MULTIPLIER THEOREMS, SQUARE FUNCTION ESTIMATES, AND BOCHNER-RIESZ MEANS ASSOCIATED WITH ROUGH DOMAINS

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A dissertation submitted in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

(Mathematics)

at the

UNIVERSITY OF WISCONSIN – MADISON

2016

Date of final oral examination: May 10, 2016

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Abstract

This thesis contains results of the author from [12], [13], [14], and [15]. In the first part of the thesis, we will prove a characterization of restricted strong type (p, p) boundedness of multiplier operators whose multiplier is a radial function on $\mathbb{R}^3$ supported compactly away from the origin, in the range $1 < p < 13/12$. This result complements a result of Heo, Nazarov, and Seeger, who obtained a characterization of radial Fourier multiplier operators bounded on $L^p(\mathbb{R}^d)$ in dimensions $d \geqslant 4$ for the range $1 < p < \frac{2d-2}{d+1}$.

In the second part of the thesis, we introduce and define Bochner-Riesz multipliers associated with convex planar domains. Such multipliers were first studied by Seeger and Ziesler, and we discuss their results as background. We then discuss new results addressing the question of sharpness of Seeger and Ziesler's theorem. We introduce the additive combinatorial notion of "additive energy" of the boundary of a convex domain which we will show gives a sufficient criteria for obtaining improved L^p bounds for Bochner-Riesz multipliers.

In the third part of the thesis, we will introduce general Fourier multipliers associated with convex planar domains and prove a criterion for L^p boundedness of the corresponding multiplier operators. The methods used to obtain multiplier theorems in this section will involve analysis of "half-wave" operators associated with convex domains.

In the fourth part of the thesis, we will discuss a related square function result and obtain new multiplier theorems as a corollary, which we will interpolate with our results from the third part of the thesis to obtain our most general quasiradial multiplier theorem.

To Andrea, Paul, Edward, and Choonie

Acknowledgments

I would first like to thank my advisor, Andreas Seeger, for his generous support and encouragement over the years, and for continually sharing his mathematical guidance and insight. I have benefited immensely from what he has taught me, both in terms of mathematical content and overall perspective. He has consistently looked out for my best interests as a young mathematician and I am most indebted to him.

I would like to thank Serguei Denissov, Betsy Stovall, Brian Street, and Andrej Zlatoš for exceedingly helpful discussions about analysis over the years, and particularly Betsy Stovall for also offering her advice and guidance. I would like to thank Xianghong Chen for many illuminating email conversations and his patience and willingness to share his knowledge.

I would like to thank my family and friends for supporting me and reminding me that (perhaps) there is more to life than math! I would especially like to thank my parents, Choonie and Edward Cladek, and my beloved brother and sister, Paul and Andrea Cladek, for their unwavering belief in me. It is not an exaggeration to say that the four of you are the world to me and you instill meaning into everything that I do.

Mom, I wouldn't be who I am without all the love that you have given me over the years. You are the most loving and supportive person that I know, and I cherish all the great times we have had together both in Madison and back home in Chicago. Dad, you have been so encouraging during these past five years and your support has meant so much. You motivate me to try to be the best version of myself in everything that I do. Paul, you are the sweetest brother that I could ask for and you always have something positive and encouraging to say. I love being around you because your cheerful attitude is infectious! Andrea, you are a hilarious and unique person, and you make me laugh like no one else on earth. You are so special to me and I have enjoyed all the great food and conversations that we've had during these past five years.

I would also like to thank Mimansa Vahia for being one of my closest friends and for staying in touch no matter where life takes us. Your friendship has truly defined my experience here at Madison.

Finally, I have been funded by a National Science Foundation Research and Training Grant during many of my semesters here at Madison, and am grateful to the National Science Foundation for its support.

Contents

Chapter 1

Overview

Fourier multiplier operators are a basic object of study in harmonic analysis. These are translation-invariant operators defined by multiplication on the Fourier side by a bounded, measurable function which is referred to as the "symbol" or "multiplier" of the transformation. More precisely, Given $m \in L^\infty(\mathbb{R}^d)$, we may define an operator T_m acting on Schwartz functions $f \in \mathcal{S}(\mathbb{R}^d)$ by

$$\mathcal{F}[T_m f](\xi) = m(\xi)\widehat{f}(\xi).$$

These may also be viewed as convolution operators, where the convolution kernel is the distribution $K = \widehat{m}$.

One is typically interested in the mapping properties of multiplier operators between various function spaces. The most basic question one may ask is for a given p whether a multiplier operator extends to a continuous mapping on $L^p(\mathbb{R}^d)$. One may easily show using Plancherel's theorem that for any multiplier $m \in L^\infty(\mathbb{R}^d)$, T_m extends to a bounded operator on $L^2(\mathbb{R}^d)$. Moreover, for $1 < p < \infty$, one may show using duality that if T_m extends to a bounded operator on $L^p(\mathbb{R}^d)$, then it also extends to a bounded operator on $L^{p'}(\mathbb{R}^d)$, where p' denotes the Hölder conjugate of p. Thus one is typically interested in the smallest (or largest) value of p for which T_m extends to a bounded operator on $L^p(\mathbb{R}^d)$.

A very difficult open question in harmonic analysis is whether there exists some kind of reasonable characterization of all multipliers $m \in L^\infty(\mathbb{R}^d)$ for which T_m extends to a bounded operator on $L^p(\mathbb{R}^d)$. That is, one may ask for a given p whether there is some kind of straightforward and useful criterion for m that determines the L^p-mapping properties of T_m. As already mentioned, if $p = 2$, T_m is always bounded on L^p since $m \in L^\infty$. If $p = 1$, it is not difficult to show that T_m is bounded on L^p if and only if the convolution kernel $K = \widehat{m}$ is a finite Borel measure. However, in the case $p \neq 1, 2$, it is widely believed that no such criterion exists for general multipliers.

It was then rather surprising that Garrigós and Seeger were able to obtain in [23] a very simple characterization of all *radial* multiplier operators acting on the space L^p_{rad} of radial L^p functions. Their characterization was quite general, since the class of radial Fourier multipliers is rather large and contains many well-studied examples, such as the Bochner-Riesz multipliers, which we will discuss later. In particular, Garrigós and Seeger showed that in the range $1 < p < \frac{2d}{d+1}$ for $d \geqslant 2$, if m is radial and compactly supported away from the origin, then T_m is bounded on L^p_{rad} if and only if the kernel $K = \widehat{m}$ is in L^p. This range of p is the largest possible range for which their result can hold, since for $p \geqslant 2d/(d+1)$ there exist radial kernels in L^p that have Fourier transforms which are compactly supported away from the origin, but are also unbounded.

In light of Garrigós and Seeger's result, is then natural to ask whether this characterization also applies to compactly supported radial multipliers on L^p rather than L^p_{rad}. Heo, Nazarov, and Seeger answered this question in the affirmative, in a breakthrough paper [28] that established that the condition $K = \widehat{m} \in L^p(\mathbb{R}^d)$ is both necessary and sufficient for T_m to be bounded on $L^p(\mathbb{R}^d)$ for m radial and supported compactly away from the origin, in the smaller range $1 < p < \frac{2d-2}{d+1}$, for $d \geqslant 4$. Heo, Nazarov, and Seeger's result for compactly supported radial multipliers may be rephrased as follows.

For a fixed p, we have that for all radial $m \in L^\infty(\mathbb{R}^d)$ supported compactly away from the origin, T_m is bounded on $L^p(\mathbb{R}^d)$ if and only if $K = \widehat{m} \in L^p(\mathbb{R}^d)$, as long as $d > \frac{2+p}{2-p}$. Thus their characterization becomes better in higher dimensions. It remains a very difficult open question as to whether the characterization extends to dimensions $d \geqslant 2$ and the best possible range $1 < p < \frac{2d}{d+1}$.

Conjecture 1.0.1. *Let $1 < p < \frac{2d}{d+1}$. If $m : \mathbb{R}^d \to \mathbb{R}$ is radial and supported compactly away from the origin, then T_m is bounded on $L^p(\mathbb{R}^d)$ if and only if $K = \widehat{m}$ is in $L^p(\mathbb{R}^d)$.*

In Chapter 2, we will discuss a new result to apear in [15] that gives a characterization of restricted strong type (p,p) estimates for operators corresponding to radial Fourier multipliers supported compactly away from the origin in three dimensions, in the range $1 < p < 13/12$. This complements Heo, Nazarov, and Seeger's result. It is also expected that this new result can be improved to a characterization of L^p boundedness for radial multipliers in three dimensions.

In studying general radial Fourier multipliers, one makes absolutely no assumptions about the smoothness of the multiplier. However, a great deal of information can typically be deduced from the regularity of a multiplier. As a general principle, there is a positive relationship between the L^p-mapping properties of a multiplier operator and the smoothness of its symbol. Regularity of a multiplier implies decay of the convolution kernel $K = \widehat{m}$, which is one reason that one would expect better L^p-mapping properties. It is natural to ask if one may quantify this relationship, and the Bochner-Riesz multipliers are a model case for studying this relationship in the case of radial multipliers. They are radial multipliers defined as

$$m_\lambda(\xi) = (1 - |\xi|)_+^\lambda$$

for $\lambda > 0$. Note that the regularity of m_λ increases as λ increases, and that m_λ converges to the characteristic function of the ball pointwise as $\lambda \to 0$.

The Bochner-Riesz Conjecture, which is one of the most well-studied open problems in harmonic analysis, is a conjecture regarding exactly how large λ needs to be in order for T_{m_λ} to be bounded on $L^p(\mathbb{R}^d)$ for a given p.

Conjecture 1.0.2. *For $\lambda > 0$, T_{m_λ} is bounded on $L^p(\mathbb{R}^d)$ if and only if $\lambda > \lambda(p) = \max(d|\frac{1}{p} - \frac{1}{2}| - \frac{1}{2}, 0)$.*

Since $\frac{2d}{d+1}$ is the critical index for the Bochner-Riesz multipliers in d dimensions, the Bochner-Riesz conjecture would in fact follow immediately from the more difficult Conjecture 1.0.1. The Bochner-Riesz Conjecture is also closely connected with a number of other important conjectures in harmonic analysis, including the celebrated Restriction, Kakeya, and Local Smoothing Conjectures. For a hierarchy of implications between these and other related conjectures, see [54]. The Bochner-Riesz conjecture was first completely solved in dimension $d = 2$ by Fefferman in [22], and then later clarified by Córdoba in [18]. The problem remains open in dimensions $d \geqslant 3$, although partial progress has been made; for recent progress see for example [33] and [6].

As already mentioned, studying L^p-mapping properties of Bochner-Riesz multipliers are a means of studying the general relationship between regularity of a radial multiplier and its L^p mapping properties. Another way one may view Bochner-Riesz operators are as multiplier operators whose symbol is a "smoothed-out" characteristic function. It was proven by Fefferman that the "ball multiplier" operator T_m, where m is the characteristic function of the unit ball, is unbounded on $L^p(\mathbb{R}^d)$ for every $p \neq 2$ when $d \geqslant 2$. The Bochner-Riesz multipliers are essentially equivalent to smoothed out versions of the characteristic function of the ball, and thus the L^p-mapping properties of Bochner-Riesz multipliers quantify the failure of the corresponding characteristic function multiplier to be bounded on L^p, in that they provide a means to measure how much additional regularity is required for L^p-boundedness.

One may also study a generalization of Bochner-Riesz multipliers where the characteristic function of a more general set plays the role of the characteristic function of the ball. Given $\Omega \subset \mathbb{R}^2$ a bounded, open, convex set containing the origin, define the associated Bochner-Riesz multipliers as

$$m_\lambda(\xi) = (1 - \rho(\xi))_+^\lambda,$$

where ρ denotes the Minkowski functional of Ω, i.e. the unique function which is identically 1 on $\partial\Omega$ and homogeneous of degree one. Of particular interest is when the boundary of Ω is not smooth (in terms of regularity, the requirement that Ω be convex only implies that the boundary is Lipschitz). The Bochner-Riesz multipliers were first studied in this generality by Seeger and Ziesler in [48], who obtained a result depending on a parameter κ_Ω similar to a notion of Minkowski dimension of the affine arclength measure of $\partial\Omega$. Interestingly, their result implies that Bochner-Riesz multipliers associated with domains with rough boundary may actually satisfy improved L^p bounds over the classical radial Bochner-Riesz multipliers. In Chapter 3, we will discuss some new results that have been posted in the preprint [13] addressing the sharpness of Seeger and Ziesler's result.

Some interesting special cases of Ω are those where $\partial\Omega$ has fractal-type structure. For example, one may consider domains whose boundary is locally parametrized by a function $\gamma : \mathbb{R} \to \mathbb{R}$, defined by

$$\gamma(t) = \int_0^t g(s)\, ds + C,$$

where C is an appropriate constant and g is a standard Cantor function, or more generally we may let g denote the distribution density of a fractal measure. In general, the study of harmonic analysis on fractals is an area of harmonic analysis which has recently attracted a great deal of attention, largely through the study of Fourier restriction phenomena to fractal measures in $\mathbb{R}$. For Fourier restriction to fractals, arithmetic structure and additive combinatorial notions such as additive energy play an important role. In Chapter 3, we introduce a notion of the "additive energy" of the boundary of a convex domain, which we will see along with Seeger and Ziesler's parameter κ_Ω also plays a role in determining L^p-boundedness of generalized Bochner-Riesz multipliers.

Since the Bochner-Riesz multipliers serve as a model case for understanding the relationship between regularity and L^p bounds for radial multipliers, one might try to extend theorems involving Bochner-Riesz multipliers to multiplier theorems for general radial multipliers. Indeed, proving certain estimates involving specific radial multipliers, such as the Bochner-Riesz multipliers or the "half-wave" multipliers $e^{i|\xi|}$, can lead to general multiplier theorems for radial multipliers. This is accomplished by means of using an appropriate "subordination formula." In the case of Bochner-Riesz multipliers, the formula to consider would be

$$m(|\xi|) = \frac{(-1)^{\lfloor \lambda \rfloor + 1}}{\Gamma(\lambda + 1)} \int_0^\infty s^\lambda m^{(\lambda+1)}(2)(1 - \frac{|\xi|}{s})_+^\lambda\, ds, \tag{1.1}$$

which may be obtained by integration by parts. In the case of the half-wave multipliers, the formula to consider woud be

$$m(|\xi|) = \frac{1}{2\pi} \int \widehat{m}(\tau) e^{i\tau|\xi|} \, d\tau, \tag{1.2}$$

which is simply the Fourier inversion formula. It turns out that the latter formula is far more "efficient", and hence to obtain stronger general multiplier theorems it is better to study the half-wave multipliers than the Bochner-Riesz multipliers.

The discussion in the previous paragraph generalizes to the case of "quasiradial" multipliers where $|\xi|$ can be replaced by a more general distance function $\rho(\xi)$. In Chapter 4, we will consider quasiradial multipliers where, as in the case of the generalized Bochner-Riesz multipliers, ρ is the Minkowski functional of a convex domain in $\mathbb{R}^2$. In view of the discussion in the previous paragraph, this will involve an analysis of the generalized half-wave multipliers $e^{i\rho(\xi)}$. By studying the half-wave multipliers, we will prove a criterion for the Fourier transform of a quasiradial multiplier of the form $m \circ \rho$ to be an L^1 kernel, which implies that $T_{m\circ\rho}$ bounded on L^p for $1 < p < \infty$. In the special case of domains with $\kappa_\Omega = 1/2$, we will further refine this to an endpoint estimate which will give us a criterion for $T_{m\circ\rho}$ to be bounded from H^1 to L^1. The results in this chapter are published in [12].

One may also obtain multiplier theorems for radial multipliers through an alternate route by proving L^p estimates for square functions associated with Bochner-Riesz means. The Bochner-Riesz square function is defined as

$$G^\lambda f(x) = \left(\int_0^\infty |\mathcal{R}_t^\lambda f(x)|^2 \, \frac{dt}{t} \right)^{1/2},$$

where

$$\mathcal{R}_t^\lambda f(x) = \frac{1}{(2\pi)^2} \int_{|\xi|\leqslant t} \left(1 - \frac{|\xi|}{t} \right)^\lambda \widehat{f}(\xi) e^{i\langle \xi, x\rangle} \, d\xi.$$

To study the maximal Bochner-Riesz operators, Carbery proved in [8] a critical $L^4(\mathbb{R}^2)$ estimate for the Bochner-Riesz square function in two dimensions. The later work of Carbery, Gasper, and Trebels in [10] was to show by means of the subordination formula (1.1) that Carbery's $L^4(\mathbb{R}^2)$ estimate for the Bochner-Riesz square function implied a sharp multiplier theorem for radial multipliers in $\mathbb{R}^2$ in the range $4/3 \leqslant p \leqslant 4$.

By an argument similar to that of [10], we may obtain multiplier theorems for quasiradial multipliers in two dimensions by proving an $L^4(\mathbb{R}^2)$ estimate for a generalized Bochner-Riesz square function, where $|\xi|$ is replaced with a generalized distance function $\rho(\xi)$, which is what we will do in Chapter 5. However, the class of distance functions ρ that we will consider will be more general than those considered in Chapter 4. Namely, we will consider ρ such that the level set $\{\xi : \rho(\xi) = 1\}$ is the boundary of a convex domain Ω containing the origin and such that ρ is homogeneous with respect to a *nonisotropic dilation group* which is compatible in a certain sense with Ω. A nonisotropic dilation group is a one-parameter family $\{t^A : t > 0\}$ where $t^A = \exp(\log(t)A)$ and A is a 2×2 matrix whose eigenvalues have positive real part. Nonisotropic dilations arise naturally in many settings in harmonic analysis, and so quasiradial multipliers of the form $m \circ \rho$ where ρ is homogeneous with respect to a nonisotropic dilation group are a very natural generalization of radial Fourier multipliers to consider. In Chapter 4, we interpolate this result with our previous multiplier theorem to obtain a more general quasiradial multiplier theorem. The results in this chapter have been posted in the preprint [14].

Chapter 2

Radial Fourier Multipliers

2.1 Introduction and statement of results

In this chapter we study radial multiplier transformations whose symbol is compactly supported away from the origin. These are operators T_m defined via the Fourier transform by

$$\mathcal{F}[T_m f](\xi) = m(\xi)\widehat{f}(\xi),$$

where $m : \mathbb{R}^d \to \mathbb{C}$ is a bounded, measurable, *radial* function supported in a compact subset of $\{\xi : 1/2 < |\xi| < 2\}$.

In the cases $p \neq 1, 2$, it is generally believed that is is impossible to give a reasonable characterization of all multiplier operators which are bounded on L^p. However, for radial Fourier multipliers, a characterization can be obtained for an appropriate range of p. In [28], Heo, Nazarov, and Seeger prove a strikingly simple characterization of radial multipliers that are bounded on $L^p(\mathbb{R}^d)$ in dimensions $d \geqslant 4$ for $1 < p < \frac{2d-2}{d+1}$.

Theorem A. *If $m : \mathbb{R}^d \to \mathbb{C}$ is radial and supported in a compact subset of $\{\xi : 1/2 < |\xi| < 2\}$, the multiplier operator T_m is bounded on $L^p(\mathbb{R}^d)$ if and only if the kernel $K = \widehat{m}$ is in $L^p(\mathbb{R}^d)$, in the range $1 < p < \frac{2d-2}{d+1}$.*

The characterization in [28] was motivated by the earlier work [23] of Garrigós and

Seeger, where the authors obtained a similar characterization of all convolution operators with radial kernels acting on the space L^p_{rad} of radial L^p functions, in the larger range $1 < p < \frac{2d}{d+1}$.

Theorem B. *If $m : \mathbb{R}^d \to \mathbb{C}$ is radial and supported in a compact subset of $\{\xi : 1/2 < |\xi| < 2\}$, the multiplier operator T_m is bounded on $L^p_{rad}(\mathbb{R}^d)$ if and only if the kernel $K = \widehat{m}$ is in $L^p(\mathbb{R}^d)$, in the range $1 < p < \frac{2d}{d+1}$.*

This range $1 < p < \frac{2d}{d+1}$ is the optimal range for their result to hold, since for $p \geqslant 2d/(d+1)$ one may construct radial kernels in L^p that have Fourier transforms which are supported compactly away from the origin but which are also unbounded. By the same reasoning, the range $1 < p < \frac{2d}{d+1}$ is also the largest possible range in which one could hope for the characterization from Theorem A to hold.

The argument of [28] did not yield any results about radial Fourier multipliers in $\mathbb{R}^3$. In this chapter, we will improve a key lemma of [28] in three dimensions to very nearly obtain a characterization of compactly supported radial Fourier multipliers m bounded on $L^p(\mathbb{R}^3)$, in the range $1 < p < \frac{13}{12}$.

Theorem 2.1.1. *Let m be a radial Fourier multiplier in $\mathbb{R}^3$ supported in $\{1/2 < |\xi| < 2\}$ and let $K = \mathcal{F}^{-1}[m]$. Then for $1 < p < 13/12$, if $K \in L^p(\mathbb{R}^3)$, then the multiplier operator T_m is restricted strong type (p,p), and moreover*

$$\|K * f\|_{L^p(\mathbb{R}^3)} \lesssim_p \|K\|_{L^p(\mathbb{R}^3)} \|f\|_{L^{p,1}(\mathbb{R}^3)} .$$

Remark 2.1.2. *Our proof will also show that $\|K * f\|_{L^p} \lesssim_p \|K\|_{L^{p,1}} \|f\|_{L^p}$, and we expect that $\|K\|_{L^{p,1}}$ could be improved to $\|K\|_{L^p}$.*

Our proof of Theorem 2.1.1 refines the arguments of [28] while simultaneously incorporating new geometric input. A key divergence of the proof of 2.1.1 from the arguments of [28] is the exploitation of the underlying "tensor product structure" inherent in the problem, a notion which will become clearer later. This, combined with a geometric argument involving sizes of multiple intersections of three-dimensional annuli, allows one to take advantage of improved scalar product estimates which were noted but not used in [28]. However, since we exploit the tensor product structure of the problem, we are currently not able to deduce any local smoothing results for the wave equation as corollaries, as was able to be done in [28].

2.2 Preliminaries and reductions

In this section we will collect some necessary preliminary results and reductions. Versions of these results can be found in [28], but we reproduce them here for completeness. In general, this section of the chapter will very closely follow [28], and for convenience we choose to adopt similar notation.

Discretization and density decomposition of sets

The first step will be to discretize our problem, and in preparation for this we will first need to introduce some notation. Let $\mathcal{Y}$ be a 1-separated set of points in $\mathbb{R}^3$ and let $\mathcal{R}$ be a 1-separated set of radii $\geqslant 1$. Let $\mathcal{E} \subset \mathcal{Y} \times \mathcal{R}$ be a finite set that is also a *product*, i.e. $\mathcal{E} = \mathcal{E}_Y \times \mathcal{E}_R$ where $\mathcal{E}_Y \subset \mathcal{Y}$ and $\mathcal{E}_R \subset \mathcal{R}$. (The assumption that $\mathcal{E}$ is a product was not used in [28], but will be crucial for our argument.)

Let

$$u \in \mathcal{U} = \{2^\nu, \nu = 0, 1, 2, \ldots\}$$

be a collection of dyadic indices. For each k, let $\mathfrak{B}_k$ denote the collection of all 4-dimensional balls of radius $\leqslant 2^k$. For a ball B, let $\mathrm{rad}(B)$ denote the radius of B. Following [28], define:

$$\mathcal{R}_k := \mathcal{R} \cap [2^k, 2^{k+1}),$$

$$\mathcal{E}_k := \mathcal{E} \cap (\mathcal{Y} \times \mathcal{R}_k),$$

$$\widehat{\mathcal{E}}_k(u) := \{(y,r) \in \mathcal{E}_k : \exists B \in \mathfrak{B}_k \text{ such that } \#(\mathcal{E}_k \cap B) \geqslant u\,\mathrm{rad}(B)\},$$

$$\mathcal{E}_k(u) = \widehat{\mathcal{E}}_k(u) \setminus \bigcup_{\substack{u' \in \mathcal{U} \\ u' > u}} \widehat{\mathcal{E}}_k(u').$$

We will refer to u as the *density* of the set $\mathcal{E}_k(u)$. Note that we have the decomposition

$$\mathcal{E}_k = \bigcup_{u \in \mathcal{U}} \mathcal{E}_k(u).$$

Let σ_r denote the surface measure on rS^2, the 2-sphere centered at the origin of radius r. Now fix a smooth, radial function ψ_0 which is supported in the ball centered at the origin of radius $1/10$ such that $\widehat{\psi_0}$ vanishes to order 40 at the origin. Let $\psi = \psi_0 * \psi_0$. For $y \in \mathcal{Y}$ and $r \in \mathcal{R}$, define

$$F_{y,r} = \sigma_r * \psi(\cdot - y).$$

For a given function $c : \mathcal{Y} \times \mathcal{R} \to \mathbb{C}$, further define

$$G_{u,k} := \sum_{(y,r) \in \mathcal{E}_k(u)} c(y,r) F_{y,r},$$

$$G_u := \sum_{k \geqslant 0} G_{u,k},$$

$$G_k := \sum_{u \in \mathcal{U}} G_{u,k}.$$

An interpolation lemma

As a preliminary tool, we will need the following dyadic interpolation lemma.

Lemma 2.2.1. *Let $0 < p_0 < p_1 < \infty$. Let $\{F_j\}_{j\in\mathbb{Z}}$ be a sequence of measurable functions on a measure space $\{\Omega, \mu\}$, and let $\{s_j\}$ be a sequence of nonnegative numbers. Assume that for all j, the inequality*

$$\|F_j\|_{p_\nu}^{p_\nu} \leqslant 2^{jp_\nu} M^{p_\nu} s_j \tag{2.1}$$

holds for $\nu = 0$ and $\nu = 1$. Then for all $p \in (p_0, p_1)$, there is a constant $C = C(p_0, p_1, p)$ such that

$$\left\| \sum_j F_j \right\|_p^p \leqslant C^p M^p \sum_j 2^{jp} s_j. \tag{2.2}$$

The discretized L^p inequality

Our goal is to prove the following proposition, which we will see implies our main result for compactly supported multipliers.

Proposition 2.2.2. *Let $\mathcal{E}$ and $\mathcal{E}_k$ be as above (recall that $\mathcal{E}$ has product structure). Let $c : \mathcal{E} \to \mathbb{C}$ be a function satisfying $|c(y, r)| \leqslant 1$ for all $(y, r) \in \mathcal{E}$. Then for $1 < p < 13/12$,*

$$\left\| \sum_{(y,r)\in\mathcal{E}} c(y, r) F_{y,r} \right\|_p^p \lesssim_p \sum_k 2^{2k} \#\mathcal{E}_k.$$

Using the dyadic interpolation lemma (Lemma 2.2.1), we obtain the following corollary.

Corollary 2.2.3. *Let E be any measurable set of finite measure, and χ_E its characteristic function. Suppose that f is a measurable function satisfying $|f| \leqslant \chi_E$. Then for*

$1 < p < 13/12$, *we have*

$$\Big\| \sum_{(y,r)\in\mathcal{Y}\times\mathcal{R}} \gamma(r) f(y) F_{y,r} \Big\|_p \lesssim \Big(\sum_{(y,r)\in\mathcal{Y}\times\mathcal{R}} |\gamma(r)\chi_E(y)|^p r^2 \Big)^{1/p}. \tag{2.3}$$

Also

$$\Big\| \int_{\mathbb{R}^3} \int_1^\infty h(r) f(y) F_{y,r}\, dr\, dy \Big\|_p \lesssim \Big(\int_{\mathbb{R}^3} \int_1^\infty |h(r)\chi_E(y)|^p r^2\, dr\, dy \Big)^{1/p}. \tag{2.4}$$

Proof that Proposition 2.2.2 implies Corollary 2.2.3. For $j \in \mathbb{Z}$, define the level sets

$$\mathcal{E}^j := \{(y,r) \in \mathcal{Y}\times\mathcal{R} :\ 2^{j-1} < |\gamma(r)\chi_E(y)| \leqslant 2^j\}.$$

Notice that $\mathcal{E}^j$ has product structure, so Proposition 2.2.2 implies that for $1 < p < 13/12$,

$$\Big\| \sum_{(y,r)\in\mathcal{E}^j} \gamma(r) f(y) F_{y,r} \Big\|_p^p \lesssim_p 2^{jp} \sum_{(y,r)\in\mathcal{E}^j} r^2.$$

Now apply Lemma 2.2.1 with $F_j = \sum_{(y,r)\in\mathcal{E}^j} \gamma(r) f(y) F_{y,r}$, $M = 1$, and $s_j = \sum_{(y,r)\in\mathcal{E}^j} r^2$ to obtain (2.3).

Now we prove (2.4). Let $y = z + w$ for $z \in \mathbb{Z}^3$ and $w \in Q_0 := [0,1)^3$ and $r = n + \tau$ for $n \in \mathbb{N}$ and $0 \leqslant \tau < 1$. By Minkowski's inequality and (2.3),

$$\begin{aligned}
&\Big\| \int_{\mathbb{R}^3} \int_1^\infty h(r) f(y) F_{y,r}\, dr\, dy \Big\|_p \\
&\qquad \lesssim_p \int\int_{Q_0\times[0,1)} \Big\| \sum_{z\in\mathbb{Z}^3} \sum_{n=1}^\infty h(n+\tau) f(z+w) F_{z+w,n+\tau} \Big\|_p\, dw\, d\tau \\
&\qquad \lesssim_p \int\int_{Q_0\times[0,1)} \Big(\sum_{z\in\mathbb{Z}^3} \sum_{n=1}^\infty |h(n+\tau)\chi_E(z+w)|^p (n+\tau)^2 \Big)^{1/p} dw\, d\tau \\
&\qquad \lesssim_p \Big(\int_{\mathbb{R}^3} \int_1^\infty |h(r)\chi_E(y)|^p r^2\, dr\, dy \Big)^{1/p},
\end{aligned}$$

where in the last step we have used Hölder's inequality. □

Support size estimates vs. L^2 inequalities

As in [28], we will show that the functions $G_{u,k}$ either have relatively small support size or satisfy relatively good L^2 bounds. We begin with a support size bound from [28] that improves as the density u increases.

Lemma C. *For all $u \in \mathcal{U}$, the Lebesgue measure of the support of $G_{u,k}$ is $\lesssim u^{-1}2^{2k}\#\mathcal{E}_k$.*

We will prove the following L^2 inequality which is in some sense an improved version of Lemma 3.6 from [28], although the hypotheses are different since it is crucial that we assume that the underlying set $\mathcal{E}$ has product structure. This inequality improves as the density u decreases. In [28], the analogous L^2 inequality proved is

$$\|G_u\|_2^2 \lesssim u^{\frac{2}{d-1}} \log(2+u) \sum_k 2^{k(d-1)} \#\mathcal{E}_k, \tag{2.5}$$

and when $d = 3$ the term $u^{\frac{2}{d-1}}$ is equal to u. One may check that combining (2.5) with Lemma C as in the proof of Lemma 2.2.5 below yields no result in three dimensions. We use geometric methods to improve on (2.5) in three dimensions.

Lemma 2.2.4. *Let $\mathcal{E}$, $\mathcal{E}_k$, and G_u be as above (recall that $\mathcal{E}$ has product structure). Assume $|c(y,r)| \leqslant 1$ for $(y,r) \in \mathcal{Y} \times \mathcal{R}$. Then for every $\epsilon > 0$,*

$$\|G_u\|_2^2 \lesssim_\epsilon u^{\frac{11}{13}+\epsilon} \sum_k 2^{2k} \#\mathcal{E}_k.$$

Combining Lemma C and Lemma 2.2.4, we obtain the following L^p bound.

Lemma 2.2.5. *For $p \leqslant 2$, for every $\epsilon > 0$,*

$$\|G_u\|_p \lesssim_\epsilon u^{-(1/p-12/13-\epsilon)} (\sum_k 2^{2k} \#\mathcal{E}_k)^{1/p}.$$

Proof of Lemma 2.2.5 given Lemma C and Lemma 2.2.4. By Hölder's inequality,

$$\|G_u\|_p \lesssim (\text{meas}(\text{supp}(G_u)))^{1/p-1/2} \|G_u\|_2$$
$$\lesssim_\epsilon u^{-1/p+1/2} u^{11/26+\epsilon} (\sum_k 2^{2k} \#\mathcal{E}_k)^{1/p}$$
$$\lesssim_\epsilon u^{12/13-1/p+\epsilon} (\sum_k 2^{2k} \#\mathcal{E}_k)^{1/p}.$$

□

Summing over $u \in \mathcal{U}$, we obtain Proposition 2.2.2. Thus to prove Proposition 2.2.2 it suffices to prove Lemma 2.2.4.

Compactly supported multipliers

Following [28], we now show how one may deduce Theorem 2.1.1 from Corollary 2.2.3. Suppose that $m : \mathbb{R}^3 \to \mathbb{C}$ is a bounded, measurable, radial function with compact support inside $\{\xi : 1/2 < |\xi| < 2\}$. Then $K = \mathcal{F}^{-1}[m]$ is radial, and so we may write $K(\cdot) = \kappa(|\cdot|)$ for some $\kappa : \mathbb{R} \to \mathbb{C}$. Fix a radial Schwartz function η_0 such that $\widehat{\eta_0}(\xi) = 1$ on $\text{supp}(m)$ and such that η_0 has Fourier support in $\{1/4 < |\xi| < 4\}$. Set $\eta = \mathcal{F}^{-1}[(\widehat{\psi})^{-1}\widehat{\eta_0}]$. We have $K * f = \eta * \psi * K * f$. Let $K_0 = K\chi_{\{x:\,|x|\leqslant 1\}}$ and write $K = K_0 + K_\infty$. Since $\|K_0\|_1 \lesssim \|K\|_p$, it suffices to show that the operator $f \mapsto \eta * \psi * K_\infty * f$ is restricted strong type (p, p) with operator norm $\lesssim_p \|K\|_p$. Let E be a measurable set of finite measure, and suppose that $|f| \leqslant \chi_E$. We may write

$$\psi * K_\infty * f = \int_1^\infty \int \psi * \sigma_r(\cdot - y)\kappa(r) f(y)\, dy\, dr.$$

By Corollary 2.2.3, we have

$$\|\eta * \psi * K_\infty * f\|_p$$
$$\lesssim_p \|\psi * K_\infty * f\|_p \lesssim_p \left(\int |\kappa(r)|^p r^2\,dr\right)^{1/p} \left(\int |\chi_E(y)|^p\,dy\right)^{1/p},$$

which implies the result of Theorem 2.1.1.

2.3 Proof of the L^2 inequality

We have shown in the Section 2.2 that to prove our main result Theorem 2.1.1 it remains to prove Lemma 2.2.4, and this section is dedicated to the proof of that lemma. The proof will rely on a geometric lemma about sizes of multiple intersections of three-dimensional annuli, which is stated and proved in Section 2.4.

Estimates for scalar products

In order to obtain the desired L^2 estimate, we need to examine pairwise interactions of the form $\langle F_{y,r}, F_{y',r'}\rangle$. By applying Plancherel's Theorem and writing $\widehat{F_{y,r}}$ and $\widehat{F_{y',r'}}$ as expressions involving Bessel functions, the authors of [28] obtained the following estimates for $|\langle F_{y,r}, F_{y',r'}\rangle|$.

Lemma 2.3.1. *For any choice of $r, r' > 1$ and $y, y' \in \mathbb{R}^3$*

$$|\langle F_{y,r}, F_{y',r'}\rangle| \lesssim \frac{(rr')}{(1 + |y - y'| + |r - r'|)}.$$

The proof of this lemma used only the decay and not the oscillation of the Bessel functions. The authors of [28] noted that by also exploiting the oscillation of the Bessel functions one may obtain the following improved bounds, which are crucial for our purposes. The following lemma was noted but not used in [28].

Lemma 2.3.2. *For any choice of* $r, r' > 1$ *and* $y, y' \in \mathbb{R}^3$ *and any* $N > 0$,

$$|\langle F_{y,r}, F_{y',r'}\rangle| \leqslant C_N(rr')(1+|y-y'|+|r-r'|)\sum_{\pm,\pm}(1+|r\pm r'\pm|y-y'||)^{-N}.$$

Another preliminary reduction

Recall that our goal is to estimate the L^2 norm of $G_u = \sum_{k\geqslant 0} G_{u,k}$. Let $N(u)$ be a sufficiently large number to be chosen later (it will be some harmless constant depending on u that is essentially $O(\log(2+u))$). As in [28], we split the sum in k as $\sum_{k\leqslant N(u)} G_{u,k} + \sum_{k>N(u)} G_{u,k}$ and apply Cauchy-Schwarz to obtain

$$\left\|\sum_k G_{u,k}\right\|_2^2 \lesssim N(u)\left[\sum_k \|G_{u,k}\|_2^2 + \sum_{k>k'>N(u)} |\langle G_{u,k'}, G_{u,k}\rangle|\right]. \tag{2.6}$$

We may thus separately estimate $\sum_k \|G_{u,k}\|_2^2$ and $\sum_{k>k'>N(u)} |\langle G_{u,k'}, G_{u,k}\rangle|$, which divides the proof of the L^2 estimate into two cases, the first being the case of "comparable radii" and the second being the case of "incomparable radii."

Comparable radii

We will first estimate $\sum_k \|G_{u,k}\|_2^2$. Our goal will be to prove the following lemma.

Lemma 2.3.3. *For every* $\epsilon > 0$,

$$\|G_{u,k}\|_2^2 \lesssim_\epsilon 2^{2k}(\#\mathcal{E}_k)u^{11/13+\epsilon}. \tag{2.7}$$

Fix k and u. As in [28], we first observe that for $(y,r),(y',r') \in \mathcal{E}_k(u)$, we have $\langle F_{y,r}, F_{y',r'}\rangle = 0$ unless $|(y,r)-(y',r')| \leqslant 2^{k+5}$. To estimate $\|G_{u,k}\|_2^2$ for a fixed k, we would thus like to bound

$$\sum_{\substack{(y,r),(y',r')\in\mathcal{E}_k(u)\\ 2^m\leqslant|(y,r)-(y',r')|\leqslant 2^{m+1}}} |\langle F_{y,r}, F_{y',r'}\rangle|$$

for all $0 \leqslant m \leqslant k+4$.

Now fix $m \leqslant k+4$. Let $\mathcal{Q}_{u,k,m}$ be a collection of almost disjoint cubes $Q \subset \mathbb{R}^4$ of sidelength 2^{m+5} such that $\mathcal{E}_k(u) \subset \bigcup_{Q\in\mathcal{Q}_{u,k,m}} Q$ and so that every Q has nonempty intersection with $\mathcal{E}_k(u)$. Let Q^* denote the 2^5-dilate of Q and $\mathcal{Q}^*_{u,k,m}$ the corresponding collection of dilated cubes. Observe that

$$\|G_{u,k}\|_2^2 \lesssim \sum_{0\leqslant m\leqslant k+4}\Bigg(\sum_{\substack{(y,r),(y',r')\in\mathcal{E}_k(u)\\ 2^m\leqslant|(y,r)-(y',r')|\leqslant 2^{m+1}}} |\langle F_{y,r}, F_{y',r'}\rangle| + \sum_{(y,r)\in\mathcal{E}_k(u)} \|F_{y,r}\|_2^2\Bigg)$$

$$\lesssim \sum_{0\leqslant m\leqslant k+4}\Bigg(\sum_{Q\in\mathcal{Q}_{u,k,m}}\Bigg(\sum_{\substack{(y,r),(y',r')\in(\mathcal{E}_k(u)\cap Q^*)\\ 2^m\leqslant|(y,r)-(y',r')|\leqslant 2^{m+1}}} |\langle F_{y,r}, F_{y',r'}\rangle|\Bigg) + \sum_{(y,r)\in\mathcal{E}_k(u)} \|F_{y,r}\|_2^2\Bigg). \quad (2.8)$$

Now we introduce some terminology that will be useful. For a subset $\mathcal{S} \subset \mathcal{Y}\times\mathcal{R}$, define its $\mathcal{Y}$- and $\mathcal{R}$-projections by

$$\mathcal{S}_Y = \{y \in \mathcal{Y} : \exists\,(y,r)\in\mathcal{S}\}$$

and

$$\mathcal{S}_R = \{r \in \mathcal{R} : \exists\,(y,r)\in\mathcal{S}\}.$$

Also define the *product-extension* $\mathcal{S}^\times$ of $\mathcal{S} \subset \mathcal{Y}\times\mathcal{R}$ to be the set $\mathcal{S}_Y \times \mathcal{S}_R$. We also define some parameters associated with a fixed $Q \in \mathcal{Q}_{u,k,m}$. Let $N_{R,Q}$ be the cardinality of the $\mathcal{R}$-projection of $\mathcal{E}_k \cap Q^*$, i.e.

$$N_{R,Q} := \#((\mathcal{E}_k \cap Q^*)_R) = \#\{r : \exists (y,r) \in \mathcal{E}_k \cap Q^*\}.$$

Similarly define

$$N_{Y,Q} := \#((\mathcal{E}_k \cap Q^*)_Y) = \#\{y : \exists (y,r) \in \mathcal{E}_k \cap Q^*\}.$$

We also note the following important observation which we will use repeatedly. Using the definition of the sets $\mathcal{E}_k(u)$ and the fact that $\mathcal{E}_k$ has product structure, one may see that if $Q \in \mathcal{Q}_{u,k,m}$ is such that $(\mathcal{E}_k(u) \cap Q^*)$ is nonempty, then

$$|N_{Y,Q} \cdot N_{R,Q}| \lesssim |\mathcal{E}_k \cap Q^*| \lesssim u2^m. \tag{2.9}$$

Now with (2.8) in mind, we will prove the following lemma.

Lemma 2.3.4. *For each $Q \in \mathcal{Q}_{u,k,m}$, we have the estimates*

$$\sum_{\substack{(y,r),(y',r') \in (\mathcal{E}_k(u) \cap Q^*) \\ 2^m \leqslant |(y,r)-(y',r')| \leqslant 2^{m+1}}} |\langle F_{y,r}, F_{y',r'} \rangle| \lesssim N_{R,Q}(\#(\mathcal{E}_k \cap Q^*))2^{2(k-m/2)}(m\log(u))\max(u^{5/6}2^{5m/6}, u2^{m/2}) \tag{2.10}$$

and

$$\sum_{\substack{(y,r),(y',r') \in (\mathcal{E}_k(u) \cap Q^*) \\ 2^m \leqslant |(y,r)-(y',r')| \leqslant 2^{m+1}}} |\langle F_{y,r}, F_{y',r'} \rangle| \lesssim 2^{2(k-m/2)}(\#(\mathcal{E}_k \cap Q^*))u2^m(N_{R,Q})^{-1}. \tag{2.11}$$

We will then choose the better estimate from Lemma 2.3.4 depending on $N_{R,Q}$ and sum over all $Q \in \mathcal{Q}_{u,k,m}$ and then over all $m \geqslant u^a$ where a is a number to be chosen later. We will then use other methods to deal with the case $m \leqslant u^a$, from which we will then obtain Lemma 2.3.3.

Proof of Lemma 2.3.4. We will first prove (2.10). By incurring a factor of $N_{R,Q}^2$, to estimate $\sum_{\substack{(y,r),(y',r')\in(\mathcal{E}_k(u)\cap Q^*)\\ 2^m\leqslant|(y,r)-(y',r')|\leqslant 2^{m+1}}} |\langle F_{y,r}, F_{y',r'}\rangle|$ it suffices to estimate for a fixed pair r_1, r_2

$$\sum_{\substack{(y,r_1),(y',r_2)\in(\mathcal{E}_k(u)^\times\cap Q^*)\\ 2^m\leqslant|(y,r_1)-(y',r_2)|\leqslant 2^{m+1}}} |\langle F_{y,r}, F_{y',r'}\rangle|,$$

i.e. to restrict (y,r) and (y',r') to lie in fixed rows of the product-extension of $\mathcal{E}_k(u)\cap Q^*$. (Our estimates will not depend on the particular choice of r_1 and r_2.)

Now, referring to the estimate in Lemma 2.3.2, we see that for a fixed y, r_1, r_2 we have that $|\langle F_{y,r_1}, F_{y',r_2}\rangle|$ decays rapidly as y' moves away from the set $\{y' : |y-y'| = |r_1-r_2| \text{ or } |y-y'| = r_1+r_2\}$, which is contained in a union of two annuli of thickness 2 and radii $|r_1-r_2|$ and r_1+r_2 centered at y.

Let $s \geqslant 0$, fix $t \leqslant 2^{m+10}$, and define $K_k(Q,s,t)$ to be the number of points $y \in (\mathcal{E}_k(u)\cap Q^*)_Y$ such that there are $\geqslant 2^s$ many points $y' \in (\mathcal{E}_k\cap Q^*)_Y$ such that y' lies in the annulus of inner radius t and thickness 3 centered at y. That is, define

$$\begin{aligned} K_k(Q,s,t) := \#\{y \in (\mathcal{E}_k(u)\cap Q^*)_Y : &\text{ there exists at least } 2^s \text{ many points} \\ & y' \in (\mathcal{E}_k\cap Q^*)_Y \text{ such that } ||y'-y| - (t+1.5)| \leqslant 1.5\}. \end{aligned}$$

In view of the observation in the previous paragraph, for a given s and a fixed number $t \leqslant 2^{m+10}$, we would like to prove a bound on $K_k(Q,s,t)$. Our bound will depend on s and m but be independent of the choice of $t \leqslant 2^{m+10}$. For this reason, we define the quantity

$$K_k^*(Q,s) := \max_{0\leqslant t\leqslant 2^{m+10}} K_k(Q,s,t),$$

and we will see that $K_k^*(Q,s)$ satisfies the same bound we prove for $K_k(Q,s,t)$. Our bound for $K_k(Q,s,t)$ will decay as 2^s gets larger and closer to $N_{Y,Q}$; in other words, "most" of the points y in $(\mathcal{E}_k(u) \cap Q^*)_Y$ cannot have a large proportion of other points in $(\mathcal{E}_k \cap Q^*)_Y$ lie in the annulus of inner radius t and thickness 3 centered at y. If we take $t = |r_1 - r_2|$ or $t = r_1 + r_2$, we see that this implies that "most" of the $F_{y,r}$ with $(y, r) \in (\mathcal{E}_k(u) \cap Q^*)_Y \times \{r_1\}$ do not "interact badly" (where by badly we mean to the worst possible extent allowed by Lemma 2.3.2, i.e. tangencies of annuli) with most of the other $F_{y',r'}$ where $(y', r') \in (\mathcal{E}_k \cap Q^*)_Y \times \{r_2\}$. This will allow us to obtain (2.10), which is a good estimate in the case that $N_{R,Q}$ is small.

More precisely, we will prove

$$K_k^*(Q, s) \lesssim \max[u2^m N_{Y,Q}^{5/3} 2^{-2s}, u2^{m/2} N_{Y,Q} 2^{-s}]. \tag{2.12}$$

Combining this with the trivial bound $K_k^*(Q, s) \lesssim N_{Y,Q}$ yields

$$K_k^*(Q, s) \lesssim \max[\min(u2^m N_{Y,Q}^{5/3} 2^{-2s}, N_{Y,Q}), \min(u2^{m/2} N_{Y,Q} 2^{-s}, N_{Y,Q})]. \tag{2.13}$$

Note that (2.12) gives decay in the number of points $K_k^*(Q, s)$ (i.e. $K_k^*(Q, s) \ll N_{Y,Q}$) if we have that

1. $N_{Y,Q}^{5/3} 2^{-2s} u 2^m \ll N_{Y,Q}$, i.e. if $2^s \gg N_{Y,Q}^{1/3} u^{1/2} 2^{m/2}$, and also

2. $N_{Y,Q} 2^{-s} u 2^{m/2} \ll N_{Y,Q}$, i.e. if $2^s \gg u 2^{m/2}$.

Using Lemma 2.3.2, we may bound

$$\sum_{\substack{(y,r),(y',r')\in(\mathcal{E}_k(u)\cap Q^*)\\ 2^m\leqslant|(y,r)-(y',r')|\leqslant 2^{m+1}}} |\langle F_{y,r},F_{y',r'}\rangle|$$

$$\lesssim \sum_{r_1,r_2\in(\mathcal{E}_k(u)\cap Q^*)_R}\Bigg(\sum_{\substack{y,y'\in(\mathcal{E}_k(u)\cap Q^*)_Y\\ 2^m\leqslant|(y,r_1)-(y',r_2)|\leqslant 2^{m+1}}} |\langle F_{y,r_1},F_{y',r_2}\rangle|\Bigg)$$

$$\lesssim 2^{2(k-m/2)}\sum_{r_1,r_2\in(\mathcal{E}_k(u)\cap Q^*)_R}\Bigg(\sum_{0\leqslant a\leqslant m+10}\Bigg(\sum_{y\in(\mathcal{E}_k(u)\cap Q^*)_Y}\sum_{\substack{y'\in(\mathcal{E}_k(u)\cap Q^*)_Y:\\ min_{\pm,\pm}(1+|r_1\pm r_2\pm|y-y'||)\approx 2^a}} 2^{-aN}\Bigg)\Bigg)$$

$$\lesssim 2^{2(k-m/2)}\sum_{r_1,r_2\in(\mathcal{E}_k(u)\cap Q^*)_R}\Bigg(\sum_{0\leqslant a\leqslant m+10}2^{-aN}\Bigg(\sum_{s\geqslant 0:2^s\leqslant 2N_{Y,Q}}K_k^*(Q,s)2^s\Bigg)\Bigg)$$

$$\lesssim 2^{2(k-m/2)}N_{Q,R}^2\sum_{s\geqslant 0:2^s\leqslant 2N_{Y,Q}}K_k^*(Q,s)2^s. \quad (2.14)$$

Assuming (2.13) holds, we have

$$\sum_{\substack{(y,r),(y',r')\in(\mathcal{E}_k(u)\cap Q^*)\\ 2^m\leqslant|(y,r)-(y',r')|\leqslant 2^{m+1}}} |\langle F_{y,r},F_{y',r'}\rangle|$$

$$\lesssim N_{R,Q}^2 2^{2(k-m/2)}\sum_{s\geqslant 0:2^s\lesssim N_{Y,Q}}\max[\min(u2^m N_{Y,Q}^{5/3}2^{-s},N_{Y,Q}2^s),$$

$$\min(u2^{m/2}N_{Y,Q},N_{Y,Q}2^s)]$$

$$\lesssim N_{R,Q}^2 2^{2(k-m/2)}\max\Bigg\{\sum_{s\geqslant 0:2^s\lesssim N_{Y,Q}}\min(u2^m N_{Y,Q}^{5/3}2^{-s},N_{Y,Q}2^s);$$

$$\sum_{s\geqslant 0:2^s\lesssim N_{Y,Q}}\min(u2^{m/2}N_{Y,Q},N_{Y,Q}2^s)\Bigg\} \quad (2.15)$$

Now, note that $u2^m N_{Y,Q}^{5/3}2^{-s}\geqslant N_{Y,Q}2^s$ if and only if $2^s\leqslant u^{1/2}2^{m/2}N_{Y,Q}^{1/3}$. Thus choosing the better estimate in the term $\min(u2^m N_{Y,Q}^{5/3}2^{-s},N_{Y,Q}2^s)$ depending on s yields that

$$\sum_{s\geqslant 0:2^s\lesssim N_{Y,Q}}\min(u2^m N_{Y,Q}^{5/3}2^{-s},N_{Y,Q}2^s)\lesssim u^{1/2}2^{m/2}N_{Y,Q}^{4/3}.$$

Note that $u2^{m/2}N_{Y,Q} \geqslant N_{Y,Q}2^s$ if and only if $2^s \leqslant u2^{m/2}$. Thus choosing the better estimate in the term $\min(u2^{m/2}N_{Y,Q}, N_{Y,Q}2^s)$ depending on s yields that

$$\sum_{s\geqslant 0: 2^s \lesssim N_{Y,Q}} \min(u2^{m/2}N_{Y,Q}, N_{Y,Q}2^s) \lesssim \log(N_{Y,Q})N_{Y,Q}\, u2^{m/2}.$$

It follows that the left hand side of (2.15) is bounded by

$$\begin{aligned} N_{R,Q}^2 2^{2(k-m/2)} N_{Y,Q} \log(N_{Y,Q}) \max(N_{Y,Q}^{1/3}u^{1/2}2^{m/2}, u2^{m/2}) \\ \lesssim N_{R,Q}^2 2^{2(k-m/2)} N_{Y,Q}(m\log(u)) \max(u^{5/6}2^{5m/6}, u2^{m/2}) \\ \lesssim N_{R,Q}(\#(\mathcal{E}_k \cap Q^*))2^{2(k-m/2)}(m\log(u)) \max(u^{5/6}2^{5m/6}, u2^{m/2}), \quad (2.16) \end{aligned}$$

which proves (2.10). This will be a good estimate when $N_{R,Q}$ is small.

Thus to prove (2.10) it remains to prove (2.12). We will in fact prove (2.12) with $K_k^*(Q,s)$ replaced by $K_k(Q,s,t)$, uniformly in $t \leqslant 2^{m+10}$. Fix $t \leqslant 2^{m+10}$ and let $j = \lceil \log_2(t) \rceil$ and cover $(\mathcal{E}_k(u) \cap Q^*)_Y$ by $\lesssim 2^{3(m-j)}$ many 3-dimensional almost disjoint balls of radius 2^{j+5}; denote this collection of balls as $\mathfrak{B} = \{B_i\}$. For each i, we define a collection of "special" points $A_{k,i}(Q,s,t)$ to be the set of all points $y \in (\mathcal{E}_k(u) \cap Q^*)_Y \cap B_i$ such that there are $\geqslant 2^s$ many points $y' \in (\mathcal{E}_k \cap Q^*)_Y$ such that y' lies in the annulus of radius t and thickness 3 centered at y. That is, we define

$$\begin{aligned} A_{k,i}(Q,s,t) := \{y \in (\mathcal{E}_k(u) \cap Q^*)_Y \cap B_i : \text{ there exist at least } 2^s \text{ many points} \\ y' \in (\mathcal{E}_k \cap Q^*)_Y \text{ such that } ||y'-y| - (t+1.5)| \leqslant 1.5\}. \end{aligned}$$

Let $K_{k,i}(Q,s,t)$ denote the cardinality of $A_{k,i}(Q,s,t)$. Now cover each B_i with $\lesssim 2^{3(j-l)}$ many almost disjoint 3-dimensional balls $B_{i,j}$ of radius 2^l for some $l \leqslant j$. Each such ball contains at most $u2^l$ many points of $A_{k,i}(Q,s,t)$, so for a fixed i there must be at least $\gtrsim K_{k,i}(Q,s,t)(u2^l)^{-1}$ many balls $B_{i,j}$ that contain at least one point in $A_{k,i}(Q,s,t)$.

Thus there must be at least $\gtrsim K_{k,i}(Q,s,t)(u2^l)^{-1}$ many such points in $B_i \cap A_{k,i}(Q,s,t)$ spaced apart by $\gtrsim 2^l$; call this set $D_{k,i}(Q,s,t)$. But by Lemma 2.4.1, which we prove later in Section 2.4 of the chapter, the size of three-fold intersections of annuli of radius $t \approx 2^j$ and thickness 3 spaced apart by $\approx 2^l$ with centers lying a ball of radius 2^{j-5} is bounded above by $2^{3(j-l)}$ provided that $l \geqslant j/2 + 20$.

It follows that if $l \geqslant j/2 + 20$, then for each of these $\approx K_{k,i}(Q,s,t)(u2^l)^{-1}$ many points $p \in D_{k,i}(Q,s,t)$, there can be at most

$$\lesssim K_{k,i}(Q,s,t)^2(u2^l)^{-2}2^{3(j-l)}$$

points lying inside the t-annulus centered at p that are simultaneously contained in at least two other different t-annuli centered at points in $D_{k,i}(Q,s,t)$. This implies that if $N_{Y,Q,i}$ denotes the cardinality of $(\mathcal{E}_k \cap Q^*)_Y \cap B_i^*$ where $B_i^* = 10B_i$, then we have

$$N_{Y,Q,i} \gtrsim K_{k,i}(Q,s,t)(u2^l)^{-1}2^s, \tag{2.17}$$

which is essentially 2^s times the number of points in $D_{k,i}(Q,s,t)$, provided that 2^s is much bigger than the total number of points lying inside a t-annulus centered at p that are simultaneously contained in at least two other different t-annuli centered at points in $D_{k,i}(Q,s,t)$, i.e. provided that

$$K_{k,i}(Q,s,t)^2(u2^l)^{-2}2^{3(j-l)} \ll 2^s \tag{2.18}$$

and

$$l \geqslant j/2 + 20.$$

Solving for 2^l in (2.18) yields

$$2^l \gg K_{k,i}(Q,s,t)^{2/5}2^{3j/5}u^{-2/5}2^{-s/5}. \tag{2.19}$$

Thus choosing a minimal l such that

$$2^l \gg \max(K_{k,i}(Q,s,t)^{2/5}2^{3j/5}u^{-2/5}2^{-s/5}, 2^{j/2})$$

for a sufficiently large implied constant and substituting into (2.17) yields

$$K_{k,i}(Q,s,t) \lesssim \max[u2^m N_{Y,Q,i}^{5/3}2^{-2s}, u2^{m/2}N_{Y,Q,i}2^{-s}], \tag{2.20}$$

and summing over all i and using the almost-disjointness of the B_i^* gives

$$K_k(Q,s,t) \lesssim \max[u2^m N_{Y,Q}^{5/3}2^{-2s}, u2^{m/2}N_{Y,Q}2^{-s}]. \tag{2.21}$$

Taking the maximum over all $0 \leqslant t \leqslant 2^{m+10}$ proves (2.12) and hence also (2.10).

It remains to prove (2.11), which will be a good estimate in the case that $N_{R,Q}$ is large. For a fixed $(y,r) \in Q^*$ and a fixed $y' \in (\mathcal{E}_k(u) \cap Q^*)_Y$, there are at most two values of r' away from which $\langle F_{y,r}, F_{y',r'}\rangle$ decays rapidly. Thus using Lemma 2.3.2 we may estimate

$$\begin{aligned}
&\sum_{\substack{(y,r),(y',r')\in(\mathcal{E}_k(u)\cap Q^*)\\ 2^m\leqslant|(y,r)-(y',r')|\leqslant 2^{m+1}}} |\langle F_{y,r}, F_{y',r'}\rangle| \\
&\lesssim \sum_{0\leqslant a\leqslant m+10}\Bigg(\sum_{(y,r)\in(\mathcal{E}_k(u)\cap Q^*)}\Bigg(\sum_{y'\in(\mathcal{E}_k(u)\cap Q^*)_Y}\Bigg(\sum_{\substack{r'\in(\mathcal{E}_k(u)\cap Q^*)_R\\ 2^m\leqslant|(y,r)-(y',r')|\leqslant 2^{m+1}\\ \min_{\pm,\pm}(1+|r\pm r'\pm|y-y'||)\approx 2^a}} 2^{-Na}2^{2(k-m/2)}\Bigg)\Bigg)\Bigg) \\
&\lesssim 2^{2(k-m/2)}(\#(\mathcal{E}_k(u)\cap Q^*))N_{Y,Q} \lesssim 2^{2(k-m/2)}(\#(\mathcal{E}_k(u)\cap Q^*))u2^m(N_{Q,R})^{-1},
\end{aligned} \tag{2.22}$$

and the proof of (2.11) is complete. □

We will now use Lemma 2.3.4 to prove Lemma 2.3.3.

Proof of Lemma 2.3.3. Fix an $a > 0$ to be determined later. As in [28], we split $G_{u,k} = \sum_\mu G_{u,k,\mu}$, where for each positive integer μ we set

$$I_{k,\mu} = [2^k + (\mu-1)u^a, 2^k + \mu u^a),$$

$$\mathcal{E}_{k,\mu}(u) = \mathcal{E}_k(u) \cap (\mathcal{Y} \times I_{k,\mu}),$$

$$G_{u,k,\mu} = \sum_{(y,r)\in\mathcal{E}_{k,\mu}(u)} c(y,r)F_{y,r},$$

and

$$G_{u,k,\mu,r} = \sum_{y\colon (y,r)\in\mathcal{E}_{k,\mu}(u)} c(y,r)F_{y,r}.$$

We have

$$\|G_{u,k}\|_2^2 \lesssim \left\|\sum_{\mu} G_{u,k,\mu}\right\|_2^2 \lesssim \sum_{\mu} \|G_{u,k,\mu}\|_2^2 + \sum_{\mu'>\mu+10} |\langle G_{u,k,\mu'}, G_{u,k,\mu}\rangle|. \tag{2.23}$$

By Cauchy-Schwarz,

$$\|G_{u,k,\mu}\|_2^2 \lesssim u^a \sum_{r\in\mathcal{I}_{k,\mu}\cap\mathcal{R}} \|G_{u,k,\mu,r}\|_2^2.$$

Write

$$G_{u,k,\mu,r} = \left(\sum_{y\colon (y,r)\in\mathcal{E}_{k,\mu}(u)} c(y,r)\psi_0(\cdot - y)\right) * (\sigma_r * \psi_0).$$

By the Fourier decay of σ_r and the order of vanishing of ψ_0 at the origin, we have

$$\left\|\widehat{\sigma_r}\widehat{\psi_0}\right\|_\infty \lesssim r.$$

Since the square of the L^2 norm of $\sum_{y\colon (y,r)\in\mathcal{E}_{k,\mu}(u)} c(y,r)\psi_0(\cdot - y)$ is $\lesssim \#\{y \in \mathcal{Y} :\ (y,r) \in \mathcal{E}_{k,\mu}(u)\}$, we have

$$\sum_{\mu} \|G_{u,k,\mu}\|_2^2 \lesssim u^a \sum_{\mu} \sum_{r\in\mathcal{I}_{k,\mu}\cap\mathcal{R}} \|G_{u,k,\mu,r}\|_2^2 \lesssim u^a 2^{2k} \#\mathcal{E}_k. \tag{2.24}$$

By (2.23), it remains to estimate $\sum_{\mu'>\mu+10} |\langle G_{u,k,\mu'}, G_{u,k,\mu}\rangle|$.

Fix $\epsilon > 0$. We will use (2.10) when $N_{R,Q} \leqslant 2^{m\epsilon}\min(u^{1/12+a/12}, u^{a/4})$ and (2.11) when $N_{R,Q} \geqslant 2^{m\epsilon}\min(u^{1/12+a/12}, u^{a/4})$. We write

$$\sum_{\substack{(y,r),(y',r')\in\mathcal{E}_k(u)\\ |(y,r)-(y',r')|\geqslant u^a}} |\langle F_{y,r}, F_{y',r'}\rangle| \lesssim \sum_{m:\,2^m\geqslant u^a}\Bigg(\sum_{\substack{(y,r),(y',r')\in\mathcal{E}_k(u)\\ |(y,r)-(y',r')|\approx 2^m}}\Bigg(\sum_{\substack{Q\in\mathcal{Q}_{u,k,m}\\ N_{R,Q}\leqslant 2^{m\epsilon}\min(u^{1/12+a/12},u^{1/4})}} |\langle F_{y,r}, F_{y',r'}\rangle| + \sum_{\substack{Q\in\mathcal{Q}_{u,k,m}\\ N_{R,Q}\geqslant 2^{m\epsilon}\min(u^{1/12+a/12},u^{1/4})}} |\langle F_{y,r}, F_{y',r'}\rangle|\Bigg)\Bigg).$$

One sees that

$$\sum_{\substack{(y,r),(y',r')\in\mathcal{E}_k(u)\\ |(y,r)-(y',r')|\geqslant u^a}} |\langle F_{y,r}, F_{y',r'}\rangle| \lesssim I + II, \tag{2.25}$$

where using (2.10) when $N_{R,Q} \leqslant 2^{m\epsilon}\min(u^{1/12+a/12}, u^{a/4})$ and summing over all $Q \in \mathcal{Q}_{u,k,m}$ and over all m such that $2^m \geqslant u^a$ we have

$$\begin{aligned} I &:= 2^{2k}(\#\mathcal{E}_k)\log(u) \\ &\quad\times \sum_{m:2^m\geqslant u^a} u^\epsilon \max\Big\{2^{-m/6+\epsilon}\min(u^{11/12+a/12}, u^{5/6+a/4}), \\ &\qquad 2^{-m/2+\epsilon}\min(u^{13/12+a/12}, u^{1+a/4})\Big\} \\ &\lesssim 2^{2k}(\#\mathcal{E}_k)u^\epsilon \max\Big\{u^{-a/6}\min(u^{11/12+a/12}, u^{5/6+a/4}), \\ &\qquad u^{-a/2}\min(u^{13/12+a/12}, u^{1+a/4})\Big\}, \end{aligned} \tag{2.26}$$

and using (2.11) when $N_{R,Q} \geqslant 2^{m\epsilon}\min(u^{1/12+a/12}, u^{a/4})$ and summing over all Q and over all m such that $2^m \geqslant u^a$ we have

$$II := 2^{2k}(\#\mathcal{E}_k)u^\epsilon \sum_{m:2^m \geqslant u^a} 2^{-m\epsilon} \max(u^{11/12-a/12}, u^{1-a/4})$$
$$\lesssim_\epsilon 2^{2k}(\#\mathcal{E}_k)u^\epsilon \max(u^{11/12-a/12}, u^{1-a/4}). \quad (2.27)$$

Combining (2.23), (2.24) and (2.25), we thus have the estimate

$$\|G_{u,k}\|_2^2 \lesssim_\epsilon 2^{2k}(\#\mathcal{E}_k)\Big[u^a + u^\epsilon \max\Big\{u^{-a/6}\min(u^{11/12+a/12}, u^{5/6+a/4}),$$
$$u^{-a/2}\min(u^{13/12+a/12}, u^{1+a/4})\Big\} + u^\epsilon \max(u^{11/12-a/12}, u^{1-a/4})\Big].$$

Choose $a = 11/13$ to obtain

$$\|G_{u,k}\|_2^2 \lesssim_\epsilon 2^{2k}(\#\mathcal{E}_k)u^{11/13+\epsilon}$$

for every $\epsilon > 0$, which is (2.7). □

Incomparable radii

We now want to estimate $\sum_{k>k'>N(u)} |\langle G_{u,k'}, G_{u,k}\rangle|$. Our estimate will be much better than in the comparable radii case. In view of (2.6), we will in fact prove the following.

Lemma 2.3.5. *Let $\epsilon > 0$. For the choice $N(u) = 100\epsilon^{-1}\log_2(2+u)$, we have*

$$\sum_{k>k'>N(u)} |\langle G_{u,k'}, G_{u,k}\rangle| \lesssim_\epsilon \sum_k 2^{2k}\#\mathcal{E}_k. \quad (2.28)$$

Fix u and k. Similar to the case of comparable radii, the first step is to cover $\mathcal{E}_k(u)$ by a collection $\mathcal{Q}_{u,k}$ of almost-disjoint cubes Q of sidelength 2^{k+5}. By the almost-disjointness of the cubes, is enough to estimate $|\langle G_{u,k'}, G_{u,k}\rangle|$ when we restrict our points in $\mathcal{E}_k(u)$ and $\mathcal{E}_{k'}(u)$ to points in a fixed Q^* and get an estimate in terms of $\#(\mathcal{E}_k \cap Q^*)$, after which we may sum in $Q \in \mathcal{Q}_{u,k}$. So fix such a cube Q, and let $N_{R,Q,k}$ denote the cardinality of $(\mathcal{E}_k \cap Q^*)_R$ and for a fixed k', let $N_{R,Q,k'}$ denote the cardinality of $(\mathcal{E}_{k'} \cap Q^*)_R$. Similarly, let $N_{Y,Q,k}$ denote the cardinality of $(\mathcal{E}_k \cap Q^*)_Y$ and for a fixed k', let $N_{Y,Q,k'}$ denote the cardinality of $(\mathcal{E}_{k'} \cap Q^*)_Y$. Next, we prove a lemma that plays a role similar to Lemma 2.3.4 in the comparable radii case.

Lemma 2.3.6. *For each $Q \in \mathcal{Q}_{u,k}$, we have the estimates*

$$\sum_{(Y,R)\in\mathcal{E}_k(u)\cap Q^*} \sum_{(y,r)\in\mathcal{E}_{k'}(u)\cap Q^*} |\langle F_{Y,R}, F_{y,r}\rangle| \lesssim R^2 \#(\mathcal{E}_k \cap Q^*) u (N_{R,Q,k'})^{-1} \tag{2.29}$$

and

$$\sum_{(Y,R)\in\mathcal{E}_k(u)\cap Q^*} \sum_{(y,r)\in\mathcal{E}_{k'}(u)\cap Q^*} |\langle F_{Y,R}, F_{y,r}\rangle|$$
$$\lesssim N_{R,Q,k'}(\#(\mathcal{E}_k \cap Q^*)) 2^k (k \log(u)) \max(u^{5/6} 2^{5k/6}, u 2^{k/2}). \tag{2.30}$$

Proof of Lemma 2.3.6. We will first prove (2.29), which will be a good estimate in the case that $N_{R,Q,k'}$ is large. For each $(Y,R) \in (\mathcal{E}_k(u) \cap Q^*)$ we need only consider $y \in (\mathcal{E}_{k'}(u) \cap Q^*)_Y$ lying in an annulus of width $2^{k'+5}$ built upon the sphere of radius R centered at Y in $\mathbb{R}^3$. Cover the intersection of this annulus with $(\mathcal{E}_{k'}(u) \cap Q^*)_Y$ by a collection $\mathcal{C}$ of $\lesssim R^2 2^{-2k'}$ 3-dimensional cubes C of sidelength $2^{k'+3}$ in $\mathbb{R}^3$ such that each $C \cap (\mathcal{E}_{k'}(u) \cap Q^*)_Y$ is nonempty. For each $C \in \mathcal{C}$, let $\tilde{C}$ denote the 4-dimensional cube $\tilde{C} = C \times [2^{k'} - 2^{k'+2}, 2^{k'} + 2^{k'+2}]$, and let $\tilde{\mathcal{C}}$ denote the corresponding collection of cubes $\tilde{C}$. Now note that $C \cap (\mathcal{E}_{k'}(u) \cap Q^*)_Y$ nonempty implies that $(\tilde{C} \cap \mathcal{E}_{k'} \cap Q^*)_R = (\mathcal{E}_{k'} \cap Q^*)_R$, and also that $\#(\tilde{C} \cap \mathcal{E}_{k'}) \lesssim u 2^{k'}$, and hence by the product structure of $\tilde{C} \cap \mathcal{E}_{k'} \cap Q^*$,

$$\#((\tilde{C} \cap \mathcal{E}_{k'} \cap Q^*)_Y) \lesssim \#(\tilde{C} \cap \mathcal{E}_{k'})(\#(\tilde{C} \cap \mathcal{E}_{k'} \cap Q^*)_R)^{-1} \lesssim u 2^{k'} (N_{R,Q,k'})^{-1}. \tag{2.31}$$

Next, note that for a fixed $Y \in (\mathcal{E}_k \cap Q^*)_Y$, a fixed $R \in (\mathcal{E}_k \cap Q^*)_R$, and a fixed $y \in (\mathcal{E}_{k'} \cap Q^*)_Y$, Lemma 2.3.2 gives rapid decay for $|\langle F_{Y,R}, F_{y,r}\rangle|$ as r moves away from two possible values of r', that is, when r moves far away from $r' = R - |Y-y|$ and $r' = |Y-y| - R$. For these values of r' we have $|\langle F_{Y,R}, F_{y,r'}\rangle| \lesssim 2^{k'}$. Using (2.31) and our bound on the size of the collection $\mathcal{C}$, we thus have

$$\sum_{(Y,R)\in\mathcal{E}_k(u)\cap Q^*}\ \sum_{(y,r)\in\mathcal{E}_{k'}(u)\cap Q^*} |\langle F_{Y,R}, F_{y,r}\rangle|$$
$$\lesssim \sum_{(Y,R)\in\mathcal{E}_k\cap Q^*}\Bigg(\sum_{\tilde{C}\in\tilde{\mathcal{C}}}\Bigg(\sum_{(y,r)\in\mathcal{E}_{k'}\cap Q^*\cap\tilde{C}} |\langle F_{Y,R}, F_{y,r}\rangle|\Bigg)\Bigg)$$
$$\lesssim \sum_{(Y,R)\in\mathcal{E}_k\cap Q^*}\Bigg(\sum_{\tilde{C}\in\tilde{\mathcal{C}}}\Bigg(\sum_{y\in(\mathcal{E}_{k'}\cap Q^*\cap\tilde{C})_Y}\Bigg(\sum_{a\in\mathbb{Z},a\geqslant 0}$$
$$\Bigg(\sum_{\substack{r\in(\mathcal{E}_{k'}\cap Q^*)_R\\ \max(|r'-r+|Y-y'||,|r'+r-|Y-y||)\approx 2^a}} 2^{-aN}2^{k'}\Bigg)\Bigg)\Bigg)\Bigg)$$
$$\lesssim R^2\#(\mathcal{E}_k\cap Q^*)(N_{R,Q,k'})^{-1}u,$$

which is (2.29).

Now we prove (2.30), which is the estimate that we will use in the case that $N_{R,Q,k'}$ is small. This estimate is similar to (2.10), and the proof is very similar with only minor modifications, but we give all the details anyways.

By incurring a factor of $N_{R,Q,k}\cdot N_{R,Q,k'}$, to estimate

$$\sum_{(Y,R)\in\mathcal{E}_k(u)\cap Q^*}\ \sum_{(y,r)\in\mathcal{E}_{k'}(u)\cap Q^*} |\langle F_{Y,R}, F_{y,r}\rangle|,$$

it suffices to estimate for a fixed pair $r_1 \in (\mathcal{E}_k \cap Q^*)_R$ and $r_2 \in (\mathcal{E}_{k'} \cap Q^*)_R$

$$\sum_{(Y,r_1)\in\mathcal{E}_k\cap Q^*}\ \sum_{(y,r_2)\in\mathcal{E}_{k'}\cap Q^*} |\langle F_{Y,r_1}, F_{y,r_2}\rangle|.$$

Similar to the proof of (2.10), for $s \geqslant 0$, let $N'_{Y,Q,k} = 2^s \leqslant N_{Y,Q,k}$ be a given dyadic number. Fix $t \leqslant 2^{k+10}$, and define $K_{k,k'}(Q,s,t)$ to be the number of points $y \in (\mathcal{E}_k(u) \cap Q^*)_Y$ such that there are $\geqslant N'_{Y,Q,k} = 2^s$ many points $y' \in (\mathcal{E}_{k'} \cap Q^*)_Y$ such that y' lies in the annulus of inner radius t and thickness 3 centered at y. That is, define

$$\begin{aligned} K_{k,k'}(Q,s,t) := \#\{y \in (\mathcal{E}_k(u) \cap Q^*)_Y &: \text{there exists at least } 2^s \text{ many points} \\ & y' \in (\mathcal{E}_{k'} \cap Q^*)_Y \text{ such that } ||y'-y|-(t+1.5)| \leqslant 1.5\}. \end{aligned}$$

Also define

$$K^*_{k,k'}(Q,s) := \max_{0 \leqslant t \leqslant 2^{k+10}} K_{k,k'}(Q,s,t).$$

Note that the product structure of $\mathcal{E}$ implies that if both $\mathcal{E}_k \cap Q^*$ and $\mathcal{E}_{k'} \cap Q^*$ are nonempty, then their $\mathcal{Y}$-projections are equal, and so (2.13) implies the bound

$$K_{k,k'}(Q,s,t) \lesssim \max\left\{ \min(u2^k N^{5/3}_{Y,Q,k} 2^{-2s}, N_{Y,Q,k}), \min(u2^{k/2} N_{Y,Q,k} 2^{-s}, N_{Y,Q,k}) \right\}. \tag{2.32}$$

Using Lemma 2.3.2, we may bound

$$\begin{aligned} &\sum_{\substack{(Y,R)\in(\mathcal{E}_k(u)\cap Q^*)\\(y,r)\in(\mathcal{E}_{k'}(u)\cap Q^*)}} |\langle F_{Y,R}, F_{y,r}\rangle| \\ &\qquad \lesssim \sum_{\substack{r_1\in(\mathcal{E}_k(u)\cap Q^*)_R\\ r_2\in(\mathcal{E}_{k'}(u)\cap Q^*)_R}} \left(\sum_{\substack{Y\in(\mathcal{E}_k(u)\cap Q^*)_Y\\ y\in(\mathcal{E}_{k'}\cap Q^*)_Y}} |\langle F_{Y,r_1}, F_{y,r_2}\rangle| \right) \\ &\lesssim 2^k \sum_{\substack{r_1\in(\mathcal{E}_k(u)\cap Q^*)_R\\ r_2\in(\mathcal{E}_{k'}(u)\cap Q^*)_R}} \left(\sum_{0\leqslant a\leqslant m+10} \left(\sum_{Y\in(\mathcal{E}_k(u)\cap Q^*)_Y} \sum_{\substack{y\in(\mathcal{E}_{k'}\cap Q^*)_Y:\\ \min_{\pm,\pm}(1+|r_1\pm r_2\pm|y-y'||)\approx 2^a}} 2^{-aN} \right)\right) \\ &\quad \lesssim 2^k \sum_{\substack{r_1\in(\mathcal{E}_k(u)\cap Q^*)_R\\ r_2\in(\mathcal{E}_{k'}(u)\cap Q^*)_R}} \left(\sum_{0\leqslant a\leqslant m+10} 2^{-aN} \left(\sum_{s\geqslant 0:2^s\leqslant 2N_{Y,Q,k}} K^*_{k,k'}(Q,s)2^s \right)\right) \\ &\qquad\qquad \lesssim 2^k N_{R,Q,k} N_{R,Q,k'} \sum_{s\geqslant 0:2^s\leqslant 2N_{Y,Q,k}} K^*_{k,k'}(Q,s)2^s. \end{aligned} \tag{2.33}$$

Applying (2.32), we have

$$\sum_{\substack{(Y,R)\in(\mathcal{E}_k(u)\cap Q^*)\\(y,r)\in(\mathcal{E}_{k'}(u)\cap Q^*)}} |\langle F_{Y,R}, F_{y,r}\rangle| \lesssim N_{R,Q,k}N_{R,Q,k'}2^k \sum_{s\geqslant 0: 2^s\lesssim N_{Y,Q,k}} \max\Big\{\min(u2^k N_{Y,Q,k}^{5/3}2^{-s}, N_{Y,Q,k}2^s), \min(u2^{k/2}N_{Y,Q,k}, N_{Y,Q,k}2^s)\Big\}. \quad (2.34)$$

Now, note that $u2^k N_{Y,Q}^{5/3}2^{-s} \geqslant N_{Y,Q}2^s$ if and only if $2^s \leqslant u^{1/2}2^{k/2}N_{Y,Q}^{1/3}$. Also note that $u2^{k/2}N_{Y,Q,k} \geqslant N_{Y,Q,k}2^s$ if and only if $2^s \leqslant u2^{k/2}$. Thus choosing the better estimate in the term $\min(u2^m N_{Y,Q}^{5/3}2^{-s}, N_{Y,Q}2^s)$ depending on s and the better estimate in the term $\min(u2^{k/2}N_{Y,Q,k}, N_{Y,Q,k}2^s)$ yields that the left hand side of (5.20) is bounded by

$$N_{R,Q,k}N_{R,Q,k'}2^k N_{Y,Q,k}\log(N_{Y,Q,k})\max(N_{Y,Q,k}^{1/3}u^{1/2}2^{k/2}, u2^{k/2}). \quad (2.35)$$

Using $N_{Y,Q,k} \lesssim u2^k$, (2.35) is bounded by

$$N_{R,Q,k}N_{R,Q,k'}2^k N_{Y,Q,k}(k\log(u))\max(u^{5/6}2^{5k/6}, u2^{k/2}) \lesssim N_{R,Q,k'}(\#(\mathcal{E}_k\cap Q^*))2^k(k\log(u))\max(u^{5/6}2^{5k/6}, u2^{k/2}),$$

which completes the proof of (2.30). □

Proof of Lemma 2.3.5. Fix $\epsilon > 0$, and set $N(u) = 100\epsilon^{-1}\log_2(2+u)$. We apply (2.29) when $N_{R,Q,k'} \geqslant 2^{k'\epsilon}$ and (2.30) when $N_{R,Q,k'} \leqslant 2^{k'\epsilon}$, and then we sum over $N(u) < k' < k$ for k fixed to obtain

$$\sum_{\substack{N(u)<k'<k\\ k \text{ fixed}}} \sum_{(Y,R)\in\mathcal{E}_k(u)\cap Q^*} \sum_{(y,r)\in\mathcal{E}_{k'}(u)\cap Q^*} |\langle F_{Y,R}, F_{y',r'}\rangle| \lesssim_\epsilon R^2\#(\mathcal{E}_k\cap Q^*)\max(1, \log(u)u^{5/6}2^{-k/6+\epsilon}, \log(u)u2^{-k/2+\epsilon}). \quad (2.36)$$

Next we sum over $Q \in \mathcal{Q}_{u,k}$ and $k > N(u)$ to obtain

$$\sum_{k} \sum_{Q\in\mathcal{Q}_{u,k}} \sum_{\substack{N(u)<k'<k \\ k \text{ fixed}}} \sum_{(Y,R)\in\mathcal{E}_k(u)\cap Q^*} \sum_{(y,r)\in\mathcal{E}_{k'}(u)\cap Q^*} |\langle F_{Y,R}, F_{y',r'}\rangle|$$

$$\lesssim_\epsilon \sum_k 2^{2k} \# \mathcal{E}_k. \quad (2.37)$$

We have thus shown that for the choice $N(u) = 100\epsilon^{-1}\log_2(2+u)$, we have

$$\sum_{k>k'>N(u)} |\langle G_{u,k'}, G_{u,k}\rangle| \lesssim_\epsilon \sum_k 2^{2k} \# \mathcal{E}_k.$$

□

Putting it together

Combining (2.6), (2.7) and (2.28), we have that for every $\epsilon > 0$,

$$\|G_u\|_2^2 = \left\| \sum_k G_{u,k} \right\|_2^2 \lesssim_\epsilon \log_2(2+u) \sum_k 2^{2k}(\#\mathcal{E}_k) u^{11/13+\epsilon}. \quad (2.38)$$

This completes the proof of Lemma 2.2.4 and hence the proof of Proposition 2.2.2.

2.4 A geometric lemma

In this section we prove the geometric lemma used in the previous section.

Lemma 2.4.1. *Fix integers j, l with $l \leqslant j$. Let $2^{j-1} \leqslant t \leqslant 2^{j+1}$. Then the size of the intersection of three annuli in $\mathbb{R}^3$ of thickness 4 and inner radius t such that the distance between the centers of any pair is at least 2^l and no greater than $2^j/10$ is $\lesssim 2^{3(j-l)}$, provided that $l \geqslant j/2 + 10$.*

We will use the following basic lemma which gives an estimate on the size of intersections of two-dimensional annuli. This is an immediate corollary of Lemma 3.1 in [57].

Lemma D. *Let A_1 and A_2 be two annuli in $\mathbb{R}^2$ of thickness 1 built upon circles C_1 and C_2 of radius R, and let d denote the distance between the centers of C_1 and C_2. If $d \leqslant R/5$, then $A_1 \cap A_2$ is contained in the 10-neighborhood of two arcs of C_1 of length $\lesssim R/d$.*

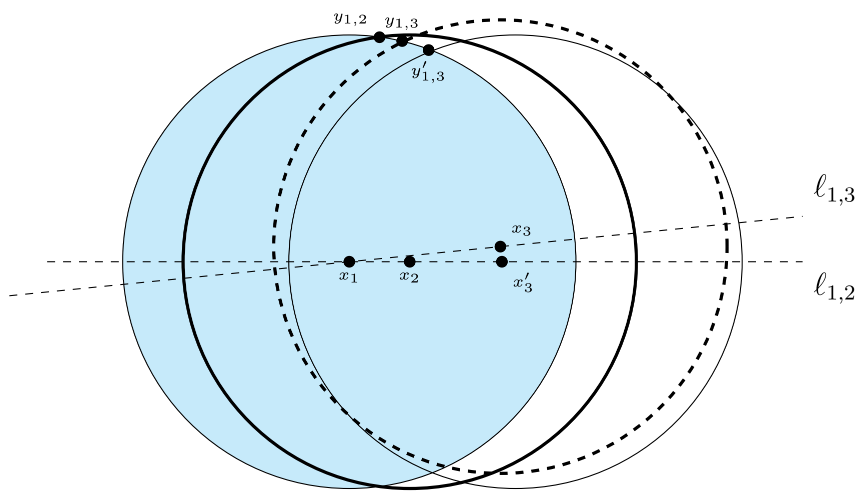

Figure 1

The circles C_1, C_2, C_3, and C'_3 in the plane P, from the proof of Lemma 2.4.1. The shaded-in circle is C_1, the thick circle is C_2, the dashed circle is C_3, and the remaining circle is C'_3.

Proof of Lemma 2.4.1. Let A_1, A_2, A_3 denote the three annuli. Let $\ell_{1,2}$ denote the line through the centers of A_1 and A_2, and let $\ell_{1,3}$ denote the line through the centers of A_1 and A_3. Let P be any plane containing both $\ell_{1,2}$ and $\ell_{1,3}$. Then $A_1 \cap A_2$ is the three dimensional solid formed by rotating the intersection of the two (circular) annuli $A_1 \cap P$ and $A_2 \cap P$ about the line $\ell_{1,2}$. Similarly, $A_1 \cap A_3$ is the three dimensional solid formed by rotating the intersection of the two (circular) annuli $A_1 \cap P$ and $A_3 \cap P$ about the line $\ell_{1,3}$.

Now, by Lemma D, $A_1 \cap A_2 \cap P$ is contained in the 10-neighborhood of two arcs of length $\lesssim 2^{j-l}$ of the circle that $A_1 \cap P$ is built upon. Rotating $A_1 \cap A_2 \cap P$ about the line $\ell_{1,2}$ to get $A_1 \cap A_2$, this implies that $A_1 \cap A_2$ is the union of $\lesssim 2^{j-l}$ many 10-neighborhoods of circles of radius $\lesssim 2^j$ lying in planes normal to the line $\ell_{1,2}$. The same holds for $A_1 \cap A_3$ with $\ell_{1,2}$ replaced by $\ell_{1,3}$. Suppose first that the angle between $\ell_{1,2}$ and $\ell_{1,3}$ is $\geqslant 2^{l-j-3}$, in radians. Then $|A_1 \cap A_2 \cap A_3|$ is bounded by $\lesssim 2^{2(j-l)}$ times the largest possible size of the intersection of two 10-neighborhoods of circular annuli, where the first lies in a plane normal to $\ell_{1,2}$ and the second lies in a plane normal to $\ell_{1,3}$. One computes that the largest possible size of such an intersection is $\lesssim 2^{j-l}$.

It remains to consider the case when the angle between $\ell_{1,2}$ and $\ell_{1,3}$ is $< 2^{l-j-3}$, in radians. We now define the following coordinates associated to the lines $\ell_{1,2}$ and $\ell_{1,3}$. Let x_1, x_2, x_3 denote the centers of A_1, A_2, A_3 respectively. For $x \in \mathbb{R}^3$, we define the $\ell_{1,2}$-coordinate

$$(x)_{1,2} = \frac{\langle x - x_1, x_2 - x_1 \rangle}{|x_2 - x_1|}.$$

Similarly define the $\ell_{1,3}$-coordinate

$$(x)_{1,3} = \frac{\langle x - x_1, x_3 - x_1 \rangle}{|x_3 - x_1|}.$$

By interchanging the order of A_1, A_2, A_3, we may assume without loss of generality that $(x_3)_{1,2} \geqslant (x_2)_{1,2} = 1$. We will show that $l \geqslant j/2 + 10$ implies that $A_1 \cap A_2$ and $A_2 \cap A_3$ are actually disjoint. Observe that since the angle between $\ell_{1,2}$ and $\ell_{1,3}$ is $< 2^{l-j-3}$, we have that $(x_3 - x_2)_{1,2} \geqslant 2^{l-1}$. Now, let x_3' be the closest point on the line $\ell_{1,2}$ whose distance from x_1 is the same as the distance from x_1 to x_3. Clearly, we also have

$(x_3' - x_2)_{1,2} \geqslant 2^{l-1}$. Let C_3 be the circle in P with center at x_3 and radius t and let C_3' be the circle in P with center at x_3' and radius t. Then if $y_{1,3}'$ denotes either of the two points in $C_1 \cap C_3'$ and $y_{1,2}$ either of the two points in $C_1 \cap C_2$, then $(x_3' - x_2)_{1,2} \geqslant 2^{l-1}$ implies that $(y_{1,3}' - y_{1,2})_{1,2} \geqslant 2^{l-2}$. This is because with respect to the $\ell_{1,2}$-coordinate, $y_{1,3}'$ lies at the midpoint of x_1 and x_3' and $y_{1,2}$ lies at the midpoint of x_1 and x_2. Note that $C_1 \cap C_3$ is the rotation within P of $C_1 \cap C_3'$ by an angle of $< 2^{l-j-3}$ where the rotation is based at x_1. This implies that if $y_{1,3}$ is either of the two points in $C_1 \cap C_3$, then $|y_{1,3}' - y_{1,3}| \leqslant 2^{l-3}$. It follows that $(y_{1,3} - y_{1,2})_{1,2} \geqslant (y_{1,3}' - y_{1,2})_{1,2} - |y_{1,3}' - y_{1,3}| \geqslant 2^{l-2} - 2^{l-3} = 2^{l-3}$.

But by Lemma D, $A_1 \cap A_2$ is the rotation in $\mathbb{R}^3$ of a 10-neighborhood of an arc of C_1 of length $\lesssim 2^{j-l}$ that contains $y_{1,2}$ about $\ell_{1,2}$, and so $A_1 \cap A_2$ lives in the slab $\{z \in \mathbb{R}^3 : |(z - y_{1,2})_{1,2}| \leqslant 2^{j-l+4}\}$. Similarly, $A_1 \cap A_3$ is the rotation in $\mathbb{R}^3$ of a 10-neighborhood of an arc of C_1 of length $\lesssim 2^{j-l}$ that contains $y_{1,3}$ about $\ell_{1,3}$, and so $A_1 \cap A_3$ lives in the half-infinite slab $\{z \in \mathbb{R}^3 : (z - y_{1,3})_{1,2} \geqslant -2^{j-l+4}\}$, and since $l \geqslant j/2 + 10$ we have $j - l + 4 \leqslant l - 10$. Since $(y_{1,3} - y_{1,2})_{1,2} \geqslant 2^{l-3}$, it follows that $A_1 \cap A_2$ and $A_2 \cap A_3$ are disjoint.

□

Chapter 3

Bochner Riesz Means associated with Rough Planar Domains

3.1 Introduction

The Bochner-Riesz operators R_λ are defined via the Fourier transform by

$$\mathcal{F}[R_\lambda f](\xi) = (1-|\xi|)_+^\lambda \widehat{f}(\xi), \qquad \lambda > 0,$$
$$\mathcal{F}[R_0 f](\xi) = \chi_{B_0(1)}(\xi)\widehat{f}(\xi),$$

where $\chi_{B_0(1)}$ denotes the characteristic function of the ball of radius 1 centered at the origin. In two dimensions the L^p mapping properties of R_λ are completely known. As first shown by Fefferman in [22] and later clarified by Córdoba in [18], if $\lambda > 0$ then R_λ is bounded on $L^p(\mathbb{R}^2)$ if and only if $\lambda > \max((|\frac{2}{p}-1|-\frac{1}{2}), 0)$. It was also shown by Fefferman in [21] that R_0 is bounded on $L^p(\mathbb{R}^2)$ if and only if $p = 2$. One may also consider the following generalization of the two-dimensional Bochner-Riesz operators. Let $\Omega \subset \mathbb{R}^2$ be a bounded, open convex set containing the origin, and let ρ be its Minkowski functional, defined as

$$\rho(\xi) = \inf\{t > 0 : t^{-1}\xi \in \Omega\}.$$

Define the generalized Bochner-Riesz operators T_λ associated to Ω by

$$\mathcal{F}[T_\lambda f](\xi) = (1-\rho(\xi))_+^\lambda \widehat{f}(\xi), \qquad \lambda > 0,$$
$$\mathcal{F}[T_0 f](\xi) = \chi_\Omega(\xi)\widehat{f}(\xi),$$

where χ_Ω denotes the characteristic function of Ω. Note that in the special case that Ω is the unit disk, T_λ is simply R_λ. We emphasize that no further regularity of $\partial\Omega$ is assumed, and for general convex domains Ω the boundary $\partial\Omega$ need only be Lipschitz.

For domains with smooth boundary, the L^p mapping properties of T_λ were shown by Sjölin in [50] to be identical to those of R_λ. However, for certain convex domains with rough boundary the L^p mapping properties of T_λ may be improved. In [43], Podkorytov showed that in the case that Ω is a polyhedron in $\mathbb{R}^d$, T_λ is bounded on L^p for $1 \leqslant p \leqslant \infty$ and for all $\lambda > 0$. In [48], Seeger and Ziesler proved a sufficient criterion for L^p boundedness of Bochner-Riesz multipliers associated to general convex domains in $\mathbb{R}^2$. Their results depended on a parameter similar to the upper Minkowski dimension of $\partial\Omega$, defined by a family of "balls", or caps, and we give a definition below. This parameter may be thought of as measuring how "curved" the boundary of Ω is.

For any $p \in \partial\Omega$, we say that a line ℓ is a *supporting line for* Ω *at* p if ℓ contains p and Ω is contained in the half plane containing the origin with boundary ℓ. Let $\mathcal{T}(\Omega, p)$ denote the set of supporting lines for Ω at p. Note that if $\partial\Omega$ is C^1, then $\mathcal{T}(\Omega, p)$ has exactly one element, the tangent line to $\partial\Omega$ at p. For any $p \in \partial\Omega$, $\ell \in \mathcal{T}(\Omega, p)$, and $\delta > 0$, define

$$B(p, \ell, \delta) = \{x \in \partial\Omega : \operatorname{dist}(x, \ell) < \delta\}. \tag{3.1}$$

Let

$$\mathcal{B}_\delta = \{B(p,\ell,\delta) : p \in \partial\Omega, \ell \in \mathcal{T}(\Omega,p)\}, \tag{3.2}$$

and let $N(\Omega,\delta)$ be the minimum number of balls $B \in \mathcal{B}_\delta$ needed to cover $\partial\Omega$. Let

$$\kappa_\Omega = \limsup_{\delta\to 0} \frac{\log N(\Omega,\delta)}{\log \delta^{-1}}. \tag{3.3}$$

It is easy to show using Cauchy-Schwarz that for any convex domain Ω, $0 \leqslant \kappa_\Omega \leqslant \frac{1}{2}$. If $\partial\Omega$ is smooth, then $\kappa_\Omega = 1/2$. This can be seen by noting that there is a point where $\partial\Omega$ has nonvanishing curvature, and near this point the contribution to $N(\Omega,\delta)$ is $\approx \delta^{-1/2}$.

We now state the main result from [48], due to Seeger and Ziesler.

Theorem A ([48]). *Suppose that $1 \leqslant p \leqslant \infty$, $\lambda > 0$ and $\lambda > \kappa_\Omega(4|1/p - 1/2| - 1)$. Then T_λ is bounded on $L^p(\mathbb{R}^2)$.*

Note that as κ_Ω gets smaller, the range of p for which T_λ is bounded improves, so for rough domains it is possible to do much better than the optimal result for domains with smooth boundary. The authors of [48] also showed that for each $\kappa \in (0,1/2)$ there is a convex domain Ω with $\kappa_\Omega = \kappa$ for which Theorem A is sharp.

Theorem B ([48]). *Let $0 < \kappa < 1/2$. Then there exists a convex domain Ω with $C^{1,\frac{\kappa}{(1-\kappa)}}$ boundary satisfying $\kappa_\Omega = \kappa$ so that for $1 \leqslant p < 4/3$ the operator T_λ associated to Ω is bounded on $L^p(\mathbb{R}^2)$ if and only if $\lambda > \kappa_\Omega(4/p - 3)$.*

We will show that for every $\kappa \in (0,1/2)$ sufficiently small there exists a convex domain Ω with $\kappa_\Omega = \kappa$ for which Theorem A is not sharp.

Theorem 3.1.1. *Let $m \geqslant 2$ be an integer. Let $\kappa \in (0, \frac{1}{4m-2}]$. Then there exists a convex domain Ω with $\kappa_\Omega = \kappa$ so that for $1 \leqslant p \leqslant \frac{2m}{2m-1}$, T_λ is bounded on $L^p(\mathbb{R}^2)$ if*

$\lambda > \kappa_\Omega(\frac{m+2}{p} - m - 1)$, and for $4/3 \leqslant p \leqslant 4$, T_λ is bounded on $L^p(\mathbb{R}^2)$ if and only if $\lambda > 0$.

Note that the case $m = 2$ above corresponds to Theorem A, and that if $m \geqslant 3$ Theorem 3.1.1 gives an improvement over Theorem A in the range $1 \leqslant p < \frac{2m}{2m-1}$ (and of course, in the dual range as well). Theorem 3.1.1 demonstrates that how "curved" the boundary of a convex planar domain is, as measured by the parameter κ_Ω, does not alone determine the L^p mapping properties of the associated Bochner-Riesz operators, but rather there must be other properties of Ω that play a role. Theorem 3.1.1 also shows that there exist domains with $\kappa_\Omega > 0$ such that $p_{\text{crit}} < 4/3$.

In the proof of Theorem B, a crucial property of the domains constructed was that their boundaries contained long arithmetic progressions at every scale, in the sense that for every $\delta > 0$ the boundary could be covered by essentially disjoint balls in $\mathcal{B}_\delta$ such that a large sequence of consecutive balls were essentially equally spaced in a single coordinate direction. We now describe a simplified version of their construction, removing the requirement that Ω has $C^{1,\frac{\kappa_\Omega}{1-\kappa_\Omega}}$ boundary in the statement of Theorem B, as well as sharpness at the endpoint $\lambda = \kappa_\Omega(\frac{4}{p} - 3)$. Choose a sequence of consecutive intervals $I_1, I_2, \ldots$ in $[0,1]$ such that I_k has length $2^{-k(1/2-\kappa_\Omega)}$. For each k, let E_k be a set of $2^{k\kappa_\Omega}$ essentially equally spaced points in I_k at a distance $\approx 2^{-k/2}$ apart. Now for each k, let Ω_k denote the convex polygon with vertices

$$\{(-1,1);(-1,-2);(0,1)\} \cup \{(x_1, x_1^2 - 2) : x_1 \in \bigcup_{1\leqslant j \leqslant k} E_j\}.$$

Let Ω be the uniform limit of $\{\Omega_k\}$ as $k \to \infty$. Then one may show using similar arguments to those presented in [48] in the proof of Theorem B that whenever $1 \leqslant p < 4/3$, T_λ is bounded on $L^p(\mathbb{R}^2)$ only if $\lambda \geqslant \kappa_\Omega(\frac{4}{p} - 3)$.

The domains we construct to prove Theorem 3.1.1 will differ from those constructed in [48] to prove Theorem B in in that they will exhibit “low n-additive energy” at every scale for some $n > 2$. To produce such domains will require a particular kind of “fast-branching” Cantor-type construction. We define the n-additive energy of $\partial\Omega$ as follows.

Definition 3.1.2. *Let $n \geqslant 2$ be an integer, and let Ω be a bounded, convex domain in $\mathbb{R}^2$. Let $\mathfrak{B}_\delta = \{B_1, B_2, \ldots, B_{N(\Omega,\delta)}\}$ be a collection of balls in $\mathcal{B}_\delta$ covering $\partial\Omega$. Let $\Xi_{\mathfrak{B}_\delta,n}$ be the smallest integer such that $\Xi_{\mathfrak{B}_\delta,n} = M_0^{2n} \cdot M_1$ and we may write $\mathfrak{B}_\delta$ as a union of M_0 subcollections $\mathfrak{B}_{\delta,1} \ldots, \mathfrak{B}_{\delta,M_0}$ such that for each $1 \leqslant k \leqslant M_0$, no point of $\mathbb{R}^2$ is contained in more than M_1 of the sets $B_{i_1} + \cdots + B_{i_n}$ where $B_{i_j} \in \mathfrak{B}_{\delta,k}$ for all j. Let $\Xi_{\delta,n} = \min_{\mathfrak{B}_\delta}(\Xi_{\mathfrak{B}_\delta,n})$, where the minimum is taken over all collections of balls in $\mathcal{B}_\delta$ covering $\partial\Omega$ with card$(\mathcal{B}_\delta) = N(\Omega,\delta)$. We define the n-additive energy of $\partial\Omega$ to be*

$$\mathcal{E}_n(\partial\Omega) = \limsup_{\delta\to 0} \frac{\log(\Xi_{n,\delta})}{\log(\delta^{-1})}.$$

As a consequence of a lemma proven in [48], we have $\mathcal{E}_2(\partial\Omega) = 0$ for all convex domains Ω. However, general convex domains fail to satisfy $\mathcal{E}_n(\partial\Omega) = 0$ for some $n > 2$, but the domains we construct will have this property.

To discuss a second important property that leads to improved L^p bounds for generalized Bochner-Riesz multipliers, we first need to associate a set of directions to Ω. Given $x \in \partial\Omega$, let θ_x, θ'_x be the slopes of two supporting lines at x with maximum difference in angle (note there is a unique choice of two lines). We will allow slopes to be infinite to include the possibility of vertical lines. Note that if we choose x so that $\partial\Omega$ may be parametrized near x by $(\alpha, \gamma(\alpha))$, then θ_x and θ'_x are simply the left and right derivatives of γ evaluated at x. Let

$$\Theta = \Theta(\Omega) = \{\theta_x, \theta_x' : x \in \partial\Omega\} \subset \mathbb{R} \cup \{\infty\}.$$

Define a sequence of Nikodym-type maximal operators $\{M_{\Theta,\delta}\}$ by

$$M_{\Theta,\delta} f(x) = \sup_{x \in R \in \mathcal{R}_\delta} \frac{1}{|R|} \int_R f(y)\, dy,$$

where $\mathcal{R}_\delta$ denotes the set of all rectangles of eccentricity $\leqslant \delta^{-1}$ with long side having slope in Θ. We will be interested in how $\|M_{\Theta,\delta}\|_{L^p \to L^p}$ behaves as $\delta \to 0$. It was shown by Bateman in [2] that if M_Θ denotes the directional maximal operator corresponding to Θ, then M_Θ is unbounded on L^p for all p such that $1 \leqslant p < \infty$ unless Θ is a union of finitely many lacunary sets of finite order, and it is easy to show that any domain Ω with $\Theta(\Omega)$ a union of finitely many lacunary sets of finite order satisfies $\kappa_\Omega = 0$. Thus for all domains with $\kappa_\Omega > 0$ we must necessarily have that $\|M_{\Theta,\delta}\|_{L^p \to L^p} \to \infty$ as $\delta \to 0$.

Definition 3.1.3. *We say that Θ is p-sparse if*

$$\|M_{\Theta,\delta}\|_{L^p \to L^p} = O(\delta^{-\epsilon})$$

for every $\epsilon > 0$.

It follows immediately by a theorem of Córdoba (see [16]) regarding the L^2 bounds for the Nikodym maximal function in $\mathbb{R}^2$ that every Θ is p-sparse for $2 \leqslant p < \infty$. We will see that if $\Theta(\Omega)$ is p-sparse for some $p < 2$, then T_λ satisfies improved L^p bounds over those stated in Theorem A. However, it is unclear whether the domains we construct are p-sparse for some $p < 2$; hence construction of domains with $\kappa_\Omega > 0$ that are p-sparse for some $p < 2$ remains an interesting open question.

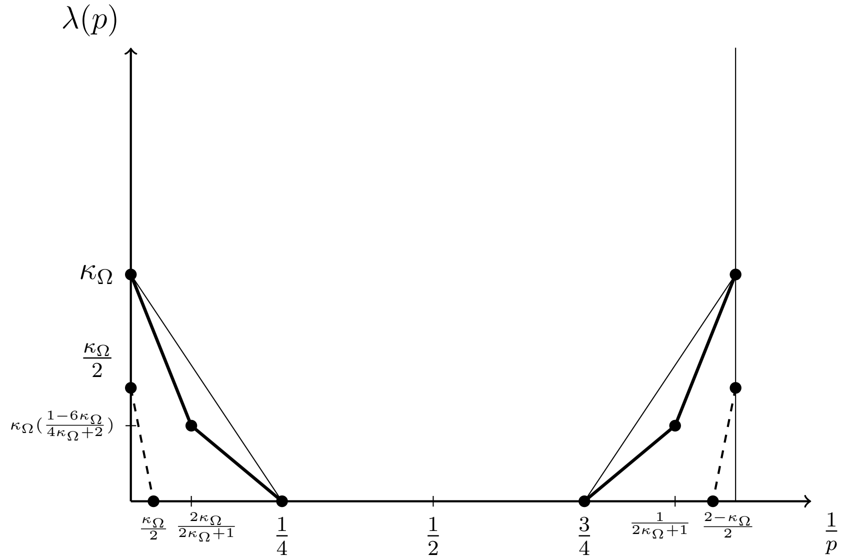

Figure 2

Here we sketch $\lambda(p)$ as a function of $\frac{1}{p}$ for certain convex domains, where T_λ is bounded on L^p for all $\lambda > \lambda(p)$. In this diagram, it is assumed that $\kappa_\Omega \leqslant \frac{1}{10}$. The thin solid lines correspond to the domains constructed in [48] in the proof of Theorem B; these domains exhibit long arithmetic progressions at every scale. The thick solid lines correspond to the domains that we construct to prove Theorem 3.1.1 using a fast-branching Cantor-type construction; these lines as drawn are only valid if $\kappa_\Omega = \frac{1}{4m-2}$ for $m \geqslant 3$ an integer. The dashed lines represent lower bounds for general convex domains. That is, for any convex domain, T_λ is unbounded on L^p if $(\frac{1}{p}, \lambda)$ lies below the dashed lines.

We now formulate a general theorem on L^p mapping properties of Bochner-Riesz means in terms of the n-additive energy of $\partial\Omega$ and the L^q-mapping properties of $M_{\Theta,\delta}$.

Theorem 3.1.4. *Let Ω be a convex domain in $\mathbb{R}^2$ containing the origin and let Θ be its associated set of directions. Let $n \geqslant 2$ be an integer. Suppose that $\mathcal{E}_n(\partial\Omega) = \alpha$ for some* *integer* $0 \leqslant \alpha \leqslant n\kappa_\Omega$ *and that*

$$\|M_{\Theta,\delta}\|_{L^{\frac{n}{n-1}}(\mathbb{R}^2)\to L^{\frac{n}{n-1}}(\mathbb{R}^2)} \leqslant C_\epsilon \delta^{-\beta-\epsilon}$$

for some $0 \leqslant \beta \leqslant \kappa_\Omega(\frac{n-2}{n})$ *and every* $\epsilon > 0$. *Then for* $1 \leqslant p \leqslant \frac{2n}{2n-1}$, T_λ *is bounded on* L^p *for* $\lambda > \kappa_\Omega(\frac{2n}{p} - 2n + 1) + (\alpha/2n + \beta/2)(\frac{2np-2n}{p})$.

Note that if $n = 2$ we recover Theorem A. One may check that if $n > 2$, $\alpha = 0$ and $\beta = \kappa_\Omega(\frac{n-2}{n})$ (i.e. β is obtained by interpolating Córdoba's estimate $\|M_{\Theta,\delta}\|_{L^2\to L^2} = O(\delta^{-\epsilon})$ with the trivial L^1 estimate $\|M_{\Theta,\delta}\|_{L^1\to L^1} = O(\delta^{-\kappa_\Omega})$), then Theorem 3.1.4 gives improved bounds over those stated in Theorem A in the range $1 \leqslant p \leqslant \frac{2n}{2n-1}$. Fix a convex domain Ω, and define

$$p_{\text{crit}} := \inf\{p : T_\lambda \text{ bounded on } L^p \text{ for all } \lambda > 0\}.$$

To achieve $p_{\text{crit}} < 4/3$ using Theorem 3.1.4 would require the construction of domains that simultaneously satisfy *both* $\alpha = 0$ and $\beta = 0$ for some $n > 2$.

Finally, in Section 3.5 we will prove the following lower bounds for T_λ for general convex domains.

Theorem 3.1.5. *Let* $1 \leqslant p \leqslant 2$. *Let* $\Omega \subset \mathbb{R}^2$ *be a convex domain containing the origin, and let* T_λ *denote the generalized Bochner-Riesz operator with exponent* λ *associated to* Ω. *Then* T_λ *is unbounded on* $L^p(\mathbb{R}^2)$ *if* $\lambda < 1 - \frac{\kappa_\Omega}{2} - \frac{1}{p}$. *In particular,* $p_{crit} \geqslant \frac{2}{2-\kappa_\Omega}$.

The proof will involve testing the operator on randomly defined functions, using Khinchine's inequality and Plancherel to estimate the L^1 and L^2 operator norms, respectively, and then interpolating.

We now give an overview of the layout of this chapter. In Section 3.2 we give useful preliminaries about convex domains in $\mathbb{R}^2$ and state some background results from [48]. In Section 3.3, we construct the convex domains which we will later prove satisfy the statement of Theorem 3.1.1, and prove some results about the n-additive energy of their boundaries. In Section 3.4 we prove Theorem 3.1.4, which gives L^p bounds for T_λ as a consequence of certain conditions on the n-additive energy of $\partial\Omega$ and range of q for which $\Theta(\Omega)$ is q-sparse. We also prove Theorem 3.1.1 as a consequence of Theorem 3.1.4. In Section 3.5 we prove Theorem 3.1.5, which gives lower L^p bounds on T_λ for general convex domains with a given value of κ_Ω. In Section 3.6 we discuss some open questions which follow naturally from the results of this chapter.

Remark 3.1.6. *All logarithms in this chapter will be assumed to be base* 2*, unless otherwise noted.*

3.2 Preliminaries on convex domains in $\mathbb{R}^2$

In this section we give some useful background about convex domains in $\mathbb{R}^2$. All results in this section can be found in [48], but we include them here for the sake of completeness. However, we will omit all proofs in this section, and the reader is encouraged to refer to [48] for proofs.

Let $\Omega \subset \mathbb{R}^2$ be a bounded, open convex set containing the origin. Assume that Ω contains the ball of radius 4 centered at the origin. Since Ω is bounded, there is an integer $M > 0$ such that

$$\{\xi : |\xi| \leqslant 4\} \subset \Omega \subset \overline{\Omega} \subset \{\xi : |\xi| < 2^M\}. \tag{3.4}$$

The following lemma is straightforward and can be proved using only elementary facts about convex functions.

Lemma C ([48])**.** *Suppose that* Ω *is a convex domain satisfying (3.4). Then* $\partial\Omega \cap \{x : -1 \leqslant x_1 \leqslant 1,\, x_2 < 0\}$ *can be parametrized by*

$$t \mapsto (t, \gamma(t)), \qquad -1 \leqslant t \leqslant 1, \tag{3.5}$$

where

1.

$$1 < \gamma(t) < 2^M, \qquad -1 \leqslant t \leqslant 1. \tag{3.6}$$

2. γ *is a convex function on* $[-1, 1]$*, so that the left and right derivatives* γ'_L *and* γ'_R *exist everywhere in* $(-1, 1)$ *and*

$$-2^{M-1} \leqslant \gamma'_R(t) \leqslant \gamma'_L(t) \leqslant 2^{M-1} \tag{3.7}$$

for $t \in [-1, 1]$*. The functions* γ'_L *and* γ'_R *are decreasing functions;* γ'_L *and* γ'_R *are right continuous in* $[-1, 1]$*.*

3. Let ℓ *be a supporting line through* $\xi \in \partial\Omega$ *and let* n *be an outward normal vector. Then*

$$|\,\langle \xi, n\rangle\,| \geqslant 2^{-M}|\xi|. \tag{3.8}$$

Decomposition of $\partial\Omega$

As another preliminary ingredient, we need the decomposition of $\partial\Omega \cap \{x : -1 \leqslant x_1 \leqslant 1,\, x_2 < 0\}$ introduced in [48]. This decomposition allows us to write $\partial\Omega$ as a disjoint union of pieces on which $\partial\Omega$ is sufficiently "flat", where the number of pieces in the decomposition is closely related to the covering numbers $N(\Omega, \delta)$. We inductively define a finite sequence of increasing numbers

$$\mathfrak{A}(\delta) = \{a_0, \dots, a_Q\}$$

as follows. Let $a_0 = -1$, and suppose $a_0, \dots, a_{j-1}$ are already defined. If

$$(t - a_{j-1})(\gamma_L'(t) - \gamma_R'(a_{j-1})) \leqslant \delta \text{ for all } t \in (a_{j-1}, 1]) \tag{3.9}$$

and $a_{j-1} \leqslant 1 - 2^{-M}\delta$, then let $a_j = 1$. If (3.9) holds and $a_{j-1} > 1 - 2^{-M}\delta$, then let $a_j = a_{j-1} + 2^{-M}\delta$. If (3.9) does not hold, define

$$a_j = \inf\{t \in (a_{j-1}, 1] : (t - a_{j-1})(\gamma_L'(t) - \gamma_R'(a_{j-1})) > \delta\}.$$

Now note that (3.9) must occur after a finite number of steps, since we have $|\gamma_L'|, |\gamma_R'| \leqslant 2^{M-1}$, which implies that $|t-s||\gamma_L'(t) - \gamma_R'(s)| < \delta$ if $|t-s| < \delta 2^{-M}$. Therefore this process must end at some finite stage $j = Q$, and so it gives a sequence $a_0 < a_1 < \cdots < a_Q$ so that for $j = 0, \dots, Q-1$

$$(a_{j+1} - a_j)(\gamma_L'(a_{j+1}) - \gamma_R'(a_j)) \leqslant \delta, \tag{3.10}$$

and for $0 \leqslant j < Q-1$,

$$(t - a_j)(\gamma_L'(t) - \gamma_R'(a_j)) > \delta \qquad \text{if } t > a_{j+1}. \tag{3.11}$$

For a given $\delta > 0$, this gives a decomposition of

$$\partial\Omega \cap \{x : -1 \leqslant x_1 \leqslant 1, x_2 < 0\}$$

into pieces

$$\bigsqcup_{n=0,1,\ldots,Q-1} \{x \in \partial\Omega :\ x_1 \in [a_n, a_{n+1}]\}.$$

The number Q in (3.10) and (3.11) is also denoted by $Q(\Omega, \delta)$. Let R_θ denote rotation by θ radians. The following lemma relates the numbers $Q(R_\theta\Omega, \delta)$ to the covering numbers $N(\Omega, \delta)$.

Lemma D ([48]). *There exists a positive constant C_M so that the following statements hold.*

1. $Q(\Omega, \delta) \leqslant C_M \delta^{-1/2}$.

2. $0 \leqslant \kappa_\Omega \leqslant 1/2$.

3. *For any θ,*

$$Q(R_\theta\Omega, \delta) \leqslant C_M N(\Omega, \delta) \log(2 + \delta^{-1}).$$

4. *For $\nu = 1, \ldots, 2^{2M}$ let $\theta_\nu = \frac{2\pi\nu}{2^{2M}}$. Then*

$$C_M^{-1} N(\Omega, \delta) \leqslant \sum_\nu Q(R_{\theta_\nu}\Omega, \delta) \leqslant C_M N(\Omega, \delta) \log(2 + \delta^{-1}).$$

Finally, we state two results from [48] that we will need later in our proof of Theorem 3.1.1. The former is an L^1 estimate for the kernels of generalized Bochner-Riesz multipliers using a decomposition analogous to the standard decomposition of the (spherical) Bochner-Riesz multipliers into annuli. The latter is an L^1 kernel estimate corresponding to a finer decomposition of the generalized Bochner-Riesz multipliers associated with the decomposition of $\partial\Omega$ introduced above, as well as a pointwise majorization of a maximal function associated with this decomposition by a related Nikodym-type maximal function.

Proposition E ([48]). *Let Ω be a convex domain containing the origin. Let β be a C^2 function supported on $(-1/2, 1/2)$ so that*

$$|\beta^k(t)| \leqslant 1, \qquad k = 0, \ldots, 4.$$

Let

$$m_{\delta,\lambda}(\xi) = \delta^\lambda \beta\big(\frac{\delta^{-1}}{2}(1 - \rho(\xi))\big).$$

Then there is some $c > 0$ such that for every $\delta > 0$ sufficiently small,

$$\left\|\mathcal{F}^{-1}[m_{\delta,\lambda}\widehat{f}]\right\|_{L^1(\mathbb{R}^2)} \lesssim \delta^\lambda \log(\delta^{-1})^c N(\Omega, \delta) \|f\|_{L^1(\mathbb{R}^2)}.$$

Proposition F ([48]). *Let Ω be a convex domain satisfying (3.4) and let $b \in C_0^\infty$ be supported in the sector $S = \{\xi : |\xi_1| \leqslant 2^{-10M}|\xi_2|, \xi_2 < 0\}$. Let $\alpha \mapsto (\alpha, \gamma(\alpha))$ be the parametrization of $\partial\Omega \cap S$ as a graph, as in Lemma C. For any subinterval I of $[-1/2, 1/2]$ denote by I^* the interval with the same center and with length $\frac{4}{3}|I|$. For $\delta < 1/2$ let $\mathfrak{J}_\delta$ be the set of open subintervals I of $[-1, 1]$ with the property that $|I| \geqslant 2^{-5M}\delta$ and*

$$(t - s)(\gamma_L'(t) - \gamma_R'(s)) \leqslant 2^5\delta \text{ for } s < t,\ s, t, \in I^*. \tag{3.12}$$

Let $\mathfrak{B}$ be the set of C^2 functions β supported on $(-1/2, 1/2)$ so that

$$|\beta^{(k)}(t)| \leqslant 1, \qquad k = 0, \ldots, 4.$$

Suppose $I = (c_I - |I|/2, c_I + |I|/2) \in \mathfrak{J}_\delta$. Let

$$m_{\beta_1,\beta_2,I}(\xi) = b(\xi)\beta_1(\frac{\delta^{-1}}{2}(1 - \rho(\xi)))\beta_2(|I|^{-1}(\xi_1 - c_I)) \tag{3.13}$$

where $\beta_1, \beta_2 \in \mathfrak{B}$*. Then for any* $\beta_1, \beta_2 \in \mathfrak{B}$ *and* $I \in \mathfrak{J}_\delta$*,*

$$\left\|F^{-1}[m_{\beta_1,\beta_2,I}]\right\|_1 \lesssim \log(\delta^{-1}). \tag{3.14}$$

Let

$$\mathfrak{M}_\delta f(x) = \sup_{\beta_1,\beta_2\in\mathfrak{B}} \sup_{I\in\mathfrak{J}_\delta} \left||\mathcal{F}^{-1}[m_{\beta_1,\beta_2,I}]| * f(x)\right|$$

and let

$$\overline{M}_\delta f(x) = \sup_{x\in R\in\mathcal{C}_\delta} \frac{1}{|R|}\int_R |f(y)|\, dy,$$

where

$$\mathcal{C}_\delta = \{R :\ R \textit{ is a rectangle of dimensions } \delta \times (a_{j+1} - a_j)$$
$$\textit{with longer side of slope } \gamma_L'(a_j),\ \textit{where } a_j, a_{j+1} \in \mathfrak{A}(\delta)\}.$$

Then

$$\mathfrak{M}_\delta f(x) \lesssim \log(\delta^{-1})\overline{M}_\delta f(x). \tag{3.15}$$

3.3 Construction of Ω and some algebraic disjointness lemmas

We will now construct domains which we will show satisfy the statement of Theorem 3.1.1. The idea is to construct a convex domain Ω such that the kernels of the pieces of the multiplier obtained by decomposing the multiplier as in Proposition F exhibit a high degree of cancellation with each other. In [48], it was shown that for abitrary convex domains that the supports of the convolution of pairs of pieces of the multiplier were more or less disjoint. This was used to prove the endpoint $p = 4/3$ estimate using duality and an L^4 argument similar to Córdoba's treatment of the (spherical) Bochner-Riesz means in $\mathbb{R}^2$ (see [18]). Here, we construct a domain so that the supports of the m-fold convolution of m-tuples of pieces of the mutiplier are more or less disjoint, which we will use to prove an L^{2m} estimate in the same vein as in [18] and [48].

Before constructing Ω, we will need the following basic lemma.

Lemma 3.3.1. *For any integer $N > 10$ and any integer $m \geqslant 1$, there exists a collection $\mathcal{I}$ of N disjoint subintervals of $[-\frac{1}{2}, \frac{1}{2}]$ each of size $\frac{N^{-(2m-1)}}{3m}$ so that*

$$\{I_1 + I_2 + \cdots + I_m\}_{I_1,\ldots,I_m \in \mathcal{I}}$$

is a pairwise disjoint collection.

Proof of Lemma 3.3.1. Let M be an integer strictly less than N. We will show that if $\mathcal{I}_M$ is a collection of M disjoint subintervals of $[-\frac{1}{2}, \frac{1}{2}]$ each of size $\frac{N^{-(2m-1)}}{3m}$ satisfying the algebraic disjointness condition of the lemma, then there is a collection $\mathcal{I}_{M+1}$ of $M + 1$ disjoint subintervals of $[-\frac{1}{2}, \frac{1}{2}]$ of size $\frac{N^{-(2m-1)}}{3m}$ satisfying the same condition.

Indeed, suppose that such a collection $\mathcal{I}_M$ exists. Suppose $I_1, \ldots, I_m \in \mathcal{I}_M$. Then given any collection of $m-1$ intervals $I_{m+1}, \ldots I_{2m-1} \in \mathcal{I}_M$, there is an interval $I_{(I_1,\ldots,I_{2m-1})} \subset [-\frac{1}{2}, \frac{1}{2}]$ of width no larger than $\frac{2N^{-(2m-1)}}{3}$ such that

$$(I_1 + \cdots + I_m) - (I_{m+1} + \cdots + I_{2m-1}) \subset I_{(I_1,\ldots,I_{2m-1})}.$$

Now define

$$E = \bigcup_{(I_1,\ldots,I_{2m-1}) \in (\mathcal{I}_M)^{2m-1}} I_{(I_1,\ldots,I_{2m-1})}.$$

Then since $\operatorname{card}((\mathcal{I}_M)^{2m-1}) = M^{2m-1}$, we have $|E| \leqslant \frac{2}{3} \cdot (\frac{M}{N})^{2m-1}$. Since $M < N$, we have $|[-\frac{1}{2}, \frac{1}{2}] \setminus E| \geqslant \frac{1}{3}$. Since E is a union of no more than M^{2m-1} disjoint intervals, the average gap length between consecutive disjoint intervals in E is at least $\frac{1}{6}M^{-7} \geqslant \frac{1}{6}N^{-7}$. Thus there exists an interval I of length $\frac{N^{-(2m-1)}}{3m}$ such that $I \subset [-\frac{1}{2}, \frac{1}{2}] \setminus E$. Now set $\mathcal{I}_{M+1} = \mathcal{I}_M \cup \{I\}$. Then $\mathcal{I}_{M+1}$ is a collection of $M+1$ disjoint subintervals of $[-\frac{1}{2}, \frac{1}{2}]$ each of size $\frac{N^{-(2m-1)}}{3m}$ satisfying the algebraic disjointness condition of the lemma. By induction on M, the proof is complete. □

Construction of Ω

We now proceed to construct the convex domain Ω which we will show satisfies the statement of Theorem 3.1.1 with $\kappa_\Omega = \frac{1}{4m-2}$. It will then be easy to explain how to modify the construction to produce a domain which satisfies the statement of Theorem 3.1.1 with $\kappa_\Omega \in [0, \frac{1}{4m-2})$.

For each integer $k \geqslant 0$, we inductively define a collection $\mathcal{I}_k$ of disjoint subintervals of $[-\frac{1}{2}, \frac{1}{2}]$. We set $\mathcal{I}_0 = \{[-\frac{1}{2}, \frac{1}{2}]\}$. For each $k \geqslant 0$, we define $\mathcal{I}_{k+1}$ to be a collection of $2^{k+4} \cdot \operatorname{card}(\mathcal{I}_k)$ subintervals of intervals in $\mathcal{I}_k$ obtained by applying Lemma 3.3.1 with $N = 2^{k+4}$ to each interval of $\mathcal{I}_k$. More precisely, if we let $\tilde{\mathcal{I}}_k$ be a collection of N disjoint subintervals of $[-\frac{1}{2}, \frac{1}{2}]$ each of size $\frac{N^{-(2m-1)}}{3m}$ given by Lemma 3.3.1 with $N = 2^{k+4}$, then for each $I \in \tilde{\mathcal{I}}_k$, let $\tilde{\mathcal{I}}_{k,I}$ be the rescaling of $\tilde{\mathcal{I}}_k$ to I, that is, if the endpoints of I are a and b with $a < b$, set $\tilde{\mathcal{I}}_{k,I} = a + (b-a)\tilde{\mathcal{I}}_k$. Then set

$$\mathcal{I}_{k+1} = \bigcup_{I \in \mathcal{I}_k} \tilde{\mathcal{I}}_{k,I}.$$

For each k, define S_k to be the set of endpoints of intervals in $\mathcal{I}_k$, and define Ω_k to be the convex polygon with vertices at

$$\{(x - \frac{1}{2}, x^2 - 8) : x \in S_k\} \cup \{(-8, 0); (-8, 8); (8, 0); (8, 8)\}.$$

Let Ω be the convex domain so that $\partial\Omega$ is the uniform limit of $\{\partial\Omega_k\}$ as $k \to \infty$. Note that Ω satisfies (3.4) with $M = 10$.

Lemma 3.3.2. *Let Ω be constructed as described previously. For every $\delta > 0$, there exist integer constants $C_1(\delta)$, $C_2(\delta)$ with $C_1(\delta) = O(\delta^{-\epsilon})$ and $C_2(\delta) = O(\delta^{-\epsilon})$ for every $\epsilon > 0$ so that if $\mathcal{J}_\delta$ denotes the collection of $Q(\Omega, \delta)$ essentially disjoint intervals obtained from the decomposition of $[-1, 1]$ as described in Section 3.2, then we can write*

$$\mathcal{J}_\delta = \bigcup_{l=1}^{C_1(\delta)} \mathcal{J}_{\delta,l}$$

such that for each l, no point of $\mathbb{R}$ is contained in more than $C_2(\delta)$ of the sets

$$\{I_1 + \cdots + I_m\}_{I_1,\ldots,I_m \in \mathcal{J}_{\delta,l}}.$$

In particular, this implies that $\mathcal{E}_m(\partial\Omega) = 0$.

Proof of Lemma 3.3.2. Given $\delta > 0$, let $K(\delta)$ be the largest integer such that each interval in $\mathcal{I}_{K(\delta)}$ has size $\geqslant \delta^{1/2}$. For each integer $k \geqslant 0$, let $\mathcal{I}'_k$ denote the set of essentially disjoint subintervals corresponding to the decomposition of $[-1/2, 1/2]$ given by the partition S_k of $[-1/2, 1/2]$. Then for every $\delta > 0$, each element of $\mathcal{J}_\delta$ intersects no more than 10 elements of $\mathcal{I}'_{K(\delta)}$, and each element of $\mathcal{I}'_{K(\delta)}$ intersects no more than 10 elements of $\mathcal{J}_\delta$. Moreover, all but at most 10 elements of $\mathcal{J}_\delta$ are covered by a union of elements of $\mathcal{I}'_{K(\delta)}$. It thus suffices to prove the lemma with $\mathcal{J}_\delta$ replaced by $\mathcal{I}'_{K(\delta)}$.

It is easy to compute that $K(\delta) \lesssim (\log(\delta^{-1}))^{1/2} = O(\delta^{-\epsilon})$ for every $\epsilon > 0$. We organize $\mathcal{I}'_{K(\delta)}$ into $K(\delta)+1$ disjoint subcollections as follows. Set $(\mathcal{I}'_{K(\delta)})_0 = \mathcal{I}_{K(\delta)}$. Set $(\mathcal{I}'_{K(\delta)})_1 = \mathcal{I}'_1 \setminus \mathcal{I}_1$ and for $1 < k \leqslant K(\delta)-1$ inductively define

$$(\mathcal{I}'_{K(\delta)})_{k+1} = \mathcal{I}'_k \setminus (\mathcal{I}_k \cup (\mathcal{I}'_{K(\delta)})_k).$$

Then

$$\mathcal{I}'_{K(\delta)} = \bigsqcup_{k=0}^{K(\delta)} (\mathcal{I}'_{K(\delta)})_k.$$

It is also easy to see that for $k > 1$, every element of $(\mathcal{I}'_{K(\delta)})_k$ is a subset of an element of $\mathcal{I}_{k-1}$. In fact, we can think of $(\mathcal{I}'_{K(\delta)})_k$ for $k > 0$ as the "gaps" leftover after subdividing $\mathcal{I}_{k-1}$.

We now show that for any $k \geqslant 0$, no point of $\mathbb{R}$ is contained in more than $(m!)^k$ of the sets $\{I_1 + \cdots + I_m\}_{I_1,\ldots,I_m \in \mathcal{I}_k}$. We prove this by induction on k. The base case is trivial. Suppose that this is true for a given k. Fix $x \in \mathbb{R}$, and suppose there are intervals $I_1, \ldots, I_m \in \mathcal{I}_{k+1}$ such that $x \in (I_1 + \cdots + I_m)$. Then there are intervals $I_{m+1}, \ldots, I_{2m} \in \mathcal{I}_k$ such that $I_1 \subset I_{m+1}$, $I_2 \subset I_{m+2}$, $\ldots$, $I_m \subset I_{2m}$. Let us count how many m-tuples $(I'_1, \ldots, I'_m)$ there are satisfying $x \in I'_1 + \cdots + I'_m$ and $I'_1 \subset I_{m+1}, I'_2 \subset I'_{m+1}, \ldots, I_m \subset I'_{2m}$. After applying an appropriate translation and dilation, this is the same as the number of ordered m-tuples of intervals whose sum contains a given point, where the intervals are restricted to a collection that satisfy the properties stated in Lemma 3.3.1 for some N. But for such a collection the number of ordered m-tuples is simply $m!$. By the inductive hypothesis, the number of choices of intervals $I_{m+1}, \ldots, I_{2m} \in \mathcal{I}_k$ is $\leqslant (m!)^k$, and therefore the number of choices of intervals $I_1, \ldots, I_m$ is $\leqslant (m!)^{k+1}$.

The above argument shows that no point of $\mathbb{R}$ is contained in more than $(m!)^{K(\delta)}$ of the sets $\{I_1 + \cdots + I_m\}_{I_1,\ldots,I_m \in (\mathcal{I}'_{K(\delta)})_0}$. Moreover, for every $0 \leqslant k \leqslant K(\delta)$ no point of

$\mathbb{R}$ is contained in more than $(m!)^{K(\delta)}$ of the sets $\{I_1+\cdots+I_m\}_{I_1,\ldots,I_m\in\mathcal{I}_k}$. Fix $k>0$, and also fix $x\in\mathbb{R}$. Given $I_1,\ldots,I_m\in\mathcal{I}_{k-1}$ with $x\in(I_1+\cdots+I_m)$, there are at most 2^{k+10} choices of intervals $I_{m+1},\ldots,I_{2m}\in(\mathcal{I}'_{K(\delta)})_k$ such that $I_1\subset I_{m+1}, I_2\subset I_{m+2},\ldots,I_m\subset I_{2m}$. It follows that x is contained in no more than $2^{K(\delta)+10}\cdot(m!)^{K(\delta)}$ of the sets $\{I_1+\cdots+I_m\}_{I_1,\ldots,I_m\in(\mathcal{I}'_{K(\delta)})_k}$.

As noted previously, $K(\delta)\lesssim(\log(\delta^{-1}))^{1/2}$, so $2^{K(\delta)+10}\cdot(m!)^{K(\delta)}=O(\delta^{-\epsilon})$ for every $\epsilon>0$. Thus we have proven the lemma with $C_1(\delta)=K(\delta)+1$ and $C_2(\delta)=2^{K(\delta)+10}\cdot 24^{K(\delta)}$. □

Lemma 3.3.3. *Let Ω be constructed as described previously. Then $\kappa_\Omega=\frac{1}{4m-2}$.*

Proof of Lemma 3.3.3. Let $K(\delta)$ be defined as in the proof of Lemma 3.3.2. $K(\delta)$ is the greatest integer such that

$$\prod_{n=1}^{K(\delta)}2^{-(2m-1)(n+4)}\geqslant\delta^{1/2}.$$

It follows that

$$\mathrm{card}(\mathcal{I}_{K(\delta)+1})=\prod_{n=1}^{K(\delta)+1}2^{(n+4)}\geqslant\delta^{-1/(4m-2)},$$

and hence

$$\delta^{-1/(4m-2)}2^{-K(\delta)-4}\leqslant\mathrm{card}(\mathcal{I}_{K(\delta)})\leqslant\delta^{-1/(4m-2)}.$$

As noted previously, $2^{-K(\delta)}=O(\delta^{-\epsilon})$ for every $\epsilon>0$, and hence by Lemma C,

$$\kappa_\Omega=\limsup_{\delta\to0}\frac{\log(N(\Omega,\delta))}{\log(\delta^{-1})}=\limsup_{\delta\to0}\frac{Q(\Omega,\delta)}{\log(\delta^{-1})}$$
$$=\limsup_{\delta\to0}\frac{\mathrm{card}(\mathcal{J}_\delta)}{\log(\delta^{-1})}=\limsup_{\delta\to0}\frac{\mathrm{card}(\mathcal{I}_{K(\delta)})}{\log(\delta^{-1})}=\frac{1}{4m-2}.$$

□

Remark 3.3.4. *Let $\kappa \in [0, \frac{1}{4m-2})$. We now describe how we may modify the construction of Ω so that it still satisfies the hypotheses of Lemma 3.3.2, but $\kappa_\Omega = \kappa$. Obviously, we may replace Lemma 3.3.1 with the weaker statement that there exists N^c (instead of N) disjoint subintervals satisfying the hypotheses of Lemma 3.3.1 with $0 \leqslant c < 1$. If we repeat the same construction of Ω described previously except applying this weaker version of Lemma 3.3.1 instead, we will produce a domain Ω with $\kappa_\Omega = \kappa$ if we choose c appropriately. Verification of the details is left to the reader.*

3.4 Proof of Theorem 3.1.4

To prove Theorem 3.1.4 in the case that $\lambda > 0$, it only remains to prove the following proposition.

Proposition 3.4.1. *Let Ω be a convex domain in $\mathbb{R}^2$ containing the origin and let Θ be its associated set of directions. Let $n \geqslant 2$ be an integer. Suppose that $\mathcal{E}_n(\partial\Omega) = \alpha$ for some integer $0 \leqslant \alpha \leqslant n\kappa_\Omega$ and that $\|M_{\Theta,\delta}\|_{L^{\frac{n}{n-1}}(\mathbb{R}^2)\to L^{\frac{n}{n-1}}(\mathbb{R}^2)} \leqslant C_\epsilon \delta^{-\beta-\epsilon}$ for some $0 \leqslant \beta \leqslant \kappa_\Omega(\frac{n-2}{n})$ and every $\epsilon > 0$. Then if $m_{\delta,\lambda}$ is as in the statement of Proposition E, there is a constant $C(\delta) = O(\delta^{-\epsilon})$ for every $\epsilon > 0$ such that*

$$\left\|\mathcal{F}^{-1}[m_{\delta,\lambda}\widehat{f}]\right\|_{L^{\frac{2n}{2n-1}}(\mathbb{R}^2)} \lesssim \delta^\lambda C(\delta)\delta^{-\frac{\alpha}{2n}-\frac{\beta}{2}} \|f\|_{L^{\frac{2n}{2n-1}}(\mathbb{R}^2)}. \tag{3.16}$$

Interpolating Proposition 3.4.1 with Proposition E gives the result of Theorem 3.1.4 for $\lambda > 0$.

Proof of Proposition 3.4.1. By duality, to prove (3.16) it suffices to prove

$$\left\|\mathcal{F}^{-1}[m_{\delta,\lambda}\widehat{f}]\right\|_{L^{2n}(\mathbb{R}^2)} \lesssim \delta^\lambda C(\delta)\delta^{-\frac{\alpha}{2n}-\frac{\beta}{2}} \|f\|_{L^{2n}(\mathbb{R}^2)}. \tag{3.17}$$

Using an appropriate partition of unity and rotation invariance, it in fact suffices to show that if $b \in C_0^\infty$ is as in the statement of Proposition F, then

$$\left\| \mathcal{F}^{-1}[b \cdot m_{\delta,\lambda} \widehat{f}] \right\|_{L^{2n}(\mathbb{R}^2)} \lesssim \delta^\lambda C(\delta) \delta^{-\frac{\alpha}{2n} - \frac{\beta}{2}} \|f\|_{L^{2n}(\mathbb{R}^2)} . \tag{3.18}$$

Let $\mathcal{J}_\delta$ denote the collection of $Q(\Omega, \delta)$ essentially disjoint intervals obtained from the decomposition of $[-1, 1]$ as described in Section 3.2. For each $I = (\alpha_0, \alpha_1) \in \mathcal{J}_\delta$, set $B(I)$ to be a rectangle that has one side parallel to $(1, \gamma'(\alpha_0))$, contains $\mathrm{supp}(b \cdot m_{\delta,\lambda}) \cap \{x : x_1 \in I\}$, and such that its 1/2-dilate is contained in $\mathrm{supp}(b \cdot m_{\delta,\lambda}) \cap \{x : x_1 \in I\}$. Since $\mathcal{E}_n(\partial\Omega) = \alpha$, there are constants $C_1(\delta)$ and $C_2(\delta)$ such that $C_1(\delta)^{2n} C_2(\delta) = O(\delta^{-\alpha-\epsilon})$ for every $\epsilon > 0$, and such that we may write $J_\delta = \bigcup_{l=1}^{C(\delta)} \mathcal{J}_{\delta,l}$ so that for each l, no point of $\mathbb{R}^2$ is contained in more than $C_2(\delta)$ of the sets

$$\{B(I_1) + \cdots + B(I_n)\}_{I_j \in \mathcal{J}_{\delta,l}}.$$

Now let $\mathfrak{J}_\delta$ be defined as in the statement of Proposition F, and let $\{\beta_i\}$ be a partition of unity of $[-\frac{1}{4}, \frac{1}{4}]$ satisfying

1. $\sum_i \beta_i$ is supported in $(-\frac{1}{2}, \frac{1}{2})$,
2. Every β_i is of the form $\beta(|I|^{-1}(\cdot - c_I))$ for some $\beta \in \mathfrak{B}$ and for some interval $I \in \mathfrak{J}_\delta$ with center c_I,
3. Each interval in $\mathcal{J}_\delta$ intersects the support of at most $(\log(\delta^{-1}))^2$ of the β_i's,
4. If the support of β_i intersects some $I \in \mathcal{J}_\delta$ then the support of β_i is contained in $10I$, where the dilation is taken from the center of I.

Set $m_i(\xi) = \beta_i(\xi_1) b(\xi) m_{\delta,\lambda}(\xi)$, and define an operator T_i by

$$T_i f(x) = \delta^{-\lambda} \mathcal{F}^{-1}[m_i \widehat{f}](x).$$

Set

$$\mathfrak{I}_1 = \{i : \text{supp}(\beta_i) \cap (\cup \mathcal{J}_{\delta,1}) \neq \emptyset\},$$

and for $l = 2, \ldots, C_1(\delta)$, set

$$\mathfrak{I}_l = \{i : \text{ supp}(\beta_i) \cap (\cup \mathcal{J}_{\delta,l-1}) = \emptyset \text{ and } \text{supp}(\beta_i) \cap (\cup \mathcal{J}_{\delta,l}) \neq \emptyset\}.$$

We write

$$\sum_i T_i f(x) = \sum_{l=1}^{C_1(\delta)} \sum_{i \in \mathfrak{I}_l} T_i f(x).$$

We now proceed with an argument similar to the familiar one from [18]. Using the triangle inequality, Hölder's inequality and Plancherel, we have

$$\begin{aligned}
\left\| \sum_i T_i f \right\|_{2n}^{2n} \lesssim \Big(\sum_{l=1}^{C_1(\delta)} \left\| \sum_{i \in \mathfrak{I}_l} T_i f \right\|_{2n} \Big)^{2n} \lesssim C_1(\delta)^{2n-1} \sum_{l=1}^{C_1(\delta)} \left\| \sum_{i \in \mathfrak{I}_l} T_i f \right\|_{2n}^{2n} \\
\lesssim C_1^{2n-1}(\delta) \sum_{l=1}^{C_1(\delta)} \int_{\mathbb{R}^2} \left| \sum_{i \in \mathfrak{I}_l} T_i f(x) \right|^{2n} dx \\
\lesssim C_1(\delta)^{2n-1} \sum_{l=1}^{C_1(\delta)} \int \left| \sum_{i_1,\ldots,i_n \in \mathfrak{I}_l} T_{i_1} f(x) T_{i_2} f(x) \cdots T_{i_n} f(x) \right|^2 dx \\
\lesssim C_1(\delta)^{2n-1} \sum_{l=1}^{C_1(\delta)} \int \left| \sum_{i_1,\ldots,i_n \in \mathfrak{I}_l} \widehat{T_{i_1} f} * \widehat{T_{i_2} f} * \cdots * \widehat{T_{i_n} f}(\xi) \right|^2 d\xi. \quad (3.19)
\end{aligned}$$

Now note that no point of $\mathbb{R}^2$ is contained in more than $C_2(\delta)$ of the sets

$$\Big\{ \text{supp}(\widehat{T_{i_1} f} * \widehat{T_{i_2} f} * \cdots * \widehat{T_{i_n} f}(\xi)) \Big\}_{i_1,\ldots,i_n \in \mathfrak{I}_l}.$$

Set $C_3(\delta) = C_1(\delta)^{2n-1} C_2(\delta) (\log(\delta^{-1}))^3$. It follows that the right hand side of (3.19) is bounded by a constant times

$$C_3(\delta) \sum_{l=1}^{C_1(\delta)} \int \sum_{i_1,\ldots,i_n \in \mathfrak{I}_l} |\widehat{T_{i_1} f} * \widehat{T_{i_2} f} * \cdots * \widehat{T_{i_n} f}(\xi)|^2 \, d\xi, \quad (3.20)$$

and by Plancherel, (3.20) is equal to

$$\begin{aligned} C_3(\delta) \sum_{l=1}^{C_1(\delta)} \int \sum_{i_1,\dots,i_n \in \mathfrak{I}_l} &|T_{i_1} f(x) T_{i_2} f(x) \cdots T_{i_n} f(x)|^2 \, dx \\ &\lesssim C_3(\delta) \sum_{l=1}^{C_1(\delta)} \int \sum_{i_1,\dots,i_n \in \mathfrak{I}_l} |T_{i_1} f(x) T_{i_2} f(x) \cdots T_{i_n} f(x)|^2 \, dx \\ &\lesssim C_3(\delta) \sum_{l=1}^{C_1(\delta)} \int \Big(\sum_{i \in \mathfrak{I}_l} |T_i f(x)|^2 \Big)^n \, dx. \end{aligned} \tag{3.21}$$

Let $\phi : [-2, 2] \to \mathbb{R}$ be a smooth function identically 1 on $[-1, 1]$. For each i, write $\beta_i = \beta(|I|^{-1}(\cdot - c_I))$ for some $\beta \in \mathfrak{B}$ and set $\psi_i(\xi) = \phi(|I|^{-1}(\xi_1 - c_I))$. Define a multiplier operator S_i by

$$S_i f = \mathcal{F}^{-1}[\psi_i \widehat{f}].$$

If K_i denotes the convolution kernel of the operator T_i, let $\tilde{T}_i$ be the operator with convolution kernel $|K_i|$. By duality, the right hand side of (3.21) is bounded by

$$\begin{aligned} C_3(\delta) \sum_{l=1}^{C_1(\delta)} &\Big(\sup_{\|w\|_{\frac{n}{n-1}} \leqslant 1} \int \sum_{i \in \mathfrak{I}_l} |T_i f(x)|^2 w(x) \, dx \Big)^n \\ &\lesssim C_3(\delta) \sum_{l=1}^{C_1(\delta)} \Big(\sup_{\|w\|_{\frac{n}{n-1}} \leqslant 1} \int \sum_{i \in \mathfrak{I}_l} |S_i f(x)|^2 (\sup_i |\tilde{T}_i w(x)|) \, dx \Big)^n \\ &\lesssim C_3(\delta) \sum_{l=1}^{C_1(\delta)} \Big\| \Big(\sum_{i \in \mathfrak{I}_l} |S_i f(x)|^2 \Big)^{1/2} \Big\|_{2n}^{2n} \sup_{\|w\|_{\frac{n}{n-1}} \leqslant 1} \Big\| \sup_i |\tilde{T}_i w| \Big\|_{\frac{n}{n-1}}^n . \end{aligned} \tag{3.22}$$

By (3.14) and the assumption $\left\| M_{\Theta(\Omega),\delta} \right\|_{L^{\frac{n}{n-1}}(\mathbb{R}^d) \to L^{\frac{n}{n-1}}(\mathbb{R}^d)} = O_\epsilon(\delta^{-\beta-\epsilon})$, we have

$$\Big\| \sup_i |\tilde{T}_i f| \Big\|_{\frac{n}{n-1}} \lesssim C(\delta) \delta^{-\beta} \|f\|_{\frac{n}{n-1}} \tag{3.23}$$

where $C(\delta) = O(\delta^{-\epsilon})$ for every $\epsilon > 0$. Moreover, since the supports of the ψ_i are $\lesssim \log(\delta^{-1})$-disjoint, by Rubio de Francia's theorem on square functions for arbitrary collections of intervals [45], we have

$$\left\| \left(\sum_{i \in \mathfrak{I}_l} |S_i f(x)|^2 \right)^{1/2} \right\|_{2n} \lesssim \log(\delta^{-1}) \, \|f\|_{2n} . \tag{3.24}$$

Set $C_4(\delta) = C(\delta)^{2n} C_1(\delta) C_3(\delta) \log(\delta^{-1})^{4n} \delta^{-\beta n}$. By (3.22), (3.23) and (3.24), we have

$$\left\| \sum_i T_i f \right\|_{2n} \lesssim_\epsilon \delta^{-\epsilon} (C_4(\delta))^{1/2n} \, \|f\|_{2n} , \tag{3.25}$$

and since $C_4(\delta) \lesssim_\epsilon C_1(\delta)^{2n} C_2(\delta) \delta^{-\beta n - \epsilon} \lesssim_\epsilon \delta^{-\alpha - \beta n - \epsilon}$ for every $\epsilon > 0$, this proves (3.18) and thus completes the proof of Proposition 3.4.1. □

It only remains to prove Theorem 3.1.1 in the case that $\lambda = 0$. This will follow fairly easily from Bateman's characterization in [2] of all planar sets of directions which admit Kakeya sets and Fefferman's proof in [21] that the ball multiplier is unbounded on $L^p(\mathbb{R}^2)$ for $p \neq 2$.

Proof of Theorem 3.1.1 in the case that $\lambda = 0$. Let Θ denote the set of all directions associated to Ω. We claim that if Θ is a union of finitely many lacunary sets of finite order, then $\kappa_\Omega = 0$. Indeed, suppose that Θ is a union of N_1 lacunary sets of order N_2. Then it is easy to see that there is a subset of $\Theta_\delta \subset \Theta$ of cardinality $\leqslant N_1 (\log(\delta^{-1}))^{N_2}$ such that every element of Θ is contained in a δ neighborhood of an element of Θ_δ. It follows that $N(\Omega, \delta) \lesssim N_1 (\log(\delta^{-1}))^{N_2}$, and hence $\kappa_\Omega = 0$.

We say that Θ *admits Kakeya sets* if for each postive integer N there is a collection $\mathcal{R}_\Theta^{(N)}$ of rectangles with longest side parallel to a direction in Θ so that

$$\left| \bigcup_{R \in \mathcal{R}_\Theta^{(N)}} R \right| \leqslant \frac{1}{N} \left| \bigcup_{R \in \mathcal{R}_\Theta^{(N)}} \tilde{R} \right| ,$$

where $\tilde{R}$ denotes the rectangle with the same center and width as R but with three times the length. In [2], the following theorem was proved.

Theorem G (Bateman, [2]). *Fix $1 < p < \infty$. The following are equivalent:*

1. *M_Θ is bounded on $L^p(\mathbb{R}^2)$;*

2. *Θ does not admit Kakeya sets;*

3. *there exist $N_1, N_2 < \infty$ such that Θ is covered by N_1 lacunary sets of order N_2.*

It follows from Theorem G that if $\kappa_\Omega > 0$, then Θ admits Kakeya sets. We will now show that if $\kappa_\Omega > 0$, then T_0 is unbounded on L^p for all $p \neq 2$. Assume that T_0 is bounded on L^p for some $p > 2$. Let $\{v_j\}$ be a sequence of unit vectors parallel to directions in Θ, and let H_j denote the half-plane $\{x \in \mathbb{R}^2 : x \cdot v_j \geqslant 0\}$. For each j, define an operator T_j by

$$\mathcal{F}[T_j f](\xi) = \chi_{H_j}(\xi)\widehat{f}(\xi).$$

Then arguing as in [21], there is an absolute constant C (independent of the choice of the sequence $\{v_j\}$) such that

$$\left\| (\sum_j |T_j f_j|^2)^{1/2} \right\|_p \leqslant C \left\| (\sum_j |f_j|^2)^{1/2} \right\|_p .$$

Since Θ admits Kakeya sets, for each $\upsilon > 0$ we may choose a sequence of unit vectors $\{v_j\}$ parallel to directions in Θ such that there is a collection of rectangles $\{R_j\}$ with the longest side of R_j parallel to v_j and so that

$$\left| \bigcup_j R_j \right| \leqslant \upsilon \left| \bigcup_j \tilde{R}_j \right|.$$

Let $E := \bigcup_j R_j$ and let $E' := \bigcup_j \tilde{R}_j$. Then arguing as in [22], we have

$$\int_E \sum_j |T_j \chi_{R_j}(x)|^2 \, dx \gtrsim \sum_j |E \cap \tilde{R}_j| \gtrsim |E'|,$$

but by Hölder's inequality

$$\int_E \sum_j |T_j \chi_{R_j}(x)|^2 \, dx \lesssim |E|^{(p-2)/p} (\sum_j |R_j|)^{2/p} \lesssim \upsilon^{(p-2)/p} |E'|.$$

Letting $\upsilon \to 0$ gives a contradiction. □

3.5 Lower bounds using Khinchine's inequality

In this section, we will prove Theorem 3.1.5, which gives lower bounds on the range of λ for which T_λ is bounded on L^p for general convex domains with a given value of κ_Ω. To prove Theorem 3.1.5, we first show that boundedness of T_λ on L^p implies (3.27), where T_λ^δ is defined below. We then test T_λ^δ on randomly defined functions and apply Khinchine's inequality to estimate the L^1 norm of these functions. After applying T_λ^δ, the randomness of these test functions will effectively "disappear" due to the test functions being essentially constant on a sequence of disjoint caps in $\mathcal{B}_\delta$. The L^2 mapping properties of T^λ acting on these functions will be easy to quantify using Plancherel. The last step is simply to interpolate between L^1 and L^2.

Proof of Theorem 3.1.5. Suppose that T_λ is bounded on L^p. Let $\phi \in C_0^\infty(\mathbb{R})$ be supported in $[-2, 2]$ and identically 1 on $[-1, 1]$. Let

$$m_\delta(s) = \phi(\delta^{-1}(1 - s))$$

and let T_λ^δ be the operator defined by

$$\mathcal{F}[T_\lambda^\delta f](\xi) = m_\delta(\rho(\xi)) \widehat{f}(\xi).$$

We will use the well-known subordination formula

$$m(\rho) = \frac{(-1)^{\lfloor\lambda\rfloor+1}}{\Gamma(\lambda+1)} \int_0^\infty s^\lambda m^{(\lambda+1)}(s)(1-\frac{\rho}{s})_+^\lambda \, ds, \tag{3.26}$$

where

$$\widehat{m^{(\gamma)}}(\tau) = (-1)^{\lfloor\gamma\rfloor}(-i\tau)^\gamma \widehat{m}(\tau).$$

See [56] for a proof of (3.26). Together, (3.26) and the L^p-boundedness of T_λ imply that

$$\left\|T_\lambda^\delta\right\|_{L^p\to L^p} \lesssim \delta^{-\lambda}. \tag{3.27}$$

Let $\mathcal{J}_\delta$ denote the collection of $Q(\Omega,\delta) \lesssim \log(2+\delta^{-1})N(\Omega,\delta)$ essentially disjoint intervals obtained from the decomposition of $[-1,1]$ into intervals with endpoints in $\mathfrak{A}(\delta) = \{a_0,\ldots,a_Q\}$ as described in Section 3.2. By rotation invariance, we may assume without loss of generality that $Q(\Omega,\delta) \gtrsim N(\Omega,\delta)$. For each $0 \leqslant j \leqslant Q-1$, let $c_j = \frac{a_j+a_{j+1}}{2}$. Now observe that for δ sufficiently small there must be $\gtrsim N(\Omega,\delta)$ indices j such that $a_{j+1} - a_j \leqslant N(\Omega,\delta)^{-1}\log(\delta^{-1})$. Thus by the pigeonhole principle there is an integer $r \geqslant \lfloor\log(N(\Omega,\delta)\log(\delta^{-1})^{-1})\rfloor$ such that there are $\gtrsim N(\Omega,\delta)(\log(\delta^{-1}))^{-1}$ indices j such that $2^{-r-1} \leqslant a_{j+1} - a_j \leqslant 2^{-r}$. Enumerate these indices as $j_1 < j_2 < \cdots < j_{Q'}$.

Let $\chi_0 \in C_0^\infty(\mathbb{R})$ with $\chi \geqslant 0$, $\chi \equiv 1$ on $[-1,1]$ and χ supported in $[-2,2]$. Set $\chi(\xi_1,\xi_2) = \chi_0(\xi_1)\chi_0(\xi_2)$. Then $|\mathcal{F}[\chi](x)| \lesssim (1+|x|)^{-2}$ and $|\mathcal{F}[\chi](x)| \geqslant 1/2$ for $x \in B_{\frac{1}{100}}(0)$. Let $\{\epsilon_i\}$ be i.i.d. random variables with $P(\epsilon_i = \pm 1) = \frac{1}{2}$ for every i. Let

$$\psi_\delta(x) = \mathcal{F}[\sum_{i\equiv 0 \mod (\lfloor\log(\delta^{-1})\rfloor)} \epsilon_i \chi(2^r(\cdot - (c_{j_i},\gamma(c_{j_i}))))](x).$$

By Plancherel,

$$\|\psi_\delta\|_2 \lesssim \left(N(\Omega,\delta)2^{-2r}\right)^{1/2} \tag{3.28}$$

and

$$\left\|T_\lambda^\delta \psi_\delta\right\|_2 \gtrsim \left(N(\Omega,\delta)(\log(\delta^{-1}))^{-1}2^{-r}\delta\right)^{1/2}. \tag{3.29}$$

By Khinchine's inequality,

$$\mathbb{E}[\|\psi_\delta\|_1] \approx Q'^{\frac{1}{2}} \lesssim \left(\log(2+\delta^{-1})N(\Omega,\delta)\right)^{1/2}. \tag{3.30}$$

Interpolating (3.28) and (3.30) yields

$$\|\psi_\delta\|_p \lesssim \log(\delta^{-1})^{\frac{2-p}{2p}} N(\Omega,\delta)^{1/2} 2^{-r(\frac{2p-2}{p})}, \qquad 1 \leqslant p \leqslant 2. \tag{3.31}$$

We now prove a lower bound for $\left\|T_\lambda^\delta \psi_\delta\right\|_1$ uniformly in the realization of the random variables $\{\epsilon_i\}$. Using homogeneous coordinates, i.e. polar coordinates associated to Ω, we write

$$T_\lambda^\delta \psi_\delta(x) = \frac{1}{(2\pi)^2} \sum_{i \equiv 0 \mod (\lfloor \log(\delta^{-1}) \rfloor)} \epsilon_i \int\int \phi(\delta^{-1}(1-s))\chi_0(2^r(s\alpha - c_{j_i}))$$
$$\times\, s(\alpha\gamma'(\alpha) - \gamma(\alpha)) e^{is(x_1\alpha + x_2\gamma(\alpha))}\, d\alpha\, ds.$$

Now note that for each i and for α in the support of $\chi_0(2^r(s\alpha - c_{j_i}))$ we have

$$e^{is(x_1\alpha+x_2\gamma(\alpha))} =$$
$$\exp\left(is(x_1 c_{j_i} + x_2\gamma(c_{j_i})) + is(\alpha - c_{j_i})(x_1 + x_2\gamma'(c_{j_i}))\right) + O(\delta|x|)$$

and

$$|\alpha\gamma'(\alpha) - \gamma(\alpha) - c_{j_i}\gamma'(c_{j_i}) + \gamma(c_{j_i})| = O(2^{-r}).$$

It follows that

$$T_\lambda^\delta \psi_\delta(x) = \frac{1}{(2\pi)^2} \sum_{i\equiv 0 \mod (\lfloor \log(\delta^{-1})\rfloor)} \epsilon_i \int\int \phi(\delta^{-1}(1-s))\chi_0(2^r(s\alpha - c_{j_i}))$$
$$\times\, s(c_{j_i}\gamma'(c_{j_i}) - \gamma(c_{j_i}))e^{is(x_1 c_{j_i} + x_2\gamma(c_{j_i})) + is(\alpha - c_{j_i})(x_1 + x_2\gamma'(c_{j_i}))}\, d\alpha\, ds$$
$$+ O(2^{-2r}\delta) + O(2^{-r}\delta^2|x|).$$

Rearranging this, we have

$$T_\lambda^\delta \psi_\delta(x) = \frac{1}{(2\pi)^2} \sum_{i\equiv 0 \mod (\lfloor \log(\delta^{-1})\rfloor)} \epsilon_i \int s\phi(\delta^{-1}(1-s))\bigg(\int \chi_0(2^r(s\alpha - c_{j_i}))$$
$$\times\, (c_{j_i}\gamma'(c_{j_i}) - \gamma(c_{j_i}))e^{is\alpha(x_1 + x_2\gamma'(c_{j_i}))}\, d\alpha\bigg)$$
$$\times\, e^{is(x_1 c_{j_i} + x_2\gamma(c_{j_i}) - c_{j_i}(x_1 + x_2\gamma'(c_{j_i})))}\, d\alpha\, ds + O(2^{-2r}\delta) + O(2^{-r}\delta^2|x|).$$

Set $\beta_i = c_{j_i}\gamma'(c_{j_i}) - \gamma(c_{j_i})$. Note that $\beta_i \approx 1$ for all i. We may rewrite this as

$$T_\lambda^\delta \psi_\delta(x) = \sum_{i\equiv 0 \mod (\lfloor \log(\delta^{-1})\rfloor)} \epsilon_i\beta_i 2^{-r}\widehat{\chi_0}(-2^{-r}(x_1 + x_2\gamma'(c_{j_i})))$$
$$\times\, \delta\cdot\widehat{\phi}(\delta(x_1 c_{j_i} + x_2\gamma(c_{j_i})))e^{i(x_1 c_{j_i} + x_2\gamma(c_{j_i}))} + O(2^{-2r}\delta) + O(2^{-r}\delta^2|x|).$$

It follows that there is a constant $C > 0$ (independent of δ) such that for each i in the sum,

$$|T_\lambda^\delta \psi_\delta(x)| \geqslant 2^{-r-10}\delta$$

whenever

$$|x\cdot(c_{j_i}, \gamma(c_{j_i}))| \leqslant C\delta^{-1},\ |x\cdot(1, \gamma'(c_{j_i}))| \leqslant C2^r.$$

It follows that

$$\left\|T_\lambda^\delta \psi_\delta\right\|_1 \gtrsim Q' = \log(\delta^{-1})^{-1} N(\Omega, \delta) \tag{3.32}$$

for $\delta > 0$ sufficiently small. Interpolating (3.29) and (3.32) gives that

$$\left\|T_\lambda^\delta \psi_\delta\right\|_p \gtrsim (\log(\delta^{-1}))^\beta N(\Omega,\delta)^{\frac{1}{p}} (2^{-r}\delta)^{\frac{p-1}{p}}, \qquad 1 \leqslant p \leqslant 2 \tag{3.33}$$

for some $\beta \in \mathbb{R}$. Together (3.31) and (3.33) imply that

$$\left\|T_\lambda^\delta\right\|_{L^p \to L^p} \gtrsim (\log(\delta^{-1}))^{\beta'} N(\Omega,\delta)^{\frac{1}{2}} \delta^{\frac{p-1}{p}} \tag{3.34}$$

for some $\beta' \in \mathbb{R}$. By (3.27), it follows that $\lambda \geqslant 1 - \frac{\kappa_\Omega}{2} - \frac{1}{p}$.

□

3.6 Concluding remarks

There are many further questions that arise naturally from the results of this chapter; we now discuss a few of them. As previously mentioned, Theorem 3.1.1 demonstrates that how "curved" the boundary of a convex planar domain is, as measured by the parameter κ_Ω, does not alone determine the L^p mapping properties of the associated Bochner-Riesz operators, but rather there must be other properties of Ω that play a role. We have seen that domains that satisfy $\mathcal{E}_n(\partial\Omega) = 0$ for some $n > 2$ can be shown to satisfy L^p mapping properties better than those proved in [48]. It would be very interesting to construct domains for which $\mathcal{E}_n(\partial\Omega) = 0$ for some $n > 2$ as well as having an associated set of directions which is q-sparse for $q = \frac{n}{n-1}$; for such domains Theorem 3.1.4 would imply that $p_{\text{crit}} < 4/3$. As a simpler preliminary question, it would be already very interesting to construct non-lacunary sets of directions that are q-sparse for some $q < 2$.

Another question one might also is if for *any* $\kappa \in (0, 1/2)$ (not just for κ sufficiently small) we can construct domains for which $p_{\text{crit}} < 4/3$. At the very least, we believe that the upper bound on κ_Ω in Theorem 3.1.1 could be significantly improved with more sophisticated algebraic disjointness constructions than the one used in the proof of Lemma 3.3.1. In particular, the domains constructed to prove Theorem 3.1.1 only exploited algebraic disjointness in one dimension, and it is quite likely that a two-dimensional approach will yield much better results. Finally, it would be interesting if one could determine whether one may prove improved L^p bounds for other certain specific examples of convex domains, such as those with associated directions lying in a standard Cantor set.

Chapter 4

Quasiradial Multiplier Thorems

4.1 Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded, open convex set such that $0 \in \Omega$, and let ρ be its Minkowski functional, given by

$$\rho(\xi) = \inf\{t > 0 |\, t^{-1}\xi \in \Omega\}.$$

Since Ω is convex, $\rho : \mathbb{R}^2 \to \mathbb{R}^+ \cup \{0\}$ is the unique function that is homogeneous of degree one and identically 1 on $\partial\Omega$. We are interested in multipliers of the form $m \circ \rho$, where $m : \mathbb{R} \to \mathbb{C}$ is a bounded, measurable function. We refer to this class of multipliers as *quasiradial multipliers*. The class of quasiradial multipliers generalizes radial multipliers on $\mathbb{R}^2$, which would correspond to the special case that Ω is the unit disc and $\rho(\xi) = |\xi|$.

As a model case for quasiradial multipliers, one can study the generalized Bochner-Riesz multipliers $(1 - \rho(\xi))_+^\lambda$ for $\lambda > 0$. We define the generalized Bochner-Riesz operators T_λ for $\lambda > 0$ by

$$\mathcal{F}[T_\lambda f](\xi) = (1 - \rho(\xi))_+^\lambda \widehat{f}(\xi).$$

These operators were introduced in Chapter 3, and we now review some essential background information, most of which was already discussed in Chapter 3.

When $\partial\Omega$ is smooth, the problem of $L^p(\mathbb{R}^2)$ boundedness of the generalized Bochner-Riesz operators is well understood. The problem was first completely solved in the special case that Ω is the unit disk by Fefferman in [22] and later clarified by Córdoba in [18], where it was proven that T_λ is bounded on $L^p(\mathbb{R}^2)$ if and only if $\lambda > \lambda_0(p) := |\frac{2}{p} - 1| - \frac{1}{2}$. This result was then generalized to domains with smooth boundary by Sjölin in [50] and Hörmander in [30].

However, for certain convex domains with rough boundary, the critical index $\lambda_0(p)$ can be improved. In [43], Podkorytov considered Bochner-Riesz means associated to polyhedra in $\mathbb{R}^d$ and showed that if ρ is the Minkowski functional of a polyhedron, then $\mathcal{F}^{-1}[(1 - \rho(\cdot))_+^\lambda] \in L^1$ for $\lambda > 0$. In [48], Seeger and Ziesler considered Bochner-Riesz means associated to general convex domains in $\mathbb{R}^2$. They obtained a result involving a parameter similar to the Minkowski dimension of $\partial\Omega$, defined by a family of "balls", or caps, and we state the definition below.

For any $p \in \partial\Omega$, we say that a line ℓ, is a *supporting line for* Ω *at* p if ℓ contains p and Ω is contained in the half plane containing the origin with boundary ℓ. Let $\mathcal{T}(\Omega, p)$ denote the set of supporting lines for Ω at p. Note that if $\partial\Omega$ is C^1, then $\mathcal{T}(\Omega, p)$ has exactly one element, the tangent line to $\partial\Omega$ at p. For any $p \in \partial\Omega$, $\ell \in \mathcal{T}(\Omega, p)$, and $\delta > 0$, define

$$B(p, \ell, \delta) = \{x \in \partial\Omega : \operatorname{dist}(x, \ell) < \delta\}. \tag{4.1}$$

Let

$$\mathcal{B}_\delta = \{B(p, \ell, \delta) : p \in \partial\Omega, \ell \in \mathcal{T}(\Omega, p)\}, \tag{4.2}$$

and let $N(\Omega, \delta)$ be the minimum number of balls $B \in \mathcal{B}_\delta$ needed to cover $\partial\Omega$. Let

$$\kappa_\Omega = \limsup_{\delta \to 0} \frac{\log N(\Omega, \delta)}{\log \delta^{-1}}. \tag{4.3}$$

The parameter κ_Ω defined in (4.3) is similar to the upper Minkowski dimension of $\partial\Omega$. It is easy to show that for any convex domain Ω, $0 \leqslant \kappa_\Omega \leqslant 1/2$ (see [48] for details). We now mention a few examples of convex domains with particular values of κ_Ω. Clearly, if Ω is a polygon, then $\kappa_\Omega = 0$. For domains with smooth boundary, $\kappa_\Omega = 1/2$. This can be seen by noting that there is a point where $\partial\Omega$ has nonvanishing curvature, and near this point the contribution to $N(\Omega, \delta)$ is $\approx \delta^{-1/2}$. One may obtain domains with intermediate values of κ_Ω by considering Lebesgue functions associated to Cantor sets with appropriate ratios of dissection. For example, let $g : [0,1] \to [0,1]$ be the Lebesgue function associated to the standard middle-thirds Cantor set, commonly referred to as the Cantor function. Define $\gamma : [0,1] \to [-1,-1/2]$ by

$$\gamma(t) = \int_0^t g(s)\, ds - 1.$$

Let Ω be the convex domain bounded by the graph of γ and the line segments connecting consecutive vertices in the set

$$\{(1,-1/2); (1,1); (-1,1); (-1,-1); (0,-1)\}.$$

Then $\kappa_\Omega = \frac{\log_3(2)}{(\log_3(2)+1)}$. One may similarly obtain a convex domain Ω with $\kappa_\Omega = \kappa$ for any $\kappa \in (0, 1/2)$ by a similar construction using a Lebesgue function corresponding to a Cantor set of an appropriate ratio of dissection.

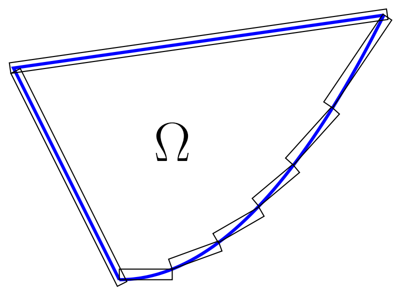

Figure 3

As an example, here Ω is a region bounded by two lines and a portion of a parabola. If we assume all rectangles have shorter sidelength equal to δ, then $N(\Omega, \delta) \leqslant 8$. Since a portion of $\partial\Omega$ is smooth with nonvanishing curvature, we have $\kappa_\Omega = 1/2$.

It was shown in [48] that T_λ is bounded on $L^p(\mathbb{R}^2)$ if $\lambda > \kappa_\Omega(|\frac{4}{p} - 2| - 1)$. In this chapter we would like to consider more general multiplier transformations. The following subordination formula from [56]

$$m(\rho(\xi)) = \frac{(-1)^{\lfloor\lambda\rfloor+1}}{\Gamma(\lambda+1)} \int_0^\infty s^\lambda m^{(\lambda+1)}(s)(1 - \frac{\rho(\xi)}{s})_+^\lambda \, ds \tag{4.4}$$

combined with the result from [48] mentioned previously immediately gives that $m \circ \rho \in M^p(\mathbb{R}^2)$ if for some $\lambda > \kappa_\Omega(|\frac{4}{p} - 2| - 1)$,

$$\int_0^\infty s^\lambda |m^{(\lambda+1)}(s)| \, ds < \infty.$$

However, this is not satisfactory as can be seen by analyzing the "localized wave multiplier" $e^{i\rho(\xi)}$. Sharp L^p estimates for this multiplier in the smooth case can be found in [4], [36], [41] and [47]. For general convex domains in $\mathbb{R}^2$, we prove the theorem below. First we make a few brief remarks regarding normalization of the domain Ω. Let Ω be a bounded, open convex set containing the origin, as above. Then Ω contains some ball centered at the origin and is also contained in some larger ball centered at the origin.

Since all results in this chapter regarding L^p boundedness of multipliers will be dilation invariant, we will assume without loss of generality that Ω contains the ball of radius 8 centered at the origin. Let $M > 0$ be an integer such that

$$\{\xi : |\xi| \leqslant 8\} \subset \Omega \subset \overline{\Omega} \subset \{\xi : |\xi| < 2^M\}. \tag{4.5}$$

We will prove

Theorem 4.1.1. *Let Ω be a convex domain satisfying (4.5) and ρ its Minkowski functional. Let $a : \mathbb{R} \to \mathbb{C}$ be a smooth function supported outside $[-2^{-2M}, 2^{-2M}]$ such that a is a symbol of order $-\kappa_\Omega - \epsilon$ for some $\epsilon > 0$, that is, for every integer $\beta \geqslant 0$,*

$$|D^\beta a(\xi)| \lesssim_\beta (1 + |\xi|)^{-\kappa_\Omega - \epsilon - \beta}.$$

Then

$$\mathcal{F}^{-1}[a(\rho(\cdot))e^{i\rho(\cdot)}] \in L^1(\mathbb{R}^2),$$

where $\left\|\mathcal{F}^{-1}[a(\rho(\cdot))e^{i\rho(\cdot)}]\right\|_{L^1(\mathbb{R}^2)}$ depends only on M, ϵ, and the quantitative estimates for a as a symbol of order $-\kappa_\Omega - \epsilon$.

The Fourier inversion formula

$$m(\rho(\xi)) = \frac{1}{2\pi} \int \widehat{m}(\tau) e^{i\tau\rho(\xi)}\, d\tau, \tag{4.6}$$

which is a more efficient subordination formula than (4.4), gives the following corollary.

Corollary 4.1.2. *Let Ω and ρ be as in the statement of Theorem 4.1.1. For $\epsilon \geqslant 0$, define*

$$\|m\|_{B(\kappa_\Omega, \epsilon)} := \int |\widehat{m}(\tau)|(1 + |\tau|)^{\kappa_\Omega + \epsilon}\, d\tau.$$

If m is a bounded, measurable function supported in $(1/2, 2)$, then

$$\|\mathcal{F}[m \circ \rho]\|_{L^1(\mathbb{R}^2)} \lesssim_{\epsilon,M} \|m\|_{B(\kappa_\Omega,\epsilon)}$$

for every $\epsilon > 0$.

Proof that Theorem 4.1.1 implies Corollary 4.1.2. Since m is supported in $(1/2, 2)$, there is a smooth cutoff $\chi : \mathbb{R}^2 \to \mathbb{R}$ supported compactly away from the origin such that

$$m(\rho(\xi)) = \frac{1}{(2\pi)^2} \int \widehat{m}(\tau)\chi(\xi)e^{i\tau\rho(\xi)}\, d\tau.$$

We then have

$$\begin{aligned}\left\|\mathcal{F}^{-1}[m \circ \rho]\right\|_{L^1(\mathbb{R}^2)} &\leqslant \frac{1}{(2\pi)^2} \int |\widehat{m}(\tau)| \left\|\mathcal{F}^{-1}[\chi(\cdot)e^{i\tau\rho(\cdot)}]\right\|_{L^1(\mathbb{R}^2)}\, d\tau \\ &= \frac{1}{(2\pi)^2} \int |\widehat{m}(\tau)| \left\|\mathcal{F}^{-1}[\chi(\frac{\cdot}{\tau})e^{i\rho(\cdot)}]\right\|_{L^1(\mathbb{R}^2)}\, d\tau.\end{aligned}$$

Now, for any $i \geqslant 0$ and for every $\epsilon > 0$,

$$|D^i_\xi[\chi(\frac{\xi}{\tau})]| \lesssim_{i,\epsilon,M} (1 + |\tau|)^{\kappa_\Omega+\epsilon}(1 + |\xi|)^{-\kappa_\Omega-\epsilon-i},$$

and thus Theorem 4.1.1 implies that

$$\left\|\mathcal{F}^{-1}[\chi(\frac{\cdot}{\tau})e^{i\rho(\cdot)}]\right\|_{L^1(\mathbb{R}^2)} \lesssim_{\epsilon,M} (1 + |\tau|)^{\kappa_\Omega+\epsilon}.$$

It follows that

$$\left\|\mathcal{F}^{-1}[m \circ \rho]\right\|_{L^1(\mathbb{R}^2)} \lesssim_{\epsilon,M} \int |\widehat{m}(\tau)|(1 + |\tau|)^{\kappa_\Omega+\epsilon}$$

for every $\epsilon > 0$. □

In the special case that $\kappa_\Omega = 1/2$, we are able to obtain the following improvement to Theorem 4.1.1.

Theorem 4.1.3. *Let Ω be a convex domain satisfying (4.5) with $\kappa_\Omega = 1/2$ and ρ its Minkowski functional. Let $a : \mathbb{R}^2 \to \mathbb{C}$ be a smooth function supported outside $[-2^{-2M}, 2^{-2M}]$ such that a is a symbol of order $-1/2$, that is, for every integer $\beta \geqslant 0$,*

$$|D^\beta a(\xi)| \lesssim_\beta (1 + |\xi|)^{-1/2-\beta}.$$

Then the operator T defined on Schwartz functions f by

$$\mathcal{F}[Tf](\xi) = a(\rho(\xi))e^{i\rho(\xi)}\mathcal{F}[f](\xi)$$

extends to a bounded linear operator from the Hardy space $H^1(\mathbb{R}^2)$ to $L^1(\mathbb{R}^2)$, where the operator norm depends only on M and the quantitative estimates for a as a symbol of order $-1/2$.

Using (4.6) gives the following corollary.

Corollary 4.1.4. *Let Ω and ρ be as in the statement of Theorem 4.1.3. Let $m : \mathbb{R} \to \mathbb{C}$ be a bounded, measurable function supported in $(1/2, 2)$. Then for $1 < p < \infty$, the operator T defined on Schwartz functions f by*

$$\mathcal{F}[Tf] = m(\rho(\xi))\mathcal{F}[f]$$

extends to a bounded operator on $L^p(\mathbb{R}^2)$, and

$$\|T\|_{H^1(\mathbb{R}^2)\to L^1(\mathbb{R}^2)} \lesssim_M \|m\|_{B_{1/2,0}}.$$

The proof that Theorem 4.1.3 implies Corollary 4.1.4 is similar to the proof that Theorem 4.1.1 implies Corollary 4.1.2, and is left to the reader.

Finally, we would like to remark that while the proof of Theorem 4.1.1 draws heavily on ideas from [48] and [47], the proof of Theorem 4.1.3 requires the introduction of new techniques.

Generalizations of Theorem 4.1.1

Theorem 4.1.1 applies only to multipliers supported compactly away from the origin. Using Calderón-Zygmund theory, we may generalize the result of Theorem 4.1.1 to multipliers with non-compact support.

Theorem 4.1.5. *Fix a smooth function ϕ supported compactly away from the origin. Let m be a measurable function on $\mathbb{R}$ with $\|m\|_\infty \leqslant 1$. Let T be the operator defined on Schwartz functions f by*

$$\mathcal{F}[Tf](\xi) = m(\rho(\xi))\mathcal{F}[f](\xi).$$

Then for every $\epsilon > 0$ and $1 < p < \infty$,

$$\|m \circ \rho\|_{M^p} \lesssim_{\epsilon,p} \sup_{t>0} \|\phi(\cdot)m(t\cdot)\|_{B_{\kappa_\Omega,\epsilon}}.$$

Theorem 4.1.5 follows immediately from Theorem 4.1.1 and the following result from [46], which we state without proof.

Proposition A (Seeger, [46])**.** *Suppose that $\sup_{t>0} \|\phi(m(t\cdot))\|_{M^p} < \infty$, for some $p \in (1, \infty)$. If for some $\epsilon > 0$, $\sup_{t>0} \|\phi(m(t\cdot))\|_{\Lambda_\epsilon} < \infty$, then $m \in M_r$, $|1/r - 1/2| < |1/p - 1/2|$.*

We will also see in Section 4.6 that $L^4(\mathbb{R}^2)$ estimates for a generalized Bochner-Riesz square function leads to a multiplier theorem for quasiradial multipliers in the range $4/3 \leqslant p \leqslant 4$. In Section 4.7, we interpolate this with the result of Theorem 4.1.5 to obtain our final, most general version of Theorem 4.1.1.

Theorem 4.1.6. *Fix a smooth function ϕ supported compactly away from the origin. Let m be a measurable function on $\mathbb{R}$ with $\|m\|_\infty \leqslant 1$. Let T be the operator defined on*

Schwartz functions f by

$$\mathcal{F}[Tf](\xi) = m(\rho(\xi))\mathcal{F}[f](\xi).$$

Let $0 \leqslant \theta \leqslant 1$*. Then for every* $\epsilon > 0$ *and* $\frac{4}{4-\theta} < p < \frac{4}{\theta}$*,*

$$\|m \circ \rho\|_{M^p} \lesssim_{\epsilon,p} \sup_{t>0} \left(\int |\mathcal{F}_{\mathbb{R}}[\phi(\cdot)m(t\cdot)](\tau)|^{\frac{2}{2-\theta}} (1+|\tau|)^{\frac{2\kappa_\Omega+\theta(1-2\kappa_\Omega)}{2-\theta}+\epsilon} \, d\tau \right)^{\frac{2-\theta}{2}}.$$

Notation

We now introduce some notation that will be used throughout the rest of the chapter. Given a function $f : X \to \mathbb{R}$ and subsets $A \subsetneq B \subset X$, we will write $A \prec f \prec B$ to indicate that f is identically 1 on A and supported in B. Many of our estimates will have constants that depend on the quantity M associated with Ω given in (4.5). For the sake of convenience, we will often choose to supress this dependence in our notation. Thus we will use the symbols $\lesssim$ and $\approx$ to denote an inequality where the implied constant possibly depends on M.

4.2 Preliminaries on convex domains in $\mathbb{R}^2$

In this section we state some useful facts about convex domains in $\mathbb{R}^2$. Most of these can be found in [48] as well as in Chapter 3, but we include them here for the sake of completeness and convenience. Let $\Omega \subset \mathbb{R}^2$ be a bounded, open convex set containing the origin and satisfying (4.5). The proof of the following lemma is straightforward and uses only elementary facts about convex functions; for more details see [48].

Lemma B (Seeger and Ziesler, [48]). $\partial\Omega \cap \{x : -1 \leqslant x_1 \leqslant 1, x_2 \leqslant 0\}$ *can be parametrized by*

$$t \mapsto (t, \gamma(t)), \qquad -1 \leqslant t \leqslant 1, \tag{4.7}$$

where

1.
$$1 < \gamma(t) < 2^M, \qquad -1 \leqslant t \leqslant 1. \tag{4.8}$$

2. γ *is a convex function on* $[-1, 1]$, *so that the left and right derivatives* γ'_L *and* γ'_R *exist everywhere in* $(-1, 1)$ *and*

$$-2^{M-1} \leqslant \gamma'_R(t) \leqslant \gamma'_L(t) \leqslant 2^{M-1} \tag{4.9}$$

for $t \in [-1, 1]$. *The functions* γ'_L *and* γ'_R *are decreasing functions;* γ'_L *and* γ'_R *are right continuous in* $[-1, 1]$.

3. *Let* ℓ *be a supporting line through* $\xi \in \partial\Omega$ *and let* n *be an outward normal vector. Then*

$$|\langle \xi, n\rangle| \geqslant 2^{-M}|\xi|. \tag{4.10}$$

Decomposition of $\partial\Omega$

As another preliminary ingredient, we need the decomposition of $\partial\Omega \cap \{x : -1 \leqslant x_1 \leqslant 1, x_2 < 0\}$ introduced in [48]. This decomposition allows us to write $\partial\Omega$ as a disjoint union of pieces on which $\partial\Omega$ is sufficiently "flat", where the number of pieces in the decomposition is closely related to the covering numbers $N(\Omega, \delta)$. We inductively define a finite sequence of increasing numbers

$$\mathfrak{A}(\delta) = \{a_0, \dots, a_Q\}$$

as follows. Let $a_0 = -1$, and suppose $a_0, \dots, a_{j-1}$ are already defined. If

$$(t - a_{j-1})(\gamma_L'(t) - \gamma_R'(a_{j-1})) \leqslant \delta \text{ for all } t \in (a_{j-1}, 1]) \tag{4.11}$$

and $a_{j-1} \leqslant 1 - 2^{-M}\delta$, then let $a_j = 1$. If (4.11) holds and $a_{j-1} > 1 - 2^{-M}\delta$, then let $a_j = a_{j-1} + 2^{-M}\delta$. If (4.11) does not hold, define

$$a_j = \inf\{t \in (a_{j-1}, 1] : (t - a_{j-1})(\gamma_L'(t) - \gamma_R'(a_{j-1})) > \delta\}.$$

Now note that (4.11) must occur after a finite number of steps, since we have $|\gamma_L'|, |\gamma_R'| \leqslant 2^{M-1}$, which implies that $|t-s||\gamma_L'(t)-\gamma_R'(s)| < \delta$ if $|t-s| < \delta 2^{-M}$. Therefore this process must end at some finite stage $j = Q$, and so it gives a sequence $a_0 < a_1 < \cdots < a_Q$ so that for $j = 0, \dots, Q-1$

$$(a_{j+1} - a_j)(\gamma_L'(a_{j+1}) - \gamma_R'(a_j)) \leqslant \delta, \tag{4.12}$$

and for $0 \leqslant j < Q - 1$,

$$(t - a_j)(\gamma_L'(t) - \gamma_R'(a_j)) > \delta \qquad \text{if } t > a_{j+1}. \tag{4.13}$$

For a given $\delta > 0$, this gives a decomposition of

$$\partial\Omega \cap \{x : -1 \leqslant x_1 \leqslant 1, x_2 < 0\}$$

into pieces

$$\bigsqcup_{n=0,1,\dots,Q-1} \{x \in \partial\Omega : x_1 \in [a_n, a_{n+1}]\}.$$

The number Q in (4.12) and (4.13) is also denoted by $Q(\Omega, \delta)$. Let R_θ denote rotation by θ radians. The following lemma relates the numbers $Q(R_\theta\Omega, \delta)$ to the covering numbers $N(\Omega, \delta)$.

Lemma C (Seeger and Ziesler, [48])**.** *There exists a positive constant C_M so that the following statements hold.*

1. $Q(\Omega, \delta) \leqslant C_M \delta^{-1/2}$.

2. $0 \leqslant \kappa_\Omega \leqslant 1/2$.

3. *For any θ,*

$$Q(R_\theta\Omega, \delta) \leqslant C_M N(\Omega, \delta) \log(2 + \delta^{-1}).$$

4. *For $\nu = 1, \ldots, 2^{2M}$ let $\theta_\nu = \frac{2\pi\nu}{2^{2M}}$. Then*

$$C_M^{-1} N(\Omega, \delta) \leqslant \sum_\nu Q(R_{\theta_\nu}\Omega, \delta) \leqslant C_M N(\Omega, \delta) \log(2 + \delta^{-1}).$$

We may think of $\mathfrak{A}(\delta)$ as a partition of $[-1, 1]$ into intervals. For the purpose of defining a partition of unity, we wish to refine this partition so that consecutive intervals have comparable length, and we construct such a refinement in the proof of the lemma below. Note the improvement to (4.15) in the special case that $\kappa_\Omega = 1/2$; this will be used later when we prove Theorem 4.1.3.

Lemma 4.2.1. *Suppose that Ω is a convex domain satisfying (4.5). Let $\delta > 0$, and let*

$$\mathfrak{A}(\delta) = \{a_0, a_1, \ldots a_Q\}$$

be the decomposition of $[-1,1]$ *constructed previously, where* $a_0 = -1$ *and* $a_1 = 1$*. There exists a refinement*

$$\tilde{\mathfrak{A}}(\delta) = \{b_0, b_1, \ldots b_{\tilde{Q}}\} \tag{4.14}$$

of $\mathfrak{A}(\delta)$ *with* $b_0 = -1$ *and* $b_{\tilde{Q}} = 1$*, and satisfying the following properties:*

1.
$$\text{card}(\tilde{\mathfrak{A}}(2^{-k})) \lesssim k^2 N(\Omega, 2^{-k}). \tag{4.15}$$

2. *Set* $I_j = [b_j, b_{j+1}]$*. For every* $1 \leqslant j \leqslant \tilde{Q}$*,*
$$(\gamma'(b_j) - \gamma'(b_{j-1}))|I_{j-1}| \leqslant 2^{-k}. \tag{4.16}$$

3. *For every* $1 \leqslant j \leqslant \tilde{Q}$*,*
$$|I_{j-1}|/8 \leqslant |I_j| \leqslant 8|I_{j-1}|. \tag{4.17}$$

4.
$$\sum_j \delta |I_j|^{-1} \lesssim 1. \tag{4.18}$$

In the special case that $\kappa_\Omega = 1/2$*, we also have*

$$\text{card}(\tilde{\mathfrak{A}}(\delta)) \lesssim \delta^{-\kappa_\Omega}. \tag{4.19}$$

Proof of Lemma 4.2.1. We construct $\tilde{\mathfrak{A}}(\delta)$ as follows. For each $0 \leqslant j \leqslant Q-1$, let $\tilde{a}_j$ be the midpoint between a_j and a_{j+1}, and consider the set

$$A := \{a_0, \tilde{a}_0, a_1, \tilde{a}_1, \ldots, \tilde{a}_{Q-1}, a_Q\}.$$

For $x \in A$, let $x^- := \max\{y \in A : y < x\}$ and $x^+ := \min\{y \in A : y > x\}$. For every $x \in A$, we define a set of points B_x as follows. If x satisfies $x^+ - x = x - x^-$, set $B_x = \{x\}$. If x satisfies $x^+ - x > x - x^-$, then iteratively define B_x to be the set of $\lesssim \log(1/\delta)$ many points $B_x = \{y_0, y_1, \ldots, y_N\}$ where y_0 is the midpoint between x and x^+, and for every $k \geqslant 0$ set y_{k+1} to be the midpoint between y_k and x, and stop at the first stage N such that $y_N - x \leqslant x - x^-$. Similarly, if x satisfies $x^+ - x < x - x^-$, then iteratively define B_x to be the set of $\lesssim \log(1/\delta)$ many points $B_x = \{y_0, y_1, \ldots, y_N\}$ where y_0 is the midpoint between x and x^-, and for every $k \geqslant 0$ set y_{k+1} to be the midpoint between y_k and x, and stop at the first stage N such that $x - y_N \leqslant x^+ - x$. Now let

$$\tilde{\mathfrak{A}}(\delta) = \bigcup_{x \in A} B_x.$$

Clearly, $\tilde{\mathfrak{A}}(\delta)$ satisfies (4.16), since any refinement of $\mathfrak{A}(\delta)$ automatically satisfies (4.16). It is also obvious that $\tilde{\mathfrak{A}}(\delta)$ satisfies (4.17). Since $\mathfrak{A}(\delta)$ satisfies (4.13), we have

$$\sum_j 2^{-k}|I_j|^{-1} \lesssim \sum_j 2^{-k}(a_{j+1} - a_j)^{-1} \lesssim \sum_j (\gamma'(a_{j+1}) - \gamma'(a_j)) \lesssim 1,$$

so $\tilde{\mathfrak{A}}(\delta)$ satisfies (4.18). By Lemma C, we have

$$\operatorname{card}(\tilde{\mathfrak{A}}(2^{-k})) = \tilde{Q} + 1 \lesssim k \cdot \operatorname{card}(\mathfrak{A}(2^{-k})) \lesssim k^2 N(\Omega, 2^{-k}). \tag{4.20}$$

and so $\tilde{\mathfrak{A}}(\delta)$ satisfies (4.15).

In the case that $\kappa_\Omega = 1/2$, we note that (4.13) implies that for any $L > 0$, the number of intervals $[a_j, a_{j+1}]$ such that $(a_{j+1} - a_j) \approx L$ is $\lesssim \min(L\delta^{-1}, L^{-1})$. Thus for any $r > 0$ the number of pairs

$\big([a_j, a_{j+1}]; [a_{j+1}, a_{j+2}]\big)$ with

$$\max\left(\frac{a_{j+2} - a_{j+1}}{a_{j+1} - a_j}, \frac{a_{j+1} - a_j}{a_{j+2} - a_{j+1}}\right) \approx r$$

is $\lesssim r^{-1}\delta^{-1/2}$. It follows that the number of points $x \in A$ with

$$\max\left(\frac{x^+ - x}{x - x^-}, \frac{x - x^-}{x^+ - x}\right) \approx r$$

is $\lesssim r^{-1}\delta^{-1/2}$. For such points x we have $\text{card}(B_x) \lesssim \log(r)$, and so summing over all dyadic $r = 2^k$ we have that

$$\sum_{k \geqslant 0} k 2^{-k} \delta^{-1/2} \lesssim \delta^{-1/2},$$

and hence $\tilde{\mathfrak{A}}(\delta)$ satisfies (4.19). □

Approximating Ω by convex domains with smooth boundary

It will be necessary to approximate Ω by a sequence of convex domains with smooth boundaries. In [48], this was done by approximating Ω by a sequence of convex polygons with sufficiently many vertices and smoothing out the boundary near the vertices. We state the following lemma from [48] without proof.

Lemma D (Seeger and Ziesler, [48])**.** *Let $\Omega \subset \mathbb{R}^2$ be an open convex domain containing the origin. There is a sequence of convex domains $\{\Omega_n\}$ containing the origin, with Minkowski functionals $\rho_n(\xi) = \inf\{t > 0 |\, \xi/t \in \Omega_n\}$, so that the following holds:*

1. *$\Omega_n \subset \Omega_{n+1} \subset \Omega$ and $\bigcup_n \Omega_n = \Omega$.*
2. *$\rho_n(\xi) \geqslant \rho_{n+1}(\xi) \geqslant \rho(\xi)$ and*

$$\frac{\rho_n(\xi) - \rho(\xi)}{\rho(\xi)} \leqslant 2^{-n-1};$$

   in particular $\lim_{n\to\infty} \rho_n(\xi) = \rho(\xi)$, with uniform convergence on compact sets.

3. *Ω_n has C^∞ boundary.*

4. If $\delta \geqslant 2^{-n+2}$ then

$$N(\Omega_n, 2\delta) \leqslant N(\Omega, \delta).$$

Computing $\nabla\rho$

Assuming that $\rho \in C^1(\mathbb{R}^2 \setminus \{0\})$, we would like to compute $\nabla\rho(\alpha, \gamma(\alpha))$ for $\alpha \in [-1, 1]$. Since $\nabla\rho$ is homogeneous of degree 0, this will actually give us $\nabla\rho(\xi)$ for any ξ in a sector of $\mathbb{R}^2 \setminus \{0\}$ bounded by rays passing through $(-1, \gamma(-1))$ and $(1, \gamma(1))$. Note that

$$\nabla\rho(\alpha, \gamma(\alpha)) \cdot (1, \gamma'(\alpha)) = 0, \tag{4.21}$$

and thus $\nabla\rho(\alpha, \gamma(\alpha))$ is parallel to $(-\gamma'(\alpha), 1)$. Differentiating the homogeneity relation

$$\rho(t(\alpha, \gamma(\alpha))) = t\rho(\alpha, \gamma(\alpha))$$

with respect to t and setting $t = 1$ yields

$$(\nabla\rho(\alpha, \gamma(\alpha))) \cdot (\alpha, \gamma(\alpha)) = 1. \tag{4.22}$$

It follows that

$$|\nabla\rho(\alpha, \gamma(\alpha))| = \frac{|(-\gamma'(\alpha), 1)|}{|\langle(\alpha, \gamma(\alpha)); (-\gamma'(\alpha), 1)\rangle|}. \tag{4.23}$$

Note that (4.5) implies that

$$|\langle(\alpha, \gamma(\alpha)); (-\gamma'(\alpha), 1)\rangle| \geqslant 2^{-4M}. \tag{4.24}$$

Together (4.21) and (4.23) imply that

$$\nabla\rho(\alpha, \gamma(\alpha)) = \frac{(\gamma'(\alpha), -1)}{\alpha\gamma'(\alpha) - \gamma(\alpha)}. \tag{4.25}$$

Note that (4.5) and (4.25) implies that

$$|\nabla\rho(\alpha, \gamma(\alpha))| \leqslant 2^{5M}. \tag{4.26}$$

4.3 L^1 kernel estimates

The goal of this section is to prove Theorem 4.1.1. Let Ω, ρ and a be as in the statement of Theorem 4.1.1. Motivated by [47], we would like to perform a dyadic decomposition of the multiplier $a(\rho(\xi))e^{i\rho(\xi)}$. Let $\{\theta_k\}_{k\geqslant 0}$ be a smooth dyadic partition of unity of $\mathbb{R}$, so that θ_0 is supported in $[-2^{-3M}, 2^{-3M}]$ and θ_k is supported in an annulus $|\xi| \approx 2^{k-3M}$ for $k > 0$. We write

$$K(x) := \mathcal{F}^{-1}[a(\rho(\cdot))e^{i\rho(\cdot)}](x) = \sum_{k\geqslant 0} K_k(x),$$

where

$$K_k(x) := \mathcal{F}^{-1}[a(\rho(\cdot))e^{i\rho(\cdot)}\theta_k(\rho(\cdot))](x). \tag{4.27}$$

It is easy to see that Theorem 4.1.1 is a consequence of the following.

Proposition 4.3.1. *Let Ω, ρ and a be as in the statement of Theorem 4.1.1. Define K_k as in (4.27). Then for $k > 0$ and for every $\epsilon > 0$,*

$$\|K_k\|_{L^1(\mathbb{R}^2)} \lesssim_\epsilon 2^{-k\epsilon/2}.$$

In order to obtain kernel estimates using techniques similar to those in [48], we want to work with domains with smooth boundaries, rather than arbitrary convex domains for which the boundary need only be Lipschitz. Thus we will use Lemma D to reduce Proposition 4.3.1 to the following.

Proposition 4.3.2. *Let Ω, ρ and a be as in the statement of Theorem 4.1.1. Fix an integer $k > 0$. Let $\tilde{\Omega}$ be a convex domain with smooth boundary such that*

$$\{\xi : |\xi| \leqslant 4\} \subset \tilde{\Omega} \subset \overline{\tilde{\Omega}} \subset \{\xi : |\xi| < 2^{M+1}\},$$

and such that

$$N(\tilde{\Omega}, 2^{-k}) \leqslant N(\Omega, 2^{-k-1}). \tag{4.28}$$

Let $\tilde{\rho}$ be the Minkowski functional of $\tilde{\Omega}$. Define

$$\tilde{K}_k(x) := \mathcal{F}^{-1}[a(\tilde{\rho}(\cdot))e^{i\tilde{\rho}(\cdot)}\theta_k(\tilde{\rho}(\cdot))](x).$$

Then for every $\epsilon > 0$,

$$\left\|\tilde{K}_k\right\|_{L^1(\mathbb{R}^2)} \lesssim_\epsilon 2^{-k\epsilon/2}.$$

Proof that Proposition 4.3.2 implies Proposition 4.3.1. Let $\{\rho_n\}$ be a sequence of Minkowski functionals approximating ρ as in Lemma D, and for each n set

$$K_{k,n}(x) := \mathcal{F}[a(\rho_n(\cdot))e^{i\rho_n(\cdot)}\theta_k(\rho_n(\cdot))](x).$$

Since $\rho_n \to \rho$ uniformly on compact sets, $K_{k,n}(x) \to K_k(x)$ pointwise almost everywhere, and so Fatou's lemma yields

$$\|K_k\|_{L^1(\mathbb{R}^2)} \leqslant \liminf_{n\to\infty} \|K_{k,n}\|_{L^1(\mathbb{R}^2)} \lesssim_\epsilon 2^{-k\epsilon/2},$$

where in the second to last step we have applied Proposition 4.3.2. □

Now that we have reduced Proposition 4.3.1 to Proposition 4.3.2 we may now work with distance functions $\tilde{\rho}$ that are smooth away from the origin, and so we may express the kernels in homogeneous coordinates (polar coordinates associated to $\tilde{\Omega}$) and integrate by parts. This is the general approach used in [48] to handle the generalized Bochner-Riesz multipliers. We emphasize that we must take care to ensure that our estimates ultimately depend only on the C^1 norm of $\partial\tilde{\Omega}$, which is bounded by 2^M (and not, for instance, the C^2 norm). That this is necessary can be seen in the statements of Theorem 4.1.1, Proposition 4.3.1 and Proposition 4.3.2, where none of the constants in the estimates to be proven depend on the C^2 norm of $\partial\tilde{\Omega}$. However, if we recall the remarks made about notation in the introduction, each of the constants in these estimates implicitly depend on M.

Proof of Proposition 4.3.2. We first note that after employing an appropriate angular partition of unity and using rotational invariance it suffices to consider $\tilde{K}_k$ multiplied by a smooth angular cutoff on the Fourier side. Thus in what follows we will instead let

$$\tilde{K}_k(x) := \mathcal{F}^{-1}[a(\tilde{\rho}(\cdot))e^{i\tilde{\rho}(\cdot)}\theta_k(\tilde{\rho}(\cdot))\chi(\cdot)](x) \tag{4.29}$$

where $\chi(\xi) = \chi_1(\frac{\xi_1}{|\xi|})\chi_2(\tilde{\rho}(\xi))$ for smooth functions $\chi_1, \chi_2 : \mathbb{R} \to \mathbb{R}$ so that $[-2^{-2M-1}, 2^{-2M-1}] \prec \chi_1 \prec [-2^{-2M}, 2^{-2M}]$, and so that χ_2 is identically 1 on the support of a and 0 in a sufficiently small ball centered at the origin. Let γ be a parametrization of $\partial\tilde{\Omega} \cap \{x : -1 \leqslant x_1 \leqslant 1,\ x_2 \leqslant 0\}$ as in Lemma B. We introduce homogeneous coordinates

$$(s, \alpha) \mapsto \xi(s, \alpha) = (s\alpha, s\gamma(\alpha)). \tag{4.30}$$

In this coordinate system, $\{(s, \alpha) : s = 1\} \subset \{\xi : \rho(\xi) = 1\}$. The map (4.30) has Jacobian

$$\det\left(\frac{\partial\xi}{\partial(s, \alpha)}\right) = s(\alpha\gamma'(\alpha) - \gamma(\alpha)).$$

Note that there is a smooth function $\tilde{\chi}_1 : \mathbb{R} \to \mathbb{R}$ so that $\chi_1(\frac{\xi_1}{|\xi|})$ in homogeneous coordinates is given by $\tilde{\chi}_1(\alpha)$. Using (4.30), we thus have

$$\begin{aligned}\tilde{K}_k(x) &= \int_{\mathbb{R}^2} e^{i\tilde{\rho}(\xi)}a(\tilde{\rho}(\xi))\theta_k(\tilde{\rho}(\xi))\chi(\xi)e^{ix\cdot\xi}\,d\xi \\ &= \int_0^\infty \int e^{is(\alpha x_1+\gamma(\alpha)x_2+1)}a(s)\theta_k(s)\tilde{\chi}_1(\alpha)s(\alpha\gamma'(\alpha) - \gamma(\alpha))\,d\alpha\,ds. \end{aligned} \tag{4.31}$$

Kernel estimates far away from the singular set

Considering the phase $ix\cdot\xi+i\tilde{\rho}(\xi)$ as a function of the variable ξ, we see that its gradient vanishes on the singular set $x \in \{-\nabla\tilde{\rho}(\xi) : \xi \in \mathbb{R}^2\}$. Since $|\nabla\tilde{\rho}| \leqslant 2^{5M}$ as noted in (4.26), we choose to separately estimate the L^1 norm of $\tilde{K}_k$ away from a sufficiently large ball (say, of radius 2^{6M}) centered at the origin. We would expect that after localization on the Fourier side, the multiplier $e^{i\tilde{\rho}(\xi)}$ acts like translation by $\nabla\tilde{\rho}(\xi_0)$ for some ξ_0, and hence we might expect any pointwise kernel estimates we obtain off of the ball of radius 2^{6M} centered at the origin to be robust under perturbations by $\nabla\tilde{\rho}(\xi_0)$. Thus we will not further decompose the multiplier $\mathcal{F}[\tilde{K}_k]$ when estimating the L^1 norm of $\tilde{K}_k$ off of this ball.

Throughout the rest of this chapter, $\phi_0 : \mathbb{R} \to \mathbb{R}$ will be a smooth function satisfying $[-1/2, 1/2] \prec \phi \prec [-1, 1]$. We set $c = c(\Omega, \epsilon) = \frac{1}{2}\max(\kappa_\Omega, \epsilon)$. We will show that

$$\int |\tilde{K}_k(x)(1 - \phi_0(2^{-6M}|x|))|\, dx \lesssim 2^{-kc}. \tag{4.32}$$

To do this we will first prove

$$\int |\tilde{K}_k(x)(\phi_0(2^{-3k-6M}|x|) - \phi_0(2^{-6M}|x|))|\, dx \lesssim 2^{-kc} \tag{4.33}$$

and then prove

$$\int |\tilde{K}_k(x)(1 - \phi_0(2^{-3k-6M}|x|))|\, dx \lesssim 2^{-k}. \tag{4.34}$$

Let $\eta : \mathbb{R} \to \mathbb{R}$ be a smooth function satisfying $[-2^{-3M-1}, 2^{-3M-1}] \prec \eta \prec [-2^{-3M}, 2^{-3M}]$. We decompose

$$\tilde{K}_k(x)(\phi_0(2^{-3k-6M}|x|) - \phi_0(2^{-6M}|x|)) = \tilde{K}_{k,1}(x) + \tilde{K}_{k,2}(x),$$

where

$$\tilde{K}_{k,1}(x) = (\phi_0(2^{-3k-6M}|x|) - \phi_0(2^{-6M}|x|)) \times \int_0^\infty \int e^{is(\alpha x_1 + \gamma(\alpha) x_2 + 1)} a(s) \eta\left(\frac{x_1 + x_2\gamma'(\alpha)}{|x|}\right) \times \theta_k(s)\tilde{\chi}_1(\alpha) s(\alpha\gamma'(\alpha) - \gamma(\alpha))\, d\alpha\, ds \quad (4.35)$$

and

$$\tilde{K}_{k,2}(x) = (\phi_0(2^{-3k-6M}|x|) - \phi_0(2^{-6M}|x|)) \times \int_0^\infty \int e^{is(\alpha x_1 + \gamma(\alpha) x_2 + 1)} a(s) \left(1 - \eta\left(\frac{x_1 + x_2\gamma'(\alpha)}{|x|}\right)\right) \times \theta_k(s)\tilde{\chi}_1(\alpha) s(\alpha\gamma'(\alpha) - \gamma(\alpha))\, d\alpha\, ds. \quad (4.36)$$

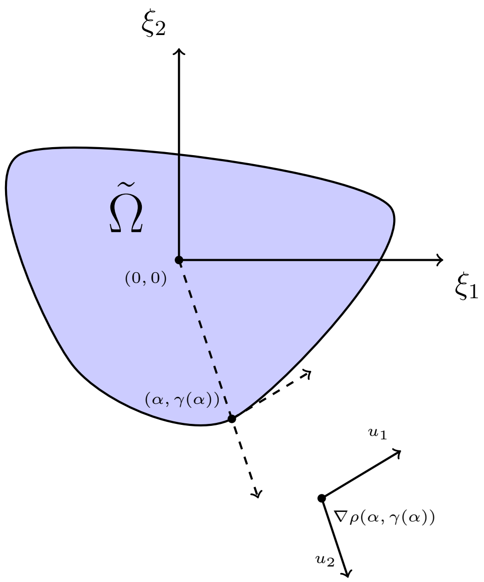

Figure 4

The coordinate system from (4.37).

Note that the coordinate system given by the change of coordinates

$$(x_1, x_2) \mapsto (u_1, u_2) := (x_1 + x_2\gamma'(\alpha), 1 + \alpha x_1 + \gamma(\alpha)x_2), \tag{4.37}$$

has Jacobian with absolute value $|\alpha\gamma'(\alpha) - \gamma(\alpha)| \approx_M 1$. It is also helpful to note that

$$x_1 + x_2\gamma'(\alpha) = [(x_1, x_2) - \nabla\rho(\alpha, \gamma(\alpha))] \cdot (1, \gamma'(\alpha))$$

and

$$1 + \alpha x_1 + \gamma(\alpha)x_2 = [(x_1, x_2) - \nabla\rho(\alpha, \gamma(\alpha))] \cdot (\alpha, \gamma(\alpha)),$$

and hence our coordinate system is centered at $\nabla\rho(\alpha, \gamma(\alpha))$ with one coordinate direction parallel to $(\alpha, \gamma(\alpha))$ and the other coordinate direction parallel to the tangent vector to $\partial\Omega$ at $(\alpha, \gamma(\alpha))$; see Figure 4. Thus by our choice of the angular cutoff χ and our choice of η, it follows that on the support of

$$(\phi_0(2^{-3k-6M}|x|) - \phi_0(2^{-6M}|x|))\eta\left(\frac{x_1 + x_2\gamma'(\alpha)}{|x|}\right)$$

we have $|x| \approx |1 + \alpha x_1 + \gamma(\alpha)x_2|$. Similarly, on the support of

$$(\phi_0(2^{-3k-6M}|x|) - \phi_0(2^{-6M}|x|))\left(1 - \eta\left(\frac{x_1 + x_2\gamma'(\alpha)}{|x|}\right)\right)$$

we have $|x| \approx |x_1 + x_2\gamma'(\alpha)|$.

Integrating (4.35) by parts three times with respect to s and using the above observations yields

$$\begin{aligned}\int |\tilde{K}_{k,1}(x)|\, dx &\lesssim 2^{-k(\kappa_\Omega+\epsilon)} \int\int \tilde{\chi}_1(\alpha) \frac{2^{2k}}{(1 + 2^k|1 + \alpha x_1 + \gamma(\alpha)x_2|)^3}\, d\alpha\, dx \\ &\lesssim 2^{-k(\kappa_\Omega+\epsilon)} \int \frac{2^{2k}}{(1 + 2^k|x|)^3}\, dx \lesssim 2^{-kc}. \end{aligned} \tag{4.38}$$

Integrating by parts (4.36) once with respect to α, we have

$$\int |\tilde{K}_{k,2}(x)|\,dx = (\phi_0(2^{-3k-6M}|x|) - \phi_0(2^{-6M}|x|)) \times \int_0^\infty \int \partial_\alpha g_k(x,\alpha) e^{is(\alpha x_1+\gamma(\alpha)x_2+1)} a(s)\theta_k(s)\,ds\,d\alpha, \quad (4.39)$$

where

$$g_k(x,\alpha) = \frac{\tilde{\chi}_1(\alpha)(\alpha\gamma'(\alpha)-\gamma(\alpha))(1-\eta(\frac{x_1+x_2\gamma'(\alpha)}{|x|}))}{x_1+x_2\gamma'(\alpha)}.$$

Integrating by parts (4.39) twice with respect to s, we have

$$|\tilde{K}_{k,2}(x)| \lesssim 2^{-k(\kappa_\Omega+\epsilon)}(\phi_0(2^{-3k-6M}|x|) - \phi_0(2^{-6M}|x|)) \times \int |\partial_\alpha g_k(x,\alpha)| \frac{2^k}{(1+2^k|\alpha x_1+\gamma(\alpha)x_2+1|)^2}\,d\alpha.$$

Note that on the support of $g_k(x,\alpha)$,

$$|\partial_\alpha g_k(x,\alpha)| \lesssim \frac{|\gamma''(\alpha)|+1}{|x_1+x_2\gamma'(\alpha)|}. \quad (4.40)$$

We apply the change of coordinates (4.37). Using (4.40), this yields

$$\begin{aligned}\int |\tilde{K}_{k,2}(x)|\,dx &\lesssim 2^{-k(\kappa_\Omega+\epsilon)} \int \left(\int_{B_{2^{3k+10M}}(0)\setminus B_1(0)} \frac{1}{|u_1|}\frac{2^k}{(1+2^k|u_2|)^2}\,du \right) \\ &\qquad \times (|\gamma''(\alpha)|+1)\tilde{\chi}_1(\alpha)d\alpha \\ &\lesssim 2^{-k(\kappa_\Omega+\epsilon)} \int_{B_{2^{3k+10M}}(0)\setminus B_1(0)} \frac{1}{|u_1|}\frac{2^k}{(1+2^k|u_2|)^2}\,du \lesssim k2^{-k(\kappa_\Omega+\epsilon)} \lesssim 2^{-kc},\end{aligned}$$

which together with (4.38) proves (4.33).

Now we prove (4.34). We will need the following lemma from [48], which we state without proof.

Lemma E (Seeger and Ziesler, [48]). *Let h be an absolutely continuous function on $[0,\infty)$ and suppose that $\lim_{t\to\infty} h(t) = 0$. Suppose that $s \mapsto sh'(s)$ defines an L^1 function on $[0,\infty)$ and let*

$$F(\tau) = \int_0^\infty h'(s)e^{is\tau}\, ds.$$

Suppose that $\mu > 0$ and that

$$|F(\tau)| + |F'(\tau)| \leqslant B(1+|\tau|)^{-\mu}.$$

Let $B(0,R)$ be the ball with radius R and center 0, and define $\mathcal{A}_l = B(0,2^l) \setminus B(0,2^{l-1})$, for $l > 0$, and $\mathcal{A}_0 = B(0,1)$. Then

$$\int_{\mathcal{A}_l} |\mathcal{F}^{-1}[h\circ\rho](x)|\, dx \lesssim_M B[2^{-l(\mu-1)} + l2^{-l}].$$

We will apply the lemma with $h(s) = e^{is}a(s)\theta(2^{-k}s)$. Then for every $N > 0$,

$$|F(\tau)| + |F'(\tau)| \leqslant 2^{k(2-\kappa_\Omega-\epsilon)}(1+|\tau|)^{-N},$$

and so we conclude that

$$\int_{\mathcal{A}_l} \mathcal{F}^{-1}[h\circ\rho](x)|\, dx \lesssim l2^{k(2-\kappa_\Omega-\epsilon)-l}.$$

Summing over $l \geqslant 10k$, we obtain (4.34) and therefore (4.32).

Remark 4.3.3. *We note that our proof of (4.32) is also valid when $\epsilon = 0$ and $\kappa_\Omega > 0$, which implies $c = \kappa_\Omega/2$. We will use this later when we prove an $H^1 \to L^1$ endpoint estimate.*

Kernel estimates near the singular set

It remains to estimate

$$\int |\tilde{K}_k(x)\phi_0(2^{-6M}|x|)|\,dx.$$

Here we will further decompose the mutiplier $\mathcal{F}[\tilde{K}_k]$ using the decomposition of $\partial\tilde{\Omega}$ from Section 4.2. Let $\mathfrak{A}(2^{-k})$ be the increasing sequence of numbers associated to $\partial\tilde{\Omega}$ as defined in Section 4.2 with $\delta = 2^{-k}$, and let $\tilde{\mathfrak{A}}(2^{-k})$ be the refinement of $\mathfrak{A}(2^{-k})$ as given by Lemma 4.2.1 and let $\{I_j\}$ be the corresponding partition of $[-1,1]$ into subintervals. We emphasize that although our collection of intervals $\{I_j\}$ is indexed only by j, it implicitly depends on k as well. Now for each such interval I_j, let I_j^* be its 25/24-dilate (dilated from the center of I_j), and let $\{\beta_{I_j}\}$ be a smooth partition of unity subordinate to $\{I_j^*\}$ such that for each $i \geqslant 0$,

$$D^i\beta_{I_j}(x) \lesssim |I_j|^{-i}.$$

The constant 25/24 is chosen so that $\{I_j^*\}$ is an almost-disjoint collection. We decompose

$$\tilde{K}_k = \sum_j \tilde{K}_{k,j},$$

where

$$\tilde{K}_{k,j}(x) = \int_0^\infty \int_{I_j^*} e^{is(\alpha x_1+\gamma(\alpha)x_2+1)}\beta_{I_j}(\alpha)\theta_k(s)a(s)s(\alpha\gamma'(\alpha)-\gamma(\alpha))\,d\alpha\,ds,$$

that is, $\tilde{K}_{k,j}$ is like $\tilde{K}_k$ with $\beta_{I_j}(\alpha)$ inserted into the integral. We may think of this decomposition on the Fourier side as a decomposition of the multiplier $\mathcal{F}[\tilde{K}_k]$ into smooth functions adapted to sectors bounded by rays originating at the origin and passing through points $(\alpha,\gamma(\alpha))$ where $\alpha \in \tilde{\mathfrak{A}}(2^{-k})$. To estimate $\int |\tilde{K}_{k,j}(x)\phi_0(2^{-6M}|x|)|\,dx$, we will further decompose

$$\tilde{K}_{k,j}(x) \cdot \phi_0(2^{-6M}|x|) = \sum_{n \geqslant 0} \tilde{K}_{k,j,n}(x),$$

where we define $\tilde{K}_{k,j,n}$ as follows. Recall that ϕ_0 is a smooth function such that $[-1/2, 1/2] \prec \phi_0 \prec [-1, 1]$, and let

$$\Phi_{k,j,0}(x, \alpha) = \phi_0(|I_j| 2^k (x_1 + x_2 \gamma'(\alpha))) \tag{4.41}$$

and for $n > 0$ let

$$\Phi_{k,j,n}(x, \alpha) = \phi_0(|I_j| 2^{k-n} (x_1 + x_2 \gamma'(\alpha))) - \phi_0(|I_j| 2^{k-n+1} (x_1 + x_2 \gamma'(\alpha))). \tag{4.42}$$

Set

$$\tilde{K}_{k,j,0}(x) := \phi_0(2^{-6M}|x|) \int_0^\infty \int_{I_j^*} e^{is(\alpha x_1 + \gamma(\alpha) x_2 + 1)} \beta_{I_j(\alpha)}$$
$$\Phi_{k,j,0}(x, \alpha) \theta_k(s) a(s) s(\alpha \gamma'(\alpha) - \gamma(\alpha)) \, d\alpha \, ds$$

and for $n > 0$ set

$$\tilde{K}_{k,j,n}(x) := \phi_0(2^{-6M}|x|) \int_0^\infty \int_{I_j^*} e^{is(\alpha x_1 + \gamma(\alpha) x_2 + 1)} \beta_{I_j(\alpha)}$$
$$\Phi_{k,j,n}(x, \alpha) \theta_k(s) a(s) s(\alpha \gamma'(\alpha) - \gamma(\alpha)) \, d\alpha \, ds,$$

that is, $\tilde{K}_{k,j,n}$ is like $\tilde{K}_{k,j}$ with $\Phi_{k,j,n}(x, \alpha)$ inserted into the integral.

To estimate $\int |\tilde{K}_{k,j,0}(x)| \, dx$, we integrate by parts in s twice to obtain

$$\int |\tilde{K}_{k,j,0}(x)| \, dx \lesssim 2^{k(1-\kappa_\Omega-\epsilon)} \int_{I_j^*} \int_{|x_1 + x_2 \gamma'(\alpha)| \leqslant |I_j|^{-1} 2^{-k}} 2^k$$
$$\times (1 + 2^k |\alpha x_1 + \gamma(\alpha) x_2 + 1|)^{-2} \, dx \, d\alpha.$$

Applying the change of coordinates (4.37) yields

$$\int |\tilde{K}_{k,j,0}(x)|\,dx \lesssim 2^{k(1-\kappa_\Omega-\epsilon)} \times \int_{I_j^*}\int_{|u_1|\leqslant|I_j|^{-1}2^{-k}} 2^k(1+2^k|u_2|)^{-2}\,du_1\,du_2\,d\alpha \lesssim 2^{-k(\kappa_\Omega+\epsilon)}.$$

By (4.15) and (4.28), we may sum in j to obtain

$$\sum_j \int |\tilde{K}_{k,j,0}(x)|\,dx \lesssim 2^{-k\epsilon/2}. \tag{4.43}$$

Now we estimate $\int |K_{k,j,n}(x)|\,dx$ for $n>0$. Observe that $\tilde{K}_{k,j,n}(x)$ is identically zero when $n \geqslant k$, so we only need consider the case $n<k$. We integrate by parts once with respect to α and then twice with respect to s. Integrating by parts with respect to α yields

$$\tilde{K}_{k,j,n}(x) = \phi_0(2^{-6M}|x|)\int_0^\infty\int_{I_j^*} \partial_\alpha g_{k,j,n}(x,\alpha)e^{is(\alpha x_1+\gamma(\alpha)x_2+1)} a(s)\theta(2^{-k}s)\,ds\,d\alpha,$$

where

$$g_{k,j,n}(x,\alpha) = \frac{\Phi_{k,j,n}(x,\alpha)\beta_{I_j}(\alpha)(\gamma(\alpha)-\alpha\gamma'(\alpha))}{x_1+x_2\gamma'(\alpha)}.$$

Integrating by parts twice with respect to s yields

$$|\tilde{K}_{k,j,n}(x)| \lesssim 2^{k(-\kappa_\Omega-\epsilon)}\phi_0(2^{-6M}|x|)\int_{I_j^*}|\partial_\alpha g_{k,j,n}(x,\alpha)| \times \frac{2^k}{(1+2^k|\alpha x_1+\gamma(\alpha)x_2+1)|)^2}\,d\alpha.$$

Observe that on the support of $\tilde{K}_{k,j,n}(x)$, $|x| \lesssim 1$, so

$$|\partial_\alpha g_{k,j,n}(x,\alpha)| \lesssim \frac{|\gamma''(\alpha)|(|I_j|2^{k-n}|x|+1)+|I_j|^{-1}}{|x_1+x_2\gamma'(\alpha)|} \lesssim \frac{|\gamma''(\alpha)|(|I_j|2^{k-n}+1)+|I_j|^{-1}}{|x_1+x_2\gamma'(\alpha)|}.$$

Thus applying the change of coordinates (4.37), we have

$$\begin{aligned}\int |\tilde{K}_{k,j,n}(x)|\,dx \lesssim 2^{k(-\kappa\Omega-\epsilon)} \int_{I_j^*} &(|\gamma''(\alpha)|(|I_j|2^{k-n}+1)+|I_j|^{-1}) \\ &\times \int_{|u_1|\approx 2^{n-k}|I_j|^{-1}} \frac{1}{|u_1|}\frac{2^k}{(1+2^k|u_2|)^2}\,du\,d\alpha \\ &\lesssim 2^{k(-\kappa\Omega-\epsilon)} \int_{I_j^*} (|\gamma''(\alpha)|(|I_j|2^{k-n}+1)+|I_j|^{-1})\,d\alpha.\end{aligned}$$

By (4.16), if we let b_j^* and b_{j+1}^* denote the endpoints of I_j^*, then we have

$$\int_{I_j^*} |\gamma''(\alpha)||I_j|\,d\alpha \lesssim (\gamma'(b_{j+1}^*)-\gamma'(b_j^*))|I_j| \lesssim 2^{-k},$$

and thus

$$\int |\tilde{K}_{k,j,n}(x)|\,dx \lesssim 2^{k(-\kappa\Omega-\epsilon)}.$$

Summing in j and n, using (4.15) and (4.28) and recalling that we only need sum over $n < k$, we obtain

$$\sum_j \sum_{n\geqslant 0} \int |\tilde{K}_{k,j,n}(x)|\,dx \lesssim k2^{-k\epsilon} \lesssim 2^{-k\epsilon/2}. \tag{4.44}$$

Combining this with our previous estimates (4.43) and (4.32), we have

$$\int |\tilde{K}_k(x)|\,dx \lesssim_\epsilon 2^{-k\epsilon/2},$$

as desired, completing the proof of Proposition 4.3.2 and hence Theorem 4.1.1. □

4.4 The $H^1 \to L^1$ endpoint estimate: preliminaries and estimate on the exceptional set

In this section, we begin the proof of Theorem 4.1.3. Throughout this section $\kappa_\Omega = 1/2$. We note that we will often continue to write κ_Ω instead of subsituting $1/2$ simply to indicate how certain quantities in our estimates arise. As in the proof of Theorem 4.1.1, the first step is to reduce Theorem 4.1.3 to a statement about convex domains with smooth boundary.

Reduction to the case of smooth boundary

We invoke Lemma D to show that it suffices to prove Theorem 4.1.3 in the special case that $\partial\Omega$ is C^∞. For any cube $Q \subset \mathbb{R}^2$, recall that an atom associated to a_Q is a bounded, measurable function supported in Q such that

$$\begin{aligned} \|a_Q\|_\infty &\leqslant |Q|^{-1}, \\ \int_Q a_Q(x)\,dx &= 0. \end{aligned}$$

Let $\phi \geqslant 0$ be a Schwartz function with compactly supported Fourier transform such that $\|\phi\|_{L^1} = 1$, and for each $m \geqslant 0$ let $\phi_m(x) = 2^{2m}\phi(2^m x)$. Then there is $N = N(M) > 0$ sufficiently large so that

$$\|T(a_Q)\|_{L^1} = \lim_{m\to\infty} \|\phi_m * (T(a_Q))\|_{L^1} = \lim_{m\to\infty} \left\| \phi_m * \Big(\sum_{k=0}^{2^m N} K_k * a_Q \Big) \right\|_{L^1},$$

where $K_k(x) = \mathcal{F}^{-1}[a(\rho(\cdot))e^{i\rho(\cdot)}\theta_k(\rho(\cdot))](x)$. Let $\{\rho_n\}$ be a sequence of Minkowski functionals approximating ρ as in Lemma D, and let $K_{k,n}(x) = \mathcal{F}^{-1}[a(\rho_n(\cdot))e^{i\rho_n(\cdot)}\theta_k(\rho_n(\cdot))](x)$. Now, assuming that Theorem 4.1.3 holds in the special case that $\partial\Omega$ is smooth, for each

m we have

$$\left\|\phi_m * \Big(\sum_{k=0}^{2^m N} K_k * a_Q\Big)\right\|_{L^1} \lesssim \liminf_{n\to\infty} \left\|\phi_m * \Big(\sum_{k=0}^{2^m N} K_{k,n} * a_Q\Big)\right\|_{L^1}$$
$$\lesssim \liminf_{n\to\infty} \left\|\sum_{k=0}^{\infty} K_{k,n} * a_Q\right\|_{L^1} \lesssim 1,$$

where in the first step above we have used the fact that $\rho_n \to \rho$ uniformly on compact sets. Thus we have shown it suffices to prove Theorem 4.1.3 in the special case that $\partial\Omega$ is C^∞.

Reduction to the case of cubes with small sidelength

We assume $\partial\Omega$ is C^∞. We need to prove that for any atom a_Q,

$$\|T(a_Q)\|_{L^1(\mathbb{R}^2)} \leqslant C, \tag{4.45}$$

where C is a constant independent of the choice of Q or a_Q.

First suppose Q has sidelength $\geqslant 1$. Let $K(x) = \mathcal{F}^{-1}[a(\rho(\cdot))e^{i\rho(\cdot)}](x)$. Recall that ϕ_0 is a smooth function such that $[-1/2, 1/2] \prec \phi_0 \prec [-1, 1]$. Let $\phi(x) = \phi_0(2^{-6M}|x|)$. Then $(K\phi) * a_Q$ is supported in $2^{6M+1}Q$, where the dilation is taken from the center of Q. Since $\widehat{K} \in L^\infty$, $\|(K\phi) * a_Q\|_2 \lesssim \|a_Q\|_2$. By Cauchy-Schwarz,

$$\|(K\phi) * a_Q\|_{L^1} \lesssim |Q|^{1/2} \|(K\phi) * a_Q\|_{L^2} \lesssim |Q|^{1/2} \|a_Q\|_{L^2} \lesssim 1. \tag{4.46}$$

As stated in Remark 4.3.3, we have already shown in Section 4.3 that

$$\|(K(1-\phi)) * a_Q\|_{L^1} \lesssim 1,$$

which proves (4.45) if the sidelength of Q is $\geqslant 1$.

Thus we have reduced Theorem 4.1.3 to the following proposition.

Proposition 4.4.1. *Let Ω be a convex domain with smooth boundary satisfying (4.5), and let ρ be its Minkowski functional. Let a and T be as in the statement of Theorem 4.1.3. Then for every cube Q of sidelength $\leqslant 1$ and for every atom a_Q associated to Q, we have*

$$\|T(a_Q)\|_{L^1(\mathbb{R}^2)} \leqslant C,$$

where the constant C depends only on M and the quantitative estimates for a as a symbol of order $-1/2$.

We now make the same observation made at the beginning of the proof of Proposition 4.3.2 and note that it is enough to prove Proposition 4.4.1 with the kernel K of the operator T redefined as

$$K(x) := \mathcal{F}^{-1}[a(\rho(\cdot))e^{i\rho(\cdot)}\theta_k(\rho(\cdot))\chi(\cdot)](x), \tag{4.47}$$

where χ is the same smooth angular cutoff as in (4.29). Thus in what follows we will take (4.47) to be our definition of K.

Estimate on the exceptional set

In what follows we assume that Q is a cube of sidelength 2^{-l} for some $l \geqslant 0$, and a_Q an atom associated to Q. To prove Proposition 4.4.1, we will define an exceptional set of sufficiently small measure off of which $T(a_Q)$ decays. Let Σ_ρ be the smooth closed curve given by

$$\Sigma_\rho := \{\xi : \xi = -\nabla\rho(\xi') \text{ for some } \xi' \in \mathbb{R}^2\}.$$

Since $\nabla\rho$ is homogeneous of degree 0, this indeed corresponds to a smooth closed curve. As noted previously, the gradient of the phase $ix \cdot \xi + i\rho(\xi)$ vanishes on the singular set

Σ_ρ. We would like to associate to Q an exceptional set $\mathcal{N}_Q$. A natural choice for $\mathcal{N}_Q$ might be

$$\{x \in \mathbb{R}^2 |\, |x - \Sigma_\rho| \leqslant C2^{-l}\}$$

for some choice of constant C. However, for technical reasons we will choose $\mathcal{N}_Q$ to be a slightly larger set. Let $\{I_j\}$ be the partition of $[-1,1]$ into subintervals corresponding to the subset $\mathfrak{A}(2^{-l})$ of $[-1,1]$, as given by Lemma 4.2.1. We emphasize that although the collection of intervals $\{I_j\}$ is indexed only by j, it implicitly depends on l as well. (Recall that Q has sidelength 2^{-l}.) For each j, choose some $\alpha_j \in I_j$. Define

$$\begin{aligned} E_{\alpha_j} := \{x :\, &|\alpha_j x_1 + \gamma(\alpha_j) x_2 + 1| \leqslant 2^{-l+15M}, \\ &|x_1 + x_2 \gamma'(\alpha_j)| \leqslant 2^{-l+15M} |I_j|^{-1}\}, \end{aligned}$$

and define

$$\mathcal{N}_Q := \bigcup_j E_{\alpha_j}.$$

Then by (4.18),

$$|\mathcal{N}_Q| \lesssim \sum_j 2^{-2l} |I_j|^{-1} \lesssim 2^{-l}.$$

We follow [47] to estimate $T(a_Q)$ on $\mathcal{N}_Q$. By the Hardy-Littlewood-Sobolev inequality,

$$\left\|(I-\Delta)^{-1/4} a_Q\right\|_2 \lesssim \|a_Q\|_{4/3}.$$

Since a is a symbol of order $-1/2$ and ρ is homogeneous of degree one, the operator $T(I-\Delta)^{-1/4}$ is bounded on L^2, and so after using Hölder's inequality twice we have

$$\begin{aligned} \|T(a_Q)\|_{L^1(\mathcal{N}_Q)} \lesssim 2^{-l/2} \|T(a_Q)\|_2 \lesssim 2^{-l/2} \left\|(I-\Delta)^{-1/4} a_Q\right\|_2 \\ \lesssim 2^{-l/2} \|a_Q\|_{4/3} \lesssim 1. \end{aligned}$$

Thus to prove Proposition 4.4.1, It remains to show

$$\|T(a_Q)\|_{L^1(\mathbb{R}^2 \setminus \mathcal{N}_Q)} \lesssim 1. \tag{4.48}$$

As noted in Remark 4.3.3, we have already shown that

$$\int |K(x)(1 - \phi_0(2^{-6M}|x|))| \, dx \lesssim 1.$$

Thus if we let S denote the operator with kernel $K(x)(\phi_0(2^{-6M}|x|)$, (4.48) reduces to proving

$$\|S(a_Q)\|_{L^1(\mathbb{R}^2 \setminus \mathcal{N}_Q)} \lesssim 1. \tag{4.49}$$

We now proceed to decompose S as a sum of operators, some of which map a_Q to a function supported inside the exceptional set $\mathcal{N}_Q$; these operators will not contribute to the left hand side of (4.49). Let S_k denote the operator with kernel $K_k(x)\phi_0(2^{-6M}|x|)$, where

$$K_k(x) = \mathcal{F}^{-1}[a(\rho(\cdot))e^{i\rho(\cdot)}\theta_k(\rho(\cdot))\chi(\cdot)](x).$$

As before, we let $\{I_j\}$ be the collection of intervals corresponding to the partition of $[-1, 1]$ given by $\tilde{\mathfrak{A}}(2^{-l})$, as defined in Section 4.2.

For each j, define

$$\Phi_{l,j,0}(x, \alpha) = \phi_0(|I_j|2^l(x_1 + x_2\gamma'(\alpha))).$$

For each j, k and for each $n > 0$, define

$$\Phi_{k,j,n}(x, \alpha) = \phi_0(|I_j|2^{k-n}(x_1 + x_2\gamma'(\alpha))) - \phi_0(|I_j|2^{k-n+1}(x_1 + x_2\gamma'(\alpha))).$$

For each $k, j, n \geqslant 0$, we consider the operators $S_{l,k,j,n}$, $\tilde{S}_{l,k,j}$ and $S'_{l,k,j}$ with kernels $L_{l,k,j,n}$, $\tilde{L}_{l,k,j}$ and $L'_{l,k,j}$, respectively, given by

$$L_{l,k,j,n} := \phi_0(2^{-6M}|x|) \int\int e^{is(\alpha x_1+\gamma(\alpha)x_2+1)} \beta_{I_j}(\alpha) \times \Phi_{k,j,n}(x,\alpha)\theta_k(s)a(s)s(\alpha\gamma'(\alpha)-\gamma(\alpha))\chi(\alpha)\, d\alpha\, ds, \quad (4.50)$$

$$\tilde{L}_{l,k,j} := \phi_0(2^{-6M}|x|) \int\int e^{is(\alpha x_1+\gamma(\alpha)x_2+1)} \beta_{I_j}(\alpha) \times \Phi_{l,j,0}(x,\alpha)(1-\phi_0(2^l(\alpha x_1+\gamma(\alpha)x_2+1))) \times \theta_k(s)a(s)s(\alpha\gamma'(\alpha)-\gamma(\alpha))\chi(\alpha)\, d\alpha\, ds \quad (4.51)$$

and

$$L'_{l,k,j} := \phi_0(2^{-6M}|x|) \int\int e^{is(\alpha x_1+\gamma(\alpha)x_2+1)} \beta_{I_j}(\alpha) \times \Phi_{l,j,0}(x,\alpha)\phi_0(2^l(\alpha x_1+\gamma(\alpha)x_2+1)) \times \theta_k(s)a(s)s(\alpha\gamma'(\alpha)-\gamma(\alpha))\chi(\alpha)\, d\alpha\, ds. \quad (4.52)$$

Note that $L_{l,k,j,n}(x)$ is like $K_k(x)\phi_0(2^{-6M}|x|)$ with

$$\beta_{I_j}(\alpha) \cdot \Phi_{k,j,n}(x,\alpha)$$

inserted into the integral, $\tilde{L}_{l,k,j}(x)$ is like $K_k(x)\phi_0(2^{-6M}|x|)$ with

$$\beta_{I_j}(\alpha) \cdot \Phi_{l,j,0} \cdot (1-\phi_0(2^l(\alpha x_1+\gamma(\alpha)x_2+1)))$$

inserted into the integral, and $L'_{l,k,j}(x)$ is like $K_k(x)\phi_0(2^{-6M}|x|)$ with

$$\beta_{I_j}(\alpha) \cdot \Phi_{l,j,0} \cdot \phi_0(2^l(\alpha x_1+\gamma(\alpha)x_2+1))$$

inserted into the integral. These kernels are most easily visualized using the coordinate system of (4.37); see Figure 5.

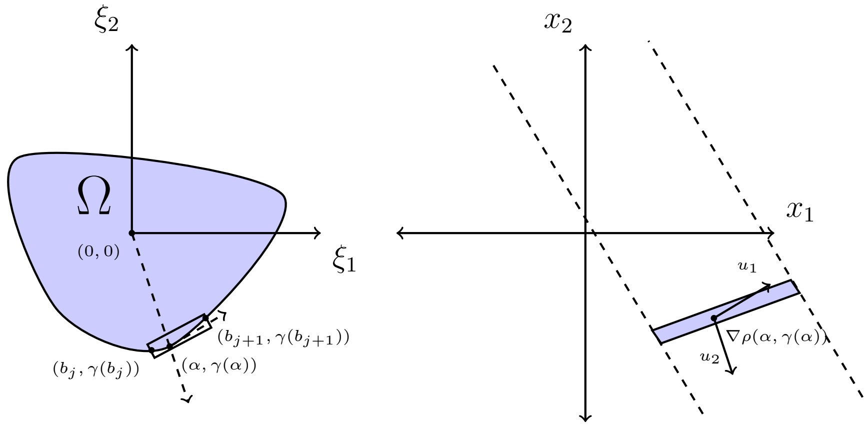

Figure 5

The domain Ω is depicted on the left, where for a fixed j a point $(\alpha, \gamma(\alpha))$ is chosen so that $\alpha \in I_j$. On the right, up to dilation by a constant, the shaded parallelogram represents the support of $\Phi_{l,j,0}(x, \alpha) \cdot \phi_0(2^l(\alpha x_1 + \gamma(\alpha)x_2 + 1))$, and up to dilation by a constant the region between the two dashed lines represents the support of $\Phi_{l,j,0}(x, \alpha) \cdot (1 - \phi_0(2^l(\alpha x_1 + \gamma(\alpha)x_2 + 1))$. The region outside the two dashed lines represents the support of $\sum_{n:\, n>k-l} \Phi_{k,j,n}(x, \alpha)$. Note that the long side of the shaded parallelogram is orthogonal to u_2, and the dashed lines are orthogonal to u_1. The short side of the parallelogram has length $\approx 2^{-l}$, and the long side has length $\approx 2^{-l}|I_j|^{-1}$.

We can write

$$S = \sum_{k:\, k<l} S_k + \sum_{k:\, k\geqslant l} \sum_{n:\, n>k-l} \sum_j S_{l,k,j,n} + \sum_{k:\, k\geqslant l} \sum_j (\tilde{S}_{l,k,j} + S'_{l,k,j}). \tag{4.53}$$

If we fix some $k \geqslant l$ and freeze all sums in k in (4.53), then we may interpret (4.53) as follows. The term $\sum_{n:\, n>k-l} \sum_j S_{l,k,j,n}$ may be thought of as the portion of the kernel of S_k supported away in the u_1-direction from the exceptional set $\mathcal{N}_Q$, with the distance from $\mathcal{N}_Q$ increasing as n increases. The term $\sum_j \tilde{S}_{l,k,j}$ may be thought of as the portion of the kernel of S_k supported away in the u_2-direction from $\mathcal{N}_Q$. We will see that the kernel of the term $\sum_j S'_{l,k,j}$ is supported in $\mathcal{N}_Q$. We prove the following lemma.

Lemma 4.4.2. *The support of*

$$\sum_{k:\, k\geqslant l} \sum_j S'_{l,k,j} a_Q$$

is contained in $\mathcal{N}_Q$.

Proof of Lemma 4.4.2. Since a_Q is supported in a cube of sidelength 2^{-l}, it suffices to show that the kernel of $\sum_{k:\, k\geqslant l} \sum_j S'_{l,k,j}$ is supported in

$$\tilde{\mathcal{N}}_Q := \bigcup_j \tilde{E}_{\alpha_j}.$$

where

$$\tilde{E}_{\alpha_j} := \{x : \, |\alpha_j x_1 + \gamma(\alpha_j) x_2 + 1| \leqslant 2^{-l+14M}, \\ |x_1 + x_2 \gamma'(\alpha_j)| \leqslant 2^{-l+14M} |I_j|^{-1}\}.$$

Observe that if we set

$$c_\alpha = \gamma'(\alpha)(\alpha\gamma'(\alpha) - \gamma(\alpha))^{-1}$$

and

$$d_\alpha = -(\alpha\gamma'(\alpha) - \gamma(\alpha))^{-1},$$

then

$$\alpha x_1 + \gamma(\alpha) x_2 + 1 = (\alpha, \gamma(\alpha)) \cdot (x_1 + c_\alpha, x_2 + d_\alpha),$$

and moreover

$$(c_\alpha, d_\alpha) \cdot (1, \gamma'(\alpha)) = 0.$$

In fact, (4.25) states that $(c_\alpha, d_\alpha) = \nabla\rho(\alpha, \gamma(\alpha))$. Now, for any $\alpha, \alpha' \in I_j^*$, (4.22) implies that we have

$$\begin{aligned}(\alpha, \gamma(\alpha)) \cdot (c_{\alpha'} - c_\alpha, d_{\alpha'} - d_\alpha) &= (c_{\alpha'}, d_{\alpha'}) \cdot (\alpha, \gamma(\alpha)) - 1 \\ &= \frac{(\gamma'(\alpha'), -1) \cdot (\alpha, \gamma(\alpha))}{(\gamma'(\alpha'), -1) \cdot (\alpha', \gamma(\alpha'))} - 1. \quad (4.54)\end{aligned}$$

By (4.12), we have that

$$|(\gamma'(\alpha'), -1) \cdot (\alpha' - \alpha, \gamma(\alpha') - \gamma(\alpha))| \leqslant 2^{-l+4}. \quad (4.55)$$

Indeed, (4.55) is equivalent to the statement that $(\alpha, \gamma(\alpha))$ is contained in a rectangle of width $\leqslant 2^{-l+4}$ containing $(\alpha', \gamma(\alpha'))$ with short side parallel to the normal to $\partial\Omega$ at $(\alpha', \gamma(\alpha'))$. That is, $(\alpha, \gamma(\alpha))$ and $(\alpha', \gamma(\alpha'))$ are contained in a single "Minkowski cap" of width $\delta \leqslant 2^{-l+4}$.

As mentioned in (4.24), $|(\gamma'(\alpha'), -1) \cdot (\alpha', \gamma(\alpha'))| \geqslant 2^{-4M}$, and so it follows from (4.54) and (4.55) that

$$\begin{aligned}|(\alpha, \gamma(\alpha)) \cdot (c_{\alpha'} - c_\alpha, d_{\alpha'} - d_\alpha)| &\leqslant \frac{(\gamma'(\alpha'), -1) \cdot (\alpha' - \alpha, \gamma(\alpha') - \gamma(\alpha))}{(\gamma'(\alpha'), -1) \cdot (\alpha', \gamma(\alpha'))} \\ &\leqslant 2^{-l+5M}. \quad (4.56)\end{aligned}$$

We also note that for any $\alpha, \alpha' \in I_j^*$,

$$\begin{aligned}|(c_{\alpha'} - c_\alpha, d_{\alpha'} - d_\alpha)| &\leqslant 2^{10M} \max(|\gamma'(\alpha) - \gamma'(\alpha')|, |\gamma(\alpha) - \gamma(\alpha')|) \\ &\leqslant 2^{10M} \max(2^{-l}|I_j|^{-1}, 2^{-l}) \leqslant 2^{-l+10M}|I_j|^{-1}, \quad (4.57)\end{aligned}$$

where in the second step we have used (4.12). It follows from (4.56) and (4.57) that for any $\alpha, \alpha' \in I_j^*$,

$$\operatorname{supp}\Big(\phi_0(|I_j|2^l(x_1 + x_2\gamma'(\alpha')))\phi_0(2^l(\alpha' x_1 + \gamma(\alpha')x_2 + 1))\Big)$$
$$\subset \{x : \ (x + (c_\alpha, d_\alpha)) \cdot (1, \gamma'(\alpha')) \leqslant 2^{-l+12M}|I_j|^{-1},$$
$$(x + (c_\alpha, d_\alpha)) \cdot (\alpha', \gamma(\alpha')) \leqslant 2^{-l+12M}\}. \quad (4.58)$$

Next, we note that (4.5) implies that for any $\alpha, \alpha' \in I_j^*$, the angle between $(\alpha, \gamma(\alpha))$ and $(\alpha', \gamma(\alpha'))$ is $\leqslant |I_j|$, and this combined with (4.58) implies that for any $\alpha, \alpha' \in I_j^*$,

$$\operatorname{supp}\Big(\phi_0(|I_j|2^l(x_1 + x_2\gamma'(\alpha')))\phi_0(2^l(\alpha' x_1 + \gamma(\alpha')x_2 + 1))\Big)$$
$$\subset \tilde{E}_\alpha := \{x : |\alpha x_1 + \gamma(\alpha)x_2 + 1| \leqslant 2^{-l+14M}, \, |x_1 + x_2\gamma'(\alpha)| \leqslant 2^{-l+14M}|I_j|^{-1}\}, \quad (4.59)$$

and taking $\alpha = \alpha_j$ completes the proof. □

We have thus reduced Proposition 4.4.1, and hence also Theorem 4.1.3, to the following proposition.

Proposition 4.4.3. *Let* $\tilde{S}_{l,k,j}$, $S_{l,k,j,n}$ *and* S_k *be as defined previously. Then*

$$\Big\|\big(\sum_{k:k\geqslant l}\sum_j(\tilde{S}_{l,k,j} + \sum_{n:n>k-l} S_{l,k,j,n})\big)(a_Q)\Big\|_{L^1(\mathbb{R}^2)} \lesssim 1 \quad (4.60)$$

and

$$\Big\|\sum_{k:k<l} S_k(a_Q)\Big\|_{L^1(\mathbb{R}^2)} \lesssim 1. \quad (4.61)$$

4.5 The $H^1 \to L^1$ endpoint estimate: estimate off the exceptional set

As in the previous section, throughout this section $\kappa_\Omega = 1/2$. We again note that we will often continue to write κ_Ω instead of subsituting 1/2 simply to indicate how certain quantities in our estimates arise. We have shown that to prove that the operator S maps a_Q into L^1, we may ignore the term $\sum_{k:k\geqslant l}\sum_j S'_{l,k,j}$ in (4.53). All other terms in (4.53) map a_Q to a function that is supported off the exceptional set. In summary, we have shown that Theorem 4.1.3 reduces to proving Proposition 4.4.3, and so this section will be devoted to proving Proposition 4.4.3.

The case $k \geqslant l$

To prove (4.60), we will first prove the following lemma.

Lemma 4.5.1. *Let $\tilde{L}_{l,k,j}$ be as defined previously. Then*

$$\sum_{k\geqslant l}\sum_j \int |\tilde{L}_{l,k,j}(x)|\, dx \lesssim 1. \tag{4.62}$$

Proof of Lemma 4.5.1. Integrating by parts (4.51) three times with respect to s yields

$$\int |\tilde{L}_{l,k,j}(x)|\, dx \lesssim 2^{k(1-\kappa_\Omega)} \int_{I_j{}^*} \int_{\substack{|x_1+x_2\gamma'(\alpha)|\leqslant |I_j|^{-1}2^{-l} \\ |\alpha x_1+\gamma(\alpha)x_2+1|\gtrsim 2^{-l}}} \frac{2^k}{(1+2^k|\alpha x_1+\gamma(\alpha)x_2+1|)^3}\, dx\, d\alpha.$$

Applying the change of coordinates (4.37) yields

$$\int |\tilde{L}_{l,k,j}(x)|\, dx \lesssim 2^{k(1-\kappa_\Omega)} \int_{I_j^*} \int_{\substack{|u_1|\leqslant |I_j|^{-1}2^{-l} \\ |u_2|\gtrsim 2^{-l}}} \frac{2^k}{(1+2^k|u_2|)^3}\, du_1\, du_2\, d\alpha$$
$$\lesssim 2^{l-k}2^{-k\kappa_\Omega}.$$

By (4.19), there are $\lesssim 2^{l/2}$ intervals I_j, so we may sum in j and then in k to obtain (4.62). $\square$

To prove (4.60), it remains to prove

Lemma 4.5.2. *Let $S_{l,k,j,n}$ be as defined previously. Then*

$$\Big\|\Big(\sum_{k:k\geqslant l}\sum_{j}\sum_{n:n>k-l} S_{l,k,j,n}(a_Q)\Big\|_{L^1(\mathbb{R}^2)} \lesssim 1. \tag{4.63}$$

Recall our treatment of the kernels $K_{k,j,n}$ in Section 4.3. In order to achieve sufficient decay in n for $\int |K_{k,j,n}(x)|\,dx$ to prove an endpoint estimate, we would have had to integrate by parts twice in the α variable. However, doing so would make our estimates for $\int |K_{k,j,n}(x)|\,dx$ ultimately depend on the C^2 norm of the graph of $\partial\Omega$. Thus in our analysis of the kernels of the operators $S_{l,k,j,n}$, we will instead opt to approximate $\partial\Omega$ by a smooth curve whose curvature is essentially constant on "Minkowski caps" of width 2^{-k}, allowing us to perform the necessary integration by parts.

Recall that $\{I_j\} = \{[b_j, b_{j+1}]\}$ is the partition of $[-1,1]$ into subintervals with endpoints in $\tilde{\mathfrak{A}}(2^{-l})$, where $\tilde{\mathfrak{A}}(2^{-l})$ is the refinement of $\mathfrak{A}(2^{-l})$ given by Lemma 4.2.1. Fix $k \geqslant l$, and let $\{J_m\} = \{[c_m, c_{m+1}]\}$ be the partition of $[-1,1]$ into subintervals with endpoints in $\mathfrak{A}(2^{-k})$. We will prove the following approximation lemma.

Lemma 4.5.3. *Fix integers $l, k \geqslant 0$ with $k \geqslant l$, and define $\{I_j\}$ and $\{J_m\}$ as above. Then there exists a smooth function $\gamma_k : [-1,1] \to \mathbb{R}$ such that for every $x \in \mathfrak{A}(2^{-k})$,*

$$\gamma_k(x) = \gamma(x), \tag{4.64}$$

$$\gamma_k'(x) = \gamma'(x), \tag{4.65}$$

and for every $\alpha \in J_m$,

$$|\gamma_k''(\alpha)| \lesssim (\gamma'(c_{m+1}) - \gamma'(c_m))|J_m|^{-1} \lesssim 2^{-k}|J_m|^{-2}, \tag{4.66}$$

and

$$\int_{J_m} |\gamma_k'''(\alpha)|\, d\alpha \lesssim 2^{-k}|J_m|^{-2}. \tag{4.67}$$

Moreover, for every j,

$$\int_{I_j^*} |I_j||\gamma_k''(\alpha)|\, d\alpha \lesssim 2^{-l} \tag{4.68}$$

and for any $\alpha \in I_j^*$,

$$|\gamma_k'(\alpha) - \gamma'(\alpha)| \lesssim 2^{-l}|I_j|^{-1}. \tag{4.69}$$

Remark 4.5.4. *Note that (4.65) and (4.66) imply that for every* $\alpha \in J_m$,

$$\begin{aligned}|\gamma(\alpha) - \gamma_k(\alpha)| &\lesssim \int_{J_m} |\gamma'(\alpha) - \gamma_k'(\alpha)|\, d\alpha \\ &\lesssim \int_{J_m}\int_{c_m}^{\alpha} (|\gamma''(t)| + |\gamma_k''(t)|)\, dt\, d\alpha \lesssim (\gamma'(c_{m+1}) - \gamma'(c_m))|J_m| \lesssim 2^{-k}, \end{aligned} \tag{4.70}$$

and

$$\begin{aligned}|\gamma'(\alpha) - \gamma_k'(\alpha)| &\lesssim \int_{J_m} (|\gamma''(\alpha)| + |\gamma_k''(\alpha)|)\, d\alpha \\ &\lesssim \gamma'(c_{m+1}) - \gamma'(c_m) \lesssim 2^{-k}|J_m|^{-1}. \end{aligned} \tag{4.71}$$

Proof of Lemma 4.5.3. The idea of the construction is to first define γ_k near each point $x \in \mathfrak{A}(2^{-k})$ so that its graph is a line segment with slope $\gamma'(x)$, to connect these line segments with curves of constant curvature, and then to smooth things out using an appropriate mollifier. We now proceed to give the details.

We first define γ_k in a neighborhood of each $x \in \mathfrak{A}(2^{-k})$. For each such x, let $J_{m(x)}$ be the element of $\{J_m\}$ whose right endpoint is x. Let O_x be the interval $[x - \frac{|J_{m(x)}|}{100}, x + \frac{|J_{m(x)+1}|}{100}]$. Define a function $\gamma_{k,x}$ on O_x so that $\{(\alpha, \gamma_{k,x}(\alpha)) : \alpha \in O_x\}$ is the graph of a line segment satisfying $\gamma_{k,x}(x) = \gamma(x)$ and $\gamma'_{k,x}(x) = \gamma'(x)$. Let x^+ be the successor of x in $x \in \mathfrak{A}(2^{-k})$. We now extend $\gamma_{k,x}$ to $\tilde{O}_x := [x - \frac{|J_{m(x)}|}{100}, x^+ - \frac{|J_{m(x)+1}|}{100}]$ by connecting the points

$$\left(x + \frac{|J_{m(x)+1}|}{100}, \gamma(x + \frac{|J_{m(x)+1}|}{100})\right); \left(x^+ - \frac{|J_{m(x)+1}|}{100}, \gamma(x^+ - \frac{|J_{m(x)+1}|}{100})\right) \tag{4.72}$$

by the unique curve of constant curvature that has slope $\gamma'(x)$ at the point

$$\left(x + \frac{|J_{m(x)+1}|}{100}, \gamma_{k,x}(x + \frac{|J_{m(x)+1}|}{100})\right).$$

Note that for α between the two points (4.72),

$$|\gamma''_{k,x}(\alpha)| \lesssim (\gamma'(c_{m+1}) - \gamma'(c_m))|J_m|^{-1} \lesssim 2^{-k}|J_{m(x)+1}|^{-2}. \tag{4.73}$$

Now define a piecewise smooth curve $\tilde{\gamma}_k : [-1, 1] \to \mathbb{R}$ by $\tilde{\gamma}_k|_{\tilde{O}_x} = \gamma_{k,x}$.

For each $x \in \mathfrak{A}(2^{-k})$, let $U_x = [x + \frac{|J_{m(x)+1}|}{200}, x^+ - \frac{|J_{m(x)+1}|}{200}]$. Let ψ_x be a smooth positive bump function supported in

$$\left[-\frac{|J_{m(x)+1}|}{800}, \frac{|J_{m(x)+1}|}{800}\right]$$

with $\int \psi_x = 1$ and satisfying

$$D^\beta \psi_x \lesssim_\beta |J_{m(x)+1}|^{-\beta-1}, \qquad \beta \geqslant 0 \text{ an integer}. \tag{4.74}$$

Define a smooth curve $\gamma_k : [-1, 1] \to \mathbb{R}$ by $\gamma_k|_{U_x} = \tilde{\gamma}_k * \psi_x$ and $\gamma_k|_{(\bigcup_x U_x)^c} = \tilde{\gamma}_k$.

By construction, γ_k satisfies (4.64) and (4.65). On $(\bigcup_x U_x)^c$, γ''_k is identically 0. Let $\tilde{\gamma}''_k$ denote the a.e. defined pointwise second derivative of $\tilde{\gamma}_k$. Let $\tilde{\gamma}'_{k,L}$ and $\tilde{\gamma}'_{k,R}$

denote the (everywhere defined) left and right derivatives of $\tilde{\gamma}_k$, respectively. Then for $\alpha \in U_x \subset J_{m(x)+1}$,

$$|\gamma_k''(\alpha)| \lesssim |(\tilde{\gamma}_k'' * \psi_x)(\alpha)| + |\tilde{\gamma}_{k,R}'(x^+ - \frac{|J_{m(x)+1}|}{100}) - \tilde{\gamma}_{k,L}'(x^+ - \frac{|J_{m(x)+1}|}{100})| \, \|\psi_x\|_\infty$$
$$\lesssim \sup_{\alpha \in U_x} |\tilde{\gamma}_k''(\alpha)| + (\gamma'(c_{m+1}) - \gamma'(c_m))|J_m|^{-1}$$
$$\lesssim (\gamma'(c_{m+1}) - \gamma'(c_m))|J_m|^{-1} \lesssim 2^{-k}|J_{m(x)+1}|^{-2}, \quad (4.75)$$

where in the second to last inequality we have used (4.73). Thus γ_k satisfies (4.66). By (4.73) and (4.74), we also have

$$\int_{J_{m(x)+1}} |\gamma_k'''(\alpha)| \, d\alpha \lesssim \int_{U_x} |(\tilde{\gamma}_k'' * \psi_x')(\alpha)| \, d\alpha$$
$$+ |\tilde{\gamma}_{k,R}'(x^+ - \frac{|J_{m(x)+1}|}{100}) - \tilde{\gamma}_{k,L}'(x^+ - \frac{|J_{m(x)+1}|}{100})| \, \|\psi_x'\|_\infty \, |J_{m(x)+1}|$$
$$\lesssim \int_{J_{m(x)+1}} 2^{-k}|J_{m(x)+1}|^{-3} \, d\alpha + 2^{-k}|J_{m(x)+1}|^{-2} \lesssim 2^{-k}|J_{m(x)+1}|^{-2},$$

and so γ_k satisfies (4.67).

Now we show that γ_k satisfies (4.68). Note that (4.66) implies that for each m,

$$\int_{J_m} |\gamma_k''(\alpha)| \, d\alpha \lesssim \gamma'(c_{m+1}) - \gamma'(c_m). \quad (4.76)$$

Given $I_j = [b_j, b_{j+1}]$, choose m, m' to the the greatest and least integers, respectively, so that $I_j^* \subset [c_m, c_{m'}]$. Let b_j^* and b_{j+1}^* denote the left and right endpoints of I_j^*, respectively. If $b_j^* - c_m \leqslant |I_j|/100$, then by (4.17) we have $b_{j-1} \leqslant c_m$, so by (4.76) we have

$$\int_{I_j^*} |\gamma_k''(\alpha)| \, d\alpha \lesssim \gamma'(c_{m'}) - \gamma'(c_m) \lesssim \gamma'(c_{m'}) - \gamma'(b_{j-1}).$$

Otherwise, $b_j^* - c_m > |I_j|/100$, and so (4.16) implies that

$$\gamma'(b_j^*) - \gamma'(c_m) \lesssim 2^{-k}|I_j|^{-1}$$

and hence

$$\int_{I_j^*} |\gamma_k''(\alpha)|\, d\alpha \lesssim \gamma'(c_{m'}) - \gamma'(c_m) \lesssim \gamma'(c_{m'}) - \gamma'(b_{j^*}) + 2^{-k}|I_j|^{-1}.$$

In either case, we have

$$\int_{I_j^*} |\gamma_k''(\alpha)|\, d\alpha \lesssim \gamma'(c_{m'}) - \gamma'(b_{j-1}) + 2^{-k}|I_j|^{-1}.$$

Arguing similarly with $c_{m'}$ and b_{j+1}^* in place of c_m and b_j^*, we may obtain

$$\int_{I_j^*} |\gamma_k''(\alpha)|\, d\alpha \lesssim \gamma'(b_{j+1}) - \gamma'(b_{j-1}) + 2^{-k}|I_j|^{-1}.$$

By (4.16) and (4.17), $\gamma'(b_{j+1}) - \gamma'(b_{j-1}) \lesssim 2^{-l}|I_j|^{-1}$, and since $k \geqslant l$ it follows that

$$\int_{I_j^*} |I_j||\gamma_k''(\alpha)|\, d\alpha \lesssim 2^{-l}.$$

Thus γ_k satisfies (4.68).

Finally, we show that γ_k satisfies (4.69). Suppose we are given some j and some $\alpha \in I_j^*$. If there exists m such that $c_m \in I_j^*$, then by (4.64) and (4.68),

$$|\gamma_k'(\alpha) - \gamma'(\alpha)| \lesssim \int_{I_j^*} (|\gamma_k''(\alpha)| + |\gamma''(\alpha)|)\, d\alpha \lesssim 2^{-l}|I_j|^{-1}.$$

Otherwise, choose m so that the distance of c_m from I_j^* is minimal. Without loss of generality, suppose $c_m < b_j^*$. Then $c_{m+1} - c_m \gtrsim |I_j|$, so by (4.66) and (4.68),

$$\begin{aligned}|\gamma_k'(\alpha) - \gamma'(\alpha)| &\lesssim \int_{[c_m, c_{m+1}] \cup I_j^*} (|\gamma_k''(\alpha)| + |\gamma''(\alpha)|)\, d\alpha \\ &\lesssim 2^{-k}|I_j|^{-1} + 2^{-l}|I_j|^{-1} \lesssim 2^{-l}|I_j|^{-1},\end{aligned}$$

and hence γ_k satisfies (4.69). $\square$

The error estimate

Define

$$B_{l,k,j,n}(x) = \phi_0(2^{-6M}|x|)\int_0^\infty \int_{I_j^*} e^{is(\alpha x_1+\gamma_k(\alpha)x_2+1)}$$

$$\beta_{I_j}(\alpha)\Phi_{k,j,n}(x,\alpha)\theta_k(s)a(s)s(\alpha\gamma_k'(\alpha)-\gamma_k(\alpha))\,d\alpha\,ds.$$

Note that $B_{l,k,j,n}$ is like $L_{l,k,j,n}$ with every occurrence of γ in the integral replaced by γ_k. We will prove

Lemma 4.5.5. *If $k \geqslant l$ and $n > k-l$, then*

$$\|L_{l,k,j,n} - B_{l,k,j,n}\|_{L^1(\mathbb{R}^2)} \lesssim 2^{-k\kappa_\Omega}(2^{n-k}|I_j|^{-1}). \tag{4.77}$$

Remark 4.5.6. *We now state a consequence of Lemma (4.5.5). By (4.19), there are $\lesssim 2^{l\kappa_\Omega}$ intervals I_j. Moreover, the presence of $\phi_0(2^{-6M}|x|)$ implies that all terms with $2^{n-k}|I_j|^{-1} \gg 1$ are identically 0, so (5.17) implies that*

$$\sum_{\substack{k:k\geqslant l,\\ j,\\ n:n>k-l}} \|L_{l,k,j,n} - B_{l,k,j,n}\|_{L^1(\mathbb{R}^2)} \lesssim 1. \tag{4.78}$$

Then (4.78) implies that it suffices to prove (4.63) with $S_{l,k,j,n}$ replaced by the operator with kernel $B_{l,k,j,n}$.

Proof of Lemma 4.5.5. The first step is to write

$$L_{l,k,j,n}(x) - B_{l,k,j,n}(x) = H_1(x) + H_2(x),$$

where

$$H_1(x) = \phi_0(2^{-6M}|x|)\int_0^\infty \int_{I_j^*} e^{is(\alpha x_1+\gamma(\alpha)x_2+1)}(1-e^{is(\gamma_k(\alpha)x_2-\gamma(\alpha)x_2)})$$

$$\beta_{I_j}(\alpha)\Phi_{k,j,n}(x,\alpha)\theta_k(s)a(s)s(\alpha\gamma'(\alpha)-\gamma(\alpha))\,d\alpha\,ds$$

and

$$H_2(x) = \phi_0(2^{-6M}|x|) \int_0^\infty \int_{I_j^*} e^{is(\alpha x_1 + \gamma_k(\alpha) x_2 + 1)} \beta_{I_j}(\alpha) \Phi_{k,j,n}(x,\alpha)$$
$$\theta_k(s) a(s) s(\alpha(\gamma'(\alpha) - \gamma_k'(\alpha)) - (\gamma(\alpha) - \gamma_k(\alpha)))\, d\alpha\, ds.$$

Note that the only places where the kernels $B_{l,k,j,n}$ and $L_{l,k,j,n}$ differ are in the complex exponential factor and the Jacobian factor in their integral representations. Here the term H_1 represents the difference in the complex exponential factor and the term H_2 represents the difference in the Jacobian factor. The estimation of $\int |H_1(x)|\, dx$ and $\int |H_2(x)|\, dx$ will share some similarities with the estimation of $\int |K_{k,j,n}(x)|\, dx$ from Section 3.

Estimation of $\int |H_1(x)|\, dx$

We observe that (4.70) implies that for s, x, α in the support of $\phi_0(2^{-6M}|x|)\theta_k(s)\beta_{I_j}(\alpha)$ and for every integer $N \geqslant 0$,

$$|\partial_s^N \partial_\alpha (1 - e^{is(\gamma_k(\alpha)x_2 - \gamma(\alpha)x_2)})| \lesssim_N 2^{-kN} 2^k |\gamma_k'(\alpha) - \gamma'(\alpha)||x|, \tag{4.79}$$

$$|\partial_s^N \partial_\alpha^2 (1 - e^{is(\gamma_k(\alpha)x_2 - \gamma(\alpha)x_2)})| \lesssim_N$$
$$2^{-kN}|x|\Big(2^{2k}|\gamma_k'(\alpha) - \gamma'(\alpha)|^2 + 2^k(|\gamma_k''(\alpha)| + |\gamma''(\alpha)|)\Big) \tag{4.80}$$

and

$$|\partial_s^N (1 - e^{is(\gamma_k(\alpha)x_2 - \gamma(\alpha)x_2)})| \lesssim_N 2^{-kN}|x|. \tag{4.81}$$

Integrating by parts H_1 once in α yields

$$H_1(x) = \phi_0(2^{-6M}|x|) \int_0^\infty \int_{I_j^*} e^{is(\alpha x_1 + \gamma(\alpha) x_2 + 1)} \partial_\alpha g_{l,k,j,n}(x, s, \alpha) \theta_k(s) a(s)\, ds$$

where

$$g_{l,k,j,n}(x, s, \alpha) = \frac{(1 - e^{is(\gamma_k(\alpha) x_2 - \gamma(\alpha) x_2)}) \beta_{I_j}(\alpha) \Phi_{k,j,n}(x, \alpha)(\alpha \gamma'(\alpha) - \gamma(\alpha))}{x_1 + x_2 \gamma'(\alpha)}.$$

Now if ∂_α hits the term $(1 - e^{is(\gamma_k(\alpha)x_2 - \gamma(\alpha)x_2)})$, then we may integrate by parts again in α, since no higher derivatives of γ or γ_k will appear. Thus we will further decompose

$$H_1(x) = H_{1,1}(x) + H_{1,2}(x),$$

where

$$H_{1,1}(x) = \phi_0(2^{-6M}|x|) \int_0^\infty \int_{I_j^*} e^{is(\alpha x_1 + \gamma(\alpha) x_2 + 1)} h_{l,k,j,n,1}(x, s, \alpha) \theta_k(s) a(s)\, ds$$

$$H_{1,2}(x) = \phi_0(2^{-6M}|x|) \int_0^\infty \int_{I_j^*} e^{is(\alpha x_1 + \gamma(\alpha) x_2 + 1)} h_{l,k,j,n,2}(x, s, \alpha) \theta_k(s) a(s)\, ds,$$

and

$$h_{l,k,j,n,1}(x, s, \alpha) = (1 - e^{is(\gamma_k(\alpha) x_2 - \gamma(\alpha) x_2)}) \partial_\alpha [\frac{\beta_{I_j}(\alpha) \Phi_{k,j,n}(x, \alpha)(\alpha \gamma'(\alpha) - \gamma(\alpha))}{x_1 + x_2 \gamma'(\alpha)}],$$

$$h_{l,k,j,n,2}(x, s, \alpha) = \partial_\alpha \left[\frac{\partial_\alpha [1 - e^{is(\gamma_k(\alpha) x_2 - \gamma(\alpha) x_2)}] \beta_{I_j}(\alpha) \Phi_{k,j,n}(x, \alpha)(\alpha \gamma'(\alpha) - \gamma(\alpha))}{s(x_1 + x_2 \gamma'(\alpha))^2} \right].$$

Here we may think of $H_{1,1}$ as representing the case when ∂_α does not hit the term $(1 - e^{is(\gamma_k(\alpha)x_2 - \gamma(\alpha)x_2)})$ when we integrate H_1 by parts with respect to α, and $H_{1,2}$ may be thought of as representing the case when ∂_α does hit $(1 - e^{is(\gamma_k(\alpha)x_2 - \gamma(\alpha)x_2)})$.

Estimation of $\int |H_{1,1}(x)|\,dx$

Observe that (4.81) with $N = 0$ implies that

$$|h_{l,k,j,n,1}(x,s,\alpha)| \lesssim \frac{|\gamma''(\alpha)|(|I_j|2^{k-n}|x|+1)+|I_j|^{-1}}{|x_1+x_2\gamma'(\alpha)|}|x|.$$

Thus integrating by parts in s three times and using (4.81) and the change of coordinates (4.37) yields

$$\begin{aligned}\int |H_{1,1}(x)|\,dx \lesssim 2^{-k\kappa_\Omega} &\int_{I_j^*} (|\gamma''(\alpha)|(|I_j|2^{k-n}+1)+|I_j|^{-1}) \\ &\times \int_{|u_1|\approx 2^{n-k}|I_j|^{-1}} \frac{1}{|u_1|}\frac{2^k}{(1+2^k|u_2|)^3}|u|\,du\,d\alpha \\ &\lesssim 2^{-k\kappa_\Omega}2^{n-k}|I_j|^{-1}\int_{I_j^*}(|\gamma''(\alpha)|(|I_j|2^{k-n}+1)+|I_j|^{-1})\,d\alpha.\end{aligned}$$

By (4.16) and (4.17), we have

$$\int_{I_j^*} |\gamma''(\alpha)||I_j|\,d\alpha \lesssim 2^{-l},$$

and so when $n > k-l$,

$$\int |H_{1,1}(x)|\,dx \lesssim 2^{-k\kappa_\Omega}2^{n-k}|I_j|^{-1}(2^{k-l-n}+1) \lesssim 2^{-k\kappa_\Omega}2^{n-k}|I_j|^{-1}. \tag{4.82}$$

Estimation of $\int |H_{1,2}(x)|\,dx$

Note that (4.79) and (4.80) with $N = 0$ implies that

$$\begin{aligned}|h_{l,k,j,n,2}(x,s,\alpha)| \lesssim\; & \\ &\frac{|\gamma''(\alpha)|(|I_j|2^{k-n}|x|+1)+|I_j|^{-1}}{|x_1+x_2\gamma'(\alpha)|}|x|\left(2^{k-n}|I_j||\gamma_k'(\alpha)-\gamma'(\alpha)|\right) \\ &+\frac{|x|}{|x_1+x_2\gamma'(\alpha)|}\left(2^{2k-n}|I_j||\gamma_k'(\alpha)-\gamma'(\alpha)|^2\right) \\ &+\frac{|x|}{|x_1+x_2\gamma'(\alpha)|}2^{k-n}|I_j|(|\gamma_k''(\alpha)|+|\gamma''(\alpha)|).\end{aligned} \tag{4.83}$$

Using (4.79), (4.81), (4.83) and the change of coordinates (4.37), we have

$$\begin{aligned}\int |H_{1,2}(x)|\,dx \lesssim&\\ &\Big(2^{-k\kappa_\Omega}\int_{I_j^*} 2^{k-n}|I_j||\gamma_k'(\alpha)-\gamma'(\alpha)|\big(|\gamma''(\alpha)|(|I_j|2^{k-n}+1)+|I_j|^{-1}\big)\\ &\qquad\times\int_{|u_1|\approx 2^{n-k}|I_j|^{-1}} \frac{1}{|u_1|}\frac{2^k}{(1+2^k|u_2|)^3}|u|\,du\,d\alpha\Big)\\ &\qquad+\Big(2^{-k\kappa_\Omega}\int_{I_j^*} 2^{2k-n}|I_j||\gamma_k'(\alpha)-\gamma'(\alpha)|^2\\ &\qquad\times\int_{|u_1|\approx 2^{n-k}|I_j|^{-1}} \frac{1}{|u_1|}\frac{2^k}{(1+2^k|u_2|)^3}|u|\,du\,d\alpha\Big)\\ &\qquad+\Big(2^{-k\kappa_\Omega}\int_{I_j^*} 2^{k-n}|I_j|(|\gamma_k''(\alpha)|+|\gamma''(\alpha)|)\\ &\qquad\qquad\times\int_{|u_1|\approx 2^{n-k}|I_j|^{-1}} \frac{1}{|u_1|}\frac{2^k}{(1+2^k|u_2|)^3}|u|\,du\,d\alpha\Big),\end{aligned}$$

and hence proceeding as in the estimation of $\int |H_{1,1}(x)|\,dx$ we have

$$\begin{aligned}\int |H_{1,2}(x)|\,dx \lesssim&\\ &\Big(2^{-k\kappa_\Omega}2^{n-k}|I_j|^{-1}\int_{I_j^*} 2^{k-n}|I_j||\gamma_k'(\alpha)-\gamma'(\alpha)||\gamma''(\alpha)|\big((|I_j|2^{k-n}+1)+|I_j|^{-1}\big)\,d\alpha\Big)\\ &\qquad+\Big(2^{-k\kappa_\Omega}2^{n-k}|I_j|^{-1}\int_{I_j^*} 2^{2k-n}|I_j||\gamma_k'(\alpha)-\gamma'(\alpha)|^2\,d\alpha\Big)\\ &\qquad\qquad+\Big(2^{-k\kappa_\Omega}2^{n-k}|I_j|^{-1}\int_{I_j^*} 2^{k-n}|I_j|(|\gamma_k''(\alpha)|+|\gamma''(\alpha)|)\,d\alpha\Big).\end{aligned}$$

Note that since $\{I_j\}$ satisfies (4.16) and (4.17), we have

$$\int_{I_j^*} |I_j||\gamma''(\alpha)|\,d\alpha \lesssim 2^{-l}.$$

As stated in (4.68), we also have

$$\int_{I_j^*} |I_j||\gamma_k''(\alpha)|\,d\alpha \lesssim 2^{-l}.$$

Thus we have

$$\int |H_{1,2}(x)|\,dx \lesssim$$
$$\left(2^{-k\kappa_\Omega}2^{n-k}|I_j|^{-1}\int_{I_j^*} 2^{k-n}|I_j||\gamma_k'(\alpha)-\gamma'(\alpha)|\big(|\gamma''(\alpha)|(|I_j|2^{k-n}+1)+|I_j|^{-1}\big)\,d\alpha\right)$$
$$+\left(2^{-k\kappa_\Omega}2^{n-k}|I_j|^{-1}\int_{I_j^*} 2^{2k-n}|I_j||\gamma_k'(\alpha)-\gamma'(\alpha)|^2\,d\alpha\right)$$
$$+\,2^{-k\kappa_\Omega}2^{n-k}|I_j|^{-1}2^{-n+(k-l)}.$$

Now we bound the integrals over I_j^* by a sum of integrals over all the J_m such that $J_m \cap I_j^* \neq \emptyset$ and use (4.71). We have

$$\int |H_{1,2}(x)|\,dx \lesssim$$
$$2^{-k\kappa_\Omega}2^{n-k}|I_j|^{-1}\sum_{m:\,J_m\cap I_j^*\neq\emptyset}\left(\int_{J_m}(2^{-n}\frac{|I_j|}{|J_m|}(|\gamma''(\alpha)|(|I_j|2^{k-n}+1)+|I_j|^{-1})\,d\alpha\right.$$
$$\left.+\int_{J_m}2^{-n}\frac{|I_j|}{|J_m|^2}\,d\alpha\right)+2^{-k\kappa_\Omega}2^{n-k}|I_j|^{-1}2^{-n+(k-l)}.$$

Using (4.16) gives

$$\int_{J_m}\left(|\gamma''(\alpha)|(|I_j|2^{k-n}+1)+|I_j|^{-1}\right)d\alpha \lesssim 2^{-n}\frac{|I_j|}{|J_m|}+\frac{|J_m|}{|I_j|}.$$

Therefore

$$\int |H_{1,2}(x)|\,dx \lesssim$$
$$2^{-k\kappa_\Omega}2^{n-k}|I_j|^{-1}\sum_{m:\,J_m\cap I_j^*\neq\emptyset}\left(2^{-2n}\frac{|I_j|^2}{|J_m|^2}+2^{-n}+2^{-n}\frac{|I_j|}{|J_m|}\right)$$
$$+\,2^{-k\kappa_\Omega}2^{n-k}|I_j|^{-1}2^{-n+(k-l)}. \quad (4.84)$$

We now proceed to bound (4.84). We will first show that for any j,

$$\operatorname{card}(\{m:\ J_m\cap I_j^*\neq\emptyset\})\lesssim 1+\operatorname{card}(\{m:\ J_m\subset I_j^*\})\lesssim 2^{(k-l)/2}. \quad (4.85)$$

By Cauchy-Schwarz, (4.13) and (4.12),

$$\operatorname{card}(\{m:\ J_m \subset I_j^*\}) \leqslant \sum_{\{m:\, J_m\subset I_j^*\}} 2^{k/2}(c_{m+1}-c_m)^{1/2}(\gamma'(c_{m+1})-\gamma'(c_m))^{1/2}$$
$$\leqslant 2^{k/2}\Bigg(\sum_{\{m:\, J_m\subset I_j^*\}} c_{m+1}-c_m\Bigg)^{1/2}\Bigg(\sum_{\{m:\, J_m\subset I_j^*\}} \gamma'(c_{m+1})-\gamma'(c_m)\Bigg)^{1/2}$$
$$\leqslant 2^{k/2}(b_{j+1}-b_j)^{1/2}(\gamma'(b_{j+1})-\gamma'(b_j))^{1/2} \leqslant 2^{(k-l)/2},$$

which proves (4.85). Using (4.85), we have

$$\sum_{m:\, J_m\cap I_j^*\neq\emptyset, |J_m|\geqslant \frac{|I_j|}{100}} \left(2^{-2n}\frac{|I_j|^2}{|J_m|^2} + 2^{-n} + 2^{-n}\frac{|I_j|}{|J_m|}\right) \lesssim 1 \tag{4.86}$$

and

$$\sum_{m:\, J_m\cap I_j^*\neq\emptyset} 2^{-n} \lesssim 1. \tag{4.87}$$

If $J_m \cap I_j^* \neq \emptyset$ and $|J_m| < \frac{|I_j|}{100}$, then $J_m \subset I_{j-1}\cup I_j \cup I_{j+1}$. We will write $\Delta_{I_j}(\gamma')$ in place of $\gamma'(b_{j+2}) - \gamma'(b_{j-1})$. Similarly define $\Delta_{J_m}(\gamma') = \gamma'(c_{m+1}) - \gamma'(c_m)$. By (4.16), we have

$$|I_j| \lesssim 2^{-l}(\Delta_{I_j}(\gamma'))^{-1}.$$

By (4.12) and (4.13), we also have

$$|J_m| \approx 2^{-k}(\Delta_{J_m}(\gamma'))^{-1}.$$

We thus have

$$\sum_{m:\, J_m\cap I_j^*\neq\emptyset,\, |J_m|<|I_j|/100} \left(2^{-2n}\frac{|I_j|^2}{|J_m|^2} + 2^{-n}\frac{|I_j|}{|J_m|}\right)$$
$$\lesssim \sum_{m:\, J_m\cap I_j^*\neq\emptyset,\, |J_m|<|I_j|/100} \left(2^{-2n}2^{2(k-l)}\left(\frac{\Delta_{J_m}(\gamma')}{\Delta_{I_j}(\gamma')}\right)^2 + 2^{-n}\left(\frac{\Delta_{J_m}(\gamma'}{\Delta_{I_j}(\gamma')}\right)\right)$$
$$\lesssim 2^{-n+k-l} \lesssim 1. \tag{4.88}$$

Together, (4.84), (4.86), (4.87) and (4.88) imply that when $n > k - l$ we have

$$\int |H_{1,2}(x)|\, dx \lesssim 2^{-k\kappa_\Omega} 2^{n-k} |I_j|^{-1}. \tag{4.89}$$

Together (4.82) and (4.89) imply that

$$\int |H_1(x)|\, dx \lesssim 2^{-k\kappa_\Omega} 2^{n-k} |I_j|^{-1}, \tag{4.90}$$

completing the estimation of $\int |H_1(x)|\, dx$.

Estimation of $\int |H_2(x)|\, dx$

Integrating by parts H_2 once in α and twice in s yields

$$\begin{aligned}\int |H_2(x)|\, dx \lesssim 2^{-k\kappa_\Omega} \int \phi_0(2^{-6M}|x|) \int_{I_j^*} &|\partial_\alpha g_{l,k,j,n}(x,\alpha)| \\ &\times \frac{2^k}{(1 + 2^k|\alpha x_1 + \gamma(\alpha)x_2 + 1)|)^2}\, d\alpha\, dx,\end{aligned}$$

where

$$g_{l,k,j,n}(x,\alpha) = \frac{\Phi_{k,j,n}(x,\alpha)\beta_{I_j}(\alpha)[\alpha(\gamma'(\alpha) - \gamma_k'(\alpha)) - (\gamma(\alpha) - \gamma_k(\alpha))]}{x_1 + x_2\gamma'(\alpha)}.$$

By (4.69) and (4.70), for α in the support of $\beta_{I_j}(\alpha)$ we have

$$|\alpha(\gamma'(\alpha) - \gamma_k'(\alpha)) - (\gamma(\alpha) - \gamma_k(\alpha))| \lesssim 2^{-l}|I_j|^{-1}. \tag{4.91}$$

It is easy to see that (4.91) implies

$$\left|\partial_\alpha \left[\alpha(\gamma'(\alpha) - \gamma_k'(\alpha)) - (\gamma(\alpha) - \gamma_k(\alpha))\right]\right| \lesssim 2^{-l}|I_j|^{-1} + |\gamma''(\alpha)| + |\gamma_k''(\alpha)|. \tag{4.92}$$

By (4.91) and (4.92), for x in the support of H_2 we have

$$|\partial_\alpha g_{l,k,j,n}(x,\alpha)| \lesssim 2^{-l}|I_j|^{-1} \frac{(|\gamma''(\alpha)| + |\gamma_k''(\alpha)|)(|I_j|2^k + 1) + |I_j|^{-1}}{|x_1 + x_2\gamma'(\alpha)|},$$

and so applying the change of coordinates (4.37) and estimating the integral using (4.16) and (4.17) as we did above in the estimation of $\int |H_1(x)|\, dx$, we obtain for $k \geqslant l$ and $n > k - l$,

$$\int |H_2(x)|\, dx \lesssim 2^{-k\kappa_\Omega} 2^{-l} |I_j|^{-1} \lesssim 2^{-k\kappa_\Omega} 2^{n-k} |I_j|^{-1}. \tag{4.93}$$

Together (4.90) and (4.93) imply that (5.17) holds whenever $n > k - l$, completing the proof of the lemma. □

Estimation of the main term

We have thus shown that to prove Lemma (4.5.2), it suffices to prove

Lemma 4.5.7. *Let $B_{l,k,j,n}$ be as defined previously. Then*

$$\Big\| \Big(\sum_{k:k\geqslant l} \sum_{j} \sum_{n:n>k-l} B_{l,k,j,n}(a_Q) \Big\|_{L^1(\mathbb{R}^2)} \lesssim 1. \tag{4.94}$$

Proof of Lemma 4.5.7. We have

$$B_{l,k,j,n}(x) = \phi_0(2^{-6M}|x|) \int_0^\infty \int_{I_j^*} e^{is(\alpha x_1 + \gamma_k(\alpha) x_2 + 1)} \times \beta_{I_j}(\alpha) \Phi_{k,j,n}(x,\alpha) \theta_k(s) a(s) s (\alpha \gamma_k'(\alpha) - \gamma_k(\alpha))\, d\alpha\, ds.$$

We integrate by parts twice in α to obtain

$$B_{l,k,j,n}(x) = \phi_0(2^{-6M}|x|) \int_0^\infty \int_{I_j^*} e^{is(\alpha x_1 + \gamma_k(\alpha) x_2 + 1)} g_{l,k,j,n}(x,\alpha) \times s^{-1} \theta_k(s) a(s)\, d\alpha\, ds.$$

where

$$g_{l,k,j,n}(x,\alpha) = \partial_\alpha \Big[\frac{1}{x_1 + x_2 \gamma_k'(\alpha)} \partial_\alpha \Big[\frac{\beta_{I_j}(\alpha) \Phi_{k,j,n}(x,\alpha)(\alpha \gamma_k'(\alpha) - \gamma_k(\alpha))}{x_1 + x_2 \gamma_k'(\alpha)} \Big]\Big].$$

Integrating by parts twice in s yields

$$\int |B_{l,k,j,n}(x)|\,dx \lesssim 2^{-k(\kappa_\Omega+1)} \int \phi_0(2^{-6M}|x|) \int_{I_j^*} |g_{l,k,j,n}(x,\alpha)| \times \frac{2^k}{(1+2^k|\alpha x_1+\gamma_k(\alpha)x_2+1|)^2}\,d\alpha\,dx.$$

Observe that for x in the support of $\phi_0(2^{-6M}|x|)$,

$$|g_{l,k,j,n}(x,\alpha)| \lesssim \frac{2^{k-n}|I_j||\gamma_k'''(\alpha)| + 2^{2(k-n)}|I_j|^2|\gamma_k''(\alpha)|^2 + |I_j|^{-2}}{|x_1+x_2\gamma_k'(\alpha)|^2}.$$

Thus using the change of coordinates

$$(x_1,x_2) \mapsto (u_1,u_2) := (x_1+x_2\gamma_k'(\alpha), 1+\alpha x_1+\gamma_k(\alpha)x_2),$$

we have

$$\begin{aligned}\int |B_{l,k,j,n}(x)|\,dx &\lesssim 2^{-k(\kappa_\Omega+1)} \int_{I_j^*} (2^{k-n}|I_j||\gamma_k'''(\alpha)| \\ &\quad + 2^{2(k-n)}|I_j|^2|\gamma_k''(\alpha)|^2 + |I_j|^{-2}) \int_{|u_1|\approx 2^{n-k}|I_j|^{-1}} \frac{1}{|u_1|^2}\frac{2^k}{(1+2^k|u_2|)^2}\,du\,d\alpha \\ &\lesssim 2^{-k(\kappa_\Omega+1)}2^{-n+k}|I_j| \int_{I_j^*} (2^{k-n}|I_j||\gamma_k'''(\alpha)| + 2^{2(k-n)}|I_j|^2|\gamma_k''(\alpha)|^2 + |I_j|^{-2})\,d\alpha.\end{aligned}$$

Since

$$2^{-k(\kappa_\Omega+1)}2^{-n+k}|I_j| \int_{|I_j|^*} |I_j|^{-2}\,d\alpha \lesssim 2^{-k\kappa_\Omega}2^{-n},$$

we have

$$\int |B_{l,k,j,n}(x)|\,dx \lesssim \Big(2^{-k(\kappa_\Omega+1)}2^{-n+k}|I_j| \int_{I_j^*} (2^{k-n}|I_j||\gamma_k'''(\alpha)| + 2^{2(k-n)}|I_j|^2|\gamma_k''(\alpha)|^2\,)d\alpha\Big) + 2^{-k\kappa_\Omega}2^{-n}.$$

Now for each m, choose $j(m)$ so that $I^*_{j(m)} \cap J_m \neq \emptyset$ and $I_{j(m)}$ has maximal length. Then using (4.19), we have

$$\begin{aligned}\sum_j \int |B_{l,k,j,n}(x)|\, dx \lesssim 2^{-n} + 2^{-n}2^{-k\kappa\Omega} \sum_j \int_{I_j^*} 2^{k-n}|I_j|^2|\gamma_k'''(\alpha)|\, d\alpha \\ + 2^{-n}2^{-k\kappa\Omega} \sum_j \int_{I_j^*} 2^{2(k-n)}|I_j|^3|\gamma_k''(\alpha)|^2\, d\alpha \\ \lesssim 2^{-n} + 2^{-n}2^{-k\kappa\Omega} \sum_m 2^{-n}\frac{|I_{j(m)}|^2}{|J_m|^2} \int_{J_m} 2^k|J_m|^2|\gamma_k'''(\alpha)|\, d\alpha \\ + 2^{-n}2^{-k\kappa\Omega} \sum_m 2^{-2n}\frac{|I_{j(m)}|^3}{|J_m|^3} \int_{J_m} 2^{2k}|J_m|^3|\gamma_k''(\alpha)|^2\, d\alpha.\end{aligned}$$

Using (4.66) and (4.67), we have

$$\sum_j \int |B_{l,k,j,n}(x)|\, dx \lesssim 2^{-n} + 2^{-n}2^{-k\kappa\Omega} \sum_m (2^{-n}\frac{|I_{j(m)}|^2}{|J_m|^2} + 2^{-2n}\frac{|I_{j(m)}|^3}{|J_m|^3}),$$

and hence using that $n > k - l$,

$$\begin{aligned}\sum_j \int |B_{l,k,j,n}(x)|\, dx \lesssim \\ 2^{-n} + 2^{2(k-l-n)}2^{-k\kappa\Omega} \sum_m (2^{-2(k-l)}\frac{|I_{j(m)}|^2}{|J_m|^2} + 2^{-3(k-l)}\frac{|I_{j(m)}|^3}{|J_m|^3}).\end{aligned}$$

Since there are at most $\lesssim 2^{l\kappa\Omega}$ intervals J_m such that for some j, $J_m \cap I_j^* \neq \emptyset$ and $|J_m| \geqslant |I_j|/100$, we have

$$\begin{aligned}2^{2(k-l-n)}2^{-k\kappa\Omega} \sum_{m:\, |J_m| \geqslant |I_{j(m)}|/100} (2^{-2(k-l)}\frac{|I_{j(m)}|^2}{|J_m|^2} + 2^{-3(k-l)}\frac{|I_{j(m)}|^3}{|J_m|^3}) \\ \lesssim 2^{(l-k)\kappa\Omega}2^{-n}. \quad (4.95)\end{aligned}$$

Note that if $|J_m| < |I_j|/100$, then $J_m \subset I_{j(m)-1} \cup I_{j(m)} \cup I_{j(m)+1}$. We will write $\Delta_{I_j}(\gamma_k')$ in place of $|\gamma_k'(b_{j+2}) - \gamma_k'(b_{j-1})|$. Similarly define $\Delta_{J_m}(\gamma_k') = |\gamma_k'(c_{m+1}) - \gamma_k'(c_m)|$. By

(4.16), (4.17) and (4.69), for every j we have

$$|I_j| \lesssim 2^{-l}(\Delta_{I_j}(\gamma_k'))^{-1}.$$

Moreover, (4.12) and (4.13) also imply that for every m

$$|J_m| \approx 2^{-k}(\Delta_{J_m}(\gamma_k'))^{-1}.$$

It follows that

$$\frac{|I_{j(m)}|}{|J_m|} \lesssim 2^{k-l}\frac{\Delta_{J_m}(\gamma_k')}{\Delta_{I_{j(m)}}(\gamma_k')},$$

and hence

$$\begin{aligned} 2^{2(k-l-n)}2^{-k\kappa_\Omega} &\sum_{m\colon |J_m|<|I_{j(m)}|/100} (2^{-2(k-l)}\frac{|I_{j(m)}|^2}{|J_m|^2} + 2^{-3(k-l)}\frac{|I_{j(m)}|^3}{|J_m|^3}) \\ &\lesssim 2^{2(k-l-n)}2^{-k\kappa_\Omega} \sum_{m\colon |J_m|<|I_{j(m)}|/100} \frac{\Delta_{J_m}(\gamma_k')}{\Delta_{I_{j(m)}}(\gamma_k')} \lesssim 2^{(l-k)\kappa_\Omega}2^{2(k-l-n)}. \end{aligned} \tag{4.96}$$

Together (4.95) and (4.96) imply that

$$\sum_j \int |B_{l,k,j,n}(x)|\,dx \lesssim 2^{-n} + 2^{(l-k)\kappa_\Omega}2^{(k-l-n)}. \tag{4.97}$$

Summing over $n > k-l$ and $k \geqslant l$ yields (4.63).

□

The case $k < l$

To prove Proposition 4.4.3, it remains to prove the following lemma.

Lemma 4.5.8. *Let S_k be defined as previously. Then*

$$\left\| \sum_{k\colon k<l} S_k(a_Q) \right\|_{L^1(\mathbb{R}^2)} \lesssim 1.$$

Proof of Lemma 4.5.8. We will need to exploit the cancellation of the atom. Since $\int a_Q = 0$, we only need prove that for $k < l$,

$$\sup_{y,y' \in Q} \int_{\mathbb{R}^2} |K_k(x-y) - K_k(x-y')|\, dx \lesssim 2^{k-l}. \tag{4.98}$$

Now,

$$\begin{aligned} \sup_{y,y' \in Q} \int_{\mathbb{R}^2} |K_k(x-y) - K_k(x-y')|\, dx &\lesssim \int \sup_{y,y' \in Q} |K_k(x-y) - K_k(x-y')|\, dx \\ &\lesssim 2^{-l} \int \sup_{y \in Q} |\nabla K_k(x-y)|\, dx, \end{aligned}$$

so to prove (4.98) it suffices to show that

$$\int \sup_{y \in Q} |\nabla K_k(x-y)|\, dx \lesssim 2^k. \tag{4.99}$$

Since $k < l$ and since $(\nabla K_k)(x) = (K_k(\cdot) * 2^{3k}\phi(2^k \cdot))(x)$ for some Schwartz function ϕ, it is easy to see that

$$\int \sup_{y \in Q} |\nabla K_k(x-y)|\, dx \lesssim 2^k \int |K_k(x)|\, dx.$$

But by the proof of (4.60) in the case that $k = l$ and the estimation of the term $K_{k,j,0}$ from Section 4.3, we have

$$\int |K_k(x)|\, dx \lesssim 1,$$

which implies (4.99) and finishes the proof. □

4.6 Estimates for a generalized Bochner-Riesz square function

In [10], Carbery, Gasper and Trebels showed that one may use the sharp L^4 estimates for the two-dimensional Bochner-Riesz square function, first obtained by Carbery in [8], to prove multiplier theorems for radial Fourier multipliers in $\mathbb{R}^2$. We are thus motivated to consider the generalized Bochner-Riesz square function

$$G^\alpha f(x) = \left(\int_0^\infty \left| \frac{\partial}{\partial t} \mathcal{R}_t^\alpha f(x) \right|^2 t\, dt \right)^{1/2}.$$

In the same vein as in [10], L^4 estimates for G^α yield a multiplier theorem for quasiradial multipliers in the range $4/3 \leqslant p \leqslant 4$, which we will then interpolate with Theorem 4.1.5. In [14], the following L^4 estimate for G^α is obtained, which will also be discussed in Chapter 5.

Proposition 4.6.1. *For $\alpha > -1/2$,*

$$\|G^\alpha f\|_4 \lesssim_M \|f\|_4 .$$

Following [10], one may then obtain the following corollary.

Corollary 4.6.2. *If $\alpha > 1/2$, then for $4/3 \leqslant p \leqslant 4$,*

$$\|m \circ \rho\|_{M^p(\mathbb{R}^2)} \lesssim \sup_{t>0} \left(\int |\mathcal{F}_\mathbb{R}[\phi(\cdot)m(t\cdot)](\tau)|^2 |\tau|^{2\alpha}\, d\tau \right)^{1/2}.$$

4.7 An interpolation argument

We now prove Theorem 4.1.6 by interpolating Corollary 4.6.2 and Theorem 4.1.5.

Proof of Theorem 4.1.6. Let $\tilde{\mathcal{S}}(\mathbb{R})$ denote the space of Schwartz functions on $\mathbb{R}$ with support in the annulus $\{x : 1/2 < |x| < 2\}$. For $s \geqslant 0$ and $1 \leqslant r \leqslant 2$ define norms $\|\cdot\|_r^s$ by

$$\|f\|_r^s = \left(\int |\widehat{f}(\tau)|^r (1 + |\tau|)^{rs}\, d\tau \right)^{1/r},$$

and let L_r^s denote the space of all measurable functions f with $\|f\|_r^s < \infty$. Let $\tilde{L}_r^s(\mathbb{R})$ denote the closure of $\tilde{\mathcal{S}}(\mathbb{R})$ in $L_r^s(\mathbb{R})$. For each integer $N \geqslant 0$, let $C_{0,N}$ denote the space of sequences with support in $[-N, N]$, and let ℓ_N^∞ denote the closure of $C_{0,N}$ in ℓ^∞. For $N \in \mathbb{N}$, define a bilinear operator T_N where $T_N : \mathcal{S}(\mathbb{R}^2) \times C_{0,N}(\tilde{S}(\mathbb{R})) \to \mathcal{S}(\mathbb{R}^2)$ by

$$\mathcal{F}[T_N(f, \{m_k\}_{k=-N}^N)(\cdot)](\xi) = \sum_{k=-N}^{N} m_k(2^{-k}\rho(\xi))\hat{f}(\xi).$$

Then Theorem 4.1.5 implies that for $s > \kappa_\Omega$ and for every N and $1 < p < \infty$, T_N extends to a bounded bilinear operator from $L^p(\mathbb{R}^2) \times \ell_N^\infty(\tilde{L}_1^s(\mathbb{R}))$ to $L^p(\mathbb{R}^2)$ with operator norm

$$\|T_N\|_{L^p(\mathbb{R}^2)\times \ell_N^\infty(\tilde{L}_1^s(\mathbb{R}))\to L^p(\mathbb{R}^2)} = C_{p,s} \tag{4.100}$$

for some constant $C_p > 0$ depending only on p and s and not on N. Corollary 4.6.2 implies that for every $\alpha > 1/2$ and for every N, T_N extends to a bounded bilinear operator from $L^{4/3}(\mathbb{R}^2) \times \ell_N^\infty(\tilde{L}_2^\alpha(\mathbb{R}))$ to $L^{4/3}(\mathbb{R}^2)$ with operator norm

$$\|T_N\|_{L^{4/3}(\mathbb{R}^2)\times \ell_N^\infty(\tilde{L}_2^\alpha(\mathbb{R}))\to L^{4/3}(\mathbb{R}^2)} = C'_\alpha \tag{4.101}$$

for some constant $C'_\alpha > 0$ depending only on α and not on N. Applying bilinear real interpolation methods (see for example [5]) to (4.100) and (4.101), we obtain for $0 \leqslant \theta \leqslant 1$,

$$\|T_N\|_{L^{q_0}(\mathbb{R}^2)\times \ell_N^\infty(\tilde{L}_{q_1}^{s_0(\epsilon)}(\mathbb{R}))\to L^{q_0}(\mathbb{R}^2)} \lesssim_{\epsilon,p,\theta} 1, \tag{4.102}$$

where

$$\frac{1}{q_0} = \frac{1-\theta}{p} + \frac{\theta}{4/3}, \qquad \frac{1}{q_1} = 1 - \frac{\theta}{2}, \qquad s_0(\epsilon) = (1-\theta)\kappa_\Omega + \frac{\theta}{2} + \epsilon. \tag{4.103}$$

Define a bilinear operator $T : \mathcal{S}(\mathbb{R}^2) \times \ell^\infty(\tilde{L}_1^0(\mathbb{R})) \to L^2(\mathbb{R}^2)$ by

$$\mathcal{F}[T(f, \{m_k\}_{k=-\infty}^\infty)(\cdot)](\xi) = \sum_{k=-\infty}^{\infty} m_k(2^{-k}\rho(\xi))\widehat{f}(\xi).$$

Using (4.102) and letting $N \to \infty$, we obtain

$$\|T\|_{L^{q_0}(\mathbb{R}^2)\times \ell^\infty(\tilde{L}^{s_0(\epsilon)}_{q_1}(\mathbb{R}))\to L^{q_0}(\mathbb{R}^2)} \lesssim_{\epsilon,p,\theta} 1,$$

for $q_0, q_1, s_0(\epsilon)$ as in (4.103). Set $s(\kappa_\Omega, \theta) = (1-\theta)\kappa_\Omega + \frac{\theta}{2}$. Since $1 < p < \infty$, we have

$$\|T\|_{L^{q_0}(\mathbb{R}^2)\times \ell^\infty(\tilde{L}^{s(\kappa_\Omega,\theta)+\epsilon}_{\frac{2}{2-\theta}}(\mathbb{R}))\to L^{q_0}(\mathbb{R}^2)} \lesssim_{\epsilon,q_0,\theta} 1, \tag{4.104}$$

for any $\frac{4}{4-\theta} < q_0 < \frac{4}{\theta}$. It is straightforward to see that (4.104) implies the result. $\square$

Chapter 5

A Generalized Bochner-Riesz Square Function

5.1 Introduction

As discussed in Chapter 1, the characterization of Fourier multiplier operators that are bounded on L^p when $p \neq 1, 2$ is a difficult open problem that has a long and rich history in harmonic analysis. A particular special case that has been especially studied is the class of radial Fourier multipliers, for which the Bochner-Riesz multipliers are prototypical examples. In [10], Carbery, Gasper and Trebels proved sufficient conditions for a radial function on $\mathbb{R}^2$ to be a Fourier multiplier on $L^p(\mathbb{R}^2)$. Their theorem can be stated as follows.

Theorem A ([10]). *Let* $m : (0, \infty) \to \mathbb{C}$ *be bounded and measurable. Then for* $4/3 \leqslant p \leqslant 4$ *and* $\alpha > 1/2$*,*

$$\|m(|\cdot|)\|_{M^p(\mathbb{R}^2)} \lesssim \sup_{t>0} \left(\int |\mathcal{F}_{\mathbb{R}}[\phi(\cdot)m(t\cdot)](\tau)|^2 |\tau|^{2\alpha} \, d\tau \right)^{1/2}.$$

Theorem A is sharp, as can be verified by comparing with the known sharp L^p bounds for Bochner-Riesz multipliers in $\mathbb{R}^2$ (see [22]). Theorem A was obtained as a consequence of a critical L^4 estimate for the Bochner-Riesz square function in $\mathbb{R}^2$, proved by Carbery in [8].

In this chapter, we extend the result of Theorem A to a class of *quasiradial* multipliers of the form $m \circ \rho$, where ρ belongs to a class of rough distance functions homogeneous with respect to a *nonisotropic* dilation group. Here we may view $\rho(\xi)$ as generalizing the function $|\xi|$, which corresponds to the special case of radial multipliers. Our consideration of such a class of distance functions is in part motivated by the work of Seeger and Ziesler in [48], where the authors consider Bochner-Riesz means of the form $(1-\rho(\xi))_+^\lambda$ where ρ is the Minkowski functional of a bounded convex domain in $\mathbb{R}^2$ containing the origin. However, the class of distance functions we work with is more general than what is considered in [48], since it also includes distance functions ρ that have nonisotropic homogeneity.

As motivated by [48], let $\Omega \subset \mathbb{R}^2$ be a bounded, open convex set containing the origin. Since the results in this chapter are dilation invariant, we will assume that Ω contains the ball of radius 8 centered at the origin. Let $M > 0$ be the smallest positive integer such that

$$\{\xi : |\xi| \leqslant 8\} \subset \Omega \subset \overline{\Omega} \subset \{\xi : |\xi| < 2^M\}. \tag{5.1}$$

This quantity M associated to such a convex domain Ω is an important parameter on which our results will depend. One may note that it determines the Lipschitz norm of parametrizations of $\partial\Omega$.

We now introduce the notion of a nonisotropic dilation group. Let A be a 2×2 matrix with eigenvalues λ_1, λ_2 (not necessarily distinct) such that $\mathrm{Re}(\lambda_1), \mathrm{Re}(\lambda_2) > 0$. A *nonisotropic dilation group* associated to A is a one-parameter family $\{t^A : t > 0\}$, where $t^A = \exp(\log(t)A)$. We say that a pair (Ω, A) is *compatible* if it satisfies the following:

1. For any $\xi \in \mathbb{R}^2 \setminus \{0\}$ the orbit $\{t^A\xi : t > 0\}$ intersects $\partial\Omega$ exactly once,

2. If $\Theta(\Omega, A)$ denotes the infimum of all angles between the tangent vector to an orbit $\{t^A\xi : t > 0\}$ at ξ and a supporting line at ξ for any $\xi \in \partial\Omega$, then $\Theta(\Omega, A) > 0$.

We associate to a compatible pair (Ω, A) a norm function $\rho \in C(\mathbb{R}^2)$, defined by setting $\rho(0) = 0$ and setting $\rho(\xi)$ to be the unique t such that $t^{-A}\xi \in \partial\Omega$ if $\xi \neq 0$. If $\partial\Omega$ is smooth, then $\rho \in C^\infty(\mathbb{R}^2 \setminus \{0\})$. To see this, apply the implicit function theorem to $F(x,t) = \operatorname{dist}(t^A x, \partial\Omega)$. Moreover, we also have $\|\rho\|_{C^{0,1}(K)} \lesssim_{K,M,\mathrm{Re}(\lambda_1),\mathrm{Re}(\lambda_2),\Theta(\Omega,A)} 1$ for any compact $K \subset \mathbb{R}^2 \setminus \{0\}$.

Note that in the special case that A is the identity, (Ω, A) is a compatible pair for any bounded, open convex set Ω satisfying (5.1), and we have $\Theta(\Omega, A) \gtrsim_M 1$. It was noted in [53] that for every A there exists a compatible pair (Ω, A) obtained by taking Ω to be the region bounded by $\{\xi \in \mathbb{R}^2 : \langle B\xi, \xi\rangle = 1\}$, where B is the positive definite symmetric matrix given by

$$B = \int_0^\infty \exp(-tA^*)\exp(-tA)\,dt.$$

See [53] for more details. In this particular case $\partial\Omega$ is smooth; however as already noted in this chapter we consider general convex domains, with special emphasis on the case when $\partial\Omega$ is rough.

Notation

Throughout the rest of the chapter, in every situation where it is clear that we have fixed a compatible pair (Ω, A), we will write $\lesssim$, $\gtrsim$ and $\approx$ to denote inequalities where the implied constant possibly depends on M, $\mathrm{Re}(\lambda_1)$, $\mathrm{Re}(\lambda_2)$, and $\Theta(\Omega, A)$. We will also assume that all explicit constants that appear possibly depend on M, $\mathrm{Re}(\lambda_1)$, $\mathrm{Re}(\lambda_2)$, and $\Theta(\Omega, A)$.

Given a compatible pair (Ω, A), define the Bochner-Riesz means $R_t^\lambda f$ associated with (Ω, A) for Schwartz functions $f \in \mathcal{S}(\mathbb{R}^2)$ by

$$\mathcal{R}_t^\lambda f(x) = \frac{1}{(2\pi)^2} \int_{|\xi| \leqslant t} \left(1 - \frac{\rho(\xi)}{t}\right)^\lambda \hat{f}(\xi) e^{i\langle \xi, x\rangle}\, d\xi.$$

Define the Bochner-Riesz square function $G^\lambda f$ associated with (Ω, A) for Schwartz functions $f \in \mathcal{S}(\mathbb{R}^2)$ by

$$G^\lambda f(x) = \left(\int_0^\infty \left|\mathcal{R}_t^\lambda f(x)\right|^2 \frac{dt}{t}\right)^{1/2}.$$

Our main result is the following critical L^4 estimate for the Bochner-Riesz square function.

Theorem 5.1.1. *Let (Ω, A) be a compatible pair, and let $G^\lambda f$ denote the Bochner-Riesz square function associated to (Ω, A). For $\lambda > -1/2$,*

$$\left\|G^\lambda f\right\|_{L^4(\mathbb{R}^2)} \lesssim \|f\|_{L^4(\mathbb{R})^2}$$

for $f \in \mathcal{S}(\mathbb{R}^2)$.

Following [10], we obtain the subsequent corollary, which is an extension of the result of Theorem A to quasiradial multipliers of the form $m \circ \rho$.

Corollary 5.1.2. *Let (Ω, A) be a compatible pair with associated norm function ρ. Let $m : \mathbb{R} \to \mathbb{C}$ be measurable function with $\|m\|_{L^\infty(\mathbb{R})} \leqslant 1$. Then for $4/3 \leqslant p \leqslant 4$ and*

$\alpha > 1/2$,

$$\|m \circ \rho\|_{M^p(\mathbb{R}^2)} \lesssim \sup_{t>0} \left(\int |\mathcal{F}_{\mathbb{R}}[\phi(\cdot)m(t\cdot)](\tau)|^2 |\tau|^{2\alpha}\, d\tau \right)^{1/2}.$$

To prove Theorem 5.1.1, we will first decompose the multiplier $(1-\rho(\xi))_+^\lambda$ in a standard fashion into smooth functions supported on "annuli" of thickness comparable to the distance from $\partial\Omega$ (for example, see [18], [8]). Theorem 5.1.1 then reduces to proving the following proposition.

Proposition 5.1.3. *Let* (Ω, A) *be a compatible pair. Fix a Schwartz function* $\Phi : \mathbb{R} \to \mathbb{R}$ *supported in* $[-1, 1]$ *with* $|\Phi| \leqslant 1$. *There is a constant* $C > 0$ *such that for every* $\epsilon > 0$ *and every* $0 < \delta < C$,

$$\left\| \left(\int_0^\infty |\psi_t * f(x)|^2 \frac{dt}{t} \right)^{1/2} \right\|_4 \lesssim_\epsilon \delta^{1/2-\epsilon} \|f\|_4 ,$$

where

$$\psi_t(x) = \mathcal{F}(\phi(\frac{\rho(\cdot)}{t}))(x), \qquad \phi(\xi) = \Phi(\frac{\xi - 1}{\delta}).$$

The overall structure of the proof of Proposition 5.1.3 will follow [8] and [48], and will draw heavily on the techniques therein. However, the presence of nonisotropic dilations and the roughness of $\partial\Omega$ introduces new difficulties to the proof since the underlying geometry becomes more complicated, requiring more intricate decompositions on the Fourier side as well as a more sophisticated use of Littlewood-Paley inequalities.

5.2 Preliminaries on convex domains in $\mathbb{R}^2$

Elementary facts about convex functions in $\mathbb{R}^2$

We note for later use the following lemma, which can be found in [48], and is also included in Chapters 3 and 4 but which we restate here for convenience. The proof is straightforward and we omit it here, and the reader is encouraged to refer to [48] for a proof.

Lemma 5.2.1 ([48]). *$\partial\Omega \cap \{x : -1 \leqslant x_1 \leqslant 1\}$ can be parametrized by*

$$t \mapsto (t, \gamma(t)), \qquad -1 \leqslant t \leqslant 1, \tag{5.2}$$

where

1.

$$1 < \gamma(t) < 2^M, \qquad -1 \leqslant t \leqslant 1. \tag{5.3}$$

2. *γ is a convex function on $[-1, 1]$, so that the left and right derivatives γ'_L and γ'_R exist everywhere in $(-1, 1)$ and*

$$-2^{M-1} \leqslant \gamma'_R(t) \leqslant \gamma'_L(t) \leqslant 2^{M-1} \tag{5.4}$$

for $t \in [-1, 1]$. The functions γ'_L and γ'_R are decreasing functions; γ'_L and γ'_R are right continuous in $[-1, 1]$.

3. *Let ℓ be a supporting line through $\xi \in \partial\Omega$ and let n be an outward normal vector. Then*

$$|\langle \xi, n \rangle| \geqslant 2^{-M}|\xi|. \tag{5.5}$$

Reduction to the case when $\partial\Omega$ is smooth.

Motivated by [48], Lemma 2.2, we will show that it suffices to prove Proposition 5.1.3 with the implied constant depending only on M (and not, for instance, the C^2 norm of local parametrizations of $\partial\Omega$) in the special case that $\partial\Omega$ is smooth. The first step is to approximate Ω by a sequence of convex domains with smooth boundaries satisfying the same quantitative estimates as Ω.

Lemma 5.2.2. *Let (Ω, A) be a compatible pair. There is a sequence of convex domains $\{\Omega_n\}$ satisfying the following:*

1. *$\partial\Omega_n$ is C^∞,*

2. *For n sufficiently large, (Ω_n, A) is a compatible pair and $\Theta(\Omega_n, A) \geqslant \Theta(\Omega, A)/2$,*

3. *For each n we have*

$$\{\xi : |\xi| \leqslant 4\} \subset \Omega_n \subset \overline{\Omega_n} \subset \{\xi : |\xi| < 2^{M+1}\},$$

4. *$\lim_{n\to\infty} \rho_n(\xi) = \rho(\xi)$ with uniform convergence on compact sets.*

Proof. We adopt the same approach as in [48], namely, approximating Ω by convex polygons and smoothing out the vertices. For each n, let P_n be the polygon with vertices $\{v_1, \ldots, v_n\}$, where v_i is the unique point on $\partial\Omega$ making an angle of $2\pi i/n$ with the ξ_2-axis. Then P_n is convex and $P_n \subset \Omega$. Choose intervals $I_n = [x_{n,0}, x_{n,1}] \subset \tilde{I}_n = (\tilde{x}_{n,0}, \tilde{x}_{n,1}) \subset \mathbb{R}$ centered at 0 such that $\partial P_n \cap \{(\xi_1, \xi_2) : \xi_1 \in I_n, \xi_2 > 0\}$ can be parametrized as $\{(\alpha, \tilde{\gamma}_n(\alpha)) : \alpha \in I_n\}$, and also so that $\{(\alpha, \tilde{\gamma}_n(\alpha) : \alpha \in \tilde{I}_n\}$ does not contain any vertices of P_n except v_1.

Now let $\eta \in C_0^\infty(\mathbb{R})$ be an even nonnegative function supported in $(-1/2, 1/2)$ so that $\int \eta(t)\,dt = 1$. Let $C_n = 100\max\{(x_{n,0} - \tilde{x}_{n,0})^{-1}, (\tilde{x}_{n,1} - x_{n,1})^{-1}\}$, and set

$$\gamma_n(\alpha) = \int C_n\, \eta(C_n t)\tilde{\gamma}_n(\alpha - t)\,dt, \qquad \alpha \in I_n.$$

By the choice of C_n, we have that $\{(\alpha, \gamma_n(\alpha)) : \alpha \in I_n\}$ coincides with P_n near the endpoints of I_n. We may thus obtain a smooth convex curve $\partial\Omega_n$ by replacing ∂P_n near v_1 with $\{(\alpha, \gamma_n(\alpha)) : \alpha \in I_n\}$, and then repeating the same procedure near the other vertices $v_2, \ldots, v_n$ after performing appropriate rotations.

It is clear that $\{\Omega_n\}$ satisfies $(1), (3)$, and (4), so it remains to show (2). Let $\epsilon_0 > 0$ be sufficiently small so that for any $\xi \in \partial\Omega$ and $s_1, s_2 \in [1-\epsilon_0, 1+\epsilon_0]$, the difference in slope between the tangent lines to the orbit $\{t^A\xi : t > 0\}$ at $s_1^A\xi$ and the tangent line at $s_2^A\xi$ is less than $\Theta(\Omega, A)/10$. Now choose $0 < \epsilon_1 < \epsilon_0$ sufficiently small so that

$$\{t^A\xi : t > 0, t \notin [1-\epsilon_0, 1+\epsilon_0], \xi \in \partial\Omega\} \cap \{t^A\xi : t \in [1-\epsilon_1, 1+\epsilon_1], \xi \in \partial\Omega\} = \emptyset. \quad (5.6)$$

Next, choose $N > 0$ sufficiently large so that whenever $n \geqslant N$, the following holds:

1. $\partial\Omega_n \subset \{\xi : 1-\epsilon_1 \leqslant \rho(\xi) \leqslant 1+\epsilon_1\}$,

2. The difference in slope between the tangent line at any point $x \in \partial\Omega_n$ and some supporting line of $\partial\Omega$ at the vertex of P_n nearest to x is less than $\Theta(\Omega, A)/10$,

3. For any $\xi \in \partial\Omega_n$, the difference in slope between the tangent vector to the orbit $\{t^A\xi : t > 0\}$ at ξ and the tangent vector to the orbit $\{t^A v_i\}$ at v_i, where v_i is the vertex of P_n nearest to ξ, is less than $\Theta(\Omega, A)/10$.

To see that we may choose N so that (1) and (3) are satisfied is fairly obvious, and to see that we may choose N so that (2) holds requires only a straightforward application of (2) from Lemma 5.2.1. It is easy to see that (2) and (3) imply that $\Theta(\Omega_n, A) > \Theta(\Omega, A)/2$. (1) and (5.6) imply that $\{t^A\xi : t > 0, t \notin [1-\epsilon_0, 1+\epsilon_0], \xi \in \partial\Omega\}$ does not intersect $\partial\Omega_n$. Given $\xi \in \partial\Omega$, let $t(\xi) > 0$ be the smallest value of t such that $t^{-A}\xi \in \partial\Omega_n$. Then $t(\xi) \in [1-\epsilon_0, 1+\epsilon_0]$. But by the choice of ϵ_0, any tangent line to $\{t^A\xi : t \in [1-\epsilon_0, 1+\epsilon_0]\}$ makes an angle of at least $\Theta(\Omega, A)/4$ with the tangent line to $\partial\Omega_n$ at $t^{-A}\xi$, and by convexity of $\partial\Omega_n$ there can be no $t > t(\xi)$ such that $t^{-A}\xi \in \partial\Omega_n$. Thus (Ω_n, A) is a compatible pair for $n \geqslant N$.

□

Lemma 5.2.3. *Suppose that Proposition 5.1.3 holds in the special case when $\partial\Omega$ is smooth, with a constant depending only on M, ϵ, $Re(\lambda_1), Re(\lambda_2)$, and $\Theta(\Omega, A)$. Then Proposition 5.1.3 holds in the full stated generality.*

Proof of Lemma 5.2.3. Let $\{\Omega_n\}$ be a sequence of convex domains approximating Ω as in Lemma 5.2.2, and suppose the statement of Proposition 5.1.3 holds in the special case of convex domains with smooth boundaries, with a constant depending only on the quantities listed in Lemma 5.2.3. Fix a Schwartz function $f \in \mathcal{S}(\mathbb{R}^2)$. Then for every $\epsilon > 0$ and every $0 < \delta < C$, for n sufficiently large we have

$$\left\| \left(\int_0^\infty |\psi_{n,t} * f(x)|^2 \frac{dt}{t} \right)^{1/2} \right\|_4 \leqslant C_{\epsilon, M, \mathrm{Re}(\lambda_1), \mathrm{Re}(\lambda_2), \Theta(\Omega, A)} \delta^{1/2-\epsilon} \|f\|_4 \,,$$

where

$$\psi_{n,t}(x) = \mathcal{F}(\phi(\frac{\rho_n(\cdot)}{t}))(x), \qquad \phi(\xi) = \Phi(\frac{\xi - 1}{\delta}).$$

Since $\phi(\frac{\rho_n(\cdot)}{t}) \to \phi(\frac{\rho(\cdot)}{t})$ uniformly as $n \to \infty$, we have that $\psi_{n,t} * f(x) \to \psi_t * f(x)$ pointwise as $n \to \infty$. By Fatou's lemma applied twice,

$$\left\| \left(\int_0^\infty |\psi_t * f(x)|^2 \frac{dt}{t} \right)^{1/2} \right\|_4 \leqslant \liminf_n \left\| \left(\int_0^\infty |\psi_{n,t} * f(x)|^2 \frac{dt}{t} \right)^{1/2} \right\|_4$$
$$\leqslant C_{\epsilon,M,\mathrm{Re}(\lambda_1),\mathrm{Re}(\lambda_2),\Theta(\Omega,A)} \delta^{1/2-\epsilon} \|f\|_4 ,$$

as desired. □

5.3 An L^2 maximal function estimate

In [16], Córdoba proved L^2 bounds for the Nikodym maximal function in $\mathbb{R}^2$. These bounds were an important ingredient in [8] to prove Proposition 5.1.3 for the special case of the classical (radial) Bochner-Riesz means. To prove Proposition 5.1.3 in the full stated generality, we need a nonisotropic version of Córdoba's result. To this end, we will closely follow [16] to prove the following proposition.

Proposition 5.3.1. *Let $N, \lambda > 0$ be real numbers, and let $\mathcal{C}$ be the collection of all rectangles in $\mathbb{R}^2$ with dimensions λ and $N\lambda$. Let*

$$\mathcal{C}_k = \{(2^k)^A R : \; R \in \mathcal{C}, k \in \mathbb{Z}\}.$$

Define a maximal operator $M_{\lambda,N}$ by

$$M_{\lambda,N} f(x) = \sup_{x \in R \in \bigcup_k \mathcal{C}_k} \frac{1}{|R|} \int_R |f(y)| \, dy.$$

Then there is a constant $\beta(Re(\lambda_1), Re(\lambda_2)) > 0$ such that for every Schwartz function $f \in \mathcal{S}(\mathbb{R}^2)$,

$$\|M_{\lambda,N} f\|_2 \lesssim_{Re(\lambda_1),Re(\lambda_2)} \log(N)^{\beta(Re(\lambda_1),Re(\lambda_2))} \|f\|_2 .$$

Proof. In what follows, for any rectangle R and any integer k, we will let $((2^k)^A R)^* := (2^k)^A(R^*)$. Here R^* denotes the double dilate of R, where the dilation is taken from the center of R. Similarly, if $\mathcal{R}$ denotes any collection of nonisotropic dilates of rectangles, then $\mathcal{R}^* := \{R^* : R \in \mathcal{R}\}$.

For each $k \in \mathbb{Z}$, define a maximal operator $M_{\lambda,N,k}$ on Schwartz functions $f \in \mathcal{S}(\mathbb{R}^2)$ by

$$M_{\lambda,N,k} f(x) = \sup_{x \in R \in \mathcal{C}_k} \frac{1}{|R|} \int_R |f(y)|\, dy. \tag{5.7}$$

It follows from rescaling the corresponding result from [16] that for every $f \in \mathcal{S}(\mathbb{R}^2)$,

$$\|M_{\lambda,N,k} f\|_2 \lesssim \log(3N)^{1/2} \|f\|_2 . \tag{5.8}$$

Now we combine the estimates for the M_k to prove an L^2 estimate for M. For each (i,k) where $1 \leqslant i \leqslant N$ and $k \in \mathbb{Z}$, define a maximal operator $T^{i,k}$ by

$$T^{i,k} f(x) = \sup_{(2^{-k})^A x \in R \in \mathcal{R}_i} \frac{1}{|(2^k)^A R|} \int_{(2^k)^A R} |f(y)|\, dy$$

where $\mathcal{R}_i$ denotes the collection of all rectangles with direction $\pi i N^{-1}$ and dimensions $\lambda \times N\lambda$. Fix a Schwartz function $f \in \mathcal{S}(\mathbb{R}^2)$, and apply a standard covering lemma to obtain for each (i,k) a sequence of rectangles $\{R_n^{i,k}\} \subset \mathcal{R}_i$ pairwise disjoint such that

$$E_\alpha^{i,k} = \{x : T^{i,k} f(x) > \alpha\} \subset \bigcup_n ((2^k)^A (R_n^{i,k})^*),$$

$$\frac{1}{|(2^k)^A R_n^{i,k}|} \int_{(2^k)^A R_n^{i,k}} |f(y)|\, dy > \alpha.$$

Then

$$E_\alpha = \{x : M_{\lambda,N} f(x) > 4\alpha\} \subset \bigcup_{i,k} E_\alpha^{i,k}.$$

Let

$$\mathcal{H} = \bigcup_{i,k,n} (2^k)^A R_n^{i,k}.$$

Let $\mathcal{H}'$ be a subcollection of $\mathcal{H}$ such that

1. There are no $R, R' \in \mathcal{H}'$ such that $R' \subset R^*$.

2. If $R \in \mathcal{H} \setminus \mathcal{H}'$, then there is $R' \in \mathcal{H}'$ such that $R \subset (R')^*$.

(To see that such a subcollection exists, we simply enumerate the rectangles in $\mathcal{H}$ as $R_1, R_2, \ldots$, and at step i we add R_i to $\mathcal{H}'$ if R_i is not contained in R_j^* for any $j < i$ such that $R_j \in \mathcal{H}'$, and in this case if $R_j \subset R_i^*$ for any $j < i$ such that $R_j \in \mathcal{H}'$, we remove R_j from $\mathcal{H}'$.) Then

$$E_\alpha \subset \bigcup_{R \in \mathcal{H}'} R^{**}. \tag{5.9}$$

Let $\mathcal{H}_k = \mathcal{H}' \cap \mathcal{C}_k$. Fix an integer $a > 0$ such that $B(0,2) \subset (2^a)^A B(0,1)$, where $B(0,r)$ denotes the (Euclidean) ball of radius r centered at the origin. Let $n_0 = \max\{k : \mathcal{H}_k \neq \emptyset\}$. For every $j \geqslant 0$, let

$$\Delta_j = \bigcup_{\substack{n_0 - (j+1)(\log N)^a \\ < k \leqslant n_0 - j(\log N)^a}} \mathcal{H}_k.$$

For each j let $A_j = \bigcup_{R \in \Delta_j} R$. Then the family of sets $\{A_j\}$ is "almost disjoint", i.e. $A_{j_1} \cap A_{j_2} = \emptyset$ if $|j_1 - j_2| > 2$. To see this, suppose that $R \in \Delta_{j_1}$ and $R' \in \Delta_{j_2}$ with $j_1 < j_2 - 2$ and $R \cap R' \neq \emptyset$. Choose k such that $R \in \mathcal{C}_k$. Then $(2^{-k})^A R \subset B(x, N\lambda)$ for some $x \in (2^{-k})^A R'$. But $((2^{-k})^A R')^* \supset B(x, N\lambda)$, and so $R \subset R'^*$, a contradiction.

Now, by (5.9) we have

$$E_\alpha \subset \bigcup_j A_j^{**}. \tag{5.10}$$

Let $f_j = f \cdot \chi_{A_j}$. Define a maximal function S_j for $g \in \mathcal{S}(\mathbb{R}^2)$ by

$$S_j g(x) = \sup_{\substack{x \in R \in \bigcup_{n_0+2-(j+1)(\log N)^a} \mathcal{C}_k \\ < k \leqslant n_0+2-j(\log N)^a}} \frac{1}{|R|} \int_R g(y)\, dy.$$

It follows from (5.8) that S_j is bounded on $L^2(\mathbb{R}^2)$ with operator norm $\lesssim (\log N)^{a+1/2}$. Now if $x \in A_j^{**}$, then there is $R \in \Delta_j$ such that $x \in R^{**}$. Then,

$$S_j f_j(x) \geqslant \frac{1}{|R^{**}|} \int_{R^{**}} |f_j(y)|\, dy \geqslant \frac{1}{16} \frac{1}{|R|} \int_R |f_j(y)|\, dy \geqslant \frac{1}{16}\alpha.$$

Thus $A_j^{**} \subset \{x : S_j f_j(x) \geqslant \frac{1}{16}\alpha\}$, and so

$$|A_j^{**}| \lesssim (\log N)^{2a+1} \frac{\|f_j\|_2^2}{\alpha^2}.$$

It follows that

$$|E_\alpha| \leqslant \sum_j |A_j^{**}| \lesssim (\log N)^{2a+1} \frac{1}{\alpha^2} \sum_j \|f_j\|_2^2 \lesssim (\log N)^{2a+1} \frac{\|f\|_2^2}{\alpha^2}. \tag{5.11}$$

To obtain a strong type L^2 estimate for $M_{\lambda,N}$ from (5.11), we will need to first prove a weak $(1,1)$ estimate for $M_{\lambda,N}$ and interpolate. By comparison with the Hardy-Littlewood maximal function and rescaling, we have for every k,

$$|\{x : M_{\lambda,N,k}(f)(x) > \alpha\}| \lesssim N \frac{\|f\|_1}{\alpha}. \tag{5.12}$$

We now repeat the above argument, using (5.12) in place of (5.8) and obtain the weak $(1,1)$ estimate

$$|\{x : M_{\lambda,N} f(x) > 4\alpha\} \lesssim N \frac{\|f\|_1}{\alpha}. \tag{5.13}$$

The result now follows by interpolation of (5.13), (5.11) and the trivial L^∞ estimate for $M_{\lambda,N}$. □

5.4 A decomposition of $\mathbb{R}^2$

In this section, we will introduce a decomposition of $\mathbb{R}^2$ that plays a similar role as the decomposition of $\mathbb{R}^2$ provided in [8]. The decomposition from [8] can be viewed more or less as a decomposition of the annulus $|\xi - 1| \leqslant \delta$ into δ-thickened caps that can be approximated by $\delta^{1/2} \times \delta$ rectangles, and dilated at different scales to cover the plane in an almost-disjoint fashion. Here we employ a different decomposition of the set $|\rho(\xi) - 1| \leqslant \delta$ into rectangles of width δ and length essentially between δ and 1, so that on each rectangle, $\partial\Omega$ may be viewed as sufficiently "flat" at scale δ. This decomposition was introduced by [48] to prove L^p bounds for Bochner-Riesz multipliers associated to convex domains. We then dilate these rectangles nonisotropically at different scales to cover the plane in an almost-disjoint fashion.

Decomposition of $\partial\Omega$

Before we describe the decomposition of $\mathbb{R}^2$, we first need to introduce a decomposition of $\partial\Omega$ from [48]. This decomposition allows us to write $\partial\Omega$ as a disjoint union of pieces on which $\partial\Omega$ is sufficiently "flat". Here, the pieces in the decomposition will play the role that the $\delta^{1/2}$-caps play in the radial case.

We inductively define a finite sequence of increasing numbers

$$\mathfrak{A}(\delta) = \{a_0, \dots, a_Q\}$$

as follows. Let $a_0 = -1$, and suppose $a_0, \dots, a_{l-1}$ are already defined. If

$$(t - a_{l-1})(\gamma_L'(t) - \gamma_R'(a_{l-1})) \leqslant \delta \text{ for all } t \in (a_{l-1}, 1]) \tag{5.14}$$

and $a_{l-1} \leqslant 1 - 2^{-M}\delta$, then let $a_l = 1$. If (5.14) holds and $a_{l-1} > 1 - 2^{-M}\delta$, then let $a_l = a_{l-1} + 2^{-M}\delta$. If (5.14) does not hold, define

$$a_l = \inf\{t \in (a_{l-1}, 1] : (t - a_{l-1})(\gamma_L'(t) - \gamma_R'(a_{l-1})) > \delta\}.$$

Now note that (5.14) must occur after a finite number of steps, since we have $|\gamma_L'|, |\gamma_R'| \leqslant 2^{M-1}$, which implies that $|t-s||\gamma_L'(t)-\gamma_R'(s)| < \delta$ if $|t-s| < \delta 2^{-M}$. Therefore this process must end at some finite stage $l = Q$, and so it gives a sequence $a_0 < a_1 < \cdots < a_Q$ so that for $l = 0, \ldots, Q-1$

$$(a_{l+1} - a_l)(\gamma_L'(a_{l+1}) - \gamma_R'(a_l)) \leqslant \delta, \tag{5.15}$$

and for $0 \leqslant j < Q-1$,

$$(t - a_l)(\gamma_L'(t) - \gamma_R'(a_l)) > \delta \qquad \text{if } t > a_{l+1}. \tag{5.16}$$

For a given $\delta > 0$, this gives a decomposition of

$$\partial\Omega \cap \{x : -1 \leqslant x_1 \leqslant 1, x_2 < 0\}$$

into pieces

$$\bigsqcup_{l=0,1,\ldots,Q-1} \{x \in \partial\Omega : x_1 \in [a_l, a_{l+1}]\}.$$

Now let $\{i_0, i_1, \ldots, a_{Q'}\}$ be a refinement of $\{a_0, a_1, \ldots, a_Q\}$ corresponding to a partition of $[-1, 1]$ into intervals $\{I_j\}$ with $I_j = [i_j, b_{j+1}]$ such that each interval $[a_l, a_{l+1}]$ is a union of $\lesssim \log(\delta^{-1})$ of the intervals I_j, and so that $|I_j|/2 \leqslant |b_{j+1}| \leqslant 2|I_j|$. We then have a decomposition

$$\partial\Omega \cap \{x : -1 \leqslant x_1 \leqslant 1, x_2 < 0\} = \bigsqcup_{j=0,1,\ldots,Q'} \{x \in \partial\Omega : x_1 \in I_j\},$$

where $Q' \lesssim \log(\delta^{-1})Q$.

Decomposition of $\mathbb{R}^2$

With the previous decomposition of $\partial\Omega$ in mind, we proceed to give a decomposition of $\mathbb{R}^2$. To begin, we define a *nonisotropic sector* to be a region bounded by the origin and two orbits $\{t^A\xi : t > 0\}$ and $\{t^A\xi' : t > 0\}$ for any $\xi, \xi' \in \mathbb{R}^2 \setminus \{0\}$. Observe there is an integer $N_M > 0$ such that

1. We can write $\mathbb{R}^2 = \bigcup_{i=0}^{N_M} \mathcal{S}_i$, where each $\mathcal{S}_i$ is a nonisotropic sector and the $\mathcal{S}_i$ are essentially disjoint.

2. For every i, there is a rotation $\mathcal{R}_i$ such that $\mathcal{R}_i(\partial\Omega \cap \mathcal{S}_i) \subset \{x : -1/2 \leqslant x_1 \leqslant 1/2\}$, $\mathcal{R}_i(\partial\Omega \cap \mathcal{S}_j) \cap \{x : -1/2 < x_1 < 1/2\} = \emptyset$ for $i \neq j$, and $\mathcal{R}_0$ is the identity map.

For each i, define $\tilde{\mathcal{S}}_i$ to be the nonisotropic sector bounded by the orbits $\{t^A\xi_i : t > 0\}$ and $\{t^A\xi_i' : t > 0\}$ where $\xi = (\xi_1, \xi_2)$ is the unique point in $\mathcal{R}_i\partial\Omega$ with $\xi_1 = -1$ and $\xi_2 > 0$, and $\xi' = (\xi_1', \xi_2')$ is the unique point in $\mathcal{R}_i\partial\Omega$ with $\xi_1 = 1$ and $\xi_2 > 0$. Clearly, $\mathcal{S}_i \subset \tilde{\mathcal{S}}_i$. Let $\{(\alpha, \gamma_i(\alpha)) : \alpha \in [-1, 1]\}$ be a parametrization of $\mathcal{R}_i(\partial\Omega \cap \tilde{\mathcal{S}}_i)$. For $0 \leqslant i \leqslant N_M$, let R_i denote the region of $\mathbb{R}^2$ bounded by the level sets $\{x : \rho(x) = 1/2\}$ and $\{x : \rho(x) = 2\}$ and the nonisotropic sector $\tilde{\mathcal{S}}_i$. Similarly, let R_i' denote the region of $\mathbb{R}^2$ bounded by the level sets $\{x : \rho(x) = 1/4\}$ and $\{x : \rho(x) = 4\}$ and the nonisotropic sector $\tilde{\mathcal{S}}_i$. Fix $\delta > 0$. Let $R_{i,\delta}$ denote the region bounded by the level sets $\{x : \rho(x) = 1 - 2\delta\}$ and $\{x : \rho(x) = 1 + 2\delta\}$ and $\tilde{\mathcal{S}}_i$. Note that $\bigcup_{i=0}^{N_M} R_{i,\delta}$ contains the support of $\mathcal{F}[\psi_1]$, where ψ_1 is as in Proposition 5.1.3.

Recall the previous decomposition of $[-1, 1]$ into intervals $\{I_j\}$. Let $B_{i,j,0,0}$ denote the region bounded by $R_{i,\delta}$ and the orbits $\{t^A\mathcal{R}_i^{-1}(i_j,$

$\gamma_i(i_j)) : t > 0\}$ and $\{t^A\mathcal{R}_i^{-1}(b_{j+1}, \gamma_i(b_{j+1})) : t > 0\}$, so that $R_{i,\delta} = \bigcup_j B_{i,j,0,0}$. For each integer m, let $B_{i,j,m,0} = (\frac{1+2\delta}{1-2\delta})^{mA} B_{i,j,0,0}$. Now let N_δ, N'_δ be integers such that

$$R_i \subset \bigcup_{j, N_\delta \leqslant m \leqslant N'_\delta} B_{i,j,m,0} \subset R'_i.$$

We are now ready to state our decomposition of $\mathbb{R}^2$. For each integer n, let $B_{i,j,m,n} = (2^{2n})^A (B_{i,j,m,0})$. Then

$$\mathcal{S}_i \subset \bigcup_{j, N_\delta \leqslant m \leqslant N'_\delta, n \in \mathbb{Z}} B_{i,j,m,n},$$

$$\mathbb{R}^2 = \bigcup_{0 \leqslant i \leqslant N_M, j, N_\delta \leqslant m \leqslant N'_\delta, n \in \mathbb{Z}} B_{i,j,m,n},$$

and there is an integer N'_M depending only on M such that every point of $\mathbb{R}^2$ lies in at most N'_M many elements of the collection $\{B_{i,j,m,n}\}$.

Some important properties of the decomposition.

We now prove some essential geometric facts regarding our decomposition; these may be viewed as analogs of Proposition 3 parts $(i) - (iii)$ from [8]. The following proposition is a key fact regarding the almost disjointness of algebraic sums of the pieces in our decomposition.

Proposition 5.4.1. *For a constant $C(M, Re(\lambda_1), Re(\lambda_2)) > 0$ depending only on M and the eigenvalues of A, let*

$$\mathcal{T}_0 = \{\xi : 1/4 \leqslant \rho(\xi) \leqslant 4, |\xi_1| \leqslant C(M, Re(\lambda_1), Re(\lambda_2))\},$$

$$\mathcal{T}_1 = \bigcup_{k \in \mathbb{Z}} (2^{4k})^A (\mathcal{T}_0).$$

For $0 < t < \infty$*, let*

$$\mathcal{A}_t = \{B \in \{B_{i,j,m,n}\} : \exists \xi \in B \text{ with } \rho(\xi) = 1/t \text{ and } \xi \in \mathcal{T}_1\}.$$

Fix positive real numbers u *and* t *satisfying* $1/2 < u/t < 2$ *with* $u \in \bigcup_{k\in\mathbb{Z}}[2^{4k-1}, 2^{4k+1}]$*, and let* $\mathcal{B}_{u,t}$ *denote the collection of all sets of the form* $\{A + B\}_{A\in\mathcal{A}_t, B\in\mathcal{A}_u}$*. Then if* $C(M, Re(\lambda_1), Re(\lambda_2))$ *is chosen sufficiently small, there exists a constant* $C'(M, Re(\lambda_1), Re(\lambda_2)) > 0$ *(depending only on* M *and the eigenvalues of* A *and independent of* δ *and the choice of* u *and* t*) such that every point of* $\mathbb{R}^2$ *is contained in at most*

$$C'(M, Re(\lambda_1), Re(\lambda_2))(\log(\delta^{-1}))^2$$

elements of $\mathcal{B}_{u,t}$*.*

Proof. Without loss of generality, assume that $u = 1$. For any $v \in A \in \mathcal{A}_t$ and $w \in B \in \mathcal{A}_u$, let $\sigma^+(v, w)$ denote the minimum nonnegative difference in slope between supporting lines to the convex curve

$$\Sigma_v := \{\xi : \rho(\xi) = \rho(v),\ \xi \in \mathcal{T}_1\}$$

at v and supporting lines to the convex curve

$$\Sigma_w := \{\xi : \rho(\xi) = \rho(w),\ \xi \in \mathcal{T}_1\}$$

at w, and $\sigma^+(v, w) := +\infty$ if no nonnegative difference exists. Let $\sigma^-(v, w)$ denote the maximum nonpositive difference in slope between supporting lines to Σ_v at v and supporting lines to Σ_w at w, and $\sigma^-(v, w) := -\infty$ if no nonpositive difference exists. Note that for every (v, w) at least one of $\sigma^+(v, w)$ and $\sigma^-(v, w)$ is finite, and if $C(M, \mathrm{Re}(\lambda_1), \mathrm{Re}(\lambda_2))$ is sufficiently small, then the slope of any supporting line is between -2^{2M} and 2^{2M}. Given $x \in \mathcal{B}_{u,t}$, we have one of three cases:

1. There is $v \in A \in \mathcal{A}_t$ and $w \in B \in \mathcal{A}_u$ with $v + w = x$ and $\sigma^+(v, w)$ finite, but $\sigma^-(v, w)$ is infinite for every pair (v', w') with $v \in A' \in \mathcal{A}_t$, $w \in B' \in \mathcal{A}_u$, and $v' + w' = x$,

2. There is $v \in A \in \mathcal{A}_t$ and $w \in B \in \mathcal{A}_u$ with $v + w = x$ and $\sigma^-(v, w)$ finite, but $\sigma^+(v, w)$ is infinite for every pair (v', w') with $v \in A' \in \mathcal{A}_t$, $w \in B' \in \mathcal{A}_u$, and $v' + w' = x$,

3. There is $v \in A \in \mathcal{A}_t$ and $w \in B \in \mathcal{A}_u$ with $v + w = x$ and $\sigma^+(v, w)$ finite, and there is $v' \in A' \in \mathcal{A}_t$ and $w' \in B' \in \mathcal{A}_u$ with $v' + w' = x$ and $\sigma^-(v, w)$ finite.

Let us assume we have case 1. Given $x \in \mathbb{R}^2$, choose $v = (v_1, v_2) \in A \in \mathcal{A}_t$ and $w = (w_1, w_2) \in B \in \mathcal{A}_u$ with $v + w = x$ minimizing $\sigma^+(v, w)$. Now suppose there is $\tilde{v} = (\tilde{v}_1, \tilde{v}_2) \in \tilde{A} \in \mathcal{A}_t$ and $\tilde{w} = (\tilde{w}_1, \tilde{w}_2) \in \tilde{B} \in \mathcal{B}_t$ such that $\tilde{v} + \tilde{w} = x$. Since Σ_v and Σ_w are convex, we have

$$\tilde{v}_1 \leqslant v_1 + C''(M, \mathrm{Re}(\lambda_1), \mathrm{Re}(\lambda_2))\delta,$$

$$\tilde{w}_1 \geqslant w_1 - C''(M, \mathrm{Re}(\lambda_1), \mathrm{Re}(\lambda_2))\delta,$$

where $C''(M, \mathrm{Re}(\lambda_1), \mathrm{Re}(\lambda_2)) > 0$ is a constant that depends only on M and the eigenvalues of A. Thus

$$v_1 - \tilde{v}_1 = \tilde{w}_1 - w_1 \geqslant -C''(M, \mathrm{Re}(\lambda_1), \mathrm{Re}(\lambda_2))\delta. \tag{5.17}$$

Choose indices i_0, j_0, m_0, n_0 and i_0', j_0', m_0', n_0' such that $B_{i_0, j_0, m_0, n_0} \ni v$ and $B_{i_0', j_0', m_0', n_0'} \ni w$. (There are $\lesssim_{M, \mathrm{Re}(\lambda_1), \mathrm{Re}(\lambda_2)}$ possible choices of indices.) Also choose indices i_1, j_1, m_1

and i_1', j_1', m_1' such that $B_{i_1,j_1,m_1,n_1} \ni \tilde{v}$ and $B_{i_1',j_1',m_1',n_1'} \ni \tilde{w}$. Note that we must necessarily have $m_1 = m_0$ and $m_1' = m_0'$, and also that $-10 \leqslant n_1, n_1' \leqslant 10$. We next observe that for some sufficiently large constant $C'''(M, \mathrm{Re}(\lambda_1), \mathrm{Re}(\lambda_2))$ we must have

$$j_0 - C'''(M, \mathrm{Re}(\lambda_1), \mathrm{Re}(\lambda_2)) \log(\delta^{-1})^2 \leqslant j_1 \leqslant j_0 + C'''(M, \mathrm{Re}(\lambda_1), \mathrm{Re}(\lambda_2)) \log(\delta^{-1})^2,$$

$$j_0' - C'''(M, \mathrm{Re}(\lambda_1), \mathrm{Re}(\lambda_2)) \log(\delta^{-1})^2 \leqslant j_1' \leqslant j_0' + C'''(M, \mathrm{Re}(\lambda_1), \mathrm{Re}(\lambda_2)) \log(\delta^{-1})^2,$$

since otherwise (5.16) and (5.17) would imply that $\tilde{v}_2 + \tilde{w}_2 < v_2 + w_2 - C''''(M, \mathrm{Re}(\lambda_1), \mathrm{Re}(\lambda_2))\delta$. This completes the proof for case 1, since we have shown that for some constant $C(M, \mathrm{Re}(\lambda_1), \mathrm{Re}(\lambda_2))$ sufficiently large there are fewer than $C(M, \mathrm{Re}(\lambda_1), \mathrm{Re}(\lambda_2)) \log(\delta^{-1})^2$ possible choices of indices i_1, j_1, m_1, n_1 and i_1', j_1', m_1', n_1' such that $B_{i_1,j_1,m_1,n_1} \ni \tilde{v}$ and $B_{i_1',j_1',m_1',n_1'} \ni \tilde{w}$. The proof for case 2 is similar.

Now let us assume we have case 3. Suppose there is $\tilde{v} = (\tilde{v}_1, \tilde{v}_2) \in \tilde{A} \in \mathcal{A}_t$ and $\tilde{w} = (\tilde{w}_1, \tilde{w}_2) \in \tilde{B} \in \mathcal{B}_t$ such that $\tilde{v} + \tilde{w} = x$. Then if $\sigma^+(\tilde{v}, \tilde{w})$ is finite, then there is a constant $C'(M, \mathrm{Re}(\lambda_1), \mathrm{Re}(\lambda_2)) > 0$ such that

$$\tilde{v}_1 \leqslant v_1 + C'(M, \mathrm{Re}(\lambda_1), \mathrm{Re}(\lambda_2))\delta, \qquad \tilde{w}_1 \geqslant w_1 - C'(b_j M, \mathrm{Re}(\lambda_1), \mathrm{Re}(\lambda_2))\delta,$$

and if $\sigma^-(\tilde{v}, \tilde{w})$ is finite, then

$$\tilde{v}_1 \leqslant v_1 + C'(M, \mathrm{Re}(\lambda_1), \mathrm{Re}(\lambda_2))\delta, \qquad \tilde{w}_1 \geqslant w_1 - C'(M, \mathrm{Re}(\lambda_1), \mathrm{Re}(\lambda_2))\delta.$$

In either case, the previous argument shows there is a constant $C = C(M, \mathrm{Re}(\lambda_1), \mathrm{Re}(\lambda_2)) > 0$ such that there are fewer than $C \log(\delta^{-1})^2$ possible choices of indices

i_1, j_1, m_1, n_1 and i_1', j_1', m_1', n_1' such that B_{i_1,j_1,m_1,n_1} $\ni \tilde{v}$ and $B_{i_1',j_1',m_1',n_1'} \ni \tilde{w}$.

□

Proposition 5.4.2. *Let $N(M, Re(\lambda_1), Re(\lambda_2))$ be a positive integer and let $\delta > 0$, and fix positive real numbers u and t satisfying $\delta^{N(M,Re(\lambda_1),Re(\lambda_2))}t > u$. Then if $N(M, Re(\lambda_1), Re(\lambda_2))$ is sufficiently large, there exists a constant $C(M, Re(\lambda_1), Re(\lambda_2)) > 0$ (independent of δ and the choice of u and t) such that no point of $\mathbb{R}^2$ is contained in more than $C(M, Re(\lambda_1), Re(\lambda_2))$ of the sets $\{A + B_\rho(0, 2/t)\}_{A\in\mathcal{A}_u}$, where $B_\rho(0, r) = \{x \in \mathbb{R}^2 : \rho(x) \leqslant r\}$.*

Proof. Without loss of generality, suppose that $u = 1$. Fix $A \in \mathcal{A}_u$, and let $x \in A$ and let $y \in B_\rho(0, 2/t)$. Choose $N(M, \mathrm{Re}(\lambda_1), \mathrm{Re}(\lambda_2))$ large enough to make $B_\rho(0, 2/t) \subset B(0, \delta^2)$, where $B(0, \delta^2)$ denotes the (Euclidean) ball of radius δ^2 centered at the origin. Assume $\delta < C'(M, \mathrm{Re}(\lambda_1), \mathrm{Re}(\lambda_2), \Theta(\Omega, A))$, where $C'(M, \mathrm{Re}(\lambda_1), \mathrm{Re}(\lambda_2), \Theta(\Omega, A)) > 0$ is chosen sufficiently small so that the minimum angle between the tangent line to $\xi \in \partial\Omega$ and any tangent line to the curve $\{t^A\xi : 1 - 10\delta \leqslant t \leqslant 1 + 10\delta\}$ is at least $\delta^{1/2}$. Now for any $\xi \in \partial\Omega$, $1 - 10\delta \leqslant t \leqslant 1 + 10\delta$, we have

$$\left|\frac{d}{dt}(t^A\xi)\right| = |t^{-1}At^A\xi| \gtrsim_{M,\mathrm{Re}(\lambda_1),\mathrm{Re}(\lambda_2)} 1,$$

and it follows that if $C'(M, \mathrm{Re}(\lambda_1), \mathrm{Re}(\lambda_2), \Theta(\Omega, A))$ is sufficiently small, the (Euclidean) distance between $t^A\xi$ and the tangent line to $\partial\Omega$ at ξ is at least $10\delta^2$. Since Ω is convex, we conclude that the distance between $\partial\Omega$ and $(1 + \delta)^A\partial\Omega$ is at least $10\delta^2$. Similarly, the distance between $\partial\Omega$ and $(1 - \delta)^A\partial\Omega$ is at least $10\delta^2$. It follows there is an absolute constant C such that for any given $\xi \in \mathbb{R}^2$, there are fewer than C possible values of m (and clearly also fewer than C possible values of n) such that $B_{i,j,m,n} + B(0, \delta^2) \ni \xi$

for some $B_{i,j,m,n} \in \mathcal{A}_1$. It remains to obtain an upper bound for the number of possible values of j. But it is clear that $\text{dist}(B_{i,j,m,n}, B_{i,j',m,n}) \geqslant \delta/10$ for $|j - j'| > 2$, and this finishes the proof.

□

Proposition 5.4.3. *There exists an absolute constant $C > 0$ such that for each fixed quadruple (i, j, m, n), the logarithmic measure of $\{t : B_{i,j,m,n} \cap \text{supp}\, \mathcal{F}[\psi_t] \neq \emptyset\}$ is less than or equal to $C\delta$.*

Proof. Immediate. □

5.5 Kernel estimates and another L^2 maximal function estimate

We note that in both [18] and [8], it was important that regarding the decomposition of the multiplier $\phi(\delta^{-1}(1-|\xi|)$ where ϕ was a smooth bump function into pieces supported on $\delta^{1/2} \times \delta$ rectangles, each piece of the multiplier had L^1 norm essentially 1. This was also true of the decomposition of $|\rho(\xi) - 1| \leqslant \delta$ introduced in [48]. In this section we prove that after the introduction of nonisotropic dilations, the same holds true.

The argument presented in [8] also used L^2 bounds for maximal functions given by the supremum of convolutions by smooth bumps supported on finitely many essentially disjoint pieces of the decomposition of $\mathbb{R}^2$ given in [8]. Since these smooth bumps could be dominated by Schwartz functions adapted to rectangles, such a maximal function could be dominated by a Nikodym maximal function. Here, as well as in [48], we do not have domination of the functions in our partition of unity by Schwartz functions adapted to rectangles, and the proof of L^1 kernel estimates is more delicate. As in [48], this also implies that the associated maximal function that we use is not simply a nonisotropic Nikodym maximal function. However, we will show that the L^2 bounds for the nonisotropic Nikodym maximal function proved earlier imply L^2 bounds for the maximal function that we are interested in, with a similar constant.

A partition of unity associated to the decomposition of $\mathbb{R}^2$

First, we need to define a partition of unity of $\mathbb{R}^2$, and as mentioned above one goal of this section is to show that each function in our partition of unity has bounded L^1 norm. Recall the decomposition

$$\mathbb{R}^2 = \bigcup_{i,j,m,n} B_{i,j,m,n}.$$

We now introduce a partition of unity $\{\sigma_{i,j,m,n}\}$ such that

1. $\sigma_{i,j,m,n} \in C^\infty(\mathbb{R}^2)$ for every (i, j, m, n),
2. $\sum_{i,j,m,n} \sigma_{i,j,m,n}(x) = 1$ for every $x \in \mathbb{R}^2$,
3. There is a constant C_M such that for every (i_0, j_0, m_0, n_0), σ_{i_0,j_0,m_0,n_0} is supported in $\bigcup_{|j|,|m|\leqslant C_M} B_{i_0,j_0+j,m_0+m,n_0}$.

Let $\phi \in C^\infty([-1, 1])$ be nonnegative and identically 1 on $[-1/2, 1/2]$, and for $n \in \mathbb{Z}$ set $\phi_n(\cdot) = \phi(2^{-n-1}\cdot) - \phi(2^{-n}\cdot)$. For each m, let $\psi_m \in C^\infty(1 - (2m + 10)\delta, 1 + (2m + 10)\delta)$ such that $\sum_m \psi_m$ is identically 1 on the support of ϕ_0, and for every k, $D^k\psi_m \lesssim_k \delta^{-k}$.

For each i, let S_i be the *isotropic* sector bounded by $|\xi| = 2, |\xi| = 2^{M+2}$, and the rays through the origin and the points ξ and ξ', where $\xi = (\xi_1, \xi_2)$ is the unique point in $\mathcal{R}_i\partial\Omega$ with $\xi_1 = -1/4$ and $\xi_2 > 0$, and $\xi' = (\xi_1', \xi_2')$ is the unique point in $\mathcal{R}_i\partial\Omega$ with $\xi_1 = 1/4$ and $\xi_2 > 0$. Let $\tilde{S}_i$ be the isotropic sector bounded by $|\xi| = 1, |\xi| = 2^{M+3}$, and the rays through the origin and the points ξ and ξ', where $\xi = (\xi_1, \xi_2)$ is the unique point in $\mathcal{R}_i\partial\Omega$ with $\xi_1 = -3/4$ and $\xi_2 > 0$, and $\xi' = (\xi_1', \xi_2')$ is the unique point in $\mathcal{R}_i\partial\Omega$ with $\xi_1 = 3/4$ and $\xi_2 > 0$. For each i, let Ψ_i be a smooth function supported in $\tilde{S}_i$ and identically 1 on S_i, such that $D^k\Psi_i \lesssim_{M,k} 1$ for all k and $\sum_i \Psi_i$ is identically 1 on the region bounded by $|\xi| = 2$ and $|\xi| = 2^{M+2}$.

Fix i, and for each j, let ℓ_{j-1}, ℓ_j, and ℓ_{j+1} be the lines through $(b_{j-1}, \gamma_i(b_{j-1}))$, $(i_j, \gamma_i(i_j))$, and $(b_{j+1}, \gamma_i(b_{j+1}))$, respectively, with slopes orthogonal to the tangent vectors $(1, \gamma_i'(b_{j-1})$, $(1, \gamma_i'(i_j))$, and $(1, \gamma_i'(b_{j+1}))$, respectively. Let e_j be a unit vector orthogonal to ℓ_j. Let α be a $C^\infty(\mathbb{R})$ function such that $0 \leqslant \alpha \leqslant 1$, $\alpha(x) = 1$ for $x \in [-1, 1]$ and $\alpha(x) = 0$ for $x \notin [-\frac{101}{100}, \frac{101}{100}]$, and set $\alpha_j(\xi) = \alpha(|I_j|^{-1}(\xi - (i_j, \gamma_i(i_j))) \cdot e_j)$. We are now ready to define the functions $\sigma_{i,j,m,n}$. Let

$$\sigma_{i,j,m,0}(\xi) = \phi_0(\rho(\xi))\Psi_i((\frac{1-2\delta}{1+2\delta})^{mA}\xi)\psi_m(\rho(\xi)) \times \alpha_j(\mathcal{R}_i(\frac{1-2\delta}{1+2\delta})^{mA}\xi)(1 - \alpha_{j+1}(\mathcal{R}_i(\frac{1-2\delta}{1+2\delta})^{mA}\xi)), \quad (5.18)$$

and

$$\sigma_{i,j,m,n}(\xi) = \sigma_{i,j,m,0}((2^{-n})^A\xi). \quad (5.19)$$

For every i and every m, we have

$$\sum_j \alpha_j(\mathcal{R}_i(\frac{1-2\delta}{1+2\delta})^{mA}\xi)(1 - \alpha_{j+1}(\mathcal{R}_i(\frac{1-2\delta}{1+2\delta})^{mA}\xi))$$

is identically 1 on the support of

$$\phi_0(\rho(\xi))\Psi_i((\frac{1-2\delta}{1+2\delta})^{mA}\xi)\psi_m(\rho(\xi)),$$

and since

$$\sum_i \sum_m \phi_0(\rho(\xi))\Psi_i((\frac{1-2\delta}{1+2\delta})^{mA}\xi)\psi_m(\rho(\xi))$$
$$= \sum_m \phi_0(\rho(\xi))\psi_m(\rho(\xi)) = \phi_0(\rho(\xi)),$$

it follows that for every $\xi \in \mathbb{R}^2$,

$$\sum_{i,j,m,n} \sigma_{i,j,m,n}(\xi) = 1.$$

Introduction of a maximal function associated with the partition of unity

Let

$$K_{i,j,m,n}(x) = \mathcal{F}[\sigma_{i,j,m,n}(\cdot)](x).$$

We define a maximal function $\overline{M}$ on $f \in \mathcal{S}(\mathbb{R}^2)$ by

$$\overline{M}f(x) = \sup_{i,j,m,n} \sup_{2^{n-10}\leqslant t\leqslant 2^{n+10}} |\psi_t * K_{i,j,m,n} * f(x)|.$$

We will prove the following L^2 bounds for $\overline{M}$.

Proposition 5.5.1. *Let $\epsilon > 0$. There is a constant $C = C(M, Re(\lambda_1), Re(\lambda_2),$ $\Theta(\Omega, A))$ such that if $0 < \delta < C$, then for $f \in \mathcal{S}(\mathbb{R}^2)$,*

$$\left\|\overline{M}f\right\|_{L^2(\mathbb{R}^2)} \lesssim_{\epsilon,M,Re(\lambda_1),Re(\lambda_2),\Theta(\Omega,A)} \delta^{-\epsilon} \left\|f\right\|_{L^2(\mathbb{R}^2)}.$$

Proof. The proof will follow [48]. First note that without loss of generality we may drop the "sup" in the i index in the definition of $\overline{M}$ and assume $i = 0$, and so in what follows we drop all i-indices. Set $l = \lceil \log(\delta^{-1}) \rceil$. We decompose $\overline{M} = M_1 + M_2$, where

$$M_1 f(x) = \sup_{j,m,n} \sup_{2^{n-10}\leqslant t\leqslant 2^{n+10}} |\psi_t * (K_{j,m,n} \cdot \chi_{|t^{A}\cdot|\geqslant 2^{10M\cdot l}}) * f(x)|,$$

$$M_2 f(x) = \sup_{j,m,n} \sup_{2^{n-10} \leqslant t \leqslant 2^{n+10}} |\psi_t * (K_{j,m,n} \cdot \chi_{|t^A \cdot| < 2^{10M \cdot l}}) * f(x)|.$$

We will first prove Proposition 5.5.1 with $\overline{M}$ replaced by M_1. Let $\sigma_j(\xi) = \mathcal{F}^{-1}[K_{j,0,0}(\cdot)](\xi)$. Note that

$$\sigma_j(\xi) = \phi_0(\rho(\xi))\psi_0(\rho(\xi))m_j(\xi), \tag{5.20}$$

where

$$m_j(\xi) = \Psi_0(\xi)\phi_0(2^{-2M}\xi)\alpha_j(\xi)(1 - \alpha_j(\xi - 2^{M+10}|I_j|(1, \gamma_0'(i_j))))$$
$$(1 - \alpha_{j+1}(\xi))(\alpha_{j+1}(\xi + 2^{M+10}|I_j|(1, \gamma_0'(b_{j+1})))). \tag{5.21}$$

Now let $\beta \in C^\infty$ be supported in $[-1, 1]$, and let $h_l(s) = \beta(2^l(1 - s))$. Note that (5.20) says that σ_j is of the form $h_l(\rho(\cdot))m_j(\cdot)$. We claim that to prove Proposition 5.5.1 with M_1 in place of $\overline{M}$, it in fact suffices to prove Proposition 5.5.1 with $\overline{M}f$ replaced by

$$\sup_{t \in (0,\infty)} |(\chi_{|t^A \cdot| \geqslant 2^{5M \cdot l}} \cdot \mathcal{F}^{-1}[h_l(t\rho(\cdot))]) * f(x)|.$$

This will follow immediately from the observation that $2^{M+10}|I_j|^{-1} << 2^{10M \cdot l}$ and that for any annulus $\mathcal{A}_k$,

$$\int_{\mathcal{A}_k} \mathcal{F}^{-1}[h_l(\rho(\cdot))](x)\, dx \lesssim 1, \tag{5.22}$$

which will be proven later.

We now prove pointwise estimates for $\mathcal{F}^{-1}[h_l(\rho(\cdot))](x)$, which we write as an integral over $\partial\Omega$ as follows:

$$(2\pi)^2 \mathcal{F}^{-1}[h_l(\rho(\cdot))](x) = \int_\Omega h_l(\rho(\xi)) e^{i\langle x, \xi \rangle}\, d\xi = -\int_\Omega e^{i\langle x, \xi \rangle} \int_{\rho(\xi)}^\infty h_l'(s)\, ds\, d\xi$$

$$= -\int_0^\infty h_l'(s) \int_{\rho(\xi)\leqslant s} e^{i\langle x,\xi\rangle}\, d\xi\, ds = -\int_0^\infty h_l'(s) \int_{\rho(\xi)\leqslant 1} e^{i\left\langle x, s^A\xi\right\rangle} |\det s^A|\, d\xi\, dx$$

$$= \int_0^\infty i|s^{A^*}x|^{-2} h_l'(s) \int_{\partial\Omega} e^{i\left\langle s^{A^*}x,\xi\right\rangle} \left\langle s^{A^*}x, n(\xi)\right\rangle\, d\sigma(\xi) |\det s^A|\, ds.$$

In the above computation, we used the divergence theorem applied to the vector field $\xi \mapsto (i|s^{A^*}x|^2)^{-1} s^{A^*} x e^{i\left\langle s^{A^*}x,\xi\right\rangle}$. For each i, let $\zeta_i \in C^\infty(\mathbb{R})$ be supported in $[-4/5, 4/5]$ and identically 1 on $[-1/3, 1/3]$ such that $\sum_i \zeta_i((\mathcal{R}_i$
$(1/\rho(\xi))^A\xi)_1) \equiv 1$. It suffices to estimate

$$\int_0^\infty i|s^{A^*}x|^{-2} h_l'(s) \int_{\partial\Omega} e^{i\left\langle s^{A^*}x,\xi\right\rangle}$$

$$\left\langle s^{A^*}x, n(\xi)\right\rangle \zeta_0((\xi)_1) d\sigma(\xi) |\det s^A|\, ds. \quad (5.23)$$

We introduce homogeneous coordinates

$$(s,\alpha) \mapsto \xi(s,\alpha) = s^A(\alpha, \gamma_0(\alpha)). \quad (5.24)$$

The Jacobian of the map (5.24) is

$$\left\langle s^A(1, \gamma_0'(\alpha)), s^{-1}As^A(\alpha,\gamma_0(\alpha))\right\rangle.$$

Using homogeneous coordinates, (5.23) can be written as

$$i\int \zeta_0(\alpha) \int_0^\infty |s^{A^*}x|^{-2} h_l'(s) e^{i\left\langle x, s^A(\alpha,\gamma_0(\alpha))\right\rangle}$$

$$\times \left\langle x, s^A(-\gamma_0'(\alpha), 1)(1 + (\gamma_0'(\alpha))^2)^{-1/2}\right\rangle$$

$$\times \left\langle s^A(1,\gamma_0'(\alpha)), s^{-1}As^A(\alpha,\gamma_0(\alpha))\right\rangle |\det s^A|\, ds\, d\alpha. \quad (5.25)$$

Let $\eta : \mathbb{R} \to \mathbb{R}$ be a smooth function supported in $[-\epsilon, \epsilon]$, where

$$\epsilon = \Theta(\Omega, A) \cdot \min(|\lambda_1|, |\lambda_2|)/(100 \cdot 2^{M+2}). \quad (5.26)$$

Then (5.25) can be written as $\tilde{K}_1(x)+\tilde{K}_2(x)$, where

$$\tilde{K}_1(x)=i\int\zeta_0(\alpha)\int_0^\infty |s^{A^*}x|^{-2}h_l'(s)e^{i\langle x,s^A(\alpha,\gamma_0(\alpha))\rangle}$$
$$\times\,\eta\big(\frac{\langle x,A(\alpha,\gamma_0(\alpha))\rangle}{|x|}\big)\,\big\langle x,s^A(-\gamma_0'(\alpha),1)(1+(\gamma_0'(\alpha))^2)^{-1/2}\big\rangle$$
$$\times\,\big\langle s^A(1,\gamma_0'(\alpha)),s^{-1}As^A(\alpha,\gamma_0(\alpha))\big\rangle\,|\det s^A|\,ds\,d\alpha,$$

$$\tilde{K}_2(x)=i\int\zeta_0(\alpha)\int_0^\infty |s^{A^*}x|^{-2}h_l'(s)e^{i\langle x,s^A(\alpha,\gamma_0(\alpha))\rangle}$$
$$\times\,\big(1-\eta\big(\frac{\langle x,A(\alpha,\gamma_0(\alpha))\rangle}{|x|}\big)\big)\,\big\langle x,s^A(-\gamma_0'(\alpha),1)(1+(\gamma_0'(\alpha))^2)^{-1/2}\big\rangle$$
$$\times\,\big\langle s^A(1,\gamma_0'(\alpha)),s^{-1}As^A(\alpha,\gamma_0(\alpha))\big\rangle\,|\det s^A|\,ds\,d\alpha.$$

To estimate $\tilde{K}_2(x)$, we integrate by parts with respect to s twice. This yields

$$\tilde{K}_2(x)=i\int\zeta_0(\alpha)\big(1-\eta\big(\frac{\langle x,A(\alpha,\gamma_0(\alpha))\rangle}{|x|}\big)\big)$$
$$\int_0^\infty g_2(x,s,\alpha)e^{i\langle x,s^A(\alpha,\gamma_0(\alpha))\rangle}\,ds\,d\alpha,$$

where

$$g_2(x,s,\alpha)=\frac{d}{ds}\Bigg(\big\langle x,s^{-1}As^A(\alpha,\gamma_0(\alpha))\big\rangle^{-1}\frac{d}{ds}\Bigg(\big\langle x,s^{-1}As^A(\alpha,\gamma_0(\alpha))\big\rangle^{-1}$$
$$\times\,|s^{A^*}x|^{-2}h_l'(s)\,\big\langle x,s^A(-\gamma_0'(\alpha),1)(1+(\gamma_0'(\alpha))^2)^{-1/2}\big\rangle$$
$$\times\,\big\langle s^A(1,\gamma_0'(\alpha)),s^{-1}As^A(\alpha,\gamma_0(\alpha))\big\rangle\,|\det s^A|\Bigg)\Bigg).$$

Note that if $0<\delta<C$ for a sufficiently small constant $C>0$, then for s in the support of $h_l(s)$ and for x in the the support of $1-\eta\big(\frac{\langle x,A(\alpha,\gamma_0(\alpha))\rangle}{|x|}\big)$, we have $\big\langle x,s^{-1}s^AA(\alpha,\gamma_0(\alpha))\big\rangle\geqslant|x|\cdot\epsilon/2$. Thus

$$|g_2(x,s,\alpha)|\lesssim|x|^{-3}|h_l''(s)|.$$

This implies that

$$|\tilde{K}_2(x)| \lesssim |x|^{-3} \int \zeta_0(\alpha) \int |h_l''(s)|\, ds\, d\alpha \lesssim 2^l |x|^{-3}. \tag{5.27}$$

To estimate $\tilde{K}_1(x)$, we integrate by parts with respect to α once and then with respect to s twice, which yields

$$\tilde{K}_1(x) = \int_0^\infty \int g_1(x,s,\alpha) e^{i\left\langle x, s^A(\alpha,\gamma_0(\alpha))\right\rangle}\, d\alpha\, ds,$$

where

$$\begin{aligned} g_1(x,s,\alpha) = -\frac{d}{ds}\Bigg(&\left\langle x, s^{-1}As^A(\alpha,\gamma_0(\alpha))\right\rangle^{-1} \frac{d}{ds}\Bigg(\left\langle x, s^{-1}As^A(\alpha,\gamma_0(\alpha))\right\rangle^{-1} \\ &\times \frac{d}{d\alpha}\Bigg(\left\langle x, s^A(1,\gamma_0'(\alpha))\right\rangle^{-1} \zeta_0(\alpha)\eta\Big(\frac{\langle x, A(\alpha,\gamma_0(\alpha))\rangle}{|x|}\Big) \\ &\qquad \times i \left\langle x, s^A(-\gamma_0'(\alpha),1)(1+(\gamma_0'(\alpha))^2)^{-1/2}\right\rangle \\ &\qquad \times \left\langle s^A(1,\gamma_0'(\alpha)), s^{-1}As^A(\alpha,\gamma_0(\alpha))\right\rangle |s^{A^*}x|^{-2} h_l'(s)|\det s^A| \Bigg)\Bigg)\Bigg). \end{aligned}$$

By the choice of ϵ, we have that for s in the support of $h_l(s)$ and for x in the the support of $\eta\big(\frac{\langle x, A(\alpha,\gamma_0(\alpha))\rangle}{|x|}\big)$, if θ denotes the angle between x and $A(\alpha,\gamma_0(\alpha))$, then $\cos(\theta) \leqslant \Omega(\Theta,A)/100$. Since $A(\alpha,\gamma_0(\alpha))$ is tangent to the orbit $\{s^A(\alpha,\gamma_0(\alpha)) : s > 0\}$ at $(\alpha,\gamma_0(\alpha))$, if $0 < \delta < C$ for a sufficiently small constant C, we have

$$|\left\langle x, s^A(1,\gamma_0'(\alpha))\right\rangle|^{-1} \geqslant (\Theta(\Omega,A)/2^{M+2}) \cdot |x|.$$

It follows that

$$|g_1(x,s,\alpha)| \lesssim |x|^{-2} |\left\langle x, s^{-1}As^A(\alpha,\gamma_0(\alpha))\right\rangle|^{-2} |h_l''(s)|\zeta_0(\alpha)(1+|\gamma''(\alpha)|),$$

and hence

$$\begin{aligned} |\tilde{K}_1(x)| \lesssim \int\int &|x|^{-2}|1 + |\left\langle x, s^{-1}As^A(\alpha,\gamma_0(\alpha))\right\rangle||^{-2} \\ &\times |h_l''(s)|\zeta_0(\alpha)|(1+|\gamma''(\alpha)|)\, d\alpha\, ds. \end{aligned} \tag{5.28}$$

It follows from (5.27) that

$$\int_{\mathcal{A}_k} |\tilde{K}_2(x)| \lesssim 2^{l-k}, \tag{5.29}$$

and it follows from (5.28) that

$$\int_{\mathcal{A}_k} |\tilde{K}_1(x)| \lesssim 2^{l-k}, \tag{5.30}$$

and (5.29) and (5.30) imply (5.22).

Now, (5.27) implies that for $0 < \delta < C$ we have that $|\tilde{K}_2 \cdot \chi_{|t^A\cdot|\geqslant 2^{5M\cdot l}}|$ is bounded above by a radial, decreasing function with L^1 norm $\lesssim 1$. It follows that there is a sequence $\{a_n\}$ with $a_n \geqslant 0$ and $\sum_{n=0}^{\infty} a_n \lesssim 1$ such that for $0 < \delta < C$,

$$\left\| \sup_{t\in(0,\infty)} |(\chi_{|t^A\cdot|\geqslant 2^{5M\cdot l}} \cdot \det(t^A)\tilde{K}_2(t^A\cdot)) * f(x)| \right\|_{L^2(\mathbb{R}^2)} \lesssim \left\| \sum_{n=0}^{\infty} a_n M_{2^{-n},1} f \right\| \lesssim \delta^{-\epsilon} \|f\|_{L^2(\mathbb{R}^2)}, \tag{5.31}$$

where we have applied Proposition 5.3.1.

We now prove a similar estimate for $\tilde{K}_2$. Observe that (5.28) implies that if $0 < \delta < C$,

$$\sup_{t\in(0,\infty)} |(\chi_{|t^A\cdot|\geqslant 2^{5M\cdot l}} \cdot \det(t^A)\tilde{K}_1(t^A\cdot)) * f(x)| \lesssim \int\int \sum_{n=0}^{\infty} 2^{-n/4} M_{2^{n/4},2^{n/4}} f(x) |h_l''(x)| \xi_0(\alpha)| (1 + |\gamma''(\alpha)| \, d\alpha \, ds,$$

and hence by Proposition 5.3.1,

$$\left\| \sup_{t\in(0,\infty)} |(\chi_{|t^A\cdot|\geqslant 2^{5M\cdot l}} \cdot \det(t^A)\tilde{K}_1(t^A\cdot)) * f(x)| \right\|_{L^2(\mathbb{R}^2)} \lesssim \delta^{-\epsilon} \|f\|_{L^2(\mathbb{R}^2)}. \tag{5.32}$$

Together (5.31) and (5.32) prove the result with $\overline{M}$ replaced by M_1.

It remains to prove the result with $\overline{M}$ replaced by M_2. Observe that

$$\sigma_j(\xi) = \phi_0(2^{-2M})\alpha_j(\xi)(1-\alpha_j(\xi - 2^{M+10}|I_j|(1,\gamma_0'(i_j)))) \\ \times (1-\alpha_{j+1}(\xi))(\alpha_{j+1}(\xi + 2^{M+10}|I_j|(1,\gamma_0'(b_{j+1})))) \cdot \tilde{m}_j(\xi),$$

where

$$\tilde{m}_j(\xi) = \Psi_0(\xi)\psi_0(\rho(\xi))\nu_j(((1/\rho(\xi))^A\xi)_1),$$

for some C^∞ function ν_j supported in an interval I_j^* of width $10|I_j|$ satisfying

$$|D^i\nu_j| \lesssim |I_j|^{-i}$$

for every integer $i \geqslant 0$. The kernel of the multiplier $\tilde{m}_j$ can be easily written as an integral in homogeneous coordinates. If we can prove that for every annulus $\mathcal{A}_k$ with $k \geqslant 0$,

$$\int_{\mathcal{A}_k} |\mathcal{F}^{-1}[\tilde{m}_j(\cdot)](x)|\, dx \lesssim l, \tag{5.33}$$

then it would follow that the desired result reduces to proving the result of the proposition with $\overline{M}f(x)$ replaced by

$$\sup_{t\in(0,\infty)} |(\chi_{|t^A\cdot|\leqslant 2^{20M\cdot l}} \cdot \mathcal{F}^{-1}[\tilde{m}_j(t^{-A}\cdot)]) * f(x)|. \tag{5.34}$$

We now proceed to prove (5.33). As before, let η be smooth and supported in $[-\epsilon,\epsilon]$, where ϵ is given by (5.26). Also, as before let $\phi \in C^\infty([-1,1])$ be nonnegative and identically 1 on $[-1/2,1/2]$, and for $n \in \mathbb{Z}$ set $\phi_n(\cdot) = \phi(2^{-n-1}\cdot) - \phi(2^{-n}\cdot)$. Define

$$\Phi_0(x,s,\alpha) = \phi_0(|I_j|\left\langle x, s^A(1,\gamma_0'(\alpha))\right\rangle)\eta(\frac{\left\langle x, s^A(1,\gamma_0'(\alpha))\right\rangle}{|x|})) \tag{5.35}$$

$$\Phi_n(x,s,\alpha) = (\phi_0(2^{-n-1}|I_j| \left\langle x, s^A(1,\gamma_0'(\alpha))\right\rangle) \\ - \phi_0(2^{-n}|I_j| \left\langle x, s^A(1,\gamma_0'(\alpha))\right\rangle))\eta\Big(\frac{\left\langle x, s^A(1,\gamma_0'(\alpha))\right\rangle}{|x|}\Big). \quad (5.36)$$

We decompose the kernel as

$$\mathcal{F}^{-1}[\tilde{m}_j(\cdot)](x) = \frac{1}{(2\pi)^2}[\tilde{K}_j(x) + \sum_{n\geqslant 0} K_{j,n}(x)], \quad (5.37)$$

where

$$K_{j,n}(x) = \int \nu_j(\alpha) \int h_l(s)\Phi_n(x,s,\alpha) e^{i\left\langle x, s^A(\alpha,\gamma_0(\alpha))\right\rangle} \\ \times \left\langle s^A(1,\gamma_0'(\alpha)), s^{-1}As^A(\alpha,\gamma_0(\alpha))\right\rangle \, ds\, d\alpha$$

and

$$\tilde{K}_j(x) = \int \nu_j(\alpha)\left(1 - \eta\Big(\frac{\left\langle x, s^A(1,\gamma_0'(\alpha))\right\rangle}{|x|}\Big)\right) \int h_l(s) e^{i\left\langle x, s^A(\alpha,\gamma_0(\alpha))\right\rangle} \\ \times \left\langle s^A(1,\gamma_0'(\alpha)), s^{-1}As^A(\alpha,\gamma_0(\alpha))\right\rangle \, ds\, d\alpha.$$

Note that the sum in (5.37) has only $\lesssim \log(1 + |I_j||x|)$ terms, since $K_{j,n}(x) = 0$ if $2^{n-10}|I_j|^{-1} \geqslant \epsilon|x|$. In particular if $x \in \mathcal{A}_k \cap \sup(K_{j,n})$ then $2^n << 2^k|I_j|$.

For $K_{j,0}$, we simply estimate $\int_{\mathbb{R}^2} |K_{j,0}(x)|\, dx$. For a given (α, s), we introduce coordinates

$$(u_1,u_2) \mapsto \xi(u_1,u_2) = u_1 s^A(1,\gamma_0'(\alpha)) + u_2 s^{-1}As^A(1,\gamma_0'(\alpha)). \quad (5.38)$$

The Jacobian of the map (5.38) is ≈ 1. Integrating by parts three times in s yields

$$|K_{j,0}(x)| \lesssim \\ \int_{s:\,|s-1|\approx 2^{-l}} \int_{\substack{\alpha\in I_j^* \\ \left\langle x, s^A(1,\gamma_0'(\alpha))\right\rangle \leqslant (2|I_j|)^{-1}}} (1 + 2^{-l}|\left\langle x, s^{-1}As^A(\alpha,\gamma_0(\alpha))\right\rangle|)^{-3}\, ds\, d\alpha, \quad (5.39)$$

and thus using the change of coordinates (5.38)

$$\int_{\mathbb{R}^2} |K_{j,0}(x)|\, dx \lesssim \int_{I_j^*} \int \int_{|u_1| \leqslant (2|I_j|)^{-1}} 2^{-l}(1+2^{-l}|u_2|)^{-3}\, du_1\, du_2\, d\alpha \lesssim 1. \quad (5.40)$$

For $n > 0$, we integrate by parts with respect to α once and then with respect to s twice, which yields

$$K_{j,n}(x) = \int \int h_l(s) g_n(x,s,\alpha) e^{i\left\langle x, s^A(\alpha,\gamma_0(\alpha))\right\rangle}\, d\alpha\, ds,$$

where

$$g_n(x,s,\alpha) = -\frac{d}{ds}\Bigg(\frac{1}{\langle x, s^{-1}As^A(\alpha,\gamma_0(\alpha))\rangle}\frac{d}{ds}\Bigg(\frac{1}{\langle x, s^{-1}As^A(\alpha,\gamma_0(\alpha))\rangle} \frac{d}{d\alpha}\Bigg(\frac{1}{\langle x, s^A(1,\gamma_0'(\alpha))\rangle}\nu_j(\alpha)\Phi_n(x,s,\alpha)\left\langle s^A(1,\gamma_0'(\alpha)), s^{-1}As^A(\alpha,\gamma_0(\alpha))\right\rangle\Bigg)\Bigg)\Bigg).$$

On the support of $h_l(s)$ we have

$$|g_n(x,s,\alpha)| \lesssim \frac{(1+|x|2^{-n}|I_j|)|\gamma_0''(\alpha)| + |I_j|^{-1}}{2^{-2l}|\left\langle x, s^{-1}As^A(1,\gamma_0'(\alpha))\right\rangle|^2|\left\langle x, s^A(1,\gamma_0'(\alpha))\right\rangle|},$$

and so

$$|K_{j,n}(x)| \lesssim \int_{s\colon |s-1|\approx 2^{-l}} \int_{\substack{\alpha\in I_j^* \\ |\left\langle x, s^A(1,\gamma_0'(\alpha))\right\rangle| \\ \approx 2^n|I_j|^{-1}}} \frac{(1+|x|2^{-n}|I_j|)|\gamma_0''(\alpha)| + |I_j|^{-1}}{|\left\langle x, s^A(1,\gamma_0'(\alpha))\right\rangle|} \frac{1}{(1+2^{-l}|\left\langle x, s^{-1}As^A(1,\gamma_0'(\alpha))\right\rangle|)^2}\, d\alpha\, ds. \quad (5.41)$$

Using the change of coordinates (5.38), it follows that

$$\int_{\mathcal{A}_k} |K_{j,n}(x)|\, dx \lesssim \int_{s\colon |s-1|\approx 2^{-l}} 2^l \int_{\alpha\in I_j^*} ((1+2^{k-n}|I_j|)|\gamma_0''(\alpha)| + |I_j|^{-1}) \times \int_{\substack{u_1\approx 2^n|I_j|^{-1} \\ |u|\approx 2^k}} |u_1|^{-1}\frac{2^{-l}}{(1+2^{-l}|u_1|)^2}\, du\, d\alpha$$

$$\lesssim \int_{I_j^*} (|\gamma_0''(\alpha)| + 2^{k-n}|I_j||\gamma_0''(\alpha)| + |I_j|^{-1})\, d\alpha.$$

By (5.15) we have $\int_{I_j^*} |I_j||\gamma_0''(\alpha)|\,d\alpha \leqslant 2^{-l}$, and so

$$\int_{\mathcal{A}_k} |K_{j,n}(x)|\,dx \lesssim \min\{2^{k-l}, 2^{l-k}\}(2^{k-l-n}+1).$$

Since $K_{j,n}$ is identically 0 on $\mathcal{A}_k$ if $n \geqslant k$, summing in n and also using (5.40) yields

$$\sum_{n\geqslant 0} \int_{\mathcal{A}_k} |K_{j,n}(x)|\,dx \lesssim k2^{-|k-l|}. \tag{5.42}$$

Now we estimate $\tilde{K}_j$. Integrating by parts once in α and then once in s yields

$$|\tilde{K}_j(x)| \lesssim \int_{s:\,|s-1|\approx 2^{-l}} \int_{I_j^*} \frac{(|I_j|^{-1} + |\gamma_0''(\alpha)|)}{|x|(1+2^{-l}|\langle x, s^{-1}As^A(\alpha,\gamma_0(\alpha))\rangle|)}\,d\alpha\,ds, \tag{5.43}$$

and so using the change of coordinates (5.38) we get

$$\int_{\mathcal{A}_k} |\tilde{K}_j(x)|\,dx \lesssim 1. \tag{5.44}$$

Combining (5.42) and (5.44) gives (5.33). We now proceed to examine

$$\sup_{t\in(0,\infty)} |(\chi_{|t^A\cdot|\leqslant 2^{20M\cdot l}} \cdot \mathcal{F}^{-1}[\tilde{m}_j(t^{-A}\cdot)]) * f(x)|.$$

By (5.39), for $0<\delta<C$ we have

$$\sup_{t\in(0,\infty)} |(\chi_{|t^A\cdot|\leqslant 2^{20M\cdot l}} \cdot K_{j,0}(t^A\cdot)) * f(x)| \lesssim$$
$$\int_{s:\,|s-1|\approx 2-l} 2^l \int_{\alpha\in|I_j|^*} |I_j|^{-1} \sum_{n=0}^{C_M\cdot l} M_{|I_j|^{-1},2^{l+n/3}|I_j|} f(x)\,d\alpha\,ds,$$

and hence by Proposition 5.3.1,

$$\left\| \sup_{t\in(0,\infty)} |(\chi_{|t^A\cdot|\leqslant 2^{20M\cdot l}} \cdot K_{j,0}(t^A\cdot)) * f(x)| \right\|_{L^2(\mathbb{R}^2)} \lesssim_\epsilon \delta^{-\epsilon} \|f\|_{L^2(\mathbb{R}^2)}. \tag{5.45}$$

Similarly examining (5.41) and (5.43) leads to

$$\left\| \sup_{t\in(0,\infty)} |(\chi_{|t^A\cdot|\leqslant 2^{20M\cdot l}} \cdot K_{j,n}(t^A\cdot)) * f(x)| \right\|_{L^2(\mathbb{R}^2)} \lesssim_\epsilon \delta^{-\epsilon} \|f\|_{L^2(\mathbb{R}^2)} \tag{5.46}$$

for $n > 0$ and

$$\left\| \sup_{t\in(0,\infty)} |(\chi_{|t^A\cdot|\leqslant 2^{20M\cdot l}} \cdot \tilde{K}_j(t^A\cdot)) * f(x)| \right\|_{L^2(\mathbb{R}^2)} \lesssim_\epsilon \delta^{-\epsilon} \|f\|_{L^2(\mathbb{R}^2)} . \tag{5.47}$$

Combining (5.45), (5.46) and (5.47) proves the result with $\overline{M}f(x)$ replaced by (5.34), and the proof of the proposition is complete. □

Finally, we note that the proof of Proposition 5.5.1 implies the following L^1 kernel estimate.

Proposition 5.5.2. *There exists a constant* $C = C(M, Re(\lambda_1), Re(\lambda_2), \Theta(\Omega, A))$ *such that for* $0 < \delta < C$*, for every* $\epsilon > 0$ *and for every quadruple* (i, j, m, n)*,*

$$\left\| \sup_{t\approx 2^n} |\psi_t * K_{i,j,m,n}| \right\|_{L^1(\mathbb{R}^2)} \lesssim_{\epsilon,M,Re(\lambda_1),Re(\lambda_2),\Theta(\Omega,A)} 1.$$

The above estimate without the supremum follows immediately from the proof of Proposition 5.5.1. We then simply note that all L^1 kernel estimates in the proof of Proposition 5.5.1 follow from pointwise estimates, which still hold uniformly in t when the kernel is convolved with ψ_t.

5.6 Littlewood-Paley Inequalities

The goal of this section is to prove the following proposition, which is an analog of Proposition 4 from [8]. As noted in the introduction, the presence of nonisotropic dilations requires a more complicated application of square function estimates than those used in [8], where Proposition 4 is proved by iteratively applying square function estimates with respect to Fourier projections to parallel strips in $\mathbb{R}^2$.

Proposition 5.6.1. *Let $\epsilon > 0$. There is $C = C(M, Re(\lambda_1), Re(\lambda_2), \Theta(\Omega, A), \epsilon)$ > 0 such that if $0 < \delta < C$, then the following holds. Let $\{\sigma_{i,j,m,n}\}$ be the partition of unity constructed in section 5.5 for the given value of δ. There are smooth functions $\{\phi_{i,j,m,n}\}$ such that $\phi_{i,j,m,n}$ is identically 1 on the support of $\sigma_{i,j,m,n}$ and so that if we define $\tilde{P}_{i,j,m,n}$ to be the convolution operator whose multiplier is $\phi_{i,j,m,n}$, then*

$$\left\| \left(\sum_{i,j,m,n} |\tilde{P}_{i,j,m,n} f|^2 \right)^{1/2} \right\|_4 \lesssim_\epsilon \delta^{-\epsilon} \|f\|_4 . \tag{5.48}$$

To prove Proposition 5.6.1, we will need the following lemma, which was originally due to Carleson. A proof can be found in [34] (Lemma 4.4). We state the lemma in full generality, although we will only need the special case $d = 2$.

Lemma 5.6.2. *Let A be an invertible linear transformation on $\mathbb{R}^d$ and A^t its transpose. Suppose that $\{m_k\}_{k \in \mathbb{N}}$ are bounded, measurable functions on $\mathbb{R}^d$ with disjoint supports. Let w be a bounded, measurable function on $\mathbb{R}^d$. Then for $s \geqslant 0$ and $f \in \mathcal{S}(\mathbb{R}^d)$,*

$$\int \sum_k |\mathcal{F}^{-1}[m_k(A^t \cdot)\hat{f}](x)|^2 w(x)\, dx$$
$$\leqslant C \sup_k \|m_k\|^2_{L^2_s(\mathbb{R}^d)} \int \int \frac{\det(A^{-1})}{(1 + |A^{-1}y|^s)^2} |f(x-y)|^2\, dy\, w(x)\, dx.$$

We state an immediate corollary of this lemma, which we will apply repeatedly in the proof of Proposition 5.6.1.

Corollary 5.6.3. *Suppose that $\{m_k\}_{k \in \mathbb{Z}}$ are disjoint translates of a smooth compactly supported function adapted to the unit cube in $\mathbb{R}^2$, with the distance between the supports of the m_k at least $O(1)$. Let R_θ be the matrix of rotation by θ degrees, and for $n \in \mathbb{Z}$ put $A_{n,\theta} = ((2^n)^A R_\theta (\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda N \end{smallmatrix}))^t$. Then for any n, θ and for any $s > 0$,*

$$\int \sum_k |\mathcal{F}^{-1}[m_k(A^t_{n,\theta} \cdot)\hat{f}](x)|^2 w(x)\, dx \leqslant C \int |f(x)|^2 \mathcal{M}_{\lambda,N} w(x)\, dx,$$

where $\mathcal{M}_{\lambda,N} := \sum_{i=0}^{\infty} 2^{-i} M_{2^i\lambda,N}$.

Proof of Proposition 5.6.1. Without loss of generality, we may restrict the sum in (5.48) to $i = 0$, and so in what follows we will assume $i = 0$ and drop the i-index. Also, in what follows we will say a collection $\mathcal{R}$ of subsets of $\mathbb{R}^2$ is *almost disjoint* if there is a constant $C = C(M, \mathrm{Re}(\lambda_1), \mathrm{Re}(\lambda_2), \Theta(\Omega, A)) > 0$ such that every point of $\mathbb{R}^2$ is contained in at most C elements of $\mathcal{R}$.

The main difficulty here introduced by nonisotropic dilations is that unlike the isotropic case, the orbits $\{t^A\xi : t > 0\}$ need not be straight lines, and thus for fixed j the supports of the $\sigma_{j,m,0}$ may only be approximated by rectangles with axes whose directions change as m varies. To deal with this difficulty, we group the supports of the $\sigma_{j,m,0}$ into nested subcollections each of which can be approximated by rectangles with long axes in a single direction, and iteratively apply Corollary 5.6.3.

Note that since $|\delta| \lesssim |I_j| \lesssim 1$, there are $\lesssim \log(1/\delta)$ dyadic intervals $[2^a, 2^{a+1}]$ with $a \leqslant 0$ and $a \in \mathbb{Z}$ such that $2^a \leqslant |I_j| \leqslant 2^{a+1}$ for some j, and so if we let $\mathfrak{J}_a = \{j : |I_j| \in [2^a, 2^{a+1}]\}$, we may restrict the sum in j in (5.48) to $\mathfrak{J}_a$ for a single fixed value of a, as long as all our estimates are uniform in a. By incurring a factor of $\delta^{-\epsilon}$, we may assume that $2^a \leqslant \delta^{-\epsilon}$.

Having fixed a, we are now ready to construct for each fixed j our nested subcollections of indices m. The idea is that for a fixed j and a fixed m, the support of $\sigma_{j,m,0}$ is essentially a $2^a \times \delta$ rectangle, and the support of $\sigma_{j,m',0}$ for m' for $|m' - m| \lesssim 2^{-a}$ is contained in a $2^a \times \delta$ rectangle whose direction differs by at most $\lesssim 2^{-a}\delta$. Thus the supports of the functions $\{\sigma_{j,m',0}\}_{|m-m'| \lesssim 2^{-a}}$ are contained in almost disjoint parallel strips of width $\approx \delta$. For such a collection of rectangles, Corollary 5.6.3 may be applied. The union of such rectangles is essentially a $2^a \times 2^{-a}\delta$ rectangle. We now iterate this process, grouping together successive $2^a \times 2^{-a}\delta$ rectangles whose direction does not change too much to obtain a rectangle of smaller eccentricity. We continue this process until we obtain a $2^a \times 2^a$ square, and then we may apply Corollary 5.6.3.

The nested subcollection of indices m will be constructed "backwards" with respect to the process described in the previous paragraph. The number of stages required by the process is N, where N is the least integer such that $2^{aN} \leqslant \delta$. For each $1 \leqslant k \leqslant N$, we will define a collection of indices m denoted by $\mathfrak{M}_{i_1,\ldots,i_k}$, so that $\mathfrak{M}_{i_1,\ldots,i_{k+1}} \subset \mathfrak{M}_{i_1,\ldots,i_k}$ and so that $\mathfrak{M}_{i_1,\ldots,i_N}$ contains at most one element. For each $(i_1,\ldots,i_k) \in \mathbb{Z}^k$, inductively define

$$\mathfrak{M}_{i_1} = \{m :\, i_1\lfloor 2^a\delta^{-1}\rfloor \leqslant m < (i_1+1)\lfloor 2^a\delta^{-1}\rfloor\},$$

$$\mathfrak{M}_{i_1,\ldots,i_k} = \mathfrak{M}_{i_1,\ldots,i_{k-1}} \cap \{m :\, \sum_{1\leqslant l\leqslant k} i_l\lfloor 2^{al}\delta^{-1}\rfloor \leqslant m \leqslant \sum_{1\leqslant l\leqslant k} i_l\lfloor 2^{al}\delta^{-1}\rfloor + \lfloor 2^{ak}\delta^{-1}\rfloor\}.$$

Then for every N-tuple $(i_1,\ldots,i_N)$, $\mathfrak{M}_{i_1,\ldots,i_N}$ contains at most one element.

Now let $C = C(M, \mathrm{Re}(\lambda_1), \mathrm{Re}(\lambda_2), \Theta(\Omega, A) > 0$ be sufficiently large. There is a collection $\{Q_{j,i_1}\}$ of almost-disjoint cubes of sidelength $C2^a$ such that if $m \in \mathfrak{M}_{i_1,\ldots,i_N}$ then the support of $\sigma_{j,m,0}$ is contained in Q_{j,i_1}. Since $\sup_{\rho(\xi)\leqslant 8} |\nabla\rho(\xi)| \lesssim 1$, there is a constant $C > 0$ such that for every j, i_1 we may cover Q_{j,i_1} with almost disjoint parallel rectangles R_{j,i_1,i_2} of width $C2^{2a}$ and length 1 so that for every i_2,

$$\bigcup_{m\in\mathfrak{M}_{i_1,i_2}} \mathrm{supp}(\sigma_{j,m,0}) \subset \bigcup_r (R_{j,i_1,i_2} \cap Q_{j,i_1}).$$

Repeating this process, for every $2 \leqslant k \leqslant N$ and every k-tuple $(i_1,\ldots,i_{k-1})$ we obtain almost disjoint parallel rectangles $R_{j,i_1,\ldots,i_k}$ of width $C2^{ka}$ and length 1 so that for every

i_k,

$$\bigcup_{m\in\mathfrak{M}_{i_1,\ldots,i_k}} \mathrm{supp}(\sigma_{j,m,0}) \subset (R_{j,i_1,\ldots,i_k} \cap \ldots \cap R_{j,i_1,i_2} \cap Q_{j,i_1}).$$

As noted previously, in the case $k = N$, $\bigcup_{m\in\mathfrak{M}_{i_1,\ldots,i_k}} \mathrm{supp}(\sigma_{j,m,0})$ contains at most one element.

Let $\phi : \mathbb{R}^2 \to [0,1]$ be a smooth function that is identically 1 on the unit cube centered at the origin and supported in its double dilate. If R is any nonisotropic dilate of a rectangle, let L_R be the affine transformation taking R to the unit cube centered at the origin. It follows that if $m \in \mathfrak{M}_{i_1,\ldots,i_N}$, then

$$\mathrm{supp}(\sigma_{j,m,0}) \subset \{x : \phi(L_{Q_{j,i_1}}x) \prod_{l=2}^{N} \phi(L_{R_{j,i_1,\ldots,i_l}}x) = 1\},$$

and so

$$\mathrm{supp}(\sigma_{j,m,n}) \subset \{x : \phi(L_{Q_{j,i_1}}(2^{-n})^A x) \prod_{l=2}^{N} \phi(L_{R_{j,i_1,\ldots,i_l}}(2^{-n})^A x) = 1\}.$$

Now for each j, i_1 let $\psi_{Q_{j,i_1}}$ be a smooth function supported in $4Q_{j,i_1}$ and identically 1 on Q_{j,i_1}, so that for each j,

$$\psi_{Q_{j,i_1}}(x) = 1, \qquad x \in \bigcup_{m\in\mathfrak{M}_{i_1}} \mathrm{supp}(\sigma_{j,m,0}).$$

For each (j,m,n), let $(i_1,\ldots,i_N)$ be the unique N-tuple such that $m \in \mathfrak{M}_{i_1,\ldots,i_N}$, and let

$$\phi_{j,m,n} = \psi_{Q_{j,i_1,r}}((2^{-n})^A x) \prod_{l=2}^{N} \phi(L_{R_{j,i_1,\ldots,i_l},r}(2^{-n})^A x)$$

Let $\tilde{P}_{j,m,n}$ denote the convolution operator with multiplier $\phi_{j,m,n}$. Let $\phi : \mathbb{R} \to [0,1]$ be a smooth function supported in $(1/4, 4)$ that is identically 1 on $[1/2, 2]$, and let P_n denote the convolution operator with multiplier $\phi(2^{-n}\rho(\cdot))$. Given an N-tuple of indices

$(i_1, \ldots, i_N)$, let $m(i_1, \ldots, i_N)$ denote the unique value of m such that $m \in \mathfrak{M}_{i_1,\ldots,i_N}$, and let $m(i_1, \ldots, i_N)$ be undefined otherwise. Let S_{j,i_1} denote the convolution operator with multiplier

$$\psi_{Q_{j,i_1,r}}((2^{-n})^A \cdot).$$

For $2 \leqslant k \leqslant N$, let $S_{j,i_1,\ldots,i_k,n}$ denote the convolution operator with multiplier

$$\psi_{Q_{j,i_1}}((2^{-n})^A L_{j,i_1,r} \cdot)\phi(L_{R_{j,i_1,\ldots,i_k}}(2^{-n})^A \cdot).$$

Then since each index m is contained in at most one N-tuple $(i_1, \ldots, i_N)$, it follows that

$$\int \sum_{j,m,n} |\tilde{P}_{j,m,n} f(x)|^2 \, w(x) \, dx = \int \sum_n \sum_j \sum_{(i_1,\ldots,i_N)} |S_{j,i_1,\ldots,i_N}(\ldots(S_{j,i_1}(P_n f(x))|^2 \, w(x) \, dx.$$

Repeatedly applying Corollary 5.6.3, we have

$$\begin{aligned}
&\int \sum_{j,m,n} |\tilde{P}_{j,m,n} f(x)|^2 \, w(x) \, dx \\
&\qquad \lesssim_\epsilon \int \sum_n \sum_j \sum_{i_1,\ldots,i_{N-1}} |S_{j,i_1,\ldots,i_{N-1}}(\cdots (S_{j,i_1}(P_n f(x)))\cdots)|^2 \\
&\qquad\qquad \times \mathcal{M}_{1,(2^{Na}\delta^{N\epsilon})^{-1}} w(x) \, dx \\
&\qquad \lesssim_\epsilon \int \sum_n \sum_j \sum_{i_1} |S_{j,i_1,}(P_n f(x))|^2 \, \mathcal{M}_{1,(2^{2a}\delta^{2\epsilon})^{-1}}(\cdots \\
&\qquad\qquad (\mathcal{M}_{1,(2^{Na}\delta^{N\epsilon})^{-1}} w(x)) \cdots) \, dx \\
&\qquad \lesssim_\epsilon \int \sum_n |P_n f(x)|^2 \, \mathcal{M}_{2^{-a}\delta^{-\epsilon},1}(\mathcal{M}_{1,(2^{2a}\delta^{2\epsilon})^{-1}}(\cdots \\
&\qquad\qquad (\mathcal{M}_{1,(2^{Na}\delta^{N\epsilon})^{-1}} w(x)) \cdots)) \, dx \\
&\qquad \lesssim_\epsilon \delta^{-\epsilon} \left\| \left(\sum_n |P_n f|^2 \right)^{1/2} \right\|_4^2 \\
&\qquad\qquad \times \left\| \mathcal{M}_{2^{-a}\delta^{0\epsilon},1}(\mathcal{M}_{1,(2^{2a}\delta^{\epsilon})^{-1}}(\cdots (\mathcal{M}_{1,(2^{Na}\delta^{N\epsilon})^{-1}} w) \cdots)) \right\|_2 . \qquad (5.49)
\end{aligned}$$

By Proposition 5.3.1, we have

$$\begin{aligned}
&\left\| \mathcal{M}_{2^{-a}\delta^{0\epsilon},1}(\mathcal{M}_{1,(2^{2a}\delta^{\epsilon})^{-1}}(\cdots (\mathcal{M}_{1,(2^{Na}\delta^{N\epsilon})^{-1}} w) \cdots)) \right\|_2 \\
&\qquad \lesssim_\epsilon \delta^{-\epsilon} \|w\|_2 . \qquad (5.50)
\end{aligned}$$

Since the operator $f \mapsto \left(\sum_n |P_n f|^2 \right)^{1/2}$ corresponds to a vector-valued singular integral on the space of homogeneous type given by nonisotropic balls and Lebesgue measure with all associated constants $\lesssim 1$, we have

$$\left\| \left(\sum_n |P_n f|^2 \right)^{1/2} \right\|_4 \lesssim \|f\|_4 . \qquad (5.51)$$

Combining (5.49), (5.50) and (5.51), we have

$$\int \sum_{j,m,n} |\tilde{P}_{j,m,n} f(x)|^2 \, w(x) \, dx \lesssim_\epsilon \delta^{-\epsilon} \, \|f\|_4^2 \, \|w\|_2 \, , \tag{5.52}$$

and the result follows by duality. $\square$

5.7 Proof of the main theorem

In this section, we combine the ingredients developed in previous sections to prove Proposition 5.1.3. The argument will closely follow [8]. As noted previously, we only need prove Proposition 5.1.3 in the case that Ω has smooth boundary, with a constant depending only on $M, \mathrm{Re}(\lambda_1), \mathrm{Re}(\lambda_2), \Theta(\Omega, A), \epsilon$.

Proof of Propostion 5.1.3. Let $Sf(x) = \left(\int_0^\infty |\psi_t * f(x)|^2 \frac{dt}{t} \right)^{1/2}$. Let $\mathfrak{S}$ be the non-isotropic sector bounded by the orbits $\{t^A \xi : t > 0\}$ and $\{t^A \xi' : t > 0\}$, where $\xi = (\xi_1, \xi_2)$ is the unique point in $\partial\Omega$ with $\xi_1 = -1/8$ and $\xi_2 > 0$ and $\xi' = (\xi_1', \xi_2')$ is the unique point in $\partial\Omega$ with $\xi_1 = 1/8$ and $\xi_2 > 0$. Assume without loss of generality that $\hat{f}$ is supported in $\mathfrak{S}$. By incurring a factor of $\log(1/\delta^{N(M,\mathrm{Re}(\lambda_1),\mathrm{Re}(\lambda_2))})$ we may restrict the domain of integration in t to the set

$$E = \bigcup_{\substack{n \equiv 0 \mod \\ \log(1/\delta^{N(M,\mathrm{Re}(\lambda_1),\mathrm{Re}(\lambda_2))})}} (2^n, 2^{n+1}],$$

where $N(M, \mathrm{Re}(\lambda_1), \mathrm{Re}(\lambda_2))$ is as in Proposition 5.4.2. Now, if $u, t \in E$ with $u < t$, then either u, t are contained in the same dyadic interval and $u/t > 1/2$, or u, t are contained in distinct dyadic intervals, and $u/t < 1/\delta^{N(M,\mathrm{Re}(\lambda_1),\mathrm{Re}(\lambda_2))}$. Using Plancherel, we have

$$\|Sf\|_4^4 = \int \left| \int_0^\infty |\psi_t * f(x)|^2 \frac{dt}{t} \right|^2 dx = \int \int_0^\infty \int_0^\infty |\psi_t * f(x)|^2 |\psi_u * f(x)|^2 \frac{dt}{t} \frac{du}{u}\, dx = \int_0^\infty \int_0^\infty \int |(\phi(\frac{\rho(\cdot)}{t})\hat{f}(\cdot)) * (\phi(\frac{\rho(\cdot)}{u})\hat{f}(\cdot))(\xi)|^2\, d\xi \frac{dt}{t}\frac{du}{u}.$$

Restricting the integration in t and u to E, we have

$$\int_E \int_E \int |(\phi(\frac{\rho(\cdot)}{t})\hat{f}(\cdot)) * (\phi(\frac{\rho(\cdot)}{u})\hat{f}(\cdot))(\xi)|^2\, d\xi \frac{dt}{t}\frac{du}{u} \lesssim \left(\int\int_{1/2<t/u<2} + \int\int_{u/t<\delta^{-N(M,\mathrm{Re}(\lambda_1),\mathrm{Re}(\lambda_2))}} \right) \int |(\phi(\frac{\rho(\cdot)}{t})\hat{f}(\cdot)) * (\phi(\frac{\rho(\cdot)}{u})\hat{f}(\cdot))(\xi)|^2\, d\xi.$$

Using Propositions 5.4.1 and 5.4.2, for every $\epsilon > 0$ we can essentially bound this by

$$\delta^{-\epsilon}\Bigg(\int_0^\infty \int_0^\infty \int \sum_{\substack{j,m,n\\ j',m',n'}} |(\sigma_{0,j,m,n}(\cdot)\phi(\frac{\rho(\cdot)}{t})\hat{f}(\cdot)) * (\sigma_{0,j',m',n'}(\cdot)\phi(\frac{\rho(\cdot)}{u})\hat{f}(\cdot))(\xi)|^2\, d\xi\, \frac{dt}{t}\, \frac{du}{u} + \int_0^\infty \int_0^\infty \int \sum_{j',m',n'} |(\phi(\frac{\rho(\cdot)}{t})\hat{f}(\cdot)) * (\sigma_{0,j',m',n'}(\cdot)\phi(\frac{\rho(\cdot)}{u})\hat{f}(\cdot))(\xi)|^2\, d\xi\, \frac{dt}{t}\, \frac{du}{u}\Bigg).$$

Let

$$Tf(x) = \left(\int_0^\infty \sum_{j,m,n} |\mathcal{F}[\sigma_{0,j,m,n}(\cdot)\phi(\frac{\rho(\cdot)}{t})\hat{f}(\cdot)](x)|^2 \frac{dt}{t} \right)^{1/2}.$$

Then the above implies that

$$\|Sf\|_4^4 \lesssim_\epsilon \delta^{-\epsilon}(\|Tf\|_4^4 + \|Sf\|_4^2\, \|Tf\|_4^2),$$

which implies

$$\|Sf\|_4 \lesssim_\epsilon \delta^{-\epsilon}\, \|Tf\|_4\,. \tag{5.53}$$

Using Proposition 5.4.3, we have

$$
\begin{aligned}
\|Tf\|_4 &= \left(\int \left| \int_0^\infty \sum_{j,m,n} |\mathcal{F}[\sigma_{0,j,m,n}(\cdot)\phi(\frac{\rho(\cdot)}{t})\hat{f}(\cdot)](x)|^2 \frac{dt}{t} \right|^2 dx \right)^{x1/4} \\
&= \left(\int \left| \sum_{j,m,n} \int_0^\infty |((\psi_t * K_{0,j,m,n}) * (\tilde{P}_{0,j,m,n} f))(x)|^2 \frac{dt}{t} \right|^2 dx \right)^{1/4} \\
&\lesssim \delta^{1/2} \left(\int \left| \sum_{j,m,n} \sup_{t \approx 2^n} |(\psi_t * K_{0,j,m,n}) * (\tilde{P}_{0,j,m,n} f)(x)|^2 \right|^2 dx \right)^{1/4} \\
&= \delta^{1/2} \left\| \left(\sum_{j,m,n} \sup_{t \approx 2^n} |(\psi_t * K_{0,j,m,n}) * (\tilde{P}_{0,j,m,n} f)(x)|^2 \right)^{1/2} \right\|_4 .
\end{aligned}
$$

Now let $\omega \in \mathcal{S}(\mathbb{R}^2)$ with $\|\omega\|_{L^2(\mathbb{R}^2)} = 1$. We have

$$
\begin{aligned}
&\int \sum_{j,m,n} \sup_{t \approx 2^n} |\psi_t * K_{0,j,m,n} * \tilde{P}_{0,j,m,n} f(x)|^2 \omega(x)\, dx \\
&\lesssim \int \sum_{j,m,n} \left\| \sup_{t \approx 2^n} |\psi_t * K_{0,j,m,n}| \right\|_1 |\tilde{P}_{0,j,m,n} f(x)|^2 | \sup_{t \approx 2^n} |\psi_t * K_{0,j,m,n} * w(x)| | \, dx \\
&\lesssim_\epsilon \delta^{-\epsilon} \int \sum_{j,m,n} |\tilde{P}_{0,j,m,n} f(x)|^2 \overline{M}\omega(x)\, dx \\
&\lesssim_\epsilon \left\| \left(\sum_{j,m,n} |\tilde{P}_{0,j,m,n} f(x)|^2 \right)^{1/2} \right\|_4^2 \left\| \overline{M} w \right\|_2 \lesssim_\epsilon \delta^{-\epsilon} \|f\|_4^2 ,
\end{aligned}
$$

where in the second inequality we have used Proposition 5.5.2 and in the last inequality we have used Propositions 5.5.1 and 5.6.1. Using (5.53) and taking the supremum over all such weights ω, we have

$$
\|Sf\|_4 \lesssim_\epsilon \delta^{-\epsilon} \|Tf\|_4 \lesssim_\epsilon \delta^{1/2-\epsilon} \|f\|_4 .
$$

□

Proof that Proposition 5.1.3 implies Theorem 5.1.1. Let C be as in the statement of Proposition 5.1.3. We will now decompose the Bochner-Riesz multipliers in a standard fashion. Let $\phi_0 : \mathbb{R} \to \mathbb{R}$ be a smooth function identically 1 on $[-1, 1]$ and supported in $[-2, 2]$ so that $\phi_0(|\cdot|)$ is a radial, decreasing function on $\mathbb{R}^2$. It is easy to see that we can find smooth functions $\phi_1 : \mathbb{R} \to \mathbb{R}$ and $\phi_2 : \mathbb{R} \to \mathbb{R}$ satisfying the following:

1. For each $k \geqslant 0$,
$$|D^k \phi_1(x)| \lesssim_k 1,$$
$$|D^k \phi_2(x)| \lesssim_k 1,$$
2. There is a constant $C' > 0$ such that ϕ_1 is supported in $[C', 1]$,
3. We can write
$$(1-\rho(\xi))^\lambda = \phi_0(2^{2M}|\xi|) + (\phi_0(2^{-2M}|\xi|) - \phi_0(2^{2M}|\xi|))\phi_1(\rho(\xi)) + \sum_{k=\lceil \log(C) \rceil}^{\infty} 2^{-k\lambda}\phi_2(2^k(1-\rho(\xi))).$$

By the triangle inequality,
$$\begin{aligned}\|G^\lambda f\|_4 \lesssim & \left\| \left(\int_0^\infty \left| \mathcal{F}^{-1}[\phi_0(2^{2M}t^{-1}|\cdot|)] * f(x) \right|^2 \frac{dt}{t} \right)^{1/2} \right\|_4 \\ & + \left\| \left(\int_0^\infty \left| \mathcal{F}^{-1}[(\phi_0(2^{-2M}t^{-1}|\cdot|) - \phi_0(2^{2M}t^{-1}|\cdot|))\phi_1(t^{-1}\rho(\cdot))] * f(x) \right|^2 \frac{dt}{t} \right)^{1/2} \right\|_4 \\ & + \sum_{k=1}^{\infty} 2^{-k\lambda} \left\| \left(\int_0^\infty \left| \mathcal{F}^{-1}[\phi_1(2^k(1-t^{-1}\rho(\cdot)))] * f(x) \right|^2 \frac{dt}{t} \right)^{1/2} \right\|_4 .\end{aligned}$$
The first term is clearly $\lesssim \|f\|_4$. By Proposition 5.1.3, the third term is also $\lesssim \|f\|_4$ if $\lambda > -1/2$. By vector-valued singular integrals, the second term is bounded by
$$\left\| \left(\int_0^\infty \left| \mathcal{F}^{-1}[\phi_1(t^{-1}\rho(\cdot))] * f(x) \right|^2 \frac{dt}{t} \right)^{1/2} \right\|_4 ,$$
and it is straightforward to adapt the proof of Proposition 5.1.3 to show that this is $\lesssim \|f\|_4$. □

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