice of subsequence but determined by w.

Now if there exists some sequence such that $\left|\pi\frac{\partial w}{\partial t}\right|$ doesn't converge to zero, we can assume it has a subsequence converging to some non-zero constant. That is because we assume finite gradient bound in Hypothesis (2). Then, we can pick a subsequence again to make it converge to a Reeb orbit and the contradiction appears. Similarly, we can do the same argument to $\lambda(\frac{\partial w}{\partial t})(\tau_k, t)$, and thus (5.11) follows.

From the Hypothesis (1), both

$$\lim_{\tau\to+\infty}\pi\frac{\partial w}{\partial \tau}=0, \quad \lim_{\tau\to+\infty}a_2=T$$

immediately follow. We only need to show

$$\lim_{\tau\to+\infty}a_1=a. \tag{5.16}$$

Assume this fails to hold, i.e., there exists some $\epsilon>0$ and a sequence (τ_k, t_k) with $\tau_k\to\infty$, such that

$$|a_1(\tau_k,\cdot)-a|>\epsilon.$$

Then we look at the translated sequence

$$w_k(\tau,t)=w(\tau+\tau_k,t)$$

again, and we get w_∞ in the same way. From (5.13), It follows that

$$0=\left|w_\infty^*\lambda\left(\frac{\partial}{\partial\tau}\right)-a\right|=\lim_{k\to\infty}\left|w_k^*\lambda\left(\frac{\partial}{\partial\tau}\right)-a\right|=\lim_{k\to\infty}|a_1(\tau_k,\cdot)-a|\geq\epsilon,$$

which gives contradiction and we are done with the proof. □

Now let (τ, t) be the coordinates of the given cylindrical metric near a given (positive) puncture of the Riemann surface (Σ, j). From now on in the rest of the paper, we will restrict ourselves to the case of vanishing charge, i.e., we put the following hypothesis

Hypothesis 5.3.1 (Charge vanishing)**.** We assume the asymptotic charges of w at all ends vanish, i.e.,

$$-a = \lim_{\tau\to\infty} \int_{\partial_\ell \Sigma(\rho)} w(\tau,0)^*\lambda \circ j = 0 \tag{5.17}$$

for all $\ell = 1, \cdots, k$ where $\rho = e^{-2\pi\tau}$.

We recall the definition of the function

$$f(\tau) = \frac{1}{2}\int_{S^1} e^\pi(\tau,t)dt, \quad e^\pi(\tau,t) = |d^\pi w|^2 = |\partial^\pi w|^2 = 2|\zeta(\tau,t)|^2.$$

Notice here, we have

$$f''(\tau) = \frac{d^2}{d\tau^2}\int_{S^1} |\zeta|^2\, dt = -\frac{1}{2}\int_{S^1} \Delta e^\pi(\tau,t)dt.$$

We remark that in this formula Δe^π is the Hodge Laplacian

$$\Delta e^\pi = -\left(\frac{\partial^2 e^\pi}{\partial\tau^2} + \frac{\partial^2 e^\pi}{\partial t^2}\right),$$

the negative of the classical Laplacian. With this being said, the following is the main result in this section.

Theorem 5.3.2. *Assume Hypotheses 4.1.2, 5.1.1 and 5.3.1. Then there exist some constant $C > 0$, $\lambda_2 > 0$ and $\tau_0 > 0$ such that*

$$\int_{S^1} |\zeta(\tau,t)|^2\, dt \leq C\, e^{-\lambda_2 \tau}$$

for all $\tau \geq \tau_0$. Here the constant C depends only on the triad (Q,λ,J) and the C^1-norm of w provided in Hypothesis 4.1.2.

The rest of the section will be occupied by the proof of this theorem by estimating of the integral

$$\frac{1}{2}\int_{S^1} \Delta e^\pi\, dt.$$

For this purpose, we need to analyze the integrands Δe^π in Theorem 3.1.9 of the integral f and give explicit estimate of each term.

We recall from Theorem 3.1.9 that for any solution w

$$\begin{aligned}\frac{1}{2}\Delta e^\pi &= -|\nabla^\pi(\partial^\pi w)|^2 - 2\delta\langle *d^{\nabla^\pi}\partial^\pi w, *\partial^\pi w\rangle + 2|\delta^{\nabla^\pi}\partial^\pi w|^2 \\ &\quad -\langle K^\pi(dw, dw)\partial^\pi w, \partial^\pi w\rangle - R|\partial^\pi w|^2. \end{aligned} \tag{5.18}$$

Here the last term drops out on the punctured neighborhood where the flat cylindrical metric is equipped.

Next, we use Theorem 3.1.9 to show the exponential decay in cylinder coordinates. We recall the operator $\bar{D}$ and Lemma 4.3.1:

$$\bar{D}\zeta = \nabla^\pi_\tau\zeta + J\nabla^\pi_t\zeta.$$

Then we have

$$\begin{aligned}\delta^{\nabla^\pi}\partial^\pi w &= -\bar{D}\zeta \\ *d^{\nabla^\pi}\partial^\pi w &= J\bar{D}\zeta. \end{aligned} \tag{5.19}$$

From the fundamental equation in the cylinder coordinate in Proposition 4.1, we have the following crucial 'on-shell' formula

$$\bar{D}\zeta = \frac{1}{2}a_2(\mathcal{L}_{X_\lambda}J)\zeta - \frac{1}{2}a_1(\mathcal{L}_{X_\lambda}J)J\zeta \tag{5.20}$$

From this expression, we obtain

Lemma 5.3.3.

$$|\bar{D}\zeta| \leq C|\zeta|$$

for some constant C which is independent of ζ but depends only on the geometry of (Q, λ, J).

Under the cylindrical coordinates (τ, t), we use the following formula for $|\nabla\alpha|$, where $\alpha = \zeta d\tau + \eta dt$ is a vector valued one-form. This is a vector-valued analog to the well-known Gärding's equality

$$|\nabla\alpha|^2 = |d\alpha|^2 + |\delta\alpha|^2$$

for ordinary real valued one-forms. We extend the wedge product $\wedge$ to the E-valued one-forms by taking the inner product in the fiber direction of $\Omega^1(E)$ and the wedge product on the base Σ for vector bundle $E \to \Sigma$.

Applying lemma 3.1.6 to our case $\nabla^\pi(\partial^\pi w)$ and using $\eta = J\zeta$ for a contact Cauchy-Riemann map, we compute the first two terms in Theorem 3.1.9 as follows.

$$\begin{aligned}
& -|\nabla^\pi(\partial^\pi w)|^2 + 2|\delta^{\nabla^\pi}\partial^\pi w|^2 \\
= \; & -|d^{\nabla^\pi}\partial^\pi w|^2 - |\delta^{\nabla^\pi}\partial^\pi w|^2 + 4\langle\nabla^\pi_\tau\zeta, J\nabla^\pi_t\zeta\rangle + 2|\delta^{\nabla^\pi}\partial^\pi w|^2 \\
= \; & 4\langle\nabla^\pi_\tau\zeta, J\nabla^\pi_t\zeta\rangle \qquad (5.21) \\
= \; & 4\langle\nabla^\pi_\tau\zeta, \bar{D}\zeta - \nabla^\pi_\tau\zeta\rangle \\
= \; & -4|\nabla^\pi_\tau\zeta|^2 + 4\langle\nabla^\pi_\tau\zeta, \bar{D}\zeta\rangle. \qquad (5.22)
\end{aligned}$$

Now, we compute the third term in Theorem 3.1.9, i.e., $\delta\langle d^{\nabla^\pi}\partial^\pi w, \partial^\pi w\rangle$. We recall that for any vector valued one-form α,

$$\delta^\nabla\alpha = -*d^\nabla*\alpha = -\frac{D}{\partial\tau}\left(\alpha\left(\frac{\partial}{\partial\tau}\right)\right) - \frac{D}{\partial t}\left(\alpha\left(\frac{\partial}{\partial\tau}\right)\right). \qquad (5.23)$$

To apply this to d^{∇^π}, δ^{∇^π} and $\alpha = \langle *d^{\nabla^\pi}\partial^\pi w, *\partial^\pi w\rangle$, we compute

$$\begin{aligned}
\alpha\left(\frac{\partial}{\partial\tau}\right) &= \left\langle *d^{\nabla^\pi}\partial^\pi w, *\partial^\pi w\right\rangle\left(\frac{\partial}{\partial\tau}\right) \\
&= \left\langle *d^{\nabla^\pi}\partial^\pi w, (*\partial^\pi w)\left(\frac{\partial}{\partial\tau}\right)\right\rangle = \langle *d^{\nabla^\pi}\partial^\pi w, -J\zeta\rangle,
\end{aligned}$$

and

$$\alpha\left(\frac{\partial}{\partial t}\right) = \langle *d^{\nabla^\pi}\partial^\pi w, *\partial^\pi w\rangle\left(\frac{\partial}{\partial t}\right)$$
$$= \left\langle *d^{\nabla^\pi}\partial^\pi w, (*\partial^\pi w)\left(\frac{\partial}{\partial t}\right)\right\rangle = \langle *d^{\nabla^\pi}\partial^\pi w, \zeta\rangle.$$

Using (5.23), we get

$$\delta\langle *d^{\nabla^\pi}\partial^\pi w, *\partial^\pi w\rangle = \frac{\partial}{\partial\tau}\langle *d^{\nabla^\pi}\partial^\pi w, J\zeta\rangle - \frac{\partial}{\partial t}\langle *d^{\nabla^\pi}\partial^\pi w, \zeta\rangle.$$

The second term will vanish after we take integral over S^1 for t parameter, so we only need to take care of the first term. By using metric property of ∇^π and (5.19), it can be written as

$$\frac{\partial}{\partial\tau}\langle *d^{\nabla^\pi}\partial^\pi w, J\zeta\rangle = \frac{\partial}{\partial\tau}\langle \bar{D}\zeta, \zeta\rangle = \langle \nabla^\pi_\tau(\bar{D}\zeta), \zeta\rangle + \langle \bar{D}\zeta, \nabla^\pi_\tau\zeta\rangle.$$

Hence together with (5.21), Theorem 3.1.9 can be written as the following expression under cylindrical coordinates.

$$\frac{1}{2}\Delta e^\pi = -4|\nabla^\pi_\tau\zeta|^2 + 2\left(\langle \nabla^\pi_\tau\zeta, \bar{D}\zeta\rangle - \langle \nabla^\pi_\tau(\bar{D}\zeta), \zeta\rangle\right)$$
$$+2\frac{\partial}{\partial t}\langle *d^{\nabla^\pi}\partial^\pi w, \zeta\rangle - \langle K^\pi(dw, dw)\partial^\pi w, \partial^\pi w\rangle. \tag{5.24}$$

We calculate the second term by looking at $\langle \nabla^\pi_\tau\zeta, \bar{D}\zeta\rangle - \langle \nabla^\pi_\tau(\bar{D}\zeta), \zeta\rangle$.

Using (5.20), we compute

$$\langle \nabla^\pi_\tau\zeta, \bar{D}\zeta\rangle = \left\langle \nabla^\pi_\tau\zeta, -\frac{1}{2}a_1(\mathcal{L}_{X_\lambda}J)J\zeta + \frac{1}{2}a_2(\mathcal{L}_{X_\lambda}J)\zeta\right\rangle$$
$$= -\frac{1}{2}a_1\left\langle \nabla^\pi_\tau\zeta, (\mathcal{L}_{X_\lambda}J)J\zeta\right\rangle + \frac{1}{2}a_2\left\langle \nabla^\pi_\tau\zeta, (\mathcal{L}_{X_\lambda}J)\zeta\right\rangle$$

and

$$\begin{aligned}\langle\nabla^\pi_\tau(\bar{D}\zeta),\zeta\rangle &= \left\langle \nabla^\pi_\tau\left(\frac{1}{2}a_2(\mathcal{L}_{X_\lambda}J)\zeta - \frac{1}{2}a_1(\mathcal{L}_{X_\lambda}J)J\zeta\right),\zeta\right\rangle \\ &= -\frac{1}{2}\left\langle(\nabla^\pi_\tau(a_1\mathcal{L}_{X_\lambda}J))\,J\zeta,\zeta\right\rangle + \frac{1}{2}\left\langle(\nabla^\pi_\tau(a_2\mathcal{L}_{X_\lambda}J))\,\zeta,\zeta\right\rangle \\ &\quad -\frac{1}{2}a_1\left\langle(\mathcal{L}_{X_\lambda}J)\nabla^\pi_\tau\zeta, J\zeta\right\rangle + \frac{1}{2}a_2\left\langle(\mathcal{L}_{X_\lambda}J)\nabla^\pi_\tau\zeta,\zeta\right\rangle\end{aligned}$$

Subtracting these two terms and noting that $\mathcal{L}_{X_\lambda}J$ is symmetric, we get

$$\begin{aligned}&\langle\nabla^\pi_\tau\zeta,\bar{D}\zeta\rangle - \langle\nabla^\pi_\tau(\bar{D}\zeta),\zeta\rangle \\ =\ & \frac{1}{2}\left\langle(\nabla^\pi_\tau(a_1\mathcal{L}_{X_\lambda}J))\,J\zeta,\zeta\right\rangle - \frac{1}{2}\left\langle(\nabla^\pi_\tau(a_2\mathcal{L}_{X_\lambda}J))\,\zeta,\zeta\right\rangle. \qquad (5.25)\end{aligned}$$

Now we estimate the two terms in (5.25).

For the first term in (5.25), we look at

$$\langle(\nabla^\pi_\tau(a_2\mathcal{L}_{X_\lambda}J))\zeta,\zeta\rangle = \frac{\partial a_2}{\partial\tau}\langle(\mathcal{L}_{X_\lambda}J)\zeta,\zeta\rangle + a_2\langle(\nabla^\pi_\tau(\mathcal{L}_{X_\lambda}J))\zeta,\zeta\rangle. \qquad (5.26)$$

The second term of (5.26) is of $o(1)|\zeta|^2$ since the operator norm

$$\|\nabla^\pi_\tau(\mathcal{L}_{X_\lambda}J)\| \le \|\nabla(\mathcal{L}_{X_\lambda}J)\|_{C^0}\left|\frac{\partial w}{\partial\tau}\right| = o(1), \qquad (5.27)$$

where we denote by $o(1)$ some function approaches zero in as $\tau\to\infty$ and $\|\nabla(\mathcal{L}_{X_\lambda}J)\|_{C^0}$ the gradient bound of the vector field $\mathcal{L}_{X_\lambda}J$ on Q which is independent of w but given by the geometry of (Q,λ,J).

We note

$$\frac{\partial a_2}{\partial\tau} - \frac{\partial a_1}{\partial t} = *d(w^*\lambda) = \frac{1}{2}|\partial^\pi w|^2 = |\zeta|^2 = o(1). \qquad (5.28)$$

The first term of (5.26),

$$\begin{aligned}&\frac{\partial a_2}{\partial\tau}\left\langle(\mathcal{L}_{X_\lambda}J)\zeta,\zeta\right\rangle \\ =\ & (*dw^*\lambda)\left\langle(\mathcal{L}_{X_\lambda}J)\zeta,\zeta\right\rangle + \frac{\partial a_1}{\partial t}\left\langle(\mathcal{L}_{X_\lambda}J)\zeta,\zeta\right\rangle \\ =\ & o(1)|\zeta|^2 + \frac{\partial}{\partial t}(a_1\left\langle(\mathcal{L}_{X_\lambda}J)\zeta,\zeta\right\rangle) - a_1\frac{\partial}{\partial t}\left\langle(\mathcal{L}_{X_\lambda}J)\zeta,\zeta\right\rangle \qquad (5.29)\end{aligned}$$

Here for the last term (5.29), we have the following lemma.

Lemma 5.3.4.

$$\frac{\partial}{\partial t}\left\langle(\mathcal{L}_{X_\lambda}J)\zeta,\zeta\right\rangle = O(1)|\zeta|^2 + O(1)|\nabla^\pi_\tau\zeta|^2.$$

Here $O(1)$ denotes some bounded functions.

Proof. We compute

$$\begin{aligned}
\frac{\partial}{\partial t}\left\langle(\mathcal{L}_{X_\lambda}J)\zeta,\zeta\right\rangle &= \langle\nabla^\pi_t((\mathcal{L}_{X_\lambda}J)\zeta),\zeta\rangle + \langle(\mathcal{L}_{X_\lambda}J)\zeta,\nabla^\pi_t\zeta\rangle \\
&= \left\langle\left(\nabla^\pi_t(\mathcal{L}_{X_\lambda}J)\right)\zeta,\zeta\right\rangle + \langle(\mathcal{L}_{X_\lambda}J)\nabla^\pi_t\zeta,\zeta\rangle + \langle(\mathcal{L}_{X_\lambda}J)\zeta,\nabla^\pi_t\zeta\rangle \\
&= \langle(\nabla^\pi_t(\mathcal{L}_{X_\lambda}J))\zeta,\zeta\rangle + 2\left\langle(\mathcal{L}_{X_\lambda}J)\nabla^\pi_t\zeta,\zeta\right\rangle \\
&= \langle(\nabla^\pi_t(\mathcal{L}_{X_\lambda}J))\zeta,\zeta\rangle + 2\left\langle J(\mathcal{L}_{X_\lambda}J)\zeta,\bar{D}\zeta\right\rangle - 2\left\langle J(\mathcal{L}_{X_\lambda}J)\zeta,\nabla^\pi_\tau\zeta\right\rangle \\
&= \langle(\nabla^\pi_t(\mathcal{L}_{X_\lambda}J))\zeta,\zeta\rangle + O(1)(|\zeta|^2 + |\nabla^\pi_\tau\zeta|^2).
\end{aligned}$$

While the first term

$$\begin{aligned}
&|\langle(\nabla^\pi_t(\mathcal{L}_{X_\lambda}J))\zeta,\zeta\rangle| \\
\le\ & \|\nabla(\mathcal{L}_{X_\lambda}J)\|_{C^0}\left|\frac{\partial w}{\partial t}\right||\zeta|^2 = O(1)|\zeta|^2,
\end{aligned}$$

and we are done with the proof. □

Thus we have established that (5.29) is of the form

$$o(1)|\zeta|^2 + o(1)|\nabla^\pi_\tau\zeta|^2 + \frac{\partial}{\partial t}(a_1\langle(\mathcal{L}_{X_\lambda}J)\zeta,\zeta\rangle).$$

For the second term of (5.25), we look at $-\langle(\nabla^\pi_\tau(a_1\mathcal{L}_{X_\lambda}J))J\zeta,\zeta\rangle$. Similarly, we have

$$\begin{aligned}
&\langle(-\nabla^\pi_\tau(a_1\mathcal{L}_{X_\lambda}J))J\zeta,\zeta\rangle \\
=\ & -\frac{\partial a_1}{\partial\tau}\left\langle(\mathcal{L}_{X_\lambda}J)J\zeta,\zeta\right\rangle - a_1\left\langle(\nabla^\pi_\tau(\mathcal{L}_{X_\lambda}J))J\zeta,\zeta\right\rangle \\
=\ & \frac{\partial a_2}{\partial t}\left\langle(\mathcal{L}_{X_\lambda}J)J\zeta,\zeta\right\rangle - a_1\left\langle(\nabla^\pi_\tau(\mathcal{L}_{X_\lambda}J))J\zeta,\zeta\right\rangle \\
=\ & \frac{\partial a_2}{\partial t}\left\langle(\mathcal{L}_{X_\lambda}J)J\zeta,\zeta\right\rangle + o(1)|\zeta|^2 \qquad (5.30)
\end{aligned}$$

by the same reason as (5.27).

Remark 5.3.5. We would like to remark that this, or more specifically the second equality, is another place where we use the closedness of $w^*\lambda \circ j$, i.e., the 'divergence free' condition of $w^*\lambda$

$$\frac{\partial a_1}{\partial \tau} + \frac{\partial a_2}{\partial t} = 0$$

in an essential way to perform the integration by parts.

The first term (5.30) is dealt with similarly as (5.29).

$$\begin{aligned}
&\frac{\partial a_2}{\partial t}\langle(\mathcal{L}_{X_\lambda}J)J\zeta,\zeta\rangle \\
= \quad & \frac{\partial}{\partial t}(a_2\langle(\mathcal{L}_{X_\lambda}J)J\zeta,\zeta\rangle) - a_2\frac{\partial}{\partial t}\langle(\mathcal{L}_{X_\lambda}J)J\zeta,\zeta\rangle \\
= \quad & \frac{\partial}{\partial t}(a_2\langle(\mathcal{L}_{X_\lambda}J)J\zeta,\zeta\rangle) - T\frac{\partial}{\partial t}\langle(\mathcal{L}_{X_\lambda}J)J\zeta,\zeta\rangle - (a_2 - T)\frac{\partial}{\partial t}\langle(\mathcal{L}_{X_\lambda}J)J\zeta,\zeta\rangle \\
= \quad & \frac{\partial}{\partial t}(a_2\langle(\mathcal{L}_{X_\lambda}J)J\zeta,\zeta\rangle) - T\frac{\partial}{\partial t}\langle(\mathcal{L}_{X_\lambda}J)J\zeta,\zeta\rangle + o(1)|\zeta|^2 + o(1)|\nabla^\pi_\tau\zeta|^2.
\end{aligned}$$

The reason for the last equality is exactly the same as the way we deal with (5.29) using Lemma 5.3.4, so we omit the details.

Above all, we now get the following estimate of energy density e^π under cylinder coordinates which is important to get L^2 exponential decay of ζ.

Theorem 5.3.6. *Let w be a contact instanton map. Then*

$$\begin{aligned}
\frac{1}{2}\Delta e^\pi \quad = \quad & -(4 - o(1))|\nabla^\pi_\tau\zeta|^2 + o(1)|\zeta|^2 \\
& +2\frac{\partial}{\partial t}\left\langle *d^{\nabla^\pi}\partial^\pi w, \zeta\right\rangle - \frac{\partial}{\partial t}\left(a_1\langle(\mathcal{L}_{X_\lambda}J)\zeta,\zeta\rangle\right) \\
& -\frac{\partial}{\partial t}\left(a_2\langle(\mathcal{L}_{X_\lambda}J)J\zeta,\zeta\rangle\right) + T\frac{\partial}{\partial t}\langle(\mathcal{L}_{X_\lambda}J)J\zeta,\zeta\rangle.
\end{aligned} \tag{5.31}$$

Notice the last two lines vanish after taking integral over S^1 for t.

Now let (τ, t) be the coordinates on the given cylindrical end and recall the function f is defined as

$$f(\tau) = \frac{1}{2}\int_{S^1} e^{\pi}(\tau, t)dt, \quad e^{\pi}(\tau, t) = |\partial^{\pi} w|^2 = 2|\zeta(\tau, t)|^2.$$

Then we have

$$f''(\tau) = \frac{1}{2}\int_{S^1} \frac{\partial^2 e^{\pi}}{\partial \tau^2}(\tau, t)dt = \frac{1}{2}\int_{S^1} -\Delta e^{\pi}(\tau, t)dt.$$

Then the above calculation together with Proposition 5.1 leads to the following proposition.

Proposition 5.3. There exist some constant $\delta > 0$ and τ_0 large such that for any $\tau > \tau_0$,

$$f(\tau)'' \geq \delta f(\tau)$$

Proof. We note

$$-\nabla_{\tau}\zeta = J\nabla_t + B(\zeta) = A_{\tau}(\zeta)$$

from the fundamental equation where B is the operator given in (5.5). On the other hand, integrating (5.31) over S^1, we have

$$f''(\tau) = (4 - o(1))\int_{S^1} |\nabla_{\tau}\zeta|^2 + o(1)\int_{S^1} |\zeta|^2$$

Applying the eigenvalue estimate in Proposition 5.1, we derive

$$f''(\tau) \geq 3\int_{S^1} |\nabla_{\tau}\zeta|^2 + o(1)\int_{S^1} |\zeta|^2 \geq \left(3(\frac{2\lambda_1}{3})^2 + o(1)\right)\int_{S^1} |\zeta|^2.$$

From here we immediately derive that there exists some τ_0 some $\delta > 0$ depending only on the first eigenvalue of the linearization operator A_z of the asymptotic orbit (or on the constant λ_1 given in Proposition 5.1 such that

$$f''(\tau) \geq \delta f(\tau)$$

for all $\tau \geq \tau_0'$. This finishes the proof. □

A well-known standard maximum principle argument and the vanishing $f(\tau) \to 0$ as $\tau \to \infty$ concludes that $f(\tau)$ exponentially decays to zero. Hence we have finished the proof of Theorem 5.3.2.

5.4 Alternating bootstrapping and C^∞ exponential convergence of dw

Recall the Hermitian connection ∇^π on the Hermitian bundle $\xi \to Q$ gives a Cauchy-Riemann operator $\bar{D}$ defined by

$$\frac{1}{2}\bar{D} = \frac{\nabla^\pi + J\nabla^\pi(\cdot) \circ j}{2},$$

which we will apply ζ.

We have the elliptic estimate for the Cauchy-Riemann operator $\overline{\partial}^\pi$ on any closed regions $[l_0+1, l_1-1] \times S^1 \subset [l_0, l_1] \times S^1$

$$\begin{aligned} & \|\zeta\|_{W^{k,2}([l_0+1,l_1-1]\times S^1)} \\ \leq \quad & c_{k,l_0,l_1}\left(\|\frac{1}{2}\bar{D}\zeta\|_{W^{k-1,2}([l_0,l_1]\times S^1)} + \|\zeta\|_{W^{k-1,2}([l_0,l_1]\times S^1)}\right) \end{aligned} \tag{5.32}$$

where c_{k,l_0,l_1} is some constant depending on k, l_0 and l_1, and $k = 1, 2, \cdots$.

We write the fundamental equation (4.12) into the form of

$$\frac{1}{2}\bar{D}\zeta + S\zeta = 0,$$

where

$$\frac{1}{2}\bar{D}\zeta = \frac{1}{2}(\nabla^\pi_\tau \zeta + J\nabla^\pi_t \zeta)$$

is a Cauchy-Riemann operator, and

$$S\zeta = \left(\frac{1}{2}a_1(\mathcal{L}_{X_\lambda}J)J - \frac{1}{2}a_2(\mathcal{L}_{X_\lambda}J)\right)\zeta.$$

Then the (5.32) gives

$$\begin{aligned} & \|\zeta\|_{W^{k,2}([l_0+1,l_1-1]\times S^1)} \\ \leq\ & c_{k,l_0,l_1}\left(\|S\zeta\|_{W^{k-1,2}([l_0,l_1]\times S^1)} + \|\zeta\|_{W^{k-1,2}([l_0,l_1]\times S^1)}\right). \end{aligned} \tag{5.33}$$

Now we proceed with several steps.

Step 1; $k = 1$ and for ζ: For $k = 1$, we estimate the right hand side on the region $[0,7]\times S^1$ by

$$\|S\zeta\|_{L^2([0,7]\times S^1)} \leq \|S\|_{C^0}\|\zeta\|_{L^2([0,7]\times S^1)}. \tag{5.34}$$

where

$$\begin{aligned} \|S\|_{C^0} &\leq \frac{1}{4}(\|\mathcal{L}_{X_\lambda}J\|_{C^0} + \|J\|_{C^0}\|\mathcal{L}_{X_\lambda}J\|_{C^0})\|\nabla w\|_{C^0} \\ &= \frac{1}{2}(\|\mathcal{L}_{X_\lambda}J\|_{C^0}\|\nabla w\|_{C^0} < \infty. \end{aligned}$$

We plug (5.34) into (5.33) and take $k = 1$. Then we get

$$\|\zeta\|_{W^{1,2}([1,6]\times S^1)} \leq c_{1,0,7}(1 + \|S\|_{C^0})\|\zeta\|_{L^2([0,7]\times S^1)}.$$

Considering the translated sequence $\zeta\circ\tau$, we get

$$\|\zeta\|_{W^{1,2}([\tau+1,\tau+6]\times S^1)} \leq c_{1,0,7}(1 + \|S\|_{C^0})\|\zeta\|_{L^2([\tau,\tau+7]\times S^1)},$$

for any $\tau\in\mathbb{R}$. Therefore, by considering $\tau\geq\tau_0$ in Theorem 5.3.2, we get

$$\|\zeta\|_{W^{1,2}([\tau+1,\tau+6]\times S^1)} \leq C'e^{-\delta'\tau},$$

where $\delta' = \frac{1}{2}\delta$ and C' is a constant given by the $W^{1,2}$ bound of ∇w and some other constants from the geometry of (Q, λ, J). They are both independent of τ but only depend on $\|\nabla w\|_{C^0}$ and the contact triad (Q, λ, J). To simplify the notation, we will always use C and δ for such kind of constants in this section. Then we get the following proposition.

Proposition 5.4. There exist some constants $C > 0$ and $\tau_0 > 0$ such that for any $\tau > \tau_0$,

$$\|\zeta\|_{W^{1,2}([\tau,+\infty)\times S^1)} \leq Ce^{-\delta\tau}.$$

Step 2; for $k = 1$ and for a_1, a_2: Next, we use Proposition 5.4 to get the L^2 exponential decay of X_λ part of dw by using the relation $*dw^*\lambda = |\zeta|^2$.

We define a complex-valued function

$$\theta(\tau, t) = (a_2 - T) - \sqrt{-1}a_1$$

and notice that θ satisfies the equation

$$\overline{\partial}\theta = \mu, \quad \mu = \frac{1}{2}(*dw^*\lambda) + \sqrt{-1}\cdot 0 = \frac{1}{2}|\zeta|^2 + \sqrt{-1}\cdot 0, \tag{5.35}$$

where $\overline{\partial} = \frac{1}{2}\left(\frac{\partial}{\partial\tau} - \sqrt{-1}\frac{\partial}{\partial t}\right)$ the standard Cauchy-Riemann operator for the standard complex structure $\sqrt{-1}$.

Notice that from Proposition 5.4 we have already established the $W^{1,2}$ exponential decay of μ. This gives rise to the L^2 exponential decay of θ as stated in the following proposition.

Proposition 5.5. There exists some constants $C > 0$ and $\delta > 0$, such that

$$\int_{S^1} |\theta|^2 dt < Ce^{-\delta\tau},$$

where $|\theta|^2 = (a_2 - T)^2 + a_1^2$.

The remaining section will be occupied by the proof of this proposition.

We start with the following general lemma which will be used several times for bootstrapping in this section. Proposition 5.5 immediately follows by considering $\theta = V$ and $\mu = W$. The proof is standard and is a much easier version of the argument used in the previous sections and so omitted.

Lemma 5.4.1. *Suppose that complex-valued functions θ and μ satisfy*

$$\overline{\partial}\theta = \mu$$

and

$$\begin{aligned} \|\mu\|_{W^{1,2}(S^1)} &\leq Ce^{-\delta\tau}, \\ \lim_{\tau\to+\infty} \theta &= 0, \end{aligned}$$

then $\|\theta\|_{L^2(S^1)} \leq Ce^{-\delta\tau}$.

Remark 5.4.2. By applying the alternating bootstrapping arguments between $w^*\lambda$ and $d^\pi w$ for the higher derivatives used in section 4.3, we can inductively obtain the bound for $|\nabla^k dw|$ in terms of $|(\nabla^\pi)^l d^\pi w|$ and $|\nabla^l w^*\lambda|$ for $l \leq k-1$. Hence in section 5.5, we directly get the exponential decay of $|\nabla^k dw|$ once we get the exponential decay of $|(\nabla^\pi)^l d^\pi w|$ and $|\nabla^l w^*\lambda|$.

We now apply the standard elliptic bootstrapping and get

$$\|\theta\|_{W^{1,2}([1,6]\times S^1)} \leq c_{1,0,7}(\|\mu\|_{L^2([0,7]\times S^1)} + \|\theta\|_{L^2([0,7]\times S^1)}) \leq Ce^{-\delta\tau}.$$

Hence by using τ translation as above and we get $W^{1,2}$ exponential decay of θ.

Also, together with $W^{1,2}$ exponential decay of ζ and by using Remark 5.4.2 (2), we have now

$$\lim_{\tau\to+\infty} \|\nabla^2 w\|_{C^0} = 0 \tag{5.36}$$

$$\lim_{\tau\to+\infty} \|\nabla a_2\|_{C^0} = 0, \quad \lim_{\tau\to+\infty} \|\nabla a_1\|_{C^0} = 0 \tag{5.37}$$

Step 3; For $k = 2$ and for ζ: We go back to the elliptic estimate for ζ on the regions $[2,5] \times S^1 \subset [1,6] \times S^1$ for $k = 2$, and get

$$\|\zeta\|_{W^{2,2}([2,5]\times S^1)} \leq c_{2,1,6} \left(\|S\zeta\|_{W^{1,2}([1,6]\times S^1)} + \|\zeta\|_{W^{1,2}([1,6]\times S^1)}\right).$$

Notice $\|S\zeta\|_{W^{1,2}([1,6]\times S^1)}$ has the following estimate

$$\|S\zeta\|_{W^{1,2}([1,6]\times S^1)} \leq 3\|S\|_{C^1}\|\zeta\|_{W^{1,2}([1,6]\times S^1)},$$

where $\|S\|_{C^1}$ is bounded by the C^1 norm of ∇w and a_1, a_2 together with C^1 norm of ∇J and $\mathcal{L}_{X_\lambda} J$ on contact manifold Q. (5.36) and (5.37) guarantee it is bounded.

Hence similar elliptic bootstrapping argument as for $W^{1,2}$, we get for $W^{2,2}$ norm

$$\|\zeta\|_{W^{2,2}([\tau+2,\tau+5]\times S^1)} \leq Ce^{-\delta\tau},$$

and further

$$\|\zeta\|_{W^{2,2}([\tau,+\infty)\times S^1)} \leq Ce^{-\delta\tau}$$

when $\tau > \tau_0$, for some constants C and $\delta > 0$ which are independent of τ.

Step 4; for $k = 2$ and for ζ, a_1, a_2: From the identity $*dw^*\lambda = |\zeta|^2$, Lemma 5.4.1 already implies that μ has $W^{2,2}$ exponential decay.

Now we differentiate equation (5.35), and get

$$\bar{\partial}\theta_\tau = \mu_\tau \tag{5.38}$$

Since μ_τ has $W^{1,2}$ exponential decay and from (5.37) we get

$$\lim_{\tau\to+\infty} \theta_\tau = 0,$$

we can apply Lemma 5.4.1 and get L^2 exponential decay of θ_τ. Then using elliptic bootstrapping to (5.38), we get $W^{1,2}$ exponential decay of θ_τ.

$W^{1,2}$ exponential decay of θ_t follows because

$$\theta_t = \sqrt{-1}(\theta_\tau - 2\mu)$$

from (5.35).

In summary, we now have $W^{2,2}$ exponential decay of both θ and ζ. By using Remark 5.4.2 (2) again, it indicates

$$\begin{aligned} \lim_{\tau\to+\infty} \|\nabla^3 w\|_{C^0} &= 0 \\ \lim_{\tau\to+\infty} \|\nabla^2 a_2\|_{C^0} = 0, \quad \lim_{\tau\to+\infty} \|\nabla^2 a_1\|_{C^0} &= 0. \end{aligned}$$

Step 5: Alternating elliptic bootstrap: Now the C^3 bound of ∇w and the C^2 bound of $\nabla^2 a_1$ and $\nabla^2 a_2$ guarantee us to do the above procedure again. For ζ, we will get

$$\|\zeta\|_{W^{3,2}([\tau+3,\tau+4]\times S^1)} \leq Ce^{-\delta\tau},$$

where C and $\delta > 0$ don't depend on τ.

By Sobolev embedding, $W^{3,2}([\tau+3,\tau+4]\times S^1)$ can be compactly embedded to $C^1([\tau+3,\tau+4]\times S^1)$, thus, we obtain the C^1 estimate

$$\|\zeta\|_{C^1([\tau+3,\tau+4]\times S^1)} \leq Ce^{-\delta\tau}.$$

Hence, further we get

$$\|\zeta\|_{C^1([\tau,+\infty)\times S^1)} \;\leq\; Ce^{-\delta\tau}$$

for $\tau > \tau_0$ when τ_0 is a large number.

Similarly, for θ, we get

$$\|\theta\|_{C^1([\tau,+\infty)\times S^1)} \;\leq\; Ce^{-\delta\tau}.$$

By induction of the above bootstrapping method, we get the following theorem for this section.

Theorem 5.4.3. *Under the same hypotheses as in Theorem 5.3.2, there exist some constants $C > 0$, $\delta > 0$ and τ_0 large such that for any $\tau > \tau_0$,*

$$\left\|\pi\frac{\partial w}{\partial \tau}\right\|_{C^\infty(S^1)} \;\leq\; Ce^{-\delta\tau} \tag{5.39}$$

$$\left\|\pi\frac{\partial w}{\partial t}\right\|_{C^\infty(S^1)} \;\leq\; Ce^{-\delta\tau} \tag{5.40}$$

$$\|a_2 - T\|_{C^\infty(S^1)} \;\leq\; Ce^{-\delta\tau} \tag{5.41}$$

$$\|a_1\|_{C^\infty(S^1)} \;\leq\; Ce^{-\delta\tau}. \tag{5.42}$$

5.5 C^0 exponential convergence of w

By recalling the C^∞ exponential decay of dw from last section, we have the following weaker statement of the L^2 (over S^1) exponential decay of dw for any order $k \geq 0$ as $\tau \to \infty$.

Proposition 5.6. There exist some constants $\delta > 0$ such that for each $k \geq 0$, whenever

$\tau > \tau_0$,

$$\int_{S^1} \left|\nabla^k \left(\frac{\partial w}{\partial \tau}\right)\right|^2 dt \;\leq\; Ce^{-\delta\tau} \tag{5.43}$$

$$\int_{S^1} \left|\nabla^k \left(\frac{\partial w}{\partial t} - TX_\lambda(w)\right)\right|^2 dt \;\leq\; Ce^{-\delta\tau}. \tag{5.44}$$

and for some universal constant $C = C_k > 0$ depending on k.

We would like to note that up to now, we have not obtained much about the C^0 asymptotic convergence of $w(\tau,\cdot)$ yet other than the subsequence asymptotic convergence in general given in Theorem 4.1.3 whose limit may depend on the choice of subsequence. Even in the current nondegenerate case, the rotation angle θ of the limit z_θ may depend on the choice of subsequences.

In this section, we finally proceed with the C^0 exponential convergence of $w(\tau,\cdot)$ to a Reeb orbit $z(\cdot)$ and also, under further assumption that $w^*\lambda\circ j$ is an exact form, i.e., there exists some function a such that $w^*\lambda\circ j = da$, $a(\tau,\cdot) \to T\tau + C$ for some constant C, provided we have subsequence convergence. (We emphasize that the a here is different from the charge in previous sections and one should not get confused here.) For this, we will see that the L^2 exponential decay of dw itself, i.e., for $k = 0$ in Proposition 5.6, is enough to give the full C^0 convergence of $w(\tau,\cdot)$ as $\tau \to \infty$. To be more specific, (5.43) for $k = 0$ is enough to give the C^0 convergence of w to the nondegenerate Reeb orbit $z(\cdot)$ which is the main proposition in the first subsection. The inequality (5.44) for $k = 0$ is enough to give the C^0 convergence of a to a linear function which is the main proposition in the second subsection.

5.5.1 C^0 exponential convergence of the map w

The following is the main proposition we prove here.

Proposition 5.7. Under Hypothesis 4.1.2, 5.1.1 and 5.3.1, there exists a unique Reeb orbit z with period $T > 0$ such that

$$\|d(w(\tau,\cdot), z(\cdot))\|_{C^0(S^1)} \to 0,$$

as $\tau \to +\infty$.

Proof. We start with claiming that for each $t \in S^1$, $w(\cdot, t)$ is a Cauchy sequence.

If this claim is not true, then there exist some $t_0 \in S^1$ and some constant $\epsilon > 0$, sequences $\{\tau_k\}$, $\{p_k\}$ such that

$$d(w(\tau_{k+p_k}, t_0), w(\tau_k, t_0)) \geq \epsilon.$$

Then from the continuity of w in t, there exists some $l > 0$ small such that

$$d(w(\tau_{k+p_k}, t), w(\tau_k, t)) \geq \frac{\epsilon}{2}, \quad |t - t_0| \leq l.$$

Hence

$$\begin{aligned}
& \int_{S^1} d(w(\tau_{k+p_k}, t), w(\tau_k, t))\ dt \\
= & \int_{|t-t_0|\leq l} d(w(\tau_{k+p_k}, t), w(\tau_k, t))\ dt + \int_{|t-t_0|>l} d(w(\tau_{k+p_k}, t), w(\tau_k, t))\ dt \\
\geq & \int_{|t-t_0|\leq l} d(w(\tau_{k+p_k}, t), w(\tau_k, t))\ dt \geq \epsilon l.
\end{aligned}$$

On the other hand, we compute

$$\begin{aligned}
& \int_{S^1} d(w(\tau_{k+p_k}, t), w(\tau_k, t))\ dt \\
\leq\ & \int_{S^1} \int_{\tau_k}^{\tau_{k+p_k}} \left| \frac{\partial w}{\partial s}(s,t) \right|\ ds\ dt \\
=\ & \int_{\tau_k}^{\tau_{k+p_k}} \int_{S^1} \left| \frac{\partial w}{\partial s}(s,t) \right|\ dt\ ds \\
\leq\ & \int_{\tau_k}^{\tau_{k+p_k}} \left(\int_{S^1} \left| \frac{\partial w}{\partial s}(s,t) \right|^2\ dt \right)^{\frac{1}{2}}\ ds \\
\leq\ & \int_{\tau_k}^{\tau_{k+p_k}} Ce^{-\delta s}\ ds \\
=\ & \frac{C}{\delta}(1 - e^{-(\tau_{k+p_k} - \tau_k)})e^{-\tau_k} \leq \frac{C}{\delta} e^{-\tau_k}.
\end{aligned}$$

Hence we can take τ_k large and get contradiction.

Now by using the subsequence convergence from Theorem 4.1.3, we can pick an arbitrary subsequence $\{\tau_k\}$ and $z \in Z$ such that

$$w(\tau_k, t) \to z(t), \quad k \to \infty$$

uniformly in t. Then immediately from the fact that $w(\cdot, t)$ is a Cauchy sequence for any t, we get for any $t \in S^1$,

$$w(\tau, t) \to z(t), \quad \tau \to \infty.$$

What left to show is just this convergence is uniform in t, i.e., it is in $C^0(S^1)$ sense. Assume this is not true. Then there exist some $\epsilon > 0$ and some sequence (τ_k, t_k) such that

$$d(w(\tau_k, t_k), z(t_k)) \geq 2\epsilon.$$

Since $t_k \in S^1$, we can further take subsequence, still denote by t_k, such that $t_k \to t_0 \in S^1$. We can take k large such that $d(z(t_k), z(t_0)) \leq \frac{1}{2}\epsilon$.

We also look at

$$d(w(\tau,t_k),w(\tau,t_0)) \leq \int_{t_0}^{t_k} \left|\frac{\partial w}{\partial t}(\tau,s)\right| ds \leq (t_k - t_0)\|\nabla w\|_{C^0},$$

and so we can make it less than $\frac{1}{2}\epsilon$ by taking k large.

On the other hand, we have

$$\begin{aligned} d(w(\tau_k,t_0),z(t_0)) &\geq d(w(\tau_k,t_k),z(t_k)) - d(w(\tau_k,t_k),w(\tau_k,t_0)) \\ &\quad -d(z(t_k),z(t_0)) \\ &\geq 2\epsilon - \frac{1}{2}\epsilon - \frac{1}{2}\epsilon = \epsilon, \end{aligned}$$

which gives contradiction to the pointwise convergence.

This finishes the proof.

□

5.5.2 C^0 exponential convergence in the symplectization case

Finally we relate our general study of contact Cauchy-Riemann map to the special exact case, i.e. the case of maps (a,w) into the symplectization $\mathbb{R} \times Q$. In this case $\widetilde{w} \equiv w$. (Here we follow the notation used by Hofer in [H1] to denote the $\mathbb{R}$ component by a, although we have used a to denote the contact charge in the previous sections, which should not confuse the readers.)

In other words, we prove the C^0 convergence of a, assuming that

$$w^*\lambda \circ j = da.$$

In this case, we have

$$a_2 = \frac{\partial a}{\partial \tau}, \quad a_1 = -\frac{\partial a}{\partial t}$$

and the pair (a, w) satisfies the standard pseudoholomorphic curve equation

$$\overline{\partial}^{\pi}_{J} w = 0, \quad w^*\lambda \circ j = da.$$

Proposition 5.8. There exists some constant C_0, such that

$$\|a(\tau, \cdot) - T\tau - C_0\|_{C^0(S^1)} \to 0,$$

as $\tau \to +\infty$.

Proof. Define $b(\tau, t) = a(\tau, t) - T\tau$. Then we have

$$\begin{aligned} \frac{\partial b}{\partial \tau} = a_2 - T \quad &\to \quad 0, \\ \frac{\partial b}{\partial t} = -a_1 \quad &\to \quad 0, \end{aligned}$$

as $\tau \to +\infty$ in $C^0(S^1)$ topology.

Define $\alpha(\tau) = \int_{S^1} b(\tau, t)dt$ and $\tilde{b}(\tau) = b(\tau, t) - \alpha(\tau)$. Then

$$|\alpha'(\tau)| = \left| \int_{S^1} \frac{\partial b}{\partial \tau} dt \right| \leq \int_{S^1} |a_2 - T| \, dt \leq Ce^{-\delta\tau},$$

which indicates that $\alpha(\tau)$ is a Cauchy sequence. Then there exists some constant C_0 such that $\alpha(\tau) \to C_0$ as $\tau \to +\infty$.

On the other hand, notice here $\int_{S^1} \tilde{b}(\tau, t)dt = 0$ and then for any τ, there exists some point $t_0 \in S^1$ such that $\tilde{b}(\tau, t_0) = 0$. Then for any τ,

$$|\tilde{b}(\tau, t)| = |\tilde{b}(\tau, t) - \tilde{b}(\tau, t_0)| \leq |t - t_0| \left\| \frac{\partial \tilde{b}}{\partial t} \right\|_{C^0(S^1)} \leq \left\| \frac{\partial \tilde{b}}{\partial t} \right\|_{C^0(S^1)} = \left\| \frac{\partial b}{\partial t} \right\|_{C^0(S^1)},$$

and thus

$$\|\tilde{b}(\tau, \cdot)\|_{C^0(S^1)} \leq \left\| \frac{\partial b}{\partial t} \right\|_{C^0(S^1)} = \|a_1\|_{C^0(S^1)} \to 0$$

as $\tau \to +\infty$. Hence

$$\|a(\tau,\cdot) - T\tau - C_0\|_{C^0(S^1)} \leq \|\tilde{b}(\tau,\cdot)\|_{C^0(S^1)} + |\alpha(\tau) - C_0| \to 0,$$

as $\tau \to +\infty$. We are done with the proof. □

Now the C^0 exponential decay immediately follows from Theorem 5.4.3 and the C^0 convergence, Proposition 5.7 and Proposition 5.8.

Theorem 5.5.1. *There exist some constants $C > 0$, $\delta > 0$ and τ_0 large such that for any $\tau > \tau_0$,*

$$\begin{aligned} \|d\,(w(\tau,\cdot), z(\cdot))\,\|_{C^0(S^1)} &\leq C\, e^{-\delta\tau} \\ \|a(\tau,\cdot) - T\tau - C_0\|_{C^0(S^1)} &\leq C\, e^{-\delta\tau} \end{aligned}$$

Proof. For any $\tau < \tau_+$, similarly as in previous proof,

$$d(w(\tau,t), w(\tau_+,t)) \leq \int_\tau^{\tau_+} \left|\frac{\partial w}{\partial\tau}(s,t)\right| ds \leq \frac{C}{\delta} e^{-\delta\tau}.$$

Take $\tau_+ \to +\infty$ and using the C^0 convergence of w part, i.e., Proposition 5.7, we get

$$d(w(\tau,t), z(t)) \leq \frac{C}{\delta} e^{-\delta\tau}.$$

This proves the first inequality.

Similarly, we have

$$|(a(\tau_+,t) - T\tau\_C_0) - (a(\tau,t) - T\tau - C_0)| \leq \int_\tau^{\tau_+} |a_2(s,t) - T|\, ds \leq \frac{C}{\delta} e^{-\delta\tau},$$

where the last inequality comes from the C^0 exponential decay of $|a_2(s,\cdot) - T|$ in 5.8. By taking $\tau_+ \to +\infty$, we are done with the proof of the second inequality. □

Chapter 6

Finite energy contact instantons III - Morse-Bott case

In this chapter, we prove the exponential decay of contact instantons with the assumption of Morse-Bott submanifold given in Section 2.3.2. We also assume that the charge a vanishes.

As stated in Section 2.3.6 and the asymptotic behavior of contact instantons studied in Section 4.1, we work in the local picture $(U_E, f\,\lambda_E, J)$ with normal form together with adapted CR-structure (U_E, λ_E, J_0). By abusing of notations, we write U_E just by E in this whole chapter.

This Chapter consists of three sections. In the first section, we give the decomposition of the contact instanton equations by the normal form.

In the second section, we study the vertical part and prove the exponential decay.

In the third section, we study the horizontal part and wrap up the exponential decay proof.

6.1 Contact instanton equations under normal form

Assume $w = (u, e) : \dot{\Sigma} \to E$. We first derive the following decomposition lemma for the almost complex structure J_0. Here ∇ is the fixed symplectic connection of the symplectic vector bundle compatible with λ_E. We refer Section 2.3.3 and Section 2.3.6 for the construction.

Lemma 6.1.1.

$$\begin{aligned}
(\pi_{\lambda_E} dw)^h &= \widetilde{\pi_\theta du} - \frac{1}{2}\widetilde{\Omega}(\vec{R}, \nabla_{du} e) X_E \\
(\pi_{\lambda_E} dw)^v &= \nabla_{du} e \\
(J_0 \pi_{\lambda_E} dw)^h &= \widetilde{J_Q \pi_\theta du} \\
(J_0 \pi_{\lambda_E} dw)^v &= J_E \nabla_{du} e \\
w^* \lambda_E &= u^*\theta + \frac{1}{2}\widetilde{\Omega}(\vec{R}, \nabla_{du} e).
\end{aligned}$$

Proof. By using the decomposition

$$\begin{aligned}
\xi_E &= \widetilde{\xi}_\theta \oplus \{(-\frac{1}{2}\widetilde{\Omega}(\vec{R}, \eta^v) X_E, \eta^v) \big| \eta^v \in VTE\} \\
TE &= \mathbb{R} \cdot X_E \oplus \xi_E,
\end{aligned}$$

we calculate

$$\begin{aligned}
\pi_{\lambda_E} dw &= \pi_{\lambda_E}(dw)^h + \pi_{\lambda_E}(dw)^v \\
&= \pi_{\lambda_E}(\widetilde{du}) + \pi_{\lambda_E}(\nabla_{du} e) \\
&= \pi_{\lambda_E}(\widetilde{\pi_\theta du} + \widetilde{u^*\theta X_\theta}) + \pi_{\lambda_E}(-\frac{1}{2}\widetilde{\Omega}(\vec{R}, \nabla_{du} e) X_E + \nabla_{du} e + \frac{1}{2}\widetilde{\Omega}(\vec{R}, \nabla_{du} e) X_E) \\
&= \pi_{\lambda_E}(\widetilde{\pi_\theta du} + u^*\theta X_E) + \pi_{\lambda_E}(-\frac{1}{2}\widetilde{\Omega}(\vec{R}, \nabla_{du} e) X_E + \nabla_{du} e + \frac{1}{2}\widetilde{\Omega}(\vec{R}, \nabla_{du} e) X_E) \\
&= \widetilde{\pi_\theta du} - \frac{1}{2}\widetilde{\Omega}(\vec{R}, \nabla_{du} e) X_E + \nabla_{du} e.
\end{aligned}$$

Hence,

$$\begin{aligned}(\pi_{\lambda_E} dw)^h &= \widetilde{\pi_\theta du} - \frac{1}{2}\widetilde{\Omega}(\vec{R}, \nabla_{du} e) X_E \\ (\pi_{\lambda_E} dw)^v &= \nabla_{du} e.\end{aligned}$$

Also, we have now

$$\begin{aligned}J_0 \pi_{\lambda_E} dw &= J_0(\widetilde{\pi_\theta du} - \frac{1}{2}\widetilde{\Omega}(\vec{R}, \nabla_{du} e) X_E + \nabla_{du} e) \\ &= J_0 \widetilde{\pi_\theta du} + J_E \nabla_{du} e \\ &= \widetilde{J_Q \pi_\theta du} + J_E \nabla_{du} e.\end{aligned}$$

Hence,

$$\begin{aligned}(J_E \pi_E dw)^h &= \widetilde{J_Q \pi du} \\ (J_E \pi_E dw)^v &= J_E \nabla_{du} e.\end{aligned}$$

At last, we calculate

$$\begin{aligned}w^* \lambda_E &= \lambda_E(dw) \\ &= \lambda_E((dw)^h + (dw)^v) \\ &= \lambda_E(\widetilde{du}) + \lambda_E(\nabla_{du} e) \\ &= \lambda_E(\widetilde{\pi_\theta du} + \widetilde{u^* \theta X_\theta}) + \lambda_E(-\frac{1}{2}\widetilde{\Omega}(\vec{R}, \nabla_{du} e) X_E + \nabla_{du} e + \frac{1}{2}\widetilde{\Omega}(\vec{R}, \nabla_{du} e) X_E) \\ &= \lambda_E(u^* \theta X_E) + \lambda_E(\frac{1}{2}\widetilde{\Omega}(\vec{R}, \nabla_{du} e) X_E) \\ &= u^* \theta + \frac{1}{2}\widetilde{\Omega}(\vec{R}, \nabla_{du} e).\end{aligned}$$

□

Recall the contact instanton is a map w from the punctured Riemann surface $(\dot{\Sigma}, j)$

to the contact triad $(E, f\lambda_E, J)$ which satisfies the equations

$$\overline{\partial}^{\pi_{f\lambda_E}}_J w = 0 \tag{6.1}$$

$$d(w^*(f\lambda_E)\circ j) = 0. \tag{6.2}$$

We are going to decompose them into the vertical and horizontal components given by the normal form (E, λ_E).

By using (2.58) and (2.59), after a straightforward calculation, we rewrite (6.1) and (6.2) into

$$\overline{\partial}^{\pi_{\lambda_E}}_{J_0} w = \lambda_E(\frac{\partial w}{\partial \tau})Y + \lambda_E(\frac{\partial w}{\partial t})JY - (J - J_0)\pi_{\lambda_E}\frac{\partial w}{\partial t} \tag{6.3}$$

$$d(w^*\lambda_E\circ j) = -dg\wedge(w^*\lambda_E\circ j), \tag{6.4}$$

where Y is short for $Y_g^{\lambda_E}$ and $g := \log f$.

Then we apply Lemma 6.1.1 to the left hand sides of (6.3) and (6.4) and get the following lemma by straightforward calculations which we omit here.

Lemma 6.1.2.

$$\begin{aligned} \nabla_\tau e + J_E\nabla_t e &= \lambda_E(\frac{\partial w}{\partial \tau})Y^v + \lambda_E(\frac{\partial w}{\partial t})J_E Y^v \\ &\quad +\lambda_E(\frac{\partial w}{\partial t})\pi_V(J - J_0)Y - \pi_V(J - J_0)\pi_{\lambda_E}\frac{\partial w}{\partial t} \end{aligned} \tag{6.5}$$

$$\begin{aligned} \pi_\theta\frac{\partial u}{\partial \tau} + J_N\pi_\theta\frac{\partial u}{\partial t} &= \frac{1}{2}\widetilde{\Omega}(\vec{R}, \nabla_\tau e)X_\theta \\ &\quad +\pi_H\left(\lambda_E(\frac{\partial w}{\partial \tau})Y + \lambda_E(\frac{\partial w}{\partial t})JY - (J - J_0)\pi_{\lambda_E}\frac{\partial w}{\partial t}\right) \end{aligned} \tag{6.6}$$

$$\begin{aligned} d(u^*\theta\circ j) &= -dg\wedge(u^*\theta\circ j) - \frac{1}{2}dg\wedge\widetilde{\Omega}(\vec{R}, \nabla_{du\circ j}e) \\ &\quad -\frac{1}{2}d\widetilde{\Omega}(\vec{R}, \nabla_{du\circ j}e), \end{aligned} \tag{6.7}$$

where $\pi_V : TE \to VTE$ denotes the projection to vertical part, and $\pi_H : TE \to HTE$ denotes the projection to horizontal part.

6.2 Exponential decay of the vertical part e

6.2.1 Derivative convergence

From Lemma 6.1.1, by recalling Proposition 5.2 and Theorem 4.3.4 we have the following corollary under the setting of normal form.

Corollary 6.2.1. *Let $w = (u, e)$ be as in the above proposition. Then*

$$\lim_{\tau\to\infty} d^{\pi_{\lambda_E}} u = 0 \tag{6.8}$$

$$\lim_{\tau\to\infty} u^*\theta = a\, d\tau + T\, dt \tag{6.9}$$

$$\lim_{\tau\to\infty} \nabla_{du} e = 0, \tag{6.10}$$

and

$$\lim_{\tau\to\infty} |\nabla^k d^{\pi_{\lambda_E}} u| = 0 \tag{6.11}$$

$$\lim_{\tau\to\infty} |\nabla^k u^*\theta| = 0 \tag{6.12}$$

$$\lim_{\tau\to\infty} \nabla^k_{du} e = 0 \tag{6.13}$$

for all $k \geq 1$.

In particular, we obtain

$$\lim_{\tau\to\infty} du = (a\, d\tau + T\, dt)\, X_\theta.$$

in C^∞ topology.

Remark 6.2.2. We expect to prove the exponential decay under the normal form setting for Morse-Bott case in an alternating bootstrapping way as in Section 5.4. However, it turns out that the L^2 exponential decay estimate we conduct as follows does need higher order decay, even for nondegenerate case. It is an interesting and delicate issue to see whether we can find a better way to combine the advantage of the two methods in the future.

In the rest of this section, we prove the exponential decay of w to some Reeb orbit for the case of vanishing charge, i.e., $a = 0$.

6.2.2 The proof of exponential decay of e

We first study the exponential decay of $|e|$ in $L^2(S^1)$ sense.

Recall the metric on the fibers is defined by $\langle \cdot, \cdot \rangle = \Omega(\cdot, J_E \cdot)$, and we calculate $\frac{\partial^2}{\partial \tau^2}|e|^2$.

$$\begin{aligned}\frac{\partial^2}{\partial \tau^2} g &= 2|\nabla_\tau e|^2 + 2\langle \nabla_\tau \nabla_\tau e, e\rangle \\ &\quad +3\Omega(\nabla_\tau e, (\nabla_\tau J_E)e) + \Omega(e, (\nabla_\tau J_E)\nabla_\tau e) + \Omega(e, (\nabla_\tau \nabla_\tau J_E)e).\end{aligned}$$

Notice the second line

$$\begin{aligned}&|3\Omega(\nabla_\tau e, (\nabla_\tau J_E)e) + \Omega(e, (\nabla_\tau J_E)\nabla_\tau e) + \Omega(e, (\nabla_\tau \nabla_\tau J_E)e)| \\ =\ & o(1)|\nabla_\tau e|^2 + o(1)|e|^2\end{aligned}$$

by using Corollary 6.2.1 under the assumption $a = 0$, so we just need to look at the first line, in particular, the second term $2\langle \nabla_\tau \nabla_\tau e, e\rangle$.

For this term, we denote the right hand side of (6.5) by S, and have

$$\begin{aligned}
&\langle \nabla_\tau \nabla_\tau e, e\rangle \\
=\ & \langle \nabla_\tau(-J_E \nabla_t e + S), e\rangle \\
=\ & -\langle (\nabla_\tau J_E)\nabla_t e, e\rangle + \langle \nabla_\tau \nabla_t e, J_E e\rangle + \langle \nabla_\tau S, e\rangle \\
=\ & -\langle (\nabla_\tau J_E) J_E(-J_E \nabla_t e + S), e\rangle + \langle (\nabla_\tau J_E) J_E S, e\rangle + \langle \nabla_\tau \nabla_t e, J_E e\rangle + \langle \nabla_\tau S, e\rangle \\
=\ & -\langle (\nabla_\tau J_E) J_E \nabla_\tau e, e\rangle + \langle (\nabla_\tau J_E) J_E S, e\rangle + \langle \nabla_\tau \nabla_t e, J_E e\rangle + \langle \nabla_\tau S, e\rangle.
\end{aligned}$$

For the same reason as before, by using Corollary 6.2.1 under the assumption $a = 0$, the first term

$$|-\langle (\nabla_\tau J_E) J_E \nabla_\tau e, e\rangle| = o(1)|\nabla_\tau e|^2 + o(1)|e|^2.$$

The second term is of $o(1)|e|^2$ because $|S| = O(1)|e|$.

Hence further, we only need to look at $\langle \nabla_\tau \nabla_t e, J_E e\rangle + \langle \nabla_\tau S, e\rangle$, and have

$$\begin{aligned}
&\langle \nabla_\tau \nabla_t e, J_E e\rangle + \langle \nabla_\tau S, e\rangle \\
=\ & \langle \nabla_t \nabla_\tau e, J_E e\rangle + \langle K(\frac{\partial u}{\partial t}, \frac{\partial u}{\partial \tau})e, J_E\rangle + \langle \nabla_\tau S, e\rangle \\
=\ & \langle \nabla_t \nabla_\tau e, J_E e\rangle + \langle \nabla_\tau S, e\rangle + o(1)|e|^2 \\
=\ & \frac{\partial}{\partial t}\langle \nabla_\tau e, J_E e\rangle + d\lambda_E(\nabla_\tau e, \nabla_t e) + \langle \nabla_\tau S, e\rangle + o(1)|e|^2 \\
=\ & -\langle \nabla_\tau e, J_E \nabla_t e\rangle + \langle \nabla_\tau S, e\rangle + \frac{\partial}{\partial t}\langle \nabla_\tau e, J_E e\rangle + o(1)|e|^2 \\
=\ & \langle \nabla_\tau e, \nabla_\tau e - S\rangle + \langle \nabla_\tau S, e\rangle + \frac{\partial}{\partial t}\langle \nabla_\tau e, J_E e\rangle + o(1)|e|^2 \\
=\ & |\nabla_\tau e|^2 - \langle \nabla_\tau e, S\rangle + \langle \nabla_\tau S, e\rangle + \frac{\partial}{\partial t}\langle \nabla_\tau e, J_E e\rangle + o(1)|e|^2,
\end{aligned}$$

where K is the curvature of the symplectic connection ∇.

Thus up to now, we have obtained

$$\frac{\partial^2}{\partial \tau^2} g = (4 + o(1))|\nabla_\tau e|^2 - 2\langle \nabla_\tau e, S\rangle + 2\langle \nabla_\tau S, e\rangle + 2\frac{\partial}{\partial t}\langle \nabla_\tau e, J_E e\rangle + o(1)|e|^2,$$

and we need to look at the two terms $-2\langle \nabla_\tau e, S\rangle + 2\langle \nabla_\tau S, e\rangle$.

For it, we need to analyze the right hand side of (6.5), i.e.,

$$\begin{aligned} S &:= \lambda_E(\frac{\partial w}{\partial \tau})Y^v + \lambda_E(\frac{\partial w}{\partial t})J_E Y^v \\ &\quad +\lambda_E(\frac{\partial w}{\partial t})\pi_V(J-J_0)Y - \pi_V(J-J_0)\pi_{\lambda_E}\frac{\partial w}{\partial t} \end{aligned}$$

Since from this lemma and (2.78), we have

$$\begin{aligned} Y^v(e) &= Y^v(0) + \bar{\nabla}_e Y^v(0) + R(e)\cdot e \\ &= -J_E Hess(g)(q)(e) + R(e)\cdot e, \end{aligned}$$

where $\bar{\nabla}$ is the Hermitian connection of the linear space (E_q, Ω, J_E) as in (2.78), and $\|R(e)\| \leq C|e|$ for some constant C. Together with Corollary 6.2.1 and the assumption that $a = 0$, we can write

$$S = T\, Hess(g)(u)(e) + B,$$

where

$$\begin{aligned} B &:= -\lambda_E(\frac{\partial w}{\partial \tau})J_E Hess(g)(u)(e) + (\lambda_E(\frac{\partial w}{\partial t}) - T)Hess(g)(u)(e) \\ &\quad -\lambda_E(\frac{\partial w}{\partial \tau})J_E R\cdot e + \lambda_E(\frac{\partial w}{\partial t})R\cdot e \\ &\quad +\lambda_E(\frac{\partial w}{\partial t})\Pi_V(J-J_0)\eta - \Pi_V(J-J_0)\pi_{\lambda_E}\frac{\partial w}{\partial t}. \end{aligned}$$

We calculate

$$\begin{aligned}
& -2\langle\nabla_\tau e, S\rangle + 2\langle\nabla_\tau S, e\rangle \\
= \; & -2T\langle\nabla_\tau e, Hess(g)(u)(e)\rangle - 2\langle\nabla_\tau e, B\rangle + 2\langle\nabla_\tau(THess(g)(u)(e) + B), e\rangle \\
= \; & -2T\langle\nabla_\tau e, Hess(g)(u)(e)\rangle + 2T\langle\nabla_\tau(Hess(g)(u)(e)), e\rangle + 2\langle\nabla_\tau B, e\rangle + o(1)|e|^2 \\
= \; & -2T\langle\nabla_\tau e, Hess(g)(u)(e)\rangle + 2T\langle Hess(g)(u)(e)\nabla_\tau e, e\rangle + 2\langle\nabla_\tau B, e\rangle + o(1)|e|^2 \\
= \; & -2T\langle\nabla_\tau e, Hess(g)(u)(e)\rangle + 2T\langle\nabla_\tau e, Hess(g)(u)(e)\rangle + 2\langle\nabla_\tau B, e\rangle + o(1)|e|^2 \\
= \; & 2\langle\nabla_\tau B, e\rangle + o(1)|e|^2
\end{aligned}$$

where the first two terms give sum zero because $Hess(g)(u)$ is a symmetric operator on e.

Now our last task is to analyze $\langle\nabla_\tau B, e\rangle$ and show it is of the order $o(1)|e|^2 + o(1)|\nabla_\tau e|^2$. The first five terms in the expression of B can be dealt with similarly as (5.26), (5.30) (actually, if we apply higher order convergence as in Corollary 6.2.1, we can make the estimates easier here), and we omit the details.

For the last term, we look at

$$\begin{aligned}
& \nabla_\tau(\Pi_V(J - J_0)\pi_{\lambda_E}\frac{\partial w}{\partial t}) \\
= \; & (\nabla_\tau\Pi_V)(J - J_0)\pi_{\lambda_E}\frac{\partial w}{\partial t} + \Pi_V(\nabla_\tau(J - J_0))\pi_{\lambda_E}\frac{\partial w}{\partial t} + \Pi_V(J - J_0)\nabla_\tau(\pi_{\lambda_E}\frac{\partial w}{\partial t}).
\end{aligned}$$

The first term and the third term are of $o(1)|e|$. For the second term, we write $J - J_0 = B_J \cdot e$ by Taylor theorem similarly as before where $\|B_J\| \leq C$ for some constant C. Then we have

$$\nabla_\tau(J - J_0) = \nabla_\tau(B_J \cdot e) = (\nabla_\tau B_J) \cdot e + B_J\nabla_\tau e,$$

which indicates that inner product of the second term above with e is of $o(1)|e|^2 + o(1)|\nabla_\tau e|^2$.

Above all, we summarize the estimates into the result that

Proposition 6.1.

$$\frac{\partial^2}{\partial \tau^2} g = (4 + o(1))|\nabla_\tau e|^2 + \frac{\partial}{\partial t}(\cdots) + o(1)|e|^2, \tag{6.14}$$

where $(\cdots)$ denotes some function of t and τ.

Recall from (6.5) and the estimates we give before, we have now

$$|\nabla_\tau e| = |J_E \nabla_t e - T\, Hess(g)(u)(e)| + o(1)|e|.$$

To further relate it to the linearized operator in Corollary 2.3.12, we need the following lemma.

Lemma 6.2.3. *Denote by ϕ_θ the flow map of Reeb vector field X_θ on N. $w = (u, e)$ the contact instanton as in this section. Denote by $z_\tau(t) := \phi_\theta^t(u(\tau, 0))$, which is a Reeb orbit in N. Then the distance*

$$d\left(u(\tau, t), z_\tau(t)\right) = o(1),$$

uniformly in $t \in S^1$.

Proof. Since we have

$$d\left(u(\tau, t), z_\tau(t)\right) \leq C\, d\left(\phi_\theta^{-t} u(\tau, t), u(\tau, 0)\right)$$

for some constant C, it is enough to look at $d\left(\phi_\theta^{-t}u(\tau,t), u(\tau,0)\right)$.

$$\begin{aligned}
d\left(\phi_\theta^{-t}u(\tau,t), u(\tau,0)\right) &\leq \int_{S^1} \left|\frac{\partial}{\partial t}\phi_\theta^{-t}u(\tau,t)\right| dt \\
&= \int_{S^1} \left|d\phi_\theta^{-t}(\frac{\partial u}{\partial t}) - T\, X_\theta(\phi_\theta^{-t}u(\tau,t))\right| dt \\
&= \int_{S^1} \left|d\phi_\theta^{-t}(\pi_\theta\frac{\partial u}{\partial t}) + (\theta(\frac{\partial u}{\partial t}) - T)\, X_\theta(\phi_\theta^{-t}u(\tau,t))\right| dt \\
&\leq \int_{S^1} \left|d\phi_\theta^{-t}(\pi_\theta\frac{\partial u}{\partial t})\right| dt + \int_{S^1} \left|(\theta(\frac{\partial u}{\partial t}) - T\right| dt \\
&= o(1),
\end{aligned}$$

by Corollary 6.2.1, and we are done with the proof. □

By using this lemma and similar proof as for Proposition 5.1, we have the following estimate that for τ is large enough,

$$\|J_E\nabla_t e - T\, Hess(g)(u)(e)\|_{L^2} \geq \frac{3}{4}\delta\|e\|_{L^2},$$

for the $\delta > 0$ is given in Corollary 2.3.12. Further, (6.14) gives rise to that there exists some constant $\delta_0 > 0$ such that for τ large enough,

$$f'' \geq \delta_0\, f,$$

where $f(\tau) := \int_{S^1} g(\tau,t)\, dt = \|e\|_{L^2}^2$. With the same argument in Section 5.3, we are done with the exponential decay of the vertical part e in L^2 sense. By the standard elliptic bootstrapping to the equation (6.5), we get the exponential decay in the sense of C^∞.

Proposition 6.2. For any $k = 0, 1, \cdots$, there exists some constant $C_k > 0$ and $\delta_k > 0$

$$|\nabla^k e| < C_k e^{-\delta_k \tau}.$$

6.3 Exponential decay of the horizontal part

From the expression of (6.6) and (6.7), we get the following immediate corollary of Proposition 6.2.

Corollary 6.3.1.

$$\pi_\theta \frac{\partial u}{\partial \tau} + J_N \pi_\theta \frac{\partial u}{\partial t} = L(\tau, t), \quad |\nabla^k L| < C\, e^{-\delta\tau} \tag{6.15}$$

$$d(u^*\theta \circ j) = \beta, \quad |\nabla^k \beta| < C\, e^{-\delta\tau},, \tag{6.16}$$

Furthermore, by recalling the prequantization $N \to P$ *and denote* $u = (\alpha, v := d\pi(u))$*, where*

$$v : [0, \infty) \times S^1 \to P,$$

if we assume J *is adapted to the prequantization in the sense of Definition 2.3.13, then* v *satisfies*

$$\frac{\partial v}{\partial \tau} + J_P \frac{\partial v}{\partial t} = L^h(\tau, t) := d\pi L(\tau, t), \quad |\nabla^k L^h| < C\, e^{-\delta\tau}. \tag{6.17}$$

We have the following exponential decay result of the horizontal component whose proof is not going to be presented in this thesis, but we refer the papers [OW3] and [OW4] in preparation currently for this part as well as more general study of the Morse-Bott case.

Proposition 6.3. There exist some constants $C > 0$, $\delta > 0$ and τ_0 large such that for any $\tau > \tau_0$,

$$\left\|\pi_\theta \frac{\partial u}{\partial \tau}\right\|_{C^\infty(S^1)} \leq Ce^{-\delta\tau} \tag{6.18}$$

$$\left\|\pi_\theta \frac{\partial u}{\partial t}\right\|_{C^\infty(S^1)} \leq Ce^{-\delta\tau} \tag{6.19}$$

$$\left\|\theta(\frac{\partial u}{\partial t}) - T\right\|_{C^\infty(S^1)} \leq Ce^{-\delta\tau} \tag{6.20}$$

$$\left\|\theta(\frac{\partial u}{\partial \tau})\right\|_{C^\infty(S^1)} \leq Ce^{-\delta\tau}. \tag{6.21}$$

After we get the exponential decay for both vertical and horizontal part, the C^0 convergence follows from the same argument as in Section 5.5.

Appendix A

Proof of (2.49)

In this appendix, we give the proof of (2.49).

By the definition of P,

$$
\begin{aligned}
& -P(Y,Z)+P(Z,Y) \\
= \; & -\frac{1}{4}\big((\nabla^{LC}_{JZ}J)Y + J((\nabla^{LC}_{Z}J)Y) + 2J((\nabla^{LC}_{Y}J)Z)\big) \\
& +\frac{1}{4}\big((\nabla^{LC}_{JY}J)Z + J((\nabla^{LC}_{Y}J)Z) + 2J((\nabla^{LC}_{Z}J)Y)\big) \\
= \; & -\frac{1}{4}\big(\nabla^{LC}_{JZ}(JY) - J\nabla^{LC}_{JZ}Y + J\nabla^{LC}_{Z}(JY) - JJ\nabla^{LC}_{Z}Y + 2J\nabla^{LC}_{Y}(JZ) - 2JJ\nabla^{LC}_{Y}Z\big) \\
& +\frac{1}{4}\big(\nabla^{LC}_{JY}(JZ) - J\nabla^{LC}_{JY}Z + J\nabla^{LC}_{Y}(JZ) - JJ\nabla^{LC}_{Y}Z + 2J\nabla^{LC}_{Z}(JY) - 2JJ\nabla^{LC}_{Z}Y\big) \\
= \; & -\frac{1}{4}\big(\nabla^{LC}_{JZ}(JY) - J\nabla^{LC}_{JZ}Y + J\nabla^{LC}_{Z}(JY) + \Pi\nabla^{LC}_{Z}Y + 2J\nabla^{LC}_{Y}(JZ) + 2\Pi\nabla^{LC}_{Y}Z\big) \\
& +\frac{1}{4}\big(\nabla^{LC}_{JY}(JZ) - J\nabla^{LC}_{JY}Z + J\nabla^{LC}_{Y}(JZ) + \Pi\nabla^{LC}_{Y}Z + 2J\nabla^{LC}_{Z}(JY) + 2\Pi\nabla^{LC}_{Z}Y\big) \\
= \; & \frac{1}{4}(\nabla^{LC}_{JY}(JZ) - \nabla^{LC}_{JZ}(JY)) - \frac{1}{4}\Pi(\nabla^{LC}_{Y}Z - \nabla^{LC}_{Z}Y) - \frac{1}{4}J(\nabla^{LC}_{JY}Z - \nabla^{LC}_{Z}(JY)) \\
& -\frac{1}{4}J(\nabla^{LC}_{Y}(JZ) - \nabla^{LC}_{JZ}Y) \\
= \; & \frac{1}{4}([JY,JZ] - \Pi[Y,Z] - J[JY,Z] - J[Y,JZ]).
\end{aligned}
$$

This finishes the proof.

Appendix B

Wedge product of vector-valued forms

In this section, we briefly recall the definition of wedge product of vector bundle-valued forms and give the proof of two lemmas whose proofs are postponed.

We first recall the E-valued differential k-form α for a vector bundle $E \to M$ in general.

Definition B.0.2. Let $E \to M$ be a vector bundle. A E-valued k-form is a section of the bundle $\Lambda^k(M) \otimes E$. We denote by $\Omega^k(E)$ the set of E-valued k-forms.

We will need to consider only the cases of zero, one and two forms and so restrict our discussion to those cases from now on.

Now suppose that E carries an inner product. Let α and β be E-valued 0-from and 1-form respectively. Then we define the inner product operation

$$(\alpha, \beta) \to \langle \alpha, \beta \rangle \in \Omega^1(M)$$

to be the one characterized by the equation

$$\langle \alpha, \beta \rangle(X) = \langle \alpha, \beta(X) \rangle \tag{B.1}$$

for any vector field X on M. Here we note that both α and $\beta(X)$ are sections of E and so the inner product is well-defined.

We now specialize to the case of $M = \Sigma$ with (Σ, j) being a Riemann surface and let h be an associated Kähler metric thereof. We denote by ∇^{LC} its Levi-Civita connection. We also assume that E carries a connection ∇ preserving the inner product on E. We define the *wedge product*, denoted by $\omega_1 \wedge \omega_2$, of two E-valued one-forms ω_1 and ω_2. This is characterized by the equation

$$\omega_1 \wedge \omega_2(X, Y) = \langle \omega_1(X), \omega_2(Y)\rangle - \langle \omega_2(X), \omega_1(Y)\rangle \tag{B.2}$$

for any two vector fields X, Y on M.

We restate Lemma 3.1.5 here.

Lemma B.0.3. *Assume α is a zero form in $\Omega^0(w^*\xi)$ and β is a one-form in $\Omega^1(w^*\xi)$. $\langle \cdot, \cdot \rangle$ is the inner production on $w^*\xi$ introduced from the metric of Q. Then we have*

$$\langle d^{\nabla^\pi}\alpha, \beta\rangle - \langle \alpha, \delta^{\nabla^\pi}\beta\rangle = -\delta\langle \alpha, \beta\rangle.$$

Proof. We compute

$$\begin{aligned} -\delta\langle \alpha, \beta\rangle &= *d^{\nabla^\pi} * \langle \alpha, \beta\rangle = *d^{\nabla^\pi}\langle \alpha, *\beta\rangle \\ &= *(d^{\nabla^\pi}\alpha \wedge *\beta) + *\langle \alpha, d^{\nabla^\pi}(*\beta)\rangle \\ &= *\langle d^{\nabla^\pi}\alpha, \beta\rangle \, d\,\mathrm{vol} + \langle \alpha, *d^{\nabla^\pi}(*\beta)\rangle \\ &= \langle d^{\nabla^\pi}\alpha, \beta\rangle \, - \langle \alpha, \delta^{\nabla^\pi}\beta\rangle. \end{aligned}$$

In the third line, we also use the fact that our connection is a Riemannian connection and here one should extend the operation $\wedge$ to the vector-forms in the way that the product is taking the inner product in the fiber direction and take the wedge product on the base. $\square$

We now restate Lemma 3.1.6 here.

Lemma B.0.4. *For any connection ∇ and vector-valued one-form α,*

$$|\nabla\alpha|^2 = |d^\nabla\alpha|^2 + |\delta^\nabla\alpha|^2 - *(\nabla\alpha \wedge \nabla_{j(\cdot)}\alpha).$$

Proof. Note that the equality is a pointwise equality. Let $z \in \Sigma$ be a given point. We choose an orthonormal local frame $\{e_1, e_2\}$ of $T\Sigma$ so that $e_2(z) = j_z e_1(z)$

$$(\nabla e_1)(z) = 0 = (\nabla e_2)(z).$$

If we denote its dual frame by $\{\theta^1, \theta^2\}$, then we have

$$(\nabla\theta^1)(z) = 0 = (\nabla\theta^2)(z).$$

We express

$$\alpha = \zeta\,\theta^1 + \eta\,\theta^2$$

for some (locally defined) sections of ξ near z. Then we have

$$(\nabla\alpha)(z) = (\nabla\zeta)(z)\,\theta^1(z) + (\nabla\eta)(z)\,(\theta^2)(z).$$

Similarly we obtain

$$(\nabla_{j(\cdot)}\alpha)(z) = (\nabla_{j(\cdot)}\zeta)(z)(\theta^1)(z) + (\nabla_{j(\cdot)}\eta)(z)\theta^2(z)$$

First we compute

$$\begin{aligned} d^\nabla\alpha &= (\nabla_{e_1}\eta - \nabla_{e_2}\zeta)\theta^1 \wedge \theta^2 \\ \delta^\nabla\alpha &= \nabla_{e_1}\zeta + \nabla_{e_2}\eta. \end{aligned}$$

Therefore we obtain

$$\begin{aligned} |d^\nabla\alpha|^2 + |\delta^\nabla\alpha|^2 &= |\nabla_{e_1}\eta|^2 + |\nabla_{e_2}\zeta|^2 + |\nabla_{e_1}\zeta|^2 + |\nabla_{e_2}\eta|^2 \\ &\quad +2(\langle\nabla_{e_1}\zeta, \nabla_{e_2}\eta\rangle - \langle\nabla_{e_1}\eta, \nabla_{e_2}\zeta\rangle) \\ &= |\nabla\alpha|^2 + 2(\langle\nabla_{e_1}\zeta, \nabla_{e_2}\eta\rangle - \langle\nabla_{e_1}\eta, \nabla_{e_2}\zeta\rangle). \end{aligned}$$

It remains to prove

$$*(\nabla\alpha \wedge \nabla_{j(\cdot)}\alpha) = 2(\langle \nabla_{e_1}\zeta, \nabla_{e_2}\eta\rangle - \langle \nabla_{e_1}\eta, \nabla_{e_2}\zeta\rangle). \tag{B.3}$$

Taking the wedge product, we get the equality

$$\nabla\alpha \wedge \nabla_{j(\cdot)}\alpha = \langle \nabla\zeta, \nabla_{j(\cdot)}\eta\rangle\theta^1 \wedge \theta^2 - \langle \nabla_{j(\cdot)}\zeta, \nabla\eta\rangle\theta^1 \wedge \theta^2$$

at z. Therefore

$$*(\nabla\alpha \wedge \nabla_{j(\cdot)}\alpha) = \langle \nabla\zeta, \nabla_{j(\cdot)}\eta\rangle - \langle \nabla_{j(\cdot)}\zeta, \nabla\eta\rangle$$

at z. But we evaluate

$$\langle \nabla\zeta, \nabla_{j(\cdot)}\eta\rangle = \langle \nabla_{e_1}\zeta, \nabla_{e_2}\eta\rangle - \langle \nabla_{e_2}\zeta, \nabla_{e_1}\eta\rangle$$

at z. Similarly we obtain the equality

$$\langle \nabla_{j(\cdot)}\zeta, \nabla\eta\rangle = \langle \nabla_{e_2}\zeta, \nabla_{e_1}\eta\rangle - \langle \nabla_{e_1}\zeta, \nabla_{e_2}\eta\rangle$$

at z. Subtracting the latter from the first, we obtain

$$*(\nabla\alpha \wedge \nabla_{j(\cdot)}\alpha) = 2\langle \nabla_{e_1}\zeta, \nabla_{e_2}\eta\rangle - 2\langle \nabla_{e_2}\zeta, \nabla_{e_1}\eta\rangle$$

at z. Since the right hand side does not depend on

Since z is arbitrary, we have finished the proof of (B.3). This finishes the proof. □

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