

Microlocal multiplicity of matroid Schubert varieties

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Abstract

Schubert varieties of hyperplane arrangements, or matroid Schubert varieties, play a similar role to ordinary Schubert varieties in the Kazhdan-Lusztig theory of matroids. Moreover, their geometry carries abundant combinatorial information about the underlying hyperplane arrangement. In this thesis, we study the multiplicity numbers of the characteristic cycles of the intersection complex of matroid Schubert varieties. We show that these numbers are combinatorial invariants and can be computed using explicit formulas. We also conjecture that their generalization to arbitrary matroids is non-negative. Along the way, we discover a simple and elegant formula for the Euler obstruction number: it equals the characteristic polynomial evaluated at 2.

Dedication

To my mom and dad. Without their support, I could not have made it till the end of my PhD. It is thier support and understanding that made my career possible. Despite our modest family financial background, they still encourage me to pursue my dream of being a mathematician.

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Chapter 1

Introduction

1.1 Matroid

A matroid is a combinatorial abstraction of “linear independence”. Given a matrix, for example,

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 2 \\ -1 & 1 & 1 & -1 & 0 \end{pmatrix},$$

one can view this matrix as a vector configuration of five vectors A_1, A_2, A_3, A_4, A_5 in \mathbb{Q}^3 , where A_i is the i -th column of A . Clearly, A has full rank. Any three vectors chosen from the set $\{A_i\}$ form a basis, except for A_3, A_4, A_5 . The matroid of the matrix A is literally the collection B of all subsets of $\{A_1, A_2, A_3, A_4, A_5\}$, such that for all $b \in B$, b forms a basis of \mathbb{Q}^3 . In this example, B contains all 3-subsets except $\{A_3, A_4, A_5\}$.

The information of all bases is adequate to determine other things related to linear independence, such as the *rank* of a given set or all *independent* subsets. It turns out that these notions are sufficient to give the definition of a matroid. In this thesis, we will use the rank function as our basic definition. In the above example, the rank of a subset is just the dimension of the linear subspace that the subset spans.

Definition 1.1. A matroid is a pair (E, rk) where E is a finite set, and $\text{rk} : 2^E \rightarrow \mathbb{Z}_{\geq 0}$ is

a function that satisfies the following axioms:

- (R1) For any subset $A \subset E$, $\text{rk}(A) \leq |A|$.
- (R2) For any two subsets A, B of E , we have $\text{rk}(A \cup B) + \text{rk}(A \cap B) \leq \text{rk}(A) + \text{rk}(B)$.
- (R3) For any subset $A \subset E$ and $x \in E$, we have $\text{rk}(A) \leq \text{rk}(A \cup \{x\}) \leq \text{rk}(A) + 1$.

The number $\text{rk}(E)$ is called the rank of the matroid M . We can define bases and independent sets using the rank function.

Definition 1.2. A subset $B \subset E$ is called a *base* if $\text{rk}(B) = \text{rk}(E)$. A subset $I \subset E$ is *independent* if $\text{rk}(I) = |I|$.

Like graphs and other combinatorial objects, there are many long-standing open problems on matroids. One of them is Dowling and Wilson's top-heavy conjecture, which was solved in the paper (Huh and Wang 2017). To introduce this conjecture, we need the concept of *flats*. Consider a set of hyperplanes $E = \{H_i\}$ in \mathbb{Q}^n , or in fancier terms, a *hyperplane arrangement*. The rank of a subset $S \subset E$ is defined to be

$$\text{rk}(S) := \text{codim} \left(\bigcap_{i \in S} H_i \right).$$

One can verify that this defines a matroid structure on E . A *flat* is an arbitrary intersection of these hyperplanes; in other words, a flat is a subspace $F \subseteq \mathbb{Q}^n$ such that there is a subset $S \subseteq E$ with $F = \bigcap_{i \in S} H_i$. In matroid language, a flat is a subset such that adding any element to it increases the rank.

Definition 1.3. Suppose $M = (E, \text{rk})$ is a matroid. A subset $F \subset E$ is a flat if for any subset G such that $F \subsetneq G$, $\text{rk}(G) > \text{rk}(F)$.

We will use the notation $\mathcal{L}(M)$ to denote the set of all flats of M . Let \mathcal{L}^p denote the set of rank p flats. The top-heavy conjecture asserts that for a matroid of rank r , the following statement is true.

(1) For every nonnegative integer p less than $\frac{r}{2}$,

$$|\mathcal{L}^p| \leq |\mathcal{L}^{r-p}|.$$

(2) For every nonnegative integer p less than $\frac{r}{2}$,

$$|\mathcal{L}^p| \leq |\mathcal{L}^{p+1}|.$$

The following famous theorem by de Bruijn and Erdős is a corollary:

Corollary. *Every finite set of points E in a projective plane determines at least $|E|$ lines, unless E is contained in a line.*

Surprisingly, the proof of this conjecture uses geometry. They constructed a (singular) algebraic variety Y , such that $|\mathcal{L}^p|$ is equal to the dimension of $\mathrm{IH}^p(Y)$, the *intersection cohomology* group of Y .

The thesis is also on this special algebraic variety Y . We call it the *matroid Schubert variety*, or the Schubert variety of a hyperplane arrangement.

1.2 Matroid Schubert variety

From now on, all matroids and hyperplane arrangements will be on \mathbb{C} .

Consider a linear space $L \subset \mathbb{C}^n$ that is not contained in any coordinate hyperplane, it defines a matroid structure on the set $E = [n] = \{1, 2, \dots, n\}$ as follows. For a subset $S \subset E$, the rank of S is given by

$$\mathrm{rk}(S) = \dim(\pi_S(L)),$$

where π_S is the natural projection $\mathbb{C}^n \rightarrow \mathbb{C}^S$. Here \mathbb{C}^S is the subspace that generated by $e_i, i \in S$, where $e_i = (0, \dots, 1, \dots, 0)$ is the i -th standard basis. We call M the associated matroid of L , or equivalently, M is realized by L .

The linear space L with the embedding in \mathbb{C}^n uniquely defines a hyperplane arrangement. Let $H_i = L \cap \{x_i = 0\}$, where x_i is the coordinate of \mathbb{C}^n , then $\{H_i\}$ is a hyperplane arrangement in L , and the matroid above is equivalent to the matroid of this hyperplane arrangement. Notice that this hyperplane arrangement is *essential*, which means $\bigcap_{i \in E} H_i = \{0\}$.

Conversely, given an *essential* hyperplane arrangement $\{H_i | i \in E\}$ in a linear space L , we can construct an embedding into \mathbb{C}^n . Let $f_i = 0$ denote the defining equation of H_i , and let $i : L \rightarrow \mathbb{C}^n$ be the map given by (f_1, f_2, \dots, f_n) . The map i is injective because the hyperplane arrangement is essential:

$$\ker i = \bigcap_{i \in E} H_i = \{0\}.$$

Definition 1.4. The Schubert variety of L is the closure of L in $(\mathbb{P}^1)^n$, denoted by Y_L .

The matroid Schubert variety is, in general, a highly singular variety, as illustrated by the following example.

Example 1.5. Consider $L \subset \mathbb{C}^3$ to be the hyperplane defined by $x_1 + x_2 + x_3 = 0$. Using $[x_i, y_i]$ to denote the coordinate of the i -th \mathbb{P}^1 , the defining equation of Y_L is

$$x_1 y_2 y_3 + y_1 x_2 y_3 + y_1 y_2 x_3 = 0.$$

To demonstrate its singularity, we examine the equation in the coordinate system with y_1, y_2, y_3 , which provides local coordinates near (∞, ∞, ∞) :

$$y_2 y_3 + y_1 y_3 + y_1 y_2 = 0.$$

This equation describes an affine cone over a quadratic plane curve.

Further examples will be discussed in subsequent chapters. Although the variety itself is singular, it admits a natural stratification. The linear space L can be viewed as an

additive group under vector addition. Since L acts on itself, it also acts on the closure Y_L .

A fundamental result is:

Proposition 1.6. *The variety Y_L has finitely many L -orbits, and these orbits are in one-to-one correspondence with the flats.*

The explicit description of these orbits will be presented in the next chapter. For now, we denote by L^F the orbit corresponding to F , and by $Y_{L,F}$ the closure of L^F .

1.3 Local Euler obstruction

The local Euler obstruction is a local invariant from the singularity theory. It was first defined by Macpherson to study Chern class of singular algebraic varieties. The precise definition will be given in Chapter 2. To get a sense of the local Euler obstruction, we list several properties. The local Euler obstruction is a constructible function on the variety. It is 1 at the smooth points. At a singular point of a singular curve, the local Euler obstruction is equal to the multiplicity of that point. Recall that for an algebraic curve, the multiplicity is defined by the degree of the local definition equation in any ambient space.

Theorem 1.11 is proved by using such invariants, and the result is interesting in its own right.

Theorem 1.7. *Given a linear subspace $L \subset \mathbb{C}^n$, let M be its corresponding matroid on $E = \{1, 2, \dots, n\}$, the local Euler obstruction of Y_L at a point of the stratum L^F is $\chi_{M_F}(2)$.*

Combining with the results on CSM classes of matroid Schubert varieties Eur, Huh, and Larson 2023, Theorem 1.15, we can write down the Chern-Mather class of the matroid Schubert variety.

Corollary 1.8. *For a rank d loopless matroid M realized by a linear subspace $L \subset \mathbb{C}^n$, the Chern-Mather class of Y_L is given by the following formula:*

$$c_{Ma}(Y_L) = \sum_{F \in \mathcal{L}(M)} 2^{\text{rk}M - \text{rk}F} y_F \in A_*(Y_L).$$

Example 1.9. Consider the case $M = U_{2,3}$, the rank 2 uniform matroid with three elements. In the affine reciprocal plane (the coordinate system about the all-infinite point), the matroid Schubert variety is the hypersurface in \mathbb{C}^3 defined by $xy + xz + yz = 0$, and the theorem above predicts that local Euler obstruction at the origin is $\chi_M(2) = 2^2 - 3 \cdot 2 + 2 = 0$. Let us verify this result by another method. The hypersurface $xy + yz + xz = 0$ is the affine cone over the smooth plane curve of degree 2. By MacPherson 1974, the local Euler obstruction in this case is $2d - d^2$, where d is the degree. Therefore, the result is $2 \cdot 2 - 2^2 = 0$, as expected.

A lot of work is done concerning the positivity of the local Euler obstruction in the settings of the ordinary Schubert variety. For example, Mihalcea and Singh proved if G/P is cominuscule then any Schubert subvariety of G/P has nonnegative local Euler obstruction function, see Mihalcea and Singh 2020. We also examined such positivity here, and to our surprise, the Euler obstruction function of Y_L is always not everywhere positive unless M is a boolean matroid.

Theorem 1.10. *For a simple rank d matroid M realized by a linear subspace L , the local Euler obstruction function of Y_L is everywhere strictly positive if and only if M is a Boolean matroid.*

1.4 Microlocal multiplicity

We need to use some singularity theory in this section, but I will try to keep things simple. For each singular quasi-projective variety Y over \mathbb{C} , we can associate a complex of constructible sheaves called the *intersection complex*, denoted by IC_Y . The hypercohomology of intersection complex is called the *intersection cohomology* which is a cohomology theory that every algebraic variety, even a singular one, has Poincare duality and Hard Lefschetz theorem. For each constructible complex \mathcal{F} on a smooth algebraic variety X , there is a conic Lagrangian cycle on the cotangent bundle of X , called its *characteristic cycle*, denoted by $CC(\mathcal{F})$. Any conic Lagrangian cycle can be written as a linear combination

of conormal varieties, which we will define in Chapter 2. For the matroid Schubert variety $Y_L \subset (\mathbb{P}^1)^n$, the characteristic cycle of the intersection complex IC_{Y_L} (viewed as a sheaf on $(\mathbb{P}^1)^n$) takes the following form:

$$\mathrm{CC}(\mathrm{IC}_{Y_L}) = \sum_{F \in \mathcal{L}(M)} m_M(F) [T_{Y_{L,F}}^*(\mathbb{P}^1)^n].$$

These coefficients $m_M(F)$ are the main objects of study in this thesis. We call them *microlocal multiplicities*. Our main result shows that these numbers depend only on $\mathcal{L}(M)$ and can be computed explicitly.

Theorem 1.11. *Given a linear subspace $L \subset \mathbb{C}^n$, let M be its corresponding matroid on $E = \{1, 2, \dots, n\}$, and let $m_M(F)$ be the microlocal multiplicity defined above.*

- (1) *The microlocal multiplicity satisfies the equation $m_M(F) = m_{M_F}(\emptyset)$. We will use m_M to denote $m_M(\emptyset)$.*
- (2) *If M is loopless, then*

$$m_M = (-1)^{\mathrm{rk}M} \sum_{F \in \mathcal{L}(M)} 2^{\mathrm{rk}F} \chi_{M^F}(1/2) P_{M_F}(1),$$

where $\mathcal{L}(M)$ denotes the set of all flats of M , χ_M is the characteristic polynomial of M , P_M is the Kazhdan-Lusztig polynomial of M , and M^F and M_F denote the localization and contraction of M at the flat F , respectively.

An important fact about the characteristic cycle is that the characteristic cycle of a perverse sheaf is effective. In particular, m_M is always nonnegative. However, the above argument only works if one can find a linear subspace $L \subset \mathbb{C}^n$ such that its associated matroid is M (such matroids are called *realizable*). For most of the matroids, this is impossible. Nevertheless, the formula for m_M above still makes sense. We propose the following conjecture.

Conjecture. *For a loopless matroid M , m_M is always nonnegative. That is,*

$$(-1)^{\text{rk}M} \sum_{F \in \mathcal{L}(M)} 2^{\text{rk}F} \chi_{M^F}(1/2) P_{M^F}(1) \geq 0.$$

Moreover, $m_M = 0$ if and only if M has a Boolean summand, that is, M is a direct sum of a Boolean matroid and another arbitrary matroid.

The last sentence is justified by the following two facts which will be proved in the subsequent chapters.

Lemma 1.12. *For any two matroids M_1, M_2 , we have $m_{M_1 \oplus M_2} = m_{M_1} m_{M_2}$.*

Lemma 1.13. *If M is a Boolean matroid (all subsets are independent), then $m_M = 0$.*

The thesis is organized as follows. In Chapter 2, we will introduce the necessary background on the intersection complex and the local Euler obstruction, as well as matroids and wonderful compactifications. In Chapter 3, we will discuss the local Euler obstruction and prove Theorem 1.7. In Chapter 4, we will discuss microlocal multiplicity.

Chapter 2

Background

We provide the necessary background knowledge in this chapter. We first introduce the tools from singularity theory and then specialize them to the case of matroid Schubert varieties.

2.1 Constructible sheaves and related concepts

Let X be a topological space. A *local system* on X is a locally constant sheaf of \mathbb{Q} -vector spaces. A \mathbb{Q} -sheaf \mathcal{F} is called *constructible* if there is a decomposition of X into locally closed subsets S_i ($X = \sqcup_i S_i$), such that $\mathcal{F}|_{S_i}$ is a local system.

Let X now be an algebraic variety. A decomposition of X into locally closed subsets is called a *stratification* if each S_i is smooth and the S_i 's satisfy the frontier condition: the closure of any S_i is a union of other S_j . A constructible sheaf \mathcal{F} is called constructible with respect to the stratification $\{S_i\}$ if $\mathcal{F}|_{S_i}$ is a local system. A function $\varphi : X \rightarrow \mathbb{Z}$ is called constructible with respect to this stratification if $\varphi^{-1}(n)$ is a union of S_i 's. Constructible functions naturally form an abelian group.

Let X be a smooth quasi-projective variety. After choosing a stratification on X , we can define the bounded derived category of \mathbb{C} -constructible sheaves $D_c^b(X)$. Given a constructible complex \mathcal{F}^\bullet , we get an associated constructible function by taking the

stalk-wise Euler characteristic:

$$p \longrightarrow \chi(\mathcal{F}_p^\bullet).$$

If we denote $F(X)$ as the group of all constructible functions on X , one can verify that we actually get a group homomorphism $K_0(D_c^b(X)) \rightarrow F(X)$, where K_0 denotes the Grothendieck group.

The characteristic cycle is another object one can associate with a constructible complex. Defining the characteristic cycle is far beyond the scope of this thesis. Nevertheless, taking the characteristic cycle defines a group homomorphism

$$K_0(D_c^b(X)) \rightarrow L(X),$$

where $L(X)$ denotes the group of all conic Lagrangian cycles in the cotangent bundle T^*X . Any irreducible conic Lagrangian subvariety on the cotangent bundle is a conormal variety to an irreducible subvariety $Z \subset X$, which is denoted by T_Z^*X . Recall that the conormal variety to Z is defined to be the closure of $T_{Z_{reg}}^*X$, the conormal bundle of Z_{reg} , where Z_{reg} is the smooth part of Z . The group of conic Lagrangian cycles $L(X)$ is generated by T_Z^*X as an abelian group.

It turns out that $L(X)$ and $F(X)$ are naturally isomorphic to each other. The isomorphism is defined via the construction of the Euler obstruction function; see MacPherson 1974. We will define them in the next section. For each T_Z^*X , we map it to $(-1)^{\dim Z} \text{Eu}_Z$, where Eu_Z is the local Euler obstruction function of Z , a constructible function on Z . This isomorphism fits into the group homomorphism from $K_0(D_c^b(X))$. More precisely, the following diagram commutes:

$$\begin{array}{ccc} & K_0(D_c^b(X)) & \\ \text{CC} \swarrow & & \searrow \chi \\ L(X) & \xrightarrow[\cong]{\text{Eu}} & F(X) \end{array}$$

In particular, it makes sense to define the characteristic cycle of a constructible func-

tion. In practice, given a constructible function φ , to get its characteristic class, we first write φ as a linear combination $\varphi = \sum_i a_i \text{Eu}_{Z_i}$, then replace Eu by the conormal varieties: $\text{CC}(\varphi) = \sum_i a_i (-1)^{\dim Z_i} [T_{Z_i}^* X]$.

2.2 Local Euler obstruction and Chern classes for singular space

In MacPherson 1974, the author defined a generalization of Chern classes to the singular variety. It is now being called the Chern-Schwartz-Macpherson class, or CSM class for short. We will briefly recall the construction in the following paragraph.

The CSM class is defined via the Chern-Mather class and the local Euler obstruction, so we will introduce these concepts first. Let X be an algebraic variety, possibly singular. Suppose there is an embedding $i : X \rightarrow M$, where M is a smooth variety. Let $f : X_{\text{reg}} \rightarrow G(\dim X, TM)$ be the map that maps $x \in X$ to $[T_x X \subset T_x M]$, where $G(\dim X, TM)$ is the Grassmannian bundle of TM . The *Nash blow-up* of X is the closure of the graph of f in $X \times G(\dim X, TM)$. We will denote it by \tilde{X} , with the projection $\nu : \tilde{X} \rightarrow X$. The pullback of the tautological bundle defines a vector bundle on \tilde{X} , which we denote by \tilde{T} .

Definition 2.1. (1) The Chern-Mather class of X is

$$c_M(X) = \nu_* (c(\tilde{T}) \cap [\tilde{X}]) \in A_*(X),$$

where c is the ordinary total Chern class. The definition does not depend on the choice of the embedding $X \rightarrow M$.

(2) For any point $p \in X$, define the local Euler obstruction $\text{Eu}_X(p)$ by the formula

$$\text{Eu}_X(p) = \int_{\nu^{-1}(p)} c(\tilde{T}|_{\nu^{-1}(p)}) \cap s(\nu^{-1}(p), \tilde{X}).$$

Again, this integer is independent of the choice of M . Here s denotes the Segre class.

The local Euler obstruction is a constructible function on X . The definition is hard to understand, but here is a list of properties of the local Euler obstruction.

Proposition 2.2 (MacPherson 1974). *The following statements are true for the local Euler obstruction:*

- (1) *If X is smooth at p , then $\text{Eu}_X(p) = 1$.*
- (2) *$\text{Eu}_{X \times X'}(p \times p') = \text{Eu}_X(p)\text{Eu}_{X'}(p')$.*
- (3) *If X is locally reducible at p and X_i are its components, then $\text{Eu}_X(p) = \sum_i \text{Eu}_{X_i}(p)$.*
- (4) *If X is a curve, then $\text{Eu}_X(p)$ is the multiplicity of X at p . If X is the cone of a smooth plane curve of degree d and p is the cone point, then $\text{Eu}_X(p) = 2d - d^2$.*

We are in a position to define the CSM class.

Definition 2.3. The Chern class transformation c_* is a group homomorphism from $F(X)$ to $A_\bullet(X)$ such that $c_*(\text{Eu}_Z) = (i_Z)_*c_M(Z)$, where Z is a subvariety of X , and i_Z is the embedding morphism. The CSM class of a subvariety Z is $c_*(1_Z)$, where 1_Z is the indicator function of Z . We will denote it by $c_{SM}(Z)$.

Recall that Eu_Z form a basis of $F(X)$, so the definition makes sense. Computing the local Euler obstruction could be difficult. Here is the tool we will use in this thesis.

Proposition 2.4 (Aluffi 2018, Proposition 3.17). *Let L be a cone over a subvariety $W \subset \mathbb{P}^m$ with vertex Λ . Let $p \in \Lambda$. Then the local Euler obstruction $\text{Eu}_V(p)$ equals*

$$\text{Eu}_V(p) = \sum_{j=0}^{\dim W} (-1)^j c_{Ma}(W)_j$$

where $c_{Ma}(W)_j$ denotes the degree of the j -dimensional component of $c_{Ma}(W)_j$.

2.3 The geometry of matroid Schubert varieties

We specialize the tools in the previous sections to the case of matroid Schubert varieties in this section. We start from the geometry of the matroid Schubert variety.

The geometry of a matroid Schubert variety is very close to an affine toric variety. They have a flat-orbit correspondence, similar to the cone-orbit correspondence of the toric variety.

Let $L \subset \mathbb{C}^n$ be a linear subspace of dimension d . Let M be the loopless matroid on $E = \{0, 1, \dots, n-1\}$ of L . Recall a subset $B \subset E$ is a base of M if and only if $L \rightarrow \mathbb{C}^E \rightarrow \mathbb{C}^B$ is an isomorphism.

The linear subspace L as an additive group acts on itself and \mathbb{C}^n by addition, so this action naturally induces an action on Y_L . Like an affine toric variety, Y_L has only finitely many L -orbits, and these orbits are one-to-one corresponding to the *flats* of M . More precisely, for a flat F of M , write $\pi_F : \mathbb{C}^n \rightarrow \mathbb{C}^F$, then the orbit corresponding to F is $\pi_F(L) \times \{\infty\}^{E \setminus F}$. The orbits give a Whitney stratification of Y_L . For more information about matroid Schubert varieties, the interested readers are directed to Crowley 2023.

The action of L on Y_L has a unique fixed point $\{\infty\}^n$. It is the orbit corresponding to the empty set, which is a flat. The intersection of Y_L and the affine chart $(\mathbb{P}^1 \setminus \{0\})^n$ is called the affine reciprocal plane of the matroid M , denoted by X_L . The stratification of Y_L induces a stratification of X_L , so the strata of X_L are also one-to-one corresponding to flats. We will use $X_{L,F}$ to denote the stratum that corresponds to the flat F . In particular, the big open stratum of X_L is $L \cap (\mathbb{C}^*)^n$ which we will denote by U_L . This is exactly the complement of the hyperplane arrangement defined by L . The following statement is proved in Elias, Proudfoot, and Wakefield 2016.

Proposition 2.5 (Elias, Proudfoot, and Wakefield 2016, Proposition 3.1). *The stratum $X_{L,F}$ is isomorphic to $U_{L,F}$, and its closure in X_L is isomorphic to $X_{L,F}$.*

Here, the notation L^F and L_F refers to the localization and the contraction of the hyperplane arrangement of L . They are $\pi_F(L) \rightarrow \mathbb{C}^F$ and $L/(L \cap \mathbb{C}^F) \rightarrow \mathbb{C}^n/\mathbb{C}^F$, respectively. On the level of matroid, we use M^F and M_F to denote the localization and the contraction of M at the flat F . See Braden et al. 2023 for the precise definitions.

Similarly, we will use the notation $Y_{L,F}$ to denote the closure of the stratum of Y_L that corresponds to the flat F . The stratum itself is literally L^F .

The local geometry of the affine reciprocal plane is described by the following theorem which roughly says that $X_{L,F}$ is an “étale slice” to the stratum $X_{L,F}$.

Theorem 2.6 (Elias, Proudfoot, and Wakefield 2016, Theorem 3.3). *Let F be a flat of M and let $x \in X_{L,F}$. There exists an open subvariety W containing x and a map $\Phi : W \rightarrow X_{L,F} \times X_{L,F}$ such that $\Phi(x) = (0, x)$ and Φ is étale at x . Here 0 is the original point in the affine chart $(\mathbb{C}^* \cup \infty)^n$.*

2.4 Wonderful model of hyperplane arrangements and CSM classes of matroids

The matroid Schubert variety Y_L admits a resolution of singularity, called the *augmented wonderful model* W_L . The projective reciprocal plane which is the decone of X_L admits a resolution of singularity called the *wonderful model* of hyperplane arrangement \underline{W}_L . They are defined in the paper De Concini and Procesi 1996. Roughly speaking, they are obtained by blowing up all the stratum (of Y_L and X_L) from of zero dimension to of top dimension. We will need the result on $c_{SM}(\mathbb{P}(U_L)) \in A_\bullet(\underline{W}_L)$, here $\mathbb{P}(U_L)$ is the quotient of U_L by the natural \mathbb{C}^* action.

We first describe $A_\bullet(\underline{W}_L)$. The following result is proved in Feichtner and Yuzvinsky 2004.

Proposition 2.7 (Feichtner and Yuzvinsky 2004). *The closed embedding $\underline{W}_L \rightarrow X_{\Sigma(\mathcal{L})}$ is a Chow equivalence.*

Here, $\Sigma(\mathcal{L})$ is the nested set fan of the matroid M , also called the *Bergman fan* in the literature. By the results on the Chow group of toric varieties (see for example Cox, Little, and Schenck 2011), we know that

$$A_k(\underline{W}_L) \simeq \text{MW}_k(\Sigma(\mathcal{L})),$$

where $\text{MW}_k(\Sigma(\mathcal{L}))$ is the group of Minkovski weight of dimension k on the fan $\Sigma(\mathcal{L})$.

A *Minkowski weight* on a fan of dimension k is an assignment of integers to the cones of dimension k that satisfies the balancing condition. A k -dimensional cone of $\Sigma(\mathcal{L})$ is labeled by a k -step flag of flats $\{\emptyset = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_k \subsetneq F_{k+1} = [n]\}$, $F_i \in \mathcal{L}(M)$.

The CSM classes of $\mathbb{P}(U_L)$ in \underline{W}_L are calculated in López de Medrano, Rincón, and Shaw 2020.

Definition 2.8. Suppose M is a rank d matroid on $[n] = \{1, 2, \dots, n\}$. For $0 \leq k \leq d - 1$, the k -dimensional *Chern-Schwartz-MacPherson (CSM) cycle* $\text{csm}_k(M)$ is the k -dimensional skeleton of $\mathcal{B}(M)$ (the Bergman fan of M) equipped with weights on its top-dimensional cones. If M is a loopless matroid, the weight of the cone $\sigma_{\mathcal{F}}$ corresponding to a flag of flats $\mathcal{F} := \{\emptyset = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_k \subsetneq F_{k+1} = [n]\}$ is

$$w(\sigma_{\mathcal{F}}) := (-1)^{d-1-k} \beta(M)[\mathcal{F}] = (-1)^{d-1-k} \prod_{i=0}^k \beta(M|_{F_{i+1}/F_i}),$$

where $M|_{F_{i+1}/F_i} = M_{F_i}^{F_{i+1}}$ denotes the minor of M obtained by localizing to F_{i+1} and contracting F_i , β is the beta invariant of M . If M has a loop then the CSM class is empty for all k .

Theorem 2.9 (López de Medrano, Rincón, and Shaw 2020). *Let \underline{W}_L be the wonderful compactification of $\mathbb{P}(U_L)$. Then*

$$c_{SM}(\mathbb{P}(U_L)) = \sum_{k=0}^{d-1} \text{csm}_k(M) \in A_*(\underline{W}_L).$$

Chapter 3

Local Euler obstruction

3.1 A recursive formula

This section is devoted to a recursive formula for the local Euler obstruction, see Proposition 3.3 below.

To start, let us fix a linear subspace $L \subset \mathbb{C}^n$ and consider the corresponding matroid Schubert variety Y_L . We stratify Y_L by L -orbits, and throughout this section all constructible functions are assumed to be constructible with respect to this stratification.

The affine reciprocal plane is an open subset of the matroid Schubert variety, so the Euler obstructions of X_L are exactly the Euler obstructions of Y_L . By definition, the Euler obstruction function is a constructible function and is therefore constant when restricted to each stratum. We denote by $\text{Eu}_M(F)$ the value of Eu_{X_L} at a point of $X_{L,F}$. By Theorem 2.6 and the fact that an étale morphism (a local isomorphism in analytic geometry) preserves the value of Eu , we conclude that

$$\text{Eu}_M(F) = \text{Eu}_{X_L}(x) = \text{Eu}_{X_{L,F}}(0)\text{Eu}_{X_{L,F}}(x) = \text{Eu}_{M_F}(\emptyset). \quad (3.1)$$

Since $X_{L,F}$ is smooth, we have $\text{Eu}_{X_{L,F}}(x) = 1$. Notice that the original point in the affine chart $(\mathbb{C}^* \cup \infty)^n$ is exactly the orbit corresponding to the empty flat \emptyset . In other words, the value at the V -fixed point $\{\infty\}^n$ is the only one we need to calculate.

There is an equivalent way of defining the affine reciprocal plane. Let

$$i : (\mathbb{C}^*)^n \rightarrow (\mathbb{C}^*)^n$$

be the map taking the inverse of each coordinate:

$$i(x_0, \dots, x_n) = (x_0^{-1}, \dots, x_n^{-1}).$$

For a linear subspace $V \subset \mathbb{C}^n$, the affine reciprocal plane of V is the closure of $i(V \cap (\mathbb{C}^*)^n)$ in \mathbb{C}^n . The subspace $V \subset \mathbb{C}^n$ is invariant under the scaling action of \mathbb{C}^* , and so is X_L . This shows that X_L is the affine cone over a projective variety $\mathbb{P}(X_L) \subset \mathbb{P}^n$, the so-called projective reciprocal plane. We are now in a position to apply Aluffi's formula, Proposition 2.4.

Lemma 3.1. *Let L be a d -dimensional subspace of \mathbb{C}^n and let M be the matroid of L . Denote by $\mathcal{L}(M)$ the lattice of flats in M . Then we have*

$$\text{Eu}_M(\emptyset) = \sum_{\emptyset \neq F \in \mathcal{L}(M)} \text{Eu}_{M_F}(\emptyset) \left(\sum_{j=0}^{d-1} (-1)^j c_{SM}(\pi(X_{L,F}))_j \right).$$

Proof. Applying Aluffi's result Proposition 2.4 to the affine reciprocal plane, we get

$$\text{Eu}_M(\emptyset) = \sum_{j=0}^{d-1} (-1)^j c_{M_a}(\mathbb{P}(X_L))_j. \quad (3.2)$$

The projective reciprocal plane has a stratification that is induced by the projection map $\pi : X_L - \{\infty\} \rightarrow \mathbb{P}(X_L)$. It is the quotient map by \mathbb{C}^* -action. Since π exhibits $X_L - \{\infty\}$ as a \mathbb{C}^* bundle over $\mathbb{P}(X_L)$, we know that $\text{Eu}_{X_L}(x) = \text{Eu}_{\mathbb{P}(X_L)}(\pi(x))$ for $x \neq \infty$. Using this fact, we can write the constructible function $\text{Eu}_{\mathbb{P}(X_L)}$ as

$$\text{Eu}_{\mathbb{P}(X_L)} = \sum_{\emptyset \neq F \in \mathcal{L}(M)} \text{Eu}_M(F) 1_{\pi(X_{L,F})}, \quad (3.3)$$

where $1_{\pi(X_{L,F})}$ is the indicator function of $\pi(X_{L,F})$, the stratum corresponding to F in the projective reciprocal plane. Putting together with Eq. (3.2), we conclude that

$$\begin{aligned}
\text{Eu}_M(\emptyset) &= \sum_{j=0}^{d-1} (-1)^j c_{Ma}(\mathbb{P}(X_L))_j \\
&= \sum_{j=0}^{d-1} (-1)^j c_* \left(\sum_{\emptyset \neq F \in \mathcal{L}(M)} \text{Eu}_M(F) 1_{\pi(X_{L,F})} \right)_j \\
&= \sum_{\emptyset \neq F \in \mathcal{L}(M)} \text{Eu}_M(F) \left(\sum_{j=0}^{d-1} (-1)^j c_*(1_{\pi(X_{L,F})})_j \right) \\
&= \sum_{\emptyset \neq F \in \mathcal{L}(M)} \text{Eu}_{M_F}(\emptyset) \left(\sum_{j=0}^{d-1} (-1)^j c_{SM}(\pi(X_{L,F}))_j \right)
\end{aligned}$$

□

The notation $c_{SM}(\pi(X_{L,F}))$ refers to the Chern-Schwartz-Macpherson class of $\pi(X_{L,F})$. Once we compute the degrees of these classes, we can inductively compute the local Euler obstruction numbers of a matroid Schubert variety. Because the closure of $X_{L,F}$ is X_{L^F} , the calculation is only necessary when $F = \emptyset$. Recall that $X_{L,\emptyset}$ is U_L , the complement of the hyperplane arrangement L , and $\mathbb{P}(U_L)$ is the complement of the projective hyperplane arrangement.

Lemma 3.2. *Let*

$$c_M := \sum_{j=0}^{d-1} (-1)^j c_{SM}(\mathbb{P}(U_L))_j,$$

then

$$c_M = (-1)^{d-1} \sum_{\mathcal{F}} \beta(M)[\mathcal{F}],$$

where \mathcal{F} ranges over all descending flags of flats. For a flag of flats $\emptyset \subset F_1 \subset \dots \subset F_l \subset M$, it is descending if $\min(F_1) > \min(F_2) > \dots > \min(F_l) > 0$, and the quantity $\beta(M)[\mathcal{F}] = \beta(M^{F_1})\beta(M_{F_1}^{F_2}) \cdots \beta(M_{F_l})$ is a product of beta invariants.

The proof of the lemma uses the wonderful compactification \underline{W}_L of the projective hyperplane arrangement $\mathbb{P}(U_L)$.

Proof of Lemma 3.2. There is a birational map p from \underline{W}_L to $\mathbb{P}(X_L)$, extended from the inclusion map $\mathbb{P}(U_L) \rightarrow \mathbb{P}(X_L)$. Since c_* commutes with proper pushforward, the class $c_{SM}(\mathbb{P}(U_L)) \in A_*(\mathbb{P}^n)$ is the pushforward of $c_{SM}(\mathbb{P}(U_L)) \in A_*(\underline{W}_L)$. Let $h \in A^1(\mathbb{P}^n)$ be the hyperplane class of \mathbb{P}^n . The sum

$$\sum_{j=0}^{d-1} (-1)^j c_{SM}(\mathbb{P}(U_L))_j$$

can be rewritten as

$$\sum_{j=0}^{d-1} (-1)^j c_{SM}(\mathbb{P}(U_L)) \cap h^j$$

in $A_0(\mathbb{P}^n) \cong \mathbb{Z}$. By the projection formula, we have

$$c_{SM}(\mathbb{P}(U_L)) \cap h^j = p_*(c_{SM}(\mathbb{P}(U_L))) \cap h^j = p_*(c_{SM}(\mathbb{P}(U_L)) \cap (p^*h)^j)$$

The pushforward p_* is the identity map on A_0 because p is birational. It remains to calculate the element

$$c_{SM}(\mathbb{P}(U_L)) \cap (p^*h)^j \in A_0(\underline{W}_L) \cong \mathbb{Z}.$$

The hyperplane class $p^*h \in A^1(\underline{W}_L)$ was studied in the papers Adiprasito, Huh, and Katz 2018 and Braden et al. 2022. We will adopt their notations. It is the beta class of M

$$\beta_M = \sum_{i \notin G} x_G \in A^1(\underline{W}_L),$$

where the sum is over all nonempty proper flats G of M not containing a given element i in E . The definition does not depend on the choice of i . Moreover, there is a combinatorial formula for β_M^j :

$$\beta_M^j = \sum_{\mathcal{F}} x_{\mathcal{F}}$$

where the sum is over all descending j -step flags of nonempty proper flags of M , and $x_{\mathcal{F}} = x_{F_1} x_{F_2} \cdots x_{F_j}$ where $\mathcal{F} : F_1 \subset F_2 \subset \cdots \subset F_j$. Recall that a flag of flats is said to be descending if $\min(F_1) > \min(F_2) > \cdots > \min(F_j) > 0$. Putting all these facts together,

we conclude that

$$\begin{aligned}
c_{SM}(\mathbb{P}(U_L)) \cap (p^*h)^j &= \text{csm}_j(M) \cap \beta_M^j \\
&= \sum_{\mathcal{F}} \text{csm}_j(M) \cap x_{\mathcal{F}} \\
&= \sum_{\mathcal{F}} \text{csm}_j(M)(\sigma_{\mathcal{F}}) \\
&= \sum_{\mathcal{F}} (-1)^{d-1-j} \beta(M)[\mathcal{F}],
\end{aligned}$$

and the alternating sum of degrees of the CSM classes is

$$\begin{aligned}
\sum_{j=0}^{d-1} (-1)^j c_{SM}(\mathbb{P}(U_L))_j &= \sum_{j=0}^{d-1} (-1)^j \sum_{\mathcal{F}} (-1)^{d-1-j} \beta(M)[\mathcal{F}] \\
&= (-1)^{d-1} \sum_{j=0}^{d-1} \sum_{\mathcal{F}} \beta(M)[\mathcal{F}],
\end{aligned}$$

where the internal sum is over all j -step descending flags, and j ranges over 0 to d . As a conclusion, the alternating sum on the CSM classes is

$$\sum_{j=0}^{d-1} (-1)^j c_{SM}(\mathbb{P}(U_L))_j = (-1)^{d-1} \sum_{\mathcal{F}} \beta(M)[\mathcal{F}] = c_M,$$

where the sum is over all descending flags of flats, from 0-step ($\emptyset \subset M$) to $(d-1)$ -step ($\emptyset \subset F_1 \subset \dots \subset F_{d-1} \subset M$). \square

Combining two lemmas, we obtain a recursive formula for Eu_M :

Proposition 3.3. *For a loopless matroid M , Eu_M can be recursively calculated by the following formula:*

$$\text{Eu}_M = \sum_{F \neq \emptyset} c_{MF} \text{Eu}_{M_F},$$

and $\text{Eu}_{\emptyset} = 1$.

We will solve this recursion in the next section. We end this section with an inductive formula for c_M 's.

Lemma 3.4. *The constant c_M is determined by the following recursive formula:*

$$c_M = (-1)^{d-1} \beta(M) + \sum_{0 \notin F \neq \emptyset} (-1)^{\text{cork}^F} c_{M^F} \beta(M^F) \quad (3.4)$$

and

$$c_\emptyset = -1.$$

Proof. By definition,

$$c_M = (-1)^{d-1} \sum_{\mathcal{F}} \beta(M)[\mathcal{F}],$$

where \mathcal{F} ranges over all descending flags of flats. The 0-step flag of flats is just $\emptyset \subset M$, therefore:

$$\begin{aligned} c_M &= (-1)^{d-1} \beta(M) + (-1)^{d-1} \sum_{k=1}^{d-1} \sum_{\emptyset \subset F_1 \subset \dots \subset F_k \subset M} \beta(M^{F_1}) \beta(M_{F_1}^{F_2}) \cdots \beta(M_{F_{k-1}}^{F_k}) \\ &= (-1)^{d-1} \beta(M) + (-1)^{d-1} \sum_{k=1}^{d-1} \sum_{0 \notin F_k \neq \emptyset} \beta(M_{F_k}) \sum_{\emptyset \subset F_1 \subset \dots \subset F_k} \beta(M^{F_1}) \beta(M_{F_1}^{F_2}) \cdots \beta(M_{F_{k-1}}^{F_k}) \\ &= (-1)^{d-1} \beta(M) + (-1)^{d-1} \sum_{0 \notin F \neq \emptyset} \beta(M_F) (-1)^{\text{rk}(F)-1} c_{M^F} \\ &= (-1)^{d-1} \beta(M) + \sum_{0 \notin F \neq \emptyset} (-1)^{\text{cork}(F)} \beta(M_F) c_{M^F}. \end{aligned}$$

Notice that the flag $\emptyset \subset F_1 \subset \dots \subset F_k$ is a descending flag of flats in the localization M^{F_k} . □

3.2 Solving the local Euler obstruction

To solve the relation in Proposition 3.3, we first have to solve for the constants c_M . The following identity is the key fact.

Lemma 3.5. *For any non-empty loopless matroid M and any element $0 \in E$, the following identity holds:*

$$\bar{\chi}_M(t) = \sum_{0 \notin F} \chi_{M^F}(t) (-t)^{\text{rk} M^F - 1} \beta(M^F) \quad (3.5)$$

where F ranges over all flats that do not contain 0 .

Throughout this section, we make the convention that the empty flat $F = \emptyset$ satisfies $0 \notin F$.

Proof. The proof is simply expanding all definitions. By definition,

$$\chi_{M^F}(t) = \sum_{G \subset F} \mu(\emptyset, G) t^{\text{rk}F - \text{rk}G},$$

and

$$\beta(M_F) = (-1)^{\text{rk}M_F} \sum_{F \subset H} \mu(F, H) (\text{rk}(H) - \text{rk}(F)),$$

so the right-hand side is

$$\sum_{\substack{G \subset F \subset H \\ 0 \notin F}} \mu(\emptyset, G) \mu(F, H) (\text{rk}H - \text{rk}F) t^{\text{rk}F - \text{rk}G} (-t)^{\text{rk}M - \text{rk}F - 1} (-1)^{\text{rk}M - \text{rk}F}$$

which simplifies to

$$- \sum_{\substack{G \subset F \subset H \\ 0 \notin F}} \mu(\emptyset, G) \mu(F, H) (\text{rk}H - \text{rk}F) t^{\text{rk}M - \text{rk}G - 1}.$$

Since

$$\sum_{H: F \subset H} \mu(F, H) = 0$$

for $F \neq M$, we can further simplify the above summation to

$$- \sum_{\substack{G \subset F \subset H \\ 0 \notin F}} \mu(\emptyset, G) \mu(F, H) \text{rk}H t^{\text{rk}M - \text{rk}G - 1}.$$

We rewrite the above summation to be $-\sum_{G \subset F \subset H} + \sum_{\substack{G \subset F \subset H \\ 0 \in F}}$. Let us consider the first

sum. By the fact (see for example Stanley 2011, Proposition 3.7.2)

$$\sum_{F:G \subset F \subset H} \mu(F, H) = \begin{cases} 1, & G = H, \\ 0, & G \neq H. \end{cases},$$

we get

$$- \sum_{G \subset F \subset H} \mu(\emptyset, G) \mu(F, H) \operatorname{rk} H t^{\operatorname{rk} M - \operatorname{rk} G - 1} = - \sum_G \mu(\emptyset, G) \operatorname{rk} G t^{\operatorname{rk} M - \operatorname{rk} G - 1}.$$

For the second sum, notice that “ $0 \in F$ and $G \subset F$ ” is equivalent to $G \cup \{0\} \subset F$, which is also equivalent to $\overline{G \cup \{0\}} \subset F$, where the overline means taking closure. Therefore, the second sum is equal to

$$\begin{aligned} & \sum_{G \subset H} \mu(\emptyset, G) \operatorname{rk} H t^{\operatorname{rk} M - \operatorname{rk} G - 1} \left(\sum_{\overline{G \cup \{0\}} \subset F \subset H} \mu(F, H) \right) \\ &= \sum_G \mu(\emptyset, G) \operatorname{rk}(\overline{G \cup \{0\}}) t^{\operatorname{rk} M - \operatorname{rk} G - 1} \\ &= \sum_{0 \in G} \mu(\emptyset, G) \operatorname{rk} G t^{\operatorname{rk} M - \operatorname{rk} G - 1} + \sum_{0 \notin G} \mu(\emptyset, G) (\operatorname{rk} G + 1) t^{\operatorname{rk} M - \operatorname{rk} G - 1} \\ &= \sum_G \mu(\emptyset, G) \operatorname{rk} G t^{\operatorname{rk} M - \operatorname{rk} G - 1} + \sum_{0 \notin G} \mu(\emptyset, G) t^{\operatorname{rk} M - \operatorname{rk} G - 1}. \end{aligned}$$

We conclude that the original sum is equal to

$$\sum_{0 \notin G} \mu(\emptyset, G) t^{\operatorname{rk} M - \operatorname{rk} G - 1}.$$

But by White 1987, Corollary 7.2.7, this is exactly $\bar{\chi}_M(t)$ if M is loopless. \square

Corollary 3.6. *For a loopless matroid M , $c_M = -2^{\operatorname{rk} M} \chi_M(1/2)$. In particular, if $M = M_1 \oplus M_2$, then $c_M = -c_{M_1} c_{M_2}$.*

Proof. If $M = \emptyset$, then both sides are equal to -1 . We only need to verify the right-hand side satisfies the inductive formula for c_M (Lemma 3.4). Lemma 3.4 can also be written

in the form

$$c_M = \sum_{0 \notin F} (-1)^{\text{rk}M_F} c_{M_F} \beta(M_F),$$

so we need to prove

$$-2^{\text{rk}M} \chi_M(1/2) = \sum_{0 \notin F} (-1)^{\text{rk}M_F} (-2^{\text{rk}F} \chi_{M_F}(1/2)) \beta(M_F),$$

equivalently,

$$\begin{aligned} \chi_M(1/2) &= \sum_{0 \notin F} (-1)^{\text{rk}M_F} (2^{-\text{rk}M_F} \chi_{M_F}(1/2)) \beta(M_F) \\ &= \sum_{0 \notin F} \chi_{M_F}(1/2) (-1/2)^{\text{rk}M_F} \beta(M_F). \end{aligned}$$

Finally, notice that $\bar{\chi}_M(1/2) = \chi_M(1/2)/(1/2 - 1) = -2\chi_M(1/2)$, we reduce the above equality to the previous theorem where we evaluate t at $1/2$. If $M = M_1 \oplus M_2$, then since $2^{\text{rk}M} = 2^{\text{rk}M_1} 2^{\text{rk}M_2}$ and $\chi_M(t) = \chi_{M_1}(t) \chi_{M_2}(t)$, we conclude that $c_M = -c_{M_1} c_{M_2}$. \square

Remark. Recall in the statement of Lemma 3.2, c_M is defined to be a huge sum of beta invariants, but this lemma states that this huge sum is equal to $-2^{\text{rk}M} \chi_M(1/2)$.

We can now calculate the value of Eu .

Lemma 3.7. *For any non-empty loopless matroid M ,*

$$\sum_{F \in \mathcal{L}(M)} t^{\text{rk}F} \chi_{M_F}(1/t) \chi_{M_F}(t) = 0.$$

Proof. Expanding everything by its definition, we get:

$$\begin{aligned} & \sum_{G \subset F \subset H} t^{\text{rk}F} \mu(\emptyset, G) (1/t)^{\text{rk}F - \text{rk}G} \mu(F, H) t^{\text{rk}M - \text{rk}H} \\ &= \sum_{G \subset F \subset G} \mu(\emptyset, G) \mu(F, H) t^{\text{rk}M - \text{rk}H + \text{rk}G} \\ &= \sum_{G \subset H} \mu(\emptyset, G) t^{\text{rk}M - \text{rk}H + \text{rk}G} \left(\sum_{F: G \subset F \subset H} \mu(F, H) \right) \\ &= \sum_G \mu(\emptyset, G) t^{\text{rk}M} = 0. \end{aligned}$$

□

Corollary 3.8. *For a loopless matroid M , $\text{Eu}_M = \chi_M(2)$. In particular, if $M = M_1 \oplus M_2$ then $\text{Eu}_M = \text{Eu}_{M_1} \text{Eu}_{M_2}$.*

Proof. If $\text{rk}M = 0$, then both sides are equal to 1. We only need to verify $\chi_M(2)$ satisfies the recursive definition of Eu_M . Namely, we want to verify

$$\sum_{F \in \mathcal{L}(M)} c_{M^F} \chi_{M^F}(2) = 0,$$

equivalently,

$$\sum_{F \in \mathcal{L}(M)} 2^{\text{rk}F} \chi_{M^F}(1/2) \chi_{M^F}(2) = 0.$$

This is the previous theorem evaluating t at 2. The case $M = M_1 \oplus M_2$ immediately follows from the general fact that the characteristic polynomial for matroids is multiplicative. □

Using this closed formula for Eu_M , we can now prove Corollary 1.8.

Proof of Corollary 1.8. Since the Euler obstruction function Eu_{Y_L} is constant on each stratum Y_F of Y , we can write Eu_{Y_L} as

$$\text{Eu}_{Y_L} = \sum_{F \in \mathcal{L}(M)} \text{Eu}_{M^F} 1_{Y_F}$$

where 1_{Y_F} is the indicator function of Y_F . According to Eur, Huh, and Larson 2023, Theorem 1.15,

$$c_*(1_{Y_L}) = c_{SM}(Y_L) = \sum_{G \in \mathcal{L}(M)} y_G.$$

Applying Macpherson's natural transform c_* to Eu_{Y_L} , we have

$$\begin{aligned}
c_{Ma}(Y_L) &= c_*(\text{Eu}_{Y_L}) \\
&= \sum_{G \in \mathcal{L}(M)} \text{Eu}_{M_G} c_*(1_{Y_G}) \\
&= \sum_{G \in \mathcal{L}(M)} \chi_{M_G}(2) \sum_{F \in \mathcal{L}(M^G)} y_F \\
&= \sum_{F \in \mathcal{L}(F)} \left(\sum_{G: G \geq F} \chi_{M_G}(2) \right) y_F \\
&= \sum_{F \in \mathcal{L}(M)} 2^{\text{rk}M - \text{rk}F} y_F.
\end{aligned}$$

□

3.3 Nonpositivity of the local Euler obstruction

As another corollary, we can now prove Theorem 1.10.

Proof of Theorem 1.10. Assume the local Euler obstruction function is everywhere positive. This implies that $\text{Eu}_M(F) > 0$ for every flat, or equivalently $\chi_{M_F}(2) > 0$ for every flat. We must show M is a Boolean matroid after simplification.

We do induction on d . When $d = 2$, the simplification of M is always a uniform matroid. Since $\chi_{U_{n,2}}(2) = 2 - n < 0$ if $n > 2$, therefore $n = 2$ and M is a Boolean matroid.

Suppose the theorem is proved for all loopless matroids of rank strictly smaller than d . We want to prove the theorem is true in the case the rank is d . Without loss of generality we will replace M by its simplification. In particular, all its rank one flats are singletons $\{i\}$ for $i = 0, 1, \dots, n - 1$.

We first claim that $n = d$. Suppose $n > d$. By relabeling elements, we claim that there are d elements $0, 1, \dots, d - 1$ such that

Any three of them are not in the same flat of rank two.

Choose any element $0 \in E$. By the induction hypothesis, the simplification of $M_{\{0\}}$ is isomorphic to a Boolean matroid. In particular, there are $d - 1$ rank two flats F_k that contains 0 , and these flats part $E \setminus \{0\}$. More precisely,

$$\bigsqcup_{k=1}^{d-1} F_k \setminus \{0\} = E \setminus \{0\}.$$

Choose an element k from $F_k \setminus \{0\}$. They must be distinct from each other because of the partition property. I claim any three of them cannot be in one flat of rank two. Suppose there is a rank two flat G that contains i, j, k . The element 0 is not in G because of the partition property. Again, by the inductive hypothesis, there is a rank three flat H such that

$$H \setminus \{0\} = F_i \setminus \{0\} \sqcup F_j \setminus \{0\}.$$

Notice that $k \notin H$. Take the intersection of H and G , we get a new flat that properly contained in H and G . Since G is of rank two, this new flat must be of rank one. But all rank one flats are singletons, and this new flat contains i and j , a contradiction.

Because $n > d$, there must be an element e in E , lies in one of the rank two flats F_k . Without loss of generality, let us assume $e \in F_1$. The above arguments can also apply to $e, 2, 3, \dots, d - 1$, so we get any three elements of them are not in the same rank two flat. Let us consider the flats that contain 2 . There are d rank two flats that contain 2 , and each of them contains exactly one of $0, 1, 3, 4, \dots, d - 1$. Since $0, e, 2, 3, \dots, d - 1$ cannot lie in the same rank two flat, e must be in the flat that contains $1, 2$. Intersecting this flat with the flat that contains $0, 1, e$, we obtain a rank one flat that contains $1, e$, a contradiction.

We conclude that there are exactly d elements in the ground set E . Therefore, E itself is an independent set, so M must be a Boolean matroid. \square

Corollary 3.9. *For $L \subset \mathbb{C}^n$ and its associated matroid M , the following statements are equivalent.*

- (1) *The matroid M is a Boolean matroid.*

(2) *The algebraic variety Y_L is smooth.*

(3) *The local Euler obstruction of Y_L is everywhere positive.*

Proof. The statement (1) implies (2) is obvious since in this case $L = \mathbb{C}^n$ and $Y_L = (\mathbb{P}^1)^n$. If Y_L is smooth, then Eu_{Y_L} is everywhere 1. The statement (3) implies (1) is exactly Theorem 1.10. □

Chapter 4

Microlocal multiplicity

We will prove Theorem 1.11 in this chapter. On a matroid Schubert variety $Y_L \subset (\mathbb{P}^1)^n$, we are interested in the characteristic cycle of the intersection complex IC_{Y_L} . Again we choose our stratification to be the one that is defined by the group orbits. Therefore, the characteristic cycle of IC_{Y_L} is a formal sum of the conic Lagrangian cycles $T_{Y_{L,F}}^*(\mathbb{P}^1)^n$, recall $Y_{L,F}$ is the closure of L^F and L^F is the L -orbit that corresponds to the flat F . We can now write the characteristic cycle of IC_{Y_L} in the form

$$\mathrm{CC}(\mathrm{IC}_{Y_L}) = \sum_F m_M(F) T_{Y_{L,F}}^*(\mathbb{P}^1)^n. \quad (4.1)$$

To calculate these coefficients, we use the Eu isomorphism described in section 2, and translate Eq. (4.1) into an equation of constructible functions. For the left-hand side, recall that for a constructible complex \mathcal{F}^\bullet on an algebraic variety X , the associated constructible function is defined by taking pointwise Euler characteristic number:

$$p \rightarrow \chi(\mathcal{F}_p^\bullet).$$

For the right-hand side, the Eu isomorphism turns every $T_{Y_{L,F}}^*(\mathbb{P}^1)^n$ on the other side of Eq. (4.1) into $(-1)^{\dim Y_{L,F}} \mathrm{Eu}_{Y_{L,F}}$. As a result, we obtain the following equation.

$$\chi(\mathrm{IC}_{Y_L,p}) = \sum_{F:p \in Y_{L,F}} m_M(F) (-1)^{\dim Y_{L,F}} \mathrm{Eu}_{Y_{L,F}}(p),$$

here p is any point in Y_L .

By Elias, Proudfoot, and Wakefield 2016, Lemma 3.8, the number $\chi((\mathrm{IC}_{Y_L})_p)$ for a point $p \in U^G$ is

$$\chi(\mathrm{IC}_{Y_L,p}) = (-1)^{\dim Y_L} P_{M_G}(1) = (-1)^{\mathrm{rk} M} P_{M_G}(1),$$

where $P_M(t)$ refers to the Kazhdan-Lusztig polynomial of M . We need the sign $(-1)^{\mathrm{rk} M}$ because we need to shift the complex by the dimension. The condition that $p \in Y_{L,F}$ is equivalent to that $F \geq G$. Therefore, we can rewrite the equation as

$$(-1)^{\mathrm{rk} M} P_{M_G}(1) = \sum_{F \geq G} m_M(F) (-1)^{\dim Y_{L,F}} \mathrm{Eu}_{Y_{L,F}}(p).$$

Moreover, $\dim Y_{L,F} = \mathrm{rk}(F)$, and $\mathrm{Eu}_{Y_{L,F}}(p) = \mathrm{Eu}_{M^F}(G) = \mathrm{Eu}_{M_G^F}$. We obtain a purely combinatorial equation:

$$(-1)^{\mathrm{rk} M} P_{M_G}(1) = \sum_{F \geq G} m_M(F) (-1)^{\mathrm{rk} F} \mathrm{Eu}_{M_G^F}. \quad (4.2)$$

We can also define the microlocal multiplicity for a general matroid M by the above equation.

Lemma 4.1. *The multiplicity number $m_M(F)$ only depends on the matroid M_F . In fact, $m_M(F) = m_{M_F}(\emptyset)$.*

Proof. Evaluating Eq. (4.2) with $M = M, G = F$ and $M = M_F, G = \emptyset$, we see that both $m_M(F)$ and $m_{M_F}(\emptyset)$ satisfy the same recursive definition, so they must be equal to each other. \square

Following the same convention in this thesis, we define $m_M = m_M(\emptyset)$. Now the definition of m_M can be clearly written as the unique solution to the following equation

for every matroid M :

$$(-1)^{\text{rk}M} P_M(1) = \sum_F m_{M_F} (-1)^{\text{rk}F} \text{Eu}_{M_F}. \quad (4.3)$$

For example, $m_\emptyset = 1$ by setting $M = \emptyset$ in this equation. We prove Theorem 1.11 by solving this linear system. Actually, Proposition 3.3 shows exactly the “inverse matrix” of Eu_{M^F} is c_{M^F} .

Proof of Theorem 1.11. Utilizing Eq. (4.3), we have

$$(-1)^{d-1} \sum_F c_{M^F} P_{M^F}(1) = (-1)^{d-1} \sum_F c_{M^F} (-1)^{\text{cork}F} \sum_{G:F \subset G} m_{M_G} (-1)^{\text{rk}G - \text{rk}F} \text{Eu}_{M_G^F}.$$

Here we change the order of summation,

$$(-1)^{d-1} \sum_G (-1)^{\text{cork}G} m_{M_G} \sum_{F:F \subset G} c_{M^F} \text{Eu}_{M_G^F},$$

we can see the right hand side is very close to Proposition 3.3. We only need to treat the case $G = \emptyset$ and $F = \emptyset$ separately. We do it first for G and get

$$(-1)^{d-1} \sum_{G \neq \emptyset} (-1)^{\text{cork}G} m_{M_G} \sum_{F \subset G} c_{M^F} \text{Eu}_{M_G^F} + (-1)^{d-1} (-1)^{\text{rk}M} m_M c_\emptyset.$$

Recall that $d = \text{rk}M$ and $c_\emptyset = -1$. Then we treat the case $F = \emptyset$:

$$m_M + (-1)^{d-1} \sum_{G \neq \emptyset} (-1)^{\text{cork}G} m_{M_G} (c_\emptyset \text{Eu}_{M^G} + \sum_{\emptyset \neq F \subset G} c_{M^F} \text{Eu}_{M_G^F}).$$

Again, $c_\emptyset = -1$. We are now in a position to apply Proposition 3.3 with $M = M^G$:

$$m_M + (-1)^{d-1} \sum_{G \neq \emptyset} (-1)^{\text{cork}G} m_{M_G} (-\text{Eu}_{M^G} + \text{Eu}_{M^G}),$$

and this is clearly m_M . Combine with Corollary 3.6, we finish the proof. \square

This closed formula for m_M also shows that it is well-behaved under the direct sum.

Corollary 4.2. *For any two matroids M_1 and M_2 , $m_{M_1 \oplus M_2} = m_{M_1} m_{M_2}$.*

Proof. A flat of $M = M_1 \oplus M_2$ is of the form $F_1 \cup F_2$, where $F_i \subset M_i$ is a flat, $i = 1, 2$. We know from Corollary 3.6 that $c_{MF} = -c_{(M_1)F_1} c_{(M_2)F_2}$, and from Elias, Proudfoot, and Wakefield 2016, Proposition 2.7 that $P_{MF}(t) = P_{(M_1)F_1}(t)P_{(M_2)F_2}(t)$. Putting all these together, we get

$$m_M = (-1)^{\text{rk}M_1 + \text{rk}M_2 - 1} \sum_{F_1, F_2} -c_{(M_1)F_1} c_{(M_2)F_2} P_{(M_1)F_1}(1) P_{(M_2)F_2}(1)$$

which is exactly the product of

$$(-1)^{\text{rk}M_i - 1} \sum_{F_i} c_{(M_i)F_i} P_{(M_i)F_i}(1),$$

$i = 1, 2$. □

Remark. This corollary is clearer when we take the geometric point of view. If $L = L_1 \oplus L_2$, then $Y_L \cong Y_{L_1} \times Y_{L_2}$, so

$$\text{IC}_{Y_L} \cong \text{IC}_{Y_{L_1}} \boxtimes \text{IC}_{Y_{L_2}}.$$

As a consequence, there is a similar exterior product relation between characteristic cycles, see Saito 2017. In particular, the coefficient $m_{M_1 \oplus M_2}$ of the conormal variety of the all-infinity point must be the product of m_{M_i} , $i = 1, 2$.

By Eq. (4.1) and the fact IC_{Y_L} is a perverse sheaf, plus the fact that the characteristic cycle of a perverse sheaf is always effective, we conclude that $m_M \geq 0$ when M is realizable over \mathbb{C} . However, it is not easy to tell the positivity of the microlocal multiplicity m of a general matroid M . Naturally, we propose the following conjecture:

Conjecture. *For a loopless matroid M , m_M is always nonnegative. Moreover, $m_M = 0$ if and only if M has a Boolean summand, that is, M is a direct sum of a Boolean matroid and another arbitrary matroid.*

Let us end the section with a corollary about the irreducibility of the characteristic cycle.

Corollary 4.3. *The characteristic cycle of the intersection complex of a matroid Schubert variety of a simple realizable matroid $M = M(L)$ is irreducible if and only if M is a Boolean matroid.*

Proof. If $\text{CC}(\text{IC}_{Y_L})$ is irreducible, then

$$\text{CC}(\text{IC}_{Y_L}) = T_{Y_L}^*(\mathbb{P}^1)^n.$$

Translate the equation into an equation of constructible functions, we get

$$(-1)^{\text{rk}M} P_{M_G}(1) = (-1)^{\text{rk}M} m_M \text{Eu}_{Y_L}(p)$$

for $p \in L^G$. Therefore,

$$\text{Eu}_M(G) = \frac{P_{M_G}(1)}{m_M} > 0$$

for every flat G . This implies M must be a Boolean matroid by Theorem 1.10. \square

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