TRIDIAGONAL PAIRS OF KRAWTCHOUK TYPE AND THEIR COMPATIBLE ELEMENTS

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Abstract

It is known that the Onsager algebra $\mathcal{O}$ can be embedded as a Lie subalgebra in the $\mathfrak{sl}_2$ loop algebra $L(\mathfrak{sl}_2)$. We give an attractive presentation of $L(\mathfrak{sl}_2)$ by generators and relations. There are three generators $\mathcal{A}, \mathcal{B}, \mathcal{H}$ and as we will see, $\mathcal{O}$ can be identified with the Lie subalgebra of $L(\mathfrak{sl}_2)$ generated by $\mathcal{A}, \mathcal{B}$. Let V denote a finite-dimensional irreducible $\mathcal{O}$-module of type $(0,0)$. It is known that the $\mathcal{O}$-action on V extends to an $L(\mathfrak{sl}_2)$-action on V. We classify the $L(\mathfrak{sl}_2)$-actions on V that extend the $\mathcal{O}$-action on V. We show that these $L(\mathfrak{sl}_2)$-actions have a certain geometric significance, which is best described using the theory of tridiagonal pairs. It is known that the $\mathcal{O}$-generators $\mathcal{A}, \mathcal{B}$ act on V as a tridiagonal pair of Krawtchouk type. A linear transformation $H : V \to V$ is said to be compatible with this tridiagonal pair whenever there exists an $L(\mathfrak{sl}_2)$-action on V that extends the $\mathcal{O}$-action on V, such that $\mathcal{H} = H$ on V. We describe the compatible elements in detail. For instance, we show that they are diagonalizable, they mutually commute, and their common eigenspaces all have dimension 1. We define an undirected graph whose vertex set consists of the common eigenspaces for the compatible elements. We describe the actions of $\mathcal{A}, \mathcal{B}$ on V in terms of this graph structure.

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Contents

Chapter 1

Introduction

We will be discussing the following related things: (i) a linear algebraic object called a tridiagonal pair of Krawtchouk type; (ii) a Lie algebra $\mathcal{O}$ called the Onsager algebra; (iii) a Lie algebra $L(\mathfrak{sl}_2)$ called the $\mathfrak{sl}_2$ loop algebra. The tridiagonal pairs were introduced in [9]. The Onsager algebra $\mathcal{O}$ was introduced in [12]. The finite-dimensional irreducible $\mathcal{O}$-modules were classified in [3]. See also [4], [5]. We will be considering a class of finite-dimensional irreducible $\mathcal{O}$-modules said to have type $(0, 0)$. In [7], B. Hartwig showed that this kind of $\mathcal{O}$-module is essentially the same thing as a tridiagonal pair of Krawtchouk type. The finite-dimensional irreducible $L(\mathfrak{sl}_2)$-modules were classified in [2].

By [3, p. 3277], $\mathcal{O}$ can be embedded as a Lie subalgebra in $L(\mathfrak{sl}_2)$. We give an attractive presentation of $L(\mathfrak{sl}_2)$ by generators and relations. There are three generators $\mathcal{A}, \mathcal{B}, \mathcal{H}$ and as we will see, $\mathcal{O}$ can be identified with the Lie subalgebra of $L(\mathfrak{sl}_2)$ generated by $\mathcal{A}, \mathcal{B}$. Let V denote a finite-dimensional irreducible $\mathcal{O}$-module of type $(0, 0)$. By [3, Theorem 6], the $\mathcal{O}$-action on V extends to an $L(\mathfrak{sl}_2)$-action on V. Moreover, [3, Proposition 5] indicates that this $L(\mathfrak{sl}_2)$-action on V is not unique in general. In this thesis, we classify the $L(\mathfrak{sl}_2)$-actions on V that extend the $\mathcal{O}$-action on V. We show that these $L(\mathfrak{sl}_2)$-actions have a certain geometric significance, which is best described using the theory of tridiagonal pairs. We will summarize our main results later in this introduction. To prepare for this, we review some preliminaries.

Let $\mathbb{F}$ denote an algebraically closed field with characteristic 0. We now recall the definition of $\mathfrak{sl}_2$ and its loop algebra $L(\mathfrak{sl}_2)$. Let $\mathfrak{sl}_2$ denote the Lie algebra over $\mathbb{F}$ with basis e, f, h and Lie bracket

$$[e, f] = h, \qquad [h, e] = 2e, \qquad [h, f] = -2f.$$

Let t denote an indeterminate, and let $\mathbb{F}[t, t^{-1}]$ denote the associative $\mathbb{F}$-algebra consisting of the Laurent polynomials in t that have all coefficients in $\mathbb{F}$. Let $L(\mathfrak{sl}_2)$ denote the Lie algebra over $\mathbb{F}$ consisting of the $\mathbb{F}$-vector space

$$\mathfrak{sl}_2 \otimes \mathbb{F}[t, t^{-1}], \qquad \otimes = \otimes_{\mathbb{F}}$$

and Lie bracket

$$[u \otimes a, v \otimes b] = [u, v] \otimes ab, \qquad u, v \in \mathfrak{sl}_2, \quad a, b \in \mathbb{F}[t, t^{-1}].$$

We call $L(\mathfrak{sl}_2)$ the $\mathfrak{sl}_2$ loop algebra.

We now recall the Onsager algebra $\mathcal{O}$. This infinite dimensional Lie algebra was introduced in [12]. By [13], $\mathcal{O}$ has a presentation by generators $\mathcal{A}, \mathcal{B}$ subject to the Dolan-Grady relations

$$[\mathcal{A}, [\mathcal{A}, [\mathcal{A}, \mathcal{B}]]] = 4[\mathcal{A}, \mathcal{B}], \qquad [\mathcal{B}, [\mathcal{B}, [\mathcal{B}, \mathcal{A}]]] = 4[\mathcal{B}, \mathcal{A}].$$

Recall the integers $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$. By [12], the $\mathbb{F}$-vector space $\mathcal{O}$ has a basis

$\{A_i, G_j \mid i, j \in \mathbb{Z}, j > 0\}$ such that

$$[A_k, A_l] = 2G_{k-l} \qquad k > l,$$

$$[G_k, A_l] = A_{l+k} - A_{l-k},$$

$$[G_k, G_l] = 0.$$

Moreover $A_0 = \mathcal{A}$ and $A_1 = \mathcal{B}$. Note that the elements $\{G_j \mid j \in \mathbb{Z}, j > 0\}$ form a basis for an abelian Lie subalgebra of $\mathcal{O}$.

By [3, p. 3277], there exists an injective homomorphism of Lie algebras $\mathcal{O} \to L(\mathfrak{sl}_2)$ that sends

$$\mathcal{A} \mapsto e \otimes 1 + f \otimes 1, \qquad \mathcal{B} \mapsto e \otimes t + f \otimes t^{-1}.$$

This homomorphism sends

$$A_i \mapsto e \otimes t^i + f \otimes t^{-i} \qquad i \in \mathbb{Z},$$

$$G_j \mapsto h \otimes (t^j - t^{-j})/2 \qquad j \in \mathbb{Z}, \quad j > 0.$$

For notational convenience, we identify $\mathcal{O}$ with its image in $L(\mathfrak{sl}_2)$ under the above injection. This embedding suggests that there exists an attractive presentation of $L(\mathfrak{sl}_2)$ that has $\mathcal{A}, \mathcal{B}$ among the generators. In this thesis, one of our results is that $L(\mathfrak{sl}_2)$ has a presentation by generators $\mathcal{A}, \mathcal{B}, \mathcal{H}$ and relations

$$[\mathcal{A}, [\mathcal{A}, \mathcal{H}]] = 4\mathcal{H}, \qquad [\mathcal{H}, [\mathcal{H}, \mathcal{A}]] = 4\mathcal{A}, \tag{1.1}$$

$$[\mathcal{B}, [\mathcal{B}, \mathcal{H}]] = 4\mathcal{H}, \qquad [\mathcal{H}, [\mathcal{H}, \mathcal{B}]] = 4\mathcal{B}, \tag{1.2}$$

$$[\mathcal{A}, [\mathcal{A}, [\mathcal{A}, \mathcal{B}]]] = 4[\mathcal{A}, \mathcal{B}], \qquad [\mathcal{B}, [\mathcal{B}, [\mathcal{B}, \mathcal{A}]]] = 4[\mathcal{B}, \mathcal{A}], \tag{1.3}$$

$$[\mathcal{H}, [\mathcal{A}, \mathcal{B}]] = 0. \tag{1.4}$$

The element $\mathcal{H}$ is $h \otimes 1$. As we will see, the elements $\mathcal{H}, \mathcal{A}$ generate a Lie subalgebra of $L(\mathfrak{sl}_2)$ that is isomorphic to $\mathfrak{sl}_2$. Similarly the elements $\mathcal{H}, \mathcal{B}$ generate a Lie subalgebra of $L(\mathfrak{sl}_2)$ that is isomorphic to $\mathfrak{sl}_2$.

Let V denote a finite-dimensional irreducible $\mathcal{O}$-module. By [7, Theorem 2.4], the $\mathcal{O}$-generators $\mathcal{A}, \mathcal{B}$ are diagonalizable on V. Furthermore there exist an integer $d \geq 0$ and scalars $\alpha, \beta \in \mathbb{F}$ such that the set of distinct eigenvalues of $\mathcal{A}$ (resp. $\mathcal{B}$) on V is $\{d - 2i + \alpha | 0 \leq i \leq d\}$ (resp. $\{d - 2i + \beta | 0 \leq i \leq d\}$) [7, Theorem 2.4]. We call the ordered pair (α, β) the type of V. Subtracting α (resp. β) times the identity from $\mathcal{A}$ (resp. $\mathcal{B}$) the type becomes $(0, 0)$.

Let V denote a finite-dimensional irreducible $L(\mathfrak{sl}_2)$-module. We restrict the $L(\mathfrak{sl}_2)$-action on V to $\mathcal{O}$ to get an $\mathcal{O}$-action on V. In [3], E. Date and S. S. Roan give necessary and sufficient conditions for the $\mathcal{O}$-module V to be irreducible. In this case, the $\mathcal{O}$-module V has type $(0, 0)$.

Let V denote a finite-dimensional irreducible $\mathcal{O}$-module of type $(0, 0)$. By [3, Theorem 6], the $\mathcal{O}$-action on V extends to an $L(\mathfrak{sl}_2)$-action on V. Moreover, [3, Proposition 5] indicates that this $L(\mathfrak{sl}_2)$-action on V is not unique in general. In this thesis, we classify the $L(\mathfrak{sl}_2)$-actions on V that extend the $\mathcal{O}$-action on V. By construction, the resulting $L(\mathfrak{sl}_2)$-module structures on V are irreducible. As we will see, these $L(\mathfrak{sl}_2)$-module structures on V are mutually non-isomorphic. We explain how these $L(\mathfrak{sl}_2)$-actions on V are related to one another. In this explanation we make use of the presentation (1.1)–(1.4) above.

We now recall the notion of a tridiagonal pair. Let V denote a vector space over $\mathbb{F}$ with finite positive dimension. Let $\mathrm{End}(V)$ denote the $\mathbb{F}$-algebra of all linear transformations from V to V. By a tridiagonal pair on V we mean an ordered pair A, B of linear transformations in $\mathrm{End}(V)$ such that (i) each of A, B is diagonalizable; (ii) there exists an ordering $\{V_i\}_{i=0}^d$ of the eigenspaces of A such that $BV_i \subseteq V_{i-1} + V_i + V_{i+1}$ for $0 \leq i \leq d$, where $V_{-1} = 0$ and $V_{d+1} = 0$; (iii) there exists an ordering $\{V_i'\}_{i=0}^{\delta}$ of the eigenspaces of B such that $AV_i' \subseteq V_{i-1}' + V_i' + V_{i+1}'$ for $0 \leq i \leq \delta$, where $V_{-1}' = 0$ and $V_{\delta+1}' = 0$; (iv) there does not exist a subspace W of V such that $AW \subseteq W$, $BW \subseteq W$, $W \neq 0$, $W \neq V$ [9].

Let A, B denote a tridiagonal pair on V. It is known that d and δ above are equal [9, Lemma 4.5]; we call this common value the diameter of A, B. An ordering of the eigenspaces of A (resp. B) will be called standard whenever it satisfies condition (ii) (resp. (iii)) above. We comment on the uniqueness of the standard ordering. Let $\{V_i\}_{i=0}^d$ denote a standard ordering of the eigenspaces of A. Then the ordering $\{V_{d-i}\}_{i=0}^d$ is standard and no other ordering is standard. A similar result holds for the eigenspaces of B. An ordering of the eigenvalues of A (resp. B) will be called standard whenever the corresponding ordering of the eigenspaces of A (resp. B) is standard. Let $\{V_i\}_{i=0}^d$ (resp. $\{V_i'\}_{i=0}^d$) denote a standard ordering of the eigenspaces of A (resp. B). For $0 \leq i \leq d$ the subspaces V_i, V_i' have the same dimension [9, Corollary 5.7]; we denote this common dimension by ρ_i. The sequence $\{\rho_i\}_{i=0}^d$ is symmetric and unimodal; that is $\rho_i = \rho_{d-i}$ for $0 \leq i \leq d$ and $\rho_{i-1} \leq \rho_i$ for $1 \leq i \leq d/2$ [9, Corollaries 5.7, 6.6]. By [11, Corollary 1.4] and since $\mathbb{F}$ is algebraically closed, $\rho_i \leq \binom{d}{i}$ for $0 \leq i \leq d$. In particular $\rho_0 = 1$. We call the sequence $\{\rho_i\}_{i=0}^d$ the shape of A, B. The tridiagonal pair A, B is called a Leonard pair whenever $\rho_i = 1$ for $0 \le i \le d$. The tridiagonal pair A, B is said to have Krawtchouk type whenever $\{d - 2i \,|\, 0 \le i \le d\}$ is a standard ordering of the eigenvalues of A and B. In this case A, B satisfy the Dolan-Grady relations [7, Corollary 2.7]. This suggests that tridiagonal pairs of Krawtchouk type are related to $\mathcal{O}$-modules. This relationship was worked out in detail by B. Hartwig [7]. We now summarize his results.

Theorem 1.1 [7, Corollary 2.7] *Let A, B denote a tridiagonal pair on V of Krawtchouk type. Then there exists a unique $\mathcal{O}$-module structure on V such that the generators $\mathcal{A}, \mathcal{B}$ act on V as A, B respectively. This $\mathcal{O}$-module is irreducible and of type* $(0, 0)$.

Theorem 1.2 [7, Corollary 2.7] *Let V denote a finite-dimensional irreducible $\mathcal{O}$-module of type* $(0, 0)$. *Then the generators $\mathcal{A}, \mathcal{B}$ act on V as a tridiagonal pair of Krawtchouk type.*

Remark 1.3 [7, Corollary 2.7] Combining the previous two theorems we obtain a bijection between the following two sets:

(i) the isomorphism classes of tridiagonal pairs over $\mathbb{F}$ that have Krawtchouk type;

(ii) the isomorphism classes of finite-dimensional irreducible $\mathcal{O}$-modules of type $(0, 0)$.

For the remainder of this section A, B will denote a tridiagonal pair on V that has Krawtchouk type. An element $H \in \mathrm{End}(V)$ is said to be compatible with A, B whenever the following relations hold:

$$[A, [A, H]] = 4H, \qquad [H, [H, A]] = 4A, \tag{1.5}$$

$$[B, [B, H]] = 4H, \qquad [H, [H, B]] = 4B, \tag{1.6}$$

$$[H, [A, B]] = 0. \tag{1.7}$$

Let $\mathrm{Com}(A, B)$ denote the set of elements in $\mathrm{End}(V)$ that are compatible with A, B.

For our tridiagonal pair A, B, consider the associated $\mathcal{O}$-module structure on V from Theorem 1.1. Comparing the relations in (1.1)–(1.4) and (1.5)–(1.7), we obtain the following results.

Lemma 1.4 *Consider an $L(\mathfrak{sl}_2)$-action on V that extends the $\mathcal{O}$-action on V. For the $L(\mathfrak{sl}_2)$-module V, the action of $\mathcal{H}$ on V is an element of* $\mathrm{Com}(A, B)$.

Lemma 1.5 *Let $H \in \mathrm{Com}(A, B)$. Then there exists a unique $L(\mathfrak{sl}_2)$-action on V that extends the $\mathcal{O}$-action on V, such that the element $\mathcal{H}$ of $L(\mathfrak{sl}_2)$ acts on V as H.*

Remark 1.6 Combining the previous two lemmas we obtain a bijection between the following two sets:

(i) $\mathrm{Com}(A, B)$;

(ii) the $L(\mathfrak{sl}_2)$-actions on V that extend the $\mathcal{O}$-action on V.

By Remark 1.6, in order to describe the $L(\mathfrak{sl}_2)$-actions on V that extend the given $\mathcal{O}$-action on V, it suffices to describe the set $\mathrm{Com}(A, B)$. We do this as follows. Let d denote the diameter of A, B. Recall the shape $\{\rho_i\}_{i=0}^{d}$ of A, B. Abbreviate $\rho = \rho_1$. We show that there exist elements $\{\mathcal{H}_i\}_{i=1}^{\rho}$ in $\mathrm{End}(V)$ such that

$$\mathrm{Com}(A, B) = \left\{ \quad \sum_{i=1}^{\rho} \varepsilon_i \mathcal{H}_i \quad \middle| \quad \varepsilon_i = \pm 1, \quad 1 \le i \le \rho \quad \right\}.$$

The elements $\{\mathcal{H}_i\}_{i=1}^{\rho}$ are uniquely determined up to sign and permutation. These elements are linearly independent, they mutually commute, and they are diagonalizable on V. Therefore the set $\mathrm{Com}(A,B)$ has cardinality 2^ρ. Moreover, the elements of $\mathrm{Com}(A,B)$ mutually commute and are diagonalizable on V. For $1 \leq i \leq \rho$ there exists an integer $d_i \geq 1$ such that the set of distinct eigenvalues of $\mathcal{H}_i$ on V is $\{d_i - 2k \mid 0 \leq k \leq d_i\}$. Let λ denote an indeterminate. We show that the sequences $\{\rho_i\}_{i=0}^d$ and $\{d_j\}_{j=1}^\rho$ determine each other via the polynomial identity

$$\sum_{i=0}^{d} \rho_i \lambda^i = \prod_{j=1}^{\rho} (1 + \lambda + \lambda^2 + \cdots + \lambda^{d_j}).$$

From this identity we see that $d = \sum_{j=1}^{\rho} d_j$. For the moment, fix an integer i $(1 \leq i \leq \rho)$. For $0 \leq k \leq d_i$ let W_k denote the eigenspace of $\mathcal{H}_i$ corresponding to eigenvalue $d_i - 2k$. We show that

$$AW_k \subseteq W_{k-1} + W_k + W_{k+1}, \qquad BW_k \subseteq W_{k-1} + W_k + W_{k+1},$$

where $W_{-1} = 0$ and $W_{d_i+1} = 0$.

Let $\mathbb{X}$ denote the set of common eigenspaces for the elements of $\mathrm{Com}(A,B)$. We show that the elements of $\mathbb{X}$ all have dimension 1. We now define an undirected graph structure on the set $\mathbb{X}$. For $1 \leq i \leq \rho$, elements $x, y \in \mathbb{X}$ are said to be i-adjacent whenever the following two conditions hold: (i) the eigenvalues of $\mathcal{H}_i$ corresponding to x and y differ by 2; (ii) for $1 \leq j \leq \rho$ such that $j \neq i$, the eigenvalues of $\mathcal{H}_j$ corresponding to x and y are equal. The elements $x, y \in \mathbb{X}$ are said to be adjacent whenever there exists $1 \leq i \leq \rho$ such that x and y are i-adjacent. The set $\mathbb{X}$ together with this adjacency relation is an undirected graph. This graph is a Cartesian product of ρ many chains, where the i^{th} chain has length d_i for $1 \leq i \leq \rho$. The graph $\mathbb{X}$ has the following property: for all $x \in \mathbb{X}$, Ax and Bx are contained in the sum of those elements of $\mathbb{X}$ that are adjacent to x.

An element $x \in \mathbb{X}$ will be called a corner whenever for $1 \leq i \leq \rho$, the eigenvalue of $\mathcal{H}_i$ on x is d_i or $-d_i$. Let Corner($\mathbb{X}$) denote the set of corners of $\mathbb{X}$. The cardinality of Corner($\mathbb{X}$) is 2^ρ.

Pick $H \in \text{Com}(A, B)$. We now describe H. The eigenvalues of H are $\{d-2i \,|\, 0 \leq i \leq d\}$. For $0 \leq i \leq d$ let U_i denote the eigenspace of H corresponding to the eigenvalue $d-2i$. The subspace U_i has dimension ρ_i. The subspace U_0 is a corner of $\mathbb{X}$. For $0 \leq i \leq d$, U_i is the sum of the elements in $\mathbb{X}$ at (path-length) distance i from U_0. We show that

$$AU_i \subseteq U_{i-1} + U_{i+1}, \qquad BU_i \subseteq U_{i-1} + U_{i+1},$$

where $U_{-1} = 0$ and $U_{d+1} = 0$.

We obtain a bijection Corner($\mathbb{X}$) $\to$ Com(A, B), $x \mapsto H_x$. For $x \in$ Corner($\mathbb{X}$), H_x is the unique element of Com(A, B) that has eigenspace x for the eigenvalue d.

We have been discussing the eigenvalues of the elements of Com(A, B). We pick a nonzero vector from each element of $\mathbb{X}$ to get an attractive basis for V. By construction, this basis consists of common eigenvectors for Com(A, B). We find the matrices that represent A, B with respect to this basis.

Let $\mathcal{C}$ denote the subspace of End(V) spanned by Com(A, B). The elements $\{\mathcal{H}_i\}_{i=1}^{\rho}$ form a basis for $\mathcal{C}$. We now describe the action of $\mathcal{C}$ on the eigenspaces of A and B. Let $\{V_i\}_{i=0}^d$ (resp. $\{V'_i\}_{i=0}^d$) denote a standard ordering of the eigenspaces of A (resp. B).

We show that for $C \in \mathcal{C}$,

$$CV_i \subseteq V_{i-1} + V_{i+1}, \qquad CV_i' \subseteq V_{i-1}' + V_{i+1}',$$

where $V_i = 0$ and $V_i' = 0$ for $i \in \{-1, d+1\}$.

Recall the elements $\{G_j \mid j \in \mathbb{Z}, j > 0\}$ of $\mathcal{O}$. We show that the actions of $\{G_i\}_{i=1}^{\rho}$ on V form a basis for $\mathcal{C}$. We display the transition matrix from the basis $\{\mathcal{H}_i\}_{i=1}^{\rho}$ to the basis $\{G_i\}_{i=1}^{\rho}$.

Pick $H \in \mathrm{Com}(A, B)$. As we saw in Lemma 2.94, there exists a unique $L(\mathfrak{sl}_2)$-module structure on V such that the $L(\mathfrak{sl}_2)$-generators $\mathcal{A}, \mathcal{B}, \mathcal{H}$ act on V as A, B, H respectively. Recall that the elements $\mathcal{H}, \mathcal{A}$ generate a Lie subalgebra of $L(\mathfrak{sl}_2)$ that is isomorphic to $\mathfrak{sl}_2$, and the elements $\mathcal{H}, \mathcal{B}$ generate a Lie subalgebra of $L(\mathfrak{sl}_2)$ that is isomorphic to $\mathfrak{sl}_2$. Restricting the $L(\mathfrak{sl}_2)$-action on V to either of these two Lie subalgebras, V becomes an $\mathfrak{sl}_2$-module. As we will see, the resulting two $\mathfrak{sl}_2$-module structures on V are isomorphic. Moreover, the isomorphism class of the $\mathfrak{sl}_2$-module V is independent of the choice of $H \in \mathrm{Com}(A, B)$. The $\mathfrak{sl}_2$-module V is a direct sum of irreducible $\mathfrak{sl}_2$-submodules. We now describe the summands. By [8, p. 31], up to isomorphism, there exists a unique irreducible $\mathfrak{sl}_2$-module of every finite positive dimension. We show that every irreducible $\mathfrak{sl}_2$-submodule of the $\mathfrak{sl}_2$-module V has dimension among $d+1, d-1, d-3, \ldots$. Moreover, for $0 \leq j \leq d/2$, the multiplicity with which the irreducible $\mathfrak{sl}_2$-module of dimension $d-2j+1$ appears in V is $\rho_j - \rho_{j-1}$, where $\rho_{-1} = 0$. We will show that on each irreducible $\mathfrak{sl}_2$-submodule of V, the pair H, A and the pair H, B act as Leonard pairs of Krawtchouk type.

We have been discussing tridiagonal pairs of Krawtchouk type. We now mention a special case in which the elements of $\mathrm{Com}(A, B)$ have an attractive interpretation. Assume $\rho_i = \binom{d}{i}$ for $0 \leq i \leq d$. In this case, $d = \rho$. Also $d_i = 1$ for $1 \leq i \leq \rho$, and the graph $\mathbb{X}$ is a d-cube. Moreover, $\mathrm{Corner}(\mathbb{X}) = \mathbb{X}$. So our earlier bijection $\mathrm{Corner}(\mathbb{X}) \to \mathrm{Com}(A, B)$ becomes a bijection $\mathbb{X} \to \mathrm{Com}(A, B)$. We remark that for every $x \in \mathbb{X}$, H_x is the dual adjacency map with respect to x in the sense of J. T. Go [6].

In Chapter 2, we prove our results concerning compatible elements. In [2], Chari classified up to isomorphism the finite-dimensional irreducible modules for the $\mathfrak{sl}_2$ loop algebra. In Chapter 3, we give an elementary version of this classification. Chapter 3 is meant for graduate students and researchers who are unfamiliar with the general representation theory of loop algebras.

Chapter 2

Tridiagonal pairs of Krawtchouk type and their compatible elements

2.1 Assumptions and preliminaries

In this section we collect some definitions and notation that will be used throughout the chapter. Recall the natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$ and the integers $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$. Let $\mathbb{F}$ denote an algebraically closed field with characteristic 0. Let V denote a vector space over $\mathbb{F}$ with finite positive dimension. Let $\mathrm{End}(V)$ denote the $\mathbb{F}$-algebra of all linear transformations from V to V. Let I denote the identity element of $\mathrm{End}(V)$. For $F \in \mathrm{End}(V)$ and $\theta \in \mathbb{F}$, define

$$V_F(\theta) = \{v \in V | Fv = \theta v\}. \tag{2.1}$$

We say that θ is an *eigenvalue* for F whenever $V_F(\theta) \neq 0$, and in this case $V_F(\theta)$ is called the *eigenspace* of F corresponding to θ. We say that F is *diagonalizable* whenever V is spanned by the eigenspaces of F.

We now turn our attention to Lie algebras. For basic definitions and facts about Lie algebras, we refer the reader to the books [1, 8]. The $\mathbb{F}$-vector space $\mathrm{End}(V)$ becomes a

Lie algebra over $\mathbb{F}$ with Lie bracket

$$[F, G] = FG - GF, \qquad F, G \in \text{End}(V).$$

This Lie algebra is often denoted by $\mathfrak{gl}(V)$, but we will not use this notation.

Lemma 2.1 *For $F, G \in \text{End}(V)$ and $\theta \in \mathbb{F}$ the following* (i), (ii) *are equivalent:*

(i) *the map $[F, [F, G]] - 4G$ vanishes on $V_F(\theta)$;*

(ii) $GV_F(\theta) \subseteq V_F(\theta - 2) + V_F(\theta + 2)$.

Proof: Let Φ denote the map in (i) and observe

$$\Phi = F^2G - 2FGF + GF^2 - 4G.$$

For $v \in V_F(\theta)$ we evaluate Φv using $Fv = \theta v$ to find

$$\begin{aligned} \Phi v &= (F^2G - 2\theta FG + \theta^2 G - 4G)v \\ &= (F - (\theta - 2)I)(F - (\theta + 2)I)Gv. \end{aligned}$$

The scalars $\theta - 2, \theta + 2$ are mutually distinct since the characteristic of $\mathbb{F}$ is 0. The result follows. $\square$

Lemma 2.2 [7, Lemma 2.1] *For $F, G \in \text{End}(V)$ and $\theta \in \mathbb{F}$ the following* (i), (ii) *are equivalent:*

(i) *the map $[F, [F, [F, G]]] - 4[F, G]$ vanishes on $V_F(\theta)$;*

(ii) $GV_F(\theta) \subseteq V_F(\theta - 2) + V_F(\theta) + V_F(\theta + 2)$.

Proof: Let Φ denote the map in (i) and observe

$$\Phi = F^3G - 3F^2GF + 3FGF^2 - GF^3 - 4FG + 4GF.$$

For $v \in V_F(\theta)$ we evaluate Φv using $Fv = \theta v$ to find

$$\begin{aligned} \Phi v &= (F^3G - 3\theta F^2G + 3\theta^2 FG - \theta^3 G - 4FG + 4\theta G)v \\ &= (F - (\theta - 2)I)(F - \theta I)(F - (\theta + 2)I)Gv. \end{aligned}$$

The scalars $\theta - 2$, θ, $\theta + 2$ are mutually distinct since the characteristic of $\mathbb{F}$ is 0. The result follows. □

We end this section with some basic facts about Lie algebras. Let L denote a Lie algebra over $\mathbb{F}$. Recall the Jacobi identity

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0, \qquad x, y, z \in L. \tag{2.2}$$

Lemma 2.3 *Let L denote a Lie algebra over $\mathbb{F}$. For all $a, b, c, d \in L$,*

$$[a, [b, [c, d]]] = [b, [d, [c, a]]] + [c, [d, [b, a]]] - [d, [c, [b, a]]] - [b, [c, [d, a]]]. \tag{2.3}$$

Proof: Observe that

$$\begin{aligned} [a, [b, [c, d]]] &= [[[c, d], b], a] \\ &= [[c, d], [b, a]] - [b, [[c, d], a]] && \text{by (2.2)} \\ &= [c, [d, [b, a]]] - [d, [c, [b, a]]] - ([b, [c, [d, a]]] - [b, [d, [c, a]]]) && \text{by (2.2).} \end{aligned}$$

□

Let V denote a vector space over $\mathbb{F}$ with finite dimension $n \geq 1$. Suppose we are given two bases for V, written $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$. By the *transition matrix* from $u_1, \ldots, u_n$ to $v_1, \ldots, v_n$, we mean the n by n matrix M with entries in $\mathbb{F}$ satisfying

$$v_j = \sum_{i=1}^{n} M_{ij} u_i \qquad (1 \leq j \leq n).$$

Throughout this thesis all unadorned tensor products are taken over $\mathbb{F}$.

2.2 Tridiagonal pairs

In this section we recall some definitions and basic facts concerning tridiagonal pairs. These results will be used throughout the chapter.

Definition 2.4 [9] Let V denote a vector space over $\mathbb{F}$ with finite positive dimension. By a *tridiagonal pair* on V we mean an ordered pair A, B of elements in $\mathrm{End}(V)$ that satisfy the following four conditions.

(i) Each of A, B is diagonalizable.

(ii) There exists an ordering $\{V_i\}_{i=0}^{d}$ of the eigenspaces of A such that

$$BV_i \subseteq V_{i-1} + V_i + V_{i+1} \qquad (0 \leq i \leq d), \qquad (2.4)$$

where $V_{-1} = 0$ and $V_{d+1} = 0$.

(iii) There exists an ordering $\{V_i'\}_{i=0}^{\delta}$ of the eigenspaces of B such that

$$AV_i' \subseteq V_{i-1}' + V_i' + V_{i+1}' \qquad (0 \leq i \leq \delta), \qquad (2.5)$$

where $V_{-1}' = 0$ and $V_{\delta+1}' = 0$.

(iv) There does not exist a subspace W of V such that $AW \subseteq W$, $BW \subseteq W$, $W \neq 0$, $W \neq V$.

We say the pair A, B is *over* $\mathbb{F}$. We call V the *vector space underlying* A, B.

Referring to the tridiagonal pair A, B in Definition 2.4, observe that B, A is also a tridiagonal pair on V. By [9, Lemma 4.5], the integers d and δ from conditions (ii) and (iii) in Definition 2.4 are equal; we call this common value the *diameter* of A, B. An ordering of the eigenspaces of A (resp. B) will be called *standard* whenever it satisfies (2.4) (resp. (2.5)). We comment on the uniqueness of the standard ordering. Let $\{V_i\}_{i=0}^d$ denote a standard ordering of the eigenspaces of A. Then the ordering $\{V_{d-i}\}_{i=0}^d$ is standard and no other ordering is standard. A similar result holds for the eigenspaces of B. An ordering of the eigenvalues of A (resp. B) will be called *standard* whenever the corresponding ordering of the eigenspaces of A (resp. B) is standard. Let $\{V_i\}_{i=0}^d$ (resp. $\{V_i'\}_{i=0}^d$) denote a standard ordering of the eigenspaces of A (resp. B). For $0 \leq i \leq d$ the spaces V_i, V_i' have the same dimension [9, Corollary 5.7]; we denote this common dimension by ρ_i. By the construction $\rho_i \neq 0$. The sequence $\{\rho_i\}_{i=0}^d$ is symmetric and unimodal; that is $\rho_i = \rho_{d-i}$ for $0 \leq i \leq d$ and $\rho_{i-1} \leq \rho_i$ for $1 \leq i \leq d/2$ [9, Corollaries 5.7, 6.6]. By [11, Corollary 1.4] and since $\mathbb{F}$ is algebraically closed, $\rho_i \leq \binom{d}{i}$ for $0 \leq i \leq d$. In particular $\rho_0 = 1$. We call the sequence $\{\rho_i\}_{i=0}^d$ the *shape* of A, B. We will often abbreviate $\rho = \rho_1$. The tridiagonal pair A, B is called a *Leonard pair* whenever $\rho_i = 1$ for $1 \leq i \leq d$.

For the remainder of this chapter, λ will denote an indeterminate. Let $\mathbb{F}[\lambda]$ denote the $\mathbb{F}$-algebra consisting of the polynomials in λ that have all coefficients in $\mathbb{F}$.

Definition 2.5 Let A, B denote a tridiagonal pair with shape $\{\rho_i\}_{i=0}^{d}$. Consider the polynomial in $\mathbb{F}[\lambda]$ given by

$$\sum_{i=0}^{d} \rho_i \lambda^i.$$

We call this the *shape polynomial* of A, B.

Example 2.6 The shape polynomial of a Leonard pair with diameter d is given by

$$1 + \lambda + \lambda^2 + \cdots + \lambda^d.$$

Definition 2.7 Let A, B and A', B' denote tridiagonal pairs over $\mathbb{F}$. By an *isomorphism of tridiagonal pairs* from A, B to A', B' we mean a vector space isomorphism γ from the vector space underlying A, B to the vector space underlying A', B' such that both

$$\gamma A = A'\gamma, \qquad \gamma B = B'\gamma.$$

2.3 Tridiagonal pairs and the Onsager algebra

In this section we consider tridiagonal pairs of Krawtchouk type and their relationship to the Onsager algebra.

Let A, B denote a tridiagonal pair with diameter d. We say that A, B has *Krawtchouk type* whenever the sequence $\{d - 2i\}_{i=0}^{d}$ is a standard ordering of the eigenvalues of A and a standard ordering of the eigenvalues of B. In this case, the tridiagonal pair B, A also has Krawtchouk type. Moreover, by Definition 2.4 and Lemma 2.2, A and B satisfy the Dolan-Grady relations

$$[A, [A, [A, B]]] = 4[A, B], \qquad [B, [B, [B, A]]] = 4[B, A].$$

Definition 2.8 [13] Let $\mathcal{O}$ denote the Lie algebra over $\mathbb{F}$ with generators $\mathcal{A}, \mathcal{B}$ and relations

$$[\mathcal{A}, [\mathcal{A}, [\mathcal{A}, \mathcal{B}]]] = 4[\mathcal{A}, \mathcal{B}], \qquad [\mathcal{B}, [\mathcal{B}, [\mathcal{B}, \mathcal{A}]]] = 4[\mathcal{B}, \mathcal{A}].$$

We call $\mathcal{O}$ the *Onsager algebra*. We call $\mathcal{A}, \mathcal{B}$ the *standard generators* for $\mathcal{O}$.

Theorem 2.9 [12] *The* $\mathbb{F}$*-vector space* $\mathcal{O}$ *has a basis* $\{A_i, G_j \,|\, i, j \in \mathbb{Z}, j > 0\}$ *such that*

$$[A_k, A_l] = 2G_{k-l} \qquad k > l,$$

$$[G_k, A_l] = A_{l+k} - A_{l-k},$$

$$[G_k, G_l] = 0.$$

Moreover $A_0 = \mathcal{A}$ *and* $A_1 = \mathcal{B}$*.*

Remark 2.10 The elements $\{G_j \,|\, j \in \mathbb{Z}, j > 0\}$ from Theorem 2.9 form a basis for an abelian Lie subalgebra of $\mathcal{O}$.

Let V denote a finite-dimensional irreducible $\mathcal{O}$-module. By [7, Theorem 2.4], the standard generators $\mathcal{A}, \mathcal{B}$ are diagonalizable on V. Furthermore there exist an integer $d \geq 0$ and scalars $\alpha, \beta \in \mathbb{F}$ such that the set of distinct eigenvalues of $\mathcal{A}$ (resp. $\mathcal{B}$) on V is $\{d - 2i + \alpha | 0 \leq i \leq d\}$ (resp. $\{d - 2i + \beta | 0 \leq i \leq d\}$) [7, Theorem 2.4]. We call the ordered pair (α, β) the *type* of V. Subtracting α (resp. β) times the identity from $\mathcal{A}$ (resp. $\mathcal{B}$) the type becomes $(0, 0)$.

The following theorems give the relationship between finite-dimensional irreducible $\mathcal{O}$-modules and tridiagonal pairs of Krawtchouk type.

Theorem 2.11 [7, Corollary 2.7] *Let A, B denote a tridiagonal pair on V of Krawtchouk type. Then there exists a unique $\mathcal{O}$-module structure on V such that the standard generators $\mathcal{A}, \mathcal{B}$ act on V as A, B respectively. This $\mathcal{O}$-module is irreducible and of type $(0, 0)$.*

Theorem 2.12 [7, Corollary 2.7] *Let V denote a finite-dimensional irreducible $\mathcal{O}$-module of type $(0, 0)$. Then the standard generators $\mathcal{A}, \mathcal{B}$ act on V as a tridiagonal pair of Krawtchouk type.*

Remark 2.13 [7, Corollary 2.7] Combining the previous two theorems we obtain a bijection between the following two sets:

(i) the isomorphism classes of tridiagonal pairs over $\mathbb{F}$ that have Krawtchouk type;

(ii) the isomorphism classes of finite-dimensional irreducible $\mathcal{O}$-modules of type $(0, 0)$.

Definition 2.14 Let V denote a finite-dimensional irreducible $\mathcal{O}$-module of type $(0, 0)$. Let A, B denote a tridiagonal pair on V that has Krawtchouk type. We say the $\mathcal{O}$-module V and the tridiagonal pair A, B are *associated* whenever the $\mathcal{O}$-generators $\mathcal{A}, \mathcal{B}$ act on V as A, B respectively. By the *diameter* (resp. *shape*) (resp. *shape polynomial*) of the $\mathcal{O}$-module V we mean the diameter (resp. shape) (resp. shape polynomial) of the associated tridiagonal pair. We abbreviate S_V for the shape polynomial of the $\mathcal{O}$-module V.

Let V denote a finite-dimensional irreducible $\mathcal{O}$-module of type $(0, 0)$. We call the $\mathcal{O}$-module V *trivial* whenever the diameter of V is zero.

Lemma 2.15 *Up to isomorphism, there exists a unique trivial finite-dimensional irreducible $\mathcal{O}$-module of type $(0, 0)$. This $\mathcal{O}$-module has dimension 1.*

Proof: Let V denote a trivial finite-dimensional irreducible $\mathcal{O}$-module of type $(0, 0)$. Since the diameter of V is zero, each of the $\mathcal{O}$-generators $\mathcal{A}, \mathcal{B}$ acts on V as the zero map. Therefore any subspace of V is an $\mathcal{O}$-submodule of V, so V has dimension 1. The result follows. □

We will return to $\mathcal{O}$ shortly.

2.4 The Lie algebra $\mathfrak{sl}_2$

In this section we recall the Lie algebra $\mathfrak{sl}_2$ and its finite-dimensional modules.

Definition 2.16 Let $\mathfrak{sl}_2$ denote the Lie algebra over $\mathbb{F}$ with basis e, f, h and Lie bracket

$$[e, f] = h, \qquad [h, e] = 2e, \qquad [h, f] = -2f.$$

We call the basis e, f, h the *Chevalley* basis of $\mathfrak{sl}_2$.

In the following two lemmas we describe the finite-dimensional $\mathfrak{sl}_2$-modules.

Lemma 2.17 [8, p. 28] *Each finite-dimensional* $\mathfrak{sl}_2$*-module is a direct sum of irreducible* $\mathfrak{sl}_2$*-modules.*

Lemma 2.18 [8, p. 31] *There exists a family*

$$\mathbb{V}_d \qquad d = 0, 1, 2, \ldots \tag{2.6}$$

of finite-dimensional irreducible $\mathfrak{sl}_2$-modules with the following property. The module $\mathbb{V}_d$ has a basis $\{v_i\}_{i=0}^d$ satisfying

$$hv_i = (d-2i)v_i \qquad (0 \le i \le d), \tag{2.7}$$

$$fv_i = (i+1)v_{i+1} \qquad (0 \le i \le d-1), \qquad fv_d = 0, \tag{2.8}$$

$$ev_i = (d+1-i)v_{i-1} \qquad (1 \le i \le d), \qquad ev_0 = 0. \tag{2.9}$$

Every finite-dimensional irreducible $\mathfrak{sl}_2$-module is isomorphic to exactly one of the modules in (3.2)*.*

We mention a fact for later use.

Lemma 2.19 [10, p. 34] *Let d denote a nonnegative integer. Let $\{\rho_i\}_{i=0}^d$ denote a sequence of positive integers such that $\rho_i = \rho_{d-i}$ for $0 \le i \le d$ and $\rho_i \le \rho_{i+1}$ for $0 \le i < d/2$. Then there exists an $\mathfrak{sl}_2$-module V satisfying the following* (i), (ii)*:*

(i) *the action of h on V is diagonalizable with eigenvalues $\{d-2i\}_{i=0}^d$;*

(ii) *for $0 \le i \le d$, $\rho_i = \dim(U_i)$, where U_i is the eigenspace for the action of h on V corresponding to the eigenvalue $d-2i$.*

The $\mathfrak{sl}_2$-module V is unique up to isomorphism. The only irreducible $\mathfrak{sl}_2$-submodules of V are

$$\mathbb{V}_d, \mathbb{V}_{d-2}, \mathbb{V}_{d-4}, \ldots .$$

Moreover, for $0 \le j \le d/2$, the multiplicity with which $\mathbb{V}_{d-2j}$ appears in V is $\rho_j - \rho_{j-1}$, where $\rho_{-1} = 0$.

Proof: For notational convenience, abbreviate $d_j = d - 2j$ and $m_j = \rho_j - \rho_{j-1}$. Consider the $\mathfrak{sl}_2$-module

$$V = \bigoplus_{j=0}^{d/2} \mathbb{V}_{d_j}^{\oplus m_j},$$

where $\mathbb{V}_{d_j}^{\oplus m_j}$ denotes the $\mathfrak{sl}_2$-module $\mathbb{V}_{d_j} \oplus \cdots \oplus \mathbb{V}_{d_j}$ (m_j times). By Lemma 2.18, h is diagonalizable on V with eigenvalues $\{d - 2i\}_{i=0}^{d}$. For $0 \leq i \leq d/2$, $\dim(U_i) = m_0 + m_1 + \cdots + m_i = \rho_i$. For $d/2 < i \leq d$, $\dim(U_i) = m_0 + m_1 + \cdots + m_{d-i} = \rho_{d-i} = \rho_i$. Therefore $\dim(U_i) = \rho_i$ for $0 \leq i \leq d$. By construction, d and the sequence $\{\rho_i\}_{i=0}^{d}$ uniquely determine the sequences $\{d_j\}_{j=0}^{d/2}$, $\{m_j\}_{j=0}^{d/2}$. The sequences $\{d_j\}_{j=0}^{d/2}$, $\{m_j\}_{j=0}^{d/2}$ uniquely determine the isomorphism type of the $\mathfrak{sl}_2$-module V. The previous two sentences together prove the uniqueness claim in the result. The remaining two claims are true by construction. □

With reference to Lemma 2.18, note that $\{v_i\}_{i=0}^{d}$ is an h-eigenbasis for $\mathbb{V}_d$.

Definition 2.20 For $d \in \mathbb{N}$ an h-eigenbasis $\{v_i\}_{i=0}^{d}$ for $\mathbb{V}_d$ will be called *normalized* whenever it satisfies (2.7)–(2.9).

Lemma 2.21 *Let $d \in \mathbb{N}$, and let $\{v_i\}_{i=0}^{d}$ denote a normalized h-eigenbasis of $\mathbb{V}_d$. Given vectors $\{u_i\}_{i=0}^{d}$ in $\mathbb{V}_d$, the following are equivalent:*

(i) *the vectors $\{u_i\}_{i=0}^{d}$ form a normalized h-eigenbasis for $\mathbb{V}_d$;*

(ii) *there exists a nonzero $s \in \mathbb{F}$ such that $u_i = s v_i$ for $0 \leq i \leq d$.*

Proof: Routine consequence of (2.7)–(2.9). □

We now give an alternate presentation of $\mathfrak{sl}_2$.

Lemma 2.22 $\mathfrak{sl}_2$ *is isomorphic to the Lie algebra over* $\mathbb{F}$ *that has generators* $\mathfrak{a}, \mathfrak{h}$ *and relations*

$$[\mathfrak{a}, [\mathfrak{a}, \mathfrak{h}]] = 4\mathfrak{h}, \qquad [\mathfrak{h}, [\mathfrak{h}, \mathfrak{a}]] = 4\mathfrak{a}.$$

An isomorphism with the presentation in Definition 2.16 *is given by*

$$\mathfrak{a} \mapsto e + f, \qquad \mathfrak{h} \mapsto h.$$

The inverse of this isomorphism is given by

$$e \mapsto \frac{[\mathfrak{h}, \mathfrak{a}] + 2\mathfrak{a}}{4}, \qquad f \mapsto \frac{[\mathfrak{a}, \mathfrak{h}] + 2\mathfrak{a}}{4}, \qquad h \mapsto \mathfrak{h}.$$

The elements $\mathfrak{a}, \mathfrak{h}, [\mathfrak{a}, \mathfrak{h}]$ *form a basis for* $\mathfrak{sl}_2$*.*

Proof: We routinely check that each map is a homomorphism of Lie algebras and that the maps are inverses. It follows that each map is an isomorphism of Lie algebras. The last assertion is routinely checked. □

Note 2.23 For notational convenience, for the rest of this chapter we identify the copy of $\mathfrak{sl}_2$ given in Definition 2.16 with the copy given in Lemma 2.22, via the isomorphism given in Lemma 2.22.

Definition 2.24 We call the elements $\mathfrak{a}, \mathfrak{h}$ from Lemma 2.22 the *alternate generators* for $\mathfrak{sl}_2$.

We now describe three automorphisms of $\mathfrak{sl}_2$.

Lemma 2.25 *The following hold.*

(i) *There exists an automorphism of* $\mathfrak{sl}_2$ *that sends* $\mathfrak{a} \mapsto \mathfrak{a}$*,* $\mathfrak{h} \mapsto -\mathfrak{h}$*.*

(ii) *There exists an automorphism of* $\mathfrak{sl}_2$ *that sends* $\mathfrak{a} \mapsto -\mathfrak{a}$, $\mathfrak{h} \mapsto \mathfrak{h}$.

(iii) *There exists an automorphism of* $\mathfrak{sl}_2$ *that sends* $\mathfrak{a} \mapsto \mathfrak{h}$, $\mathfrak{h} \mapsto \mathfrak{a}$.

Each of the automorphisms from (i)–(iii) *has order* 2.

Proof: Clear by the first assertion in Lemma 2.22. □

The automorphisms of $\mathfrak{sl}_2$ from Lemma 2.25 do the following to the Chevalley basis.

Lemma 2.26 *The following hold.*

(i) *The automorphism of* $\mathfrak{sl}_2$ *from Lemma* 2.25(i) *sends* $e \mapsto f$, $f \mapsto e$, $h \mapsto -h$.

(ii) *The automorphism of* $\mathfrak{sl}_2$ *from Lemma* 2.25(ii) *sends* $e \mapsto -e$, $f \mapsto -f$, $h \mapsto h$.

(iii) *The automorphism of* $\mathfrak{sl}_2$ *from Lemma* 2.25(iii) *sends*

$$e \mapsto \frac{f - e + h}{2}, \qquad f \mapsto \frac{e - f + h}{2}, \qquad h \mapsto e + f.$$

Proof: Routine. □

Lemma 2.27 *For each* $d \in \mathbb{N}$ *the actions of the alternate generators* $\mathfrak{a}, \mathfrak{h}$ *on a normalized* h*-eigenbasis* $\{v_i\}_{i=0}^d$ *of* $\mathbb{V}_d$ *are as follows:*

$$\mathfrak{a}v_i = (d + 1 - i)v_{i-1} + (i + 1)v_{i+1} \quad (1 \leq i \leq d - 1), \qquad \mathfrak{a}v_0 = v_1, \qquad \mathfrak{a}v_d = v_{d-1},$$

$$\mathfrak{h}v_i = (d - 2i)v_i \quad (0 \leq i \leq d).$$

Proof: Routine consequence of (2.7)–(2.9). □

Lemma 2.28 *Let* $d \in \mathbb{N}$. *Let* $\{v_i\}_{i=0}^d$ *be vectors in* $\mathbb{V}_d$, *not all zero. Then* $\{v_i\}_{i=0}^d$ *form a normalized* h*-eigenbasis of* $\mathbb{V}_d$ *if and only if the following* (i) *and* (ii) *hold:*

(i) $\mathfrak{h}v_i = (d-2i)v_i$ *for* $0 \leq i \leq d$*;*

(ii) *the sum* $\sum_{i=0}^{d} v_i$ *is an eigenvector for* $\mathfrak{a}$ *with eigenvalue* d*.*

Proof: Routine consequence of (2.7)–(2.9). □

Interchanging the roles of $\mathfrak{a}, \mathfrak{h}$ in Lemma 2.27 we obtain the following result.

Lemma 2.29 *For each* $d \in \mathbb{N}$ *the* $\mathfrak{sl}_2$*-module* $\mathbb{V}_d$ *has a basis* $\{w_i\}_{i=0}^{d}$ *such that*

$$\mathfrak{h}w_i = (d+1-i)w_{i-1} + (i+1)w_{i+1} \quad (1 \leq i \leq d-1), \qquad \mathfrak{h}w_0 = w_1, \qquad \mathfrak{h}w_d = w_{d-1},$$

$$\mathfrak{a}w_i = (d-2i)w_i \quad (0 \leq i \leq d).$$

Proof: Routine using Lemma 2.27, Lemma 2.25(iii), and the last assertion in Lemma 2.18. □

Lemma 2.30 *Let* V *denote a finite-dimensional irreducible* $\mathfrak{sl}_2$*-module. Then the alternate generators* $\mathfrak{a}, \mathfrak{h}$ *of* $\mathfrak{sl}_2$ *act on* V *as a Leonard pair of Krawtchouk type.*

Proof: By the last assertion in Lemma 2.18, there exists $d \in \mathbb{N}$ such that the $\mathfrak{sl}_2$-modules V and $\mathbb{V}_d$ are isomorphic. With respect to the basis $\{v_i\}_{i=0}^{d}$ from Lemma 2.27, the matrix representing $\mathfrak{a}$ is irredicuble tridiagonal and the matrix representing $\mathfrak{h}$ is diagonal. With respect to the basis $\{w_i\}_{i=0}^{d}$ from Lemma 2.29, the matrix representing $\mathfrak{h}$ is irreducible tridiagonal and the matrix representing $\mathfrak{a}$ is diagonal. By Lemmas 2.27 and 2.29, the sequence $\{d-2i\}_{i=0}^{d}$ is a standard ordering of the eigenvalues of $\mathfrak{a}$ and a standard ordering of the eigenvalues of $\mathfrak{h}$. The result follows. □

Recall the notation $V_F(\theta)$ from line (2.1).

Lemma 2.31 *Let V denote a finite-dimensional $\mathfrak{sl}_2$-module. Then each of the alternate generators $\mathfrak{a}, \mathfrak{h}$ is diagonalizable on V. Let A, H denote the actions of $\mathfrak{a}, \mathfrak{h}$ on V respectively. For $\theta \in \mathbb{F}$ the spaces $V_A(\theta), V_H(\theta)$ have the same dimension.*

Proof: If the $\mathfrak{sl}_2$-module V is irreducible, then the result holds by Lemma 2.30. The general case follows by Lemma 3.6. □

2.5 The $\mathfrak{sl}_2$ loop algebra

Definition 2.32 Let t denote an indeterminate, and let $\mathbb{F}[t, t^{-1}]$ denote the $\mathbb{F}$-algebra consisting of the Laurent polynomials in t that have all coefficients in $\mathbb{F}$. Let $L(\mathfrak{sl}_2)$ denote the Lie algebra over $\mathbb{F}$ consisting of the $\mathbb{F}$-vector space $\mathfrak{sl}_2 \otimes \mathbb{F}[t, t^{-1}]$ and Lie bracket

$$[u \otimes a, v \otimes b] = [u, v] \otimes ab, \qquad u, v \in \mathfrak{sl}_2, \qquad a, b \in \mathbb{F}[t, t^{-1}]. \tag{2.10}$$

We call $L(\mathfrak{sl}_2)$ the $\mathfrak{sl}_2$ *loop algebra.*

Observe that $\{t^i\}_{i \in \mathbb{Z}}$ is a basis of the $\mathbb{F}$-vector space $\mathbb{F}[t, t^{-1}]$. Therefore the following is a basis for $L(\mathfrak{sl}_2)$:

$$e \otimes t^i, \qquad f \otimes t^i, \qquad h \otimes t^i \qquad i \in \mathbb{Z}. \tag{2.11}$$

The $\mathfrak{sl}_2$ loop algebra is related to the Kac-Moody algebra [10] associated with the Cartan matrix

$$C := \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}.$$

This is made clear in the following lemma.

Lemma 2.33 [10, p. 100] *The loop algebra* $L(\mathfrak{sl}_2)$ *is isomorphic to the Lie algebra over* $\mathbb{F}$ *that has generators* $e_i, f_i, h_i,\ i \in \{0,1\}$ *and the following relations:*

$$h_0 + h_1 = 0, \qquad [e_i, f_j] = \delta_{ij} h_j,$$

$$[h_i, e_j] = C_{ij} e_j, \qquad [h_i, f_j] = -C_{ij} f_j,$$

$$[e_i, [e_i, [e_i, e_j]]] = 0, \qquad [f_i, [f_i, [f_i, f_j]]] = 0, \qquad i \neq j.$$

An isomorphism is given by

$$e_1 \mapsto e \otimes 1, \qquad f_1 \mapsto f \otimes 1, \qquad h_1 \mapsto h \otimes 1,$$

$$e_0 \mapsto f \otimes t^{-1}, \qquad f_0 \mapsto e \otimes t, \qquad h_0 \mapsto -h \otimes 1.$$

We now give an alternate presentation of $L(\mathfrak{sl}_2)$.

Theorem 2.34 *The loop algebra* $L(\mathfrak{sl}_2)$ *is isomorphic to the Lie algebra over* $\mathbb{F}$ *that has generators* $\mathcal{A}, \mathcal{B}, \mathcal{H}$ *and relations*

$$[\mathcal{A}, [\mathcal{A}, \mathcal{H}]] = 4\mathcal{H}, \qquad [\mathcal{H}, [\mathcal{H}, \mathcal{A}]] = 4\mathcal{A}, \tag{2.12}$$

$$[\mathcal{B}, [\mathcal{B}, \mathcal{H}]] = 4\mathcal{H}, \qquad [\mathcal{H}, [\mathcal{H}, \mathcal{B}]] = 4\mathcal{B}, \tag{2.13}$$

$$[\mathcal{A}, [\mathcal{A}, [\mathcal{A}, \mathcal{B}]]] = 4[\mathcal{A}, \mathcal{B}], \qquad [\mathcal{B}, [\mathcal{B}, [\mathcal{B}, \mathcal{A}]]] = 4[\mathcal{B}, \mathcal{A}], \tag{2.14}$$

$$[\mathcal{H}, [\mathcal{A}, \mathcal{B}]] = 0. \tag{2.15}$$

An isomorphism with the presentation from Lemma 2.33 *is given by*

$$\mathcal{A} \mapsto e_1 + f_1, \qquad \mathcal{B} \mapsto e_0 + f_0, \qquad \mathcal{H} \mapsto h_1. \tag{2.16}$$

The inverse of this isomorphism is given by

$$e_1 \mapsto \frac{[\mathcal{H}, \mathcal{A}] + 2\mathcal{A}}{4}, \qquad f_1 \mapsto \frac{[\mathcal{A}, \mathcal{H}] + 2\mathcal{A}}{4}, \qquad h_1 \mapsto \mathcal{H}, \tag{2.17}$$

$$e_0 \mapsto \frac{[\mathcal{B}, \mathcal{H}] + 2\mathcal{B}}{4}, \qquad f_0 \mapsto \frac{[\mathcal{H}, \mathcal{B}] + 2\mathcal{B}}{4}, \qquad h_0 \mapsto -\mathcal{H}. \tag{2.18}$$

The proof of Theorem 2.34 is given in Section 2.14.

Remark 2.35 We have used the same symbol to refer to the element $\mathcal{A}$ (resp. $\mathcal{B}$) of $\mathcal{O}$ from Definition 2.8 and the element $\mathcal{A}$ (resp. $\mathcal{B}$) of $L(\mathfrak{sl}_2)$ from Theorem 2.34. Our justification for doing this will become clear prior to Note 2.39. As we proceed, it should be clear from the context whether we are discussing $\mathcal{O}$ or $L(\mathfrak{sl}_2)$.

In Lemma 2.33, we gave an isomorphism from the copy of $L(\mathfrak{sl}_2)$ given in Definition 3.12 to the copy in Lemma 2.33. In Theorem 2.34, we gave an isomorphism from the copy of $L(\mathfrak{sl}_2)$ given in Lemma 2.33 to the copy in Theorem 2.34. Composing those isomorphisms we get an isomorphism from the copy of $L(\mathfrak{sl}_2)$ given in Definition 3.12 to the copy in Theorem 2.34. This isomorphism is described as follows.

Corollary 2.36 *The composition of the isomorphisms from Lemma* 2.33 *and Theorem* 2.34 *sends*

$$\mathcal{A} \mapsto e \otimes 1 + f \otimes 1, \qquad \mathcal{B} \mapsto e \otimes t + f \otimes t^{-1}, \qquad \mathcal{H} \mapsto h \otimes 1.$$

Proof: Immediate from Lemma 2.33 and Theorem 2.34. □

Note 2.37 For notational convenience, for the rest of this chapter we identify the copies of $L(\mathfrak{sl}_2)$ given in Definition 3.12, Lemma 2.33, Theorem 2.34, via the isomorphisms given in Lemma 2.33, Theorem 2.34, Corollary 2.36.

By [3, p. 3277], there exists a homomorphism of Lie algebras $\mathcal{O} \to L(\mathfrak{sl}_2)$ that sends

$$\mathcal{A} \mapsto e \otimes 1 + f \otimes 1, \qquad \mathcal{B} \mapsto e \otimes t + f \otimes t^{-1}.$$

Moreover, this map is injective [3, p. 3277]. If we look at this map from the point of view of the presentation in Theorem 2.34, then we see that it sends

$$\mathcal{A} \mapsto \mathcal{A}, \qquad \mathcal{B} \mapsto \mathcal{B}. \tag{2.19}$$

We call the above homomorphism $\mathcal{O} \to L(\mathfrak{sl}_2)$ *natural*.

Recall the basis $\{A_i, G_j \,|\, i, j \in \mathbb{Z}, j > 0\}$ of $\mathcal{O}$ from Theorem 2.9.

Lemma 2.38 [3, p. 3277] *The natural homomorphism $\mathcal{O} \to L(\mathfrak{sl}_2)$ sends*

$$A_i \mapsto e \otimes t^i + f \otimes t^{-i} \qquad i \in \mathbb{Z},$$

$$G_j \mapsto h \otimes (t^j - t^{-j})/2 \qquad j \in \mathbb{Z}, \quad j > 0.$$

Note 2.39 For notational convenience, for the rest of this chapter we identify $\mathcal{O}$ with its image in $L(\mathfrak{sl}_2)$ under the natural homomorphism.

We now describe two automorphisms of $L(\mathfrak{sl}_2)$.

Lemma 2.40 *There exists an automorphism ϑ of $L(\mathfrak{sl}_2)$ that sends*

$$\mathcal{A} \mapsto \mathcal{A}, \qquad \mathcal{B} \mapsto \mathcal{B}, \qquad \mathcal{H} \mapsto -\mathcal{H}.$$

Moreover, there exists an automorphism τ of $L(\mathfrak{sl}_2)$ that sends

$$\mathcal{A} \mapsto \mathcal{B}, \qquad \mathcal{B} \mapsto \mathcal{A}, \qquad \mathcal{H} \mapsto \mathcal{H}.$$

The automorphisms ϑ and τ satisfy $\vartheta\tau = \tau\vartheta$, $\vartheta^2 = 1$, $\tau^2 = 1$.

Proof: The first and second assertions are clear by Theorem 2.34. The last assertion is easily checked. □

By the last assertion in Lemma 2.40, ϑ and τ induce an action of the Klein-four group $\mathbb{Z}_2 \times \mathbb{Z}_2$ on $L(\mathfrak{sl}_2)$ as a group of automorphisms.

Lemma 2.41 *The automorphisms ϑ and τ of $L(\mathfrak{sl}_2)$ do the following to the generators e_i, f_i, h_i, $i \in \{0, 1\}$ of $L(\mathfrak{sl}_2)$ from Lemma 2.33.*

(i) *The map ϑ sends $e_0 \leftrightarrow f_0$, $e_1 \leftrightarrow f_1$, $h_0 \leftrightarrow h_1$.*

(ii) *The map τ sends $e_0 \leftrightarrow f_1$, $e_1 \leftrightarrow f_0$, $h_0 \mapsto h_0$, $h_1 \mapsto h_1$.*

(iii) *The composition $\vartheta\tau$ sends $e_0 \leftrightarrow e_1$, $f_0 \leftrightarrow f_1$, $h_0 \leftrightarrow h_1$.*

Proof: Routine using Theorem 2.34 and Lemma 2.40. □

Lemma 2.42 *For the automorphism ϑ of $L(\mathfrak{sl}_2)$ from Lemma 2.40, we have $\vartheta = \vartheta_1 \otimes \vartheta_2$, where ϑ_1 denotes the automorphism of $\mathfrak{sl}_2$ from Lemma 2.25(i), and ϑ_2 denotes the automorphism of $\mathbb{F}[t, t^{-1}]$ that sends $t \mapsto t^{-1}$.*

Proof: Routine using Lemma 2.25(i), Corollary 2.36, and Lemma 2.40. □

Lemma 2.43 [3, p. 3277] *Pick $x \in L(\mathfrak{sl}_2)$. Then $x \in \mathcal{O}$ if and only if $\vartheta(x) = x$.*

Recall the element $\mathfrak{a}$ of $\mathfrak{sl}_2$ from Lemma 2.22. For later use, we mention some elements of $L(\mathfrak{sl}_2)$ that are contained in $\mathcal{O}$.

Lemma 2.44 For all $k \in \mathbb{Z}$, the element $\mathfrak{a} \otimes (t^k + t^{-k})$ of $L(\mathfrak{sl}_2)$ is contained in $\mathcal{O}$.

Proof: Use Lemmas 2.42 and 2.43. □

Note that there exists an injection of Lie algebras

$$\begin{aligned} \mathfrak{sl}_2 &\to L(\mathfrak{sl}_2) \\ x &\mapsto x \otimes 1. \end{aligned} \tag{2.20}$$

If we look at this map from the point of view of the presentation of $\mathfrak{sl}_2$ in Lemma 2.22 and the presentation of $L(\mathfrak{sl}_2)$ in Theorem 2.34, we obtain the following result.

Lemma 2.45 *The injection of Lie algebras $\mathfrak{sl}_2 \to L(\mathfrak{sl}_2)$ from* (2.20) *sends*

$$\mathfrak{a} \mapsto \mathcal{A}, \qquad \mathfrak{h} \mapsto \mathcal{H}. \tag{2.21}$$

Moreover, the composition of the injection from (2.20) *and the automorphism τ of $L(\mathfrak{sl}_2)$ from Lemma* 2.40 *sends*

$$\mathfrak{a} \mapsto \mathcal{B}, \qquad \mathfrak{h} \mapsto \mathcal{H}. \tag{2.22}$$

Proof: By Lemma 2.22, Note 2.37, and Lemma 2.40. □

Recall the notation $V_F(\theta)$ from line (2.1).

Lemma 2.46 *Let V denote a finite-dimensional $L(\mathfrak{sl}_2)$-module. Then each of $\mathcal{A}, \mathcal{B}, \mathcal{H}$ is diagonalizable on V. Let A, B, H denote the actions of $\mathcal{A}, \mathcal{B}, \mathcal{H}$ on V respectively. For $\theta \in \mathbb{F}$ the spaces $V_A(\theta), V_B(\theta), V_H(\theta)$ have the same dimension.*

Proof: Consider the $L(\mathfrak{sl}_2)$-module V. Pull back the $L(\mathfrak{sl}_2)$-action via the homomorphism given by (2.21). Then V becomes an $\mathfrak{sl}_2$-module on which the $\mathfrak{sl}_2$-generators $\mathfrak{a}, \mathfrak{h}$ act as A, H respectively. By Lemma 2.31, each of A, H is diagonalizable, and the spaces

$V_A(\theta), V_H(\theta)$ have the same dimension. By a similar argument, B is diagonalizable, and the spaces $V_B(\theta), V_H(\theta)$ have the same dimension. □

We finish this section with a comment. Up to isomorphism, there exists a unique $L(\mathfrak{sl}_2)$-module of dimension 1. On this module every element of $L(\mathfrak{sl}_2)$ acts as the zero map. We call this $L(\mathfrak{sl}_2)$-module *trivial*.

2.6 Evaluation modules for $L(\mathfrak{sl}_2)$

In this section we discuss a type of $L(\mathfrak{sl}_2)$-module called an evaluation $L(\mathfrak{sl}_2)$-module.

Definition 2.47 For nonzero $a \in \mathbb{F}$, define a map $EV_a : L(\mathfrak{sl}_2) \to \mathfrak{sl}_2$ by

$$EV_a(u \otimes g(t)) = g(a)u, \qquad u \in \mathfrak{sl}_2, \quad g(t) \in \mathbb{F}[t, t^{-1}].$$

The map EV_a is a homomorphism of Lie algebras.

With reference to Definition 3.15, we routinely check that EV_a is surjective and its kernel is $\mathfrak{sl}_2 \otimes (t-a)\mathbb{F}[t, t^{-1}]$.

Definition 2.48 For a finite-dimensional $\mathfrak{sl}_2$-module V and for $0 \neq a \in \mathbb{F}$, we pull back the $\mathfrak{sl}_2$-action via EV_a to obtain an $L(\mathfrak{sl}_2)$-action on V. We denote the resulting $L(\mathfrak{sl}_2)$-module by $V(a)$.

Definition 2.49 With reference to Lemma 2.18 and Definition 3.16, by an *evaluation $L(\mathfrak{sl}_2)$-module* we mean an $L(\mathfrak{sl}_2)$-module $\mathbb{V}_d(a)$, where d is a positive integer and $0 \neq a \in \mathbb{F}$. By construction the evaluation $L(\mathfrak{sl}_2)$-module $\mathbb{V}_d(a)$ is nontrivial and irreducible. We call a the *evaluation parameter* of $\mathbb{V}_d(a)$. Note that $\mathbb{V}_d(a)$ has dimension $d+1$.

Lemma 2.50 *For a positive integer* d *and nonzero* $a \in \mathbb{F}$*, the evaluation* $L(\mathfrak{sl}_2)$*-module* $\mathbb{V}_d(a)$ *is described as follows. Let* $\{v_i\}_{i=0}^d$ *denote a normalized* h*-eigenbasis of the* $\mathfrak{sl}_2$*-module* $\mathbb{V}_d$*. The elements* (3.6) *of* $L(\mathfrak{sl}_2)$ *act on* $\mathbb{V}_d(a)$ *as follows. For* $k \in \mathbb{Z}$*,*

$$(h \otimes t^k)\, v_i = (d-2i)a^k v_i \qquad (0 \le i \le d),$$

$$(f \otimes t^k)\, v_i = (i+1)a^k v_{i+1} \qquad (0 \le i \le d-1), \qquad (f \otimes t^k)\, v_d = 0,$$

$$(e \otimes t^k)\, v_i = (d+1-i)a^k v_{i-1} \qquad (1 \le i \le d), \qquad (e \otimes t^k)\, v_0 = 0.$$

Note 2.51 With reference to Lemma 3.19, for any $k \in \mathbb{Z}$, v_0 spans the eigenspace of $h \otimes t^k$ corresponding to eigenvalue $a^k d$.

Lemma 2.52 *The evaluation* $L(\mathfrak{sl}_2)$*-modules* $\mathbb{V}_d(a)$ *and* $\mathbb{V}_{d'}(a')$ *are isomorphic if and only if* $d = d'$ *and* $a = a'$*.*

Proof: Suppose $\mathbb{V}_d(a)$ and $\mathbb{V}_d(a')$ are isomorphic. Isomorphic modules have the same dimension, so $d = d'$. Considering the action of $h \otimes t$, we see by Note 3.20 that $ad = a'd$. Since d is positive, we have $a = a'$. This proves the lemma in one direction. The proof for the other direction is immediate. □

Note 2.53 With reference to Lemma 3.19, the $L(\mathfrak{sl}_2)$-generators $\mathcal{A}, \mathcal{B}, \mathcal{H}$ act as follows:

$$\mathcal{A}v_i = (d+1-i)v_{i-1} + (i+1)v_{i+1} \qquad (1 \le i \le d-1), \quad \mathcal{A}v_0 = v_1, \qquad \mathcal{A}v_d = v_{d-1},$$

$$\mathcal{B}v_i = (d+1-i)av_{i-1} + (i+1)a^{-1}v_{i+1} \quad (1 \le i \le d-1), \quad \mathcal{B}v_0 = a^{-1}v_1, \quad \mathcal{B}v_d = av_{d-1},$$

$$\mathcal{H}v_i = (d-2i)v_i \qquad (0 \le i \le d).$$

Lemma 2.54 *Let* V *denote an evaluation* $L(\mathfrak{sl}_2)$*-module of dimension* $d+1$*. The following hold for* $Z \in \{\mathcal{A}, \mathcal{B}, \mathcal{H}\}$*.*

(i) *Z is diagonalizable on V.*

(ii) *The set of distinct eigenvalues of Z on V is $\{d - 2i | 0 \leq i \leq d\}$.*

(iii) *The eigenspaces for Z all have dimension* 1.

Proof: Write $V = \mathbb{V}_d(a)$. The result holds for $Z = \mathcal{H}$ by the last line in Note 2.53. For the other cases use Lemma 2.46. $\square$

We mention two facts for later use.

Note 2.55 With reference to Lemma 3.19, define $w = \sum_{i=0}^{d} v_i$. Then w spans the eigenspace of $\mathcal{A}$ corresponding to eigenvalue d. Moreover, for all $k \in \mathbb{Z}$, w is an eigenvector for $\mathfrak{a} \otimes (t^k + t^{-k})$ with eigenvalue $(a^k + a^{-k})d$.

Note 2.56 Observe that $[\mathcal{A}, \mathcal{B}] = h \otimes (t^{-1} - t)$. With reference to Lemma 3.19, we have $[\mathcal{A}, \mathcal{B}]v_i = (a^{-1} - a)(d - 2i)v_i$ for $0 \leq i \leq d$.

2.7 Twisting $L(\mathfrak{sl}_2)$-modules

Recall the automorphism ϑ of $L(\mathfrak{sl}_2)$ from Lemma 2.40. In this section we discuss how to twist an $L(\mathfrak{sl}_2)$-module via ϑ.

Definition 2.57 Let V denote an $L(\mathfrak{sl}_2)$-module. There exists an $L(\mathfrak{sl}_2)$-module structure on V, called V *twisted via* ϑ, that behaves as follows: for all $x \in L(\mathfrak{sl}_2)$ and $v \in V$, the vector xv computed in V twisted via ϑ coincides with the vector $\vartheta(x)v$ computed in the original $L(\mathfrak{sl}_2)$-module V. We abbreviate ${}^{\vartheta}V$ for V twisted via ϑ.

With reference to Definition 2.57, we emphasize that for all $x \in L(\mathfrak{sl}_2)$ the following are the same:

(i) the action of x on the $L(\mathfrak{sl}_2)$-module ${}^{\vartheta}V$;

(ii) the action of $\vartheta(x)$ on the $L(\mathfrak{sl}_2)$-module V.

Special cases of particular interest are given in the following two lemmas.

Lemma 2.58 *Let V denote an $L(\mathfrak{sl}_2)$-module. For all $x \in \mathcal{O}$ the following are the same:*

(i) *the action of x on the $L(\mathfrak{sl}_2)$-module ${}^{\vartheta}V$;*

(ii) *the action of x on the $L(\mathfrak{sl}_2)$-module V.*

Proof: By Lemma 2.43. □

Lemma 2.59 *Let V denote an $L(\mathfrak{sl}_2)$-module. The following are the same:*

(i) *the action of $\mathcal{H}$ on the $L(\mathfrak{sl}_2)$-module ${}^{\vartheta}V$;*

(ii) *the action of $-\mathcal{H}$ on the $L(\mathfrak{sl}_2)$-module V.*

Proof: By Lemma 2.40. □

Lemma 2.60 *Let $\mathbb{V}_d(a)$ denote an evaluation $L(\mathfrak{sl}_2)$-module. Then the $L(\mathfrak{sl}_2)$-modules ${}^{\vartheta}\mathbb{V}_d(a)$ and $\mathbb{V}_d(a^{-1})$ are isomorphic. Let $\{v_i\}_{i=0}^{d}$ denote a normalized h-eigenbasis of the $\mathfrak{sl}_2$-module $\mathbb{V}_d$. There exists an isomorphism of $L(\mathfrak{sl}_2)$-modules ${}^{\vartheta}\mathbb{V}_d(a) \to \mathbb{V}_d(a^{-1})$ that sends $v_i \mapsto v_{d-i}$ for $0 \leq i \leq d$.*

Proof: Let γ denote the $\mathbb{F}$-linear transformation ${}^{\vartheta}\mathbb{V}_d(a) \to \mathbb{V}_d(a^{-1})$ that sends $v_i \mapsto v_{d-i}$ for $0 \leq i \leq d$. By construction γ is an isomorphism of vector spaces. To show that γ is a homomorphism of $L(\mathfrak{sl}_2)$-modules, we show that for $Z \in \{\mathcal{A}, \mathcal{B}, \mathcal{H}\}$,

$$\gamma(Z.v_i) = Z.(\gamma(v_i)) \tag{2.23}$$

for $0 \leq i \leq d$, where the action on the left in (2.23) is the ${}^{\vartheta}\mathbb{V}_d(a)$-action and the one on the right in (2.23) is the $\mathbb{V}_d(a^{-1})$-action. We routinely verify (2.23) using Lemma 2.58, Lemma 2.59, and the data in Note 2.53. □

2.8 Evaluation modules for $\mathcal{O}$

In this section we discuss a type of $\mathcal{O}$-module called an evaluation $\mathcal{O}$-module. This is an $\mathcal{O}$-module that is obtained from an evaluation $L(\mathfrak{sl}_2)$-module by restricting the $L(\mathfrak{sl}_2)$-action to $\mathcal{O}$.

Definition 2.61 Let V and V' denote $L(\mathfrak{sl}_2)$-modules. We restrict the $L(\mathfrak{sl}_2)$-action on V (resp. V') to $\mathcal{O}$ to get an $\mathcal{O}$-action on V (resp. V'). The $L(\mathfrak{sl}_2)$-modules V and V' are said to be *related* whenever the resulting $\mathcal{O}$-modules are isomorphic.

Consider the set $\mathbb{F}\backslash\{0\}$ of nonzero scalars in $\mathbb{F}$.

Definition 2.62 We define a binary relation $\sim$ on $\mathbb{F}\backslash\{0\}$ as follows. Let $a, a' \in \mathbb{F}\backslash\{0\}$. Then $a \sim a'$ whenever $a = a'$ or $aa' = 1$. Observe that $\sim$ is an equivalence relation. Let $\mathbb{E}$ denote the set of equivalence classes for $\sim$ on $\mathbb{F}\backslash\{0\}$. For $a \in \mathbb{F}\backslash\{0\}$ let $\overline{a} \in \mathbb{E}$ denote the equivalence class of $\sim$ that contains a. Note that the equivalence classes $\overline{1}$ and $\overline{-1}$ have cardinality one, and every other equivalence class has cardinality two. The equivalence classes of cardinality two will be called *feasible*.

Lemma 2.63 *Let $\mathbb{V}_d(a)$ and $\mathbb{V}_d(a')$ denote evaluation $L(\mathfrak{sl}_2)$-modules. Then these $L(\mathfrak{sl}_2)$-modules are related in the sense of Definition* 2.61 *if and only if $a \sim a'$.*

Proof: Suppose $a \sim a'$. We show that the $L(\mathfrak{sl}_2)$-modules $\mathbb{V}_d(a)$ and $\mathbb{V}_d(a')$ are related. This is trivial if $a = a'$, so suppose $aa' = 1$. By Lemma 2.60, the $L(\mathfrak{sl}_2)$-modules ${}^{\vartheta}\mathbb{V}_d(a)$ and $\mathbb{V}_d(a')$ are isomorphic. Therefore the $L(\mathfrak{sl}_2)$-modules ${}^{\vartheta}\mathbb{V}_d(a)$ and $\mathbb{V}_d(a')$ are related. By Lemma 2.58, the $L(\mathfrak{sl}_2)$-modules $\mathbb{V}_d(a)$ and ${}^{\vartheta}\mathbb{V}_d(a)$ are related. Therefore the $L(\mathfrak{sl}_2)$-modules $\mathbb{V}_d(a)$ and $\mathbb{V}_d(a')$ are related.

Now suppose that the $L(\mathfrak{sl}_2)$-modules $\mathbb{V}_d(a)$ and $\mathbb{V}_d(a')$ are related. We show that $a \sim a'$. Recall the element $\mathfrak{a} \otimes (t + t^{-1}) \in \mathcal{O}$ from Lemma 2.44. Considering the action of $\mathfrak{a} \otimes (t + t^{-1})$, we see by Note 2.55 that $(a + a^{-1})d = (a' + (a')^{-1})d$. Since d is positive, we have $a + a^{-1} = a' + (a')^{-1}$. Rewriting this equation we get $(a - a')(aa' - 1) = 0$. Therefore $a \sim a'$. □

The following definition is motivated by Lemma 2.63.

Definition 2.64 Let d denote a positive integer, and let $b \in \mathbb{E}$. By Lemma 2.63, up to isomorphism there exists a unique $\mathcal{O}$-module $\mathbb{V}_d(b)$ with the following property. For every $a \in b$ the restriction of the $L(\mathfrak{sl}_2)$-module $\mathbb{V}_d(a)$ to $\mathcal{O}$ is isomorphic to $\mathbb{V}_d(b)$.

Definition 2.65 By an *evaluation $\mathcal{O}$-module* we mean an $\mathcal{O}$-module $\mathbb{V}_d(b)$, where $b \in \mathbb{E}$ and d is a positive integer. We call b the *evaluation parameter* of $\mathbb{V}_d(b)$.

We now emphasize a few facts about evaluation $\mathcal{O}$-modules.

Lemma 2.66 *Let $\mathbb{V}_d(b)$ and $\mathbb{V}_{d'}(b')$ denote evaluation $\mathcal{O}$-modules. Then these $\mathcal{O}$-modules are isomorphic if and only if $d = d'$ and $b = b'$.*

Proof: Suppose the $\mathcal{O}$-modules $\mathbb{V}_d(b)$ and $\mathbb{V}_{d'}(b')$ are isomorphic. Isomorphic modules have the same dimension, so $d = d'$. Pick $a \in b$ and $a' \in b'$. Without loss we may assume that the $\mathcal{O}$-action on $\mathbb{V}_d(b)$ (resp. $\mathbb{V}_d(b')$) is obtained by restricting the $L(\mathfrak{sl}_2)$-action on $\mathbb{V}_d(a)$ (resp. $\mathbb{V}_d(a')$) to $\mathcal{O}$. Note that the $L(\mathfrak{sl}_2)$-modules $\mathbb{V}_d(a)$ and $\mathbb{V}_d(a')$ are related. By Lemma 2.63, we get $a \sim a'$. Therefore $b = b'$. This proves the lemma in one direction. The other direction is proved in a similar fashion using Lemma 2.63. □

Let $a \in \mathbb{F}$ be nonzero and recall the Lie algebra homomorphism $EV_a : L(\mathfrak{sl}_2) \to \mathfrak{sl}_2$ from Definition 3.15. Let ev_a denote the restriction of EV_a to $\mathcal{O}$. Then $ev_a : \mathcal{O} \to \mathfrak{sl}_2$ is a Lie algebra homomorphism.

Lemma 2.67 *Let $a \in \mathbb{F}$ be nonzero. Then the map ev_a is surjective if and only if $a \neq \pm 1$.*

Proof: Note that the image of $\mathcal{O}$ under ev_a is the Lie subalgebra of $\mathfrak{sl}_2$ that is generated by the elements $ev_a(\mathcal{A}), ev_a(\mathcal{B})$. These elements are $e + f, ae + a^{-1}f$ respectively. We routinely check that these elements generate $\mathfrak{sl}_2$ if and only if $a \neq \pm 1$. The result follows. □

Lemma 2.68 *Let $b \in \mathbb{E}$. Let V denote an evaluation $\mathcal{O}$-module with evaluation parameter b. Then the $\mathcal{O}$-module V is irreducible if and only if b is feasible in the sense of Definition 2.62. In this case the $\mathcal{O}$-module V has type $(0, 0)$.*

Proof: Write $V = \mathbb{V}_d(b)$ and pick $a \in b$. Without loss we may assume that the $\mathcal{O}$-action on V is obtained by restricting the $L(\mathfrak{sl}_2)$-action on $\mathbb{V}_d(a)$ to $\mathcal{O}$. Observe that the $\mathcal{O}$-action on V is obtained by pulling back the $\mathfrak{sl}_2$-action on $\mathbb{V}_d$ via the map ev_a.

Suppose that b is feasible, so that $a \neq \pm 1$. By Lemma 2.67, the map ev_a is surjective. Since the $\mathfrak{sl}_2$-module $\mathbb{V}_d$ is irreducible, we find that the $\mathcal{O}$-module V is irreducible.

Now suppose that b is not feasible, so that $a = \pm 1$. We show that the $\mathcal{O}$-module V is reducible by displaying a nonzero proper $\mathcal{O}$-submodule of V. By Note 2.53, $\mathcal{A} = \pm \mathcal{B}$ on V. By Definition 2.8, $\mathcal{A}, \mathcal{B}$ generate $\mathcal{O}$. Therefore any eigenvector of $\mathcal{A}$ spans a nonzero proper $\mathcal{O}$-submodule of V.

The last assertion follows by Lemma 2.54(ii). □

Lemma 2.69 *Let $\mathbb{V}_d(b)$ denote an evaluation $\mathcal{O}$-module, with b feasible. With reference to Definition* 2.14*, the $\mathcal{O}$-module $\mathbb{V}_d(b)$ has diameter d and shape polynomial*

$$1 + \lambda + \lambda^2 + \cdots + \lambda^d.$$

Proof: Pick $a \in b$. Without loss we may assume that the $\mathcal{O}$-action on $\mathbb{V}_d(b)$ is obtained by restricting the $L(\mathfrak{sl}_2)$-action on $\mathbb{V}_d(a)$ to $\mathcal{O}$. The result follows by Lemma 2.54. □

Lemma 2.70 *Let V denote an evaluation $\mathcal{O}$-module with feasible evaluation parameter. Then the $\mathcal{O}$-action on V can be extended to precisely two $L(\mathfrak{sl}_2)$-actions on V. The resulting two $L(\mathfrak{sl}_2)$-module structures on V are non-isomorphic. Each of these two $L(\mathfrak{sl}_2)$-module structures is obtained from the other by twisting the $L(\mathfrak{sl}_2)$-action via the automorphism ϑ of $L(\mathfrak{sl}_2)$ from Lemma* 2.40*.*

Proof: Let A, B denote the actions of $\mathcal{A}, \mathcal{B}$ on V respectively. Suppose we are given an $L(\mathfrak{sl}_2)$-action on V that extends the $\mathcal{O}$-action on V. For this $L(\mathfrak{sl}_2)$-action, let H denote the $\mathcal{H}$-action. By Lemma 2.68, the $\mathcal{O}$-module V is irreducible and has type $(0, 0)$. Let d denote the diameter of the $\mathcal{O}$-module V. By construction the set of distinct eigenvalues of A is $\{d-2i|0 \leq i \leq d\}$. By Lemma 2.46, the set of distinct eigenvalues of H is $\{d-2i|0 \leq i \leq d\}$. Recall the notation $V_F(\theta)$ from (2.1). By (2.12) and Lemma 2.1,

$$AV_H(d) \subseteq V_H(d-2), \qquad AV_H(-d) \subseteq V_H(2-d), \tag{2.24}$$

$$AV_H(d-2i) \subseteq V_H(d-2i-2) + V_H(d-2i+2) \qquad (1 \leq i \leq d-1). \tag{2.25}$$

Let b denote the evaluation parameter for the $\mathcal{O}$-module V, so that $V = \mathbb{V}_d(b)$. Pick $a \in b$. Without loss we may assume that the $\mathcal{O}$-action on V is obtained by restricting the $L(\mathfrak{sl}_2)$-action on $\mathbb{V}_d(a)$ to $\mathcal{O}$. Let $\{v_i\}_{i=0}^d$ denote a normalized h-eigenbasis of the $\mathfrak{sl}_2$-module $\mathbb{V}_d$. By construction b is feasible, so $a \neq \pm 1$, and consequently $a \neq a^{-1}$. By this and Note 2.56, for $0 \leq i \leq d$, v_i spans the eigenspace of $[A, B]$ corresponding to eigenvalue $(a^{-1} - a)(d - 2i)v_i$. By (2.15), H commutes with $[A, B]$. Therefore $\{v_i\}_{i=0}^d$ are eigenvectors for H. By Note 2.53,

$$Av_0 = v_1, \qquad Av_d = v_{d-1}, \tag{2.26}$$

$$Av_i = (d+1-i)v_{i-1} + (i+1)v_{i+1} \qquad (1 \leq i \leq d-1). \tag{2.27}$$

Comparing lines (2.24), (2.25) with lines (2.26), (2.27), we see that either v_i is a basis for $V_H(d-2i)$ for $0 \leq i \leq d$, or v_i is a basis for $V_H(2i-d)$ for $0 \leq i \leq d$. Consequently either $Hv_i = (d-2i)v_i$ for $0 \leq i \leq d$, or $Hv_i = (2i-d)v_i$ for $0 \leq i \leq d$. In the former case, H is the $\mathcal{H}$-action on $\mathbb{V}_d(a)$, by Note 2.53. In the latter case, H is the $\mathcal{H}$-action on ${}^{\vartheta}\mathbb{V}_d(a)$, by Lemma 2.59. The $L(\mathfrak{sl}_2)$-modules $\mathbb{V}_d(a)$ and ${}^{\vartheta}\mathbb{V}_d(a)$ are non-isomorphic by Lemma 3.21 and Lemma 2.60. The result follows. $\square$

2.9 Finite-dimensional irreducible modules for $\mathcal{O}$ and $L(\mathfrak{sl}_2)$

In Section 2.7 we discussed evaluation $L(\mathfrak{sl}_2)$-modules. In Section 2.8 we discussed evaluation $\mathcal{O}$-modules, and we discussed how these are related to evaluation $L(\mathfrak{sl}_2)$-modules. In this section we consider general finite-dimensional irreducible $L(\mathfrak{sl}_2)$-modules and $\mathcal{O}$-modules. We then discuss how these $L(\mathfrak{sl}_2)$-modules and $\mathcal{O}$-modules are related. First we make a comment. Let L denote a Lie algebra over $\mathbb{F}$, and let U, V denote L-modules. By [8, p. 26], $U \otimes V$ has an L-module structure such that

$$x(u \otimes v) = (xu) \otimes v + u \otimes (xv) \qquad x \in L, \quad u \in U, \quad v \in V. \tag{2.28}$$

The following lemma is routinely checked.

Lemma 2.71 *Let L denote a Lie algebra over $\mathbb{F}$, and let U, V denote L-modules. Then the following hold.*

(i) *There exists an L-module isomorphism $U \otimes V \to V \otimes U$ that sends $u \otimes v \mapsto v \otimes u$ for all $u \in U$ and $v \in V$.*

(ii) *Assume the L-module $U \otimes V$ is irreducible. Then U and V are irreducible.*

The classification of the finite-dimensional irreducible $L(\mathfrak{sl}_2)$-modules is given in the following theorem.

Theorem 2.72 [2] *Every finite-dimensional irreducible $L(\mathfrak{sl}_2)$-module is isomorphic to a tensor product of evaluation $L(\mathfrak{sl}_2)$-modules. Two such tensor products are isomorphic as $L(\mathfrak{sl}_2)$-modules if and only if one can be obtained from the other by permuting the* *factors in the tensor product. A tensor product of evaluation $L(\mathfrak{sl}_2)$-modules $\otimes_{i=1}^n \mathbb{V}_{d_i}(a_i)$ is irreducible if and only if the $\{a_i\}_{i=1}^n$ are mutually distinct.*

We now give the classification of the finite-dimensional irreducible $\mathcal{O}$-modules. By the comments in Section 2.3, it suffices to classify the finite-dimensional irreducible $\mathcal{O}$-modules of type $(0,0)$.

Theorem 2.73 [3, Proposition 5, Theorem 6] *Every finite-dimensional irreducible $\mathcal{O}$-module of type $(0,0)$ is isomorphic to a tensor product of evaluation $\mathcal{O}$-modules. Two such tensor products are isomorphic as $\mathcal{O}$-modules if and only if one can be obtained from the other by permuting the factors in the tensor product. A tensor product of evaluation $\mathcal{O}$-modules $\otimes_{i=1}^n \mathbb{V}_{d_i}(b_i)$ is irreducible if and only if the $\{b_i\}_{i=1}^n$ are mutually distinct and feasible.*

Definition 2.74 Let V denote a finite-dimensional irreducible $L(\mathfrak{sl}_2)$-module (resp. $\mathcal{O}$-module of type $(0,0)$). By Theorem 2.72 (resp. Theorem 2.73) there exists a unique $n \in \mathbb{N}$ such that the $L(\mathfrak{sl}_2)$-module (resp. $\mathcal{O}$-module) V is isomorphic to a tensor product of n evaluation $L(\mathfrak{sl}_2)$-modules (resp. $\mathcal{O}$-modules). We call n the *tensor degree* of V. If V is the trivial $L(\mathfrak{sl}_2)$-module or the trivial $\mathcal{O}$-module, we interpret the tensor degree to be zero.

In Theorems 2.72 and 2.73 we discussed the finite-dimensional irreducible modules for $L(\mathfrak{sl}_2)$ and $\mathcal{O}$. We now discuss how these $\mathcal{O}$-modules and $L(\mathfrak{sl}_2)$-modules are related.

Definition 2.75 Let V denote a finite-dimensional irreducible $L(\mathfrak{sl}_2)$-module. By Theorem 2.72, the $L(\mathfrak{sl}_2)$-module V is isomorphic to a tensor product $\otimes_{i=1}^n \mathbb{V}_{d_i}(a_i)$ of evaluation $L(\mathfrak{sl}_2)$-modules. The $L(\mathfrak{sl}_2)$-module V is said to be *inverse-free* whenever $a_i a_j \neq 1$ for $1 \leq i, j \leq n$.

Note 2.76 Referring to Definition 2.75, the $L(\mathfrak{sl}_2)$-module V is inverse-free if and only if the $\{\overline{a_i}\}_{i=1}^n$ are mutually distinct and feasible.

Proposition 2.77 *The following hold.*

(i) *Let V denote a finite-dimensional irreducible $L(\mathfrak{sl}_2)$-module that is inverse-free. When the $L(\mathfrak{sl}_2)$-action on V is restricted to $\mathcal{O}$, the resulting $\mathcal{O}$-module is irreducible, with type $(0,0)$. Moreover, the $L(\mathfrak{sl}_2)$-module structure on V and the $\mathcal{O}$-module structure on V have the same tensor degree.*

(ii) *Let V denote a finite-dimensional irreducible $\mathcal{O}$-module, with type $(0,0)$ and tensor degree n. Then the $\mathcal{O}$-action on V can be extended to precisely 2^n $L(\mathfrak{sl}_2)$-actions on V. The resulting 2^n $L(\mathfrak{sl}_2)$-module structures on V are irreducible, inverse-free, and mutually non-isomorphic.*

The proof of Proposition 2.77(i) is routine using Theorem 2.72, Theorem 2.73, and Note 2.76. The proof of Proposition 2.77(ii) will be completed shortly. This proof will involve the following lemma.

Lemma 2.78 *Let V denote a finite-dimensional irreducible $\mathcal{O}$-module of type $(0,0)$. Assume we are given two $L(\mathfrak{sl}_2)$-actions on V that extend the $\mathcal{O}$-action on V. Then the following are equivalent:*

(i) *the two $L(\mathfrak{sl}_2)$-module structures on V are isomorphic;*

(ii) *the two $L(\mathfrak{sl}_2)$-actions on V are the same.*

Proof: (i) $\Rightarrow$ (ii) Let A, B denote the actions of $\mathcal{A}, \mathcal{B}$ on V respectively. Let H, H' denote the $\mathcal{H}$-actions on V afforded by the given $L(\mathfrak{sl}_2)$-actions. We show that $H = H'$. By construction, there exists a vector space isomorphism $\gamma : V \to V$ such that γ commutes with each of A, B and

$$\gamma H = H'\gamma. \tag{2.29}$$

Let $W \subseteq V$ denote an eigenspace of γ. By Definition 2.8, $\mathcal{A}, \mathcal{B}$ generate $\mathcal{O}$. Therefore W is a nonzero $\mathcal{O}$-submodule of V. The $\mathcal{O}$-module V is irreducible, so $W = V$. Therefore γ is a scalar multiple of the identity element of $\mathrm{End}(V)$. Since γ is invertible, that scalar is nonzero. Combining this with (2.29) we get $H = H'$. The result follows since $\mathcal{A}, \mathcal{B}, \mathcal{H}$ generate the Lie algebra $L(\mathfrak{sl}_2)$.

(ii) $\Rightarrow$ (i) Immediate. □

Proof of Proposition 2.77(ii)*:* Invoking Theorem 2.73, we identify the $\mathcal{O}$-module V with a tensor product $\otimes_{i=1}^{n} \mathbb{V}_{d_i}(b_i)$ of evaluation $\mathcal{O}$-modules, with the $\{b_i\}_{i=1}^{n}$ mutually distinct and feasible. By construction $\{b_i\}_{i=1}^{n}$ are mutually disjoint and each b_i has cardinality two. Consider the $L(\mathfrak{sl}_2)$-modules

$$\otimes_{i=1}^{n} \mathbb{V}_{d_i}(a_i) \qquad\qquad a_i \in b_i, \qquad 1 \leq i \leq n. \tag{2.30}$$

For each $L(\mathfrak{sl}_2)$-module in (2.30) the restriction to $\mathcal{O}$ is isomorphic to the $\mathcal{O}$-module V. By Theorem 2.72, the $L(\mathfrak{sl}_2)$-modules (2.30) are irreducible and mutually non-isomorphic. By Note 2.76, each of these $L(\mathfrak{sl}_2)$-modules is inverse-free. Suppose we are given an $L(\mathfrak{sl}_2)$-action on V that extends the $\mathcal{O}$-action on V. By Theorem 2.72, the resulting $L(\mathfrak{sl}_2)$-module V is isomorphic to a tensor product $\otimes_{j=1}^{n'} \mathbb{V}_{d'_j}(a'_j)$ of evaluation $L(\mathfrak{sl}_2)$-modules. By Theorem 2.73, we have $n = n'$, and up to a permutation of $\{\mathbb{V}_{d'_i}(a'_i)\}_{i=1}^{n}$ we have $d_i = d'_i$ and $a'_i \in b_i$ for $1 \leq i \leq n$. Therefore the $L(\mathfrak{sl}_2)$-module V is isomorphic to one of the $L(\mathfrak{sl}_2)$-modules (2.30). The result follows routinely using

Lemma 2.78. □

Definition 2.79 Let V denote a finite-dimensional irreducible $L(\mathfrak{sl}_2)$-module that is inverse-free. When the $L(\mathfrak{sl}_2)$-action on V is restricted to $\mathcal{O}$, the resulting $\mathcal{O}$-module is irreducible, with type $(0,0)$, by Proposition 2.77(i). We say this $\mathcal{O}$-module is *associated* with the $L(\mathfrak{sl}_2)$-module V. By the *diameter* (resp. *shape*) (resp. *shape polynomial*) of the $L(\mathfrak{sl}_2)$-module V we mean the diameter (resp. shape) (resp. shape polynomial) of the associated $\mathcal{O}$-module, in the sense of Definition 2.14.

2.10 Finite-dimensional irreducible modules for $\mathcal{O}$ and $L(\mathfrak{sl}_2)$; the shape polynomial

In this section we continue to discuss a finite-dimensional irreducible $\mathcal{O}$-module V of type $(0,0)$. We will obtain some results about the shape polynomial S_V from Definition 2.14. We then discuss the relationship between the shape and the tensor degree of V. At the end of the section, we take the results earlier in the section and apply them to obtain results for finite-dimensional irreducible $L(\mathfrak{sl}_2)$-modules.

Let V, V' denote finite-dimensional irreducible $\mathcal{O}$-modules of type $(0,0)$ such that the $\mathcal{O}$-module $V \otimes V'$ is irreducible. In this case the $\mathcal{O}$-module $V \otimes V'$ has type $(0,0)$. We are going to show that $S_{V\otimes V'} = S_V S_{V'}$. We will use the following three lemmas.

Lemma 2.80 *Let V, V' denote finite-dimensional irreducible $\mathcal{O}$-modules of type $(0,0)$ such that the $\mathcal{O}$-module $V \otimes V'$ is irreducible. Let d (resp. d') denote the diameter of V (resp. V'). Then the diameter of $V \otimes V'$ is $d + d'$.*

Proof: Routine using (3.7). □

Lemma 2.81 *Adopt the notation of Lemma* 2.80*. For* $X \in \{\mathcal{A}, \mathcal{B}\}$ *the eigenspaces of the action of* X *on* $V \otimes V'$ *are given as follows. For* $0 \leq r \leq d$ *(resp.* $0 \leq s \leq d'$*) let* V_r *(resp.* V'_s*) denote the eigenspace of the action of* X *on* V *(resp.* V'*) corresponding to the eigenvalue* $d - 2r$ *(resp.* $d' - 2s$*). For* $0 \leq n \leq d + d'$ *the eigenspace of the action of* X *on* $V \otimes V'$ *corresponding to the eigenvalue* $d + d' - 2n$ *is given by*

$$\sum_{r,s} V_r \otimes V'_s, \tag{2.31}$$

where the sum is over all ordered pairs r, s *such that* $0 \leq r \leq d$*,* $0 \leq s \leq d'$*,* $r + s = n$*.*

Proof: Routine using (3.7). □

Lemma 2.82 *Adopt the notation of Lemmas* 2.80*,* 2.81*. Let* $\{\rho_r\}_{r=0}^{d}$ *(resp.* $\{\rho'_s\}_{s=0}^{d'}$*) denote the shape of the* $\mathcal{O}$*-module* V *(resp.* V'*). Then the vector space in line* (2.31) *has dimension*

$$\sum_{r,s} \rho_r \rho'_s,$$

where the sum is over all ordered pairs r, s *such that* $0 \leq r \leq d$*,* $0 \leq s \leq d'$*,* $r + s = n$*.*

Proof: The sum in (2.31) is direct. The result follows. □

Proposition 2.83 *Let* V, V' *denote finite-dimensional irreducible* $\mathcal{O}$*-modules of type* $(0, 0)$ *such that the* $\mathcal{O}$*-module* $V \otimes V'$ *is irreducible. Then*

$$S_{V \otimes V'} = S_V S_{V'}.$$

Proof: Routine using Lemmas 2.80–2.82. □

Proposition 2.84 *Let V denote a finite-dimensional irreducible $\mathcal{O}$-module, with type* $(0,0)$*, shape* $\{\rho_i\}_{i=0}^{d}$*, and tensor degree* n*. By Theorem* 2.73*, the $\mathcal{O}$-module V is isomorphic to a tensor product* $\otimes_{j=1}^{n}\mathbb{V}_{d_j}(b_j)$ *of evaluation $\mathcal{O}$-modules. Then*

$$\sum_{i=0}^{d}\rho_i\lambda^i = \prod_{j=1}^{n}(1+\lambda+\lambda^2+\cdots+\lambda^{d_j}). \tag{2.32}$$

Proof: By construction $S_V = \sum_{i=0}^{d}\rho_i\lambda^i$. Combine Lemma 2.69 and Proposition 2.83. □

Remark 2.85 With reference to Proposition 2.84, we have $d = \sum_{j=1}^{n} d_j$.

Let V denote a nontrivial finite-dimensional irreducible $\mathcal{O}$-module of type $(0,0)$. We now discuss the relationship between the shape of V and the tensor degree of V. Note that the diameter of V is at least one since V is nontrivial.

Corollary 2.86 *For a nontrivial finite-dimensional irreducible $\mathcal{O}$-module of type* $(0,0)$ *with shape* $\{\rho_i\}_{i=0}^{d}$ *the tensor degree is given by* ρ_1.

Proof: Adopt the notation in Proposition 2.84. Comparing the coefficient of λ from each side of (2.32) we get $\rho_1 = n$. The result follows. □

Proposition 2.84, Remark 2.85, and Corollary 2.86 are about finite-dimensional irreducible $\mathcal{O}$-modules of type $(0,0)$. We now obtain similar results for finite-dimensional irreducible $L(\mathfrak{sl}_2)$-modules that are inverse-free.

Proposition 2.87 *Let V denote a finite-dimensional irreducible $L(\mathfrak{sl}_2)$-module that is inverse-free, with shape* $\{\rho_i\}_{i=0}^{d}$ *and tensor degree* n*. By Theorem* 2.72*, the $L(\mathfrak{sl}_2)$-module V is isomorphic to a tensor product* $\otimes_{j=1}^{n}\mathbb{V}_{d_j}(a_j)$ *of evaluation $L(\mathfrak{sl}_2)$-modules. Then*

$$\sum_{i=0}^{d}\rho_i\lambda^i = \prod_{j=1}^{n}(1+\lambda+\lambda^2+\cdots+\lambda^{d_j}). \tag{2.33}$$

Proof: Combine Definition 2.79 and Proposition 2.84. □

Remark 2.88 With reference to Proposition 2.87, we have $d = \sum_{j=1}^{n} d_j$.

Corollary 2.89 *For a nontrivial finite-dimensional irreducible $L(\mathfrak{sl}_2)$-module that is inverse-free, with shape $\{\rho_i\}_{i=0}^{d}$, the tensor degree is given by ρ_1.*

Proof: Similar to the proof of Corollary 2.86. □

2.11 Compatible elements

For the moment let V denote a finite-dimensional irreducible $\mathcal{O}$-module, with type $(0,0)$ and tensor degree n. By Proposition 2.77(ii), the $\mathcal{O}$-action on V can be extended to precisely 2^n $L(\mathfrak{sl}_2)$-actions on V. Our next general goal is to describe in detail how these 2^n extensions are related to one another. In this description we make use of Theorem 2.34. To aid in this description we introduce the notion of a compatible element for a tridiagonal pair of Krawtchouk type.

Definition 2.90 Let V denote a vector space over $\mathbb{F}$ with finite positive dimension. Let A, B denote a tridiagonal pair on V that has Krawtchouk type. An element $H \in \mathrm{End}(V)$ is said to be *compatible* with A, B whenever the following relations hold:

$$[A,[A,H]] = 4H, \qquad [H,[H,A]] = 4A, \tag{2.34}$$

$$[B,[B,H]] = 4H, \qquad [H,[H,B]] = 4B, \tag{2.35}$$

$$[H,[A,B]] = 0. \tag{2.36}$$

Let $\mathrm{Com}(A, B)$ denote the set of elements in $\mathrm{End}(V)$ that are compatible with A, B.

Remark 2.91 Referring to Definition 2.90, let $H \in \text{Com}(A, B)$. Then H commutes with $[A, B]$ by (2.36).

Remark 2.92 Let A, B denote a tridiagonal pair of Krawtchouk type. We mentioned at the beginning of Section 2.3 that B, A is a tridiagonal pair of Krawtchouk type. By Definition 2.90, $\text{Com}(A, B) = \text{Com}(B, A)$.

From now until the end of Remark 2.95, V will denote a finite-dimensional irreducible $\mathcal{O}$-module of type $(0, 0)$. Let A, B denote the tridiagonal pair that is associated with the $\mathcal{O}$-module V, in the sense of Definition 2.14. Consider the set $\text{Com}(A, B)$. We now explain how the elements of $\text{Com}(A, B)$ are related to the $L(\mathfrak{sl}_2)$-actions on V that extend the $\mathcal{O}$-action on V.

Lemma 2.93 *Consider an $L(\mathfrak{sl}_2)$-action on V that extends the $\mathcal{O}$-action on V. For the $L(\mathfrak{sl}_2)$-module V, the action of $\mathcal{H}$ on V is an element of* $\text{Com}(A, B)$.

Proof: Compare (2.12), (2.13), (2.15) with (2.34), (2.35), (2.36). □

Lemma 2.94 *Let $H \in \text{Com}(A, B)$. Then there exists a unique $L(\mathfrak{sl}_2)$-action on V that extends the $\mathcal{O}$-action on V, such that the element $\mathcal{H}$ of $L(\mathfrak{sl}_2)$ acts on V as H.*

Proof: Compare (2.12), (2.13), (2.15) with (2.34), (2.35), (2.36). □

Remark 2.95 Combining Lemmas 2.93 and 2.94 we obtain a bijection between the following two sets:

(i) $\text{Com}(A, B)$;

(ii) the $L(\mathfrak{sl}_2)$-actions on V that extend the $\mathcal{O}$-action on V.

In view of Remark 2.95, we will describe the $L(\mathfrak{sl}_2)$-actions on V that extend the $\mathcal{O}$-action on V by describing the set $\mathrm{Com}(A, B)$.

Definition 2.96 For a tridiagonal pair of Krawtchouk type, we define its *tensor degree* to be the tensor degree of the associated $\mathcal{O}$-module from Definition 2.14.

Remark 2.97 Let A, B denote a tridiagonal pair that has Krawtchouk type and diameter d. First assume $d = 0$. By Definitions 2.74 and 2.96, the tensor degree of A, B is zero. Now assume $d \geq 1$. Let $\{\rho_i\}_{i=0}^{d}$ denote the shape of A, B. By Definition 2.96 and Corollary 2.86, the tensor degree of A, B is equal to ρ_1.

Lemma 2.98 *Let A, B denote a tridiagonal pair that has Krawtchouk type and tensor degree ρ. Then $\mathrm{Com}(A, B)$ has cardinality 2^ρ.*

Proof: Combine Proposition 2.77(ii) and Remark 2.95. □

Remark 2.99 Referring to Remark 2.97 and Lemma 2.98, if the diameter of A, B is zero, then the tensor degree of A, B is zero, so the set $\mathrm{Com}(A, B)$ has cardinality 1. In this case, $\mathrm{Com}(A, B)$ consists of the zero map on V. In order to avoid trivialities, for the remainder of this section we will assume that the diameter of A, B is at least 1.

Theorem 2.100 *Let V denote a vector space over $\mathbb{F}$ with finite positive dimension. Let A, B denote a tridiagonal pair on V that has Krawtchouk type, with diameter at least 1 and tensor degree ρ. Then there exist elements $\{\mathcal{H}_i\}_{i=1}^{\rho}$ in $\mathrm{End}(V)$ such that*

$$\mathrm{Com}(A, B) = \left\{ \sum_{i=1}^{\rho} \varepsilon_i \mathcal{H}_i \;\middle|\; \varepsilon_i = \pm 1, \quad 1 \leq i \leq \rho \right\}. \tag{2.37}$$

The elements $\{\mathcal{H}_i\}_{i=1}^{\rho}$ are uniquely determined up to sign and permutation. These elements are linearly independent, they mutually commute, and they are diagonalizable on

V. Moreover, the elements of $\mathrm{Com}(A, B)$ *mutually commute and are diagonalizable on V.*

We will prove Theorem 2.100 shortly. For the rest of this paragraph, we will set some notation that will remain in effect for the rest of this section. Let V denote a vector space over $\mathbb{F}$ with finite positive dimension. Let A, B denote a tridiagonal pair on V of Krawtchouk type, with diameter $d \geq 1$ and shape $\{\rho_i\}_{i=0}^d$. Abbreviate $\rho = \rho_1$. By Remark 2.97, the tensor degree of A, B is equal to ρ. Fix $H \in \mathrm{Com}(A, B)$. By Lemma 2.94, there exists a unique $L(\mathfrak{sl}_2)$-module structure on V such that the $L(\mathfrak{sl}_2)$-generators $\mathcal{A}, \mathcal{B}, \mathcal{H}$ act on V as A, B, H respectively. Invoking Theorem 2.72, we identify the $L(\mathfrak{sl}_2)$-module V with a tensor product of evaluation $L(\mathfrak{sl}_2)$-modules:

$$V = \mathbb{V}_{d_1}(a_1) \otimes \cdots \otimes \mathbb{V}_{d_\rho}(a_\rho). \tag{2.38}$$

Recall the indeterminate λ.

Proposition 2.101 *The sequences $\{\rho_i\}_{i=0}^d$ and $\{d_j\}_{j=1}^\rho$ determine each other via the polynomial identity*

$$\sum_{i=0}^{d} \rho_i \lambda^i = \prod_{j=1}^{\rho} (1 + \lambda + \lambda^2 + \cdots + \lambda^{d_j}).$$

Proof: Use Proposition 2.87. □

Remark 2.102 With reference to Proposition 2.101, we have $d = \sum_{j=1}^{\rho} d_j$.

Next we define elements $\{\mathcal{H}_i\}_{i=1}^\rho$ of $\mathrm{End}(V)$ that satisfy (2.37).

Definition 2.103 For $Z \in L(\mathfrak{sl}_2)$ and $1 \leq i \leq \rho$, define Z_i to be the element of $\mathrm{End}(V)$ that sends

$$u_1 \otimes \cdots \otimes u_\rho \mapsto u_1 \otimes \cdots \otimes u_{i-1} \otimes Z u_i \otimes u_{i+1} \otimes \cdots \otimes u_\rho,$$

where $u_j \in \mathbb{V}_{d_j}(a_j)$ for $1 \leq j \leq \rho$. We call Z_i the i^{th} *part* of Z.

Remark 2.104 Referring to Definition 2.103, the element $Z \in L(\mathfrak{sl}_2)$ acts on V as $\sum_{i=1}^{\rho} Z_i$.

Lemma 2.105 *For $Z \in L(\mathfrak{sl}_2)$ the elements $\{Z_i\}_{i=1}^{\rho}$ mutually commute.*

Proof: Routine using Definition 2.103. □

For the element $\mathcal{H}$ of $L(\mathfrak{sl}_2)$, consider the corresponding elements $\{\mathcal{H}_i\}_{i=1}^{\rho}$ from Definition 2.103. Recall that $\mathcal{H}$ acts on V as H. By Remark 2.104, $\mathcal{H}$ acts on V as $\sum_{i=1}^{\rho} \mathcal{H}_i$. Therefore $H = \sum_{i=1}^{\rho} \mathcal{H}_i$. Note that by Lemma 2.105, the $\{\mathcal{H}_i\}_{i=1}^{\rho}$ mutually commute.

Lemma 2.106 *The elements $\{\mathcal{H}_i\}_{i=1}^{\rho}$ satisfy* (2.37).

Proof: Recall the automorphism ϑ of $L(\mathfrak{sl}_2)$ from Lemma 2.40. For $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_\rho) \in \{\pm 1\}^\rho$ define the $L(\mathfrak{sl}_2)$-module V_ε by

$$V_\varepsilon = U_1 \otimes \cdots \otimes U_\rho, \qquad (2.39)$$

where for $1 \leq i \leq \rho$, $U_i = \mathbb{V}_{d_i}(a_i)$ if $\varepsilon_i = 1$ and $U_i = {}^{\vartheta}\mathbb{V}_{d_i}(a_i)$ if $\varepsilon_i = -1$. By Lemma 2.60 and Theorem 2.72, the $L(\mathfrak{sl}_2)$-modules

$$V_\varepsilon \qquad\qquad \varepsilon \in \{\pm 1\}^\rho \qquad (2.40)$$

are mutually non-isomorphic. By Lemma 2.58, the elements $\mathcal{A}, \mathcal{B}$ of $L(\mathfrak{sl}_2)$ act as A, B on each of the $L(\mathfrak{sl}_2)$-modules in (2.40). By Lemma 2.59, we find that for $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_\rho) \in \{\pm 1\}^\rho$, the element $\mathcal{H}$ acts on V_ε as $\sum_{i=1}^{\rho} \varepsilon_i \mathcal{H}_i$. Equation (2.37) follows from this along with Proposition 2.77(ii) and Remark 2.95. □

Refer to (2.38). For $1 \le i \le \rho$ let $\{v_j^{(i)}\}_{j=0}^{d_i}$ denote a normalized h-eigenbasis for the $\mathfrak{sl}_2$-module $\mathbb{V}_{d_i}$, in the sense of Definition 2.20. Let $\mathbb{I}$ denote the set of ρ-tuples $(k_1, \dots, k_\rho)$ of integers such that $0 \le k_i \le d_i$ for $1 \le i \le \rho$. For $k = (k_1, \dots, k_\rho) \in \mathbb{I}$ define the vector $v_k \in V$ by

$$v_k = v_{k_1}^{(1)} \otimes v_{k_2}^{(2)} \otimes \cdots \otimes v_{k_\rho}^{(\rho)}.$$

Note that the elements $\{v_k\}_{k \in \mathbb{I}}$ form a basis for V. For $1 \le i \le \rho$, let $\mathbf{e}_i$ denote the element of $\mathbb{F}^\rho$ with a 1 in the i^{th} coordinate and 0 in all other coordinates. For all $k \in \mathbb{F}^\rho$ we define $v_k = 0$ whenever $k \notin \mathbb{I}$.

Lemma 2.107 *The following* (i)–(iii) *hold for all* $k = (k_1, \dots, k_\rho) \in \mathbb{I}$.

(i) $Av_k = \sum_{i=1}^{\rho} (d_i + 1 - k_i) v_{k-\mathbf{e}_i} + (k_i + 1) v_{k+\mathbf{e}_i}$.

(ii) $Bv_k = \sum_{i=1}^{\rho} (d_i + 1 - k_i) a_i v_{k-\mathbf{e}_i} + (k_i + 1) a_i^{-1} v_{k+\mathbf{e}_i}$.

(iii) $Hv_k = \sum_{i=1}^{\rho} (d_i - 2k_i) v_k$.

Proof: Routine using Note 2.53 and (3.7). □

Lemma 2.108 *For* $k = (k_1, \dots, k_\rho) \in \mathbb{I}$, v_k *is a common eigenvector for* $\{\mathcal{H}_i\}_{i=1}^{\rho}$. *The corresponding eigenvalues are as follows. For* $1 \le i \le \rho$, v_k *is an eigenvector for* $\mathcal{H}_i$ *with eigenvalue* $d_i - 2k_i$.

Proof: Routine using Note 2.53 and Definition 2.103. □

Lemma 2.109 *The following* (i), (ii) *hold for* $1 \le i \le \rho$.

(i) $\mathcal{H}_i$ *is diagonalizable.*

(ii) *The set of distinct eigenvalues of* $\mathcal{H}_i$ *is* $\{d_i - 2j \,|\, 0 \le j \le d_i\}$.

Proof: Use Lemma 2.108. □

Lemma 2.110 *The common eigenspaces for* $\{\mathcal{H}_i\}_{i=1}^{\rho}$ *all have dimension* 1*. These common eigenspaces are* $\{\mathbb{F}v_k\}_{k \in \mathbb{I}}$.

Proof: Use Lemma 2.108. □

Lemma 2.111 *The elements* $\{\mathcal{H}_i\}_{i=1}^{\rho}$ *are linearly independent.*

Proof: Let $\{s_i\}_{i=1}^{\rho}$ denote scalars in $\mathbb{F}$, and assume

$$\sum_{1 \le i \le \rho} s_i \mathcal{H}_i = 0. \tag{2.41}$$

We prove $s_i = 0$ for $1 \le i \le \rho$. Let i be given. Consider the elements $k = (0, \ldots, 0)$ and $k' = (0, \ldots, 0, d_i, 0, \ldots, 0)$ of $\mathbb{I}$. By applying both sides of (2.41) to each of $v_k, -v_{k'}$ we see that

$$\sum_{1 \le r \le \rho} s_r d_r = 0, \qquad 2 s_i d_i - \sum_{1 \le r \le \rho} s_r d_r = 0. \tag{2.42}$$

In (2.42) we add the two equations to get $s_i d_i = 0$. Recall that d_i is positive, so $s_i = 0$. □

We are now ready to prove Theorem 2.100.

Proof of Theorem 2.100*:* Recall the elements $\{\mathcal{H}_i\}_{i=1}^{\rho}$ from above Lemma 2.106. By Lemma 2.106 and the comments immediately preceding it, the $\{\mathcal{H}_i\}_{i=1}^{\rho}$ mutually commute and satisfy (2.37). By Lemmas 2.109 and 2.111, the $\{\mathcal{H}_i\}_{i=1}^{\rho}$ are linearly independent and diagonalizable on V. Moreover, the elements of $\mathrm{Com}(A, B)$ mutually commute and are diagonalizable on V. It remains to prove the uniqueness claim. Define the set

$$S = \{(H - X)/2 \,|\, X \in \mathrm{Com}(A, B)\}.$$

Using (2.37) one routinely checks that

$$S = \left\{ \quad \sum_{i=1}^{\rho} s_i \mathcal{H}_i \quad \middle| \quad s_i \in \{0, 1\}, \quad 1 \leq i \leq \rho \quad \right\}. \tag{2.43}$$

Let $\{\mathcal{H}'_i\}_{i=1}^{\rho}$ denote elements in $\mathrm{End}(V)$ satisfying (2.37). Changing the signs of $\{\mathcal{H}'_i\}_{i=1}^{\rho}$ if necessary, we may assume that $H = \sum_{i=1}^{\rho} \mathcal{H}'_i$. We will show that the sequence $\{\mathcal{H}'_i\}_{i=1}^{\rho}$ is a permutation of the sequence $\{\mathcal{H}_i\}_{i=1}^{\rho}$. Observe that

$$S = \left\{ \quad \sum_{i=1}^{\rho} s_i \mathcal{H}'_i \quad \middle| \quad s_i \in \{0, 1\}, \quad 1 \leq i \leq \rho \quad \right\}. \tag{2.44}$$

Comparing (2.43) and (2.44) one routinely checks that $\{\mathcal{H}_i\}_{i=1}^{\rho}$ and $\{\mathcal{H}'_i\}_{i=1}^{\rho}$ span the same subspace of $\mathrm{End}(V)$. Since the elements $\{\mathcal{H}_i\}_{i=1}^{\rho}$ are linearly independent, so are the elements $\{\mathcal{H}'_i\}_{i=1}^{\rho}$. Let M denote the transition matrix from $\{\mathcal{H}_i\}_{i=1}^{\rho}$ to $\{\mathcal{H}'_i\}_{i=1}^{\rho}$. By (2.43) and (2.44), each entry of M is either 0 or 1. Observe that

$$\sum_{i=1}^{\rho} \mathcal{H}_i = H = \sum_{i=1}^{\rho} \mathcal{H}'_i = \sum_{i=1}^{\rho} \sum_{j=1}^{\rho} M_{ij} \mathcal{H}_i. \tag{2.45}$$

For $1 \leq i \leq \rho$, we compare the coefficients of $\{\mathcal{H}_i\}_{i=1}^{\rho}$ in (2.45) to obtain $\sum_{j=1}^{\rho} M_{ij} = 1$. Therefore each row of M has exactly one entry equal to 1. Since the $\{\mathcal{H}'_i\}_{i=1}^{\rho}$ are all nonzero, each column of M has at least one entry equal to 1. Therefore M is a permutation matrix. This shows that the sequence $\{\mathcal{H}'_i\}_{i=1}^{\rho}$ is a permutation of the sequence $\{\mathcal{H}_i\}_{i=1}^{\rho}$. □

For $1 \leq i \leq \rho$, we now consider the action of A, B on the eigenspaces of $\mathcal{H}_i$.

Proposition 2.112 *For* $1 \leq i \leq \rho$ *and* $0 \leq j \leq d_i$*, let* W_j *denote the eigenspace of* $\mathcal{H}_i$ *corresponding to the eigenvalue* $d_i - 2j$*. Then*

$$AW_j \subseteq W_{j-1} + W_j + W_{j+1}, \qquad BW_j \subseteq W_{j-1} + W_j + W_{j+1} \qquad (1 \leq j \leq d_i),$$

where $W_{-1} = 0$ *and* $W_{d_i+1} = 0$*.*

Proof: By Lemma 2.108, W_j is spanned by the vectors v_k such that $k = (k_1, \ldots, k_\rho) \in \mathbb{I}$ and $k_i = j$. The result follows by Lemma 2.107. □

Definition 2.113 Let $\mathcal{C}$ denote the subspace of $\mathrm{End}(V)$ spanned by $\mathrm{Com}(A, B)$.

Remark 2.114 By Theorem 2.100 and Definition 2.113, the elements $\{\mathcal{H}_i\}_{i=1}^{\rho}$ form a basis for $\mathcal{C}$.

Corollary 2.115 *The common eigenspaces for* $\mathcal{C}$ *are the same as the common eigenspaces for* $\{\mathcal{H}_i\}_{i=1}^{\rho}$ *discussed in Lemma* 2.110*. In particular these common eigenspaces all have dimension* 1*.*

Proof: Routine using Remark 2.114. □

We now describe the action of $\mathcal{C}$ on the eigenspaces of A and B.

Proposition 2.116 *Let* $\{V_i\}_{i=0}^{d}$ *(resp.* $\{V_i'\}_{i=0}^{d}$*) denote a standard ordering of the eigenspaces of* A *(resp.* B*). Then for all* $C \in \mathcal{C}$*,*

$$CV_i \subseteq V_{i-1} + V_{i+1}, \qquad CV_i' \subseteq V_{i-1}' + V_{i+1}' \qquad (0 \leq i \leq d),$$

where $V_j = 0$ *and* $V_j' = 0$ *for* $j \in \{-1, d+1\}$*.*

Proof: Without loss of generality, $C \in \mathrm{Com}(A,B)$. The result follows by Lemma 2.1 and the equations on the left in lines (2.34) and (2.35). $\square$

Recall from Theorem 2.9 the elements $\{G_j \mid j \in \mathbb{Z}, j > 0\}$ of $\mathcal{O}$. Recall the equivalence relation on $\mathbb{F}\backslash\{0\}$ from Definition 2.62. Recall the evaluation parameters $\{a_i\}_{i=1}^{\rho}$ from (2.38). By Theorem 2.73, the $\{\overline{a_i}\}_{i=1}^{\rho}$ are mutually distinct and feasible, in the sense of Definition 2.62.

Proposition 2.117 *For all integers $j > 0$, the following holds on V:*

$$G_j = \sum_{i=1}^{\rho} \mathcal{H}_i \left(a_i^j - a_i^{-j}\right)/2.$$

Proof: For $1 \leq i \leq \rho$, consider the i^{th} part of G_j, in the sense of Definition 2.103. Using Lemma 2.38 and Definition 3.18, one routinely checks that the i^{th} part of G_j is equal to $\mathcal{H}_i \left(a_i^j - a_i^{-j}\right)/2$. The result follows by Remark 2.104. $\square$

Proposition 2.118 *The actions of $\{G_i\}_{i=1}^{\rho}$ on V form a basis for $\mathcal{C}$.*

Proof: Let M denote the ρ by ρ matrix whose (i,j)-entry is equal to $a_i^j - a_i^{-j}$ for $1 \leq i, j \leq \rho$. By Proposition 2.117, it suffices to show that the matrix M is invertible. One routinely checks that the determinant of M is equal to

$$\prod_{1 \leq k \leq \rho} (a_k - a_k^{-1}) \prod_{1 \leq i < j \leq \rho} a_i^{-1} a_j^{-1} (a_i a_j - 1)(a_j - a_i). \qquad (2.46)$$

The scalar (2.46) is nonzero because the $\{\overline{a_i}\}_{i=1}^{\rho}$ are mutually distinct and feasible. Therefore M is invertible. $\square$

2.12 The graph $\mathbb{X}$

Throughout this section the following notation will be in effect. Let V denote a vector space over $\mathbb{F}$ with finite positive dimension. Let A, B denote a tridiagonal pair on V of Krawtchouk type, with diameter $d \geq 1$ and shape $\{\rho_i\}_{i=0}^d$. Abbreviate $\rho = \rho_1$. By Remark 2.97, the tensor degree of A, B is equal to ρ. By Theorem 2.100, the elements of $\mathrm{Com}(A, B)$ mutually commute and are diagonalizable on V. In this section we discuss the eigenvalues and the common eigenspaces for the elements of $\mathrm{Com}(A, B)$. These common eigenspaces all have dimension 1. We view these common eigenspaces as vertices of a certain undirected graph. We describe the action of A, B in terms of this graph structure.

Proposition 2.119 *Pick* $H \in \mathrm{Com}(A, B)$. *The eigenvalues of* H *are* $\{d - 2i \,|\, 0 \leq i \leq d\}$. *For* $0 \leq i \leq d$, *the eigenspace of* H *corresponding to eigenvalue* $d - 2i$ *has dimension* ρ_i.

Proof: By Lemma 2.94, there exists a unique $L(\mathfrak{sl}_2)$-module structure on V such that the $L(\mathfrak{sl}_2)$-generators $\mathcal{A}, \mathcal{B}, \mathcal{H}$ act on V as A, B, H respectively. By construction, the eigenvalues of A are $\{d - 2i \,|\, 0 \leq i \leq d\}$, and for $0 \leq i \leq d$ the eigenspace of A corresponding to eigenvalue $d - 2i$ has dimension ρ_i. The result follows by Lemma 2.46.

□

Definition 2.120 Let $\mathbb{X}$ denote the set of common eigenspaces for the elements of $\mathrm{Com}(A, B)$. By Corollary 2.115, the elements of $\mathbb{X}$ all have dimension 1.

Remark 2.121 Pick $H \in \mathrm{Com}(A, B)$. Consider the basis $\{v_k\}_{k \in \mathbb{I}}$ of V from Section 2.11. Using Theorem 2.100 and Lemma 2.108, one routinely checks that the elements of $\mathbb{X}$ are $\{\mathbb{F} v_k\}_{k \in \mathbb{I}}$.

We now define an undirected graph with vertex set $\mathbb{X}$.

Definition 2.122 Let $x, y \in \mathbb{X}$. Then for $1 \leq i \leq \rho$, the elements x, y are said to be *i-adjacent* whenever the following two conditions hold:

(i) the eigenvalues of $\mathcal{H}_i$ corresponding to x and y differ by 2;

(ii) for $1 \leq j \leq \rho$ such that $j \neq i$, the eigenvalues of $\mathcal{H}_j$ corresponding to x and y are equal.

The elements x, y are said to be *adjacent* whenever there exists $1 \leq i \leq \rho$ such that x and y are i-adjacent. The set $\mathbb{X}$ together with this adjacency relation is an undirected graph.

We will be discussing the (path-length) distance function for the graph from Definition 2.122.

Remark 2.123 Referring to Definition 2.122, the graph $\mathbb{X}$ is a Cartesian product of ρ many chains, where the i^{th} chain has diameter d_i for $1 \leq i \leq \rho$. The graph $\mathbb{X}$ has diameter d.

Definition 2.124 An element $x \in \mathbb{X}$ will be called a *corner* whenever for $1 \leq i \leq \rho$, the eigenvalue of $\mathcal{H}_i$ on x is d_i or $-d_i$. Let $\mathrm{Corner}(\mathbb{X})$ denote the set of corners of $\mathbb{X}$. Note that the cardinality of $\mathrm{Corner}(\mathbb{X})$ is 2^ρ.

Proposition 2.125 *Pick $H \in \mathrm{Com}(A, B)$. For $0 \leq i \leq d$, let U_i denote the eigenspace of H corresponding to the eigenvalue $d - 2i$. The subspace U_0 is a corner of $\mathbb{X}$. For $0 \leq i \leq d$, U_i is the sum of the elements in $\mathbb{X}$ at distance i from U_0.*

Proof: Adopt the notation in Remark 2.121. By Lemma 2.107, U_i is spanned by the set of all vectors v_k ($k \in \mathbb{I}$) such that the coordinates of k sum to i. The result follows using Lemma 2.108, Definition 2.122, and Definition 2.124. □

Proposition 2.126 *Pick $x \in \mathrm{Corner}(\mathbb{X})$. Then there exists a unique $H_x \in \mathrm{Com}(A, B)$ that has eigenspace x for the eigenvalue d.*

Proof: Adopt the notation in Remark 2.121. Recall the corresponding $\{\mathcal{H}_i\}_{i=1}^{\rho}$ from Definition 2.103. Since $x \in \mathbb{X}$, there exists $k = (k_1, \ldots, k_\rho) \in \mathbb{I}$ such that $x = \mathbb{F}v_k$. Since $x \in \mathrm{Corner}(\mathbb{X})$, $k_i \in \{0, d_i\}$ for $1 \leq i \leq \rho$ by Lemma 2.108. Define $H_x = \sum_{i=1}^{\rho} s_i \mathcal{H}_i$, where for $1 \leq i \leq \rho$, $s_i = 1$ if $k_i = 0$ and $s_i = -1$ if $k_i = d_i$. By Theorem 2.100, $H_x \in \mathrm{Com}(A, B)$. By Lemma 2.108, x is the eigenspace of H_x for the eigenvalue d. The uniqueness claim follows from Proposition 2.125. □

Remark 2.127 Combining Propositions 2.125 and 2.126 we obtain a bijection

$$\mathrm{Corner}(\mathbb{X}) \to \mathrm{Com}(A, B)$$

$$x \mapsto H_x.$$

In the next two propositions, we discuss the action of A, B on various subspaces of V. We start by considering the action of A, B on the elements of $\mathbb{X}$.

Proposition 2.128 *For all $x \in \mathbb{X}$, Ax and Bx are contained in the sum of those elements of $\mathbb{X}$ that are adjacent to x.*

Proof: Adopt the notation in Remark 2.121. The result follows from Lemma 2.107, Lemma 2.108, and Definition 2.122. □

We now discuss the action of A, B on the eigenspaces of the elements of $\mathrm{Com}(A, B)$.

Proposition 2.129 *Pick $H \in \mathrm{Com}(A,B)$. For $0 \le i \le d$, let U_i denote the eigenspace of H corresponding to eigenvalue $d-2i$. Then*

$$AU_i \subseteq U_{i-1} + U_{i+1}, \qquad BU_i \subseteq U_{i-1} + U_{i+1},$$

where $U_{-1} = 0$ and $U_{d+1} = 0$.

Proof: Use Proposition 2.125 and Proposition 2.128. □

2.13 Compatible elements, $\mathfrak{sl}_2$-modules, and Leonard pairs of Krawtchouk type

Throughout this section the following notation will be in effect. Let V denote a vector space over $\mathbb{F}$ with finite positive dimension. Let A, B denote a tridiagonal pair on V of Krawtchouk type, with diameter $d \ge 1$ and shape $\{\rho_i\}_{i=0}^d$. Abbreviate $\rho = \rho_1$. By Remark 2.97, the tensor degree of A, B is equal to ρ. In this section we discuss a relationship between the elements of $\mathrm{Com}(A,B)$, $\mathfrak{sl}_2$-modules, and Leonard pairs. Fix $H \in \mathrm{Com}(A,B)$. As we saw in Lemma 2.94, there exists a unique $L(\mathfrak{sl}_2)$-module structure on V such that the $L(\mathfrak{sl}_2)$-generators $\mathcal{A}, \mathcal{B}, \mathcal{H}$ act on V as A, B, H respectively. By Lemma 2.45, the elements $\mathcal{H}, \mathcal{A}$ generate a Lie subalgebra of $L(\mathfrak{sl}_2)$ that is isomorphic to $\mathfrak{sl}_2$, and the elements $\mathcal{H}, \mathcal{B}$ generate a Lie subalgebra of $L(\mathfrak{sl}_2)$ that is isomorphic to $\mathfrak{sl}_2$. Restricting the $L(\mathfrak{sl}_2)$-action on V to either of these two Lie subalgebras, V becomes an $\mathfrak{sl}_2$-module.

Proposition 2.130 *The two $\mathfrak{sl}_2$-module structures on V defined in the previous paragraph are isomorphic. Moreover, the isomorphism class of the $\mathfrak{sl}_2$-module V is independent of the choice of $H \in \mathrm{Com}(A,B)$.*

Proof: Use Lemmas 2.19 and 2.31. □

By Lemma 3.6, the $\mathfrak{sl}_2$-module V is a direct sum of irreducible $\mathfrak{sl}_2$-submodules. We now describe the summands.

Proposition 2.131 *The only irreducible $\mathfrak{sl}_2$-submodules of the $\mathfrak{sl}_2$-module V are*

$$\mathbb{V}_d, \mathbb{V}_{d-2}, \mathbb{V}_{d-4}, \ldots.$$

Moreover, for $0 \leq j \leq d/2$, the multiplicity with which $\mathbb{V}_{d-2j}$ appears in V is $\rho_j - \rho_{j-1}$, where $\rho_{-1} = 0$.

Proof: Use Lemmas 2.19 and 2.31. □

Corollary 2.132 *The $\mathfrak{sl}_2$-module V is irreducible if and only if $\rho = 1$.*

Proof: Use Proposition 2.131. □

Proposition 2.133 *On each irreducible $\mathfrak{sl}_2$-submodule of the $\mathfrak{sl}_2$-module V, the pair H, A and the pair H, B act as Leonard pairs of Krawtchouk type.*

Proof: Use Lemmas 2.18, 2.116, and 2.129. □

Proposition 2.134 *The following* (i)–(iii) *are equivalent:*

(i) *the pair H, A and the pair H, B act on V as Leonard pairs of Krawtchouk type;*

(ii) *the pair H, A and the pair H, B act on V as tridiagonal pairs of Krawtchouk type;*

(iii) $\rho = 1$.

Proof: Immediate from Corollary 2.132 and Proposition 2.133. □

2.14 The proof of Theorem 2.34

In this section we give a proof of Theorem 2.34.

Definition 2.135 Let $\mathcal{L}$ denote the Lie algebra over $\mathbb{F}$ with generators $\mathcal{A}, \mathcal{B}, \mathcal{H}$ and relations

$$[\mathcal{A}, [\mathcal{A}, \mathcal{H}]] = 4\mathcal{H}, \qquad [\mathcal{H}, [\mathcal{H}, \mathcal{A}]] = 4\mathcal{A}, \tag{2.47}$$

$$[\mathcal{B}, [\mathcal{B}, \mathcal{H}]] = 4\mathcal{H}, \qquad [\mathcal{H}, [\mathcal{H}, \mathcal{B}]] = 4\mathcal{B}, \tag{2.48}$$

$$[\mathcal{A}, [\mathcal{A}, [\mathcal{A}, \mathcal{B}]]] = 4[\mathcal{A}, \mathcal{B}], \qquad [\mathcal{B}, [\mathcal{B}, [\mathcal{B}, \mathcal{A}]]] = 4[\mathcal{B}, \mathcal{A}], \tag{2.49}$$

$$[\mathcal{H}, [\mathcal{A}, \mathcal{B}]] = 0. \tag{2.50}$$

We will show that the Lie algebras $\mathcal{L}$ and $L(\mathfrak{sl}_2)$ are isomorphic, with the isomorphism as given in Theorem 2.34. First we show that there exists a homomorphism of Lie algebras $\mathcal{L} \to L(\mathfrak{sl}_2)$ that satisfies (2.16). By Lemma 2.33, it suffices to check that the elements

$$e \otimes 1 + f \otimes 1, \qquad e \otimes t + f \otimes t^{-1}, \qquad h \otimes 1$$

of $L(\mathfrak{sl}_2)$ satisfy the defining relations (2.47)–(2.50) of $\mathcal{L}$. This is routine using Definition 2.16 and (2.10). To illustrate, we verify the relation on the left in (2.48). First observe that

$$\begin{aligned} [e \otimes t + f \otimes t^{-1}, h \otimes 1] &= [e, h] \otimes t + [f, h] \otimes t^{-1} \\ &= -2e \otimes t + 2f \otimes t^{-1}. \end{aligned}$$

Therefore

$$
\begin{aligned}
&[\, e \otimes t + f \otimes t^{-1} \,,\, [e \otimes t + f \otimes t^{-1}, h \otimes 1] \,] \\
&\qquad = [e \otimes t + f \otimes t^{-1}, -2e \otimes t + 2f \otimes t^{-1}] \\
&\qquad = -2[e,e] \otimes t^2 + 2[e,f] \otimes 1 - 2[f,e] \otimes 1 + 2[f,f] \otimes t^{-2} \\
&\qquad = 4h \otimes 1.
\end{aligned}
$$

We have now verified the relation on the left in (2.48). The other relations in (2.47)–(2.50) are verified in a similar fashion. We have shown that there exists a homomorphism $\mathcal{L} \to L(\mathfrak{sl}_2)$ that satisfies (2.16).

Our next general goal is to show that there exists a homomorphism $L(\mathfrak{sl}_2) \to \mathcal{L}$ that satisfies (2.17), (2.18). The following definition is for notational convenience.

Definition 2.136 For $i \in \{0, 1\}$ define $E_i, F_i, H_i \in \mathcal{L}$ as follows:

$$
E_1 = \frac{[\mathcal{H}, \mathcal{A}] + 2\mathcal{A}}{4}, \qquad F_1 = \frac{[\mathcal{A}, \mathcal{H}] + 2\mathcal{A}}{4}, \qquad H_1 = \mathcal{H},
$$

$$
E_0 = \frac{[\mathcal{B}, \mathcal{H}] + 2\mathcal{B}}{4}, \qquad F_0 = \frac{[\mathcal{H}, \mathcal{B}] + 2\mathcal{B}}{4}, \qquad H_0 = -\mathcal{H}.
$$

To show that there exists a homomorphism $L(\mathfrak{sl}_2) \to \mathcal{L}$ that satisfies (2.17), (2.18), it suffices to prove that the elements of $\mathcal{L}$ from Definition 2.136 satisfy the relations in Lemma 2.33. This proof will be completed in Lemma 2.142. To prepare for this we first describe two automorphisms of $\mathcal{L}$. We then establish some relations involving the generators $\mathcal{A}, \mathcal{B}, \mathcal{H}$ of $\mathcal{L}$.

Lemma 2.137 *There exists an automorphism ϑ of $\mathcal{L}$ that sends*

$$
\mathcal{A} \mapsto \mathcal{A}, \qquad \mathcal{B} \mapsto \mathcal{B}, \qquad \mathcal{H} \mapsto -\mathcal{H}. \tag{2.51}
$$

Moreover, there exists an automorphism τ *of* $\mathcal{L}$ *that sends*

$$\mathcal{A} \mapsto \mathcal{B}, \qquad \mathcal{B} \mapsto \mathcal{A}, \qquad \mathcal{H} \mapsto \mathcal{H}. \tag{2.52}$$

The automorphisms ϑ *and* τ *satisfy* $\vartheta\tau = \tau\vartheta$, $\vartheta^2 = 1$, $\tau^2 = 1$.

Proof: The first and second assertions are clear by Definition 2.135. The last assertion is easily checked. □

By the last assertion in Lemma 2.137, ϑ and τ induce an action of the Klein-four group $\mathbb{Z}_2 \times \mathbb{Z}_2$ on $\mathcal{L}$ as a group of automorphisms.

Lemma 2.138 *The automorphisms* ϑ *and* τ *of* $\mathcal{L}$ *do the following to the elements of* $\mathcal{L}$ *from Definition* 2.136.

(i) *The map* ϑ *sends* $E_0 \leftrightarrow F_0$, $E_1 \leftrightarrow F_1$, $H_0 \leftrightarrow H_1$.

(ii) *The map* τ *sends* $E_0 \leftrightarrow F_1$, $E_1 \leftrightarrow F_0$, $H_0 \mapsto H_0$, $H_1 \mapsto H_1$.

(iii) *The composition* $\vartheta\tau$ *sends* $E_0 \leftrightarrow E_1$, $F_0 \leftrightarrow F_1$, $H_0 \leftrightarrow H_1$.

Proof: Routine using Definition 2.136 and Lemma 2.137. □

Lemma 2.139 *For the Lie algebra* $\mathcal{L}$,

(i) $[\mathcal{A}, [\mathcal{B}, \mathcal{H}]] = [\mathcal{B}, [\mathcal{A}, \mathcal{H}]]$;

(ii) $[\mathcal{H}, [\mathcal{A}, [\mathcal{B}, \mathcal{H}]]] = 0$;

(iii) $[[\mathcal{H}, \mathcal{A}], [\mathcal{B}, \mathcal{H}]] = 4[\mathcal{A}, \mathcal{B}]$.

Proof: (i) Use (2.2) and (2.50).

(ii) In the equation on the right in (2.47), take the Lie bracket of each side with $\mathcal{B}$ to get

$$[\mathcal{B}, [\mathcal{H}, [\mathcal{H}, \mathcal{A}]]] = 4[\mathcal{B}, \mathcal{A}]. \tag{2.53}$$

By (2.3), the left-hand side of (2.53) equals

$$2[\mathcal{H}, [\mathcal{A}, [\mathcal{H}, \mathcal{B}]]] - [\mathcal{A}, [\mathcal{H}, [\mathcal{H}, \mathcal{B}]]] - [\mathcal{H}, [\mathcal{H}, [\mathcal{A}, \mathcal{B}]]]. \tag{2.54}$$

Consider the three terms in (2.54). The term on the left equals $-2[\mathcal{H}, [\mathcal{A}, [\mathcal{B}, \mathcal{H}]]]$, the term in the middle equals $4[\mathcal{B}, \mathcal{A}]$ by (2.48), and the term on the right is zero by (2.50). The result follows.

(iii) By (2.2), $[[\mathcal{H}, \mathcal{A}], [\mathcal{B}, \mathcal{H}]] = [\mathcal{H}, [\mathcal{A}, [\mathcal{B}, \mathcal{H}]]] - [\mathcal{A}, [\mathcal{H}, [\mathcal{B}, \mathcal{H}]]]$. In this equation evaluate the right-hand side using part (ii) of this lemma and the equation on the right in (2.48). The result follows. □

We will show that the elements E_i, F_i, H_i of $\mathcal{L}$ satisfy the relations in Lemma 2.33. We will do this in two steps. The first step will be accomplished in Lemma 2.140, and the second step will be accomplished in Lemma 2.142.

Lemma 2.140 *Referring to Definition* 2.136*,*

$$H_0 + H_1 = 0, \qquad [E_i, F_j] = \delta_{ij} H_j, \tag{2.55}$$

$$[H_i, E_j] = C_{ij} E_j, \qquad [H_i, F_j] = -C_{ij} F_j, \tag{2.56}$$

where C is the Cartan matrix immediately preceding Lemma 2.33*.*

Proof: The equation on the left in (2.55) is immediate from Definition 2.136. Consider the equations on the right in (2.55). For $i = j$ these relations are checked using the relations on the left in (2.47), (2.48). For $i \neq j$ these relations are checked using parts (i), (iii) of Lemma 2.139. We routinely check (2.56) using the relations on the right in (2.47), (2.48). □

Lemma 2.141 *For the Lie algebra* $\mathcal{L}$*,*

(i) $[\mathcal{A}, [\mathcal{A}, [\mathcal{A}, [\mathcal{B}, \mathcal{H}]]]] = 4[\mathcal{A}, [\mathcal{B}, \mathcal{H}]]$*;*

(ii) $[\mathcal{B}, [\mathcal{B}, [\mathcal{B}, [\mathcal{A}, \mathcal{H}]]]] = 4[\mathcal{B}, [\mathcal{A}, \mathcal{H}]]$.

Proof: (i) In the equation on the left in (2.49), take the Lie bracket of each side with $\mathcal{H}$, and evaluate the right-hand side using (2.50) to get

$$[\mathcal{H}[\mathcal{A}, [\mathcal{A}, [\mathcal{A}, \mathcal{B}]]]] = 0. \tag{2.57}$$

Abbreviate $D = [\mathcal{A}, \mathcal{B}]$. By (2.3), the left-hand side of (2.57) equals

$$2[\mathcal{A}, [D, [\mathcal{A}, \mathcal{H}]]] - [D, [\mathcal{A}, [\mathcal{A}, \mathcal{H}]]] - [\mathcal{A}, [\mathcal{A}, [D, \mathcal{H}]]]. \tag{2.58}$$

Consider the three terms in (2.58). The term in the middle is zero by (2.47) and (2.50), and the term on the right is zero by (2.50). Next we evaluate the term on the left. By (2.2), $[D, [\mathcal{A}, \mathcal{H}]] = [\mathcal{A}, [\mathcal{B}, [\mathcal{A}, \mathcal{H}]]] - [\mathcal{B}, [\mathcal{A}, [\mathcal{A}, \mathcal{H}]]]$. In this equation evaluate the right-hand side using Lemma 2.139(i) and (2.47) to get

$$[D, [\mathcal{A}, \mathcal{H}]] = [\mathcal{A}, [\mathcal{A}, [\mathcal{B}, \mathcal{H}]]] - 4[\mathcal{B}, \mathcal{H}]. \tag{2.59}$$

To evaluate the term on the left in (2.58), take the Lie bracket of each side of (2.59) with $\mathcal{A}$. The result follows.

(ii) Apply the automorphism τ to (i). □

Lemma 2.142 *The elements* E_0, E_1, F_0, F_1 *of* $\mathcal{L}$ *from Definition* 2.136 *satisfy the following relations:*

$$[E_i, [E_i, [E_i, E_j]]] = 0, \qquad i \neq j, \tag{2.60}$$

$$[F_i, [F_i, [F_i, F_j]]] = 0, \qquad i \neq j. \tag{2.61}$$

Proof: By Lemma 2.138, it suffices to prove (2.60) for $(i, j) = (1, 0)$. Add 1/2 times the equation on the left in (2.49) to 1/4 times the equation in Lemma 2.141(i). This yields

$$[\mathcal{A}, [\mathcal{A}, [\mathcal{A}, E_0]]] = 4[\mathcal{A}, E_0]. \tag{2.62}$$

Note $\mathcal{A} = E_1 + F_1$ by Definition 2.136. By the equation on the right in (2.55), we find that $[\mathcal{A}, E_0] = [E_1, E_0]$. Therefore the left-hand side of (2.62) equals

$$[E_1, [E_1, [E_1, E_0]]] + [E_1, [F_1, [E_1, E_0]]] + [F_1, [E_1, [E_1, E_0]]] + [F_1, [F_1, [E_1, E_0]]]. \tag{2.63}$$

Consider the four terms in (2.63). Our goal is to show that the first term from the left is zero. Consider the other three terms. Using (2.2) and Lemma 2.140, we routinely check that

$$[F_1, [E_1, E_0]] = 2E_0. \tag{2.64}$$

In (2.64), take the Lie bracket of each side with E_1 to get

$$[E_1, [F_1, [E_1, E_0]]] = 2[E_1, E_0]. \tag{2.65}$$

In (2.64), take the Lie bracket of each side with F_1, and evaluate the right-hand side using the equation on the right in (2.55) to get

$$[F_1, [F_1, [E_1, E_0]]] = 0. \tag{2.66}$$

By (2.3), $[F_1, [E_1, [E_1, E_0]]]$ equals

$$2[E_1, [E_0, [E_1, F_1]]] - [E_0, [E_1, [E_1, F_1]]] - [E_1, [E_1, [E_0, F_1]]]. \tag{2.67}$$

Consider the three terms in (2.67). Using Lemma 2.140, we routinely check that the term on the left is $4[E_1, E_0]$, the term in the middle is $-2[E_1, E_0]$, and the term on the right is zero. Therefore we get

$$[F_1, [E_1, [E_1, E_0]]] = 2[E_1, E_0]. \tag{2.68}$$

Evaluate (2.62) using (2.65), (2.66), (2.68) to get $[E_1, [E_1, [E_1, E_0]]] = 0$. This proves (2.60) for $(i, j) = (1, 0)$. The result follows. □

We are ready to complete the proof of Theorem 2.34. We have shown that there exists a homomorphism $\mathcal{L} \to L(\mathfrak{sl}_2)$ that satisfies (2.16). We have shown that there exists a homomorphism $L(\mathfrak{sl}_2) \to \mathcal{L}$ that satisfies (2.17), (2.18). We routinely check that these homomorphisms are inverses. It follows that each map is an isomorphism of Lie algebras.

Chapter 3

The classification of the finite-dimensional irreducible modules for the $\mathfrak{sl}_2$ loop algebra

3.1 Introduction

In [2], Chari classified up to isomorphism the finite-dimensional irreducible modules for the $\mathfrak{sl}_2$ loop algebra. Our purpose in this chapter is to give an elementary version of this classification. This chapter is meant for graduate students and researchers who are unfamiliar with the general representation theory of loop algebras.

We now recall the definition of $\mathfrak{sl}_2$ and its loop algebra $L(\mathfrak{sl}_2)$. Let $\mathbb{F}$ denote an algebraically closed field with characteristic 0. Let $\mathfrak{sl}_2$ denote the Lie algebra over $\mathbb{F}$ with basis e, f, h and Lie bracket

$$[e, f] = h, \qquad [h, e] = 2e, \qquad [h, f] = -2f.$$

Let t denote an indeterminate, and let $\mathbb{F}[t, t^{-1}]$ denote the $\mathbb{F}$-algebra consisting of the Laurent polynomials in t that have all coefficients in $\mathbb{F}$. Let $L(\mathfrak{sl}_2)$ denote the Lie algebra over $\mathbb{F}$ consisting of the $\mathbb{F}$-vector space

$$\mathfrak{sl}_2 \otimes \mathbb{F}[t, t^{-1}], \qquad \otimes = \otimes_{\mathbb{F}}$$

and Lie bracket

$$[u \otimes a, v \otimes b] = [u, v] \otimes ab, \qquad u, v \in \mathfrak{sl}_2, \quad a, b \in \mathbb{F}[t, t^{-1}].$$

We are going to classify up to isomorphism the finite-dimensional irreducible modules for $L(\mathfrak{sl}_2)$. This classification is stated in Theorems 3.23–3.25.

3.2 Assumptions and preliminaries

In this section we collect some definitions and basic facts that will be used throughout this chapter. Recall the natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$ and the integers $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$. Let $\mathbb{F}$ denote an algebraically closed field with characteristic 0. Let V denote a vector space over $\mathbb{F}$ with finite positive dimension. Let $\mathrm{End}(V)$ denote the $\mathbb{F}$-algebra of all linear transformations from V to V. For $A \in \mathrm{End}(V)$ and $\theta \in \mathbb{F}$, define

$$V_A(\theta) = \{v \in V | Av = \theta v\}. \tag{3.1}$$

We say that θ is an *eigenvalue* for A whenever $V_A(\theta) \neq 0$, and in this case $V_A(\theta)$ is called the *eigenspace* of A corresponding to θ. We say that A is *diagonalizable* whenever V is spanned by the eigenspaces of A.

We now turn our attention to Lie algebras. For basic definitions and facts about Lie algebras, we refer the reader to the books [1, 8]. The $\mathbb{F}$-vector space $\mathrm{End}(V)$ becomes a

Lie algebra over $\mathbb{F}$ with Lie bracket

$$[X, Y] = XY - YX, \qquad X, Y \in \mathrm{End}(V).$$

This Lie algebra is often denoted $\mathfrak{gl}(V)$, but we will not use this notation.

Lemma 3.1 *For $X, Y \in \mathrm{End}(V)$ and $\theta \in \mathbb{F}$, the following are equivalent.*

(i) *The map $[X, Y] - 2Y$ vanishes on $V_X(\theta)$.*

(ii) $YV_X(\theta) \subseteq V_X(\theta + 2)$.

Proof: Abbreviate Φ for $[X, Y] - 2Y$. Let $v \in V_X(\theta)$. It suffices to show $\Phi v = 0$ if and only if $Yv \in V_X(\theta + 2)$. Using $Xv = \theta v$ we find $\Phi v = (X - (\theta + 2))Yv$. The result follows. □

Lemma 3.2 *For $X, Y \in \mathrm{End}(V)$ and $\theta \in \mathbb{F}$, the following are equivalent.*

(i) *The map $[X, Y] + 2Y$ vanishes on $V_X(\theta)$.*

(ii) $YV_X(\theta) \subseteq V_X(\theta - 2)$.

Proof: Similar to the proof of Lemma 3.1. □

Let L denote a Lie algebra over $\mathbb{F}$. There exists an L-module of dimension 1 on which every element of L acts as zero. This L-module is unique up to isomorphism. We call this L-module *trivial*. The proofs of the following basic facts are left as an exercise.

Lemma 3.3 *Let L_1, L_2 denote Lie algebras over $\mathbb{F}$. Assume $\mu : L_1 \to L_2$ is a surjective Lie algebra homomorphism. Let V denote an L_2-module. Pulling back the L_2-action on V via μ we turn V into an L_1-module. Then V is irreducible as an L_1-module if and only if V is irreducible as an L_2-module.*

Lemma 3.4 *Let L_1, L_2 denote Lie algebras over $\mathbb{F}$. Assume $\mu : L_1 \to L_2$ is a surjective Lie algebra homomorphism. Let V, V' denote L_2-modules. Pulling back the L_2-action on V (resp. V') via μ we turn V (resp. V') into an L_1-module. Given an $\mathbb{F}$-linear map $X : V \to V'$, the following are equivalent.*

(i) *X is a homomorphism of L_1-modules.*

(ii) *X is a homomorphism of L_2-modules.*

Let L denote a Lie algebra over $\mathbb{F}$. Let K, K' denote subspaces of L. Define $[K, K']$ to be the subspace of L spanned by the elements $[k, k']$ with $k \in K$ and $k' \in K'$.

Throughout this chapter all unadorned tensor products are taken over $\mathbb{F}$.

3.3 The Lie algebra $\mathfrak{sl}_2$

In this section we recall the Lie algebra $\mathfrak{sl}_2$ and its finite-dimensional modules.

Definition 3.5 Let $\mathfrak{sl}_2$ denote the Lie algebra over $\mathbb{F}$ with basis e, f, h and Lie bracket

$$[e, f] = h, \qquad [h, e] = 2e, \qquad [h, f] = -2f.$$

We call the basis e, f, h the *Chevalley* basis of $\mathfrak{sl}_2$.

Let L denote a Lie algebra over $\mathbb{F}$. Observe that the subspace $[L, L]$ is an ideal of L. The Lie algebra L is called *abelian* whenever $[L, L] = 0$. L is called *simple* whenever it is not abelian and the only ideals of L are 0 and L. In this case, the center of L is 0 and $[L, L] = L$ [8, p. 6]. The Lie algebra $\mathfrak{sl}_2$ is simple [8, p. 6].

In the following two lemmas we describe the finite-dimensional $\mathfrak{sl}_2$-modules.

Lemma 3.6 [8, p. 28] *Each finite-dimensional $\mathfrak{sl}_2$-module is a direct sum of irreducible $\mathfrak{sl}_2$-modules.*

Lemma 3.7 [8, p. 31] *There exists a family*

$$\mathbb{V}_d \qquad\qquad d = 0, 1, 2, \ldots \tag{3.2}$$

of finite-dimensional irreducible $\mathfrak{sl}_2$-modules with the following property. The module $\mathbb{V}_d$ has a basis $\{v_i\}_{i=0}^d$ satisfying

$$hv_i = (d - 2i)v_i \qquad\qquad (0 \le i \le d), \tag{3.3}$$

$$fv_i = v_{i+1} \qquad\qquad (0 \le i \le d-1), \quad fv_d = 0, \tag{3.4}$$

$$ev_i = i(d + 1 - i)v_{i-1} \qquad\qquad (1 \le i \le d), \qquad ev_0 = 0. \tag{3.5}$$

Every finite-dimensional irreducible $\mathfrak{sl}_2$-module is isomorphic to exactly one of the modules in (3.2).

Note that $\mathbb{V}_0$ is the trivial $\mathfrak{sl}_2$-module.

Definition 3.8 With reference to Lemma 3.7, a basis $\{v_i\}_{i=0}^d$ of $\mathbb{V}_d$ satisfying (3.3)–(3.5) is said to be *standard*.

Lemma 3.9 *Let $d \in \mathbb{N}$, and let $\{v_i\}_{i=0}^d$ denote a standard basis of $\mathbb{V}_d$. Given vectors $\{u_i\}_{i=0}^d$ in $\mathbb{V}_d$, the following are equivalent.*

(i) *The vectors $\{u_i\}_{i=0}^d$ form a standard basis of $\mathbb{V}_d$.*

(ii) *There exists a nonzero $s \in \mathbb{F}$ such that $u_i = sv_i$ for $0 \le i \le d$.*

Proof: Routine. □

Lemma 3.10 *The $\mathfrak{sl}_2$-action on $\mathbb{V}_d$ is faithful provided that d is at least 1.*

Proof: Let K denote the kernel of the $\mathfrak{sl}_2$-action on $\mathbb{V}_d$, and note that K is an ideal of $\mathfrak{sl}_2$. Since $\mathfrak{sl}_2$ is simple, either $K = 0$ or $K = \mathfrak{sl}_2$. By (3.3) and since d is nonzero, we see $h \notin K$. Therefore $K \neq \mathfrak{sl}_2$, so $K = 0$. The result follows. □

Lemma 3.11 [10, p. 31] *Let V denote a finite-dimensional $\mathfrak{sl}_2$-module. Let $0 \neq v \in V$ denote an eigenvector of h such that $ev = 0$. Then the eigenvalue for h corresponding to v is a nonnegative integer. Denote this eigenvalue by d. The elements $\{f^i v\}_{i=0}^d$ form a standard basis for an $\mathfrak{sl}_2$-submodule of V that is isomorphic to $\mathbb{V}_d$.*

3.4 The $\mathfrak{sl}_2$ loop algebra and its irreducible modules

In this section we discuss the $\mathfrak{sl}_2$ loop algebra $L(\mathfrak{sl}_2)$ and its finite-dimensional irreducible modules. First we recall the definition of $L(\mathfrak{sl}_2)$. We then discuss some basic results about $L(\mathfrak{sl}_2)$-modules, with an emphasis on a special case called an evaluation module. At the end of this section we give three theorems, which taken together amount to a classification of the finite-dimensional irreducible $L(\mathfrak{sl}_2)$-modules. These theorems are the main results of the chapter.

Definition 3.12 Let t denote an indeterminate, and let $\mathbb{F}[t, t^{-1}]$ denote the $\mathbb{F}$-algebra consisting of the Laurent polynomials in t that have all coefficients in $\mathbb{F}$. Let $L(\mathfrak{sl}_2)$ denote the Lie algebra over $\mathbb{F}$ consisting of the $\mathbb{F}$-vector space $\mathfrak{sl}_2 \otimes \mathbb{F}[t, t^{-1}]$ and Lie bracket

$$[u \otimes a, v \otimes b] = [u, v] \otimes ab, \qquad u, v \in \mathfrak{sl}_2, \quad a, b \in \mathbb{F}[t, t^{-1}].$$

We call $L(\mathfrak{sl}_2)$ the $\mathfrak{sl}_2$ *loop algebra*.

Observe that $\{t^i\}_{i\in\mathbb{Z}}$ is a basis of the $\mathbb{F}$-vector space $\mathbb{F}[t,t^{-1}]$. By construction, the following is a basis of $L(\mathfrak{sl}_2)$.

$$e\otimes t^i, \qquad f\otimes t^i, \qquad h\otimes t^i \qquad\qquad i\in\mathbb{Z} \tag{3.6}$$

Lemma 3.13 *We have* $L(\mathfrak{sl}_2) = [L(\mathfrak{sl}_2), L(\mathfrak{sl}_2)]$.

Proof: This follows from Definition 3.12 and the fact that $\mathfrak{sl}_2 = [\mathfrak{sl}_2, \mathfrak{sl}_2]$. □

Lemma 3.14 *Every* $L(\mathfrak{sl}_2)$*-module of dimension* 1 *is trivial.*

Proof: Let W denote an $L(\mathfrak{sl}_2)$-module of dimension 1. We show that every element of $L(\mathfrak{sl}_2)$ is zero on W. Observe that each element of $L(\mathfrak{sl}_2)$ acts on W as a scalar multiple of the identity. Therefore any two elements of $L(\mathfrak{sl}_2)$ commute on W, so every element of $[L(\mathfrak{sl}_2), L(\mathfrak{sl}_2)]$ is zero on W. The result follows by Lemma 3.13. □

Definition 3.15 For nonzero $a\in\mathbb{F}$, define a map $EV_a : L(\mathfrak{sl}_2)\to\mathfrak{sl}_2$ by

$$EV_a(u\otimes\eta(t)) = \eta(a)u, \qquad u\in\mathfrak{sl}_2, \quad \eta(t)\in\mathbb{F}[t,t^{-1}].$$

The map EV_a is a homomorphism of Lie algebras.

With reference to Definition 3.15, one routinely checks that EV_a is surjective and its kernel is $\mathfrak{sl}_2\otimes(t-a)\mathbb{F}[t,t^{-1}]$.

Definition 3.16 For a finite-dimensional $\mathfrak{sl}_2$-module V and for $0\neq a\in\mathbb{F}$, we pull back the $\mathfrak{sl}_2$-action via EV_a to obtain an $L(\mathfrak{sl}_2)$-action on V. We denote the resulting $L(\mathfrak{sl}_2)$-module by $V(a)$.

Lemma 3.17 *Let V denote a finite-dimensional $\mathfrak{sl}_2$-module, and let $a \in \mathbb{F}$ be nonzero. Then the $L(\mathfrak{sl}_2)$-module $V(a)$ is irreducible if and only if the $\mathfrak{sl}_2$-module V is irreducible.*

Proof: By Lemma 3.3. □

Definition 3.18 With reference to Lemma 3.7 and Definition 3.16, by an *evaluation module* for $L(\mathfrak{sl}_2)$ we mean an $L(\mathfrak{sl}_2)$-module $\mathbb{V}_d(a)$, where d is a positive integer and $0 \neq a \in \mathbb{F}$. By construction, the evaluation module $\mathbb{V}_d(a)$ is nontrivial and irreducible.

From the construction, we have the following description of the evaluation modules for $L(\mathfrak{sl}_2)$.

Lemma 3.19 *For a positive integer d and nonzero $a \in \mathbb{F}$, the evaluation module $\mathbb{V}_d(a)$ is described as follows. Let $\{v_i\}_{i=0}^d$ denote a standard basis of the $\mathfrak{sl}_2$-module $\mathbb{V}_d$. The elements* (3.6) *of $L(\mathfrak{sl}_2)$ act on $\mathbb{V}_d(a)$ as follows. For $k \in \mathbb{Z}$,*

$$
\begin{aligned}
&(h \otimes t^k)\, v_i = (d-2i)a^k v_i && (0 \le i \le d), && \\
&(f \otimes t^k)\, v_i = a^k v_{i+1} && (0 \le i \le d-1), && (f \otimes t^k)\, v_d = 0, \\
&(e \otimes t^k)\, v_i = i(d+1-i)a^k v_{i-1} && (1 \le i \le d), && (e \otimes t^k)\, v_0 = 0.
\end{aligned}
$$

Note 3.20 With reference to Lemma 3.19, for any $k \in \mathbb{Z}$, v_0 spans the eigenspace of $h \otimes t^k$ corresponding to eigenvalue $a^k d$.

Lemma 3.21 *The evaluation modules $\mathbb{V}_d(a)$ and $\mathbb{V}_{d'}(a')$ are isomorphic if and only if $d = d'$ and $a = a'$.*

Proof: Suppose $\mathbb{V}_d(a)$ and $\mathbb{V}_{d'}(a')$ are isomorphic. Isomorphic modules have the same dimension, so $d = d'$. Considering the action of $h \otimes t$, we see by Note 3.20 that $ad = a'd$. Since d is positive, we have $a = a'$.

The converse is immediate. $\square$

Let L denote a Lie algebra over $\mathbb{F}$, and let U, V denote L-modules. By [8, p. 26], $U \otimes V$ has an L-module structure given by

$$x(u \otimes v) = (xu) \otimes v + u \otimes (xv) \qquad x \in L, \quad u \in U, \quad v \in V. \tag{3.7}$$

Let U', V' denote L-modules such that U and U' are isomorphic and V and V' are isomorphic. Then the L-modules $U \otimes V$ and $U' \otimes V'$ are isomorphic. The following lemma is routinely checked.

Lemma 3.22 *Let L denote a Lie algebra over $\mathbb{F}$, and let U, V denote L-modules. Then the following hold.*

(i) *There exists an L-module isomorphism $U \otimes V \to V \otimes U$ that sends $u \otimes v \mapsto v \otimes u$ for all $u \in U$ and $v \in V$.*

(ii) *Assume the L-module $U \otimes V$ is irreducible. Then U and V are irreducible.*

The classification of finite-dimensional irreducible $L(\mathfrak{sl}_2)$-modules is stated in the following three theorems. We acknowledge that these theorems are a reformulation of the classification of finite-dimensional irreducible $L(\mathfrak{sl}_2)$-modules from [2].

Theorem 3.23 *Let N denote a positive integer. For $1 \leq i \leq N$ let $\mathbb{V}_{d_i}(a_i)$ denote an evaluation module for $L(\mathfrak{sl}_2)$. Then the $L(\mathfrak{sl}_2)$-module $\otimes_{i=1}^{N} \mathbb{V}_{d_i}(a_i)$ is irreducible if and only if $\{a_i\}_{i=1}^{N}$ are mutually distinct.*

Theorem 3.24 *Let N, N' denote positive integers. For $1 \leq i \leq N$ and $1 \leq j \leq N'$ let U_i and U'_j denote evaluation modules for $L(\mathfrak{sl}_2)$. Consider the $L(\mathfrak{sl}_2)$-modules*

$V = \otimes_{i=1}^{N} U_i$ and $V' = \otimes_{j=1}^{N'} U'_j$, and assume V and V' are irreducible. Then the following are equivalent.

(i) *The $L(\mathfrak{sl}_2)$-modules V and V' are isomorphic.*

(ii) *$N = N'$, and up to a permutation of $\{U'_i\}_{i=1}^{N}$, for $1 \leq i \leq N$ the $L(\mathfrak{sl}_2)$-modules U_i and U'_i are isomorphic.*

Theorem 3.25 *Every nontrivial finite-dimensional irreducible $L(\mathfrak{sl}_2)$-module is isomorphic to a tensor product of evaluation modules.*

Our goal for the rest of the chapter is to prove these theorems. The proofs of Theorems 3.23 and 3.24 will be completed in Section 3.13. The proof of Theorem 3.25 will be completed in Section 3.14.

3.5 The Lie algebra $\mathfrak{g}$

In this section we bring in a certain Lie algebra $\mathfrak{g}$ that will play an important role in our description of $L(\mathfrak{sl}_2)$-modules. First we make a comment. Let L_1, L_2 denote Lie algebras over $\mathbb{F}$. Recall that the direct sum $L_1 \oplus L_2$ becomes a Lie algebra over $\mathbb{F}$ with Lie bracket

$$[(u_1, u_2), (v_1, v_2)] = ([u_1, v_1], [u_2, v_2]), \qquad u_1, v_1 \in L_1, \quad u_2, v_2 \in L_2.$$

Fix a positive integer N and consider the Lie algebra

$$\mathfrak{g} = \mathfrak{sl}_2 \oplus \cdots \oplus \mathfrak{sl}_2 \quad (N \text{ copies}). \tag{3.8}$$

Our next general goal is to classify up to isomorphism the finite-dimensional irreducible $\mathfrak{g}$-modules. The proof of this classification will be completed in Section 3.8. Beginning in Section 3.12, we will consider how $\mathfrak{g}$-modules and $L(\mathfrak{sl}_2)$-modules are related.

For $1 \leq i \leq N$ and $x \in \mathfrak{sl}_2$, define $x_i \in \mathfrak{g}$ by

$$x_i = (0, \ldots, 0, x, 0, \ldots, 0),$$

where x above is in the i^{th} coordinate.

For $1 \leq i \leq N$ there exists a Lie algebra homomorphism

$$\begin{aligned} \mathfrak{sl}_2 &\to \mathfrak{g} \\ x &\mapsto x_i. \end{aligned} \tag{3.9}$$

The homomorphism (3.9) is injective. Let $\mathfrak{g}_i$ denote the image of $\mathfrak{sl}_2$ under (3.9). By construction, $\mathfrak{g}_i$ is a Lie subalgebra of $\mathfrak{g}$ that is isomorphic to $\mathfrak{sl}_2$. Note that $\mathfrak{g}_i$ is simple since $\mathfrak{sl}_2$ is simple. Let e, f, h denote the Chevalley basis of $\mathfrak{sl}_2$. The elements e_i, f_i, h_i form a basis for $\mathfrak{g}_i$. We have

$$[\mathfrak{g}_i, \mathfrak{g}_j] = 0, \quad \text{if} \quad i \neq j \qquad (1 \leq i, j \leq N). \tag{3.10}$$

Observe that

$$\mathfrak{g} = \mathfrak{g}_1 + \cdots + \mathfrak{g}_N \qquad \text{(direct sum)}, \tag{3.11}$$

so $\{e_i, f_i, h_i \,|\, i = 1, \ldots, N\}$ is a basis for $\mathfrak{g}$. Combining (3.10) and (3.11) we see that $\mathfrak{g}_i$ is an ideal of $\mathfrak{g}$ for $1 \leq i \leq N$.

Lemma 3.26 *Given a subspace K of $\mathfrak{g}$ the following are equivalent.*

(i) *K is an ideal of $\mathfrak{g}$.*

(ii) *There exists a subset S of $\{1,\dots,N\}$ such that $K=\sum_{r\in S}\mathfrak{g}_r$.*

Proof: (i) $\Rightarrow$ (ii) Define $S=\{r\,|\,1\le r\le N, \mathfrak{g}_r\subseteq K\}$. By construction, $K\supseteq\sum_{r\in S}\mathfrak{g}_r$, so it suffices to show that $K\subseteq\sum_{r\in S}\mathfrak{g}_r$. Let $k\in K$. By (3.11), there exists $g_i\in\mathfrak{g}_i$ $(1\le i\le N)$ such that $k=\sum_{i=1}^N g_i$. We now show that $g_i=0$ for all $i\notin S$ $(1\le i\le N)$. Let i be given. Since K and $\mathfrak{g}_i$ are ideals of $\mathfrak{g}$, we find $[K,\mathfrak{g}_i]$ is contained in the ideal $K\cap\mathfrak{g}_i$ of $\mathfrak{g}_i$. Recall $\mathfrak{g}_i$ is simple, so $K\cap\mathfrak{g}_i$ is either 0 or $\mathfrak{g}_i$. Note $K\cap\mathfrak{g}_i\neq\mathfrak{g}_i$; otherwise $\mathfrak{g}_i\subseteq K$, so $i\in S$ for a contradiction. Therefore $K\cap\mathfrak{g}_i=0$. Consequently $[K,\mathfrak{g}_i]=0$, so $[k,g]=0$ for all $g\in\mathfrak{g}_i$. Now using (3.10) we find $[g_i,g]=[k,g]=0$ for all $g\in\mathfrak{g}_i$. Therefore g_i is in the center of $\mathfrak{g}_i$. The center of $\mathfrak{g}_i$ is 0 since $\mathfrak{g}_i$ is simple. So $g_i=0$. This proves $k\in\sum_{r\in S}\mathfrak{g}_r$. The result follows.

(ii) $\Rightarrow$ (i) Routine. $\square$

We now discuss a way to construct $\mathfrak{g}$-modules by taking tensor products of $\mathfrak{sl}_2$-modules. For $1\le i\le N$ let V_i denote a finite-dimensional $\mathfrak{sl}_2$-module, and consider the $\mathbb{F}$-vector space $V=\otimes_{i=1}^N V_i$. By [1, p. 85], V has a $\mathfrak{g}$-module structure which is described as follows. Given $u=\oplus_{i=1}^N u^{(i)}\in\mathfrak{g}$ and $w_1\otimes\cdots\otimes w_N\in V$,

$$u\,(w_1\otimes\cdots\otimes w_N)=\sum_{i=1}^N w_1\otimes\cdots\otimes w_{i-1}\otimes u^{(i)}w_i\otimes w_{i+1}\otimes\cdots\otimes w_N. \tag{3.12}$$

Lemma 3.27 *For $1\le i\le N$ let V_i denote a finite-dimensional $\mathfrak{sl}_2$-module. Consider the $\mathfrak{g}$-module $V=\otimes_{i=1}^N V_i$. Then the following are equivalent.*

(i) *The $\mathfrak{g}$-action on V is faithful.*

(ii) *For $1\le i\le N$ the $\mathfrak{sl}_2$-action on V_i is faithful.*

Proof: For $1 \leq i \leq N$ the $\mathfrak{g}$-module V is a $\mathfrak{g}_i$-module by restriction. Observe that for $x \in \mathfrak{sl}_2$, x is zero on V_i if and only if x_i is zero on V. Therefore the $\mathfrak{sl}_2$-action on V_i is faithful if and only if the $\mathfrak{g}_i$-action on V is faithful.

(i) $\Rightarrow$ (ii) Since the $\mathfrak{g}$-action on V is faithful, the $\mathfrak{g}_i$-action on V is faithful. The result follows by the above comments.

(ii) $\Rightarrow$ (i) Let K denote the kernel of the $\mathfrak{g}$-action on V, and note that K is an ideal of $\mathfrak{g}$. For $1 \leq i \leq N$ the $\mathfrak{g}_i$-action on V is faithful, since the $\mathfrak{sl}_2$-action on V_i is faithful. Therefore $K \cap \mathfrak{g}_i = 0$ for $1 \leq i \leq N$. By Lemma 3.26, $K = 0$. The result follows. $\square$

3.6 $\mathfrak{g}$-modules

Fix a positive integer N, and recall the corresponding Lie algebra $\mathfrak{g}$ from (3.8). In this section we consider the finite-dimensional $\mathfrak{g}$-modules.

Throughout this section V will denote a $\mathfrak{g}$-module of finite positive dimension.

Lemma 3.28 *For $1 \leq i \leq N$ the element h_i is diagonalizable on V, and the elements e_i and f_i are nilpotent on V.*

Proof: Consider the $\mathfrak{g}$-module V. Pull back the $\mathfrak{g}$-action via the homomorphism in line (3.9). Then V becomes an $\mathfrak{sl}_2$-module on which x acts as x_i for all $x \in \mathfrak{sl}_2$. Combining Lemma 3.6 and Lemma 3.7 we find the element h is diagonalizable on V, and the elements e and f are nilpotent on V. The result follows. $\square$

Recall the notation from line (3.1). For $\lambda = (\lambda_1, \ldots, \lambda_N) \in \mathbb{F}^N$ define

$$V(\lambda) = V_{h_1}(\lambda_1) \cap \cdots \cap V_{h_N}(\lambda_N). \tag{3.13}$$

By Lemma 3.28, each of $\{h_i\}_{i=1}^N$ is diagonalizable on V. By (3.10), $\{h_i\}_{i=1}^N$ mutually commute. Therefore

$$V = \sum_{\lambda \in \mathbb{F}^N} V(\lambda) \qquad \text{(direct sum)}. \tag{3.14}$$

Observe that the nonzero summands in (3.14) are the common eigenspaces for $\{h_i\}_{i=1}^N$. A nonzero element of $V(\lambda)$ is said to be a *weight vector* with *weight* λ.

For $1 \leq i \leq N$ let ε_i denote the element of $\mathbb{F}^N$ with a 1 in the i^{th} coordinate and 0 in all other coordinates. Let I denote the identity element of $\text{End}(V)$.

Lemma 3.29 *The following hold for* $1 \leq i \leq N$ *and* $\lambda = (\lambda_1, \ldots, \lambda_N) \in \mathbb{F}^N$.

(i) $e_i V(\lambda) \subseteq V(\lambda + 2\varepsilon_i)$.

(ii) $f_i V(\lambda) \subseteq V(\lambda - 2\varepsilon_i)$.

(iii) $(h_i - \lambda_i I)V(\lambda) = 0$.

Proof: (i) Use $[h_i, e_i] = 2e_i$, Lemma 3.1, and (3.10).
(ii) Use $[h_i, f_i] = -2f_i$, Lemma 3.2, and (3.10).
(iii) By construction. □

Definition 3.30 A given vector $v \in V$ is said to be a *highest weight vector* whenever v is a weight vector and $e_i v = 0$ for $1 \leq i \leq N$.

Lemma 3.31 *V has a highest weight vector.*

Proof: By (3.14), there exists $\lambda \in \mathbb{F}^N$ with $V(\lambda) \neq 0$ and $V(\lambda + 2\varepsilon_i) = 0$ for $1 \leq i \leq N$. By Lemma 3.29(i), $e_i V(\lambda) = 0$ for $1 \leq i \leq N$. Therefore any nonzero element of $V(\lambda)$ is a highest weight vector. □

Until further notice v will denote a highest weight vector in V. Consider the $\mathfrak{g}$-module V. For $1 \leq i \leq N$ pull back the $\mathfrak{g}$-action via the homomorphism in line (3.9). Then V becomes an $\mathfrak{sl}_2$-module on which x acts as x_i for all $x \in \mathfrak{sl}_2$. Invoking Lemma 3.11, we obtain the following results. There exists $d_i \in \mathbb{N}$ such that $h_i v = d_i v$. The elements $\{f_i^r v\}_{r=0}^{d_i}$ form a standard basis for an $\mathfrak{sl}_2$-submodule of V that is isomorphic to $\mathbb{V}_{d_i}$. By construction, $f_i^{d_i+1} v = 0$ for $1 \leq i \leq N$. Define $d = (d_1, \ldots, d_N)$, and note that v has weight d. By (3.10), the elements $\{f_i\}_{i=1}^N$ mutually commute. For $k = (k_1, \ldots, k_N) \in \mathbb{N}^N$ define

$$v_k = f_1^{k_1} \cdots f_N^{k_N} v. \tag{3.15}$$

By Lemma 3.29(ii),

$$v_k \in V(d - 2k) \qquad (k \in \mathbb{N}^N). \tag{3.16}$$

Lemma 3.32 *The following hold for $1 \leq i \leq N$ and $k = (k_1, \ldots, k_N) \in \mathbb{N}^N$.*

(i) $h_i v_k = (d_i - 2k_i) v_k$.

(ii) $f_i v_k = v_{k+\varepsilon_i}$.

(iii) $e_i v_k = k_i (d_i + 1 - k_i) v_{k-\varepsilon_i}$, *where* $v_{k-\varepsilon_i} = 0$ *if* $k_i = 0$.

Proof: (i) By (3.16).

(ii) By construction.

(iii) Consider the $\mathfrak{g}$-module V. Pull back the $\mathfrak{g}$-action via the homomorphism in line

(3.9). Then V becomes an $\mathfrak{sl}_2$-module on which x acts as x_i for all $x \in \mathfrak{sl}_2$. By the comments following Lemma 3.31, the elements $\{f_i^r v\}_{r=0}^{d_i}$ form a standard basis for an $\mathfrak{sl}_2$-submodule of V that is isomorphic to $\mathbb{V}_{d_i}$. Therefore

$$e_i f_i^r v = r(d_i + 1 - r) f_i^{r-1} v \qquad (r \in \mathbb{N}, r \geq 1), \qquad e_i v = 0. \tag{3.17}$$

The result follows by lines (3.10), (3.15), (3.17). □

We now define a partial order on $\mathbb{N}^N$.

Notation 3.33 For $1 \leq i \leq N$ let $k_i, m_i \in \mathbb{N}$. Write $k = (k_1, \ldots, k_N)$ and $m = (m_1, \ldots, m_N)$. We define $k \leq m$ whenever $k_i \leq m_i$ for $1 \leq i \leq N$.

Lemma 3.34 *For $k \in \mathbb{N}^N$ the following are equivalent.*

(i) $v_k \neq 0$.

(ii) $k \leq d$.

Proof: Write $k = (k_1, \ldots, k_N)$.

(i) $\Rightarrow$ (ii) We assume that $k \not\leq d$ and show that $v_k = 0$. There exists $1 \leq i \leq N$ such that $k_i > d_i$. By the comments following Lemma 3.31, we have $f_i^{d_i+1} v = 0$, so $f_i^{k_i} v = 0$. Since $\{f_i\}_{i=1}^N$ mutually commute, we get $v_k = f_1^{k_1} \cdots f_{i-1}^{k_{i-1}} f_{i+1}^{k_{i+1}} \cdots f_N^{k_N} f_i^{k_i} v = 0$.

(ii) $\Rightarrow$ (i) By construction, $k_i \leq d_i$ for $1 \leq i \leq N$. By Lemma 3.32(iii), the vector $e_1^{k_1} \cdots e_N^{k_N} v_k$ is a nonzero scalar multiple of the highest weight vector v. Therefore $v_k \neq 0$. □

Let $w \in V$. We will be discussing the $\mathfrak{g}$-submodule of V generated by w. By definition this is the intersection of all $\mathfrak{g}$-submodules of V that contain w.

Lemma 3.35 *Let v denote a highest weight vector in V. Recall the vectors v_k from* (3.15)*. Then the vectors*

$$v_k \qquad\qquad k \in \mathbb{N}^N, \quad k \leq d \tag{3.18}$$

form a basis for the $\mathfrak{g}$-submodule of V generated by v.

Proof: Abbreviate U for the subspace of V spanned by the vectors (3.18). Note U is a $\mathfrak{g}$-module by Lemma 3.32 and Lemma 3.34. The vectors (3.18) are linearly independent by Lemma 3.34 and lines (3.14), (3.16). Therefore the vectors (3.18) form a basis for the $\mathfrak{g}$-module U. Abbreviate W for the $\mathfrak{g}$-submodule of V generated by v. We now prove that $W = U$. Note $v \in U$, so $W \subseteq U$. We now prove that $W \supseteq U$. It suffices to show $v_k \in W$ for $k \in \mathbb{N}^N$. Since W is a $\mathfrak{g}$-module and $v \in W$, we get $v_k \in W$ by (3.15). This shows $W = U$. The result follows. □

Lemma 3.36 *Let v denote a highest weight vector in V. Let W denote the $\mathfrak{g}$-submodule of V generated by v. Given $w \in W$ the following are equivalent.*

(i) *w is a highest weight vector.*

(ii) *There exists a nonzero $s \in \mathbb{F}$ such that $w = sv$.*

Proof: (i) ⇒ (ii) By Lemma 3.35, the vectors (3.18) form a basis for W. Note w is a weight vector in W by construction. By Lemma 3.32(i), w is a nonzero scalar multiple of some vector in (3.18). The only highest weight vector in (3.18) is v. Therefore w is a nonzero scalar multiple of v.

(ii) ⇒ (i) Routine. □

3.7 Highest weight modules for $\mathfrak{g}$

Fix a positive integer N, and recall the corresponding Lie algebra $\mathfrak{g}$ from (3.8). In Section 3.6 we proved some results about finite-dimensional $\mathfrak{g}$-modules. We now restrict our attention to a special case, called a highest weight module.

Definition 3.37 A $\mathfrak{g}$-module is said to be *highest weight* whenever it is generated by a highest weight vector.

Let V denote a finite-dimensional highest weight $\mathfrak{g}$-module. By construction, V is generated by a highest weight vector. By Lemma 3.36, any two highest weight vectors in V have the same weight. We define the *highest weight* of V to be the weight of a highest weight vector in V.

Lemma 3.38 *For* $1 \leq i \leq N$ *let* V_i *denote an irreducible* $\mathfrak{sl}_2$*-module of finite dimension* $d_i + 1$*. Then the* $\mathfrak{g}$*-module* $\otimes_{i=1}^N V_i$ *is highest weight with highest weight* $(d_1, \ldots, d_N)$*.*

Proof: For $1 \leq i \leq N$ pick $0 \neq u_i \in V_i$ such that $hu_i = d_i u_i$. Define $v = \otimes_{i=1}^N u_i$, and write $\lambda = (d_1, \ldots, d_N)$. Abbreviate V for the $\mathfrak{g}$-module $\otimes_{i=1}^N V_i$. One routinely checks that $v \in V(\lambda)$ is a highest weight vector and that V is generated by v. $\square$

Lemma 3.39 *For a finite-dimensional* $\mathfrak{g}$*-module* V *the following are equivalent.*

(i) V *is highest weight.*

(ii) V *is irreducible.*

Proof: (i) $\Rightarrow$ (ii) By construction, V is generated by a highest weight vector $v \in V$. Note $V \neq 0$ because $0 \neq v \in V$. Let U be a nonzero $\mathfrak{g}$-submodule of V. We now prove that $U = V$. It suffices to show that $v \in U$. By Lemma 3.31 applied to U, there exists a highest weight vector $u \in U$. Note u is a highest weight vector in V by construction. By Lemma 3.36, u is a nonzero scalar multiple of v, so $v \in U$. This proves $U = V$. Therefore V is irreducible.

(ii) $\Rightarrow$ (i) By Lemma 3.31, there exists a highest weight vector $v \in V$. Let W denote the submodule of V that is generated by v. Note that $W \neq 0$ because $0 \neq v \in W$. Therefore $W = V$ by the irreducibility of V. □

Lemma 3.40 *Let V, V' denote finite-dimensional highest weight $\mathfrak{g}$-modules with highest weights d, d', respectively. Then the $\mathfrak{g}$-modules V and V' are isomorphic if and only if $d = d'$.*

Proof: Combine Lemma 3.32 and Lemma 3.35. □

3.8 The classification of the finite-dimensional irreducible $\mathfrak{g}$-modules

Fix a positive integer N, and recall the corresponding Lie algebra $\mathfrak{g}$ from (3.8). We now classify up to isomorphism the finite-dimensional irreducible $\mathfrak{g}$-modules. This classification is given in the following three theorems.

Theorem 3.41 *For $1 \leq i \leq N$ let V_i denote a finite-dimensional irreducible $\mathfrak{sl}_2$-module. Then the $\mathfrak{g}$-module $\otimes_{i=1}^N V_i$ is irreducible.*

Proof: Combine Lemma 3.38 and Lemma 3.39. □

Theorem 3.42 *For $1 \leq i \leq N$ let V_i and V_i' denote finite-dimensional irreducible $\mathfrak{sl}_2$-modules. Then the $\mathfrak{g}$-modules $\otimes_{i=1}^N V_i$ and $\otimes_{i=1}^N V_i'$ are isomorphic if and only if the $\mathfrak{sl}_2$-modules V_i and V_i' are isomorphic for $1 \leq i \leq N$.*

Proof: For $1 \leq i \leq N$ denote the dimensions of V_i, V_i' by $d_i + 1, d_i' + 1$, respectively. By Lemma 3.38, the $\mathfrak{g}$-modules $V = \otimes_{i=1}^N V_i$ and $V' = \otimes_{i=1}^N V_i'$ are highest weight with highest weights $d = (d_1, \ldots, d_N)$ and $d' = (d_1', \ldots, d_N')$, respectively. By Lemma 3.40, the $\mathfrak{g}$-modules V and V' are isomorphic if and only if $d = d'$. For $1 \leq i \leq N$ the $\mathfrak{sl}_2$-modules V_i and V_i' are isomorphic if and only if $d_i = d_i'$. The result follows. □

Theorem 3.43 *Let V denote a finite-dimensional irreducible $\mathfrak{g}$-module. Then for $1 \leq i \leq N$ there exists a finite-dimensional irreducible $\mathfrak{sl}_2$-module V_i such that the $\mathfrak{g}$-modules V and $\otimes_{i=1}^N V_i$ are isomorphic.*

Proof: V is highest weight by Lemma 3.39. Let $d = (d_1, \ldots, d_N)$ denote the highest weight of V. Consider the $\mathfrak{g}$-module $V' = \otimes_{i=1}^N \mathbb{V}_{d_i}$. By Lemma 3.38, V' is highest weight with highest weight d. By Lemma 3.40, the $\mathfrak{g}$-modules V and V' are isomorphic. □

3.9 The ideals of $L(\mathfrak{sl}_2)$

We now turn our attention back to the Lie algebra $L(\mathfrak{sl}_2)$. In this section we describe the ideals of $L(\mathfrak{sl}_2)$.

Lemma 3.44 *Let K denote an ideal of $L(\mathfrak{sl}_2)$ and $p \in \mathbb{F}[t, t^{-1}]$. Then $\mathfrak{sl}_2 \otimes p$ is either contained in K or has zero intersection with K.*

Proof: We assume that $\mathfrak{sl}_2 \otimes p$ has nonzero intersection with K and show $\mathfrak{sl}_2 \otimes p$ is contained in K. Define a subspace W of $\mathfrak{sl}_2$ by $W = \{y \in \mathfrak{sl}_2 \,|\, y \otimes p \in K\}$. Note $W \neq 0$. We show W is an ideal of $\mathfrak{sl}_2$. Suppose $w \in W$, $z \in \mathfrak{sl}_2$. We show $[w, z] \in W$. Since $w \in W$, we have $w \otimes p \in K$. Since K is an ideal, $[w \otimes p, z \otimes 1] \in K$. But $[w \otimes p, z \otimes 1] = [w, z] \otimes p$, so $[w, z] \in W$. Therefore W is an ideal of $\mathfrak{sl}_2$. Since $W \neq 0$ and $\mathfrak{sl}_2$ is simple, we must have $W = \mathfrak{sl}_2$. The result follows. □

Lemma 3.45 *Let K denote an ideal of $L(\mathfrak{sl}_2)$. Define $J = \{p \in \mathbb{F}[t, t^{-1}] \,|\, \mathfrak{sl}_2 \otimes p \subseteq K\}$. Then J is an ideal of $\mathbb{F}[t, t^{-1}]$.*

Proof: By construction, J is a subspace of $\mathbb{F}[t, t^{-1}]$. Note that $\mathbb{F}[t, t^{-1}]$ is generated by t and t^{-1}. To prove that J is an ideal it suffices to show $t^{\varepsilon} J \subseteq J$ for $\varepsilon = \pm 1$. By construction, $\mathfrak{sl}_2 \otimes J \subseteq K$. Therefore $[\mathfrak{sl}_2 \otimes J, \mathfrak{sl}_2 \otimes t^{\varepsilon}] \subseteq K$ since K is an ideal of $L(\mathfrak{sl}_2)$. Note $[\mathfrak{sl}_2 \otimes J, \mathfrak{sl}_2 \otimes t^{\varepsilon}] = [\mathfrak{sl}_2, \mathfrak{sl}_2] \otimes t^{\varepsilon} J$, and $[\mathfrak{sl}_2, \mathfrak{sl}_2] = \mathfrak{sl}_2$, so $\mathfrak{sl}_2 \otimes t^{\varepsilon} J \subseteq K$. Therefore $t^{\varepsilon} J \subseteq J$. The result follows. □

We now describe the ideals of $L(\mathfrak{sl}_2)$.

Theorem 3.46 *Given a subspace K of $L(\mathfrak{sl}_2)$ the following are equivalent.*

(i) *K is an ideal of $L(\mathfrak{sl}_2)$.*

(ii) *There exists an ideal J of $\mathbb{F}[t, t^{-1}]$ such that $\mathfrak{sl}_2 \otimes J = K$.*

Suppose (i) *and* (ii) *hold. Then J is uniquely determined by K.*

Proof: (i) $\Rightarrow$ (ii) Consider the ideal J of $\mathbb{F}[t, t^{-1}]$ from Lemma 3.45. We show that $\mathfrak{sl}_2 \otimes J = K$. By construction, $\mathfrak{sl}_2 \otimes J \subseteq K$, so it suffices to show that

$\mathfrak{sl}_2 \otimes J \supseteq K$. Let $u \in K$. Write $u = e \otimes g_1 + f \otimes g_2 + h \otimes g_3$, where $g_1, g_2, g_3 \in \mathbb{F}[t, t^{-1}]$ and e, f, h is the Chevalley basis of $\mathfrak{sl}_2$. We show that $g_1, g_2, g_3 \in J$. Observe that $[u, h \otimes 1] \in K$, so $e \otimes g_1 - f \otimes g_2 \in K$. We have $[e \otimes g_1 - f \otimes g_2, f \otimes 1] \in K$, so $h \otimes g_1 \in K$. Consequently $g_1 \in J$ by Lemma 3.44. Similarly $[e \otimes g_1 - f \otimes g_2, e \otimes 1] \in K$, so $h \otimes g_2 \in K$. Consequently $g_2 \in J$ by Lemma 3.44. By the above comments K contains $e \otimes g_1$ and $f \otimes g_2$. Therefore $h \otimes g_3 = u - e \otimes g_1 - f \otimes g_2 \in K$. Consequently $g_3 \in J$ by Lemma 3.44. We have shown $g_1, g_2, g_3 \in J$, so $u \in \mathfrak{sl}_2 \otimes J$. The result follows.

(ii) $\Rightarrow$ (i) Routine.

The last assertion is routinely checked. $\square$

Theorem 3.46 motivates us to describe the ideals of the algebra $\mathbb{F}[t, t^{-1}]$. In Section 3.10 we will describe the ideals and quotients of $\mathbb{F}[t, t^{-1}]$. In Section 3.11 we will use Theorem 3.46 and the results in Section 3.10 to describe the quotients of the Lie algebra $L(\mathfrak{sl}_2)$.

3.10 The ideals and quotients of $\mathbb{F}[t, t^{-1}]$

In this section we describe the ideals and quotients of the algebra $\mathbb{F}[t, t^{-1}]$.

Throughout this section let J denote a nonzero proper ideal of $\mathbb{F}[t, t^{-1}]$. Consider the subalgebra $\mathbb{F}[t]$ of $\mathbb{F}[t, t^{-1}]$. The algebra $\mathbb{F}[t]$ is a principal ideal domain, and $J \cap \mathbb{F}[t]$ is an ideal of $\mathbb{F}[t]$. Therefore there exists $g \in \mathbb{F}[t]$ such that $J \cap \mathbb{F}[t] = g\,\mathbb{F}[t]$. We now prove that $J = g\,\mathbb{F}[t, t^{-1}]$. Since J is an ideal of $\mathbb{F}[t, t^{-1}]$ and $g \in J$, we have $J \supseteq g\,\mathbb{F}[t, t^{-1}]$. We now show that $J \subseteq g\,\mathbb{F}[t, t^{-1}]$. Let $\eta \in J$. There exists $i \in \mathbb{N}$ such that $t^i \eta \in \mathbb{F}[t]$. Since J is an ideal of $\mathbb{F}[t, t^{-1}]$ and $\eta \in J$, we have that $t^i \eta \in J$. Therefore $t^i \eta \in J \cap \mathbb{F}[t] = g\,\mathbb{F}[t]$, so $\eta \in gt^{-i}\,\mathbb{F}[t] \subseteq g\,\mathbb{F}[t,t^{-1}]$. We have now shown that $J = g\,\mathbb{F}[t,t^{-1}]$. The polynomial g is nonzero because J is nonzero. Without loss we can assume that g is monic. Let l be the minimal degree of a nonzero polynomial in $J \cap \mathbb{F}[t]$. By construction, g is the unique monic polynomial in $J \cap \mathbb{F}[t]$ with degree l. We call g the *standard generator* of J. The constant term of g is nonzero; otherwise $g = th$ for some $h \in \mathbb{F}[t]$, and $h = t^{-1}g \in J \cap \mathbb{F}[t]$ has lower degree than g, which contradicts our earlier comments.

Let N denote the degree of g. Since J is properly contained in $\mathbb{F}[t,t^{-1}]$, J does not contain any units of $\mathbb{F}[t,t^{-1}]$. Therefore N is at least one. Factor the polynomial g as

$$g = (t-a_1)(t-a_2)\cdots(t-a_N), \tag{3.19}$$

where $a_i \in \mathbb{F}$ for $1 \le i \le N$. Note that the constant term of g is $(-1)^N a_1 a_2 \cdots a_N$. This constant term is nonzero, so $a_i \neq 0$ for $1 \le i \le N$.

Consider the quotient algebra $\mathbb{F}[t,t^{-1}]/J$. Let T denote the image of t under the canonical homomorphism $\mathbb{F}[t,t^{-1}] \to \mathbb{F}[t,t^{-1}]/J$. Recall $J = g\,\mathbb{F}[t,t^{-1}]$ and g has minimal degree among all the nonzero elements of $J \cap \mathbb{F}[t]$. Using this we find that $g(T) = 0$, and we find that the elements $1, T, \ldots, T^{N-1}$ form a basis of the $\mathbb{F}$-vector space $\mathbb{F}[t,t^{-1}]/J$. We now give another basis of $\mathbb{F}[t,t^{-1}]/J$ that will be more convenient for us in later sections. For $0 \le i \le N-1$ define p_i to be the polynomial

$$p_i(t) = (t-a_1)(t-a_2)\cdots(t-a_i).$$

Note that $p_0 = 1$. Observe that the degree of p_i is i for $0 \le i \le N-1$. Therefore $\{p_i(T)\}_{i=0}^{N-1}$ is a basis of $\mathbb{F}[t,t^{-1}]/J$.

3.11 The quotients of $L(\mathfrak{sl}_2)$

In this section we describe the quotients of the Lie algebra $L(\mathfrak{sl}_2)$.

Throughout this section let K denote a nonzero proper ideal of $L(\mathfrak{sl}_2)$. By Theorem 3.46, there exists an ideal J of $\mathbb{F}[t, t^{-1}]$ such that $K = \mathfrak{sl}_2 \otimes J$. Note that J is nonzero and properly contained in $\mathbb{F}[t, t^{-1}]$.

Consider the quotient algebra $\mathbb{F}[t, t^{-1}]/J$. Observe that the $\mathbb{F}$-vector space $\mathfrak{sl}_2 \otimes (\mathbb{F}[t, t^{-1}]/J)$ becomes a Lie algebra over $\mathbb{F}$ with Lie bracket

$$[u \otimes a, v \otimes b] = [u, v] \otimes ab, \qquad u, v \in \mathfrak{sl}_2, \quad a, b \in \mathbb{F}[t, t^{-1}]/J.$$

Let $\varphi : \mathbb{F}[t, t^{-1}] \to \mathbb{F}[t, t^{-1}]/J$ denote the canonical homomorphism. Consider the quotient Lie algebra $L(\mathfrak{sl}_2)/K$.

Lemma 3.47 *The Lie algebra homomorphism*

$$L(\mathfrak{sl}_2) \to \mathfrak{sl}_2 \otimes \big(\mathbb{F}[t, t^{-1}]/J\big)$$
$$u \otimes a \mapsto u \otimes \varphi(a)$$

is surjective and its kernel is K. Therefore the homomorphism

$$\begin{aligned} L(\mathfrak{sl}_2)/K &\to \mathfrak{sl}_2 \otimes \big(\mathbb{F}[t, t^{-1}]/J\big) \\ u \otimes a + K &\mapsto u \otimes \varphi(a) \end{aligned} \tag{3.20}$$

is an isomorphism of Lie algebras.

Proof: Routine. □

We identify $L(\mathfrak{sl}_2)/K$ with $\mathfrak{sl}_2 \otimes (\mathbb{F}[t,t^{-1}]/J)$ via the isomorphism (3.20).

Recall the standard generator (3.19) of J and the basis $\{p_i(T)\}_{i=0}^{N-1}$ of $\mathbb{F}[t,t^{-1}]/J$ from Section 3.10. By Lemma 3.47, the quotient Lie algebra $L(\mathfrak{sl}_2)/K$ has a basis $\{E_i, F_i, H_i \,|\, i = 1, \ldots, N\}$ where

$$E_i = e \otimes p_{i-1}(T), \quad F_i = f \otimes p_{i-1}(T), \quad H_i = h \otimes p_{i-1}(T), \tag{3.21}$$

for $1 \le i \le N$.

One routinely checks that the following hold in $L(\mathfrak{sl}_2)/K$ for all $1 \le k, r \le N$.

$$[E_k, F_r] \in \mathrm{Span}\{H_1, \ldots, H_N\}, \tag{3.22}$$

$$[H_k, E_r] \in \mathrm{Span}\{E_1, \ldots, E_N\}, \tag{3.23}$$

$$[H_k, F_r] \in \mathrm{Span}\{F_1, \ldots, F_N\}, \tag{3.24}$$

$$[E_k, E_r] = 0, \qquad [F_k, F_r] = 0, \qquad [H_k, H_r] = 0, \tag{3.25}$$

$$[E_1, F_r] = H_r, \qquad [H_1, E_r] = 2E_r, \qquad [H_1, F_r] = -2F_r. \tag{3.26}$$

We abbreviate

$$E = E_1, \qquad F = F_1, \qquad H = H_1. \tag{3.27}$$

In view of relations (3.26) with $r = 1$, there exists an injection of Lie algebras $\mathfrak{sl}_2 \to L(\mathfrak{sl}_2)/K$ that sends

$$e \mapsto E, \qquad f \mapsto F, \qquad h \mapsto H. \tag{3.28}$$

The image of $\mathfrak{sl}_2$ under this map is the Lie subalgebra of $L(\mathfrak{sl}_2)/K$ with basis E, F, H.

3.12 The homomorphism $\psi : L(\mathfrak{sl}_2) \to \mathfrak{g}$

Fix a positive integer N, and recall the corresponding Lie algebra $\mathfrak{g}$ from (3.8). In this section we consider a homomorphism $\psi : L(\mathfrak{sl}_2) \to \mathfrak{g}$. We use this homomorphism to describe a relationship between $\mathfrak{g}$-modules and $L(\mathfrak{sl}_2)$-modules.

For $1 \leq i \leq N$ let $a_i \in \mathbb{F}$ be nonzero and recall the Lie algebra homomorphism $EV_{a_i} : L(\mathfrak{sl}_2) \to \mathfrak{sl}_2$ from Definition 3.15. We define the Lie algebra homomorphism $\psi : L(\mathfrak{sl}_2) \to \mathfrak{g}$ by

$$\psi(u) = (EV_{a_1}(u), \ldots, EV_{a_N}(u)), \qquad u \in L(\mathfrak{sl}_2). \tag{3.29}$$

Let $\mathcal{S}$ denote the set of distinct elements among $\{a_i\}_{i=1}^N$. Define G to be the polynomial

$$G(t) = \prod_{a \in \mathcal{S}} (t - a). \tag{3.30}$$

Lemma 3.48 *The kernel of ψ is $\mathfrak{sl}_2 \otimes G\,\mathbb{F}[t, t^{-1}]$, where G is from* (3.30).

Proof: Observe that

$$\begin{aligned} \ker(\psi) &= \bigcap_{i=1}^{N} \ker(EV_{a_i}) \\ &= \bigcap_{a \in \mathcal{S}} \ker(EV_a) \\ &= \bigcap_{a \in \mathcal{S}} \mathfrak{sl}_2 \otimes (t-a)\mathbb{F}[t, t^{-1}] \\ &= \mathfrak{sl}_2 \otimes G\,\mathbb{F}[t, t^{-1}]. \end{aligned}$$

□

Lemma 3.49 *The map ψ is surjective if and only if $\{a_i\}_{i=1}^N$ are mutually distinct.*

Proof: Let n denote the cardinality of $\mathcal{S}$. Abbreviate K for $\mathfrak{sl}_2 \otimes G\,\mathbb{F}[t,t^{-1}]$, where G is from (3.30). By Lemma 3.48, K is the kernel of ψ. Note G is the standard generator of the ideal $G\,\mathbb{F}[t,t^{-1}]$ of $\mathbb{F}[t,t^{-1}]$. By construction, G has degree n. By the comments in Section 3.11, the quotient Lie algebra $L(\mathfrak{sl}_2)/K$ has dimension $3n$. The Lie algebra $\mathfrak{g}$ has dimension $3N$. Therefore the map ψ is surjective if and only if $n = N$. The result follows. □

Lemma 3.50 *For* $1 \leq i \leq N$ *let* $d_i \in \mathbb{N}$. *Consider the* $L(\mathfrak{sl}_2)$*-module* $W = \otimes_{i=1}^{N} \mathbb{V}_{d_i}(a_i)$. *Then the following hold.*

(i) *The* $L(\mathfrak{sl}_2)$*-action on* W *is obtained by pulling back the* $\mathfrak{g}$*-action on* $\otimes_{i=1}^{N} \mathbb{V}_{d_i}$ *via the map* ψ.

(ii) *Assume that* $d_i \neq 0$ *for* $1 \leq i \leq N$. *Then the kernel of the* $L(\mathfrak{sl}_2)$*-action on* W *is* $\mathfrak{sl}_2 \otimes G\,\mathbb{F}[t,t^{-1}]$, *where* G *is from* (3.30).

Proof: (i) Routine.

(ii) By Lemma 3.48, the kernel of ψ is $\mathfrak{sl}_2 \otimes G\,\mathbb{F}[t,t^{-1}]$. By Lemmas 3.10 and 3.27, the $\mathfrak{g}$-action on $\otimes_{i=1}^{N} \mathbb{V}_{d_i}$ is faithful. Combining these facts with part (i) of this lemma gives the result. □

3.13 The proof of Theorems 3.23 and 3.24

We are now ready to prove Theorems 3.23 and 3.24.

Proof of Theorem 3.23*:* Abbreviate V for the $L(\mathfrak{sl}_2)$-module $\otimes_{i=1}^{N} \mathbb{V}_{d_i}(a_i)$.

Suppose $\{a_i\}_{i=1}^N$ are mutually distinct. We will show that the $L(\mathfrak{sl}_2)$-module V is irreducible. Consider the corresponding Lie algebra $\mathfrak{g}$ from (3.8), and recall the map $\psi : L(\mathfrak{sl}_2) \to \mathfrak{g}$ from (3.29). By Lemma 3.49, ψ is surjective, and by Lemma 3.50(i), the $L(\mathfrak{sl}_2)$-action on V is obtained by pulling back the $\mathfrak{g}$-action on $\otimes_{i=1}^N \mathbb{V}_{d_i}$ via the map ψ. By Theorem 3.41, the $\mathfrak{g}$-module $\otimes_{i=1}^N \mathbb{V}_{d_i}$ is irreducible, so by Lemma 3.3, the $L(\mathfrak{sl}_2)$-module V is irreducible.

Now suppose $\{a_i\}_{i=1}^N$ are not mutually distinct. Then N is at least 2. We will show that the $L(\mathfrak{sl}_2)$-module V is reducible. By Lemma 3.22(i), we may assume without loss of generality that $a_1 = a_2$. Denote this common value by a. Consider the $\mathfrak{sl}_2$-action on $\mathbb{V}_{d_1} \otimes \mathbb{V}_{d_2}$ given by (3.7). By construction, d_1 and d_2 are positive. Using (3.7) one can check that the $\mathfrak{sl}_2$-module $\mathbb{V}_{d_1} \otimes \mathbb{V}_{d_2}$ is reducible. One routinely checks that the $L(\mathfrak{sl}_2)$-action on $\mathbb{V}_{d_1}(a) \otimes \mathbb{V}_{d_2}(a)$ is obtained by pulling back the $\mathfrak{sl}_2$-action on $\mathbb{V}_{d_1} \otimes \mathbb{V}_{d_2}$ via the map $EV_a : L(\mathfrak{sl}_2) \to \mathfrak{sl}_2$. By these comments and Lemma 3.3, the $L(\mathfrak{sl}_2)$-module $\mathbb{V}_{d_1}(a) \otimes \mathbb{V}_{d_2}(a)$ is reducible. By Lemma 3.22(ii), the $L(\mathfrak{sl}_2)$-module V is reducible. $\square$

Proof of Theorem 3.24*:* Write $U_i = \mathbb{V}_{d_i}(a_i)$ for $1 \le i \le N$ and $U'_j = \mathbb{V}_{d'_j}(a'_j)$ for $1 \le j \le N'$.

(i) $\Rightarrow$ (ii) By Theorem 3.23, $\{a_i\}_{i=1}^N$ are mutually distinct and $\{a'_j\}_{j=1}^{N'}$ are mutually distinct. By Lemma 3.50(ii), the kernel of the $L(\mathfrak{sl}_2)$-action on V is $\mathfrak{sl}_2 \otimes G\,\mathbb{F}[t, t^{-1}]$, where the polynomial $G = (t - a_1)\cdots(t - a_N)$. Similarly, the kernel of the $L(\mathfrak{sl}_2)$-action on V' is $\mathfrak{sl}_2 \otimes G'\,\mathbb{F}[t, t^{-1}]$, where the polynomial $G' = (t - a'_1)\cdots(t - a'_{N'})$. Since the $L(\mathfrak{sl}_2)$-modules V, V' are isomorphic, the kernels of the $L(\mathfrak{sl}_2)$-actions on V and V' are the same. Therefore $\mathfrak{sl}_2 \otimes G\,\mathbb{F}[t, t^{-1}] = \mathfrak{sl}_2 \otimes G'\,\mathbb{F}[t, t^{-1}]$. Invoking the last assertion in Theorem 3.46 we get $G\,\mathbb{F}[t, t^{-1}] = G'\,\mathbb{F}[t, t^{-1}]$. For this common ideal of $\mathbb{F}[t, t^{-1}]$, both

G and G' are the standard generator, so $G = G'$. Therefore $N = N'$, and $a'_1, a'_2, \ldots, a'_N$ is a permutation of $a_1, a_2, \ldots, a_N$. After permuting $\{U'_i\}_{i=1}^N$ we may assume $a_i = a'_i$ for $1 \leq i \leq N$. So far we have $U'_i = \mathbb{V}_{d'_i}(a_i)$ for $1 \leq i \leq N$. We now show that $d_i = d'_i$ for $1 \leq i \leq N$. Consider the corresponding Lie algebra $\mathfrak{g}$ from (3.8). The map ψ from Lemma 3.49 is surjective. By Lemma 3.50(i), the $L(\mathfrak{sl}_2)$-action on V (resp. V') is obtained by pulling back the $\mathfrak{g}$-action on $\otimes_{i=1}^N \mathbb{V}_{d_i}$ (resp. $\otimes_{i=1}^N \mathbb{V}_{d'_i}$) via the map ψ. By Lemma 3.4, the $\mathfrak{g}$-modules $\otimes_{i=1}^N \mathbb{V}_{d_i}$ and $\otimes_{i=1}^N \mathbb{V}_{d'_i}$ are isomorphic. By Theorem 3.42, $d_i = d'_i$ for $1 \leq i \leq N$. The result follows.

(ii) $\Rightarrow$ (i) By Lemma 3.22(i). $\square$

3.14 The proof of Theorem 3.25

Let V denote a nontrivial finite-dimensional irreducible $L(\mathfrak{sl}_2)$-module. Let K denote the kernel of the $L(\mathfrak{sl}_2)$-action on V, and observe that K is an ideal of $L(\mathfrak{sl}_2)$. We shall view V as a module for $L(\mathfrak{sl}_2)/K$. Note that this module is faithful and irreducible. Observe that K is nonzero because $\mathrm{End}(V)$ is finite-dimensional while $L(\mathfrak{sl}_2)$ is not. Furthermore K is properly contained in $L(\mathfrak{sl}_2)$ because V is nontrivial and irreducible. By Theorem 3.46 there exists an ideal J of $\mathbb{F}[t, t^{-1}]$ such that $K = \mathfrak{sl}_2 \otimes J$. Note that J is nonzero and properly contained in $\mathbb{F}[t, t^{-1}]$. Recall the standard generator g of J from Section 3.10.

In this section we have two related goals. We will invoke the results in Section 3.8 and Section 3.12 to prove Theorem 3.25. In order to do this we first need to show that g has no repeated roots.

Consider the basis (3.21) of $L(\mathfrak{sl}_2)/K$, and recall the notation from (3.27).

Lemma 3.51 *H is diagonalizable on V.*

Proof: Consider the $L(\mathfrak{sl}_2)/K$-module V. Pull back the $L(\mathfrak{sl}_2)/K$-action via the homomorphism in line (3.28). Then V becomes an $\mathfrak{sl}_2$-module on which h acts as H. Combining Lemma 3.6 and Lemma 3.7 we find h is diagonalizable on V. The result follows. □

Recall the notation from line (3.1).

Lemma 3.52 *The following hold for $\mu \in \mathbb{F}$ and $1 \leq i \leq N$.*

(i) $E_i V_H(\mu) \subseteq V_H(\mu + 2)$.

(ii) $F_i V_H(\mu) \subseteq V_H(\mu - 2)$.

(iii) $H_i V_H(\mu) \subseteq V_H(\mu)$.

Proof: (i) Use Lemma 3.1 and relations (3.26).

(ii) Use Lemma 3.2 and relations (3.26).

(iii) Use relations (3.25). □

Lemma 3.53 *For $1 \leq i \leq N$ the elements E_i and F_i are nilpotent on V.*

Proof: Combine Lemma 3.51 and Lemma 3.52. □

Definition 3.54 A given vector $v \in V$ is said to be a *highest weight vector* whenever $v \neq 0$, $E_i v = 0$ for $1 \leq i \leq N$, and v is a common eigenvector of $\{H_i\}_{i=1}^N$.

Lemma 3.55 *V has a highest weight vector.*

Proof: Since V has finite dimension, there exists $\mu \in \mathbb{F}$ with $V_H(\mu) \neq 0$ and $V_H(\mu+2) = 0$. By Lemma 3.52(i), we get that $E_i V_H(\mu) = 0$ for $1 \leq i \leq N$. Since $\{H_i\}_{i=1}^N$ mutually commute by (3.25), there exists $v \in V_H(\mu)$ that it is a common eigenvector of $\{H_i\}_{i=1}^N$. By construction, $\{E_i\}_{i=1}^N$ annihilate v. Therefore v is a highest weight vector. □

Let $\mathcal{F}$ denote the subalgebra of $\mathrm{End}(V)$ generated by the actions of $\{F_i\}_{i=1}^N$. By (3.25), $\mathcal{F}$ is spanned by the actions of

$$F_1^{n_1} F_2^{n_2} \cdots F_N^{n_N}, \qquad n_1, \ldots, n_N \in \mathbb{N}.$$

Lemma 3.56 *Let $v \in V$ denote a highest weight vector. Then $V = \mathcal{F}v$.*

Proof: Abbreviate W for $\mathcal{F}v$. By construction, W is a subspace of V, and $W \neq 0$ because $0 \neq v \in W$. By the irreducibility of the $L(\mathfrak{sl}_2)/K$-module V, it suffices to show that W is a submodule of V. To show this it suffices to check that W is closed under the action of E_k, F_k, H_k for $1 \leq k \leq N$. By construction, W is closed under the action of F_k for $1 \leq k \leq N$. Next we show that W is closed under the action of H_k for $1 \leq k \leq N$. Let k be given. By the comments immediately preceding the statement of the lemma, it suffices to show that

$$H_k F_1^{n_1} F_2^{n_2} \cdots F_N^{n_N} v \in W, \qquad n_1, \ldots, n_N \in \mathbb{N}. \tag{3.31}$$

The proof of (3.31) is by induction on $n_1 + \cdots + n_N$. For the case $n_1 + \cdots + n_N = 0$, the claim is true because v is a common eigenvector of $\{H_i\}_{i=1}^N$. Now suppose that $n_1 + \cdots + n_N > 0$. Therefore there exists $1 \leq r \leq N$ such that $n_r > 0$. By (3.25),

$$H_k F_1^{n_1} F_2^{n_2} \cdots F_N^{n_N} v = H_k F_r F_1^{n_1} F_2^{n_2} \cdots F_r^{n_r - 1} \cdots F_N^{n_N} v. \tag{3.32}$$

The right hand side of (3.32) equals

$$F_r H_k F_1^{n_1} F_2^{n_2} \cdots F_r^{n_r - 1} \cdots F_N^{n_N} v + [H_k, F_r] F_1^{n_1} F_2^{n_2} \cdots F_r^{n_r - 1} \cdots F_N^{n_N} v. \qquad (3.33)$$

In line (3.33), the term on the left is in W by the induction hypothesis, and the term on the right is in W by (3.24). We have now shown (3.31) for $1 \leq k \leq N$. Now we show that W is closed under the action of E_k for $1 \leq k \leq N$. Let k be given. It suffices to show that

$$E_k F_1^{n_1} F_2^{n_2} \cdots F_N^{n_N} v \in W, \qquad n_1, \ldots, n_N \in \mathbb{N}. \qquad (3.34)$$

The proof of (3.34) is by induction on $n_1 + \cdots + n_N$. For the case $n_1 + \cdots + n_N = 0$, the claim is true because v is annihilated by each of $\{E_i\}_{i=1}^N$. Now suppose that $n_1 + \cdots + n_N > 0$. Therefore there exists $1 \leq r \leq N$ such that $n_r > 0$. By (3.25),

$$E_k F_1^{n_1} F_2^{n_2} \cdots F_N^{n_N} v = E_k F_r F_1^{n_1} F_2^{n_2} \cdots F_r^{n_r - 1} \cdots F_N^{n_N} v. \qquad (3.35)$$

The right hand side of (3.35) equals

$$F_r E_k F_1^{n_1} F_2^{n_2} \cdots F_r^{n_r - 1} \cdots F_N^{n_N} v + [E_k, F_r] F_1^{n_1} F_2^{n_2} \cdots F_r^{n_r - 1} \cdots F_N^{n_N} v. \qquad (3.36)$$

In line (3.36), the term on the left is in W by the induction hypothesis. The term on the right is in W by (3.22) and since W is closed under the action of each of $\{H_i\}_{i=1}^N$. This shows (3.34) for $1 \leq k \leq N$. The result follows. □

Theorem 3.57 *The polynomial g has no repeated roots.*

Proof: Recall the roots $\{a_i\}_{i=1}^N$ of g from line (3.19). We will show that $\{a_i\}_{i=1}^N$ are mutually distinct. Suppose by way of contradiction that g has a repeated root. Then

N is at least 2. Relabeling the roots of g if necessary, we may assume that $a_1 = a_N$. We first show that the following relations hold in $L(\mathfrak{sl}_2)/K$ for $2 \leq r \leq N$.

$$[E_N, F_r] = 0, \quad [F_N, E_r] = 0, \quad [H_N, E_r] = 0, \tag{3.37}$$

$$[E_N, H_r] = 0, \quad [F_N, H_r] = 0, \quad [H_N, F_r] = 0. \tag{3.38}$$

We check the relation on the left in (3.37). Observe $[E_N, F_r] = [e, f] \otimes p_{N-1}(T)p_{r-1}(T)$. Note that $p_{N-1}(T)p_{r-1}(T)$ is equal to

$$(T - a_1)(T - a_2) \cdots (T - a_{N-1})(T - a_1) \cdots (T - a_{r-1}).$$

Keeping in mind that $a_1 = a_N$ and $r \geq 2$, we see that

$$(T - a_1)(T - a_2) \cdots (T - a_N)$$

is a factor of $p_{N-1}(T)p_{r-1}(T)$. But this factor is $g(T) = 0$, so $p_{N-1}(T)p_{r-1}(T) = 0$. The relation on the left in (3.37) now follows. The other relations in (3.37) and (3.38) are proved in a similar fashion.

Fix a highest weight vector v of V, and consider $F_N v$. We will show that

$$F_N v = 0. \tag{3.39}$$

Suppose by way of contradiction that $F_N v \neq 0$. We now show that $F_N v$ is a highest weight vector.

First we check that $F_N v$ is an eigenvector of H_1. By construction, $v \in V_H(\mu)$ for some $\mu \in \mathbb{F}$. By Lemma 3.52(ii), $F_N v \in V_H(\mu - 2)$. Thus $F_N v$ is an eigenvector of H_1.

Now we check that $F_N v$ is an eigenvector of H_r for $2 \leq r \leq N$. Fix r with $2 \leq r \leq N$. Recall that H_r and F_N commute by (3.38). So $F_N v$ must be an eigenvector of H_r since v is an eigenvector of H_r.

Next we check that E_1 annihilates $F_N v$. First we need to make a few observations. By construction, $H_N v = \lambda v$ for some $\lambda \in \mathbb{F}$. We claim that for all $j \in \mathbb{N}$ the following holds.

$$E_1 F_N^{j+1} v = (j+1)\lambda F_N^j v. \tag{3.40}$$

To show (3.40), we proceed by induction on j. Observe that

$$E_1 F_N v = F_N E_1 v + H_N v = \lambda v,$$

where the first equality holds by (3.26) and the second since $E_1 v = 0$ and $H_N v = \lambda v$. This shows (3.40) holds when $j = 0$. Now suppose $j \geq 1$, and observe that

$$\begin{aligned} E_1 F_N^{j+1} v &= F_N E_1 F_N^j v + H_N F_N^j v \\ &= j\lambda F_N^j v + H_N F_N^j v \\ &= j\lambda F_N^j v + F_N^j H_N v \\ &= (j+1)\lambda F_N^j v, \end{aligned}$$

where the first equality holds by (3.26), the second by the induction hypothesis, and the third by (3.38). This shows (3.40) holds for all $j \in \mathbb{N}$. Now we show that

$$\lambda = 0. \tag{3.41}$$

To see why (3.41) holds we argue as follows. By Lemma 3.53, F_N is nilpotent on V. By this and since $v \neq 0$, there exists $M \in \mathbb{N}$ such that $F_N^M v \neq 0$ while $F_N^{M+1} v = 0$. Setting

$j = M$ in (3.40) we obtain $\lambda = 0$. This shows that (3.41) holds. If we set $j = 0$ in (3.40) and we combine this with (3.41), then we get that E_1 annihilates $F_N v$, as desired.

Finally, we check that E_r annihilates $F_N v$ for $2 \le r \le N$. Fix r with $2 \le r \le N$. Recall that E_r and F_N commute by (3.37) and that E_r annihilates v. It follows that $E_r F_N v = F_N E_r v = 0$. So E_r annihilates $F_N v$.

We have shown that $F_N v$ is a highest weight vector. We apply Lemma 3.56 to the highest weight vector $F_N v$ to get that

$$V = \mathcal{F} F_N v. \tag{3.42}$$

Recall that $v \in V_H(\mu)$. Therefore the right hand side of (3.42) is contained in the space

$$V_H(\mu - 2) + V_H(\mu - 4) + \cdots$$

by Lemma 3.52(ii). So v is contained in the left hand side of (3.42) but not in the right hand side of (3.42), which is a contradiction. In conclusion, we have seen that assuming $F_N v \neq 0$ has led to a contradiction, so (3.39) holds.

To finish the proof of Theorem 3.57, we note the following. By (3.39), (3.25), and Lemma 3.56, applied to the highest weight vector v, we see that F_N vanishes on V. However the action of $L(\mathfrak{sl}_2)/K$ on V is faithful. This gives a contradiction. Therefore our assumption that g has a repeated root is false. This proves the result. $\square$

We are now ready to prove Theorem 3.25.

Proof of Theorem 3.25*:* Let V denote a nontrivial finite-dimensional irreducible $L(\mathfrak{sl}_2)$-module. We will show that V is isomorphic to a tensor product of evaluation modules. We will be referring to the discussion in the first paragraph of this section, in particular the ideals J and K and the standard generator g of J. By Theorem 3.57, the polynomial g has no repeated roots. In other words, the roots $\{a_i\}_{i=1}^N$ of g from (3.19) are mutually distinct. For that N, we consider the Lie algebra $\mathfrak{g}$ from (3.8). By Lemma 3.48 and Lemma 3.49, the Lie algebra homomorphism ψ from (3.29) is surjective and its kernel is K. By the construction, the homomorphism

$$\begin{aligned} L(\mathfrak{sl}_2)/K &\rightarrow \mathfrak{g} \\ u + K &\mapsto (EV_{a_1}(u), \ldots, EV_{a_N}(u)) \end{aligned}$$

is an isomorphism of Lie algebras. Therefore there exists an irreducible $\mathfrak{g}$-module structure on V such that the $L(\mathfrak{sl}_2)$-action on V is obtained by pulling back the $\mathfrak{g}$-action on V via the map ψ. By Theorem 3.43, there exists $d_i \in \mathbb{N}$ $(1 \leq i \leq N)$ such that the $\mathfrak{g}$-modules V and $\otimes_{i=1}^N \mathbb{V}_{d_i}$ are isomorphic. By Lemma 3.50(i), the $L(\mathfrak{sl}_2)$-action on $\otimes_{i=1}^N \mathbb{V}_{d_i}(a_i)$ is obtained by pulling back the $\mathfrak{g}$-action on $\otimes_{i=1}^N \mathbb{V}_{d_i}$ via the map ψ. Combining the above observations with Lemma 3.4 we get that the $L(\mathfrak{sl}_2)$-modules V and $\otimes_{i=1}^N \mathbb{V}_{d_i}(a_i)$ are isomorphic. Finally, note that the $\mathfrak{g}$-action on $\otimes_{i=1}^N \mathbb{V}_{d_i}$ is faithful since the $L(\mathfrak{sl}_2)/K$-action on V is faithful. By Lemma 3.27, d_i is postive for $1 \leq i \leq N$. This proves that the $L(\mathfrak{sl}_2)$-module V is isomorphic to a tensor product of evaluation modules. □

Bibliography

[1] R. Carter. *Lie algebras of finite and affine type.* Cambridge University Press, Cambridge, 2005.

[2] V. Chari. Integrable representations of affine Lie algebras. *Invent. Math.* **85** (1986), 317–335.

[3] E. Date and S. S. Roan. The structure of quotients of the Onsager algebra by closed ideals. *J. Phys. A: Math. Gen.* **33** (2000) 3275–3296.

[4] B. Davies. Onsager's algebra and superintegrability. *J. Phys. A: Math. Gen.* **23** (1990), 2245–2261.

[5] B. Davies. Onsager's algebra and the Dolan-Grady condition in the non-self-dual case. *J. Math. Phys.* **32** (1991), 2945–2950.

[6] J. Go. The Terwilliger algebra of the hypercube. *European J. Combin.* **23** (2002) 399–429.

[7] B. Hartwig. The tetrahedron algebra and its finite-dimensional irreducible modules. *Linear Algebra Appl.* **422** (2007) 219–235. `arXiv:math.RT/0606197`.

[8] J. Humphreys. *Introduction to Lie algebras and representation theory.* Springer-Verlag, New York, NY, 1972.

[9] T. Ito, K. Tanabe, P. Terwilliger. Some algebra related to P- and Q-polynomial association schemes, in: Codes and Association Schemes (Piscataway NJ, 1999), Amer. Math. Soc., Providence RI, 2001, pp. 167–192; `arXiv:math/0406556`.

[10] V. Kac. *Infinite dimensional Lie algebras.* Cambridge University Press, Cambridge, 1990.

[11] K. Nomura, P. Terwilliger. On the shape of a tridiagonal pair. *Linear Algebra Appl.* **432** (2010) 615–636.

[12] L. Onsager. Crystal statistics. I. A two-dimensional model with an order-disorder transition. *Phys. Rev.* **65** (1944) 117–149.

[13] J. H. H. Perk. Star-triangle relations, quantum Lax pairs, and higher genus curves. *Proceedings of Symposia in Pure Mathematics* **49** 341–354. Amer. Math. Soc., Providence, RI, 1989.