Some results involving the positive part of the quantized enveloping algebra for affine $\mathfrak{sl}_2$

by

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Abstract

The q-deformed enveloping algebra $U_q(\widehat{\mathfrak{sl}}_2)$ has a subalgebra U_q^+, called the positive part. The algebra U_q^+ has a presentation with two generators A, B and two relations called the q-Serre relations. The literature contains at least three PBW bases for U_q^+, called the Damiani, Beck, and alternating PBW bases. This thesis is about the three PBW bases and related topics. In our investigation, we will adopt the following approach. Let $\mathbb{V}$ denote the free algebra with two generators x, y. In 1995, Rosso introduced another algebra structure on $\mathbb{V}$, called the q-shuffle algebra. Rosso gave an injective algebra homomorphism from U_q^+ to the q-shuffle algebra $\mathbb{V}$ that sends $A \mapsto x$ and $B \mapsto y$. Let U denote the image of U_q^+ under this injective homomorphism. Our research is focused on U. This thesis consists of three main parts, which we now summarize.

The first part concerns the alternating PBW basis. In 2019, Terwilliger introduced the alternating words $\{W_{-n}\}_{n\in\mathbb{N}}$, $\{W_{n+1}\}_{n\in\mathbb{N}}$, $\{G_{n+1}\}_{n\in\mathbb{N}}$, $\{\tilde{G}_{n+1}\}_{n\in\mathbb{N}}$ of U. He showed that the alternating words $\{W_{-n}\}_{n\in\mathbb{N}}$, $\{W_{n+1}\}_{n\in\mathbb{N}}$, $\{\tilde{G}_{n+1}\}_{n\in\mathbb{N}}$ form a PBW basis for U, and he expressed $\{G_{n+1}\}_{n\in\mathbb{N}}$ in this alternating PBW basis. In his calculation, Terwilliger used some elements $\{D_n\}_{n\in\mathbb{N}}$ with the following property: the generating function $D(t) = \sum_{n\in\mathbb{N}} D_n t^n$ is the multiplicative inverse of the generating function $\tilde{G}(t) = \sum_{n\in\mathbb{N}} \tilde{G}_n t^n$ with respect to the q-shuffle product, where $\tilde{G}_0 = 1$. Terwilliger defined $\{D_n\}_{n\in\mathbb{N}}$ recursively; we will express the elements $\{D_n\}_{n\in\mathbb{N}}$ in closed form.

We now describe the second part. It is known that the Damiani, Beck, and alternating PBW bases are related via exponential formulas. We will introduce an exponential generating function whose argument is a power series involving the Beck PBW basis and an integer parameter m. The cases $m = 2$ and $m = -1$ yield the known exponential formulas for the Damiani and alternating PBW bases, respectively. The case $m = 1$ involves the elements $\{D_n\}_{n\in\mathbb{N}}$. We will give a comprehensive study of the generating function for an arbitrary integer m. We have two main results in this part. The first main result gives a factorization of the generating function. In the second main result, we express the coefficients of the generating function in closed form.

The third part is motivated by a basis for $\mathbb{V}$, called the standard basis. The standard basis consists of all the words in $\mathbb{V}$. It is known that the subalgebra U is properly contained in $\mathbb{V}$. We will classify the words contained in U. The classification shows that such words are one of three types. Words of the first type are powers of x or y with respect to the free product. Words of the second type are exactly the alternating words. Words of the third type are said to be doubly alternating. In addition to the classification, we will express the doubly alternating words in terms of the alternating words.

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Chapter 1

Introduction

This thesis involves the q-deformed enveloping algebra $U_q(\widehat{\mathfrak{sl}}_2)$ [8]. The algebra $U_q(\widehat{\mathfrak{sl}}_2)$ appears in the topics of combinatorics [14, 16, 18, 19], quantum algebras [2, 5, 10, 11, 15, 30], and representation theory [1, 6, 12, 17, 31]. The algebra $U_q(\widehat{\mathfrak{sl}}_2)$ is associative, noncommutative, and infinite-dimensional.

The algebra $U_q(\widehat{\mathfrak{sl}}_2)$ has a subalgebra U_q^+ called the positive part [8, 20]. The algebra U_q^+ has a presentation with two generators A, B and two relations

$$A^3B - [3]_qA^2BA + [3]_qABA^2 - BA^3 = 0,$$

$$B^3A - [3]_qB^2AB + [3]_qBAB^2 - AB^3 = 0,$$

where $[3]_q = (q^3 - q^{-3})/(q - q^{-1})$. The above relations are called the q-Serre relations.

We will be discussing the notion of a Poincaré-Birkhoff-Witt (or PBW) basis for U_q^+; see Definition 2.2.1 below.

In [9], Damiani obtained a PBW basis for U_q^+. The PBW basis elements $\{E_{n\delta+\alpha_0}\}_{n\in\mathbb{N}}$, $\{E_{n\delta+\alpha_1}\}_{n\in\mathbb{N}}$, $\{E_{(n+1)\delta}\}_{n\in\mathbb{N}}$ were defined recursively using a braid group action.

In [3], Beck obtained a PBW basis for U_q^+ from the Damiani PBW basis, by replacing $E_{n\delta}$ with an element $E_{n\delta}^{\mathrm{Beck}}$ for $n \geq 1$. In [4], Beck, Chari, and Pressley showed that the elements $\{E_{(n+1)\delta}\}_{n\in\mathbb{N}}$ and $\{E^{\text{Beck}}_{(n+1)\delta}\}_{n\in\mathbb{N}}$ are related via an exponential formula.

In [22, 23], Rosso introduced an embedding of U_q^+ into a q-shuffle algebra. In Section 2.1 we will review this algebra in detail, and for now we give a quick preview. Let $\mathbb{V}$ denote the free associative algebra with two generators x, y. A free product of generators is called a word. The vector space $\mathbb{V}$ has a basis consisting of the words, called the standard basis. In [22, 23], Rosso introduced a second algebra structure on $\mathbb{V}$, called the q-shuffle algebra. He then gave an embedding of U_q^+ into the q-shuffle algebra $\mathbb{V}$. This embedding sends $A \mapsto x$ and $B \mapsto y$. Let U denote the image of U_q^+ under this embedding.

In [27], Terwilliger used the Rosso embedding to obtain a closed form for the Damiani PBW basis elements. This closed form involves some words of a certain type, said to be Catalan. Let $\overline{x} = 1$ and $\overline{y} = -1$. A word $a_1a_2\cdots a_n$ is Catalan whenever $\overline{a}_1 + \overline{a}_2 + \cdots + \overline{a}_i \geq 0$ for $1 \leq i \leq n-1$ and $\overline{a}_1 + \overline{a}_2 + \cdots + \overline{a}_n = 0$. The length of a Catalan word is even. For $n \geq 0$, Terwilliger introduced the element C_n in $\mathbb{V}$. He defined C_n as a linear combination of the Catalan words of length $2n$. For a Catalan word $a_1a_2\cdots a_{2n}$, its coefficient in C_n is

$$\prod_{i=1}^{2n}[1 + \overline{a}_1 + \overline{a}_2 + \cdots + \overline{a}_i]_q.$$

In [27, Theorem 1.7], Terwilliger showed that the Rosso embedding sends

$$E_{n\delta+\alpha_0} \mapsto q^{-2n}(q-q^{-1})^{2n}xC_n, \qquad E_{n\delta+\alpha_1} \mapsto q^{-2n}(q-q^{-1})^{2n}C_ny$$

for $n \geq 0$, and

$$E_{n\delta} \mapsto -q^{-2n}(q-q^{-1})^{2n-1}C_n$$

for $n \geq 1$.

In [29], Terwilliger used the Rosso embedding to obtain a closed form for the elements $\{E^{\text{Beck}}_{(n+1)\delta}\}_{n\in\mathbb{N}}$. He introduced the elements $\{xC_ny\}_{n\in\mathbb{N}}$ in $\mathbb{V}$. In [29, Theorem 7.1], Terwilliger showed that the Rosso embedding sends

$$E_{n\delta}^{\mathrm{Beck}} \mapsto \frac{[2n]_q}{n} q^{-2n}(q-q^{-1})^{2n-1} x C_{n-1} y$$

for $n \geq 1$.

In [26], Terwilliger introduced the alternating PBW basis for U. He introduced the words of the form $\cdots xyxyxy \cdots$, said to be alternating. The alternating words are named as follows:

$$\begin{aligned}
&W_0 = x, \quad W_{-1} = xyx, \quad W_{-2} = xyxyx, \quad W_{-3} = xyxyxyx, \quad \ldots \\
&W_1 = y, \quad W_2 = yxy, \quad W_3 = yxyxy, \quad W_4 = yxyxyxy, \quad \ldots \\
&G_1 = yx, \quad G_2 = yxyx, \quad G_3 = yxyxyx, \quad G_4 = yxyxyxyx, \quad \ldots \\
&\tilde{G}_1 = xy, \quad \tilde{G}_2 = xyxy, \quad \tilde{G}_3 = xyxyxy, \quad \tilde{G}_4 = xyxyxyxy, \quad \ldots
\end{aligned}$$

In [26, Theorem 10.1], Terwilliger showed that the alternating words are contained in U. Moreover, the words $\{W_{-n}\}_{n\in\mathbb{N}}$, $\{W_{n+1}\}_{n\in\mathbb{N}}$, $\{\tilde{G}_{n+1}\}_{n\in\mathbb{N}}$ form a PBW basis for U. In [26, Theorem 9.15], he gave $\{G_{n+1}\}_{n\in\mathbb{N}}$ in terms of $\{W_{-n}\}_{n\in\mathbb{N}}$, $\{W_{n+1}\}_{n\in\mathbb{N}}$, $\{\tilde{G}_{n+1}\}_{n\in\mathbb{N}}$.

In order to illuminate the algebraic structure of the above PBW bases, we consider the generating functions

$$C(t) = \sum_{n\in\mathbb{N}} C_n t^n, \qquad \tilde{G}(t) = \sum_{n\in\mathbb{N}} \tilde{G}_n t^n,$$

where $\tilde{G}_0 = 1$.

In [26, Section 9], Terwilliger considered the multiplicative inverse of $\tilde{G}(t)$ with respect to the q-shuffle product; this inverse is denoted by $D(t)$. Following [26, Definition 9.11] we write

$$D(t) = \sum_{n\in\mathbb{N}} D_n t^n.$$

In [24], we obtained a closed form for the elements $\{D_n\}_{n\in\mathbb{N}}$. We showed that for $n \geq 0$, the element D_n is a linear combination of the Catalan words of length $2n$. Moreoever, for a Catalan word $a_1 a_2 \cdots a_{2n}$, its coefficient in D_n is

$$(-1)^n \prod_{i=1}^{2n} [\overline{a}_1 + \overline{a}_2 + \cdots + \overline{a}_{i-1} + (\overline{a}_i + 1)/2]_q.$$

The main purpose of Chapter 3 is to present and prove this closed form.

As mentioned earlier, the elements $\{E_{(n+1)\delta}\}_{n\in\mathbb{N}}$ and $\{E^{\mathrm{Beck}}_{(n+1)\delta}\}_{n\in\mathbb{N}}$ are related via an exponential formula. In [29, Section 8], Terwilliger used the Rosso embedding to reformulate this exponential formula as follows:

$$C(t) = \exp\left(\sum_{n=1}^{\infty} \frac{[2n]_q}{n} x C_{n-1} y t^n\right). \tag{1.1}$$

In [29, Section 9] Terwilliger showed

$$D(t) = \exp\left(\sum_{n=1}^{\infty} \frac{(-1)^n [n]_q}{n} x C_{n-1} y t^n\right), \tag{1.2}$$

$$\tilde{G}(t) = \exp\left(-\sum_{n=1}^{\infty} \frac{(-1)^n [n]_q}{n} x C_{n-1} y t^n\right). \tag{1.3}$$

To summarize (1.1)–(1.3) in a uniform way, for an integer m we consider the generating function

$$\exp\left(\sum_{n=1}^{\infty} \frac{[mn]_q}{n} x C_{n-1} y t^n\right). \tag{1.4}$$

Setting $m = 2, m = 1, m = -1$ in (1.4), we get $C(t)$, $D(-t)$, $\tilde{G}(-t)$ respectively.

Motivated by this observation, in [25] we investigated the generating function (1.4) for an arbitrary integer m and obtained a uniform approach to (1.1)–(1.3). We have two main results about this topic. The first main result gives a factorization of (1.4). The factors involve $D(t)$ if m is positive, and $\tilde{G}(t)$ if m is negative. In the second main result, we expressed (1.4) explicitly as a linear combination of Catalan words, and we gave the coefficients. As we will see, for a Catalan word $a_1 a_2 \cdots a_{2n}$, its coefficient in (1.4) is equal to t^n times

$$\prod_{i=1}^{2n}[\overline{a}_1 + \overline{a}_2 + \cdots + \overline{a}_{i-1} + m(\overline{a}_i + 1)/2]_q.$$

The main purpose of Chapter 4 is to give the uniform approach and to prove the two main results above. Our proof is from scratch; we do not invoke earlier results in the literature.

Recall that the vector space $\mathbb{V}$ has a basis consisting of the words. It is known that the subalgebra U is properly contained in $\mathbb{V}$. Therefore, it is natural to ask which words are contained in U. We will obtain a classification of the words in U. To display this classification, we call a word of the form $\cdots xxyyxxyyxxyy \cdots$ doubly alternating. As we will see, the words in U fall into three types:

(i) the words x^n, y^n for $n \in \mathbb{N}$;

(ii) the alternating words;

(iii) the doubly alternating words.

The main purpose of Chapter 5 is to prove the above classification and to study the doubly alternating words. There will be formulas expressing the doubly alternating words in terms of the alternating words with respect to the q-shuffle product. We will write out these formulas explicitly.

This thesis is organized as follows. In Chapter 2, we recall the necessary background knowledge. In Chapter 3, we present the closed form for the elements $\{D_n\}_{n\in\mathbb{N}}$. In Chapter 4, we study the generating function (1.4) and present the uniform approach to (1.1)–(1.3). In Chapter 5, we classify all the words in U and study the doubly alternating words. In Chapter 6, we discuss some future projects.

The main results of this thesis are Theorems 3.1.8, 4.1.6, 4.1.7, 5.1.6. The related results Propositions 3.3.3, 3.3.4, 4.10.7, 4.10.8 and the formulas in Sections 5.3, 5.4 may be of independent interest.

Chapter 2

Background

2.1 The algebra U_q^+ and its Rosso embedding U

In this section, we recall the algebra U_q^+ and its Rosso embedding.

First, we establish some conventions and notation that will be used throughout the thesis.

Recall the integers $\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$ and the natural numbers $\mathbb{N} = \{0, 1, 2, \dots\}$. Let $\mathbb{F}$ denote a field of characteristic zero. All algebras in this thesis are associative, over $\mathbb{F}$, and have a multiplicative identity. Let q denote a nonzero scalar in $\mathbb{F}$ that is not a root of unity. For $n \in \mathbb{Z}$, define

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}.$$

Next we recall the positive part U_q^+ of the q-deformed enveloping algebra $U_q(\widehat{\mathfrak{sl}}_2)$ [8, 20]. The algebra U_q^+ is defined by the generators A, B and the q-Serre relations

$$A^3B - [3]_q A^2BA + [3]_q ABA^2 - BA^3 = 0, \tag{2.1}$$

$$B^3A - [3]_q B^2AB + [3]_q BAB^2 - AB^3 = 0. \tag{2.2}$$

We will be discussing a q-shuffle algebra $\mathbb{V}$. Let x, y denote noncommuting indeterminates, and let $\mathbb{V}$ denote the free algebra generated by x, y. We call x and y *letters*. For $n \in \mathbb{N}$, the product of n letters is called a *word* of *length* n. The word of length 0 is called *trivial* and denoted by $\mathbb{1}$. The words form a basis for the vector space $\mathbb{V}$; this basis is called *standard*. The vector space $\mathbb{V}$ admits another algebra structure, called the q-shuffle algebra [22, 23]. The q-shuffle product is denoted by $\star$. The following recursive definition of $\star$ is adopted from [13].

- For $v \in \mathbb{V}$,

$$\mathbb{1} \star v = v \star \mathbb{1} = v.$$

- For the letters u, v,

$$u \star v = uv + vuq^{\langle u,v \rangle},$$

  where

$$\langle x, x \rangle = \langle y, y \rangle = 2, \qquad \langle x, y \rangle = \langle y, x \rangle = -2.$$

- For a letter u and a nontrivial word $v = v_1 v_2 \cdots v_n$ in $\mathbb{V}$,

$$u \star v = \sum_{i=0}^{n} v_1 \cdots v_i u v_{i+1} \cdots v_n q^{\langle u,v_1 \rangle + \cdots + \langle u,v_i \rangle},$$

$$v \star u = \sum_{i=0}^{n} v_1 \cdots v_i u v_{i+1} \cdots v_n q^{\langle u,v_n \rangle + \cdots + \langle u,v_{i+1} \rangle}.$$

- For nontrivial words $u = u_1 u_2 \cdots u_r$ and $v = v_1 v_2 \cdots v_s$ in $\mathbb{V}$,

$$u \star v = u_1((u_2 \cdots u_r) \star v) + v_1(u \star (v_2 \cdots v_s))q^{\langle v_1,u_1 \rangle + \cdots + \langle v_1,u_r \rangle},$$

$$u \star v = (u \star (v_1 \cdots v_{s-1}))v_s + ((u_1 \cdots u_{r-1}) \star v)u_r q^{\langle u_r,v_1 \rangle + \cdots + \langle u_r,v_s \rangle}.$$

It was shown in [22, 23] that the vector space $\mathbb{V}$, together with the q-shuffle product $\star$, forms an algebra. This is the q-shuffle algebra.

Next we recall an embedding of U_q^+ into the q-shuffle algebra $\mathbb{V}$. This embedding is due to Rosso [22, 23]. He showed that x, y satisfy

$$x \star x \star x \star y - [3]_q x \star x \star y \star x + [3]_q x \star y \star x \star x - y \star x \star x \star x = 0,$$

$$y \star y \star y \star x - [3]_q y \star y \star x \star y + [3]_q y \star x \star y \star y - x \star y \star y \star y = 0.$$

Consequentially there exists an algebra homomorphism $\natural$ from U_q^+ to the q-shuffle algebra $\mathbb{V}$ that sends $A \mapsto x$ and $B \mapsto y$. It was shown in [23, Theorem 15] that $\natural$ is injective. Let U denote the image of U_q^+ under $\natural$. Observe that U is the subalgebra of the q-shuffle algebra $\mathbb{V}$ generated by x, y. We remark that this subalgebra is proper. Throughout this thesis, we identify U_q^+ with U via $\natural$.

2.2 Three PBW bases for U

Definition 2.2.1. By a *Poincaré-Birkhoff-Witt basis* (or *PBW basis*) for U we mean a subset $\Omega \subseteq U$ and a linear ordering $<$ on Ω, such that the following elements form a basis for U:

$$\omega_1 \omega_2 \cdots \omega_n,$$

$$n \in \mathbb{N},$$

$$\omega_1, \omega_2, \ldots, \omega_n \in \Omega,$$

$$\omega_1 \leq \omega_2 \leq \cdots \leq \omega_n.$$

In this section, we recall three PBW basis for U. In order to do this, we first discuss several types of words in $\mathbb{V}$ that will be used later.

Definition 2.2.2. (See [27, Definition 1.3].) A word $a_1 a_2 \cdots a_n$ in $\mathbb{V}$ is said to be *balanced* whenever

$$|\{i \mid 1 \leq i \leq n, a_i = x\}| = |\{i \mid 1 \leq i \leq n, a_i = y\}|.$$

In this case n is even.

Example 2.2.3. We list the balanced words of length ≤ 4.

$$\mathbb{1}, \qquad xy, \quad yx,$$

$$xxyy, \quad xyxy, \quad xyyx, \quad yxxy, \quad yxyx, \quad yyxx.$$

Definition 2.2.4. (See [27, Definition 1.3].) For notational convenience, to each letter a we assign a weight $\overline{a}$ as follows:

$$\overline{x} = 1, \qquad \overline{y} = -1.$$

Lemma 2.2.5. (See [27, Definition 1.3].) A word $a_1 a_2 \cdots a_n$ is balanced if and only if $\overline{a}_1 + \overline{a}_2 + \cdots + \overline{a}_n = 0$.

Proof. Follows from Definitions 2.2.2 and 2.2.4. □

We will be discussing a certain type of balanced word, said to be Catalan.

Definition 2.2.6. (See [27, Definition 1.3].) A word $a_1 a_2 \cdots a_n$ is *Catalan* whenever $\overline{a}_1 + \overline{a}_2 + \cdots + \overline{a}_i \geq 0$ for $1 \leq i \leq n-1$ and $\overline{a}_1 + \overline{a}_2 + \cdots + \overline{a}_n = 0$. A Catalan word is balanced. The length of a Catalan word is even. For $n \in \mathbb{N}$, let Cat_n denote the collection of Catalan words of length $2n$.

Example 2.2.7. We list the Catalan words of length ≤ 6.

$$\mathbb{1}, \qquad xy, \qquad xyxy, \quad xxyy,$$

$$xyxyxy, \quad xxyyxy, \quad xyxxyy, \quad xxyxyy, \quad xxxyyy.$$

Lemma 2.2.8. Let $w = a_1 a_2 \cdots a_{2n}$ be a nontrivial Catalan word. Then $a_1 = x$ and $a_{2n} = y$.

Proof. Follows from Definition 2.2.6. □

Definition 2.2.9. (See [27, Definition 1.5].) For $n \in \mathbb{N}$, define

$$C_n = \sum_{a_1 a_2 \cdots a_{2n} \in \mathrm{Cat}_n} a_1 a_2 \cdots a_{2n} \prod_{i=1}^{2n} [1 + \overline{a}_1 + \overline{a}_2 + \cdots + \overline{a}_i]_q.$$

We interpret $C_0 = \mathbb{1}$. We call C_n the n^{th} Catalan element.

Example 2.2.10. We list C_n for $0 \leq n \leq 3$.

$$C_0 = \mathbb{1}, \qquad C_1 = [2]_q xy, \qquad C_2 = [2]_q^2 xyxy + [2]_q^2 [3]_q xxyy,$$

$$C_3 = [2]_q^3 xyxyxy + [2]_q^3 [3]_q xxyyxy + [2]_q^3 [3]_q xyxxyy + [2]_q^3 [3]_q^2 xxyxyy + [2]_q^2 [3]_q^2 [4]_q xxxyyy.$$

Next we recall three PBW basis for U. We will encounter the elements xC_n, $C_n y$, $xC_{n-1}y$. We emphasize that this notation refers to the free product on $\mathbb{V}$.

Proposition 2.2.11. (See [29, Corollary 6.8].) The elements $\{xC_n\}_{n\in\mathbb{N}}$, $\{C_n y\}_{n\in\mathbb{N}}$, $\{C_{n+1}\}_{n\in\mathbb{N}}$ form a PBW basis for U under the linear ordering

$$xC_0 < xC_1 < xC_2 < \cdots < C_1 < C_2 < C_3 < \cdots < C_2 y < C_1 y < C_0 y.$$

We remark that up to normalization, the above PBW basis is the one given by Damiani in [9].

Proposition 2.2.12. (See [29, Corollary 8.3].) The elements $\{xC_n\}_{n\in\mathbb{N}}$, $\{C_n y\}_{n\in\mathbb{N}}$, $\{xC_n y\}_{n\in\mathbb{N}}$ form a PBW basis for U under the linear ordering

$$xC_0 < xC_1 < xC_2 < \cdots < xC_0 y < xC_1 y < xC_2 y < \cdots < C_2 y < C_1 y < C_0 y.$$

We remark that up to normalization, the above PBW basis is the one given by Beck in [3].

Definition 2.2.13. (See [26, Definition 5.2].) For $n \in \mathbb{N}$, define

$$W_{-n} = (xy)^n x, \qquad W_{n+1} = y(xy)^n, \qquad G_{n+1} = (yx)^{n+1}, \qquad \tilde{G}_{n+1} = (xy)^{n+1}.$$

The above exponents are with respect to the free product.

Example 2.2.14. We list W_{-n}, W_{n+1}, G_{n+1}, $\tilde{G}_{n+1}$ for $0 \le n \le 3$.

$$W_0 = x, \qquad W_{-1} = xyx, \qquad W_{-2} = xyxyx, \qquad W_{-3} = xyxyxyx;$$
$$W_1 = y, \qquad W_2 = yxy, \qquad W_3 = yxyxy, \qquad W_4 = yxyxyxy;$$
$$G_1 = yx, \qquad G_2 = yxyx, \qquad G_3 = yxyxyx, \qquad G_4 = yxyxyxyx;$$
$$\tilde{G}_1 = xy, \qquad \tilde{G}_2 = xyxy, \qquad \tilde{G}_3 = xyxyxy, \qquad \tilde{G}_4 = xyxyxyxy.$$

The words $\{W_{-n}\}_{n\in\mathbb{N}}$, $\{W_{n+1}\}_{n\in\mathbb{N}}$, $\{G_{n+1}\}_{n\in\mathbb{N}}$, $\{\tilde{G}_{n+1}\}_{n\in\mathbb{N}}$ are called *alternating*. For notational convenience, we define $G_0 = \mathbb{1}$ and $\tilde{G}_0 = \mathbb{1}$.

Proposition 2.2.15. (See [26, Theorems 10.1 and 10.2].) Each of the following (i), (ii) forms a PBW basis for U under an appropriate linear ordering.

(i) $\{W_{-n}\}_{n\in\mathbb{N}}$, $\{W_{n+1}\}_{n\in\mathbb{N}}$, $\{\tilde{G}_{n+1}\}_{n\in\mathbb{N}}$;

(ii) $\{W_{-n}\}_{n\in\mathbb{N}}$, $\{W_{n+1}\}_{n\in\mathbb{N}}$, $\{G_{n+1}\}_{n\in\mathbb{N}}$.

2.3 Elevation sequences, Dyck paths, and profiles

In this section, we recall a few notions related to the Catalan words. These notions will be used later in the thesis.

Definition 2.3.1. (See [27, Definition 2.6].) For $n \in \mathbb{N}$ and a word $w = a_1 a_2 \cdots a_n$, the *elevation sequence* of w is the sequence $(e_0, e_1, \ldots, e_n)$, where $e_i = \overline{a}_1 + \overline{a}_2 + \cdots + \overline{a}_i$ for $0 \le i \le n$.

Example 2.3.2. In the table below, we list the Catalan words w of length ≤ 6 and the corresponding elevation sequences.

Referring to Definition 2.3.1, we have $e_0 = 0$.

Lemma 2.3.3. Let w be a word with elevation sequence $(e_0, e_1, \ldots, e_n)$. The word w is Catalan if and only if $e_i \ge 0$ for $1 \le i \le n-1$ and $e_n = 0$.

w	elevation sequence of w
$\mathbb{1}$	(0)
xy	$(0,1,0)$
$xyxy$	$(0,1,0,1,0)$
$xxyy$	$(0,1,2,1,0)$
$xyxyxy$	$(0,1,0,1,0,1,0)$
$xxyyxy$	$(0,1,2,1,0,1,0)$
$xyxxyy$	$(0,1,0,1,2,1,0)$
$xxyxyy$	$(0,1,2,1,2,1,0)$
$xxxyyy$	$(0,1,2,3,2,1,0)$

Table 2.1: Example of elevation sequences

Proof. Follows from Definitions 2.2.6 and 2.3.1. □

Next we give a way to visualize a word using its elevation sequence.

Definition 2.3.4. (See [7, Section 8.5].) Let $n \in \mathbb{N}$. For a word with elevation sequence $(e_0, e_1, \dots, e_n)$, the corresponding *Dyck path* is a diagonal lattice path with $n+1$ vertices, where for $0 \le i \le n$ the i-th vertex is the lattice point (i, e_i).

Example 2.3.5. In the picture below, we give the Dyck paths for the Catalan words of length ≤ 6.

Observe in Example 2.3.5 that the Dyck path of a word is uniquely determined by its peaks and valleys. This observation is captured in the following definition.

Definition 2.3.6. (See [27, Definition 2.8].) For a word w with elevation sequence $(e_0, e_1, \dots, e_n)$, the *profile* of w is the subsequence of $(e_0, e_1, \dots, e_n)$ obtained by removing all the e_i such that $1 \le i \le n-1$ and $e_i - e_{i-1} = e_{i+1} - e_i$.

In other words, the profile of a word w is the subsequence of the elevation sequence of w consisting of the end points and turning points.

By a *profile* we mean the profile of a word.

By a *nontrivial profile* (resp. *Catalan profile*) we mean the profile of a nontrivial word (resp. Catalan word).

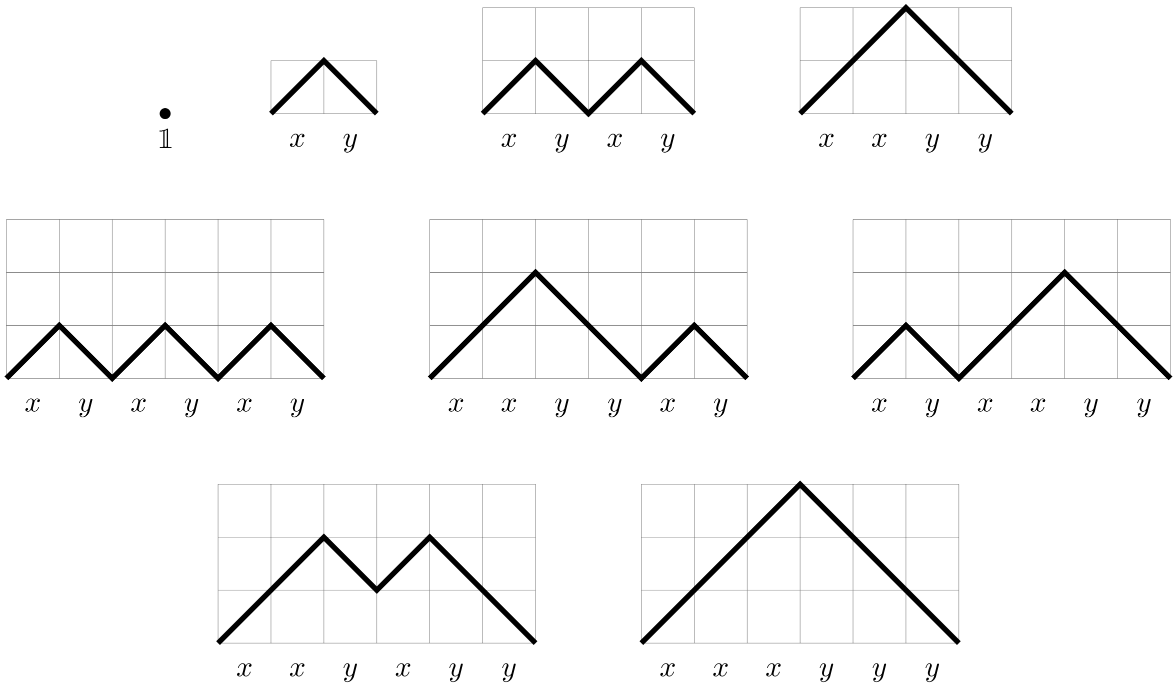

Figure 2.1: Example of Dyck paths

Example 2.3.7. In the table below, we list the Catalan words w of length ≤ 6 and the corresponding profiles.

w	profile of w
$\mathbb{1}$	(0)
xy	$(0,1,0)$
$xyxy$	$(0,1,0,1,0)$
$xxyy$	$(0,2,0)$
$xyxyxy$	$(0,1,0,1,0,1,0)$
$xxyyxy$	$(0,2,0,1,0)$
$xyxxyy$	$(0,1,0,2,0)$
$xxyxyy$	$(0,2,1,2,0)$
$xxxyyy$	$(0,3,0)$

Table 2.2: Example of profiles

Lemma 2.3.8. A profile $(l_0, h_1, l_1, \ldots, h_r, l_r)$ is Catalan if and only if $l_i \geq 0$ for $1 \leq i \leq r-1$ and $l_r = 0$.

Proof. Follows from Lemma 2.3.3 and Definition 2.3.6. □

We end this section with an observation.

Lemma 2.3.9. Let $n \in \mathbb{N}$. For a word w of length $2n$, the following are equivalent:

(i) $w = \tilde{G}_n$;

(ii) each entry in the elevation sequence of w is either 0 or 1.

Proof. Follows from Definitions 2.2.13 and 2.3.1. □

Chapter 3

The elements $\{D_n\}_{n\in\mathbb{N}}$ of U

3.1 Statement of the main results

In this section, we will motivate and state our main results of this chapter.

Recall that Proposition 2.2.15 gives two alternating PBW bases for U. In [26, Section 9] it is explained how the two PBW bases are related. This relationship is described using some generating functions. We now review these generating functions.

For the rest of this thesis, let t denote an indeterminate. We will discuss generating functions in the variable t.

Definition 3.1.1. Define the generating function

$$\tilde{G}(t) = \sum_{n\in\mathbb{N}} \tilde{G}_n t^n.$$

We will be discussing the multiplicative inverse of $\tilde{G}(t)$ with respect to $\star$. We now introduce this inverse.

Definition 3.1.2. (See [26, Definition 9.5].) We define the elements $\{D_n\}_{n\in\mathbb{N}}$ of U in the following recursive way:

$$D_0 = \mathbb{1}, \qquad D_n = -\sum_{k=0}^{n-1} D_k \star \tilde{G}_{n-k} \quad (n \geq 1). \tag{3.1}$$

Define the generating function

$$D(t) = \sum_{n \in \mathbb{N}} D_n t^n.$$

Lemma 3.1.3. (See [28, Lemma 4.1].) The generating function $D(t)$ is the multiplicative inverse of $\tilde{G}(t)$ with respect to $\star$. In other words,

$$\tilde{G}(t) \star D(t) = \mathbb{1} = D(t) \star \tilde{G}(t). \tag{3.2}$$

Proof. The relation (3.2) can be checked routinely using (3.1). □

For $n \in \mathbb{N}$ we can calculate D_n recursively using (3.1).

Example 3.1.4. We list D_n for $0 \leq n \leq 3$.

$$D_0 = \mathbb{1}, \qquad D_1 = -xy, \qquad D_2 = xyxy + [2]_q^2 xxyy,$$

$$D_3 = -xyxyxy - [2]_q^2 xxyyxy - [2]_q^2 xyxxyy - [2]_q^4 xxyxyy - [2]_q^2 [3]_q^2 xxxyyy.$$

Observe that for $0 \leq n \leq 3$, each D_n is a linear combination of Catalan words of length $2n$. We now show that this observation is true for all $n \in \mathbb{N}$.

Proposition 3.1.5. For $n \in \mathbb{N}$, D_n is contained in the span of Cat_n.

Proof. For $n \in \mathbb{N}$, by Definition 2.2.13 we have that $\tilde{G}_n = xyxy \cdots xy$ where the xy is repeated n times. The word $\tilde{G}_n$ is Catalan by Definition 2.2.6. Note that the q-shuffle product of two Catalan words is a linear combination of Catalan words. The result follows by (3.1) and induction on n. □

Definition 3.1.6. For $n \in \mathbb{N}$ and a word $w \in \mathrm{Cat}_n$, let $(-1)^n D(w)$ denote the coefficient of w in D_n. In other words,

$$D_n = (-1)^n \sum_{w \in \mathrm{Cat}_n} D(w) w. \tag{3.3}$$

Example 3.1.7. In the table below, we list the Catalan words w of length ≤ 6 and the corresponding $D(w)$.

w	$\mathbb{1}$	xy	$xyxy$	$xxyy$	$xyxyxy$	$xxyyxy$	$xyxxyy$	$xxyxyy$	$xxxyyy$
$D(w)$	1	1	1	$[2]_q^2$	1	$[2]_q^2$	$[2]_q^2$	$[2]_q^4$	$[2]_q^2[3]_q^2$

Table 3.1: Example of $D(w)$

By (3.3), our goal of finding a closed formula for D_n reduces to finding a closed formula for $D(w)$ where w is Catalan. The following is the main theorem of this chapter.

Theorem 3.1.8. For $n \in \mathbb{N}$ and a word $w = a_1 \cdots a_{2n} \in \mathrm{Cat}_n$, we have

$$D(w) = \prod_{i=1}^{2n} \left[\overline{a}_1 + \cdots + \overline{a}_{i-1} + (\overline{a}_i + 1)/2\right]_q . \tag{3.4}$$

Moreover,

$$D(w) = E(w)^2, \tag{3.5}$$

where

$$E(w) = \prod_{\substack{1 \leq i \leq 2n \\ a_i = x}} [\overline{a}_1 + \cdots + \overline{a}_i]_q = \prod_{\substack{1 \leq i \leq 2n \\ a_i = y}} [\overline{a}_1 + \cdots + \overline{a}_{i-1}]_q. \tag{3.6}$$

Remark 3.1.9. There is a striking resemblance between (3.4) and [27, Definition 2.5]. While not explicitly used in our proofs, this resemblance did motivate our proof techniques and our interest in this entire topic.

3.2 The proof of Theorem 3.1.8

In this section, we prove Theorem 3.1.8.

Definition 3.2.1. For $n \in \mathbb{N}$ and a word $w = a_1 \cdots a_{2n} \in \mathrm{Cat}_n$, we define

$$\mathcal{D}(w) = \prod_{i=1}^{2n} \left[\overline{a}_1 + \cdots + \overline{a}_{i-1} + (\overline{a}_i + 1)/2\right]_q ,$$

$$\mathcal{D}_x(w) = \prod_{\substack{1\le i\le 2n\\ a_i=x}} [\overline{a}_1+\cdots+\overline{a}_i]_q,$$

$$\mathcal{D}_y(w) = \prod_{\substack{1\le i\le 2n\\ a_i=y}} [\overline{a}_1+\cdots+\overline{a}_{i-1}]_q.$$

In order to prove Theorem 3.1.8, we establish the following for all Catalan words w:

(i) $\mathcal{D}(w) = D(w)$;

(ii) $\mathcal{D}(w) = \mathcal{D}_x(w)\mathcal{D}_y(w)$;

(iii) $\mathcal{D}_x(w) = \mathcal{D}_y(w)$.

Item (i) will be achieved in Theorem 3.2.19.

Item (ii) will be achieved in Lemma 3.2.2.

Item (iii) will be achieved in Lemma 3.2.4.

We first show item (ii).

Lemma 3.2.2. For any Catalan word w, we have

$$\mathcal{D}(w) = \mathcal{D}_x(w)\mathcal{D}_y(w).$$

Proof. Note that $(\overline{x}+1)/2 = 1$ and $(\overline{y}+1)/2 = 0$, so the result follows by Definition 3.2.1. □

Next we show item (iii). For notational convenience, for $n \in \mathbb{N}$ we define

$$[n]_q^! = [n]_q[n-1]_q\cdots[1]_q.$$

We interpret $[0]_q^! = 1$.

Lemma 3.2.3. For a Catalan word w with profile $(l_0, h_1, l_1, \ldots, h_r, l_r)$, we have

$$\mathcal{D}_x(w) = \frac{[h_1]_q^!\cdots[h_r]_q^!}{[l_0]_q^!\cdots[l_r]_q^!},$$

$$\mathcal{D}_y(w) = \frac{[h_1]_q^! \cdots [h_r]_q^!}{[l_0]_q^! \cdots [l_r]_q^!}.$$

Proof. Follows from [27, Lemma 2.10] by direct computation. □

Lemma 3.2.4. For any Catalan word w, we have

$$\mathcal{D}_x(w) = \mathcal{D}_y(w).$$

Proof. Follows from Lemma 3.2.3. □

Lemma 3.2.5. For $n \in \mathbb{N}$ and a word $w \in \mathrm{Cat}_n$ with profile $(l_0, h_1, l_1, \ldots, h_r, l_r)$, we have

$$\mathcal{D}(w) = \mathcal{D}_x(w)^2 = \mathcal{D}_y(w)^2 = \left(\frac{[h_1]_q^! \cdots [h_r]_q^!}{[l_0]_q^! \cdots [l_r]_q^!} \right)^2.$$

Proof. Follows from Lemmas 3.2.2, 3.2.3, 3.2.4. □

Motivated by Lemma 3.2.5, we make the following definition.

Definition 3.2.6. Given a Catalan profile $(l_0, h_1, l_1, \ldots, h_r, l_r)$, define

$$\mathcal{D}(l_0, h_1, l_1, \ldots, h_r, l_r) = \left(\frac{[h_1]_q^! \cdots [h_r]_q^!}{[l_0]_q^! \cdots [l_r]_q^!} \right)^2.$$

Definition 3.2.7. For $n \in \mathbb{N}$, we define

$$\mathcal{D}_n = (-1)^n \sum_{w \in \mathrm{Cat}_n} \mathcal{D}(w) w.$$

We interpret $\mathcal{D}_0 = \mathbb{1}$.

Next we will achieve a recurrence relation involving the $\mathcal{D}_n$. This will be accomplished in Proposition 3.2.13.

Lemma 3.2.8. For a Catalan profile $(l_0, h_1, l_1, \ldots, h_r, l_r)$ with $r \geq 1$,

$$\begin{aligned}&\mathcal{D}(l_0, h_1, l_1, \ldots, h_r, l_r)\\&= \sum_{j=\xi}^{r-1} \mathcal{D}(l_0, h_1, l_1, \ldots, h_j, l_j, h_{j+1}-1, l_{j+1}-1, \ldots, l_{r-1}-1, h_r-1, l_r)\left([h_{j+1}]_q^2 - [l_j]_q^2\right),\end{aligned}$$

where $\xi = \max\{j \mid 0 \leq j \leq r-1, l_j = 0\}$.

Proof. To prove the above equation, consider the quotient of the right-hand side divided by the left-hand side. We will show that this quotient is equal to 1.

By Definition 3.2.6, the above quotient is equal to

$$\begin{aligned}&\sum_{j=\xi}^{r-1} \frac{[l_{j+1}]_q^2 \cdots [l_{r-1}]_q^2}{[h_{j+1}]_q^2 \cdots [h_r]_q^2}\left([h_{j+1}]_q^2 - [l_j]_q^2\right)\\&= \frac{1}{[h_{\xi+1}]_q^2 \cdots [h_r]_q^2} \sum_{j=\xi}^{r-1} [h_{\xi+1}]_q^2 \cdots [h_j]_q^2 [l_{j+1}]_q^2 \cdots [l_{r-1}]_q^2 \left([h_{j+1}]_q^2 - [l_j]_q^2\right)\\&= \frac{1}{[h_{\xi+1}]_q^2 \cdots [h_r]_q^2} \sum_{j=\xi}^{r-1} \left([h_{\xi+1}]_q^2 \cdots [h_{j+1}]_q^2 [l_{j+1}]_q^2 \cdots [l_{r-1}]_q^2 - [h_{\xi+1}]_q^2 \cdots [h_j]_q^2 [l_j]_q^2 \cdots [l_{r-1}]_q^2\right)\\&= \frac{1}{[h_{\xi+1}]_q^2 \cdots [h_r]_q^2}\left([h_{\xi+1}]_q^2 \cdots [h_r]_q^2 - [l_\xi]_q^2 \cdots [l_{r-1}]_q^2\right)\\&= 1,\end{aligned}$$

where the last step follows from $l_\xi = 0$. □

Lemma 3.2.9. For any Catalan word $w = a_1 \cdots a_m$, we have

$$\frac{qx \star w - q^{-1} w \star x}{q - q^{-1}} = \sum_{i=0}^{m} a_1 \cdots a_i x a_{i+1} \cdots a_m [1 + 2\overline{a}_1 + \cdots + 2\overline{a}_i]_q.$$

Proof. By the definition of the q-shuffle product, we have

$$\begin{aligned}
&\frac{qx \star w - q^{-1}w \star x}{q - q^{-1}} \\
&= \sum_{i=0}^{m} a_1 \cdots a_i x a_{i+1} \cdots a_m \, \frac{q^{1+2\overline{a}_1+\cdots+2\overline{a}_i} - q^{-1+2\overline{a}_{i+1}+\cdots+2\overline{a}_m}}{q - q^{-1}} \\
&= \sum_{i=0}^{m} a_1 \cdots a_i x a_{i+1} \cdots a_m \, \frac{q^{1+2\overline{a}_1+\cdots+2\overline{a}_i} - q^{-1-2\overline{a}_1-\cdots-2\overline{a}_i}}{q - q^{-1}} \\
&= \sum_{i=0}^{m} a_1 \cdots a_i x a_{i+1} \cdots a_m [1 + 2\overline{a}_1 + \cdots + 2\overline{a}_i]_q.
\end{aligned}$$

□

For notation convenience, we bring in a bilinear form on $\mathbb{V}$.

Definition 3.2.10. (See [27, Page 6].) Let $(\, , \,) : \mathbb{V} \times \mathbb{V} \to \mathbb{F}$ denote the bilinear form given by $(w, w) = 1$ for a word $w \in \mathbb{V}$ and $(w, v) = 0$ for distinct words $w, v \in \mathbb{V}$.

One can rountinely check that $(\, , \,)$ is symmetric and nondegenerate. For a word $w \in \mathbb{V}$ and any $u \in \mathbb{V}$, the scalar (w, u) is the coefficient of w in u.

Lemma 3.2.11. For any word v and any Catalan word $w = a_1 \cdots a_m$, consider the scalar

$$\left(\frac{(qx \star w - q^{-1}w \star x)y}{q - q^{-1}}, v \right). \tag{3.7}$$

(i) If v is Catalan and of length $m + 2$, then the scalar (4.12) is equal to

$$\sum_i [1 + 2\overline{a}_1 + \cdots + 2\overline{a}_i]_q,$$

where the sum is over all i $(0 \leq i \leq m)$ such that $v = a_1 \cdots a_i x a_{i+1} \cdots a_m y$.

(ii) If v is not Catalan or is not of length $m + 2$, then the scalar(4.12) is equal to 0.

Proof. Follows from Lemma 4.5.2. □

Lemma 3.2.12. For $n \geq 1$ and a word $v \in \mathrm{Cat}_n$, we have

$$\mathcal{D}(v) = \sum_{w \in \mathrm{Cat}_{n-1}} \mathcal{D}(w) \left(\frac{(qx \star w - q^{-1} w \star x)y}{q - q^{-1}}, v \right).$$

Proof. By Lemma 4.5.3, it suffices to show that $\mathcal{D}(v)$ is equal to

$$\sum_{w,i} \mathcal{D}(w)[1 + 2\overline{a}_1 + \cdots + 2\overline{a}_i]_q, \tag{3.8}$$

where the sum is over all ordered pairs (w, i) such that $w = a_1 \cdots a_{2n-2} \in \mathrm{Cat}_{n-1}$ and $v = a_1 \cdots a_i x a_{i+1} \cdots a_{2n-2} y$.

Let $(l_0, h_1, l_1, \ldots, h_r, l_r)$ denote the profile of v and let $\xi = \max\{j \mid 0 \leq j \leq r-1, l_j = 0\}$.

To compute the sum (3.8), we study what kind of words w are being summed over and what is the coefficient for each corresponding $\mathcal{D}(w)$.

For any w being summed over in (3.8), its profile must be of the form

$$(l_0, h_1, l_1, \ldots, h_j, l_j, h_{j+1} - 1, l_{j+1} - 1, \ldots, l_{r-1} - 1, h_r - 1, l_r)$$

for some j such that $\xi \leq j \leq r - 1$. (If $j < \xi$, then the profile of w contains $l_\xi - 1 = -1$, which means w is not Catalan.)

For such w, the coefficient of $\mathcal{D}(w)$ in (3.8) is

$$\sum_{s=l_j}^{h_{j+1}-1} [1 + 2s]_q,$$

which is equal to

$$[h_{j+1}]_q^2 - [l_j]_q^2$$

by direct computation.

Therefore, by Lemma 3.2.8 we have

$$\begin{aligned}
&\sum_{w,i}\mathcal{D}(w)[1+2\overline{a}_1+\cdots+2\overline{a}_i]_q\\
&=\sum_{j=\xi}^{r-1}\mathcal{D}(l_0,h_1,l_1,\ldots,h_j,l_j,h_{j+1}-1,l_{j+1}-1,\ldots,l_{r-1}-1,h_r-1,l_r)\left([h_{j+1}]_q^2-[l_j]_q^2\right)\\
&=\mathcal{D}(l_0,h_1,l_1,\ldots,h_r,l_r)\\
&=\mathcal{D}(v).
\end{aligned}$$

□

Proposition 3.2.13. For $n \geq 1$,

$$\mathcal{D}_n = \frac{(q^{-1}\mathcal{D}_{n-1}\star x - qx\star\mathcal{D}_{n-1})y}{q-q^{-1}}. \tag{3.9}$$

Proof. Given any word v, we will show that its inner product with the right-hand side of (3.9) coincides with $(\mathcal{D}_n, v)$.

If v does not have length $2n$, then the two inner products are both 0.

If v is not Catalan, then $(\mathcal{D}_n, v) = 0$ by Definition 3.2.7, and

$$\left(\frac{(q^{-1}\mathcal{D}_{n-1}\star x - qx\star\mathcal{D}_{n-1})y}{q-q^{-1}}, v\right) = 0$$

by Definition 3.2.7 and Lemma 4.5.3.

If $v \in \mathrm{Cat}_n$, then by Definition 3.2.7 and Lemma 3.2.12,

$$\begin{aligned}
&\left(\frac{(q^{-1}\mathcal{D}_{n-1}\star x - qx\star\mathcal{D}_{n-1})y}{q-q^{-1}}, v\right)\\
&=(-1)^n\sum_{w\in\mathrm{Cat}_{n-1}}\mathcal{D}(w)\left(\frac{(qx\star w-q^{-1}w\star x)y}{q-q^{-1}}, v\right)\\
&=(-1)^n\mathcal{D}(v)\\
&=(\mathcal{D}_n, v).
\end{aligned}$$

□

Definition 3.2.14. (See [26, Definition 9.11].) We define a generating function

$$\mathcal{D}(t) = \sum_{n \in \mathbb{N}} \mathcal{D}_n t^n,$$

where $\mathcal{D}_n$ is from Definition 3.2.7.

Next we will show that $\mathcal{D}(t) = D(t)$. To do this, we will show that $\mathcal{D}(t)$ is the multiplicative inverse of $\tilde{G}(t)$ with respect to $\star$. This will be accomplished in Proposition 3.2.18.

Lemma 3.2.15. For $k \in \mathbb{N}$, we have

$$q\tilde{G}_k \star x = (q - q^{-1})W_{-k} + q^{-1} x \star \tilde{G}_k.$$

Proof. Follows from the definition of $\star$ by direct computation. □

Lemma 3.2.16. For $n \geq 1$,

$$\mathcal{D}_n = -\sum_{k=1}^{n} \tilde{G}_k \star \mathcal{D}_{n-k}. \tag{3.10}$$

Proof. We use induction on n.

First assume that $n = 1$. Then (3.10) holds because

$$\mathcal{D}_0 = \mathbb{1}, \qquad \mathcal{D}_1 = -xy, \qquad \tilde{G}_1 = xy.$$

Next assume that $n \geq 2$. By induction,

$$\mathcal{D}_{n-1} = -\sum_{k=1}^{n-1} \tilde{G}_k \star \mathcal{D}_{n-1-k}. \tag{3.11}$$

In order to prove (3.10), it suffices to show

$$\sum_{k=1}^{n-1} \tilde{G}_k \star \mathcal{D}_{n-k} = -\mathcal{D}_n - \tilde{G}_n. \tag{3.12}$$

For $1 \leq k \leq n-1$ we examine the k-summand in (3.12). We use the following notation: for a word w ending with the letter y, the word wy^{-1} is obtained from w by removing the rightmost y. Furthermore, for a linear combination A of words ending in y, the element Ay^{-1} is obtained from A by removing the rightmost y of each word in the linear combination.

Note that $\tilde{G}_k$ is a word ending in y, and $\mathcal{D}_{n-k}$ is a linear combination of Catalan words which end in y by Definition 2.2.6, so

$$\tilde{G}_k \star \mathcal{D}_{n-k} = (\tilde{G}_k y^{-1} \star \mathcal{D}_{n-k})y + (\tilde{G}_k \star \mathcal{D}_{n-k}y^{-1})y. \tag{3.13}$$

We focus on the second term of the right-hand side of (3.13). By Proposition 3.2.13 and Lemma 3.2.15, we have

$$\begin{aligned}
&\tilde{G}_k \star \mathcal{D}_{n-k}y^{-1} \\
&= -\frac{1}{q-q^{-1}}\tilde{G}_k \star (qx \star \mathcal{D}_{n-k-1} - q^{-1}\mathcal{D}_{n-k-1} \star x) \\
&= -\frac{q}{q-q^{-1}}\tilde{G}_k \star x \star \mathcal{D}_{n-k-1} + \frac{q^{-1}}{q-q^{-1}}\tilde{G}_k \star \mathcal{D}_{n-k-1} \star x \\
&= -W_{-k} \star \mathcal{D}_{n-k-1} - \frac{q^{-1}}{q-q^{-1}} x \star \tilde{G}_k \star \mathcal{D}_{n-k-1} + \frac{q^{-1}}{q-q^{-1}}\tilde{G}_k \star \mathcal{D}_{n-k-1} \star x.
\end{aligned}$$

By the above comment, and since $\tilde{G}_k y^{-1} = W_{-k+1}$, we can write (3.13) as

$$\begin{aligned}
&\tilde{G}_k \star \mathcal{D}_{n-k} \\
&= (W_{-k+1} \star \mathcal{D}_{n-k})y - (W_{-k} \star \mathcal{D}_{n-k-1})y \\
&\qquad - \frac{q^{-1}}{q-q^{-1}}(x \star \tilde{G}_k \star \mathcal{D}_{n-k-1})y + \frac{q^{-1}}{q-q^{-1}}(\tilde{G}_k \star \mathcal{D}_{n-k-1} \star x)y.
\end{aligned}$$

We now sum the above equation over k from 1 to $n-1$, using (3.11) and Proposition

3.2.13. We have

$$\begin{aligned}
&\sum_{k=1}^{n-1} \tilde{G}_k \star \mathcal{D}_{n-k} \\
&= (W_0 \star \mathcal{D}_{n-1})y - (W_{-n+1} \star \mathcal{D}_0)y + \frac{q^{-1}}{q-q^{-1}}(x \star \mathcal{D}_{n-1})y - \frac{q^{-1}}{q-q^{-1}}(\mathcal{D}_{n-1} \star x)y \\
&= (x \star \mathcal{D}_{n-1})y - \tilde{G}_n + \frac{q^{-1}}{q-q^{-1}}(x \star \mathcal{D}_{n-1})y - \frac{q^{-1}}{q-q^{-1}}(\mathcal{D}_{n-1} \star x)y \\
&= \frac{q}{q-q^{-1}}(x \star \mathcal{D}_{n-1})y - \frac{q^{-1}}{q-q^{-1}}(\mathcal{D}_{n-1} \star x)y - \tilde{G}_n \\
&= -\mathcal{D}_n - \tilde{G}_n.
\end{aligned}$$

We have verified (3.12), and (3.10) follows. □

Definition 3.2.17. (See [27, Page 5].) Let $\zeta : \mathbb{V} \to \mathbb{V}$ denote the $\mathbb{F}$-linear map such that

- $\zeta(x) = y$,
- $\zeta(y) = x$,
- For any word $a_1 \cdots a_m$,

$$\zeta(a_1 \cdots a_m) = \zeta(a_m) \cdots \zeta(a_1).$$

By the above definition, ζ is an antiautomorphism on the free algebra $\mathbb{V}$. One can routinely check using the definition of $\star$ that ζ is also an antiautomorphism on the q-shuffle algebra $\mathbb{V}$. Moreover, ζ fixes $\tilde{G}_n$ and $\mathcal{D}_n$ for all $n \in \mathbb{N}$.

Proposition 3.2.18. We have

$$\tilde{G}(t) \star \mathcal{D}(t) = \mathbb{1} = \mathcal{D}(t) \star \tilde{G}(t).$$

Proof. We have $\tilde{G}_0 = \mathbb{1}$ and $\mathcal{D}_0 = \mathbb{1}$. By Lemma 3.2.16, for any $n \geq 1$ we have

$$\sum_{k=0}^{n} \tilde{G}_k \star \mathcal{D}_{n-k} = 0.$$

By these comments,

$$\tilde{G}(t) \star \mathcal{D}(t) = \mathbb{1}\,. \tag{3.14}$$

Applying ζ to (3.14), we have

$$\mathcal{D}(t) \star \tilde{G}(t) = \mathbb{1}\,.$$

□

Theorem 3.2.19. The following hold.

(i) $\mathcal{D}(t) = D(t)$.

(ii) $\mathcal{D}_n = D_n$ for any $n \in \mathbb{N}$.

(iii) $\mathcal{D}(w) = D(w)$ for any Catalan word w.

Proof. Comparing Lemma 3.1.3 and Proposition 3.2.18, we obtain item (i). Item (ii) follows from item (i) by Definitions 3.1.2 and 3.2.14. Item (iii) follows from item (ii) by Definitions 3.1.6 and 3.2.7. □

This finishes our proof of Theorem 3.1.8.

3.3 Some facts about $\{D_n\}_{n\in\mathbb{N}}$

In this section, we state some facts about $\{D_n\}_{n\in\mathbb{N}}$ that we find attractive.

Proposition 3.3.1. (See [26, Lemma 9.7].) For $n \geq 1$,

- D_n is a polynomial in $\tilde{G}_1, \ldots, \tilde{G}_n$ of degree n, where each $\tilde{G}_i$ is given the degree i,
- $\tilde{G}_n$ is a polynomial in $D_1, \ldots, D_n$ of degree n, where each D_i is given the degree i.

Proposition 3.3.2. (See [26, Lemma 9.10].) For $n, m \in \mathbb{N}$,

$$D_n \star \tilde{G}_m = \tilde{G}_m \star D_n, \qquad D_n \star D_m = D_m \star D_n.$$

Proposition 3.3.3. For $n \geq 1$,

$$D_n = \frac{(q^{-1}D_{n-1} \star x - qx \star D_{n-1})y}{q - q^{-1}}. \tag{3.15}$$

Proof. Follows from Proposition 3.2.13 and Theorem 3.2.19. □

Proposition 3.3.4. For $n \geq 1$,

$$D_n = \frac{x(q^{-1}y \star D_{n-1} - qD_{n-1} \star y)}{q - q^{-1}}.$$

Proof. Apply the antiautomorphism ζ to each side of (3.15), and note that D_n is invariant under ζ. □

Recall that for a linear combination A of words ending in y, the element Ay^{-1} is obtained from A by removing the rightmost y of each word. We make a similar notation that for a linear combination B of words starting with x, the element $x^{-1}B$ is obtained from B by removing the leftmost x of each word.

Proposition 3.3.5. For $n \geq 2$,

$$x^{-1}D_n y^{-1} + D_{n-1} = \frac{q^{-1}x^{-1}D_{n-1} \star x - q^3 x \star x^{-1}D_{n-1}}{q - q^{-1}}. \tag{3.16}$$

Proof. By the definition of the q-shuffle product, we have

$$x \star D_{n-1} = xD_{n-1} + q^2 x(x \star x^{-1}D_{n-1}),$$

$$D_{n-1} \star x = xD_{n-1} + x(x^{-1}D_{n-1} \star x).$$

The result follows from Proposition 3.3.3 and the two equations above. □

Proposition 3.3.6. For $n \geq 2$,

$$x^{-1}D_n y^{-1} + D_{n-1} = \frac{q^{-1}y \star D_{n-1}y^{-1} - q^3 D_{n-1}y^{-1} \star y}{q - q^{-1}}.$$

Proof. Apply the antiautomorphism ζ to each side of (3.16), and note that D_n is invariant under ζ. □

Next we mention some PBW bases for U_q^+ that involve $\{D_{n+1}\}_{n\in\mathbb{N}}$. The readers may refer to [26, Definition 2.1] for a formal definition of a PBW basis.

Proposition 3.3.7. The elements $\{W_{-n}\}_{n\in\mathbb{N}}$, $\{D_{n+1}\}_{n\in\mathbb{N}}$, $\{W_{n+1}\}_{n\in\mathbb{N}}$ form a PBW basis for U_q^+ in any linear order that satisfies one of the following:

(i) $W_{-i} < D_{j+1} < W_{k+1}$ for $i, j, k \in \mathbb{N}$;

(ii) $W_{k+1} < D_{j+1} < W_{-i}$ for $i, j, k \in \mathbb{N}$;

(iii) $W_{k+1} < W_{-i} < D_{j+1}$ for $i, j, k \in \mathbb{N}$;

(iv) $W_{-i} < W_{k+1} < D_{j+1}$ for $i, j, k \in \mathbb{N}$;

(v) $D_{j+1} < W_{k+1} < W_{-i}$ for $i, j, k \in \mathbb{N}$;

(vi) $D_{j+1} < W_{-i} < W_{k+1}$ for $i, j, k \in \mathbb{N}$.

Proof. Follows from [26, Theorem 10.1] and Propositions 3.3.1, 3.3.2. □

Proposition 3.3.8. The elements $\{E_{n\delta+\alpha_0}\}_{n\in\mathbb{N}}$, $\{D_{n+1}\}_{n\in\mathbb{N}}$, $\{E_{n\delta+\alpha_1}\}_{n\in\mathbb{N}}$ form a PBW basis for U_q^+ in the following linear order:

$$E_{\alpha_0} < E_{\delta+\alpha_0} < E_{2\delta+\alpha_0} < \cdots < D_1 < D_2 < D_3 < \cdots < E_{2\delta+\alpha_1} < E_{\delta+\alpha_1} < E_{\alpha_1}.$$

Proof. Follows from [9, Section 5], [27, Theorem 1.7], [26, Proposition 11.9], and Proposition 3.3.2. □

Chapter 4

A uniform approach to three PBW bases for U

4.1 Statement of the main results

To motivate our main results in this chapter, we recall some relations involving the elements $\{\tilde{G}_n\}_{n\in\mathbb{N}}$, $\{D_n\}_{n\in\mathbb{N}}$, $\{C_n\}_{n\in\mathbb{N}}$, $\{xC_ny\}_{n\in\mathbb{N}}$. Recall that we have discussed the generating functions $\tilde{G}(t), D(t)$ in Chapter 3. We now bring in another generating function.

Definition 4.1.1. Define the generating function

$$C(t) = \sum_{n\in\mathbb{N}} C_n t^n.$$

We will be discussing the exponential function

$$\exp(z) = \sum_{n\in\mathbb{N}} \frac{z^n}{n!}.$$

Theorem 4.1.2. (See [29, Corollaries 8.1, 8.4 and Proposition 9.11].) The following (i)–(iv) hold.

(i) The elements $\{xC_ny\}_{n\in\mathbb{N}}$ mutually commute with respect to the q-shuffle product.

(ii) $C(t) = \exp\left(\sum_{n=1}^{\infty} \frac{[2n]_q}{n} xC_{n-1}yt^n\right)$.

(iii) $\tilde{G}(t) = \exp\left(-\sum_{n=1}^{\infty} \frac{(-1)^n[n]_q}{n} xC_{n-1}yt^n\right)$.

(iv) $D(t) = \exp\left(\sum_{n=1}^{\infty} \frac{(-1)^n[n]_q}{n} xC_{n-1}yt^n\right)$.

In the above equations, the exponential power series is computed with respect to the q-shuffle product.

In the next result we give a variation on the exponential formulas from Theorem 4.1.2.

Proposition 4.1.3. For $n \geq 1$,

(i) $C_n = \frac{1}{n}\sum_{k=1}^{n}[2k]_q xC_{k-1}y \star C_{n-k}$;

(ii) $\tilde{G}_n = -\frac{1}{n}\sum_{k=1}^{n}(-1)^k[k]_q xC_{k-1}y \star \tilde{G}_{n-k}$;

(iii) $D_n = \frac{1}{n}\sum_{k=1}^{n}(-1)^k[k]_q xC_{k-1}y \star D_{n-k}$.

Proof. (i) Taking the derivative with respect to t on both sides of Theorem 4.1.2(ii) gives

$$\sum_{n=1}^{\infty} nC_nt^{n-1} = \left(\sum_{n=1}^{\infty}[2n]_q xC_{n-1}yt^{n-1}\right) \star C(t).$$

Comparing the coefficients of t^{n-1} in the above equation gives the desired result.

(ii), (iii) Similar to the proof of (i). □

The formulas in Proposition 4.1.3 can be used to recursively compute $\{C_n\}_{n=1}^{\infty}$, $\{\tilde{G}_n\}_{n=1}^{\infty}$, $\{D_n\}_{n=1}^{\infty}$ in terms of $\{xC_ny\}_{n\in\mathbb{N}}$.

The following propositions are immediate consequences of Theorem 4.1.2.

Proposition 4.1.4. (See [27, Corollary 1.8], [26, Proposition 5.10 and Lemma 9.10], [29, Corollary 8.4].) The elements in the set

$$\{C_n\}_{n\in\mathbb{N}} \cup \{\tilde{G}_n\}_{n\in\mathbb{N}} \cup \{D_n\}_{n\in\mathbb{N}} \cup \{xC_ny\}_{n\in\mathbb{N}}$$

mutually commute with respect to the q-shuffle product.

Proof. Follows from Theorem 4.1.2. □

Proposition 4.1.5. (See [26, Proposition 11.8].) We have

$$C(-t) = D(qt) \star D(q^{-1}t).$$

Proof. Follows from Theorem 4.1.2. □

In this chapter we have two main results. We now state our first main result.

Theorem 4.1.6. For $m \geq 1$,

$$\tilde{G}(-q^{m-1}t) \star \tilde{G}(-q^{m-3}t) \star \cdots \star \tilde{G}(-q^{1-m}t) = \exp\left(-\sum_{n=1}^{\infty} \frac{[mn]_q}{n} xC_{n-1}yt^n\right); \tag{4.1}$$

$$D(-q^{m-1}t) \star D(-q^{m-3}t) \star \cdots \star D(-q^{1-m}t) = \exp\left(\sum_{n=1}^{\infty} \frac{[mn]_q}{n} xC_{n-1}yt^n\right). \tag{4.2}$$

In the above equations, the exponential power series is computed with respect to the q-shuffle product.

We now state our second main result.

Theorem 4.1.7. For $m \geq 1$ the following (i), (ii) hold.

(i) For $n \in \mathbb{N}$, the coefficient of t^n in either side of (4.1) is

$$\sum_{a_1a_2\cdots a_{2n} \in \mathrm{Cat}_n} a_1a_2\cdots a_{2n} \prod_{i=1}^{2n} [\overline{a}_1 + \overline{a}_2 + \cdots + \overline{a}_{i-1} - m(\overline{a}_i + 1)/2]_q; \tag{4.3}$$

(ii) For $n \in \mathbb{N}$, the coefficient of t^n in either side of (4.2) is

$$\sum_{a_1a_2\cdots a_{2n} \in \mathrm{Cat}_n} a_1a_2\cdots a_{2n} \prod_{i=1}^{2n} [\overline{a}_1 + \overline{a}_2 + \cdots + \overline{a}_{i-1} + m(\overline{a}_i + 1)/2]_q. \tag{4.4}$$

In this chapter, we will prove Theorems 4.1.6 and 4.1.7. Our proof is from scratch. We will not cite earlier results from the literature. The proof for Theorems 4.1.6 and 4.1.7 takes up most of this chapter. In the course of this proof, we will prove Theorem 4.1.2 from scratch, without citing earlier results from the literature.

In order to prove Theorems 4.1.2, 4.1.6, 4.1.7, we will use the following strategy. We will introduce two types of elements in $\mathbb{V}$, denoted by $\{\Delta_n^{(m)}\}_{m\in\mathbb{Z},n\in\mathbb{N}}$ and $\{\nabla_n^{(m)}\}_{m\in\mathbb{Z},n\geq 1}$. We will develop a uniform theory of these elements. This theory will be used to establish Theorems 4.1.2, 4.1.6, 4.1.7 from scratch. The elements $\{\Delta_n^{(m)}\}_{m\in\mathbb{Z},n\in\mathbb{N}}$ and $\{\nabla_n^{(m)}\}_{m\in\mathbb{Z},n\geq 1}$ will be introduced in Sections 4.2 and 4.3.

4.2 The elements $\Delta_n^{(m)}$

In this section, we introduce the elements $\Delta_n^{(m)}$. These elements are inspired by (4.3) and (4.4).

In (4.3) and (4.4) it was assumed that $m \geq 1$. In the following definition, it is convenient to assume $m \in \mathbb{Z}$.

Definition 4.2.1. Let $m \in \mathbb{Z}$. For $n \in \mathbb{N}$ and a Catalan word $w = a_1a_2\cdots a_{2n}$, define the scalar

$$\Delta^{(m)}(w) = \prod_{i=1}^{2n}[\overline{a}_1 + \overline{a}_2 + \cdots + \overline{a}_{i-1} + m(\overline{a}_i + 1)/2]_q.$$

For $n = 0$, we have $w = \mathbb{1}$. In this case, $\Delta^{(m)}(w) = 1$.

The following formula will be useful.

Lemma 4.2.2. Let $m \in \mathbb{Z}$. For $n \in \mathbb{N}$ and a Catalan word $w = a_1a_2\cdots a_{2n}$,

$$\Delta^{(m)}(w) = \left(\prod_{\substack{1\leq i\leq 2n\\ \overline{a}_i = x}}[\overline{a}_1 + \overline{a}_2 + \cdots + \overline{a}_{i-1} + m]_q\right)\left(\prod_{\substack{1\leq i\leq 2n\\ \overline{a}_i = y}}[\overline{a}_1 + \overline{a}_2 + \cdots + \overline{a}_{i-1}]_q\right). \quad (4.5)$$

Proof. Follows from Definition 4.2.1. □

Example 4.2.3. In the table below, we list the Catalan words w of length ≤ 6 and the corresponding scalars $\Delta^{(m)}(w)$ for $-3 \leq m \leq 3$.

w	$\Delta^{(-3)}(w)$	$\Delta^{(-2)}(w)$	$\Delta^{(-1)}(w)$	$\Delta^{(0)}(w)$	$\Delta^{(1)}(w)$	$\Delta^{(2)}(w)$	$\Delta^{(3)}(w)$
$\mathbb{1}$	1	1	1	1	1	1	1
xy	$-[3]_q$	$-[2]_q$	-1	0	1	$[2]_q$	$[3]_q$
$xyxy$	$[3]_q^2$	$[2]_q^2$	1	0	1	$[2]_q^2$	$[3]_q^2$
$xxyy$	$[2]_q^2[3]_q$	$[2]_q^2$	0	0	$[2]_q^2$	$[2]_q^2[3]_q$	$[2]_q[3]_q[4]_q$
$xyxyxy$	$-[3]_q^3$	$-[2]_q^3$	-1	0	1	$[2]_q^3$	$[3]_q^3$
$xxyyxy$	$-[2]_q^2[3]_q^2$	$-[2]_q^3$	0	0	$[2]_q^2$	$[2]_q^3[3]_q$	$[2]_q[3]_q^2[4]_q$
$xyxxyy$	$-[2]_q^2[3]_q^2$	$-[2]_q^3$	0	0	$[2]_q^2$	$[2]_q^3[3]_q$	$[2]_q[3]_q^2[4]_q$
$xxyxyy$	$-[2]_q^4[3]_q$	$-[2]_q^3$	0	0	$[2]_q^4$	$[2]_q^3[3]_q^2$	$[2]_q^2[3]_q[4]_q^2$
$xxxyyy$	$-[2]_q^2[3]_q^2$	0	0	0	$[2]_q^2[3]_q^2$	$[2]_q^2[3]_q^2[4]_q$	$[2]_q[3]_q^2[4]_q[5]_q$

Table 4.1: Example of $\Delta^{(m)}(w)$

Lemma 4.2.4. Let $m \leq -1$. For $n \in \mathbb{N}$ and a Catalan word $w = a_1 a_2 \cdots a_{2n}$, we have $\Delta^{(m)}(w) \neq 0$ if and only if $\overline{a}_1 + \overline{a}_2 + \cdots + \overline{a}_i \leq |m|$ for $0 \leq i \leq 2n$.

Proof. Note that there exists an integer i $(0 \leq i \leq 2n)$ such that $\overline{a}_1 + \overline{a}_2 + \cdots + \overline{a}_i \geq |m| + 1$ if and only if there exists an integer j $(1 \leq j \leq 2n)$ such that $a_j = x$ and $\overline{a}_1 + \overline{a}_2 + \cdots + \overline{a}_{j-1} = |m|$. Evaluating (4.5) using this comment, we obtain the result. □

Definition 4.2.5. For $m \in \mathbb{Z}$ and $n \in \mathbb{N}$, define

$$\Delta_n^{(m)} = \sum_{w \in \mathrm{Cat}_n} \Delta^{(m)}(w) w.$$

We remark that $\Delta_0^{(m)} = \mathbb{1}$.

Next, we examine the cases $-1 \leq m \leq 2$ in Definition 4.2.5. For convenience, we do this in order $m = 2, 1, -1, 0$.

Lemma 4.2.6. For $n \in \mathbb{N}$,

$$\Delta_n^{(2)} = C_n.$$

Proof. Follows from Definitions 2.2.9 and 4.2.5. □

Example 4.2.7. We will show later in this chapter that for $n \in \mathbb{N}$,

$$\Delta_n^{(1)} = (-1)^n D_n.$$

Lemma 4.2.8. For $n \in \mathbb{N}$,

$$\Delta_n^{(-1)} = (-1)^n \tilde{G}_n.$$

Proof. For $w \in \mathrm{Cat}_n$, by Lemmas 2.3.9 and 4.2.4 we have $\Delta^{(-1)}(w) \neq 0$ if and only if the elevation sequence of w consists of 0 and 1 if and only if $w = \tilde{G}_n$. In this case, $\Delta^{(-1)}(w) = (-1)^n$, and the result follows by Definition 4.2.5. □

Lemma 4.2.9. For $n \in \mathbb{N}$,

$$\Delta_n^{(0)} = \begin{cases} \mathbb{1}, & \text{if } n = 0; \\ 0, & \text{if } n \geq 1. \end{cases}$$

Proof. The case $n = 0$ is trivial.

Now assume $n \geq 1$. For a Catalan word $w = a_1 a_2 \cdots a_{2n}$, by Definition 4.2.1 we have

$$\Delta^{(0)}(w) = \prod_{i=1}^{2n} [\overline{a}_1 + \overline{a}_2 + \cdots + \overline{a}_{i-1}]_q = 0,$$

where the second equality holds because the first factor in the product is $[0]_q$, and $[0]_q = 0$. The result follows by Definition 4.2.5. □

4.3 The elements $\nabla_n^{(m)}$

In this section, we introduce a variation on $\Delta_n^{(m)}$ called $\nabla_n^{(m)}$.

Definition 4.3.1. Let $m \in \mathbb{Z}$. For $n \geq 1$ and a Catalan word $w = a_1 a_2 \cdots a_{2n}$, define the scalar

$$\nabla^{(m)}(w) = \prod_{i=2}^{2n} [\overline{a}_1 + \overline{a}_2 + \cdots + \overline{a}_{i-1} + m(\overline{a}_i + 1)/2]_q.$$

We emphasize that $\nabla^{(m)}(w)$ is not defined for $w = \mathbb{1}$.

Example 4.3.2. In the table below, we list the nontrivial Catalan words w of length ≤ 6 and the corresponding scalars $\nabla^{(m)}(w)$ for $-3 \leq m \leq 3$.

w	$\nabla^{(-3)}(w)$	$\nabla^{(-2)}(w)$	$\nabla^{(-1)}(w)$	$\nabla^{(0)}(w)$	$\nabla^{(1)}(w)$	$\nabla^{(2)}(w)$	$\nabla^{(3)}(w)$
xy	1	1	1	1	1	1	1
$xyxy$	$-[3]_q$	$-[2]_q$	-1	0	1	$[2]_q$	$[3]_q$
$xxyy$	$-[2]_q^2$	$-[2]_q$	0	$[2]_q$	$[2]_q^2$	$[2]_q[3]_q$	$[2]_q[4]_q$
$xyxyxy$	$[3]_q^2$	$[2]_q^2$	1	0	1	$[2]_q^2$	$[3]_q^2$
$xxyyxy$	$[2]_q^2[3]_q$	$[2]_q^2$	0	0	$[2]_q^2$	$[2]_q^2[3]_q$	$[2]_q[3]_q[4]_q$
$xyxxyy$	$[2]_q^2[3]_q$	$[2]_q^2$	0	0	$[2]_q^2$	$[2]_q^2[3]_q$	$[2]_q[3]_q[4]_q$
$xxyxyy$	$[2]_q^4$	$[2]_q^2$	0	$[2]_q^2$	$[2]_q^4$	$[2]_q^2[3]_q^2$	$[2]_q^2[4]_q^2$
$xxxyyy$	$[2]_q^2[3]_q$	0	0	$[2]_q^2[3]_q$	$[2]_q^2[3]_q^2$	$[2]_q[3]_q^2[4]_q$	$[2]_q[3]_q[4]_q[5]_q$

In the following lemma, we compare $\nabla^{(m)}(w)$ and $\Delta^{(m)}(w)$ for a nontrivial Catalan word w.

Lemma 4.3.3. For $m \in \mathbb{Z}$ and a nontrivial Catalan word w,

$$\Delta^{(m)}(w) = [m]_q \nabla^{(m)}(w).$$

Proof. Follows from Definitions 4.2.1 and 4.3.1. □

Definition 4.3.4. For $m \in \mathbb{Z}$ and $n \geq 1$, define

$$\nabla_n^{(m)} = \sum_{w \in \mathrm{Cat}_n} \nabla^{(m)}(w) w.$$

We have a comment on Definition 4.3.4.

Lemma 4.3.5. For $m \in \mathbb{Z}$,

$$\nabla_1^{(m)} = xy.$$

Proof. Follows from Definitions 4.3.1 and 4.3.4. □

In the following lemma we compare $\nabla_n^{(m)}$ and $\Delta_n^{(m)}$.

Lemma 4.3.6. For $m \in \mathbb{Z}$ and $n \geq 1$,

$$\Delta_n^{(m)} = [m]_q \nabla_n^{(m)}.$$

Proof. Follows from Definitions 4.2.5, 4.3.4 and Lemma 4.3.3. □

We have seen that for $m \neq 0$, the elements $\Delta_n^{(m)}$ and $\nabla_n^{(m)}$ agree up to a nonzero scalar factor. Let us examine the case $m = 0$. The elements $\Delta_n^{(0)}$ are given in Lemma 4.2.9. Next we display $\nabla_n^{(0)}$.

Lemma 4.3.7. For $n \geq 1$,

$$\nabla_n^{(0)} = xC_{n-1}y.$$

Proof. For a word $w \in \mathrm{Cat}_n$, by Lemma 2.2.8 we can write $w = xvy$ where v is a word of length $2n-2$. Write $v = a_1 a_2 \cdots a_{2n-2}$.

By Definition 4.3.1,

$$\nabla^{(0)}(w) = \prod_{i=1}^{2n-2} [1 + \overline{a}_1 + \overline{a}_2 + \cdots + \overline{a}_i]_q. \tag{4.6}$$

Moreover, $\nabla^{(0)}(w) = 0$ if and only if there exists an integer i $(1 \leq i \leq 2n-2)$ such that $\overline{a}_1 + \overline{a}_2 + \cdots + \overline{a}_i = -1$ if and only if v is not Catalan.

By the above comment, the map

$$\begin{aligned} \mathrm{Cat}_{n-1} &\to \{w \in \mathrm{Cat}_n \mid \nabla^{(0)}(w) \neq 0\} \\ v &\mapsto xvy \end{aligned}$$

is a bijection.

Therefore, by Definition 4.3.4 we have

$$\begin{aligned}\nabla_n^{(0)} &= \sum_{w\in \mathrm{Cat}_n} \nabla^{(0)}(w)w \\ &= \sum_{v\in \mathrm{Cat}_{n-1}} \nabla^{(0)}(xvy)xvy \\ &= xC_{n-1}y,\end{aligned}$$

where the last step follows by (4.6) and Definition 2.2.9. □

4.4 Some properties of $\nabla^{(m)}(w)$

Let $m \in \mathbb{Z}$ and let w denote a nontrivial Catalan word. In this section, we express $\nabla^{(m)}(w)$ in terms of the profile of w.

Lemma 4.4.1. Let $m \in \mathbb{Z}$. For $n \geq 1$ and a Catalan word $w = a_1a_2\cdots a_{2n}$,

$$\nabla^{(m)}(w) = \left(\prod_{\substack{2\le i\le 2n\\ a_i = x}} [\overline{a}_1 + \overline{a}_2 + \cdots + \overline{a}_{i-1} + m]_q\right)\left(\prod_{\substack{2\le i\le 2n\\ a_i = y}} [\overline{a}_1 + \overline{a}_2 + \cdots + \overline{a}_{i-1}]_q\right). \quad (4.7)$$

Proof. Follows from Definition 4.3.1. □

Observe that in (4.7), the first product depends on m and the second product does not depend on m. The following definition is motivated by this observation.

Definition 4.4.2. Let $m \in \mathbb{Z}$. For $n \geq 1$ and a Catalan word $w = a_1a_2\cdots a_{2n}$, define

$$\nabla_x^{(m)}(w) = \prod_{\substack{2\le i\le 2n\\ a_i = x}} [\overline{a}_1 + \overline{a}_2 + \cdots + \overline{a}_{i-1} + m]_q,$$

$$\nabla_y(w) = \prod_{\substack{2\le i\le 2n\\ a_i = y}} [\overline{a}_1 + \overline{a}_2 + \cdots + \overline{a}_{i-1}]_q.$$

Lemma 4.4.3. For $m \in \mathbb{Z}$ and a nontrivial Catalan word w,

$$\nabla^{(m)}(w) = \nabla_x^{(m)}(w)\nabla_y(w).$$

Proof. Follows from Lemma 4.4.1 and Definition 4.4.2. □

Lemma 4.4.4. Let $k \in \mathbb{Z}$ and let $n \geq 1$. For a balanced word $a_1 a_2 \cdots a_{2n}$, the following sets have the same cardinality:

$$\{1 \leq i \leq 2n \mid a_i = x \text{ and } \overline{a}_1 + \overline{a}_2 + \cdots + \overline{a}_i = k\}, \tag{4.8}$$

$$\{1 \leq i \leq 2n \mid a_i = y \text{ and } \overline{a}_1 + \overline{a}_2 + \cdots + \overline{a}_{i-1} = k\}. \tag{4.9}$$

Proof. We consider the Dyck path of the word $a_1 a_2 \cdots a_{2n}$. We interpret (4.8) as the set of edges in the Dyck path that rise from elevation $k-1$ to elevation k. Similarly, we interpret (4.9) as the set of edges in the Dyck path that fall from elevation k to elevation $k-1$. Recall that the Dyck path of the balanced word $a_1 a_2 \cdots a_{2n}$ starts and ends at elevation 0. The result follows. □

Lemma 4.4.5. For a nontrivial Catalan word w,

$$\nabla_y(w) = \nabla_x^{(1)}(w).$$

Proof. Write $w = a_1 a_2 \cdots a_{2n}$ with $n \geq 1$.

By Definition 4.4.2 with $m = 1$, we have

$$\nabla_x^{(1)}(w) = \prod_{\substack{2 \leq i \leq 2n \\ a_i = x}} [\overline{a}_1 + \overline{a}_2 + \cdots + \overline{a}_i]_q,$$

$$\nabla_y(w) = \prod_{\substack{2 \leq i \leq 2n \\ a_i = y}} [\overline{a}_1 + \overline{a}_2 + \cdots + \overline{a}_{i-1}]_q.$$

Since $a_1 = x$ and $\overline{x} = 1$, we can write

$$\nabla_x^{(1)}(w) = \prod_{\substack{1 \le i \le 2n \\ a_i = x}} [\overline{a}_1 + \overline{a}_2 + \cdots + \overline{a}_i]_q, \tag{4.10}$$

$$\nabla_y(w) = \prod_{\substack{1 \le i \le 2n \\ a_i = y}} [\overline{a}_1 + \overline{a}_2 + \cdots + \overline{a}_{i-1}]_q. \tag{4.11}$$

Comparing (4.10) and (4.11) using Lemma 4.4.4, we obtain the result. □

Corollary 4.4.6. For a nontrivial Catalan word w,

$$\nabla^{(m)}(w) = \nabla_x^{(m)}(w)\nabla_x^{(1)}(w).$$

Proof. Follows from Lemmas 4.4.3 and 4.4.5. □

Corollary 4.4.7. Let $m \in \mathbb{Z}$. For $n \ge 1$ and a Catalan word $w = a_1 a_2 \cdots a_{2n}$,

$$\nabla^{(m)}(w) = \prod_{\substack{2 \le i \le 2n \\ a_i = x}} \Big([\overline{a}_1 + \overline{a}_2 + \cdots + \overline{a}_{i-1} + m]_q [\overline{a}_1 + \overline{a}_2 + \cdots + \overline{a}_{i-1} + 1]_q \Big).$$

Proof. Follows from Definition 4.4.2 and Corollary 4.4.6. □

The following definition is for notational convenience.

Definition 4.4.8. For $m, n \in \mathbb{Z}$, define

$$\left\{ \begin{matrix} m \\ n \end{matrix} \right\}_q = \begin{cases} [m]_q [m-1]_q \cdots [m-n+1]_q, & \text{if } n \ge 1; \\ 1, & \text{if } n = 0; \\ 0, & \text{if } n \le -1. \end{cases}$$

Lemma 4.4.9. For $m \in \mathbb{Z}$ and a nontrivial Catalan word w with profile $(l_0, h_1, l_1, \ldots, h_r, l_r)$,

$$\nabla^{(m)}(w)$$
$$= \begin{Bmatrix} h_1+m-1 \\ h_1-l_0-1 \end{Bmatrix}_q \begin{Bmatrix} h_1 \\ h_1-l_0-1 \end{Bmatrix}_q \begin{Bmatrix} h_2+m-1 \\ h_2-l_1 \end{Bmatrix}_q \begin{Bmatrix} h_2 \\ h_2-l_1 \end{Bmatrix}_q \cdots \begin{Bmatrix} h_r+m-1 \\ h_r-l_{r-1} \end{Bmatrix}_q \begin{Bmatrix} h_r \\ h_r-l_{r-1} \end{Bmatrix}_q.$$

Proof. Follows from Corollary 4.4.7 and Definition 4.4.8. □

Motivated by the above lemma, we make a definition.

Definition 4.4.10. For $m \in \mathbb{Z}$ and a sequence $(l_0, h_1, l_1, \ldots, h_r, l_r)$ of integers with $r \geq 1$, define

$$\nabla^{(m)}(l_0, h_1, l_1, \ldots, h_r, l_r)$$
$$= \begin{Bmatrix} h_1+m-1 \\ h_1-l_0-1 \end{Bmatrix}_q \begin{Bmatrix} h_1 \\ h_1-l_0-1 \end{Bmatrix}_q \begin{Bmatrix} h_2+m-1 \\ h_2-l_1 \end{Bmatrix}_q \begin{Bmatrix} h_2 \\ h_2-l_1 \end{Bmatrix}_q \cdots \begin{Bmatrix} h_r+m-1 \\ h_r-l_{r-1} \end{Bmatrix}_q \begin{Bmatrix} h_r \\ h_r-l_{r-1} \end{Bmatrix}_q.$$

We list four identities for $[n]_q$ that will be useful.

Lemma 4.4.11. The following identities hold.

(i) For $a, b, c \in \mathbb{Z}$,

$$[a+c]_q[b+c]_q - [a]_q[b]_q = [c]_q[a+b+c]_q.$$

(ii) For $a, b, c \in \mathbb{Z}$,

$$[a]_q[b-c]_q + [b]_q[c-a]_q + [c]_q[a-b]_q = 0.$$

(iii) For $a, b, c, d \in \mathbb{Z}$,

$$[a]_q[b]_q[c-d]_q + [b]_q[c]_q[d-a]_q + [c]_q[d]_q[a-b]_q + [d]_q[a]_q[b-c]_q = 0.$$

(iv) For $a, b, c, d \in \mathbb{Z}$,

$$[a]_q[b]_q[a-b]_q + [b]_q[c]_q[b-c]_q + [c]_q[d]_q[c-d]_q + [d]_q[a]_q[d-a]_q = [a-c]_q[b-d]_q[a+c-b-d]_q.$$

Proof. By direct computation. □

4.5 A recurrence relation for $\nabla_n^{(m)}$

Let $m \in \mathbb{Z}$ and $n \geq 1$. In this section, we obtain a recurrence relation that gives $\nabla_{n+1}^{(m)}$ in terms of $\nabla_n^{(m)}$. This will be achieved in Proposition 4.5.5.

We begin with a useful formula about nontrivial Catalan words. To avoid a degenerate situation, we exclude the Catalan word xy.

Lemma 4.5.1. Consider a nontrivial Catalan word other than xy, and let $(l_0, h_1, l_1, \ldots, h_r, l_r)$ denote its profile. Then for $m \in \mathbb{Z}$ we have

$$\begin{aligned}
&\nabla^{(m)}(l_0, h_1, l_1, \ldots, h_r, l_r) \\
&= \sum_{j=\xi}^{r-1} \nabla^{(m)}(l_0, h_1, l_1, \ldots, h_j, l_j, h_{j+1}-1, \ldots, h_r-1, l_r) \\
&\qquad\qquad\qquad \times \Big([h_{j+1}]_q[h_{j+1}+m-1]_q - [l_j]_q[l_j+m-1]_q\Big),
\end{aligned}$$

where $\xi = \max\{j \mid 0 \leq j \leq r-1, l_j = 0\}$.

Proof. If $\xi = r-1$, the result follows by Definition 4.4.10 and the fact that $[l_\xi]_q = 0$.

If $\xi < r-1$, by Definition 4.4.10 and the fact that $[l_\xi]_q = 0$, the right-hand side of the above equation is equal to

$$\begin{aligned}
&\nabla^{(m)}(l_0, h_1, l_1, \ldots, h_\xi, l_\xi, h_{\xi+1}, l_{\xi+1}-1, \ldots, h_r-1, l_r) \\
&+ \sum_{j=\xi+1}^{r-2} \Big(\nabla^{(m)}(l_0, h_1, l_1, \ldots, h_j, l_j, h_{j+1}, l_{j+1}-1, \ldots, h_r-1, l_r) \\
&\qquad\qquad - \nabla^{(m)}(l_0, h_1, l_1, \ldots, h_j, l_j-1, \ldots, h_r-1, l_r)\Big) \\
&+ \Big(\nabla^{(m)}(l_0, h_1, l_1, \ldots, h_r, l_r) - \nabla^{(m)}(l_0, h_1, l_1, \ldots, h_{r-1}, l_{r-1}-1, h_r-1, l_r)\Big).
\end{aligned}$$

The above sum is telescoping. Upon cancellation we obtain the result. □

To see why the word xy is excluded from Lemma 4.5.1, we consider the following interpretation of this lemma. Let $m \in \mathbb{Z}$. For $n \geq 1$ and a word $v \in \mathrm{Cat}_{n+1}$, Lemma 4.5.1 writes $\nabla^{(m)}(v)$ in terms of some $\nabla^{(m)}(w)$ where $w \in \mathrm{Cat}_n$. To include the word xy, we need to extend the result to $n = 0$. But this is not possible since $\nabla^{(m)}(\mathbb{1})$ is not defined.

Lemma 4.5.2. Let $m \in \mathbb{Z}$. For $n \geq 1$ and a Catalan word $w = a_1 a_2 \cdots a_{2n}$,

$$\frac{q^m x \star w - q^{-m} w \star x}{q - q^{-1}} = \sum_{i=0}^{2n} a_1 \cdots a_i x a_{i+1} \cdots a_{2n} [m + 2\overline{a}_1 + 2\overline{a}_2 + \cdots + 2\overline{a}_i]_q.$$

Proof. By the definition of the q-shuffle product in Section 2.1, we have

$$\begin{aligned}
&\frac{q^m x \star w - q^{-m} w \star x}{q - q^{-1}} \\
&= \sum_{i=0}^{2n} a_1 \cdots a_i x a_{i+1} \cdots a_{2n} \frac{q^{m+2\overline{a}_1+\cdots+2\overline{a}_i} - q^{-m+2\overline{a}_{i+1}+\cdots+2\overline{a}_{2n}}}{q - q^{-1}} \\
&= \sum_{i=0}^{2n} a_1 \cdots a_i x a_{i+1} \cdots a_{2n} \frac{q^{m+2\overline{a}_1+\cdots+2\overline{a}_i} - q^{-m-2\overline{a}_1-\cdots-2\overline{a}_i}}{q - q^{-1}} \\
&= \sum_{i=0}^{2n} a_1 \cdots a_i x a_{i+1} \cdots a_{2n} [m + 2\overline{a}_1 + 2\overline{a}_2 + \cdots + 2\overline{a}_i]_q.
\end{aligned}$$

□

For notational convenience, we will be using the bilinear form (,) defined in Definition 3.2.10.

Lemma 4.5.3. Let $m \in \mathbb{Z}$ and let $n \geq 1$. For a word v and a Catalan word $w = a_1 a_2 \cdots a_{2n}$, consider the scalar

$$\left(\frac{(q^m x \star w - q^{-m} w \star x) y}{q - q^{-1}}, v \right). \tag{4.12}$$

(i) If v is Catalan and of length $2n + 2$, then the scalar (4.12) is equal to

$$\sum_i [m + 2\overline{a}_1 + 2\overline{a}_2 + \cdots + 2\overline{a}_i]_q,$$

where the sum is over all i $(0 \leq i \leq 2n)$ such that $v = a_1 \cdots a_i x a_{i+1} \cdots a_{2n} y$.

(ii) If v is not Catalan or is not of length $2n+2$, then the scalar (4.12) is equal to 0.

Proof. Follows from Lemma 4.5.2. □

Lemma 4.5.4. Let $m \in \mathbb{Z}$. For $n \geq 1$ and a word $v \in \mathrm{Cat}_{n+1}$, we have

$$\nabla^{(m)}(v) = \sum_{w \in \mathrm{Cat}_n} \nabla^{(m)}(w) \left(\frac{(q^m x \star w - q^{-m} w \star x) y}{q - q^{-1}}, v \right).$$

Proof. By Lemma 4.5.3, it suffices to show that $\nabla^{(m)}(v)$ is equal to

$$\sum_w \sum_i \nabla^{(m)}(w)[m + 2\overline{a}_1 + 2\overline{a}_2 + \cdots + 2\overline{a}_i]_q, \tag{4.13}$$

where the first sum is over all $w = a_1 a_2 \cdots a_{2n} \in \mathrm{Cat}_n$ such that $v = a_1 \cdots a_i x a_{i+1} \cdots a_{2n} y$ for some i $(0 \leq i \leq 2n)$, and the second sum is over all such i.

Let $(l_0, h_1, l_1, \ldots, h_r, l_r)$ denote the profile of v. Since v is nontrivial, we have $r \geq 1$. Let $\xi = \max\{j \mid 0 \leq j \leq r-1, l_j = 0\}$.

To compute the sum (4.13), we consider what kind of words w are being summed over, and for such w what is the corresponding sum over i.

For any word w being summed over in (4.13), there exists an integer j $(\xi \leq j \leq r-1)$ such that the following hold:

(i) if $l_j < h_{j+1} - 1$ and $j < r-1$, then the profile of w is given by

$$(l_0, h_1, l_1, \ldots, h_j, l_j, h_{j+1} - 1, \ldots, h_r - 1, l_r);$$

(ii) if $l_j < h_{j+1} - 1$ and $j = r-1$, then the profile of w is given by

$$(l_0, h_1, l_1, \ldots, h_j, l_j, h_r - 1, l_r);$$

(iii) if $l_j = h_{j+1} - 1$ and $j < r - 1$, then the profile of w is given by

$$(l_0, h_1, l_1, \ldots, h_j, l_{j+1} - 1, \ldots, h_r - 1, l_r);$$

(iv) if $l_j = h_{j+1} - 1$ and $j = r - 1$, then the profile of w is given by

$$(l_0, h_1, l_1, \ldots, h_j, l_r).$$

For each of the cases above, by Definitions 4.4.8 and 4.4.10 we have that

$$\nabla^{(m)}(w) = \nabla^{(m)}(l_0, h_1, l_1, \ldots, h_j, l_j, h_{j+1} - 1, \ldots, h_r - 1, l_r).$$

For such w, the corresponding sum over i in (4.13) is equal to

$$\begin{aligned}
&\sum_i [m + 2\overline{a}_1 + 2\overline{a}_2 + \cdots + 2\overline{a}_i]_q \\
&= \sum_{s=l_j}^{h_{j+1}-1} [m + 2s]_q \\
&= \sum_{s=l_j}^{h_{j+1}-1} \Big([s+1]_q [s+m]_q - [s]_q [s+m-1]_q \Big) \\
&= [h_{j+1}]_q [h_{j+1} + m - 1]_q - [l_j]_q [l_j + m - 1]_q,
\end{aligned}$$

where the second step follows by setting $a = s, b = s + m - 1, c = 1$ in Lemma 4.4.11(i).

Using the above comments, we now evaluate the entire sum in (4.13). We have

$$\begin{aligned}
&\sum_w \sum_i \nabla^{(m)}(w)[m + 2\overline{a}_1 + 2\overline{a}_2 + \cdots + 2\overline{a}_i]_q \\
&= \sum_{j=\xi}^{r-1} \nabla^{(m)}(l_0, h_1, l_1, \ldots, h_j, l_j, h_{j+1} - 1, \ldots, h_r - 1, l_r) \\
&\qquad\qquad \times \Big([h_{j+1}]_q [h_{j+1} + m - 1]_q - [l_j]_q [l_j + m - 1]_q\Big) \\
&= \nabla^{(m)}(l_0, h_1, l_1, \ldots, h_r, l_r) \\
&= \nabla^{(m)}(v),
\end{aligned}$$

where the second step follows by Lemma 4.5.1. □

Proposition 4.5.5. For $m \in \mathbb{Z}$ and $n \geq 1$,

$$\nabla^{(m)}_{n+1} = \frac{\Big(q^m x \star \nabla^{(m)}_n - q^{-m} \nabla^{(m)}_n \star x\Big)\, y}{q - q^{-1}}. \tag{4.14}$$

Proof. Let v denote a word. We will show that the inner products of v with the both sides of (4.14) coincide.

If v does not have length $2n + 2$, then the two inner products are both 0.

If v is not Catalan, then the two inner products are both 0 by Definition 4.3.4 and Lemma 4.5.3.

If $v \in \mathrm{Cat}_{n+1}$, then by Definition 4.3.4 and Lemma 4.5.4,

$$\begin{aligned}
&\left(\frac{\Big(q^m x \star \nabla^{(m)}_n - q^{-m} \nabla^{(m)}_n \star x\Big)\, y}{q - q^{-1}}, v\right) \\
&= \sum_{w \in \mathrm{Cat}_n} \nabla^{(m)}(w) \left(\frac{(q^m x \star w - q^{-m} w \star x) y}{q - q^{-1}}, v\right) \\
&= \nabla^{(m)}(v) \\
&= \left(\nabla^{(m)}_{n+1}, v\right).
\end{aligned}$$

□

We consider the case $m = 0$ in Proposition 4.5.5. Recall that in Lemma 4.3.7 we have showed that $\nabla_n^{(0)} = xC_{n-1}y$ for all $n \geq 1$.

Corollary 4.5.6. (See [29, Lemma 7.3].) For $n \geq 1$,

$$xC_n = \frac{x \star xC_{n-1}y - xC_{n-1}y \star x}{q - q^{-1}}.$$

Proof. Setting $m = 0$ in Proposition 4.5.5 and applying Lemma 4.3.7 gives

$$xC_n y = \frac{(x \star xC_{n-1}y - xC_{n-1}y \star x)\, y}{q - q^{-1}}.$$

In the above equation, on each side remove the y on the right to obtain the result. □

Next we will show that the elements $\{\nabla_n^{(m)}\}_{m \in \mathbb{Z}, n \geq 1}$ mutually commute with repect to the q-shuffle product. This will be achieved in three steps. Before giving the steps, we have some comments. Recall that above Corollary 4.5.6 we mentioned that $\nabla_n^{(0)} = xC_{n-1}y$ for all $n \geq 1$. Also, by Lemma 4.3.5 we have $\nabla_1^{(m)} = xy$ for all $m \in \mathbb{Z}$. We now give the three steps.

(i) In Section 4.6, we will show that $\nabla_n^{(m)}$ commutes with xy for $m \in \mathbb{Z}$ and $n \geq 1$.

(ii) In Section 4.7, we will show that the elements $\{\nabla_n^{(0)}\}_{n \geq 1}$ mutually commute.

(iii) In Section 4.9, we will show that the elements $\{\nabla_n^{(m)}\}_{m \in \mathbb{Z}, n \geq 1}$ mutually commute.

4.6 $\nabla_n^{(m)}$ commutes with xy

Let $m \in \mathbb{Z}$ and let $n \geq 1$. In this section, we show that $\nabla_n^{(m)}$ commutes with xy with respect to the q-shuffle product.

The following results will be useful.

Lemma 4.6.1. Let $n, k \in \mathbb{N}$. If $v \in \mathrm{Cat}_n$ and $w \in \mathrm{Cat}_k$, then $v \star w$ is a linear combination of words in Cat_{n+k}.

Proof. Follows from the definition of the q-shuffle product in Section 2.1. $\square$

Lemma 4.6.2. Let $m \in \mathbb{Z}$ and let $(l_0, h_1, l_1, \ldots, h_r, l_r)$ denote a sequence of integers with $r \geq 1$. If there exists an integer i $(1 \leq i \leq r-1)$ such that $l_i = -1 \neq h_{i+1}$, then

$$\nabla^{(m)}(l_0, h_1, l_1, \ldots, h_r, l_r) = 0.$$

Proof. Follows from Definitions 4.4.8 and 4.4.10. $\square$

Lemma 4.6.3. Let $m \in \mathbb{Z}$. For $n \geq 1$ and $w \in \mathrm{Cat}_{n+1}$ with profile $(l_0, h_1, l_1, \ldots, h_r, l_r)$, the coefficient of w in $\dfrac{xy \star \nabla_n^{(m)} - \nabla_n^{(m)} \star xy}{q - q^{-1}}$ is equal to

$$\sum_{0 \leq i < j \leq r} \nabla^{(m)}(l_0, h_1, l_1, \ldots, l_i, h_{i+1} - 1, \ldots, h_j - 1, l_j, \ldots, h_r, l_r)$$
$$\times [l_i - h_{i+1}]_q [l_j - h_j]_q [l_i + h_{i+1} - l_j - h_j]_q.$$

Proof. To prove our result, we consider what are the words $v \in \mathrm{Cat}_n$ such that $(xy \star v, w)$ or $(v \star xy, w)$ is nonzero, and for such v what is the coefficient of w in $\dfrac{xy \star v - v \star xy}{q - q^{-1}}$.

For any $v \in \mathrm{Cat}_n$ such that $(xy \star v, w)$ or $(v \star xy, w)$ is nonzero, let us compare the profile of v with the profile of w. By construction, there exists integers i, j $(0 \leq i < j \leq r)$ such that $l_k \geq 1$ for $i < k < j$ and the following hold:

(i) if $l_i < h_{i+1} - 1$, $h_j - 1 > l_j$, and $i < j - 1$, then the profile of v is given by

$$(l_0, h_1, l_1, \ldots, l_i, h_{i+1} - 1, \ldots, h_j - 1, l_j, \ldots, h_r, l_r);$$

(ii) if $l_i < h_{i+1} - 1$, $h_j - 1 > l_j$, and $i = j - 1$, then the profile of v is given by

$$(l_0, h_1, l_1, \ldots, l_i, h_{i+1} - 1, l_j, \ldots, h_r, l_r);$$

(iii) if $l_i = h_{i+1} - 1$, $h_j - 1 > l_j$, and $i < j - 1$, then the profile of v is given by

$$(l_0, h_1, l_1, \ldots, h_i, l_{i+1} - 1, \ldots, h_j - 1, l_j, \ldots, h_r, l_r);$$

(iv) if $l_i = h_{i+1} - 1$, $h_j - 1 > l_j$, and $i = j - 1$, then the profile of v is given by

$$(l_0, h_1, l_1, \ldots, h_i, l_j, \ldots, h_r, l_r);$$

(v) if $l_i < h_{i+1} - 1$, $h_j - 1 = l_j$ and $i < j - 1$, then the profile of v is given by

$$(l_0, h_1, l_1, \ldots, l_i, h_{i+1} - 1, \ldots, l_{j-1} - 1, h_{j+1}, \ldots, h_r, l_r);$$

(vi) if $l_i < h_{i+1} - 1$, $h_j - 1 = l_j$ and $i = j - 1$, then the profile of v is given by

$$(l_0, h_1, l_1, \ldots, l_i, h_{j+1}, \ldots, h_r, l_r);$$

(vii) if $l_i = h_{i+1} - 1$, $h_j - 1 = l_j$, and $i < j - 1$, then the profile of v is given by

$$(l_0, h_1, l_1, \ldots, h_i, l_{i+1} - 1, \ldots, l_{j-1} - 1, h_{j+1}, \ldots, h_r, l_r);$$

(viii) if $l_i = h_{i+1} - 1$, $h_j - 1 = l_j$, and $i = j - 1$, then the profile of v is given by

$$(l_0, h_1, l_1, \ldots, h_i, l_i, h_{j+1}, \ldots, h_r, l_r).$$

For each of the cases above, by Definitions 4.4.8 and 4.4.10 we have that

$$\nabla^{(m)}(v) = \nabla^{(m)}(l_0, h_1, l_1, \ldots, l_i, h_{i+1} - 1, \ldots, h_j - 1, l_j, \ldots, h_r, l_r).$$

For such v, we have that

$$\left(\frac{xy \star v - v \star xy}{q - q^{-1}}, w\right) = \sum_{i,j} [l_i - h_{i+1}]_q [l_j - h_j]_q [l_i + h_{i+1} - l_j - h_j]_q,$$

where the sum is over all i, j such that the profile of v is given in one of the cases (i)-(viii) above.

By the above comments, the coefficient of w in $\dfrac{xy \star \nabla_n^{(m)} - \nabla_n^{(m)} \star xy}{q - q^{-1}}$ is equal to

$$\sum_{\substack{0 \le i < j \le r \\ l_k \ge 1 \text{ for } i<k<j}} \nabla^{(m)}(l_0, h_1, l_1, \ldots, l_i, h_{i+1} - 1, \ldots, h_j - 1, l_j, \ldots, h_r, l_r)$$
$$\times [l_i - h_{i+1}]_q [l_j - h_j]_q [l_i + h_{i+1} - l_j - h_j]_q.$$

By Lemma 4.6.2 the condition

$$l_k \ge 1 \text{ for } i < k < j$$

in the above sum can be dropped. The result follows. □

Lemma 4.6.4. Let $m \in \mathbb{Z}$. For $n \ge 1$ and $w \in \mathrm{Cat}_{n+1}$ with profile $(l_0, h_1, l_1, \ldots, h_r, l_r)$, the coefficient of w in $\dfrac{xy \star \nabla_n^{(m)} - \nabla_n^{(m)} \star xy}{q - q^{-1}}$ is equal to

$$\sum_{1 \le i < j \le r-1} \nabla^{(m)}(l_0, h_1, l_1, \ldots, l_{i-1}, h_i - 1, l_i, h_{i+1} - 1, \ldots, h_j - 1, l_j, h_{j+1} - 1, l_{j+1}, \ldots, h_r, l_r)$$
$$\times [l_i]_q [l_j]_q [h_i]_q [h_{j+1}]_q$$
$$\times \Big([h_i + m - 1]_q [h_{j+1} + m - 1]_q [l_i - l_j]_q + [h_{j+1} + m - 1]_q [l_i + m - 1]_q [l_j - h_i]_q$$
$$+ [l_i + m - 1]_q [l_j + m - 1]_q [h_i - h_{j+1}]_q + [l_j + m - 1]_q [h_i + m - 1]_q [h_{j+1} - l_i]_q \Big)$$
$$+ \sum_{i=1}^{r-1} \nabla^{(m)}(l_0, h_1, l_1, \ldots, l_{i-1}, h_i - 1, l_i, h_{i+1} - 1, l_{i+1}, \ldots, h_r, l_r)$$
$$\times [l_i]_q [h_i]_q [h_{i+1}]_q$$
$$\times \Big([h_{i+1} + m - 1]_q [l_i - h_i]_q + [l_i + m - 1]_q [h_i - h_{i+1}]_q + [h_i + m - 1]_q [h_{i+1} - l_i]_q \Big).$$

Proof. By setting $a = l_i, b = l_j, c = h_{i+1}, d = h_j$ in Lemma 4.4.11(iv), we have

$$[l_i - h_{i+1}]_q[l_j - h_j]_q[l_i + h_{i+1} - l_j - h_j]_q$$
$$= [l_i]_q[l_j]_q[l_i - l_j]_q + [l_j]_q[h_{i+1}]_q[l_j - h_{i+1}]_q + [h_{i+1}]_q[h_j]_q[h_{i+1} - h_j]_q + [h_j]_q[l_i]_q[h_j - l_i]_q.$$

Applying the above equation to Lemma 4.6.3 and shifting indices (note that the boundary terms that come up are always equal to 0) shows that the coefficient of w in $\dfrac{xy \star \nabla_n^{(m)} - \nabla_n^{(m)} \star xy}{q - q^{-1}}$ is equal to

$$\sum_{1 \le i < j \le r-1} \nabla^{(m)}(l_0, h_1, l_1, \ldots, l_i, h_{i+1} - 1, \ldots, h_j - 1, l_j, \ldots, h_r, l_r)[l_i]_q[l_j]_q[l_i - l_j]_q \quad (4.15)$$

$$+ \sum_{1 \le i \le j \le r-1} \nabla^{(m)}(l_0, h_1, l_1, \ldots, l_{i-1}, h_i - 1, \ldots, h_j - 1, l_j, \ldots, h_r, l_r)[l_j]_q[h_i]_q[l_j - h_i]_q \quad (4.16)$$

$$+ \sum_{1 \le i \le j \le r-1} \nabla^{(m)}(l_0, h_1, l_1, \ldots, l_{i-1}, h_i - 1, \ldots, h_{j+1} - 1, l_{j+1}, \ldots, h_r, l_r)[h_i]_q[h_{j+1}]_q[h_i - h_{j+1}]_q \quad (4.17)$$

$$+ \sum_{1 \le i \le j \le r-1} \nabla^{(m)}(l_0, h_1, l_1, \ldots, l_i, h_{i+1} - 1, \ldots, h_{j+1} - 1, l_{j+1}, \ldots, h_r, l_r)[h_{j+1}]_q[l_i]_q[h_{j+1} - l_i]_q. \quad (4.18)$$

For $1 \le i < j \le r - 1$ we examine the (i, j)-summand in the sums (4.15)–(4.18). We will express each of these summands in terms of

$$\nabla^{(m)}(l_0, h_1, l_1, \ldots, l_{i-1}, h_i - 1, l_i, h_{i+1} - 1, \ldots, h_j - 1, l_j, h_{j+1} - 1, l_{j+1}, \ldots, h_r, l_r). \quad (4.19)$$

Using Definition 4.4.10 we obtain:

- the (i, j)-summand in (4.15) is equal to (4.19) times

$$[l_i]_q[l_j]_q[h_i]_q[h_{j+1}]_q[h_i + m - 1]_q[h_{j+1} + m - 1]_q[l_i - l_j]_q;$$

- the (i,j)-summand in (4.16) is equal to (4.19) times

$$[l_i]_q[l_j]_q[h_i]_q[h_{j+1}]_q[h_{j+1}+m-1]_q[l_i+m-1]_q[l_j-h_i]_q;$$

- the (i,j)-summand in (4.17) is equal to (4.19) times

$$[l_i]_q[l_j]_q[h_i]_q[h_{j+1}]_q[l_i+m-1]_q[l_j+m-1]_q[h_i-h_{j+1}]_q;$$

- the (i,j)-summand in (4.18) is equal to (4.19) times

$$[l_i]_q[l_j]_q[h_i]_q[h_{j+1}]_q[l_j+m-1]_q[h_i+m-1]_q[h_{j+1}-l_i]_q.$$

By these comments, For $1 \le i < j \le r-1$ the combined (i,j)-summand in the sums (4.15)–(4.18) is equal to the (i,j)-summand in first sum of the lemma statement.

Next, for $1 \le i = j \le r-1$ we examine the (i,j)-summand in the sums (4.16)–(4.18). We will express each of these summands in terms of

$$\nabla^{(m)}(l_0,h_1,l_1,\ldots,l_{i-1},h_i-1,l_i,h_{i+1}-1,l_{i+1},\ldots,h_r,l_r). \tag{4.20}$$

Using Definition 4.4.10 we obtain:

- the (i,j)-summand in (4.16) is equal to (4.20) times

$$[l_i]_q[h_i]_q[h_{i+1}]_q[h_{i+1}+m-1]_q[l_i-h_i]_q;$$

- the (i,j)-summand in (4.17) is equal to (4.20) times

$$[l_i]_q[h_i]_q[h_{i+1}]_q[l_i+m-1]_q[h_i-h_{i+1}]_q;$$

- the (i, j)-summand in (4.18) is equal to (4.20) times

$$[l_i]_q[h_i]_q[h_{i+1}]_q[h_i + m - 1]_q[h_{i+1} - l_i]_q.$$

By these comments, for $1 \leq i = j \leq r - 1$ the combined (i, j)-summand in the sums (4.16)–(4.18) is equal to the i-summand in the second sum of the lemma statement. □

Proposition 4.6.5. For $m \in \mathbb{Z}$ and $n \geq 1$,

$$xy \star \nabla_n^{(m)} = \nabla_n^{(m)} \star xy.$$

Proof. Referring to either of the sums in Lemma 4.6.4, we will show that for each summand the big parenthesis is equal to 0.

For $1 \leq i < j \leq r-1$, we set $a = h_i + m - 1, b = h_{j+1} + m - 1, c = l_i + m - 1, d = l_j + m - 1$ in Lemma 4.4.11(iii). This yields

$$\begin{aligned}
&[h_i + m - 1]_q[h_{j+1} + m - 1]_q[l_i - l_j]_q + [h_{j+1} + m - 1]_q[l_i + m - 1]_q[l_j - h_i]_q \\
&\qquad + [l_i + m - 1]_q[l_j + m - 1]_q[h_i - h_{j+1}]_q + [l_j + m - 1]_q[h_i + m - 1]_q[h_{j+1} - l_i]_q \\
&= 0.
\end{aligned}$$

For $1 \leq i \leq r - 1$, we set $a = h_{i+1} + m - 1, b = l_i + m - 1, c = h_i + m - 1$ in Lemma 4.4.11(ii). This yields

$$[h_{i+1} + m - 1]_q[l_i - h_i]_q + [l_i + m - 1]_q[h_i - h_{i+1}]_q + [h_i + m - 1]_q[h_{i+1} - l_i]_q = 0.$$

By Lemma 4.6.1, we have that $xy \star \nabla_n^{(m)} - \nabla_n^{(m)} \star xy$ is a linear combination of words in Cat_{n+1}. The result follows. □

Corollary 4.6.6. For $n \geq 1$,

$$xy \star xC_{n-1}y = xC_{n-1}y \star xy.$$

Proof. Setting $m = 0$ in Proposition 4.6.5 and applying Lemma 4.3.7, we obtain the result. $\square$

4.7 The elements $\{\nabla_n^{(0)}\}_{n\geq 1}$ mutually commute

In this section, we show that the elements $\{\nabla_n^{(0)}\}_{n\geq 1}$ mutually commute with respect to the q-shuffle product. Recall from Lemma 4.3.7 that $\nabla_n^{(0)} = xC_{n-1}y$ for all $n \geq 1$.

The following definition is for notational convenience.

Definition 4.7.1. (See [21, Lemma 4.3].) For $n \geq 1$ and a word $w = a_1a_2\cdots a_n$, define

$$wy^{-1} = \begin{cases} 0, & \text{if } a_n = x; \\ a_1a_2\cdots a_{n-1}, & \text{if } a_n = y. \end{cases}$$

We further define $\mathbb{1}\, y^{-1} = 0$. For $v \in \mathbb{V}$, we define vy^{-1} in a linear way.

Example 4.7.2. We have

$$(xxyy - 6xyxy + 2xyyx + 3yxxy - 5yxyx - 4yyxx)y^{-1} = xxy - 6xyx + 3yxx.$$

We make the convention that the y^{-1} notation has higher priority than the q-shuffle product.

Example 4.7.3. We have

$$xy \star xyxyy^{-1} = xy \star xyx, \qquad xyy^{-1} \star xyxy = x \star xyxy.$$

The following result about y^{-1} will be useful.

Lemma 4.7.4. (See [21, Lemma 8.3].) Let v, w be balanced words. Then

$$(v \star w)y^{-1} = vy^{-1} \star w + v \star wy^{-1}.$$

Proof. Follows from the definition of the q-shuffle product in Section 2.1. $\square$

Lemma 4.7.5. For $m \in \mathbb{Z}$ and $n \geq 1$,

$$x \star \nabla_n^{(m)} - \nabla_n^{(m)} \star x = \nabla_n^{(m)} y^{-1} \star xy - xy \star \nabla_n^{(m)} y^{-1}.$$

Proof. In the equation of Proposition 4.6.5, on both sides apply y^{-1} on the right and evaluate the result using Lemma 4.7.4. This yields

$$x \star \nabla_n^{(m)} + xy \star \nabla_n^{(m)} y^{-1} = \nabla_n^{(m)} y^{-1} \star xy + \nabla_n^{(m)} \star x.$$

In this equation, we rearrange the terms to obtain the result. $\square$

Lemma 4.7.6. (See [27, Proposition 2.20].) For $n \geq 1$,

(i) $\nabla_{n+1}^{(0)} y^{-1} = \dfrac{x \star \nabla_n^{(0)} - \nabla_n^{(0)} \star x}{q - q^{-1}}$;

(ii) $\nabla_{n+1}^{(0)} y^{-1} = \dfrac{\nabla_n^{(0)} y^{-1} \star xy - xy \star \nabla_n^{(0)} y^{-1}}{q - q^{-1}}$.

Proof. In (4.14), set $m = 0$ and on both sides apply y^{-1} on the right. This yields (i). Evaluating (i) using Lemma 4.7.5, we obtain (ii). $\square$

Lemma 4.7.7. (See [29, Corollary 8.5].) For $n, k \geq 1$,

$$\nabla_{n+k}^{(0)} y^{-1} = \frac{\nabla_n^{(0)} y^{-1} \star \nabla_k^{(0)} - \nabla_k^{(0)} \star \nabla_n^{(0)} y^{-1}}{q - q^{-1}}. \tag{4.21}$$

Proof. We use induction on n.

First assume $n = 1$. In Lemma 4.7.6(i), substitute n with k and evaluate the result using Lemma 4.3.5. This yields (4.21).

Now assume $n \geq 2$. By induction,

$$\nabla_{n+k-1}^{(0)} y^{-1} = \frac{\nabla_{n-1}^{(0)} y^{-1} \star \nabla_k^{(0)} - \nabla_k^{(0)} \star \nabla_{n-1}^{(0)} y^{-1}}{q - q^{-1}}. \tag{4.22}$$

Using in order Lemma 4.7.6(ii), (4.22), Proposition 4.6.5, and Lemma 4.7.6(ii), we have

$$
\begin{aligned}
&\nabla^{(0)}_{n+k} y^{-1} \\
&= \frac{\nabla^{(0)}_{n+k-1} y^{-1} \star xy - xy \star \nabla^{(0)}_{n+k-1} y^{-1}}{q - q^{-1}} \\
&= \frac{\left(\nabla^{(0)}_{n-1} y^{-1} \star \nabla^{(0)}_{k} - \nabla^{(0)}_{k} \star \nabla^{(0)}_{n-1} y^{-1}\right) \star xy - xy \star \left(\nabla^{(0)}_{n-1} y^{-1} \star \nabla^{(0)}_{k} - \nabla^{(0)}_{k} \star \nabla^{(0)}_{n-1} y^{-1}\right)}{(q - q^{-1})^2} \\
&= \frac{\left(\nabla^{(0)}_{n-1} y^{-1} \star xy - xy \star \nabla^{(0)}_{n-1} y^{-1}\right) \star \nabla^{(0)}_{k} - \nabla^{(0)}_{k} \star \left(\nabla^{(0)}_{n-1} y^{-1} \star xy - xy \star \nabla^{(0)}_{n-1} y^{-1}\right)}{(q - q^{-1})^2} \\
&= \frac{\nabla^{(0)}_{n} y^{-1} \star \nabla^{(0)}_{k} - \nabla^{(0)}_{k} \star \nabla^{(0)}_{n} y^{-1}}{q - q^{-1}}.
\end{aligned}
$$

□

Lemma 4.7.8. (See [29, Corollary 8.4].) For $n, k \geq 1$,

$$\nabla^{(0)}_{n} \star \nabla^{(0)}_{k} = \nabla^{(0)}_{k} \star \nabla^{(0)}_{n}.$$

Proof. Swap n and k in (4.21) and compare the result with (4.21). This yields

$$\nabla^{(0)}_{n} y^{-1} \star \nabla^{(0)}_{k} - \nabla^{(0)}_{k} \star \nabla^{(0)}_{n} y^{-1} = \nabla^{(0)}_{k} y^{-1} \star \nabla^{(0)}_{n} - \nabla^{(0)}_{n} \star \nabla^{(0)}_{k} y^{-1}.$$

Rearranging the terms, we have

$$\nabla^{(0)}_{n} y^{-1} \star \nabla^{(0)}_{k} + \nabla^{(0)}_{n} \star \nabla^{(0)}_{k} y^{-1} = \nabla^{(0)}_{k} y^{-1} \star \nabla^{(0)}_{n} + \nabla^{(0)}_{k} \star \nabla^{(0)}_{n} y^{-1}.$$

Evaluating the above equation using Lemma 4.7.4, we have

$$\left(\nabla^{(0)}_{n} \star \nabla^{(0)}_{k}\right) y^{-1} = \left(\nabla^{(0)}_{k} \star \nabla^{(0)}_{n}\right) y^{-1}.$$

Using Definition 4.3.4, we expand out $\nabla^{(0)}_{n} \star \nabla^{(0)}_{k}$ and $\nabla^{(0)}_{k} \star \nabla^{(0)}_{n}$ and express them as linear combinations of words. Each word is Catalan by Lemma 4.6.1. Each word ends with y by Lemma 2.2.8. The result follows. □

4.8 A relation involving $\{\Delta_n^{(m)}\}_{m\in\mathbb{Z},n\in\mathbb{N}}$ and $\{\nabla_n^{(0)}\}_{n\geq 1}$

In this section, we return our attention to the elements $\{\Delta_n^{(m)}\}_{m\in\mathbb{Z},n\in\mathbb{N}}$ introduced in Section 4.2. We will obtain a relation involving these elements and the elements $\{\nabla_n^{(0)}\}_{n\geq 1}$. This relation will be used in the next section.

To begin, we use Lemma 4.3.6 to reformulate Propositions 4.5.5, 4.6.5 and Lemma 4.7.5 in terms of $\{\Delta_n^{(m)}\}_{m\in\mathbb{Z},n\in\mathbb{N}}$.

Proposition 4.8.1. For $m \in \mathbb{Z}$ and $n \in \mathbb{N}$,

$$\Delta_{n+1}^{(m)} = \frac{\left(q^m x \star \Delta_n^{(m)} - q^{-m}\Delta_n^{(m)} \star x\right) y}{q - q^{-1}}. \tag{4.23}$$

Proof. The case $n = 0$ is routine, and the case $n \geq 1$ follows from Lemma 4.3.6 and Proposition 4.5.5. □

Proposition 4.8.2. For $m \in \mathbb{Z}$ and $n \in \mathbb{N}$,

$$xy \star \Delta_n^{(m)} = \Delta_n^{(m)} \star xy.$$

Proof. The case $n = 0$ is routine, and the case $n \geq 1$ follows from Lemma 4.3.6 and Proposition 4.6.5. □

Lemma 4.8.3. For $m \in \mathbb{Z}$ and $n \in \mathbb{N}$,

$$x \star \Delta_n^{(m)} - \Delta_n^{(m)} \star x = \Delta_n^{(m)} y^{-1} \star xy - xy \star \Delta_n^{(m)} y^{-1}.$$

Proof. The case $n = 0$ is routine, and the case $n \geq 1$ follows from Lemmas 4.3.6 and 4.7.5. □

Lemma 4.8.4. For $m \in \mathbb{Z}$ and $n \in \mathbb{N}$,

(i) $\Delta_{n+1}^{(m)} y^{-1} = [m]_q \sum_{k=0}^{n} q^{-mk} \nabla_{k+1}^{(0)} y^{-1} \star \Delta_{n-k}^{(m)};$

(ii) $\Delta_{n+1}^{(m)} y^{-1} = [m]_q \sum_{k=0}^{n} q^{mk} \Delta_{n-k}^{(m)} \star \nabla_{k+1}^{(0)} y^{-1}$.

Proof. (i) We use induction on n.

The case $n = 0$ is routine.

Now assume $n \geq 1$. By induction,

$$\Delta_n^{(m)} y^{-1} = [m]_q \sum_{k=0}^{n-1} q^{-mk} \nabla_{k+1}^{(0)} y^{-1} \star \Delta_{n-1-k}^{(m)}. \tag{4.24}$$

By Lemma 4.7.6(ii) and Proposition 4.8.2,

$$\begin{aligned}
&[m]_q \sum_{k=0}^{n} q^{-mk} \nabla_{k+1}^{(0)} y^{-1} \star \Delta_{n-k}^{(m)} \\
&= [m]_q x \star \Delta_n^{(m)} + [m]_q \sum_{k=1}^{n} q^{-mk} \nabla_{k+1}^{(0)} y^{-1} \star \Delta_{n-k}^{(m)} \\
&= [m]_q x \star \Delta_n^{(m)} + \frac{[m]_q}{q - q^{-1}} \sum_{k=1}^{n} q^{-mk} \left(\nabla_k^{(0)} y^{-1} \star \Delta_{n-k}^{(m)} \star xy - xy \star \nabla_k^{(0)} y^{-1} \star \Delta_{n-k}^{(m)} \right) \\
&= [m]_q x \star \Delta_n^{(m)} + \frac{q^{-m}[m]_q}{q - q^{-1}} \sum_{k=0}^{n-1} q^{-mk} \left(\nabla_{k+1}^{(0)} y^{-1} \star \Delta_{n-k-1}^{(m)} \star xy - xy \star \nabla_{k+1}^{(0)} y^{-1} \star \Delta_{n-k-1}^{(m)} \right) \\
&= [m]_q x \star \Delta_n^{(m)} + \frac{q^{-m}}{q - q^{-1}} \left(\Delta_n^{(m)} y^{-1} \star xy - xy \star \Delta_n^{(m)} y^{-1} \right),
\end{aligned}$$

where the last step follows by (4.24).

Applying Lemma 4.8.3 to the above result, we have

$$\begin{aligned}
&[m]_q \sum_{k=0}^{n} q^{-mk} \nabla_{k+1}^{(0)} y^{-1} \star \Delta_{n-k}^{(m)} \\
&= [m]_q x \star \Delta_n^{(m)} + \frac{q^{-m}}{q - q^{-1}} \left(x \star \Delta_n^{(m)} - \Delta_n^{(m)} \star x \right) \\
&= \frac{q^m x \star \Delta_n^{(m)} - q^{-m} \Delta_n^{(m)} \star x}{q - q^{-1}} \\
&= \Delta_{n+1}^{(m)} y^{-1},
\end{aligned}$$

where the last step follows by (4.23).

(ii) Similar to the proof of (i). □

The following definition is for notational convenience.

Definition 4.8.5. For $m \in \mathbb{Z}$, define the generating function

$$\Delta^{(m)}(t) = \sum_{n \in \mathbb{N}} \Delta_n^{(m)} t^n.$$

We also define the generating function

$$\nabla^{(0)}(t) = \sum_{n=1}^{\infty} \nabla_n^{(0)} t^n.$$

Proposition 4.8.6. For $m \in \mathbb{Z}$,

(i) $\Delta^{(m)}(t)y^{-1} = q^m [m]_q \nabla^{(0)}(q^{-m}t)y^{-1} \star \Delta^{(m)}(t)$;

(ii) $\Delta^{(m)}(t)y^{-1} = q^{-m} [m]_q \Delta^{(m)}(t) \star \nabla^{(0)}(q^m t)y^{-1}$.

Proof. This is Lemma 4.8.4 expressed in terms of generating functions. □

Referring to Proposition 4.8.6, part (ii) will not be used later in the thesis. It is included for the sake of completeness.

4.9 An exponential formula

Let $m \in \mathbb{Z}$. In this section, we first obtain an exponential formula relating $\Delta^{(m)}(t)$ and $\nabla^{(0)}(t)$. This formula is shown in Theorem 4.9.6. Using this formula, we obtain multiple corollaries that confirm our main results stated in Section 4.1.

Lemma 4.9.1. We have

$$\nabla^{(0)}(t)y^{-1} = tx + \frac{tx \star \nabla^{(0)}(t) - \nabla^{(0)}(t) \star tx}{q - q^{-1}}.$$

Proof. This is Lemma 4.7.6(i) expressed in terms of generating functions. □

We have a comment on Proposition 4.8.1. In (4.23), on both sides apply y^{-1} on the right. This shows that for $m \in \mathbb{Z}$ and $n \in \mathbb{N}$,

$$\Delta_{n+1}^{(m)} y^{-1} = \frac{q^m x \star \Delta_n^{(m)} - q^{-m} \Delta_n^{(m)} \star x}{q - q^{-1}}. \tag{4.25}$$

Lemma 4.9.2. For $m \in \mathbb{Z}$,

$$\Delta^{(m)}(t) y^{-1} = \frac{q^m tx \star \Delta^{(m)}(t) - q^{-m} \Delta^{(m)}(t) \star tx}{q - q^{-1}}.$$

Proof. This is (4.25) expressed in terms of generating functions. □

Lemma 4.9.3. For $m \in \mathbb{Z}$,

$$\left(\frac{d}{dt}\Delta^{(m)}(t)\right) y^{-1} = t^{-1}\Delta^{(m)}(t) y^{-1} + \frac{q^m tx \star \left(\frac{d}{dt}\Delta^{(m)}(t)\right) - q^{-m}\left(\frac{d}{dt}\Delta^{(m)}(t)\right) \star tx}{q - q^{-1}}.$$

Proof. In the equation of Lemma 4.9.2, on both sides take the derivative with respect to t and evaluate the result using the product rule. This yields

$$\begin{aligned}\left(\frac{d}{dt}\Delta^{(m)}(t)\right) y^{-1} = {} & \frac{q^m x \star \Delta^{(m)}(t) - q^{-m}\Delta^{(m)}(t) \star x}{q - q^{-1}} \\ & + \frac{q^m tx \star \left(\frac{d}{dt}\Delta^{(m)}(t)\right) - q^{-m}\left(\frac{d}{dt}\Delta^{(m)}(t)\right) \star tx}{q - q^{-1}}.\end{aligned}$$

Evaluate this equation using Lemma 4.9.2 to obtain the result. □

Lemma 4.9.4. For $m \in \mathbb{Z}$,

(i) $\Big(\nabla^{(0)}(q^m t) \star \Delta^{(m)}(t)\Big) y^{-1} = q^m tx \star \Delta^{(m)}(t)$
$\quad + \dfrac{q^m}{q - q^{-1}} tx \star \nabla^{(0)}(q^m t) \star \Delta^{(m)}(t) - \dfrac{q^{-m}}{q - q^{-1}} \nabla^{(0)}(q^m t) \star \Delta^{(m)}(t) \star tx;$

(ii) $\Big(\nabla^{(0)}(q^{-m} t) \star \Delta^{(m)}(t)\Big) y^{-1} = q^{-m}\Delta^{(m)}(t) \star tx$
$\quad + \dfrac{q^m}{q - q^{-1}} tx \star \nabla^{(0)}(q^{-m} t) \star \Delta^{(m)}(t) - \dfrac{q^{-m}}{q - q^{-1}} \nabla^{(0)}(q^{-m} t) \star \Delta^{(m)}(t) \star tx.$

Proof. (i) Apply Lemma 4.7.4 to the left-hand side and evaluate the result using Lemmas 4.9.1 and 4.9.2.

(ii) Let $\Psi(t)$ denote the left-hand side minus the right-hand side.

Apply Lemma 4.7.4 to $\Psi(t)$ and evaluate the result using Lemmas 4.9.1 and 4.9.2. This yields

$$\begin{aligned}\Psi(t) = q^{-m} tx \star \Delta^{(m)}(t) - q^{-m}\Delta^{(m)}(t) \star tx \\ + [m]_q \Big(\nabla^{(0)}(q^{-m}t) \star tx - tx \star \nabla^{(0)}(q^{-m}t) \Big) \star \Delta^{(m)}(t).\end{aligned}$$

Evaluate the above big parenthesis using Lemma 4.9.1. This yields

$$\begin{aligned}\Psi(t) &= q^{-m} tx \star \Delta^{(m)}(t) - q^{-m}\Delta^{(m)}(t) \star tx + (q^m - q^{-m}) \Big(tx - q^m \nabla^{(0)}(q^{-m}t) y^{-1} \Big) \star \Delta^{(m)}(t) \\ &= q^{m} tx \star \Delta^{(m)}(t) - q^{-m}\Delta^{(m)}(t) \star tx - (q - q^{-1}) q^m [m]_q \nabla^{(0)}(q^{-m}t) y^{-1} \star \Delta^{(m)}(t).\end{aligned}$$

In the above result, evaluate the first two terms using Lemma 4.9.2 and the third term using Proposition 4.8.6(i). This yields

$$\Psi(t) = (q - q^{-1})\Delta^{(m)}(t) y^{-1} - (q - q^{-1})\Delta^{(m)}(t) y^{-1} = 0.$$

□

Lemma 4.9.5. For $m \in \mathbb{Z}$,

$$\frac{d}{dt}\Delta^{(m)}(t) = \frac{\nabla^{(0)}(q^m t) - \nabla^{(0)}(q^{-m}t)}{(q - q^{-1})t} \star \Delta^{(m)}(t).$$

Proof. Define

$$\Phi(t) = \frac{d}{dt}\Delta^{(m)}(t) - \frac{\nabla^{(0)}(q^m t) - \nabla^{(0)}(q^{-m}t)}{(q - q^{-1})t} \star \Delta^{(m)}(t). \tag{4.26}$$

We will show $\Phi(t) = 0$.

Write

$$\Phi(t) = \sum_{n \in \mathbb{N}} \Phi_n t^n.$$

We will show $\Phi_n = 0$ for $n \in \mathbb{N}$.

We claim that

$$\Phi(t)y^{-1} = \frac{q^m tx \star \Phi(t) - q^{-m}\Phi(t) \star tx}{q - q^{-1}}. \tag{4.27}$$

To verify the claim, in (4.27) eliminate $\Phi(t)$ everywhere using (4.26) and evaluate the result using Lemmas 4.9.2–4.9.4. We have proved the claim.

We can now easily show $\Phi_n = 0$ for $n \in \mathbb{N}$. We do this by induction on n.

First we examine the constant term in (4.26). Recall that $\Delta_0^{(m)} = \mathbb{1}$, $\Delta_1^{(m)} = [m]_q xy$, $\nabla_1^{(0)} = xy$. Therefore, $\Phi_0 = 0$.

Next we show that $\Phi_n = 0$ implies $\Phi_{n+1} = 0$ for $n \in \mathbb{N}$.

Let n be given. We compare the coefficient of t^{n+1} on both sides in (4.27). This yields

$$\Phi_{n+1}y^{-1} = \frac{q^m x \star \Phi_n - q^{-m}\Phi_n \star x}{q - q^{-1}} = 0.$$

In (4.26), compare the coefficient of t^{n+1} on both sides. This yields

$$\Phi_{n+1} = (n+2)\Delta_{n+2}^{(m)} - \sum_{k=0}^{n+1} [m(k+1)]_q \nabla_{k+1}^{(0)} \star \Delta_{n+1-k}^{(m)}.$$

We evaluate the right-hand side of the above equation using Definitions 4.2.5 and 4.3.4 and expand out the result as a linear combination of words. Each word is contained in Cat_{n+2} by Lemma 4.6.1. Each word ends with y by Lemma 2.2.8.

By the above comment,

$$\Phi_{n+1} = \Phi_{n+1}y^{-1}y = 0y = 0.$$

We have shown that $\Phi_n = 0$ for $n \in \mathbb{N}$, so $\Phi(t) = 0$. The result follows. □

Theorem 4.9.6. For $m \in \mathbb{Z}$,

$$\Delta^{(m)}(t) = \exp\left(\sum_{n=1}^{\infty} \frac{[mn]_q}{n} \nabla_n^{(0)} t^n\right).$$

In the above equation, the exponential power series is computed with respect to the q-shuffle product.

Proof. Define

$$\Theta(t) = \exp\left(-\sum_{n=1}^{\infty} \frac{[mn]_q}{n} \nabla_n^{(0)} t^n\right).$$

Then $\Theta(t)$ is invertible, with inverse

$$(\Theta(t))^{-1} = \exp\left(\sum_{n=1}^{\infty} \frac{[mn]_q}{n} \nabla_n^{(0)} t^n\right).$$

We will show that $\Delta^{(m)}(t)$ is equal to the inverse of $\Theta(t)$. To do this, it suffices to show that $\Theta(t) \star \Delta^{(m)}(t) = \mathbb{1}$.

By Lemma 4.7.8, the elements $\{\nabla_n^{(0)}\}_{n\geq 1}$ mutually commute. By the chain rule we have

$$\frac{d}{dt}\Theta(t) = \frac{\nabla^{(0)}(q^{-m}t) - \nabla^{(0)}(q^m t)}{(q-q^{-1})t} \star \Theta(t),$$

$$\frac{d}{dt}\Theta(t) = \Theta(t) \star \frac{\nabla^{(0)}(q^{-m}t) - \nabla^{(0)}(q^m t)}{(q-q^{-1})t}. \tag{4.28}$$

By the product rule and (4.28) we have

$$\begin{aligned}
\frac{d}{dt}\left(\Theta(t) \star \Delta^{(m)}(t)\right) &= \frac{d}{dt}\Theta(t) \star \Delta^{(m)}(t) + \Theta(t) \star \frac{d}{dt}\Delta^{(m)}(t) \\
&= \Theta(t) \star \frac{\nabla^{(0)}(q^{-m}t) - \nabla^{(0)}(q^m t)}{(q-q^{-1})t} \star \Delta^{(m)}(t) + \Theta(t) \star \frac{d}{dt}\Delta^{(m)}(t) \\
&= 0,
\end{aligned}$$

where the last step follows by Lemma 4.9.5.

Therefore, $\Theta(t) \star \Delta^{(m)}(t) \in \mathbb{V}$. Since both $\Theta(t)$ and $\Delta^{(m)}(t)$ have constant term $\mathbb{1}$, we have $\Theta(t) \star \Delta^{(m)}(t) = \mathbb{1}$. The result follows. □

Corollary 4.9.7. The elements in the set

$$\{\Delta_n^{(m)}\}_{m\in\mathbb{Z}, n\in\mathbb{N}} \cup \{\nabla_n^{(m)}\}_{m\in\mathbb{Z}, n\geq 1}$$

mutually commute with respect to the q-shuffle product.

Proof. By Theorem 4.9.6, for $m \in \mathbb{Z}$ and $n \geq 1$ the element $\Delta_n^{(m)}$ is a polynomial in the elements $\nabla_1^{(0)}, \nabla_2^{(0)}, \ldots, \nabla_n^{(0)}$. The result follows by Lemmas 4.3.6 and 4.7.8. □

Corollary 4.9.8. For $m \in \mathbb{Z}$,

$$\Delta^{(-m)}(t) \star \Delta^{(m)}(t) = \mathbb{1} = \Delta^{(m)}(t) \star \Delta^{(-m)}(t).$$

Proof. Follows from Theorem 4.9.6. □

Corollary 4.9.9. For $n \in \mathbb{N}$,

$$\Delta_n^{(1)} = (-1)^n D_n.$$

Proof. Setting $m = 1$ in Corollary 4.9.8 gives

$$\Delta^{(-1)}(t) \star \Delta^{(1)}(t) = \mathbb{1} = \Delta^{(1)}(t) \star \Delta^{(-1)}(t).$$

Recall that $D(t)$ is defined to be the inverse of $\tilde{G}(t)$ and that $\Delta_n^{(-1)} = (-1)^n \tilde{G}_n$ for $n \in \mathbb{N}$. The result follows. □

We now explain how the above results imply Theorems 4.1.2, 4.1.6, 4.1.7. In order to do this, we first recall the results of Lemmas 4.2.6, 4.2.8, 4.2.9, 4.3.7 and Corollary 4.9.9. For $n \in \mathbb{N}$,

$$\Delta_n^{(-1)} = (-1)^n \tilde{G}_n, \qquad \Delta_n^{(1)} = (-1)^n D_n, \qquad \Delta_n^{(2)} = C_n,$$

$$\Delta_n^{(0)} = \delta_{n,0}\, \mathbb{1}, \qquad \nabla_{n+1}^{(0)} = x C_n y.$$

In terms of generating functions, we have

$$\Delta^{(-1)}(t) = \tilde{G}(-t), \qquad \Delta^{(1)}(t) = D(-t), \qquad \Delta^{(2)}(t) = C(t),$$

$$\Delta^{(0)}(t) = \mathbb{1}, \qquad \nabla^{(0)}(t) = \sum_{n=1}^{\infty} xC_{n-1}yt^n.$$

Proof of Theorem 4.1.2. Follows by Theorem 4.9.6 and Corollary 4.9.7. □

Proof of Theorem 4.1.6. By direct computation we have that for $n \geq 1$,

$$(q^n)^{m-1} + (q^n)^{m-3} + \cdots + (q^n)^{1-m} = \frac{q^{mn} - q^{-mn}}{q^n - q^{-n}}. \tag{4.29}$$

In order to show (4.1), eliminate each term on the left using Theorem 4.1.2 and evaluate the result using (4.29).

We can show (4.2) using the same method. □

Proof of Theorem 4.1.7. Follows by Theorems 4.1.6 and 4.9.6. □

4.10 Some additional results

For the sake of completeness, we give some additional results involving the elements $\{\nabla_n^{(m)}\}_{m\in\mathbb{Z},n\geq 1}$ and $\{\Delta_n^{(m)}\}_{m\in\mathbb{Z},n\in\mathbb{N}}$.

The following definition will be useful.

Definition 4.10.1. (See [27, Page 5].) Let $\zeta : \mathbb{V} \to \mathbb{V}$ denote the $\mathbb{F}$-linear map such that

- $\zeta(x) = y$ and $\zeta(y) = x$;
- for a word $a_1 \cdots a_n$,

$$\zeta(a_1 \cdots a_n) = \zeta(a_n) \cdots \zeta(a_1).$$

By the above definition, the map ζ is an antiautomorphism on the free algebra $\mathbb{V}$. Moreover, by the definition of the q-shuffle product in Section 2.1, the map ζ is an antiautomorphism on the q-shuffle algebra $\mathbb{V}$. Thus for $v, w \in \mathbb{V}$ we have

$$\zeta(vw) = \zeta(w)\zeta(v), \qquad \zeta(v \star w) = \zeta(w) \star \zeta(v).$$

Lemma 4.10.2. For a Catalan word w the following hold:

(i) $\zeta(w)$ is Catalan;

(ii) assuming that w is nontrivial, then $\nabla^{(m)}(w) = \nabla^{(m)}(\zeta(w))$ for $m \in \mathbb{Z}$;

(iii) $\Delta^{(m)}(w) = \Delta^{(m)}(\zeta(w))$ for $m \in \mathbb{Z}$.

Proof. (i) Follows by Definitions 2.2.6.

(ii) Follows by Lemmas 4.4.1 and 4.4.4.

(iii) The case $w = \mathbb{1}$ is routine, and the case $w \neq \mathbb{1}$ follows from (ii) and Lemma 4.3.3. □

Lemma 4.10.3. The map ζ fixes $\nabla_n^{(m)}$ for $m \in \mathbb{Z}$ and $n \geq 1$. Moreover, ζ fixes $\Delta_n^{(m)}$ for $m \in \mathbb{Z}$ and $n \in \mathbb{N}$.

Proof. Follows from Definitions 4.2.5, 4.3.4 and Lemma 4.10.2. □

Recall the y^{-1} notation from Definition 4.7.1. We now give a similar notation involving x.

Definition 4.10.4. (See [21, Lemma 4.3].) For $n \geq 1$ and a word $w = a_1 a_2 \cdots a_n$, define

$$x^{-1}w = \begin{cases} a_2 a_3 \cdots a_n, & \text{if } a_1 = x; \\ 0, & \text{if } a_1 = y. \end{cases}$$

We further define $x^{-1}\,\mathbb{1} = 0$. For $v \in \mathbb{V}$, we define $x^{-1}v$ in a linear way.

Lemma 4.10.5. For $v \in \mathbb{V}$,

$$\zeta(vy^{-1}) = x^{-1}\zeta(v).$$

Proof. Follows from Definitions 4.7.1 and 4.10.4. □

Using the map ζ and Lemmas 4.10.3, 4.10.5, we obtain the following collection of identities. For completeness, we restate some identities proved earlier in the chapter.

Proposition 4.10.6. For $m \in \mathbb{Z}$ and $n \geq 1$,

$$\nabla_{n+1}^{(m)} = \frac{\left(q^m x \star \nabla_n^{(m)} - q^{-m} \nabla_n^{(m)} \star x\right) y}{q - q^{-1}}, \tag{4.30}$$

$$\nabla_{n+1}^{(m)} = \frac{x \left(q^m \nabla_n^{(m)} \star y - q^{-m} y \star \nabla_n^{(m)}\right)}{q - q^{-1}}. \tag{4.31}$$

Proof. Equation (4.30) is from Proposition 4.5.5. Equation (4.31) is obtained by applying ζ to (4.30). □

Proposition 4.10.7. For $m \in \mathbb{Z}$ and $n \in \mathbb{N}$,

$$\Delta_{n+1}^{(m)} = \frac{\left(q^m x \star \Delta_n^{(m)} - q^{-m} \Delta_n^{(m)} \star x\right) y}{q - q^{-1}}, \tag{4.32}$$

$$\Delta_{n+1}^{(m)} = \frac{x \left(q^m \Delta_n^{(m)} \star y - q^{-m} y \star \Delta_n^{(m)}\right)}{q - q^{-1}}. \tag{4.33}$$

Proof. Equation (4.32) is from Proposition 4.8.1. Equation (4.33) is obtained by applying ζ to (4.32). □

Proposition 4.10.8. For $m \in \mathbb{Z}$,

$$\Delta^{(m)}(t) y^{-1} = q^m [m]_q \nabla^{(0)}(q^{-m} t) y^{-1} \star \Delta^{(m)}(t), \tag{4.34}$$

$$\Delta^{(m)}(t) y^{-1} = q^{-m} [m]_q \Delta^{(m)}(t) \star \nabla^{(0)}(q^m t) y^{-1}, \tag{4.35}$$

$$x^{-1} \Delta^{(m)}(t) = q^m [m]_q \Delta^{(m)}(t) \star x^{-1} \nabla^{(0)}(q^{-m} t), \tag{4.36}$$

$$x^{-1} \Delta^{(m)}(t) = q^{-m} [m]_q x^{-1} \nabla^{(0)}(q^m t) \star \Delta^{(m)}(t). \tag{4.37}$$

Proof. Equations (4.34), (4.35) are from Proposition 4.8.6. Equations (4.36), (4.37) are obtained by applying ζ to (4.34), (4.35) respectively. □

Chapter 5

The doubly alternating words

5.1 Classification of all the words in U

In this section, we will list all the words in U.

The following lemma will be useful.

Lemma 5.1.1. Let w denote a word in $\mathbb{V}$. Then $w \in U$ if and only if w does not contain any of the following segments

$$xxxy, xxyx, xyxx, yxxx, yyyx, yyxy, yxyy, xyyy.$$

Proof. Follows from [21, Lemma 6.5]. $\square$

Using Lemma 5.1.1, we obtain some examples of words in U. We remark that the exponents in these examples are with respect to the free product.

Example 5.1.2. For $n \in \mathbb{N}$, the following words are in U.

$$x^n, \qquad y^n. \tag{5.1}$$

We call the words listed in (5.1) *single*.

Example 5.1.3. For $n \in \mathbb{N}$, the following words are in U.

$$(xy)^{n+1}, \qquad (yx)^{n+1}, \qquad (xy)^n x, \qquad (yx)^n y. \tag{5.2}$$

The words listed in (5.2) are alternating; see Definition 2.2.13.

Example 5.1.4. For $n \in \mathbb{N}$, the following words are in U.

$$(xxyy)^{n+1}, \qquad (yyxx)^{n+1}, \qquad (xxyy)^n xx, \qquad (yyxx)^n yy; \tag{5.3}$$

$$xyy(xxyy)^n, \qquad yxx(yyxx)^n, \qquad x(yyxx)^n, \qquad y(xxyy)^n; \tag{5.4}$$

$$(xxyy)^n xxy, \qquad (yyxx)^n yyx, \qquad (xxyy)^n x, \qquad (yyxx)^n y; \tag{5.5}$$

$$x(yyxx)^n y, \qquad y(xxyy)^n x, \qquad xyy(xxyy)^n x, \qquad yxx(yyxx)^n y. \tag{5.6}$$

Observe that the words listed in (5.3)–(5.6) share a common pattern. Motivated by this observation, we make the following definition.

Definition 5.1.5. A word of the form

$$\cdots xxyyxxyyxxyy \cdots$$

is said to be *doubly alternating.*

There are 16 families of doubly alternating words, depending on the choices of the first and last two letters. These families are listed in (5.3)–(5.6).

Theorem 5.1.6. All the words in U appear in (5.1)–(5.6).

Proof. Let w denote a word in U.

If w has length at most 1, one can check the result routinely. For the rest of this proof, we assume that w has length at least 2.

If w contains only one of the letters x, y, then w is listed in (1). For the rest of this proof, we assume that w contains both the letters x, y. By Lemma 5.1.1, this assumption implies that w does not contain either of the segments xxx, yyy.

If w contains one of the segments xyx, yxy, then w is listed in (5.2) by Lemma 5.1.1. For the rest of this proof, we assume that w does not contain either of the segments xyx, yxy.

We consider the first and last two letters in w. There are $2^4 = 16$ choices of the four letters. Using the above assumptions and Lemma 5.1.1, one can routinely check that these 16 choices are in one-to-one correspondence with the 16 families of words listed in (5.3)–(5.6). □

Theorem 5.1.6 tells us that each word in U is single, alternating, or doubly alternating. The alternating words were studied comprehensively in [26]. We will study the doubly alternating words in the following sections.

5.2 Commutator relations

In this section, we list the commutator relations involving one of x, y and one doubly alternating word with respect to the q-shuffle product.

For $u, v \in \mathbb{V}$ and an invertible scalar $c \in \mathbb{F}$, we write

$$[u, v]_c = cu \star v - c^{-1} v \star u.$$

Proposition 5.2.1. Let $n \in \mathbb{N}$. The following relations hold in U.

$$[(xxyy)^n, x]_{q^2} = (q^2 - q^{-2})(xxyy)^n x; \tag{5.7}$$

$$[x, (yyxx)^n]_{q^2} = (q^2 - q^{-2}) x (yyxx)^n; \tag{5.8}$$

$$[x, (xxyy)^n xx] = 0; \tag{5.9}$$

$$[(yyxx)^n yy, x] = (1 - q^{-4}) \left((yyxx)^n yyx - xyy(xxyy)^n \right). \tag{5.10}$$

Proof. Routine computation. □

Proposition 5.2.2. Let $n \in \mathbb{N}$. The following relations hold in U.

$$[y, (xxyy)^n]_{q^2} = (q^2 - q^{-2})y(xxyy)^n; \tag{5.11}$$

$$[(yyxx)^n, y]_{q^2} = (q^2 - q^{-2})(yyxx)^n y; \tag{5.12}$$

$$[y, (xxyy)^n xx] = (1 - q^{-4})\left(yxx(yyxx)^n - (xxyy)^n xxy\right); \tag{5.13}$$

$$[(yyxx)^n yy, y] = 0. \tag{5.14}$$

Proof. Routine computation. □

Proposition 5.2.3. Let $n \in \mathbb{N}$. The following relations hold in U.

$$[xyy(xxyy)^n, x]_q = (q - q^{-3})\left(xyy(xxyy)^n x - (xxyy)^{n+1}\right); \tag{5.15}$$

$$[x, yxx(yyxx)^n]_q = 0; \tag{5.16}$$

$$[x, x(yyxx)^n]_q = (q^3 - q^{-1})(xxyy)^n xx; \tag{5.17}$$

$$[y(xxyy)^n, x]_q = (q - q^{-3})y(xxyy)^n x. \tag{5.18}$$

Proof. Routine computation. □

Proposition 5.2.4. Let $n \in \mathbb{N}$. The following relations hold in U.

$$[xyy(xxyy)^n, y]_q = 0; \tag{5.19}$$

$$[yxx(yyxx)^n, y]_q = (q - q^{-3})\left(yxx(yyxx)^n y - (yyxx)^{n+1}\right); \tag{5.20}$$

$$[x(yyxx)^n, y]_q = (q - q^{-3})x(yyxx)^n y; \tag{5.21}$$

$$[y, y(xxyy)^n]_q = (q^3 - q^{-1})(yyxx)^n yy. \tag{5.22}$$

Proof. Routine computation. □

Proposition 5.2.5. Let $n \in \mathbb{N}$. The following relations hold in U.

$$[(xxyy)^n xxy, x]_q = 0; \tag{5.23}$$

$$[x, (yyxx)^n yyx]_q = (q - q^{-3}) \left(xyy(xxyy)^n x - (yyxx)^{n+1} \right); \tag{5.24}$$

$$[(xxyy)^n x, x]_q = (q^3 - q^{-1})(xxyy)^n xx; \tag{5.25}$$

$$[x, (yyxx)^n y]_q = (q - q^{-3}) x (yyxx)^n y. \tag{5.26}$$

Proof. Routine computation. □

Proposition 5.2.6. Let $n \in \mathbb{N}$. The following relations hold in U.

$$[y, (xxyy)^n xxy]_q = (q - q^{-3}) \left(yxx(yyxx)^n y - (xxyy)^{n+1} \right); \tag{5.27}$$

$$[(yyxx)^n yyx, y]_q = 0; \tag{5.28}$$

$$[y, (xxyy)^n x]_q = (q - q^{-3}) y (xxyy)^n x; \tag{5.29}$$

$$[(yyxx)^n y, y]_q = (q^3 - q^{-1})(yyxx)^n yy. \tag{5.30}$$

Proof. Routine computation. □

Proposition 5.2.7. Let $n \in \mathbb{N}$. The following relations hold in U.

$$[x, x(yyxx)^n y] = (q^2 - q^{-2})(xxyy)^n xxy; \tag{5.31}$$

$$[y(xxyy)^n x, x] = (q^2 - q^{-2}) yxx(yyxx)^n; \tag{5.32}$$

$$[x, xyy(xxyy)^n x] = (q^2 - q^{-2}) \left((xxyy)^{n+1} x - x(yyxx)^{n+1} \right); \tag{5.33}$$

$$[x, yxx(yyxx)^n y] = 0. \tag{5.34}$$

Proof. Routine computation. □

Proposition 5.2.8. Let $n \in \mathbb{N}$. The following relations hold in U.

$$[x(yyxx)^n y, y] = (q^2 - q^{-2})xyy(xxyy)^n; \tag{5.35}$$

$$[y, y(xxyy)^n x] = (q^2 - q^{-2})(yyxx)^n yyx; \tag{5.36}$$

$$[xyy(xxyy)^n x, y] = 0; \tag{5.37}$$

$$[yxx(yyxx)^n y, y] = (q^2 - q^{-2})\left(y(xxyy)^{n+1} - (yyxx)^{n+1}y\right). \tag{5.38}$$

Proof. Routine computation. □

5.3 The doubly alternating words in terms of the alternating words

In this section, we will write the doubly alternating words as polynomials in the alternating words with respect to the q-shuffle product.

The following lemma will be useful.

Lemma 5.3.1. Let $v_1, v_2 \in \mathbb{V}$. We have $v_1 = v_2$ if and only if at least one of the following conditions hold:

(i) $v_1 x^{-1} = v_2 x^{-1}$ and $v_1 y^{-1} = v_2 y^{-1}$;

(ii) $x^{-1} v_1 = x^{-1} v_2$ and $y^{-1} v_1 = y^{-1} v_2$.

Proof. Routine. □

Proposition 5.3.2. For $n \in \mathbb{N}$,

$$\sum_{k=0}^{2n} (-1)^k \tilde{G}_k \star \tilde{G}_{2n-k} = (-1)^n [2]_q^{2n} (xxyy)^n, \tag{5.39}$$

$$\sum_{k=0}^{2n}(-1)^k G_k \star G_{2n-k} = (-1)^n [2]_q^{2n} (yyxx)^n, \tag{5.40}$$

$$\sum_{k=0}^{2n}(-1)^k W_{-k} \star W_{k-2n} = (-1)^n q[2]_q^{2n+1} (xxyy)^n xx, \tag{5.41}$$

$$\sum_{k=0}^{2n}(-1)^k W_{k+1} \star W_{2n+1-k} = (-1)^n q[2]_q^{2n+1} (yyxx)^n yy. \tag{5.42}$$

Proof. We first show (5.39) and (5.41) by induction on n.

Clearly (5.39) and (5.41) hold for $n = 0$. Next, we will show the following results for $n \in \mathbb{N}$:

(i) if (5.41) holds for n, then (5.39) holds for $n+1$;

(ii) if (5.39) holds for $n+1$, then (5.41) holds for $n+1$.

We first show (i). Assume that (5.41) holds for n. To show that (5.39) holds for $n+1$, by Lemma 5.3.1(i) it suffices to show that

$$\sum_{k=0}^{2n+1}(-1)^k \tilde{G}_k \star W_{k-2n-1} + \sum_{k=1}^{2n+2}(-1)^k W_{1-k} \star \tilde{G}_{2n+2-k} = (-1)^{n+1} [2]_q^{2n+2} (xxyy)^n xxy. \tag{5.43}$$

To show (5.43), by Lemma 5.3.1(i) it suffices to show that

$$\sum_{k=0}^{2n+1}(-1)^k \tilde{G}_k \star \tilde{G}_{2n+1-k} + \sum_{k=1}^{2n+2}(-1)^k \tilde{G}_{k-1} \star \tilde{G}_{2n+2-k} = 0 \tag{5.44}$$

and

$$\sum_{k=1}^{2n+1}(-1)^k \left(q^{-2} W_{1-k} \star W_{k-2n-1} + W_{1-k} \star W_{k-2n-1}\right) = (-1)^{n+1} [2]_q^{2n+2} (xxyy)^n xx. \tag{5.45}$$

Note that (5.44) can be checked by routine computation and (5.45) follows from the inductive hypothesis. Consequently (5.43) holds and (5.39) holds for $n+1$. We have proved (i).

We now show (ii). Assume that (5.39) holds for $n+1$. To show that (5.41) holds for $n+1$, by Lemma 5.3.1(i) it suffices to show that

$$\sum_{k=0}^{2n+2}(-1)^k\left(W_{-k}\star\tilde{G}_{2n+2-k}+q^2\tilde{G}_k\star W_{k-2n-2}\right)=(-1)^{n+1}q[2]_q^{2n+3}(xxyy)^{n+1}x. \quad (5.46)$$

To show (5.46), by Lemma 5.3.1(i) it suffices to show that

$$\sum_{k=0}^{2n+2}(-1)^k\left(\tilde{G}_k\star\tilde{G}_{2n+2-k}+q^2\tilde{G}_k\star\tilde{G}_{2n+2-k}\right)=(-1)^{n+1}q[2]_q^{2n+3}(xxyy)^{n+1} \quad (5.47)$$

and

$$\sum_{k=0}^{2n+1}(-1)^kW_{-k}\star W_{k-2n-1}+\sum_{k=1}^{2n}(-1)^kW_{1-k}\star W_{k-2n-2}=0. \quad (5.48)$$

Note that (5.47) follows from the inductive hypothesis and (5.48) can be checked by routine computation. Consequently (5.46) holds and (5.41) holds for $n+1$. We have proved (ii).

By the above discussions, we have proved (5.39) and (5.41). Similarly, we can prove (5.40) and (5.42) using Lemma 5.3.1(ii) and induction on n. □

Proposition 5.3.3. For $n\in\mathbb{N}$,

$$\sum_{k=0}^{2n+1}(-1)^k\tilde{G}_k\star\tilde{G}_{2n+1-k}=0, \quad (5.49)$$

$$\sum_{k=0}^{2n+1}(-1)^kG_k\star G_{2n+1-k}=0, \quad (5.50)$$

$$\sum_{k=0}^{2n+1}(-1)^kW_{-k}\star W_{k-2n-1}=0, \quad (5.51)$$

$$\sum_{k=0}^{2n+1}(-1)^kW_{k+1}\star W_{2n+2-k}=0. \quad (5.52)$$

Proof. Routine computation. □

Proposition 5.3.4. For $n \in \mathbb{N}$,

$$\begin{aligned}\sum_{k=0}^{2n}(-1)^k W_{-k} \star \tilde{G}_{2n-k} &= (-1)^n [2]_q^{2n} (xxyy)^n x \\ &= \sum_{k=0}^{2n}(-1)^k \tilde{G}_{2n-k} \star W_{-k},\end{aligned} \tag{5.53}$$

$$\begin{aligned}q^{-1}\sum_{k=0}^{2n+1}(-1)^k W_{-k} \star \tilde{G}_{2n+1-k} &= (-1)^n [2]_q^{2n+1} (xxyy)^n xxy \\ &= q\sum_{k=0}^{2n+1}(-1)^k \tilde{G}_{2n+1-k} \star W_{-k}.\end{aligned} \tag{5.54}$$

Proof. Note that

$$\begin{aligned}&\left(\sum_{k=0}^{2n}(-1)^k W_{-k} \star \tilde{G}_{2n-k}\right) x^{-1} \\ &= \sum_{k=0}^{2n}(-1)^k \tilde{G}_{k} \star \tilde{G}_{2n-k} \\ &= (-1)^n [2]_q^{2n} (xxyy)^n\end{aligned}$$

by (5.39), and

$$\begin{aligned}&\left(\sum_{k=0}^{2n}(-1)^k W_{-k} \star \tilde{G}_{2n-k}\right) y^{-1} \\ &= \sum_{k=0}^{2n-1}(-1)^k W_{-k} \star W_{k-2n+1} \\ &= 0\end{aligned}$$

by (5.51).

By Lemma 5.3.1(i), we have proved the first equality in (5.53). The remaining equalities can be proved in a similar way. □

Proposition 5.3.5. For $n \in \mathbb{N}$,

$$\begin{aligned}\sum_{k=0}^{2n}(-1)^k W_{-k} \star G_{2n-k} &= (-1)^n [2]_q^{2n} x(yyxx)^n \\ &= \sum_{k=0}^{2n}(-1)^k G_{2n-k} \star W_{-k},\end{aligned} \tag{5.55}$$

$$\begin{aligned}q\sum_{k=0}^{2n+1}(-1)^k W_{-k} \star G_{2n+1-k} &= (-1)^n [2]_q^{2n+1} yxx(yyxx)^n \\ &= q^{-1}\sum_{k=0}^{2n+1}(-1)^k G_{2n+1-k} \star W_{-k}.\end{aligned} \tag{5.56}$$

Proof. Similar to the proof of Proposition 5.3.4. □

Proposition 5.3.6. For $n \in \mathbb{N}$,

$$\begin{aligned}\sum_{k=0}^{2n}(-1)^k W_{k+1} \star \tilde{G}_{2n-k} &= (-1)^n [2]_q^{2n} y(xxyy)^n \\ &= \sum_{k=0}^{2n}(-1)^k \tilde{G}_{2n-k} \star W_{k+1},\end{aligned} \tag{5.57}$$

$$\begin{aligned}q\sum_{k=0}^{2n+1}(-1)^k W_{k+1} \star \tilde{G}_{2n+1-k} &= (-1)^n [2]_q^{2n+1} xyy(xxyy)^n \\ &= q^{-1}\sum_{k=0}^{2n+1}(-1)^k \tilde{G}_{2n+1-k} \star W_{k+1}.\end{aligned} \tag{5.58}$$

Proof. Similar to the proof of Proposition 5.3.4. □

Proposition 5.3.7. For $n \in \mathbb{N}$,

$$\begin{aligned}\sum_{k=0}^{2n}(-1)^k W_{k+1} \star G_{2n-k} &= (-1)^n [2]_q^{2n} (yyxx)^n y \\ &= \sum_{k=0}^{2n}(-1)^k G_{2n-k} \star W_{k+1},\end{aligned} \tag{5.59}$$

$$q^{-1}\sum_{k=0}^{2n+1}(-1)^k W_{k+1}\star G_{2n+1-k} = (-1)^n[2]_q^{2n+1}(yyxx)^n yyx$$
$$= q\sum_{k=0}^{2n+1}(-1)^k G_{2n+1-k}\star W_{k+1}. \tag{5.60}$$

Proof. Similar to the proof of Proposition 5.3.4. □

Proposition 5.3.8. For $n \in \mathbb{N}$,

$$\sum_{k=0}^{2n+2}(-1)^k G_k\star \tilde{G}_{2n+2-k} = (-1)^{n+1}[2]_q^{2n+1}\big(q^{-1}xyy(xxyy)^n x + qy(xxyy)^n xxy\big), \tag{5.61}$$

$$\sum_{k=0}^{2n+2}(-1)^k \tilde{G}_{2n+2-k}\star G_k = (-1)^{n+1}[2]_q^{2n+1}\big(qxyy(xxyy)^n x + q^{-1}y(xxyy)^n xxy\big), \tag{5.62}$$

$$\sum_{k=0}^{2n+1}(-1)^k G_k\star \tilde{G}_{2n+1-k} = (-1)^n[2]_q^{2n}\big(x(yyxx)^n y - y(xxyy)^n x\big), \tag{5.63}$$

$$\sum_{k=0}^{2n+1}(-1)^k \tilde{G}_{2n+1-k}\star G_k = (-1)^n[2]_q^{2n}\big(x(yyxx)^n y - y(xxyy)^n x\big). \tag{5.64}$$

Proof. Similar to the proof of Proposition 5.3.4. □

Proposition 5.3.9. For $n \in \mathbb{N}$,

$$\sum_{k=0}^{2n}(-1)^k W_{k+1}\star W_{k-2n} = (-1)^n[2]_q^{2n}\big(q^{-2}x(yyxx)^n y + y(xxyy)^n x\big), \tag{5.65}$$

$$\sum_{k=0}^{2n}(-1)^k W_{k-2n}\star W_{k+1} = (-1)^n[2]_q^{2n}\big(x(yyxx)^n y + q^{-2}y(xxyy)^n x\big), \tag{5.66}$$

$$\sum_{k=0}^{2n+1}(-1)^k W_{k+1}\star W_{k-2n-1} = (-1)^n q^{-1}[2]_q^{2n+1}\big(xyy(xxyy)^n x - yxx(yyxx)^n y\big), \tag{5.67}$$

$$\sum_{k=0}^{2n+1}(-1)^k W_{k-2n-1}\star W_{k+1} = (-1)^n q^{-1}[2]_q^{2n+1}\big(xyy(xxyy)^n x - yxx(yyxx)^n y\big). \tag{5.68}$$

Proof. Similar to the proof of Proposition 5.3.4. □

Note that (5.39)–(5.42) and (5.53)–(5.60) write the doubly alternating words listed in

(5.3)–(5.5) as polynomials in the alternating words. Next, we will write the doubly alternating words listed in (5.6) as polynomials in the alternating words.

Corollary 5.3.10. Let $n \in \mathbb{N}$. We have

$$(-1)^n[2]_q^{2n+1}x(yyxx)^n y = (1-q^{-2})^{-1}\sum_{k=0}^{2n}(-1)^k[W_{k-2n}, W_{k+1}]_q, \tag{5.69}$$

$$(-1)^n[2]_q^{2n+1}y(xxyy)^n x = (1-q^{-2})^{-1}\sum_{k=0}^{2n}(-1)^k[W_{k+1}, W_{k-2n}]_q, \tag{5.70}$$

$$\begin{aligned}\sum_{k=0}^{2n+1}(-1)^k G_k \star \tilde{G}_{2n+1-k} &= (1-q^{-2})^{-1}\sum_{k=0}^{2n}(-1)^k[W_{k-2n}, W_{k+1}] \\ &= \sum_{k=0}^{2n+1}(-1)^k \tilde{G}_{2n+1-k} \star G_k.\end{aligned} \tag{5.71}$$

There are infinitely many ways to write the words $x(yyxx)^n y$ and $y(xxyy)^n x$ as polynomials in the alternating words. These polynomials can be obtained using (5.69)–(5.71).

Proof. Follows from (5.63)–(5.66). □

Corollary 5.3.11. Let $n \in \mathbb{N}$. We have

$$(-1)^{n+1}[2]_q^{2n+2}xyy(xxyy)^n x = (q-q^{-1})^{-1}\sum_{k=0}^{2n+2}(-1)^k[\tilde{G}_{2n+2-k}, G_k]_q, \tag{5.72}$$

$$(-1)^{n+1}[2]_q^{2n+2}yxx(yyxx)^n y = (q-q^{-1})^{-1}\sum_{k=0}^{2n+2}(-1)^k[G_k, \tilde{G}_{2n+2-k}]_q. \tag{5.73}$$

$$\begin{aligned}\sum_{k=0}^{2n+1}(-1)^k W_{k+1} \star W_{k-2n-1} &= (q^2-1)^{-1}\sum_{k=0}^{2n+2}(-1)^k[G_k, \tilde{G}_{2n+2-k}] \\ &= \sum_{k=0}^{2n+1}(-1)^k W_{k-2n-1} \star W_{k+1}.\end{aligned} \tag{5.74}$$

There are infinitely many ways to write the words $xyy(xxyy)^n x$ and $yxx(yyxx)^n y$ as polynomials in the alternating words. These polynomials can be obtained using (5.72)–(5.74).

Proof. Follows from (5.61), (5.62), (5.67), (5.68). □

5.4 Generating functions

In this section, we write the results in Section 5.3 in terms of generating functions.

Proposition 5.4.1. We have

$$\tilde{G}(-t) \star \tilde{G}(t) = \sum_{n\in\mathbb{N}} (-1)^n [2]_q^{2n} (xxyy)^n t^{2n}, \tag{5.75}$$

$$G(-t) \star G(t) = \sum_{n\in\mathbb{N}} (-1)^n [2]_q^{2n} (yyxx)^n t^{2n}, \tag{5.76}$$

$$W^-(-t) \star W^-(t) = \sum_{n\in\mathbb{N}} (-1)^n q [2]_q^{2n+1} (xxyy)^n xx t^{2n}, \tag{5.77}$$

$$W^+(-t) \star W^+(t) = \sum_{n\in\mathbb{N}} (-1)^n q [2]_q^{2n+1} (yyxx)^n yy t^{2n}. \tag{5.78}$$

Proof. This is Propositions 5.3.2 and 5.3.3 in terms of generating functions. □

Proposition 5.4.2. We have

$$\begin{aligned} W^-(-t) \star \tilde{G}(t) = &\sum_{n\in\mathbb{N}} (-1)^n [2]_q^{2n} (xxyy)^n x t^{2n} \\ &+ q \sum_{n\in\mathbb{N}} (-1)^n [2]_q^{2n+1} (xxyy)^n xxy t^{2n+1}, \end{aligned} \tag{5.79}$$

$$\begin{aligned} \tilde{G}(t) \star W^-(-t) = &\sum_{n\in\mathbb{N}} (-1)^n [2]_q^{2n} (xxyy)^n x t^{2n} \\ &+ q^{-1} \sum_{n\in\mathbb{N}} (-1)^n [2]_q^{2n+1} (xxyy)^n xxy t^{2n+1}. \end{aligned} \tag{5.80}$$

Proof. This is Propositions 5.3.4 in terms of generating functions. □

Proposition 5.4.3. We have

$$\begin{aligned} W^-(-t) \star G(t) &= \sum_{n\in\mathbb{N}} (-1)^n [2]_q^{2n} x(yyxx)^n t^{2n} \\ &\quad + q^{-1} \sum_{n\in\mathbb{N}} (-1)^n [2]_q^{2n+1} yxx(yyxx)^n t^{2n+1}, \end{aligned} \tag{5.81}$$

$$\begin{aligned} G(t) \star W^-(-t) &= \sum_{n\in\mathbb{N}} (-1)^n [2]_q^{2n} x(yyxx)^n t^{2n} \\ &\quad + q \sum_{n\in\mathbb{N}} (-1)^n [2]_q^{2n+1} yxx(yyxx)^n t^{2n+1}. \end{aligned} \tag{5.82}$$

Proof. This is Propositions 5.3.5 in terms of generating functions. □

Proposition 5.4.4. We have

$$\begin{aligned} W^+(-t) \star \tilde{G}(t) &= \sum_{n\in\mathbb{N}} (-1)^n [2]_q^{2n} y(xxyy)^n t^{2n} \\ &\quad + q^{-1} \sum_{n\in\mathbb{N}} (-1)^n [2]_q^{2n+1} xyy(xxyy)^n t^{2n+1}, \end{aligned} \tag{5.83}$$

$$\begin{aligned} \tilde{G}(t) \star W^+(-t) &= \sum_{n\in\mathbb{N}} (-1)^n [2]_q^{2n} y(xxyy)^n t^{2n} \\ &\quad + q \sum_{n\in\mathbb{N}} (-1)^n [2]_q^{2n+1} xyy(xxyy)^n t^{2n+1}. \end{aligned} \tag{5.84}$$

Proof. This is Propositions 5.3.6 in terms of generating functions. □

Proposition 5.4.5. We have

$$\begin{aligned} W^+(-t) \star G(t) &= \sum_{n\in\mathbb{N}} (-1)^n [2]_q^{2n} (yyxx)^n y t^{2n} \\ &\quad + q \sum_{n\in\mathbb{N}} (-1)^n [2]_q^{2n+1} (yyxx)^n yyx t^{2n+1}, \end{aligned} \tag{5.85}$$

$$\begin{aligned} G(t) \star W^+(-t) &= \sum_{n\in\mathbb{N}} (-1)^n [2]_q^{2n} (yyxx)^n y t^{2n} \\ &\quad + q^{-1} \sum_{n\in\mathbb{N}} (-1)^n [2]_q^{2n+1} (yyxx) yyx^n t^{2n+1}. \end{aligned} \tag{5.86}$$

Proof. This is Propositions 5.3.7 in terms of generating functions. □

Proposition 5.4.6. We have

$$\begin{aligned} G(-t) \star \tilde{G}(t) = &\sum_{n\in\mathbb{N}} (-1)^n [2]_q^{2n-1} \big(q^{-1} xyy(xxyy)^{n-1}x + qy(xxyy)^{n-1}xxy\big) t^{2n} \\ &+ \sum_{n\in\mathbb{N}} (-1)^n [2]_q^{2n} \big(x(yyxx)^n y - y(xxyy)^n x\big) t^{2n+1}, \end{aligned} \tag{5.87}$$

$$\begin{aligned} \tilde{G}(t) \star G(-t) = &\sum_{n\in\mathbb{N}} (-1)^n [2]_q^{2n-1} \big(qxyy(xxyy)^{n-1}x + q^{-1} y(xxyy)^{n-1}xxy\big) t^{2n} \\ &+ \sum_{n\in\mathbb{N}} (-1)^n [2]_q^{2n} \big(x(yyxx)^n y - y(xxyy)^n x\big) t^{2n+1}. \end{aligned} \tag{5.88}$$

Proof. This is Proposition 5.3.8 in terms of generating functions. □

Proposition 5.4.7. We have

$$\begin{aligned} W^+(-t) \star W^-(t) = &\sum_{n\in\mathbb{N}} (-1)^n [2]_q^{2n} \big(q^{-2} x(yyxx)^n y + y(xxyy)^n x\big) t^{2n} \\ &+ q^{-1} \sum_{n\in\mathbb{N}} (-1)^n [2]_q^{2n+1} \big(xyy(xxyy)^n x - yxx(yyxx)^n y\big) t^{2n+1}, \end{aligned} \tag{5.89}$$

$$\begin{aligned} W^-(t) \star W^+(-t) = &\sum_{n\in\mathbb{N}} (-1)^n [2]_q^{2n} \big(x(yyxx)^n y + q^{-2} y(xxyy)^n x\big) t^{2n} \\ &+ q^{-1} \sum_{n\in\mathbb{N}} (-1)^n [2]_q^{2n+1} \big(xyy(xxyy)^n x - yxx(yyxx)^n y\big) t^{2n+1}. \end{aligned} \tag{5.90}$$

Proof. This is Proposition 5.3.9 in terms of generating functions. □

5.5 New relations involving the alternating words

Recall the formulas (5.53)–(5.60), (5.71), (5.74). These formulas give new relations involving the alternating words. In this section, we will give alternative proofs to them without using the doubly alternating words.

Proposition 5.5.1. For $n \in \mathbb{N}$,

$$\sum_{k=0}^{2n} (-1)^k W_{-k} \star \tilde{G}_{2n-k} = \sum_{k=0}^{2n} (-1)^k \tilde{G}_{2n-k} \star W_{-k}, \tag{5.91}$$

$$q^{-1} \sum_{k=0}^{2n+1} (-1)^k W_{-k} \star \tilde{G}_{2n+1-k} = q \sum_{k=0}^{2n+1} (-1)^k \tilde{G}_{2n+1-k} \star W_{-k}. \tag{5.92}$$

Proof. By [26, (43)] we have

$$\sum_{k=0}^{n-1} (-1)^k \left([W_{-k}, \tilde{G}_{2n-k}] + [\tilde{G}_{k+1}, W_{k+1-2n}] \right) = 0.$$

Rearranging the terms yields (5.91).

Similarly (5.92) follows from [26, (49)]. □

Proposition 5.5.2. For $n \in \mathbb{N}$,

$$\sum_{k=0}^{2n} (-1)^k W_{-k} \star G_{2n-k} = \sum_{k=0}^{2n} (-1)^k G_{2n-k} \star W_{-k}, \tag{5.93}$$

$$q \sum_{k=0}^{2n+1} (-1)^k W_{-k} \star G_{2n+1-k} = q^{-1} \sum_{k=0}^{2n+1} (-1)^k G_{2n+1-k} \star W_{-k}. \tag{5.94}$$

Proof. Similar to the proof of Proposition 5.5.1. □

Proposition 5.5.3. For $n \in \mathbb{N}$,

$$\sum_{k=0}^{2n} (-1)^k W_{k+1} \star \tilde{G}_{2n-k} = \sum_{k=0}^{2n} (-1)^k \tilde{G}_{2n-k} \star W_{k+1}, \tag{5.95}$$

$$q \sum_{k=0}^{2n+1} (-1)^k W_{k+1} \star \tilde{G}_{2n+1-k} = q^{-1} \sum_{k=0}^{2n+1} (-1)^k \tilde{G}_{2n+1-k} \star W_{k+1}. \tag{5.96}$$

Proof. Similar to the proof of Proposition 5.5.1. □

Proposition 5.5.4. For $n \in \mathbb{N}$,

$$\sum_{k=0}^{2n} (-1)^k W_{k+1} \star G_{2n-k} = \sum_{k=0}^{2n} (-1)^k G_{2n-k} \star W_{k+1}, \tag{5.97}$$

$$q^{-1} \sum_{k=0}^{2n+1} (-1)^k W_{k+1} \star G_{2n+1-k} = q \sum_{k=0}^{2n+1} (-1)^k G_{2n+1-k} \star W_{k+1}. \tag{5.98}$$

Proof. Similar to the proof of Proposition 5.5.1. □

Proposition 5.5.5. For $n \in \mathbb{N}$,

$$\begin{aligned} \sum_{k=0}^{2n+1} (-1)^k G_k \star \tilde{G}_{2n+1-k} &= (1 - q^{-2})^{-1} \sum_{k=0}^{2n} (-1)^k [W_{k-2n}, W_{k+1}] \\ &= \sum_{k=0}^{2n+1} (-1)^k \tilde{G}_{2n+1-k} \star G_k. \end{aligned} \tag{5.99}$$

Proof. By [26, (47)] we have

$$\sum_{k=0}^{n} (-1)^k \left([\tilde{G}_k, G_{2n+1-k}] + [G_k, \tilde{G}_{2n+1-k}] \right) = 0.$$

Rearranging the terms yields

$$\sum_{k=0}^{2n+1} (-1)^k G_k \star \tilde{G}_{2n+1-k} = \sum_{k=0}^{2n+1} (-1)^k \tilde{G}_{2n+1-k} \star G_k. \tag{5.100}$$

By [26, (52)] we have

$$\begin{aligned} &\sum_{k=0}^{n-1} (-1)^{k+1} \left([G_{k+1}, \tilde{G}_{2n-k}]_q - [G_{2n-k}, \tilde{G}_{k+1}]_q \right) \\ &\qquad = q \sum_{k=0}^{n-1} (-1)^{k+1} \Big([W_{k+1-2n}, W_{k+2}] - [W_{-k}, W_{2n+1-k}] \Big). \end{aligned}$$

Rearrange the terms and apply [26, (37)]. This yields

$$\sum_{k=0}^{2n+1}(-1)^k[G_k,\tilde{G}_{2n+1-k}]_q = q\sum_{k=0}^{2n}(-1)^k[W_{k-2n},W_{k+1}]. \tag{5.101}$$

The result follows from (5.100) and (5.101). □

Proposition 5.5.6. For $n \in \mathbb{N}$,

$$\begin{aligned}\sum_{k=0}^{2n+1}(-1)^k W_{k+1}\star W_{k-2n-1} &= (q^2-1)^{-1}\sum_{k=0}^{2n+2}(-1)^k[G_k,\tilde{G}_{2n+2-k}] \\ &= \sum_{k=0}^{2n+1}(-1)^k W_{k-2n-1}\star W_{k+1}.\end{aligned} \tag{5.102}$$

Proof. Similar to the proof of Proposition 5.5.5. □

Chapter 6

Future Projects

6.1 Topics involving the uniform approach

In this section, we describe some future research projects involving the uniform approach to the Damiani, Beck, and alternating PBW bases for U_q^+. This uniform approach was discussed in Chapter 4.

Problem 6.1.1. In the early 1990's, Kashiwara and Lusztig independently introduced the *canonical basis* for U_q^+. Later Leclerc introduced the *dual canonical basis* for U_q^+. In 2003, Leclerc studied these bases using the Rosso embedding. He gave a nonconstructive formula for the canonical basis and a recursive algorithm to compute the dual canonical basis. We may use the Rosso embedding to express these bases in closed form and fit them into the uniform approach if possible.

Problem 6.1.2. There is a basis for U_q^+ due to Ito and Terwilliger, called the *zigzag basis.* We may use the Rosso embedding to express this basis in closed form and fit it into the uniform approach if possible.

Problem 6.1.3. Recall the elements $\{\Delta_n^{(m)}\}_{n\in\mathbb{N}}$ where $m \in \mathbb{Z}$. The case $m = 2$ (resp. $m = -1$) leads to the Damiani (resp. alternating) PBW basis for U_q^+. We hope to show that, for an arbitary $m \in \mathbb{Z}$, a similar PBW basis can be constructed. For any two such PBW bases, we may seek to express the transition matrix in closed form.

Problem 6.1.4. Recall the alternating words $\{W_{-n}\}_{n\in\mathbb{N}}$, $\{W_{n+1}\}_{n\in\mathbb{N}}$, $\{\tilde{G}_{n+1}\}_{n\in\mathbb{N}}$, $\{G_{n+1}\}_{n\in\mathbb{N}}$ from Definition 2.2.13. In [26] Terwilliger obtained a recurrence relation for these words with respect to the q-shuffle product. He then showed that these words are polynomials in the generators x, y of U with respect to the q-shuffle product. We may seek to express these polynomials in closed form.

Problem 6.1.5. Recall the elements $\{C_n\}_{n\in\mathbb{N}}$, $\{xC_n\}_{n\in\mathbb{N}}$, $\{C_ny\}_{n\in\mathbb{N}}$, $\{xC_ny\}_{n\in\mathbb{N}}$ from Section 2.2. In [29] Terwilliger used earlier results by Damiani and Beck to obtain a recurrence relation for these elements with respect to the q-shuffle product. He then showed that these elements are polynomials in x, y with respect to the q-shuffle product. We may seek to express these polynomials in closed form.

Problem 6.1.6. Recently Terwilliger introduced the *alternating central extension* of U_q^+. We may explore how to extend the uniform approach to this alternating central extension.

Problem 6.1.7. The finite-dimensional irreducible $U_q(\widehat{\mathfrak{sl}}_2)$-modules are classified by Chari and Pressley, using the *Drinfeld presentation* of $U_q(\widehat{\mathfrak{sl}}_2)$. We may explore how the Damiani, Beck, and alternating PBW bases act on the finite-dimensional irreducible $U_q(\widehat{\mathfrak{sl}}_2)$-modules. We hope to investigate these modules from the point of view of the *Drinfeld-Jimbo presentation* or the *equitable presentation* of $U_q(\widehat{\mathfrak{sl}}_2)$. We may also explore how to extend the uniform approach to $U_q(\widehat{\mathfrak{sl}}_2)$.

Problem 6.1.8. Recently Pascal Baseilhac obtained a presentation for U_q^+ of *Freidel-Maillet type*. The defining relation in this presentation is a *reflection equation* of the form RKR_0K. Under the Rosso embedding, the K-matrix is given in the alternating words $\{W_{-n}\}_{n\in\mathbb{N}}$, $\{W_{n+1}\}_{n\in\mathbb{N}}$, $\{\tilde{G}_{n+1}\}_{n\in\mathbb{N}}$, $\{G_{n+1}\}_{n\in\mathbb{N}}$. We may use the uniform approach to contruct a universal K-matrix satisfying a universal version of the above reflection equation.

6.2 Topics involving the doubly alternating words

In this section, we describe some future research projects involving the doubly alternating words. These words were discussed in Chapter 5.

Problem 6.2.1. In Section 5.2 we listed the commutator relations involving one of x, y and one doubly alternating word with respect to the q-shuffle product. We may explore the commutator relations involving two doubly alternating words with respect to the q-shuffle product.

Problem 6.2.2. In [26] Terwilliger showed that alternating words are polynomials in the generators x, y of U with respect to the q-shuffle product. However, these polynomials have no known closed form. In Section 5.3 we showed that the doubly alternating words are polynomials in the alternating words with respect to the q-shuffle product. We may seek to express the doubly alternating words explicitly as polynomials in x, y with respect to the q-shuffle product.

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