STOCHASTIC MODELS FOR STRATEGIC SOURCING IN ENERGY INDUSTRY

by

Ashesh Kumar Sinha

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The dissertation is approved by the following members of the Final Oral Committee:

Ananth Krishnamurthy, Associate Professor, Industrial and System Engineering

Oguzhan Alagoz, Professor, Industrial and System Engineering

Jeffrey Linderoth, Professor, Industrial and System Engineering

Dharmaraj Veeramani, Professor, Industrial and System Engineering

Gregory DeCroix, Professor, Wisconsin School of Business

To my parents

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ABSTRACT

The supply chain involved in the manufacturing of equipment for oil and gas (O&G) industry faces several challenges due to fluctuations in demand and custom-engineered nature of components. This research develops stochastic models to address important problems related to production and subcontracting in this supply chain.

This research analyzes production and subcontracting policies for two types of components: standard-type components and knowledge-type components. Standard-type components do not have proprietary designs, and are often supported using Make-to-Stock (MTS) or Assemble-to-Order (ATO) policies. In contrast, knowledge-type components are highly custom-engineered components that use proprietary designs, and are often supported using Make-to-Order (MTO) policies.

For standard-type components, we first analyze single product ATO system with capacity constraints and stochastic lead times. We assume that component replenishment is carried out by orders placed to an internal manufacturing facility and/or an external subcontractor, and component stock levels at the manufacturer are determined by dual index based policies. Using queuing models, we analyze the tradeoffs related to internal manufacturing versus subcontracting under different types of dual index policies. We use matrix geometric methods to conduct an exact analysis for systems with two components and develop a decomposition based algorithm to analyze the performance of systems with more than two components.

Numerical studies provide useful insights on the performance of various dual index policies.

Next, we analyze manufacturing system with multiple components where individual components are made to stock through production either at a shared in-house manufacturing facility or at facilities of external subcontractors dedicated to individual components. The manufacturer and the subcontractor differ in terms of costs, production capacities, rates, and service level capabilities. Using Markov decision process models, we determine the optimal policy and characterize its structure. To address the curse of dimensionality, we derive a set of conditions that partitions the state space into regions and characterize optimal policies in each region. We consider several special scenarios and prove that the optimal policy has a multi-index type structure in each of these settings with dual index type structure as a special case in some settings. Next, we extend the analysis to multi-product ATO systems where individual components can be made either at a shared manufacturing facility or at facilities of external subcontractors dedicated to particular components. We develop an iterative procedure that exploits solution characteristics of subsystems to reduce the action space and use the procedure to determine optimal policies for the original ATO system.

For knowledge-type components, we analyze strategic production and subcontracting decisions for a system with centralized control and decentralized control. Since, these knowledge-type components are often made to order, they pose different challenges especially in terms of capacity investments and demand variations. In both up-markets and down-markets, manufacturer must balance capacity investments, subcontracting production to certified subcontractors, and cost of unused capacity. We study this problem in both a centralized and a decentralized setting using Markov decision process models and stochastic game formulations. We analytically provide optimal capacity investment and production strategies for both the manufacturer and the subcontractor, and show the impact of unused capacity on such decisions. Using numerical studies, we analyze the inefficiencies of operating in a decentralized setting.

Chapter 1

Introduction

For the foreseeable future, clean oil and gas (O&G) is likely to remain the biggest and the most economical source of energy. O&G industries contribute about $1.2 trillion to the US economy and support roughly 10 million jobs (Anonymous (2013)). Despite the importance, its manufacturing and supply chain challenges have been vastly under-appreciated. This thesis develops stochastic models to address several important problems in these supply chains.

1.1 Supply Chain for Oil Drilling Equipment

Broadly speaking, the supply chain for oil drilling equipment (see Figure 1.1) is comprised of mainly three phases: (1) exploration phase, (2) production phase, and (3) processing and distribution phase. In the exploration phase, companies such as Exxon Mobil, BP, Shell, etc identify potential drilling sites and oil wells. These companies hire drilling contractors such as Seadrill, Transocean, etc, to drill oil using rigs that could be land rigs, floating rigs, offshore rigs, or inland barge rigs. In the production phase, O&G equipment manufacturers such as National Oilwell Varco, GE Oil and Gas, Cameron, Schlumberger, etc provide custom-engineered equipment to support the oil drilling operations. These equipment can be large in size and thus are transported to the drilling site as separate parts and assembled at the drilling site. Once, the oil rig is fully functional, the safety and environmental authorities conduct a thorough check to prevent any accidents during the drilling process. Next, drilling contractors pump up oil. Finally, in the processing and distribution phase, logistic and distribution systems transport these crude oil to operating companies for refining and distribution.

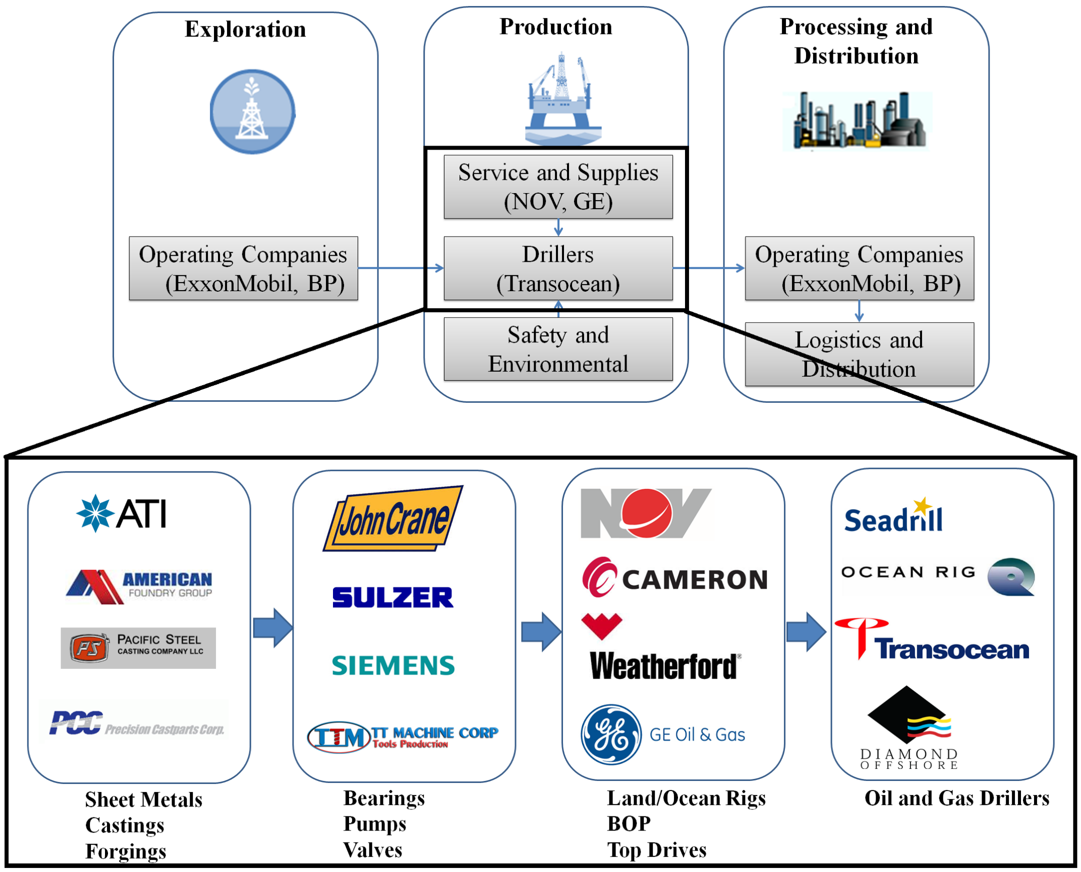

Figure 1.1 Overview of Supply Chain in O&G Industries

Inefficiencies in this supply chain can be very expensive. For example, manufacturing delays in delivery of the equipment could cost up to $10,000 per day. This implies that the equipment manufacturers lie on the critical path of a supply chain that consists of drilling contractors (Transocean, Seadrill), manufacturers of equipment (drawworks, top drives), suppliers of parts (pumps, bearings), and suppliers of raw materials (castings, forgings).

A typical oil rig is mainly comprised of many equipment such as power system, drawworks, top drives, etc as shown in Figure 1.2. Much of this equipment is custom-engineered for specific drilling applications. Power system comprises of combustion engines and transmission system, and provides power to run various equipments on the rig. A top drive is a mechanical equipment that is located on the oil rig (either on the land or under the ocean) to facilitate the drilling process. Top drives are usually assembled from different components such as hydraulic motors, main body, shaft, pipe handlers, etc. Similarly, drawworks is a heavy equipment that wraps the wire-rope drilling line. It consists of a main drum to spool the wire-rope with the help of powerful motors, a brake system to stop the spooling process, a main body to provide the structure, and skids to support the weight of the drawworks.

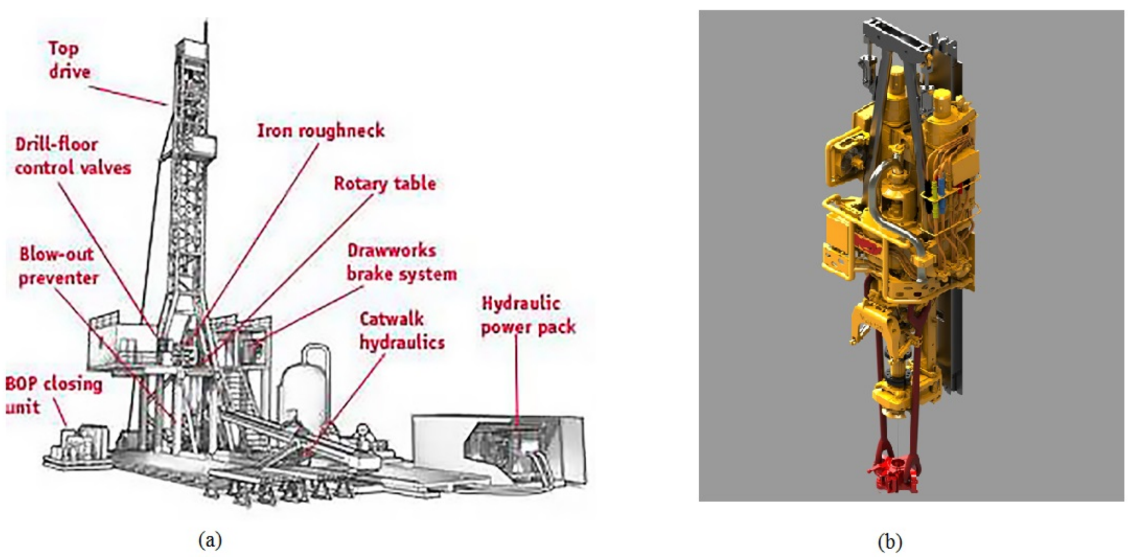

Figure 1.2 Oil Drilling Equipment (a) Land Rig, and (b) Top Drive

1.2 Challenges in the Supply Chain

The supply chain involved in the manufacturing of drilling equipment faces several challenges:

Custom-engineered equipment: Oil drilling equipment is highly custom-engineered and could easily require over 10,000 hours in engineering and 50,000 hours in manufacturing. For instance, drawworks could require around 20,000 components in the assembly process. This requires the supply chain to have efficient assemble-to-order, make-to-order, or engineer-to-order strategies.

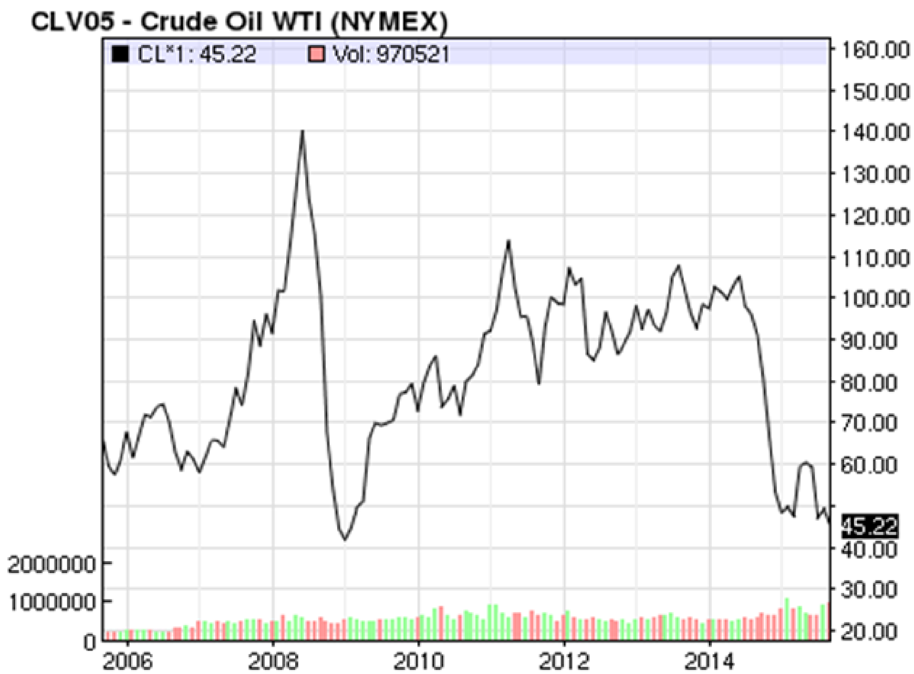

Figure 1.3 Variation in Crude Oil Prices (Anonymous (2015a))

Market fluctuations: The demand of oil drilling equipment directly depend on the price of the crude oil. Figure 1.3 shows the price trend of the crude oil over a 10 year period (2006 - 2015) (Anonymous (2015a)). We observe a highly variable trend in the price which directly impacts the demand of oil drilling equipment. These price variations drive corresponding fluctuations in demand for the drilling equipment used in this industry (Damodaran (2009)). These demand fluctuations have a significant cascading impact throughout this supply chain. High retail price of the crude oil triggers drilling more oil to increase supply and reduce prices. This results in more orders for drilling equipment. Demands could increase by over 50% within a span of a year. This sudden spike in demands over a short time span puts great stress on the supply chain. Similarly, low retail price of the crude oil results in fewer orders for drilling equipment. For instance, demands could drop by over 50% in a span of two years. This sudden drop in demand affects the entire supply chain and sometimes threaten the survival of equipment manufacturers. Companies like Samson Oil and Gas and Sabine Oil and Gas, that barely survived the market downturn in year 2009, could not survive the downturn in year 2015 (Anonymous (2015b,c); Stenner (2015)). Even larger companies like Schlumberger, National Oilwell Varco, and General Electric have been forced to layoff thousands of employees (Anonymous (2009); Merette (2009); Eaton (2015); Long (2016)). Even during up-markets, manufacturers struggle to ramp up production capacities at the pace necessary, and resulting product delays cost over $10,000 per day.

Manufacturing capacity: Oil drilling equipment supply chain manufactures massive equipment which consumes significant manufacturing resources. For example, top drives and drawworks can consume more than 2000 hours of manufacturing resources which is equivalent to an years worth of manufacturing on a single machine. The limited internal manufacturing capacity at equipment manufacturers impose a pressing challenge to meet the customer demand. During up-market, manufacturers face capacity issues and struggle to keep up with demand and avoid component shortages. For instance, drilling motors could be delayed due to shortages of 20% of their components. Similarly, during down-market, manufacturers struggle to keep the manufacturing resources busy which results in high unused capacity costs (or overhead absorption risks).

Subcontracting: Since, capital equipment needed for manufacturing of oil drilling equipment is expensive (often costing $2-3M for a single machine). Manufacturer often needs to subcontract significant manufacturing operations to certified external subcontractors either because the manufacturer does not have the required capacity or because the manufacturing cost is lower at a subcontractor. For instance, as stated earlier, drawworks is assembled from various parts such as drum assembly, brake system, main body, skids, etc. To satisfy the demand of the drawworks and overcome internal capacity limitations, a manufacturer might subcontract few parts such as drum assembly and skids to various external vendors. This enables them to meet customer demand and reduce cost.

Knowledge versus standard components: Oil drilling equipment are mainly classified into two categories: knowledge-type components and standard-type components. Knowledge-type components are highly custom-engineered parts that use proprietary designs, and are often supported using Make-to-Order (MTO) policies. However, these components require high capital investment and have high costs associated with unused capacity. In contrast, the standard-type components do not have proprietary designs, and are often supported using Make-to-Stock (MTS) or Assemble-to-Order (ATO) policies. These two categories drive different supply chain partnerships and impose challenge in decision making process.

In the next section, we describe some of the research issues in this supply chain.

1.3 Research Issues and Questions

We develop a set of stochastic optimization models to derive insights that will address the key supply chain challenges faced by equipment manufacturers for the O&G industry. From the components perspective, we focus on two categories: standard-type components and knowledge-type components. We describe research issues and questions in the subsequent sections.

1.3.1 Subcontracting Strategies for Standard-type Components

At first, we analyze subcontracting issues in a manufacturing system where a single end product is assembled to customer specification from multiple standard-type components that are held in stock (See Figure 1.4). Each component is either manufactured in-house or sourced from subcontractors. For instance, although the manufacturer could manufacture components at a faster rate using in-house manufacturing capacity, they might choose to reserve that capacity for other products and decide to subcontract production to a subcontractor that might have a lower production rate. However, the availability of components in stock is critical to assemble the final product and satisfy the demand in a timely manner. Under this setting, we analyze the potential of dual index based production and stocking policies that can be used by the manufacturer. We analyze the optimal thresholds for dual index policies and provide answers to the following research questions:

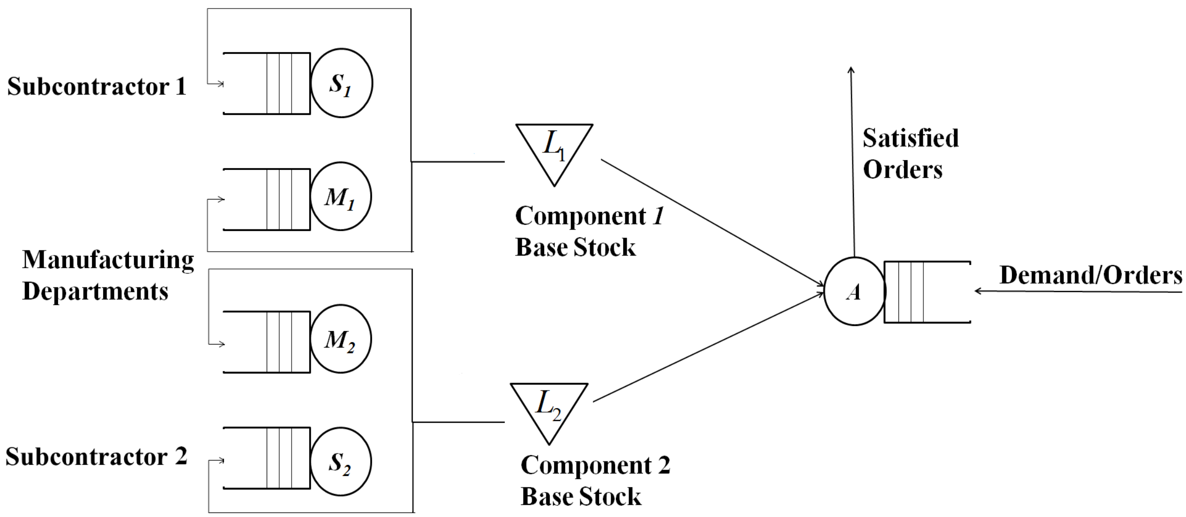

Figure 1.4 Subcontracting Strategies for Single Product Made of Standard-type Components

RQ1: What are the optimal thresholds and production quantities for the in-house manufacturer and the subcontractor?

RQ2: Under what conditions would certain types of dual index policies outperform other dual index policies? How do these thresholds impact total cost, expected inventory, and backorders?

In this thesis, we answer these research questions in Chapter 3.

Next, we analyze a make-to-stock (MTS) system comprising of multiple standard-type components as shown in Figure 1.5. These components require special equipment that cannot be dedicated to serve a specific component. So, the individual standard-type components can be made either at a shared in-house manufacturing facility or at dedicated facilities of external subcontractors. Therefore, the supply chain manager has to make decisions such as when and how much capacity at the manufacturer should be dedicated to a given component and when and how much production of a given component needs to be subcontracted. We investigate the following research questions:

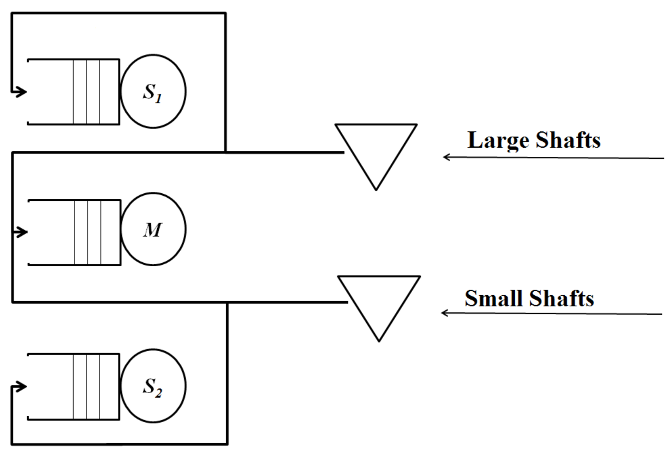

Figure 1.5 Subcontracting Strategies for Make-to-Stock Standard-type Components

RQ3: How do differences in capabilities, costs and service level expectations impact the optimal production and capacity utilization strategies? How do such tradeoffs depend on the differences in production costs?

RQ4: Do optimal policies have a easily describable structure that can be friendly for industry implementation?

In this thesis, we answer these research questions in Chapter 4.

Next, we extend our research to systems with multiple end products assembled from standard-type components. Here, multiple products are assembled from various standard-type components as shown in Figure 1.6. Again, the manufacturer could produce components using shared internal manufacturing capacity or choose to reserve that capacity for other products, and instead subcontract production to a subcontractor that might have a lower production rate. Now, multiple manufacturing resources are shared to make two or more components. This increases the complexity of the problem. We investigate the following research questions:

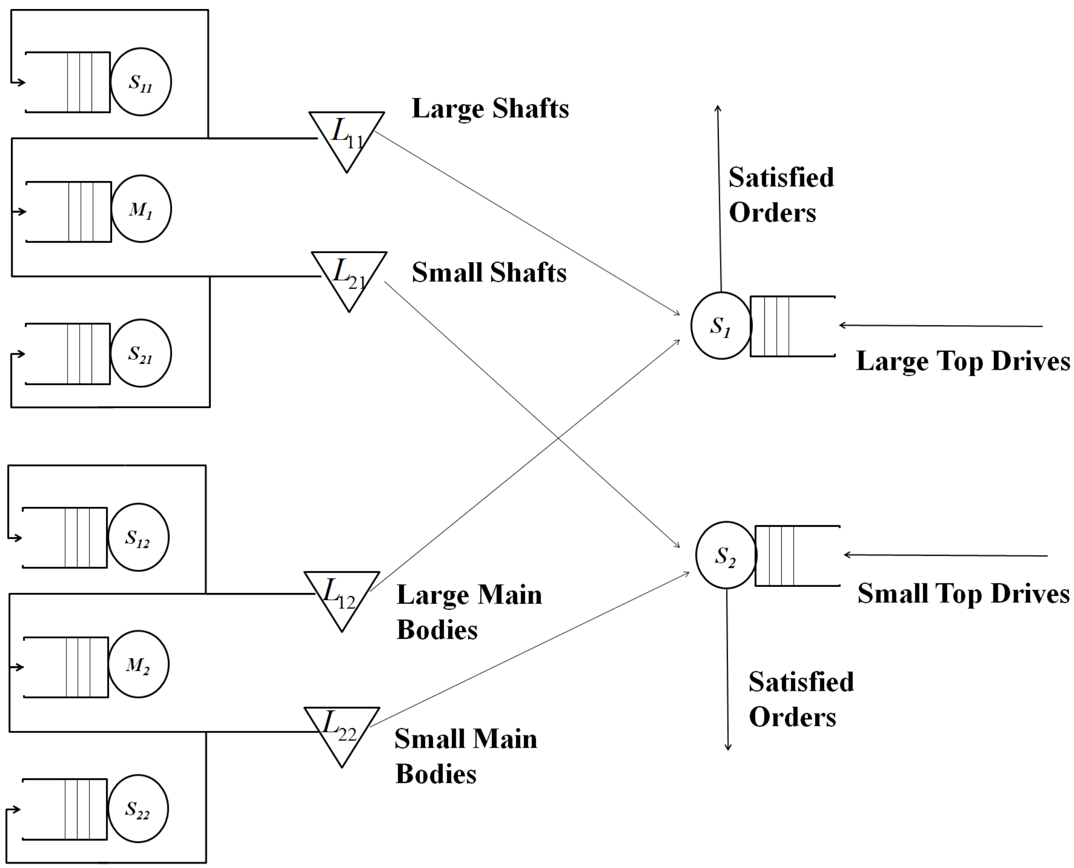

Figure 1.6 Subcontracting Strategies for Multi-product Made of Standard-type Components

RQ5: How could we address the state space complexity associated with determining the optimal policies for multi-product systems? Are there efficient algorithms to resolve state space complexity?

RQ6: What is the optimal use of in-house manufacturing capacity? What is the structure of the optimal policy?

In this thesis, we answer these research questions in Chapter 5.

1.3.2 Subcontracting Strategies for Knowledge-type Components

We analyze subcontracting decisions for knowledge-type components in a multi-period setting with non-stationary demands. Recall that knowledge-type components have proprietary design and are often made to order. However, such components require high capital investment and the cost of under utilizing the available capacity is significant. In such cases, knowledge-type components might need to be strategically subcontracted to vendors to either to exploit available capacity at the subcontractor or to reduce the costs associated with unused capacity. We analyze a manufacturing system under centralized control consisting of a manufacturer and a subcontractor as shown in Figure 1.7. In each time period, the manufacturer and the subcontractor needs to balance tradeoffs related to production costs and unused capacity costs to determine the optimal production and capacity investment decisions. Under centralized setting, we aim to provide answers to the following research questions.

RQ7: When and how much capacity should the manufacturer and the subcontractor invest in and utilize during each time period?

RQ8: What is the structure of the optimal policy and how does the unused capacity impact the optimal production and subcontracting decisions?

Next, we analyze the tradeoffs under decentralized setting consisting of autonomous manufacturer and subcontractor as shown in Figure 1.8. The subcontractor provides the pricing scheme and capacity availability, and the manufacturer decides their capacity, production,

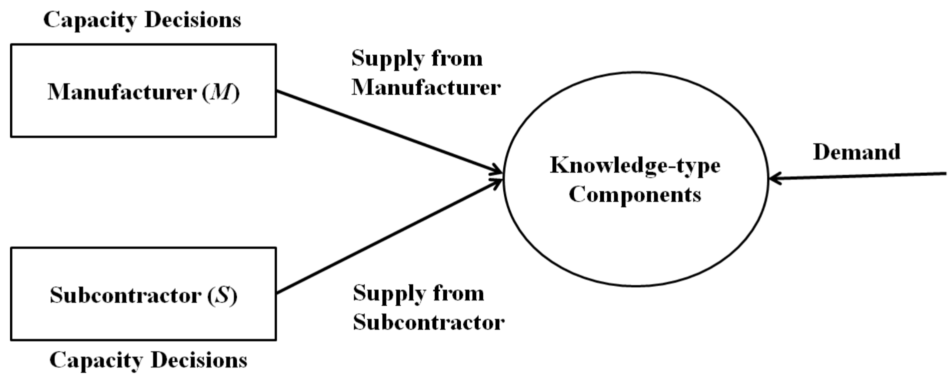

Figure 1.7 Capacity and Sourcing Decisions for Components in Centralized Setting

and subcontracting decisions. Under this setting, we aim to provide answers to the following research questions.

RQ9: How do optimal capacity, manufacturing, and subcontracting decisions depend on pricing scheme?
RQ10: How can we reduce the gap between the system with centralized control and a system with decentralized control?

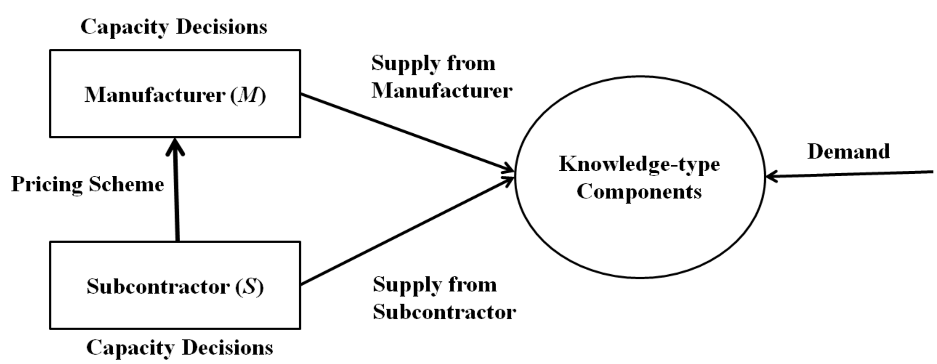

Figure 1.8 Capacity and Sourcing Decisions for Components in Decentralized Setting

In this thesis, we answer these research questions in Chapter 6.

1.4 Research Methodology

We approach this research through university-industry collaboration. We develop analytical methods to provide structural insights of the models, and validate our models with industry partners.

1.4.1 University-Industry Collaboration

This research is part of a multi-year, multi-university collaboration between National Oilwell Varco (NOV), a leading O&G equipment manufacturer, and three universities (University of Wisconsin-Madison, Texas A&M University, Pennsylvania State University). Each of the research problem listed have been motivated and addressed in collaboration with our industry partners. For instance, in the manufacturing of drilling motors at one of NOV manufacturing facility, we observed the issues related to the standard-type components such as rotor, stator, etc. In this setting, the facility could either subcontract the manufacturing of rotor, stator, etc to external subcontractors at a slower production rate and lower costs, or manufacturer these components in-house at a faster production rate and higher production costs. This motivated our research on single product ATO system with standard-type components (**RQ1, RQ2**). Our model and insights for standard-type components have been validated by the industry partners.

Next, while analyzing components/operations such as shafts, wire harnessing, PCB assembly, we observe that some of these components require capacity on a special equipment. This prevented the manufacturer from dedicating such equipment to a specific group of components, and multiple components share the available capacity. This motivated our research on make-to-stock systems and assembly systems comprising of multiple components (**RQ3, RQ4, RQ5, RQ6**).

Finally, while analyzing knowledge-type components such as blowout preventers, we observed that these components have proprietary designs and are made-to-order. These components require special equipment that require high capital investments. The manufacturer also incurs penalty costs (overhead costs) for any under utilized capacity. These components are strategically subcontracted to the external subcontractors to better balance capacity with demand variations in different time periods. However, the industry partner is also concerned with the overhead costs associated with unused capacity at the in-house manufacturing facility. This motivated our research for knowledge-type components where we investigate capacity and production decisions (**RQ7, RQ8, RQ9, RQ10**). In the next section, we discuss analytical methods for subcontracting strategies for standard-type components and knowledge-type components.

1.4.2 Analytical Approach and Thesis Outline

We develop stochastic models to analyze production, subcontracting, and capacity decisions for standard-type and knowledge-type components. We derive theoretical results related to optimal policies and costs, and validate the results using numerical computation and discussions with industry partners.

At first, we analyze production and subcontracting decisions in assemble to order (ATO) system for standard-type components. We use concepts of queuing theory and Markov decision process to analyze ATO system with single product (Chapter 3) and ATO systems with multiple products (Chapter 5). For ATO system with single products, we propose exact method that uses queuing theory and matrix geometric approach to identify optimal thresholds, and production quantities for the in-house manufacturer and the subcontractor. We also propose a novel approximation to solve large systems and provide error bounds using numerical studies (**RQ1**). Next, we compare multiple dual index policies under different parameter settings to obtain conditions under which one dual index policy outperform other policies (**RQ2**).

In Chapter 4 of the thesis, we analyze MTS system with multiple components that share the same manufacturing resource. Using Markov decisions process models and efficient action elimination techniques, we determine the structure of the optimal policy. Numerical experiments validate the theoretical results and highlight the impact of costs and service rates on the optimal production decisions (**RQ3**). Using conditions on the cost and service rates, we partition the state space into various regions and show that the optimal policy has a simple characterization in each region (**RQ4**).

In Chapter 5 of the thesis, we extend our analysis to ATO system with multiple products that share the same manufacturing resources. We address the state space complexity associated with determining the optimal policies for ATO systems with multiple products by using a decomposition based Markov decision process model, and provide the structure of the optimal solution. Note that, in this case we model a shared resource where one could manufacture multiple types of components on the same resources. We also provide insights on the use of the shared resources and validate these observations using numerical studies (**RQ5, RQ6**).

In Chapter 6 of this thesis, we analyze subcontracting strategies for knowledge-type components. In this case, we analyze the structure of the optimal capacity and production decision in the centralized system using Markov decisions process models, and analyze the structure of optimal capacity and production decision in the decentralized system using stochastic game models (**RQ7**). We analytically show the impact of unused capacity on optimal capacity decision at the manufacturer and the subcontractor, and support this analysis with numerical experiments (**RQ8**). Using numerical experiments, we also compare the gap between centralized system and decentralized system and analyze the impact of pricing parameters set by the subcontractor on this gap, and production and subcontracting decisions (**RQ9, RQ10**).

Chapter 2

Literature Review

In this chapter, we review relevant literature. This chapter in categorized into four sections. Section 2.1 focuses on subcontracting strategies in manufacturing systems. Section 2.2 focuses on the single product and multiple product systems. Section 2.3 focuses on the capacity investment models. Section 2.4 focuses on supply chain competition for pricing, capacity and production decisions.

2.1 Subcontracting Strategies in Manufacturing Systems

Studies on subcontracting often focus on how it enables manufacturing firms to improve service levels (Li and Kouvelis (1999); Jiang et al. (2006); Yao et al. (2010)). However, subcontracting strategies have lead time and cost implications as well. Lee and Zipkin (1989) analyze make-or-buy decision in a capacitated system where the manufacture satisfies demand *either* through available internal capacity *or* through unrestricted purchasing from the subcontractor. They assume zero replenishment lead time from both the sources and provide the optimal make-or-buy quantities using a dynamic programming algorithm. They also determine conditions where the manufacturer satisfies the demand through (1) only in-house manufacturing (2) only purchasing from the external supplier, and (3) both in-house manufacturing and purchasing from the external subcontractor. Additionally, the model is extended to include backordering and bounded inventory while minimizing the total costs of production and subcontracting. Van Mieghem (1999) analyzes a two stage capacity acquisition model where the in-house manufacturer and the subcontractor coordinate capacity using game theory approach. They develop a two stage stochastic capacity game where in the first stage the manufacturer and the subcontractor independently and simultaneously decide their capacity levels while in the second stage the manufacturer and the subcontractor independently and simultaneously decide their production levels to satisfy customer demand. Platts et al. (2002) assume constant lead times for procurement and develop a subcontracting framework to determine the quantity to be produced in-house and purchased from external suppliers. Sethi et al. (2003) analyze manufacturing systems with subcontractors that differ in delivery rates and costs and determine optimal (s,S) policies for these settings.

Next, we summarize the literature that studies dual index policies. Bradley (2005) analyzes an in-house production and subcontracting model with exponential processing times for orders and Poisson demand arrival process and shows that the stationary dual base stock policy for component replenishment is optimal. They consider the setting where unit production costs at the subcontractor exceeds the in-house manufacturing variable cost and derive a closed-form structure for the optimal threshold and show that it is a dual base stock type policy. The dual base stock policy specifies one threshold that separates the region where the low production rate is used from the region where the high production rate is used. When the low production rate is used, the production is carried out only by the in-house manufacturer, while when the high production rate is used, production is carried simultaneously by both the subcontractor and the manufacturer. Our research focuses on make and buy decisions (as opposed to make versus buy) i.e., the manufacturing facility primarily procures components but reserves the option to make parts in-house to meet service level obligations. Further, our focus on an ATO system makes our analysis more complex than the study in Bradley (2005). Veeraraghavan and Scheller-Wolf (2008) determine optimal order quantities for both the subcontractor as well as the in-house manufacturer under the assumption of deterministic lead times. They propose a dual index policy that is near optimal.

Our work builds on the dual base stock and dual index policies discussed in the literature, but extends their application to a broad class of ATO systems with multiple components. In particular, depending on the sourcing and replenishment decisions, the replenishment lead times of these components could vary with the workload at both in-house manufacturing and local subcontracting facilities. ATO systems that operate under dual index policies recognize the sensitivity of lead time to workloads at the in-house manufacturing facility and local subcontractor to suitably adapt their production and subcontracting decisions to improve system performance.

In the subsequent sections, we review the literature related to the component replenishment policies. Many studies on ATO systems build on the classical results reported in Rosling (1989) and Clark and Scarf (1960). These studies analyze a multi-stage assembly system and show that base stock policy is optimal when the system does not have any capacity constraints. We analyze two streams of literature on ATO system: (1) system operating under ATO system with single end product, and (2) system operating under ATO system with multiple products.

2.2 Assemble-to-Order Systems

Single Product Systems: Several researchers have analyzed ATO system with a single end product. Studies on ATO systems with single product focus on the impact of optimal decisions and system parameters. One stream of literature focuses on base stock control models. Song and Yao (2002) model a single product ATO system where the final product is assembled from components or sub-assemblies that are made to stock. The paper assumes that customer orders follows a Poisson process and the final product is assembled components is negligible time if all components or sub-assemblies are available, otherwise the final product is backordered with positive backordering cost. The paper assumes that component replenishment lead times are independent and identically distributed and the authors analyze the performance of the system using $M/G/\infty$ queuing system. Using greedy-type algorithm, the paper evaluates the impact on system parameters on the performance measures and also determine a easy to compute performance bounds on the average backorders and component inventory. Gallien and Wein (2001) analyze a single product ATO system with independent and non-identical replenishment lead times using queuing theory and provide an approximate solution to determine component replenishments.

Ko et al. (2011) model a single product ATO system with Poisson demand arrival for the assembled product and exponential service times for the components. In this case, each component is produced at a production facility (in-house or subcontractor). They assume base stock policy for inventory replenishment and derive a closed-form expression using linear bounds for lead times of components. Karaarslan et al. (2013) consider single product ATO system and derive the optimality condition for ATO system under two variations of the pure base stock policy.

Another stream of literature focuses on continuous-review models. Glasserman and Wang (1998) analyze a ATO system with a single product assembled from multiple components. They assumes a continuous review base stock policy and establish trade-offs between delivery performance and the average on-hand inventory of components in ATO systems. Song (2002) considers continuous review model of single product ATO system with multiple components and develop an efficient algorithm to analyze the performance of the ATO system.

All of these studies share two similarities: (i) the component stock replenishment is done using a base stock policy, and (ii) the lead time distribution for component replenishment is known. In Chapter 3 of this thesis, we analyze a ATO system with single end product. Our research also assumes that the component stock replenishment is done using a base stock policy, but in contrast it assumes that the replenishment can be done either in-house or at a subcontractor facility based on the stock level. This additional flexibility in turn influences the distribution of lead time for component replenishment orders.

Multi-product Systems: Studies including Song (1998, 2000); Lu and Song (2005); Zhou and Chao (2012) analyze the performance of ATO systems with multiple products. Song (1998) is the first to analyze the order fill rate in a multi-component ATO system operating under a base stock policy. The demand process is modeled as a multivariate compound Poisson process where several types of customer arrives and orders a certain subset of components. The demand of each component is then superimposed to obtain a compound Poisson process that models the demand for each product. The paper assumes that the unfulfilled demand is backlogged at positive cost. However, the paper also assumes that upon demand arrival, if some of the components are unavailable, then the in-stock components are shipped to the customer and the customer only waits for the out-of stock components. The paper derives a structured expression to determine the optimal order fill rate for multi-component system and shows that the fill rate of an individual component is not a good indicator of the order fill rate. Song (2000) analyzes a ATO system with multiple products where customer orders arrives in batches of different sizes. They assume constant replenishment lead time and compound Poisson process for customer demand. They present a model that estimates the order fill rate for the final product. Lu and Song (2005) analyze an multi-product ATO system operating under a base stock policy and develop simple bounds and approximations to evaluate the expected backorders. Zhou and Chao (2012) analyze multi-product ATO system using simple Stein-Chen approximation. They assume arbitrary distributed component replenishment lead times and provide error bounds on the optimal order fill rate.

ATO systems with multiple products has been widely analyzed using numerical methods. Zhao and Simchi-Levi (2006) develop an efficient numerical method based on Monte-Carlo simulation to analyze large multi-product ATO system with batch ordering. They assume the demand arrival follows Poisson process and determine the optimal order fill rate. El-Hafsi et al. (2008) develop heuristics to analyze production and inventory policies in a multi-product ATO system. Under the assumption of independent lead time of component replenishment and lost sales, they show that the base stock and inventory rationing are the optimal production and inventory policies. Later, Zhao (2009) extends Zhao and Simchi-Levi (2006) research to a general class of ATO systems that includes both non-split orders and split-orders. The paper develop an exact method as well as an efficient sampling method to determine the order fill rate in an ATO system. However, none of these studies analyze production and subcontracting decisions while considering interaction in the component replenishment in a multi-product, multi-component ATO system. In many settings, subcontractors present an alternate production capability that could be used to reduce average backorders.

Several studies have analyzed MTS system with multiple products. Ha (1997) studies the optimal production scheduling in a facility that manufacturers two products on a shared manufacturing resource. For the special case where both products have equal service rates, they develop a linear switching rule for production scheduling. Benjaafar et al. (2004) develop a non-linear formulation for multi-product, multiple flexible resources demand allocation problem and use branch and bound algorithm to determine stationary long-run fractions for each resource and product. Gurvich et al. (2008) and Hu and Benjaafar (2009) analyze system with multiple fully flexible resources, multiple demand classes and use queuing theory to derive stationary policies to reduce number of resources, and wait time of customers respectively. In contrast, Chapter 4 of this thesis analyzes partially flexible resources (dedicated subcontractors and shared manufacturer) and determine the structure of the optimal production scheduling and subcontracting decisions. Our research also requires that the components have different service rates and can be made at one or more facilities.

In Chapter 5 of this thesis, we analyze ATO system with multiple products. We consider dedicated subcontractor and shared in-house manufacturing facility for component replenishment. Using stochastic models, we analyze the interactions in components during the manufacturing process and provide insights on optimal component replenishment strategy.

Subsequent section provides an survey of literature on strategic outsourcing in supply chain. Research usually involves a game theoretic formulations involving either Cournot competitions (Varian (2006)), Bertrand competitions (Bertrand (1883)), and/or Stackelberg games (von Stackelberg (2011)).

We analyze two streams of literature: (1) capacity investment models, and (2) supply chain competition in assembly systems.

2.3 Capacity Investment Models

Several studies in the literature derive capacity acquisition strategies to set optimal production/capacity levels (Van Mieghem (1999); Atamturk and Hochbaum (2001); Rajagopalan and Swaminathan (2001); Bish et al. (2005); Niroomand and Hochbaum (2012)). Van Mieghem (1999) analyzes a single period centralized system and a decentralized system where the manufacturer and the subcontractor decides the capacity levels at the beginning of the time period, followed by production decisions when the demand is realized. The subcontractor optimally decides the production quantities to satisfy its own market demand and also to supply products to the manufacturer. The paper analyzes various price contracts and show that the lower price contracts with the supplier could decrease the overall profit of the manufacturer. Next, Atamturk and Hochbaum (2001) develop a multi-period deterministic linear capacity acquisition and subcontracting model for non-stationary demand. They also provide insights on the tradeoffs to balance insufficient capacity and excess capacity in any time period to minimize the total cost of capacity acquisition, production, subcontracting, and inventory decisions. Bradley and Glynn (2002) develop a Brownian motion model to analyze capacity investment and inventory decisions for a single firm. Niroomand and Hochbaum (2012) develop a mixed-integer formulation to identify the optimal capacity allocation in manufacturing systems.

There have been several studies that use game theory models to determine the optimal capacity and production levels under uncertain demand (Van Mieghem (1999); Wang and Gerchak (2003); Iyer and Jain (2004); Bernstein et al. (2011)). Wang and Gerchak (2003) consider a decentralized system where the components can be replenished through a secondary source and the manufacturer acts as a leader. The manufacturer can either invest on capacity in-house or use external subcontractor to satisfy the demand. They assume stochastic demand and use Stackelberg game to provide insights on the optimal capacity and pricing levels of the components. Bernstein et al. (2011) develop a Stackelberg game to analyze multi-product system where a single firms decides the capacity levels prior to demand realization. Li and Debo (2009) analyze a two period model where one supplier invests on non disposable capacity to satisfy demand of the downstream firm for the first time period, and two suppliers could invest on non disposable capacity to satisfy demand of the downstream firm for the second time period. They assume zero unused capacity cost and show that both suppliers should produce in the second time period under increasing demand case. Swinney et al. (2011) analyze capacity investment timing model where the firms can invest on capacity at two times: (i) invest on capacity early at a lower price, when the demand uncertainty is not resolved (ii) invest on capacity late at a higher price when the demand uncertainty is resolved. They use game theory to show that under high demand uncertainty, new firms should make early investment on capacity and the more established firm should make late invest on capacity.

2.4 Supply Chain Competition in Assembly Systems

Another stream of work relates to supply chain competition in assembly systems (Gerchak and Wang (2004), Bernstein and DeCroix (2006), Zhang (2006); Zhang et al. (2008); Jiang and Wang (2010)). For instance, Gerchak and Wang (2004) consider a decentralized assembly system with uncertain demand and analyze two types of supply chain contracts between the assembler and the subcontractors. First contract considers quantity game similar to Stackelberg game between the assembler and the subcontractors to maximize revenue while the second contract considers a price game similar to Bertrand game between multiple suppliers to select the best wholesale price for the product. Bernstein and DeCroix (2006) analyze a decentralized assembly system with stochastic and stationary demand where a single product is assembled from two components. The components are replenished using base stock policy. Using game theory, they analyze equilibrium base stock levels for each component. Zhang (2006) analyzes a decentralized assembly system that consists of a manufacturer and multiple subcontractors. Using stochastic game formulation, they determine equilibrium base stock levels for components under random demand case. Zhang et al. (2008) analyze a decentralized assembly system where the manufacturer and subcontractors are involved in the quantity game, and the manufacturer also provides the wholesale price for the components according to a push and pull system. Jiang and Wang (2010) analyze a decentralized assembly system where components are sourced from multiple suppliers. In this case, suppliers are involved in Bertrand price competition to decide the price for the components. Li (2002) considers a two-level supply chain consisting of an upstream manufacturer and multiple downstream retailers, and analyze a setting with demand and cost information leakage. They assume demand uncertainty and analyze a Stackelberg game between the manufacturer and retailers where the manufacturer sets the pricing scheme and the retailers decide the production quantity, and a Cournot competition between retailers to sell the product at a constant cost. They identify the equilibrium price decision by the upstream manufacturer and the equilibrium quantity decision made by retailers. Zhang (2002) extends Li (2002) research to include Bertrand price competition for the downstream firms and show that the type of game at the downstream level does not impact the optimal strategy.

Cachon and Lariviere (2001) model a Stackelberg game between the manufacturer and single supplier where the manufacturer is the leader and determine the production capacity levels. Bernstein and DeCroix (2004) analyze a modular assembly system where the components are purchased from different subcontractors. They analyze pricing and capacity games between the assembler (leader) and the subcontractors and show that in equilibrium, the assembler tends to set the price of the components such that the subcontractor always produces at the same capacity level. Anand and Goyal (2009) use Stackelberg game to analyze demand information leakage between the incumbent and the entrant in a supply chain with demand uncertainty.

These studies do not consider the impact of unused capacity on the optimal capacity and production decisions. In addition, studies on decentralized system usually assume single time period. In Chapter 6 of this thesis, we consider a multi-time period problem and analyze the effect of unused capacity cost at the manufacturer and the subcontractor on the optimal capacity, production, and pricing decisions.

Chapter 3

Single Product Systems with Standard-type Components

3.1 Introduction

In order to cut costs and reduce lead times, many manufacturers design their products and processes so that the final product can be quickly assembled from its components. In the literature, these systems are commonly referred to as assemble-to-order (ATO) systems. ATO systems combine the benefits of make-to-order (MTO) systems and make-to-stock (MTS) systems to provide custom products at short lead times. The strategy initially found popularity in the computer industry, and since then the concept has gained acceptance in several other industries. Our research is motivated by collaborations with a large manufacturer of custom drilling motors for industrial applications. These drilling motors vary significantly in terms of their power requirement, motor speed, motor size and shape. A typical drilling motor is assembled from various components such as rotor, stator, shaft, and connection box that have different specifications and ratings. Since the manufacturing of some of these components takes considerable amount of time and machining resources, the manufacturer often builds the critical components to stock. On receipt of an order, the drilling motor is assembled to the required specification from the components in stock. Therefore, the availability of components is critical to guarantee high service levels and short lead times.

In such a setting the manufacturer could either subcontract some components to a local vendor or produce them in-house to minimize component stock outs. Consequently, the manufacturer needs to balance the tradeoffs (in-house production costs, subcontracting costs, on-hand inventory costs, and backordering costs) and determine when and how much quantity of components need to be made in-house versus at the subcontractor. For example, if the manufacturer can produce the components at shorter lead times, the benefits of short lead times might outweigh the higher in-house production costs. In contrast, if the manufacturer is constrained by capacity, they might subcontract manufacturing of components to a local subcontractor and incur the subcontracting costs. Understandably, the manufacturer needs to analyze these tradeoffs while making production and subcontracting decisions.

In this chapter, we analyze an ATO system that uses a combination of stocking policies and subcontracting strategies to improve component availability. The ATO system assembles a single end-product from N components that are build to stock. Production and stocking decisions are made based on one of three dual index policies namely, the dual base stock policy (DB policy), the on-hand inventory based policy (OH policy), and the lead time based policy (LT policy), respectively. The stock for the components are replenished either from a local subcontractor or by the in-house manufacturing facility. Both facilities, have finite production capacity and stochastic lead times. We use the Matrix Geometric approach described in Neuts (1981) and exploit the structure in the sparse transition matrix to provide an exact solution to estimate system performance in moderately sized systems with two components ($N = 2$).

For larger systems with more than two components ($N > 2$), state space explosion prevents an exact analysis. We overcome this challenge through a novel approximation method that uses decomposition of the Markov chain to efficiently evaluate the system performance. The approximation method has several advantages. First, the approach scales well with increase in the number of components (N). Second, the approach can be easily adapted to analyze system performance under various dual index policies. Third, the approach yields reasonably accurate estimates of performance to guide managerial decisions.

Using numerical studies we illustrate the performance of our approach for a system operating under three dual index policies (namely, DB, OH, and LT). We point out an operational ambiguity that arises when a system operates under the DB policy and then show that the OH policy and LT policy provide insights into how this ambiguity can be resolved in the DB policy to realize its benefits in practice.

The rest of the chapter is organized as follows. Section 3.2 presents the Markov chain formulation of the proposed system. Using these formulations, we develop an exact solution methodology for solving ATO systems and analyze its computational challenge for large scale problems. Section 3.3 presents an approximation method to solve large systems for dual index polices. We also extend approximation method for an ATO system with multiple components. Section 3.4 summarizes numerical studies for the proposed policies. Finally, Section 3.5 summarizes model insights and conclusions.

3.2 Manufacturing System with Production and Subcontracting

Figure 3.1 illustrates an ATO system that assembles a single product with two components. Component $k, k = 1, 2$; can be manufactured by the the in-house manufacturing facility M_k and the local subcontractor S_k. The components are stored at inventory location L_k and are assembled at station A to satisfy the demand for the final product. We assume that the customer orders for the final product arrive according to a Poisson process $N(t), t \geq 0$ with rate λ and are satisfied on a first-come-first serve (FCFS) basis at assembly station A. Assembly operations of this station are instantaneous, i.e. if both components are available at the demand arrival epoch, then the demand for the final product is immediately satisfied. If one or more component is unavailable, then the demand for the final product is backordered and the customer order stays in the queue at station A. We model the local subcontractor S_k and the in-house manufacturing facility M_k, $k = 1, 2$ as a single server queue with exponentially distributed service time with mean $\mu_{s,k}^{-1}$ and $\mu_{m,k}^{-1}$ respectively. This allows us to model the effect of workload on lead times at these facilities. The production cost per unit is $c_{s,k}$ and $c_{m,k}$ for component k at S_k and M_k respectively. Without loss of generality, we assume that $\mu_{s,k} < \mu_{m,k}$ and $c_{s,k} < c_{m,k}, k = 1, 2$.

We assume that the system maintains a base stock level z_k, for each component k, i.e., we ensure that the net inventory position is z_k through orders for replenishing inventory placed at demand arrival epochs. Let $O_{m,k}(t)$, $O_{s,k}(t)$, $I_k(t)$, $B_k(t)$ denote in-house manufacturer's on-orders, subcontractor's on-orders, on-hand inventory quantity and backorders for component k respectively at decision epoch t. Then, since the system maintains a base stock policy for each component, the following equation holds:

$$z_k = O_{m,k}(t) + O_{s,k}(t) + I_k(t) - B_k(t), k = 1, 2, \forall t \tag{3.1}$$

Note that at anytime t, $I_k(t)B_k(t) = 0$. For this system, we analyze system performance under three ordering policies, namely dual base stock policy (or DB policy), on-hand inventory base policy (or OH policy), and lead time based policy (or LT policy).

Dual Base Stock (DB) Policy:

Under the dual base stock policy, if at any instant t corresponding to a demand arrival, $I_k(t) < e_k$ (where e_k is a predefined inventory threshold limit), then the manufacturer uses all available capacity at its internal manufacturing facility, M_k and the local subcontractor, S_k to replenish the inventory for component k. If instead at the demand arrival epoch t, $z_k > I_k(t) \geq e_k$, the manufacturer places an order to replenish inventory for component k only to its local subcontractor S_k. If $I_k(t) = z_k$; at demand arrival epoch, no replenishment order is placed for component k. Note that when $I_k(t) < e_k$, the dual base stock policy does not specify who should get the order (M_k or S_k) as long as the order ensures that both the internal manufacturing facility M_k and local subcontractor S_k are busy. However, from an

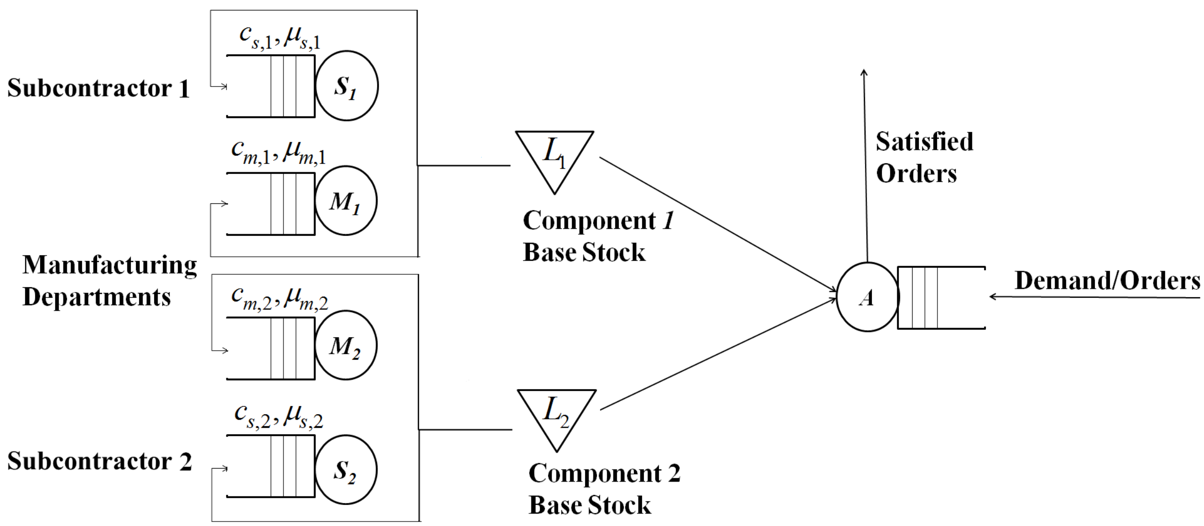

Figure 3.1 Supply Chain Model for Single Product System

order fulfillment point of view, it is important to determine how workload of replenishing inventory must be distributed between M_k and S_k. Therefore, we will consider two variations of the dual base stock policy, namely the OH policy and the LT policy, that specifies who gets the order when $I_k(t) < e_k$.

1. *On-hand inventory based (OH) policy*: Under the OH policy, if at a demand arrival epoch t, $I_k(t) < j_k$ (where j_k is a predefined inventory threshold limit), then the manufacturer places the order for component k order to its internal manufacturing facility M_k. If at the demand arrival epoch t, $z_k > I_k(t) \geq j_k$, the manufacturer places the order for component k to the local subcontractor S_k.

2. *Leadtime based (LT) policy*: Under the LT policy, the manufacturer first determines the estimates of lead time $\hat{L}_{m,k}(t) = O_{m,k}(t)/\mu_{m,k}$ for the in-house manufacturing facility and $\hat{L}_{s,k}(t) = O_{s,k}(t)/\mu_{s,k}$ for the local subcontractor of component k, respectively. Then, if at the demand arrival epoch t, $\hat{L}_{m,k}(t) < l_k \hat{L}_{s,k}(t)$ (where l_k is a predefined lead time threshold limit), the manufacturer places the order for component k to its internal manufacturing facility M_k, and to the local subcontractor S_k when $\hat{L}_{m,k}(t) \geq l_k \hat{L}_{s,k}(t)$.

Note that we intentionally use separate notations, e_k, j_k, and l_k to denote the thresholds corresponding to the DB, OH, and LT policies, as these thresholds could be different from each other. In the next section, we present an exact analysis of this system under these three ordering policies. We shall show through numerical studies in Section 3.5, that despite the operational ambiguities in the DB policy, the policy actually yields better performance. The performance of OH and LT policy then provide an intuitive explanation for this superior performance and also resolve this operational dilemma of who should get specific orders in the DB policy.

3.3 Exact Analysis of System with Two Components

This section presents exact approach to determine the steady state probabilities Π for an ATO system with subcontracting flexibility.

3.3.1 Exact Analysis under DB Policy

Under dual base stock policy, let each state in the state space, Σ^{DB} be defined as $\sigma^{DB} = (I_1, I_2)$; where, I_k is the inventory position of component $k, k = 1, 2$. Then, the system evolution can be modeled as a Markov chain. Let Π^{DB} denote the steady state probability vector and, $\pi^{DB}(I_1, I_2)$ denote the steady state probability of state (I_1, I_2). Let N_k^{DB} denote the possible values of I_k under the dual base stock policy and B_{max} denote the finite maximum limit for backorders of any component $k, k = 1, 2$. Then, the total number of states in Σ^{DB} is $N_1^{DB} N_2^{DB}$ and under the dual base stock policy, $N_k^{DB} = (z_k + B_{max}), k = 1, 2$. Note that our assumption that B_{max} is finite is not restrictive and the analysis in the sections below can be extended to the case where $B_{max} = \infty$ with minimal modifications. However, setting B_{max} to be finite allows us to see the impact of various policies on the structure of the transition probability matrices, and limits computations to finite matrices.

The state space Σ^{DB} can be re-written as $\Sigma^{DB} = \mathbb{A}_1^{DB} \times \mathbb{A}_2^{DB}$ where $\mathbb{A}_k^{DB}, k = 1, 2;$ is the set of all possible values of the (m_k, s_k) pair. To construct the transition matrix $\mathbb{Q}^{DB}$, we exploit the similarities in the transition probabilities for states belonging to a particular set within each $\mathbb{A}_k^{DB}$. Each set $\mathbb{A}_k^{DB}, k = 1, 2;$ can be further partitioned into 5 mutually exclusive subsets, $\mathbb{A}_{k,i}^{DB} \subset \mathbb{A}_k^{DB}, i = 1, 2, ..., 5$ where $\cup_i \mathbb{A}_{k,i}^{DB} = \mathbb{A}_k^{DB}$ and

$A_{k,1}^{DB} = \{I_k : I_k = -B_{max}\}$
$A_{k,2}^{DB} = \{I_k : 1 - B_{max} \leq I_k \leq e_k - 1\}$
$A_{k,3}^{DB} = \{I_k : I_k = e_k\}$
$A_{k,4}^{DB} = \{I_k : e_k + 1 \leq I_k \leq z_k - 1\}$
$A_{k,5}^{DB} = \{I_k : I_k = z_k\}$

Note that if $B_{max} = \infty$, $A_{k,1}^{DB}$ and $A_{k,2}^{DB}$ merge into one subset. For notational simplicity, we drop the superscript DB in the rest of this section. Using these subsets $\mathbb{A}_{k,i} \subset \mathbb{A}_k, i = 1, 2, ..., 5$, Chapman-Kolmogorov (C-K) equations can be written. For instance, for $I_1 \in \mathbb{A}_{1,2}$ and $I_2 \in \mathbb{A}_2$ the C-K equations are written as follows:

For $I_1 \in \mathbb{A}_{1,2}$ and $I_2 \in \mathbb{A}_{2,1}$:

$$\begin{aligned}(\mu_{s,1} + \mu_{m,1} + \mu_{s,2} + \mu_{m,2})\pi(I_1, I_2) &= \lambda\pi(I_1 + 1, I_2 + 1) \\ &\quad + (\mu_{m,1} + \mu_{s,1})\pi(I_1 - 1, I_2) \qquad (3.2)\end{aligned}$$

For $I_1 \in \mathbb{A}_{1,2}$ and $I_2 \in \mathbb{A}_{2,2}$:

$$\begin{aligned}(\lambda + \mu_{s,1} + \mu_{m,1} + \mu_{s,2} + \mu_{m,2})\pi(I_1, I_2) &= \lambda\pi(I_1 + 1, I_2 + 1) + (\mu_{m,1} + \mu_{s,1})\pi(I_1 - 1, I_2) \\ &\quad + (\mu_{m,2} + \mu_{s,2})\pi(I_1, I_2 - 1) \qquad (3.3)\end{aligned}$$

For $I_1 \in \mathbb{A}_{1,2}$ and $I_2 \in \mathbb{A}_{2,3}$:

$$\begin{aligned}(\lambda + \mu_{s,1} + \mu_{m,1} + \mu_{s,2})\pi(I_1, I_2) &= \lambda\pi(I_1 + 1, I_2 + 1) + (\mu_{m,1} + \mu_{s,1})\pi(I_1 - 1, I_2) \\ &\quad + (\mu_{m,2} + \mu_{s,2})\pi(I_1, I_2 - 1) \qquad (3.4)\end{aligned}$$

For $I_1 \in \mathbb{A}_{1,2}$ and $I_2 \in \mathbb{A}_{2,4}$:

$$\begin{aligned}(\lambda + \mu_{s,1} + \mu_{m,1} + \mu_{s,2})\pi(I_1, I_2) &= \lambda\pi(I_1 + 1, I_2 + 1) + (\mu_{m,1} + \mu_{s,1})\pi(I_1 - 1, I_2) \\ &\quad + \mu_{s,2}\pi(I_1, I_2 - 1) \end{aligned} \tag{3.5}$$

For $I_1 \in \mathbb{A}_{1,2}$ and $I_2 \in \mathbb{A}_{2,5}$:

$$(\lambda + \mu_{s,1} + \mu_{m,1} + \mu_{s,2})\pi(I_1, I_2) = (\mu_{m,1} + \mu_{s,1})\pi(I_1 - 1, I_2) + \mu_{s,2}\pi(I_1, I_2 - 1) \tag{3.6}$$

The C-K equations for other pairs $I_1 \in \mathbb{A}, i = 1, ..., 5$ and $I_2 \in \mathbb{A}_{k,j}$ can be written in a similar way. Unfortunately, these C-K equations yield a large and sparse transition matrix $\mathbb{Q}$. However, we can exploit the structural properties of the transition matrix using Matrix-Geometric representation. We discuss the details below.

Let, $\mathbb{C} = diag(0, \lambda, \lambda, \lambda)$. Define $\mathbb{I}_1$ as an identity matrix of size $N_1 \times N_1$, $\mathbb{B}_{2,1} = \mathbb{B}_{1,1} + \mathbb{C} - (\mu_{m,2} + \mu_{s,2})\mathbb{I}_1$, $\mathbb{B}_{3,1} = \mathbb{B}_{1,1} - (\mu_{m,2} + \mu_{s,2})\mathbb{I}_1$, and $\mathbb{B}_{4,1} = \mathbb{B}_{1,1} - (\mu_{s,2})\mathbb{I}_1$. The corresponding matrices ($\mathbb{B}_{2,2}, \mathbb{B}_{3,2}$, and $\mathbb{B}_{4,2}$) for component 2 are defined in a similar way. The matrices, $\mathbb{B}_{1,k}, k = 1, 2$ and $\mathbb{D}$ are defined as follows:

$$\mathbb{D} = \begin{vmatrix} 0 & 0 & 0 & 0 \\ \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \end{vmatrix}$$

$$\mathbb{B}_{1,k} = \begin{vmatrix} -(\mu_{m,k} + \mu_{s,k}) & (\mu_{m,k} + \mu_{s,k}) & 0 & 0 \\ 0 & -(\lambda + \mu_{m,k} + \mu_{s,k}) & (\mu_{m,k} + \mu_{s,k}) & 0 \\ 0 & 0 & -(\lambda + \mu_{m,k} + \mu_{s,k}) & \mu_{s,k} \\ 0 & 0 & 0 & -(\lambda + \mu_{m,k} + \mu_{s,k}) \end{vmatrix}$$

Then the transition matrix, $\mathbb{Q}$ can be constructed using the above mentioned matrices as shown in Equation (3.7) and the steady state probabilities can be calculated using the system of Equations (3.8) and (3.9) and the Matrix-Geometric technique described in Neuts (1981).

$$\mathbb{Q} = \begin{array}{c|cccc} & \overrightarrow{(0,*)}^{\top} & \overrightarrow{(1,*)}^{\top} & \overrightarrow{2,*)}^{\top} & \overrightarrow{(3,*)}^{\top} \\ \hline \overrightarrow{(0,*)} & \mathbb{B}_{2,2} & (\mu_{m,1}+\mu_{s,1})\mathbb{I}_2 & & \\ \overrightarrow{(1,*)} & \mathbb{D} & \mathbb{B}_{3,2} & (\mu_{m,1}+\mu_{s,1})\mathbb{I}_2 & \\ \overrightarrow{(2,*)} & & \mathbb{D} & \mathbb{B}_{4,2} & \mu_{s,1}\mathbb{I}_2 \\ \overrightarrow{(0,*)} & & & \mathbb{D} & \mathbb{B}_{1,2} \end{array} \quad (3.7)$$

$$\Pi\mathbb{Q} = \mathbf{0} \quad (3.8)$$

$$\Pi\mathbf{e} = 1 \quad (3.9)$$

Here, $\mathbf{e}$ =[1,...,1] of size $1 \times (N_1 \times N_2)$ and $\overrightarrow{(0,*)}$ represents a vector with all states having $I_1 = 0$ in the dual base stock policy. From the solutions to Equations (3.8) and (3.9), the expected on-hand inventory levels $E[I_1]$ and expected backorders $E[B_1]$ for component 1 can be calculated using Equations (3.10) and (3.11). The performance measures for the component 2 can be calculated in a similar way. The in-house throughput, $TH_{m,1}$ and the supplier's throughput, $TH_{s,1}$ for component 1 are also computed using Equation (3.12) and (3.13). Similarly, we can define the performance measures for component 2. Note that in these equations $\pi(*,*)$ denotes the steady state probability at the particular state.

$$E[B_1] = \sum_{I_1} max(-I_1,0)\pi(I_1,*) \quad (3.10)$$

$$E[I_1] = \sum_{I_1} max(I_1,0)\pi(I_1,*) \quad (3.11)$$

$$TH_{m,1} = \sum_{I_1<e_1} \mu_{m,1}\pi(I_1,*) \quad (3.12)$$

$$TH_{s,1} = \sum_{I_1<z_k} \mu_{s,1}\pi(I_1,*) \quad (3.13)$$

3.3.2 Exact Analysis under OH and LT Policy

When a system operates under policy, $\mathcal{P}$, where $\mathcal{P} \in \{OH, LT\}$, the state of the system is completely defined only if $O_{m,k}(t)$ and $O_{s,k}(t)$ are known for each component $k, k = 1, 2$ at any time t. Thus, we define a four-dimensional state variable to describe the system state under these policies. Let, $\sigma^{\mathcal{P}}$ denote a state in the state space $\Sigma^{\mathcal{P}}$, where $\sigma^{\mathcal{P}} = (m_1, s_1, m_2, s_2)$ and m_k (or s_k) represents the on-order quantity of component k at in-house manufacturer M_k (or local subcontractor S_k). Then, the system evolution can be modeled as a Markov chain. Let $\Pi^{\mathcal{P}}$ denote the steady state probability vector and $\pi^{\mathcal{P}}(m_1, s_1, m_2, s_2)$ denote the steady state probability of state (m_1, s_1, m_2, s_2). Let, $N_k^{\mathcal{P}}$ denote the possible values in the tuple (m_k, s_k). Then, the total number of states in $\Sigma^{\mathcal{P}}$ is $N_1^{\mathcal{P}} N_2^{\mathcal{P}}$. Under the OH policy, $N_k^{OH} = (z_k - j_k + 2)(B_{max} + (z_k + j_k + 1)/2)$ and under the LT policy, $N_k^{LT} = (z_k + B_{max})(z_k + B_{max} + 1)/2$ for $k = 1, 2$.

The state space $\Sigma^{\mathcal{P}}$ can be re-written as $\Sigma^{\mathcal{P}} = \mathbb{A}_1^{\mathcal{P}} \times \mathbb{A}_2^{\mathcal{P}}$ where, $\mathbb{A}_k^{\mathcal{P}}, k = 1, 2$; is the set of all possible values of the (m_k, s_k) pair. To construct the transition matrix $\mathbb{Q}^{\mathcal{P}}$ we exploit the similarities in the transition probabilities for states belonging to a particular sets within each $\mathbb{A}_k^{\mathcal{P}}$. For instance, each set $\mathbb{A}_k^{OH}, k = 1, 2$; can be further partitioned into 11 mutually exclusive subsets, $\mathbb{A}_{k,i}^{OH} \subset \mathbb{A}_k^{OH}, i = 1, 2, ..., 11$ where $\cup_i \mathbb{A}_{k,i}^{OH} = \mathbb{A}_k^{OH}$ and

$A_{k,1}^{OH} = \{(m_k, s_k) : m_k = 0,\ s_k = 0\}$
$A_{k,2}^{OH} = \{(m_k, s_k) : m_k = 0,\ 1 \le s_k \le z_k - j_k\}$
$A_{k,3}^{OH} = \{(m_k, s_k) : m_k = 0,\ s_k = z_k - j_k + 1\}$
$A_{k,4}^{OH} = \{(m_k, s_k) : q_k \le m_k \le B_{max} + j_k + q_k - 3,\ s_k = z_k - j_k - q_k + 2,\ q_k = \{1, ..., z_k - j_k + 1\}\}$
$A_{k,5}^{OH} = \{(m_k, s_k) : m_k = B_{max} + j_k + q_k - 2,\ s_k = z_k - j_k - q_k + 2,\ q_k = \{1, ..., z_k - j_k + 1\}\}$
$A_{k,6}^{OH} = \{(m_k, s_k) : 1 \le m_k \le z_k - j_k,\ s_k = 0\}$
$A_{k,7}^{OH} = \{(m_k, s_k) : 1 \le m_k \le z_k - j_k,\ 1 \le s_k \le z_k - j_k - q_k + 1,\ q_k = \{2, ..., z_k - j_k + 1\}\}$
$A_{k,8}^{OH} = \{(m_k, s_k) : 1 \le m_k \le z_k - j_k,\ s_k = z_k - j_k - q_k + 2,\ q_k = \{2, ..., z_k - j_k + 1\}\}$
$A_{k,9}^{OH} = \{(m_k, s_k) : m_k = z_k - j_k + 1,\ s_k = 0\}$

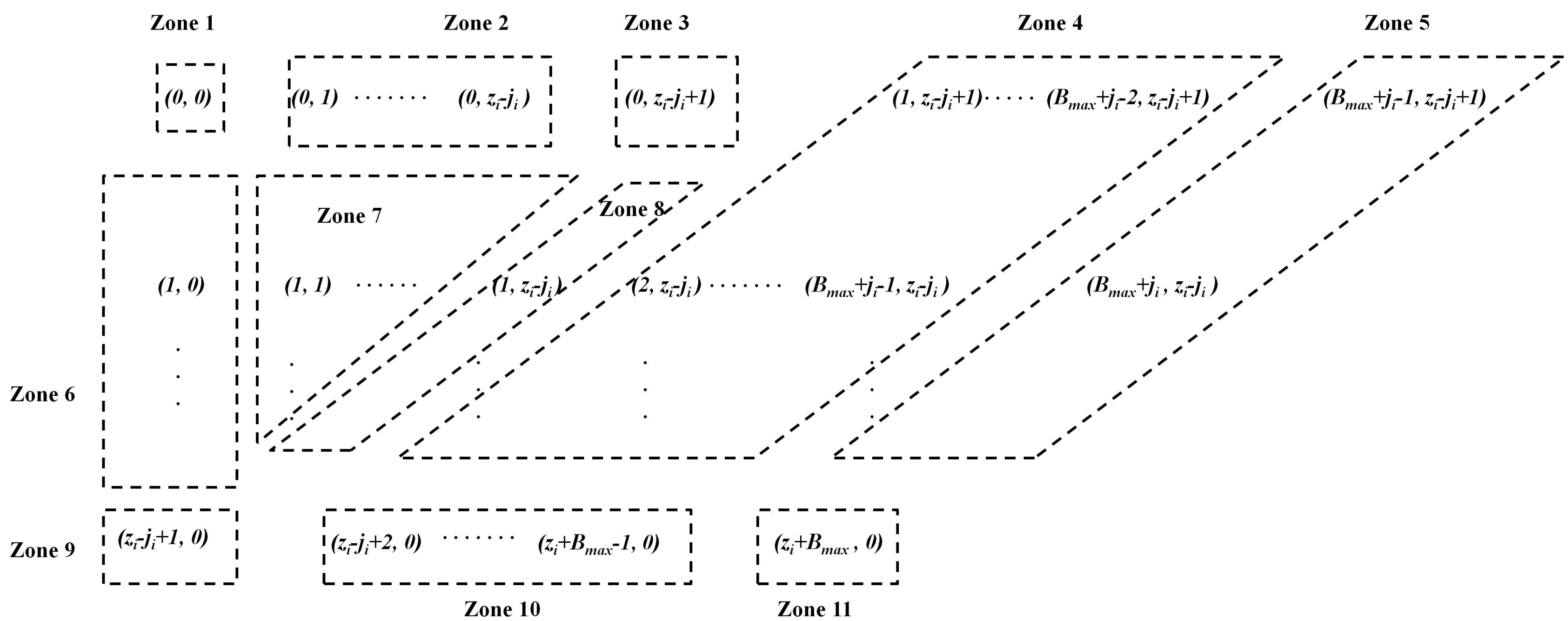

Figure 3.2 Subset Classification of the States for Component k under OH Policy

$A_{k,10}^{OH} = \{(m_k, s_k) : z_k - j_k + 2 \leq m_k \leq z_k + B_{max} - 1,\ s_k = 0\}$
$A_{k,11}^{OH} = \{(m_k, s_k) : m_k = z_k + B_{max},\ s_k = 0\}$

Note that if $B_{max} = \infty$, subsets $A_{k,4}^{OH}$ and $A_{k,5}^{OH}$ merge into one subset, and subsets $A_{k,10}^{OH}$ and $A_{k,11}^{OH}$ merge into another subset. Figure 3.2 represents these subsets pictorially. For notational simplicity, we drop the superscript OH in the rest of the section. Using these subsets $\mathbb{A}_{k,i} \subset \mathbb{A}_k, i = 1, 2, ..., 11$, Chapman-Kolmogorov (C-K) equations can be written. For instance, for $(m_1, s_1) \in \mathbb{A}_{1,4}$ and $(m_2, s_2) \in \mathbb{A}_2$ the C-K equations are written as follows:

For $(m_1, s_1) \in A_{1,4}$ and $(m_2, s_2) \in A_{2,1}$:

$$(\lambda + \mu_{s,1} + \mu_{m,1})\pi(m_1, s_1, m_2, s_2) = \mu_{m,1}\pi(m_1 + 1, s_1, m_2, s_2) + \mu_{s,2}\pi(m_1, s_1, m_2, s_2 + 1) \quad (3.14)$$

For $(m_1, s_1) \in A_{1,4}$ and $(m_2, s_2) \in A_{2,2}$:

$$\begin{aligned}(\lambda + \mu_{s,1} + \mu_{m,1} + \mu_{s,2})\pi(m_1, s_1, m_2, s_2) &= \lambda\pi(m_1 - 1, s_1, m_2, s_2 - 1) \\ &+\mu_{m,1}\pi(m_1 + 1, s_1, m_2, s_2) \\ &+\mu_{m,2}\pi(m_1, s_1, m_2 + 1, s_2) \\ &+\mu_{s,2}\pi(m_1, s_1, m_2, s_2 + 1) \qquad (3.15)\end{aligned}$$

For $(m_1, s_1) \in A_{1,4}$ and $(m_2, s_2) \in A_{2,3}$:

$$\begin{aligned}(\lambda + \mu_{s,1} + \mu_{m,1} + \mu_{s,2})\pi(m_1, s_1, m_2, s_2) &= \lambda\pi(m_1 - 1, s_1, m_2, s_2 - 1) \\ &+\mu_{m,1}\pi(m_1 + 1, s_1, m_2, s_2) \\ &+\mu_{m,2}\pi(m_1, s_1, m_2 + 1, s_2) \qquad (3.16)\end{aligned}$$

For $(m_1, s_1) \in A_{1,4}$ and $(m_2, s_2) \in A_{2,4}$:

$$\begin{aligned}(\lambda + \mu_{s,1} + \mu_{m,1} + \mu_{s,2} + \mu_{m,2})\pi(m_1, s_1, m_2, s_2) &= \lambda\pi(m_1 - 1, s_1, m_2 - 1, s_2) \\ &+\mu_{m,1}\pi(m_1 + 1, s_1, m_2, s_2) \\ &+\mu_{m,2}\pi(m_1, s_1, m_2 + 1, s_2) \qquad (3.17)\end{aligned}$$

For $(m_1, s_1) \in A_{1,4}$ and $(m_2, s_2) \in A_{2,5}$:

$$\begin{aligned}(\mu_{s,1} + \mu_{m,1} + \mu_{s,2} + \mu_{m,2})\pi(m_1, s_1, m_2, s_2) = \lambda\pi(m_1 - 1, s_1, m_2 - 1, s_2) \\ +\mu_{m,1}\pi(m_1 + 1, s_1, m_2, s_2)\end{aligned} \quad (3.18)$$

For $(m_1, s_1) \in A_{1,4}$ and $(m_2, s_2) \in A_{2,6}$:

$$\begin{aligned}(\lambda + \mu_{s,1} + \mu_{m,1} + \mu_{m,2})\pi(m_1, s_1, m_2, s_2) &= \mu_{m,1}\pi(m_1 + 1, s_1, m_2, s_2) \\ &+\mu_{m,2}\pi(m_1, s_1, m_2 + 1, s_2) \\ &+\mu_{s,2}\pi(m_1, s_1, m_2, s_2 + 1)\end{aligned} \quad (3.19)$$

For $(m_1, s_1) \in A_{1,4}$ and $(m_2, s_2) \in A_{2,7}$:

$$\begin{aligned}(\lambda + \mu_{s,1} + \mu_{m,1} + \mu_{m,2} + \mu_{s,2})\pi(m_1, s_1, m_2, s_2) = \lambda\pi(m_1 - 1, s_1, m_2, s_2 - 1) \\ +\mu_{m,1}\pi(m_1 + 1, s_1, m_2, s_2) \\ +\mu_{m,2}\pi(m_1, s_1, m_2 + 1, s_2) \\ +\mu_{s,2}\pi(m_1, s_1, m_2, s_2 + 1)\end{aligned} \quad (3.20)$$

For $(m_1, s_1) \in A_{1,4}$ and $(m_2, s_2) \in A_{2,8}$:

$$\begin{aligned}(\lambda + \mu_{s,1} + \mu_{m,1} + \mu_{m,2} + \mu_{s,2})\pi(m_1, s_1, m_2, s_2) = \lambda\pi(m_1 - 1, s_1, m_2, s_2 - 1) \\ +\mu_{m,1}\pi(m_1 + 1, s_1, m_2, s_2) \\ +\mu_{m,2}\pi(m_1, s_1, m_2 + 1, s_2)\end{aligned} \quad (3.21)$$

For $(m_1, s_1) \in A_{1,4}$ and $(m_2, s_2) \in A_{2,9}$:

$$\begin{aligned}(\lambda + \mu_{s,1} + \mu_{m,1} + \mu_{m,2})\pi(m_1, s_1, m_2, s_2) &= \mu_{m,1}\pi(m_1 + 1, s_1, m_2, s_2) \\ &+\mu_{m,2}\pi(m_1, s_1, m_2 + 1, s_2)\end{aligned} \quad (3.22)$$

For $(m_1, s_1) \in A_{1,4}$ and $(m_2, s_2) \in A_{2,10}$:

$$\begin{aligned}(\lambda + \mu_{s,1} + \mu_{m,1} + \mu_{m,2})\pi(m_1, s_1, m_2, s_2) &= \lambda\pi(m_1 - 1, s_1, m_2 - 1, s_2) \\ &+\mu_{m,1}\pi(m_1 + 1, s_1, m_2, s_2) \\ &+\mu_{m,2}\pi(m_1, s_1, m_2 + 1, s_2) \\ &+\mu_{s,2}\pi(m_1, s_1, m_2, s_2 + 1)\end{aligned} \quad (3.23)$$

For $(m_1, s_1) \in A_{1,4}$ and $(m_2, s_2) \in A_{2,11}$:

$$\begin{aligned}(\mu_{s,1} + \mu_{m,1} + \mu_{m,2})\pi(m_1, s_1, m_2, s_2) &= \lambda\pi(m_1 - 1, s_1, m_2 - 1, s_2) \\ &\quad + \mu_{m,1}\pi(m_1 + 1, s_1, m_2, s_2) \qquad (3.24)\end{aligned}$$

The C-K equation for other states where $(m_1, s_1) \in \mathbb{A}_{k,i}, i = 1, ..., 11$ and $(m_2, s_2) \in \mathbb{A}_{k,j}$ for can be written in a similar way. As in the case of the dual base stock policy, we can exploit structural properties using the Matrix-Geometric representation of $\mathbb{Q}^{OH}$. Let, $\mathbb{C} = diag(\lambda, \lambda, \lambda, 0, \lambda, \lambda, 0)$ and define $\mathbb{I}_k$ as an identity matrix of size N_k^{OH} x N_k^{OH}. Also define $\mathbb{B}_{3,k} = \mathbb{B}_{1,k} - \mu_{s,1}\mathbb{I}_k$, $\mathbb{B}_{4,k} = \mathbb{B}_{1,k} - (2\mu_{s,1}\mathbb{I}_k + \mu_{m,1}\mathbb{I}_k)$, $\mathbb{B}_{5,k} = \mathbb{B}_{1,k} - (\mathbb{C} + 2\mu_{s,1}\mathbb{I}_k + \mu_{m,1}\mathbb{I}_k)$, $\mathbb{B}_{9,k} = \mathbb{B}_{1,k} - \mu_{m,1}\mathbb{I}_k - \mu_{s,1}\mathbb{I}_k$ and $\mathbb{B}_{11,k} = \mathbb{B}_{1,k} - (\mathbb{C} + \mu_{m,1}\mathbb{I}_k) + \mu_{s,1}\mathbb{I}_k)$, where $\mathbb{B}_{1,k}$ for $k = 1, 2$ and $\mathbb{D}$ are defined as follows:

$$\mathbb{D} = \begin{vmatrix} 0 & \lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{vmatrix}$$

$$\mathbb{B}_{1,k} = \begin{vmatrix} -\lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ \mu_{s,k} & -(\lambda + \mu_{s,k}) & 0 & 0 & 0 & 0 & 0 \\ 0 & \mu_{m,k} & -(\lambda + \mu_{s,k} + \mu_{m,k}) & 0 & \mu_{s,k} & 0 & 0 \\ 0 & 0 & \mu_{m,k} & -(\mu_{s,k} + \mu_{m,k}) & 0 & \mu_{m,k} & 0 \\ \mu_{m,k} & 0 & 0 & 0 & -(\lambda + \mu_{m,k}) & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu_{m,k} & -(\lambda + \mu_{m,k}) & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu_{m,k} & -\mu_{m,k} \end{vmatrix}$$

Then, we can construct the transition matrix, $\mathbb{Q}^{OH}$ using the above mentioned matrices as shown in Equation (3.25) and compute the steady state probabilities by solving the system of equations Equation (3.26) and Equation (3.27) using the Matrix-Geometric technique described in Neuts (1981).

$$\mathbb{Q}^{OH} = \begin{array}{c|ccccccc} & \overrightarrow{(0,0,*,*)}^{\top} & \overrightarrow{(0,1,*,*)}^{\top} & \overrightarrow{(1,1,*,*)}^{\top} & \overrightarrow{(2,1,*,*)}^{\top} & \overrightarrow{(1,0,*,*)}^{\top} & \overrightarrow{(2,0,*,*)}^{\top} & \overrightarrow{(3,0,*,*)}^{\top} \\ \hline \overrightarrow{(0,0,*,*)} & \mathbb{B}_{1,2} & \mathbb{D} & & & & & \\ \overrightarrow{(0,1,*,*)} & \mu_{s,1}\mathbb{I}_2 & \mathbb{B}_{3,2} & \mathbb{D} & & & & \\ \overrightarrow{(1,1,*,*)} & & \mu_{m,1}\mathbb{I}_2 & \mathbb{B}_{4,2} & \mathbb{D} & \mu_{s,1}\mathbb{I}_2 & & \\ \overrightarrow{(2,1,*,*)} & & & \mu_{m,1}\mathbb{I}_2 & \mathbb{B}_{5,2} & & \mu_{s,1}\mathbb{I}_2 & \\ \overrightarrow{(1,0,*,*)} & \mu_{m,1}\mathbb{I}_2 & & & & \mathbb{B}_{9,2} & \mathbb{D} & \\ \overrightarrow{(2,0,*,*)} & & & & & \mu_{m,1}\mathbb{I}_2 & \mathbb{B}_{9,2} & \mathbb{D} \\ \overrightarrow{(3,0,*,*)} & & & & & & \mu_{m,1}\mathbb{I}_2 & \mathbb{B}_{11,2} \end{array} \tag{3.25}$$

$$\Pi\mathbb{Q}^{OH} = \mathbf{0} \tag{3.26}$$

$$\Pi\mathbf{e} = 1 \tag{3.27}$$

Here, $\mathbf{e}$ =[1,...,1] of size 1 $\times$ ($N_1^{OH} \times N_2^{OH}$) and $\overrightarrow{(0,0,*,*)}$ denotes a vector with all states having $m_1 = s_1 = 0$ in OH policy. The expected on-hand inventory levels $E[I_1]$ and expected backorders $E[B_1]$ for component 1 are also calculated using Equations (3.28) and (3.29). The performance measures for component 2 can be calculated in a similar way. Note that in these equations, $\pi(*,*,*,*)$ denote the steady state probability of the particular state.

$$E[B_1] = \sum_{m_1+s_1>z_1} (m_1 + s_1 - z_1)\pi(m_1, s_1, *, *) \tag{3.28}$$

$$E[I_1] = \sum_{m_1+s_1\leq z_1} (z_1 - m_1 - s_1)\pi(m_1, s_1, *, *) \tag{3.29}$$

$$TH_{m,1} = \sum_{m_1>0} \mu_{m,k}\pi(m_1, *, *, *) \tag{3.30}$$

$$TH_{s,1} = \sum_{s_1>0} \mu_{s,k}\pi(*, s_1, *, *) \tag{3.31}$$

Similarly, we can conduct the analysis for the LT policy. In Section 3.5, we conduct numerical analysis for these systems under the three polices (DB, OH and LT policy) respectively.

3.4 Approximate Analysis

The exact analysis of ATO systems described above become computationally challenging even with the Matrix-Geometric representation of the transition matrices. As we go from a 2-component ATO system to a 4-component ATO system, the total number of unique C-K equations increases from 25 to 625 for the DB policy, and from 121 to 14641 in OH policy. This limits the use of Matrix-Geometric approach to analyze large systems. To overcome this issue, we propose an approximation method that uses decomposition (see Figure 3.3). The key idea is to split the original Markov chain for a system with N components into N independent Markov chains, each corresponding to a subsystem that models the evolution of one of the components. However, for the decomposition technique to be accurate, in the Markov chain for component k, the effect of the other components need to be accounted for appropriately by using the effective demand arrival rate λ_k.

Note that in the original system described in Section 3.2, external orders are lost when backorders due to one or more components reach B_{max}. However, in the decomposition analysis, demands for component k arrive at subsystem k and queue at station A as long as the backorders for that component is less than B_{max} i.e. the decomposition ignores the fact that orders could be lost because backorders for one or more of the other components, i $(i \neq k)$ might reach B_{max}. Therefore, the effective effective demand arrival rate λ_k for component k needs to be set to recognize this possibility. Let X_i denote the event that the backorders at subsystem i is equal to B_{max}, and let P_i denote the probability of this event. Then assuming that X_i is independent of X_j for every pair, i and j, then, $\prod_{i \neq k} (1 - P_i)$ is the probability that the backorders at all of the others subsystem i is less than B_{max}. Then $[1 - \prod_{i \neq k} (1 - P_i)][1 - P_k]$ is the probability that the backorders at one or more of the others subsystem $i \neq k$ is equal to B_{max}, while the backorders at subsystem k is less than B_{max}.

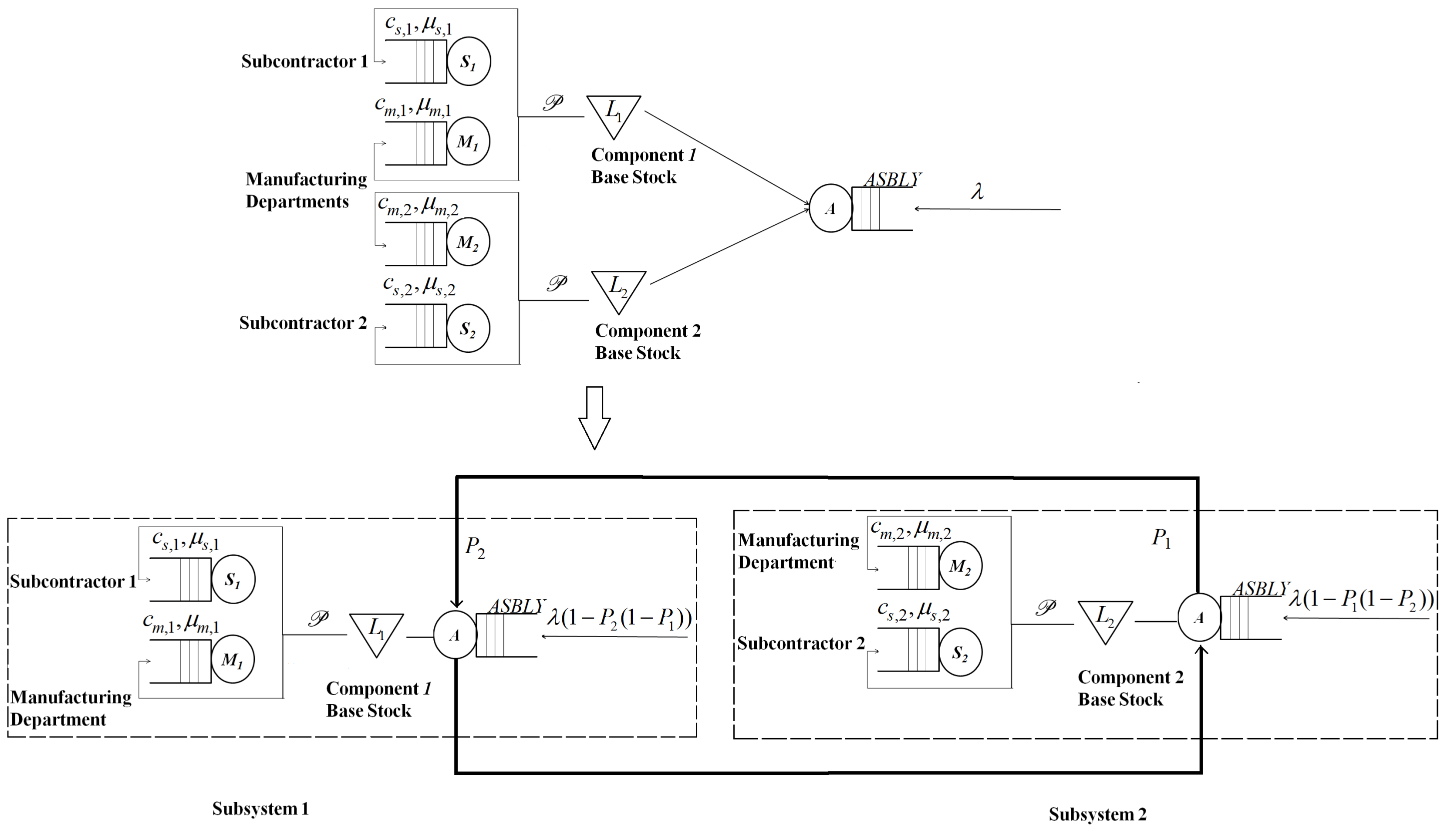

Figure 3.3 Illustration of Solution Approach for $N = 2$

This implies that effective demand arrival rate λ_k for component k in the analysis of subsystem for component k is given by

$$\lambda_k = \lambda \left(1 - [1 - \prod_{i \neq k} (1 - P_i)][1 - P_k] \right) \tag{3.32}$$

Clearly, the solution to subsystem k requires estimates of $P_i, \forall i \in 1, 2, ..., N$ and $i \neq k$. In Section 3.4.1 and 3.4.2 we characterize the subsystems under the DB and OH policy, and in Section 3.4.3 we present the approximate solution algorithm.

3.4.1 Characterizing Subsystems under DB Policy

Let $\mathbb{Q}_k^{DB}$ denote the transition matrix for subsystem corresponding to component k under dual base stock based policy. Let λ_k denote the demand arrival rate corresponding to component k in the decomposition approach, and let $\pi_k^{DB}(I_k)$ denote the corresponding steady state probabilities. For notational simplicity, we drop the superscript DB in the rest of the section. Then, $\mathbb{Q}_k$ can be written using the Chapman-Kolmogorov (C-K) equations shown below:

For $I_k \in \mathbb{A}_{k,1}$:

$$(\mu_{s,k} + \mu_{m,k})\pi_k(I_k) = \lambda_k \pi_k(I_k + 1) \tag{3.33}$$

For $I_k \in \mathbb{A}_{k,2}$:

$$(\lambda + \mu_{s,k} + \mu_{m,k})\pi_k(I_k) = \lambda_k \pi_k(I_k + 1) + (\mu_{m,k} + \mu_{s,k})\pi_k(I_k - 1) \tag{3.34}$$

For $I_k \in \mathbb{A}_{k,3}$:

$$(\lambda + \mu_{s,k})\pi_k(I_k) = \lambda_k \pi_k(I_k + 1) + (\mu_{m,k} + \mu_{s,k})\pi_k(I_k - 1) \tag{3.35}$$

For $I_k \in \mathbb{A}_{k,4}$:

$$(\lambda + \mu_{s,k})\pi_k(I_k) = \lambda_k \pi_k(I_k + 1) + \mu_{s,k}\pi_k(I_k - 1) \tag{3.36}$$

For $I_k \in \mathbb{A}_{k,5}$:

$$(\lambda + \mu_{s,k})\pi_k(I_k) \quad = \quad \mu_{s,k}\pi_k(I_k - 1) \tag{3.37}$$

These equations are solved using iterative algorithm described in Section 3.4.3.

3.4.2 Characterizing Subsystems under OH Policy

Let $\mathbb{Q}_k^{OH}$ denote the transition matrix for the subsystem corresponding to component k under the on-hand inventory based policy. Let λ_k denote the demand arrival rate corresponding to component k in the decomposition approach, and let $\pi_k^{OH}(m_k, s_k)$ denote the corresponding steady state probabilities. For notational simplicity, we drop the superscript OH in the rest of the section. Then, $\mathbb{Q}_k$ can be written using the Chapman-Kolmogorov (C-K) equations shown below:

For $(m_k, s_k) \in \mathbb{A}_{k,1}$:

$$\lambda_k\pi_k(m_k, s_k) = \mu_{s,k}\pi_k(m_k, s_k + 1) \tag{3.38}$$

For $(m_k, s_k) \in \mathbb{A}_{k,2}$:

$$\begin{aligned}(\lambda_k + \mu_{s,k})\pi_k(m_k, s_k) \quad &= \quad \lambda_k\pi_k(m_k, s_k - 1) + \mu_{m,k}\pi_k(m_k + 1, s_k) \\ &\quad +\mu_{s,k}\pi_k(m_k, s_k + 1)\end{aligned} \tag{3.39}$$

For $(m_k, s_k) \in \mathbb{A}_{k,3}$:

$$(\lambda_k + \mu_{s,k})\pi_k(m_k, s_k) \quad = \quad \lambda_k\pi_k(m_k, s_k - 1) + \mu_{m,k}\pi_k(m_k + 1, s_k) \tag{3.40}$$

For $(m_k, s_k) \in \mathbb{A}_{k,4}$:

$$(\lambda_k + \mu_{s,k} + \mu_{m,k})\pi_k(m_k, s_k) \quad = \quad \lambda_k\pi_k(m_k - 1, s_k) + \mu_{m,k}\pi_k(m_k + 1, s_k) \tag{3.41}$$

For $(m_k, s_k) \in \mathbb{A}_{k,5}$:

$$(\mu_{s,k} + \mu_{m,k})\pi_k(m_k, s_k) = \lambda_k\pi_k(m_k - 1, s_k) \tag{3.42}$$

For $(m_k, s_k) \in \mathbb{A}_{k,6}$:

$$(\lambda_k + \mu_{m,k})\pi_k(m_k, s_k) = \mu_{m,k}\pi_k(m_k + 1, s_k) + \mu_{s,k}\pi_k(m_k, s_k + 1) \tag{3.43}$$

For $(m_k, s_k) \in \mathbb{A}_{k,7}$:

$$\begin{aligned}(\lambda_k + \mu_{m,k} + \mu_{s,k})\pi_k(m_k, s_k) &= \lambda_k\pi_k(m_k, s_k - 1) + \mu_{m,k}\pi_k(m_k + 1, s_k) \\ &\quad + \mu_{s,k}\pi_k(m_k, s_k + 1)\end{aligned} \tag{3.44}$$

For $(m_k, s_k) \in \mathbb{A}_{k,8}$:

$$(\lambda_k + \mu_{m,k} + \mu_{s,k})\pi_k(m_k, s_k) = \lambda_k\pi_k(m_k, s_k - 1) + \mu_{m,k}\pi_k(m_k + 1, s_k) \tag{3.45}$$

For $(m_k, s_k) \in \mathbb{A}_{k,9}$:

$$(\lambda_k + \mu_{m,k})\pi_k(m_k, s_k) = \mu_{m,k}\pi_k(m_k + 1, s_k) \tag{3.46}$$

For $(m_k, s_k) \in \mathbb{A}_{k,10}$:

$$\begin{aligned}(\lambda_k + \mu_{m,k})\pi_k(m_k, s_k) &= \lambda_k\pi_k(m_k - 1, s_k) + \mu_{m,k}\pi_k(m_k + 1, s_k) \\ &\quad + \mu_{s,k}\pi_k(m_k, s_k + 1)\end{aligned} \tag{3.47}$$

For $(m_k, s_k) \in \mathbb{A}_{k,11}$:

$$(\lambda_k + \mu_{m,k})\pi_k(m_k, s_k) = \lambda_k\pi_k(m_k - 1, s_k) \tag{3.48}$$

These equations are solved using iterative algorithm described in Section 3.4.3.

3.4.3 Solution Algorithm and Performance Measures

Recall that the solution to subsystem k requires the estimates of $P_i, \forall i, i \neq k$ must be obtained from the solution to subsystem $\forall i, i \neq k$ and vice versa. Therefore, we use an iterative approach. The steps of the iterative approach are shown below:

Step 0: Initialize $P_k^{(0)} = 0$, $\epsilon = 10^{-5}$, $\delta_k = 1$, and calculate an estimate of $\lambda_k^{(0)}$ using Equation (3.32), for $k = 1, ..., N$. At each iteration n, $n = 1, ...,$ while $\delta_k \geq \epsilon$, *Step 1:* Solve subsystem $k, k = 1, ..., N$ using $\lambda_k^{(n-1)}, i > 1$

- For DB policy: Solve Equations (3.33) - (3.37).
- For OH policy: Solve Equations (3.38) - (3.48).
- Compute estimates of $P_k^{(n)}$ from the steady state probabilities.

Step 2: Calculate new estimate of $\lambda_k^{(n)}$ using $P_k^{(n)}$.

Step 3: Compute $\delta_k = |P_k^{(n)} - P_k^{(n-1)}|$.

If $\delta_k < \epsilon, \forall k$, stop. Else, repeat steps $1 - 3$.

Using a similar approach, we can analyze the performance of an ATO system operating under LT policy as well.

The expected on-hand quantities $E[I_k]$, expected backorders $E[B_k]$, the throughput $TH_{m,k}$ for component k at the manufacturing facility, and $TH_{s,k}$ at the subcontractor facility are calculated using Equations (3.49) - (3.52) for DB policy.

$$E[B_k] = \sum_{I_k} max(-I_k, 0)\pi_k(I_k) \tag{3.49}$$

$$E[I_k] = \sum_{I_k} max(I_k, 0)\pi_k(I_k) \tag{3.50}$$

$$TH_{m,k} = \sum_{I_k < e_k} \mu_{m,k}\pi_k(I_k) \tag{3.51}$$

$$TH_{s,k} = \sum_{I_k < z_k} \mu_{s,k}\pi_k(I_k) \tag{3.52}$$

The corresponding equations for the OH policy are given by Equations (3.53) - (3.56).

$$
\begin{aligned}
E[B_k] &= \sum_{m_k+s_k>z_k} (m_k+s_k-z_k)\pi_k(m_1,s_1) && (3.53)\\
E[I_k] &= \sum_{m_k+s_k\leq z_k} (z_k-m_k-s_k)\pi_k(m_k,s_k) && (3.54)\\
TH_{m,k} &= \sum_{m_k>0} \mu_{m,k}\pi_k(m_k,*) && (3.55)\\
TH_{s,k} &= \sum_{s_k>0} \mu_{s,k}\pi_k(*,s_k) && (3.56)
\end{aligned}
$$

Note that the performance measures obtained using the above equations use approximation method. Alternatively, for smaller systems, these performance measures could be obtained from exact solutions using Equations (3.2) - (3.6) for DB policy and Equations (3.14) - (3.24) for OH policy respectively. For larger systems where exact solutions of the Markov chain becomes computationally challenging, we derive results from detailed simulation models. We compare the accuracy of these estimates as part of our numerical studies in Section 3.5.

3.5 Numerical Comparison of Dual Index Policies

In this section, we discuss the numerical experiments conducted to compare the performance of policies under different scenarios. We define the total cost function $TC = \sum_k (c_{m,k}TH_{m,k} + c_{s,k}TH_{s,k} + b_kE[B_k] + h_kE[I_k])$ where, b_k is the cost of backordering per unit of component k and h_k is holding cost per unit for component k. We conduct three sets of experiments. The first two experiments compare the performance of DB, OH, and LT policies (see Sections 3.5.1 and 3.5.2), while the third experiment investigates the performance of the decomposition algorithm for ATO systems with $N \geq 2$ (see Section 3.5.3). In all our experiments, we observed that the algorithm converges within 0.5 seconds on a personal computer with an Intel core i5 processor. Further, we always obtained a unique solution to the system. Although we do not have a proof for the convergence, in our numerical computations, we find that the time for convergence does not increase with increase in the number of components.

3.5.1 Performance Comparison of DB, OH, and LT Policy

We present the properties and the comparison of the expected costs for system operating under DB policy, OH policy, and LT policy using Matrix-Geometric approach. We analyze an ATO system with 2 components and use Matrix-Geometric approach to solve and compare the costs in DB policy, LT policy, and OH policy. We consider a symmetric case where all the parameters for component 1 are equal to that of component 2 (see Table 3.1). We set B_{max} to be large enough so that the probability of lost sales is less than 10^{-3} in all cases. For $z_k = 10$, we vary the threshold limits e_k, j_k, and l_k from 1 to z_k and calculate in each case the total costs for each policy.

Table 3.1 System Parameters and Costs for Dual Index Policies

System Parameters		**Costs**	
λ	1.5	$c_{m,k}, k = 1, 2$	10
$z_k, k = 1, 2$	10	$c_{s,k}, k = 1, 2$	5
$\mu_{m,k}, k = 1, 2$	2	$b_k, k = 1, 2$	20
$\mu_{s,k}, k = 1, 2$	1	$h_k, k = 1, 2$	1

Figure 3.4 plots different costs (in-house throughput costs, subcontractor's throughput costs, on-hand inventory costs, and backordering costs) vs threshold limit (e_k, j_k, and l_k) for DB, OH, and LT policy. In Figure 3.4(a) we observe that as the threshold (e_k, j_k, l_k) increases, the throughput cost at the manufacturing facility increases under all three policies (DB, OH, and LT policy). This is to be expected because in these policies when the threshold is high, orders for component k are placed more to the manufacturing facility than external subcontractor. This results in the increase in the throughput cost at the manufacturing facility. However under LT policy, the in-house throughput cost increases at low threshold and becomes flat at high threshold. Correspondingly, in Figure 3.4(b) as the threshold (e_k, j_k, l_k) increases, we observe that throughput cost at the external subcontractor

decreases under all three policies.

In Figure 3.4(c) as the threshold (e_k, j_k, l_k) increases, we observe that the on-hand inventory cost increases under DB, OH, and LT policy. In these three policies as the threshold increases, more orders for component k are placed to the manufacturing facility, which has a faster production rate and therefore replenishes the component inventory at a faster rate. In Figure 3.4(d) as the threshold increases, we observe that the backordering cost for the system operating under DB, OH, and LT policy is convex. The backordering costs are convex due to the effects of increased queue length and lead times at the manufacturing facility at higher thresholds.

We can get the total cost, TC by adding the above mentioned costs. Our solution algorithm can then be used to identify this optimal threshold e_k^*, j_k^*, and l_k^* numerically that minimizes the total cost TC. We compare the total cost for these three policies and find DB policy as the best among the three policies.

3.5.2 Comparison of Optimal Costs under DB, OH, and LT Policy

We analyze an ATO system with 2 components and use decomposition approach to solve and compare the optimal costs in DB policy, LT policy, and OH policy. The results are shown in Figure 3.5. We consider a symmetric case shown in Table 3.1 where all the parameters for component 1 are equal to that of component 2. In this experiment, we vary $z_k = 2$ to 25, and for each z_k we determine the corresponding optimal threshold limit (e_k^* for DB policy, j_k^* for OH policy, and l_k^* for LT policy). Then, we calculate the optimal cost TC^* and compare this cost across the three policies. Therefore, determining each point in the corresponding plots in Figure 3.5 itself requires a search procedure.

In this subsection, we discuss insights related to the optimal solution for each policy using the same system parameters and costs listed in Table 3.1 and varying $z_k = 2$ to 25. Figure

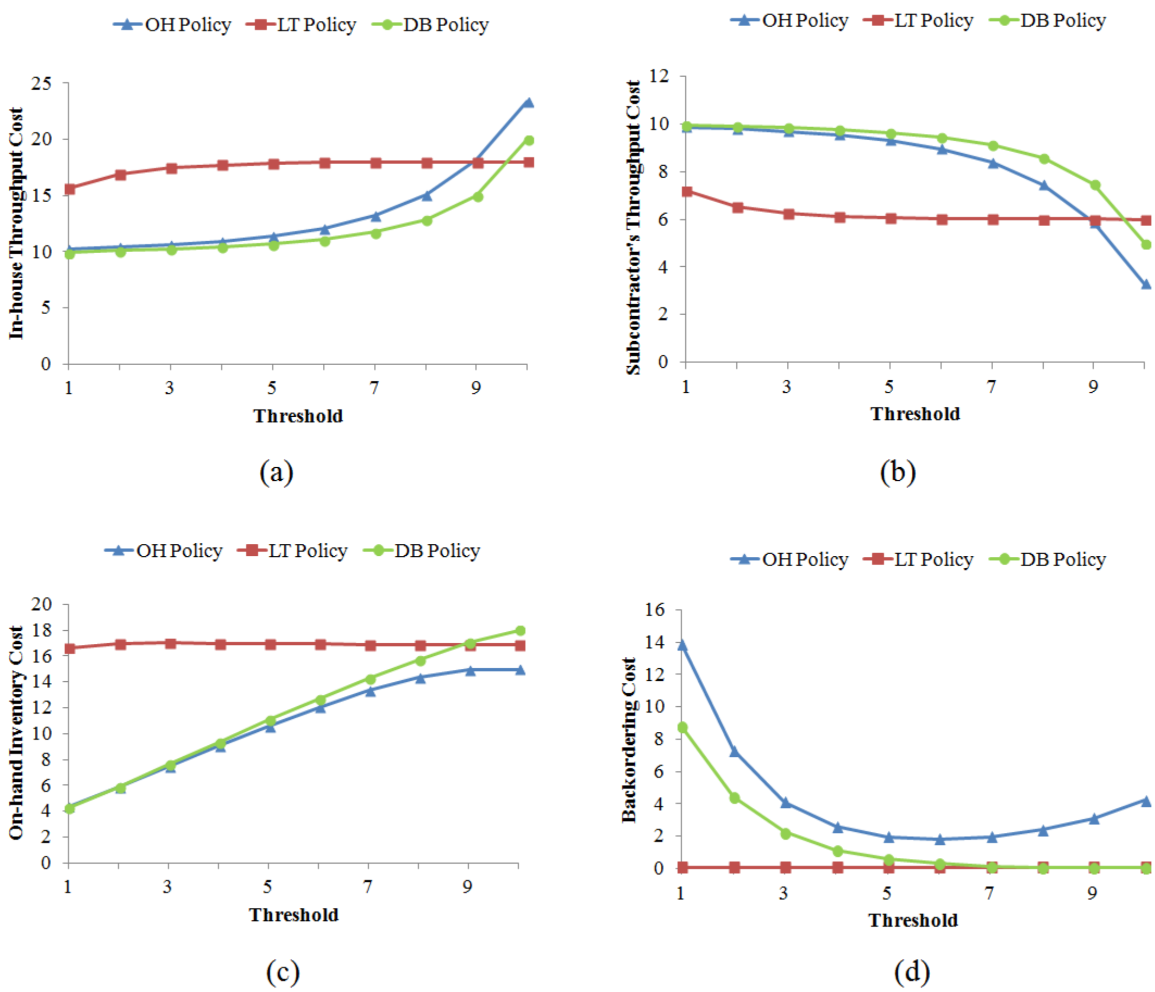

Figure 3.4 Estimated Costs in Policies at $z_k = 10$

3.5 shows the optimal costs for both OH and LT policy with increasing z_k under different zones. In zone 1, the expected inventory cost is less than the expected backordering cost. Note that in LT policy, orders for component k are placed based on the lead time estimates $(\hat{L}_{m,k}(t), \hat{L}_{s,k}(t))$. Therefore, the LT policy encourages reduction in backordering cost as opposed to inventory costs. However in the OH policy, orders for component k are placed based on the inventory levels $(I_k(t))$. Therefore, the OH policy encourages reduction in inventory cost as opposed to backordering costs. In zone 1, the expected inventory cost is

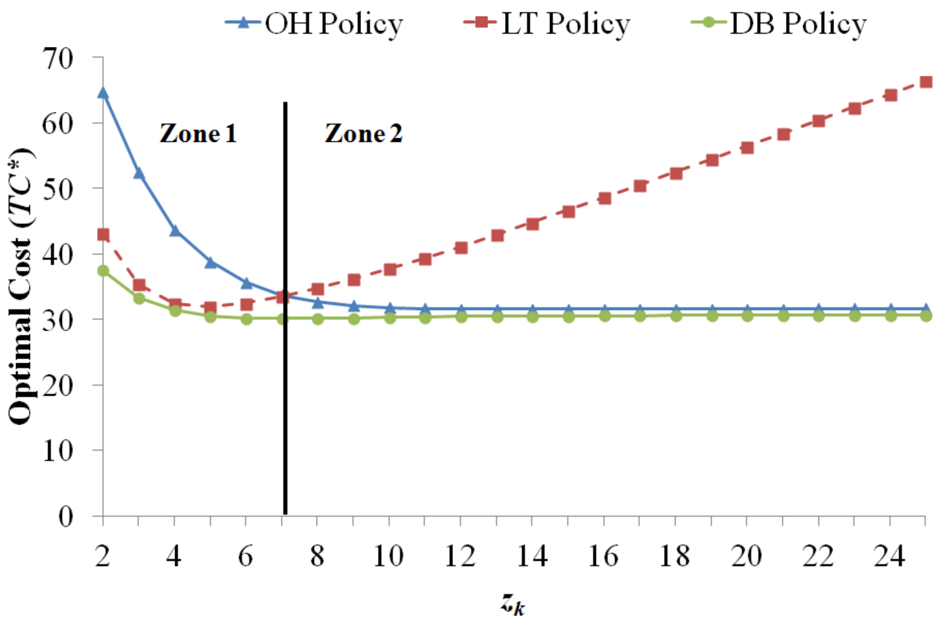

Figure 3.5 Optimal Cost Comparison for Policies

less significant than the expected backorder cost. Thus, LT policy outperforms OH policy in zone 1. Similarly in zone 2, the expected inventory cost is more significant than the expected backorder cost. Thus, OH policy outperforms LT policy in zone 2. However, the dual base stock policy always outperforms OH policy and LT policy as it has the most flexibility. In the dual base stock policy, in zone 1 if the inventory level is above a certain threshold , orders for component k are placed to only external subcontractor. This results in the replenishment of the components at a slower rate ($\mu_{s,k}$), which in turn reduces the expected inventory cost. Similarly in zone 2, if the inventory level is below a certain threshold, orders for component k are placed to both the in-house manufacturer and the external subcontractor. This results in the replenishment of the components at a faster rate ($\mu_{s,k} + \mu_{m.k}$), which in turn reduces the expected backorder cost.

Although, the results for $z_k = 0$ and $z_k = M$ (M is significantly large, say $M = 1000$) are not shown in these numerical experiments, we observe that when $z_k = 0$, the LT policy converges to the DB policy, and for large values of z_k, say $z_k = 1000$, the OH policy converges to the DB policy. This provides an intuitive solution to the operational ambiguity inherent in the DB policy, suggesting that the LT policy should be preferred at lower z_k and OH policy should be preferred at higher z_k.

3.5.3 Performance of a System with N Components

In this subsection, we analyze the approximate method for N-component ATO system and provide insights on its performance. We analyze the performance of the decomposition based approximation for both the symmetric and asymmetric cases.

Under DB policy, we analyze an ATO system with N components ($N = 2, 4, 8, 16$) and compare the numerical accuracy of the decomposition based approach with the exact solution for three choices of service time distributions, namely exponential (E), shifted-exponential (S), and triangular (T) distribution. We use simulation models to obtain the exact solutions for ATO systems with $N \geq 2$. We use 5 replications and a 99.99% confidence interval in our simulations and ensure that the half width was less than 0.001% in all cases. For ease of readability, we do not report the half-width intervals in the paper. The results from the decomposition are then compared with the exact results for several inputs. Table 3.2 presents the processing time parameters used for exponential, shifted-exponential, and triangular distribution under symmetric case. For sake of comparison, we ensure that the mean processing time is the same for all three distributions.

Table 3.2 Distribution Parameters of a System with N Components

Distribution	Processing time at M_k	Processing time at S_k
Exponential (E)	$EXP(0.5)$	$EXP(1)$
Shifted-Exponential (S)	$0.1 + EXP(0.4)$	$0.1 + EXP(0.9)$
Triangular (T)	$TRIAG(0, 0.5, 1)$	$TRIAG(0, 1, 2)$

We define $\Delta[TH^E_{m,k}] = (TH^A_{m,k} - TH^E_{m,k})/TH^E_{m,k}$ as the error in the estimate of the throughput at the manufacturing facility M_k, and is computed as the relative difference between estimates from the approximation solution and the exact solution for the case of with exponentially distributed processing times. Similarly, we define the error measures for other performance measures and distributions. We analyze the performance of the approximate method under two cases: (1) symmetry with respect to service rate for each component i.e. $\mu_{s,i+1} = \mu_{s,i}$ and $\mu_{m,i+1} = \mu_{m,i}, i = 1, 2, ..., N-1$, and (2) asymmetry with respect to service rate for each component with $\mu_{s,i+1} = 1.1\mu_{s,i}$ and $\mu_{m,i+1} = 1.1\mu_{m,i}, i = 1, 2, ..., N-1$. The rest of the parameters are same as described in Table 6.1.

Table 3.3 shows the error in $TH_{s,1}$ obtained from the decomposition-based approach for $2, 4, 8,$ and 16 component symmetric ATO system under dual base stock policy. We observe that the estimates of subcontractor's throughput, $TH_{s,1}$ and manufacturer's throughput, $TH_{m,1}$ are within 2% for the exponential (E) and shifted-exponential (S) case, and within 4% for the triangular distribution (T) case.

Table 3.4 reports the error in $E[I_1]$ obtained from the decomposition approach for a symmetric ATO system with $2, 4, 8,$ and 16 components under dual base stock policy. We observe that the expected on-hand inventory, $E[I_1]$ is within 3% for all cases, except for the low threshold case under the triangular distribution (T) where the error is less than 9%. Similar performance is observed for expected backorders $E[B_1]$ as well.

Table 3.5 reports the error in $TH_{s,1}$ in the approximation for $2, 4, 8,$ and 16 component asymmetric ATO system under dual base stock policy. We observe that the estimates of subcontractor's throughput, $TH_{s,1}$ and manufacturer's throughput, $TH_{m,1}$ are within 4% of the exact values for most of the cases.

Table 3.3 Performance of $TH_{s,1}$ under DB Policy for the Symmetric Case

N	e_1	$TH^A_{s,1}$	$TH^E_{s,1}$	$\Delta[TH^E_{s,1}]$	$TH^S_{s,1}$	$\Delta[TH^S_{s,1}]$	$TH^T_{s,1}$	$\Delta[TH^T_{s,1}]$
2	2	0.990	0.989	0.07%	0.993	-0.29%	0.999	-0.90%
	4	0.977	0.976	0.12%	0.982	-0.52%	0.995	-1.80%
	6	0.945	0.942	0.31%	0.952	-0.74%	0.978	-3.32%
	8	0.857	0.850	0.81%	0.861	-0.45%	0.892	-3.86%
4	2	0.990	0.989	0.06%	0.993	-0.29%	0.999	-0.92%
	4	0.977	0.976	0.13%	0.982	-0.50%	0.995	-1.81%
	6	0.945	0.942	0.32%	0.952	-0.72%	0.978	-3.31%
	8	0.857	0.850	0.84%	0.861	-0.40%	0.892	-3.86%
8	2	0.989	0.989	0.01%	0.992	-0.29%	0.999	-0.97%
	4	0.977	0.975	0.13%	0.981	-0.46%	0.995	-1.83%
	6	0.945	0.941	0.39%	0.952	-0.72%	0.977	-3.31%
	8	0.857	0.850	0.85%	0.860	-0.38%	0.891	-3.84%
16	2	0.989	0.987	0.22%	0.992	-0.25%	0.999	-0.97%
	4	0.977	0.973	0.34%	0.981	-0.43%	0.995	-1.83%
	6	0.945	0.941	0.47%	0.951	-0.60%	0.977	-3.31%
	8	0.857	0.848	1.04%	0.860	-0.31%	0.891	-3.83%

Table 3.4 Performance of $E[I_1]$ under DB Policy for the Symmetric Case

N	e_1	$E[I^A_1]$	$E[I^E_1]$	$\Delta[E[I^E_1]]$	$E[I^S_1]$	$\Delta[E[I^S_1]]$	$E[I^T_1]$	$\Delta[E[I^T_1]]$
2	2	2.945	2.988	-1.45%	2.939	0.20%	2.707	8.76%
	4	4.670	4.623	1.02%	4.613	1.25%	4.593	1.69%
	6	6.353	6.225	2.05%	6.258	1.51%	6.452	-1.55%
	8	7.860	7.635	2.96%	7.728	1.71%	8.083	-2.76%
4	2	2.964	3.022	-1.91%	2.982	-0.60%	2.708	9.48%
	4	4.675	4.674	0.01%	4.618	1.23%	4.607	1.47%
	6	6.354	6.244	1.76%	6.259	1.51%	6.458	-1.62%
	8	7.860	7.643	2.84%	7.728	1.71%	8.089	-2.83%
8	2	3.001	3.050	-1.61%	3.049	-1.57%	2.760	8.72%
	4	4.684	4.682	0.04%	4.656	0.60%	4.609	1.64%
	6	6.356	6.246	1.75%	6.273	1.31%	6.460	-1.62%
	8	7.861	7.673	2.45%	7.731	1.68%	8.090	-2.83%
16	2	3.001	3.091	-2.89%	3.064	-2.03%	2.764	8.60%
	4	4.684	4.815	-2.71%	4.720	-0.75%	4.609	1.63%
	6	6.356	6.302	0.85%	6.310	0.72%	6.460	-1.62%
	8	7.861	7.689	2.24%	7.746	1.48%	8.080	-2.72%

Table 3.5 Performance of $TH_{s,1}$ under DB Policy for the Asymmetric Case

N	e_1	$TH^A_{s,1}$	$TH^E_{s,1}$	$\Delta[TH^E_{s,1}]$	$TH^S_{s,1}$	$\Delta[TH^S_{s,1}]$	$TH^T_{s,1}$	$\Delta[TH^T_{s,1}]$
2	2	0.990	0.989	0.06%	0.993	-0.29%	0.999	-0.88%
	4	0.977	0.976	0.10%	0.982	-0.50%	0.995	-1.79%
	6	0.945	0.943	0.23%	0.952	-0.74%	0.978	-3.32%
	8	0.857	0.850	0.81%	0.861	-0.45%	0.892	-3.88%
4	2	0.990	0.989	0.06%	0.993	-0.30%	0.999	-0.89%
	4	0.977	0.976	0.10%	0.982	-0.50%	0.995	-1.79%
	6	0.945	0.943	0.23%	0.952	-0.74%	0.978	-3.30%
	8	0.857	0.850	0.81%	0.861	-0.45%	0.892	-3.88%
8	2	0.990	0.989	0.06%	0.993	-0.30%	0.999	-0.89%
	4	0.977	0.976	0.10%	0.982	-0.50%	0.995	-1.79%
	6	0.945	0.943	0.23%	0.952	-0.74%	0.892	6.00%
	8	0.857	0.850	0.81%	0.861	-0.45%	0.892	-3.88%
16	2	0.990	0.989	0.06%	0.993	-0.30%	0.999	-0.89%
	4	0.977	0.976	0.10%	0.982	-0.50%	0.995	-1.79%
	6	0.945	0.943	0.23%	0.952	-0.74%	0.978	-3.30%
	8	0.857	0.845	1.43%	0.861	-0.45%	0.892	-3.88%

Table 3.6 reports the error in $E[I_1]$ obtained from the approximation for an asymmetric ATO system with $2, 4, 8$, and 16 components under dual base stock policy. We observe that the expected on-hand inventory, $E[I_1]$ is within 3% of the exact values for all cases, except for the low threshold case under the triangular distribution (T) where the error is within than 9%. Similar performance is observed for expected backorders $E[B_1]$ as well.

For a given e_k as the number of components increases, we observe a decrease in the throughput at the external subcontractor, and an increase in the expected inventory. This is because, as components increase, backorders increase and more orders for component k are placed to the internal manufacturing facility. This reduces throughput at the external subcontractor, and increases the expected inventory. Again for any given N, as e_k increases, we observe a decrease in the throughput at the external subcontractor, and an increase in the expected inventory. This is because, with increase in the threshold, e_k, more orders for component k

Table 3.6 Performance of $E[I_1]$ under DB Policy for the Asymmetric Case

N	e_1	$E[I_1^A]$	$E[I_1^E]$	$\Delta[E[I_1^E]]$	$E[I_1^S]$	$\Delta[E[I_1^S]]$	$E[I_1^T]$	$\Delta[E[I_1^T]]$
2	2	2.939	3.027	-2.89%	2.905	1.17%	2.695	9.06%
	4	4.669	4.624	0.98%	4.594	1.62%	4.603	1.44%
	6	6.352	6.220	2.13%	6.255	1.56%	6.452	-1.54%
	8	7.860	7.632	2.99%	7.725	1.75%	8.082	-2.74%
4	2	2.942	3.030	-2.91%	2.925	0.57%	2.697	9.10%
	4	4.669	4.624	0.97%	4.602	1.47%	4.603	1.45%
	6	6.352	6.228	1.99%	6.258	1.51%	6.452	-1.54%
	8	7.860	7.633	2.97%	7.726	1.73%	8.083	-2.76%
8	2	2.942	3.033	-2.99%	2.930	0.43%	2.698	9.07%
	4	4.669	4.628	0.90%	4.604	1.41%	4.603	1.44%
	6	6.352	6.229	1.98%	6.263	1.42%	6.460	-1.67%
	8	7.860	7.635	2.95%	7.726	1.73%	8.088	-2.81%
16	2	2.942	3.033	-3.00%	2.930	0.41%	2.705	8.80%
	4	4.669	4.629	0.88%	4.607	1.36%	4.603	1.44%
	6	6.352	6.229	1.97%	6.267	1.35%	6.460	-1.67%
	8	7.860	7.636	2.93%	7.727	1.73%	8.088	-2.82%

are placed to the internal manufacturing facility. This reduces the throughput at the external subcontractor, and increases the expected inventory.

Our results suggests that the decomposition approach is fairly accurate for various choices of service time distributions. Further, in terms of computational effort, it should be noted that all these experiments were performed on a personal computer with a 1.6GHz Intel Core i5 Processor. We observe that the run time for the decomposition approach ranges from 0-20 seconds, and did not increase significantly as we varied N from 2 to 16. This suggests that the approach can be used to analyze fairly large systems. In contrast, the run time for obtaining exact results using simulation ranged from 5-10 minutes for each case.

3.6 Conclusions

We consider a single product ATO system where the product is assembled from multiple components. The components can be manufactured in-house or purchased from the local subcontractor with different system parameters and costs. We analyze dual index policies (DB, OH, and LT policy) using Matrix-Geometric approach for moderately sized systems, and using a decomposition based approximation for large systems. The OH policy uses thresholds limits on the inventory levels of both the components where as the LT policy uses thresholds limits on the on-order levels of both the components. The performance of these policies are compared to determine regions in the system design space where they each perform well. We observe that LT policy works well at low base stock levels, while OH policy works well at high base stock levels. The DB policy outperforms other two policies although, it lacks details necessary for implementation. However, we observe that in particular settings, the performance of OH policy and LT policy closely resembles with DB policy and also provides the clarity needed for implementation. This suggests using a combination of LT policy and OH policy to overcome the operational ambiguity in DB policy. For an ATO system with $N > 2$ components, we face computational challenges with the exact approach used to analyze smaller systems. However, our proposed approximations exploits structural characteristics of the system to address this challenge. The approach not only provides reasonably accurate estimates of performance measures for large systems, but also scales well in terms of computational effort. Developing similar decomposition based approaches for ATO systems with both multiple components and end products seems to be a promising area of future research.

Chapter 4

Production Systems with Multiple Standard-type Components

4.1 Introduction

This research is motivated by production and capacity utilization issues observed in supply chains involved in the manufacturing of complex engineered components for drilling systems. Components of top drives (used to drill oil), blowout preventers (a safety mechanism), drilling subs, mud pumps, control systems that form core components of oil drilling systems all involve several thousands of hours of engineering and hundreds of hours of machining. In this industry, strategic collaborations with pre-certified subcontractors is essential to providing high customer service levels at reasonable cost. For instance, manufacturing of components of top drives and blowout preventers requires special purpose equipment which is very expensive. Consequently, capacity on this equipment is often shared across multiple types of top drives. When demands for particular top drives are high or when service levels expectations extend beyond internal capabilities, production is strategically subcontracted to pre-certified subcontractors. Although, these subcontractors may have higher costs and/or slower production rates, the belief is that subcontracting under these odds relieve capacity on critical internal resources that can be used for other components, resulting in overall economic benefits. Therefore, supply chain managers need to decide when and how much capacity, the manufacturer should dedicate to a given component, and when and how much production of a given component needs to be subcontracted. It is important to understand how differences in capabilities, costs, and service level expectations impact the optimal production and capacity utilization strategies for the manufacturer. In this chapter, our research investigates whether optimal policies have a easily describable structure that can be friendly for industry implementation and investigates the efficiency of simple control limit policies in this environment. Although, our motivating industry are manufacturers of oil drilling equipment, our model and insights extend into other environments, such as manufacturers of power equipment (thermal, nuclear, hydroelectric) and equipment used in chemical and process industries (condensers, reactors, turbines).

Some of the key studies in the literature that address the relevant questions include Ha (1997), Bradley (2005), and Huh et al. (2013). Ha (1997) studies the optimal production scheduling in a facility that manufacturers two components on a shared manufacturing resource. For the special case where both components have equal service rates, they develop a linear switching rule for production scheduling. In contrast, our research setting assumes that the components have different service rates and can be made at one or more facilities (in-house manufacturer or subcontractors). Bradley (2005) analyzes a single component system with dual-sourcing (in-house manufacturer and subcontractors) and shows that the dual base stock policy for component replenishment is optimal. Our research is an expansion of this problem setting as it considers multi-component system that share a manufacturing resource. We attempt to characterize the structure of the optimal policy in this general setting and explore whether policies have a dual index structure at least in some problem settings. Huh et al. (2013) analyze capacity decisions using a finite horizon model with multiple demand classes requiring different resources sequentially. For this setting they provide bounds on the structure of the optimal policy and argue that the characterization of the optimal policy is challenging. In contrast, our research analyzes infinite horizon model and allows components to be produced in parallel using capacity available at the manufacturer and the subcontractor. However, like Huh et al. (2013), we also find that characterizing optimal policy is hard, but we are able to characterize optimal policy in various regions of the state space.

While the Markov decision process (MDP) framework provides a convenient methodology to study the underlying production and capacity utilization problem, it suffers from the curse of dimensionality, often the state space description requires us to keep track of stocking levels of each component. In addition, in our case, the action space involves decisions for both shared manufacturing facility and each subcontracting facility. As a result, the underlying production-inventory problem can be very challenging in terms of the size of the underlying state and action space for even small problems. Further, since the manufacturer needs to balance the tradeoffs in costs due to production, backordering, and inventory of components, deriving the relevant monotonicity results become non-trivial. To address this complexity, we derive various conditions in terms of costs, capabilities, and production rates and use them to partition the state space into regions. Within each region, we are then able to show dominance of certain actions thereby reducing the relevant action space for that region. Further, for a complete symmetric system (with respect to costs and service rates of the products), we show that the optimal policy is of dual index type. Exploiting relations between costs and service rates, we further reduce the action space and analytically derive other settings where the optimal policy has a multi-index type structure. This implies that in these settings, the components may not be manufactured at all or only manufactured at the subcontractor or they might be manufactured simultaneously by both the manufacturer and the subcontractor.

The rest of the chapter is organized as follows. Section 4.2 describes the model of the system with multiple components and presents the MDP formulation of the system with multiple components. Section 4.3 derives conditions under which particular actions are optimal. Section 4.4 derives conditions under which optimal policy is multi-index type. Section 4.5

provides numerical studies and Section 4.6 summarizes model insights and conclusions.

4.2 Mathematical Model

We analyze a system that manufacturers two components $C_i, i = 1, 2$ to stock as shown in Figure 4.1. The demand for each component C_i is assumed to be a Poisson process $N_i(t), t \geq 0$ with rate λ_i and is satisfied from stock whenever possible; and otherwise backordered.

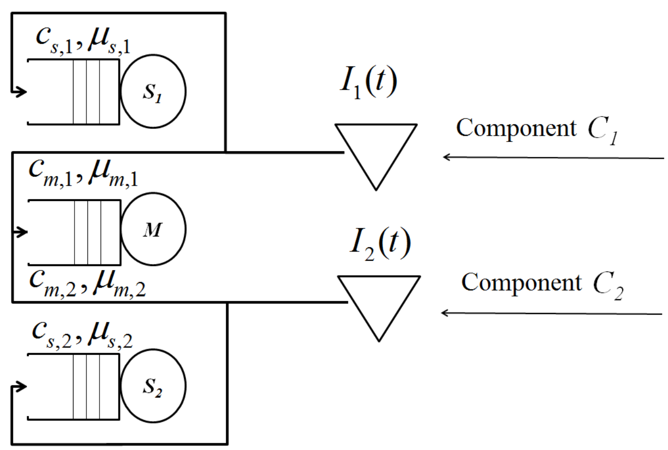

Figure 4.1 Manufacturing System

We let $I_i(t)$ denote the net inventory (on-hand minus backorders) for component C_i at time t. The manufacturer has the option to replenish inventory for component C_i either using capacity available at the external subcontractor, S_i or using capacity available at the in-house manufacturing facility M. We model the external subcontractor, $S_i, i = 1, 2$ and the internal manufacturing facility M as single server queues and assume that they have exponential service time for component C_i with mean $\mu_{s,i}^{-1}$ and $\mu_{m,i}^{-1}$ respectively. Further, let $c_{s,i}$ and $c_{m,i}$ denote the unit cost rate to manufacture component C_i at the external subcontractor and in-house manufacturing facility respectively. We let h_i denote the unit inventory holding cost rate for component C_i, and b_i denote the unit backordering cost rate of component C_i. The key elements in Markov decision process problem for determining the optimal production and inventory strategy are as follows:

Decision epoch: We analyze the problem in the continuous time domain and assume that actions are taken at epochs corresponding to state change, i.e at demand arrival and service completion epochs.

State space, Σ: The state of the system at any time t is described as $\sigma = (I_1, I_2)$, where $I_i, i = 1, 2$ is the net inventory position of component C_i and $\sigma \in \Sigma$.

$\mathbb{A}$	M	S_1	S_2
a_1	1	1	2
a_2	1	0	2
a_3	1	1	0
a_4	1	0	0
a_5	2	1	2
a_6	2	0	2
a_7	2	1	0
a_8	2	0	0
a_9	0	1	2
a_{10}	0	0	2
a_{11}	0	1	0
a_{12}	0	0	0

Figure 4.2 Action Space for the System

Action space, $\mathbb{A}$: The set of actions $\mathbb{A}$ is defined by $a_j = (m, s_1, s_2), j = 1, ..., 12$. In any action a_j, m takes the value i if the action corresponds to manufacturing of component C_i at the in-house manufacturing facility M and takes the value 0 if the action corresponds to being idle. Similarly, $s_i, i = 1, 2$ takes the value i when the action corresponds to manufacturing of component C_i at the external subcontractor S_i and takes the value 0 if the action corresponds to the external subcontractor being idle. Figure 4.2 lists the 12 possible actions available for the decision maker. For instance, action a_1 implies that component C_1 is manufactured at both the in-house manufacturer M and the external subcontractor S_1, while component C_2 is manufactured only at the external subcontractor S_2. Note that the action space can be partitioned into $\mathbb{A}_1 = \{a_1, a_3, a_5, a_6, a_9, a_{10}, a_{11}, a_{12}\}$ and $\mathbb{A}_2 = \{a_2, a_4, a_7, a_8\}$ where actions in $\mathbb{A}_2$ imply that the manufacturer uses the capacity to produce one of the components $C_i, i = 1, 2$ when the corresponding subcontractor S_i is idle. Further, actions in $\mathbb{A}_1$ can be grouped based on how the action impacts the component rate used to replenish the stock for component $C_i, i = 1, 2$ (see Figure 4.3).

Production rate for C_2 \ **Production rate for C_1**	**0**	$\mu_{s,1}$	$\mu_{s,1} + \mu_{m,1}$
$\mu_{s,2} + \mu_{m,2}$	a_6	a_5	-
$\mu_{s,2}$	a_{10}	a_9	a_1
0	a_{12}	a_{11}	a_3

Figure 4.3 Actions in $\mathbb{A}_1$

Transition probabilities: Define $p(\sigma'|\sigma, a_j)$ as the transition probability from state $\sigma = (I_1, I_2)$ to state $\sigma' = (I'_1, I'_2)$ under action $a_j \in \mathbb{A}$. Let $\nu = \sum_{i=1}^{2} \lambda_i + \sum_{i=1}^{2}(\mu_{m,i} + \mu_{s,i}) + A$ denote the normalizing factor used for uniformization (Lippman (1975)). Then, the transition probabilities $p(\sigma'|\sigma, a_j)$ are defined as follows:

Demand arrival for component C_i: Then $I'_i = I_i - 1, i = 1, 2$; and the corresponding $p(\sigma'|\sigma, a_j)$ is given by:

$$p(\sigma'|\sigma, a_j) = \lambda_i/\nu, \forall i = 1, 2$$

Service completion for component C_1: Then $I_1' = I_1 + 1$; and the corresponding $p(\sigma'|\sigma, a_j)$ is given by:

$$p(\sigma'|\sigma, a_j) = (\mathbb{1}_{m,1,j}\mu_{m,1} + \mathbb{1}_{s,1,j}\mu_{s,1})/\nu$$

where $\mathbb{1}_{m,i,j}$ and $\mathbb{1}_{s,i,j}, i = 1, 2$ are indicator functions that takes the value 1 if manufacturer M and subcontractor S_i respectively are producing component C_i under action a_j, and 0 otherwise.

Service completion for component C_2: Then $I_2' = I_2 + 1$; and the corresponding $p(\sigma'|\sigma, a_j)$ is given by:

$$p(\sigma'|\sigma, a_j) = (\mathbb{1}_{m,2,j}\mu_{m,2} + \mathbb{1}_{s,2,j}\mu_{s,2})/\nu$$

Finally, $I_i' = I_i, i = 1, 2$; and the corresponding $p(\sigma'|\sigma, a_j)$ is given by:

$$p(\sigma'|\sigma, a_j) = (\nu - \sum_i(\lambda_i + \mathbb{1}_{m,i,j}\mu_{m,i} + \mathbb{1}_{s,i,j}\mu_{s,i}))/\nu$$

Cost function: Define $h(\sigma) = h_1 \max(I_1, 0) + h_2 \max(I_2, 0)$ as the total inventory holding cost rate and $b(\sigma) = b_1 \max(-I_1, 0) + b_2 \max(-I_2, 0)$ as the total backordering cost rate. Let $c(a_j) = \sum_i(c_{m,i}\mathbb{1}_{m,i,j} + c_{s,i}\mathbb{1}_{s,i,j})$ represent the production cost rate for action a_j. Let $r(\sigma, a_j)$ denote the immediate cost function at state σ for action a_j. This means that $r(\sigma, a_j) = h(\sigma) + b(\sigma) + c(a_j)$. Let $V_t(\sigma)$ denote the value function at state σ at time t. For simplicity, we normalize and set $\nu = 1$ and $A = 0$. Equation (4.1) defines the standard Bellman cost equation with value function, $V_t(\sigma)$ at state σ and decision epoch t with discount factor $\eta, \eta \in (0, 1)$.

$$V_t(I_1, I_2) = \min_{a_j \in \mathbb{A}}[h(\sigma) + b(\sigma) + c(a_j) + \eta \sum_{\sigma'} p(\sigma'|\sigma, a_j)V_t(\sigma')] \tag{4.1}$$

The system described above presents challenges in terms of structural analysis of the optimal policy. First, the size of the state space Σ and action space $\mathbb{A}$ increases the complexity of the analysis. For example, with $I_i, i = 1, 2$ varying from -500 to 500, the model has 1 million states and 12 actions. Second, the optimal value function $V_t^*(I_1, I_2)$ may not be convex in

I_i and the transition probabilities may not have sub-additivity or super-additivity property with respect to I_i in the state space Σ and action space $\mathbb{A}$ because of high action space.

Despite of the above mentioned challenges, in the next section (Section 4.3) we describe the characteristics of the optimal policy and the optimal value function. We use efficient action comparison and action elimination techniques to develop conditions that relate change in value function to production cost rates and service rates, and show that under these conditions particular actions are optimal. Then in Section 4.4, using action elimination conditions, we significantly reduce the action space which helps us to prove when simple multi-index policies are optimal. For notational simplicity where possible, we suppress subscript t in subsequent sections.

4.3 Characteristics of Optimal Solution

We analyze the formulation described in Section 4.2 and determine the characteristics of the optimal solution. Section 4.3.1 provides some necessary preliminaries required to show the main results of Sections 4.3.2 and 4.3.3 respectively.

4.3.1 Preliminaries

We develop set of conditions to characterize the structure of the optimal policy for both the products. Let $\Delta_1(I_1, I_2) = V_{t+1}(I_1 + 1, I_2) - V_{t+1}(I_1, I_2)$ be the first difference of the value function with respect to I_1, and $\Delta_2(I_1, I_2) = V_{t+1}(I_1, I_2 + 1) - V_{t+1}(I_1, I_2)$ be the first difference of the value function with respect to I_2. Table 4.1 defines a set of conditions used in subsequent sections to characterize the optimal policy.

These conditions provide the relationship between rate of change of the value function per unit of a component and the unit production and subcontracting costs. For example, condition $\mathcal{A}_{1s}(I_1, I_2)$ holds if the expected cost at state $(I_1 + 1, I_2)$ with a unit change in the

Table 4.1 Preliminary Conditions

Notation	Definition
$\mathcal{A}_{1s}(I_1, I_2)$	$\Delta_1(I_1, I_2) > -\frac{c_{s,1}}{\mu_{s,1}\eta}$
$\hat{\mathcal{A}}_{1s}(I_1, I_2)$	$\Delta_1(I_1, I_2) < -\frac{c_{s,1}}{\mu_{s,1}\eta}$
$\mathcal{A}_{1m}(I_1, I_2)$	$\Delta_1(I_1, I_2) > -\frac{c_{m,1}}{\mu_{m,1}\eta}$
$\hat{\mathcal{A}}_{1m}(I_1, I_2)$	$\Delta_1(I_1, I_2) < -\frac{c_{m,1}}{\mu_{m,1}\eta}$
$\mathcal{A}_{2s}(I_1, I_2)$	$\Delta_2(I_1, I_2) > -\frac{c_{s,2}}{\mu_{s,2}\eta}$
$\hat{\mathcal{A}}_{2s}(I_1, I_2)$	$\Delta_2(I_1, I_2) < -\frac{c_{s,2}}{\mu_{s,2}\eta}$
$\mathcal{A}_{2m}(I_1, I_2)$	$\Delta_2(I_1, I_2) > -\frac{c_{m,2}}{\mu_{m,2}\eta}$
$\hat{\mathcal{A}}_{2m}(I_1, I_2)$	$\Delta_2(I_1, I_2) < -\frac{c_{m,2}}{\mu_{m,2}\eta}$
$\mathcal{B}_1(I_1, I_2)$	$\frac{c_{m,2}}{\eta} + \mu_{m,2}\Delta_2(I_1, I_2) < \frac{c_{m,1}}{\eta} + \mu_{m,1}\Delta_1(I_1, I_2)$
$\mathcal{B}_2(I_1, I_2)$	$\frac{c_{m,2}}{\eta} + \mu_{m,2}\Delta_2(I_1, I_2) > \frac{c_{m,1}}{\eta} + \mu_{m,1}\Delta_1(I_1, I_2)$

inventory I_1 while incurring a unit production cost $\frac{c_{s,1}}{\mu_{s,1}\eta}$ for component C_1 at the subcontractor is greater than the expected cost at state (I_1, I_2). Condition $\mathcal{B}_2(I_1, I_2)$ has a useful interpretation when $\mu_{m,1} = \mu_{m,2}$, it implies that the expected cost at state $(I_1, I_2 + 1)$ with a unit change in the inventory I_2 while incurring a unit production cost $\frac{c_{m,2}}{\mu_{m,2}\eta}$ for component C_2 at the manufacturer is greater than the the expected cost at state $(I_1 + 1, I_2)$ with a unit change in the inventory I_1 while incurring a unit production cost $\frac{c_{m,1}}{\mu_{m,1}\eta}$ for component C_1 at the manufacturer. In the next section, we analyze the conditions for optimal decision using these conditions.

4.3.2 Characteristics of Optimal Policy

We use the conditions defined in Section 4.3.1 to show that certain actions are optimal. For notational simplicity, we suppress (I_1, I_2) in the conditions, but note that these conditions hold for each state (I_1, I_2). Theorem 4.1 provides relationship between the change in the value function, costs, and service rates for which the optimal action is that neither the manufacturer nor the subcontractor produces a given component C_i.

Theorem 4.1. *For system in state σ:*

(1) if conditions $\mathcal{A}_{1s} \wedge (\mathcal{A}_{1m} \vee \mathcal{B}_1)$ hold, then the optimal action implies that neither the manufacturer M nor the subcontractor S_1 should manufacture component C_1.

(2) if conditions $\mathcal{A}_{2s} \wedge (\mathcal{A}_{2m} \vee \mathcal{B}_2)$ hold, then the optimal action implies that neither the manufacturer M nor the subcontractor S_2 should manufacture component C_2.

Proof. We prove Theorem 4.1 in two parts. In the first part, we consider component C_2 and show that under conditions $(\Delta_2 > -\frac{c_{s,2}}{\mu_{s,2}\eta}) \wedge ((\Delta_2 > -\frac{c_{m,2}}{\mu_{m,2}\eta}) \vee (\mu_{m,2}\Delta_2 - \mu_{m,1}\Delta_1 > \frac{c_{m,1}-c_{m,2}}{\eta}))$, actions from the set $\{a_1, a_2, a_5, a_6, a_7, a_8, a_9, a_{10}\}$ have higher costs than that of at least one action from the set $\{a_3, a_4, a_{11}, a_{12}\}$. Since, the optimal action is from the set $\{a_3, a_4, a_{11}, a_{12}\}$, the result of Theorem 4.1, part (2) holds. The proof of part (1) follows along similar lines by considering component C_1 instead of component C_2. Refer to Appendix for the proof of Theorem 4.1. □

Next, Theorem 4.2 provides the relationship between the change in the value function, costs, and service rates for which the optimal action is that only the external subcontractor should use available capacity to produce a given component C_i.

Theorem 4.2. *For system in state σ:*

(1) if conditions $\hat{\mathcal{A}}_{1s} \wedge (\mathcal{A}_{1m} \vee \mathcal{B}_1)$ hold, then the optimal action implies that only the subcontractor S_1 should manufacture component C_1.

(2) if conditions $\hat{\mathcal{A}}_{2s} \wedge (\mathcal{A}_{2m} \vee \mathcal{B}_2)$ hold, the optimal action implies that only the subcontractor S_2 should manufacture component C_2.

Proof. We prove Theorem 4.2 in two parts. In the first part, we consider component C_2 and show that under conditions $(\Delta_2 < -\frac{c_{s,2}}{\mu_{s,2}\eta}) \wedge ((\Delta_2 > -\frac{c_{m,2}}{\mu_{m,2}\eta}) \vee (\mu_{m,2}\Delta_2 - \mu_{m,1}\Delta_1 > \frac{c_{m,1}-c_{m,2}}{\eta}))$, actions from the set $\{a_3, a_4, a_5, a_6, a_7, a_8, a_{11}, a_{12}\}$ have higher costs than that of at least one action from the set $\{a_1, a_2, a_9, a_{10}\}$. Since, the optimal action is from the set $\{a_1, a_2, a_9, a_{10}\}$, the result of Theorem 4.2, part (2) holds. The proof of part (1) follows along similar lines by considering component C_1 instead of component C_2. Refer to Appendix for the proof of Theorem 4.2.

□

Next, Theorem 4.3 provides relationship between the change in the value function, costs, and service rates for which the optimal action is that only the manufacturer and not the corresponding subcontractor should use the available capacity to produce a given component C_i.

Theorem 4.3. *For system in state σ:*
(1) if conditions $\mathcal{A}_{1s} \wedge \hat{\mathcal{A}}_{1m} \wedge \mathcal{B}_2$ hold, then the optimal action implies that only the manufacturer M should manufacture component C_1.
(2) if conditions $\mathcal{A}_{2s} \wedge \hat{\mathcal{A}}_{2m} \wedge \mathcal{B}_1$ hold, then the optimal action implies that only the manufacturer M should manufacture component C_2.

Proof. We prove Theorem 4.3 in two parts. In the first part, we consider component C_2 and show that under conditions $(\Delta_2 > -\frac{c_{s,2}}{\mu_{s,2}\eta}) \wedge (\Delta_2 < -\frac{c_{m,2}}{\mu_{m,2}\eta}) \wedge (\mu_{m,2}\Delta_2 - \mu_{m,1}\Delta_1 < \frac{c_{m,1}-c_{m,2}}{\eta})$, actions from the set $\{a_1, a_2, a_3, a_4, a_5, a_6, a_9, a_{10}, a_{11}, a_{12}\}$ have higher costs than that of at least one action from the set $\{a_7, a_8\}$. Since, the optimal action is from the set $\{a_7, a_8\}$, the result of Theorem 4.3, part (2) holds. The proof of part (1) follows along similar lines by considering component C_1 instead of component C_2. Refer to Appendix for the proof of Theorem 4.3.

□

Corollary 4.1. *If $\frac{c_{m,i}}{\mu_{m,i}} \geq \frac{c_{s,i}}{\mu_{s,i}}, i = 1, 2$, then the optimal action does not belong to the set $\mathbb{A}_2 = \{a_2, a_4, a_7, a_8\}$.*

Proof. The proof follows immediately from the proof of Theorem 4.3. □

Corollary 4.1 provides the conditions when it could never be optimal for the manufacturer to invest capacity in manufacturing a component while the relevant subcontractor is idle. Finally, Theorem 4.4 provides relationship between the change in the value function, costs, and service rates for which the optimal action is that both the external subcontractor and manufacturer should use their respective capacity to produce a given component C_i.

Theorem 4.4. *For system in state σ:*

(1) if conditions $\hat{\mathcal{A}}_{1s} \wedge \hat{\mathcal{A}}_{1m} \wedge \mathcal{B}_2$ hold, then the optimal action implies that both the manufacturer M and the subcontractor S_1 should manufacture component C_1.

(2) if conditions $\hat{\mathcal{A}}_{2s} \wedge \hat{\mathcal{A}}_{2m} \wedge \mathcal{B}_1$ hold, then the optimal action implies that both the manufacturer M and the subcontractor S_2 should manufacture component C_2.

Proof. We prove Theorem 4.4 in two parts. In the first part, we consider component C_2 and show that under conditions $(\Delta_2 < -\frac{c_{m,2}}{\mu_{m,2}\eta}) \wedge (\Delta_2 < -\frac{c_{s,2}}{\mu_{s,2}\eta}) \wedge (\mu_{m,2}\Delta_2 - \mu_{m,1}\Delta_1 < \frac{c_{m,1}-c_{m,2}}{\eta})$, actions from the set $\{a_1, a_2, a_3, a_4, a_7, a_8, a_9, a_{10}, a_{11}, a_{12}\}$ have higher costs than that of at least one action from the set $\{a_5, a_6\}$. Since, the optimal action is from the set $\{a_5, a_6\}$, the result of Theorem 4.4, part (2) holds. The proof of part (1) follows along similar lines by considering component C_1 instead of component C_2. Refer to Appendix for the proof of Theorem 4.4. $\square$

Production rate for C_2			
$\boldsymbol{\mu_{s,2} + \mu_{m,2}}$	$\mathcal{A}_{1s} \wedge (\mathcal{A}_{1m} \vee \mathcal{B}_1)$ $\wedge \hat{\mathcal{A}}_{2s} \wedge \hat{\mathcal{A}}_{2m} \wedge \mathcal{B}_1$ **(Action a_6)**	$\hat{\mathcal{A}}_{1s} \wedge (\mathcal{A}_{1m} \vee \mathcal{B}_1)$ $\wedge \hat{\mathcal{A}}_{2s} \wedge \hat{\mathcal{A}}_{2m} \wedge \mathcal{B}_1$ **(Action a_5)**	-
$\boldsymbol{\mu_{s,2}}$	$\mathcal{A}_{1s} \wedge (\mathcal{A}_{1m} \vee \mathcal{B}_1)$ $\wedge \hat{\mathcal{A}}_{2s} \wedge (\mathcal{A}_{2m} \vee \mathcal{B}_2)$ **(Action a_{10})**	$\hat{\mathcal{A}}_{1s} \wedge (\mathcal{A}_{1m} \vee \mathcal{B}_1)$ $\wedge \hat{\mathcal{A}}_{2s} \wedge (\mathcal{A}_{2m} \vee \mathcal{B}_2)$ **(Action a_9)**	$\hat{\mathcal{A}}_{1s} \wedge \hat{\mathcal{A}}_{1m} \wedge \mathcal{B}_2$ $\wedge \hat{\mathcal{A}}_{2s} \wedge (\mathcal{A}_{2m} \vee \mathcal{B}_2)$ **(Action a_1)**
0	$\mathcal{A}_{1s} \wedge (\mathcal{A}_{1m} \vee \mathcal{B}_1)$ $\wedge \mathcal{A}_{2s} \wedge (\mathcal{A}_{2m} \vee \mathcal{B}_2)$ **(Action a_{12})**	$\hat{\mathcal{A}}_{1s} \wedge (\mathcal{A}_{1m} \vee \mathcal{B}_1)$ $\wedge \mathcal{A}_{2s} \wedge (\mathcal{A}_{2m} \vee \mathcal{B}_2)$ **(Action a_{11})**	$\hat{\mathcal{A}}_{1s} \wedge \hat{\mathcal{A}}_{1m} \wedge \mathcal{B}_2$ $\wedge \mathcal{A}_{2s} \wedge (\mathcal{A}_{2m} \vee \mathcal{B}_2)$ **(Action a_3)**
	0	$\boldsymbol{\mu_{s,1}}$	$\boldsymbol{\mu_{s,1} + \mu_{m,1}}$
		Production rate for C_1	

Figure 4.4 Conditions for Optimal Actions

Figure 4.4 summarizes the insights from Theorems 4.1, 4.2, 4.3, 4.4. For instance, if $\mathcal{A}_{1s} \wedge (\mathcal{A}_{1m} \vee \mathcal{B}_1) \wedge \hat{\mathcal{A}}_{2s} \wedge \hat{\mathcal{A}}_{2m} \wedge \mathcal{B}_1$, then the component C_2 is produced by both manufacturer

M and subcontractor S_2, and component C_1 is produced by only subcontractor S_1 which corresponds to action a_5. Note that the conditions presented in Section 4.3.1 exclude the equality cases, such as $\Delta_1(I_1, I_2) = -\frac{c_{s,1}}{\mu_{s,1}\eta}$, etc since in these cases multiple actions could be optimal. Theorems 4.1 - 4.4 provide a strong characterization of optimal policies for this production inventory problem. For a general system, the conditions allow us to partition the state space into regions where the optimal policy has a simple characterization. Further insights on the characteristics of the optimal policy are obtained by considering symmetric setting of the problem parameters. We elaborate on this in Section 4.3.3.

4.3.3 Optimal Policy for Symmetric Systems

We consider two types of symmetry, $M - S$ symmetry and complete symmetry. For an $M - S$ symmetric system, the costs and service rates for a given component C_i for the manufacturer M is the same as that of the subcontractor S_i, i.e. $c_{m,i} = c_{s,i}$, and $\mu_{m,i} = \mu_{s,i}$. Complete symmetry corresponds to a special case of $M - S$ symmetric system where the production costs and the service rates are same across the components. Figure 4.5 summarizes the conditions under which particular actions are optimal. Although, Figure 4.5 presents a simple version of the conditions presented in Figure 4.4, it reveals a useful insight: in a symmetric system, action a_9 is never optimal, i.e. both the components will not be simultaneously manufactured only by the respective external subcontractors. This happens because condition $\hat{\mathcal{A}}_{2s}$ and $\mathcal{A}_{2m}$ can never be simultaneously satisfied under $M - S$ symmetry.

Next, we consider complete symmetric system. For this case, first we use Theorem 4.5 and further reduce the optimal action space from 12 actions to only 5 actions.

Theorem 4.5. *If $c_{m,i} = c_{s,i}$, and $\mu_{m,i} = \mu_{s,i}, i = 1, 2$, then optimal action belongs to the set $\{a_1, a_3, a_5, a_6, a_{12}\}$.*

Proof. From results for $M - S$ symmetric case, it follows that actions in the set $\{a_1, a_3, a_5, a_6, a_{10}, a_{11}, a_{12}\}$ could be optimal. Next, from Theorem 4.4.1, if $\Delta_2 < -\frac{c_{m,2}}{\mu_{m,2}\eta}$, then action a_6 results in lower cost than action a_{10}. Thus, action a_{10} cannot be optimal. Similarly, if

Production rate for C_2 \ Production rate for C_1	$\mathbf{0}$	$\boldsymbol{\mu_{s,1}}$	$\boldsymbol{2\mu_{s,1}}$
$\boldsymbol{2\mu_{s,2}}$	$\mathcal{A}_{1s} \wedge \hat{\mathcal{A}}_{2s} \wedge \mathcal{B}_1$ (**Action a_6**)	$\hat{\mathcal{A}}_{1s} \wedge \mathcal{B}_1 \wedge \hat{\mathcal{A}}_{2s}$ (**Action a_5**)	-
$\boldsymbol{\mu_{s,2}}$	$\mathcal{A}_{1s} \wedge \hat{\mathcal{A}}_{2s} \wedge \mathcal{B}_2$ (**Action a_{10}**)	-	$\hat{\mathcal{A}}_{1s} \wedge \mathcal{B}_2 \wedge \hat{\mathcal{A}}_{2s}$ (**Action a_1**)
$\mathbf{0}$	$\mathcal{A}_{1s} \wedge \mathcal{A}_{2s}$ (**Action a_{12}**)	$\hat{\mathcal{A}}_{1s} \wedge \mathcal{B}_1 \wedge \mathcal{A}_{2s}$ (**Action a_{11}**)	$\hat{\mathcal{A}}_{1s} \wedge \mathcal{A}_{2s}$ (**Action a_3**)

Figure 4.5 Conditions for Optimal Actions under $M - S$ Symmetry

$\Delta_1 > -\frac{c_{m,1}}{\mu_{m,1}\eta}$, then action a_3 results in lower cost than action a_{11}. Thus, action a_{11} cannot be optimal. This concludes the proof. $\square$

Note that under complete symmetric system with respect to costs and service rates of the components, the optimal policy suggests that either one of the components should be always produced at the fastest rate or none of the components should be produced. This is because actions a_{10} and a_{11} are never optimal. Figure 4.4 and 4.5 demonstrate dual index characterization of the optimal policy; i.e., under certain condition, there exists thresholds in I_1 and I_2 at which optimal production rates for particular component structures. Such simple characterization of the optimal policy can be very useful in practice. However, due to the structure of the value function, it is hard to show exhaustively set of conditions under which dual index policies are optimal. However, in the next section we prove conditions and instances where multi-index policies can be demonstrated to be optimal.

4.4 Optimality of Multi-index Policies

In this section, we derive conditions when multi-index policies are optimal. Proposition 4.1 derives a condition and a subset $\mathbb{S}$ over which the optimal value function $V^*(I_1, I_2)$ can be proved to be convex.

Proposition 4.1. *For system with state $\sigma \in \mathbb{S} = \{(I_1, I_2)|I_1 + I_2 = K\}$, where K is constant, the optimal value function $V^*(I_1, I_2)$ is convex over $\mathbb{S}$.*

Proof. We prove this proposition in two steps. First, we show that the cost function $r(\sigma, a_j)$ is convex over $\mathbb{S}$, and then we show the required properties of the transition probabilities. Refer to Appendix for the proof of Proposition 4.1. □

Further, the transition probabilities are sub-additive or super-additive in $\mathbb{A} \times \mathbb{S}$. We make no claim that the value function is convex only over this set. In fact, it can be shown that the value function is convex over other subset of the state space. For example, $V^*(I_1, I_2)$ is convex with respect to I_1 for state $\sigma \in \mathbb{S}_1 = \{(I_1, I_2)|I_2 = K\}$. However, we do not observe monotonicity property in the optimal actions because the transition probabilities are neither sub-additive nor super-additive in $\mathbb{A} \times \mathbb{S}_1$.

We consider three cases and show optimality of multi-index policies in each case.

Case 1: Negligible production costs: Typically, certain components/operations such as mechanical assembly, PCB assembly, wire harnesses assembly, etc have negligible production costs as compared to corresponding material costs/inventory costs. Under this setting, the production cost could be assumed to be zero. Theorem 4.6 shows that if the production costs are zero then the optimal decisions are non-increasing in the service rate of component C_1 with respect to $I_1, (I_1, I_2) \in \mathbb{S}$.

Theorem 4.6. *For system with state $\sigma \in \mathbb{S}$, $\mathbb{S} = \{(I_1, I_2)|I_1 + I_2 = K\}$ where, K is constant, if $c_{m,i} = c_{s,i} = 0, i = 1, 2$ and $\Delta_1 \neq 0, \Delta_2 \neq 0$, then the optimal action a^* is non-increasing in service rates for component C_1 with respect to increasing I_1.*

Proof. We prove this theorem in two parts. At first, we consider the case where $V^*(I_1, I_2)$ is non-decreasing with respect to I_1 and show that the optimal actions are monotone with respect to I_1. Next, we we consider the case where $V^*(I_1, I_2)$ is non-decreasing with respect to I_2 and show that the optimal actions are monotone with respect to I_2. $\square$

Theorem 4.6 implies that the supply chain manager could keep track of the stock levels of components and follow the dual index type optimal policy, i.e. there exists thresholds k_1 and $k_2, k_2 > k_1$ such that the component C_1 (i) is not manufactured if $I_1 > k_2$, or (ii) is only be produced by the corresponding subcontractor if $k_1 < I_1 < k_2$ to replenish inventory at a slower rate, or (iii) is produced at both the manufacturer and the corresponding subcontractor if $I_1 < k_1$ to replenish inventory at a faster rate.

Case 2: Manufacturer is cheaper: When demand is high and exceeds internal capacity, the supply chain manager could subcontract components to the external subcontractor to alleviate the production and capacity burden at the manufacturer, even though the subcontractor is more expensive than the manufacturer. For example, blowout preventers (prominent energy product) vary significantly in size and require special equipment. For this component, the production cost at the subcontractor could be more than the production cost at the manufacturer. Theorem 4.7 shows that if the production cost per unit at the manufacturer is less than the production cost per unit at the subcontractor, service rate of the component C_1 is significantly more than the service rate of component C_2, then optimal decisions are non-increasing in the service rate of component C_1 with respect to $I_1, (I_1, I_2) \in \mathbb{S}$.

Theorem 4.7. *For system with state $\sigma \in \mathbb{S}$, $\mathbb{S} = \{(I_1, I_2) | I_1 + I_2 = K\}$ where, K is constant and $I_1 > K$, if:*
(1) $\frac{c_{m,i}}{\mu_{m,i}} < \frac{c_{s,i}}{\mu_{s,i}}, i = 1, 2$
(2) $\mu_{s,1} > \mu_{s,2}$
(3) $\mu_{m,1} > \mu_{m,2} + \mu_{s,1} + \mu_{s,2}$

Then the optimal action a^* is non-increasing in service rates for component C_1 with respect to increasing I_1.

Proof. We prove this theorem in three steps. First, we show that the optimal value function, $V^*(I_1, I_2)$ is non-decreasing over $I_1, I_1 > K$. Next, we show that if $\frac{c_{m,i}}{\mu_{m,i}} < \frac{c_{s,i}}{\mu_{s,i}}, i = 1, 2$ then actions from the set $\{a_9, a_{10}, a_{11}\}$ are not optimal. Finally, we show that $q((I_1'', I_2'')|(I_1, I_2), a_j)$ is sub-additive with respect to non-decreasing I_1 and action $\mathbb{A}$. The details of the proof are in the Appendix. $\square$

Theorem 4.7 implies that a multi-index policy with three thresholds k_1, k_2, and $k_3, k_3 > k_2 > k_1$ is optimal. With respect to component C_1, this multi-index policy implies that component C_1 (i) is not manufactured if $I_1 > k_3$, (ii) is only be produced by the corresponding subcontractor if $k_2 < I_1 < k_3$, (ii) is only be produced by the manufacturer if $k_1 < I_1 < k_2$, (iii) is produced at the manufacturer and the corresponding subcontractor if $I_1 < k_1$.

Case 3: Subcontractor is cheaper: For example, components of top drives are often expensive to produce using capacity only available at the manufacturer. Using capacity available at the subcontractor is often cheaper. However, if the subcontractor has a higher lead time, this could lead to high backorders and poor service levels. So, the supply chain manager needs to balance the tradeoffs in cost and delivery performance to decide on the production and subcontracting decisions. Theorem 4.8 shows that if the production cost per unit at the manufacturer is more than the production cost per unit at the subcontractor, service rate of the component C_1 is significantly more than the service rate of component C_2, then optimal decisions are non-increasing in the service rate of component C_1 with respect to $I_1, (I_1, I_2) \in \mathbb{S}$.

Theorem 4.8. *For system with state $\sigma \in \mathbb{S}$, $\mathbb{S} = \{(I_1, I_2)|I_1 + I_2 = K\}$ where, K is constant and $I_1 > K$, if:*

(1) $\frac{c_{m,i}}{\mu_{m,i}} > \frac{c_{s,i}}{\mu_{s,i}}, i = 1, 2$

(2) $\mu_{m,1} > \mu_{m,2} + \mu_{s,1}$

(3) $\mu_{s,1} > \mu_{m,2} + \mu_{s,2}$

Then the optimal action a^* *is non-increasing in service rates for component* C_1 *with respect to increasing* I_1.

Proof. We prove this theorem in three steps. First, we show that the optimal value function, $V^*(I_1, I_2)$ is non-decreasing over $I_1, I_1 > K$. Next, we show that if $\frac{c_{m,i}}{\mu_{m,i}} < \frac{c_{s,i}}{\mu_{s,i}}, i = 1, 2$ then using Corollary 4.1, actions from the set $\{a_2, a_4, a_7, a_8\}$ are not optimal. Finally, we show that $q((I_1'', I_2'')|(I_1, I_2), a_j)$ is sub-additive with respect to non-decreasing I_1 and action $\mathbb{A}$. The details of the proof are in the Appendix. □

Theorem 4.8 implies that the supply chain manager could keep track of the stock levels of components and follow the dual index type optimal policy, i.e. there exists thresholds k_1 and $k_2, k_2 > k_1$ such that the component C_1 (i) is not manufactured if $I_1 > k_2$, or (ii) is only be produced by the corresponding subcontractor if $k_1 < I_1 < k_2$ to replenish inventory at a slower rate, or (iii) is produced at both the manufacturer and the corresponding subcontractor if $I_1 < k_1$ to replenish inventory at a faster rate.

Having shown that the multi-index policy is optimal under certain conditions on the service rate and costs, in the next section we use numerical studies to validate these observations and also demonstrate other situations where multi-index policies are optimal.

4.5 Numerical Studies

This section presents numerical studies to provide insights on the characteristics of the optimal solution. Section 4.5.1 demonstrates multi-index policies while Section 4.5.2 analyzes the impact of service rates on the optimal policy.

4.5.1 Demonstration of Multi-index Policy

In this experiment, we numerically validate the results shown in Theorem 4.7 where the unit production cost at the manufacturer is less than the unit production cost at the subcontractor, i.e. $\frac{c_{m,i}}{\mu_{m,i}} < \frac{c_{s,i}}{\mu_{s,i}}, i = 1, 2$. Table 4.2 presents the system and cost parameters for this experiment.

Table 4.2 System Parameters and Costs for Multi-index Policy

Subcontractor's Parameters		**Manufacturer's Parameters**	
$c_{s,1}$	15	$c_{m,1}$	30
$c_{s,2}$	10	$c_{m,2}$	20
$\mu_{s,1}$	1.5	$\mu_{m,1}$	4
$\mu_{s,2}$	0.5	$\mu_{m,2}$	1.1
System Parameters		**Other Costs**	
B_{max}	5	$b_i, i = 1, 2$	40
$\lambda_i, i = 1, 2$	1.5	$h_i, i = 1, 2$	2

Here, the manufacturer is twice as expensive ($c_{m,1} = 2c_{s,1}$) as the external subcontractor. The service rates for the in-house manufacturing department and the external subcontractor are set to satisfy the conditions given in Theorem 4.7. For $B_{max} = 5$, we fix any $K, K \in \{-10, -9, ..., 9, 10\}$ and analyze the optimal solution for monotonicity property in the service rate.

Table 4.3 presents the optimal actions for system corresponding to each state σ. The optimal policy contains actions in the set $\{a_1, a_3, a_4, a_5, a_6, a_{12}\}$. For fixed total inventory position, $I_1 + I_2$, we observe a monotone property in service rates of components with increasing I_1 or I_2. For instance, for $I_1 + I_2 = -8$, the production rate of component C_1 is always $\mu_{m,1} + \mu_{s,1}$ as I_1 increases, meaning that both the manufacturer M and the corresponding subcontractor S_1 are producing component C_1. Similarly, for $I_1 + I_2 = 0$, (i) if $I_1 < 0$, the production rate of component C_1 is $\mu_{m,1} + \mu_{s,1}$, meaning that both the manufacturer M and the corresponding subcontractor S_1 are producing component C_1, (ii) if $0 \leq I_1 < 5$, the production rate of component C_1 decreases from $\mu_{m,1} + \mu_{s,1}$ to $\mu_{s,1}$, meaning that only the subcontractor S_1 is producing component C_1, (iii) if $I_1 = 5$, the production rate of component C_1 is 0, meaning that the component C_1 is neither manufactured by the manufacturer M nor by the subcontractor S_1. Finally, for $I_1 + I_2 = 7$, (i) if $I_1 < 3$, the production rate of component C_1 is $\mu_{m,1}$, meaning that the manufacturer M is producing component C_1, (ii) if $3 \leq I_1 < 5$, the production rate of component C_1 decreases from $\mu_{m,1} + \mu_{s,1}$ to $\mu_{s,1}$, meaning that only the subcontractor S_1 is producing component C_1, (iii) if $I_1 = 5$, the production rate of component C_1 is 0, meaning that the component C_1 is neither manufactured by the manufacturer M nor by the subcontractor S_1. This demonstrates the result in Theorem 4.7 where the optimal policy is multi-index index type with three thresholds.

Table 4.3 Optimal Actions Corresponding to Each State σ for $B_{max} = 5$

I_1/I_2	-5	-4	-3	-2	-1	0	1	2	3	4	5
-5	a_1	a_1	$\boldsymbol{a_1}$	a_1	a_1	a_1	a_1	a_1	a_1	a_1	$\boldsymbol{a_3}$
-4	a_1	$\boldsymbol{a_1}$	a_1	a_1	a_1	a_1	a_1	a_1	a_1	$\boldsymbol{a_1}$	a_3
-3	$\boldsymbol{a_1}$	a_1	a_1	a_1	a_1	a_1	a_1	a_1	$\boldsymbol{a_1}$	a_1	a_3
-2	a_1	a_1	a_1	a_1	a_1	a_1	a_1	$\boldsymbol{a_1}$	a_1	a_1	a_3
-1	a_1	a_1	a_1	a_1	a_1	a_1	$\boldsymbol{a_1}$	a_1	a_1	a_1	a_3
0	a_1	a_5	a_5	a_5	a_5	$\boldsymbol{a_5}$	a_5	a_5	a_1	a_1	a_3
1	a_1	a_5	a_5	a_5	$\boldsymbol{a_5}$	a_5	a_5	a_5	a_5	a_5	a_3
2	a_5	a_5	a_5	$\boldsymbol{a_5}$	a_5	a_5	a_5	a_5	a_5	a_5	$\boldsymbol{a_4}$
3	a_5	a_5	$\boldsymbol{a_5}$	a_5	a_5	a_5	a_5	a_5	a_5	$\boldsymbol{a_5}$	a_4
4	a_5	$\boldsymbol{a_5}$	a_5	a_5	a_5	a_5	a_5	a_5	$\boldsymbol{a_5}$	a_6	a_4
5	$\boldsymbol{a_6}$	a_6	a_6	a_6	a_6	a_6	a_6	$\boldsymbol{a_6}$	a_6	a_6	a_{12}

4.5.2 Impact of Service Rates on the Optimal Policy

In this experiment, we analyze the impact of service rates on the optimal policy. Table 4.4 presents the system and cost parameters for this experiment. We analyze two cases.

Table 4.4 System Parameters and Costs for Sensitivity Analysis

Subcontractor's Parameters		**Manufacturer's Parameters**	
$c_{s,1}$	15	$c_{m,1}$	30
$c_{s,2}$	10	$c_{m,2}$	20
$\mu_{s,1}$	1.1	$\mu_{m,1}$	1,2,3,4
$\mu_{s,2}$	1.1	$\mu_{m,2}$	1,2,3,4
System Parameters		**Other Costs**	
B_{max}	5	$b_i, i = 1, 2$	40
$\lambda_i, i = 1, 2$	1.5	$h_i, i = 1, 2$	2

Case 1: Manufacturer is cheaper: The scenario where the unit production cost at the manufacturer is less than the unit production cost at the subcontractor, i.e. $\frac{c_{m,i}}{\mu_{m,i}} < \frac{c_{s,i}}{\mu_{s,i}}, i = 1, 2$ is analyzed by considering the case where $\mu_{m,1} = 3, 4$ and $\mu_{m,2} = 3, 4$. For $B_{max} = 10$ and $I_1 + I_2 = 0$, Figure 4.6 shows the impact of service rates on the optimal decision if the manufacturer is cheaper. We observe that the service rate of the component C_1 is non-increasing with increasing inventory position I_1. This implies that the optimal policy is of dual index type. For example, for $\mu_{m,1} = 3$ and $\mu_{m,2} = 3$, the optimal service rate of component C_1 at inventory position $I_1 = 1$ changes from 4.1 to 1.1 implying that only the subcontractor S_1 is producing component C_1 at $I_1 = 1$. Next, the component C_1 is neither produced by the manufacturer M nor by the subcontractor S_1 at inventory position $I_1 > 4$. We also observe that the increase in the service rate of the manufacturer results in (i) non-decreasing threshold to switch the service rate from $\mu_{m,1} + \mu_{s,1}$ to $\mu_{s,1}$, (ii) non-increasing threshold to switch the service rate from $\mu_{s,1}$ to 0. For example, one of the thresholds for scenario with

$\mu_{m,1} = 3, \mu_{m,2} = 4$ (see Figure 4.6(b)) is at $I_1 = 0$ and switches to $I_1 = 1$ as the service rate changes from $\mu_{m,1} = 3$ to $\mu_{m,1} = 4$.

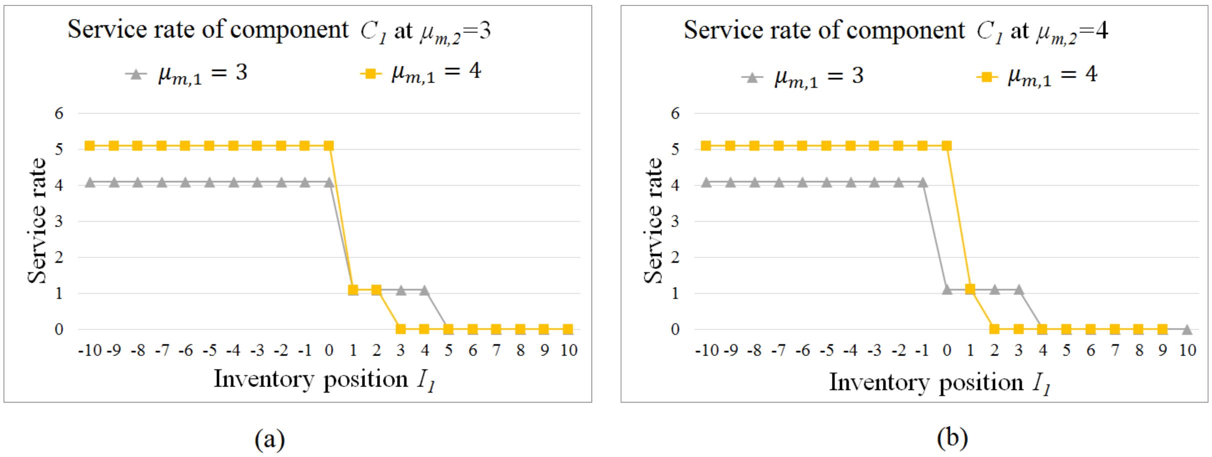

Figure 4.6 Optimal Service Rate of Component C_1 at (a) $\mu_{m,2} = 3$, (b) $\mu_{m,2} = 4$

Similarly, another threshold for scenario with $\mu_{m,1} = 3, \mu_{m,2} = 4$ is at $I_1 = 4$ and switches to $I_1 = 2$ as the service rate changes from $\mu_{m,1} = 3$ to $\mu_{m,1} = 4$. This happens because if $\mu_{m,1} = 4, \mu_{m,2} = 4$ then the component C_1 is being produced at a faster rate longer than compared to scenario where $\mu_{m,1} = 4, \mu_{m,2} = 3$, replenishing inventory at a faster rate. So, it is not optimal to produce the component longer at the expensive subcontractor.

Case 2: Subcontractor is cheaper: The scenario where the unit production cost at the manufacturer is more than the unit production cost at the subcontractor, i.e. $\frac{c_{m,i}}{\mu_{m,i}} > \frac{c_{s,i}}{\mu_{s,i}}, i = 1, 2$ is analyzed by considering the case where $\mu_{m,1} = 1, 2$ and $\mu_{m,2} = 1, 2$. For $B_{max} = 10$ and $I_1 + I_2 = 0$, Figure 4.7 shows the impact of service rates on the optimal decision if the subcontractor is cheaper. We observe that the service rate of the component C_1 is non-increasing with increasing inventory position I_1. This implies that the optimal policy is of dual index type. For example, for $\mu_{m,1} = 1$ and $\mu_{m,2} = 1$, the optimal service rate of component C_1 at inventory position $I_1 = 0$ changes from 2.1 to 1.1 implying that only the subcontractor S_1 is producing component C_1 at $I_1 = 0$.

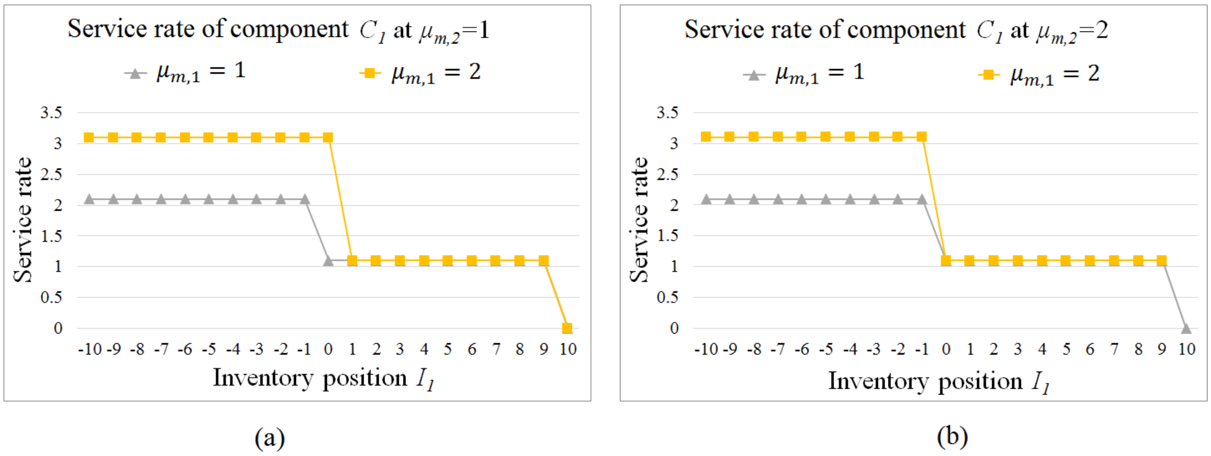

Figure 4.7 Optimal Service Rate of Component C_1 at (a) $\mu_{m,2} = 1$, (b) $\mu_{m,2} = 2$

Next, the component C_1 is neither produced by the manufacturer M nor by the subcontractor S_1 at inventory position $I_1 = 10$. We also observe that the increase in the service rate of the manufacturer results in non-decreasing thresholds. For example, the threshold switches from $I_1 = 0$ to $I_1 = 1$ when the service rate of the manufacturer for component C_1 changes from $\mu_{m,1} = 1$ to $\mu_{m,1} = 2$. We observe similar results when $\mu_{m,2} = 2$.

4.6 Conclusions

We consider a system with multiple components where components can be produced using capacity available at the shared manufacturing resource and using capacity available at dedicated subcontractors. We develop Markov decision process formulation of the system that captures subcontracting and scheduling of shared resource for multiple components, and provide insights on the optimal policies.

We analytically provide exhaustive sets of conditions to characterize the structure of the optimal policy and optimal value function. We show conditions based on first difference of value function and unit production cost under which the optimal action is that (i) a component is neither produced by the manufacturer nor by the subcontractor, (ii) a component is only produced by the manufacturer, (iii) a component is only produced by the corresponding subcontractor, (iv) a component is produced by both the manufacturer and corresponding subcontractor. For $M-S$ symmetric system, we show that it is never optimal to keep the manufacturer idle if subcontractors are producing their corresponding components. For complete symmetric system (with respect to cost and service rates), we show that the optimal policy is of dual index type, i.e. it suggests that either one of the components should be always produced at the fastest rate or none of the components should be produced.

Next, we consider three cases using simple conditions on the costs and service rate. Using these conditions we show that the multi-index policy is optimal under three cases. In the first case, if the unit production costs are negligible which is typical in manual assembly, we show that the optimal policy is of dual index type , i.e. there exists thresholds k_1 and $k_2, k_2 > k_1$ such that the component C_1 (i) is not manufactured if $I_1 > k_2$, or (ii) is only be produced by the corresponding subcontractor if $k_1 < I_1 < k_2$, or (iii) is produced at both the manufacturer and the corresponding subcontractor if $I_1 < k_1$. In the second case, if the components vary significantly in size and the unit production cost at the manufacturer is less than that of subcontractor, we show that multi-index type policy with three thresholds k_1, k_2, and $k_3, k_3 > k_2 > k_1$ is optimal, i.e. component C_1 (i) is not manufactured if $I_1 > k_3$, (ii) is only be produced by the corresponding subcontractor if $k_2 < I_1 < k_3$, (ii) is only be produced by the manufacturer if $k_1 < I_1 < k_2$, (iii) is produced at both the manufacturer and the corresponding subcontractor if $I_1 < k_1$. Finally, in the third case, if the components vary significantly in size and the unit production cost at the manufacturer is more than that of subcontractor, we again show that dual index policy is optimal.

In the next chapter, we use insights obtained in this chapter to derive optimal policies for assembly systems with multiple standard-type components.

4.7 Appendix

Proof of Theorem 4.1: We prove Theorem 4.1 in two parts. In the first part, we consider component C_2 and show that under conditions $(\Delta_2 > -\frac{c_{s,2}}{\mu_{s,2}\eta}) \wedge ((\Delta_2 > -\frac{c_{m,2}}{\mu_{m,2}\eta}) \vee (\mu_{m,2}\Delta_2 - \mu_{m,1}\Delta_1 > \frac{c_{m,1}-c_{m,2}}{\eta}))$, actions from the set $\{a_1, a_2, a_5, a_6, a_7, a_8, a_9, a_{10}\}$ have higher costs than that of at least one action from the set $\{a_3, a_4, a_{11}, a_{12}\}$. Since, the optimal action is from the set $\{a_3, a_4, a_{11}, a_{12}\}$, the result of Theorem 4.1, part (2) holds. Let $f_{j,t}(\sigma) = c(a_j) + \eta \sum_{\sigma'} p(\sigma'|\sigma, a_j)V_t(\sigma')$ denote the sum of total production cost rate and discounted cost at state σ after taking action a_j. Then, $f_{j,t}(\sigma)$ at each state $\sigma = (I_1, I_2)$ for each action a_j is defined as follows:

$$
\begin{aligned}
f_{1,t}(I_1,I_2) &= c_{m,1} + c_{s,1} + c_{s,2} + (\mu_{m,1} + \mu_{s,1})\eta V_t(I_1+1,I_2) + \mu_{s,2}\eta V_t(I_1,I_2+1) \\
&\quad + \lambda_1\eta V_t(I_1-1,I_2) + \lambda_2\eta V_t(I_1,I_2-1) + \mu_{m,2}\eta V_t(I_1,I_2) \\
f_{2,t}(I_1,I_2) &= c_{m,1} + c_{s,2} + \mu_{m,1}\eta V_t(I_1+1,I_2) + \mu_{s,2}\eta V_t(I_1,I_2+1) \\
&\quad + \lambda_1\eta V_t(I_1-1,I_2) + \lambda_2\eta V_t(I_1,I_2-1) + (\mu_{s,1} + \mu_{m,2})\eta V_t(I_1,I_2) \\
f_{3,t}(I_1,I_2) &= c_{m,1} + c_{s,1} + (\mu_{m,1} + \mu_{s,1})\eta V_t(I_1+1,I_2) \\
&\quad + \lambda_1\eta V_t(I_1-1,I_2) + \lambda_2\eta V_t(I_1,I_2-1) + (\mu_{m,2} + \mu_{s,2})\eta V_t(I_1,I_2) \\
f_{4,t}(I_1,I_2) &= c_{m,1} + \mu_{m,1}\eta V_t(I_1+1,I_2) \\
&\quad + \lambda_1\eta V_t(I_1-1,I_2) + \lambda_2\eta V_t(I_1,I_2-1) + (\mu_{s,1} + \mu_{m,2} + \mu_{s,2})\eta V_t(I_1,I_2) \\
f_{5,t}(I_1,I_2) &= c_{m,2} + c_{s,1} + c_{s,2} + \mu_{s,1}\eta V_t(I_1+1,I_2) + (\mu_{m,2} + \mu_{s,2})\eta V_t(I_1,I_2+1) \\
&\quad + \lambda_1\eta V_t(I_1-1,I_2) + \lambda_2\eta V_t(I_1,I_2-1) + \mu_{m,1}\eta V_t(I_1,I_2) \\
f_{6,t}(I_1,I_2) &= c_{m,2} + c_{s,2} + (\mu_{m,2} + \mu_{s,2})\eta V_t(I_1,I_2+1) \\
&\quad + \lambda_1\eta V_t(I_1-1,I_2) + \lambda_2\eta V_t(I_1,I_2-1) + (\mu_{m,1} + \mu_{s,1})\eta V_t(I_1,I_2) \\
f_{7,t}(I_1,I_2) &= c_{s,1} + c_{m,2} + \mu_{s,1}\eta V_t(I_1+1,I_2) + \mu_{m,2}\eta V_t(I_1,I_2+1) \\
&\quad + \lambda_1\eta V_t(I_1-1,I_2) + \lambda_2\eta V_t(I_1,I_2-1) + (\mu_{m,1} + \mu_{s,2})\eta V_t(I_1,I_2) \\
f_{8,t}(I_1,I_2) &= c_{m,2} + \mu_{m,2}\eta V_t(I_1,I_2+1) \\
&\quad + \lambda_1\eta V_t(I_1-1,I_2) + \lambda_2\eta V_t(I_1,I_2-1) + (\mu_{m,1} + \mu_{s,1} + \mu_{s,2})\eta V_t(I_1,I_2)
\end{aligned}
$$

$$
\begin{aligned}
f_{9,t}(I_1,I_2) &= c_{s,1}+c_{s,2}+\mu_{s,1}\eta V_t(I_1+1,I_2)+\mu_{s,2}\eta V_t(I_1,I_2+1)\\
&\quad+\lambda_1\eta V_t(I_1-1,I_2)+\lambda_2\eta V_t(I_1,I_2-1)+(\mu_{m,1}+\mu_{m,2})\eta V_t(I_1,I_2)\\
f_{10,t}(I_1,I_2) &= c_{s,2}+\mu_{s,2}\eta V_t(I_1,I_2+1)\\
&\quad+\lambda_1\eta V_t(I_1-1,I_2)+\lambda_2\eta V_t(I_1,I_2-1)+(\mu_{m,1}+\mu_{m,2}+\mu_{s,1})\eta V_t(I_1,I_2)\\
f_{11,t}(I_1,I_2) &= c_{s,1}+\mu_{s,1}\eta V_t(I_1+1,I_2)\\
&\quad+\lambda_1\eta V_t(I_1-1,I_2)+\lambda_2\eta V_t(I_1,I_2-1)+(\mu_{m,1}+\mu_{m,2}+\mu_{s,2})\eta V_t(I_1,I_2)\\
f_{12,t}(I_1,I_2) &= \lambda_1\eta V_t(I_1-1,I_2)+\lambda_2\eta V_t(I_1,I_2-1)\\
&\quad+(\mu_{m,1}+\mu_{m,2}+\mu_{s,1}+\mu_{s,2})\eta V_t(I_1,I_2) \qquad (4.2)
\end{aligned}
$$

So, Equation (4.1) can rewritten as:

$$
V_t(I_1,I_2) = h(\sigma)+b(\sigma)+\min_{a_j\in\mathbb{A}}[f_{j,t}(I_1,I_2)] \qquad (4.3)
$$

At first, we show that if $\Delta_2 > -\frac{c_{s,2}}{\mu_{s,2}\eta}$, then actions from the set $\{a_1,a_2,a_5,a_9,a_{10}\}$ have higher costs than that of at least one action from the set $\{a_3,a_4,a_{11},a_{12}\}$. To show that a_1 is not optimal, we show that $f_{1,t}(I_1,I_2)-f_{3,t}(I_1,I_2)>0$.

$$
\begin{aligned}
f_{1,t}(I_1,I_2)-f_{3,t}(I_1,I_2) &= c_{m,1}+c_{s,1}+c_{s,2}+(\mu_{m,1}+\mu_{s,1})\eta\Delta_1+\mu_{s,2}\eta\Delta_2\\
&\quad-(c_{m,1}+c_{s,1}+(\mu_{m,1}+\mu_{s,1})\eta\Delta_1(I_1,I_2))\\
&= c_{s,2}+\mu_{s,2}\eta\Delta_2 \qquad (4.4)
\end{aligned}
$$

If $\Delta_2 > -\frac{c_{s,2}}{\mu_{s,2}\eta}$, then from Equation (4.4) $f_{1,t}(I_1,I_2)-f_{3,t}(I_1,I_2)>0$. So, action a_1 results in higher cost than actions a_3. To show that a_2 is not optimal, we show that $f_{2,t}(I_1,I_2)-f_{4,t}(I_1,I_2)>0$.

$$
\begin{aligned}
f_{2,t}(I_1,I_2)-f_{4,t}(I_1,I_2) &= c_{m,1}+c_{s,2}+\mu_{m,1}\eta\Delta_1(I_1,I_2)+\mu_{s,2}\eta\Delta_2(I_1,I_2)\\
&\quad-(c_{m,1}+\mu_{m,1}\eta\Delta_1(I_1,I_2))\\
&= c_{s,2}+\mu_{s,2}\eta\Delta_2 \qquad (4.5)
\end{aligned}
$$

If $\Delta_2 > -\frac{c_{s,2}}{\mu_{s,2}\eta}$, then from Equation (4.5) $f_{2,t}(I_1,I_2)-f_{4,t}(I_1,I_2)>0$. So, action a_2 results in higher cost than actions a_4. Similarly, if $\Delta_2 > -\frac{c_{s,2}}{\mu_{s,2}\eta}$, we can show that action a_9 results in higher cost than action a_{11}, and action a_{10} results in higher cost than action a_{12}.

Next, we show that actions from the set $\{a_5, a_6, a_7, a_8\}$ results in higher cost than at least one action from the set $\{a_3, a_4, a_9, a_{10}, a_{11}, a_{12}\}$. So, actions from the set $\{a_5, a_6, a_7, a_8\}$ could have higher costs than at least one action from the set $\{a_3, a_4\}$ or could have higher cost than at least one action from the set $\{a_9, a_{10}, a_{11}, a_{12}\}$.

To show that a_7 is not optimal, we show that $f_{7,t}(I_1, I_2) - f_{3,t}(I_1, I_2) > 0$.

$$\begin{aligned} f_{7,t}(I_1, I_2) - f_{3,t}(I_1, I_2) &= c_{m,2} + c_{s,1} + \mu_{s,1}\eta\Delta_1 + \mu_{m,2}\eta\Delta_2 \\ &\quad -(c_{m,1} + c_{s,1} + (\mu_{m,1} + \mu_{s,1})\eta\Delta_1) \\ &= c_{m,2} - c_{m,1} + \mu_{m,2}\eta\Delta_2 - \mu_{m,1}\eta\Delta_1 \end{aligned} \tag{4.6}$$

If $\mu_{m,2}\Delta_2 - \mu_{m,1}\Delta_1 > \frac{c_{m,1}-c_{m,2}}{\eta}$, then from Equation (4.6) $f_{7,t}(I_1, I_2) - f_{3,t}(I_1, I_2) > 0$. So, action a_7 results in higher cost than actions a_3.

To show that a_8 is not optimal, we show that $f_{8,t}(I_1, I_2) - f_{4,t}(I_1, I_2) > 0$.

$$f_{8,t}(I_1, I_2) - f_{4,t}(I_1, I_2) = c_{m,2} + \mu_{m,2}\eta\Delta_2 - (c_{m,1} + \mu_{m,1}\eta\Delta_1) \tag{4.7}$$

If $\mu_{m,2}\Delta_2 - \mu_{m,1}\Delta_1 > \frac{c_{m,1}-c_{m,2}}{\eta}$, then from Equation (4.7) $f_{8,t}(I_1, I_2) - f_{4,t}(I_1, I_2) > 0$. So, action a_8 results in higher cost than actions a_4. Similarly, if $\mu_{m,2}\Delta_2 - \mu_{m,1}\Delta_1 > \frac{c_{m,1}-c_{m,2}}{\eta}$, we can show that action a_5 results in higher cost than action a_3, and action a_6 results in higher cost than action a_4.

To show that a_5 is not optimal, we show that $f_{5,t}(I_1, I_2) - f_{9,t}(I_1, I_2) > 0$.

$$\begin{aligned} f_{5,t}(I_1, I_2) - f_{9,t}(I_1, I_2) &= c_{m,2} + c_{s,1} + c_{s,2} + \mu_{s,1}\eta\Delta_1 + (\mu_{m,2} + \mu_{s,2})\eta\Delta_2 \\ &\quad -(c_{s,1} + c_{s,2} + \mu_{s,1}\eta\Delta_1 + \mu_{s,2}\eta\Delta_2) \\ &= c_{m,2} + \mu_{m,2}\eta\Delta_2 \end{aligned} \tag{4.8}$$

If $\Delta_2 > -\frac{c_{m,2}}{\mu_{m,2}\eta}$, then from Equation (4.8) $f_{5,t}(I_1, I_2) - f_{9,t}(I_1, I_2) > 0$. So, action a_5 results in higher cost than actions a_9.

To show that a_6 is not optimal, we show that $f_{6,t}(I_1, I_2) - f_{10,t}(I_1, I_2) > 0$.

$$f_{5,t}(I_1, I_2) - f_{9,t}(I_1, I_2) \quad = \quad c_{m,2} + c_{s,2} + (\mu_{m,2} + \mu_{s,2})\eta\Delta_2 - (c_{s,2} + \mu_{s,2}\eta\Delta_2) \qquad (4.9)$$

If $\Delta_2 > -\frac{c_{m,2}}{\mu_{m,2}\eta}$, then from Equation (4.9) $f_{6,t}(I_1, I_2) - f_{10,t}(I_1, I_2) > 0$. So, action a_6 results in higher cost than actions a_{10}. Similarly, if $\Delta_2 > -\frac{c_{m,2}}{\mu_{m,2}\eta}$, we can show that action a_7 results in higher cost than action a_{11}, and action a_8 results in higher cost than action a_{12}. This concludes the proof.

Proof of Theorem 4.2: We prove Theorem 4.2 in two parts. In the first part, we consider component C_2 and show that under conditions $(\Delta_2 < -\frac{c_{s,2}}{\mu_{s,2}\eta}) \wedge ((\Delta_2 > -\frac{c_{m,2}}{\mu_{m,2}\eta}) \vee (\mu_{m,2}\Delta_2 - \mu_{m,1}\Delta_1 > \frac{c_{m,1}-c_{m,2}}{\eta}))$, actions from the set $\{a_3, a_4, a_5, a_6, a_7, a_8, a_{11}, a_{12}\}$ have higher costs than that of at least one action from the set $\{a_1, a_2, a_9, a_{10}\}$. Since, the optimal action is from the set $\{a_1, a_2, a_9, a_{10}\}$, the result of Theorem 4.2, part (2) holds. Using logic used in the proof of Theorem 4.1, we can show that if $\Delta_2 < -\frac{c_{s,2}}{\mu_{s,2}\eta}$, action a_3 results in higher cost than action a_1, action a_4 results in higher cost than action a_2, action a_{11} results in higher cost than action a_9, and action a_{12} results in higher cost than action a_{10}. Again, we show that if $\Delta_2 > -\frac{c_{m,2}}{\mu_{m,2}\eta}$, action a_5 results in higher cost than action a_9, action a_6 results in higher cost than action a_{10}, action a_7 results in higher cost than action a_{11}, and action a_8 results in higher cost than action a_{12}. If $\mu_{m,2}\Delta_2 - \mu_{m,1}\Delta_1 > \frac{c_{m,1}-c_{m,2}}{\eta}$, action a_5 results in higher cost than action a_1, action a_6 results in higher cost than action a_2, action a_7 results in higher cost than action a_3, action a_8 results in higher cost than action a_4. This concludes the proof.

Proof of Theorem 4.3: We prove Theorem 4.3 in two parts. In the first part, we consider component C_2 and show that under conditions $(\Delta_2 > -\frac{c_{s,2}}{\mu_{s,2}\eta}) \wedge (\Delta_2 < -\frac{c_{m,2}}{\mu_{m,2}\eta}) \wedge (\mu_{m,2}\Delta_2 -$

$\mu_{m,1}\Delta_1 < \frac{c_{m,1}-c_{m,2}}{\eta}$), any actions in the set $\{a_1, a_2, a_3, a_4, a_5, a_6, a_9, a_{10}, a_{11}, a_{12}\}$ have higher costs than that of at least one action in the set $\{a_7, a_8\}$. Since, the optimal action is from the set $\{a_7, a_8\}$, the result of Theorem 4.3, part (2) holds. Using logic used in the proof of Theorem 4.1, we can show that if $\Delta_2 > -\frac{c_{s,2}}{\mu_{s,2}\eta}$, action a_5 results in higher cost than action a_7, action a_6 results in higher cost than action a_8. Again, we show that if $\Delta_2 < -\frac{c_{m,2}}{\mu_{m,2}\eta}$, action a_9 results in higher cost than action a_5, action a_{10} results in higher cost than action a_6, action a_{11} results in higher cost than action a_7, and action a_{12} results in higher cost than action a_8. Finally, if $\mu_{m,2}\Delta_2 - \mu_{m,1}\Delta_1 < \frac{c_{m,1}-c_{m,2}}{\eta}$, action a_3 results in higher cost than action a_7, action a_4 results in higher cost than action a_8, action a_1 results in higher cost than action a_5, and action a_2 results in higher cost than action a_6. This concludes the proof.

Proof of Theorem 4.4: We prove Theorem 4.4 in two parts. In the first part, we consider component C_2 and show that under conditions $(\Delta_2 < -\frac{c_{m,2}}{\mu_{m,2}\eta}) \wedge (\Delta_2 < -\frac{c_{s,2}}{\mu_{s,2}\eta}) \wedge (\mu_{m,2}\Delta_2 - \mu_{m,1}\Delta_1 < \frac{c_{m,1}-c_{m,2}}{\eta})$, actions from the set $\{a_1, a_2, a_3, a_4, a_7, a_8, a_9, a_{10}, a_{11}, a_{12}\}$ have higher costs than that of at least one action from the set $\{a_5, a_6\}$. Since, the optimal action is from the set $\{a_5, a_6\}$, the result of Theorem 4.4, part (2) holds. Using logic used in the proof of Theorem 4.1, we can show that if $\Delta_2 < -\frac{c_{s,2}}{\mu_{s,2}\eta}$, action a_3 results in higher cost than action a_1, action a_4 results in higher cost than action a_2, action a_7 results in higher cost than action a_5, and action a_8 results in higher cost than action a_6. Again, we show that if $\Delta_2 < -\frac{c_{m,2}}{\mu_{m,2}\eta}$, action a_9 results in higher cost than action a_5, action a_{10} results in higher cost than action a_6, action a_{11} results in higher cost than action a_7, and action a_{12} results in higher cost than action a_8. Finally, if $\mu_{m,2}\Delta_2 - \mu_{m,1}\Delta_1 < \frac{c_{m,1}-c_{m,2}}{\eta}$, action a_1 results in higher cost than action a_5, and action a_2 results in higher cost than action a_6. This concludes the proof.

Proof of Proposition 4.1: We prove this proposition in two steps. First, we show that the cost function $r(\sigma, a_j)$ is convex over $\mathbb{S}$, and then we show the required properties of the transition probabilities. Note that $h_1 \max(I_1, 0)$ is non-decreasing with I_1, and $b_1 \max(-I_1, 0)$ is non-increasing with I_1. Thus, $h_1 \max(I_1, 0) + b_1 \max(-I_1, 0)$ is convex with respect to I_1.

Next, since $I_1+I_2 = K$, $h_2 \max(K-I_1, 0)$ is non-increasing with I_1, and $b_2 \max(-K+I_1, 0)$ is non-decreasing with I_1. Thus, $h_2 \max(K-I_1, 0)+b_2 \max(-K+I_1, 0)$ is convex with respect to I_1 and since $(h_1 \max(I_1, 0) + b_1 \max(-I_1, 0)) + (h_2 \max(K - I_1, 0) + b_2 \max(-K + I_1, 0))$ is a sum of convex functions, it is also convex. Finally, since the total production cost rate, $c(a_j)$ at action a_j does not depend on I_1, the cost function $r(\sigma, a) = h(\sigma) + b(\sigma) + c(a_j)$ is convex with respect to I_1.

Next, we show that $q((I''_1, I''_2)|(I_1, I_2), a_j)$ is non-decreasing over over $\mathbb{S}$, where $q((I''_1, I''_2)|(I_1, I_2), a_j) = \sum_{I'_1=I''_1}^{\infty} p((I'_1, I'_2)|(I_1, I_2), a_j)$, $(I'_1, I'_2) \in \mathbb{S}$, and $(I''_1, I''_2) \in \mathbb{S}$. From the definition of transition probabilities, we have

$$
\begin{aligned}
&q((I''_1, I''_2)|(I_1, I_2), a_1) = \mu_{m,2}/\nu, \text{ if } I''_1 \leq I_1, \text{ and 0 otherwise.}\\
&q((I''_1, I''_2)|(I_1, I_2), a_2) = (\mu_{m,2} + \mu_{s,1})/\nu, \text{ if } I''_1 \leq I_1, \text{ and 0 otherwise.}\\
&q((I''_1, I''_2)|(I_1, I_2), a_3) = (\mu_{m,2} + \mu_{s,2})/\nu, \text{ if } I''_1 \leq I_1, \text{ and 0 otherwise.}\\
&q((I''_1, I''_2)|(I_1, I_2), a_4) = (\mu_{m,2} + \mu_{s,1} + \mu_{s,2})/\nu, \text{ if } I''_1 \leq I_1, \text{ and 0 otherwise.}\\
&q((I''_1, I''_2)|(I_1, I_2), a_5) = \mu_{m,1}/\nu, \text{ if } I''_1 \leq I_1, \text{ and 0 otherwise.}\\
&q((I''_1, I''_2)|(I_1, I_2), a_6) = (\mu_{m,1} + \mu_{s,1})/\nu, \text{ if } I''_1 \leq I_1, \text{ and 0 otherwise.}\\
&q((I''_1, I''_2)|(I_1, I_2), a_7) = (\mu_{m,1} + \mu_{s,2})/\nu, \text{ if } I''_1 \leq I_1, \text{ and 0 otherwise.}\\
&q((I''_1, I''_2)|(I_1, I_2), a_8) = (\mu_{m,1} + \mu_{s,1} + \mu_{s,2})/\nu, \text{ if } I''_1 \leq I_1, \text{ and 0 otherwise.}\\
&q((I''_1, I''_2)|(I_1, I_2), a_9) = (\mu_{m,1} + \mu_{m,2})/\nu, \text{ if } I''_1 \leq I_1, \text{ and 0 otherwise.}\\
&q((I''_1, I''_2)|(I_1, I_2), a_{10}) = (\mu_{m,1} + \mu_{m,2} + \mu_{s,1})/\nu, \text{ if } I''_1 \leq I_1, \text{ and 0 otherwise.}\\
&q((I''_1, I''_2)|(I_1, I_2), a_{11}) = (\mu_{m,1} + \mu_{m,2} + \mu_{s,2})/\nu, \text{ if } I''_1 \leq I_1, \text{ and 0 otherwise.}\\
&q((I''_1, I''_2)|(I_1, I_2), a_{12}) = (\mu_{m,1} + \mu_{m,2} + \mu_{s,1} + \mu_{s,2})/\nu, \text{ if } I''_1 \leq I_1, \text{ and 0 otherwise.} \qquad (4.10)
\end{aligned}
$$

For for any state $(I_1 + d, I_2 - d) \in \mathbb{S}$ with discrete $d = 1, 2, ...$, $p((I_1 + d, I_2 - d)|(I_1, I_2), a_j) = 0, \forall a_j$ because each transition impacts at most inventory position I_1 or I_2 but not both. Thus, $q((I''_1, I''_2)|(I_1, I_2), a_j) \geq 0$ if $I''_1 \leq I_1$ and 0 otherwise. This implies that $q((I''_1, I''_2)|(I_1, I_2), a_j)$ is non-decreasing in I_1. Since, $r(\sigma, a_j)$ is convex over $\mathbb{S}$ $\forall a_j$, and $q((I''_1, I''_2)|(I_1, I_2), a_j)$ is non-decreasing over $\mathbb{S}$ $\forall a_j$, from Proposition 4.7.3 from Puterman (1994), it follows that the optimal value function $V^*(I_1, I_2)$ is convex over $\mathbb{S}$. This concludes the proof.

Proof of Theorem 4.6: We prove this theorem in two parts. At first, we consider the case where $V^*(I_1, I_2)$ is non-decreasing with respect to I_1 and show that the optimal actions are monotone with respect to I_1. Next, we we consider the case where $V^*(I_1, I_2)$ is non-decreasing with respect to I_2 and show that the optimal actions are monotone with respect to I_2.

If $c_{m,i} = 0, c_{s,i} = 0, i = 1, 2$, then using Theorem 4.3, action a_2 is optimal if $\Delta_2 < 0$ and $\Delta_2 > 0$, which is not feasible. So, action a_2 is not optimal. Similarly, we can show using Theorem 4.1 - 4.4 that actions from the set $\{a_2, a_4, a_7, a_8, a_9, a_{10}, a_{11}\}$ are not optimal. Next, if $V^*(I_1, I_2)$ is non-decreasing with respect to $I_1, (I_1, I_2) \in \mathbb{S}$, then $\Delta_1 - \Delta_2 > 0$. From Figure 4.4, if action a_3 is optimal then $\Delta_1 < 0$ and $\Delta_2 > 0$. This implies that $\Delta_1 - \Delta_2 < 0$. This contradicts our assumption and therefore action a_3 is not optimal. Next, using definition of $q((I_1'', I_2'')|(I_1, I_2), a_j)$, if $\mu_{m,1} > \mu_{m,2}$ then $q((I_1'', I_2'')|(I_1, I_2), a_1) < q((I_1'', I_2'')|(I_1, I_2), a_5)$. Again from Figure 4.4, if $\mu_{m,1} > \mu_{m,2}$ then action a_1 is not optimal. So, for non-increasing sequence of service rate for component C_1 that are defined by the sequence $\{a_5, a_6, a_{12}\}$, $q(\sigma''|\sigma, a_j)$ is sub-additive with respect to non-decreasing I_1 and action space $\mathbb{A}$. Similarly, we consider $V^*(I_1, I_2)$ to be non-decreasing with respect to $I_2, (I_1, I_2) \in \mathbb{S}$, and prove that the optimal actions are monotone with respect to I_2 for the other half of the state space. This concludes the proof.

Proof of Theorem 4.7: We prove this theorem in two parts. At first, we consider the case where $V^*(I_1, I_2)$ is non-decreasing with respect to I_1 and show that the optimal actions are monotone with respect to I_1. Next, we we consider the case where $V^*(I_1, I_2)$ is non-decreasing with respect to I_2 and show that the optimal actions are monotone with respect to I_2. If $(I_1 + I_2) = K$ and $I_{1,} > \max(0, K)$, then $h_1 \max(I_1, 0)$ is non-decreasing with I_1, $b_1 \max(-I_1, 0)$ is zero, $h_2 \max(K - I_1, 0)$ is zero, and $b_2 \max(-K + I_1, 0)$ is non-decreasing with I_1. Thus, $r(I_1, I_2)$ is non-decreasing for $I_{1,} > \max(0, K)$. Also, from Proposition 4.1, $q((I_1'', I_2'')|(I_1, I_2), a_j)$ is non-decreasing in I_1. Thus, $V^*(I_1, I_2)$ is non-decreasing over I_1.

Next, using Theorem 4.1, 4.2, 4.3, and 4.4, we observe that if $\frac{c_{m,i}}{\mu_{m,i}} < \frac{c_{m,i}}{\mu_{m,i}}, i = 1, 2$, then any action in the set $\{a_9, a_{10}, a_{11}\}$ is not optimal. This implies that optimal action belongs to the set $\{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_{12}\}$. Next, using definition of $q((I_1'', I_2'')|(I_1, I_2), a_j)$, if $\mu_{s,1} > \mu_{s,2}$ then $q((I_1'', I_2'')|(I_1, I_2), a_3) < q((I_1'', I_2'')|(I_1, I_2), a_2)$, if $\mu_{m,1} > \mu_{m,2} + \mu_{s,1} + \mu_{s,2}$ then $q((I_1'', I_2'')|(I_1, I_2), a_4) < q((I_1'', I_2'')|(I_1, I_2), a_5)$. So, for non-increasing sequence of service rate for component C_1 that are defined by the sequence $\{a_1, a_3, a_2, a_4, a_5, a_7, a_6, a_8, a_{12}\}$, $q(\sigma''|\sigma, a_j)$ is sub-additive with respect to non-decreasing I_1 and action space $\mathbb{A}$. Thus, from Theorem 4.7.4 of Puterman (1994), we can conclude that the optimal action a^* is non-increasing in service rates for component C_1 with respect to increasing I_1, i.e. there exists thresholds k_1, k_2, and $k_3, k_3 > k_2 > k_1$ such that the component C_1 (i) is not manufactured if $I_1 > k_3$, (ii) is only be produced by the corresponding subcontractor if $k_2 < I_1 < k_3$, (ii) is only be produced by the manufacturer if $k_1 < I_1 < k_2$, (iii) is produced at the manufacturer and the corresponding subcontractor if $I_1 < k_1$.

Proof of Theorem 4.8: We prove this theorem in three steps. First, we show that the optimal value function, $V^*(I_1, I_2)$ is non-decreasing over $I_1, I_1 > K$. Next, we show that if $\frac{c_{m,i}}{\mu_{m,i}} < \frac{c_{s,i}}{\mu_{s,i}}, i = 1, 2$ then using Corollary 4.1, actions from the set $\{a_2, a_4, a_7, a_8\}$ are not optimal. Finally, we show that $q((I_1'', I_2'')|(I_1, I_2), a_j)$ sub-additive with respect to non-decreasing I_1 and action $\mathbb{A}$. In Theorem 4.7, we show that $V^*(I_1, I_2)$ is non-decreasing over $I_1, (I_1, I_2) \in \mathbb{S}$. Next, using Corollary 4.1, we observe that if $\frac{c_{m,i}}{\mu_{m,i}} > \frac{c_{m,i}}{\mu_{m,i}}, i = 1, 2$, then actions from the set $\{a_2, a_4, a_7, a_8\}$ is not optimal. This implies that optimal action belongs to the set $\{a_1, a_3, a_5, a_6, a_9, a_{10}, a_{11}, a_{12}\}$. Next, using definition of $q((I_1'', I_2'')|(I_1, I_2), a_j)$, if $\mu_{m,1} > \mu m, 2 + \mu_{s,1}$ then $q((I_1'', I_2'')|(I_1, I_2), a_3) < q((I_1'', I_2'')|(I_1, I_2), a_5)$, if $\mu_{s,1} > \mu_{s,2} + \mu_{m,2} + \mu_{s,2}$ then $q((I_1'', I_2'')|(I_1, I_2), a_{11}) < q((I_1'', I_2'')|(I_1, I_2), a_6)$. So, for non-increasing sequence of service rate for component C_1 that are defined by the sequence $\{a_1, a_3, a_5, a_9, a_{11}, a_6, a_{10}, a_{12}\}$, $q(\sigma''|\sigma, a_j)$

is sub-additive with respect to non-decreasing I_1 and action space $\mathbb{A}$. Thus, from Theorem 4.7.4 of Puterman (1994), we can conclude that the optimal action a^* is non-increasing in service rates for component C_1 with respect to increasing I_1, i.e. there exists thresholds k_1 and $k_2, k_2 > k_1$ such that the component C_1 (i) is not manufactured if $I_1 > k_2$, or (ii) is only be produced by the corresponding subcontractor if $k_1 < I_1 < k_2$, or (iii) is produced at the manufacturer and the corresponding subcontractor if $I_1 < k_1$.

Chapter 5

Assembly Systems with Multiple Standard-type Components

5.1 Introduction

In this chapter, we model a multi-product system where products are assembled from its respective components. The components are made to stock with inventory being replenished from both the subcontractor and the in-house manufacturing facility. The subcontractor and the manufacturing facility have finite production capacity and stochastic lead times. However, the manufacturing facility is shared across components needed for multiple products. Further, both the subcontractor and the manufacturing facility have stochastic lead times.

In practice, multiple components might require similar operations on a specialized equipment. However, such equipment might require high capital investment. This prevents the manufacturer from dedicating this equipment to a single product and often the available capacity is shared across multiple products and components. Sometimes, the manufacturer could subcontract the production of these components to external subcontractor to meet expectations of high demands. In this context, we assume the following two research questions in this chapter: (i) What is the structure of the optimal policy for the manufacturer? (ii) How should the components be scheduled on shared resources?

We analyze a system with multiple products using stochastic models. However, the state space of multi-product system requires us to keep track of stocking levels of each component. In addition, action space involves decisions for manufacturing and each supplier. Thus, deriving the monotonicity results for optimal policy is non-trivial. This poses additional research challenges related to the state space complexity associated with determining optimal policies for multi-product systems.

We use efficient action elimination techniques to determine the exact solution of the multi-product system for small sized problems. Additionally, we propose a fairly accurate approach that combines decomposition and Markov decision process (MDP) to address larger sized problems. In this approach, we first decompose the system with multiple products into two equivalent subsystems that involve a component for each end product. Next, using an iterative approach, we determine the optimal production and subcontracting policy for the original system.

The rest of the chapter is organized as follows. Section 5.2 describes the model of the system with multiple products. Section 5.3 describes the MDP formulations of the multi-product system. Section 5.4 provide structural results on the optimal policy for a subsystem. Section 5.5 presents a decomposition based approach to solve multi-product system. Section 5.6 provides numerical studies to validate the structure of the optimal policy. Finally, Section 5.7 summarizes model insights and conclusions.

5.2 System Model

We analyze a manufacturing system that assembles two products $i = 1, 2$, each from its respective two components $C_{ij}, j = 1, 2$ as shown in Figure 5.1. We assume that each assembly needs one unit of each component. For example, the products $i = 1, 2$ could correspond to transmissions (large and small) that are assembled from gears and housings (large and small respectively). In Figure 5.1, L_{ij} represents the storage location of component $C_{ij}, j = 1, 2;$ of product $i, i = 1, 2$. Product i is assembled at assembly station S_i and we assume that assembly time for product i at station S_i is negligible. The demand for product i is assumed to be a Poisson process $N_i(t), t \geq 0$ with rate λ_i. If at the arrival epoch of the demand for product i, both components $C_{ij}, j = 1, 2;$ are available, then the demand for product i is immediately satisfied. If one or more components required for product i is unavailable, then the demand for product i is backordered.

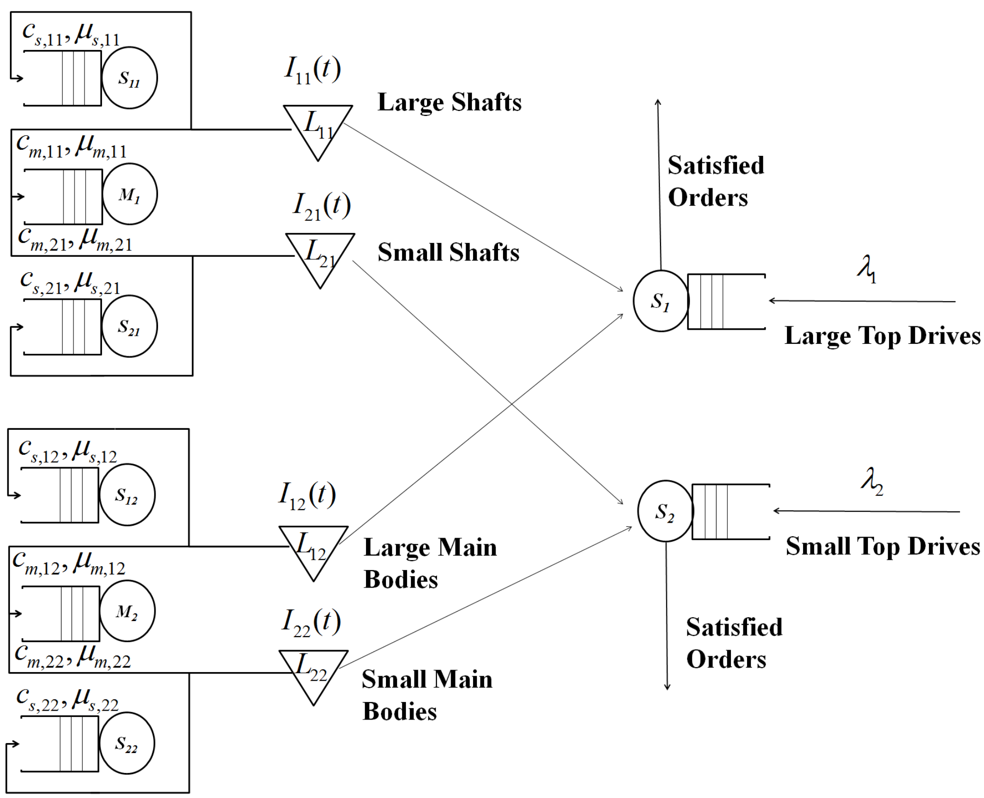

Figure 5.1 Supply Chain Model for System with Two Products

Let $I_{ij}(t)$ denote the net inventory for component C_{ij} which is defined as on-hand inventory minus backorders of component C_{ij} at time t. The manufacturer has the option to replenish inventory for component j using available capacity at the external subcontractor, S_{ij} or use capacity from the in-house manufacturing facility M_j that can be used to manufacturer either component C_{1j} or C_{2j}.

We model the external subcontractor, S_{ij} and the internal manufacturing facility M_j as single server queues with exponential service time with mean $\mu_{s,ij}^{-1}$ and $\mu_{m,ij}^{-1}$ respectively and $c_{s,ij}$ and $c_{m,ij}$ denote the unit cost rate to manufacture component C_{ij} at the external subcontractor and in-house manufacturing facility receptively. We let h_{ij} denote the unit inventory holding cost rate for component C_{ij} and b_i denote the unit backordering cost rate of product i. Next section presents the Markov decision process formulation to determine the optimal production and subcontracting decisions.

5.3 Markov Decision Process Formulation

We develop a continuous-time Markov chain to capture the dynamics of the system. For analysis purposes, we define two subsystems $\chi_j, j = 1, 2$ as subsystem j that corresponds to the manufacturing of components C_{1j} and C_{2j} (see Figure 5.2). This subsystem includes external subcontractors S_{1j} and S_{2j}, and in-house manufacturing department M_j. Note that this subsystem has similarities to the system analyzed in Chapter 4.

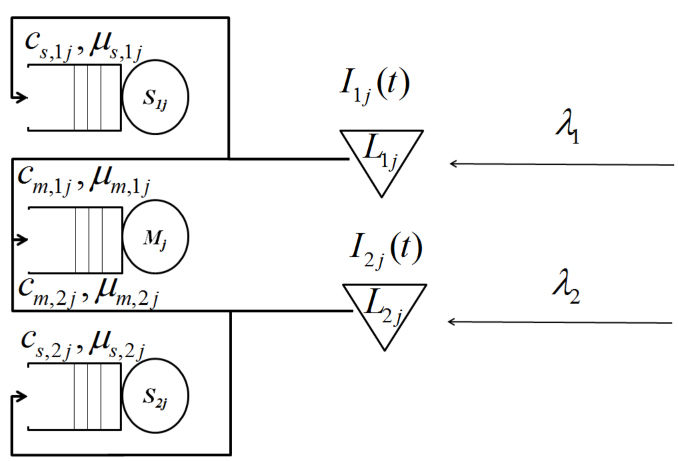

Figure 5.2 Subsystem $\chi_j, j = 1, 2$

Following are the key elements in the infinite horizon Markov decision process formulation:

Decision epoch: In the proposed system, actions are taken at a state change, i.e upon demand arrival and service completion epochs.

State space, Σ: The state of the system can be completely described with a 4-tupled state space with state $\sigma = (I_{11}, I_{21}, I_{12}, I_{22}), \sigma \in \Sigma$ where I_{ij} is the net inventory position of component C_{ij}.

Action space, $\mathbb{A}$: We define the action space, $\mathbb{A} = \mathbb{A}_1 \times \mathbb{A}_2$ where $\mathbb{A}_j, j = 1, 2$; represents the set of actions for subsystem χ_j with $a_{j,k_j} = (m_j, s_{1j}, s_{2j}), a_{j,k_j} \in \mathbb{A}_j, k_j = 1, ..., 12$. Here, m_j takes the value i if action corresponds to manufacturing of component C_{ij} at the in-house manufacturing facility M_j and takes the value 0 the action corresponds to being idle. Similarly, $s_{ij}, i = 1, 2$ takes the value i when the action corresponds to manufacturing of component C_{ij} at the external supplier S_{ij} and takes the value 0 if the action corresponds to being idle. Table 5.1 defines the 12 possible action available for subsystem χ_j. Note that since $\mathbb{A} = \mathbb{A}_1 \times \mathbb{A}_2$, there are 144 potential actions $a_k = (a_{1,k_1}, a_{2,k_2}) \in \mathbb{A}$ to choose from at each state.

Transition probabilities: Define $p(\sigma'|\sigma, a_{1,k_1}, a_{2,k_2})$ as the transition probability for any state $\sigma = (I_{11}, I_{21}, I_{12}, I_{22})$ to state $\sigma' = (I'_{11}, I'_{21}, I'_{12}, I'_{22})$ under actions $a_{1,k_1} \in \mathbb{A}_1, a_{2,k_2} \in \mathbb{A}_2$. We define $\nu = \sum_{i=1}^{2} \lambda_i + \sum_{i=1}^{2} \sum_{j=1}^{2} (\mu_{m,ij} + \mu_{s,ij}) + A$ as the normalizing factor for the uniformization technique described in Lippman (1975). Then the transition probabilities $p(\sigma'|\sigma, a_{1,k_1}, a_{2,k_2})$ are defined as follows:

Demand arrival for product i: Then $I'_{ij} = I_{ij} - 1, j = 1, 2$; and the corresponding transition probability $p(\sigma'|\sigma, a_{1,k_1}, a_{2,k_2})$ is given by:

$$p(\sigma'|\sigma, a_{1,k_1}, a_{2,k_2}) = \lambda_i/\nu, \forall i = 1, 2$$

Table 5.1 Action Space for Subsystem χ_j

$\mathbb{A}_j$	M_j	S_{1j}	S_{2j}
$a_{j,1}$	1	1	2
$a_{j,2}$	1	0	2
$a_{j,3}$	1	1	0
$a_{j,4}$	1	0	0
$a_{j,5}$	2	1	2
$a_{j,6}$	2	0	2
$a_{j,7}$	2	1	0
$a_{j,8}$	2	0	0
$a_{j,9}$	0	1	2
$a_{j,10}$	0	0	2
$a_{j,11}$	0	1	0
$a_{j,12}$	0	0	0

Service completion of component C_{1j}: Then $I'_{1j} = I_{1j} + 1$; and the corresponding transition probability $p(\sigma'|\sigma, a_{1,k_1}, a_{2,k_2})$ is given by:

$$p(\sigma'|\sigma, a_{1,k_1}, a_{2,k_2}) = \sum_j (\mathbb{1}_{m,1,k_j}\mu_{m,1j} + \mathbb{1}_{s,1,k_j}\mu_{s,1j})/\nu$$

where $\mathbb{1}_{m,i,k_j}$ and $\mathbb{1}_{s,i,k_j}$, $i = 1, 2$ are indicator functions that takes the value 1 if manufacturer M_j and subcontractor S_{ij} respectively are producing component C_{ij} under action a_{j,k_j}, and 0 otherwise.

Service completion of component C_{2j}: Then $I'_{2j} = I_{2j} + 1$; and the corresponding transition probability $p(\sigma'|\sigma, a_{1,k_1}, a_{2,k_2})$ is given by:

$$p(\sigma'|\sigma, a_{1,k_1}, a_{2,k_2}) = \sum_j (\mathbb{1}_{m,2,k_j}\mu_{m,2j} + \mathbb{1}_{s,2,k_j}\mu_{s,2j})/\nu$$

Finally, $I'_{ij} = I_{ij}, \forall i, j = 1, 2$; and the transition probability $p(\sigma'|\sigma, a_{1,k_1}, a_{2,k_2})$ is given by:

$$p(\sigma'|\sigma, a_{1,k_1}, a_{2,k_2}) = (\nu - \sum_{i,j}(\lambda_i + \mathbb{1}_{m,i,k_j}\mu_{m,ij} + \mathbb{1}_{s,i,k_j}\mu_{s,ij}))/\nu$$

Cost equation: Define $h(\sigma) = \sum_j \sum_i h_{ij} \max(I_{ij}, 0)$ as the total inventory holding cost and $b(\sigma) = \sum_i b_i \max_j \max(-I_{ij}, 0)$ as the total backordering cost. Let, $c(a_{1,k_1}, a_{2,k_2}) = \sum_{i,j}(c_{m,ij}\mathbb{1}_{m,i,k_j} + c_{s,ij}\mathbb{1}_{s,i,k_j})$ represents the total production cost for action a_{1,k_1} and a_{2,k_2}, where $\mathbb{1}_{m,i,k_j}$ (or $\mathbb{I}_{s,i,k_j}$) is a binary variable that takes the value 1 if the in-house manufacturer (or the subcontractor) is producing component C_{ij}, and takes the value 0 otherwise . For instance, if $a_{1,2} = (1, 0, 2)$ and $a_{2,2} = (1, 0, 2)$; then $C(a_{1,2}, a_{2,2}) = (c_{m,11} + c_{s,21} + c_{m,12} + c_{s,22})$. This implies that system incurs production cost at a facility only if action suggests to produce at the facility. Then, we construct a standard Bellman cost equation with value function, $V_t(\sigma)$ at state σ and decision epoch t. Equation (5.1) defines the value function $V_t(\sigma)$ at state σ. We use the discount factor $\eta \in (0, 1)$ in the optimization.

$$\begin{aligned} V_t(\sigma) \quad &= \quad h(\sigma) + b(\sigma) \\ &\quad +min_{(a_{1,k_1}, a_{2,k_2}) \in \mathbb{A}}[c(a_1, a_2) + \eta \sum_{\sigma'} p(\sigma'|\sigma, a_{1,k_1}, a_{2,k_2}) V_{t+1}(\sigma')], \forall \sigma \in \Sigma \quad (5.1) \end{aligned}$$

The objective minimizes the value function, $V_t(\sigma)$ at each state σ and determine the optimal action $(a^*_{1,k_1}, a^*_{2,k_2})$. The system described above presents challenges in terms of structural analysis of the optimal policy. First, the size of the state space Σ and action space $\mathbb{A}$ increases the complexity of the analysis. For example, with $I_{ij}, i, j = 1, 2$ varying from -100 to 100, the model has 1 billion states and 144 actions. Second, the optimal value function $V^*_t(\sigma)$ may not be convex in $I_{1j}, j = 1, 2$ and the transition probabilities may not have sub-additivity or super-additivity property with respect to I_{ij} in the state space Σ and action space $\mathbb{A}$ because of high action space.

Despite of the above mentioned challenges, we are able to analyze the original problem by decomposing the original system into two subsystems as presented in the next section. Each subsystem $\chi_j, j = 1, 2$ only models components C_{1j} and C_{2j}, requiring 2-tupled state space and 12 actions. This decreases the complexity of the model. Further, we use efficient action comparison and action elimination techniques to significantly reduce the action space which helps us to prove that simple multi-index policies are optimal when the sum of inventory position of product C_{1j} and product C_{2j} are constant.

Next, we describe the characteristics of the optimal policy. We present the characteristics of the optimal value function of the subsystem $\chi_j, j = 1, 2$.

5.4 Characteristics of the Optimal Policy for Subsystem χ_j

We model each subsystem χ_j as a Markov decision process model. In each subsystem χ_j, actions are taken at epochs corresponding to the state change. Note that, the state of subsystem χ_j is two tupled and is described in a similar way as the system presented in Chapter 4. We define state $\sigma_j = (I_{1j}, I_{2j})$, where $I_{ij}, i = 1, 2$ is the net inventory position of component C_{ij} and $\sigma_j \in \Sigma_j$. Next, the set of actions $\mathbb{A}_j$ is defined by $a_{j,k_j} = (m_j, s_{1j}, s_{2j}), k_j = 1, ..., 12$. Next, we use definitions of transition probabilities and immediate cost function to define the equivalent transition probabilities $p(\sigma'_j|\sigma_j, a_{j,k_j})$ and immediate cost function $r_j(\sigma_j, a_{j,k_j})$ for subsystem χ_j. This means that $r_j(\sigma_j, a_{j,k_j}) = h_j(\sigma_j) + b_j(\sigma_j) + c_j(a_{j,k_j}), j = 1, 2$. Note that $h_j(\sigma_j) = \sum_i h_{ij} \max(I_{ij}, 0)$ is the total inventory holding cost, $b_j(\sigma_j) = \sum_i b_{ij} \max(-I_{ij}, 0)$ is the total backordering cost, and $c_j(a_{j,k_j}) = \sum_i (c_{m,ij} \mathbb{1}_{m,i,k_j} + c_{s,ij} \mathbb{1}_{s,i,k_j})$ represents the total production cost for action a_{j,k_j}. Finally, we construct a standard Bellman cost equation for subsystem χ_j with value function, $V_{t,j}(\sigma_j)$ at state σ_j and decision epoch t as shown in Equation (5.2). For simplicity, we normalize and set $\nu_j = 1$ and $A_j = 0$.

$$
\begin{aligned}
V_{t,j}(I_{1j}, I_{2j}) \quad = \quad & h_j(\sigma_j) + b_j(\sigma_j) \\
& + min_{a_{j,k_j} \in \mathbb{A}_j} [c_j(a_{j,k_j}) + \eta_j \sum_{\sigma'_j} p_j(\sigma'_j|\sigma_j, a_{j,k_j}) V_{t+1,j}(\sigma'_j)]
\end{aligned}
\tag{5.2}
$$

We analyze a complete symmetric system (with respect to costs and service rates of the products), and show that the optimal policy is of dual index type, i.e. it suggests that either one of the products should be always produced at the fastest rate or none of the products should be produced. In other words, actions $a_{j,10}$ and $a_{j,11}$ are never optimal. Next, we analyze asymmetric systems (with respect to costs and service rates) and partition the action space into regions based on the unit production cost of the manufacturer and the subcontractor, and show that the multi-index policies could be optimal under specific conditions on the service rates. In this section, we use Proposition 4.1 from Chapter 4 to summarize optimal policies for several cases of subsystem χ_j.

5.4.1 Optimal Policy for Symmetric Systems

We consider two types of symmetry, $M-S$ symmetry and complete symmetry. For an $M-S$ symmetric system, the costs and service rates for a given component C_{ij} for the manufacturer M_j is the same as that of the subcontractor S_{ij}, i.e. $c_{m,ij} = c_{s,ij}$, and $\mu_{m,ij} = \mu_{s,ij}$. Complete symmetry corresponds to a special case of $M - S$ symmetric system where the production costs and the service rates are same across the products. Under $M-S$ symmetry , we observe that action $a_{j,9}$ is never optimal, i.e. both the products will not be simultaneously manufactured only by the respective external subcontractors. This happens because the action $a_{j,9}$ has higher costs than action $a_{j,5}$.

Next, we consider complete symmetric system. For this case, first we use Proposition 5.1 and further reduce the optimal action space from 12 actions to only 5 actions.

Proposition 5.1. *If $c_{m,ij} = c_{s,ij}$, and $\mu_{m,ij} = \mu_{s,ij}, i = 1, 2$, then optimal action belongs to the set $\{a_{j,1}, a_{j,3}, a_{j,5}, a_{j,6}, a_{j,12}\}$.*

Proof. The results follows from Theorem 4.5 in Chapter 4. □

Further, we show that under complete symmetric system with respect to costs and service rates of the products, the optimal policy suggests that either one of the products should be always produced at the fastest rate or none of the products should be produced. In other words, actions $a_{j,10}$ and $a_{j,11}$ are never optimal.

5.4.2 Optimal Policy for Asymmetric Systems

For the asymmetric, we analyze three cases : (1) system with negligible production costs, (2) system with cheaper manufacturer, (3) system with cheaper subcontractor.

Case 1: Negligible production costs: Typically, in the energy industry, certain products/ operations such as manual assembly, PCB assembly, wire harnesses, etc have negligible production costs as compared to material costs or inventory costs. Under this setting, the production cost could be assumed to be zero. Proposition 5.2 shows that if the production costs are zero then the optimal action belongs to the set $\{a_{j,1}, a_{j,3}, a_{j,5}, a_{j,6}, a_{j,12}\}$ and the decisions are non-increasing in the service rate of product C_{1j} with respect to $I_1, (I_1, I_2) \in \mathbb{S}_j$. Note that this case significantly reduces the action space for subsystem χ_j from 12 actions to 5 actions, and thereby reducing the computational complexity in the original system.

Proposition 5.2. *For subsystem χ_j with state $\sigma_j \in \mathbb{S}_j = \{(I_{1j}, I_{2j})|I_{1j} + I_{2j} = K_j\}$ where, K_j is constant, if $c_{m,ij} = c_{s,ij} = 0, i = 1, 2$, then the optimal action a^*_{j,k_j}*
(1) belongs to the set $\{a_{j,1}, a_{j,3}, a_{j,5}, a_{j,6}, a_{j,12}\}$.
(2) is non-increasing in service rates for product C_{1j} with respect to increasing I_{1j}.

Proof. Refer to Theorem 4.6 in Chapter 4 for the proof. □

Case 2: Manufacturer is cheaper: When demand is high and exceeds internal capacity, the supply chain manager could subcontract products to the external subcontractor to alleviate the production and capacity burden at the manufacturer, even though the subcontractor is more expensive than the manufacturer. However, for example, blowout preventers (prominent energy product) vary significantly in size and require special equipment. For this product, the production cost at the subcontractor could be more than the production cost at the manufacturer. Proposition 5.3 shows that if the production cost per unit at the manufacturer is less than the production cost per unit at the subcontractor, then the optimal actions belongs to the set $\{a_{j,1}, a_{j,2}, a_{j,3}, a_{j,4}, a_{j,5}, a_{j,6}, a_{j,7}, a_{j,8}, a_{j,12}\}$. Additionally, if service rate of the product C_{1j} is significantly more than the service rate of product C_{2j} which is typical for products which high variation in size, then optimal decisions are non-increasing in the service rate of product C_{1j} with respect to $I_{1j}, (I_{1j}, I_{2j}) \in \mathbb{S}_j$. Note that this case reduces the action space for subsystem χ_j from 12 actions to 9 actions.

Proposition 5.3. *For subsystem χ_j with state $\sigma_j \in \mathbb{S}_j = \{(I_{1j}, I_{2j}) | I_{1j} + I_{2j} = K_j\}$ where, K_j is constant $\frac{c_{m,i}}{\mu_{m,i}} < \frac{c_{s,i}}{\mu_{s,i}}, i = 1, 2$, then*
(1) the optimal action a^*_{j,k_j} belongs to the set $\{a_{j,1}, a_{j,2}, a_{j,3}, a_{j,4}, a_{j,5}, a_{j,6}, a_{j,7}, a_{j,8}, a_{j,12}\}$.
(2) if $I_{1j} > K_j$, $\mu_{s,1j} > \mu_{s,2j}$, and $\mu_{m,1j} > \mu_{m,2j} + \mu_{s,1j} + \mu_{s,2j}$, then the optimal action a^*_{j,k_j} is non-increasing in service rates for product C_{1j} with respect to increasing I_{1j}.

Proof. Refer to Theorem 4.7 in Chapter 4 for the proof. □

Case 3: Subcontractor is cheaper: For example, products such as top drives are often expensive to produce using capacity available at the manufacturer. Using capacity available at the subcontractor is often cheaper. However, if the subcontractor has a higher lead time, this could lead to high backorders and poor service levels. So, the supply chain manager needs to balance the tradeoffs in cost and delivery performance to decide on the production and subcontracting decisions. Proposition 5.4 shows that if the production cost per unit at the manufacturer is more than the production cost per unit at the subcontractor, then the optimal actions belongs to the set $\{a_{j,1}, a_{j,3}, a_{j,5}, a_{j,6}, a_{j,9}, a_{j,10}, a_{j,11}, a_{j,12}\}$. Additionally, if the service rate of the product C_{1j} is significantly more than the service rate of product C_{2j}, then optimal decisions are non-increasing in the service rate of product C_{1j} with respect to $I_{1j}, (I_{1j}, I_{2j}) \in \mathbb{S}_j$. Note that this case reduces the action space for subsystem χ_j from 12 actions to 8 actions.

Proposition 5.4. *For subsystem χ_j with state $\sigma_j \in \mathbb{S}_j = \{(I_{1j}, I_{2j}) | I_{1j} + I_{2j} = K_j\}$ where, K_j is constant $\frac{c_{m,i}}{\mu_{m,i}} > \frac{c_{s,i}}{\mu_{s,i}}, i = 1, 2$, then*
(1) the optimal action a^*_{j,k_j} belongs to the set $\{a_{j,1}, a_{j,3}, a_{j,5}, a_{j,6}, a_{j,9}, a_{j,10}, a_{j,11}, a_{j,12}\}$.
(2) if $I_{1j} > K_j$, $\mu_{m,1j} > \mu_{m,2j} + \mu_{s,1j}$, and $\mu_{s,1j} > \mu_{m,2j} + \mu_{s,2j}$, then the optimal action a^*_{j,k_j} is non-increasing in service rates for product C_{1j} with respect to increasing I_{1j}.

Proof. Refer to Theorem 4.8 in Chapter 4 for the proof. □

Next, we use the characteristics of the optimal policy of the subsystem χ_j to propose an approximate analysis of the optimal policy for the original multi-product system.

5.5 Approximate Analysis

Recall that the system described in Section 5.3 presents challenges in terms the size of the state space Σ and action space $\mathbb{A}$. For example, with $I_{ij}, i, j = 1, 2$ varying from -100 to 100, the model has 1 billion states and 144 actions. So, we leverage results summarized in Section 5.4 for subsystem χ_j to develop efficient approximate solution for the original system. This section outlines the approximate analysis of the original system. We develop a decomposition based approach which considers two Markov chains corresponding to two subsystems, $\chi_j, j = 1, 2$.

In the original multi-product system described in Section 5.2, for states $I_{1j}, I_{2j}, j = 1, 2$ where, $I_{1j} > -B_{max}, I_{2j} > -B_{max}, j = 1, 2$, transition probabilities and behavior of χ_j is exactly the same as in the original system. The difference only occurs when I_{1j} or I_{2j} reaches $-B_{max}$ in one of the subsystems. For instance, if either I_{11} or I_{21} is equal to $-B_{max}$ in χ_1; in the original system, demands would be backordered. But subsystem χ_2 cannot record this information and demands continue to queue in χ_2. To account for this in the subsystem χ_j, we define the surrogate demand arrival rate, λ_{ij} corresponding to component C_{ij} for product $i, i = 1, 2$. In the decomposition, we adjust the demand arrival rate λ_{1j} and λ_{2j}, $j = 1, 2$ to ensure that each subsystem χ_j models the behavior of the original two-product system. For subsystem χ_1, $\lambda_{i1} = \lambda_i(1 - P_{i2} + P_{i1}P_{i2})$ where, P_{i2} is the probability that the backorders for component C_{12} and C_{22} in subsystem χ_2 are equal to B_{max}, and P_{i1} is the probability that the backorders for component C_{11} and C_{21} in subsystem χ_1 are equal to B_{max}. Similarly, for subsystem χ_2, $\lambda_{i2} = \lambda_i(1 - P_{i1} + P_{i1}P_{i2})$.

Clearly, the solution to subsystem χ_1 requires the estimates of P_{i2} that is obtained from the optimal action set for subsystem χ_2 and vice versa. This suggests an iterative approach as shown in Figure 5.3. We analyze the optimal action for each subsystem using policy iteration algorithm within a larger iterative approach. Let $Q_j(a^{(*)}_{j,k_j})$ denote the transition probability matrix and $\pi_j(I_{1j}, I_{2j}, a^{(*)}_{j,k_j}) \in \Pi_j(a^{(*)}_{j,k_j})$ denotes the corresponding steady state probabilities for subsystem χ_j. Then, in system χ_1, we first assume $P_{i2} = 0$ and determine the optimal optimal action $a^{(*)}_{1,k_1}$ for subsystem χ_1 using policy iteration. Then, we construct the Markov chain model for subsystem χ_1 with $a^{(*)}_{1,k_1}$ and determine the steady state probabilities $\pi_1(I_{11}, I_{21}, a^{(*)}_{1,k_1})$ by solving the Chapman-Kolmogorov equations. Next, we use steady state probabilities to evaluate the estimates of P_{i1} (see Figure 5.3). Subsystem χ_2 uses this estimate of P_{i1} to calculate the surrogate demand arrival rate λ_{i2}. Next, we use this demand arrival rate λ_{i2} and determine the optimal optimal action $a^{(*)}_{2,k_2}$ for subsystem χ_2 using policy iteration. Then, we construct the Markov chain model for subsystem χ_2 with $a^{(*)}_{2,k_2}$ and determine the steady state probabilities $\pi_2(I_{12}, I_{22}, a^{(*)}_{2,k_2})$ by solving the Chapman-Kolmogorov equations. Next, we use steady state probabilities to evaluate the estimates of P_{i2}. This iterative process continues till the convergence is achieved for $P_{ij}, i, j = 1, 2$.

Executing this iterative procedure therefore, has two key steps: i) solving the simplified Markov decision process formulations for each subsystem to obtain the optimal action for the subsystem for a given estimates P_{i1} and P_{i2}. and ii) solving the Markov chain for each subsystem under current estimates of optimal action to obtain new estimates P_{i1} and P_{i2}.

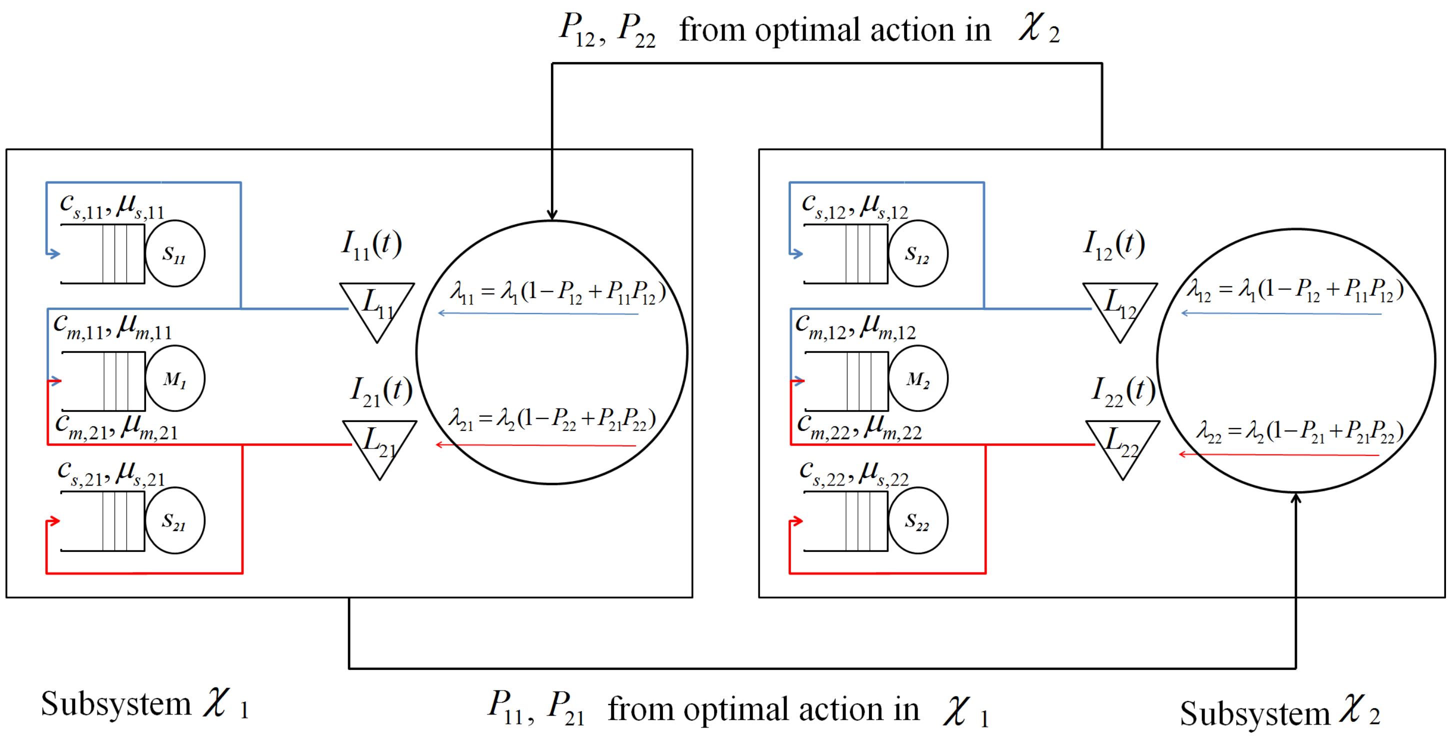

Figure 5.3 Approximation Algorithm

Section 5.4 already described the formulation of subsystem $\chi_j, j = 1, 2$, and the structural properties of optimal value function and optimal policy. In the following paragraph, we describe the details of Markov chain formulation used to determine the estimates of $P_{ij}, i, j = 1, 2$. Although, we have not proved the convergence of this algorithm, we observed that it always converged in our numerical experiments. In the iterative procedure, policy iteration on subsystem χ_j for a given estimate of P_{i1} and P_{i2} provides the optimal actions corresponding to each state $\sigma_j, j = 1, 2$. We use the optimal actions $a^{(*)}_{j,k_j}$ obtained at each iteration to develop a Markov chain formulation for the subsystem χ_j under these optimal actions.

Let $p_j(\sigma'_j|\sigma_j, a^{(*)}_{j,k_j})$ denote the transition probability from state $\sigma_j = (I_{1j}, I_{2j})$ to state $\sigma'_j = (I'_{1j}, I'_{2j})$ by taking action $a^{(*)}_{j,k_j}$, and let $\pi_j(I_{1j}, I_{2j}, a^{(*)}_{j,k_j}) \in \Pi_j(a^{(*)}_{j,k_j})$ denotes the corresponding steady state probabilities. Then, the Chapman Kolmogorov (CK) equations for all states in subsystem χ_j with given action $a^{(*)}_{j,k_j}$ is shown as follows:

- If $-B_{max} < I_{1j} < max(I_{1j})$ and $-B_{max} < I_{2j} < max(I_{2j})$, $\forall j = 1, 2$:

$$\begin{aligned}(\lambda_{1j} + \lambda_{2j} + \mu_{m,1j} + \mu_{s,1j} + \mu_{s,2j})\pi_j(I_{1j}, I_{2j}, a^{(*)}_{j,k_j}) \;=\;& \lambda_{1j}\pi_j(I_{1j} + 1, I_{2j}, a^{(*)}_{j,k_j}) \\ &+\lambda_{2j}\pi_j(I_{1j}, I_{2j} + 1, a^{(*)}_{j,k_j}) \\ &+\mu_{s,2j}\pi_j(I_{1j}, I_{2j} - 1, a^{(*)}_{j,k_j}) \\ &+(\mu_{m,1j} + \mu_{s,1j})\pi_j(I_{1j} - 1, I_{2j}, a^{(*)}_{j,k_j})\end{aligned} \tag{5.3}$$

- If $-B_{max} < I_{1j} < max(I_{1j})$ and $I_{2j} = max(I_{2j})$, $\forall j = 1, 2$:

$$\begin{aligned}(\lambda_{1j} + \lambda_{2j} + \mu_{m,1j} + \mu_{s,1j})\pi_j(I_{1j}, I_{2j}, a^{(*)}_{j,k_j}) \;=\;& \lambda_{1j}\pi_j(I_{1j} + 1, I_{2j}, a^{(*)}_{j,k_j}) \\ &+\mu_{s,2j}\pi_j(I_{1j}, I_{2j} - 1, a^{(*)}_{j,k_j}) \\ &+(\mu_{m,1j} + \mu_{s,1j})\pi_j(I_{1j} - 1, I_{2j}, a^{(*)}_{j,k_j})\end{aligned} \tag{5.4}$$

- If $-B_{max} < I_{1j} < max(I_{1j})$ and $I_{2j} = -B_{max}$, $\forall j = 1,2$:

$$\begin{aligned}(\lambda_{1j} + \mu_{m,1j} + \mu_{s,1j} + \mu_{s,2j})\pi_j(I_{1j}, I_{2j}, a^{(*)}_{j,k_j}) &= \lambda_{1j}\pi_j(I_{1j}+1, I_{2j}, a^{(*)}_{j,k_j}) \\ &+\lambda_{2j}\pi_j(I_{1j}, I_{2j}+1, a^{(*)}_{j,k_j}) \\ &+(\mu_{m,1j}+\mu_{s,1j})\pi_j(I_{1j}-1, I_{2j}, a^{(*)}_{j,k_j})\end{aligned} \tag{5.5}$$

- If $I_{1j} = -B_{max}$ and $-B_{max} < I_{2j} < max(I_{2j})$, $\forall j = 1,2$:

$$\begin{aligned}(\lambda_{2j} + \mu_{m,1j} + \mu_{s,1j} + \mu_{s,2j})\pi_j(I_{1j}, I_{2j}, a^{(*)}_{j,k_j}) &= \lambda_{1j}\pi_j(I_{1j}+1, I_{2j}, a^{(*)}_{j,k_j}) \\ &+\lambda_{2j}\pi_j(I_{1j}, I_{2j}+1, a^{(*)}_{j,k_j}) \\ &+\mu_{s,2j}\pi_j(I_{1j}, I_{2j}-1, a^{(*)}_{j,k_j})\end{aligned} \tag{5.6}$$

- If $I_{1j} = -B_{max}$ and $I_{2j} = max(I_{2j})$, $\forall j = 1,2$:

$$\begin{aligned}(\lambda_{2j} + \mu_{m,1j} + \mu_{s,1j})\pi_j(I_{1j}, I_{2j}, a^{(*)}_{j,k_j}) &= \lambda_{1j}\pi_j(I_{1j}+1, I_{2j}, a^{(*)}_{j,k_j}) \\ &+\mu_{s,2j}\pi_j(I_{1j}, I_{2j}-1, a^{(*)}_{j,k_j})\end{aligned} \tag{5.7}$$

- If $I_{1j} = -B_{max}$ and $I_{2j} = -B_{max}$, $\forall j = 1,2$:

$$\begin{aligned}(\mu_{m,1j} + \mu_{s,1j} + \mu_{s,2j})\pi_j(I_{1j}, I_{2j}, a^{(*)}_{j,k_j}) &= \lambda_{1j}\pi_j(I_{1j}+1, I_{2j}, a^{(*)}_{j,k_j}) \\ &+\lambda_{2j}\pi_j(I_{1j}, I_{2j}+1, a^{(*)}_{j,k_j})\end{aligned} \tag{5.8}$$

- If $I_{1j} = max(I_{1j})$ and $-B_{max} < I_{2j} < max(I_{2j})$, $\forall j = 1,2$:

$$\begin{aligned}(\lambda_{1j} + \lambda_{2j} + \mu_{s,2j})\pi_j(I_{1j}, I_{2j}, a^{(*)}_{j,k_j}) &= \lambda_{2j}\pi_j(I_{1j}, I_{2j}+1, a^{(*)}_{j,k_j}) \\ &+\mu_{s,2j}\pi_j(I_{1j}, I_{2j}-1, a^{(*)}_{j,k_j})\end{aligned} \tag{5.9}$$

- If $I_{1j} = max(I_{1j})$ and $I_{2j} = max(I_{2j})$, $\forall j = 1,2$:

$$(\lambda_{1j} + \lambda_{2j})\pi_j(I_{1j}, I_{2j}, a^{(*)}_{j,k_j}) = \mu_{s,2j}\pi_j(I_{1j}, I_{2j}-1, a^{(*)}_{j,k_j}) \tag{5.10}$$

- If $I_{1j} = max(I_{1j})$ and $I_{2j} = -B_{max}$, $\forall j = 1,2$:

$$(\lambda_{1j} + \mu_{s,2j})\pi_j(I_{1j}, I_{2j}, a^{(*)}_{j,k_j}) = \lambda_{2j}\pi_j(I_{1j}, I_{2j}+1, a^{(*)}_{j,k_j}) \tag{5.11}$$

The CK equations for other actions $a_{j,k_j}^{(*)} \in A_j$ can be written in similar way. Then, the steady state probabilities $\pi_j(I_{1j}, I_{2j}, a_{j,k_j}^{(*)})$ corresponding to the optimal action are obtained using Equation (5.12) and Equation (5.13).

$$\Pi_j(a_{j,k_j})Q_j(a_{j,k_j}^{(*)}) = \mathbf{0} \tag{5.12}$$

$$\sum_{I_{1j}}\sum_{I_{2j}} \pi_j(I_{1j}, I_{2j}, a_{j,k_j}^{(*)}) = 1 \tag{5.13}$$

Using these steady state probabilities, we estimate $P_{ij}, \forall i = 1, 2; j = 1, 2$ as shown in Equation (5.14) and Equation (5.15).

$$P_{1j} = \sum_{I_{2j}=-B_{max}}^{I_{2j}=B_{max}} \pi_j(-B_{max}, I_{2j}, a_{j,k_j}^{(*)}) \tag{5.14}$$

$$P_{2j} = \sum_{I_{1j}=-B_{max}}^{I_{1j}=B_{max}} \pi_j(I_{1j}, -B_{max}, a_{j,k_j}^{(*)}) \tag{5.15}$$

5.6 Numerical Studies

This section presents numerical studies of the proposed multi-product system to provide insights on the characteristics of the optimal solution. We conduct three experiments: Experiment 1, Experiment 2, and Experiment 3. Experiment 1 discussed in Section 5.6.1 considers zero production costs, Experiment 2 discussed in Section 5.6.2 considers cheaper manufacturer, and Experiment 3 discussed in Section 5.6.3 considers expensive manufacturer. Finally, in Section 5.6.4, we analyze the accuracy of the decomposition approach for these three experiments. We let B_{max} denote the maximum backordering limit. Thus, we have $-B_{max} \leq I_{ij} \leq \infty, i, j = 1, 2$.

5.6.1 Experiment 1: Zero Production Costs

In Experiment 1, we consider a case where the manufacturer and the subcontractor have zero costs i.e. $c_{m,ij} = 0, c_{s,ij} = 0,\ i = 1, 2,\ j = 1, 2$. Using Proposition 5.2, the optimal action belongs to the set $\{a_{j,1}, a_{j,3}, a_{j,5}, a_{j,6}, a_{j,12}\}$, reducing the total action space from 144 actions to 25 actions. Table 5.2 presents the system and cost parameters for Experiment 1. Here, the manufacturer is twice as faster ($\mu_{m,1j} = 2\mu_{s,1j}$) as the external subcontractor. We perform an exact analysis and analyze characteristics of the optimal solution for the original system.

Table 5.2 System Parameters and Costs for Experiment 1

Subcontractor's Parameters		**Manufacturer's Parameters**	
$c_{s,1j}, j = 1, 2$	0	$c_{m,1j}, j = 1, 2$	0
$c_{s,2j}, j = 1, 2$	0	$c_{m,2j}, j = 1, 2$	0
$\mu_{s,i1}, i = 1, 2$	1	$\mu_{m,i1}, i = 1, 2$	2
$\mu_{s,i2}, i = 1, 2$	1.5	$\mu_{m,i2}, i = 1, 2$	3
System Parameters		**Other Costs**	
B_{max}	5	$b_i, i = 1, 2$	80
$\lambda_i, i = 1, 2$	1.5	$h_{ij}, i, j = 1, 2$	2

For $B_{max} = 5$, we analyze the characteristics of the optimal solution. Table 5.3 presents the optimal actions for subsystem χ_1 corresponding to each state $(\sigma_1, \sigma_2), \sigma_2 = (0, 0)$. For instance, if $I_{11} = I_{21} = -5$, the optimal action $a_{j,5}$ corresponding to state $(-5, -5, 0, 0)$. Under given parameters and $\sigma_1 = (-5, -5)$, the optimal decision suggests to use in-house manufacturing department M_j as well as external subcontractor S_{2j} to manufacture component C_{2j} and only use external subcontractor S_{1j} to manufacture component C_{1j}. Similarly, the optimal actions for other states are defined as well. The optimal policy contains actions

$\{a_{j,1}, a_{j,3}, a_{j,5}, a_{j,6}, a_{j,12}\}$.

For a fixed total inventory position, $I_{11} + I_{21}$, we observe a monotone property in service rates of components with increasing I_{11} or I_{21}. For instance, if $I_{11} + I_{21} = 0$, the total service rate for component C_{11} is non-increasing with increasing inventory position I_{11}. In contrast, if $I_{11} + I_{21} = -6$, the total service rate for component C_{11} is non-decreasing with increasing inventory position I_{11}. Under the given parameter setting, these results suggest a dual index type policy for component replenishment for a fixed total inventory position $I_{11} + I_{21}$. Although, the results shown in Table 5.3 is for $\sigma_2 = (0, 0)$, we observe this property to hold for all other values of σ_2. This means that for every state in the subsystem χ_2, there is dual index type structure of state in subsystem χ_1 if $I_{11} + I_{21} = K$, where is K is constant.

Table 5.3 Optimal Actions Corresponding to Each State σ_1 in Experiment 1

I_{11}/I_{21}	-5	-4	-3	-2	-1	0	1	2	3	4	5
-5	$a_{1,5}$	$a_{1,5}$	$a_{1,5}$	$a_{1,5}$	$\boldsymbol{a_{1,5}}$	$a_{1,1}$	$a_{1,1}$	$a_{1,1}$	$a_{1,3}$	$a_{1,3}$	$\boldsymbol{a_{1,3}}$
-4	$a_{1,1}$	$a_{1,5}$	$a_{1,5}$	$\boldsymbol{a_{1,5}}$	$a_{1,1}$	$a_{1,1}$	$a_{1,1}$	$a_{1,1}$	$a_{1,3}$	$\boldsymbol{a_{1,3}}$	$a_{1,3}$
-3	$a_{1,1}$	$a_{1,1}$	$\boldsymbol{a_{1,1}}$	$a_{1,1}$	$a_{1,1}$	$a_{1,1}$	$a_{1,1}$	$a_{1,1}$	$\boldsymbol{a_{1,3}}$	$a_{1,3}$	$a_{1,3}$
-2	$a_{1,1}$	$\boldsymbol{a_{1,1}}$	$a_{1,5}$	$a_{1,5}$	$a_{1,1}$	$a_{1,1}$	$a_{1,1}$	$\boldsymbol{a_{1,3}}$	$a_{1,3}$	$a_{1,3}$	$a_{1,3}$
-1	$\boldsymbol{a_{1,1}}$	$a_{1,5}$	$a_{1,5}$	$a_{1,5}$	$a_{1,5}$	$a_{1,1}$	$\boldsymbol{a_{1,1}}$	$a_{1,3}$	$a_{1,3}$	$a_{1,3}$	$a_{1,3}$
0	$a_{1,5}$	$a_{1,5}$	$a_{1,5}$	$a_{1,5}$	$a_{1,5}$	$\boldsymbol{a_{1,1}}$	$a_{1,1}$	$a_{1,3}$	$a_{1,3}$	$a_{1,3}$	$a_{1,3}$
1	$a_{1,5}$	$a_{1,5}$	$a_{1,5}$	$a_{1,5}$	$\boldsymbol{a_{1,5}}$	$a_{1,5}$	$a_{1,5}$	$a_{1,3}$	$a_{1,3}$	$a_{1,3}$	$a_{1,3}$
2	$a_{1,5}$	$a_{1,5}$	$a_{1,5}$	$\boldsymbol{a_{1,6}}$	$a_{1,6}$	$a_{1,6}$	$a_{1,6}$	$a_{1,12}$	$a_{1,12}$	$a_{1,12}$	$a_{1,12}$
3	$a_{1,6}$	$a_{1,6}$	$\boldsymbol{a_{1,6}}$	$a_{1,6}$	$a_{1,6}$	$a_{1,6}$	$a_{1,6}$	$a_{1,12}$	$a_{1,12}$	$a_{1,12}$	$a_{1,12}$
4	$a_{1,6}$	$\boldsymbol{a_{1,6}}$	$a_{1,6}$	$a_{1,6}$	$a_{1,6}$	$a_{1,6}$	$a_{1,6}$	$a_{1,12}$	$a_{1,12}$	$a_{1,12}$	$a_{1,12}$
5	$\boldsymbol{a_{1,6}}$	$a_{1,6}$	$a_{1,6}$	$a_{1,6}$	$a_{1,6}$	$a_{1,6}$	$a_{1,6}$	$a_{1,12}$	$a_{1,12}$	$a_{1,12}$	$a_{1,12}$

For $I_{11} + I_{21} = 0$, Figure 5.4 shows the characteristics of the optimal solution for $\sigma_2 = (0,0), \sigma_2 = (-5,-5), \sigma_2 = (5,5)$. We observe that the service rate of the component C_{11} is non-increasing with increasing inventory position I_{11}. Under all states of the subsystem χ_2, these results suggest a dual index type policy for product replenishment for a fixed total inventory position $I_{11} + I_{21} = 0$. For example, for $\sigma_2 = (-5,-5)$, the optimal service rate of component C_{11} at inventory position $I_{11} = 1$ changes from 3 to 1 meaning that only the subcontractor S_{11} is producing product C_{11} at $I_{11} = 1$. Next, the component C_{11} is neither produced by the manufacturer M_1 nor by the subcontractor S_{11} at inventory position $I_1 = 3$. We observe similar results for $I_1 + I_2 = K$, $K = -10, ..., 10$. However, the optimal policy might not be dual index type for other parameters. We also observe that there is no monotone property in service rate of a given component C_{11} if $I_{11} = K$ or $I_{21} = K$.

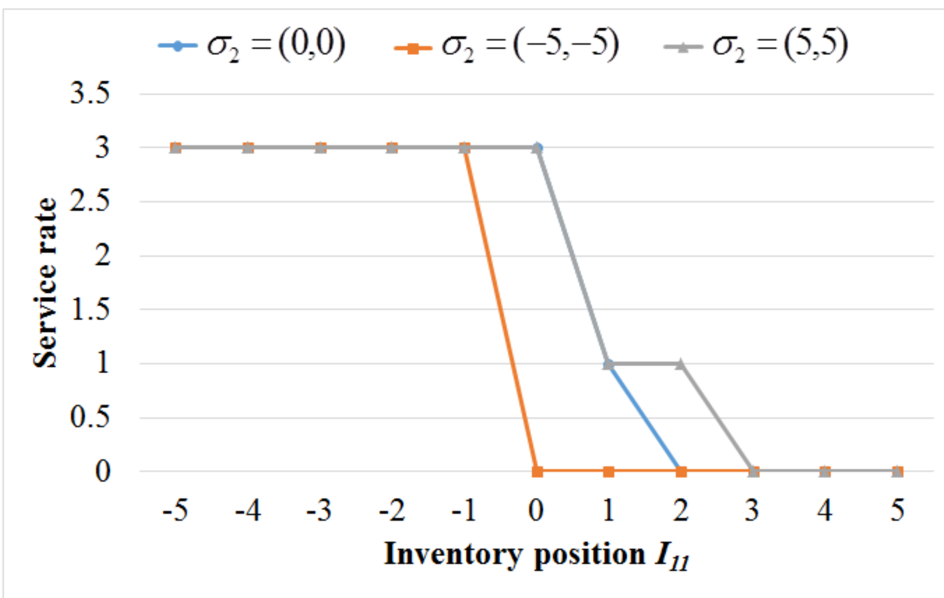

Figure 5.4 Optimal Service Rate of Component C_{11} for Experiment 1

5.6.2 Experiment 2: Manufacturer is Cheaper than Subcontractor

In Experiment 2, we consider that the manufacturer is cheaper than the external subcontractor. In this setting, the component $C_{2j}, j = 1, 2$ is more expensive than the component $C_{1j}, j = 1, 2$. Using Proposition 5.3, the optimal action belongs to the set $\{a_{j,1}, a_{j,2}, a_{j,3}, a_{j,4}, a_{j,5}, a_{j,6}, a_{j,7}, a_{j,8}, a_{j,12}\}$, reducing the total action space from 144 actions to 81 actions. Note that in Experiment 2, we have more than twice the actions as compared to Experiment 1 which increases the computational complexity of this experiment. Table 5.4 presents the system and cost parameters for Experiment 2. Here again, the manufacturer is twice as faster ($\mu_{m,1j} = 2\mu_{s,1j}$) as the external subcontractor. We perform an exact analysis and analyze characteristics of the optimal solution for the original system.

Table 5.4 System Parameters and Costs for Experiment 2

Subcontractor's Parameters		**Manufacturer's Parameters**	
$c_{s,1j}, j = 1, 2$	35	$c_{m,1j}, j = 1, 2$	15
$c_{s,2j}, j = 1, 2$	25	$c_{m,2j}, j = 1, 2$	10
$\mu_{s,i1}, i = 1, 2$	1	$\mu_{m,i1}, i = 1, 2$	2
$\mu_{s,i2}, i = 1, 2$	1.5	$\mu_{m,i2}, i = 1, 2$	3
System Parameters		**Other Costs**	
B_{max}	5	$b_i, i = 1, 2$	80
$\lambda_i, i = 1, 2$	1.5	$h_{ij}, i = 1, 2$	2

For $B_{max} = 5$, we analyze the characteristics of the optimal solution. Table 5.5 presents the optimal actions for subsystem χ_1 corresponding to each state $(\sigma_1, \sigma_2), \sigma_2 = (0, 0)$. For instance, if $I_{11} = I_{21} = -5$, the optimal action $a_{j,6}$ corresponding to state $(-5, -5, 0, 0)$. Under given parameters and $\sigma_1 = (-5, -5)$, the optimal decision suggests to use manufacturer M_j as well as external subcontractor S_{2j} to manufacture component C_{2j} and do not manufacture component C_{1j}. Similarly, the optimal actions for other states are defined as well. The optimal policy contains actions $\{a_{j,1}, a_{j,3}, a_{j,4}, a_{j,5}, a_{j,6}, a_{j,8}, a_{j,12}\}$.

For a fixed total inventory position, $I_{11} + I_{21}$, we sometime observe a monotone property in service rates of components with increasing I_{11} or I_{21} but sometime do not observe monotone property. For instance, if $I_{11} + I_{21} = 0$, the total service rate for component C_{11} is not always non-increasing with increasing inventory position I_{11}. In contrast, if $I_{11} + I_{21} = -6$, the total service rate for component C_{11} is non-decreasing with increasing inventory position I_{11}. This suggests that the optimal policy may not be dual index type. But, if we consider condition on service rates as given in Proposition 5.3 then we do observe dual index type policy to be optimal at fixed total inventory position.

Table 5.5 Optimal Actions Corresponding to Each State σ_1 in Experiment 2

I_{11}/I_{21}	-5	-4	-3	-2	-1	0	1	2	3	4	5
-5	$a_{1,6}$	$a_{1,6}$	$a_{1,6}$	$a_{1,6}$	$\boldsymbol{a_{1,6}}$	$a_{1,4}$	$a_{1,4}$	$a_{1,4}$	$a_{1,4}$	$a_{1,4}$	$\boldsymbol{a_{1,4}}$
-4	$a_{1,1}$	$a_{1,6}$	$a_{1,6}$	$\boldsymbol{a_{1,6}}$	$a_{1,6}$	$a_{1,3}$	$a_{1,4}$	$a_{1,4}$	$a_{1,4}$	$\boldsymbol{a_{1,4}}$	$a_{1,4}$
-3	$a_{1,1}$	$a_{1,5}$	$\boldsymbol{a_{1,5}}$	$a_{1,5}$	$a_{1,1}$	$a_{1,3}$	$a_{1,3}$	$a_{1,3}$	$\boldsymbol{a_{1,3}}$	$a_{1,3}$	$a_{1,3}$
-2	$a_{1,1}$	$\boldsymbol{a_{1,1}}$	$a_{1,5}$	$a_{1,5}$	$a_{1,1}$	$a_{1,4}$	$a_{1,4}$	$\boldsymbol{a_{1,4}}$	$a_{1,4}$	$a_{1,4}$	$a_{1,4}$
-1	$\boldsymbol{a_{1,1}}$	$a_{1,5}$	$a_{1,5}$	$a_{1,5}$	$a_{1,6}$	$a_{1,4}$	$\boldsymbol{a_{1,4}}$	$a_{1,4}$	$a_{1,4}$	$a_{1,4}$	$a_{1,4}$
0	$a_{1,6}$	$a_{1,6}$	$a_{1,6}$	$a_{1,6}$	$a_{1,6}$	$\boldsymbol{a_{1,8}}$	$a_{1,12}$	$a_{1,12}$	$a_{1,12}$	$a_{1,12}$	$a_{1,12}$
1	$a_{1,6}$	$a_{1,6}$	$a_{1,6}$	$a_{1,6}$	$\boldsymbol{a_{1,8}}$	$a_{1,8}$	$a_{1,12}$	$a_{1,12}$	$a_{1,12}$	$a_{1,12}$	$a_{1,12}$
2	$a_{1,6}$	$a_{1,6}$	$a_{1,6}$	$\boldsymbol{a_{1,6}}$	$a_{1,8}$	$a_{1,8}$	$a_{1,12}$	$a_{1,12}$	$a_{1,12}$	$a_{1,12}$	$a_{1,12}$
3	$a_{1,6}$	$a_{1,6}$	$\boldsymbol{a_{1,6}}$	$a_{1,6}$	$a_{1,8}$	$a_{1,8}$	$a_{1,12}$	$a_{1,12}$	$a_{1,12}$	$a_{1,12}$	$a_{1,12}$
4	$a_{1,6}$	$\boldsymbol{a_{1,6}}$	$a_{1,6}$	$a_{1,6}$	$a_{1,8}$	$a_{1,8}$	$a_{1,12}$	$a_{1,12}$	$a_{1,12}$	$a_{1,12}$	$a_{1,12}$
5	$\boldsymbol{a_{1,6}}$	$a_{1,6}$	$a_{1,6}$	$a_{1,6}$	$a_{1,8}$	$a_{1,8}$	$a_{1,12}$	$a_{1,12}$	$a_{1,12}$	$a_{1,12}$	$a_{1,12}$

For $I_{11} + I_{21} = 0$, Figure 5.5 shows the characteristics of the optimal solution for $\sigma_2 = (0,0), \sigma_2 = (-5,-5), \sigma_2 = (5,5)$. We observe that the service rate of the component C_{11} is not always non-increasing with increasing inventory position I_{11}. For example, for $\sigma_2 = (5, 5)$, the optimal service rate of product C_{11} at inventory position $I_{11} = -4$ changes from 2 to 3 meaning that the manufacturer M_j and the subcontractor S_{11} is producing component C_{11} at $I_{11} = 1$ then the service rate again changes to 2 at $I_{11} = -2$ and 0 at $I_{11} = 0$.

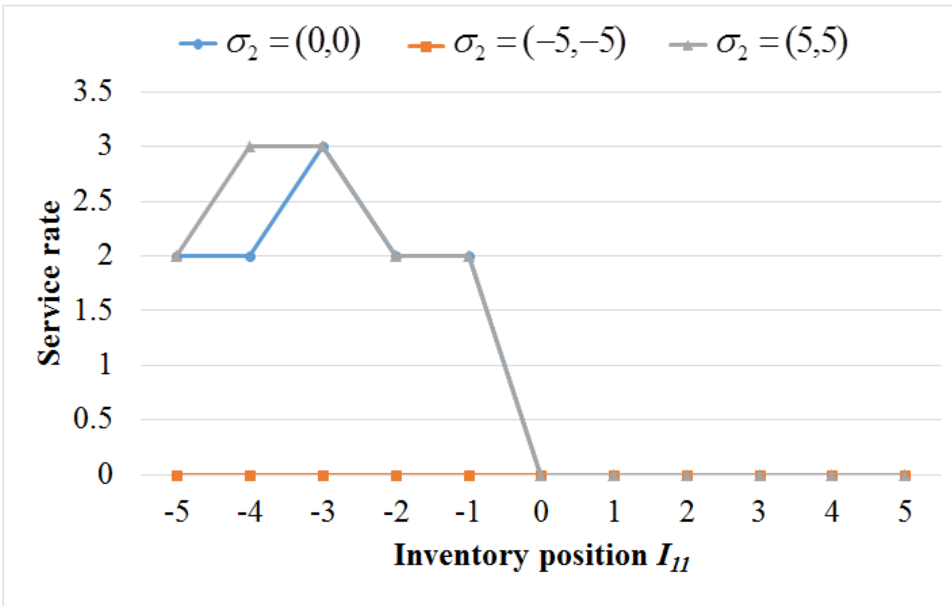

Figure 5.5 Optimal Service Rate of Component C_{11} for Experiment 2

5.6.3 Experiment 3: Subcontractor is Cheaper than Manufacturer

In Experiment 3, we consider a case where the subcontractor is cheaper than the manufacturer. In this setting, the component $C_{2j}, j = 1, 2$ is more expensive than the component $C_{1j}, j = 1, 2$. Using Proposition 5.4, the optimal action belongs to the set $\{a_{j,1}, a_{j,3}, a_{j,5}, a_{j,6}, a_{j,9}, a_{j,10}, a_{j,11}, a_{j,12}\}$, reducing the total action space from 144 actions to 64 actions. Note that in Experiment 3, we have more than twice the actions as compared to Experiment 1 but less number of actions as compared to Experiment 2. Table 5.6 presents the system and cost parameters for Experiment 3. Here, the manufacturer is twice as faster $(\mu_{m,1j} = 2\mu_{s,1j})$ as the external subcontractor. We perform an exact analysis and analyze characteristics of the optimal solution for the original system.

Table 5.6 System Parameters and Costs for Experiment 3

Subcontractor's Parameters		**Manufacturer's Parameters**	
$c_{s,1j}, j = 1, 2$	15	$c_{m,1j}, j = 1, 2$	35
$c_{s,2j}, j = 1, 2$	10	$c_{m,2j}, j = 1, 2$	25
$\mu_{s,i1}, i = 1, 2$	1	$\mu_{m,i1}, i = 1, 2$	2
$\mu_{s,i2}, i = 1, 2$	1.5	$\mu_{m,i2}, i = 1, 2$	3
System Parameters		**Other Costs**	
B_{max}	5	$b_i, i = 1, 2$	80
$\lambda_i, i = 1, 2$	1.5	$h_{ij}, i, j = 1, 2$	2

For $B_{max} = 5$, we analyze the characteristics of the optimal solution. Table 5.7 presents the optimal actions for subsystem χ_1 corresponding to each state $(\sigma_1, \sigma_2), \sigma_2 = (0, 0)$. The optimal policy contains actions $a_{j,1}, a_{j,3}, a_{j,5}, a_{j,6}, a_{j,12}$. The results look very similar to Experiment 1. Although, we did not observe actions $a_{j,9}, a_{j,10}, a_{j,11}$ for $\sigma_2 = (0, 0)$, we do observe these action in the optimal policy for other states of subsystem χ_2.

For a fixed total inventory position, $I_{11}+I_{21}$, we observe a monotone property in service rates of components with increasing I_{11} or I_{21}. For instance, if $I_{11} + I_{21} = 0$, the total service rate for component C_{11} is non-increasing with increasing inventory position I_{11} where at $I_{11} < 0$ both the manufacturer and the subcontractor is producing component C_{11}, and at $I_{11} \geq 0$ component C_{11} is not manufactured. In contrast, if $I_{11} + I_{21} = -7$, the total service rate for component C_{11} is non-decreasing with increasing inventory position I_{11}. Under the given parameter setting, these results suggest a dual index type policy for component replenishment for a fixed total inventory position $I_{11}+I_{21}$. Although, we show this result for $\sigma_2 = (0, 0)$, we observe this property to hold for all other values of σ_2 whenever the subcontractor is cheaper

Table 5.7 Optimal Actions Corresponding to Each State σ_1 in Experiment 3

I_{11}/I_{21}	-5	-4	-3	-2	-1	0	1	2	3	4	5
-5	$a_{1,5}$	$a_{1,5}$	$a_{1,5}$	$a_{1,5}$	$\boldsymbol{a_{1,5}}$	$a_{1,3}$	$a_{1,3}$	$a_{1,3}$	$a_{1,3}$	$a_{1,3}$	$\boldsymbol{a_{1,3}}$
-4	$a_{1,1}$	$a_{1,5}$	$a_{1,5}$	$\boldsymbol{a_{1,5}}$	$a_{1,1}$	$a_{1,3}$	$a_{1,3}$	$a_{1,3}$	$a_{1,3}$	$\boldsymbol{a_{1,3}}$	$a_{1,3}$
-3	$a_{1,1}$	$a_{1,5}$	$\boldsymbol{a_{1,5}}$	$a_{1,5}$	$a_{1,1}$	$a_{1,3}$	$a_{1,3}$	$a_{1,3}$	$\boldsymbol{a_{1,3}}$	$a_{1,3}$	$a_{1,3}$
-2	$a_{1,1}$	$\boldsymbol{a_{1,5}}$	$a_{1,5}$	$a_{1,5}$	$a_{1,1}$	$a_{1,3}$	$a_{1,3}$	$\boldsymbol{a_{1,3}}$	$a_{1,3}$	$a_{1,3}$	$a_{1,3}$
-1	$\boldsymbol{a_{1,5}}$	$a_{1,5}$	$a_{1,5}$	$a_{1,5}$	$a_{1,5}$	$a_{1,3}$	$\boldsymbol{a_{1,3}}$	$a_{1,3}$	$a_{1,3}$	$a_{1,3}$	$a_{1,3}$
0	$a_{1,6}$	$a_{1,6}$	$a_{1,6}$	$a_{1,6}$	$a_{1,6}$	$\boldsymbol{a_{1,12}}$	$a_{1,12}$	$a_{1,12}$	$a_{1,12}$	$a_{1,12}$	$a_{1,12}$
1	$a_{1,6}$	$a_{1,6}$	$a_{1,6}$	$a_{1,6}$	$\boldsymbol{a_{1,6}}$	$a_{1,12}$	$a_{1,12}$	$a_{1,12}$	$a_{1,12}$	$a_{1,12}$	$a_{1,12}$
2	$a_{1,6}$	$a_{1,6}$	$a_{1,6}$	$\boldsymbol{a_{1,6}}$	$a_{1,6}$	$a_{1,12}$	$a_{1,12}$	$a_{1,12}$	$a_{1,12}$	$a_{1,12}$	$a_{1,12}$
3	$a_{1,6}$	$a_{1,6}$	$\boldsymbol{a_{1,6}}$	$a_{1,6}$	$a_{1,6}$	$a_{1,12}$	$a_{1,12}$	$a_{1,12}$	$a_{1,12}$	$a_{1,12}$	$a_{1,12}$
4	$a_{1,6}$	$\boldsymbol{a_{1,6}}$	$a_{1,6}$	$a_{1,6}$	$a_{1,6}$	$a_{1,12}$	$a_{1,12}$	$a_{1,12}$	$a_{1,12}$	$a_{1,12}$	$a_{1,12}$
5	$\boldsymbol{a_{1,6}}$	$a_{1,6}$	$a_{1,6}$	$a_{1,6}$	$a_{1,6}$	$a_{1,12}$	$a_{1,12}$	$a_{1,12}$	$a_{1,12}$	$a_{1,12}$	$a_{1,12}$

than the manufacturer. This means that for every state in the subsystem χ_2, there is dual index type structure of state in subsystem χ_1 if $I_{11} + I_{21} = K$, where is K is constant.

For $I_{11} + I_{21} = 0$, Figure 5.6 shows the characteristics of the optimal solution for $\sigma_2 = (0, 0), \sigma_2 = (-5, -5), \sigma_2 = (5, 5)$ when the subcontractor is cheaper than the manufacturer. We again observe that the service rate of the component C_{11} is non-increasing with increasing inventory position I_{11}. Under all states of the subsystem χ_2, these results suggest a dual index type policy for component replenishment for a fixed total inventory position $I_{11} + I_{21} = 0$. For example, for $\sigma_2 = (-5, -5)$, the optimal service rate of component C_{11} at inventory position $I_{11} = 1$ changes from 3 to 1 meaning that only the subcontractor S_{11} is producing component C_{11} at $I_{11} = 1$. Next, the component C_{11} is neither produced by the manufacturer M_1 nor by the subcontractor S_{11} at inventory position $I_1 = 3$. We observe similar results for $I_1 + I_2 = K$, $K = -10, ..., 10$. However, the optimal policy might not be dual index type for

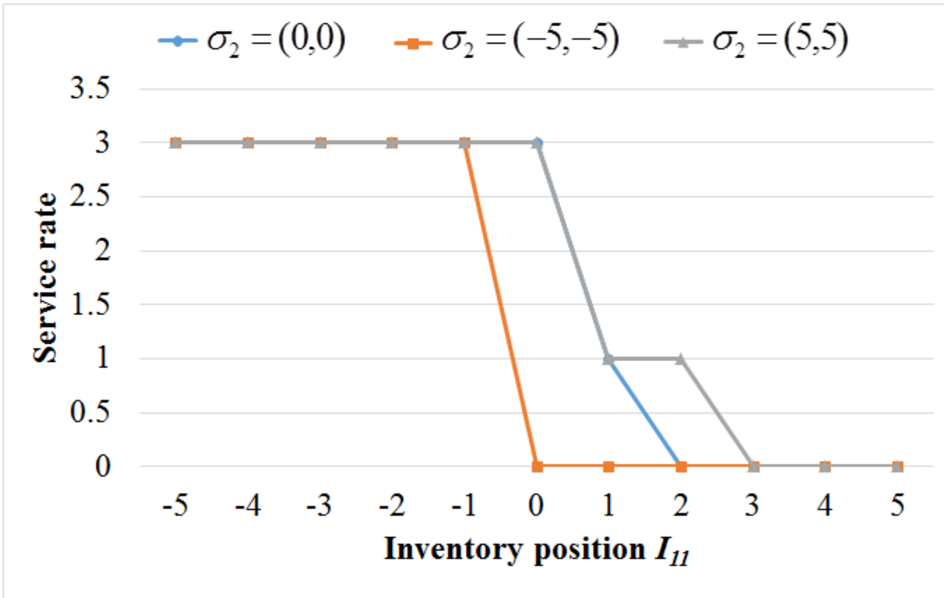

Figure 5.6 Optimal Service Rate of Component C_{11} for Experiment 3

other parameters. We also observe that there is no monotone property in service rate of a given product C_{11} if $I_{11} = K$ or $I_{21} = K$.

5.6.4 Accuracy of the Decomposition Approach

In this section, we present the accuracy of the decomposition method to solve multi-product system. We define $V(\sigma)^{E*}$ as the optimal value function at state $\sigma = (\sigma_1, \sigma_2)$ for the original system, and define $V(\sigma)^{A*} = V_1(\sigma_1)^* + V_2(\sigma_2)^*$ as the sum of optimal value functions for subsystems χ_1 and χ_2 respectively. Next, let $Er(\sigma) = (1 - V(\sigma)^{A*}/V(\sigma)^{E*}) * 100$ be the error percentage in the approximate method for state σ. To measure the accuracy of the decomposition method, we define $b(\sigma) = \sum_{ij} b_{ij} \max(-I_{ij}, 0)$ as the total backordering cost where $b_{ij} = 40, i, j = 1, 2$.

Table 5.8 presents the error range in the decomposition and the number states that have error in the value function in the corresponding error range. For example, in Experiment 1,

Table 5.8 Accuracy of the Decomposition Approach

Experiment	$Er(\sigma)$ Range	Number of States
Experiment 1	$0 - 5\%$	1420
	$5 - 10\%$	6641
	$10 - 15\%$	6206
	$> 15\%$	374
Experiment 2	$0 - 5\%$	7371
	$5 - 10\%$	5090
	$10 - 15\%$	1078
	$> 15\%$	1102
Experiment 3	$0 - 5\%$	5682
	$5 - 10\%$	6723
	$10 - 15\%$	903
	$> 15\%$	1333

1420 states falls under $0 - 5\%$ error range with respect to value functions. When the production costs are zero, then more than 50% of states have less than 10% error range leading to an average error of 9.5%. Next, when the manufacturer is cheaper, then more than 85% of states have less than 10% error range leading to an average error of 6.7%. Finally, when the subcontractor is cheaper, then more than 85% of states have less than 10% error range leading to an average error of 8.1%.

5.7 Conclusions

In this chapter, we consider a multi-product system where two products are assembled from two components. we assume that components are made to stock and can manufactured from dedicated external subcontractor and shared in-house manufacturing department. We develop an approximate method that uses decomposition of the original system into component based subsystems and uses iterative procedure to determine the solution. For each subsystem, we leverage results from Chapter 4 to significantly reduce the action space of the original system. For instance, for a subsystem with negligible production costs, the action space for each subsystem reduces from 12 actions to 5 actions. Similarly, we reduce the action space for subsystems where the manufacturer is cheaper than the subcontractor, and subsystems where the manufacturer is expensive than the subcontractor. Next, using iterative approach, we determine the optimal solution of the original system. We develop numerical experiments that provide insights on the structure the optimal solution for the original system. If the productions costs of the manufacturer and the subcontractor are zero or the manufacturer is expensive than the subcontractor then we observe that for every state in subsystem χ_2 there is a dual index type policy in subsystem χ_1. Numerically, we show that the approximate method is fairly accurate in certain cases.

Chapter 6

Capacity and Production Decision for Knowledge-type Components

6.1 Introduction

In the energy equipment industry, knowledge-type components could account for 30-50% of the bill of materials of the final product and contribute to 20-40% of the product revenue (in terms of sales from original equipment and aftermarket). Knowledge-type involve involve proprietary intellectual property (in the form of proprietary designs, manufacturing processes, or both).

Knowledge-type components are made to order (in contrast to standard-type components that are made to stock); based on the customers' unique requirements (performance criteria, operating environment, etc). These components often help manufacturers differentiate their components from those manufactured by competitors and help to gain a competitive edge in the market. For instance, blow out preventer (BOP) valves are considered as knowledge-type components by several rig manufacturers like Cameron, National Oilwell Varco, and Schlumberger. The proprietary designs for these components provide unique safety and reliability ratings for the rigs and are sometimes the deciding factor in awarding contracts. However, such components require high capital investment and the cost to under utilize the available capacity is significant. In such cases, knowledge-type components might need to be strategically subcontracted to vendors for various reasons: (i) the subcontractor has available capacity that helps the manufacturer gain more revenue during the market up-cycles, or

(ii) the manufacturer has high unused capacity costs (overhead costs) and balances tradeoffs between unused capacity costs and production costs during market down-cycles.

In this chapter, we consider a supply chain setting where the manufacturer could subcontract manufacturing of knowledge-type components to external vendors. However, we consider the scenario where the subcontractor has cost associated with unused capacity. This leads to the following research questions: (i) When and how much capacity should the manufacturer and the subcontractor invest on? (ii) What is the structure of the optimal policy and how does the unused capacity impact the optimal policy? (iii) How can we reduce the gap between the system with centralized control and system if decentralized control. We develop Markov decision process model for centralized system and stochastic game model for decentralized system, and analytically analyze the structure of the optimal policy under both settings.

The rest of the chapter is organized as follows: Section 6.2 describes the system model and assumptions for the centralized system and presents the Markov decision process model for the system. Section 6.3 describes the optimal production and capacity decisions for one time period model. Section 6.4 describes the optimal production and capacity decisions for multiple time period model. Section 6.5 describes the system model and assumptions for decentralized system and presents the stochastic game formulation of the system. Finally, Section 6.6 summarizes the findings.

6.2 Capacity and Production System Model

We analyze a make-to-order manufacturing system producing knowledge-type components in a multi-period setting under centralized decision making. At each time $t, t = 1, ..., T$, components can be manufactured either by using available capacity $C_{m,t}$ at the manufacturer M, or by using available capacity $C_{s,t}$ at the subcontractor S (see Figure 6.1). Typically, the time periods could be in years. We assume that the capacities $C_{m,t}$ and $C_{s,t}$ are associated with long term investment on a special purpose capital equipment that cannot be disposed or decreased in the later time periods. We assume that the demand at time t is denoted by d_t and can take values $d_{low,t}$ or $d_{high,t}$ with probability q and $(1-q)$ respectively.

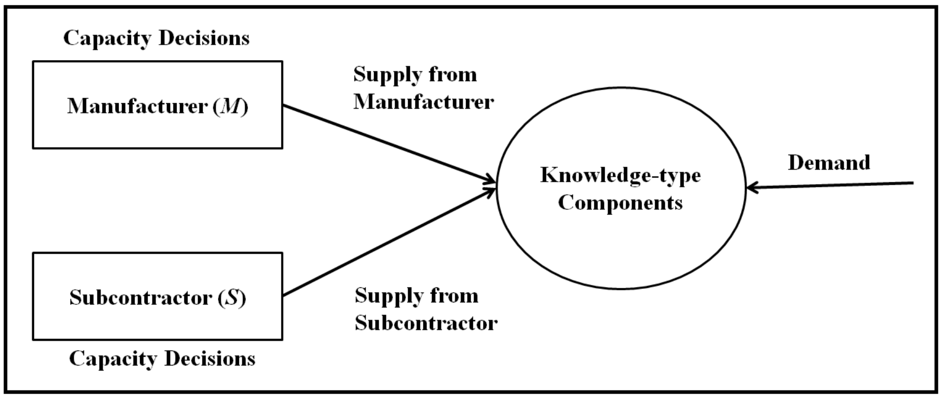

Figure 6.1 Manufacturing System with Knowledge-type Components

At the beginning of each time period t, both the manufacturer M and the subcontractor S decide if they want to invest in additional capacity $c_{m,t}$ and $c_{s,t}$ respectively. We assume that the lead time to make these capital investments is negligible and hence a decision to invest increases the capacity level of the manufacturer and subcontractor from $C_{m,t-1}$ to $C_{m,t} = C_{m,t-1} + c_{m,t-1}$, and $C_{s,t-1}$ to $C_{s,t} = C_{s,t-1} + c_{s,t-1}$ respectively. Next, the demand d_t is realized. Demands are in the terms of confirmed order of components with promised delivery dates in the future. After observing the demand, the manufacturer M decides to produce $x_{m,t}$ quantity of the component at a production cost of f_m per unit, and subcontract $x_{s,t}$ quantity to the subcontractor S at a subcontracting cost of f_s per unit. Note that excess production at time period t cannot be used to satisfy demands for later time periods as the demand is associated with knowledge-type components and are make-to-order. Therefore, any excess inventory is disposed at zero cost. Since, the capacity investments correspond to capital equipment, both the manufacturer and the subcontractor incur unused capacity costs o_m and o_s respectively to absorb the relevant overheads. The total unused capacity cost incurred by the manufacturer and the subcontractor in time period t is given by $o_m(C_{m,t} + c_{m,t} - x_{m,t})$ and $o_s(C_{s,t} + c_{s,t} - x_{s,t})$ respectively. At the end of time period t, manufacturer generates a per unit revenue $w_t = A - e_m \min(d_t, x_{m,t} + x_{s,t})$ by satisfying demands to the extent possible through components manufactured in that time period. Note that, A and e_m control the intersection and slope of the revenue function with respect to the total production.

We use Markov decision process formulation to determine the optimal capacity and production decisions of the system. The key elements in the Markov decision process formulation are as follows:

Decision epoch: In the manufacturing system, actions are taken at epochs corresponding to every time periods $t = 1, 2, ..., T$.

State space, Σ: The state of the system is described with a 2-tupled state space with state $\sigma = (C_{m,t}, C_{s,t}), \sigma \in \Sigma$, where $C_{m,t}$ and $C_{s,t}$ are the capacity levels at the manufacture M and the subcontractor S respectively at the beginning of time t.

Action space, $\mathbb{A}$: The action space $\mathbb{A}$ represents the set of actions with $a = (c_{m,t}, c_{s,t}), a \in \mathbb{A}$. Here, $c_{m,t}$ is the available capacity choices for the manufacturer M and can take values $0, c, 2c, ..., nc$, and $c_{s,t}$ is the available capacity choices for the subcontractor S and can take values $0, c, 2c, ..., nc$. Note that the manufacturer and the subcontractor, each has n capacity level choices.

Transition probabilities: Define $p(\sigma'|\sigma, a)$ as the transition probability for any state $\sigma = (C_{m,t}, C_{s,t})$ to state $\sigma' = (C'_{m,t}, C'_{s,t})$ corresponding to action $a \in \mathbb{A}$. Then, the transition probability $p(\sigma'|\sigma, a) = 1$ if $C'_{m,t} = C_{m,t} + c_{m,t}$ and $C'_{s,t} = C_{s,t} + c_{s,t}$, 0 otherwise.

Cost equation: Define $w(\sigma, a, x_{m,t}, x_{s,t}) = w_t \min(d_t, x_{m,t} + x_{s,t})$ as the total expected revenue, $f(\sigma, a, x_{m,t}, x_{s,t}) = f_m x_{m,t} + f_s x_{s,t}$ as the total production cost, and $o(\sigma, a, x_{m,t}, x_{s,t}) = o_m(C_{m,t} + c_{m,t} - x_{m,t}) + o_s(C_{s,t} + c_{s,t} - x_{s,t})$ as the total unused capacity cost. We let $g_{\sigma,a}(x_{m,t}, x_{s,t}) = w(\sigma, a, x_{m,t}, x_{s,t}) + f(\sigma, a, x_{m,t}, x_{s,t}) + o(\sigma, a, x_{m,t}, x_{s,t})$ denote the expected reward at state $\sigma = (C_{m,t}, C_{s,t})$, action $a = (c_{m,t}, c_{s,t})$, and production quantities $x_{m,t}, x_{s,t}$. Let $r(\sigma, a)$ denote the immediate reward function at state σ for action a, defined as $r(\sigma, a) = \max_{x_{m,t}, x_{s,t}} \mathbb{E}[g_{\sigma,a}(x_{m,t}, x_{s,t})]$, where $x^*_{m,t}$ and $x^*_{s,t}$ correspond to optimal production quantities of the manufacturer and the subcontractor respectively that maximizes the expected reward $\mathbb{E}[g_{\sigma,a}(x_{m,t}, x_{s,t})]$. We construct a standard Bellman cost equation for the system with value function, $V_t(\sigma)$ at state σ and decision epoch t. Equation (6.1) defines the value function $V_{t+1}(\sigma)$ at state $\sigma = (C_{m,t}, C_{s,t})$ and discount factor $\eta, \eta \in (0, 1)$.

$$V_t(C_{m,t}, C_{m,t}) \quad = \quad \max_{a \in \mathbb{A}} [r(\sigma, a) + \eta \sum_{\sigma'} p(\sigma' | \sigma, a) V_{t+1}(\sigma')] \tag{6.1}$$

Note that our formulation does not include cost of capacity investment, but only costs/revenues associated with use of invested capacity for two reasons: (i) capacity investments often are paid from strategic cost pool; our focus is on the operational costs/revenues from the use of this capacity, (ii) considering investments confound the operational challenge of how capacity must be used with the strategic question of payback on capacity investment.

The underlying problem has two-tupled state space and action space with $(n+1)^2$ states and $(n+1)^2$ actions, increasing the complexity to characterize the structure of the optimal policy. In the next section, we analyze one time period problem to study the impact of tradeoffs in unused capacity costs and productions costs on the optimal production and capacity decisions.

6.3 Optimal Solution for One Time Period Problem

We analyze a one time period problem ($n = 1$) where the manufacturer and the subcontractor have 0 capacity at the beginning of $t = 1$, i.e. $C_{m,t} = C_{s,t} = 0$. The manufacturer M and subcontractor S, each has two capacity choices $c_{m,t} = \{0, c\}$ and $c_{s,t} = \{0, c\}$ respectively. Note that for one time period problem and for any state $\sigma = (C_{m,2}, C_{s,2})$, $V_2(\sigma) = 0$ in the Equation (6.1). If the manufacturer and the subcontractor, each invest in capacity $c_{m,t}$ and $c_{s,t}$ respectively, then we define the supply chain capacity to be $C_{m,t} + c_{m,t} + C_{s,t} + c_{s,t}$ or equivalently, $c_{m,t} + c_{s,t}$. Further, we assume that parameters A and e_m take values such that the total profit at optimal decision is always positive.

For this two-tupled state space and action space, determining the optimal capacity and production decisions for one time period problem is non-trivial. Typically, for a system with no unused capacity costs (i.e. $o_m = o_s = 0$), the optimal decision should recommend capacity investment at whoever has the lower production cost. Similarly, for a system with equal production costs (i.e. $f_m = f_s$), the optimal decision should recommend capacity investment at whoever has the lower unused capacity cost. However, if the productions costs and the unused capacity costs are distinct and non-zero then the tradeoffs between production costs and unused capacity costs make the capacity investment decisions non-trivial. Further, understanding the tradeoffs in the context of a single time period setting can be very useful to determine the optimal decisions for the multi-time period problem (Section 6.4). In the subsequent sections, we analyze the structure of the optimal policy and reward function for one time period problem.

6.3.1 Properties of the Reward Function

In this section, we present properties of the expected reward function $r(\sigma, a)$. Note that $C_{m,t+1}$ and $C_{s,t+1}$ correspond to the capacity of the manufacturer and the subcontractor respectively after making capacity investments at time t, i.e $C_{m,t+1} = C_{m,t} + c_{m,t}$ and $C_{s,t+1} = C_{s,t} + c_{s,t}$. We assume that $o_m < f_m$ and $o_s < f_s$. This ensures that the optimal production

$x^*_{m,t} + x^*_{s,t}$ at time t is less than the maximum demand $d_{high,t}$. Further, we assume that $d_{low,t} = 0, t = 1, ..., T$. Then, the expected reward function is defined as:

$$\begin{aligned} r(\sigma, a) &= \mathbb{E}[w(\sigma, a, x^*_{m,t}, x^*_{s,t}) + f(\sigma, a, x^*_{m,t}, x^*_{s,t}) + o(\sigma, a, x^*_{m,t}, x^*_{s,t})] \\ &= \mathbb{E}[(A - e_m \min(d_t, x^*_{m,t} + x^*_{s,t})) \min(d_t, x^*_{m,t} + x^*_{s,t}) + f_m x^*_{m,t} + f_s x^*_{s,t} \\ &\quad + o_m(C_{m,t} + c_{m,t} - x^*_{m,t}) + o_s(C_{s,t} + c_{s,t} - x^*_{s,t})] \end{aligned} \tag{6.2}$$

If we have $x^*_{m,t} + x^*_{s,t} < d_{high,t}$, then $\mathbb{E}[(A - e_m \min(d_t, x^*_{m,t} + x^*_{s,t})) \min(d_t, x^*_{m,t} + x^*_{s,t})] = A(1-q)(x^*_{m,t} + x^*_{s,t}) - e_m(1-q)^2(x^*_{m,t} + x^*_{s,t})^2$. Similarly, if we have $x^*_{m,t} + x^*_{s,t} \geq d_{high,t}$, then $\mathbb{E}[(A - e_m \min(d_t, x^*_{m,t} + x^*_{s,t})) \min(d_t, x^*_{m,t} + x^*_{s,t})] = A(1-q)d_{high,t} - e_m(1-q)^2 d^2_{high,t}$. Then, for instance, if $x^*_{m,t} + x^*_{s,t} < d_{high,t}$, Equation (6.2) can be further simplified as:

$$\begin{aligned} r(\sigma, a) &= A(1-q)(x^*_{m,t} + x^*_{s,t}) - e_m(1-q)^2(x^*_{m,t} + x^*_{s,t})^2 \\ &\quad + f_m x^*_{m,t} + f_s x^*_{s,t} + o_m(C_{m,t} + c_{m,t} - x^*_{m,t}) + o_s(C_{s,t} + c_{s,t} - x^*_{s,t}) \\ &= -e_m(1-q)^2(x^*_{m,t} + x^*_{s,t})^2 + (A(1-q) + o_m - f_m)x^*_{m,t} \\ &\quad + (A(1-q) + o_s - f_s)x^*_{s,t} - o_m(C_{m,t} + c_{m,t}) - o_s(C_{s,t} + c_{s,t}) \\ &= -\alpha(x^*_{m,t} + x^*_{s,t})^2 + \beta x^*_{m,t} + \gamma x^*_{s,t} + \delta \end{aligned} \tag{6.3}$$

Where $\alpha = e_m(1-q)^2$, $\beta = A(1-q) + o_m - f_m$, $\gamma = A(1-q) + o_s - f_s$, and $\delta = -o_m(C_{m,t} + c_{m,t}) - o_s(C_{s,t} + c_{s,t})$. Lemma 6.1 provides the optimal production quantity for the manufacturer and the subcontractor for a given state σ and action a at time t.

Lemma 6.1. *For the system with expected reward* $g_{\sigma,a}(x_{m,t}(\sigma, a), x_{s,t}(\sigma, a)) = -\alpha(x_{m,t}(\sigma, a) + x_{s,t}(\sigma, a))^2 + \beta x_{m,t}(\sigma, a) + \gamma x_{s,t}(\sigma, a) + \delta$ *at state* $\sigma = (C_{m,t}, C_{s,t})$ *and action* $a = (c_{m,t}, c_{s,t})$, *and for* $d_{high,t} > \min(\frac{\beta}{2\alpha}, \frac{\gamma}{2\alpha})$,
(1) if $\beta < \gamma$, *then* $x^*_{m,t}(\sigma, a) = \min(\max(\beta/2\alpha - x^*_{s,t}(\sigma, a), 0), C_{m,t} + c_{m,t})$ *and* $x^*_{s,t}(\sigma, a) = \min(\gamma/2\alpha, C_{s,t} + c_{s,t})$.
(2) if $\beta > \gamma$, *then* $x^*_{m,t}(\sigma, a) = \min(\beta/2\alpha, C_{m,t} + c_{m,t})$ *and* $x^*_{s,t}(\sigma, a) = \min(\max(\gamma/2\alpha - x^*_{m,t}(\sigma, a), 0), C_{s,t} + c_{s,t})$.

Proof. To prove Lemma 6.1, we take partial derivatives of $g_{\sigma,a}(x_{m,t}(\sigma,a), x_{s,t}(\sigma,a))$ with respect to $x_{m,t}(\sigma,a)$ and $x_{s,t}(\sigma,a)$ and show the desired results. The details of the proof are in the Appendix. □

Note that the expected reward function $g_{\sigma,a}(x_{m,t}, x_{s,t})$ is concave with respect to $x_{m,t}$ and $x_{s,t}$, and $x^*_{m,t} = \beta/2\alpha$ and $x^*_{s,t} = \gamma/2\alpha$ are the individual saddle points of the maximization equation $g_{\sigma,a}(x_{m,t}, x_{s,t}) = -\alpha(x_{m,t} + x_{s,t})^2 + \beta x_{m,t} + \gamma x_{s,t} + \delta$. We observe that the optimal production decision depends on the relative difference between β and γ. Define $\Delta o = (o_s - o_m)$ and $\Delta f = (f_s - f_m)$. Then $\beta < \gamma$ implies $o_m - f_m < o_s - f_s$ or equivalently $\Delta f < \Delta o$. Similarly, $\beta > \gamma$ implies $\Delta f > \Delta o$. In Lemma 6.1, we observe that for the case of equal production cost i.e. $f_m = f_s$, if the unused capacity cost at the subcontractor, o_s is more than the unused capacity cost at the manufacturer, o_m then the optimal production quantity at the supplier is more than that of the manufacturer. This happens to reduce the total unused capacity cost at the subcontractor. Then, if $\beta < \gamma$, $\beta/2\alpha < C_{m,t} + c_{m,t}$, and $\gamma/2\alpha < C_{s,t} + c_{s,t}$, then $x^*_{s,t}(\sigma,a) = \gamma/2\alpha$ and $x^*_{m,t}(\sigma,a) = 0$. However, if $\beta < \gamma$, $\beta/2\alpha < C_{m,t} + c_{m,t}$, and $\gamma/2\alpha > C_{s,t} + c_{s,t}$, then $x^*_{s,t}(\sigma,a) = C_{s,t} + c_{s,t}$ and $x^*_{m,t}(\sigma,a) = \min(\max(\beta/2\alpha - x^*_{s,1}(\sigma), 0), C_{m,t} + c_{m,t})$.

Further note that, based on relative difference of demand and saddle points, we get three cases. For example, when $\beta < \gamma$:

Case (i): demand is more than the saddle point of the subcontractor, i.e. $d_{high,t} > \gamma/2\alpha > \beta/2\alpha$ then the optimal production quantities are given by Lemma 6.1 and illustrated in Figure 6.2(a).

Case (ii): demand is more than the saddle point of the manufacturer but less than the saddle point of the subcontractor, i.e. $\beta/2\alpha < d_{high,t} < \gamma/2\alpha$ then we use Lemma 6.1 to determine production quantities, we find that both the manufacturer and the subcontractor should produce at most up to the capacity, or their respective saddle points, or up to the demand.

Case (iii): demand is less than the saddle point of the manufacturer, i.e. $d_{high,t} < \beta/2\alpha < \gamma/2\alpha$ then the optimal production quantity of the subcontractor is the minimum of the current capacity and the demand (Figure 6.2(b)).

If $\beta > \gamma$, then based on relative difference of $d_{high,t}, \beta/2\alpha, \gamma/2\alpha$, we get three similar cases.

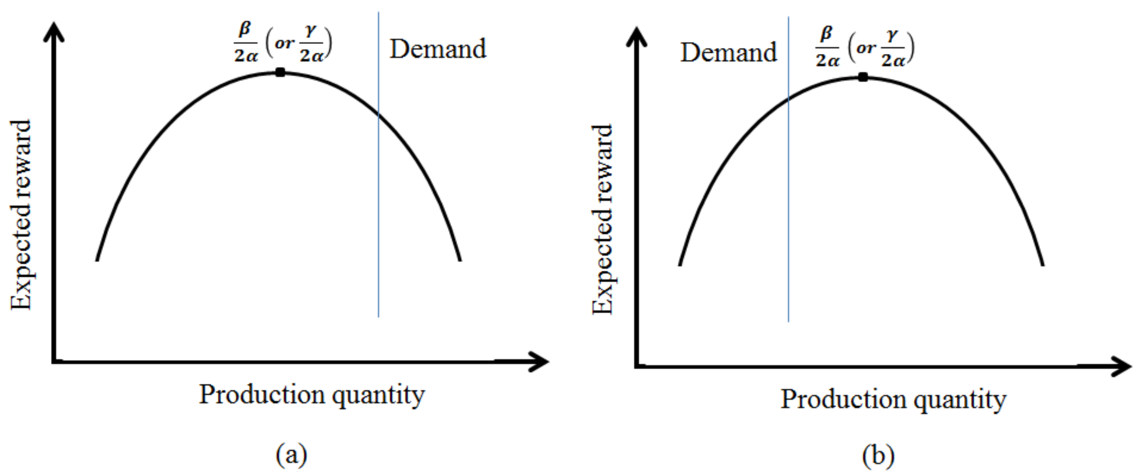

Figure 6.2 Reward Function with Respect to Production Quantity

So far we considered optimal production decisions for a given capacity decision. In Section 6.3.2 and Section 6.3.3, we analyze optimal capacity decisions. First for the case of system with sufficient capacity where the saddle point corresponding to the manufacturer and the subcontractor is less than the capacity c , i.e. $\beta/2\alpha < c$, $\gamma/2\alpha < c$ in Section 6.3.2, and then for the case of system with insufficient capacity with respect to the subcontractor where the saddle point for manufacturer is less than the maximum capacity at manufacturer, but the saddle point at the subcontractor is more than the maximum capacity at subcontractor, i.e $\beta/2\alpha < c$, $\gamma/2\alpha > c$ in Section 6.3.3.

6.3.2 System with Sufficient Capacity

We define a system with sufficient capacity as a system where $\beta/2\alpha < c$, $\gamma/2\alpha < c$. We define it in terms of the saddle points and not in the traditional terms of demand and capacity because in this setting, the manufacturer and the subcontractor are going to make no more than their optimal production quantities defined by $\beta/2\alpha$ and $\gamma/2\alpha$. Theorem 6.1 provides the conditions under which either the subcontractor invests in capacity c or the manufacturer invests in capacity c. Recall that $\Delta o = (o_s - o_m)$ and $\Delta f = (f_s - f_m)$. The capacity investment decision depends on the relative difference between Δo and Δf, and the difference in the maximum unused capacity cost $c\Delta o$.

Theorem 6.1. *For the system with sufficient capacity,*
(1) if $(\Delta o - \Delta f) > k_1 \Delta o$, where $k_1 = \frac{c}{\min(\frac{\beta+\gamma}{4\alpha}, d_{high,1})}$ then the optimal policy suggests that only the subcontractor should invest in capacity c.
(2) if $(\Delta o - \Delta f) < k_1 \Delta o$ then the optimal policy suggests that only the manufacturer should invest in capacity c.
(3) if $o_m > 0, o_s > 0$, it is not optimal for the supply chain capacity to take value of 0 or $2c$.

Proof. We prove Theorem 6.1 separately in three parts. For part (1) of the theorem, we show that if $(\Delta o - \Delta f) > k_1 \Delta o$, then the profit when only the subcontractor invests in capacity c is more than the profit when only the manufacturer invests in capacity c or when the supply chain capacity is 0 or $2c$. Similarly, we prove results for other parts. The details of the proof are in the Appendix. □

We observe that if $\Delta o - \Delta f$ is more than a threshold $k_1 \Delta o$, then only the subcontractor should invest in capacity c. Similarly, if $\Delta o - \Delta f$ is less than a threshold $k_1 \Delta o$, then only the manufacturer should invest in capacity c. Table 6.1 summarizes the conditions for Theorem 6.1. We observe that if the difference in the unused capacity cost, Δo is more than the difference in the production cost, Δf and if the demand $d_{high,1}$ decreases, then threshold k_1 increases and only the manufacturer invests in capacity c as opposed to the subcontractor. We also observe that it is not optimal for both the manufacturer and the subcontractor to invest in capacity at the same time, i.e. the supply chain capacity will never take the value $2c$ where the manufacturer and the subcontractor, each investing in a capacity c.

Table 6.1 Conditions for Optimal Capacity Levels for Theorem 6.1

Capacity at Subcontractor			
	c	$(\Delta o - \Delta f) > k_1 \Delta o$	$-$
	0	$-$	$(\Delta o - \Delta f) < k_1 \Delta o$
		0	c
		Capacity at Manufacturer	

If the unused capacity cost at the manufacturer is more than unused capacity cost at the subcontractor, i.e $\Delta o < 0$, and the difference in unused capacity cost, Δo is more than the difference in production cost, Δf, then conditions presented in Theorem 6.1 part (1) always hold, and suggests that only the subcontractor should invest in capacity c. Similarly, if the unused capacity cost at the manufacturer is less than unused capacity cost at the subcontractor, i.e $\Delta o > 0$, and the difference in unused capacity cost, Δo is less than the difference in production cost, Δf, then conditions presented in Theorem 6.1 part (2) always hold, and suggests that only the manufacturer should invest in capacity c.

Next, consider the special case where the unused capacity costs are negligible, i.e. $o_m = 0$ and $o_s = 0$. Then from Theorem 6.1 part (1), if the production cost at the manufacturer, f_m is more than the production cost at the subcontractor, f_s then only the subcontractor should invest in capacity c. Similarly, from Theorem 6.1 part (2), if the production cost at the manufacturer, f_m is less than the production cost at the subcontractor, f_s then only the manufacturer should invest in capacity c. These results are shown in Corollary 6.1.

Corollary 6.1. *For the system with sufficient capacity and* $o_m = o_s = 0$*,*

(1) if $\Delta f < 0$ *then the optimal policy suggests that either only the subcontractor should invest in capacity* c*.*

(2) if $\Delta f > 0$ *then the optimal policy suggests that either only the manufacturer should invest in capacity* c*.*

Proof. Proof of Corollary 6.1 follows directly from Theorem 6.1 by setting $o_m = 0, o_s = 0$. □

Next note that, if $o_m = o_s = 0$, then the optimal decision could suggest that both the manufacturer and the subcontractor should invest in capacity, which is not practical and hence we exclude this result from Corollary 6.1.

Note that under sufficient capacity, $\min(\frac{\beta+\gamma}{4\alpha}, d_{high,1}) < c$, and if the production costs are the same, i.e. $\Delta f = 0$, then from Theorem 6.1 part (1) if the unused capacity cost at the manufacturer, o_m is more than the unused capacity cost at the subcontractor, o_s then only the subcontractor should invest in capacity c. Similarly, from Theorem 6.1 part (2) if the unused capacity cost at the manufacturer, o_m is less than the unused capacity cost at the subcontractor, o_s then only the manufacturer should invest in capacity c. These results are shown in Corollary 6.2. Table 6.2 summarizes these conditions for special case presented in Corollary 6.1 and Corollary 6.2.

Corollary 6.2. *For the system with sufficient capacity and* $\Delta f = 0$*,*

(1) if $\Delta o < 0$ *then the optimal policy suggests that only the subcontractor should invest in capacity* c*.*

(2) if $\Delta o > 0$ *then the optimal policy suggests that only the manufacturer should invest in capacity* c*.*

Proof. Proof of Corollary 6.2 follows directly from Theorem 6.1. □

Table 6.2 Conditions for Optimal Capacity Levels for Corollary 6.1 and 6.2

Capacity at Subcontractor			
	c	$((o_m = o_s = 0) \wedge (\Delta f < 0)) \vee ((\Delta f = 0) \wedge (\Delta o < 0))$	$-$
	0	$-$	$((o_m = o_s = 0) \wedge (\Delta f > 0)) \vee ((\Delta f = 0) \wedge (\Delta o > 0))$
		0	c
		Capacity at Manufacturer	

6.3.3 System with Insufficient Capacity at Subcontractor

In this section, we present analysis for a system with insufficient capacity at the *subcontractor*, i.e. we assume that the saddle point of the subcontractor is more than the capacity (or $\gamma/2\alpha > c$). However, we do assume that for the saddle point of the manufacturer is less than the capacity (or $\beta/2\alpha < c$). Note that for the other case, for a system with insufficient capacity at the *manufacturer*, i.e. when the saddle point of the manufacturer is more than the capacity (or $\beta/2\alpha > c$), and saddle point of the subcontractor is less than the capacity (or $\gamma/2\alpha < c$), the analysis is very similar. The results can be obtained in similar way by interchanging the parameters of the manufacturer and the subcontractor. Theorem 6.2 provides conditions under which at optimal, only the manufacturer should invest in capacity c, or only the subcontractor should invest in capacity c. Table 6.3 summarizes the conditions in Theorem 6.2.

Theorem 6.2. *For the system with insufficient capacity,*
(1) if $\frac{\gamma}{2\alpha} < d_{high,1}$ *and* $(\Delta o - \Delta f) > k_2 \Delta o + k_3$*, where* $k_2 = \frac{c}{\frac{\beta+\gamma}{4\alpha}}$ *and* $k_3 = \frac{4(\frac{\gamma^2}{4\alpha^2} - c^2)}{\beta+\gamma}$ *then the optimal policy suggests that only the subcontractor should invest in capacity* c*.*

(2) if $\frac{\gamma}{2\alpha} < d_{high,1}$ *and* $(\Delta o - \Delta f) < k_2\Delta o + k_3$ *then the optimal policy suggests that only the manufacturer should invest in capacity* c*.*

Proof. To prove Theorem 6.2, we compare the profits when either manufacturer or subcontractor or both are investing in capacity and show the desired conditions. For part (1) of the theorem, we show that if $\frac{\gamma}{2\alpha} < d_{high,1}$ and $(\Delta o - \Delta f) > k_2\Delta o + k_3$, then the profit when only the subcontractor invests in capacity c is more than the profit when either only the manufacturer invests in capacity c or both the manufacturer and the subcontractor invest in capacity c. Similarly, we prove part (2). The details of the proof are in the Appendix. □

Note that Theorem 6.2 considers the case where $\frac{\gamma}{2\alpha} < d_{high,1}$. However, if $\frac{\gamma}{2\alpha} \geq d_{high,1}$, then the optimal production at the subcontractor and the manufacturer should either sum to the demand or should be up to their respective capacities. In this case, results can be obtained by following the similar procedure presented in the proof of Theorem 6.2. The condition presented in Theorem 6.2 does not consider equality such as $(\Delta o - \Delta f) = k_2\Delta o + k_3$, since in this case multiple decisions could be optimal where either only the manufacturer invests in capacity or only the subcontractor invests in capacity.

Table 6.3 Conditions for Optimal Capacity Levels for Theorem 6.2

Capacity at Subcontractor \ Capacity at Manufacturer	0	c
c	$(\Delta o - \Delta f) > k_2\Delta o + k_3$	–
0	–	$(\Delta o - \Delta f) < k_2\Delta o + k_3$

As a special case, note that if the unused capacity cost at the manufacturer is more than the unused capacity cost at the subcontractor, i.e $\Delta o < 0$, and the difference in unused capacity costs, Δo is more than the difference in production costs, Δf, then conditions presented in Theorem 6.2 part (1) always hold, and suggests that only the subcontractor should invest in capacity c. Similarly, if the unused capacity cost at the manufacturer is less than the unused capacity cost at the subcontractor, i.e $\Delta o > 0$, and the difference in unused capacity costs, Δo is less than the difference in production costs, Δf, then conditions presented in Theorem 6.2 part (2) always hold, and suggests that only the manufacturer should invest in capacity c. Similarly, for the system with zero unused capacity costs, the optimal capacity levels follow same conditions as in Corollary 6.1.

Next, if the production costs are the same, i.e. $\Delta f = 0$, then from Theorem 6.2 part (1), if $\Delta o > k_2 \Delta o + k_3$ then only the subcontractor should invest in capacity c. Similarly, from Theorem 6.2 part (2), if $\Delta o < k_2 \Delta o + k_3$ then only the manufacturer should invest in capacity c. These results are shown in Corollary 6.3.

Corollary 6.3. *For the system with insufficient capacity and $f_m = f_s$ (or $\Delta f = 0$),*
(1) if $\frac{\gamma}{2\alpha} < d_{high,1}$ and $\Delta o > k_2 \Delta o + k_3$ then the optimal policy suggests that only the subcontractor should invest in capacity c.
(2) if $\frac{\gamma}{2\alpha} < d_{high,1}$ and $\Delta o < k_2 \Delta o + k_3$ then the optimal policy suggests that only the manufacturer should invest in capacity c.

Proof. Proof of Corollary 6.3 follows directly from Theorem 6.2. □

Note that the conditions presented in Section 6.3 exclude the equality cases, such as $\Delta o = \Delta f$, etc since in these cases multiple decisions could be optimal. For one time period problem, we observe that the characterization of the optimal capacity and production decisions depend on not just productions costs (f_m, f_s) or unused capacity costs (o_m, o_s), but on the the difference in unused capacity cost, Δo and the difference in production cost, Δf. These results provides useful insights on the structure of the optimal policy and the results can be further extended to multi-time period model as described in the next section.

6.4 Optimal Solution for Multiple Time Period Problem

We first consider a two time period problem and make few observations. In contrast to the one time period problem, two time period problem poses additional challenges. First, under the assumption that capacity once acquired cannot be disposed, any capacity investment decisions in the first time period affects the capacity decisions and costs in the second time period. For example, under market down-cycles, capacity investments made at time $t = 1$ could result in high unused capacity costs at time $t = 2$. So, the resulting problem cannot be decomposed into multiple one-time period problems. Second, the optimal decision depends on the saddle points, $\beta/2\alpha, \gamma/2\alpha$ and demands, $d_{high,1}, d_{high,2}$. Note that, in the one time period problem, we observed three cases with respect to demand and saddle points. Using similar approach, we get six demand cases for two time period problem (three case for each time period). For example, if $d_{high,1} > \gamma/2\alpha > \beta/2\alpha$ and $d_{high,2} > \gamma/2\alpha > \beta/2\alpha$, then the optimal production quantity of the manufacturer and the subcontractor at both time periods should be up to capacity or saddle points $\beta/2\alpha, \gamma/2\alpha$ respectively. Similarly, if $d_{high,1} < \gamma/2\alpha < \beta/2\alpha$ and $d_{high,2} < \gamma/2\alpha < \beta/2\alpha$, then the optimal production quantity of the manufacturer and the subcontractor at time period t should be up to capacity or demand $d_{high,t}$. Other demand cases can be defined in similar way.

We analyze a two time period problem ($n = 2$) and provide structure of the optimal policy. We assume that the manufacturer and the subcontractor have 0 capacity at the beginning of time $t = 1$, i.e. $C_{m,t} = C_{s,t} = 0$ and can make capacity investment $c_{m,t} = \{0, c\}$ and $c_{s,t} = \{0, c\}$ respectively at time $t = 1, 2$. For notational simplicity, Table 6.4 defines several conditions that will be subsequently used to determine optimal actions. Each condition listed in Table 6.4 depends on the difference in unused capacity costs, Δo, difference in production costs, Δf, maximum unused capacity costs, $o_m c, o_s c$, and demands, $d_{high,1}, d_{high,2}$.

Table 6.4 Preliminary Conditions for Centralized System

$\mathcal{O}$	$(\Delta o - \Delta f) > 0$
$\hat{\mathcal{O}}$	$(\Delta o - \Delta f) < 0$
$\mathcal{M}_1$	$(\Delta o - \Delta f)\min(\frac{\beta+\gamma}{4\alpha}, d_{high,1}) > c\Delta o - o_m c$
$\hat{\mathcal{M}}_1$	$(\Delta o - \Delta f)\min(\frac{\beta+\gamma}{4\alpha}, d_{high,1}) < c\Delta o - o_m c$
$\mathcal{M}_2$	$(\Delta o - \Delta f)\min(\frac{\beta+\gamma}{4\alpha}, d_{high,2}) > o_s c$
$\hat{\mathcal{M}}_2$	$(\Delta o - \Delta f)\min(\frac{\beta+\gamma}{4\alpha}, d_{high,2}) < o_s c$
$\mathcal{S}_1$	$(\Delta o - \Delta f)\min(\frac{\beta+\gamma}{4\alpha}, d_{high,1}) < c\Delta o - o_s c$
$\hat{\mathcal{S}}_1$	$(\Delta o - \Delta f)\min(\frac{\beta+\gamma}{4\alpha}, d_{high,1}) > c\Delta o - o_s c$
$\mathcal{S}_2$	$(\Delta o - \Delta f)\min(\frac{\beta+\gamma}{4\alpha}, d_{high,2}) < -o_m c$
$\hat{\mathcal{S}}_2$	$(\Delta o - \Delta f)\min(\frac{\beta+\gamma}{4\alpha}, d_{high,2}) > -o_m c$
$\mathcal{MS}$	$(\Delta o - \Delta f)(\min(\frac{\beta+\gamma}{4\alpha}, d_{high,1}), \min(\frac{\beta+\gamma}{4\alpha}, d_{high,2})) > 2c\Delta o$
$\hat{\mathcal{MS}}$	$(\Delta o - \Delta f)(\min(\frac{\beta+\gamma}{4\alpha}, d_{high,1}), \min(\frac{\beta+\gamma}{4\alpha}, d_{high,2})) < 2c\Delta o$

We observe that conditions $\mathcal{M}_1$, $\hat{\mathcal{M}}_1$, $\mathcal{S}_1$, $\hat{\mathcal{S}}_1$ define the relationship between difference in unused capacity costs, difference in production costs, demand, and maximum unused capacity costs for time period $t = 1$. For instance, condition $\hat{\mathcal{M}}_1$ could hold for low demand $d_{high,1}$. Similarly, conditions $\mathcal{M}_2$, $\hat{\mathcal{M}}_2$, $\mathcal{S}_2$, $\hat{\mathcal{S}}_2$ define the relationship between difference in unused capacity costs, difference in production costs, demand, and maximum unused capacity costs for time period $t = 2$. For instance, condition $\hat{\mathcal{M}}_2$ could hold for low demand $d_{high,2}$. Finally, conditions $\mathcal{MS}$ and $\hat{\mathcal{MS}}$ connect demand and costs across the two time periods. In Section 6.4.1, we characterize the structure of the optimal decisions for two time periods under sufficient capacity case.

6.4.1 System with Sufficient Capacity

We define a system with sufficient capacity as a system where $\beta/2\alpha < c$, $\gamma/2\alpha < c$ for each time period $t, t = 1, 2$. Under system with sufficient capacity, Theorem 6.3 provides the optimal capacity decisions. These conditions depend on the demands $d_{high,1}$, $d_{high,2}$, and the relative difference between Δo and Δf.

Theorem 6.3. *For the system with sufficient capacity,*
(1) if conditions $((\mathcal{O} \wedge \hat{\mathcal{M}}_2 \wedge \hat{\mathcal{MS}}) \vee (\mathcal{M}_2 \wedge \mathcal{M}_1)) \vee (\hat{\mathcal{O}} \wedge \hat{\mathcal{S}}_2 \wedge \hat{\mathcal{MS}})$ *hold, then the optimal policy suggests that only the subcontractor should invest in capacity* c *at time period* $t = 1$ *and should maintain that capacity at time* $t = 2$.
(2) if conditions $(\mathcal{O} \wedge \hat{\mathcal{M}}_2 \wedge \mathcal{MS}) \vee ((\hat{\mathcal{O}} \wedge \hat{\mathcal{S}}_2 \wedge \mathcal{MS}) \vee (\mathcal{S}_2 \wedge \mathcal{S}_1))$ *hold, then the optimal policy suggests that only the manufacturer should invest in capacity* c *at time period* $t = 1$ *and should maintain that capacity at time* $t = 2$.
(3) if conditions $(\mathcal{O} \wedge \mathcal{M}_2 \wedge \hat{\mathcal{M}}_1)$ *hold, then the optimal policy suggests that the manufacturer should invest in capacity* c *at time period* $t = 1$ *and the subcontractor should invest in capacity* c *at time* $t = 2$.
(4) if conditions $(\hat{\mathcal{O}} \wedge \mathcal{S}_2 \wedge \hat{\mathcal{S}}_1)$ *hold, then the optimal policy suggests that only the subcontractor should invest in capacity* c *at time period* $t = 1$ *and the manufacturer should invest in capacity* c *at time* $t = 2$.

Proof. We prove each part of Theorem 6.3 separately. To prove part (1), we show that if conditions $((\mathcal{O} \wedge \hat{\mathcal{M}}_2 \wedge \hat{\mathcal{MS}}) \vee (\mathcal{M}_2 \wedge \mathcal{M}_1)) \vee (\hat{\mathcal{O}} \wedge \hat{\mathcal{S}}_2 \wedge \hat{\mathcal{MS}})$ hold, then the the profit when only subcontractor invests in capacity c at $t = 1$ and should maintain that capacity at $t = 2$ is more than the profit at any other actions. Similarly, we prove other parts. The details of the proof are in the Appendix. □

We observe that at optimal, the supply chain capacity can take the value of c at time $t = 1$, where either only the manufacturer invests in capacity or only the subcontractor invests in capacity. However, at time $t = 2$ the supply chain capacity can take the c or $2c$. If condition $\mathcal{O} : (\Delta o - \Delta f) > 0$ holds, or equivalently if $\Delta o > \Delta f$, and if conditions $(\mathcal{M}_2 \wedge \hat{\mathcal{M}}_1)$ hold, then the manufacturer invests in capacity c at time $t = 1$ and the subcontractor invests in capacity c at time $t = 2$. This happens because if $\Delta o > \Delta f$, then Lemma 6.1 suggests that producing components using available capacity of the subcontractor could be cheaper than that of the manufacturer. So, for some thresholds on $\Delta o - \Delta f$, we observe that the subcontractor invests in capacity at $t = 2$. Similarly, if condition $\hat{\mathcal{O}} : (\Delta o - \Delta f) < 0$ holds, or equivalently if $\Delta o < \Delta f$, then Lemma 6.1 suggests that producing components using available capacity of the manufacturer could be cheaper than that of the subcontractor. So, for some thresholds on $\Delta o - \Delta f$, we observe that the manufacturer invests in capacity at $t = 2$.

We observe that if the difference in the unused capacity cost, Δo is more than the difference in the production cost, Δf, then Lemma 6.1 suggests that producing components using available capacity of the subcontractor could be cheaper than that of the manufacturer, and at low demand $d_{high,2}$, there is enough capacity at the subcontractor to satisfy the demand. So, the optimal decision could recommend capacity investment at the subcontractor instead at the manufacturer. Similarly, if the difference in the unused capacity cost Δo is less than the difference in the production cost Δf, then Lemma 6.1 suggests that producing components using available capacity of the manufacturer could be cheaper than that of the subcontractor, and at low demand $d_{high,2}$, there is enough capacity at the manufacturer to satisfy the demand. So, the optimal decision could recommend capacity investment at the manufacturer instead at the subcontractor. Table 6.5 summarizes the results of Theorem 6.3 where we only show states with optimal solution.

Table 6.5 Conditions for Optimal Capacity Levels for Theorem 6.3

Time $t = 1$	Time $t = 2$	Conditions
$(0, c)$	$(0, c)$	$((\mathcal{O} \wedge \hat{\mathcal{M}}_2 \wedge \hat{\mathcal{M}\mathcal{S}}) \vee (\mathcal{M}_2 \wedge \mathcal{M}_1)) \vee (\hat{\mathcal{O}} \wedge \hat{\mathcal{S}}_2 \wedge \hat{\mathcal{M}\mathcal{S}})$
$(c, 0)$	$(c, 0)$	$(\mathcal{O} \wedge \hat{\mathcal{M}}_2 \wedge \mathcal{M}\mathcal{S}) \vee ((\hat{\mathcal{O}} \wedge \hat{\mathcal{S}}_2 \wedge \mathcal{M}\mathcal{S}) \vee (\mathcal{S}_2 \wedge \mathcal{S}_1))$
$(c, 0)$	(c, c)	$(\mathcal{O} \wedge \mathcal{M}_2 \wedge \hat{\mathcal{M}}_1)$
$(0, c)$	(c, c)	$(\hat{\mathcal{O}} \wedge \mathcal{S}_2 \wedge \hat{\mathcal{S}}_1)$

Conditions presented in Theorem 6.3 depend primarily on the relative difference between Δo and Δf, and certain threshold values. Using Corollary 6.4, we show a sufficient condition which depends on the relative difference between Δo and Δf.

Corollary 6.4. *For the system with sufficient capacity,*
(1) if $\Delta o < 0$ and $(\Delta o - \Delta f) > k_s$ where k_s is a non-negative threshold then the optimal policy suggests that only the subcontractor should invest in capacity c at time period $t = 1$ and maintain that capacity at time $t = 2$.
(2) if $\Delta o > 0$ and $(\Delta f - \Delta o) > k_m$ where k_m is a non-negative threshold then the optimal policy suggests that only the manufacturer should invest in capacity c at time period $t = 1$ and maintain that capacity at time $t = 2$.
(3) if $k_s < (\Delta o - \Delta f) < k_{ms}$ where k_{ms} is some threshold then the optimal policy suggests that only the manufacturer should invest in capacity c at time period $t = 1$ and subcontractor should invest in capacity c at time $t = 2$.
(4) if $k_m < (\Delta f - \Delta o) < k_{sm}$ where k_{ms} is some threshold then the optimal policy suggests that only the subcontractor should invest in capacity c at time period $t = 1$ and manufacturer should invest in capacity c at time $t = 2$.

Proof. Proof of Corollary 6.4 follows directly from Theorem 6.3. □

We observe that if the unused capacity cost at the manufacturer is more than the unused capacity cost at the subcontractor, and $\Delta o - \Delta f$ is more than a non-negative threshold $k_s = \min(\frac{(o_s - 2o_m)c}{\min(\frac{\beta+\gamma}{4\alpha}, d_{high,1})}, \frac{o_s c}{\min(\frac{\beta+\gamma}{4\alpha}, d_{high,2})})$, then from Theorem 6.3 part (1), the optimal policy suggests that only the subcontractor should invest in capacity c at time $t = 1$ and maintain that capacity at time $t = 2$. Similarly, if the unused capacity cost at the manufacturer is less than at the subcontractor, and $\Delta f - \Delta o$ is more than a non-negative threshold $k_m = \min(\frac{(o_m - 2o_s)c}{\min(\frac{\beta+\gamma}{4\alpha}, d_{high,1})}, \frac{o_m c}{\min(\frac{\beta+\gamma}{4\alpha}, d_{high,2})})$ then from Theorem 6.3 part (2) the optimal policy suggests that only the manufacturer should invest in capacity c at time $t = 1$ and maintain that capacity at time $t = 2$.

Again, if $\Delta o - \Delta f$ is bounded by thresholds, k_s and $k_{ms} = \frac{(o_s - 2o_m)c}{\min(\frac{\beta+\gamma}{4\alpha}, d_{high,1})}$ then the optimal policy suggests that only the manufacturer should invest in capacity c at time $t = 1$ and subcontractor should invest in capacity c at time $t = 2$. Similarly, if $\Delta f - \Delta o$ is bounded by thresholds, k_m and $k_{sm} = \frac{o_m c}{\min(\frac{\beta+\gamma}{4\alpha}, d_{high,2})}$ then the optimal policy suggests that only the subcontractor should invest in capacity c at time $t = 1$ and the manufacturer should invest in capacity c at time $t = 2$. Table 6.6 summarizes the results of Corollary 6.4.

Table 6.6 Conditions for Optimal Capacity Levels for Corollary 6.4

Time $t = 1$	Time $t = 2$	Conditions
$(0, c)$	$(0, c)$	$(\Delta o < 0) \wedge (\Delta o - \Delta f) > k_s$
$(c, 0)$	$(c, 0)$	$(\Delta o > 0) \wedge (\Delta f - \Delta o) > k_m$
$(0, c)$	(c, c)	$k_m < (\Delta f - \Delta o) < k_{sm}$
$(c, 0)$	(c, c)	$k_s < (\Delta o - \Delta f) < k_{ms}$

For the system with zero unused capacity costs, the optimal capacity levels for two time period follow same conditions as in Corollary 6.1. In Section 6.4.2, we numerically analyze the impact of unused capacity costs on the optimal decisions. Note that the conditions presented in Section 6.4 exclude the equality cases, such as $\Delta o = \Delta f$, etc since in these cases multiple decisions could be optimal.

6.4.2 Effect of Unused Capacity Cost

Using numerical studies, we analyze the effect of unused capacity on the optimal decision. Table 6.7 presents the system and production parameters for the experiment. We assume that the production cost at the manufacturer is more than the production cost at the subcontractor, and the unused capacity cost at the manufacturer is less than the unused capacity cost at the subcontractor. To analyze the effect of unused capacity, we consider that the unused capacity cost at the subcontractor can take values of 3, 3.5, and 4. Note that if $o_s = 3$ then conditions presented in Theorem 6.3 part (1) hold, if $o_s = 3.5$ then conditions presented in Theorem 6.3 part (3) hold, and if $o_s = 4$ then conditions presented in Theorem 6.3 part (2) hold.

Table 6.7 Parameters for Centralized System

System Parameters	Production Parameters
$T = 2$	$A = 150$
$q = 0.5$	$e_m = 20$
$d_{high,t} = 2, 8$	$o_m = 1$
$d_{low,t} = 0, 0$	$o_s = 3, 3.5, 4$
$a_m = \{0, 15\}$	$f_m = 25$
$a_s = \{0, 15\}$	$f_s = 18$

Figure 6.3 shows the optimal capacity levels for two time period model. We observe that as the unused capacity cost at the subcontractor increases then the production gradually shifts to the manufacturer to reduce the total cost due to unused capacity. For instance, if $o_s = 3$ then we observe that only the subcontractor invests in capacity at $t = 1$ and keep the same capacity at $t = 2$, validating the claim in Theorem 6.3 part (1). Next, if $o_s = 3.5$ then we observe that the subcontractor invests in capacity at $t = 1$ and the manufacturer invests in capacity at $t = 2$, validating the claim in Theorem 6.3 part (3). Finally, if $o_s = 4$ then we observe that the manufacturer invests in capacity at $t = 1$ and keep the same capacity at $t = 2$, validating the claim in Theorem 6.3 part (2).

Note that the centralized system prevents manufacturer and the subcontractor to make their own respective decisions to maximize individual profits. Typically in industry, it is common for multiple facilities (manufacturers or subcontractors) to make their own decisions. However, such decision could potentially lead to lower overall profit of the supply chain. In the next section, we present the decentralized system and study the impact of unused capacity

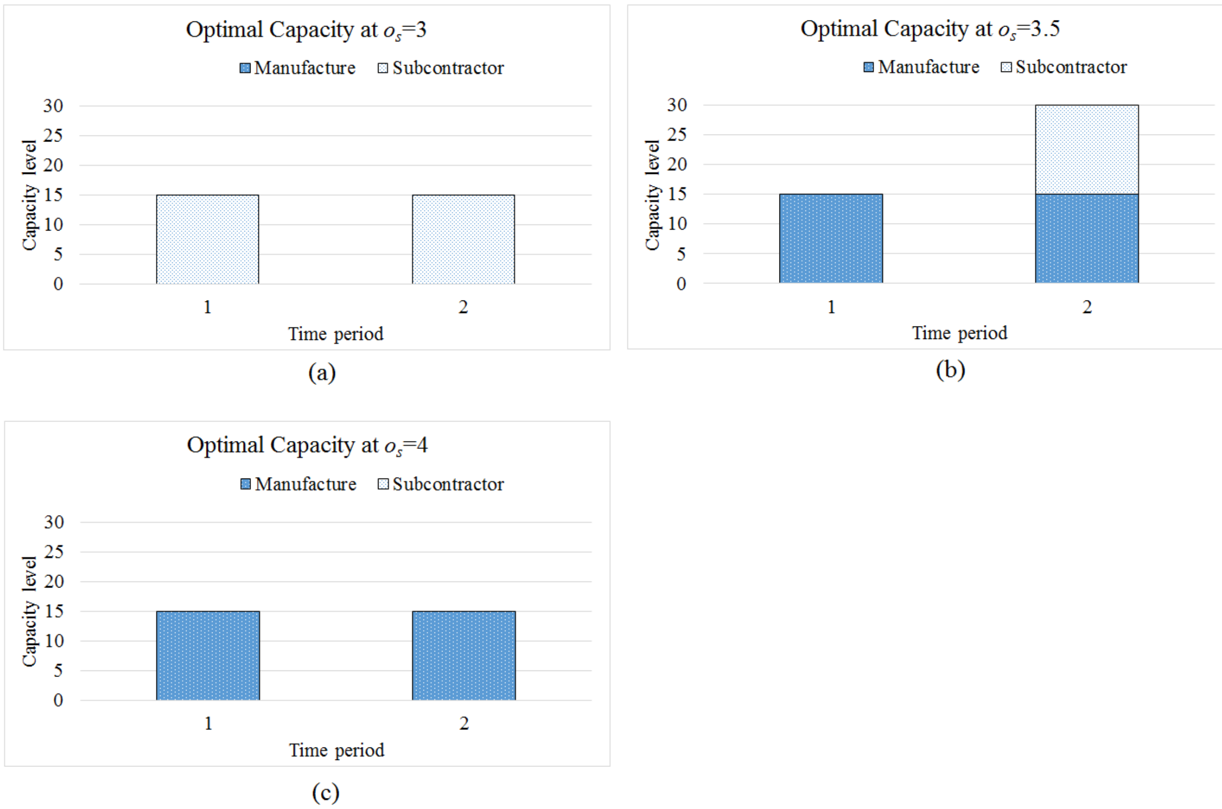

Figure 6.3 Optimal Capacity Decisions at (a) $o_s = 3$, (b) $o_s = 3.5$, (c) $o_s = 4$

costs on the characteristics of the optimal decision.

6.5 Decentralized System

We consider a decentralized supply chain setting with two autonomous firms, the manufacturer, M and the supplier, S that collaborate in the production of a knowledge-type component over a finite time horizon with distinct time periods, $1, \ldots, T$. In this setting, the manufacturer (follower) and the subcontractor (leader) indulge in a sequence of capacity-production-price game to produce a knowledge-type component (Cachon and Lariviere (2001); Wang and Gerchak (2003); Savaskan et al. (2004); Bernstein and DeCroix

(2004)). Figure 6.4 describes the sequence of events in the game.

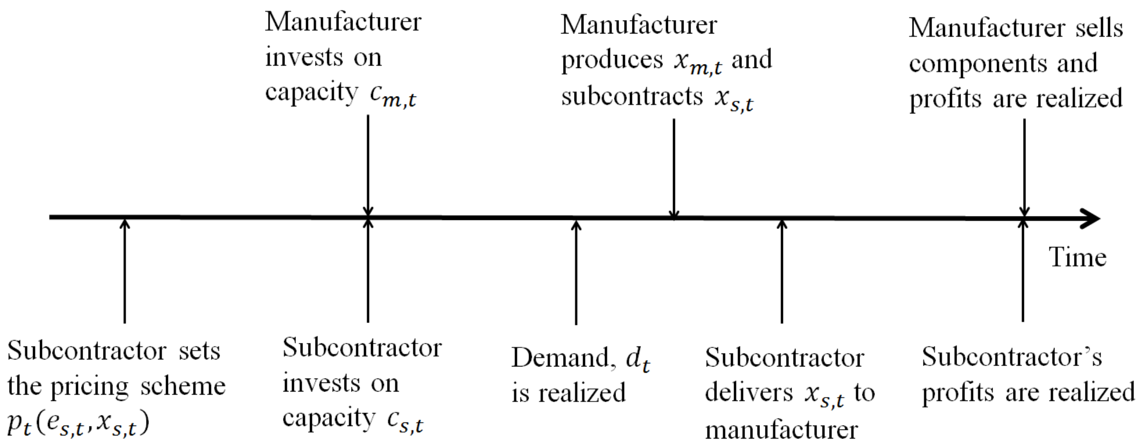

Figure 6.4 Sequence of Events

At the beginning of each time period t, the subcontractor shares the unit price, $p_t(e_{s,t}, x_{s,t}) = (b_0 - e_{s,t}x_{s,t})$ with the manufacturer, where b_0 and $e_{s,t}$ are the intercept and the slope of the price function with respect to production $x_{s,t}$. The manufacturer M and the subcontractor S decide to invest in capacity $c_{m,t}$ and $c_{s,t}$ respectively. This changes the current capacity levels of the manufacturer and the subcontractor to $C_{m,t} = C_{m,t-1} + c_{m,t-1}$ and $C_{s,t} = C_{s,t-1} + c_{s,t-1}$ respectively. Next, the demand d_t is realized which takes the values $d_{low,t}$ or $d_{high,t}$ with probability q and $(1-q)$ respectively. In response to the subcontractor S, the manufacturer M decides to produce $x_{m,t}$ units where, $x_{m,t} \in [0, C_{m,t}]$, and incurs a total production cost of $f_m(x_{m,t}) = f_m x_{m,t}$. The manufacturer also decides to subcontract $x_{s,t}$ units where, $x_{s,t} \in [0, C_{s,t}]$, and the subcontractor incurs a total production cost of $f_s(x_{s,t}) = f_s x_{s,t}$. Note that, any unused capacity incurs a penalty cost $o_{i,t}, i = M, S$ per unit capacity resulting in a cost $h_i(x_{i,t}) = o_{i,t}(C_{i,t} + c_{i,t} - x_{i,t}), i = M, S$ for the unused capacity. Unused capacity is of concern, since, capital investments are expensive and there is a pressure to recover the investments on assets. The market is characterized by diminishing return on the production quantity $x_{i,t}$ with revenue function $w_t(x_{m,t} + x_{s,t})$ per unit with $w_t(x_{m,t} + x_{s,t}) = A - e_m \min(d_t, x_{m,t} + x_{s,t})$ being a special case of this function. At the end of time period t, profits of manufacturer M,

$\pi_{m,t}$ and subcontractor S, $\pi_{s,t}$ are realized as shown in Equation (6.4) and (6.5) respectively.

$$\pi_{m,t} = w_t(x_{m,t}+x_{s,t})\min(d_t, x_{m,t}+x_{s,t}) - f_m(x_{m,t}) - h_m(x_{m,t}) - p_t(e_{s,t}, x_{s,t})x_{s,t} \quad (6.4)$$

$$\pi_{s,t} = p_t(e_{s,t}, x_{s,t})x_{s,t} - f_s(x_{s,t}) - h_s(x_{s,t}) \quad (6.5)$$

We analyze this system as a finite horizon stochastic game consisting of sequence of capacity-production-price type competition between the manufacturer and the subcontractor. The key elements of the stochastic game are:

Decision epoch: The manufacturer and the subcontractor take decisions at every time period $t, t = 1, ..., T$.

State space, Σ: We define state, $\sigma = (C_{m,t}, C_{s,t}), \sigma \in \Sigma$, where $C_{m,t}$ and $C_{s,t}$ are the capacities of manufacturer and subcontractor respectively at the beginning of time t.

Action space, $\mathbb{A}$: Let the action space $\mathbb{A} = \mathbb{A}_m \times \mathbb{A}_s$, where $a_m = (c_{m,t}), a_m \in \mathbb{A}_m$ denote the action action taken by manufacturer M, and $a_s = (c_{s,t}, e_{s,t}), a_S \in \mathbb{A}_S$ denote the action action taken by supplier S, where $e_{s,t}$ is the pricing scheme parameter.

Transition probabilities: Let $p(\sigma'|\sigma, a_m, a_s)$ denote the probability of transitioning from state $\sigma = (C_{m,t}, C_{s,t})$ to state $\sigma' = (C'_{m,t}, C'_{s,t})$ under actions a_m and a_s. Then, the transition probabilities are defined as: $p(\sigma'|\sigma, a_m, a_s) = 1$, if, $C'_{m,t} = C_{m,t} + c_{m,t}$ and $C'_{s,t} = C_{s,t} + c_{s,t}$, and $p(\sigma'|\sigma, a_m, a_s) = 0$, otherwise.

Profit function: Let $\pi_{m,t}(\sigma, a_m, a_s, x_{m,t}, x_{s,t})$ denote the profit function for manufacturer M at time t for state σ, actions a_m, a_s, and production quantities $x_{m,t}, x_{s,t}$, as given by Equation (6.6). The profit function of the manufacturer comprises of the following terms: revenue on the knowledge part $(w_t(x_{m,t}+x_{s,t})\min(d_t, x_{m,t}+x_{s,t}))$, production cost $(f_m(x_{m,t}))$, unused capacity cost function $(h_m(x_{m,t}))$, and price paid to the subcontractor $(p_t(e_{s,t}, x_{s,t})x_{s,t})$.

Similarly, let $\pi_{s,t}(\sigma, a_m, a_s, x_{m,t}, x_{s,t})$ denote the profit function for subcontractor S at state σ, actions a_m, a_s, and production quantities $x_{m,t}, x_{s,t}$, as given by Equation (6.7). The profit function of the subcontractor comprises of the following terms: revenue from the manufacturer ($p_t(e_{s,t}, x_{s,t})x_{s,t}$), production cost ($f_s(x_{s,t})$), and unused capacity cost function ($h_s(x_{s,t})$). We assume that the capacity investment cost is normalized within the unused capacity cost. Note that $\pi_{m,t}(\sigma, a_m, a_s, x_{m,t}, x_{s,t}) + \pi_{s,t}(\sigma, a_m, a_s, x_{m,t}, x_{s,t})$ is same as the reward $g_{\sigma,a}(x_{m,t}, x_{s,t})$ function presented in Section 6.2. However, in contrast to centralized system, in the decentralized system the manufacturer and the subcontractor makes their own capacity and production decisions.

$$\begin{aligned} \pi_{m,t}(\sigma, a_m, a_s, x_{m,t}, x_{s,t}) &= w_t(x_{s,t} + x_{m,t}) \min(d_t, x_{m,t} + x_{s,t}) - f_m(x_{m,t}) - h_m(x_{m,t}) \\ &\quad - p_t(e_{s,t}, x_{s,t})x_{s,t} \qquad (6.6) \\ \pi_{s,t}(\sigma, a_m, a_s, x_{m,t}, x_{s,t}) &= p_t(e_{s,t}, x_{s,t})x_{s,t} - f_s(x_{s,t}) - h_s(x_{s,t})) \qquad (6.7) \end{aligned}$$

Expected utility: Let $\mathcal{U}_{m,t}(\sigma)$ and $\mathcal{U}_{s,t}(\sigma)$ denote the expected utility of manufacturer M and subcontractor S respectively at state σ and decision epoch t. Then we can write the expected utilities for both firms as follows:

$$\begin{aligned} \mathcal{U}_{i,t}(\sigma) &= \max_{a_m, a_s} \Big(\max_{x_{s,t}, x_{m,t}} \pi_{i,t}(\sigma, a_m, a_s, x_{m,t}, x_{s,t}) \\ &\quad + \eta \sum_{\sigma' \in \Sigma} p(\sigma' | \sigma, a_m, a_s) \mathcal{U}_{i,t+1}(\sigma') \Big), \forall i = M, S \qquad (6.8) \end{aligned}$$

The above decentralized formulation has 2-dimensional states and 3-dimensional action which increases the complexity of the problem. In the next section, we use numerical examples to study the inefficiencies in the system due to decentralized control.

6.5.1 Inefficiencies due to Decentralized Control

Using numerical studies, we analyze the inefficiencies in the system due to decentralized control as compared to the system with centralized control. We assume that the subcontractor is the leader and the manufacturer is the follower. Table 6.8 presents the system and production parameters for the experiment. We consider a 5 period problem with 3 capacity level choices for the manufacturer. We assume that the production cost at the manufacturer is more than the production cost at the subcontractor, and the unused capacity cost at the manufacturer is less than the unused capacity cost at the subcontractor. we conduct this experiment in two steps. In the first step, we vary the price parameter b_0 from 10 to 30 to determine the optimal pricing parameter b_0 that results in lowest deviation in the total expected utility function (for the states transitioned at optimal) as compared to total expected value function (for the states transitioned at optimal) in the centralized system.

Table 6.8 Parameters for Centralized and Decentralized System

System Parameters	Production Parameters
$T = 5$	$A = 150$
$q = 0.5$	$e_m = 20$
$d_{high,t} = 2, 4, 8, 10, 12$	$o_m = 1$
$d_{low,t} = 0, 0, 0, 0, 0$	$o_s = 3.5$
$c_m = \{0, 2, 4\}$	$f_m = 25$
$c_s = \{0, 2, 4\}$	$f_s = 18$
$e_{s,t} = \{0, 1, 2\}$	
$b_0 = 10$ to 30	

Figure 6.5 shows the optimal total expected value function in the centralized system and the total expected utility function in the decentralized system. For example, at $b_0 = 10$, the total expected value function for the transitioned states is 2416 while the total expected utility function for the transitioned states is 1120. We observe that b_0 results in the lower difference between the total expected value function and utility function. We also, observe that the subcontractor should set $e^*_{s,t} = 0, t = 1, ..., 5$ meaning that the price increases linearly with the production quantity.

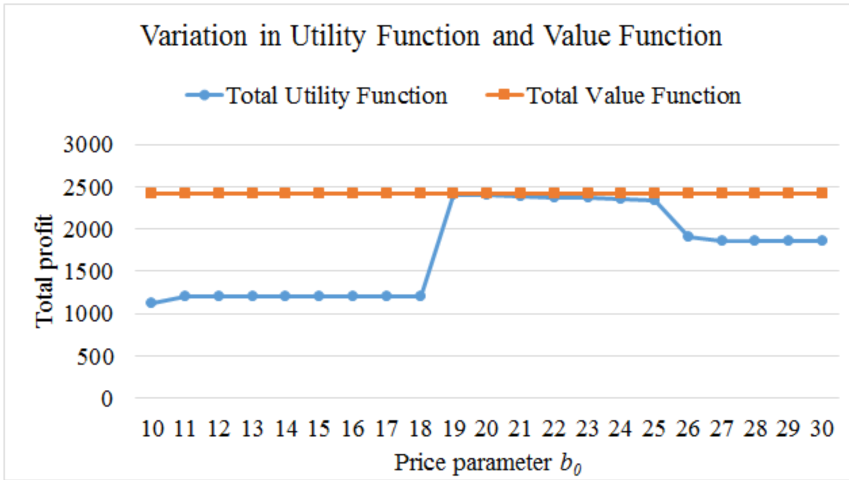

Figure 6.5 Variation in the Value Function and Utility Function with Respect to b_0

Next, Figure 6.6 presents the optimal capacity decisions for the system with centralized control and the corresponding system with decentralized control with $b_0^* = 19$.

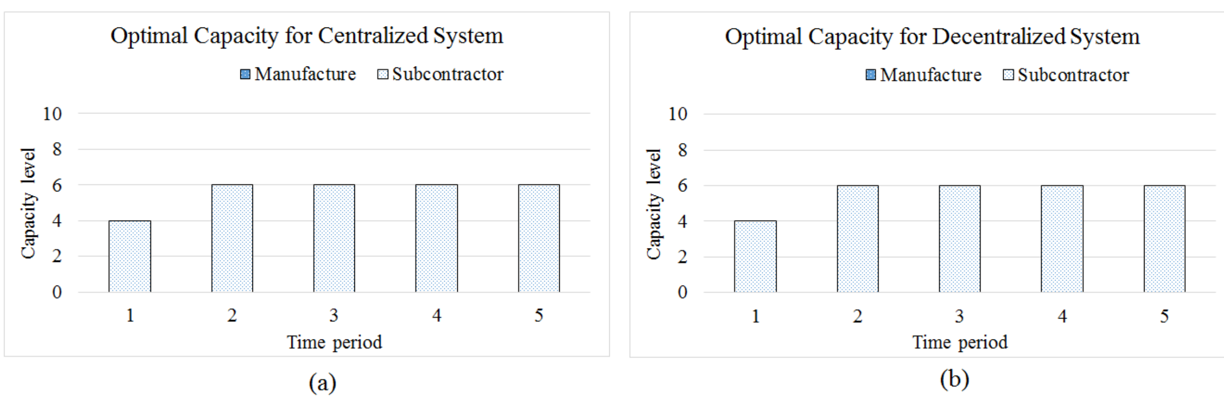

Figure 6.6 Optimal Capacity Decisions for (a) Centralized System (b) Decentralized System

We observe that only the subcontractor makes capacity investment, and the optimal capacity decisions for the system with decentralized control is same as the optimal decision for the system with centralized control. This happens because the optimal pricing parameter b_0^* makes the total expected utility function very close to the total expected value function. However, in reality the subcontractor may choose a different pricing parameter which could significantly change the optimal decisions.

6.6 Conclusions

We analyze a centralized system and a corresponding decentralized system consisting of a manufacturer and a subcontractor that balances tradeoffs between unused capacity costs and production costs to produce knowledge-type components. In the centralized system, the manufacturer and the subcontractor makes capacity investment and production decisions that maximizes the total profit of the system. Using Markov decision process, we analyze single period and multi-period problem and provide conditions to determine when and how much capacity should the manufacturer and the subcontractor invest. We observe that the optimal capacity decision depends on the relative difference between the unused capacity costs, Δo and production costs, Δf, and the relative difference between the maximum unused capacity at the manufacturer, $o_m c$ and maximum unused capacity at the subcontractor $o_s c$. In the decentralized system, the manufacturer and the subcontractor makes capacity investment and production decisions that maximizes their individual profits. Using game theory, we analyze single period and multi-period problem and provide conditions to determine when and how much capacity should the manufacturer and the subcontractor invest. we observe that the optimal decision depends on the relative difference between pricing parameter b_0, and the difference in the production cost and unused capacity cost at the manufacturer $f_m - o_m$. Using numerical experiments, we analyze the gap in the decentralized system as opposed to centralized system and determine the optimal pricing parameters to reduce the gap.

6.7 Appendix

Proof of Lemma 6.1: To prove Lemma 6.1, we first consider state $\sigma = (C_{m,t}, C_{s,t})$ and action $a = (c_{m,t}, c_{s,t})$, and take partial derivatives of $g_{\sigma,a}(x_{m,t}(\sigma,a), x_{s,t}(\sigma,a))$ with respect to $x_{m,t}(\sigma,a)$ and $x_{s,t}(\sigma,a)$ and show the desired results. Let $g_{\sigma,a}(x_{m,t}(\sigma,a), x_{s,t}(\sigma,a))$ be the expected reward function defined by:

$$g_{\sigma,a}(x_{m,t}(\sigma,a), x_{s,t}(\sigma,a)) = -\alpha(x_{m,t}(\sigma,a) + x_{s,t}(\sigma,a))^2 + \beta x_{m,t}(\sigma,a) + \gamma x_{s,t}(\sigma,a) + \delta$$

Now taking partial derivative of $g_{\sigma,a}(x_{m,t}(\sigma,a), x_{s,t}(\sigma,a))$ with respect to $x_{m,t}(\sigma,a)$ and $x_{s,t}(\sigma,a)$ we get

$$\frac{\partial g_{\sigma,a}(x_{m,t}(\sigma,a), x_{s,t}(\sigma,a))}{\partial x_{m,t}(\sigma,a)} = -2\alpha(x^*_{m,t}(\sigma,a) + x'_{s,t}(\sigma,a)) + \beta \quad (6.9)$$

$$\frac{\partial g_{\sigma,a}(x_{m,t}(\sigma,a), x_{s,t}(\sigma,a))}{\partial x_{s,t}(\sigma,a)} = -2\alpha(x'_{m,t}(\sigma,a) + x^*_{s,t}(\sigma,a)) + \gamma \quad (6.10)$$

Here, $x'_{s,t}(\sigma,a)$ represents the production quantity of the subcontractor when the expected reward function $g_{\sigma,a}(x_{m,t}(\sigma,a), x_{s,t}(\sigma,a))$ is maximized with respect to $x_{m,t}(\sigma,a)$. Similarly, $x'_{m,t}(\sigma,a)$ represents the production quantity of the manufacturer when the expected reward function $g_{\sigma,a}(x_{m,t}(\sigma,a), x_{s,t}(\sigma,a))$ is maximized with respect to $x_{s,t}(\sigma,a)$. Again, taking double derivative of $g_{\sigma,a}(x_{m,t}(\sigma,a), x_{s,t}(\sigma,a))$ with respect to $x_{m,t}(\sigma,a)$ and $x_{s,t}(\sigma,a)$ we get

$$\frac{\partial^2 g_{\sigma,a}(x_{m,t}(\sigma,a), x_{s,t}(\sigma,a))}{\partial x^2_{m,t}(\sigma,a)} = -2\alpha \quad (6.11)$$

$$\frac{\partial^2 g_{\sigma,a}(x_{m,t}(\sigma,a), x_{s,t}(\sigma,a))}{\partial x^2_{s,t}(\sigma,a)} = -2\alpha \quad (6.12)$$

Note that from Equation (6.11), we get $\frac{\partial^2 g_{\sigma,a}(x_{m,t}(\sigma,a), x_{s,t}(\sigma,a))}{\partial x^2_{m,t}(\sigma,a)} < 0$, from Equation (6.12), we get $\frac{\partial^2 g_{\sigma,a}(x_{m,t}(\sigma,a), x_{s,t}(\sigma,a))}{\partial x^2_{s,t}(\sigma,a)} < 0$ suggesting that the expected reward is concave with respect to $x_{m,t}(\sigma,a)$ and $x_{s,t}(\sigma,a)$.

Now, we consider a case where $C_{m,t} + c_{m,t} = c$ and $C_{s,t} + c_{s,t} = c, t = 1, ..., T$. If $\beta/2\alpha < c$ and $\gamma/2\alpha < c$, then by using Equation (6.9), the optimal reward function $g_\sigma(x^*_{m,t}(\sigma,a), x'_{s,t}(\sigma,a))$

with respect to $x_{m,t}(\sigma, a)$ can be written as:

$$
\begin{aligned}
g_{\sigma,a}(x^*_{m,t}(\sigma,a), x'_{s,t}(\sigma,a)) &= -\alpha(\beta/2\alpha)^2 + \beta^2/2\alpha + (\gamma-\beta)x'_{s,t}(\sigma,a) + \delta \\
&= \beta^2/4\alpha + (\gamma-\beta)x'_{s,t}(\sigma,a) + \delta \qquad (6.13)
\end{aligned}
$$

From Equation (6.13), if $\gamma < \beta$ then $x'_{s,t}(\sigma, a) = 0$ to maximize the expected reward. Similarly, using Equation (6.10), the optimal reward function $g_{\sigma,a}(x'_{m,t}(\sigma,a), x^*_{s,t}(\sigma,a))$ with respect to $x_{s,t}(\sigma, a)$ can be written as:

$$
\begin{aligned}
g_{\sigma,a}(x'_{m,t}(\sigma,a), x^*_{s,t}(\sigma,a)) &= -\alpha(\gamma/2\alpha)^2 + \gamma^2/2\alpha + (\beta-\gamma)x'_{m,t}(\sigma,a) + \delta \\
&= \gamma^2/4\alpha + (\beta-\gamma)x'_{m,t}(\sigma,a) + \delta \qquad (6.14)
\end{aligned}
$$

From Equation (6.14), if $\gamma > \beta$ then $x'_{m,t}(\sigma, a) = 0$ to maximize the total profit. Let $\Delta g_{\sigma,a} = g_{\sigma,a}(x^*_{m,t}(\sigma,a), x'_{s,t}(\sigma,a)) - g_{\sigma,a}(x'_{m,t}(\sigma,a), x^*_{s,t}(\sigma,a))$ then

$$
\begin{aligned}
\Delta g_{\sigma,a} &= \beta^2/4\alpha + (\gamma-\beta)x'_{s,t}(\sigma,a) - \gamma^2/4\alpha - (\beta-\gamma)x'_{m,t}(\sigma,a) \\
&= (\beta^2-\gamma^2)/4\alpha - (\beta-\gamma)(x'_{m,t}(\sigma,a) + x'_{s,t}(\sigma,a)) \\
&= (\beta-\gamma)((\beta+\gamma)/4\alpha - (x'_{m,t}(\sigma,a) + x'_{s,t}(\sigma,a))) \qquad (6.15)
\end{aligned}
$$

If $\beta > \gamma$ then from Equation (6.13), $x'_{s,t}(\sigma, a) = 0$ and from Equation (6.14), $x'_{m,t}(\sigma, a) = \gamma/2\alpha$. So Equation (6.10) implies that $\Delta g_{\sigma,a} = (\beta+\gamma)/4\alpha - (\gamma/2\alpha)$ or $\Delta g_{\sigma,a} = (\beta - \gamma)/4\alpha > 0$. Thus, $g_{\sigma,a}(x^*_{m,t}(\sigma,a), x'_{s,t}(\sigma,a)) > g_{\sigma,a}(x'_{m,t}(\sigma,a), x^*_{s,t}(\sigma,a))$, $x^*_{m,t}(\sigma,a) = \beta/2\alpha$, and $x^*_{s,t}(\sigma, a) = 0$. Similarly, if $\beta < \gamma$ then from Equation (6.14), $x'_{m,t}(\sigma, a) = 0$ and from Equation (6.13), $x'_{s,t}(\sigma,a) = \beta/2\alpha$. So Equation (6.10) implies that $\Delta g_{\sigma,a} = (\beta+\gamma)/4\alpha - (\beta/2\alpha)$ or $\Delta g_{\sigma,a} = (\gamma-\beta)/4\alpha > 0$. Thus, $g_{\sigma,a}(x'_{m,t}(\sigma,a), x^*_{s,t}(\sigma,a)) > g_{\sigma,a}(x^*_{m,t}(\sigma,a), x'_{s,t}(\sigma,a))$, $x^*_{s,t}(\sigma,a) = \gamma/2\alpha$, and $x^*_{m,t}(\sigma,a) = 0$. This concludes the proof of the case where $C_{m,t}+c_{m,t} = c$ and $C_{s,t} + c_{s,t} = c, t = 1, ..., T$..

Next, we consider other states and actions and use the above results to prove this lemma. From above, if $\beta > \gamma$ then $g_{\sigma,a}(x^*_{m,t}(\sigma,a), x'_{s,t}(\sigma,a)) > g_{\sigma,a}(x'_{m,t}(\sigma,a), x^*_{s,t}(\sigma,a))$ and if $\beta/2\alpha > c$ then $x^*_{m,t}(\sigma,a) = C_{m,t}+c_{m,t}$. So, maximizing expected reward $g_{\sigma,a}(x_{m,t}(\sigma,a), x_{s,t}(\sigma,a))$ with respect to $x_{s,t}(\sigma, a)$, using Equation (6.10) we get $x^*_{s,t}(\sigma, a) = \max(\gamma/2\alpha - x^*_{m,t}(\sigma, a), 0)$. We know that $x^*_{s,t}(\sigma, a) < C_{s,t} + c_{s,t}$, so $x^*_{s,t}(\sigma, a) = \min(\max(\gamma/2\alpha - x^*_{m,t}(\sigma, a), 0), C_{s,t} + c_{s,t})$. Similarly, if $\beta < \gamma$ then $g_{\sigma,a}(x^*_{m,t}(\sigma, a), x'_{s,t}(\sigma, a)) < g_{\sigma,a}(x'_{m,t}(\sigma, a), x^*_{s,t}(\sigma, a))$ and if $\gamma/2\alpha > c$ then $x^*_{s,t}(\sigma, a) = C_{s,t} + c_{s,t}$. Now, maximizing expected reward $g_{\sigma,a}(x_{m,t}(\sigma, a), x_{s,t}(\sigma, a))$ with respect to $x_{s,t}(\sigma, a)$, we get $x^*_{m,t}(\sigma, a) = \max(\beta/2\alpha - x^*_{s,t}(\sigma, a), 0)$ using Equation (6.9). We know that $x^*_{m,t}(\sigma, a) < C_{m,t} + c_{m,t}$, so $x^*_{m,t}(\sigma, a) = \min(\max(\beta/2\alpha - x^*_{s,t}(\sigma, a), 0), C_{m,t} + c_{m,t})$. This concludes the proof.

Proof of Theorem 6.1: We prove Theorem 6.1 separately in three parts. For part (1) of the theorem, we show that if $(\Delta o - \Delta f) > k_1 \Delta o$, then the profit when only the subcontractor invests in capacity c is more than the profit when only the manufacturer invests in capacity c or when the supply chain capacity is 0 or $2c$.

We assume $\beta < \gamma$, then from Lemma 6.1 we know that if $\beta/2\alpha < c$ and $\gamma/2\alpha < c$, then $x^*_{m,t} = \beta/2\alpha$ and $x^*_{s,t} = 0$. Then the expected reward when only manufacturer invests in capacity, and expected reward when both the manufacture and the subcontractor invest in capacity is given by:

$$r((0,0),(c,0)) \quad = \quad \beta^2/4\alpha - o_m c \tag{6.16}$$

$$r((0,0),(c,c)) \quad = \quad \beta^2/4\alpha - o_m c - o_s c \tag{6.17}$$

Next, if $\beta > \gamma$ then from Lemma 6.1 we know that if $\beta/2\alpha < c$ and $\gamma/2\alpha < c$, then $x^*_{s,t} = \gamma/2\alpha$ and $x^*_{m,t} = 0$. Then the expected reward when only subcontractor invests in capacity, and expected reward when both the manufacture and the subcontractor invest in capacity is given by:

$$r((0,0),(0,c)) \quad = \quad \gamma^2/4\alpha - o_s c \tag{6.18}$$

$$r((0,0),(c,c)) \quad = \quad \gamma^2/4\alpha - o_m c - o_s c \tag{6.19}$$

Next, considering various demand cases. If $\max(\frac{\beta}{2\alpha}, \frac{\gamma}{2\alpha}) < d_{high,1}$ then from Equations (6.16) and (6.18):

$$\begin{aligned} r((0,0),(0,c)) - r((0,0),(c,0)) &= (\gamma^2 - \beta^2)/4\alpha - (o_s c - o_m c) \\ &= (\gamma - \beta)(\gamma + \beta)/4\alpha - (o_s c - o_m c) \\ &= ((o_s - o_m) - (f_s - f_m))(\gamma + \beta)/4\alpha - (o_s c - o_m c) \end{aligned}$$

If $((o_s - o_m) - (f_s - f_m))\frac{\beta+\gamma}{4\alpha} > o_s c - o_m c$, then $r((0,0),(0,c)) > r((0,0),(c,0))$. This concludes the proof of part (1). Next, for part (2) of the theorem, we show that if $\min(\frac{\beta}{2\alpha}, \frac{\gamma}{2\alpha}) > d_{high,1}$ and $((o_s - o_m) - (f_s - f_m))d_{high,1} > o_s c - o_m c$, then the profit when the supply chain capacity is c is more than the profit when the supply chain capacity is c. If $\min(\frac{\beta}{2\alpha}, \frac{\gamma}{2\alpha}) > d_{high,1}$ then the expected profit at state $(0,c)$ and $(c,0)$ is given by:

$$r(0,c) = -\alpha d_{high,1}^2 + \gamma d_{high,1} - o_s c \tag{6.20}$$

$$r(c,0) = -\alpha d_{high,1}^2 + \beta d_{high,1} - o_m c \tag{6.21}$$

Then $r((0,0),(0,c)) - r((0,0),(c,0)) = (\gamma - \beta)d_{high,1} - (o_s c - o_m c)$ or $r((0,0),(0,c)) - r((0,0),(c,0)) = ((o_s - o_m) - (f_s - f_m))d_{high,1} - (o_s c - o_m c)$. If $((o_s - o_m) - (f_s - f_m))d_{high,1} > o_s c - o_m c$, then $r((0,0),(0,c)) > r((0,0),(c,0))$. We use similar approach described above to prove other demand cases. This concludes the proof of part (1). Similarly, we can prove part (2) by reversing the inequality in the conditions in part (1). If $o_s > 0$ then from Equations (6.16) and (6.17), $r((0,0),(c,0)) > r((0,0),(c,c))$. If $o_m > 0$ then from Equations (6.18) and (6.19), $r((0,0),(0,c)) > r((0,0),(c,c))$. This concludes proof of part (3).

Proof of Theorem 6.2: To prove Theorem 6.2, we compare the profits when either the manufacturer or the subcontractor or both invest in capacity and show the desired conditions. For part (1) of the theorem, we show that if $\frac{\gamma}{2\alpha} < d_{high,1}$ and $(\Delta o - \Delta f)\frac{\beta+\gamma}{4\alpha} > c\Delta o + \alpha(\frac{\gamma^2}{4\alpha^2} - c^2)$, then the profit when only the subcontractor invests in capacity c is more than the profit when either only the manufacturer invests in capacity c or both the manufacturer and the subcontractor invest in capacity c.

If $\frac{\gamma}{2\alpha} < d_{high,1}$ and $\beta < \gamma$, then using Lemma 6.1 expected optimal rewards are given by:

$$r((0,0),(0,c)) = -\alpha c^2 + \gamma c - o_s c \tag{6.22}$$

$$r((0,0),(c,0)) = \beta^2/4\alpha - o_m c \tag{6.23}$$

$$r((0,0),(c,c)) = -\alpha c^2 + \gamma c - o_m c - o_s c \tag{6.24}$$

Therefore,

$$\begin{aligned} r((0,0),(0,c)) - r((0,0),(c,0)) &= -\alpha c^2 + \gamma c - o_s c - \beta^2/4\alpha + o_m c \\ &> -\alpha c^2 + \gamma^2/2\alpha - o_s c - \beta^2/4\alpha + o_m c \\ &= (\gamma^2/4\alpha - \alpha c^2) + \gamma^2/4\alpha - o_s c - \beta^2/4\alpha + o_m c \\ &= \alpha((\frac{\gamma}{2\alpha})^2 - c^2) + (\gamma - \beta)\frac{\gamma + \beta}{4\alpha} - (o_s c - o_m c) \end{aligned}$$

If $((o_m - o_s) - (f_m - f_s))\frac{\beta+\gamma}{4\alpha} < (o_m c - o_s c) + \alpha(\frac{\gamma^2}{4\alpha^2} - c^2)$ then $r((0,0),(0,c)) > r((0,0),(c,0))$. Also $r((0,0),(0,c)) > r((0,0),(c,c))$. Similarly, if $\beta > \gamma$, then using Lemma 6.1 expected optimal rewards are given by:

$$r((0,0),(0,c)) = -\alpha c^2 + \gamma c - o_s c \tag{6.25}$$

$$r((0,0),(c,0)) = \beta^2/4\alpha - o_m c \tag{6.26}$$

$$r((0,0),(c,c)) = \beta^2/4\alpha - o_m c - o_s c \tag{6.27}$$

Therefore,

$$r((0,0),(0,c)) - r((0,0),(c,0)) > \alpha((\frac{\gamma}{2\alpha})^2 - c^2) + (\gamma - \beta)\frac{\gamma + \beta}{4\alpha} - (o_s c - o_m c)$$

This concludes the proof for part (1). Similarity, we can prove part(2) where if $((o_m - o_s) - (f_m - f_s))\frac{\beta+\gamma}{4\alpha} > (o_m c - o_s c) + \alpha(\frac{\gamma^2}{4\alpha^2} - c^2)$, then $r((0,0),(0,c)) < r((0,0),(c,0))$. This concludes the proof of part (1) and part (2).

Proof of Theorem 6.3: We prove each part of Theorem 6.3 separately. To prove part (1), we show that if conditions $((\mathcal{O} \wedge \hat{\mathcal{M}}_2 \wedge \hat{\mathcal{M}}\mathcal{S}) \vee (\mathcal{M}_2 \wedge \mathcal{M}_1)) \vee (\hat{\mathcal{O}} \wedge \hat{\mathcal{S}}_2 \wedge \hat{\mathcal{M}}\mathcal{S})$ hold, then the

the profit when only subcontractor invests in capacity c at $t = 1$ and maintain that capacity at $t = 2$ is more than the profit at any other actions.

At first we consider a case where $\Delta o > \Delta f$ and show that if conditions $((\mathcal{O} \wedge \hat{\mathcal{M}}_2 \wedge \hat{\mathcal{M}}\mathcal{S}) \vee (\mathcal{M}_2 \wedge \mathcal{M}_1))$ hold, then the the profit when only subcontractor invests in capacity c at $t = 1$ and maintain that capacity at $t = 2$ is more than the profits at any other actions. We analyze a demand case where $\min(\frac{\beta}{2\alpha}, \frac{\gamma}{2\alpha}) \geq d_{high,1}$ and $\max(\frac{\beta}{2\alpha}, \frac{\gamma}{2\alpha}) < d_{high,2}$.

Let $V_{1,(a_m,a_s)}(0,0)$ be the value function at time $t = 1$ after taking actions a_m, a_s. Now, if $\beta < \gamma$, $\min(\frac{\beta}{2\alpha}, \frac{\gamma}{2\alpha}) \geq d_{high,1}$, $\max(\frac{\beta}{2\alpha}, \frac{\gamma}{2\alpha}) < d_{high,2}$ then the optimal value function can be written as:

$$\begin{aligned}
V_1^*(0,0) &= \max[V_{1,(0,0)}(0,0), V_{1,(0,c)}(0,0), V_{1,(c,0)}(0,0), V_{1,(c,c)}(0,0)] \\
&= \max[V_2^*(0,0), \\
&\quad -\alpha d_{high,1}^2 + \gamma d_{high,1} - o_s c + V_2^*(0,c), \\
&\quad -\alpha d_{high,1}^2 + \beta d_{high,1} - o_m c + V_2^*(c,0), \\
&\quad -\alpha d_{high,1}^2 + \gamma d_{high,1} - o_s c - o_m c + V_2^*(c,c)]
\end{aligned}$$

From the assumption of positive revenue at demand satisfaction, $V_2^*(0,0) < V_2^*(0,c)$, and $-\alpha d_{high,1}^2 + \gamma d_{high,1} - o_s c > 0$. This implies that $V_{1,(0,0)}(0,0) < V_{1,(0,c)}(0,0)$ and action $(0,0)$ is not optimal at time $t = 1$. Next, if $\beta < \gamma$ and $\gamma/2\alpha < c$ then using Equations (6.25) and (6.27), $r((0,0),(0,c)) > r((0,0),(c,c))$ resulting in $V_2^*(c,c) < V_2^*(0,c)$. This implies that $V_{1,(c,c)}(0,0) < V_{1,(0,c)}(0,0)$ and action (c,c) is not optimal at time $t = 1$. Next, we derive conditions under which $V_{1,(c,0)}(0,0)$ is less than $V_{1,(c,c)}(0,0)$.

We know that only actions $(0,0)$ and $(0,c)$ are available at state $(c,0)$, i.e. either neither the manufacturer nor the subcontractor invests on additional capacity at time $t = 2$, or only the subcontractor invests in capacity c at time $t = 2$. So, if $\beta < \gamma$, $\beta/2\alpha < c$, and $\gamma/2\alpha < c$ then using Lemma 6.1, the optimal action at state $(c,0)$ at time $t = 2$, i.e. $argmax\{V_2^*(c,0)\}$ is defined as:

$$argmax\{V_2^*(c,0)\} = argmax\{\beta^2/4\alpha - o_m c, \gamma^2/4\alpha - o_m c - o_s c\} \tag{6.28}$$

We consider two cases where $argmax\{V_2^*(c,0)\} = (0,0)$ or $argmax\{V_2^*(c,0)\} = (0,c)$. In the first case, if $\beta < \gamma$ and $\gamma^2 - \beta^2 < 4\alpha o_s c$, then from Equation (6.28) $argmax\{V_2^*(c,0)\} = (0,0)$. So, $V_{1,(0,c)}(0,0) - V_{1,(c,0)}(0,0)$ can be written as:

$$\begin{aligned} V_{1,(0,c)}(0,0) - V_{1,(c,0)}(0,0) &= (\gamma-\beta)d_{high,1} + \gamma^2/4\alpha - 2o_s c - (\beta^2/4\alpha - 2o_m c) \\ &= (\gamma-\beta)(\frac{\beta+\gamma}{4\alpha} + d_{high,1}) - 2(o_s c - o_m c) \end{aligned} \tag{6.29}$$

Equation (6.29) implies that if $(\beta-\gamma)(\frac{\beta+\gamma}{4\alpha} + d_{high,1}) < 2(o_m c - o_s c)$ then $V_{1,(0,c)}(0,0) > V_{1,(c,0)}(0,0)$. In the second case, if $\beta < \gamma$ and $\gamma^2 - \beta^2 > 4\alpha o_s c$, then $argmax\{V_2^*(c,0)\} = (0,c)$. So, $V_{1,(0,c)}(0,0) - V_{1,(c,0)}(0,0)$ can be written as:

$$\begin{aligned} V_{1,(0,c)}(0,0) - V_{1,(c,0)}(0,0) &= (\gamma-\beta)d_{high,1} + \gamma^2/4\alpha - 2o_s c - (\gamma^2/4\alpha - o_s c - 2o_m c) \\ &= (\gamma-\beta)(d_{high,1}) - (o_s c - 2o_m c) \end{aligned} \tag{6.30}$$

Equation (6.30) implies that if $(\beta-\gamma)d_{high,1} < (2o_m c - o_s c)$ then $V_{1,(0,c)}(0,0) > V_{1,(c,0)}(0,0)$. Using the similar approach, we can show condition for other demand cases. Next, we consider a case where $\Delta o < \Delta f$ and using similar approach described above, we show that if conditions $(\hat{\mathcal{O}} \wedge \hat{\mathcal{S}}_2 \wedge \hat{\mathcal{M}}\mathcal{S})$ hold, then the the profit when only subcontractor invests in capacity c at $t=1$ and maintain that capacity at $t=2$ is more than the profit at any other actions. This concludes proof of part (1) of the theorem.

Next, to prove part (2), we show that if conditions $(\mathcal{O} \wedge \hat{\mathcal{M}}_2 \wedge \mathcal{MS}) \vee ((\hat{\mathcal{O}} \wedge \hat{\mathcal{S}}_2 \wedge \mathcal{MS}) \vee (\mathcal{S}_2 \wedge \mathcal{S}_1))$ hold, then the the profit when only manufacturer invests in capacity c at $t=1$ and maintain that capacity at $t=2$ is more than the profits at any other actions. Note that $V_{1,(0,c)}(0,0) - V_{1,(c,0)}(0,0)$ is only defined as Equation (6.29) if $\beta < \gamma$ and $\gamma^2 - \beta^2 < 4\alpha o_s c$. From Equation (6.29), if $(\beta-\gamma)(\frac{\beta+\gamma}{4\alpha} + d_{high,1}) > 2(o_m c - o_s c)$ then $V_{1,(0,c)}(0,0) < V_{1,(c,0)}(0,0)$. This concludes the proof of part (2) of the theorem.

Next, to prove part (3), we show that if conditions $(\mathcal{O} \wedge \mathcal{M}_2 \wedge \hat{\mathcal{M}}_1)$ hold, then the the profit when the manufacturer invests in capacity in time $t = 1$ and the subcontractor invest in capacity at time $t = 2$ is more than the profits at any other actions. Note that $V_{1,(0,c)}(0,0) - V_{1,(c,0)}(0,0)$ is only defined as Equation (6.30) if $\beta < \gamma$ and $\gamma^2 - \beta^2 > 4\alpha o_s c$. If $\gamma^2 - \beta^2 > 4\alpha o_s c$ and $(\beta - \gamma)d_{high,1} > (2o_m c - o_s c)$ then from Equation (6.30), $V_{1,(0,c)}(0,0) < V_{1,(c,0)}(0,0)$. This concludes the proof of part (3) of the theorem. In the similar way described above, we prove part (4) for the case where condition $\hat{\mathcal{O}}$ holds.

Chapter 7

Research Summary and Extensions

In this chapter, we discuss model summaries, insights, conclusions, and potential extensions of the research.

7.1 Research Summaries and Insights

In Chapter 3 of this thesis, we consider a single product ATO system where individual product is assembled from multiple standard-type components that are made to stock. The supply chain manager can decide to replenish these components at a lower cost by using capacity available at the external subcontractor with high lead times. Additionally, the supply chain manager can also leverage capacity available at the in-house manufacturer to replenish components at a higher cost and faster service rate, reducing the lead time of components. We assume that the demand of the final product is random, and at the demand arrival if all components are available then the customer orders are satisfied, otherwise the demand is backordered. We analyze three dual index policies that are common in practice: (i) base stock policy (DB policy), (ii) on-hand inventory based policy (OH policy), and (iii) lead time based policy (LT policy) and determine the optimal thresholds for these policies. Next, we use Matrix-geometric approach to exploit the structure of sparse matrix and provide exact solution for the single product ATO system with two components. However, for large systems, we propose an efficient decomposition based approach that decomposes the original system into component based subsystems.

We observe that the DB policy outperforms OH policy and LT policy. However, DB policy has operational ambiguity which is resolved by using combination of OH policy and LT policy. Next, the OH policy works well at high base stock levels, while the LT policy works well at low base stock levels. Finally, decomposition provides accurate results with error $< 2\%$ in most cases.

In Chapter 4 of this thesis, we consider a make-to-stock system with multiple standard-type components. These components require capacity on a special equipment that cannot be dedicated to serve a specific component, and the components share the same manufacturing resource. However, the supply chain manager can also decide be replenished these components by using capacity available at the dedicated external subcontractor. The number of components and the size of state space and action space increases the complexity of the problem and the the underlying problem is hard to solve. However, we use efficient action elimination techniques that partitions the action space into three regions: (i) zero production costs, (ii) manufacturer is cheaper, (iii) subcontractor is cheaper, and using Markov decisions process models, we analyze the structure of the optimal policy in each region. We analytically provide an exhaustive set of conditions that depends on the value function, costs, and service rate, under which each actions are optimal.

We observe that for a complete symmetric system (with respect to cost and service rates), the optimal policy is of dual index type, i.e. it suggests that either one of the components should be always produced at the fastest rate or none of the components should be produced. We analyze three cases: (i) if production costs are zero, then the optimal policy is dual index type whenever the sum of inventory positions is constant, (ii) next, if the manufacture is cheaper, then the optimal policy is multi-index type with three thresholds whenever the sum of inventory positions is constant and the service rates satisfy specific conditions, (iii) finally, if the subcontractor is cheaper, then the optimal policy is dual index type whenever the sum of inventory positions is constant and the service rates satisfy specific conditions.

In Chapter 5 of this thesis, we analyze a multi-product ATO system where the products are assembled from multiple standard-type components that share the same manufacturing resource. In this case, the supply chain manager could replenish components using capacity available at the in-house manufacturer and using capacity available at the dedicated external subcontractors. We propose a fairly accurate approach that combines decomposition and Markov decision process where, we decompose the ATO system with multiple products into two equivalent subsystems that characterize a component for each product. For a subsystem, we leverage results from Chapter 4 to determine the structure of the optimal policy. Next, using iterative algorithm for subsystems, we provide optimal solutions to the original multi-product ATO system.

We observe that if production costs are zero or the subcontractor is cheaper, then for each state of one subsystem there exists a dual index type policy in another subsystem. Next, if the manufacture is cheaper, then the dual index type policy might not be optimal. We also compare results from exact analysis and decomposition approach for three cases: negligible production costs, manufacturer is cheaper, and subcontractor is cheaper, and observe that the decomposition approach is fairly accurate specially if the manufacture is cheaper.

In Chapter 6 of this thesis, we analyze make to order system with knowledge-type components. Knowledge-type components require high capital investment that have costs associated with unused capacity. The supply chain manager could subcontract these components to external subcontractors to gain additional capacity. Additionally, the supply chain manager could also produce these components in-house to absorb overhead costs associated with under utilized capacity. We analyze two system : centralized system and decentralized system. Using Markov decision process, we analyze the system with centralized control and analytically characterize the structure of the optimal capacity and production decisions. Using stochastic games formulations, we analyze the system with decentralized control. We also analyze the gap in the decentralized system as compared to centralized system.

We observe that for equal production cost, if the unused capacity cost at the subcontractor, is more than the unused capacity cost at the manufacturer, then the optimal production quantity at the supplier is more than the manufacturer. Next, the optimal capacity and production decisions depend on the relative difference between the difference in the cost of unused capacity (manufacturer and the subcontractor) and difference in the production costs.

7.2 Research Extensions

In this section, we discuss potential research extensions of this thesis.

Dynamic production, capacity and sourcing models: With changing production environment, certified subcontractors' pool, product mix, etc, the industry faces issues with maintaining the existing optimization and production planning system. This could even result in the change of the entire model altogether. This problem can be divided into two categories: (i) maintaining stable connection with the existing supply chain data sources and dynamically update the model with the changing data, (ii) analyzing the effect of change in system parameters such as service rates, demand distribution, capabilities and service requirements on system performance and optimal decisions. Using concepts of robust optimization paired with advanced data analytics, we plan to invest such problems and provide solutions in an dynamic environment.

Reliability models: In addition to the supply chain and manufacturing issues prevalent in O&G industries, reliability of equipments is emerging as one of the critical aspects of O&G industries. Scheduling issues related to mandatory third party inspection presents complex challenges due to the multiple iterations needed before approval. Thus, determining the optimal scheduling policy for third party inspection that balances tradeoffs in quality, costs, and lead times is an important question. We plan to investigate such third party inspection scheduling and reliability concerns using Markov decision process models that capture real-world restrictions and uncertainty in inspection and production decisions.

R&D investment strategy: With depleting fossil fuel reserves, advancement of renewable energy sources present exciting new opportunities. These advancements emphasize complex equipments (wind turbines, generators, etc) which promotes advanced manufacturing capabilities with targeted R&D investments by renewable energy industries. Determining the optimal R&D investment and process improvement strategies for such industries and estimating the impact off these strategies on quality and costs is an important research area. Using Markov decision process models, We plan to investigate a holistic approach to R&D investments that includes capability investment decisions, manufacturing process improvement and reliability decisions.

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