Hyperplane arrangements and compactifications of vector groups

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Abstract

Schubert varieties of hyperplane arrangements, also known as matroid Schubert varieties, play an essential role in the proof of the Dowling-Wilson conjecture and in Kazhdan-Lusztig theory for matroids. We begin by studying these varieties as equivariant compactifications of affine spaces, and give necessary and sufficient conditions to characterize them. Next, we generalize the theory to include partial compactifications and morphisms between them. This theory resembles the correspondence between toric varieties and polyhedral fans. Finally, in joint work with Connor Simpson and Botong Wang, we introduce a new family of equivariant compactifications of affine spaces associated to arrangements of linear subspaces of higher codimension. The associated combinatorics is that of polymatroids.

Dedication

To my family. My wonderful parents, who encouraged my interests when I was younger and gave me guidance when I was older. Karen and Jake for their support and for sharing war stories from their PhDs. Sam and Leah for cheering me on. And finally Sarah, for being an invaluable friend and confidant during this chapter of my life.

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Chapter 1

Introduction

The Schubert variety of a hyperplane arrangement is a compactification of the ambient vector space of the arrangement, which is analogous to a classical Schubert variety in the role it plays in Kazhdan-Lusztig theory for matroids (Braden, Huh, Matherne, Proudfoot, and Wang 2020). Given an arrangement $H_1, \ldots, H_n$ of linear hyperplanes in a finite dimensional complex vector space V, whose common intersection is zero, the Schubert variety is the closure of V in $(\mathbb{P}^1)^n$ via the embedding

$$V \subseteq V/H_1 \times \ldots \times V/H_n \subseteq \mathbb{P}^1 \times \ldots \times \mathbb{P}^1.$$

The Schubert variety of a hyperplane arrangement was first studied by (Ardila and Boocher 2016), where the authors showed that the combinatorics of the matroid associated to the arrangement determined much of the geometry of the Schubert variety. The intersection cohomology of the Schubert variety was used in (Huh and Wang 2017) to prove Dowling and Wilson's Top Heavy conjecture for matroids in the realizable case.

The affine reciprocal plane of an essential hyperplane arrangement is the intersection of the Schubert variety with the affine chart $(\mathbb{C}^\times \cup \{\infty\})^n \subseteq (\mathbb{P}^1)^n$. Reciprocal planes are studied in (Terao 2002; Proudfoot and Speyer 2006; Elias, Proudfoot, and Wakefield 2016; Kummer and Vinzant 2019), were the authors also observe a two way street between the combinatorics of arrangements and the geometry of their reciprocal planes. The intersection cohomology of the projectivized reciprocal plane was used in (Elias, Proudfoot, and Wakefield 2016) to prove that the coefficients of the Kazhdan-Lusztig polynomial of a realizable matroid are non-negative.

In this thesis we study the geometry of Schubert varieties of hyperplane arrangements through the lens of equivariant compactifications. If G is an algebraic group, then an equivariant compactification of G is a proper variety X containing G as a dense open set, and an action $G \times X \to X$ extending the group law $G \times G \to G$. With the word "proper" omitted, we call X an equivariant partial compactification. For example, a toric variety is by definition an equivariant partial compactification of the algebraic torus $T = (\mathbb{C}^\times)^d$. One of the main theorems in toric geometry states that all normal toric varieties arise from polyhedral fans. The Schubert variety of a hyperplane arrangement is an equivariant compactification of the additive group $V = \mathbb{C}^d$, which we will call a *vector group*.

In Chapter 2 we prove the following characterization of which equivariant compactifications of vector groups arise as Schubert varieties of hyperplane arrangements.

Theorem 1.0.1. *An equivariant compactification Y of the vector group $V = \mathbb{C}^d$ is isomorphic to the Schubert variety of a hyperplane arrangement if and only if Y is normal as a variety, Y has only finitely many orbits, and each orbit contains a point which can be reached by a limit $\lim_{t\to\infty} tv$, for $v \in V$.*

The limit condition in the above theorem is analogous to the fact that any orbit in a normal toric variety can be reached by a one-parameter subgroup of the torus. Because the fan corresponding to a normal toric variety is constructed by considering the limits of one-parameter subgroups, it is natural to look for an analogous correspondence only for equivariant compactifications of V where every orbit is reached by a one-variable limit.

In the course of proving the above theorem, we give another characterization in which the limit condition is replaced by the stronger condition that each orbit admits a normal slice satisfying certain properties. The second characterization resembles a key geometric property of Schubert varieties of hyperplane arrangements: for each flat in a hyperplane arrangement, the Schubert variety of the restriction is a normally nonsingular slice through the corresponding orbit (Braden, Huh, Matherne, Proudfoot, and Wang 2020).

In Chapter 3, we prove an equivalence of categories which generalizes both characterizations to include partial compactifications as well as morphisms between them. The objects in the first category are equivariant partial compactifications of V satisfying the conditions of the above theorem, or the stronger formulation involving normal slices. The objects in the second category we call *partial hyperplane arrangements*, which include all essential hyperplane arrangements as examples. The construction of varieties from combinatorial data and vice versa in Chapter 2 and Chapter 3 have remarkably similar formulas to those in toric geometry. We further explain this analogy in Section 3.1.1.

Chapter 4 is reproduced from forthcoming joint work with Connor Simpson and Botong Wang, where we consider a generalized class of finite-orbit equivariant compactifications, where the limit condition of Theorem 1.0.1 fails. We call these varieties *polymatroid schubert varieties*, because the poset of orbits is determined by an associated integer polymatroid.

Polymatroids axiomatize the combinatorics of arrangements of linear subspaces, much like matroids axiomatize the combinatorics of hyperplane arrangements. Given a finite collection of linear subspaces $V_1, \ldots, V_n \subseteq V$, the associated polymatroid is defined as the rank function

$$\mathrm{rk} : 2^{\{1,\ldots,n\}} \to \mathbb{Z}_{\geq 0}, \quad S \mapsto \operatorname{codim} \bigcap_{i \in S} V_i.$$

The most direct way to generalize the construction of matroid Schubert variety would be to take the closure of V in the product of the projective closures of V/V_i. This construction does give an equivariant compactification of V, however it does not have finitely many V-orbits. The presence of infinitely many orbits means that (1) there is no natural decomposition into affine cells, so the arguments of (Björner and Ekedahl 2009; Huh and Wang 2017) do not generalize, and (2) there is no distinguished "most singular" point, so one cannot define a Kazhdan-Lusztig polynomial for polymatroids in the same way as (Elias, Proudfoot, and Wakefield 2016).

Our innovation is to replace the usual inclusion of V/V_i into its projective completion with the unique finite-orbit action of $\mathbb{C}^r$ on $\mathbb{P}^r$, described in (Hassett and Tschinkel 1999). We construct an equivariant compactification $\mathbb{C}^d \subseteq Y$ which depends on the arrangement $V_1, \ldots, V_n \subseteq V$ as well as a generic choice of coordinates on V/V_i, and prove:

Theorem 1.0.2. *Y has finitely many orbits, and the orbit poset of Y is isomorphic to the poset of combinatorial flats of the associated polymatroid, which we introduce in Section 4.2.*

Chapter 2

Matroid Schubert varieties

2.1 Introduction and summary

We assume all varieties are irreducible and separated over $\mathbb{C}$. Suppose that G is a commutative linear algebraic group, acting on a variety X. Given a point $x \in X$, we write $G \cdot x \subseteq X$ for the orbit and $G_x \subseteq G$ for the stabilizer.

The main tool we will use throughout the chapter is the following notion of a *slice* (Definition 2.2.4), which is standard for actions of Lie groups and algebraic groups (Gleason 1950; Mostow 1957; Montgomery and Yang 1957; Palais 1961). A (Zariski) slice through $x \in X$ is a G_x-stable subvariety $Z_x \subseteq X$ containing x, such that $G \cdot Z_x \subseteq X$ is an open set, and

$$G \cdot Z_x \cong G \times Z_x / \sim, \quad \text{where } (gh, z) \sim (g, hz) \text{ for all } g \in G, h \in G_x, z \in Z_x.$$

Geometrically, Z_x is a normal slice through the orbit $G \cdot x$, and $G \cdot Z_x$ is neighborhood of $G \cdot x$ that admits a product structure, similar to a tubular neighborhood. We use the words "Zariski slice" to emphasize the difference between the above notion and that of an étale slice.

In order to state our results, we make the following abbreviations. Suppose now that X is an equivariant partial compactification of G. We say X satisfies

- **FO** (**F**inite **o**rbits) if X has finitely many G-orbits,
- **OP** (**O**ne-**p**arameter subgroups) if for every G-orbit $G\cdot x \subseteq X$, there is a one dimensional algebraic subgroup of G whose closure in X intersects $G \cdot x$, and
- **SL** (**Sl**ices) if there exists a Zariski slice through every point of X.

The following is our main result on Schubert varieties of hyperplane arrangements, which implies Theorem 1.0.1.

Theorem A. Suppose that Y is an equivariant compactification of the vector group $V = \mathbb{C}^d$. Then the following are equivalent.

i. Y is equivariantly isomorphic to the Schubert variety of an essential hyperplane arrangement in V.

ii. Y is normal and satisfies **FO** and **SL**.

iii. Y is normal and satisfies **FO** and **OP**.

The original aim of this project was to prove that the third statement implies the first, however we have found that it is most natural to prove that the third statement implies second, and then prove that the second implies the first. For this reason, we view the existence of slices as a more fundamental property of Schubert varieties of hyperplane arrangements.

The study of equivariant compactifications of vector groups was initiated by (Hassett and Tschinkel 1999), and we recommend (Arzhantsev and Zaitseva 2020) for a survey. We will see in Section 3.1.1 that Schubert varieties of hyperplane arrangements show many parallels to toric varieties, however the study of general equivariant compactifications of vector groups has little in common with toric geometry (Arzhantsev and Zaitseva 2020). In particular, toric varieties satisfy **FO**, **OP**, and **SL**, whereas these conditions need not hold for equivariant compactifications of vector groups, as the following examples show.

Example 2.1.1. Consider the action of a vector group V of dimension at least two on its projective closure $\mathbb{P}(V \oplus \mathbb{C})$, where the action on the boundary is trivial. This compactification satisfies **SL** and **OP**, but not **FO**.

Example 2.1.2 ((Hassett and Tschinkel 1999)). Consider the action of the two-dimensional vector group $\mathbb{C}^2$ on $\mathbb{P}^2$ where (a_1, a_2) acts as

$$\exp\left(\begin{bmatrix} 0 & 0 & 0 \\ a_1 & 0 & 0 \\ a_2 & a_1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 \\ a_1 & 1 & 0 \\ a_2 + \frac{1}{2}a_1^2 & a_1 & 1 \end{bmatrix}.$$

This action has one two-dimensional orbit (with which we can identify $\mathbb{C}^2$), one one-dimensional orbit, and one zero-dimensional orbit, so **FO** holds. However **SL** and **OP** fail for the one dimensional orbit.

2.2 Slices

2.2.1 Slices of group actions

We begin by reviewing the definition of homogeneous fiber spaces, following (Shafarevich 1994, Chapter II.4.8). Suppose that G is an algebraic group, H is an algebraic subgroup, and Z is a quasiprojective variety on which H acts. Then there exists a variety $G *_H Z$ called the *homogeneous fiber space*, which parameterizes equivalence classes in $G \times Z$, where

$$(gh, z) \sim (g, hz), \quad \text{for all } g \in G, h \in H, z \in Z.$$

There is a canonical map $G \times Z \to G *_H Z$ sending a point (g, z) to its equivalence class, which we write as $[g, z]$. The universal property which characterizes $G *_H Z$ is the following: if $\pi : G \times Z \to X$ is a map of varieties such that $\pi(gh, z) = \pi(g, hz)$ for all $g \in G, h \in H, z \in Z$, then there is a unique factorization as follows.

$$\begin{tikzcd} G \times Z \arrow[rr, "\pi"] \arrow[dr] & & X \\ & G *_H Z \arrow[ur] & \end{tikzcd}$$

From the universal property, we see that there is a canonical map

$$\tau : G *_H Z \to G/H, \quad [g, z] \mapsto gH,$$

with each fiber isomorphic to Z. We call τ the *canonical fibration*. If $H \subseteq G$ is normal and has a splitting $s : G \to H$, then there is a G-equivariant isomorphism

$$G *_H Z \cong G/H \times Z, \quad [g, z] \mapsto (gH, s(g) \cdot z),$$

which makes the following diagram commute, where pr_1 is the projection.

$$\begin{tikzcd} G *_H Z \arrow[r, "\cong"] \arrow[dr, "\tau"'] & G/H \times Z \arrow[d, "\mathrm{pr}_1"] \\ & G/H \end{tikzcd}$$

Remark 2.2.1. For the remainder of the chapter we will take G to be commutative. Therefore it is possible for us to avoid defining $G *_H Z$ by choosing splittings and working with $G/H \times Z$ instead. While $G/H \times Z$ is a simpler construction, we have found that thinking in terms of the more canonical construction $G *_H Z$ shortens and clarifies the rest of the chapter enough to make it worthwhile.

We need the following lemmas, which are formal consequence of the definitions.

Lemma 2.2.2 (Associativity of $*$)**.** *Suppose that $H' \subseteq H \subseteq G$ are closed subgroups and H' acts on a variety Z'. Then there is a natural isomorphism*

$$G *_H (H *_{H'} Z') \cong G *_{H'} Z', \quad [g, [h, z]] \mapsto [gh, z].$$

Lemma 2.2.3 (Orbits and stabilizers of $G *_H Z$)**.**

i. There is a one-to-one correspondence between G-orbits in $G *_H Z$ and H-orbits in Z which sends $G \cdot [v, z]$ to $H \cdot z$.

ii. Suppose that G is commutative, and $x = [v, z] \in G *_H Z$. Then the stabilizers G_x and H_z coincide as subgroups of G.

Definition 2.2.4. Suppose that X is an algebraic variety with a G-action. If $x \in X$ is a point with stabilizer G_x, we say that a G_x-stable locally closed subvariety $Z_x \subseteq X$ containing x is a **(Zariski) slice** at x if the natural map

$$G *_{G_x} Z_x \to X, \quad [g, z] \mapsto g \cdot z$$

is a G-equivariant Zariski open embedding.

The point x is in the image of $G *_{G_x} Z_x$, so we have that $G *_{G_x} Z_x$ is identified with a G-stable neighborhood of the orbit $G \cdot x$.

We will often use the following criterion for open embeddings to prove the existence of slices.

Theorem 2.2.5 (Zariski's main theorem)**.** *Suppose that $\pi : X \to Y$ is a morphism of varieties which is birational and injective on closed point, and that Y is normal. Then π is an open embedding.*

For the above formulation, we refer to (Vakil n.d., Exercise 29.6.D) and the surrounding discussion. For our purposes, checking injectivity on closed points can be rephrased as follows.

Lemma 2.2.6. *Suppose that $x \in X$ and $Z_x \subseteq X$ is a G_x-stable subvariety containing x. Then $G *_{G_x} Z_x \to X$ is injective on closed points if $g_1 \cdot z_1 = g_2 \cdot z_2$ implies $g_2^{-1} g_1 \in G_x$ for all $g_1, g_2 \in G$ and $z_1, z_2 \in Z_x$.*

2.2.2 Partial compactifications with slices

For this subsection let G be a commutative linear algebraic group, and X a normal equivariant partial compactification of G such that **FO** and **SL** hold. We will first collect some basic consequences.

Lemma 2.2.7. *If $x \in X$ has a slice Z_x, then $Z_x \cap G$ is a coset of G_x.*

Proof. We have that G is contained in any invariant open neighborhood of X, and the natural map

$$G *_{G_x} Z_x \to X$$

is a G-equivariant open embedding, so there must be $[v, z] \in G *_{G_x} Z_x$ mapping to G. Therefore $Z_x \cap G \neq \emptyset$. We also have that $G *_{G_x} Z_x \cong G/G_x \times Z_x$ embeds as a Zariski open set in the variety X, so Z_x and thus $Z_x \cap G$ is irreducible of dimension $\dim G - \dim G/G_x = \dim G_x$. Finally we note that the only irreducible G_x-invariant closed subsets of G of dimension $\dim G_x$ are cosets. □

There is a special point in each orbit corresponding to the trivial coset:

Definition 2.2.8. We say that a point $x \in X$ is **distinguished** if it has a slice containing the identity of G.

It follows from Lemma 2.2.7 that every orbit contains a distinguished point, and we will see in Lemma 2.2.11 that every orbit contains at most one distinguished point.

The orbits of X form a finite stratification, so each orbit $G \cdot x$ has a unique smallest G-invariant open neighborhood defined as follows.

Definition 2.2.9. The **minimal G-invariant neighborhood** U_x of $x \in X$ is given by the union of all orbits $G \cdot y$ such that $G \cdot x \subset \overline{G \cdot y}$.

There exists a unique slice through $x \in X$ contained in U_x, defined as follows:

Definition 2.2.10. The **minimal slice** through $x \in X$ is $Z_x \cap U_x$, where Z_x is any slice of x.

It follows by (Shafarevich 1994, Proposition II.4.21) that the minimal slice through x is indeed a slice, and uniqueness of the minimal slice for distinguished points (the case of an arbitrary point follows easily) comes from the following observation:

Lemma 2.2.11. *The minimal slice Z_x at a distinguished point $x \in X$ is the closure of G_x in U_x.*

Proof. Because x is distinguished, Z_x contains the identity of G. Since $G \cap Z_x$ is a coset of G_x by Lemma 2.2.7, $G \cap Z_x = G_x$. Thus the closure of G_x in U_x is contained in Z_x. Because $G *_{G_x} Z_x \cong G/G_x \times Z_z$ embeds as an open set in X, we get that Z_x is irreducible of dimension $\dim G - \dim G/G_x = \dim G_x$, and closed in U_x. Therefore the closure of G_x in U_x equals Z_x, since they are both irreducible closed subvarieties of U_x of the same dimension. □

Example 2.2.12 (Distinguished points and minimal slices of $\mathbb{P}^1$).

i. Consider $\mathbb{P}^1 = \mathbb{C}^\times \cup \{0\} \cup \{\infty\}$ as an equivariant compactification of $\mathbb{C}^\times$. The distinguished points are $1, 0$, and ∞ with minimal invariant open neighborhoods $\mathbb{C}^\times, \mathbb{C}^\times \cup \{0\}$, and $\mathbb{C}^\times \cup \{\infty\}$ respectively. The minimal slices are $\{1\}, \mathbb{C}^\times \cup \{0\}$, and $\mathbb{C}^\times \cup \{\infty\}$ respectively. Note that $\mathbb{P}^1$ is a non minimal slice through both 0 and ∞.

ii. Consider $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ as an equivariant compactification of $\mathbb{C}$. The distinguished points are 0 and ∞ with minimal invariant open neighborhoods $\mathbb{C}$ and $\mathbb{P}^1$ respectively. The minimal slices are $\{0\}$ and $\mathbb{P}^1$ respectively.

In Section 2.5 we demonstrate the notions developed in this section for the Schubert variety of a hyperplane arrangement.

We now prove that the class of varieties we are working with is closed under taking slices:

Lemma 2.2.13. *If $x \in X$ is a distinguished point, then the minimal slice Z_x is a normal partial compactification of G_x satisfying* ***FO*** *and* ***SL****.*

Proof. By Lemma 2.2.11, Z_x is an equivariant partial compactification of G_x, and Z_x is normal by (Shafarevich 1994, Proposition II.4.22). By Lemma 2.2.3, the G_x-orbits of Z_x correspond to the G-orbits of an open set in X, so Z_x has finitely many G_x-orbits. Finally we check that Z_x has slices. For a point $y \in Z_x$, simply take the slice Z_y through y in X.

Since $y \in Z_x \subseteq U_x$, $G \cdot x \subseteq \overline{G \cdot y}$ by definition of U_x. Therefore $G_y \subseteq G_x$, so $Z_y \subseteq Z_x$ by Lemma 2.2.11. Now consider the diagram

$$\begin{array}{ccc} G *_{G_y} Z_y & \longrightarrow & X \\ \uparrow & & \uparrow \\ G_x *_{G_y} Z_y & \longrightarrow & Z_x \end{array}$$

where the horizontal maps are the open embeddings $[v, z] \mapsto v \cdot z$, and the left vertical map is given by $[v, z] \mapsto [v, z]$. We wish to show that the bottom arrow is an open embedding, for which we use Theorem 2.2.5. The bottom arrow restricts to an isomorphism $G_x *_{G_y} G_y \cong G_x$, so it is birational. All three arrows except the bottom one are already known to be injective on closed points, so the bottom arrow is injective on closed points. $\square$

Remark 2.2.14. It follows from (Shafarevich 1994, Proposition II.4.21) that every orbit closure satisfies **SL** for the group G/G_x. So modulo normality, the class of varieties studied in this section is also closed under taking orbit closures.

2.2.3 Topology of orbit stratification

In the previous section we studied partial compactifications of tori and vector groups simultaneously. In this section, we will use properties of vector groups which fail for tori.

Proposition 2.2.15. *If X is an equivariant partial compactification of a vector group V satisfying **FO** and **SL**, and $x \in X$ is a distinguished point, then the minimal slice Z_x is proper and has x as the unique V_x-fixed point.*

The proof follows from a general topological observation about varieties stratified into affine spaces. We say that an *algebraic cell decomposition* of a variety X is a partition $X = \sqcup_\alpha S_\alpha$ into finitely many locally closed subvarieties S_α called *cells*, such that each cell is isomorphic to an affine space and the closure of a cell is a union of cells.

Lemma 2.2.16. *Suppose that Z is a connected variety with an algebraic cell decomposition that has at least one zero dimensional cell. Then Z is proper and has exactly one zero dimensional cell.*

Proof. Consider the singular cohomology with compact support $H_c^i(Z;\mathbb{Q})$. Since Z is connected, we have that $H_c^0(Z;\mathbb{Q})$ is zero if Z is not proper and one dimensional if Z is proper. The lemma follows from the well known fact that

$$\dim H_c^{2i}(Z;\mathbb{Q}) = \#\{i\text{-dimensional cells in } Z\}.$$

One way to prove the above equation is by inducting on the number of cells as follows. Suppose S is a cell of lowest dimension in Z and U is its open complement. Since $S \cong \mathbb{A}^r$ for some r, $H_c^{2r}(S;\mathbb{Q}) = \mathbb{Q}$ and $H_c^i(S;\mathbb{Q}) = 0$ for $i \neq 2r$. Then the above equation follows from induction using long exact sequence

$$\ldots \to H_c^i(U;\mathbb{Q}) \to H_c^i(Z;\mathbb{Q}) \to H_c^i(S;\mathbb{Q}) \to H_c^{i+1}(U;\mathbb{Q}) \to \ldots. \qquad \square$$

Proof of Proposition 2.2.15. We have by Lemma 2.2.13 that Z_x has finitely many V_x orbits. Each orbit of V_x is isomorphic to an affine space, and as is true of any algebraic group action, orbits are locally closed and the closure of an orbit is a union of orbits. Thus the V_x-orbits of Z_x form an algebraic cell decomposition. Since $x \in Z_x$ is a zero dimensional cell, the proposition follows from Lemma 2.2.16. $\square$

Remark 2.2.17. In the notation of Proposition 2.2.15, it follows that the minimal slice through x is the unique slice through x, as opposed to the torus case. See Example 2.2.12.

Remark 2.2.18. In the case where X is a toric variety with torus T, the minimal slice Z_x through a point $x \in X$ is not proper but rather affine. However Z_x still has x as the unique T_x-fixed point for a different reason. This is due to the fact that disjoint T-invariant closed sets in an affine T-variety can be separated by an invariant function (Dolgachev 2003, Lemma 6.1). Since Z_x has a dense T_x-orbit, all invariant functions are constant. Therefore all invariant closed sets intersect, so x is the only T_x-fixed point.

2.2.4 Slices and one-parameter subgroups

In this section we prove that Item ii and Item iii in Theorem B are equivalent. We break the proof into two lemmas.

Definition 2.2.19. Suppose that X is an equivariant partial compactification of a vector group V, and $x \in X$. Define

$$V_x^\circ = V_x \setminus \bigcup V_y$$

where the union is over $y \in X$ such that $V_y \subsetneq V_x$.

Lemma 2.2.20. *Suppose X is a normal equivariant partial compactification of a vector group V, satisfying **FO** and **SL**. Let $x \in X$ be a distinguished point, and let $v \in V$. Then $\lim_{t\to\infty} tv = x$ if and only if $v \in V_x^\circ$. In particular, X satisfies **OP**.*

Proof. Suppose that $v \in V_x^\circ$. By Proposition 2.2.15, Z_x is proper, so $\lim_{t\to\infty} tv$ must converge to a boundary point of $Z_x \supseteq V_x$. In addition, v lies in the stabilizer of $\lim_{t\to\infty} tv$, so $\lim_{t\to\infty} tv$ be a V_x-fixed point. By Proposition 2.2.15, x is the unique V_x-fixed point in Z_x, so $\lim_{t\to\infty} tv = x$. To prove the other direction, we note that V is partitioned into sets of the form V_y° for $y \in X$, so if $v \notin V_x^\circ$ then $\lim_{t\to\infty} tv = y$ for some $y \neq x$. □

Lemma 2.2.21. *Suppose that X is a normal equivariant partial compactification of V satisfying **FO** and **OP**. Then X satisfies **SL**.*

Proof. We wish to construct a slice through a point $x \in X$. We first explain why it is enough to show that the quotient map

$$V \to V/V_x$$

extends to a V-equivariant map

$$\tau : U_x \to V/V_x,$$

where U_x is the minimal invariant open neighborhood of Definition 2.2.9. We can assume without loss of generality that $\tau(x) = 0$, since the translation of a slice is a slice. Setting

$Z_x := \tau^{-1}(0)$, we have that V_x acts on Z_x and $x \in Z_x$. To show that Z_x is a slice, we must check that the natural map

$$V *_{V_x} Z_x \to X, \quad [v, z] \mapsto v \cdot z$$

is an open embedding. For this we use Theorem 2.2.5. As before we have that $V *_{V_x} Z_x \to X$ restricts to an isomorphism $V *_{V_x} V_x \cong V$, so it is birational. By Lemma 2.2.6 we must show that if $v_1, v_2 \in V$ and $z_1, z_2 \in Z_x$ such that

$$v_1 \cdot z_2 = v_1 \cdot z_2,$$

then $v_1 - v_2 \in V_x$. We check this by applying τ:

$$\begin{aligned} \tau(v_1) \cdot \tau(z_1) = \tau(v_2) \cdot \tau(z_2) & \qquad \text{by equivariance of } \tau, \\ \tau(v_1) = \tau(v_2) & \qquad \text{because } z_1, z_2 \in Z_x = \tau^{-1}(0), \\ v_1 - v_2 \in V_x & \qquad \text{because } \tau \text{ extends } V \to V/V_x. \end{aligned}$$

Next we show how to construct τ. We wish to construct a V-equivariant map

$$\mathrm{Sym}(V/V_x)^\vee \to H^0(U_x, \mathcal{O}_X),$$

so it is enough to show that if $f \in V^\vee$ vanishes on V_x, then f can extend to U_x. Since U_x is normal, it suffices to show that f does not have a pole along any codimension one orbit. Let $L \subseteq V$ be a one dimensional vector subspace of V, and let y be the boundary point of L in X. Assume that $y \in U_x$. By our assumption that X satisfies **OP**, it is enough to show that f does not have a pole along $V \cdot y$. Since the action of L fixes the boundary of L, $L \subseteq V_y$. We also have $V_y \subseteq V_x$ by definition of U_x. Therefore f vanishes on L. Now let L' denote the translation of L by a generic vector, and y' the boundary point of L'. Since f is linear, f is constant and nonzero on L'. Thus f^{-1} is constant and nonzero on L'. If f^{-1} is undefined at y', then since y' is generic in $V \cdot y$, f has a zero along $V \cdot y$. If on the other hand f^{-1} is defined at y', then f^{-1} cannot vanish at y' by continuity, so f does not have a pole along $V \cdot y$. □

This completes the proof that the statements Item ii and Item iii in Theorem B are equivalent.

2.3 The orbit-flat correspondence

Now that we have proved the equivalence of the statements Item ii and Item iii in Theorem B, we will refer to an equivariant partial compactification of a vector group V which satisfies either of these conditions as a *linear V-variety.* In Lemma 2.2.20, we showed that if X is a linear V-variety and $x \in X$ is a distinguished point, then $v \in V_x^\circ$ (Definition 2.2.19) if and only if $\lim_{t\to\infty} tv = x$. As a consequence, we have:

Corollary 2.3.1. *If X is a linear V-variety, then there is a canonical bijection between any two of the following sets.*

- *Orbits of X*
- *Distinguished points of X*
- *Stabilizers of points of X*

Moreover, each of the above sets is functorial on the category of normal equivariant partial compactifications of vector groups, and the bijections between them are natural.

Proof. The correspondence between orbits and distinguished points is automatic, and we have by Lemma 2.2.20 that any distinguished point $x \in X$ can be recovered from its stabilizer V_x by taking the limit $\lim_{t\to\infty} tv$ for $v \in V_x^\circ$. Thus all three sets are in correspondence.

Suppose that T is a morphism of linear vector group varieties. It is automatic that orbits are mapped inside of orbits and stabilizers are mapped inside of stabilizers. Since distinguished points are the set of points that arise as limits of one-parameter subgroups, distinguished points are mapped to distinguished points. Naturality of these correspondences follows formally. □

Definition 2.3.2. Let X be a linear V-variety. The **partial hyperplane arrangement** $\mathscr{L}(X)$ **associated to** X is the collection of stabilizers of points in X.

To justify the definition of $\mathscr{L}(X)$, we will prove:

Proposition 2.3.3.

(i) If X is a proper linear V-variety, then $\mathscr{L}(X)$ is the collection of flats of an essential hyperplane arrangement in V.

(ii) If X is a linear V-variety then $\mathscr{L}(X)$ is a partial hyperplane arrangement in V.

By combining Corollary 2.3.1 and Proposition 2.3.3 we have a natural one-to-one correspondence between the orbits of X and the flats of its relative hyperplane arrangement $\mathscr{L}(X)$, as described in Section 3.1.1.

Lemma 2.3.4. *If X is a linear V-variety, then $\mathscr{L}(X)$ is closed under intersections.*

Proof. Suppose that $x, y \in X$ are distinguished points, and consider the action of $V_x \cap V_y$ on the closure $\overline{V_x \cap V_y}$ in X. Let Z_x be the minimal slice through x. Then $V_x \subseteq Z_x$ by Lemma 2.2.11, so $\overline{V_x \cap V_y}$ is a closed subvariety of Z_x. Then by Proposition 2.2.15, Z_x is proper, so $\overline{V_x \cap V_y}$ is proper. By the Borel fixed point theorem (Humphreys 1975, Chapter 21.2), there exists a $(V_x \cap V_y)$-fixed point $z \in \overline{V_x \cap V_y}$. Thus $V_x \cap V_y \subseteq V_z$. To show the opposite inclusion, note that

$$z \in \overline{V_x \cap V_y} \subseteq Z_x \cap Z_y \subseteq U_x \cap U_y,$$

where U_x is the minimal invariant neighborhood. Therefore by definition of U_x, we have $x, y \in \overline{V \cdot z}$, and thus $V_z \subseteq V_x \cap V_y$. □

To prove Proposition 2.3.3 (i), it now suffices to prove that any stabilizer $V_x \subsetneq V$ is the intersection of the codimension one stabilizers containing it. For this we need the following lemmas.

Lemma 2.3.5. *Suppose that G is a linear algebraic group acting on a variety X, and Z_x is a slice through $x \in X$. Then any regular function on $G \cdot x$ extends to the neighborhood $G *_{G_x} Z_x \subseteq X$.*

Proof. This follows from the universal property of $G *_{G_x} Z_x$ applied to the map $G \times Z_x \to \mathbb{C}$ given by $(g, z) \mapsto f(g \cdot x)$ where f is a regular function on $G \cdot x$. □

Lemma 2.3.6. *If U is a connected algebraic variety which has nonconstant global regular functions, and $U \subseteq K$ is a compactification, then the boundary $K \setminus U$ has has an irreducible component of codimension one in K.*

Proof. Assume for a contradiction that every component of $K \setminus U$ has codimension at least two. Consider the inclusion of the normalizations $\tilde{U} \subseteq \tilde{K}$. Since U has nonconstant global regular functions, then so must $\tilde{U}$. The normalization map is finite and therefore preserves the codimension of the boundary, so $\tilde{K} \setminus \tilde{U}$ is a closed set of codimension at least two. Thus any regular function on $\tilde{U}$ extends to $\tilde{K}$, so we get that the proper variety $\tilde{K}$ has nonconstant global regular functions, which is a contradiction. □

Proof of Proposition 2.3.3 (i). Suppose that $x \in X$ is a distinguished point with stabilizer V_x of codimension at least two in V. We wish to show that V_x is the intersection of the codimension one stabilizers containing it, so by induction it is enough to find $y, z \in X$ such that

$$V_x = V_y \cap V_z, \quad \dim V_y = \dim V_z = \dim V_x + 1.$$

By Corollary 2.3.1, $V \cdot y \neq V \cdot z$ implies $V_y \neq V_z$. Therefore it suffices to show that the orbit closure $\overline{V \cdot x}$ contains two distinct orbits of codimension one in $\overline{V \cdot x}$. Suppose that $V \cdot y \subseteq \overline{V \cdot x}$ is an orbit of codimension one in $\overline{V \cdot x}$. Since $V_x \subseteq V$ is codimension at least two, $\dim V \cdot y > 0$, and so we can choose a nonconstant regular function f on $V \cdot y$. By Lemma 2.3.5, f extends to a regular function on the minimal invariant neighborhood $U_y \supseteq V \cdot x \cup V \cdot y$, which is nonconstant when restricted to $V \cdot x \cup V \cdot y$. Therefore by Lemma 2.3.6 with $U = V \cdot x \cup V \cdot y$ and $K = \overline{V \cdot x}$, there is another orbit $V \cdot z \subseteq \overline{V \cdot x}$ of codimension one. □

Proof of Proposition 2.3.3 (ii). We will apply Proposition 2.3.3 (i) to the slices of X. We have that $\{0\} \in \mathscr{L}(X)$, and by Lemma 2.3.4, $\mathscr{L}(X)$ is closed under intersections. It remains to show that for $F \in \mathscr{L}(X)$, $\{G \in \mathscr{L}(X) : G \subseteq F\}$ is the collection of flats of a partial hyperplane arrangement in F. Suppose that F is the stabilizer of the distinguished point $x \in X$. Then the slice Z_x is a proper linear F-variety by Lemma 2.2.13 and Proposition 2.2.15, so the set of stabilizer $\mathscr{L}(Z_x)$ is the collection of flats of an essential hyperplane arrangement in F by Proposition 2.3.3 (i). Then by Lemma 2.2.3, $\mathscr{L}(Z_x) = \{G \in \mathscr{L}(X) : G \subseteq F\}$. □

2.4 Proof of Theorem A

Let X be an equivariant compactification of V, such that X is normal and satisfies **FO** and **SL**. We have shown in Lemma 2.2.20 and Lemma 2.2.21 that this is equivalent to assuming X is normal and satisfies **FO** and **OP**. In Corollary 2.5.2 we show that the Schubert variety of a hyperplane arrangement satisfies **FO** and **SL**, and normality of the Schubert variety follows from (Brion 2003, Theorem 1) together with (Ardila and Boocher 2016, Theorem 1.3(c)). Thus it only remains to show that X is equivariantly isomorphic to the Schubert variety of a hyperplane arrangement in V.

By Proposition 2.3.3, there exists an essential hyperplane arrangement $\mathscr{A} = \{H_1, \ldots, H_n\}$ in V whose lattice of flats is the collection of stabilizers of X. We write

$$\Phi_{\mathscr{A}} : V \to V/H_1 \times \ldots \times V/H_n,$$

for the induced linear embedding, and $Y_{\mathscr{A}}$ for the Schubert variety of $\mathscr{A}$. Our goal is to show that there exists an isomorphism

$$T : X \to Y_{\mathscr{A}(X)}$$

extending the isomorphism

$$\Phi_{\mathscr{A}} : V \to \Phi_{\mathscr{A}}(V).$$

For each hyperplane H_i, we denote by x_i, Z_i, and $V *_{H_i} Z_i \subseteq X$ the corresponding distinguished point, slice, and minimal V-invariant open neighborhood, respectively. Explicitly, Z_i is the closure of H_i in X, x_i is the H_i-fixed point in Z_i, and $V *_{H_i} Z_i$ is embedded in X as the union of all V-orbits in X which intersect Z_i (see Section 2.2.2). Because Z_i is proper (Proposition 2.2.15) and X is separated, $Z_i \subseteq X$ is closed. Recall from Section 2.2.1 that Z_i is the fiber of the trivial V-equivariant fibration

$$\tau : V *_{H_i} Z_i \to V/H_i.$$

Therefore Z_i is a prime Cartier divisor in X, so there is an associated line bundle $\mathcal{O}_X(Z_i)$. Fix a linearization of $\mathcal{O}_X(Z_i)$, which exists by (Dolgachev 2003, Theorem 7.2). We then have a linear action

$$V \curvearrowright H^0(X, \mathcal{O}_X(Z_i)).$$

Because τ is V-equivariant, the translations of Z_i under the action of V are the fibers of τ, so letting $Z_i' \neq Z_i$ be any such translation, we have that Z_i and Z_i' are linearly equivalent and disjoint. Thus $\mathcal{O}_X(Z_i)$ is globally generated, since the sections (up to scaling) corresponding to Z_i and Z_i' have no common zeros. So far, we have that Z_i defines a V-equivariant morphism

$$T_i : X \to \mathbb{P}(H^0(X, \mathcal{O}_X(Z_i))^\vee).$$

Finally, we have that the target of T_i is $\mathbb{P}^1$ from a general observation:

Lemma 2.4.1. *Suppose X is a proper normal variety, and Z and Z' are prime Cartier divisors which are linearly equivalent and such that $Z \cap Z' = \emptyset$. Then the space of global sections of $\mathcal{O}_X(Z)$ is two dimensional.*

Proof. Let $i : Z \to X$ denote the inclusion, and consider the short exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(Z) \longrightarrow \mathcal{O}_X(Z) \otimes i_*\mathcal{O}_Z \longrightarrow 0.$$

By the projection formula, the sheaf on the right is isomorphic to $i_*(i^*\mathcal{O}_X(Z) \otimes \mathcal{O}_Z)$.

However the restriction of $\mathcal{O}_X(Z)$ to Z is trivial because Z can be moved to the disjoint divisor Z'. Thus $\mathcal{O}_X(Z) \otimes i_*\mathcal{O}_Z \cong i_*\mathcal{O}_Z$. Now take the long exact sequence in cohomology.

$$0 \longrightarrow H^0(X, \mathcal{O}_X) \longrightarrow H^0(X, \mathcal{O}_X(Z)) \longrightarrow H^0(X, i_*\mathcal{O}_Z) \longrightarrow \dots$$

Since X and Z are proper and irreducible, $\dim H^0(X, \mathcal{O}_X) = \dim H^0(X, i_*\mathcal{O}_Z) = 1$. Therefore $\dim H^0(X, \mathcal{O}_X(Z)) \leq 2$. We also have that the sections (up to scaling) corresponding to Z and Z' are independent, so $\dim H^0(X, \mathcal{O}_X(Z)) = 2$. $\square$

Let us choose coordinates on the target of T_i:

$$s_0, s_1 \in H^0(X, \mathcal{O}_X(Z_i)), \quad \mathrm{div}(s_0) = Z_i, \quad s_1 \neq 0 \text{ is } V\text{-fixed.}$$

The section s_1 exists because V is unipotent. For any isomorphism between $\mathcal{O}_X(Z_i)|_V$ and $\mathcal{O}_V$, we have that $s_0|_V$ is sent to a linear form vanishing on H_i, and $s_1|_V$ is sent to a constant since s_0 is V-fixed. Thus there is a commutative square

$$\begin{tikzcd} X \arrow[r, "T_i"] & \mathbb{P}(V/H_i \oplus \mathbb{C}) \\ V \arrow[u, hook] \arrow[r] & V/H_i \arrow[u, hook] \end{tikzcd}$$

where the right vertical arrow is the embedding

$$V/H_i \to \mathbb{P}^1, \quad v \mapsto [s_0(v) : s_1(v)].$$

From this it follows that the product map $X \to \prod_{i=1}^n \mathbb{P}(V/H_i \oplus \mathbb{C})$ extends $\Phi_{\mathscr{A}}$, and thus we can define a morphism

$$T : X \to Y_{\mathscr{A}(X)}, \quad T := (T_1, \dots, T_n).$$

Since T is birational, by Theorem 2.2.5 we can show that T is an isomorphism by showing that it is bijective on closed points. Since T extends $\Phi_{\mathscr{A}}$, it is a morphism of linear V-varieties. The set of stabilizers of X is the lattice of flats of $\mathscr{A}$ by assumption, and one can prove in coordinates that the set of stabilizers of $Y_{\mathscr{A}}$ is the image under $\Phi_{\mathscr{A}}$ of the lattice of flats of $\mathscr{A}$ (see Corollary 2.5.2 (iii)). Thus T carries the set of stabilizers of X bijectively onto the set of stabilizers of $Y_{\mathscr{A}}$. Furthermore, T carries the distinguished points of X bijectively onto the distinguished points of $Y_{\mathscr{A}}$ by Corollary 2.3.1. Let $x \in X$ be a distinguished point with stabilizer V_x, and let $T(x) \in Y_{\mathscr{A}}$ be the corresponding distinguished point with stabilizer $\Phi_{\mathscr{A}}(V_x)$. We have the following commutative square relating the orbit $V \cdot x$ in X and the corresponding orbit $\Phi_{\mathscr{A}}(V) \cdot T(x)$ in $Y_{\mathscr{A}}$.

$$\begin{tikzcd} V \cdot x \arrow[r, "T"] \arrow[d, "\cong"] & \Phi_{\mathscr{A}}(V) \cdot T(x) \arrow[d, "\cong"] \\ V/V_x \arrow[r] & \Phi_{\mathscr{A}}(V)/\Phi_{\mathscr{A}}(V_x) \end{tikzcd}$$

Since the bottom arrow is an isomorphism, it follows that the top arrow is an isomorphism, and we take the disjoint union over all distinguished points to obtain that T is a bijection on closed points.

This completes the proof of Theorem A.

2.5 Schubert varieties of hyperplane arrangements in coordinates

In this section we give a coordinate formula for the Schubert variety of a hyperplane arrangement as a closed subset of $(\mathbb{P}^1)^n$, as well as coordinate formulas for the orbits, distinguished points, stabilizers, minimal V-invariant open neighborhoods, and slices. All of the following formulas are consequences of the defining multihomogeneous equations of the Schubert variety given in (Ardila and Boocher 2016), and they will be familiar to the experts.

We fix the following notation.

- **Coordinates:** Set $E = \{1, \dots, n\}$, and let $\mathscr{A} = \{H_1, \dots, H_n\}$ be an essential hyperplane arrangement in V. In order to work with coordinates, let us fix an isomorphism of V/H_i with $\mathbb{C}$ for each i, and thus we can consider $V \subseteq \mathbb{C}^E$.

- **Group action:** Let us identify $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ set theoretically, writing z for $[z : 1]$

and ∞ for $[1:0]$. In this notation, the action of $\mathbb{C}^E$ on $(\mathbb{P}^1)^E$ is given by coordinate wise addition, using the rule $z+\infty=\infty$ for all $z \in \mathbb{C}$. Since the action of V on $Y_{\mathscr{A}}$ is restricted from the action of V on $(\mathbb{P}^1)^E$, we have that the action of V on $Y_{\mathscr{A}}$ is also given by coordinate wise addition.

- **Projections:** Given $S \subseteq E$, write $\pi_S : (\mathbb{P}^1)^E \to (\mathbb{P}^1)^S$ for the coordinate projection. Because we consider $\mathbb{C}^E \subseteq (\mathbb{P}^1)^E$, we will also write $\pi_S(V)$ for the coordinate projection of V onto $\mathbb{C}^S \times \{0\}^{E\setminus S}$.

- **Matroid flats:** A *flat* of the matroid associated to $V \subseteq \mathbb{C}^E$ is a subset $F \subseteq E$ such that $F = \{i \in E : v_i = 0\}$ for some $v \in V$. Write $\mathscr{F}$ for the collection of flats of the matroid associated to V. There is a natural bijection between $\mathscr{F}$ and the lattice of flats of $\mathscr{A}$ given by sending $F \in \mathscr{F}$ to $\cap_{i\in F} H_i$.

We begin with a set theoretic description of $Y_{\mathscr{A}}$:

Proposition 2.5.1 ((Proudfoot, Xu, and Young 2018) Lemma 7.5 and Lemma 7.6)**.** *Write $Y_{\mathscr{A}}$ as the closure of the linear subspace $V \subseteq \mathbb{C}^E$. Fix a point $x \in (\mathbb{P}^1)^E$, and write $F \subseteq E$ for the set of indices corresponding to non-infinite entries of x. Then $x \in Y_{\mathscr{A}}$ if and only if $F \in \mathscr{F}$ and $\pi_F(x) \in \pi_F(V)$. Equivalently,*

$$Y_{\mathscr{A}} = \bigcup_{F\in\mathscr{F}} \pi_F(V) \times \{\infty\}^{E\setminus F} \subseteq (\mathbb{P}^1)^E.$$

Let us now demonstrate in explicit coordinates the objects defined in Section 2.2.2.

Corollary 2.5.2. *Let $x, y \in Y_{\mathscr{A}}$, and write $F, G \subseteq E$ for the set of indices corresponding to non-infinite entries of x and y respectively.*

(i) The V-orbit of x is $V \cdot x = \pi_F(V) \times \{\infty\}^{E\setminus F}$.

(ii) The distinguished point in the V-orbit of x is $x_F = \{0\}^F \times \{\infty\}^{E\setminus F}$.

(iii) The stabilizer of x is $V_x = V \cap (\{0\}^F \times \mathbb{C}^{E\setminus F})$.

(iv) The minimal V-invariant open neighborhood U_x of x contains y if and only if $F \subseteq G$. Equivalently,

$$U_x = \bigcup_{G \in \mathscr{F}, F \subseteq G} \pi_G(V) \times \{\infty\}^{E \setminus G}.$$

(v) The minimal slice Z_x through x contains y if and only if $F \subseteq G$ and $\pi_F(x) = \pi_F(y)$. Equivalently

$$Z_x = \bigcup_{G \in \mathscr{F}, F \subseteq G} \left(\pi_G(V) \cap \left(\pi_F(x) \times \mathbb{C}^{G \setminus F}\right)\right) \times \{\infty\}^{E \setminus G}.$$

(vi) Set $Y_{\mathscr{A}_F}$ equal to the closure of $V \cap \mathbb{C}^{E \setminus F}$ in $(\mathbb{P}^1)^{E \setminus F}$. The minimal slice through the distinguish point $x_F = \{0\}^F \times \{\infty\}^{E \setminus G}$ is $Z_{x_F} = \{0\}^F \times Y_{\mathscr{A}_F}$.

Proof. We begin with Corollary 2.5.2 (i). Since $x_i = \infty$ for $i \notin F$, we have that $v \in V$ acts on x via $(v \cdot x)_i = v_i + x_i$ for $i \in F$ and $(v \cdot x)_i = \infty$ for $i \notin F$. Thus $V \cdot x \subseteq \pi_F(V) \times \{\infty\}^{E \setminus F}$. For the reverse inclusion, let $y \in \pi_F(V) \times \{\infty\}^{E \setminus F}$, and choose $v \in V$ such that $\pi_F(v) = \pi_F(y) - \pi_F(x)$. Then $v \cdot x = y$.

We have that $v \cdot x = x$ if and only $v_i + x_i = x_i$ for all $i \in F$, so Corollary 2.5.2 (iii) follows.

To prove Corollary 2.5.2 (iv), let $y \in Y_{\mathscr{A}}$ and write $G = \{i \in E : y_i \neq \infty\}$. We wish to show that the set of y for which $F \subseteq G$ is equal to the minimal open neighborhood U_x. We first note that the set of y such that $F \subseteq G$ is a V-stable open set of $V \cdot x$, so it contains the minimal one. To show the reverse inclusion we must check that if $F \subseteq G$, then $\pi_F(V) \times \{\infty\}^{E \setminus F} \subseteq \overline{\pi_G(V) \times \{\infty\}^{E \setminus G}}$. Since $F \in \mathscr{F}$ is a flat, we may choose a vector $v \in V$ such that $v_i = 0$ for $i \in F$ and $v_i \neq 0$ for $i \notin F$. Then for each value of $t \in \mathbb{C}$, $\pi_G(tv) \times \{\infty\}^{E \setminus G} \in \pi_G(V) \times \{\infty\}^{E \setminus G}$, but as $t \to \infty$, the limit lies in $\pi_F(V) \times \{\infty\}^{E \setminus F}$.

We now turn to Corollary 2.5.2 (v). We have that

$$Z_x := \{y \in Y_{\mathscr{A}} : F \subseteq G,\ \pi_F(x) = \pi_F(y)\}$$

is contained in U_x by Corollary 2.5.2 (iv), and so we just need to check that it is a slice.

We have that V_x acts on Z_x, so we use Lemma 2.2.6 to show that the induced map $V *_{V_x} Z_x \to Y_{\mathscr{A}}$ is injective on closed points. Suppose that $v \cdot z = v' \cdot z'$ for $v, v' \in V$ and $z, z' \in Z_x$. Then $\pi_F(v) + \pi_F(z) = \pi_F(v') + \pi_F(z')$, however $\pi_F(z) = \pi_F(z') = \pi_F(x)$, so $\pi_F(v - v') = 0$ as required. Since $V \cap Z_x$ is a coset of V_x, we have that $V *_{V_x} Z_x \to Y_{\mathscr{A}}$ is birational, and thus an open embedding by Theorem 2.2.5.

Now Corollary 2.5.2 (ii) and Corollary 2.5.2 (vi) follow from knowing the slice through x_F. $\square$

Chapter 3

Partial compactifications and morphisms

3.1 Introduction and summary

We now describe how Theorem A extends to equivariant partial compactifications of vector groups, as well as morphisms between them. We define a *morphism of equivariant compactifications of vector groups* to be a map of varieties, which restricts to a linear map from the first vector group to the second.

Let us first review some hyperplane arrangement terminology. We will work only with arrangements of linear hyperplanes in a finite dimensional complex vector space. We do not consider arrangements of affine hyperplanes. We say that a hyperplane arrangement is *essential* if the common intersection of the hyperplanes is zero. A *flat* of a hyperplane arrangement is a linear subspace of the ambient vector space which can be written as the intersection of several hyperplanes. We consider the ambient vector space to be a flat, because it arises from the empty intersection of hyperplanes.

Remark 3.1.1. Following the standard convention, we equip the collection of flats with the partial order given by reverse inclusion, writing $F \leq G$ if F and G are flats such that $G \subseteq F$. When the arrangement is essential, this partial order gives the collection of flats the structure of a finite geometric lattice, or equivalently, a simple matroid. For our purposes, the partial order will only be used in Example 3.1.3 and in the fact that will refer to the flats of an essential hyperplane arrangement as the "lattice of flats."

Viewing an essential hyperplane arrangement as its lattice of flats, we make the following generalization:

Definition 3.1.2. A **partial hyperplane arrangement** in $V = \mathbb{C}^d$ is a finite collection $\mathscr{L}$ of vector subspaces of V, such that

i. $\{0\} \in \mathscr{L}$,

ii. if $F, G \in \mathscr{L}$ then $F \cap G \in \mathscr{L}$,

iii. for each $F \in \mathscr{L}$, the set $\{G \in \mathscr{L} : G \subseteq F\}$ is the lattice of flats of an essential hyperplane arrangement in the vector space F.

Example 3.1.3. Suppose that $\mathscr{L}$ is the lattice of flats of an essential hyperplane arrangement, and $\mathscr{P} \subseteq \mathscr{L}$ is an order filter (i.e. an upward closed set under the partial order of reverse inclusion.) Then $\mathscr{P}$ is a partial hyperplane arrangement.

Example 3.1.4. Here we give an example of a partial hyperplane arrangement in $\mathbb{C}^4$. Let $\mathscr{L}$ consist of the zero subspace of $\mathbb{C}^4$, together with the affine cones of the points, lines, and planes in $\mathbb{P}^3$ listed in Fig. 3.1.

Points	Lines	Planes
$A = [0:0:1:1]$	AB	$ABCD$
$B = [0:1:0:1]$	AC	$BCDE$
$C = [0:0:0:1]$	AD	
$D = [0:-1:0:1]$	BE	
$E = [1:0:0:1]$	CE	
	DE	
	AE	
	BCD	

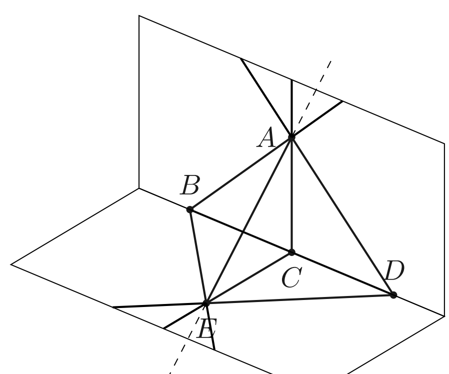

Figure 3.1: The projectivization of a partial hyperplane arrangement in $\mathbb{C}^4$.

Example 3.1.5. Here we give an example of a partial hyperplane arrangement which cannot be realized as an order filter in the lattice of flats of a hyperplane arrangement. Consider the partial hyperplane arrangement $\mathscr{L}$ in $\mathbb{C}^3$ consisting of the proper coordinate subspaces, and a general line. Any hyperplane passing through the general line will intersect one of the coordinate hyperplanes in a non-coordinate line. Therefore if there is hyperplane arrangement whose lattice of flats contains $\mathscr{L}$, then $\mathscr{L}$ cannot be upward closed.

Definition 3.1.6. Suppose that $\mathscr{L}_i$ is a partial hyperplane arrangement in a vector group V_i for $i = 1, 2$, and $T : V_1 \to V_2$ is a linear map. Then we say T is a **morphism of partial hyperplane arrangements** if

i. for each $F_1 \in \mathscr{L}_1$ there exists $F_2 \in \mathscr{L}_2$ such that $T(F_1) \subseteq F_2$,

ii. for each $F_1 \in \mathscr{L}_1$ and $F_2 \in \mathscr{L}_2$, $T^{-1}(F_2) \cap F_1 \in \mathscr{L}_1$.

Example 3.1.7. In the case where $\mathscr{L}_i$ is the lattice of flats of a hyperplane arrangement $\mathscr{A}_i$, then T is a morphism of partial hyperplane arrangements if and only if the preimage of each hyperplane in $\mathscr{A}_2$ is either a hyperplane in $\mathscr{A}_1$ or is V_1.

The following is our main result, in maximal generality.

Theorem B. There is a fully faithful embedding of categories from partial hyperplane arrangements to equivariant partial compactifications of vector groups, such that the following are equivalent for an equivariant partial compactification Y.

i. Y arises from a partial hyperplane arrangement.

ii. Y is normal and satisfies **FO** and **SL**.

iii. Y is normal and satisfies **FO** and **OP**.

3.1.1 Analogy with toric varieties

For the remainder of the introduction, we say that an equivariant partial compactification of V satisfying any of the equivalent conditions of Theorem B is a *linear V-variety*. In

Toric varieties		Linear V-varieties	
Combinatorics	Geometry	Combinatorics	Geometry
N	T	V	V
Full dimensional cones in $N_{\mathbb{R}}$	Affine toric varieties with no torus factors	Essential hyperplane arrangements in V	Schubert varieties of hyperplane arrangements
Cones in $N_{\mathbb{R}}$	Affine toric varieties	Essential hyperplane arrangements in $F \subseteq V$	Simple linear V-varieties
Fans in $N_{\mathbb{R}}$	Toric varieties	Partial hyperplane arrangements in V	Linear V-varieties
Cones in a fan Σ	Orbits, distinguished points, and invariant affine opens in X_Σ	Flats in a partial hyperplane arrangement $\mathscr{L}$	Orbits, distinguished points, and invariant simple opens in $Y_{\mathscr{L}}$

Figure 3.2: Correspondences for toric varieties versus correspondences for linear V-varieties.

this section, we explain the analogy between Theorem B and the correspondence between normal toric varieties and polyhedral fans. From now on, we assume all toric varieties are normal.

Toric varieties satisfies **FO**, **SL**, and **OP**, and once we impose these conditions onto an equivariant partial compactification of a vector group, there is dictionary (Fig. 3.2) which is similar to the dictionary between toric varieties and fans. In both cases the idea is to cover the variety with “simple” open sets. The main difference is that these open sets are affine in the torus case and non-affine in the vector group case.

One-parameter subgroups

Let T be an algebraic torus. A one-parameter subgroup of T is an algebraic group homomorphism from $\mathbb{C}^\times$ to T. The one-parameter subgroups of T form a finitely generated free abelian group N, and write $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ for the corresponding real vector space.

Let V be a vector group. By a one-parameter subgroup of V, we mean an algebraic group homomorphism from $\mathbb{C}$ to V. The one-parameter subgroups of V naturally correspond to the elements of V, so V will play the role of both T and N.

Full dimensional cones and essential hyperplane arrangements

The toric varieties arising from full dimensional cones are exactly the affine toric varieties that have no torus factors. If $\sigma \subseteq N_{\mathbb{R}}$ is full dimensional (strictly convex rational polyhedral) cone, then there is a canonical embedding of tori $T \subseteq \prod_{u \in \mathscr{H}} \mathbb{C}^{\times}$ where $\mathscr{H}$ is the unique minimal basis of the dual semigroup. Note that this embedding is only canonical when σ is full dimensional. The corresponding toric variety is the closure of T in $\prod_{u \in \mathscr{H}}(\mathbb{C}^{\times} \cup \{0\})$. If $\mathscr{A}$ is an essential hyperplane arrangement in V, then there is a canonical embedding of vector groups $V \subseteq \prod_{H \in \mathscr{A}} V/H$. The corresponding Schubert variety is the closure of V in $\prod_{H \in \mathscr{A}}(V/H \cup \{\infty\})$.

Simple partial compactifications

Suppose an algebraic group G acts on a variety X with finitely many orbits. We say that X is *simple* if there is a unique closed orbit. Since the orbits form a finite stratification, X can be covered with simple G-stable open sets. Simple toric varieties are exactly affine toric varieties by (Sumihiro 1974, Corrollary 2), and every affine toric variety arises from a sublattice $N' \subseteq N$ and a full dimensional cone $\sigma \subseteq N' \otimes_{\mathbb{Z}} \mathbb{R}$. Simple linear V-varieties are not affine, however by Proposition 2.2.15 and Theorem A, every simple linear V-variety arises from a vector subspace $F \subseteq V$ and an essential hyperplane arrangement in F.

Partial compactifications

Toric varieties are constructed from affine toric varieties by gluing according to the fan, and likewise linear V-varieties are constructed from simple linear V-varieties by gluing according to the partial hyperplane arrangement. See Section 3.2.3 for more details.

Orbit correspondences

Let Σ be a fan in $N_{\mathbb{R}}$ corresponding to a toric variety X_Σ. Let σ° denote the relative interior of a cone $\sigma \in \Sigma$. That is, $\sigma^\circ = \sigma \setminus \bigcup \tau$ where the union is over $\tau \in \Sigma$ such that $\tau \subsetneq \sigma$. The cones $\sigma \in \Sigma$ are in one-to-one correspondence with distinguished points $x_\sigma \in X_\Sigma$, given by

$$N \cap \sigma^\circ = \{\lambda \in N : \lim_{t \to 0} \lambda(t) = x_\sigma\}.$$

Let $\mathscr{L}$ be a partial hyperplane arrangement corresponding to a linear V-variety $Y_{\mathscr{L}}$. Given $F \in \mathscr{L}$, write $F^\circ = F \setminus \bigcup G$ where the union is over $G \in \mathscr{L}$ such that $G \subsetneq F$. The flats $F \in \mathscr{L}$ are in one-to-one correspondence with distinguished points $y_F \in Y_{\mathscr{L}}$, given by

$$F^\circ = \{v \in V : \lim_{t \to \infty} tv = y_F\}.$$

In both cases, each orbit contains exactly one distinguished point. Therefore cones (resp. flats) also correspond to orbits, and to simple invariant open sets in X_Σ (resp. $Y_{\mathscr{L}}$). See Section 2.3 for more details.

3.2 Proof of Theorem B

3.2.1 Overview

In this section $V = \mathbb{C}^d$ will denote a vector group, and we will use the term *linear V-variety* to mean a normal equivariant partial compactification $V \subseteq X$, which has finitely many orbits and a slice through every point. We have proved in Section 2.2.4 that Item ii and Item iii in Theorem B are equivalent. To prove Theorem B, it remains to show the following.

Theorem 3.2.1. *Given a linear V-variety X, let $\mathscr{L}(X)$ denote the associated partial hyperplane arrangement of Definition 2.3.2.*

(i) ***Functoriality:*** *If X_i is a linear V_i-variety for $i = 1, 2$ and $T : X_1 \to X_2$ is a morphism, then the restricted linear map $T : V_1 \to V_2$ is a morphism of partial*

hyperplane arrangements $\mathscr{L}(X_1) \to \mathscr{L}(X_2)$.

(ii) ***Full faithfulness:*** *If* X_i *is a linear* V_i*-variety for* $i = 1, 2$ *and* $T : V_1 \to V_2$ *is a morphism of partial hyperplane arrangements* $\mathscr{L}(X_1) \to \mathscr{L}(X_2)$*, then* T *extends uniquely to a morphism of linear* V*-varieties* $X_1 \to X_2$.

(iii) ***Essential surjectivity:*** *If* $\mathscr{L}$ *is an essential hyperplane arrangement in* V*, then there exists a linear* V*-variety* X *such that* $\mathscr{L}(X) \cong \mathscr{L}$.

The proof of essential surjectivity describes how to construct the linear V-variety associated to a partial hyperplane arrangement.

3.2.2 Morphisms

In this section we prove functoriality and full faithfulness. To prove functoriality, we generalize the proof of Lemma 2.3.4, appealing again to the Borel fixed point theorem.

Proof of Theorem 3.2.1 (i). Since the stabilizer of $x \in X_1$ is mapped into the stabilizer of $T(x) \in X_2$, it follows that for each flat of $\mathscr{L}(X_1)$ is mapped into a flat of $\mathscr{L}(X_2)$, as required in Item i of Definition 3.1.6. If $F_1 \in \mathscr{L}(X_1)$ and $F_2 \in \mathscr{L}(X_2)$, it remains to show that $T^{-1}(F_2) \cap F_1 \in \mathscr{L}(X_1)$. Write Z_1 for the minimal slices through the distinguished point $x_1 \in X_1$ corresponding to F_1. Then Z_1 is proper by Proposition 2.2.15, so $T^{-1}(F_2) \cap Z_1$ is proper. By the Borel fixed point theorem, there exists $z \in T^{-1}(x_2) \cap Z_1$ such that

$$T^{-1}(F_2) \cap F_1 \subseteq (V_1)_z.$$

We now show the opposite inclusion. By Proposition 2.2.15 there is a unique F_1-fixed point in Z_1, and $\overline{F_1 \cdot z}$ contains a F_1-fixed point, so $x_1 \in \overline{F_1 \cdot z}$. Therefore $(V_1)_z \subseteq F_1$. On the other hand, $z \in T^{-1}(x_2)$, so $(V_1)_z \subseteq T^{-1}(F_2)$. □

Let us start by proving full faithfulness in the compact case.

Lemma 3.2.2. *Suppose that* $\mathscr{A}_i$ *is an essential hyperplane arrangement in the vector group* V_i *for* $i = 1, 2$*, and* $T : V_1 \to V_2$ *is a linear map such that the preimage of each* *hyperplane in* $\mathscr{A}_2$ *is either a hyperplane in* $\mathscr{A}_1$ *or is* V_1*. Then* T *extends to a morphism between Schubert varieties* $Y_{\mathscr{A}_1} \to Y_{\mathscr{A}_2}$.

Proof. Since $Y_{\mathscr{A}_i}$ is the closure of $V_i \subseteq \prod_{H \in \mathscr{A}_i} \mathbb{P}(V_i/H \oplus \mathbb{C})$, we are reduced to showing that T extends to a map

$$\prod_{H_1 \in \mathscr{A}_1} \mathbb{P}(V_1/H_1 \oplus \mathbb{C}) \to \prod_{H_2 \in \mathscr{A}_2} \mathbb{P}(V_2/H_2 \oplus \mathbb{C}).$$

Fix $H_2 \in \mathscr{A}_2$. If $T^{-1}(H_2) = V_1$, then the H_2 component of the displayed function can be defined to be constant with value $0 \in V_2/H_2$. Otherwise $T^{-1}(H_2) \in \mathscr{A}_1$, in which case the H_2 component of the displayed function can be defined as projection onto $\mathbb{P}(V_1/T^{-1}(H_2) \oplus \mathbb{C})$ followed by the map

$$[v + T^{-1}(H_2) : z] \mapsto [T(v) + H_2 : z].$$

$\square$

To prove full faithfulness in general, we apply Lemma 3.2.2 to the slices in a linear V-variety, which are Schubert varieties of hyperplane arrangements by Proposition 2.2.15 and Theorem A.

Proof of Theorem 3.2.1 (ii). Since $V_1 \subseteq X_1$ is dense, uniqueness follows immediately. We now argue that an extension $T : X_1 \to X_2$ exists.

Suppose that $F_i \in \mathscr{L}(X_i)$ for $i = 1, 2$ such that $T(F_1) \subseteq F_2$, and denote by $Z_i \subseteq X_i$ the slices through the corresponding distinguished points. By Proposition 2.2.15, Lemma 2.2.13, and Theorem A, Z_i is the Schubert variety of a hyperplane arrangement in F_i. By Lemma 2.2.3, the hyperplane arrangement corresponding to Z_i is given by those flats of $\mathscr{L}(X_i)$ which are contained in F_i. Therefore, since T is a morphism of partial hyperplane arrangements, the hypotheses of Lemma 3.2.2 are satisfied for the restriction $T|_{F_1} : F_1 \to F_2$. Thus $T|_{F_1}$ extends to a morphism $\overline{T|_{F_1}} : Z_1 \to Z_2$.

We can now extend T to the open set $V_1 *_{F_1} Z_1 \subseteq X_1$ by sending $[v, z]$ to $[T(v), \overline{T|_{F_1}}(z)] \in$

$V_2 *_{F_2} Z_2$. As F_1 and F_2 vary, the open sets $V_1 *_{F_1} Z_1$ cover X_1 (here we are using Item i in Definition 3.1.6), and the extensions of T to $V_1 *_{F_1} Z_1$ are unique and thus compatible on intersections, so they define a global extension $X_1 \to X_2$. □

3.2.3 Construction of linear V-varieties

Now we turn to essential surjectivity. Let $\mathscr{L}$ be a partial hyperplane arrangement in V.

A diagram of hyperplane arrangements

From Definition 3.1.2 we have that for every $F \in \mathscr{L}$, there is an essential hyperplane arrangement $\mathscr{A}_F$ in the vector space F whose lattice of flats is $\{G \in \mathscr{L} : G \subseteq F\}$. It follows immediately that $\mathscr{A}_F$ is unique. If $G \subseteq F$ are elements of $\mathscr{L}$, then $\mathscr{A}_G$ is called the *restriction* of $\mathscr{A}_F$ to the flat G.

A diagram of Schubert varieties

For each $F \in \mathscr{L}$, we have the Schubert variety $Y_{\mathscr{A}_F}$ associated to the hyperplane arrangement $\mathscr{A}_F$. If $G \in \mathscr{L}$ is contained in F, then G is a flat of $\mathscr{A}_F$. Therefore $Y_{\mathscr{A}_G}$ is the slice through a distinguished point $x_G \in Y_{\mathscr{A}_F}$ by the coordinate formula given in Corollary 2.5.2 (vi). So $\mathscr{L}$ indexes a diagram of Schubert varieties of hyperplane arrangements, where each arrow is the inclusion of a slice.

A diagram of open embeddings

Given $G \subseteq F$ elements of $\mathscr{L}$, we have an open embedding $F *_G Y_{\mathscr{A}_G} \subseteq Y_{\mathscr{A}_F}$ because $Y_{\mathscr{A}_G}$ is a slice through x_G. It is straight forward to check that $V *_F -$ preserves open embeddings, so by the associativity property of Lemma 2.2.2 we have an open embedding

$$V *_G Y_{\mathscr{A}_G} \cong V *_F (F *_G Y_{\mathscr{A}_G}) \subseteq V *_F Y_{\mathscr{A}_F}.$$

Therefore $\mathscr{L}$ indexes a diagram of open embeddings between the varieties $V *_F Y_{\mathscr{A}_F}$ for $F \in \mathscr{L}$. By Lemma 2.2.3, $V *_F Y_{\mathscr{A}_F}$ has finitely many V-orbits, and again by the associativity property of Lemma 2.2.2 each point $[v, y] \in V *_F Y_{\mathscr{A}_F}$ has a slice $V *_F Z_y$, where Z_y is a slice through $y \in Y_{\mathscr{A}_F}$. Thus $V *_F Y_{\mathscr{A}}$ is a linear V-variety.

Cocycle condition

To verify that the $V *_F Y_{\mathscr{A}_F}$ glue together, we must check the cocycle condition (Hartshorne 1977, Exercise II.2.12). This reduces to the following lemma.

Lemma 3.2.3. *If $G, H \subseteq F$ and $I = G \cap H$ are elements of $\mathscr{L}$, then*

$$V *_G Y_{\mathscr{A}_G} \cap V *_H Y_{\mathscr{A}_H} = V *_I Y_{\mathscr{A}_I}$$

considered as open sets in $V *_F Y_{\mathscr{A}_F}$.

Proof. A point $[v, z] \in V *_F Y_{\mathscr{A}_F}$ lies in $V *_G Y_{\mathscr{A}_G}$ if and only if $z \in Y_{\mathscr{A}_G}$, so we are reduced to showing that $Y_{\mathscr{A}_G} \cap Y_{\mathscr{A}_H} = Y_{\mathscr{A}_I}$, considered as closed sets in $Y_{\mathscr{A}_F}$. To check this one can use the set theoretic formula of Proposition 2.5.1. □

Separation

We now prove that the variety $Y_{\mathscr{L}}$ glued from the $V *_F Y_{\mathscr{A}_F}$ is separated. By (Hartshorne 1977, Corralary II.4.2), checking that the diagonal morphism $Y_{\mathscr{L}} \to Y_{\mathscr{L}} \times Y_{\mathscr{L}}$ is a closed embedding reduces to the following lemma.

Lemma 3.2.4. *Suppose that $F = G \cap H$ where $F, G, H \in \mathscr{L}$, and write*

$$i : V *_F Y_{\mathscr{A}_F} \to V *_G Y_{\mathscr{A}_G} \times V *_H Y_{\mathscr{A}_H}, \quad [v, y] \mapsto [v, y] \times [v, y].$$

Then i has a closed image.

Proof. Consider the canonical fibration defined in Section 2.2.1,

$$\tau_F : V *_F Y_{\mathscr{A}_F} \to V/F, \quad [v, y] \mapsto v + F.$$

We get that the following square is commutative.

$$\begin{tikzcd}
V *_F Y_{\mathscr{A}_F} \arrow[r, "i"] \arrow[d, "\tau_F"] \arrow[dr, "f"] & V *_G Y_{\mathscr{A}_G} \times V *_H Y_{\mathscr{A}_H} \arrow[d, "\tau_G \times \tau_H"] \\
V/F \arrow[r, "j"] & V/G \times V/H
\end{tikzcd}$$

Notice that j is a closed embedding, since $F = G \cap H$. We also have that τ_F is proper (it's conjugate to the projection $V/F \times Y_{\mathscr{A}_F} \to V/F$), so f is proper. Then i is proper by (Hartshorne 1977, Corralary II.4.8(e)), and thus closed. $\square$

Linearity

Since the action of V extends to each open set in a cover of $Y_{\mathscr{L}}$ and is compatible on intersections, the action extends to $Y_{\mathscr{L}}$. From the fact that $Y_{\mathscr{L}}$ is glued from linear V-varieties with maps that preserve V, it follows that $Y_{\mathscr{L}}$ is normal, that it has finitely many V-orbits, and that every point has a slice. Thus $Y_{\mathscr{L}}$ is a linear V-variety.

Proof of Theorem 3.2.1 (iii). We have constructed a linear V-variety $Y_{\mathscr{L}}$, and we wish to show that the collection of stabilizers $\mathscr{L}(Y_{\mathscr{L}})$ coincides with $\mathscr{L}$.

First we check that each flat of $\mathscr{L}$ is the stabilizer of a point in $Y_{\mathscr{L}}$. Suppose that $F \in \mathscr{L}$. Then consider the point $[0, x_F] \in V *_F Y_{\mathscr{A}_F} \subseteq Y_{\mathscr{L}}$, where x_F is the unique fixed point of $Y_{\mathscr{A}_F}$. By Lemma 2.2.3, F is the stabilizer of $[0, x_F] \in Y_{\mathscr{L}}$.

Now we check that the stabilizer of each point of $Y_{\mathscr{L}}$ is a flat of $\mathscr{L}$. From the construction of $Y_{\mathscr{L}}$, we have that every point is contained in an open set isomorphic to $V *_F Y_{\mathscr{A}_F}$ for $F \in \mathscr{L}$. Suppose that $[v, y] \in V *_F Y_{\mathscr{A}_F}$. Then the stabilizer of $[v, y]$ with respect to the action of V is equal the stabilizer of $y \in Y_{\mathscr{A}_F}$ with respect to the action of F by Lemma 2.2.3, and is therefore equal to a flat in $\mathscr{A}_F$ by Corollary 2.5.2 (iii). $\square$

This completes the proof of Theorem B.

Comparison with toric varieties

We conclude this section by explaining how the construction of a toric variety form a polyhedral fan can be made to look like the construction above. In order to be consistent with (Cox, Little, and Schenck 2011), we will use n for the dimension of a toric variety rather than d as we have been doing so far. We will use d for the dimension of a cone. Let us fix the following notation.

- $N \cong \mathbb{Z}^n$, a lattice of dimension n,
- $T = \operatorname{Spec} \mathbb{C}[N^\vee]$, the corresponding n-dimensional torus,
- Σ, a fan of strictly convex rational polyhedral cones in $N \otimes_{\mathbb{Z}} \mathbb{R}$,
- $\sigma \in \Sigma$, a cone of dimension d,
- $U_{\sigma,N}$, the toric variety of dimension n corresponding to σ considered as a cone in $N \otimes_{\mathbb{Z}} \mathbb{R}$ (Cox, Little, and Schenck 2011, Theorem 1.2.18),
- $x_\sigma \in U_{\sigma,N}$, the distinguished point in the minimal T-orbit (Cox, Little, and Schenck 2011, Chapter 3.2),
- $N_\sigma = \operatorname{span}_{\mathbb{Z}}(\sigma \cap N)$, the sublattice of dimension d generated by σ,
- $T_\sigma = \operatorname{Spec} \mathbb{C}[N_\sigma^\vee]$, the d-dimensional torus corresponding to N_σ,
- U_{σ,N_σ}, the toric variety of dimension d corresponding to σ considered as a cone in $N_\sigma \otimes_{\mathbb{Z}} \mathbb{R}$,
- $\mathscr{H}_\sigma$, the unique minimal basis (see (Cox, Little, and Schenck 2011, Proposition 1.2.23)) for the semigroup

$$S_{\sigma,N_\sigma} = \{u \in \operatorname{Hom}_{\mathbb{Z}}(N_\sigma, \mathbb{Z}) : u \text{ is nonnegative on } \sigma\}.$$

The fan Σ indexes a commutative diagram of inclusions among its cones, as in Section 3.2.3. The cone σ defines an embedding of the torus T_σ in the larger torus $(\mathbb{C}^\times)^{\mathscr{H}_\sigma}$ (see (Cox, Little, and Schenck 2011, Definition 1.1.7)), and the closure of $T_\sigma \subseteq (\mathbb{A}^1)^{\mathscr{H}_\sigma}$ is U_{σ,N_σ} (see (Cox, Little, and Schenck 2011, Theorem 1.1.17)), similar to the definition of the Schubert variety of a hyperplane arrangement. Given $\tau \subseteq \sigma$ elements of Σ, the natural homomorphism of tori $T_\tau \subseteq T_\sigma$ extends to a morphism of toric varieties $U_{\tau,N_\tau} \subseteq U_{\sigma,N_\sigma}$, and one can check that U_{τ,N_τ} is the minimal slice through the distinguished point $x_\tau \in U_{\sigma,N_\sigma}$, as in Section 3.2.3. As in Section 3.2.3, for elements $\tau \subseteq \sigma$ of Σ, we have a natural open embedding

$$T *_{T_\tau} U_{\tau,N_\tau} \cong T *_{T_\sigma} (T *_{T_\tau} U_{\tau,N_\tau}) \subseteq T *_{T_\sigma} U_{\sigma,N_\sigma}.$$

One can verify that U_{σ,N_σ} is the minimal slice through the distinguished point $x_\sigma \in U_{\sigma,N}$, so there is natural isomorphism $T *_{T_\sigma} U_{\sigma,N_\sigma} \cong U_{\sigma,N}$. Thus the above diagram of open embeddings is isomorphic to the usual diagram of open embeddings $U_{\tau,N} \subseteq U_{\sigma,N}$ (see (Cox, Little, and Schenck 2011, Section 3.1)), whose colimit is the toric variety X_Σ.

Chapter 4

Subspace arrangements

4.1 Introduction and summary

Let us begin by recalling the main result of Chapter 2:

Theorem 1.0.1. *An equivariant compactification Y of the vector group $V = \mathbb{C}^d$ is isomorphic to the Schubert variety of a hyperplane arrangement if and only if Y is normal as a variety, Y has only finitely many orbits, and each orbit contains a point which can be reached by a limit $\lim_{t\to\infty} tv$, for $v \in V$.*

We observed in Example 2.1.2 that there is an additive action of $\mathbb{C}^2$ on $\mathbb{P}^2$ which has finitely many orbits. Since $\mathbb{P}^2$ is not the Schubert variety of hyperplane arrangement (it cannot embedded into $(\mathbb{P}^1)^n$ for any n), this shows that the limit condition is necessary in the above theorem statement. In fact, projective space of any dimension admits an additive action with finitely many orbits:

Theorem 4.1.1. *(Hassett and Tschinkel 1999, Proposition 3.7) Over a field of characteristic zero, the projection space $\mathbb{P}^n$ admits a unique (up to isomorphism) action by $\mathbb{C}^n$ with finitely many orbits.*

In coordinates, the action is given by the faithful representation $\rho_n : \mathbb{C}^n \to \mathrm{Aut}(\mathbb{P}^n)$

$$a = (a_1, \ldots, a_n) \mapsto \exp \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ a_1 & 0 & 0 & \ddots & \vdots \\ a_2 & a_1 & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ a_n & \cdots & a_2 & a_1 & 0 \end{pmatrix}.$$

Notice that the matrix exponential in the displayed formula has polynomial entries, so ρ_n is an algebraic action.

In this chapter, we construct a class of equivariant compactifications containing both Schubert varieties of hyperplane arrangements and the finite-orbit actions on projective spaces considered by (Hassett and Tschinkel 1999).

Definition 4.1.2. Given an arrangement of linear subspaces $V_1, \ldots, V_n \subseteq V = \mathbb{C}^d$, write V_i as the intersection of $c_i := \mathrm{codim}(V_i)$ generic hyperplanes defined by linear forms $a_{ij} : V \to \mathbb{C}$. Let $A = \{a_{ij}\}_{ij}$. We define the **polymatroid Schubert variety** Y_A to be the orbit closure of the origin in $\prod_{i=1}^n \mathbb{P}^{c_i}$ under the V-action given by

$$V \to \prod_{i=1}^{n} \mathrm{Aut}(\mathbb{P}^{c_i}), \quad v \mapsto \rho_{c_i}(a_{ij}(v)).$$

If $V_1, \ldots, V_n$ are codimension one, then Y_A coincides with the matroid Schubert variety, and if $n = 1$ then Y_A is a projective space with V action as in Theorem 4.1.1. Because Y_A above is defined as an orbit closure, it automatically has the structure of an equivariant compactification of V. The main result of this chapter is the following.

Theorem 4.1.3. *Y_A has finitely many orbits, and the orbit poset of Y_A is isomorphic to the poset of combinatorial flats of the associated polymatroid, which we introduce in Section 4.2.*

Since Y_A has finitely many V-orbits, the orbits form an algebraic cell decomposition. It then follows by the arguments of (Björner and Ekedahl 2009; Huh and Wang 2017) that the orbit poset of Y_A is always top-heavy. One way to see that the orbit poset of Y_A cannot be the usual lattice of flats of the polymatroid, is that the usual lattice of flats is not top heavy in general.

4.1.1 Future directions

Motivated by Theorem 1.0.1, it would be interesting to classify all finite-orbit equivariant compactifications of V. Towards this direction, we pose the following questions.

Question 4.1.4.

i. Is every polymatroid Schubert variety normal?

ii. Suppose that Y is a normal equivariant compactification of $V = \mathbb{C}^d$, with finitely many orbits. Is Y isomorphic to a polymatroid Schubert variety?

At the time of writing this thesis, the author does not know the answer to either question.

In the realizable case, the Kazhdan-Lusztig polynomial and Z-polynomail of a matroid (Elias, Proudfoot, and Wakefield 2016; Proudfoot, Xu, and Young 2018) measure the local intersection cohomology (at the V-fixed point) and global intersection cohomology of a matroid Schubert variety respectively. Since the polymatroid Schubert variety Y_A has a unique fixed point, we can formulate the following question:

Question 4.1.5. *Can one define a Kazhdan-Lusztig polynomail and a Z polynomial for polymatroids by studying the (local) intersection cohomology of Y_A?*

4.2 Polymatroids

A **polymatroid** on a finite set $E = \{1, \ldots, n\}$ is the data of a **rank** function $\mathrm{rk} : 2^E \to \mathbb{Z}_{\geq 0}$ that is

normalized $\mathrm{rk}(\emptyset) = 0$,

increasing if $A \subseteq B$, then $\mathrm{rk}(A) \leq \mathrm{rk}(B)$, and

submodular if $A, B \subseteq E$, then $\mathrm{rk}(A \cup B) + \mathrm{rk}(A \cap B) \leq \mathrm{rk}(A) + \mathrm{rk}(B)$.

A polymatroid is a **matroid** if its rank function satisfies $\mathrm{rk}(A) \leq |A|$ for any $A \subseteq E$.

Let P be a polymatroid on E. A **flat** of P is a subset $F \subseteq E$ maximal among those of its rank. The **closure** of $S \subseteq E$ is the unique minimal flat containing S, denoted $\overline{S}$. The **deletion** of a subset $T \subseteq E$ is the polymatroid $P \setminus T$ on $E \setminus T$ defined by $\mathrm{rk}_{P\setminus T}(S) := \mathrm{rk}_P(S)$ for $S \subseteq E \setminus T$. The **contraction** P/T is the polymatroid on $E \setminus T$ defined by $\mathrm{rk}_{P/T}(S) := \mathrm{rk}_P(S \cup T) - \mathrm{rk}_P(T)$.

Prototypical examples of polymatroids arise from linear subspaces.

Example 4.2.1. Let $V_1, \ldots, V_n \subseteq V$ be an essential arrangement of linear subspaces in a complex vector space V of dimension d. By **essential** we mean that $\bigcap_{i\in E} V_i = \{0\}$. The associated polymatroid P is defined by the rank function

$$\mathrm{rk}_P(S) := \operatorname{codim} \bigcap_{i\in S} V_i.$$

We will often consider $V \subseteq \prod_{i\in E} V/V_i$ as a linear subspace. Writing

$$\pi_S : \prod_{i\in E} V/V_i \to \prod_{i\in S} V/V_i$$

for the projection, we have that $\mathrm{rk}_P(S) = \dim \pi_S(V)$. A polymatroid that arises in this way is **realizable**, and $V_1, \ldots, V_n \subseteq V$ is a **realization**. Let us write $V_T := \bigcap_{i\in T} V_i$. Then the deletion $P \setminus T$ is realized by the subspace arrangement

$$V_i/V_{E\setminus T} \subseteq V/V_{E\setminus T} \quad \text{for each } i \in E \setminus T,$$

and the contraction P/T is realized by the subspace arrangement

$$V_i \cap V_T \subseteq V_T \quad \text{for each } i \in E \setminus T.$$

4.2.1 Lifts

Let P be a polymatroid on $E = \{1, \ldots, n\}$, and $\widetilde{E}$ a finite set, equipped with a surjection $\nu : \widetilde{E} \to E$ such that $|\nu^{-1}(i)| \geq \mathrm{rk}_P(i)$ for each $i \in E$. The **multisymmetric lift** of P associated to ν is the matroid P_ν on $\widetilde{E}$ defined by

$$\mathrm{rk}_{P_\nu}(S) = \min\{\mathrm{rk}_P(A) + |S \setminus \nu^{-1}(A)| : A \subseteq E\}.$$

The lift carries a natural action by the product of symmetric groups $\mathfrak{S} = \prod_{i=1}^{n} \mathfrak{S}_{\nu^{-1}(i)}$. Any flat stable under the action of $\mathfrak{S}$ is **geometric**. Each flat F of P_ν contains a unique maximal geometric flat $F^{\mathrm{geo}} = \bigcap_{\sigma \in \mathfrak{S}} \sigma \cdot F$.

Lemma 4.2.2 ((Crowley, Huh, Larson, Simpson, and Wang 2020))**.** *If F is a flat of P_ν, then there is a unique maximal geometric flat F^{geo} contained in F, and*

$$\mathrm{rk}_{P_\nu}(F) = \mathrm{rk}_{P_\nu}(F^{\mathrm{geo}}) + |F \setminus F^{\mathrm{geo}}|.$$

Lemma 4.2.3 ((Crowley, Huh, Larson, Simpson, and Wang 2020))**.** *If $S \subseteq \widetilde{E}$, then for each $i \in E$, either $\overline{S} \cap \nu^{-1}(i) = S \cap \nu^{-1}(i)$ or $\overline{S} \cap \nu^{-1}(i) = \nu^{-1}(i)$.*

Remark 4.2.4. Suppose P is a polymatroid realized by $V_1, \ldots, V_n \subseteq V$, and let P_ν be a lift. Choosing generic coordinates $a_{i1}, \ldots, a_{i\,\mathrm{codim}\,V_i}$ on each V/V_i gives an isomorphism $\prod_{i \in E} V/V_i \cong \mathbb{C}^{\widetilde{E}}$, and the resulting embedding $V \subseteq \mathbb{C}^{\widetilde{E}}$ is a realization of P_ν.

4.2.2 Combinatorial flats

A **multiset** on a finite set E is an element of $\mathbb{Z}^{E}_{\geq 0}$. If $\nu : \widetilde{E} \to E$ is a surjection, then $S \subseteq \widetilde{E}$ **represents** a multiset $\mathbf{s} = (s_i)_{i \in E}$ if $s_i = |\nu^{-1}(i) \cap S|$ for all $i \in E$.

A **combinatorial flat** of a polymatroid P on E is any multiset represented by a flat of P_ν. If $\mathbf{s}$ is represented by S, then the **closure** of $\mathbf{s}$ is the multiset $\overline{\mathbf{s}} = (\bar{s}_i)_{i \in E}$ represented by $\overline{S}$, the closure of S in P_ν, and the **rank** of $\mathbf{s}$ is $\mathrm{rk}(\mathbf{s}) := \mathrm{rk}_{P_\nu}(S)$. The $\mathfrak{S}$-symmetry of P_ν guarantees that both $\overline{\mathbf{s}}$ and $\mathrm{rk}(\mathbf{s})$ are independent of S.

We can give a more direct description of the combinatorial flats of a polymatroid, which does not involve the lift P_ν. Let $C \subseteq \mathbb{Z}_{\geq 0}^E$ be the rectangle $\{\mathbf{s} : s_i \leq \mathrm{rk}_P(i)\}$. Then the independence polytope of P is given by

$$\mathrm{Ind}_P = \{\mathbf{s} \in C : \sum_{i \in A} s_i \leq \mathrm{rk}_P(A) \text{ for all } A \subseteq E\} \subseteq C$$

Lemma 4.2.5. $\mathbf{s} \in$ *Ind*$_P$ *if and only if there exists a set* $\widetilde{I} \subseteq \widetilde{E}$*, independent in* P_ν*, such that* $\#(\widetilde{I} \cap \nu^{-1}(i)) = s_i$*.*

Proof. Let $\mathbf{s} \in C$, and let $\widetilde{I} \subseteq \widetilde{E}$ be any set such that $\#(\widetilde{I} \cap \nu^{-1}(i)) = s_i$. We will show that $\mathbf{s} \in \mathrm{Ind}_P$ if and only if $\widetilde{I}$ is independent. By definition of P_ν, the rank $\widetilde{I}$ is equal to its size if and only if for all $A \subseteq E$, $\#(\widetilde{I} \cap \nu^{-1}(A)) \geq \mathrm{rk}_P(A \cap \nu(\widetilde{I}))$. This condition is equivalent to $\sum_{i \in A} s_i \leq \mathrm{rk}_P(A)$ for all $A \subseteq E$. □

Proposition 4.2.6. *As above, let Ind*$_P$ *be the independence polytope of* P*, inscribed in the rectangle* $C \subseteq \mathbb{Z}_{\geq 0}^E$*.*

i. For each $\mathbf{s} \in C$*,* $\mathrm{rk}(\mathbf{s}) = \max\{\sum_{i \in E} s'_i : \mathbf{s}' \in$ *Ind*$_P$ *and* $\mathbf{s}' \leq \mathbf{s}$ *coordinate wise*$\}$*.*

ii. $\mathbf{s} \in C$ *is a combinatorial flat if and only if it is coordinate-wise maximal with respect to its rank.*

Proof. Let $S \subseteq \widetilde{E}$ be a set representing $\mathbf{s} \in C$. By Lemma 4.2.5, the maximum in Item i is equal to the maximum size of an independent subset of S. This proves Item i. Item ii follows by definition of $\mathrm{rk}(\mathbf{s})$ as $\mathrm{rk}(S)$. □

We a lemma which we will need in the following sections.

Lemma 4.2.7. *Let* P *be the polymatroid. If* $\mathbf{s}$ *is not a combinatorial flat of* P *and* $\overline{s}_n < \nu^{-1}(n)$*, then* $\mathbf{s}' = (s_1, \ldots, s_{n-1})$ *is not a combinatorial flat of* $P \setminus n$*.*

Proof. Let rk be the rank function of P. Choose $S \subseteq \widetilde{E}$ representing $\mathbf{s}$. Since $\mathbf{s}$ is not a combinatorial flat, $\overline{S}$ properly contains S. In fact, $\overline{S} \setminus \nu^{-1}(n)$ properly contains $S \setminus \nu^{-1}(n)$:

this is because $|\overline{S} \cap \nu^{-1}(n)| < |\nu^{-1}(n)|$, so $\overline{S} \cap \nu^{-1}(n) = S \cap \nu^{-1}(n)$ by Lemma 4.2.3. By Lemma 4.2.2,

$$\operatorname{rk}(S \setminus \nu^{-1}(n)) = \operatorname{rk}(S) - s_n = \operatorname{rk}(\overline{S}) - s_n = \operatorname{rk}(\overline{S} \setminus \nu^{-1}(n)).$$

Since $S \setminus \nu^{-1}(n)$ is properly contained in $\overline{S} \setminus \nu^{-1}(n)$, this means $S \setminus \nu^{-1}(n)$ is not a flat of $P_\nu \setminus \nu^{-1}(n)$. By (Crowley, Huh, Larson, Simpson, and Wang 2020), $P_\nu \setminus \nu^{-1}(n) = (P \setminus n)_\nu$, so $S \setminus \nu^{-1}(n)$ is also not a flat of $(P \setminus n)_\nu$. Since $S \setminus \nu^{-1}(n)$ represents $\mathbf{s}'$, this means $\mathbf{s}'$ is not a combinatorial flat of $P \setminus n$. □

4.3 The finite-orbit equivariant structure on $\mathbb{P}^n$

4.3.1 Group actions on varieties

Following (Hassett and Tschinkel 1999), let G be a connected linear algebraic group. A G-**structure** on a variety X is a left action of G on X such that the stabilizer of a generic point is trivial and the orbit of a generic point in X is dense. A G-variety is a variety with a fixed G-structure. A **morphism** of G-varieties is a map $X_1 \to X_2$ that commutes with the G-action. An **isomorphism** of G-varieties is a commuting square

$$\begin{array}{ccc} G \times X_1 & \xrightarrow{(\alpha,j)} & G \times X_2 \\ \downarrow & & \downarrow \\ X_1 & \xrightarrow{j} & X_2 \end{array}$$

where α is an automorphism of G, and j is an isomorphism.

4.3.2 The finite-orbit action on $\mathbb{P}^n$

We now specialize to consider the vector group $\mathbb{C}^n$.

Theorem 4.3.1. *(Hassett and Tschinkel 1999, Proposition 3.7) The projection space $\mathbb{P}^n$ admits a unique (up to isomorphism of $\mathbb{C}^n$-varieties) $\mathbb{C}^n$ structure with finitely many orbits.*

In coordinates, the action is given by the faithful representation $\rho_n : \mathbb{C}^n \to \mathrm{Aut}(\mathbb{P}^n)$

$$a = (a_1, \ldots, a_n) \mapsto \exp \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ a_1 & 0 & 0 & \ddots & \vdots \\ a_2 & a_1 & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ a_n & \cdots & a_2 & a_1 & 0 \end{pmatrix}.$$

Let $b_0, \ldots, b_n$ be the coordinates on the projective space $\mathbb{P}^n$, and let the boundary (resp. interior) of $\mathbb{P}^n$ be the locus given by $b_0 = 0$ (resp. $b_0 \neq 0$).

We will only need the following properties of the action ρ_n, which can be derived directly from the definition.

Lemma 4.3.2. *The vector group $\mathbb{C}^n$ is mapped isomorphically onto the interior of $\mathbb{P}^n$ by*

$$(a_1, \ldots, a_n) \mapsto \rho_n(a_1, \ldots, a_n) \cdot [1 : 0 : \ldots : 0],$$

and $\rho_n(a_1, \ldots, a_n)$ acts on the boundary of $\mathbb{P}^n$ as $\rho_{n-1}(a_1, \ldots, a_{n-1})$.

Proof. We expand the matrix exponential and apply it to the origin in the interior of $\mathbb{P}^n$:

$$\rho_n(a_1, \ldots, a_n) \cdot [1 : 0 : \ldots : 0] = [1 : a_1 : a_2 + p_2(a_1) : a_3 + p_3(a_1, a_2) : \ldots : a_n + p_n(a_1, \ldots, a_{n-1})],$$

where p_i are polynomials. Therefore $a_n = b_n - p_n(a_1, \ldots, a_{n-1})$, and each a_i for $i < n$ can be written as a polynomial in the b_i by induction. This proves that $\mathbb{C}^n$ is embedded as the interior of $\mathbb{P}^n$. For the second statement, note that $\rho_n(a_1, \ldots, a_n)$ acts on an element of the boundary as the submatrix obtained by deleting the first row and column. □

Lemma 4.3.3. *The action of $\mathbb{C}^n$ on $\mathbb{P}^n$ partitions $\mathbb{P}^n$ into orbits*

$$O_k = \{b_0 = b_1 = \cdots = b_{n-k-1} = 0,\, b_{n-k} \neq 0\}, \quad 0 \leq k \leq n.$$

The stabilizer of any point of O_k is

$$W_k = \{a_1 = a_2 = \cdots = a_k = 0\} \subseteq \mathbb{G}_a^n,$$

and the closure of O_k in $\mathbb{P}^n$ is $\overline{O}_k = O_k \cup O_{k-1} \cup \cdots \cup O_0$.

Proof. By Lemma 4.3.2 O_n is an orbit with stabilizer W_n and closure $\overline{O_n} = \mathbb{P}^n$. The other statements follow by induction since $\rho_n(a_1, \ldots, a_n)$ acts on the boundary of $\mathbb{P}^n$ as $\rho_{n-1}(a_1, \ldots, a_{n-1})$. □

Example 4.3.4. On $\mathbb{P}^2$, we have

$$(a_1, a_2)\cdot[b_0 : b_1 : b_2] = \begin{pmatrix} 1 & 0 & 0 \\ a_1 & 1 & 0 \\ \frac{1}{2}a_1^2 + a_2 & a_1 & 1 \end{pmatrix} \cdot [b_0 : b_1 : b_2] = \left[b_0 : a_1 b_0 + b_1 : \tfrac{1}{2}a_1^2 b_0 + a_2 b_0 + a_1 b_1 + b_2\right].$$

4.4 Polymatroid Schubert varieties

Let $\nu : \widetilde{E} \to E = \{1, \ldots, n\}$ be a surjection of finite sets, and let $V_1, \ldots, V_n \subseteq V$ be an essential subspace arrangement with $c_i := \operatorname{codim} V_i = |\nu^{-1}(i)|$ for $1 \leq i \leq n$. Write P for the associated polymatroid. Choose c_i general[1] hyperplanes containing V_i defined by linear forms $a_{i1}, \ldots, a_{ic_i}$ for $1 \leq i \leq n$. These coordinates define maps

$$\iota_i : V \to V/V_i \to \mathbb{P}^{c_i}, \quad v \mapsto \rho_{c_i}(a_{ij}(v)) \cdot [1 : 0 : 0 : \cdots : 0].$$

Set $A = \{a_{ij}\}_{ij}$ and $\iota_A = (\iota_1, \ldots, \iota_n)$. The **Schubert variety** defined by these choices is

$$Y_A := \overline{\operatorname{img}\left(V \xrightarrow{\iota} \prod_i \mathbb{P}^{c_i}\right)}.$$

[1]Here, "general" means that the coordinates define a realization of P_ν, as in Remark 4.2.4.

For each vector $\mathbf{s} = (s_1, \ldots, s_n)$ with $0 \leq s_i \leq c_i$ for all i, set

$$\mathbb{C}^{\mathbf{s}} := \textstyle\prod_{i=1}^n \mathbb{C}^{s_i} \cong \prod_{i=1}^n O_{s_i}, \quad \mathbb{P}^{\mathbf{s}} := \overline{\mathbb{C}^{\mathbf{s}}}, \quad \text{and} \quad W_{\mathbf{s}} := \prod_{i=1}^n W_{s_i} \subseteq \prod_{i=1}^n V_i.$$

By Lemma 4.3.2, Y_A is a compactification of V, and since ι_A is a V-equivariant map, Y_A is a V-equivariant compactification.

Proposition 4.4.1. *Suppose $Y_A \cap \mathbb{C}^{\mathbf{s}}$ is non-empty.*

i. $Y_A \cap \mathbb{C}^{\mathbf{s}}$ *is a (perhaps infinite) union translates of* $V/\iota^{-1}(W_{\mathbf{s}})$.

ii. $\dim Y_A \cap \mathbb{C}^{\mathbf{s}} \geq \dim \operatorname{span}_{\mathbb{C}}(a_{ij} : 1 \leq i \leq n,\, 1 \leq j \leq s_i) = \operatorname{rk}(\mathbf{s})$.

If we further suppose $\mathbf{s}$ *is a combinatorial flat, then*

i'. $Y_A \cap \mathbb{C}^{\mathbf{s}}$ *is a translate of* $V/\iota^{-1}(W_{\mathbf{s}})$

ii'. $\dim Y_A \cap \mathbb{C}^{\mathbf{s}} = \dim \operatorname{span}_{\mathbb{C}}(a_{ij} : 1 \leq i \leq n,\, 1 \leq j \leq s_i) = \operatorname{rk}(\mathbf{s})$.

Proof. The first statement follows immediately from the equivariance of Y_A and our description of the stabilizer of a point in $\mathbb{C}^{\mathbf{s}}$. Combining these two facts with the generality of the coordinates $\{a_{ij}\}$ yields the second.

For the latter two statements, we proceed by induction on n. When $n = 1$, $Y_A \subseteq \mathbb{P}(\mathbb{C} \oplus V/V_1)$ and its combinatorial flats are $(0), (1), \ldots, (\dim V - 1)$ and (c_1). If $\mathbf{s} = (c_1)$, then $Y_A \cap \mathbb{C}^{\mathbf{s}} = \iota(V)$, so (i') and (ii') hold. These statements also hold for the remaining combinatorial flats because for any $0 \leq i \leq \dim V - 1$, (ii) implies the orbit of a point in $\mathbb{C}^i$ under V is equal to $\mathbb{C}^i$.

We now consider $n > 1$. If all coordinates of $\mathbf{s}$ are maximal, then $Y_A \cap \mathbb{C}^{\mathbf{s}} = \iota(V)$, so (i') and (ii') hold. Otherwise, we may assume (without loss of generality) that s_n is non-maximal. Set $A' := \{a_{ij} \in A : i \neq n\}$ and $\mathbf{s}' := (s_1, \ldots, s_{n-1})$. Let $\pi : \mathbb{C}^{\mathbf{s}} \to \mathbb{C}^{\mathbf{s}'}$ be the restriction of the projection $\prod_{i=1}^n \mathbb{P}^{c_i} \to \prod_{i=1}^{n-1} \mathbb{P}^{c_i}$ to $\mathbb{C}^{\mathbf{s}}$.

The projection induces a surjection $Y_A \to Y_{A'}$. In particular, $\pi(Y_A \cap \mathbb{C}^{\mathbf{s}}) \subseteq Y_{A'} \cap \mathbb{C}^{\mathbf{s}'}$, so the nonemptiness of $Y_A \cap \mathbb{C}^{\mathbf{s}}$ implies $Y_{A'} \cap \mathbb{C}^{\mathbf{s}'}$ is also nonempty. This allows us to apply the induction hypothesis in the following chain of inequalities:

$$
\begin{aligned}
\dim Y_A \cap \mathbb{C}^{\mathbf{s}} &\leq \dim \pi^{-1}(Y_{A'} \cap \mathbb{C}^{\mathbf{s}'}) \\
&= \dim \operatorname{span}_{\mathbb{C}}\langle a_{ij} : 1 \leq i \leq n-1,\ 1 \leq j \leq s_i \rangle + s_n && \text{by induction hypothesis} \\
&= \dim \operatorname{span}_{\mathbb{C}}\langle a_{ij} : 1 \leq i \leq n,\ 1 \leq j \leq s_i \rangle && \text{by Lemmas 4.2.2 \& 4.2.4.}
\end{aligned}
$$

The above inequalities and (ii) imply (ii'). From (ii'), we obtain

$$Y_A \cap \mathbb{C}^{\mathbf{s}} = \pi^{-1}(Y_{A'} \cap \mathbb{C}^{\mathbf{s}'}) = (Y_{A'} \cap \mathbb{C}^{\mathbf{s}'}) \times \mathbb{C}^{s_n}.$$

By the induction hypothesis, the right-most term is connected, and by (ii'), $Y_A \cap \mathbb{C}^{\mathbf{s}}$ consists of finitely many translates of $V/\iota^{-1}(W_{\mathbf{s}})$, which proves (i'). $\square$

Proposition 4.4.2. *If* $\mathbf{s}$ *is not a combinatorial flat, then* $Y_A \cap \mathbb{C}^{\mathbf{s}} = \emptyset$.

Proof. We first establish the following claim.

Claim. If $\mathbf{s}$ is not a combinatorial flat and $\operatorname{rk}(\mathbf{s}) = \dim V$, then $Y_A \cap \mathbb{C}^{\mathbf{s}} = \emptyset$.

Proof of claim. Suppose towards a contradiction that $Y_A \cap \mathbb{C}^{\mathbf{s}}$ is nonempty. By Item ii of Proposition 4.4.1, $\dim Y_A \cap \mathbb{C}^{\mathbf{s}} \geq \operatorname{rk}(\mathbf{s}) = \dim Y_A$. However, $\mathbf{s}$ is not a combinatorial flat, so $Y_A \cap \mathbb{C}^{\mathbf{s}}$ is contained in the proper closed subvariety $Y_A \cap \mathbb{P}^{\mathbf{s}}$ of Y_A. In particular, $\dim Y_A \cap \mathbb{C}^{\mathbf{s}} < \dim Y_A$, a contradiction. $\diamond$

We finish the proof with induction on n. If $n = 1$, then $Y_A \subseteq \mathbb{P}(\mathbb{C} \oplus V_1)$, and the multisets that are not combinatorial flats of P are $(\dim V), (\dim V + 1), \cdots, (c_1 - 1)$. All of these multisets have rank $\dim V$, so the desired result holds by the preceding claim.

If $n > 1$ and $\operatorname{rk}(\mathbf{s}) = \dim V$, then $Y_A \cap \mathbb{C}^{\mathbf{s}} = \emptyset$ by the claim. Otherwise, if $\operatorname{rk}(\mathbf{s}) < \dim V$, we may assume without loss of generality that the final coordinate of $\overline{\mathbf{s}}$ is not maximal. Let $A' = \{a_{ij} \in A : i \neq n\}$ and $\mathbf{s}' = (s_1, \ldots, s_{n-1})$. The projection $\prod_{i=1}^{n} \mathbb{P}^{c_i} \to \prod_{i=1}^{n-1} \mathbb{P}^{c_i}$ induces a map $\pi : Y_A \cap \mathbb{C}^{\mathbf{s}} \to Y_{A'} \cap \mathbb{C}^{\mathbf{s}'}$. By Lemma 4.2.7, $\mathbf{s}'$ is not a combinatorial flat of $P \setminus n$, so $Y_{A'} \cap \mathbb{C}^{\mathbf{s}'} = \emptyset$ by the induction hypothesis. This implies $Y_A \cap \mathbb{C}^{\mathbf{s}} = \emptyset$, too. $\square$

Proposition 4.4.3. *If* $\mathbf{s}$ *is a combinatorial flat, then* $Y_A \cap \mathbb{C}^{\mathbf{s}}$ *is nonempty.*

To prove the proposition, we will need the following dimension counting lemma. The proof is not specific to the automorphism $V/V_i \to V/V_i$ given by $a_{ij} \mapsto b_{ij}(a_{ij})$, but rather works for any regular automorphism of V/V_i, as long as the linear forms a_{ij} are chosen generically.

Lemma 4.4.4. *Let* $\mathbf{s}$ *be a multiset and consider the coordinate projection*

$$\beta_{\mathbf{s}} : \prod_{i=1}^{n} V/V_i \to W_{\mathbf{s}}, \quad v \mapsto (a_{ij}(v) : 1 \leq i \leq n, s_i + 1 \leq j \leq c_i).$$

Then $\dim \beta_{\mathbf{s}}(\iota(V)) = \mathrm{rk}(\mathbf{s})$.

Proof. To make use of the genericity assumption, suppose that $V \subseteq \prod V/V_i$ with coordinates x_{ij}, and let $x_{ij} \mapsto a_{ij}(x_{ij})$ be a generic map of graded vector spaces. We have the following diagram.

$$\begin{tikzcd}
V \arrow[r, hook] \arrow[drrr, two heads, "\beta''_{\mathbf{s}}"'] & \prod V/V_i \arrow[r, "x\mapsto a(x)"] \arrow[drr, two heads, "\beta'_{\mathbf{s}}"] & \prod V/V_i \arrow[r, "a\mapsto b(a)"] \arrow[dr, two heads, "\beta_{\mathbf{s}}"] & \prod V/V_i \arrow[d, two heads] \\
 & & & W_{\mathbf{s}}
\end{tikzcd}$$

We have that $\dim \beta''_{\mathbf{s}}(V)$ is equal to the rank of the derivative of $\beta''_{\mathbf{s}}$ at a generic point. However, since the derivative of $\beta_{\mathbf{s}}$ is surjective at a generic point of $\prod V/V_i$, and $x \mapsto a(x)$ is a generic linear map, it follows that the derivative of $\beta'_{\mathbf{s}}$ at a generic point is a generic map of graded vector spaces. Thus, $\dim \beta''_{\mathbf{s}}(V)$ is equal to the rank in the linear matroid M associated to

$$V \subseteq \prod V_i, \quad x \mapsto a_{ij}(x)$$

of a set $S \subseteq \widetilde{E}$ such that $\nu^{-1}(i) = s_i$. However, M is the multisymmetric lift of the polymatroid P, so $\mathrm{rk}_M(S) = \dim \beta_{\mathbf{s}}(\iota(V))$, and as discussed in Item ii, $\mathrm{rk}(\mathbf{s}) = \mathrm{rk}_M(S)$. $\square$

proof of Proposition 4.4.3. Suppose that $\mathbf{s}$ is a combinatorial flat. Let $\beta_{\mathbf{s}}$ be as in Lemma 4.4.4, and let $Z \subseteq \iota(V)$ denote a generic fiber of $\beta_{\mathbf{s}}|_{\iota(V)} : \iota(V) \to W_{\mathbf{s}}$. Let $\mathbb{C}^{\mathbf{s}'} \subseteq Y_A$ be a minimal orbit which intersects $\overline{Z} \subseteq Y_A$. We will show that $\mathbf{s} = \mathbf{s}'$.

First, we argue that $\mathbf{s} \leq \mathbf{s}'$. By the assumption that Z projects to a point in $W_{\mathbf{s}}$, we have that Z projects to a point in $W_{s_i} \subseteq \mathbb{P}(\mathbb{C} \oplus V/V_i)$. Therefore, the coordinate functions $b_{i,s_i+1}/b_{i,0}, b_{i,s_i+2}/b_{i,0}, \ldots, b_{i,c_i}/b_{i,0}$ are constant on Z for all i. So the coordinates $b_{i,s_i+1}, b_{i,s_i+2}, \ldots, b_{i,c_i}$ vanish on the boundary of $\overline{Z}$, meaning that $\overline{Z}$ is contained in $\bigcup_{\mathbf{s} \leq \mathbf{s}'} \mathbb{C}^{\mathbf{s}'}$.

Next, we argue that $\operatorname{corank}(\mathbf{s}) \leq \operatorname{corank}(\mathbf{s}')$. By assumption that $\mathbb{C}^{\mathbf{s}'}$ is a minimal orbit intersecting $\overline{Z}$, we have that $\overline{Z} \cap \overline{\mathbb{C}^{\mathbf{s}'}} = \overline{Z} \cap \mathbb{C}^{\mathbf{s}'}$. Thus, $\overline{Z} \cap \mathbb{C}^{\mathbf{s}'}$ is closed in both Y_A and $\mathbb{C}^{\mathbf{s}'}$, so it is both affine and projective, and is therefore zero-dimensional. Therefore the codimension of $\mathbb{C}^{s'} \cap Y_A$ in Y_A is at least $\dim Z$, which is equal to $\operatorname{corank}(\mathbf{s})$ by Lemma 4.4.4. $\square$

proof of Theorem 4.1.3. By Proposition 4.4.2 and Proposition 4.4.3, we have that Y_A is the union of $Y_A \cap \mathbb{C}^{\mathbf{s}}$ where $\mathbf{s}$ runs over all combinatorial flats of the polymatroid, and each $Y_A \cap \mathbb{C}^{\mathbf{s}}$ is a V-orbit by Proposition 4.4.1. This proves that Y_A has finitely many V-orbits, indexed by the combinatorial flats. By Proposition 4.4.1, the set of V-orbits in the closure of $\mathbb{C}^{\mathbf{s}}$ for a combinatorial flat $\mathbf{s}$ is identified with the combinatorial flats of the polymatroid P' of $V/\iota^{-1}(W_{\mathbf{s}})$. Therefore to show that the orbit-poset of Y_A is the lattice of combinatorial flats, it suffices to show that the combinatorial flats of P' are identified with the lower interval at $\mathbf{s}$ in the lattice of combinatorial flats of P. To show this, note that the matroid of $V/\iota^{-1}(W_{\mathbf{s}})$ is the matroid restriction of the multisymmetric lift P_ν to $S \subseteq \widetilde{E}$, for some set S representing $\mathbf{s}$. Since combinatorial flat $\mathbf{s}'$ satisfies $\mathbf{s} \leq \mathbf{s}'$ if and only if $\mathbf{s}'$ can be represented by a flat $S' \subseteq \widetilde{E}$ of P_ν such that $S' \subseteq S$, the theorem follows from the fact that $P_\nu|_S$ is a multisymmetric lift of the polymatroid of $V/\iota^{-1}(W_{\mathbf{s}})$. $\square$

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