Aspects of the geometry of polymatroids

by
Connor Simpson

A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
Mathematics

at the
University of Wisconsin-Madison
2024

Date of Final Oral Exam: 04/26/24
The dissertation is approved by the following members of the Final Oral Committee:

Botong Wang, Professor, Mathematics
Jose Rodriguez, Professor, Mathematics
Laurentiu Maxim, Professor, Mathematics
Sean Paul, Professor, Mathematics

Aspects of the geometry of polymatroids

Connor Simpson

Abstract

This thesis is compiled from two works exploring the combinatorics and algebraic geometry of polymatroids. We first review some known results on matroids, then describe various combinatorial constructions using matroids. We next describe the "smooth geometry of polymatroids", also known as augmented wonderful compactifications. Finally, we describe polymatroid Schubert varieties, singular spaces that encode polymatroid combinatorics.

Acknowledgements

Many people deserve my thanks. For revealing to me math's depths, I thank Severin Drix, Sergio Da Silva, Balázs Elek, Hung Tran, and Mary Ann Huntley, whose work made the IHS math seminar possible. Early guidance from Karola Mészáros, Allen Knutson, Irena Peeva, and Vic Reiner begat the interests I pursue today. I am also grateful to mentors, collaborators, and colleagues, including Spencer Backman, Cashous Bortner, Sam Coskey, Chris Eur, Thomas Hameister, Xuhua He, June Huh, Asvin G, Matt Larson, Sujit Rao, Jörg Schürmann, Yiyu Wang, Zoe Wellner, and Kaitao Xie, for advice, conversations, ideas, and insights I would not have had alone. Among this group, Colin Crowley deserves special mention for his constant mathematical companionship throughout the past several years—we learned better together than we might have done apart. Finally, I thank my advisor, Botong Wang, for his patient and thoughtful support throughout my gradaute studies.

Contents

Chapter 1

The world of matroids

We introduce some basics facts about matroids, then introduce two algebraic varieties that underlie recent breakthroughs.

1.1 Matroids

Matroids were invented by Nakasawa Nakasawa 1935 and Whitney Whitney 1935 to capture the combinatorics of linear independence. They have since found broad application across pure and applied mathematics Rosen, Sidman, and Theran 2020; Recski 2011; Iri 1983, and recent years have seen the development of a particularly close relationship with algebraic geometry.

Following Whitney, we introduce matroids in terms of their rank functions (though not with precisely his axioms).

Definition 1.1.1. Let E be a finite set. Let $\mathrm{rk} : 2^E \to \mathbb{N}$ such that

1. $\mathrm{rk}(\emptyset) = 0$,
2. for all $A \subset B$, $\mathrm{rk}(A) \leq \mathrm{rk}(B)$, and
3. $\mathrm{rk}(A \cap B) + \mathrm{rk}(A \cup B) \leq \mathrm{rk}(A) + \mathrm{rk}(B)$.

The data of such a function is a **polymatroid**. If additionally $\mathrm{rk}(A) \leq |A|$ for all $A \subset E$, then rk is the data of a **matroid**.

1.1.1 Examples of matroids...

...from linear algebra

The prototypical example of a matroid, cited by both Nakasawa and Whitney, comes from linear algebra. Let $E = \{v_1, \ldots, v_n\}$ be a set of vectors in a d-dimensional vector space V. One checks quickly that

$$\mathrm{rk}(S) := \dim \mathrm{span}\{v_i : i \in S\}$$

is a matroid rank function. Any matroid that arises in this way is called **realizable**.

For our purposes, it will be more useful to think of realizable matroids via a dual formulation. Let $\mathbb{K}$ be a field and E a finite set. A d-dimensional linear subspace $V \subset \mathbb{K}^E$ has an associated matroid, defined by

$$\mathrm{rk}(S) := \dim \pi_S(V),$$

where $\pi_S : \mathbb{K}^E \to \mathbb{K}^S$ is the coordinate projection. Plainly, the matroid associated to V is the same as the matroid of the vectors $\{a_i|_V\}_{i \in E} \subset V^\vee$, where $\{a_i : i \in E\}$ are the coordinate functions of $\mathbb{K}^E$, so matroids arising from linear subspaces are precisely realizable matroids.

...from graph theory

A finite graph G with edge set E has an associated **graphic matroid** on E, defined by

$$\mathrm{rk}(S) := \text{number of edges of a maximal acyclic subgraph of } G \text{ using only edges in } S.$$

In fact, graphic matroids are merely a special kind of realizable matroids—if the vertices of G are enumerated $1, \ldots, n$, then the matroid of G is realized by the vectors $\{\mathbf{e}_i - \mathbf{e}_j : ij \in E\}$ in $\mathbb{K}^n$—but their introduction gives a path to an important matroid invariant.

1.1.2 The characteristic polynomial

The **characteristic polynomial** of a matroid M on E is

$$\chi_M(t) = \sum_{S \subset E} (-1)^{|S|} t^{\mathrm{rk}(M) - \mathrm{rk}(S)} = \sum_i w_i t^{\mathrm{rk}(M) - i}.$$

Each of the two classical incarnations of matroids discussed above gives a way of motivating this invariant.

Graphs

If G is a graph, then we can consider its **characteristic polynomial**

$$\kappa_G(t) = \#\big\{\text{vertex colorings with } t \text{ colors so that no edge's endpoints are the same color}\big\}.$$

In fact, κ_G is almost entirely determined by the matroid M of G. The two invariants are related by the well-known formula (White 1987, Proposition 7.5.1)

$$\kappa_G(t) = \chi_M(t) t^c,$$

where c is the number of connected components of G.

Linear algebra

Hyperplane arrangements give another motivation. Suppose that $V \subset \mathbb{C}^E$ is a linear subspace not contained in any coordinate subspace of $\mathbb{C}^E$. Let $\{H_i\}_{i \in E}$ be the hyperplanes of V determined by the coordinates of $\mathbb{C}^E$, and let $U = V \setminus \cup_i V_i$. A natrual question for one interested in the topology of U is: what the Betti numbers of U?

This question is again answered by the characteristic polynomial (Orlik and Solomon 1980)

$$\sum_{i=0}^{\dim V} \dim H^i(U; \mathbb{C})(-t)^i = t^{\dim V} \chi_M(-1/t).$$

Example 1.1.2. A small, random sample of characteristic polynomials includes:

- $t^4 - 6t^3 + 13t^2 - 12t + 4$ (from a rank 4 matroid on 7 elements)
- $t^3 - 5t^2 + 10t - 6$ (from a rank 3 matroid on 7 elements)
- $t^4 - 7t^3 + 19t^2 - 24t + 11$ (from rank 4 on 8 elements)

$\diamondsuit$

Visible in the Example 1.1.2 are some striking numerical patterns. First, the coefficients of $\chi_M(t)$ always alternate in sign. Second, their absolute values appear to be unimodal; that is, there is i such that $|w_0| \leq |w_1| \leq \cdots \leq |w_i| \geq |w_{i+1}| \geq \cdots \geq |w_{\mathrm{rk}(M)}|$. The first observation is nontrivial, and can be proved using tools from "poset topology" (Stanley 2011, Proposition 3.10.1). The latter observation is a consequence of a more subtle property: the coefficients are **log-concave**, meaning $w_i^2 \geq w_{i-1}w_{i+1}$. This fact was subject of conjectures by Rota, Welsh, Read, and Heron (see Oxley 2006, Chapter 15.2 for history), and was established in celebrated work of (Adiprasito, Huh, and Katz 2018).

Theorem 1.1.3. *(Adiprasito, Huh, and Katz 2018) The coefficients of χ_M are log-concave.*

While we will not fully explain the proof of Theorem 1.1.3, it will be helpful for us to speak a little of some of the tools involved later. In order to understand these, a new piece of combinatorial data is needed.

1.2 Flats

Definition 1.2.1. A **flat** of a matroid M on E is $F \subset E$ such that $\mathrm{rk}(F \cup i) > \mathrm{rk}(F)$ for all $i \in E \setminus F$.

When ordered by inclusion, flats form a graded lattice, with meet given by intersection. We will denote by $\mathscr{L}_M$ the lattice of flats of M. It is possible to have two matroids M and M' such that $\mathscr{L}_M \cong \mathscr{L}_{M'}$ as posets, but the failure of injectivity is not too severe.

Call M **simple** if $\{i\}$ is a flat of positive rank for all $i \in E$.

Proposition 1.2.2. *(see, e.g. Stanley 2011, Chapter 3.3) If M is any matroid, then there is a unique simple matroid M^{sim} such that $\mathscr{L}_{M^{\mathrm{sim}}} \cong \mathscr{L}_M$.*

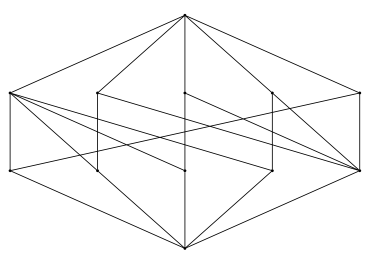
Lattice of flats also admit a relatively axiomatization.

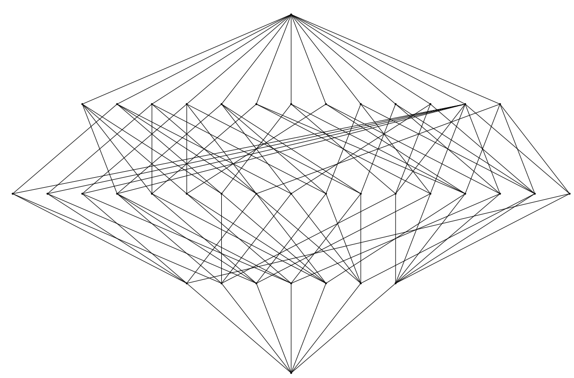
Proposition 1.2.3. *(see, e.g. Stanley 2011, Chapter 3.3) A poset is the lattice of flats of a matroid if and only if it is a* ***geometric lattice****. Explicitly, if and only if it is a graded lattice in which every element can be obtained as a join of atoms, with rank function satisfying*

$$r(x) + r(y) \geq r(x \vee y) + r(x \wedge y).$$

We will later see generalizations of both of these to polymatroids.

Example 1.2.4. Two lattices of flats. The first froma rank 3 matroid on 5 elements; the second from a rank 4 matroid on 7 elements.





$\diamondsuit$

Let f_i be the number of rank-i flats of M. From Example 1.2.4, one sees a numerical pattern of "top-heaviness", i.e. $f_i \leq f_j$ for $i \leq j \leq \mathrm{rk}(M) - i$. This statement was conjectured by Dowling & Wilson (Dowling and Wilson 1974) and was recently proved.

Theorem 1.2.5. *(Braden et al. 2020) The f_i's are top-heavy.*

For realizable matroids, a geometric proof is possible (Huh and Wang 2017). We will shortly discuss some of the pieces involved.

Remark 1.2.6. For historical remarks on Theorem 1.2.5, see (Oxley 2006, Chapter 15.2), (White 1987, Chapter 8), and (Dowling and Wilson 1974).

1.3 The geometry of matroids

For realizable matroids, both Theorem 1.2.5 and Theorem 1.1.3 have geometric proofs. Here, we briefly introduce the spaces involved.

1.3.1 Wonderful compactifications

Let $V \subset \prod_{i\in E} \mathbb{C}^E$ be a subspace contained in no coordinate subspace, $\{H_i\}_{i\in E}$ the hyperplanes in V defined by the coordinates of $\mathbb{C}^E$. Let M be the associated matroid.

Definition 1.3.1. A **building set** for $\mathscr{L}_M$ is a collection $\mathcal{G} \supset \{E\}$ of nonempty flats of M such that for all $F \in \mathscr{L}_M$, there is an isomorphism

$$\prod_{G\in \max \mathcal{G}_{\leq F}} [\emptyset, G] \to [\emptyset, F]$$

sending $(\emptyset, \ldots, G, \ldots, \emptyset)$ to G for each $G \in \mathcal{G}_{\leq F}$.

Building sets of $\mathscr{L}_M$ give compactifications of $U := V \setminus \cup_{i\in E} H_i$.

Theorem/Definition 1.3.2. *(Concini and Procesi 1995) The* ***wonderful compactification*** *of U with respect to the building set $\mathcal{G}$ is*

$$W_{\mathcal{G}} := \overline{\operatorname{im}(U \to \prod_{G\in\mathcal{G}} \mathbb{P}(V/\cap_{i\in G} H_i))}.$$

The wonderful compactification is a smooth projective variety, and $W_{\mathcal{G}} \setminus U$ is a normal-crossings divisor.

Now, suppose that $\mathcal{G}$ consists of all nonempty flats of M and write $W = W_{\mathcal{G}}$ The wonderful compactification is equipped with maps $\alpha : W \to \mathbb{P}(V)$ and $\beta : W \to \frac{1}{V}$, where

$$\frac{1}{V} := \overline{\operatorname{im}\left(U \to \mathbb{P}(\mathbb{C}^E) \overset{[1/x_0:\cdots:1/x_N]}{\dashrightarrow} \mathbb{P}(\mathbb{C}^E)\right)}$$

is the **reciprocal space** of V. An important fact in the proof of Theorem 1.1.3 is that top-degree products of the pullbacks of the hyperplane class of $\mathbb{P}(\mathbb{C}^E)$ along α and β give coefficients of $\chi_M(t)/(t-1)$ (Adiprasito, Huh, and Katz 2018, Section 9). After observing this, log-concaivty of the coefficients of $\chi_M(t)/(t-1)$ (which implies Theorem 1.1.3) can then be deduced from the the fact that $H^*(W)$ has the **Kähler package**. Explicitly, this means there is an isomorphism $\deg : H^{\dim W}(W) \to \mathbb{R}$ such that for any ample $\ell \in H^2(W;\mathbb{R})$,

- Poincaré duality: The pairing

  $$H^i(W) \times H^i(W) \to \mathbb{R}, \quad (a,b) \mapsto \deg(a\ell^{\mathrm{rk}(M)-2i}b)$$

  is perfect.

- Hard Lefschetz: The map $H^i(W) \to H^{\dim(W)-i-1}(W)$ given by $a \mapsto \ell^{\mathrm{rk}(M)-1-2i}a$ is an ismorphism.

- Hodge-Riemann: The quadratic form $H^i(W) \times H^i(W) \to \mathbb{R}$ defined by $(a,b) \mapsto (-1)^i \deg(a\ell^{\mathrm{rk}(M)-1-2i}b)$ is positive definite on $\{x \in H^i(W) : \ell^{\dim(W)-2i}x = 0\}$.

To carry out this proof for *all* matroids, rather than only realizable ones, Adiprasito, Huh, and Katz 2018 construct a purely combinatorial **Chow ring** $A(M)$ for every matroid M, which is isomorphic to $H^*(W)$ when M is realized by V. The Chow ring of a matroid is not in general the Chow ring of a smooth projective variety, but the main result of (Adiprasito, Huh, and Katz 2018) says that $A(M)$ always has the Kähler package, allowing one to deduce log-cocnavity.

1.3.2 Matroid Schubert varieties

While wonderful cmpactifications are at the heart of Theorem 1.1.3, a different variety hides within Theorem 1.2.5. Let $V \subset \mathbb{C}^E$. Via the decomposition $\mathbb{P}^1 = \mathbb{C} \cup \infty$, we may think of $\mathbb{C}^E$ as sitting inside $(\mathbb{P}^1)^E$.

Definition 1.3.3. The **matroid Schubert variety** of V is $Y_V := \overline{V} \subset (\mathbb{P}^1)^E$.

Matroid Schubert varieties were defined by (Ardila and Boocher 2016). They contain the affine cone over $\frac{1}{V}$ as an open neighborhood, which was studied in (Proudfoot and Speyer 2006).

The following statement is key to the proof of Theorem 1.2.5 for realizable matroids given in (Huh and Wang 2017).

Proposition 1.3.4. *(Ardila and Boocher 2016) Y_V admits a decomposition $Y_V = \sqcup_F \mathbb{C}^{\mathrm{rk}(F)}$, where F runs over all flats of the matroid M of V.*

Ardila and Boocher 2016 use algebraic methods to show Proposition 1.3.4. One can also see it more goemetrically: $\mathbb{C}$ acts on $\mathbb{P}^1$ by addition, fixing ∞, so $\mathbb{C}^n$ acts on $(\mathbb{P}^1)^n$, decomposing it into orbits of the form $\mathbb{C}^S \times \infty^{E \setminus S}$ for $S \subset E$. The subgroup $V \subset \mathbb{C}^n$ acts on Y_V, and likewise decmoposes it into orbits, which refine those of $\mathbb{C}^n$ on $(\mathbb{P}^1)^n$. In fact, $Y_V \cap \mathbb{C}^F \times \infty^{E \setminus F}$ is a single V-orbit, isomorphic to $\mathbb{C}^{\mathrm{rk}(F)}$. We will later see these facts generalized to realizable polymatroids.

Remark 1.3.5. While this fact will not be required here, it bears mention that matroid Schubert varieties and wonderful compactifications are closely related. A resolution of singuarities of Y_V is obtained by considering an "augmented wonderful compactification" of V, on which the usual wonderful compactification appears as a divisor.

Chapter 2

Polymatroids: combinatorics

This chapter is comprised of combinatorial results on polymatroids. Section 2.2 is condensed from (Crowley, Simpson, and Wang 2024), and both Section 2.3 & Section 2.5 are drawn directly from the same. Section 2.4 is condensed from (Crowley, Huh, et al. 2022). Aside from combinatorial interest, Sections 2.4 and 2.5 will be useful later when discussing Chow rings of polymatroids & polymatroid Schubert varieties.

2.1 The pathologies of flats

Recall from Definition 1.1.1 that a polymatroid on E is the data of an increasing subomdular function $\mathrm{rk} : 2^E \to \mathbb{N}$. Geometrically, polymatroids arise as follows. Let $\widetilde{E} = E_1 \sqcup \cdots \sqcup E_N$ be a finite set and let

$$V \subset \mathbb{K}^{E_1} \times \mathbb{K}^{E_2} \times \cdots \times \mathbb{K}^{E_N}.$$

be a linear subspace. For $S \subset \{1, \ldots, N\}$, write π_S for the projection to $\prod_{i \in S} \mathbb{K}^{E_i}$. The polymatroid realized by V is given by

$$\mathrm{rk}(S) := \dim \pi_S(V).$$

Any polymatroid that arises this way is **realizable**.

Flats of a polymatroid defined exactly as for matroids. Flats are in order-reverseing bijection with the subspaces of V obtained by intersecting V with collections of subspaces of the form $\prod_{i \in E} \mathbb{K}^{E_i} \times 0$. Ordering the flats by inclusion yields a lattice $\mathscr{L}_P$. Unlike for matroids, $\mathscr{L}_P$ may fail to be graded or atomic, and the poset $\mathscr{L}_P$ alone is insufficient to recover much data about P.

Example 2.1.1. Let $V = \mathbb{C}^1 \times \mathbb{C}^2 \subset \mathbb{C}^1 \times \mathbb{C}^2$. The lattice of flats of the polymatroid associated to V is the same as the lattice of flats of the matroid of $\mathbb{C}^2 \subset \mathbb{C}^1 \times \mathbb{C}^1$. This shows that for polymatroids, injectivity of the map $P \mapsto \mathscr{L}_P$ fails much more dramatically than for matroids. ♢

Example 2.1.2. Consider the polymatroid (from (Pagaria and Pezzoli 2021, Section 7)) on $E = \{1, 2, 3\}$ with rank function $\mathrm{rk}(1) = \mathrm{rk}(2) = 2$, $\mathrm{rk}(12) = \mathrm{rk}(3) = 4$, and $\mathrm{rk}(13) = \mathrm{rk}(23) = \mathrm{rk}(123) = 5$. The corresponding lattice of flats is not graded. ♢

Example 2.1.3. Consider the polymatroid on $E = \{1, 2\}$ with $\mathrm{rk}(12) = \mathrm{rk}(1) = 2$, $\mathrm{rk}(2) = 1$. The lattice of flats is not atomic. ♢

In the remainder of this chapter, we discuss how to fix the various pathologies of polymatroid flats.

2.2 Multisymmetric matroids & lifts

2.2.1 Multisymmetric matroids

A matroid M on $\widetilde{E} = E_1 \sqcup \cdots \sqcup E_N$ is **multisymmetric** if the permutation action of the group $\mathfrak{S} = \mathfrak{S}_{E_1} \times \cdots \times \mathfrak{S}_{E_N}$ on $\widetilde{E}$ takes flats to flats. The **geometric part** of $S \subset \widetilde{E}$ is $S^{\mathrm{geo}} := \bigcap_{\sigma \in \mathfrak{S}} \sigma(S)$, and S is **geometric** if $S = S^{\mathrm{geo}}$.

A fundamental property of multisymmetric matroids is that they are determined by the ranks of geometric sets.

Proposition 2.2.1. *(Crowley, Huh, et al. 2020, Lemma 2.9) Two multisymmetric matroids on $E_1 \sqcup \cdots \sqcup E_N$ are isomorphic if and only if their geometric sets have the same ranks.*

Proof of this fact depends upon understanding the rank function of a multisymmetric matroid.

Lemma 2.2.2. *(Crowley, Huh, et al. 2020, Lemma 2.8) Let M be a multisymmetric matroid on $\widetilde{E} = E_1 \sqcup \cdots \sqcup E_N$. The following equivalent statements all hold:*

- *If F is a flat of M, then $\mathrm{rk}(F) = \mathrm{rk}(F^{\mathrm{geo}}) + |F \setminus F^{\mathrm{geo}}|$.*
- *If $A \subset \widetilde{E}$, then for each $1 \leq i \leq N$, either $\overline{A} \cap E_i = A \cap E_i$ or $\overline{A} \supset E_i$.*
- *If F is a flat of M, then $F \setminus e$ is a flat of rank $\mathrm{rk}(F) - 1$ for any $e \in F \setminus F^{\mathrm{geo}}$.*

Proof of Proposition 2.2.1. Let M and M' be two multisymmetric matroids on $\widetilde{E} = \widetilde{E}_1 \sqcup \cdots \sqcup \widetilde{E}_n$, and suppose that for all $A \subset \{1, \ldots, n\}$, $\mathrm{rk}_M(\cup_{i\in A}\widetilde{E}_i) = \mathrm{rk}_{M'}(\cup_{i\in A}\widetilde{E}_i)$. If F is a flat of M, and F' is the closure of F in M', then

$$\mathrm{rk}_{M'}(F) \leq \mathrm{rk}_{M'}(F^{\mathrm{geo}}) + \mathrm{rk}_{M'}(F \setminus F^{\mathrm{geo}}) \leq \mathrm{rk}_M(F^{\mathrm{geo}}) + |F \setminus F^{\mathrm{geo}}| = \mathrm{rk}_M(F)$$

by Lemma 2.2.2. Symmetrically, $\mathrm{rk}_M(F') \leq \mathrm{rk}_{M'}(F')$, so

$$\mathrm{rk}_{M'}(F) \leq \mathrm{rk}_M(F) \leq \mathrm{rk}_M(F') \leq \mathrm{rk}_{M'}(F').$$

The left- and rightmost terms are equal, so $F = F'$ because F is a flat of M. This shows that M and M' have the same flats, and that their flats have the same ranks, so M and M' are equal. □

2.2.2 Lifts

A **cage** for a polymatroid P on E is a multiset $\mathbf{n} \in \mathbb{N}^E$ such that $n_i \geq \mathrm{rk}(i)$ for all $i \in E$. A pair $(P, \mathbf{n})$ with $\mathbf{n}$ a cage for P is a **caged polymatroid**. Caged polymatroid are equivalent to multisymmetric matroids. To see this, we employ an oft-rediscovered (Helgason 1972; Nguyen 1986; Lovász 1977; McDiarmid 1975; Bonin, Chun, and Fife 2023; Crowley, Huh, et al. 2020) construction, which we render in the language of (Crowley, Huh, et al. 2022).

Theorem/Definition 2.2.3. *Fix a multiset* $\mathbf{n} = (n_1, \ldots, n_N)$*, and let* $\widetilde{E} = E_1 \sqcup \cdots \sqcup E_N$ *be a set with* $|E_i| = n_i$*. There is a bijection*

$$\{\textit{caged polymatroids } (P, \mathbf{n})\} \xrightarrow{\sim} \{\textit{multisymmetric matroids on } \widetilde{E}\}$$
$$(P, \mathbf{n}) \longmapsto \widetilde{P},$$

where $\widetilde{P}$ *is the* ***multisymmetric lift*** *of* P*, defined by*

$$\mathrm{rk}_{\widetilde{P}}(A) := \min\{\mathrm{rk}_P(B) + |A \setminus \cup_{i\in B} E_i| : B \subset E\}, \quad A \subset \widetilde{E}.$$

The lift can also be described with bases.

Proposition 2.2.4. *Let* $(P, \mathbf{n})$ *be a caged polymatroid with multisymmetric lift* $\widetilde{P}$*. The bases of* $\widetilde{P}$ *are the multisets* $\mathbf{b} \in \{0,1\}^{\widetilde{E}}$ *with* $|\mathbf{b}| = \mathrm{rk}(P)$ *such that for all* $A \subset E$*,*

$$|\pi_{\cup_{i\in A} E_i}(\mathbf{b})| \leq \mathrm{rk}_P(A).$$

Equivalently, $\mathbf{b}$ *is a basis of* $\widetilde{P}$ *if and only if* $(|\pi_{E_i}(\mathbf{b})|)_{i=1}^N$ *is a basis of* P*.*

We demonstrate below that lifts generally interact well with matroid & polymatroid operations.

Remark 2.2.5. If P is the polymatroid realized by $V \subset \prod_i \mathbb{K}^{n_i}$, then $\widetilde{P}$ is realized by a general translate of V by $\prod_i \mathrm{GL}(n_i)$.

2.3 Lifts and operations

2.3.1 Deletion

The **deletion** of $A \subset E$ is the polymatroid $P \setminus A$ obtained by restricting rk_P to subsets of $E \setminus A$. The **restriction** of P to A is $P|_A := P \setminus (E \setminus A)$. We were unable to find a reference for the following well-known description of the flats of $P \setminus A$.

Lemma 2.3.1. *The flats of $P \setminus A$ are all sets of the form $F \setminus A$ such that F is a flat of P.*

Proof. A set $G \subset E \setminus A$ is a flat of $P \setminus A$ if and only if for all $i \in E \setminus A$, $\mathrm{rk}_P(G \cup i) > \mathrm{rk}_P(G)$. Equivalently, the closure of G in P is a flat $F \subset G \cup A$, which must satisfy $G = F \setminus A$. □

Deletion commutes with lifts.

Lemma 2.3.2. *(Crowley, Huh, et al. 2020, Lemma 2.14) For any subset $A \subset E$,*

$$\widetilde{P \setminus A} = \widetilde{P} \setminus \cup_{i \in A} E_i,$$

where the left-hand lift is taken with respect to $\pi_{E \setminus A}(\mathbf{n})$.

2.3.2 Truncation

The **truncation** of P at a set $S \subset E$ is the polymatroid $T_S P$ on E, with rank function

$$\mathrm{rk}_{T_S P}(A) := \begin{cases} \mathrm{rk}_P(A) - 1, & \text{if } \mathrm{rk}_P(A) = \mathrm{rk}_P(A \cup S) \\ \mathrm{rk}_P(A), & \text{otherwise.} \end{cases}$$

Since $\mathrm{rk}_P(A \cup S) = \mathrm{rk}_P(\overline{A \cup S}) = \mathrm{rk}_P(A \cup \overline{S})$, $T_S P$ depends only on the closure of S.

Lemma 2.3.3. *(Crowley, Simpson, and Wang 2024) If $S \subset E$, then*

$$\widetilde{T_S P} = T_{\cup_{i \in S} E_i} \widetilde{P}.$$

Proof. It suffices to give a proof for $F := \overline{S}$. Since $\cup_{i \in F} E_i$ is a geometric set of $\widetilde{P}$, the truncation $T_{\cup_{i\in F}E_i}\widetilde{P}$ is also a multisymmetric matroid on $\widetilde{E}$. For $A \subset E$,

$$\begin{aligned}\mathrm{rk}_{T_{\cup_{i\in F}E_i}\widetilde{P}}(\cup_{i\in A}E_i) &= \begin{cases}\mathrm{rk}_{\widetilde{P}}(\cup_{i\in A}E_i)-1, & \text{if } \mathrm{rk}_{\widetilde{P}}(\cup_{i\in A}E_i)=\mathrm{rk}_{\widetilde{P}}(\cup_{i\in A\cup F}E_i)\\ \mathrm{rk}_{\widetilde{P}}(\cup_{i\in A}E_i), & \text{otherwise}\end{cases}\\ &= \begin{cases}\mathrm{rk}_P(A)-1, & \text{if } \mathrm{rk}_P(A)=\mathrm{rk}_P(A\cup F)\\ \mathrm{rk}_P(A), & \text{otherwise}\end{cases}\\ &= \mathrm{rk}_{T_FP}(A).\end{aligned}$$

Consequently, $\widetilde{T_FP} = T_{\cup_{i\in F}E_i}\widetilde{P}$ by Proposition 2.2.1. □

The following description of the bases of a truncation will later be useful for studying the topology of polymatroid Schubert varieties.

Lemma 2.3.4. *(Crowley, Simpson, and Wang 2024) Let P be a polymatroid on E and $S \subset E$. The bases of T_SP are*

$$\{\mathbf{b}-\mathbf{e}_i : i \in \overline{S},\ b_i > 0,\ \textit{and } \mathbf{b} \textit{ is a basis of } P\}.$$

Proof. We reduce to the case when P is a matroid, which is proved in Brylawski 1986, Proposition 7.4.9.

Let $\mathbf{b} \in \mathbb{N}^E$ be a mutiset with $|\mathbf{b}| = \mathrm{rk}(P)$. By Proposition 2.2.4, $\mathbf{b}$ is a basis of P if and only if each multiset $\mathbf{b}' \in \{0,1\}^{\widetilde{E}}$ with $\sum_{j\in E_i} b'_j = b_i$ for all $1 \le i \le N$ is a basis of $\widetilde{P}$. By Brylawski 1986, Proposition 7.4.9, $\mathbf{b}'$ is a basis of $\widetilde{P}$ if and only if $\mathbf{b}' - \mathbf{e}_j$ is a basis of $T_{\cup_{i\in S}E_i}\widetilde{P}$ for any $j \in \cup_{i\in\overline{S}}E_i$ such that $\mathbf{b}'_j > 0$. Lemma 2.3.3 and a second application of Proposition 2.2.4 complete the proof. □

2.3.3 Reduction

The **reduction** of P at an element $i \in E$ of positive rank is the polymatroid R_iP defined by

$$\mathrm{rk}_{R_iP}(A) := \begin{cases} \mathrm{rk}_P(A) - 1, & \mathrm{rk}_P(A) = \mathrm{rk}_P(A \setminus i) + \mathrm{rk}_P(i) \\ \mathrm{rk}_P(A), & \text{otherwise.} \end{cases}$$

The **reduction** of a caged polymatroid is defined by

$$R_i(P, \mathbf{n}) := \begin{cases} (P, \mathbf{n} - \mathbf{e}_i), & n_i > \mathrm{rk}_P(i) \\ (R_iP, \mathbf{n} - \mathbf{e}_i), & n_i = \mathrm{rk}_P(i). \end{cases}$$

Lemma 2.3.5. *(Crowley, Simpson, and Wang 2024) The multisymmetric lift of $R_i(P, \mathbf{n})$ is isomorphic to $\widetilde{P} \setminus j$, where j is any element of E_i.*

Proof. We may think of both matroids as being multisymmetric on $\widetilde{E} \setminus j$, so it suffices by Proposition 2.2.1 to check that their geometric sets have the same ranks. The geometric sets of these matroids are all of the form $(\cup_{k \in A} E_k) \setminus j$, with $A \subset E$. If $\mathrm{rk}_P(A) < \mathrm{rk}_P(A \setminus i) + n_i$, then by Lemma 2.2.2,

$$\mathrm{rk}_{\widetilde{P} \setminus e}((\cup_{k \in A} E_k) \setminus j) = \mathrm{rk}_{\widetilde{P}}((\cup_{k \in A} E_k) \setminus j) = \mathrm{rk}_{\widetilde{P}}(\cup_{k \in A} E_k) = \mathrm{rk}_P(A).$$

This completes the proof when $n_i > \mathrm{rk}_P(i)$. When $n_i = \mathrm{rk}_P(i)$, it also shows that geometric sets with $\mathrm{rk}_P(A) < \mathrm{rk}_P(A \setminus i) + \mathrm{rk}_P(i)$ have the correct rank. It remains to check when $n_i = \mathrm{rk}_P(i)$ and $\mathrm{rk}_P(A) = \mathrm{rk}_P(A \setminus i) + \mathrm{rk}_P(i)$. In this case, $i \in A$ and every element of E_i is a coloop of $\cup_{k \in A} E_i$ in $\widetilde{P}$, hence

$$\mathrm{rk}_{\widetilde{P} \setminus e}((\cup_{k \in A} E_k) \setminus j) = \mathrm{rk}_{\widetilde{P}}((\cup_{k \in A} E_k) \setminus j) = \mathrm{rk}_{\widetilde{P}}(\cup_{k \in A} E_k) - 1 = \mathrm{rk}_P(A) - 1,$$

as desired. $\square$

2.4 Lifts & building sets

We summarize results on building sets of polymatroids from (Crowley, Huh, et al. 2020), which will later useful in our discussion of polymatroid Chow rings later.

Definition 2.4.1. Let P be a loopless polymatroid with lattice of flats $\mathscr{L}_P$. A **geometric building set** for $\mathscr{L}_P$ is a collection of nonempty flats $\mathcal{G} \supset \{E\}$ such that for all $F \in \mathscr{L}_P$ there is an isomorphism

$$\phi : \prod_{G \in \max(\mathcal{G} \cap [\emptyset, F])} [\emptyset, G] \to [\emptyset, x]$$

with $\phi(\emptyset, \dots, G, \dots, \emptyset) = G$ for each $G \in \max(\mathcal{G} \cap [\emptyset, G])$, and $\sum_{G \in \max(\mathcal{G} \cap [\emptyset, F])} \mathrm{rk}(G) = \mathrm{rk}(F)$.. A set $\mathcal{N} \subset \mathcal{G}$ is **nested** if all sets of pairwise incomparable elements $\{F_1, \dots, F_k\} \subset \mathcal{N}$ with $k \geq 2$ satisfy $F_1 \vee F_2 \vee \cdots \vee F_k \notin \mathcal{G}$.

The key fact is that building sets of $\mathscr{L}_P$ lift to building sets of $\mathscr{L}_{\widetilde{P}}$.

Proposition 2.4.2. *(Crowley, Huh, et al. 2020, Lemmas 3.3 & 3.4) If $\mathcal{G}$ is a building set for $\mathscr{L}_P$, then*

$$\widetilde{\mathcal{G}} := \{\cup_{i \in F} E_i : F \in \mathcal{G}\} \cup \{\textit{atoms of } \mathscr{L}_{\widetilde{P}}\}$$

is a building set for $\mathscr{L}_{\widetilde{P}}$. Moreover, $\mathcal{N} \subset \mathcal{G}$ is nested if and only if $\widetilde{\mathcal{N}} := \{\cup_{i \in F} E_i : F \in \mathcal{N}\}$ is nested with respect to $\widetilde{\mathcal{G}}$.

2.5 Combinatorial flats

We introduce the combinatorial flats of a polymatroid, which will be used to describe polymatroid Schubert varieties. The lattice of combinatorial flats is better-behaved than the lattice of ordinary flats of a polymatroid. Throughout this subsection, we fix notations

- $(P, \mathbf{n})$ is a caged polymatroid on $E = \{1, \dots, N\}$, and
- $\widetilde{P}$ is the multisymmetric lift of $(P, \mathbf{n})$, on ground set $\widetilde{E} = E_1 \sqcup \cdots E_N$.

All results of this section are from (Crowley, Simpson, and Wang 2024).

2.5.1 First properties

Definition 2.5.1. A **combinatorial flat** of $(P, \mathbf{n})$ is a multiset $\mathbf{s} \in \mathbb{N}^E$ that is represented by a flat of $\widetilde{P}$.

Suppose $\mathbf{s} \leq \mathbf{n}$ is a multiset represented by $S \subset \widetilde{E}$. The **rank** of $\mathbf{s}$ is defined to be $\mathrm{rk}_{\widetilde{P}}(S)$, and is denoted $\mathrm{rk}_{P,\mathbf{n}}(\mathbf{s})$. The **closure** of $\mathbf{s}$ is the multiset $\overline{\mathbf{s}}$ represented by $\overline{S}$, and $\mathbf{s}^{\mathrm{geo}}$ is the multiset represented by S^{geo}. Multisymmetry guarantees these definitions do not depend on choice of S. From Lemma 2.2.2, we learn:

Lemma 2.5.2.

- *If* $\mathbf{s}$ *is a combinatorial flat of* $(P, \mathbf{n})$*, then* $\mathrm{rk}(\mathbf{s}) = \mathrm{rk}(\mathbf{s}^{\mathrm{geo}}) + |\mathbf{s} - \mathbf{s}^{\mathrm{geo}}|$.
- *If* $\mathbf{a} \leq \mathbf{n}$ *is a multiset, then for each* $1 \leq i \leq N$*, either* $\overline{a}_i = a_i$ *or* $\overline{a}_i = n_i$.
- *If* $\mathbf{s}$ *is a combinatorial flat of* $(P, \mathbf{n})$ *and* $0 < s_i < n_i$*, then* $\mathbf{s} - \mathbf{e}_i$ *is also a combinatorial flat.*

Let $\mathcal{L}_{P,\mathbf{n}}$ denote the poset of combinatorial flats of $(P, \mathbf{n})$, ordered by inclusion. If $\mathbf{s}$ and $\mathbf{s}'$ are multisets, set $\mathbf{s} \vee \mathbf{s}' := \overline{(\max(s_i, s_i'))_i}$ and $\mathbf{s} \wedge \mathbf{s}' := (\min(s_i, s_i'))_i$.

Proposition 2.5.3. $\mathcal{L}_{P,\mathbf{n}}$ *is a lattice with join and meet given by* $\vee$ *and* $\wedge$.

Proof. Given combinatorial flats $\mathbf{s}$ and $\mathbf{s}'$, any multiset containing both $\mathbf{s}$ and $\mathbf{s}'$ must contain $(\max(s_i, s_i'))_i$, and any multiset contained in both $\mathbf{s}$ and $\mathbf{s}'$ must be contained in $\mathbf{s} \wedge \mathbf{s}'$. It remains to show that $\mathbf{s} \vee \mathbf{s}'$ and $\mathbf{s} \wedge \mathbf{s}'$ are combinatorial flats. For $\mathbf{s} \vee \mathbf{s}'$, this is clear from the definition. For $\mathbf{s} \wedge \mathbf{s}'$, we may use multisymmetry to pick flats S and S' of $\widetilde{P}$ representing $\mathbf{s}$ and $\mathbf{s}'$, respectively, such that $S \cap S'$ represents $\mathbf{s} \wedge \mathbf{s}'$. □

Proposition 2.5.4. *For any two multisets* $\mathbf{s}$ *and* $\mathbf{s}'$ *contained in* $\mathbf{n}$*,*

$$\mathrm{rk}_{P,\mathbf{n}}(\mathbf{s} \wedge \mathbf{s}') + \mathrm{rk}_{P,\mathbf{n}}(\mathbf{s} \vee \mathbf{s}') \leq \mathrm{rk}_{P,\mathbf{n}}(\mathbf{s}) + \mathrm{rk}_{P,\mathbf{n}}(\mathbf{s}').$$

Proof. Apply submodularity of matroid rank functions to sets $S, S' \subset \widetilde{E}$ representing $\mathbf{s}$ and $\mathbf{s}'$, respectively, such that $S \cap S'$ represents $\mathbf{s} \wedge \mathbf{s}'$ and $S \cup S'$ represents $(\max(s_i, s_i'))_i$. □

We next consider the effects of polymatroid operations on $\mathcal{L}_{P,\mathbf{n}}$.

Lemma 2.5.5. *Let $i \in E$. The combinatorial flats of $(P \setminus i, \pi_{E\setminus i}(\mathbf{n}))$ are of the form $\pi_{E\setminus i}(\mathbf{s})$ with $\mathbf{s} \in \mathcal{L}_{P,\mathbf{n}}$.*

Proof. Apply Lemma 2.3.1 and Lemma 2.3.2. □

Lemma 2.5.6. *Let $i \in E$. If $\mathbf{s} \leq \mathbf{n}$ is a multiset with $\overline{s}_i < n_i$, then $\pi_{E\setminus i}(\mathbf{s})$ is a combinatorial flat of $(P \setminus i, \pi_{E\setminus i}(\mathbf{n}))$ if and only if $\mathbf{s}$ is a combinatorial flat of $(P, \mathbf{n})$.*

Proof. The "if" direction is Lemma 2.5.5. For the "only if", suppose that $\pi_{E\setminus i}(\mathbf{s})$ is a combinatorial flat of $(P \setminus i, \pi_{E\setminus i}(\mathbf{n}))$. We will show that $\mathbf{s}$ must be a flat by proving $\mathbf{s} = \overline{\mathbf{s}}$.

By Lemma 2.5.5, there is a combinatorial flat $\mathbf{s}'$ of $(P, \mathbf{n})$ with $\pi_{E\setminus i}(\mathbf{s}') = \pi_{E\setminus i}(\mathbf{s})$. If $s_i < s'_i$, then $\mathbf{s} < \mathbf{s}'$, so $\mathbf{s} \leq \overline{\mathbf{s}} \leq \mathbf{s}'$. This means that $s_j = \overline{s}_j = s'_j$ for all $j \neq i$, and $s_i = \overline{s}_i$ by hypothesis; hence $\mathbf{s} = \overline{\mathbf{s}}$. Otherwise, $s_i \geq s'_i$, in which case $\mathbf{s} \geq \mathbf{s}'$. By Lemma 2.5.2, $\mathbf{s}'' = \mathbf{s}' - s'_i\mathbf{e}_i$ is also a combinatorial flat. By Proposition 2.5.4 and Lemma 2.5.2,

$$\operatorname{rk}(\mathbf{s}) = \operatorname{rk}(\mathbf{s}'' \vee s_i\mathbf{e}_i) \leq \operatorname{rk}(\mathbf{s}'') + \operatorname{rk}(s_i\mathbf{e}_i) = \operatorname{rk}(\mathbf{s}') - s'_i + \operatorname{rk}(s_i\mathbf{e}_i) \leq \operatorname{rk}(\mathbf{s}') + (s_i - s'_i),$$

so $\operatorname{rk}(\mathbf{s}) - (s_i - s'_i) \leq \operatorname{rk}(\mathbf{s}')$. On the other hand, by Lemma 2.5.2, $\mathbf{s} - (s_i - s'_i)\mathbf{e}_i$ is a combinatorial flat containing $\mathbf{s}'$ with rank $\operatorname{rk}(\mathbf{s}) - (s_i - s'_i)$. Since $\mathbf{s}'$ is a combinatorial flat, this means $\mathbf{s}' = \mathbf{s} - (s_i - s'_i)\mathbf{e}_i$. Combined with the inequality $\mathbf{s}' \leq \mathbf{s} \leq \overline{\mathbf{s}}$, we see that $s'_j = s_j = \overline{s}_j$ for each $j \neq i$, and $s_i = \overline{s}_i$ by hypothesis, so $\mathbf{s} = \overline{\mathbf{s}}$. □

Lemma 2.5.7. *Let $(P, \mathbf{n})$ be a caged polymatroid on E. Let $F \subset E$ be a flat, and $\mathbf{f} = \sum_{i\in F} n_i\mathbf{e}_i$. The combinatorial flats of $T_F P$ are the combinatorial flats $\mathbf{s}$ of P that either contain $\mathbf{f}$ or satisfy $\overline{\mathbf{s} + \mathbf{e}_i} \not\geq \mathbf{f}$ for every $i \in E$ such that $s_i < n_i$. In the former case, $\operatorname{rk}_{T_F P}(\mathbf{s}) = \operatorname{rk}_P(\mathbf{s}) - 1$, and in the latter, $\operatorname{rk}_{T_F P}(\mathbf{s}) = \operatorname{rk}_P(\mathbf{s})$.*

Proof. For matroids and ordinary flats, this lemma is proved in Brylawski 1986, Proposition 7.4.9. Explicitly, it says that if P is a matroid, then the flats of $T_F P$ are the flats S of P that either contain F or satisfy $\overline{S \cup i} \not\supset F$ for every $i \in E \setminus S$.

By the matroid case and Lemma 2.3.3, a set S is a flat of $\widetilde{T_F P}$ if and only if S is a flat of $\widetilde{P}$ that either contains $\cup_{i\in F} E_i$ or satisfies $\overline{S \cup i} \not\supset \cup_{i\in F} E_i$ for all $i \in \widetilde{E} \setminus S$. In the former case, $\mathrm{rk}_{\widetilde{T_F P}}(S) = \mathrm{rk}_{\widetilde{P}}(S) - 1$; in the latter $\mathrm{rk}_{\widetilde{T_F P}}(S) = \mathrm{rk}_{\widetilde{P}}(S)$. Taking $\mathbf{s}$ to be the multiset represented by S completes the proof. □

2.5.2 Combinatorial flats and simple polymatroids

In the remainder of this section, we show that $\mathcal{L}_{P,\mathbf{n}}$ depends only on the P, then generalize the well-known correspondence between simple matroids and geometric lattices to polymatroids. A polymatroid is **loopless** if it assigns no nonempty set rank 0. Call P **simple** if every one-element subset of its ground set is a flat of positive rank.

Lemma 2.5.8. *Suppose that P is loopless, that $n_i = \mathrm{rk}_P(i)$, and that $\{i\} \subset E$ is not a flat. If $j \in \overline{\{i\}} \setminus i$, $e \in E_i$, and $e' \in E_j$, then* $\mathrm{rk}_{\widetilde{P}}((E_i \setminus e) \cup e') \geq \mathrm{rk}_P(i)$.

Proof. From the definition of $\widetilde{P}$, there is $A \subset E$ such that

$$\mathrm{rk}_{\widetilde{P}}((E_i \setminus e) \cup e') = \mathrm{rk}_P(A) + |((E_i \setminus e) \cup e') \setminus \cup_{k\in A} E_k|.$$

If $i \in A$, then we are done. If $i \notin A$, then

$$\mathrm{rk}_{\widetilde{P}}((E_i \setminus e) \cup e') = \mathrm{rk}_P(A) + |E_i \setminus e| + |\{e'\} \setminus \cup_{k\in A} E_k|.$$

If $A = \emptyset$, then the right-hand side is equal to $|E_i| = \mathrm{rk}_P(i)$. Otherwise, if A is nonempty, then $\mathrm{rk}_P(A) > 0$ because P is loopless, so the right-hand side remains at least $|E_i| = \mathrm{rk}_P(i)$. □

Lemma 2.5.9. *Suppose P is loopless. If $\{i\} \subset E$ is not a flat of P with rank n_i, then* $\mathcal{L}_{R_i(P,\mathbf{n})} \cong \mathcal{L}_{P,\mathbf{n}}$.

Proof. Define $\psi : \mathcal{L}_{P,\mathbf{n}} \to \mathcal{L}_{R_i(P,\mathbf{n})}$ by

$$\psi(\mathbf{s}) = \begin{cases} \mathbf{s}, & s_i < n_i \\ \mathbf{s} - \mathbf{e}_i, & s_i = n_i. \end{cases}$$

Multisets in the image of ψ are combinatorial flats of $\mathcal{L}_{R_i(P,\mathbf{n})}$ by Lemma 2.3.5.

For surjectivity, suppose that $\mathbf{s}' \in \mathcal{L}_{R_i(P,\mathbf{n})}$ and choose a flat S' of $\widetilde{R_i(P,\mathbf{n})}$. By Lemma 2.3.5 and Lemma 2.3.1, there is an element $e \in E_i \setminus S'$ such that $S' \cup e$ or S' is a flat of $\widetilde{P}$. First suppose $S' \cup e$ is a flat of $\widetilde{P}$, representing $\mathbf{s}' + \mathbf{e}_i$, and that $s'_i + 1 = n_i$. In this case, then $\psi(\mathbf{s}' + \mathbf{e}_i) = \mathbf{s}'$, so we are done. Otherwise, if $s'_i + 1 < n_i$ or if S' is a flat of $\widetilde{P}$, then $\mathbf{s}'$ is a combinatorial flat of $(P, \mathbf{n})$, and $\psi(\mathbf{s}') = \mathbf{s}'$.

For injectivity, we break into two cases. If $\mathrm{rk}(i) < n_i$, then P is the underlying polymatroid of both $R_i(P,\mathbf{n})$ and $(P,\mathbf{n})$, so $\mathrm{rk}_{R_i(P,\mathbf{n})}(\psi(\mathbf{s})^{\mathrm{geo}}) = \mathrm{rk}_{P,\mathbf{n}}(\mathbf{s}^{\mathrm{geo}})$. Combining this fact with Lemma 2.5.2, we learn that ψ is rank-preserving. Now, two preimages of the same element of $\mathcal{L}_{R_i(P,\mathbf{n})}$ can differ in at most one coordinate, hence must be comparable. Since ψ preserves rank, they must in fact be equal.

Otherwise, we are in the case where $\mathrm{rk}(i) = n_i$, but $\{i\}$ is not a flat of P. Let $\mathbf{s}, \mathbf{s}' \in \mathcal{L}_{P,\mathbf{n}}$ be two preimages of $\mathbf{s}'' \in \mathcal{L}_{R_i(P,\mathbf{n})}$. The only nontrivial case is when $s'_i = n_i$, which entails $s_i \in \{n_i, n_i - 1\}$. By multisymmetry, we may choose flats $S \subset S'$ of $\widetilde{P}$ representing $\mathbf{s}$ and $\mathbf{s}'$, along with $e \in E_i$ such that $S \supset S' \setminus e$. It suffices to show that $e \in S$.

Since S' is a flat containing E_i, it contains $T := \cup_{j \in \overline{\{i\}}} E_j$. Since $\{i\}$ is not a flat and P is loopless, $T \setminus E_i$ is nonempty, Lemma 2.5.8 informs us that $\mathrm{rk}_{\widetilde{P}}(T \setminus e) \geq \mathrm{rk}_P(i) = \mathrm{rk}_{\widetilde{P}}(T)$. This means $T = \overline{T \setminus e}$, and the latter set is contained in S, so $e \in S$ as desired. □

Corollary 2.5.10. *The poset $\mathcal{L}_{P,\mathbf{n}}$ does not depend on $\mathbf{n}$.*

Proof. No matter the cage $\mathbf{n}$, $\mathcal{L}_{P,\mathbf{n}} \cong \mathcal{L}_{P,(\mathrm{rk}_P(1),\ldots,\mathrm{rk}_P(N)))}$ by Lemma 2.5.9. □

In view of Corollary 2.5.10, we write $\mathcal{L}_P$ for the **poset of combinatorial flats** of P, and define it to be any of the isomorphic posets $\mathcal{L}_{P,\mathbf{n}}$.

Corollary 2.5.11. *There is a unique simple polymatroid P^{sim} such that $\mathcal{L}_P \cong \mathcal{L}_{P^{\mathrm{sim}}}$.*

Proof. Deleting loops from P plainly does not change $\mathcal{L}_P$, so we may assume P it loopless. Now, let $i \in E$ such that $\{i\}$ is not a flat of P. Directly from the definition, one checks that $\{i\}$ is a flat of R_iP and that if $\{j\}$ was a flat of P, then it remains a flat of R_iP. We obtain P^{sim} by repeating this construction for each singleton that is not a flat of P (deleting loops as needed between reductions), then apply Lemma 2.5.9 to obtain $\mathcal{L}_P \cong \mathcal{L}_{P^{\text{sim}}}$.

For uniqueness, note the ground set of P^{sim} is in bijection with maximal join-irreducible elements of $\mathcal{L}_{P^{\text{sim}}}$. We recover the rank function of P^{sim} by taking ranks of joins of these elements in $\mathcal{L}_{P^{\text{sim}}}$. □

2.5.3 Axioms

We provide an abstract characterization for lattices of combinatorial flats, generalizing the notion of geometric lattice. Let $\mathcal{L}$ be a graded lattice with minimum element $\hat{0}$. The **nullity** of $e \in \mathcal{L}$ is $\text{null}(e) := \sum_{e' \leq e} \text{rk}_{\mathcal{L}}(e') - \text{rk}_{\mathcal{L}}(e)$, where e' runs over maximal irreducibles of the interval $[\hat{0}, e]$. The **geometric part** of e, denoted e^{geo}, is the join of all maximal irreducibles of $\mathcal{L}$ that lie in $[\hat{0}, e]$.

Theorem 2.5.12. *A poset $\mathcal{L}$ is the lattice of combinatorial flats of a polymatroid P if and only if*

- *$\mathcal{L}$ is a semimodular lattice,*
- *the join-irreducibles of $\mathcal{L}$ form a downward-closed set, and*
- *for any $e \in \mathcal{L}$, $\text{null}(e) = \text{null}(e^{\text{geo}})$.*

Proof. If P is a (without loss of generality) simple polymatroid, then the join irreducibles of $\mathcal{L}_P$ are the combinatorial flats of the form $s\mathbf{e}_i$ for some $i \in E$ and $s \in \mathbb{N}$. The remainder of the properties are verified by Lemma 2.5.2, Proposition 2.5.3, and Proposition 2.5.4.

Conversely, suppose that $\mathcal{L}$ has the enumerated properties. Let E be the set of maximal join-irreducibles of L and define a polymatroid P by $\text{rk}_P(A) := \text{rk}_{\mathcal{L}}(\vee_{i \in A} i)$. This defines a simple polymatroid because $\mathcal{L}$ is a submodular lattice. Let $\mathbf{n} = (\text{rk}_{\mathcal{L}}(i))_{i \in E}$.

Since the irreducibles of $\mathcal{L}$ are downward closed, for each $i \in E$, $[\hat{0}, i] = \{\hat{0} < x_{i,1} < \cdots < x_{i,n_i} = i\}$. By construction of P, the following are equivalent:

- $F \subset E$ is a flat of rank r
- $\{i : x_{i,n_i} \leq \vee_{j \in F} x_{j,n_j}\} = F$ and $\mathrm{rk}_{\mathcal{L}}(\vee_{j \in F} x_{j,n_j}) = r$,
- and $\sum_{j \in F} n_j \mathbf{e}_j$ is a combinatorial flat of rank r.

Define maps

$$\varphi : \mathcal{L}_P \cong \mathcal{L}_{P,\mathbf{n}} \to \mathcal{L}, \quad \varphi(\mathbf{s}) = \vee_{i \in E}\, x_{i,s_i}$$
$$\varphi' : \mathcal{L} \to \mathcal{L}_{P,\mathbf{n}} \cong \mathcal{L}_P, \quad x \mapsto \vee_{x_{i,j} \leq x}\, j\mathbf{e}_i.$$

Plainly, the compositions $\varphi \circ \varphi'$ and $\varphi' \circ \varphi$ are increasing maps. To finish, we show that both φ and φ' decrease rank. Let $\mathbf{s}$ be a combinatorial flat and let $F = \{i : s_i = n_i\}$, so that $\mathbf{s}^{\mathrm{geo}} = \sum_{i \in F} n_i \mathbf{e}_i$. Using submodularity of $\mathrm{rk}_{\mathcal{L}}$, our discussion of flats of P, and Lemma 2.5.2, we obtain

$$\mathrm{rk}_{\mathcal{L}}(\varphi(\mathbf{s})) \leq \mathrm{rk}_{\mathcal{L}}(\vee_{i \in F} x_{i,n_i}) + \sum_{i \in E \setminus F} s_j = \mathrm{rk}_P(F) + \sum_{i \in E \setminus F} s_j = \mathrm{rk}_P(\mathbf{s}^{\mathrm{geo}}) + |\mathbf{s} - \mathbf{s}^{\mathrm{geo}}| = \mathrm{rk}_P(\mathbf{s}).$$

This shows that φ decreases rank. On the other hand, if $x \in \mathcal{L}$ and $F = \{i : x_{i,n_i} \leq x\}$, then

$$\sum_{i \in E} \max\{j : x_{ij} \leq x\} - \mathrm{rk}_{\mathcal{L}}(x) = \mathrm{null}(x) = \mathrm{null}(x^{\mathrm{geo}}) = \sum_{i : x_{i,n_i} \leq x} n_i - \mathrm{rk}_{\mathcal{L}}(x^{\mathrm{geo}}),$$

so

$$\begin{aligned} \mathrm{rk}_{\mathcal{L}}(x) &= \mathrm{rk}_{\mathcal{L}}(x^{\mathrm{geo}}) + \sum_{i : x_{i,n_i} \not\leq x} \max\{j : x_{i,j} \leq x\} \\ &= \mathrm{rk}_P\left(\sum_{i \in F} n_i \mathbf{e}_i\right) + \sum_{i : x_{i,n_i} \not\leq x} \max\{j : x_{i,j} \leq x\} \geq \mathrm{rk}_P(\vee_{x_{i,j} \leq x} j\mathbf{e}_j) = \mathrm{rk}_P(\varphi'(x)). \end{aligned}$$

This shows that φ' also decreases rank. Hence, both $\varphi \circ \varphi'$ and $\varphi' \circ \varphi$ are increasing maps that decrease rank, which means they must be the identity. $\square$

2.6 A remark on the characteristic polynomial

Let $\mathcal{L}$ be a finite lattice with maximum element $\hat{1}$, minimum element $\hat{0}$, equipped with a function $r : \mathcal{L} \to \mathbb{N}$. The **characteristic polynomial** of $(\mathcal{L}, r)$ is

$$\chi_{\mathcal{L}}(t) = \sum_{x \in \mathcal{L}} \mu(x) t^{r(\hat{1}) - r(x)} = \sum_{k=0}^{r(\hat{1})} w_k t^{r(\hat{1}) - k},$$

where μ is defined recursively by $\mu(\hat{0}) = 1$ and $\mu(x) = -\sum_{x' < x} \mu(x')$. If $\mathcal{L}$ is graded and r is its rank function, then the coefficients of $\chi_{\mathcal{L}}$ alternate in sign (Stanley 2011, Proposition 3.10.1). If $\mathcal{L}$ is the lattice of flats of a matroid, then this definnition agrees with the one given in Section 1.1.2, so its coefficients are both alternating and log-concave. On the other hand, the characteristic polynomial of the lattice of flats of a polymatroid is quite badly behaved, displaying neither log-concave coefficients nor even alternating signs (Pagaria and Pezzoli 2021, Remark 6.6).

Proposition 2.6.1. *Let P be a polymatroid. The coefficients of $\chi_{\mathcal{L}_P}$ are alternating and log-concave.*

Proof. Without loss of generality, we may assume P is simple on groundset E. By Rota's crosscut theorem (Rota 1964, Theorem 3), if $\mathbf{s}$ is a combinatorial flat, then

$$\mu_{\mathcal{L}_P}(\mathbf{s}) = \sum_{\substack{S \subset E \\ \vee_{i \in S} \mathbf{e}_i = \mathbf{s}}} (-1)^{|S|}.$$

In particular, if $\mathbf{s}$ cannot be written as a join of atoms of $\mathcal{L}_P$, then $\mu_{\mathcal{L}_P}(\mathbf{s}) = 0$.

This discussion shows that if $\mathcal{L}'$ is the subposet of $\mathcal{L}_P$ consisting of all joins of $\mathbf{e}_i$'s, then

$$\chi_{\mathcal{L}_P}(t) = \sum_{\mathbf{s} \in \mathcal{L}'} \mu(\mathbf{s}) t^{\mathrm{rk}(P) - \mathrm{rk}(\mathbf{s})} = t^{\mathrm{rk}(P) - \mathrm{rk}((1,1,\ldots,1))} \chi_{\mathcal{L}'}(t).$$

We show that $\mathcal{L}'$ is a lattice. It is plainly has joins, maximum element, and minimum element, hence also has meets. It is also graded: let $e = \mathrm{rk}_P(\vee_{i\in E}\mathbf{e}_i)$ and suppose there is a chain $\mathbf{s}_* = \{\mathbf{0} < \mathbf{s}_1 < \mathbf{s}_2 < \cdots < \mathbf{s}_k = \vee_{i\in E}\mathbf{e}_i\}$ with $k < e$. Then there is i such that $\mathrm{rk}_P(\mathbf{s}_i) - \mathrm{rk}_P(\mathbf{s}_{i-1}) > 1$. Choose some j such that $s_{i,j} > s_{i-1,j}$. Then $\mathrm{rk}_P(\mathbf{s}_i) > \mathrm{rk}_P(\mathbf{s}_{i-1} + \mathbf{e}_j)$ by Proposition 2.5.4, so $\mathbf{s}_*$ is not a maximal chain. Hence, all maximal chains in $\mathcal{L}$ have length e. Combined with Proposition 2.5.4, this shows that $\mathcal{L}'$ is a geometric lattice, i.e. the lattice of flats of a matroid, so the desired results follow from applying (Adiprasito, Huh, and Katz 2018) and (Stanley 2011, Proposition 3.10.1). $\square$

Chapter 3

Smooth geometry of polymatroids

This short chapter comprises a condensed account of Chow rings for polymatroids and the fan that underlies them. Material is drawn from (Crowley, Huh, et al. 2020), except where otherwise noted. We have chosen to elide certain details.

3.1 The Chow ring of a polymatroid

Let P be a polymatroid and $\mathcal{G}$ a geometric building set for $\mathscr{L}_P$. Pagaria and Pezzoli 2021 introduced the Chow ring of a polymatroid.

Definition 3.1.1. Let P be a polymatroid and $\mathcal{G} \subseteq \mathscr{L}_P$ a geometric building set. The **Chow ring** of $(P, \mathcal{G})$ is

$$\mathrm{DP}(P, \mathcal{G}) \coloneqq \mathbb{Z}[x_F : F \in \mathcal{G}]/I,$$

where I is the ideal generated by

$$x_{G_1} \cdots x_{G_k} \Big(\sum_{H \geq G} x_H \Big)^b$$

for $G \in \mathcal{G}$, $\mathcal{S} = \{G_1, \ldots, G_k\} \subseteq \mathcal{G}$, and $b \geq \mathrm{rk}_P(G) - \mathrm{rk}_P(\cup \mathcal{S}_{<G})$. **Nested sets** with respect to $\mathcal{G}$ are defined as in Definition 2.4.1.

Let $V \subset \prod_i \mathbb{C}^{n_i}$, and let H_i be the subspace of V defined by setting all coordinates of

$\mathbb{C}^{n_i}$ equal to zero. Let $U = V \setminus \cup_i H_i$. Via the obvious generalization of the construction in Definition 3.1.1, each geometric building set of $\mathscr{L}_P$ gives rise to a wonderful compactification $W_{\mathcal{G}}$ of U. In this case, $\mathrm{DP}(P)$ is isomorphic to the cohomology ring of $W_{\mathcal{G}}$. Moreover, if P is a matroid, then $\mathrm{DP}(P)$ is isomorphic to the Chow ring $A(M)$.

Recall from Section 1.3.1 that $A(M)$ has the Kähler package, even when it is not the cohomology ring of a projective variety. A main result of (Pagaria and Pezzoli 2021) generalized this.

Theorem 3.1.2. *(Pagaria and Pezzoli 2021)* $\mathrm{DP}(P, \mathcal{G})$ *has the Kähler package.*

At the time of its appearance, Theorem 3.1.2 was somewhat mysterious. Recall from Section 1.3.1 that when M is a realizable matroid, $A(M)$ is isomorphic to the cohomology ring of a smooth projective variety. If M is not realizable, this is no longer the case; however, $A(M)$ *can* be viewed as the Chow ring of a simplicial fan Σ_M, called the **Bergman fan** of M. The support of the Bergman fan is a **tropical linear space** in the sense of (Maclagan and Sturmfels 2021, Definition 4.2.5), so one can sensibly view $A(M)$ as the cohomology ring of some tropical space. This view was reinforced by:

Theorem 3.1.3. *(Ardila, Denham, and Huh 2023) If* Σ *and* Σ' *are two simplicial fans whose combinatorial ample cones are both nonempty, then the Chow ring of* Σ *has the Kähler package if and only if the Chow ring of* Σ' *does.*

Hence, the Kähler package for $A(M)$ is morally a property of the tropical linear space that is the support of Σ_M.

In contrast to this view of $A(M)$ as a "tropical cohomology ring", the polymatroid Chow ring (Pagaria and Pezzoli 2021) seemed to have the Kähler package without a tropical space in evidence. We now explain how (Crowley, Huh, et al. 2020) resolves this discrepancy.

3.2 The Bergman fan of a polymatroid

Let P be a polymatroid on $E = \{1, \dots, N\}$, and $\widetilde{P}$ on $\widetilde{E} = E_1 \sqcup \cdots \sqcup E_N$ its lift with respect to $\mathbf{n} = (\mathrm{rk}_P(i))_{i \in E}$. Let $\mathbb{R}^{\widetilde{E}}$ be the vector space spanned by $\mathbf{e}_i$ for $i \in \widetilde{E}$, and write $\mathbf{e}_S \coloneqq \sum_{i \in S} \mathbf{e}_i$ for $S \subseteq \widetilde{E}$. If $\mathcal{S} \subseteq 2^{\widetilde{E}}$ is a collection of subsets, write

$$\sigma_{\mathcal{S}} \coloneqq \mathrm{cone}(\mathbf{e}_S : S \in \mathcal{S}) \subseteq \mathbb{R}^{\widetilde{E}}/\mathbb{R}(1, 1, \dots, 1).$$

Recall from Section 2.4 that each building set of $\mathscr{L}_P$ lifts to a building set of $\mathscr{L}_{\widetilde{P}}$.

Definition 3.2.1. Let $\mathcal{G}$ be a geometric building set for $\mathscr{L}_P$. The **Bergman fan** associated to $(P, \mathcal{G})$ is $\Sigma_{P,\mathcal{G}} \coloneqq \{\sigma_{\mathcal{N}}\}_{\mathcal{N}}$, where $\mathcal{N}$ ranges over all $\widetilde{\mathcal{G}}$-nested sets of $\widetilde{P}$ such that $\widetilde{E} \notin \mathcal{N}$.

Definition 3.2.2. The **Chow ring** of $\Sigma_{P,\mathcal{G}}$ is

$$A(\Sigma_{P,\mathcal{G}}) = \mathbb{Z}[z_G : G \in \widetilde{\mathcal{G}} \setminus \{\widetilde{E}\}]/I_{P,\mathcal{G}}$$

where $I_{P,\mathcal{G}}$ is the ideal generated by

$$\begin{aligned} z_{G_1} \cdots z_{G_k} \quad & \text{for any collection } \{G_1, \dots, G_k\} \text{ that is not } \widetilde{\mathcal{G}}\text{-nested}, \\ \sum_{i \in G} z_G - \sum_{j \in F} z_F \quad & \text{for any } i, j \in \widetilde{E}. \end{aligned}$$

Two facts make $\Sigma_{P,\mathcal{G}}$ a promising candidate for a "space" to explain the Kähler package of $\mathrm{DP}(P, \mathcal{G})$. First, the support of $\Sigma_{P,\mathcal{G}}$ is equal to that of $\Sigma_{\widetilde{P}}$, so $\Sigma_{P,\mathcal{G}}$ is supported on a tropical linear space. By Theorem 3.1.3, this means that $A(\Sigma_{P,\mathcal{G}})$ has the Kähler package (assuming its combinatorial ample cone is nonempty).

Second, suppose that P is realized by a linear subspace $V \subset \prod_{i=0}^{N} \mathbb{C}^{n_i}$. Let $U \subset V$ be the complement of the N coordinate subspaces (as in Section 3.1). Let $U' \subset V$ be the complement of the $n_1 + \cdots + n_N$ coordinate hyperplanes in V. The wonderful compactification $W_{\widetilde{\mathcal{G}}}$ of U' with respect to $\widetilde{\mathcal{G}}$ is the same as the wonderful compactification

$W_{\widetilde{\mathcal{G}}}$ of U with respect to $\mathcal{G}$. By results of (Feichtner and Yuzvinsky 2004), it follows that there are isomorphisms

$$\mathrm{DP}(P, \mathcal{G}) \cong H^*(W_{\mathcal{G}}) \cong H^*(W_{\widetilde{\mathcal{G}}}) \cong A(\Sigma_{P,\mathcal{G}}).$$

The composite isomoprhism $\mathrm{DP}(P, \mathcal{G}) \cong A(\Sigma_{P,\mathcal{G}})$ extends to polymatroids in general. To prove this, we make use of a slightly different presentation for $A(\Sigma_{P,\mathcal{G}})$, originating in (Feichtner and Yuzvinsky 2004). For $F \in \widetilde{\mathcal{G}}$, set

$$y_F = \begin{cases} -\sum_{i \in G} z_G, & F = \widetilde{E} \\ z_F, & \text{otherwise.} \end{cases}$$

In terms of the y_F's, $A(\Sigma_{P,\mathcal{G}})$ is defined by the ideal I_{FY} generated by

$$\begin{aligned} &y_{G_1} \cdots y_{G_k}, \quad \{G_1, \ldots, G_k\} \text{ not } \widetilde{\mathcal{G}}\text{-nested and} \\ &\sum_{i \in G} y_G, \quad i \in \widetilde{E}. \end{aligned}$$

Theorem 3.2.3. *(Crowley, Huh, et al. 2020) There is an isomorphism*

$$\mathrm{DP}(P, \mathcal{G}) \to A(\Sigma_{P,\widetilde{P}}), \quad x_F \mapsto z_{\cup_{i \in F} E_i}.$$

Proof. Let $I_{DP} \subseteq \mathbb{Z}[x_F : F \in \mathcal{G}]$ be the defining ideal of $\mathrm{DP}(P, \mathcal{G})$, and let $I_{FY} \subseteq \mathbb{Z}[y_F : F \in \widetilde{\mathcal{G}}]$ be the defining ideal of $A(\Sigma_{P,\mathcal{G}})$ as in (Feichtner and Yuzvinsky 2004). We define the following map on polynomial rings.

$$\varphi \colon \mathbb{Z}[x_F : F \in \mathcal{G}] \to \mathbb{Z}[y_F : F \in \widetilde{\mathcal{G}}], \quad x_F \mapsto y_{\pi^{-1}(F)}$$

First we show that $\varphi(I_{DP}) \subseteq I_{FY}$. Write $f \in I_{DP}$ for one of the defining relations of I_{DP}:

$$f = \Big(\prod_{F \in \mathcal{S}} x_F\Big)\Big(\sum_{\mathcal{G} \ni H \geq G} x_H\Big)^b.$$

By (Feichtner and Yuzvinsky 2004, Theorems 1 and 3), I_{FY} contains the following two types of polynomials:

$$\prod_{F\in\mathcal{S}} y_F, \qquad \mathcal{S} \text{ not } \widetilde{\mathcal{G}}\text{-nested},$$

$$\prod_{F\in\mathbb{N}} y_F\Big(\sum_{H\geq G} y_G\Big)^d, \quad \mathbb{N} \text{ a nested antichain, } \cup\mathbb{N} < G, \text{ and } d = \mathrm{rk}(G) - \mathrm{rk}(\cup N).$$

If $\mathcal{S}$ is not $\mathcal{G}$-nested, then $\tilde{\mathcal{S}} := \{\pi^{-1}(F) : F \in \mathcal{S}\}$ is not $\tilde{\mathcal{G}}$-nested by Proposition 2.4.2. Hence, $\varphi(f)$ is divisible by a relation of the first type. Otherwise, $\mathcal{S}$ is $\mathcal{G}$-nested, so $\tilde{\mathcal{S}}$ is $\widetilde{\mathcal{G}}$-nested. In this case, $\varphi(f)$ is divisible by a relation of the second type because

$$b \geq \mathrm{rk}_P(G) - \mathrm{rk}_P(\cup\mathcal{S}_{<G}) = \mathrm{rk}_{\tilde{P}}(\pi^{-1}(G)) - \mathrm{rk}_{\tilde{P}}(\cup\tilde{\mathcal{S}}_{<\pi^{-1}(G)}) = \mathrm{rk}_{\tilde{P}}(\pi^{-1}(G)) - \mathrm{rk}_{\tilde{P}}(\cup \max \tilde{\mathcal{S}}_{<\pi^{-1}(G)}).$$

This proves that $\varphi(I_{DP}) \subseteq I_{FY}$, so φ descends to $\bar{\varphi}\colon \mathrm{DP}(P,\mathcal{G}) \to A(\Sigma_{P,\mathcal{G}})$.

If $F \in \widetilde{\mathcal{G}}$ is a flat of rank greater than 1, then y_F is in the image of $\bar{\varphi}$. By the linear relation $\sum_{i\in G} y_G = 0$, it follows that y_i is also in the image of $\bar{\varphi}$. Therefore, $\bar{\varphi}$ is surjective. It remains to show that $\bar{\varphi}$ is injective. By (Feichtner and Yuzvinsky 2004, Theorem 2), the generators of I_{FY} in the previous paragraph are a Gröbner basis with respect to any lexicographic monomial order $<$ in which $F_1 \subseteq F_2$ implies $y_{F_1} > y_{F_2}$. Any such order is an elimination order with respect to $\{y_i : i \in \widetilde{E}\}$. By (Eisenbud 1995, Proposition 15.29)[1], the generators of I_{FY} in the previous paragraph that do not involve any y_i, $i \in \widetilde{E}$, are a Gröbner basis for $\mathrm{im}(\varphi) \cap I_{FY}$. Any such polynomial is the image of a generator of I_{DP}, so $\varphi^{-1}(I_{FY}) = I_{DP}$. This implies $\bar{\varphi}$ is an isomorphism. □

Corollary 3.2.4. *The Chow ring of a polymatroid with respect to any geometric building set has the Kähler package.*

Proof. The combinatorial ample cones of $\Sigma_{P,\mathcal{G}}$ and $\Sigma_{\widetilde{P}}$ are nonempty by (Crowley, Huh, et al. 2020, Lemma 3.9) and (Adiprasito, Huh, and Katz 2018, Proposition 2.4). Since

[1]Eisenbud's proof of this statement works over $\mathbb{Z}$ because all leading coefficients in our Gröbner basis are 1.

$A(\widetilde{P}) \cong A(\Sigma_{\widetilde{P}})$ has the Kähler package, so does $A(\Sigma_{P,\mathcal{G}}) \cong \mathrm{DP}(P, \mathcal{G})$ by Theorem 3.1.3. $\square$

Chapter 4

Singular geometry of polymatroids

Over the course of this chapter, we develop a generalization of matroid Schubert varieties. The new varieties, called **polymatroid Schubert varieties** are equivariant compactifications of the additive group of a vector space. The group action partitions them into orbits corresponding to the combinatorial flats of a polymatroid. All content in this chapter is reproduced from (Crowley, Simpson, and Wang 2024).

Throughout, we take $\mathbb{K}$ to be an algebraically closed field of characteristic zero.

4.1 The Hassett-Tschinkel compactification

We begin our study of polymatroid Schubert varieties by reviewing a construction of Hassett and Tschinkel Hassett and Tschinkel 1999, which we will eventually think of as the variety associated to a polymatroid on one element. Loosely speaking, Hassett and Tschinkel show that there is only one action of the n-dimensional additive group $\mathbb{G}_a^n$ on $\mathbb{P}^n$ with finitely many orbits. More precisely:

Theorem 4.1.1. *Hassett and Tschinkel 1999, Proposition 3.7 Over an algebraically closed field of characteristic 0, there is a left action of* $\mathbb{G}_a^n$ *on* $\mathbb{P}^n$ *with finitely many orbits and such that the stabilizer of a general point of* $\mathbb{P}^n$ *is trivial. The action is unique up to isomorphism of* $\mathbb{G}_a^n$*-variety structures.*

Since there are finitely many orbits, there is a unique n-dimensional orbit. The condition on the stabilizer implies this orbit is isomorphic to $\mathbb{G}_a^n$, so Theorem 4.1.1 allows us to view $\mathbb{P}^n$ as an equivariant compactification of $\mathbb{G}_a^n$.

To make matters concrete, choose coordinates $a_1, \ldots, a_n$ on $\mathbb{G}_a^n$. The action on $\mathbb{P}^n$ is given by the faithful representation $\rho : \mathbb{G}_a^n \to \mathrm{Aut}(\mathbb{P}^n)$,

$$(a_1, \ldots, a_n) \mapsto \exp \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ a_1 & 0 & 0 & \ddots & \vdots \\ a_2 & a_1 & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ a_n & \cdots & a_2 & a_1 & 0 \end{pmatrix}.$$

The entries in each northwest-to-southeast diagonal of $\rho(a)$ are all equal to one another. The (i, j) entry of $\rho(a_1, \ldots, a_n)$ is

$$(\rho(a_1, \ldots, a_n))_{i,j} = \begin{cases} \frac{1}{(i-j)!} B_{i-j}(1!a_1, 2!a_2, \ldots, (i-j)!a_{i-j}), & i \geq j \\ 0, & \text{otherwise,} \end{cases}$$

where B_k is the (exponential) **Bell polynomial** of degree k, defined by the recurrence $B_0 = 1$ and

$$B_{k+1}(x_1, \ldots, x_{k+1}) = \sum_{\ell=0}^{k} \binom{k}{\ell} B_{k-\ell}(x_1, \ldots, x_{k-\ell}) x_{\ell+1}.$$

This action gives an embedding $\iota : \mathbb{G}_a^n \to \mathbb{P}^n$ defined by $\iota(a_1, \ldots, a_n) = \rho(a_1, \ldots, a_n) \cdot [1 : 0 : 0 : \cdots : 0]$. On the image of ι, the coordinates $b_0, \ldots, b_n$ of $\mathbb{P}^n$ are related to $a_1, \ldots, a_n$ by the formula

$$b_i/b_0 = \sum_{j_1 + 2j_2 + \cdots + ij_i = i} \frac{a_1^{j_1} a_2^{j_2} \cdots a_i^{j_i}}{j_1! j_2! \cdots j_i!}.$$

The explicit descriptions above make evident the following facts.

Proposition 4.1.2. *The action of $\mathbb{G}_a^n$ on $\mathbb{P}^n$ partitions $\mathbb{P}^n$ into orbits*

$$O_k := \{b_0 = b_1 = \cdots = b_{n-k-1} = 0,\, b_{n-k} \neq 0\}, \quad 0 \leq k \leq n.$$

The stabilizer of any point of O_k is

$$S_k := \{a_1 = a_2 = \cdots = a_k = 0\} \subset \mathbb{G}_a^n,$$

and the closure of O_k in $\mathbb{P}^n$ is $\overline{O_k} = \cup_{i \leq k} O_i$.

Example 4.1.3. The action of $\mathbb{G}_a^2$ on $\mathbb{P}^2$ is

$$(a_1, a_2)\cdot[b_0 : b_1 : b_2] = \begin{pmatrix} 1 & 0 & 0 \\ a_1 & 1 & 0 \\ \frac{1}{2}a_1^2 + a_2 & a_1 & 1 \end{pmatrix} \cdot [b_0 : b_1 : b_2] = \left[b_0 : a_1 b_0 + b_1 : \tfrac{1}{2}a_1^2 b_0 + a_2 b_0 + a_1 b_1 + b_2\right].$$

◊

Remark 4.1.4. Consider the semidirect product $\mathbb{G}_a^n \rtimes \mathbb{G}_m$, where $\mathbb{G}_m$ acts on $\mathbb{G}_a^n$ by $t \cdot \mathbf{e}_i = t^i \mathbf{e}_i$. The identity of $\mathbb{G}_a^n$ is the unique fixed point of this $\mathbb{G}_m$-action. If we define

$$\lambda : \mathbb{G}_m \to \mathrm{Aut}(\mathbb{P}^n), \quad t \mapsto \begin{pmatrix} 1 & & & & \\ & t & & & \\ & & t^2 & & \\ & & & \ddots & \\ & & & & t^n \end{pmatrix} \in \mathrm{Aut}(\mathbb{P}^n),$$

then ρ extends to an injective homomorphism

$$\mathbb{G}_a^n \rtimes \mathbb{G}_m \hookrightarrow \mathrm{Aut}(\mathbb{P}^n), \quad (a, t) \mapsto \rho(a)\lambda(t),$$

in which the image of λ normalizes the image of ρ. The orbits of $\mathbb{G}_a^n \rtimes \mathbb{G}_m$ on $\mathbb{P}^n$ are the same as those of $\mathbb{G}_a^n$, but each $\mathbb{G}_a^n$-orbit O_i contains a unique $\mathbb{G}_m$ fixed point. Consequently, we may canonically identify O_i with $\mathbb{G}_a^n/S_i$.

4.2 Polymatroid Schubert varieties

We now consider polymatroid Schubert varities in general. Let $V \subset \mathbb{K}^{\mathbf{n}}$ be a linear subspace realizing a polymatroid P. Choose coordinates $a_{i,1}, \ldots, a_{i,n_i}$ on $\mathbb{K}^{n_i}$ for each $1 \leq i \leq N$. The Hassett-Tschinkel embeddings

$$\mathbb{K}^{n_i} = \mathbb{G}_a^{n_i} \hookrightarrow \mathbb{P}^{n_i}, \quad (a_1, \ldots, a_{n_i}) \mapsto \rho(a_{i1}, \ldots, a_{i,n_i}) \cdot [1 : 0 : 0 : \cdots : 0]$$

assemble to $\iota : \mathbb{K}^{\mathbf{n}} \hookrightarrow \mathbb{P}^{\mathbf{n}}$. For each $\mathbf{0} \leq \mathbf{s} \leq \mathbf{n}$, there is an orbit $O_{\mathbf{s}} := O_{s_1} \times \cdots \times O_{s_N} \subset \mathbb{P}^{\mathbf{n}}$, whose closure is a product of projectivized coordinate subspaces $\mathbb{P}^{\mathbf{s}} \subset \mathbb{P}^{\mathbf{n}}$. All points in $O_{\mathbf{s}}$ have the same stabilizer, denoted by $S_{\mathbf{s}}$. The description of stabilizers in Section 4.1 shows that $S_{\mathbf{s}}$ is the coordinate subspace of $\mathbb{K}^{\mathbf{n}}$ defined by

$$a_{i,1} = a_{i,2} = \cdots = a_{i,s_i} = 0, \quad 1 \leq i \leq N.$$

Definition 4.2.1. The **Schubert variety** of $V \subset \mathbb{K}^{\mathbf{n}}$ is $Y_V := \overline{\iota(V)}$, the closure of $\iota(V)$ in $\mathbb{P}^{\mathbf{n}}$.

The Schubert variety Y_V is V-equivariant, hence is a union of V-orbits. For general V, there may be infinitely many orbits. This pathology is avoided by imposing a mild genericity condition.

Call V **polymatroid general** (henceforth, **p.g.**) if for each combinatorial flat $\mathbf{s}$ of $(P, \mathbf{n})$,

$$\mathrm{rk}(\mathbf{s}) = \mathrm{codim}_V\, V \cap S_{\mathbf{s}}.$$

Remark 2.2.5 shows that any realizable polymatroid has a p.g. realization.

Proposition 4.2.2. *If V is p.g. and $i \in E$, then $\pi_{E\setminus i}(V)$ is p.g.*

Proof. Let $\mathbf{s}'$ be a combinatorial flat of $V' = \pi_{E\setminus i}(V)$ and let $S_{\mathbf{s}'} \subset \mathbb{K}^{\pi_{E\setminus i}(\mathbf{n})}$ be the corresponding stabilizer. Let $\mathbf{s} = (s'_1, \ldots, s'_{i-1}, 0, s'_{i+1}, \ldots, s'_N)$. There is a combinatorial flat of V that projects to $\mathbf{s}'$ by Lemma 2.5.5, and $\mathbf{s}$ must be contained in any such combinatorial flat. By Lemma 2.5.2, this implies that $\pi_{E\setminus i}(\overline{\mathbf{s}}) = \mathbf{s}'$. In either case, $\mathrm{rk}_P(\overline{\mathbf{s}}) = \mathrm{rk}_P(\mathbf{s}) = \mathrm{rk}_{P\setminus i}(\mathbf{s}')$ and $V \cap S_{\overline{\mathbf{s}}} = V \cap S_{\mathbf{s}}$.

Since V is p.g. and $S_{\mathbf{s}} \supset \ker(\pi_{E\setminus i})$,

$$\begin{aligned}\mathrm{rk}_{P\setminus i}(\mathbf{s}') = \mathrm{rk}_P(\overline{\mathbf{s}}) &= \dim V - \dim V \cap S_{\overline{\mathbf{s}}} = \dim V - \dim V \cap S_{\mathbf{s}} \\ &= (\dim V - \dim V \cap \ker(\pi_{E\setminus i})) - (\dim V \cap S_{\mathbf{s}} - \dim V \cap \ker(\pi_{E\setminus i})) \\ &= \dim V' - \dim \pi_{E\setminus i}(V \cap S_{\mathbf{s}}) \\ &= \dim V' - \dim V' \cap S_{\mathbf{s}'}\end{aligned}$$

where the final equality holds because $\pi_{E\setminus i}^{-1}(S_{\mathbf{s}'}) = S_{\mathbf{s}}$ and V surjects onto V'. □

Proposition 4.2.3. *Let F be a flat of p.g. subspace V. If H is the preimage in $\mathbb{K}^{\mathbf{n}}$ of a general hyperplane in $\mathbb{K}^{\pi_F(\mathbf{n})}$, then $V \cap H$ is p.g.*

Proof. Let $\mathbf{s}$ be a combinatorial flat of $V \cap H$. By Lemma 2.5.7, $\mathbf{s}$ is also a combinatorial flat of V. Since V is p.g., it suffices to show that H contains $V \cap S_{\mathbf{s}}$ if and only if $F \subset \{i : s_i = n_i\}$. If $F \subset \{i : s_i = n_i\}$, then $H \supset S_{\mathbf{s}} \supset V \cap S_{\mathbf{s}}$. Otherwise, suppose that H contains $V \cap S_{\mathbf{s}}$. Since H is general, this means that $\pi_F(V \cap S_{\mathbf{s}}) = 0$. Since $\mathbf{s}$ is a combinatorial flat, this means that $F \subset \{i : s_i = n_i\}$. □

Some notation

Throughout the remainder of this section, fix notation:

- $\mathbf{n} = (n_1, \ldots, n_N)$ is a cage for a polymatroid P on $E = \{1, \ldots, N\}$
- $V \subset \mathbb{K}^{\mathbf{n}}$ is a p.g. realization of P
- $\widetilde{P}$ is the lift of P with respect to $\mathbf{n}$, on ground set $\widetilde{E} = E_1 \sqcup \cdots \sqcup E_N$.

Phrases like "combinatorial flat of V" should be taken to mean "combinatorial flat of $(P, \mathbf{n})$."

4.3 Strata

The main result of this section describes the poset of strata of Y_V.

Theorem 4.3.1. *For a p.g. subspace V, the stratum $Y_V \cap O_{\mathbf{s}}$ is nonempty if and only if $\mathbf{s}$ is a combinatorial flat of V, and in this case, it is a single V-orbit of dimension* $\operatorname{rk}(\mathbf{s})$*. The poset of orbit-closures is isomorphic to the lattice of combinatorial flats of V.*

We use the combinatorics developed in Section 2.5 to establish Theorem 4.3.1 via a series of lemmas.

Lemma 4.3.2. *If $\mathbf{s}$ is not a combinatorial flat of V, then $Y_V \cap O_{\mathbf{s}}$ is empty.*

Proof. We induct on N. For any point $x \in O_{\mathbf{s}}$,

$$\dim(V \cdot x) = \operatorname{codim}_V(V \cap S_{\mathbf{s}}) = \operatorname{rk}(\mathbf{s}). \tag{$*$}$$

Since Y_V is a union of V-orbits, the desired statement follows immediately from dimensional considerations when $N = 1$.

Otherwise, suppose $N > 1$. If $\operatorname{rk}(\mathbf{s}) = d$, then the dimension formula implies $Y_V \cap O_{\mathbf{s}}$ is nonempty if and only if $\mathbf{s} = (n_1, \ldots, n_N)$. Otherwise, if $\operatorname{rk} \mathbf{s} < d$, then there is $i \in E$ such that $\overline{s}_i < n_i$. The projection $Y_V \to Y_{\pi_{E \setminus i}(V)}$ sends $Y_V \cap O_{\mathbf{s}}$ into $Y_{\pi_{E\setminus i}(V)} \cap O_{\pi_{E\setminus i}(\mathbf{s})}$. By Lemma 2.5.6, $\pi_{E\setminus i}(\mathbf{s})$ is not a combinatorial flat of $\pi_{E\setminus i}(V)$, so $Y_{\pi_{E\setminus i}(V)} \cap O_{\pi_{E\setminus i}(\mathbf{s})}$ is empty by induction on N. This means $Y_V \cap O_{\mathbf{s}}$ is empty too. □

Lemma 4.3.3. *If $\mathbf{s}$ is a combinatorial flat of V, then $Y_V \cap O_{\mathbf{s}}$ is nonempty.*

Proof. We induct on $\dim V$. If $\dim V = 1$, let $\mathbf{s}$ be its minimal combinatorial flat. Since V is just a line, one sees that $O_{\mathbf{s}} \subset Y_V$ using the explicit description of the Bell polynomials in Section 4.1.

Otherwise, suppose $\dim V > 1$. Let r be the rank of $\mathbf{s}$ and let $F = \{i : s_i = n_i\}$. Let H be the preimage in $\mathbb{K}^{\mathbf{n}}$ of a general hyperplane in $\prod_{i \in F} \mathbb{K}^{n_i}$ and set $V' = V \cap H$. By Lemma 2.5.7, $\mathbf{s}$ is a combinatorial flat of V', and V' is p.g. by Proposition 4.2.3. By the induction hypothesis, $Y_{V'} \cap O_{\mathbf{s}}$ is nonempty, so $Y_V \cap O_{\mathbf{s}}$ is also nonempty. □

The following result gives a sharper description of strata than Theorem 4.3.1.

Lemma 4.3.4. *If* $\mathbf{s}$ *is a combinatorial flat of* V *and* $V' = \pi_{\{i:s_i=n_i\}}(V)$*, then* $Y_V \cap O_{\mathbf{s}} = \iota(V') \times \prod_{s_i<n_i} O_{s_i}$*. Consequently,* $\overline{Y_V \cap O_{\mathbf{s}}} = Y_{V'} \times \prod_{s_i<n_i} \mathbb{P}^{s_i}$*.*

Proof. The statement is clear when $\mathbf{s} = \mathbf{n}$. Otherwise, if $\mathbf{s}$ is a proper combinatorial flat, then $Y_V \cap O_{\mathbf{s}}$ is nonempty by Lemma 4.3.3. The V-orbit of any point in $O_{\mathbf{s}}$ has dimension $\mathrm{rk}(\mathbf{s})$ and Y_V is V-equivariant; therefore, $Y_V \cap O_{\mathbf{s}}$ has dimension at least $\mathrm{rk}(\mathbf{s})$.

We complete the proof for proper combinatorial flats by inducting on N. When $N = 1$, the lower bound on dimension implies $Y_V \cap O_i = O_i$ for $0 \le i \le d-1$, so the proposition holds.

Otherwise, suppose $N > 1$. Since $\mathbf{s}$ is proper, there is $j \in E$ such that $s_j < n_j$. Set $V' = \pi_{E \setminus j}(V)$ and $\mathbf{s}' = \pi_{E\setminus j}(\mathbf{s})$. The projection $\pi_{E\setminus j} : \mathbb{P}^{\mathbf{n}} \to \mathbb{P}^{\pi_{E\setminus i}(\mathbf{n})}$ induces a surjection $Y_V \to Y_{V'}$. This map sends $Y_V \cap O_{\mathbf{s}}$ into $Y_{V'} \cap O_{\mathbf{s}'}$, so by the induction hypothesis

$$Y_V \cap O_{\mathbf{s}} \subset O_{s_j} \times (Y_{V'} \cap O_{\mathbf{s}'}) = O_{s_j} \times \iota(\pi_{\cup_{s_i'=n_i} E_i}(V)) \times \prod_{s_i'<n_i} O_{s_i} = \iota(\pi_{\cup_{s_i=n_i} E_i}(V)) \times \prod_{s_i<n_i} O_{s_i}.$$

The leftmost and rightmost sets are both closed subsets of $O_{\mathbf{s}}$ of dimension $\mathrm{rk}(\mathbf{s})$, and the rightmost set is irreducible. This means the two sides are equal, proving the first assertion; the latter follows immediately. □

Proof of Theorem 4.3.1. The characterization of when Y_V intersects $O_{\mathbf{s}}$ is Lemma 4.3.2 and Lemma 4.3.3. If $\mathbf{s}$ is a combinatorial flat, then by Lemma 4.3.4, $Y_V \cap O_{\mathbf{s}}$ is a connected set of dimension $\mathrm{rk}(\mathbf{s})$. The V-orbit of any point of $O_{\mathbf{s}}$ also has dimension $\mathrm{rk}(\mathbf{s})$, so $Y_V \cap O_{\mathbf{s}}$ consists of a single V-orbit.

Finally, we prove $\overline{Y_V \cap O_{\mathbf{s}}} = \cup_{\mathbf{s}' \le \mathbf{s}} Y_V \cap O_{\mathbf{s}'}$ when $\mathbf{s}$ is a combinatorial flat of V. The only nonempty terms of the union are those indexed by combinatorial flats $\mathbf{s}'$.

By Lemma 2.5.6, $\mathbf{s}' \leq \mathbf{s}$ is a combinatorial flat of V if and only if $\mathbf{s}'_i \leq \mathbf{s}_i$ for all i and $(\mathbf{s}'_j)_{j:s_j=n_j}$ is a combinatorial flat of $V' := \pi_{\{j:s_j=n_j\}}(V)$. Such multisets $\mathbf{s}'$ are precisely those that index the nonempty strata of $\overline{Y_V \cap O_{\mathbf{s}}}$, since $\overline{Y_V \cap O_{\mathbf{s}}} = Y_{V'} \times \prod_{s_i<n_i} \mathbb{P}^{s_i}$ by Lemma 4.3.4. Lemma 4.3.4 also implies $O_{\mathbf{s}'} \cap (Y_{V'} \times \prod_{s_i<n_i} \mathbb{P}^{s_i}) = O_{\mathbf{s}'} \cap Y_V$ whenever both sides are nonempty, which completes the proof. □

Remark 4.3.5. Given a linear subspace $W \subset \mathbb{K}^n$, the matroid Schubert variety Y_W is defined to be the closure of W in $(\mathbb{P}^1)^n$. If we change W by rescaling each coordinate, then the isomorphism class of Y_W does not change. Similarly, given a subspace $V \subset \mathbb{K}^{\mathbf{n}} = \mathbb{K}^{n_1} \times \cdots \times \mathbb{K}^{n_N}$, if we perform a weighted rescaling of each factor $\mathbb{K}^{n_i}$ as described in Remark 4.1.4, then the isomorphism class of the polymatroid Schubert variety Y_V does not change either.

4.4 Multidegree & cohomology

Next, we establish our topological results, Theorem 4.4.1 and Theorem 4.4.2.

Theorem 4.4.1. *If V is p.g. and $\mathbf{s}$ is a combinatorial flat of V, then $[\overline{Y_V \cap O_{\mathbf{s}}}] = \sum_{\mathbf{b}} c_{\mathbf{b}}[\mathbb{P}^{\mathbf{b}}]$, where $\mathbf{b}$ runs over all bases of $\mathbf{s}$, and every coefficient $c_{\mathbf{b}}$ is positive.*

Theorem 4.4.2. *If V is a p.g. subspace defined over $\mathbb{K} = \mathbb{C}$, then the singular cohomology ring of Y_V is the graded vector space*

$$H^*(Y_V; \mathbb{Q}) \cong \mathbb{Q}\langle\, y_{\mathbf{s}} : \mathbf{s} \textit{ is a combinatorial flat} \rangle$$

where $\deg y_{\mathbf{s}} = \operatorname{rk}(\mathbf{s})$ and the multiplication is given by the following formula

$$c_{\mathbf{b}} c_{\mathbf{b}'} y_{\mathbf{s}} y_{\mathbf{s}'} = \begin{cases} c_{\mathbf{b}+\mathbf{b}'} y_{\mathbf{s}\vee\mathbf{s}'}, & \textit{if } \operatorname{rk}(\mathbf{s}\vee\mathbf{s}') = \operatorname{rk}(\mathbf{s}) + \operatorname{rk}(\mathbf{s}') \\ 0, & \textit{otherwise,} \end{cases}$$

with $\mathbf{b}$, $\mathbf{b}'$, and $\mathbf{b}+\mathbf{b}'$ bases for $\mathbf{s}$, $\mathbf{s}'$, and $\mathbf{s}\vee\mathbf{s}'$, respectively.

Given a variety X, let $A_d(X)$ be the Chow group of X spanned by d-dimensional cycles.

Lemma 4.4.3. *The Chow class of Y_V in $A_d(\mathbb{P}^{\mathbf{n}})$ is of the form*

$$[Y_V] = \sum_{\mathbf{b} \text{ basis of } P} c_{\mathbf{b}}\,[\mathbb{P}^{\mathbf{b}}].$$

Proof. Write $[Y_V] = \sum_{|\mathbf{t}|=d} c_{\mathbf{t}}[\mathbb{P}^{\mathbf{t}}]$. Let $\mathbf{s}$ be a multiset with $|\mathbf{s}| = d$ that is not a basis of P. Since $\mathbf{s}$ is not a basis, $T = \{i : \overline{s}_i < n_i\}$ is nonempty. Let $\mathbf{s}'$ be the multiset with $s'_i = s_i$ when $i \in T$ and $s'_i = 0$ if $i \notin T$. We have

$$\dim \pi_T(Y_V) = \mathrm{rk}_P(T) = \mathrm{rk}(\overline{\mathbf{s}}) - \textstyle\sum_{i \notin T} s_i < \mathrm{rk}(P) - \sum_{i \notin T} s_i = \sum_{i \in T} s_i,$$

so

$$c_{\mathbf{s}}\,[\mathrm{pt}] = [\mathbb{P}^{\mathbf{n}-\mathbf{s}}] \cdot [Y_V] = [\mathbb{P}^{\mathbf{n}-(\mathbf{s}-\mathbf{s}')}] \cdot ([\mathbb{P}^{\mathbf{n}-\mathbf{s}'}] \cdot [Y_V]) = [\mathbb{P}^{\mathbf{n}-(\mathbf{s}-\mathbf{s}')}] \cdot 0 = 0. \qquad \square$$

The following lemma is the base case for our proof that every coefficient $c_{\mathbf{b}}$ is positive.

Lemma 4.4.4. *If $\dim V = 1$, then $[Y_V] = \sum_{i:\mathrm{rk}_P(i)=1} n_i[\mathbb{P}^{\mathbf{e}_i}]$.*

Proof. By Lemma 4.4.3, $[Y_V] = \sum_{i:\mathrm{rk}_P(i)=1} c_{\mathbf{e}_i}[\mathbb{P}^{\mathbf{e}_i}]$. Write $V = \mathbb{K} \cdot (v_{ij})_{ij}$. If $\mathrm{rk}_P(i) = i$, then $\mathrm{codim}_V(V \cap S_{\mathbf{e}_i}) = 1$, so $v_{i1} \neq 0$. Let H be the preimage in $\prod_{i=1}^N \mathbb{P}^{n_i}$ of a general hyperplane in $\mathbb{P}^{n_i}$, such that $O_{\mathbf{0}}$ does not intersect H. The preimage of $H \cap O_{\mathbf{n}}$ in $\prod_{i=1}^N \mathbb{K}^{n_i}$ is an affine hypersurface defined by an equation of the form

$$\frac{1}{n_i!} a_{i1}^{n_i} + (\text{lower order terms}).$$

Such an equation has n_i solutions in V, so $c_{\mathbf{e}_i} = n_i$. $\square$

Proof of Theorem 4.4.1. In view of Lemma 4.4.3, it remains to show that all coefficients in the expansion $[Y_V] = \sum_{\mathbf{b}} c_{\mathbf{b}}[\mathbb{P}^{\mathbf{b}}]$ are positive. We induct on $\dim V$. The base case, when $\dim V = 1$, is Lemma 4.4.4.

Otherwise, suppose $\dim V > 1$. Pick $i \in E$ of positive rank, and let $H \subset \prod_{i=1}^{N} \mathbb{K}^{n_i}$ be the preimage of a general hyperplane in $\mathbb{K}^{n_i}$. From the explicit description of the Bell polynomials given in Section 4.1, we see that Y_H is defined by an equation of multidegree $(0, \ldots, 0, n_i, 0, \ldots, 0)$, so $[Y_H] = n_i[\mathbb{P}^{\mathbf{n}-\mathbf{e}_i}]$. The following claim identifies the irreducible components of $Y_V \cap Y_H$.

Claim.

$$Y_V \cap Y_H = Y_{V\cap H} \cup \left(\bigcup_{\mathbf{s}} Y_V \cap \mathbb{P}^{\mathbf{s}}\right),$$

where $\mathbf{s}$ runs over all corank 1 combinatorial flats $\mathbf{s}$ of V such that $\mathbf{s}$ is a combinatorial flat of H, but is not a combinatorial flat of $V \cap H$.

Proof of claim. To identify the irreducible components of $Y_V \cap Y_H$, we examine each stratum $Y_V \cap Y_H \cap O_{\mathbf{s}}$ with $\mathbf{s}$ a combinatorial flat of both V and H. Explicitly, this means $\mathbf{s}$ is a combinatorial flat of V with $s_i \neq n_i - 1$.

First suppose that $\mathbf{s}$ is also a combinatorial flat of $V \cap H$. We will show that $Y_V \cap Y_H \cap O_{\mathbf{s}} \subset Y_{V\cap H}$. By Lemma 2.5.7, either $s_i = n_i$ or $\overline{\mathbf{s} + \mathbf{e}_i} \not\geq n_i\mathbf{e}_i$. In the former case, Lemma 4.3.4 shows that $Y_V \cap Y_H \cap O_{\mathbf{s}} = Y_{V\cap H} \cap O_{\mathbf{s}}$. In the latter case, $\mathbf{s}$ has the same rank with respect to both V and $V \cap H$, and has rank $|\mathbf{s}|$ with respect to H, hence $Y_{V\cap H} \cap O_{\mathbf{s}} = Y_V \cap O_{\mathbf{s}} = Y_V \cap (Y_H \cap O_{\mathbf{s}})$.

Otherwise, suppose that $\mathbf{s}$ is not a combinatorial flat of $V \cap H$. In this case, $s_i \leq n_i - 2$ and $\overline{\mathbf{s} + \mathbf{e}_i} \geq n_i\mathbf{e}_i$ by Lemma 2.5.7. By the former inequality and Lemma 4.3.4, $Y_V \cap Y_H \cap O_{\mathbf{s}} = Y_V \cap O_{\mathbf{s}}$.

By Theorem 4.3.1, it now suffices to show there is a corank 1 combinatorial flat $\mathbf{s}' \geq \mathbf{s}$ of P such that $\mathbf{s}'$ is a combinatorial flat of H, but not $V \cap H$. Let S be a flat of $\widetilde{P}$ representing $\mathbf{s}$. We may write S as an intersection of corank 1 flats of $\widetilde{P}$, $S = S_1 \cap \cdots \cap S_k$. Since $s_i < n_i$, there is j such that $|S_j \cap E_i| < n_i$. Let $\mathbf{s}'$ be the multiset represented by S_j. If $s'_i = n_i - 1$, then we arrive at a contradiction because

$$\mathbf{s}' = \mathbf{s} \vee \mathbf{s}' \geq \overline{\mathbf{s} + \mathbf{e}_i} \geq n_i\mathbf{e}_i.$$

Hence, $s'_i \leq n_i - 2$ and $\overline{\mathbf{s}' + \mathbf{e}_i} \geq n_i\mathbf{e}_i$. This means that $\mathbf{s}'$ is a combinatorial flat of both P and H, but not of $V \cap H$. $\diamond$

To finish, consider the intersection product $[Y_V] \cdot [Y_H]$. On one hand,

$$[Y_V] \cdot [Y_H] = n_i[Y_V] \cdot [\mathbb{P}^{\mathbf{n}-\mathbf{e}_i}] = n_i \sum_{\mathbf{b}\,:\,b_i>0} c_{\mathbf{b}}[\mathbb{P}^{\mathbf{b}-\mathbf{e}_i}],$$

where $\mathbf{b}$ runs over bases of V. On the other hand, by the claim,

$$[Y_V] \cdot [Y_H] = m[Y_{V\cap H}] + \sum_{\mathbf{s}} m_{\mathbf{s}}[Y_V \cap \mathbb{P}^{\mathbf{s}}],$$

where $\mathbf{s}$ runs over corank 1 combinatorial flats of V that combinatorial flats of H but not of $V \cap H$. Since Y_V and Y_H are generically transverse along $Y_{V\cap H}$, $m = 1$. The remaining multiplicities are all positive, and by Kleiman's theorem, the expansion of $[Y_V \cap \mathbb{P}^{\mathbf{s}}]$ in the basis $\{[\mathbb{P}^{\mathbf{c}}] : |\mathbf{c}| = \dim V - 1\}$ has non-negative coefficients. Consequently, there is a coefficient-wise inequality

$$n_i \sum_{\mathbf{b}:b_i>0} c_{\mathbf{b}}[\mathbb{P}^{\mathbf{b}-\mathbf{e}_i}] \geq \sum_{\mathbf{b}'} c'_{\mathbf{b}'}[\mathbb{P}^{\mathbf{b}'}] = [Y_{V\cap H}],$$

where $\mathbf{b}$ runs over bases of V and $\mathbf{b}'$ runs over bases of $V \cap H$. The coefficients $c'_{\mathbf{b}'}$ are all strictly positive by the induction hypothesis. Moreover, recall from Section 2.3 that the bases of $V \cap H$ are exactly

$$\{\mathbf{b} - \mathbf{e}_i : \mathbf{b} \text{ is a basis of } V \text{ with } b_i > 0\},$$

so $n_i c_{\mathbf{b}} \geq c'_{\mathbf{b}-\mathbf{e}_i} > 0$ for each basis $\mathbf{b}$ of P with $b_i > 0$. Every basis of P has some coordinate positive, so this completes the proof. $\square$

A **basis** of a combinatorial flat $\mathbf{s}$ is an independent multiset maximal among those contained in $\mathbf{s}$.

Corollary 4.4.5. *If* $\mathbf{s}$ *is a combinatorial flat of* V*, then*

$$[Y_V \cap \mathbb{P}^{\mathbf{s}}] = \sum_{\mathbf{b}} c_{\mathbf{b}} [\mathbb{P}^{\mathbf{b}}]$$

where $\mathbf{b}$ *runs over all bases of* $\mathbf{s}$*, and all coefficients are positive.*

Proof. Let $A = \{i : s_i = n_i\}$. Let $V' = \pi_A(V)$ and $\mathbf{n}' = \pi_A(\mathbf{n})$. By Lemma 4.3.4, $Y_V \cap \mathbb{P}^{\mathbf{s}} = Y_{V'} \times \prod_{j \in E \setminus A} \mathbb{P}^{s_j}$, so there is a commuting square

$$\begin{array}{ccc} Y_V \cap \mathbb{P}^{\mathbf{s}} & \longrightarrow & \mathbb{P}^{\mathbf{n}} \\ \downarrow & & \downarrow \\ Y_{V'} & \longrightarrow & \mathbb{P}^{\mathbf{n}'} \end{array} \quad \text{which induces} \quad \begin{array}{ccc} A_{\mathrm{rk}(\mathbf{s})}(Y_V \cap \mathbb{P}^{\mathbf{s}}) & \longrightarrow & A_{\mathrm{rk}(\mathbf{s})}(\mathbb{P}^{\mathbf{n}}) \\ \uparrow & & \uparrow \\ A_{\mathrm{rk}(\mathbf{s}^{\mathrm{geo}})}(Y_{V'}) & \longrightarrow & A_{\mathrm{rk}(\mathbf{s}^{\mathrm{geo}})}(\mathbb{P}^{\mathbf{n}'}) \end{array},$$

where the two horizontal arrows are proper pushforward and the two vertical arrows are flat pullback. By Theorem 4.4.1 and the latter square's commutativity, $[Y_V \cap \mathbb{P}^{\mathbf{s}}] = \sum_{\mathbf{b}'} c_{\mathbf{b}'} [\mathbb{P}^{\widehat{\mathbf{b}'}}]$, where $\mathbf{b}'$ ranges over all bases of V' and $\widehat{\mathbf{b}'}$ is the multiset with $\widehat{b'_i} = b'_i$ if $s_i = n_i$ and $\widehat{b'_i} = s_i$ otherwise. Using Lemma 2.5.2, one sees that the multisets $\widehat{\mathbf{b}'}$ are exactly the bases of $\mathbf{s}$. □

For an independent multiset $\mathbf{b}$, let $c_{\mathbf{b}} > 0$ be the coefficient of $[\mathbb{P}^{\mathbf{b}}]$ in $[Y_V \cap \mathbb{P}^{\overline{\mathbf{b}}}]$.

Proposition 4.4.6. *If* V *is defined over the complex numbers, then its singular cohomology ring* $H^*(Y_V, \mathbb{Q})$ *is isomorphic to* $\mathbb{Q}[y_1, \ldots, y_N]/I$*, where* I *is the ideal generated by*

$$\begin{array}{rl} c_{\mathbf{b}'} y_1^{b_1} \cdots y_N^{b_N} - c_{\mathbf{b}} y_1^{b'_1} \cdots y_N^{b'_N}, & \mathbf{b} \text{ and } \mathbf{b}' \text{ are independent multisets of } P \text{ with } \overline{\mathbf{b}} = \overline{\mathbf{b}'}, \text{ and} \\ y_1^{d_1} \cdots y_N^{d_N}, & \mathbf{d} \text{ is dependent.} \end{array}$$

Proof. Let $\kappa : Y_V \to \mathbb{P}^{\mathbf{n}}$ denote the inclusion. By Corollary 4.4.5, the classes $[Y_V \cap \mathbb{P}^{\mathbf{s}}]$ all have pairwise disjoint supports, so the pushforward $A_* Y_V \to A_* \mathbb{P}^{\mathbf{n}}$ is injective.

The sets $O_{\mathbf{s}}$ and $Y_V \cap O_{\mathbf{s}}$ comprise a cell decompositions of $\mathbb{P}^{\mathbf{n}}$ and Y_V, respectively. Hence, the Chow groups of these varieties are naturally isomorphic to their Borel-Moore homology groups via the cycle class map Fulton 1998, Example 19.1.11. Since $\mathbb{P}^{\mathbf{n}}$ and Y_V are compact, their Borel-Moore homology is equal to their singular homology. Summing up, we've learned that the singular homology groups of Y_V are all free and that the homology pushforward $\kappa_* : H_*(Y_V;\mathbb{Z}) \to H_*(\mathbb{P}^{\mathbf{n}};\mathbb{Z})$ is injective. Applying the Universal Coefficient Theorem, we learn that $\kappa^* : H^*(\mathbb{P}^{\mathbf{n}};\mathbb{Q}) \to H^*(Y_V;\mathbb{Q})$ is surjective.

The cohomology ring of $\mathbb{P}^{\mathbf{n}}$ is isomorphic to $R := \mathbb{Q}[y_1,\dots,y_N]/(y_1^{n_1+1},\dots,y_N^{n_N+1})$, where y_i represents a hyperplane pulled back from $\mathbb{P}^{n_i}$. The monomials $y_i^{n_i+1}$ are among the claimed relations because $n_i \geq \mathrm{rk}_P(i)$ for all $1 \leq i \leq N$.

Let us verify that the remaining generators of I are in $\ker(\kappa^*)$. By the Universal Coefficient Theorem, the pullback of $\alpha \in H^k(\mathbb{P}^{\mathbf{n}};\mathbb{Q})$ is zero if and only if its cap product with any element of $H_k(Y_V;\mathbb{Q})$ is zero. By the projection formula and the injectivity of κ_*, this is equivalent to $\alpha \frown [Y_V \cap \mathbb{P}^{\mathbf{s}}] = 0$ for all rank k combinatorial flats $\mathbf{s}$ of P. By Theorem 4.4.1,

$$y_1^{b_1}\cdots y_N^{b_N} \frown [Y_V \cap \mathbb{P}^{\mathbf{s}}] = \begin{cases} c_{\mathbf{b}}, & \text{if } \mathbf{b} \text{ is independent and } \overline{\mathbf{b}} = \mathbf{s} \\ 0, & \text{otherwise,} \end{cases}$$

so all claimed relations are in $\ker(\kappa^*)$. Moreover, the dimension of the degree k homogeneous component of R/I is plainly equal to $\dim H^k(Y_V;\mathbb{Q})$, which completes the proof. □

Proof of Theorem 4.4.2. Let $S = \mathbb{Q}[y_{\mathbf{s}} : \mathbf{s} \text{ combinatorial flat}]$. Using the notation of Proposition 4.4.6, define a map

$$\phi : S \to H^*(Y_V,\mathbb{Q}), \quad y_{\mathbf{s}} \mapsto \frac{1}{c_{\mathbf{b}}} y_1^{b_1}\cdots y_N^{b_N}$$

where $\mathbf{b}$ is any basis of $\mathbf{s}$. The image of $y_{\mathbf{s}}$ does not depend on $\mathbf{b}$ by Proposition 4.4.6.

Suppose $\mathbf{s}$ and $\mathbf{s}'$ are two combinatorial flats. If $\mathrm{rk}(\mathbf{s} \vee \mathbf{s}') < \mathrm{rk}(\mathbf{s}) + \mathrm{rk}(\mathbf{s}')$, then the sum of any pair of bases of $\mathbf{s}$ and $\mathbf{s}'$ is a dependent multiset, so $\phi(y_{\mathbf{s}}y_{\mathbf{s}'}) = 0$. On the other hand, if $\mathrm{rk}(\mathbf{s} \vee \mathbf{s}') = \mathrm{rk}(\mathbf{s}) + \mathrm{rk}(\mathbf{s}')$, then there are bases $\mathbf{b}$ and $\mathbf{b}'$ such that $\mathbf{b} + \mathbf{b}'$ is a basis for $\mathbf{s} \vee \mathbf{s}'$, so $c_{\mathbf{b}}c_{\mathbf{b}'}\phi(y_{\mathbf{s}}y_{\mathbf{s}'}) = c_{\mathbf{b}+\mathbf{b}'}\phi(y_{\mathbf{s}\vee\mathbf{s}'})$. Hence, $J \subset \ker\phi^*$.

Surjectivity of ϕ follows from the fact that $\phi(y_{\mathbf{s}}) \frown [Y_V \cap \mathbb{P}^{\mathbf{s}'}]$ is 1 if $\mathbf{s} = \mathbf{s}'$ and 0 otherwise. Finally, it is evident from the relations that the degree k part of S/J has dimension at most $\dim H^k(Y_V; \mathbb{Q})$, so $J = \ker \phi$. □

Remark 4.4.7 (Top-heaviness of combinatorial flats). Let P be a polymatroid of rank d, and let f_i be the number of combinatorial flats of P that have rank i. The work of Braden et al. 2020 implies that these numbers are top-heavy; that is, $f_i \leq f_j$ for each $i \leq j \leq d-i$. To see this, let $\widetilde{P}$ be a lift of P, carrying an action by the product of symmetric groups $\mathfrak{S}$. The **graded Möbius algebra** of $\widetilde{P}$ is a graded vector space $H^*(\widetilde{P}) = \bigoplus_F y_F\mathbb{Q}$, where F runs over all flats of $\widetilde{P}$ and $y_F\mathbb{Q}$ is a summand in degree $\mathrm{rk}_{\widetilde{P}}(F)$. There is a natural $\mathfrak{S}$-action on $H^*(\widetilde{P})$, and the space of invariants in degree i has dimension f_i. By Braden et al. 2020, Theorem 1.1 there are $\mathfrak{S}$-equivariant injections $H^i(\widetilde{P}) \to H^j(\widetilde{P})$ for each $i \leq j \leq d-i$. These injections induce injections on the invariants in degrees $i \leq j \leq d-i$, hence the top-heaviness.

If P is realized by a linear space V, our work gives an independent proof of top-heaviness. By Theorem 4.3.1, polymatroid Schubert varieties admit an algebraic cell decomposition. It follows by arguments of Huh and Wang 2017; Björner and Ekedahl 2009 that there is an injection of $H^*(Y_V)$-modules $H^*(Y_V) \hookrightarrow IH^*(Y_V)$, where $IH^*(Y_V)$ is the intersection cohomology of Y_V. Intersection cohomology has the Hard Lefschetz property, which implies that for any ample class $L \in H^2(Y_V)$ and $i \leq j \leq d-i$, the maps

$$IH^i(Y_V) \to IH^j(Y_V), \quad \alpha \mapsto L^{d-i+j}\alpha$$

are injective. It follows that the restrictions $H^i(Y_V) \to H^j(Y_V)$ are also injective. Theorem 4.4.2 states that $\dim H^k(Y_V) = f_k$ for all k, so we obtain the desired inequality.

Remark 4.4.8. In a matroid Schubert variety Y_V, there is a neighborhood of $(\infty, \infty, \cdots, \infty)$ isomorphic to the affine cone over the reciprocal plane $\frac{1}{V}$. One resolves the singularities of Y_V by using the **augmented wonderful compactification** of V, on which the usual wonderful compactification appears as a divisor. In the polymatroid case, we are still unsure how to resolve singularities. While defining the augmented wonderful model of a subspace arrangement is straightforward, there is not an obvious map to our polymatroid Schubert varieties.

Bibliography

Nakasawa, T. (1935). "Zur axiomatik der linearen abhängigkeit". In: *Science Reports of the Tokyo Bunrika Daigaku* 2.43.

Whitney, H. (1935). "On abstract properties of linear dependence". In: *Amer. J. Math.* 3, pp. 509–533.

Rosen, Zvi, Jessica Sidman, and Louis Theran (2020). "Algebraic matroids in action". In: *The American Mathematical Monthly* 127.3, pp. 199–216. arXiv: 1809.00865 [math.CO].

Recski, AndrÁs (2011). "Engineering Applications of Matroids - A Survey". In: *Matroid Theory and its Applications*. Ed. by A. Barlotti. Berlin, Heidelberg: Springer Berlin Heidelberg, pp. 300–321. ISBN: 978-3-642-11110-5.

Iri, M. (1983). "Applications of Matroid Theory". In: *Mathematical Programming The State of the Art: Bonn 1982*. Ed. by Achim Bachem, Bernhard Korte, and Martin Grötschel. Berlin, Heidelberg: Springer Berlin Heidelberg, pp. 158–201. ISBN: 978-3-642-68874-4.

White, Neil (1987). *Combinatorial geometries*. 29. Cambridge University Press.

Orlik, Peter and Louis Solomon (1980). "Combinatorics and topology of complements of hyperplanes". In: *Inventiones mathematicae* 56.2, pp. 167–189.

Stanley, Richard P. (2011). *Enumerative Combinatorics*. 2nd ed. Vol. 1. Cambridge University Press.

Oxley, J.G. (2006). *Matroid Theory*. Oxford graduate texts in mathematics. Oxford University Press. ISBN: 9780199202508. URL: https://books.google.com/books?id=puKta1Hdz-8C.

Adiprasito, Karim, June Huh, and Eric Katz (2018). "Hodge theory for combinatorial geometries". In: *Annals of Mathematics* 188.2, pp. 381–452.

Dowling, Thomas A and Richard M Wilson (1974). "The slimmest geometric lattices". In: *Transactions of the American Mathematical Society* 196, pp. 203–215.

Braden, Tom et al. (Oct. 2020). "Singular Hodge theory for combinatorial geometries". In: *arXiv e-prints*. arXiv: 2010.06088 [math.CO].

Huh, June and Botong Wang (2017). "Enumeration of points, lines, planes, etc". In: *Acta Math.* 218.2, pp. 297–317. ISSN: 0001-5962.

Concini, C. de and C. Procesi (1995). "Wonderful models of subspace arrangements". In: *Selecta Mathematica* 1.3.

Ardila, Federico and Adam Boocher (2016). "The closure of a linear space in a product of lines". In: *J. Algebraic Combin.* 43.1, pp. 199–235. ISSN: 0925-9899.

Proudfoot, Nicholas and David Speyer (2006). "A broken circuit ring". In: *Beitraäge zue algebra and geometrie* 47.1, pp. 161–166.

Crowley, Colin, Connor Simpson, and Botong Wang (2024). "Polymatroid Schubert varieties". In: *forthcoming*.

Crowley, Colin, June Huh, et al. (2022). "The Bergman fan of a polymatroid". In: *arXiv preprint arXiv:2207.08764*.

Pagaria, Roberto and Gian Marco Pezzoli (May 2021). "Hodge Theory for Polymatroids". In: *arXiv e-prints*, arXiv:2105.04214, arXiv:2105.04214. arXiv: `2105.04214 [math.AG]`.

Crowley, Colin, June Huh, et al. (July 2020). "The Bergman fan of a polymatroid". In: *arXiv e-prints*, arXiv:2207.08764v1.

Helgason, Thorkell (1972). "Aspects of the theory of hypermatroids". In: *Hypergraph Seminar of Ohio State University*. Springer-Verlag.

Nguyen, Hien Q. (1986). "Submodular functions". In: *Theory of matroids*. Ed. by Neil White. Cambridge University Press. Chap. 10, pp. 272–297.

Lovász, L. (1977). "Flats in matroids and geometric graphs". In: *Combinatorial surveys (Proc. Sixth British Combinatorial Conf., Royal Holloway Coll., Egham, 1977)*, pp. 45–86.

McDiarmid, Colin (1975). "Rado's theorem for polymatroids". In: *Math. Proc. Camb. Phil. Soc.* 78.263, pp. 263–281.

Bonin, Joseph E, Carolyn Chun, and Tara Fife (2023). "The natural matroid of an integer polymatroid". In: *SIAM Journal on Discrete Mathematics* 37.3, pp. 1751–1770.

Brylawski, Thomas (1986). "Constructions". In: *Theory of matroids*. Ed. by Neil White. Cambridge University Press. Chap. 7, pp. 127–223.

Rota, Gian-Carlo (1964). "On the foundations of combinatorial theory: I. Theory of Möbius functions". In: *Classic Papers in Combinatorics*. Springer, pp. 332–360.

Maclagan, Diane and Bernd Sturmfels (2021). *Introduction to tropical geometry*. Vol. 161. American Mathematical Society.

Ardila, Federico, Graham Denham, and June Huh (2023). "Lagrangian geometry of matroids". In: *J. Amer. Math. Soc.* 36.3, pp. 727–794. ISSN: 0894-0347,1088-6834. DOI: `10.1090/jams/1009`. URL: `https://doi.org/10.1090/jams/1009`.

Feichtner, Eva Maria and Sergey Yuzvinsky (2004). "Chow rings of toric varieties defined by atomic lattices". In: *Invent. Math.* 155.3, pp. 515–536. ISSN: 0020-9910.

Eisenbud, David (1995). *Commutative algebra with a view to algebraic geometry*. Graduate texts in mathematics. Springer-Verlag New York.

Hassett, Brendan and Yuri Tschinkel (1999). "Geometry of equivariant compactifications of G_a^n". In: *IMRN* 22.

Fulton, William (1998). *Intersection theory*. 2nd ed. Springer-Verlag New York.

Björner, Anders and Torsten Ekedahl (2009). "On the shape of Bruhat intervals". In: *Annals of Mathematics* 170, pp. 799–817.