LIMITING FREE ENERGY AND SCALING EXPONENTS FOR DIRECTED POLYMERS WITH INHOMOGENEOUS PARAMETERS

By

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Abstract

Limiting free energy and scaling exponents for directed polymers with inhomogeneous parameters

We consider directed polymer models where a fluctuating path is coupled with a random environment. Our focus is on the models with a random environment given by inhomogeneous parameters. We study the limiting free energy of fairly general inhomogeneous models. First, we derive the existence and basic properties of the limiting free energy for an asymptotically mean stationary model. Second, we apply our results to the exactly solvable log-gamma polymer. We give a variational formula for the point to point free energy in terms of the marginal distributions of the parameters. We identify critical angles at which the free energy transitions from strictly concave to linear. We also obtain explicit formulas for some special distributions of the parameters. Third, we study the fluctuation of free energy around some limit shape. We give scaling exponents for the log-gamma polymer. The KPZ exponent 1/3 appears in the concave sector and the diffusive exponent 1/2 in the flat region.

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Chapter 1

Introduction

1.1 Directed polymer models

We study a model called *directed polymer in a random environment* where a fluctuating path is coupled with a random environment (see [20, 21, 33] for related results and notations). We consider directed paths in the nonnegative orthant $\mathbb{Z}_+^d$ of the d-dimensional integer lattice. Let $\mathbf{x}_\cdot = (\mathbf{x}_k)_{k\geq 0}$ denote the directed path started at the origin: $\mathbf{x}_k \in \mathbb{Z}_+^d, \mathbf{x}_0 = \mathbf{0}$, and $\mathbf{x}_k - \mathbf{x}_{k-1} \in \mathcal{R} = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_d\}$ where $\mathbf{e}_i$ $(1 \leq i \leq d)$ are the standard basis vectors of $\mathbb{R}^d$. Let Π_n^{p2l} be the set of admissible paths $\mathbf{x}_\cdot = (\mathbf{x}_i)_{0\leq i\leq n}$ that start at the origin. (Here p2l stands for "point to line".) The path $\mathbf{x}_\cdot$ represents the directed polymer. The environment $\omega = \{\omega_\mathbf{x} : \mathbf{x} \in \mathbb{Z}_+^d\}$ is a collection of real-valued weights.

For a path segment $\mathbf{x}_{0,n} = (\mathbf{x}_0, \ldots, \mathbf{x}_n)$, $H_n(\mathbf{x}_{0,n})$ is the total weight collected by the walk up to time n: $H_n(\mathbf{x}_\cdot) = H_n(\mathbf{x}_{0,n}) = \sum_{i=1}^n \omega_{\mathbf{x}_i}$. The quenched polymer distribution on paths, in environment ω and at inverse temperature $\beta > 0$, is the probability measure defined by

$$Q_n^\omega(\mathbf{x}_\cdot) = \frac{1}{Z_n^\omega} \exp\{\beta H_n(\mathbf{x}_{0,n})\} \tag{1.1}$$

where

$$Z_n^\omega = \sum_{\mathbf{x}_{0,n}\in\Pi_n^{p2l}} e^{\beta H_n(\mathbf{x}_{0,n})}$$

is a normalization factor (partition function). Our primary subjects are asymptotic behavior of $\log Z_n^\omega$ (free energy) and its fluctuation around the limiting value as n goes to infinity. This model is for directed polymers with free endpoints. Another model considered is the directed polymer with constrained endpoints. Relevant definitions are as follows.

For $\mathbf{u} \le \mathbf{v}$ (coordinatewise inequality) in $\mathbb{Z}_+^d$ let $\Pi_{\mathbf{u},\mathbf{v}}$ denote the set of admissible lattice paths $\mathbf{x}_\cdot = (\mathbf{x}_i)_{0\le i\le n}$ with $n = |\mathbf{v}-\mathbf{u}|_1$ that satisfy $\mathbf{x}_0 = \mathbf{u}$, $\mathbf{x}_i - \mathbf{x}_{i-1} \in \mathcal{R}$, $\mathbf{x}_n = \mathbf{v}$. Define point-to-point polymer partition functions for $\mathbf{u} \le \mathbf{v}$ in $\mathbb{Z}_+^d$ by

$$Z_{\mathbf{u},\mathbf{v}} = Z_{\mathbf{u},\mathbf{v}}^\omega = \sum_{\mathbf{x}_\cdot\in\Pi_{\mathbf{u},\mathbf{v}}} \exp\{\beta \sum_{i=1}^{|\mathbf{v}-\mathbf{u}|_1} \omega_{\mathbf{x}_i}\} = \sum_{\mathbf{x}_\cdot\in\Pi_{\mathbf{u},\mathbf{v}}} \exp\{\beta H_{|\mathbf{v}-\mathbf{u}|_1}(\mathbf{x}_\cdot)\} \tag{1.2}$$

and the polymer measure on the set of paths $\Pi_{\mathbf{u},\mathbf{v}}$ by

$$Q_{\mathbf{u},\mathbf{v}}\{\mathbf{x}_\cdot\} = Q_{\mathbf{u},\mathbf{v}}^\omega\{\mathbf{x}_\cdot\} = \frac{1}{Z_{\mathbf{u},\mathbf{v}}} \exp\{\beta \sum_{i=1}^{|\mathbf{v}-\mathbf{u}|_1} \omega_{\mathbf{x}_i}\}, \quad \mathbf{x}_\cdot \in \Pi_{\mathbf{u},\mathbf{v}}. \tag{1.3}$$

The environment ω is typically assumed to be i.i.d. random variables and subadditive ergodic theorem plays an essential role to prove the existence of limiting free energy. In the present work, we focus on the environment driven by nonstationary distributions. Our goal is to derive limit theorems similar to those of i.i.d. cases, in inhomogeneous settings. However, there is no hope to derive limit theorems under arbitrary inhomogeneous distributions. Thus we require some sort of stationarity. Our results are based on the asymptotically mean stationary setting in a sense made precise below.

Let $(\Omega, \mathcal{F})$ be a measurable space and $T : \Omega \to \Omega$ a measurable transformation. A probability measure P is called *asymptotically mean stationary* (AMS) (relative to T), if there is a probability measure $\bar{P}$ on $(\Omega, \mathcal{F})$ such that

$$\forall B \in \mathcal{F} : \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} P(T^{-i}B) = \bar{P}(B). \tag{1.4}$$

In this setting the measure $\bar{P}$ is stationary and it is therefore called the *stationary mean of* P, see [24, 25] for details. The invariant σ algebra $\mathcal{I}$ is the set of all invariant events ($T^{-1}B = B, \forall B \in \mathcal{I}$). A probability measure P is said to be *ergodic* if $P(B) = 0$ or 1 for $B \in \mathcal{I}$. AMS measure is ergodic if and only if its stationary mean is (Lemma 7.13 [24]). Examples of AMS measures are given in Chapters 2 and 3.

1.2 Background

Polymer models were first introduced by Huse and Henley in 1985 in statistical physics context [26]. Since then rigorous mathematical research started [27]. When $\beta = 0$, the polymer model becomes a simple random walk, more precisely, a rotated version of SRW. The diagonal axis plays the role of the time axis. If β goes to ∞, that is, if we take the zero temperature limit, our model converges to the last-passage percolation model or corner growth model. Our main interest is in the effect of random environment on the behavior of the polymer at a positive temperature $0 < \beta < \infty$. For $d \geq 4$ and small $\beta > 0$, these models show diffusive behavior and converge to Brownian motion if suitably scaled [8]. These results were obtained through the observation that $W_n = Z_n/\mathbb{E}Z_n$ is a martingale under i.i.d. random environment. The limit $W_\infty = \lim W_n$ is either almost surely 0 or almost surely positive by a zero-one law. The case $W_\infty > 0$ is called *weak disorder*. Note that $\beta = 0$ case gives $W_\infty = 1$ and the disordered noise driven by random environment has no effect on the behavior of polymer. The case $W_\infty = 0$ is called *strong disorder* since the disordered noise has a strong effect. It is known that there is a critical value β_c such that weak disorder appears for $\beta < \beta_c$ and strong for $\beta > \beta_c$. For $d = 2$ and $d = 3$, $\beta_c = 0$ [12]. Since the β parameter plays no role in the present work, we fix its value at $\beta = 1$.

$d = 2$ cases have received active research attention regarding the KPZ (Kardar-Paris-Zhang) universality class (see the survey [13] and references therein). Both polymer model and corner growth model are believed to belong to the KPZ class. In many cases, both models share similar properties and proof methodologies. KPZ class is characterized by its statistics: the fluctuations around limiting quantities (limiting free energy, time constant, etc.) are the order of $n^{1/3}$ and appropriately rescaled random variables converge to some Tracy-Widom distributions. These conjectured exponents and limiting distributions are proved for some *exactly solvable* models where explicit formulas are available for precise analysis. For corner growth models, not only i.i.d. cases but also inhomogeneous cases are studied ([17, 22, 23]). Our application to a log-gamma model in the later part of this paper is inspired by Emrah's work [17]. We adapted some notations and reasoning from [17]. In polymer models, most researches were carried out with i.i.d. setting. Some inhomogeneous parameter models for a log-gamma model are considered in [9, 14] in the course of deriving the results for i.i.d. cases. Their works are useful if one studies the Tracy-Widom distributional limit of inhomogeneous models. In this thesis, we consider the limiting free energy and fluctuation exponents of exactly solvable log-gamma model under inhomogeneous settings.

1.3 Log-gamma polymer models

In this section, we introduce the inhomogeneous 2-dimensional log-gamma polymer which can be explicitly solvable. The log-gamma polymer was first introduced in [33] and exact formulas of limits are derived under i.i.d. assumptions. This model belongs to the KPZ universality class and conjectured scaling exponents with limiting Tracy-Widom distribution were derived in [9, 33].

We change the picture slightly. Our model lives in $\mathbb{N}^2$. Choose parameters $\Lambda = \{\lambda_i\}_{i=1}^{\infty}$ and $\Theta = \{\theta_j\}_{j=1}^{\infty}$. (Λ, Θ) is in $S_0 \triangleq [a_0, a_1]^{\mathbb{N}} \times [b_0, b_1]^{\mathbb{N}} \simeq ([a_0, a_1] \times [b_0, b_1])^{\mathbb{N}}$ for some $a_0 < a_1$ and $b_0 < b_1$ in $\mathbb{R}$ with $a_0 + b_0 > 0$. S_0 is equipped with the product Borel σ-algebra $\mathcal{G}_0$ generated by coordinate projections. We give weight parameters at site $(i, j) \in \mathbb{N}^2$ by $\rho_{i,j} = \lambda_i + \theta_j$. Let $\rho = \{\rho_{\mathbf{x}}\}_{\mathbf{x} \in \mathbb{N}^2}$. The distribution $\mathbb{P}^{\rho}$ of ω given ρ is a product measure over $\mathbf{x} = (i, j) \in \mathbb{N}^2$ with

$$e^{-\omega(i,j)} \sim \text{Gamma}(\rho_{i,j}), \tag{1.5}$$

where the density of the Gamma$(\rho_{i,j})$ is given in Table 1. We call the distribution of $-\omega(i, j)$ the log-gamma$(\rho_{i,j})$ distribution. See A.2 for the properties of the log-gamma distribution. We consider either deterministic or random parameters Λ and Θ. In case of random parameters, $\mathbb{P}^{\rho}$ is the conditional distribution of ω given (Λ, Θ) and the (unconditional) distribution of ω is denoted by $\mathbb{P}$. Hence $\mathbb{P} = \int \mathbb{P}^{\rho(\Lambda,\Theta)}\, d\mathcal{Q}(\Lambda, \Theta)$, where $\mathcal{Q}$ is the distribution of (Λ, Θ).

It turns out that, for some AMS choices of parameters, the log-gamma polymer models have properties that allow precise analysis. In particular, we can compute the limiting free energy and the scaling exponents for the fluctuation. As a simple example of parameters, one can take i.i.d. sequences of Λ and Θ. More detailed formulations of the model and results can be found in Chapters 3 and 4.

1.4 Overview of the main results

This thesis consists of three main chapters. Each chapter includes more detailed information and results. In this section, we state some selected results of our work. A summary of the organization of the results obtained in this thesis is as follows.

Chapter 2 considers general inhomogeneous polymer models. In Chapter 2 we show the existence of the limiting free energy under a quite general AMS setting. For $\mathbf{x} \in \mathbb{R}^d_+$, a quantity, if exists,

$$\lim_{n\to\infty} \frac{1}{n} \log Z^{\omega}_{\mathbf{0},\lfloor n\mathbf{x}\rfloor} \tag{1.6}$$

is called the *limiting point-to-point free energy*. The *limiting point-to-line free energy* is defined by

$$\lim_{n\to\infty} \frac{1}{n} \log Z^{\omega}_{n} \tag{1.7}$$

if the limit exists.

Positive temperature polymer models were studied by several authors. In [19], large deviations and law of large numbers for various polymer related quantities are derived in general i.i.d. setting. They proved large deviations of free energy and quenched large deviations for the exit point of the polymer chain. In this thesis we focus on law of large numbers for the free energy of polymers with inhomogeneous parameters. The existence of limiting free energy was proved in [19] using standard subadditive ergodic theorems. We employ some ideas from [19] using a nonstationary subadditive ergodic theorem. However the setting and argument are quite complicated due to nonstationarity. There is a one difference between i.i.d. weights and AMS weights. For i.i.d. weights, the limiting free energy is deterministic and continuous on $\mathbb{R}^d_+$. However, in general AMS setting the limiting free energy is deterministic and continuous only on $\mathbb{R}^d_{>0}$. The limiting free energy is random on the boundary (see Theorem 3.6).

In Section 2.1 precise assumptions on the distribution of weights are given, and main results are stated. In Section 2.3 AMS measures are fully studied, and a nonstationary subadditive ergodic theorem is established. A nonstationary subadditive ergodic theorem is an essential tool to establish the existence of the limiting free energy with inhomogeneous parameters. We devote Section 2.4 to technical results for the proof of the main results.

In Chapter 3, we restrict our discussion to 2-dimensional polymers. For $2d$-polymers, we provide more natural and general weight assumptions to guarantee the existence of limiting free energy (see Assumption 3.1). We apply these results to explicitly solvable model, log-gamma polymer. Moreover, we show that the limiting point-to-point free energy has a variational formula

$$\bar{\phi}(x,y) = \lim_{n\to\infty} \frac{1}{n} \log Z^{\omega}_{\mathbf{0},(\lfloor nx\rfloor,\lfloor ny\rfloor)} = \inf_{-a_0<z<b_0} \{xA(z) + yB(z)\},$$

where A and B are convex functions on the interval $(-a_0, b_0)$ defined by

$$A(z) = -\int_{(0,\infty)} \Psi_0(z+\lambda)\,\alpha(d\lambda),$$
$$B(z) = -\int_{(0,\infty)} \Psi_0(-z+\theta)\,\beta(d\theta),$$

where Ψ_0 is the digamma function (see Table 1), and α, β are some distributions of parameters. Based on this formula, we give more precise picture of limiting free energy. Figure 1 shows a possible level curve of limiting free energy. As we can see, we have flat regions S_1 and S_2 near coordinate axes. We give conditions for the existence of these flat regions in terms of A and B. In principle, we can obtain similar variational formulas for general polymers. However precise information about A and B are not known in general, it is hard to find conditions for these regions in terms of inhomogeneous parameter distributions.

Emrah derived some explicit formulas for some inhomogeneous corner growth models and identified some conditions for flat regions in [17]. We adapt the zero-temperature argument of [17] to positive temperature polymers. In Section 3.4 the inhomogeneous log-gamma model is analyzed, and certain formulas are derived.

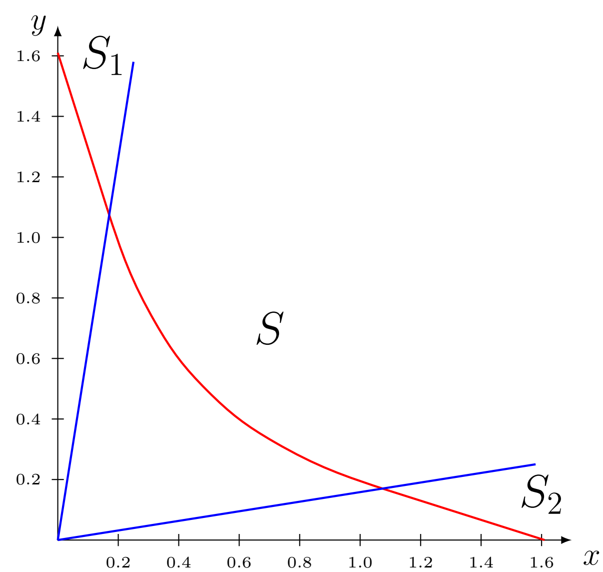


Figure 1: A level curve of limiting free energy (red).

Chapter 4 is concerned about the fluctuation around the limiting free energy. We restrict to a specific model, the log-gamma polymer. With slightly less restrictive assumptions on the parameters, scaling exponents are derived. Indeed we only assume some weak convergence conditions instead of much stronger AMS conditions. This weakening is possible due to explicit formulas which enable precise analysis. We focus on quenched (a fixed realization of parameters) fluctuation. Due to the inhomogeneity of parameters, we study fluctuations around *quenched shape* $\phi_{m,n}$ (see (4.19)) instead of *annealed shape* $\bar{\phi}(x, y)$. The quenched shape itself converges to the annealed shape as (m, n) grows along the direction (x, y). We show that the fluctuation of the free energy around $\phi_{m,n}$ is of order $n^{1/3}$ in the region S, and of order $n^{1/2}$ in S_1 and S_2. We will give the precise meaning of *fluctuation* in Chapter 4 and derive various quantities to prove this assertion. The key feature that enables precise analysis is the existence of *stationary* processes, which have the *Burke property* that gives exact formulas for the expectation and variance of free energy. We discuss these formulas at length in Chapter 4.

1.5 Notations and conventions

Some notations used in this paper are provided below.

Table 1: Notations

Notation	**Definition**
$\mathbb{N}$	the set of natural numbers $\{1, 2, 3, \dots\}$
$\mathbb{Z}_+$	the set of nonnegative integers $\{0, 1, 2, \dots\}$
$\mathbb{R}_+$	the set of nonnegative real numbers
$\mathbb{R}_{>0}$	the set of positive real numbers
$[n, m]$	the set $\{n, \dots, m\}$ $(n \leq m)$ for $n, m \in \mathbb{N}$
$x \vee y$	$\max\{x, y\}$ for x, $y \in \mathbb{R}$
$x \wedge y$	$\min\{x, y\}$ for x, $y \in \mathbb{R}$

Continued on next page

Continuation of Table 1

Notation	Definition
$\lfloor x \rfloor$	the largest integer less than or equal to $x \in \mathbb{R}$
$\lceil x \rceil$	the least integer greater than or equal to $x \in \mathbb{R}$
$\lvert \mathbf{x} \rvert_p$	the ℓ^p norm $(1 \le p < \infty)$ of $\mathbf{x} \in \mathbb{R}^d$. Equals $(\lvert x_1\rvert^p + \cdots + \lvert x_d\rvert^p)^{1/p}$
$\lvert \mathbf{x} \rvert_\infty$	the ℓ^∞ norm of $\mathbf{x} \in \mathbb{R}^d$. Equals $\max_{1 \le i \le d} \lvert x_i \rvert$
$\mathbf{x} \cdot \mathbf{y}$	the inner product in $\mathbb{R}^d$. Equals $x_1 y_1 + \cdots + x_d y_d$
$\mathbf{0}$	the zero vector $(0, \ldots, 0)$ in $\mathbb{R}^d$
$\mathbf{1}$	the one vector $(1, \ldots, 1)$ in $\mathbb{R}^d$
$\mathbf{e}_i$	the i-th standard coordinate vector $(0, \ldots, 1, \ldots, 0)$ of $\mathbb{R}^d$
$\mathbf{x}_{\bullet}$	a path $\mathbf{x}_{\bullet} = (\mathbf{x}_k)_{k=0}^n$ in $\mathbb{Z}_+^d$ with steps $\mathbf{z}_k = \mathbf{x}_k - \mathbf{x}_{k-1} \in \mathcal{R} = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_d\}$
$\lvert A \rvert = \mathbf{card}(A)$	the cardinality of a set A
$\Gamma(\rho)$	the usual gamma function for $\rho > 0$. $\Gamma(\rho) = \int_0^\infty x^{\rho-1} e^{-x}\,dx$
Gamma(α,β)	the gamma distribution on $\mathbb{R}_+$ with the density $\Gamma(\alpha)^{-1} \beta^\alpha x^{\alpha-1} e^{-\beta x}\,dx$
Gamma(ρ)	the gamma distribution Gamma(ρ,1)
Ψ_0	the digamma function Γ'/Γ
Ψ_1	the trigamma function Ψ_0'

Vector notations: elements of $\mathbb{R}^d$ and $\mathbb{Z}^d$ are occasionally written as $\mathbf{v} = (v_1, v_2, \ldots, v_d)$ to emphasize $\mathbf{v}$ is a vector. We understand some vector operations and relations coordinatewise. Here are examples. Inequalities are coordinatewise: $\mathbf{x} \le \mathbf{y}$ if $x_i \le y_i$ and $\mathbf{x} < \mathbf{y}$ if $x_i < y_i$ for $1 \le i \le d$. We also understand positive and negative parts and the absolute value of vectors coordinatewise: $\mathbf{x}^\pm = (x_1^\pm, \ldots, x_d^\pm)$, $\lvert \mathbf{x} \rvert = (\lvert x_1 \rvert, \ldots, \lvert x_d \rvert)$. For $\mathbf{x} \in \mathbb{R}^d$, $\lfloor \mathbf{x} \rfloor = (\lfloor x_1 \rfloor, \ldots, \lfloor x_d \rfloor)$ is a *floor* of $\mathbf{x}$.

Shift maps $T_{\mathbf{v}}$ act on suitably indexed configurations $w = (w_{\mathbf{x}})$ by $(T_{\mathbf{v}}w)_x = w_{\mathbf{v}+\mathbf{x}}$. We employ compact expressions like T-invariant, T-ergodic, and T-AMS.

Chapter 2

Directed polymers with inhomogeneous parameters

2.1 Introduction

We present our results in this chapter and explain precise assumptions on the distribution of weights. Refer to Section 1.1 for the definition of polymer models and free energy. A natural nonstationary condition for weights would be independent but not identically distributed weights. The distribution of weights is given by inhomogeneous parameters. Let $(\mathbf{S}, \mathcal{S})$ be a measurable space. Let $(\Omega_0, \mathcal{G}_0)$ be the space of parameters where $\Omega_0 = \mathbf{S}^{\mathbb{Z}_+^d}$ and $\mathcal{G}_0$ the product σ-algebra generated by coordinate projections. Let $\Omega_1 = \mathbb{R}^{\mathbb{Z}_+^d}$ be the space of weights with the product Borel σ-algebra $\mathcal{G}_1$. To control errors of estimations, we will assume, throughout this paper, the following:

Assumption 2.1. *(a) There is a nonnegative random variable η_0 which has a cumulative distribution function (CDF) F with the property*

$$\int_0^\infty (1 - F(x))^{1/d} dx < \infty. \tag{2.1}$$

(b) A function $F_2 : \mathbf{S} \times \mathbb{R} \to [0, 1]$ is given. For fixed $\rho_0 \in \mathbf{S}$, $F_2(\rho_0, \cdot)$ is a CDF. If a random variable ω_0 has the CDF $F_2(\rho_0, \cdot)$, then $|\omega_0|$ is stochastically dominated by

η_0 $(|\omega_0| \leq_{ST} \eta_0)$.

We consider the following weight distributions.

Definition 2.2. *For given parameters* $\rho = \{\rho_{\mathbf{x}} : \mathbf{x} \in \mathbb{Z}_+^d\} \in \Omega_0$, *define a distribution of weights* $\{\omega_{\mathbf{x}} : \mathbf{x} \in \mathbb{Z}_+^d\} \in \Omega_1$ *by a product measure using* F_2 *in Assumption 2.1 :*

$$\mathbb{P}^\rho = \otimes_{\mathbf{x} \in \mathbb{Z}_+^d} F_2(\rho_{\mathbf{x}}, \cdot). \tag{2.2}$$

As explained in Introduction 1.1 we use the AMS setting to guarantee the existence of limits. To apply the AMS setting to our models we need more definitions. Suppose $(\Omega, \mathcal{F})$ is a measurable space. Let $\hat{T} = \{T_{\mathbf{u}}\}_{\mathbf{u} \in \mathbb{Z}_+^d}$ be a semigroup of measurable transformations of Ω such that $T_{\mathbf{0}} = Id|_\Omega$ and $T_{\mathbf{u}} \circ T_{\mathbf{v}} = T_{\mathbf{u}+\mathbf{v}}$ for all $\mathbf{u}, \mathbf{v} \in \mathbb{Z}_+^d$. A probability measure P on Ω is called *stationary* with respect to $\hat{T}$ (or $\hat{T}$-stationary) if $P = P \circ T_{\mathbf{u}}^{-1}$ for each $\mathbf{u} \in \mathbb{Z}_+^d$. We say P is *totally ergodic* with respect to $\hat{T}$ (or $\hat{T}$-totally ergodic) if $\mathcal{I}_{\mathbf{u}} \subseteq \mathcal{F}$, the invariant σ-field of $T_{\mathbf{u}}$, is trivial for each $\mathbf{u} \in \mathbb{N}^d$: $P(A) \in \{0, 1\}$ for $A \in \mathcal{I}_{\mathbf{u}}$, $\mathbf{u} \in \mathbb{N}^d$. Note that this definition refers only to the bulk directions $\mathbf{u} > \mathbf{0}$ and do not require $\hat{T}$-stationarity. To state assumptions of main results we give the following definition.

Definition 2.3. *Let* Ω *and* $\hat{T}$ *be as above. A probability measure* $\mathcal{Q}$ *is called AMS with respect to* $\hat{T}$ *(or* $\hat{T}$*-AMS) if the following conditions are satisfied.*

(a) There is a reference measure $\mathcal{Q}_0$ *on* Ω *such that* $\mathcal{Q}_0$ *is stationary and totally ergodic with respect to* $\hat{T}$.

(b) For each $\mathbf{u} \in \mathbb{N}^d$, $\mathcal{Q}$ *is AMS relative to* $T_{\mathbf{u}}$ *with stationary mean* $\mathcal{Q}_0$ *:*

$$\forall B \in \mathcal{F} : \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathcal{Q}((T_{\mathbf{u}})^{-i} B) = \mathcal{Q}_0(B). \tag{2.3}$$

We call $\mathcal{Q}_0$ the *stationary mean* of $\mathcal{Q}$. $\mathcal{Q}$ is also $\hat{T}$-totally ergodic by Lemma 7.13 of [24]. Note that we do not require that $\mathcal{Q}$ is AMS relative to $T_{\mathbf{u}}$ if $u \notin \mathbb{N}^d$.

In this chapter we select a particular semigroup as follows: Let $T_{\mathbf{u}} : \Omega_0 \to \Omega_0$ be a translation operator given by $T_{\mathbf{u}}(\rho)(\mathbf{v}) = \rho(\mathbf{v} + \mathbf{u})$ for $\rho \in \Omega_0$ and $\mathbf{u}, \mathbf{v} \in \mathbb{Z}_+^d$. Then $\hat{T} = \{T_{\mathbf{u}}\}_{\mathbf{u} \in \mathbb{Z}_+^d}$ forms a semigroup. The following assumption is used for the main results of this chapter.

Assumption 2.4. *We consider either deterministic or random parameters $\rho \in \Omega_0$. When $\rho_{\mathbf{x}}$ are random, we denote the distribution of $\{\rho_{\mathbf{x}} : \mathbf{x} \in \mathbb{Z}_+^d\}$ by $\mathcal{Q}$. We assume $\mathcal{Q}$ is $\hat{T}$-AMS. When parameters $\rho_{\mathbf{x}}$ are chosen according to $\mathcal{Q}$, the conditional law of weights $\omega = \{\omega_{\mathbf{x}} : \mathbf{x} \in \mathbb{Z}_+^d\}$ given ρ is given by (2.2): $P(\omega \in \cdot \mid \rho) = \mathbb{P}^\rho$.*

Remark 2.5. *Deterministic parameter also can be handled in this framework. $\mathcal{Q}$ is given by the Dirac measure $\mathcal{Q} = \delta_\rho$. The distribution of ω on $\Omega_1 = \mathbb{R}^{Z_+^d}$ is denoted by $\mathbb{P}$. Hence $\mathbb{P} = E^{\mathcal{Q}}\mathbb{P}^\rho$. Note that for deterministic parameters, $\mathbb{P} = \mathbb{P}^\rho$. Under $\mathbb{P}$ ($\mathbb{P}^\rho$), the expectation is denoted by $\mathbb{E}$ ($\mathbb{E}^\rho$). We write ν to denote the joint distribution of (ρ, ω) on $\Omega_0 \times \Omega_1$. Hence P above is simply ν. $\mathcal{Q}$ and $\mathbb{P}$ are marginal distributions of ν. We adopt this notation to remove confusion.*

In this thesis, we are interested in the scaling limits of free energy. To obtain limit theorems, we use a nonstationary subadditive ergodic theorem. A subadditive ergodic theorem requires further restrictions on $\mathcal{Q}$ and F_2 since in general, the subadditive ergodic theorem does not hold in the AMS setting. Therefore we add the following.

Assumption 2.6. *The distribution of parameters $\mathcal{Q}$ and F_2 satisfy one of the following conditions.*

(a) $\mathcal{Q} \ll \mathcal{Q}_0$.

(b) *We assume that* $\mathbf{S}$ *is equipped with a partial order* $\preceq$. (*Typically* $\mathbf{S}$ *is a Polish space with a closed ordering so that all intervals are measurable [29].*) *Give a coordinatewise partial order on* Ω_0. *Now assume* $\mathcal{Q}$ *is stochastically smaller than* $\mathcal{Q}_0$ *on* Ω_0, *that is,* $\int g\, d\mathcal{Q} \le \int g\, d\mathcal{Q}_0$ *for all bounded monotonically increasing function* $g : \Omega_0 \to \mathbb{R}$. *In this case, we assume that the function* F_2 *in Assumption 2.1 is monotonically decreasing function in the first variable:* $F_2(s_1, x) \ge F_2(s_2, x)$ *if* $s_1 \preceq s_2$.

Remark 2.7. *If two measurable maps* $\rho, \rho^0 : (\Omega, \mathbf{P}) \to \Omega_0$ *satisfy* $\rho_{\mathbf{x}} \preceq \rho^0_{\mathbf{x}}$ *for all* $\mathbf{x} \in \mathbb{Z}^d_+$, *then the distribution of* ρ *is stochastically smaller than the distribution of* ρ^0.

Here are some examples of $\mathcal{Q}$ and $\mathcal{Q}_0$.

Example 2.8. (a) *Let* $L \in \mathbb{N}$ *be fixed. Suppose* $\rho \in \Omega_0$ *is deterministic and* $\rho_{\mathbf{x}+L\mathbf{e}_i} = \rho_{\mathbf{x}}$ *for all* $\mathbf{x} \in \mathbb{Z}^d_+$ *and* $\mathbf{e}_i$, $i = 1, \ldots, d$. *Then*

$$\mathcal{Q}_0 = \frac{1}{L^d} \sum_{\mathbf{0} \le \mathbf{x} \le (L-1)\mathbf{1}} \delta_{T_{\mathbf{x}}(\rho)}$$

is $\hat{T}$*-stationary and ergodic. In this case, we take* $\mathcal{Q} = \delta_\rho$. *Hence periodic weights belong to our model. Note that an i.i.d. environment* ω *belongs to this example with* $L = 1$.

(b) $\mathcal{Q}_0 = \alpha^{\otimes \mathbb{Z}^d_+}$ *where* α *is a probability measures on* $\mathbf{S}$. $\mathcal{Q}_0$ *is totally ergodic with respect to* $\hat{T}$ *by Kolmogorov 0-1 law. For* $\mathcal{Q}$, *choose any nonnegative function* $f : \Omega_0 \to \mathbb{R}$ *with* $E^{\mathcal{Q}_0} f = 1$ *and take* $d\mathcal{Q} = f d\mathcal{Q}_0$.

(c) *Let* $\mathbf{S} = \mathbb{R}$ *with the standard ordering. Fix* $N \in \mathbb{N}$, $a_0 \in \mathbb{R}$ *and* $\mathbf{u} \in \mathbb{N}^d$. *Let* $\mathcal{Q}_0$ *be any probability measure on* $[a_0, \infty)^{\mathbb{Z}^d_+} \subseteq \Omega_0$ *that is* $\hat{T}$*-stationary and ergodic.*

Let $\rho^0 : (\Omega, \mathbf{P}) \to \Omega_0$ *be a measurable map with the distribution* $\mathcal{Q}_0$*. Suppose a measurable map* $\rho : \Omega \to \Omega_0$ *satisfies* $\rho_{\mathbf{x}} = \rho^0_{\mathbf{x}}$ *for* $\mathbf{x} \geq N\mathbf{u}$ *and* $\rho_{\mathbf{x}} \leq a_0$ *for other* $\mathbf{x} \in \mathbb{Z}^d_+$*. Then* $\mathcal{Q}$*, the distribution of* ρ*, is stochastically smaller than* $\mathcal{Q}_0$*. One can easily show that* $\mathcal{Q}$ *is* $\hat{T}$*-AMS with stationary mean* $\mathcal{Q}_0$*.*

(d) *Assumption 2.6 (b) can be satisfied with deterministic parameters. If we combine (a) and (c), we can construct an example: Let* $\mathbf{S} = (0, \infty)$ *with the standard ordering. Fix* $N \in \mathbb{N}$*,* $b > a > 0$ *in* $\mathbf{S}$*. Set* $\rho^0_{\mathbf{x}} = b$ *for all* $\mathbf{x} \in \mathbb{Z}^d_+$*. Set* $\rho_{\mathbf{x}} = b$ *for all* $\mathbf{x} \geq N\mathbf{1}$ *and* $\rho_{\mathbf{x}} = a$ *for other points. Take* F_2 *the cdf of log-gamma random variables defined by* $F_2(r, x) = F(r, 1, x)$ *using* (A.13) *(*$r \in \mathbf{S}$*,* $x \in \mathbb{R}$*). Then the distributions of* ρ^0 *and* ρ *become an example.*

2.2 Results

We begin by defining the concept of scaling limits. For a given function $\phi : \mathbb{Z}^d_+ \to \mathbb{R}$, the *scaling limit* of ϕ is a function $\bar{\phi}$ whose domain is a subset of $\mathbb{R}^d_+$ and defined by $\bar{\phi}(\mathbf{x}) = \lim_{n\to\infty} \frac{1}{n}\phi(\lfloor n\mathbf{x} \rfloor)$, provided the limit exists. The basic properties of scaling limits are developed in Section 2.4. The limiting point-to-point free energy is a scaling limit of the point-to-point free energy. Now we can state the main result of our work. First, we show that limiting point-to-point free energy exists. Recall the definition of $\mathbb{P}^\rho$ and $\mathbb{P}$ in Remark 2.5.

Theorem 2.9. *Suppose Assumptions 2.4 and 2.6 hold. Then, there exists a unique deterministic function* $\bar{\phi}$ *defined on* $\mathbb{R}^d_+$ *and an event* $\Omega'_1 \subseteq \Omega_1$ *such that the following hold.*

(a) *The limit* $\lim_{n\to\infty} n^{-1}\log Z^{\omega}_{\mathbf{0},\lfloor n\mathbf{x}\rfloor} = \bar\phi(\mathbf{x})$ *for all* $\mathbf{x}\in\mathbb{R}^d_{>0}$ *simultaneously if* $\omega\in\Omega_1'$. *Furthermore, for* $\mathcal{Q}$*-almost every choice of* ρ, $\mathbb{P}^{\rho}(\Omega_1')=1$.

(b) $\bar\phi(\mathbf{x})$ *satisfies for* $\mathbf{x}\in\mathbb{R}^d_{>0}$

$$\bar\phi(\mathbf{x}) = \lim_{n\to\infty}\mathbb{E}\big(\frac{1}{n}\log Z_{\mathbf{0},\lfloor n\mathbf{x}\rfloor}\big) = \lim_{n\to\infty}\frac{1}{n}\int\big(\mathbb{E}^{\rho}\log Z_{\mathbf{0},\lfloor n\mathbf{x}\rfloor}\big)\,\mathcal{Q}_0(d\rho).$$

(c) $\bar\phi$ *is continuous, positive-homogeneous, superadditive, concave on* $\mathbb{R}^d_+$:

For $\mathbf{x}$, $\mathbf{y}\in\mathbb{R}^d_+$, $0\le s\le 1$ *and* $c>0$,

$$\bar\phi(c\mathbf{x}) = c\bar\phi(\mathbf{x}) \tag{2.4}$$

$$\bar\phi(\mathbf{x})+\bar\phi(\mathbf{y}) \le \bar\phi(\mathbf{x}+\mathbf{y}) \tag{2.5}$$

$$s\bar\phi(\mathbf{x})+(1-s)\bar\phi(\mathbf{y}) \le \bar\phi(s\mathbf{x}+(1-s)\mathbf{y}). \tag{2.6}$$

Note that in (a) and (b) we do not require the convergence occurs on the boundary of $\mathbb{R}^d_+$ even though $\bar\phi$ is defined on $\mathbb{R}^d_+$. Using this result, we obtain a limit theorem for the point-to-line free energy.

Corollary 2.10. *Under the same assumptions, the limit* $\lim_{n\to\infty} n^{-1}\log Z^{\omega}_n$ *exists if* $\omega\in\Omega_1'$, *and is given by* $\lim_{n\to\infty} n^{-1}\log Z^{\omega}_n = \sup_{\substack{\mathbf{x}\ge\mathbf{0}\\ |\mathbf{x}|_1=1}}\bar\phi(\mathbf{x})$.

Organization of Chapter 2. Before we prove the main results, we collect technical results first. The crux of the proof of the main theorems is a combination of a nonstationary subadditive ergodic theorem and concentration inequalities. This proof strategy is somewhat unusual. In Section 2.3, nonstationary subadditive ergodic theorems are established. In Section 2.4, we collect useful concentration inequalities. We use concentration inequalities to show that deviation of free energy around its mean is small enough. And then a nonstationary subadditive ergodic theorem is used to show that (inhomogeneous) mean of free energy, when normalized by scaling parameter n, converges. Combination of these two results gives the proof of the main results. Theorem 2.9 and Corollary 2.10 are proved in Section 2.4.

2.3 Nonstationary subadditive ergodic theorems

In this section, we investigate conditions that guarantee the existence of limiting free energy. The main topic is asymptotically mean stationary (AMS) measures. After we develop a nonstationary subadditive ergodic theorem, which is the most important theorem in this section, we prove the main results in the next section. Let $(\Omega, \mathcal{F})$ be a measurable space and $T : \Omega \to \Omega$ a measurable transformation. Recall that a probability measure P is AMS relative to T (T-AMS), if there is a probability measure $\bar{P}$ on $(\Omega, \mathcal{F})$ such that

$$\forall B \in \mathcal{F} : \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} P(T^{-i}B) = \bar{P}(B).$$

We say a process is AMS if its distribution is AMS. Here are some examples of nonstationary yet AMS measures [25, 30].

Example 2.11. (a) *If μ, η are probability measures such that η is stationary and $\mu \ll \eta$, then μ is AMS (see the next Theorem).*

(b) *A time homogeneous irreducible Markov chain with a stationary distribution π is AMS for any initial distribution P_0.*

(c) *If μ is stationary with respect to T^N for some integer N (or N-periodic), then μ is AMS with respect to T with stationary mean $\bar{\mu}(F) = N^{-1} \sum_{i=0}^{N-1} \mu(T^{-i}F)$.*

Theorem 2.12. *Let $(\Omega, \mathcal{F})$ be a measurable space and $T : \Omega \to \Omega$ a measurable transformation. Let μ, η be probability measures on $(\Omega, \mathcal{F})$.*

(a) *μ is AMS if and only if for every bounded measurable $f : \Omega \to \mathbb{R}$, $\frac{1}{n}\sum_{i=0}^{n-1} f \circ T^i$ converges μ-a.s. as $n \to \infty$. In that case, the ergodic theorem holds and the limit is given by $E^{\bar{\mu}}(f|\mathcal{I})$.*

(b) *If η is stationary and dominates μ asymptotically, then μ is AMS: η dominate μ asymptotically if $B \in \mathcal{F}$ and $\eta(B) = 0$ implies that $\lim_{n\to\infty} \mu(T^{-n}B) = 0$.*

(c) *Let $\mathcal{T} = \cap_{n\geq 0} T^{-n}\mathcal{F}$ be the tail σ-field. If η is stationary, the following are equivalent:*

(1) η dominates μ asymptotically.

(2) If $F \in \mathcal{I}$ and $\eta(F) = 0$, then $\mu(F) = 0$.

(3) If $F \in \mathcal{T}$ and $\eta(F) = 0$, then $\mu(F) = 0$.

Proof. See Theorems 1, 2, and 3 in [25]. □

From the above theorem, if $\mu \ll \eta$ and η is stationary, μ is AMS. One may be tempted to conclude that η is the stationary mean of μ. However, we need ergodicity to reach that conclusion as the following Corollary shows.

Corollary 2.13. *If $\mu \ll \eta$ and η is a stationary and ergodic probability measure on $(\Omega, \mathcal{F})$ relative to T, then μ is AMS and its stationary mean is η.*

Proof. We show that η is the stationary mean of μ. If $d\mu = f d\eta$ for some nonnegative

$f:\Omega\to\mathbb{R}$ with $E^\eta f=1$ then for $A\in\mathcal{F}$,

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\mu(T^{-i}(A)) &= \lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\int_\Omega 1_A\circ T^i\,d\mu \\ = \lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\int_\Omega(1_A\circ T^i)f\,d\eta &= \int_\Omega(\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}1_A\circ T^i)f\,d\eta \\ = \int\eta(A)f\,d\eta = \eta(A).\end{aligned} \tag{2.7}$$

The equality in the last line comes from ergodic theorem and ergodicity of η. □

Lemma 2.14. *Let $(S,\mathfrak{S})$ be a measurable space and $\tau:S\to S$ a measurable map. Let $\gamma:S\to\Omega$ be a measurable map and $\gamma\circ\tau=T\circ\gamma$. Suppose P and P_0 are probability measures on $(S,\mathfrak{S})$ and $Q=\gamma_\#(P)$, $Q_0=\gamma_\#(P_0)$ are pushforward measures on $(\Omega,\mathcal{F})$ of P and P_0 under γ. Then the following hold.*

(a) *If $P\ll P_0$, then $Q\ll Q_0$.*

(b) *If P_0 is τ-invariant, then Q_0 is T-invariant .*

(c) *If P is τ-AMS with stationary mean P_0, then Q is T-AMS with stationary mean Q_0.*

(d) *If P_0 is τ-ergodic, then Q_0 is T-ergodic.*

Proof. (a) If $A\in\mathcal{F}$ and $Q_0(A)=0$ then $P_0(\gamma^{-1}(A))=Q_0(A)=0$. Since $P\ll P_0$ we have $Q(A)=P(\gamma^{-1}(A))=0$.

(b) This is a direct consequence of the property

$$\gamma\circ\tau=T\circ\gamma. \tag{2.8}$$

For $A\in\mathcal{F}$,

$$Q_0(T^{-1}(A))=P_0(\gamma^{-1}(T^{-1}(A)))=P_0(\tau^{-1}(\gamma^{-1}(A)))=P_0(\gamma^{-1}(A))=Q_0(A).$$

(c)

$$\begin{aligned}\forall A\in\mathcal{F},\ \lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}Q(T^{-i}(A)) &= \lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}P(\gamma^{-1}(T^{-i}(A)))\\ &= \lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}P(\tau^{-i}(\gamma^{-1}(A))) = P_0(\gamma^{-1}(A)) = Q_0(A)\end{aligned}\tag{2.9}$$

(d) If $T^{-1}(A)=A$ ($A\in\mathcal{F}$), then

$$\tau^{-1}(\gamma^{-1}(A))=\gamma^{-1}(T^{-1}(A))=\gamma^{-1}(A).$$

Hence $Q_0(A)=P_0(\gamma^{-1}(A))=0$ or 1.

□

Lemma 2.14 says AMS property is preserved under pushforwards of a measurable map that intertwines *translation maps*. We have a similar result for stochastic kernels. Let $(S,\mathfrak{S})$ be a measurable space and $\tau: S\to S$ a measurable map. A stochastic kernel from S to Ω is a measurable map $\kappa: S\to\mathcal{M}_1(\Omega)$, where $\mathcal{M}_1(\Omega)$ is the set of probability measures on Ω equipped with the σ-algebra induced by the mappings $\pi_B:\mu\mapsto\mu(B)$, $B\in\mathcal{F}$ (Chapters 1 and 5 of [28]). Suppose that κ intertwines τ with T: $\kappa(s,T^{-1}(B))=\kappa(\tau(s),B)$ for $s\in S$, $B\in\mathcal{F}$. Then we have $\int_\Omega g(T(\omega))\ \kappa(s,d\omega)=\int_\Omega g(\omega)\ \kappa(\tau(s),d\omega)$ for an integrable function g (g is $\kappa(\tau(s),\cdot)$-integrable if and only if $g\circ T$ is $\kappa(s,\cdot)$-integrable). For a probability measure P on S, a measure $Q=P\kappa$ is defined by $Q(B)=\int_S\kappa(s,B)\,P(ds)$. The integral of $g:\Omega\to\mathbb{R}$ under Q is given by $\int g\,dQ=\int_S\left(\int_\Omega g(\omega)\,\kappa(s,d\omega)\right)P(ds)$. We say that κ is ergodic relative to T if $\kappa(s,A)=0$ or 1 for $s\in S$ and $A\in\mathcal{F}$ with $T^{-1}(A)=A$. Note that the value 0 or 1 may depends on s.

Lemma 2.15. *For probability measures P and P_0 on S, consider $Q = P\kappa$ and $Q_0 = P_0\kappa$. If κ intertwines τ with T then the following hold.*

(a) *If $P \ll P_0$, then $Q \ll Q_0$.*

(b) *If P_0 is τ-invariant, then Q_0 is T-invariant.*

(c) *If P is τ-AMS with stationary mean P_0, then Q is T-AMS with stationary mean Q_0.*

(d) *If P_0 is τ-ergodic and κ is T-ergodic, then Q_0 is T-ergodic.*

Proof. The proof is almost similar to the proof of Lemma 2.14.

(a) If $A \in \mathcal{F}$ and $Q_0(A) = 0$ then $\kappa(s, A) = 0$ for P_0-a.s. s. Thus $\kappa(s, A) = 0$ for P-a.s. s since $P \ll P_0$ and we have $Q(A) = \int_S \kappa(s, A)\, P(ds) = 0$.

(b) This comes immediately from the intertwining property.

(c)

$$\begin{aligned}\forall A \in \mathcal{F},\ \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} Q(T^{-i}(A)) &= \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \int_S \left(\int_\Omega 1_A \circ T^i(\omega)\, \kappa(s, d\omega) \right) P(ds) \\ &= \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \int_S \left(\int_\Omega 1_A(\omega)\, \kappa(\tau^i(s), d\omega) \right) P(ds) \qquad (2.10) \\ &= \int_S \left(\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} g(\tau^i(s)) \right) P(ds) = \int_S g(s)\, P_0(ds) = Q_0(A),\end{aligned}$$

where $g(s) = \int_\Omega 1_A(\omega)\, \kappa(s, d\omega) = \kappa(s, A)$ and the last line is from Theorem 2.12 (a).

(d) If $T^{-1}(A) = A$ ($A \in \mathcal{F}$), then $\kappa(s, A) = \kappa(s, T^{-1}(A)) = \kappa(\tau(s), A)$ for all s. Therefore $g(s) = \kappa(s, A)$ is a τ-invariant function. Since P_0 is ergodic relative to τ, g is a constant P_0-a.s. $g(s)$ can take only 0 or 1 since κ is ergodic relative to T. Hence $g \equiv 0$ or 1 P_0-a.s. Therefore $Q_0(A) = \int_S g\, dP_0 = 0$ or 1.

□

For i.i.d. cases the subadditive ergodic theorem is used to prove the existence of various limits. Unlike the ergodic theorem, the subadditive ergodic theorem does not easily generalize to AMS measures. For a counterexample, see Theorem 8.5 in [24]. However, we have some partial results. A sequence of functions $\{f_n : \Omega \to \mathbb{R} \,|n = 1, 2, \dots\}$ satisfying the relation $f_{m+n} \geq f_m + f_n \circ T^m$ for all $m, n \in \mathbb{Z}_+$ is called *superadditive* ($f_0 = 0$). Define $\overline{f}(\omega) = \limsup_{n\to\infty} \frac{1}{n} f_n(\omega)$ and $\underline{f}(\omega) = \liminf_{n\to\infty} \frac{1}{n} f_n(\omega)$. We say a measurable function $g : \Omega \to S$ is *μ-almost surely invariant* if $\mu(g \circ T^k = g\,;\, k = 1, 2, \dots) = 1$ (see Lemma 7.6 in [24]). An event F is said to be μ-almost invariant if 1_F is.

Lemma 2.16. *Let $(\Omega, \mathcal{F}, \mu, T)$ be an AMS system with stationary mean $\bar{\mu}$ and S a standard Borel space (or nice space). If $f : \Omega \to S$ is μ-almost surely invariant and $h : S \to S'$ is measurable, then $h \circ f$ is μ-almost surely invariant. If $f : \Omega \to \mathbb{R}$ is μ-almost surely invariant, then f is μ-integrable if and only if f is $\bar{\mu}$-integrable. In that case $E^{\mu} f = E^{\bar{\mu}} f$.*

Proof. The first statement is obvious from the definition of almost surely invariance. For any μ-almost surely invariant set F, $\mu(F) = E^{\mu} 1_F = E^{\mu}(1_F \circ T^k) = \mu(T^{-k}F)$. So by the definition of $\bar{\mu}$ we have $\mu(F) = \bar{\mu}(F)$. Therefore $E^{\mu} f = E^{\bar{\mu}} f$ holds for indicator functions of μ-almost surely invariant sets and their linear combinations. For general functions approximation by simple functions gives the result. Let $f \geq 0$ be a μ-almost surely invariant function. Define f_n by $f_n = \displaystyle\sum_{k=0}^{2^{2n}-1} \frac{k}{2^n} 1_{(\frac{k}{2^n} \leq f < \frac{k+1}{2^n})}$. Then $f_n \uparrow f$ and f_n is μ-almost surely invariant from the first claim. Now monotone convergence theorem gives

$$E^{\mu} f = \lim_{n\to\infty} E^{\mu} f_n = \lim_{n\to\infty} E^{\bar{\mu}} f_n = E^{\bar{\mu}} f.$$

For general f, write $f = f^+ - f^-$ and use the result of a nonnegative case. This proof also shows that f is μ integrable if and only if f is $\bar{\mu}$ integrable when f is μ-almost surely invariant. □

Theorem 2.17. *Let $(\Omega, \mathcal{F}, \mu, T)$ be an AMS system with stationary mean $\bar{\mu}$. Suppose that $\{f_n : n = 1, 2, \dots\}$ is a superadditive sequence of $\bar{\mu}$ integrable random variables and $\sup_n E^{\bar{\mu}} f_n/n < \infty$. Suppose one of the following conditions is satisfied :*

(a) *$\overline{f}$ is μ-almost surely invariant.*

(b) *$\mu \ll \bar{\mu}$.*

(c) *Ω is equipped with a partial order $\preceq$. f_n are increasing functions (i.e., $x \preceq y$ implies $f_n(x) \leq f_n(y)$). Finally, μ is stochastically smaller than $\bar{\mu}$, that is, $\int g\, d\mu \leq \int g\, d\bar{\mu}$ for all bounded measurable increasing functions g.*

Then there is an invariant function $\phi : \Omega \to \mathbb{R}$ and an event $\Omega_0 \subseteq \Omega$ such that $\lim_{n\to\infty} \frac{1}{n} f_n(\omega) = \phi(\omega)$ $(\omega \in \Omega_0)$, and $\mu(\Omega_0) = \bar{\mu}(\Omega_0) = 1$. The function $\phi(\omega)$ can be chosen as

$$\phi = \sup_n \frac{1}{n} E^{\bar{\mu}}(f_n|\mathcal{I}). \tag{2.11}$$

ϕ satisfies

$$E^{\mu}\phi = E^{\bar{\mu}}\phi = \sup_n \frac{1}{n} E^{\bar{\mu}} f_n = \lim_{n\to\infty} \frac{1}{n} E^{\bar{\mu}} f_n. \tag{2.12}$$

Proof. To prove the result under this condition, we first establish some general properties of superadditive sequences. Let $F_n = \sum_{i=0}^{n-1} f_1 \circ T^i$. Then $\{F_n\}_{n\geq 1}$ is additive and $f_n \geq F_n$ by superadditivity. Since the theorem holds for an additive sequence by Theorem 2.12 (a) we only need to prove the assertion for $f_n - F_n$. Hence we may assume that $f_n \geq 0$.

We show that $\underline{f} \geq \phi$ holds μ-a.s.. Fix m. For each i ($0 \leq i < m$) write $n = i + m\ell + r$ ($0 \leq r < m$). By superadditivity and positivity of f_n

$$\begin{aligned} f_n &\geq f_i + f_m \circ T^i + f_m \circ T^{i+m} + f_m \circ T^{i+2m} + \cdots + f_m \circ T^{i+(\ell-1)m} + f_r \circ T^{i+\ell m} \\ &\geq f_i + f_m \circ T^i + f_m \circ T^{i+m} + f_m \circ T^{i+2m} + \cdots + f_m \circ T^{i+(\ell-1)m} = f_i + \sum_{j=0}^{\ell-1} f_m \circ T^{i+jm} \end{aligned}$$

Summing over i gives

$$m f_n \geq \sum_{i=0}^{m-1} f_i + \sum_{j=0}^{m\ell-1} f_m \circ T^j$$

Dividing by mn and AMS ergodic theorem give $\underline{f} \geq \frac{1}{m} E^{\bar{\mu}}(f_m | \mathcal{I})$. Since this holds for all m, we have $\overline{f} \geq \underline{f} \geq \phi$ μ-a.s. Now we are in a position to prove the claim.

(a) Since $\overline{f}$ is μ-almost surely invariant and ϕ is invariant, from Lemma 2.16,

$$E^{\bar{\mu}}\phi = E^{\bar{\mu}}\overline{f} = E^{\mu}\overline{f} \geq E^{\mu}\underline{f} \geq E^{\mu}\phi = E^{\bar{\mu}}\phi \tag{2.13}$$

The first equality holds by the subadditive ergodic theorem for stationary measures. Therefore we have $\mu(\overline{f} = \underline{f} = \phi) = 1$.

(c) Since f_n are increasing functions, $\overline{f}$ and $\underline{f}$ are also increasing.

$$E^{\bar{\mu}}\phi = E^{\bar{\mu}}\overline{f} \geq E^{\mu}\overline{f} \geq E^{\mu}\underline{f} \geq E^{\mu}\phi = E^{\bar{\mu}}\phi \tag{2.14}$$

The first equality holds by the subadditive ergodic theorem for stationary measures and the last equality comes from the invariance of ϕ. In particular, we have $E^{\mu}\overline{f} = E^{\mu}\underline{f} = E^{\mu}\phi$ which implies $\mu(\overline{f} = \underline{f} = \phi) = 1$.

(b) The conclusion follows immediately from the subadditive ergodic theorem for stationary measures since $\bar{\mu}(\{\overline{f} = \underline{f} = \phi\}) = 1$.

Lemma 2.16 and the subadditive ergodic theorem for stationary measures give (2.12).

□

2.4 Technical results and proof of main theorems

This section is devoted to the proof of the main results Theorem 2.9 and Corollary 2.10. We develop some technicalities in section 2.4.1 used throughout the paper.

2.4.1 Preliminaries

In this section, we record results that are used throughout. First, we establish some conventions. On a given measurable space we consider several probability measures simultaneously. To reduce confusion and to ease proofs, we mainly use the following conventions. Weights ω_x with the distribution in Definition 2.2 are realized as functions of i.i.d. uniform random variables and coupling with η is used:

$$\omega_{\mathbf{x}} = F_{\mathbf{x}}^{-1}(U_{\mathbf{x}}), \quad \eta_{\mathbf{x}} = F^{-1}(U_{\mathbf{x}}), \quad \mathbf{x} \in \mathbb{Z}_+^d \tag{2.15}$$

where $\{U_{\mathbf{x}} : \mathbf{x} \in \mathbb{Z}_+^d\}$ are i.i.d. Uniform(0,1) random variables, $F_{\mathbf{x}}$ the CDF of $\omega_{\mathbf{x}}$ and $F^{-1}(t) = \inf\{s \in \mathbb{R} : F(s) \geq t\}$, $(0 < t < 1)$, the generalized inverse of F [6].

In the remaining work, we use the following conventions. $(\Omega, \mathcal{F}, \mathbf{P})$ is a probability space where most random variables are defined. A generic element in Ω is denoted by $\mathbf{w}$. Recall that $\Omega_1 = \mathbb{R}^{\mathbb{Z}_+^d}$ equipped with the σ-algebra $\mathcal{G}_1$ generated by the coordinate projections is used for sets of configurations. In particular ω, η, and U in (2.15) are measurable maps from Ω to Ω_1. Sometimes we use the notations ω, η, and U for configurations in Ω_1. The context should indicate which one is meant: measurable map from Ω to Ω_1 or element of Ω_1.

For Assumption 2.4 we continue to use the couplings in (2.15) and assume $\rho = \{\rho_{\mathbf{x}} : \mathbf{x} \in \mathbb{Z}_+^d\} : \Omega \to \Omega_0$ are independent of uniform random variables $\{U_{\mathbf{x}}\}$ in (2.15). Recall that for given parameters $\rho = \{\rho_{\mathbf{x}} : \mathbf{x} \in \mathbb{Z}_+^d\}$ we write $\mathbb{P}^\rho$ for the conditional distribution of ω given ρ (see (2.2)). Note that $\mathbb{E}^\rho(X \circ T_{\mathbf{u}}) = \mathbb{E}^{T_{\mathbf{u}}\rho} X$ holds for a given random variable $X : \Omega_1 \to \mathbb{R}$, configuration of parameters ρ, and shift map $T_{\mathbf{u}}$. (Here we slightly abused notation: The first $T_{\mathbf{u}}$ is defined on Ω_1 and the second $T_{\mathbf{u}}$ is defined on Ω_0.)

Next, basic properties of the free energy are considered; their connection to last passage percolation and greedy lattice animals. For $\mathbf{u} \leq \mathbf{v}$ in $\mathbb{Z}_+^d$, the last-passage times are defined by

$$G_{\mathbf{u},\mathbf{v}} = G^\omega_{\mathbf{u},\mathbf{v}} = \max_{\mathbf{x}_\cdot \in \Pi_{\mathbf{u},\mathbf{v}}} \sum_{i=1}^{|\mathbf{v}-\mathbf{u}|_1} \omega_{\mathbf{x}_i}, \qquad \mathbf{u} \leq \mathbf{v} \text{ in } \mathbb{Z}_+^d. \tag{2.16}$$

A finite path $(\mathbf{x}_i)_{0 \leq i \leq n}$ in $\Pi_{\mathbf{u},\mathbf{v}}$ is a *geodesic* between $\mathbf{u}$ and $\mathbf{v}$ if it is the maximizing path that realizes $G_{\mathbf{u},\mathbf{v}}$, namely, $G_{\mathbf{u},\mathbf{v}} = \sum_{i=1}^n \omega_{\mathbf{x}_i}$. When paths start at the origin we drop $\mathbf{u}$ from the notation;

$$\Pi_{\mathbf{v}} = \Pi_{\mathbf{0},\mathbf{v}}, \; Z_{\mathbf{u}} = Z_{\mathbf{0},\mathbf{u}}, \; \text{ and } G_{\mathbf{v}} = G_{\mathbf{0},\mathbf{v}}. \tag{2.17}$$

For a finite subset ξ of $\mathbb{Z}_+^d$, the *weight* $H(\xi)$ of ξ is defined by $H(\xi) = \sum_{\mathbf{v}\in\xi} \omega_{\mathbf{v}}$. A *lattice animal* [31] is a finite connected subset of $\mathbb{Z}^d$. Let $A(n)$ be the set of lattice animals of size n which contain the origin. A *greedy lattice animal of size* n is a connected subset of size n containing the origin, whose weight is maximal among all such sets. Let $N(n)$ be this maximum weight. We have

$$N(n) = N^\omega(n) = \max_{\xi \in A(n)} H(\xi). \tag{2.18}$$

It is convenient to construct the above objects as functions of coordinate variables. Let B be a finite subset of $\mathbb{Z}^d$ and $\mathbf{A} \subseteq \{0,1\}^B$ be a subset of the power set of B or collection of indicator functions supported on subsets of B. We may consider $\mathbf{A}$ as a subset of $\mathbb{R}^B$. Here we regard $\mathbb{R}^B$ as $\mathbb{R}^n$ with $n = |B|$. Let

$$\|\mathbf{A}\|_p = \max_{\mathbf{a}\in\mathbf{A}} |\mathbf{a}|_p \; (1 \leq p \leq \infty) \quad \text{and} \quad |\mathbf{A}| = \mathbf{card}(\mathbf{A}).$$

For a set B and a subset $\mathbf{A}$ of $\{0,1\}^B$, define $f_{B,\mathbf{A}} : \mathbb{R}^B \longrightarrow \mathbb{R}$ by

$$f_{B,\mathbf{A}}(X) = \log \sum_{\mathbf{a}\in\mathbf{A}} e^{\mathbf{a}\cdot X} \quad \text{for } X \in \mathbb{R}^B. \tag{2.19}$$

Similarly define $G_{B,\mathbf{A}} : \mathbb{R}^B \longrightarrow \mathbb{R}$ by

$$G_{B,\mathbf{A}}(X) = \max_{\mathbf{a}\in\mathbf{A}} \mathbf{a}\cdot X \quad \text{for } X \in \mathbb{R}^B. \tag{2.20}$$

Typical examples of $\mathbf{A}$ are formed from a set of paths: Let Π be a collection of paths in B. Then $\mathbf{a}_\pi = 1_{\{\mathbf{x}_1,\ldots,\mathbf{x}_n\}}$ for $\pi = \mathbf{x}_{\boldsymbol{\cdot}} = (\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n)$. ($\mathbf{x}_0$ is excluded from $\mathbf{a}_\pi$.) $\mathbf{A}$ is given by $\mathbf{A} = \{\mathbf{a}_\pi\}_{\pi\in\Pi}$.

We customize these definitions to our models. For $\mathbf{u} \le \mathbf{v}$ in $\mathbb{Z}_+^d$, write

$$B = [\mathbf{u},\mathbf{v}] = \prod_{i=1}^d [u_i, v_i] \subseteq \mathbb{Z}_+^d, \quad \Pi_B = \Pi_{[\mathbf{u},\mathbf{v}]} = \{\mathbf{a}_{\mathbf{x}_{\boldsymbol{\cdot}}} : \mathbf{x}_{\boldsymbol{\cdot}} \in \Pi_{\mathbf{u},\mathbf{v}}\} \subseteq \mathbb{R}^B.$$

Note the difference between $\Pi_{\mathbf{u},\mathbf{v}}$ and $\Pi_{[\mathbf{u},\mathbf{v}]}$. The former is a set of paths, and the latter is a set of indicator functions. For $\omega \in \mathbb{R}^{\mathbb{Z}_+^d}$ write its restriction to B by $\omega_B \in \mathbb{R}^B$. We write f_B and G_B for $f_{B,\mathbf{A}}$, $G_{B,\mathbf{A}}$ when $\mathbf{A} = \Pi_B$. We also write f_n and G_n for $f_{B,\mathbf{A}}$, $G_{B,\mathbf{A}}$ when $\mathbf{A} = \{\mathbf{a}_\pi\}_{\pi\in\Pi_n^{p2l}}$. Under these conventions, we have the following representations of free energy, last passage time, and weight of greedy lattice animal.

$$\begin{aligned} \log Z^\omega_{\mathbf{u},\mathbf{v}} &= f_{[\mathbf{u},\mathbf{v}]}(\omega_{[\mathbf{u},\mathbf{v}]}) \\ \log Z^\omega_n &= f_n(\omega_{[0,n]^d}) \\ G^\omega_{\mathbf{u},\mathbf{v}} &= G_{[\mathbf{u},\mathbf{v}]}(\omega_{[\mathbf{u},\mathbf{v}]}) \\ N^\omega(n) &= G_{B_n,\tilde{A}(n)}(\omega_{B_n}) \end{aligned} \tag{2.21}$$

where $B_n = [-n,n]^d$ and $\tilde{A}(n) = \{1_\xi : \xi \in A(n)\}$.

We close this section with the following lemma.

Lemma 2.18. *Suppose* $\mathbf{A}$ *satisfies* $|\mathbf{A}| \le C_0^{\|\mathbf{A}\|_1}$ *for some positive constant* C_0. *Then the following hold.*

(a) $f_{B,\mathbf{A}}$ *and* $G_{B,\mathbf{A}}$ *are convex and nondecreasing functions on* $\mathbb{R}^B$.

(b) $G_{B,\mathbf{A}} \le f_{B,\mathbf{A}} \le G_{B,\mathbf{A}} + \log|\mathbf{A}| \le G_{B,\mathbf{A}} + \|\mathbf{A}\|_1 \log C_0$.

(c) $|G_{B,\mathbf{A}}(X)| \le G_{B,\mathbf{A}}(|X|) \le \|\mathbf{A}\|_2 \cdot |X|_2$.

(d) $|f_{B,\mathbf{A}}(Y) - f_{B,\mathbf{A}}(X)| \vee |G_{B,\mathbf{A}}(Y) - G_{B,\mathbf{A}}(X)| \le G_{B,\mathbf{A}}(|Y - X|)$.

(e) $f_{B,\mathbf{A}}$ *and* $G_{B,\mathbf{A}}$ *are Lipschitz functions on* $\mathbb{R}^B$ *with respect to the* ℓ^2 *norm with the Lipschitz constants* $\le \|\mathbf{A}\|_2 \le \sqrt{\|\mathbf{A}\|_1}$.

Proof. (a) Since $\mathbf{a} \ge \mathbf{0}$, $f_{B,\mathbf{A}}$ and $G_{B,\mathbf{A}}$ are nondecreasing functions. $G_{B,\mathbf{A}}$ is the maximum of linear functions so that it is convex. Let $\alpha, \beta > 0$ and $\alpha + \beta = 1$.

$$
\begin{aligned}
& f_{B,\mathbf{A}}(\alpha X_1 + \beta X_2) = \log \sum_{\mathbf{a}\in\mathbf{A}} e^{\mathbf{a}\cdot(\alpha X_1 + \beta X_2)} \\
& \le \log \left(\sum_{\mathbf{a}\in\mathbf{A}} e^{\mathbf{a}\cdot X_1} \right)^{\alpha} \left(\sum_{\mathbf{a}\in\mathbf{A}} e^{\mathbf{a}\cdot X_2} \right)^{\beta} \quad \text{(Hölder's ineqaulity)} \\
& = \alpha \log \sum_{\mathbf{a}\in\mathbf{A}} e^{\mathbf{a}\cdot X_1} + \beta \log \sum_{\mathbf{a}\in\mathbf{A}} e^{\mathbf{a}\cdot X_2} = \alpha f_{B,\mathbf{A}}(X_1) + \beta f_{B,\mathbf{A}}(X_2)
\end{aligned}
$$

(b) Trivial.

(c) From (a) $G_{B,\mathbf{A}}$ is nondecreasing so that $|G_{B,\mathbf{A}}(X)| \le G_{B,\mathbf{A}}(|X|)$.

$$
G_{B,\mathbf{A}}(|X|) = \max_{\mathbf{a}\in\mathbf{A}} \mathbf{a} \cdot |X| \le \max_{\mathbf{a}\in\mathbf{A}} |\mathbf{a}|_2 |X|_2 \le \|\mathbf{A}\|_2 \cdot |X|_2.
$$

(d) Since $\mathbf{a} \geq \mathbf{0}$,

$$
\begin{aligned}
f_{B,\mathbf{A}}(Y) &= \log \sum_{\mathbf{a}\in\mathbf{A}} e^{\mathbf{a}\cdot Y} \leq \log \sum_{\mathbf{a}\in\mathbf{A}} e^{\mathbf{a}\cdot X} e^{\mathbf{a}\cdot(Y-X)^+} \\
&\leq \log \left(\left(\sum_{\mathbf{a}\in\mathbf{A}} e^{\mathbf{a}\cdot X} \right) e^{\max_{\mathbf{a}\in\mathbf{A}} \mathbf{a}\cdot(Y-X)^+} \right) \qquad (2.22) \\
&= f_{B,\mathbf{A}}(X) + G_{B,\mathbf{A}}((Y-X)^+) \leq f_{B,\mathbf{A}}(X) + G_{B,\mathbf{A}}(|Y-X|).
\end{aligned}
$$

Changing the role of X, Y, we obtain $|f_{B,\mathbf{A}}(Y) - f_{B,\mathbf{A}}(X)| \leq G_{B,\mathbf{A}}(|Y-X|)$. The inequality for $G_{B,\mathbf{A}}$ is proved similarly.

(e) This follows from (c) and (d). Note that for $\mathbf{a} \in \mathbf{A}$, $|\mathbf{a}|_2 = \sqrt{|\mathbf{a}|_1}$ since $\mathbf{a}_{(i,j)} = 0, 1$. □

2.4.2 Concentration inequalities

In this subsection, we collect useful concentration inequalities.

Theorem 2.19. *Let $\mathbf{X} = (X_1, \ldots, X_n)$ be a random variable with independent components taking values in $[0, R]$. Let $\mathbf{F} : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a convex L-Lipschitz function with respect to the ℓ^2 norm. Let $M\mathbf{F}$ be a median of $\mathbf{F}(\mathbf{X})$. Then for all $t \geq 0$*

$$
\begin{aligned}
\mathbf{P}(|\mathbf{F}(\mathbf{X}) - M\mathbf{F}| \geq t) &\leq 4e^{-t^2/(4L^2R^2)} \\
\mathbf{P}(|\mathbf{F}(\mathbf{X}) - \mathbb{E}\mathbf{F}(\mathbf{X})| \geq t) &\leq e^{64} e^{-t^2/(16L^2R^2)}
\end{aligned}
\qquad (2.23)
$$

Proof. For the first inequality, see [10] Theorem 7.12. For the second, see the proof of [31] Lemma 5.1. □

Theorem 2.20. *There exists a universal constant $c_0 < \infty$ which does not depends on the distribution (but depends on d) such that if F satisfies (2.1) and weights are F-i.i.d. then there exists a deterministic N with*

$$
\frac{N(n)}{n} \to N \text{ almost surely and in } L^1(\Omega) \qquad (2.24)
$$

as $n \to \infty$, and

$$N \le c_0 \int_0^\infty (1 - F(x))^{1/d} dx. \tag{2.25}$$

The same c_0 satisfies

$$\sup_n E^F \frac{N(n)}{n} \le c_0 \int_0^\infty (1 - F(x))^{1/d} dx. \tag{2.26}$$

Proof. See [31] Theorems 1.1 and 2.3. □

Theorem 2.21. *Suppose F in (2.1) is supported on $[0, R]$ and weights ω has a distribution $\mathbb{P}^\rho$. Then for all $t \ge 0$, $n \in \mathbb{N}$, and $\mathbf{u} \le \mathbf{v}$ in $\mathbb{Z}_+^d$*

$$\mathbb{P}^\rho(|\log Z_n - \mathbb{E}^\rho \log Z_n| \ge nt) \le e^{64} e^{-nt^2/(64R^2)} \tag{2.27}$$

and

$$\mathbb{P}^\rho(|\log Z_{n\mathbf{u},n\mathbf{v}} - \mathbb{E}^\rho \log Z_{n\mathbf{u},n\mathbf{v}}| \ge nt) \le e^{64} e^{-nt^2/(64R^2|\mathbf{v}-\mathbf{u}|_1)}. \tag{2.28}$$

Proof. These are direct consequences of Lemma 2.18 and Theorem 2.19. Note that $\omega_{\mathbf{x}}$ are supported on $[-R, R]$. Lipschitz constants are bounded by $\sqrt{n}$ and $\sqrt{n|\mathbf{v} - \mathbf{u}|_1}$ respectively. □

Concentration inequalities give the following theorems. Recall that at some places ω and η are considered as measurable maps from $(\Omega, \mathcal{F}, \mathbf{P})$ to Ω_1 (see the paragraph below (2.15)). Expectation under $\mathbf{P}$ is denoted by $\mathbf{E}$.

For $y > 0$ consider the "y-truncated" weight ω^y given by

$$\omega_{\mathbf{x}}^y = \omega_{\mathbf{x}} 1_{(|\omega_{\mathbf{x}}| \le y)} + y 1_{(\omega_{\mathbf{x}} > y)} - y 1_{(\omega_{\mathbf{x}} < -y)}. \tag{2.29}$$

Theorem 2.22. *For a fixed $L \in \mathbb{N}$, suppose $\mathbf{u}_n \le \mathbf{v}_n \le Ln\mathbf{1}$ for all $n \in \mathbb{N}$. Then $\mathbb{P}^\rho$-a.e. ω,*

$$\limsup_{n\to\infty} \frac{1}{n} \log Z_{\mathbf{u}_n,\mathbf{v}_n}^\omega = \lim_{y\to\infty} \limsup_{n\to\infty} \frac{1}{n} \log Z_{\mathbf{u}_n,\mathbf{v}_n}^{\omega^y} \tag{2.30}$$

and

$$\liminf_{n\to\infty}\frac{1}{n}\log Z^{\omega}_{\mathbf{u}_n,\mathbf{v}_n}=\lim_{y\to\infty}\liminf_{n\to\infty}\frac{1}{n}\log Z^{\omega^y}_{\mathbf{u}_n,\mathbf{v}_n}. \tag{2.31}$$

The same identity holds for the last passage time G.

Proof. Note that $|\omega_{\mathbf{x}}-\omega^y_{\mathbf{x}}|=(|\omega_{\mathbf{x}}|-y)^+\le(\eta_{\mathbf{x}}-y)^+$ (η in (2.15)) . Now the random variables $(\eta_{\mathbf{x}}-y)^+$, $\mathbf{x}\in\mathbb{Z}^d_+$, are i.i.d. and non-negative with distribution F^y, where $F^y(t)=F(t+y)$, $t\ge 0$. We have

$$k_y\triangleq\int_0^\infty(1-F^y(t))^{1/d}dt=\int_y^\infty(1-F(t))^{1/d}dt. \tag{2.32}$$

From (2.21) and Lemma 2.18,

$$\begin{aligned}&|\log Z^{\omega}_{\mathbf{u}_n,\mathbf{v}_n}-\log Z^{\omega^y}_{\mathbf{u}_n,\mathbf{v}_n}|\vee|G^{\omega}_{\mathbf{u}_n,\mathbf{v}_n}-G^{\omega^y}_{\mathbf{u}_n,\mathbf{v}_n}|\\ \le\quad & G_{\mathbf{u}_n,\mathbf{v}_n}(|\omega-\omega^y|)\le N^{(\eta-y)^+}(Ln+1).\end{aligned} \tag{2.33}$$

Divide both sides by n and take $n\to\infty$. Theorem 2.20 applied to the situation where the distribution F is replaced by F^y and the bounded case give

$$\limsup_{n\to\infty}\left|\frac{\log Z^{\omega}_{\mathbf{u}_n,\mathbf{v}_n}}{n}-\frac{\log Z^{\omega^y}_{\mathbf{u}_n,\mathbf{v}_n}}{n}\right|\le Lc_0k_y$$

$\mathbb{P}^\rho$-almost surely. By Lemma 2.18(a), the limits on the right-hand side of (2.30) and (2.31) exist for any ω. If we let $y\to\infty$ along a countable sequence, $\mathbb{P}^\rho$-a.s. convergence holds.

□

Theorem 2.23. *We have for $\mathbb{P}^\rho$-a.e. ω,*

$$\lim_{n\to\infty}\left(\frac{\log Z^\omega_n}{n}-\frac{\mathbb{E}^\rho\log Z_n}{n}\right)=0. \tag{2.34}$$

Proof. If $|\omega_{\mathbf{x}}|$ is bounded by $y > 0$ for all $\mathbf{x} \in \mathbb{Z}_+^d$, Borel-Cantelli lemma and Theorem 2.21 give the result. For general weights, consider ω^y in (2.29).

From (2.21) and Lemma 2.18,

$$\begin{aligned} &|\log Z_n^\omega - \mathbb{E}^\rho \log Z_n| = |f_n(\omega) - \mathbf{E} f_n(\omega)| \\ &\le |f_n(\omega) - f_n(\omega^y)| + |f_n(\omega^y) - \mathbf{E} f_n(\omega^y)| + |\mathbf{E} f_n(\omega^y) - \mathbf{E} f_n(\omega)| \\ &\le |G_n(|\omega - \omega^y|)| + |f_n(\omega^y) - \mathbf{E} f_n(\omega^y)| + \mathbf{E} G_n(|\omega - \omega^y|) \\ &\le N^{(\eta - y)^+}(n+1) + |f_n(\omega^y) - \mathbf{E} f_n(\omega^y)| + \mathbf{E} N^{(\eta-y)^+}(n+1). \end{aligned} \tag{2.35}$$

Divide both sides by n and take $n \to \infty$. Theorem 2.20 applied to the situation where the distribution F is replaced by F^y and the bounded case give

$$\limsup_{n\to\infty} \left|\frac{\log Z_n^\omega}{n} - \frac{\mathbb{E}^\rho \log Z_n}{n}\right| \le 2c_0 k_y$$

almost surely. $y \to \infty$ along a countable sequence proves (2.34).

□

Corollary 2.24. *Suppose the distribution of ρ is $\mathcal{Q}$ and the conditional law of ω given ρ is $\mathbb{P}^\rho$. Denote the joint distribution of (ρ, ω) by ν. Then for ν-a.e. ρ and ω,*

$$\lim_{n\to\infty} \left(\frac{\log Z_n^\omega}{n} - \frac{\mathbb{E}^\rho \log Z_n}{n}\right) = 0. \tag{2.36}$$

Proof. Let $A \in \mathcal{G}_0 \otimes \mathcal{G}_1$ be an event

$$A = \{(\rho, \omega) \in \Omega_0 \times \Omega_1 \,:\, \lim_{n\to\infty} \left(\frac{\log Z_n^\omega}{n} - \frac{\mathbb{E}^\rho \log Z_n}{n}\right) = 0\}$$

and its section

$$A_\rho = \{\omega \in \Omega_1 \,:\, \lim_{n\to\infty} \left(\frac{\log Z_n^\omega}{n} - \frac{\mathbb{E}^\rho \log Z_n}{n}\right) = 0\}.$$

Then for $\mathcal{Q}$-a.e. ρ, $\mathbb{P}^\rho(A_\rho) = 1$ by Theorem 2.23. Therefore

$$\nu(A) = \int_{\Omega_0} \mathbb{P}^\rho(A_\rho)\, \mathcal{Q}(d\rho) = 1.$$

□

Theorem 2.25. *Let $L \in \mathbb{N}$ and ρ be fixed. Then $\mathbb{P}^\rho$-a.e. ω, we have*

$$\lim_{n\to\infty} \frac{1}{n} \max\left\{\left|\log Z^{\omega}_{\mathbf{u},\mathbf{v}} - \mathbb{E}^\rho \log Z_{\mathbf{u},\mathbf{v}}\right| : \mathbf{u} \le \mathbf{v} \le Ln\mathbf{1} \quad in \quad \mathbb{Z}^d_+\right\} = 0. \tag{2.37}$$

Proof. Suppose $\omega_{\mathbf{x}}$, $\mathbf{x} \in \mathbb{Z}^d_+$ are bounded by $y > 0$. Then by (2.28) for $t > 0$,

$$\mathbb{P}^\rho(|\log Z^{\omega}_{\mathbf{u},\mathbf{v}} - \mathbb{E}^\rho \log Z_{\mathbf{u},\mathbf{v}}| \ge nt) \le e^{64} e^{-nt^2/(64y^2L)}.$$

Therefore, counting $\mathbf{u}$ and $\mathbf{v}$, we have

$$\mathbb{P}^\rho\left(\max_{\mathbf{u},\mathbf{v}}\left\{\left|\log Z^{\omega}_{\mathbf{u},\mathbf{v}} - \mathbb{E}^\rho \log Z_{\mathbf{u},\mathbf{v}}\right|\right\} \ge nt\right) \le (Ln+1)^{2d} e^{64} e^{-nt^2/(64y^2L)}.$$

Borel-Cantelli lemma gives (2.37).

For more general weights, consider y-truncated weights as in (2.29). From (2.21) and Lemma 2.18

$$\begin{aligned}
&|\log Z^{\omega}_{\mathbf{u},\mathbf{v}} - \mathbb{E}^\rho \log Z_{\mathbf{u},\mathbf{v}}| \\
\le& |\log Z^{\omega}_{\mathbf{u},\mathbf{v}} - \log Z^{\omega^y}_{\mathbf{u},\mathbf{v}}| + |\log Z^{\omega^y}_{\mathbf{u},\mathbf{v}} - \mathbb{E}^\rho \log Z^{\omega^y}_{\mathbf{u},\mathbf{v}}| + |\mathbb{E}^\rho \log Z^{\omega^y}_{\mathbf{u},\mathbf{v}} - \mathbb{E}^\rho \log Z^{\omega}_{\mathbf{u},\mathbf{v}}| \\
\le& G_{\mathbf{u},\mathbf{v}}(|\omega - \omega^y|) + \max_{\mathbf{u}_1,\mathbf{v}_1} |\log Z^{\omega^y}_{\mathbf{u}_1,\mathbf{v}_1} - \mathbb{E}^\rho \log Z^{\omega^y}_{\mathbf{u}_1,\mathbf{v}_1}| + \mathbb{E}^\rho G_{\mathbf{u},\mathbf{v}}(|\omega - \omega^y|) \\
\le& N^{(\eta-y)^+}(Ln+1) + \max_{\mathbf{u}_1,\mathbf{v}_1} |\log Z^{\omega^y}_{\mathbf{u}_1,\mathbf{v}_1} - \mathbb{E}^\rho \log Z^{\omega^y}_{\mathbf{u}_1,\mathbf{v}_1}| + \mathbb{E}^\rho N^{(\eta-y)^+}(Ln+1).
\end{aligned} \tag{2.38}$$

Therefore from Theorem 2.20, for $\mathbb{P}^\rho$-a.e. ω,

$$\limsup_{n\to\infty} \frac{1}{n} \max_{\mathbf{u},\mathbf{v}} |\log Z^{\omega}_{\mathbf{u},\mathbf{v}} - \mathbb{E}^\rho \log Z_{\mathbf{u},\mathbf{v}}| \le 2Lc_0 k_y, \tag{2.39}$$

where k_y is as in (2.32). If we let $y \to \infty$ along a countable sequence, then we have (2.37). □

The following corollary is one of the key elements for the main results. The a.s. convergence happens simultaneously for all $\mathbf{x} \le \mathbf{y}$ in $\mathbb{R}^d_+$.

Corollary 2.26. *Let ρ be fixed, for $\mathbb{P}^\rho$-a.e. ω, we have for all $\mathbf{x} \leq \mathbf{y}$ in $\mathbb{R}^d_+$*

$$\lim_{n\to\infty}\Big(\frac{\log Z^\omega_{\lfloor n\mathbf{x}\rfloor,\lfloor n\mathbf{y}\rfloor}}{n} - \frac{\mathbb{E}^\rho \log Z_{\lfloor n\mathbf{x}\rfloor,\lfloor n\mathbf{y}\rfloor}}{n}\Big) = 0. \tag{2.40}$$

Corollary 2.27. *Suppose the distribution of ρ is $\mathcal{Q}$ and the conditional law of ω given ρ is $\mathbb{P}^\rho$. Denote the joint distribution of (ρ, ω) by ν. Then for ν-a.e. ρ and ω, we have for all $\mathbf{x} \leq \mathbf{y}$ in $\mathbb{R}^d_+$*

$$\lim_{n\to\infty}\Big(\frac{\log Z^\omega_{\lfloor n\mathbf{x}\rfloor,\lfloor n\mathbf{y}\rfloor}}{n} - \frac{\mathbb{E}^\rho \log Z_{\lfloor n\mathbf{x}\rfloor,\lfloor n\mathbf{y}\rfloor}}{n}\Big) = 0 \tag{2.41}$$

Proof. The proof is the same as in Corollary 2.24. □

Remark 2.28. *These Theorems also hold for the last-passage times $G_{u,v}$ because we only used general concentration inequalities applicable also to last-passage times in the proof.*

2.4.3 Properties of limits

We develop some general theory of the scaling limit. Let $\phi : \mathbb{Z}^d_+ \to \mathbb{R}$ be a function. $\bar{\phi} : \mathfrak{D}_\phi \subseteq \mathbb{R}^d_{>0} \to (-\infty, \infty]$ is defined as $\bar{\phi}(\mathbf{x}) = \lim_{n\to\infty} \frac{1}{n}\phi(\lfloor n\mathbf{x}\rfloor)$ provided the limit exists and $\mathfrak{D}_\phi$ denotes the set of all points in $\mathbb{R}^d_{>0}$ where the limit exists. We call $\bar{\phi}$ the scaling limit of ϕ. Note that ∞ is allowed as a limit.

Proposition 2.29. *Suppose ϕ is nondecreasing $(\mathbf{u} \leq \mathbf{v} \Rightarrow \phi(\mathbf{u}) \leq \phi(\mathbf{v}))$ and $\mathbb{N}^d \subseteq \mathfrak{D}_\phi$. Then the following hold :*

(a) $\mathbf{x} \in \mathfrak{D}_\phi$, $a > 0 \Rightarrow a\mathbf{x} \in \mathfrak{D}_\phi$, $\bar{\phi}(a\mathbf{x}) = a\bar{\phi}(\mathbf{x})$. *$\bar{\phi}$ is nonnegative and nondecreasing on $\mathfrak{D}_\phi$.*

(b) $\mathfrak{D}_\phi = \mathbb{R}^d_{>0}$.

(c) *If* $\lim_{n\to\infty} \mathbf{x}_n/n = \mathbf{x}$, $(\mathbf{x}_n \in \mathbb{Z}^d_+, \mathbf{x} \in \mathbb{R}^d_{>0})$ *then* $\lim_{n\to\infty} \frac{1}{n}\phi(\mathbf{x}_n) = \bar{\phi}(\mathbf{x})$.

(d) *If* $\bar{\phi}$ *attains* ∞ *at some point in* $\mathbb{R}^d_{>0}$, $\bar{\phi} \equiv \infty$ *on* $\mathbb{R}^d_{>0}$.

(e) *If* $\bar{\phi}$ *is finite then* $\bar{\phi}$ *is continuous.* $\bar{\phi}$ *extends continuously to* $\mathbb{R}^d_+$. *If we compute* $\bar{\phi}$ *on the boundary of* $\mathbb{R}^d_+$ *directly from* ϕ *we have* $\bar{\phi} \leq$ *the continuous extension of* $\bar{\phi}$ *on the boundary.*

Proof. (a) Since $\lfloor \lfloor na \rfloor \mathbf{x} \rfloor \leq \lfloor na\mathbf{x} \rfloor \leq \lfloor \lceil na \rceil \mathbf{x} \rfloor$ we obtain

$$a\bar{\phi}(\mathbf{x}) = \liminf_{n\to\infty} \frac{\lfloor na \rfloor}{n} \frac{\phi(\lfloor \lfloor na \rfloor \mathbf{x})\rfloor}{\lfloor na \rfloor} \leq \liminf_{n\to\infty} \frac{\phi(\lfloor na\mathbf{x} \rfloor)}{n}$$
$$\leq \limsup_{n\to\infty} \frac{\phi(\lfloor na\mathbf{x} \rfloor)}{n} \leq \limsup_{n\to\infty} \frac{\lceil na \rceil}{n} \frac{\phi(\lfloor \lceil na \rceil \mathbf{x})\rfloor}{\lceil na \rceil} = a\bar{\phi}(\mathbf{x}).$$

Clearly $\bar{\phi}$ is nondecreasing. Hence for $0 < r < 1$, $\bar{\phi}(\mathbf{x}) \geq \bar{\phi}(r\mathbf{x}) = r\bar{\phi}(\mathbf{x})$ and letting $r \downarrow 0$ proves nonnegativeness.

(b) From (a) and $\mathbb{N}^d \subseteq \mathfrak{D}_\phi$, $\mathbb{Q}^d_{>0} \subseteq \mathfrak{D}_\phi$. Let $\mathbf{x} \in \mathbb{R}^d_{>0}$. Choose sequences $\{\mathbf{x}_k\}$ of $\mathbb{Q}^d_{>0}$ with $\mathbf{x}_k \uparrow \mathbf{x}$ and $\epsilon_k \downarrow 0$ such that $\mathbf{x}(1-\epsilon_k) < \mathbf{x}_k < \mathbf{x}(1+\epsilon_k)$. This is possible since $\mathbf{x} > \mathbf{0}$ and $\mathbb{Q}^d_{>0}$ is dense. We obtain $\mathbf{x}_k/(1+\epsilon_k) < \mathbf{x} < \mathbf{x}_k/(1-\epsilon_k)$, from which

$$\frac{1}{1+\epsilon_k}\bar{\phi}(\mathbf{x}_k) \leq \liminf_{n\to\infty} \frac{1}{n}\phi(\lfloor n\mathbf{x} \rfloor) \leq \limsup_{n\to\infty} \frac{1}{n}\phi(\lfloor n\mathbf{x} \rfloor) \leq \frac{1}{1-\epsilon_k}\bar{\phi}(\mathbf{x}_k)$$

Since $\bar{\phi}(\mathbf{x}_k)$ is nondecreasing letting $k \to \infty$ gives $\lim_{n\to\infty} \frac{1}{n}\phi(\lfloor n\mathbf{x} \rfloor) = \lim_{k\to\infty} \bar{\phi}(\mathbf{x}_k)$.

(c) Fix $0 < a < 1 < b$. For all sufficiently large n, $\lfloor na\mathbf{x} \rfloor < \mathbf{x}_n < \lfloor bn\mathbf{x} \rfloor$. Hence (a) and (b) give

$$a\bar{\phi}(\mathbf{x}) \leq \liminf_{n\to\infty} \frac{1}{n}\phi(\mathbf{x}_n) \leq \limsup_{n\to\infty} \frac{1}{n}\phi(\mathbf{x}_n) \leq b\bar{\phi}(\mathbf{x}).$$

$a, b \to 1$ proves (c).

(d) Suppose $\bar{\phi}(\mathbf{x}_0) = \infty$. Given $\mathbf{x}$, considering a ray connecting the origin and $\mathbf{x}$, choose $\mathbf{y}$ on the ray with $\mathbf{x}_0 \le \mathbf{y}$. Since $\bar{\phi}$ is nondecreasing and homogeneous we have $\bar{\phi}(\mathbf{x}) = (|\mathbf{x}|_1/|\mathbf{y}|_1)\bar{\phi}(\mathbf{y}) = \infty$.

(e) Let $\mathbf{x} \in \mathbb{R}^d_{>0}$ and $\epsilon > 0$. Consider an open box centered at $\mathbf{x}$, $\quad B_\epsilon = \{\mathbf{y} \in \mathbb{R}^d_{>0} : \mathbf{x}(1-\epsilon) < \mathbf{y} < \mathbf{x}(1+\epsilon)\}$. $\sup_{B_\epsilon} \bar{\phi} - \inf_{B_\epsilon} \bar{\phi} = \bar{\phi}(\mathbf{x}(1+\epsilon)) - \bar{\phi}(\mathbf{x}(1-\epsilon)) = 2\epsilon\bar{\phi}(\mathbf{x})$. $\epsilon \downarrow 0$ implies the continuity of $\bar{\phi}$ at $\mathbf{x}$. Define $[\bar{\phi}](\mathbf{x}) = \inf\{\bar{\phi}(\mathbf{y}) : \mathbf{y} > \mathbf{x}\}$ for $\mathbf{x} \in \mathbb{R}^d_+$. Then $[\bar{\phi}] = \bar{\phi}$ on $\mathbb{R}^d_{>0}$ by continuity and monotonicity of $\bar{\phi}$. $[\bar{\phi}]$ is nondecreasing and positive-homogeneous on $\mathbb{R}^d_+$. For $\mathbf{x} \in \partial\mathbb{R}^d_+$ let

$$C_\epsilon = \{\mathbf{y} \in \mathbb{R}^d_+ : \mathbf{x}(1-\epsilon) < \mathbf{y} < \mathbf{x} + \epsilon\mathbf{1}\}$$

where $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^d_+$. Since $[\bar{\phi}]$ is nondecreasing and positive-homogeneous,

$$\sup_{C_\epsilon}[\bar{\phi}] - \inf_{C_\epsilon}[\bar{\phi}] = [\bar{\phi}](\mathbf{x}+\epsilon\mathbf{1}) - [\bar{\phi}](\mathbf{x}(1-\epsilon)) = \bar{\phi}(\mathbf{x}+\epsilon\mathbf{1}) - [\bar{\phi}](\mathbf{x}) + \epsilon[\bar{\phi}](\mathbf{x}).$$

Let $\epsilon \downarrow 0$ and use the definition of $[\bar{\phi}]$ to get continuity at $\mathbf{x}$.

The second assertion is obvious.

□

We extend this result to the superadditive case.

Proposition 2.30. *Suppose a doubly indexed real sequence* $x_{\mathbf{u},\mathbf{v}}$, $(\mathbf{u} \le \mathbf{v} \in \mathbb{Z}^d_+)$ *satisfies:*

(a) *x is superadditive.*

$$x_{\mathbf{0},\mathbf{u}+\mathbf{v}} \ge x_{\mathbf{0},\mathbf{u}} + x_{\mathbf{u},\mathbf{u}+\mathbf{v}}, \tag{2.42}$$

(b) *There is a linear function $\psi : \mathbb{R}^d \to \mathbb{R}$ such that $x_{\mathbf{u},\mathbf{u}+\mathbf{v}} \ge \psi(\mathbf{v})$ for $\mathbf{u}$, $\mathbf{v} \in \mathbb{Z}^d_+$.*

Define $\phi(\mathbf{u}) = x_{\mathbf{0},\mathbf{u}}$. Then the following hold.

(1) $\phi - \psi$ is nondecreasing.

(2) If $\mathbb{N}^d \subseteq \mathfrak{D}_\phi$, then $\bar{\phi}$ satisfies the properties of Proposition 2.29 except the monotonicity. (But $\bar{\phi} - \psi$ is nondecreasing and $\bar{\phi} \geq \psi$.)

(3) Moreover if $\bar{\phi}$ is superadditive on $\mathbb{N}^d$: $(\bar{\phi}(\mathbf{u}+\mathbf{v}) \geq \bar{\phi}(\mathbf{u}) + \bar{\phi}(\mathbf{v}))$, then $\bar{\phi}$ is superadditive, concave on $\mathbb{R}^d_{>0}$.

Proof. (1) We prove $\phi - \psi$ is nondecreasing. From (b) we have

$$(\phi(\mathbf{u}+\mathbf{v}) - \psi(\mathbf{u}+\mathbf{v})) - (\phi(\mathbf{u}) - \psi(\mathbf{u})) = (\phi(\mathbf{u}+\mathbf{v}) - \phi(\mathbf{u})) - (\psi(\mathbf{u}+\mathbf{v}) - \psi(\mathbf{u}))$$

$$\geq x_{\mathbf{u},\mathbf{u}+\mathbf{v}} - \psi(\mathbf{v}) \geq 0$$

Therefore it follows that $\phi - \psi$ is nondecreasing.

(2) This part is an immediate consequence of (1) and Proposition 2.29. For part (3), from continuity and homogeneity of $\bar{\phi}$, we have

$$\bar{\phi}(\mathbf{x}+\mathbf{y}) = \lim_{n\to\infty} \frac{1}{n}\bar{\phi}(\lfloor n\mathbf{x}\rfloor + \lfloor n\mathbf{y}\rfloor)$$
$$\geq \lim_{n\to\infty} \frac{1}{n}\bar{\phi}(\lfloor n\mathbf{x}\rfloor) + \lim_{n\to\infty} \frac{1}{n}\bar{\phi}(\lfloor n\mathbf{y}\rfloor) = \bar{\phi}(\mathbf{x}) + \bar{\phi}(\mathbf{y}) \quad \text{on } \mathbb{R}^d_{>0}. \tag{2.43}$$

Superadditivity and positive-homogeneity give concavity. □

2.4.4 Proofs of the main results

We apply the previous development to prove Theorem 2.9 and Corollary 2.10. The conditional law of ω conditioned on ρ defines a stochastic kernel from Ω_0 to Ω_1 by $\kappa(\rho) = \mathbb{P}^\rho$. κ is an ergodic kernel relative to $T_\mathbf{u}$ for all $\mathbf{u} > \mathbf{0}$ in $\mathbb{Z}^d_+$ by the Kolmogorov 0-1 law. Therefore by Lemma 2.15, $\mathbb{P}$ is totally ergodic with respect to $\hat{T}$ and $\mathbb{P}$ has stationary mean $\mathbb{P}_0 = \mathcal{Q}_0\kappa$ under Assumption 2.4. Hence we may apply Theorem 2.17 to $\mathbb{P}$ to prove limit theorems under Assumption 2.6. However we do not take this approach because to establish simultaneous limit theorems for uncountably many directions is more difficult in this setting and, more importantly, we cannot use this approach if $\mathcal{Q}$ does not satisfy Assumptions 2.4 and 2.6 (see Theorem 3.17). Therefore first we prove limit theorems for parameters ρ using the nonstationary subadditive ergodic theorem and then combine this with results from concentration inequalities to obtain limit theorems for ω.

Theorem 2.31. *Suppose Assumptions 2.4 and 2.6 hold. Define* $Y_{\mathbf{u},\mathbf{v}}(\rho) = \mathbb{E}^\rho \log Z_{\mathbf{u},\mathbf{v}}$ *for* $\mathbf{u} \le \mathbf{v}$ *in* $\mathbb{Z}_+^d$ *and* $\rho \in \Omega_0$*. Then* $\mathcal{Q}$*-a.s. the scaling limit of* $\phi_Y(\mathbf{u}) = Y_{\mathbf{0},\mathbf{u}}$ $(\mathbf{u} \in \mathbb{Z}_+^d)$ *exists for all* $\mathbf{x} \in \mathbb{R}_{>0}^d$ *and is deterministic. The scaling limit* $\bar{\phi}_Y$ *satisfies*

$$\bar{\phi}_Y(\mathbf{x}) = \lim_{n\to\infty} \frac{1}{n} E^{\mathcal{Q}_0}\big(Y_{\mathbf{0},\lfloor n\mathbf{x}\rfloor}\big) = \lim_{n\to\infty} \frac{1}{n} E^{\mathcal{Q}}\big(Y_{\mathbf{0},\lfloor n\mathbf{x}\rfloor}\big).$$

This limit function $\bar{\phi}_Y$ *is continuous, positive-homogeneous, superadditive, concave on* $\mathbb{R}_{>0}^d$ *and extends continuously to* $\mathbb{R}_+^d$*.*

Proof. For fixed $\mathbf{u} \in \mathbb{N}^d$ define $X_n(\omega) = \log Z^\omega_{\mathbf{0},n\mathbf{u}}$ $(\omega \in \Omega_0)$. Then this sequence is superadditive:

$$X_{n+m}(\omega) = \log Z^\omega_{\mathbf{0},(n+m)\mathbf{u}} \ge \log Z^\omega_{\mathbf{0},m\mathbf{u}} + \log Z^\omega_{m\mathbf{u},(m+n)\mathbf{u}} = X_m(\omega) + X_n(T^m_{\mathbf{u}}\omega) \quad (2.44)$$

Averaging this sequence with respect to $\mathbb{P}^\rho$ $(\rho \in \Omega_0)$ we obtain $Y_n(\rho) \triangleq \mathbb{E}^\rho X_n$. The new sequence is also superadditive :

$$Y_{n+m}(\rho) \ge Y_m(\rho) + Y_n(T^m_{\mathbf{u}}\rho) \quad (2.45)$$

Since we assumed that $|\omega_{\mathbf{x}}|$ are stochastically bounded by η_0 (see the condition (b) below (2.1)), Lemma 2.18(b) and (2.21) give

$$-E^F N(n|\mathbf{u}|_1 + 1) \le Y_n(\rho) \le n|\mathbf{u}|_1 \log d + E^F N(n|\mathbf{u}|_1 + 1), \quad (2.46)$$

where F is the CDF of η_0 and E^F refers to $F^{\otimes \mathbb{Z}_+^d}$. By Theorem 2.20, $Y_n(\rho)/n$ is bounded below with lower bound $-C|u|_1$ for some constant C and bounded above with upper bound $(C + \log d)|u|_1$. $\mathcal{Q}$ is AMS with stationary mean $\mathcal{Q}_0$ relative to $T_{\mathbf{u}}$ by Assumption 2.4. We can apply Theorem 2.17 to ρ and $T_{\mathbf{u}}$. Note that in the case of Assumption 2.6(b), $Y_{\mathbf{u},\mathbf{v}}(\rho)$ is a monotone increasing function of ρ by Lemma 2.18(a). Therefore $\mathcal{Q}(A) = 1$ where $A = \{\rho \in \Omega_0 : n^{-1}\mathbb{E}^\rho \log Z_{\mathbf{0},n\mathbf{u}}$ converges for all $\mathbf{u} \in \mathbb{N}^d\}$. Since $\mathcal{Q}_0$ is assumed to be totally ergodic with respect to $\hat{T}$, these limits are deterministic and satisfy

$$\bar{\phi}_Y(\mathbf{u}) = \lim_{n\to\infty} E^{\mathcal{Q}_0}\big(\frac{1}{n}Y_{\mathbf{0},n\mathbf{u}}\big) = \lim_{n\to\infty} \frac{1}{n}Y_{\mathbf{0},n\mathbf{u}} = \lim_{n\to\infty} E^{\mathcal{Q}}\big(\frac{1}{n}Y_{\mathbf{0},n\mathbf{u}}\big)$$

for $\mathbf{u} \in \mathbb{N}^d$ $\mathcal{Q}$-a.s. by Theorem 2.17. The last equality comes from the dominated convergence theorem. Note that $\bar{\phi}_Y$ is superadditive on $\mathbb{N}^d$ since its formula is given by a $\hat{T}$-stationary measure $\mathcal{Q}_0$. Therefore we can use Proposition 2.30 since $\mathcal{Q}$-a.s., for all $\mathbf{u}, \mathbf{v}$, $Y_{\mathbf{u},\mathbf{v}}$ are bounded below by $-C|\mathbf{v}-\mathbf{u}|_1$ (by the same reasoning as in (2.46)). This yields the stated properties for $\bar{\phi}_Y$. □

Proof of Theorem 2.9. Define $X_{\mathbf{u},\mathbf{v}}(\omega) = \log Z^\omega_{\mathbf{u},\mathbf{v}}$ and $\phi(\mathbf{u}) = X_{\mathbf{0},\mathbf{u}}$ for $\mathbf{u} \le \mathbf{v}$ in $\mathbb{Z}_+^d$ and $\omega \in \Omega_1$. In Corollary 2.27 we showed that for ν-a.e. (ρ, ω), for all $\mathbf{x} \le \mathbf{y}$ in $\mathbb{R}_+^d$

$$\lim_{n\to\infty}\Big(\frac{\log Z^\omega_{\lfloor n\mathbf{x}\rfloor,\lfloor n\mathbf{y}\rfloor}}{n} - \frac{\mathbb{E}^\rho \log Z_{\lfloor n\mathbf{x}\rfloor,\lfloor n\mathbf{y}\rfloor}}{n}\Big) = 0, \tag{2.47}$$

and in Theorem 2.31 we showed $\mathcal{Q}$-a.s. and hence ν-a.s.,

$$\lim_{n\to\infty} \frac{\mathbb{E}^\rho \log Z^\omega_{\mathbf{0},\lfloor n\mathbf{x}\rfloor}}{n} = \bar{\phi}_Y(\mathbf{x}). \tag{2.48}$$

for a deterministic function $\bar{\phi}_Y$ and $\mathbf{x} > \mathbf{0}$ in $\mathbb{R}_+^d$. Therefore $\bar{\phi}$ as claimed in Theorem 2.9 exists ν-a.s. for all $\mathbf{x} > \mathbf{0}$ and agrees with $\bar{\phi}_Y$. The properties of $\bar{\phi}$ are given in Theorem 2.31. □

Proof of Corollary 2.10. Let ϵ be a positive rational number. Then choose rational points $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_M\}$ on the hyperplane $|\mathbf{x}|_1 = 1+\epsilon$ and $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_M\}$ on the hyperplane $|\mathbf{x}|_1 = 1-\epsilon$ so that $|\mathbf{v}_i - \mathbf{u}_i|_1 = 2\epsilon$ and boxes $[\mathbf{u}_i, \mathbf{v}_i]$ cover the hyperplane $|\mathbf{x}|_1 = 1$ in the first orthant ($\mathbf{0} \le \mathbf{u}_i \le \mathbf{v}_i$, $\mathbf{v}_i > \mathbf{0}$). Then $\{[n\mathbf{u}_i, n\mathbf{v}_i]\}$ cover the hyperplane $|\mathbf{x}|_1 = n$. For points $\mathbf{x} \in \mathbb{Z}_+^d$ in $[n\mathbf{u}_i, n\mathbf{v}_i]$ we have $\lfloor n\mathbf{u}_i \rfloor \le \mathbf{x} \le \lfloor n\mathbf{v}_i \rfloor$ and $\log Z^\omega_{\mathbf{0},\mathbf{x}} \le \log Z^\omega_{\mathbf{0},\lfloor n\mathbf{v}_i\rfloor} + G^{|\omega|}_{\lfloor n\mathbf{u}_i\rfloor,\lfloor n\mathbf{v}_i\rfloor}$.

Therefore

$$\begin{aligned}\log Z^\omega_n &= \log \sum_{\mathbf{x}\in\mathbb{Z}_+^d : |\mathbf{x}|_1 = n} Z^\omega_{\mathbf{0},\mathbf{x}} \le \log\big((n+1)^d \max_{\mathbf{x}\in\mathbb{Z}_+^d : |\mathbf{x}|_1 = n} Z^\omega_{\mathbf{0},\mathbf{x}}\big) \\ &\le d\log(n+1) + \max_{1\le i\le M} \log Z^\omega_{\mathbf{0},\lfloor n\mathbf{v}_i\rfloor} + \max_{1\le i\le M} G^{|\omega|}_{\lfloor n\mathbf{u}_i\rfloor,\lfloor n\mathbf{v}_i\rfloor}.\end{aligned} \tag{2.49}$$

Divide by n, and take $n\to\infty$ to conclude that ν-a.s.,

$$\begin{aligned}\limsup_{n\to\infty} \frac{\log Z^\omega_n}{n} &\le \max_{1\le i\le M} \lim_{n\to\infty} \frac{\log Z^\omega_{\mathbf{0},\lfloor n\mathbf{v}_i\rfloor}}{n} + \max_{1\le i\le M} \limsup_{n\to\infty} \frac{G^{|\omega|}_{\lfloor n\mathbf{u}_i\rfloor,\lfloor n\mathbf{v}_i\rfloor}}{n} \\ &\le \max_{1\le i\le M} \bar{\phi}(\mathbf{v}_i) + \max_{1\le i\le M} c_0 k |\mathbf{v}_i - \mathbf{u}_i|_1 \\ &\le (1+\epsilon) \max_{\mathbf{x}:|\mathbf{x}|_1=1} \bar{\phi}(\mathbf{x}) + 2\epsilon c_0 k\end{aligned} \tag{2.50}$$

where

$$k = \int_0^\infty (1-F(t))^{1/d} dt.$$

The inequality $\limsup_{n\to\infty} n^{-1} G^{|\omega|}_{\lfloor n\mathbf{u}_i\rfloor,\lfloor n\mathbf{v}_i\rfloor} \le c_0 k |\mathbf{v}_i - \mathbf{u}_i|_1$ is justified as follows. From the coupling (2.15) we have $G^{|\omega|}_{\lfloor n\mathbf{u}_i\rfloor,\lfloor n\mathbf{v}_i\rfloor} \le G^{\eta}_{\lfloor n\mathbf{u}_i\rfloor,\lfloor n\mathbf{v}_i\rfloor}$ and

$$\limsup_{n\to\infty} \frac{G^{|\omega|}_{\lfloor n\mathbf{u}_i\rfloor,\lfloor n\mathbf{v}_i\rfloor}}{n} \le \lim_{n\to\infty} \Big(\frac{G^{\eta}_{\lfloor n\mathbf{u}_i\rfloor,\lfloor n\mathbf{v}_i\rfloor}}{n} - \frac{E^F G_{\lfloor n\mathbf{u}_i\rfloor,\lfloor n\mathbf{v}_i\rfloor}}{n}\Big) + \lim_{n\to\infty} \frac{E^F G_{\mathbf{0},\lfloor n\mathbf{v}_i\rfloor - \lfloor n\mathbf{u}_i\rfloor}}{n}. \tag{2.51}$$

Corollary 2.27 applied to η, G, and F (see Remark 2.28) with Theorem 2.20 gives the result.

For the lower bound note that for $\mathbf{x} \in \mathbb{R}^d_{>0}$ with $|\mathbf{x}|_1 = 1$,

$$\log Z^{\omega}_{\mathbf{0},\lfloor n\mathbf{x}\rfloor} + \log Z^{\omega}_{\lfloor n\mathbf{x}\rfloor,\mathbf{u}} \leq \log Z^{\omega}_n$$

where $\mathbf{u} \in \mathbb{Z}^d_+$ and $\lfloor n\mathbf{x}\rfloor \leq \mathbf{u} \leq \lceil n\mathbf{x}\rceil$ with $|\mathbf{u}|_1 = n$. Hence we have

$$\lim_{n\to\infty} \frac{\log Z^{\omega}_{\mathbf{0},\lfloor n\mathbf{x}\rfloor}}{n} \leq \liminf_{n\to\infty} \frac{\log Z^{\omega}_n}{n} + \lim_{n\to\infty} \frac{E^F G_{\mathbf{0},\lceil n\mathbf{x}\rceil - \lfloor n\mathbf{x}\rfloor}}{n}$$

ν-a.s. The second term comes as in (2.51). Therefore

$$\bar{\phi}(\mathbf{x}) \leq \liminf_{n\to\infty} \frac{1}{n} \log Z^{\omega}_n.$$

and, since $\mathbf{x}$ is arbitrary we have $\lim_{n\to\infty} \frac{1}{n} \log Z^{\omega}_n = \max_{\mathbf{x}|\mathbf{x}|_1=1} \bar{\phi}(\mathbf{x})$ ν-a.s. □

Chapter 3

Limiting free energy for two-dimensional polymers

3.1 Introduction

In this chapter, we focus on 2-dimensional directed polymers. We give a more precise picture of the limit shape $\bar{\phi}$ and show the existence of limiting free energy for some special types of models. In two dimension, we have conditions to guarantee the existence of limiting free energy which are more natural and useful than Assumption 2.6.

For a random variable X with the distribution $F_2(r, \cdot)$, let $F_2(r)$, $r \in \mathbf{S}$, be the expectation of X:

$$F_2(r) = EX = \int_{\mathbb{R}} x F_2(r, dx). \tag{3.1}$$

Assumption 3.1. *Let $\mathcal{Q}$ be the distribution of parameters $\rho = \{\rho_{i,j}\} \in \Omega_0 = \mathbf{S}^{\mathbb{Z}_+^2}$ and F_2 be as in Assumption 2.1. For $\mathcal{Q}$-a.e. ρ, the limits*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} F_2(\rho_{i,k}), \quad \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} F_2(\rho_{k,j}) \tag{3.2}$$

exist for all i, $j \in \mathbb{Z}_+$ and satisfy

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} F_2(\rho_{i,k}) \leq \limsup_{m\to\infty} \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} F_2(\rho_{m,k}) \tag{3.3}$$

and

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}F_2(\rho_{k,j})\le\limsup_{m\to\infty}\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}F_2(\rho_{k,m})\tag{3.4}$$

for all i and $j\in\mathbb{Z}_+$.

Remark 3.2. *A subadditive ergodic theorem is applicable to these models as we will see soon. In Definition 2.3, $\mathcal{Q}$ is $T_{\mathbf{u}}$-AMS only for $\mathbf{u}>\mathbf{0}$ with a common stationary mean $\mathcal{Q}_0$. (3.2) imposes some sort of AMS properties for $\mathbf{u}=\mathbf{e}_1$ and $\mathbf{e}_2$. These conditions are quite natural since if $\mathcal{Q}$ is stationary, these conditions are satisfied not only for F_2 but also for any bounded function $f:\mathbf{S}\to\mathbb{R}$.*

We give some models that satisfy Assumptions 2.4 and 3.1. We change the picture slightly. Our model lives in $\mathbb{N}^2$. Choose parameters $\Lambda=\{\lambda_i\}_{i=1}^{\infty}$ and $\Theta=\{\theta_j\}_{j=1}^{\infty}$. (Λ,Θ) is in $S_0\triangleq[a_0,a_1]^{\mathbb{N}}\times[b_0,b_1]^{\mathbb{N}}\simeq([a_0,a_1]\times[b_0,b_1])^{\mathbb{N}}$ for some $a_0<a_1$ and $b_0<b_1$ in $\mathbb{R}$. S_0 is equipped with the product Borel σ-algebra $\mathcal{G}_0$ generated by coordinate projections. We introduce notations to fit this example into our general setting. Shift maps $\tau_{k,l}:S_0\to S_0$ act on S_0 for $k,l\in\mathbb{Z}_+$ by $(\{\lambda_i\}_{i=1}^{\infty},\{\theta_j\}_{j=1}^{\infty})\mapsto(\{\lambda_{i+k}\}_{i=1}^{\infty},\{\theta_{j+l}\}_{j=1}^{\infty})$. Then $\hat{\tau}=\{\tau_{\mathbf{x}}\}_{\mathbf{x}\in\mathbb{Z}_+^2}$ is a semigroup.

Assumption 3.3. *We consider parameters $(\Lambda,\Theta)\in S_0$. We denote the distribution of (Λ,Θ) by $\mathcal{Q}$. $\mathcal{Q}$ and the conditional law of weights $\omega=\{\omega_{\mathbf{x}}:\mathbf{x}\in\mathbb{N}^2\}$ given (Λ,Θ) satisfy the following:*

(a) $\mathcal{Q}$ is AMS with respect to $\hat{\tau}$ (see Definition 2.3). We denote the stationary mean of $\mathcal{Q}$ by $\mathcal{Q}_0$.

(b) The weight parameters $\rho=\{\rho_{\mathbf{x}}:\mathbf{x}\in\mathbb{N}^2\}\in\mathbb{R}^{\mathbb{N}^2}$ are determined by (Λ,Θ): The weight parameter at site $(i,j)\in\mathbb{N}^2$ is $\rho_{i,j}=\gamma(\lambda_i,\theta_j)$, where $\gamma:[a_0,a_1]\times[b_0,b_1]\to$

$\mathbb{R}$ is a fixed function. We assume that γ is a continuous function that monotonically increases in each coordinate ($x \le y$ implies $\gamma(x,z) \le \gamma(y,z)$ and $\gamma(z,x) \le \gamma(z,y)$).

(c) The conditional law of ω given (Λ, Θ) is given by (2.2):

$$P(\omega \in \cdot \,|\, \Theta, \Lambda) = \mathbb{P}^{\rho} = \otimes_{\mathbf{x} \in \mathbb{N}^2} F_2(\rho_{\mathbf{x}}, \cdot).$$

(d) F_2 in part (c) satisfies the following. $F_2 : \mathbb{R} \times \mathbb{R} \to [0,1]$ is a monotonically increasing function in the first variable. Let α and β denote the distributions of λ_1 and θ_1 under $\mathcal{Q}_0$, respectively. We assume $a_0 = \inf \operatorname{supp} \alpha$ and $b_0 = \inf \operatorname{supp} \beta$. Here, for a probability measure P, supp P is the support of P. If $\alpha(\{a_0\}) = 0$ then we require that $\lambda_i > a_0$ for all i, $\mathcal{Q}$-a.s. Similarly if $\beta(\{b_0\}) = 0$ then we require that $\theta_j > b_0$ for all j, $\mathcal{Q}$-a.s.

(d') Alternatively, F_2 is a monotonically decreasing function in the first variable, and we assume $a_1 = \sup \operatorname{supp} \alpha$ and $b_1 = \sup \operatorname{supp} \beta$. We have similar conditions on boundary points as (d).

We give more details of how this procedure connects our main results and these two-dimensional models in Section 3.3. Here are some examples of $\mathcal{Q}_0$ and $\mathcal{Q}$.

Example 3.4 (Examples of AMS measures and their stationary means)**.** (a) *Let*

$$\mathcal{Q}_0 = \alpha^{\otimes \mathbb{N}} \otimes \beta^{\otimes \mathbb{N}}$$

where α and β are probability measures on $[a_0, a_1]$ and $[b_0, b_1]$ respectively. $\mathcal{Q}_0$ is totally ergodic with respect to $\hat{\tau}$ by Kolmogorov 0-1 law. For $\mathcal{Q}$, choose any nonnegative function $f : S_0 \to \mathbb{R}$ with $E^{\mathcal{Q}_0} f = 1$ and take $d\mathcal{Q} = f d\mathcal{Q}_0$.

(b) *Suppose* (Λ, Θ) *is deterministic and* L*-periodic:* $L \in \mathbb{N}$ *and*

$$\tau_{L,0}(\Lambda, \Theta) = \tau_{0,L}(\Lambda, \Theta) = (\Lambda, \Theta).$$

Then

$$\mathcal{Q}_0 = \frac{1}{L^2} \sum_{0 \le k,l < L} \delta_{\tau_{k,l}(\Lambda,\Theta)}$$

is stationary and totally ergodic with respect to $\hat{\tau}$*. In this case take* $\mathcal{Q} = \delta_{(\Lambda,\Theta)}$*. Note that an i.i.d. environment* ω *belongs to this example with* $L = 1$*.*

(c) *Consider independent time homogeneous irreducible, aperiodic Markov chains* X_n *on a countable set* $S_1 \subseteq [a_0, a_1]$ *and* Y_n *on a countable set* $S_2 \subseteq [b_0, b_1]$ *with transition probabilities* p *and* q *respectively* $(n \geq 1)$*. Assume* p *and* q *have stationary distributions* π_X*,* π_Y *and initial distributions of* X_n*,* Y_n *are* π_X*,* π_Y *respectively. Let* $Z_n = (X_n, Y_n)$ *then* Z_n *is a Markov chain on* $S_1 \times S_2$ *with the transition probability* $\bar{p}((x_1, y_1), (x_2, y_2)) = p(x_1, x_2)q(y_1, y_2)$*.* $\bar{p}$ *has a stationary distribution* $\pi = \pi_X \otimes \pi_Y$ *Let* $\mathcal{Q}_0$ *be the distribution of* $(\{X_n\}_{n\geq 1}, \{Y_n\}_{n \geq 1}) \simeq Z_n$*. Then clearly* $\mathcal{Q}_0$ *is stationary. We claim that* $\mathcal{Q}_0$ *is ergodic with respect to* τ_u *for any* $u = (u_1, u_2) > 0$*. The proof is similar to that given in Example 7.1.7 [16].*

Let $\mathcal{F}_n = \sigma(X_1, \ldots, X_{nu_1}, Y_1, \ldots, Y_{nu_2})$*. Let* $W_n = (X_{nu_1}, Y_{nu_2})$*.* W_n *is a Markov chain on* $S_1 \times S_2$ *with respect to* $\mathcal{F}_n$ *and its transition probability is given by* $\bar{p}_W((x_1, y_1), (x_2, y_2)) = p^{u_1}(x_1, x_2)q^{u_2}(y_1, y_2)$*. Then* W_n *is irreducible since* X_n*,* Y_n *are aperiodic. And* $\bar{p}_W$ *has a stationary distribution* π*. (See the proof of Theorem 6.6.4.[16]) If* $A \in \mathcal{I}_u$*,* $1_A \circ \tau_u^n = 1_A$*. So the shift invariance of* 1_A*, independence of* X_n *and* Y_n*, and the Markov property imply*

$$E^{P_0}(1_A|\mathcal{F}_n) = E^{\pi}(1_A \circ \tau_u^n|\mathcal{F}_n) = h(W_n)$$

where $h(x) = E^x 1_A$, $x \in S_1 \times S_2$, *and* P^x *is the distribution of* Z_n *started at* x. (*Note that we do not use the distribution of* W_n.) *Lévy's 0-1 law gives that the left-hand side converges to* 1_A *as* $n \to \infty$. (*Here we used* $u > 0$.) *On the other hand since* W_n *is irreducible and recurrent* $\mathcal{Q}_0$*-a.s., for any* $y \in S_1 \times S_2$ *the right-hand side of the above equation is* $h(y)$ *i.o., so either* $h(x) \equiv 0$ *or* $h(x) \equiv 1$, *and* $\mathcal{Q}_0(A) =$ *0 or 1. In this case, we may take* $\mathcal{Q}$ *as the distribution of* $(\{X_n\}_{n\geq 1}, \{Y_n\}_{n\geq 1})$ *for any initial distributions.*

(d) *Take* $\mathcal{Q}_0$ *as in preceding examples. For* $a_2 > a_1$, $b_2 > b_1$ *and* $N > 0$ *take any probability measure* $\mathcal{Q}_1$ *on* $([a_1, a_2] \times [b_1, b_2])^N$. *Let* $\mathcal{Q} = \mathcal{Q}_1 \otimes \mathcal{Q}_0$. *Note that* $\mathcal{Q}$ *is not absolutely continuous with respect to* $\mathcal{Q}_0$ *but it is stochastically larger than* $\mathcal{Q}_0$.

3.2 Results

In this section, we present our results: First, the existence of the limiting free energy for models introduced in the previous section. Second, boundary values of limiting free energy. Finally, examples of log-gamma polymer with a variational characterization of the shape of the limiting free energy. At the end of this section, we briefly explain how and where these results are proved.

For models satisfying Assumption 3.1, a subadditive ergodic theorem can be used without Assumption 2.6.

Theorem 3.5. *Suppose Assumptions 2.4 and 3.1 holds. Then the conclusion of Theorem 2.9 holds.*

Theorem 3.6. *Suppose Assumption 3.3 holds. Then the conclusion of Theorem 2.9* *holds. Furthermore, the continuous extension of $\bar{\phi}$ satisfies the boundary condition in terms of essential supremums under $\mathcal{Q}_0$ in Assumption 3.3:*

$$\bar{\phi}(1,0) = \beta\text{-}\operatorname*{ess\,sup}_{\theta_1} \int F_2(\gamma(\lambda, \theta_1))\, \alpha(d\lambda) \tag{3.5}$$

and

$$\bar{\phi}(0,1) = \alpha\text{-}\operatorname*{ess\,sup}_{\lambda_1} \int F_2(\gamma(\lambda_1, \theta))\, \beta(d\theta) \tag{3.6}$$

where α and β are distributions of λ_1 and θ_1 under $\mathcal{Q}_0$.

Finally, we give results for the inhomogeneous log-gamma polymer.

Theorem 3.7. *Consider the log-gamma polymer introduced in Section 1.3. Suppose $\mathcal{Q}$, the distribution of (Λ, Θ), is $\hat{\tau}$-AMS and we also have boundary conditions on a_0 and b_0 as explained in Assumption 3.3(d). Then Theorem 3.6 is applicable. Furthermore, a variational formula holds*

$$\lim_{n\to\infty} \frac{1}{n} \log Z^{\omega}_{(1,1),(\lfloor nx\rfloor, \lfloor ny\rfloor)} = \bar{\phi}(x,y) = \inf_{-a_0<z<b_0} \{xA(z) + yB(z)\} \tag{3.7}$$

for any x, $y > 0$. $A(z)$ and $B(z)$ are defined on $(-a_0, b_0)$ and

$$\begin{aligned} A(z) &= -\int_{(0,\infty)} \Psi_0(z+\lambda)\, \alpha(d\lambda) \\ B(z) &= -\int_{(0,\infty)} \Psi_0(-z+\theta)\, \beta(d\theta). \end{aligned} \tag{3.8}$$

From these explicit formulas, we obtain a more precise picture of limiting shape. Especially, we have some moment conditions for the existence of flat regions S_1 and S_2. We borrow notations from [17].

Let S denote the sector of the first quadrant ($\mathbb{R}^2_{>0}$) on which

$$-B'(-a_0)/A'(-a_0) < x/y < -B'(b_0)/A'(b_0). \tag{3.9}$$

holds, Note that A and B are infinitely many differentiable functions. Let S_1 and S_2 denote the sectors defined by the inequalities $x/y \leq -B'(-a_0)/A'(-a_0)$ and $x/y \geq -B'(b_0)/A'(b_0)$, respectively. We have $\mathbb{R}^2_{>0} = S_1 \cup S_2 \cup S$. Possibly, $S_1 = \emptyset$ or $S_2 = \emptyset$. The following Corollary is analogous to Corollary 2.3 of [17].

Corollary 3.8. *$\bar{\phi}$ extends to $\mathbb{R}^2_+$ and the following hold.*

(a) $\bar{\phi}(x,y) = xA(-a_0) + yB(-a_0)$ *for* $(x,y) \in S_1$.

(b) $\bar{\phi}(x,y) = xA(b_0) + yB(b_0)$ *for* $(x,y) \in S_2$.

(c) $\bar{\phi}(cx_1 + (1-c)x_2, cy_1 + (1-c)y_2) > c\bar{\phi}(x_1,y_1) + (1-c)\bar{\phi}(x_2,y_2)$ *for* $0 < c < 1$ *and* (x_1,y_1), $(x_2,y_2) \in S$ *that are nonparallel.*

(d) *For* $(x,y) \in S$, *there is a unique minimizer* $\zeta \in (-a_0, b_0)$ *in* (3.7). ζ *depends on* x/y *and given by the inverse function of* $-B'(z)/A'(z) > 0$ *for* $z \in (-a_0, b_0)$:

$$-\frac{B'(\zeta(x/y))}{A'(\zeta(x/y))} = \frac{x}{y}. \tag{3.10}$$

(e) $\bar{\phi}$ *is continuously differentiable.*

(f) $$S_1 \neq \emptyset \Leftrightarrow \int \frac{1}{(\lambda - a_0)^2}\,\alpha(d\lambda) < \infty, \quad S_2 \neq \emptyset \Leftrightarrow \int \frac{1}{(\theta - b_0)^2}\,\beta(d\theta) < \infty.$$

For certain choices of α and β, more tractable formulas are possible. One can derive the following formulas from (3.10). First, we introduce some generalized polygamma functions to represent our formulas. Define

$$\Psi_{-1}(x) = \log\Gamma(x), \quad \Psi_{-n}(x) = \frac{1}{(n-2)!}\int_0^x (x-t)^{n-2}\log\Gamma(t)\,dt \text{ for } n \geq 2.$$

These functions are called negapolygamma functions [1] and satisfy $\Psi'_n = \Psi_{n+1}$ for all $n \in \mathbb{Z}$. See A.1 for more properties of polygamma functions. Using integration by parts,

we have

$$A(n,L,y) \triangleq -\frac{n+1}{L^{n+1}}\int_0^L x^n \Psi_0(x+y)\,dx$$
$$= \frac{(-1)^n(n+1)!}{L^{n+1}}\Psi_{-n-1}(y) + (n+1)!\sum_{k=1}^{n+1}\frac{(-1)^k L^{-k}}{(n+1-k)!}\Psi_{-k}(y+L)$$

and

$$A'(n,L,y) \triangleq -\frac{n+1}{L^{n+1}}\int_0^L x^n \Psi_1(x+y)\,dx$$
$$= \frac{(-1)^n(n+1)!}{L^{n+1}}\Psi_{-n}(y) + (n+1)!\sum_{k=1}^{n+1}\frac{(-1)^k L^{-k}}{(n+1-k)!}\Psi_{-k+1}(y+L)$$

for $y, L > 0$ and $n \geq 0$.

Example 3.9. *[Explicit formulas] Some level curves are illustrated in Figure 2 and Figure 3. In Figure 3, the level curve of (a) is strictly convex but not tangential to the axes, and the level curve of (b) is strictly convex only in the middle sector S and flat on the edges.*

(a) $\alpha = \beta = \delta_c$ *for some* $c > 0$.

$$\bar{\phi}(x,y) = -x\Psi_0(\zeta_c(x/y)+c) - y\Psi_0(-\zeta_c(x/y)+c),$$

where ζ_c *is the inverse function of the map* $z \mapsto \frac{\Psi_1(-z+c)}{\Psi_1(z+c)}$, $|z| < c$. *Note that this model is for the i.i.d. environment with a parameter* $\mu = 2c$. *This formula is given in (2.16) of [33].*

(b) α *and* β *are uniform measures on the interval* $[c, c+L]$ *for some* c, $L > 0$.

$$\bar{\phi}(x,y) = -\,x\frac{\log\Gamma\big(\zeta_0(x/y)+c+L\big) - \log\Gamma\big(\zeta_0(x/y)+c\big)}{L}$$
$$-\,y\frac{\log\Gamma\big(-\zeta_0(x/y)+c+L\big) - \log\Gamma\big(-\zeta_0(x/y)+c\big)}{L},$$

where ζ_0 *is the inverse function of the map* $z \mapsto \dfrac{\Psi_0(-z+c+L)-\Psi_0(-z+c)}{\Psi_0(z+c+L)-\Psi_0(z+c)}$, $|z|<c$. *Note that when* $L \to 0$ *we obtain the result in (a).*

(c) *For* $n \geq 1$, $\alpha = \beta = (n+1)(x-c)^n/L^{n+1}\,dx$ *on the interval* $[c, c+L]$ *for some* c, $L > 0$.

$$\bar{\phi}(x,y) = xA_n(\zeta_n(x/y)) + yA_n(-\zeta_n(x/y)) \text{ on } S,$$

where $A_n(z) = A(n,L,c+z)$ *and* ζ_n *is the inverse function of the map* $z \mapsto \dfrac{A'(n,L,c-z)}{A'(n,L,c+z)}$, $|z|<c$. . *For* $n \geq 2$, S_1, S_2 *are nonempty.*

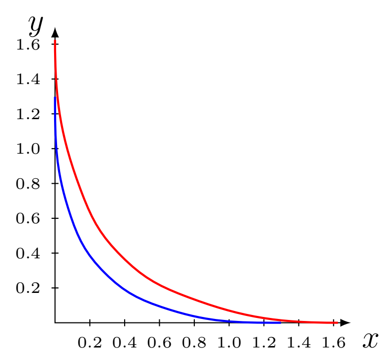
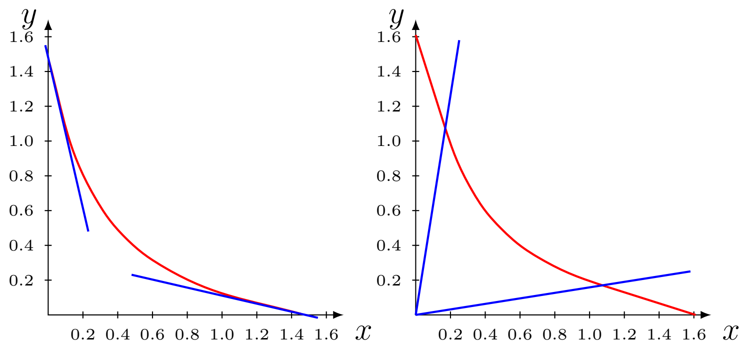


Figure 2: The level curve $\bar{\phi} = 2$ (blue) with $c = 0.3$ and level curve $\bar{\phi} = 2$ (red) with $c = 0.3$, $L = 0.2$ (Uniform measure) in Example 3.9 (a) and (b).

Organization of Chapter 3. Theorems 3.5 and 3.6 are proved in Section 3.3. We show that a nonstationary subadditive ergodic theorem is applicable under assumptions in this chapter. In Section 3.4 we prove Theorem 3.7 and Corollary 3.8. Explicit formulas are derived by precise analysis of *stationary process with boundary conditions* and coupling with these processes are used.



(a) $c = 0.2$, $L = 0.4$, $n = 1$. (b) $c = 0.2$, $L = 0.4$, $n = 2$.

Figure 3: The level curves $\bar{\phi} = 2$ (red) with various α and β in Example 3.9 (c).

3.3 The existence of the limiting free energy

In this section, we prove Theorem 3.5 and Theorem 3.6. The difficult part when we apply a nonstationary subadditive ergodic theorem is that not every superadditive sequence is almost surely invariant. Hence we used conditions (b) and (c) in Theorem 2.17 when we proved Theorem 2.9. For 2-dimensional polymers, it is much easier to prove that the superadditive sequence obtained from free energy is almost surely invariant under weaker assumptions like Assumption 3.1. The key observation is that free energy formed by bulk weights is not much different from the free energy formed by weights including boundary weights as Lemma 3.10 shows.

Let $\mathbf{x}_{\cdot} = (\mathbf{x}_k)_{k\geq 0}$ be a directed path in $\mathbb{Z}_+^2$. We consider $\mathbf{x}_{\cdot}$ as a Markov chain with transition probability $p(\mathbf{x}, \mathbf{x}+\mathbf{e}_1) = p(\mathbf{x}, \mathbf{x}+\mathbf{e}_2) = 1/2$ for $\mathbf{x} \in \mathbb{Z}_+^2$. Denote the distribution of $\mathbf{x}_{\cdot}$ by P. We write $P^{\mathbf{x}}$ for P if $P(\mathbf{x}_0 = \mathbf{x}) = 1$.

For a path segment $\mathbf{x}_{m:n} = (\mathbf{x}_m, \dots, \mathbf{x}_n)$, let $H(\mathbf{x}_{m:n}) = \sum_{m<k\leq n} \omega(\mathbf{x}_k)$. Let $H_{m,n}(\mathbf{x}_{\cdot}) = H(\mathbf{x}_{m:n})$. If $\mathbf{x} \leq \mathbf{y}$ and there exists a unique directed path connecting two points, we denote that path by $\mathbf{x} \to \mathbf{y}$. Concatenation of $\mathbf{x} \to \mathbf{y}$ and $\mathbf{y} \to \mathbf{z}$ is

denoted by $\mathbf{x} \to \mathbf{y} \to \mathbf{z}$.

With this convention, we have, for $\mathbf{u} \leq \mathbf{v} \in \mathbb{Z}_+^2$,

$$\begin{aligned} Z_{\mathbf{u},\mathbf{v}} =& 2^{n-m} E^{\mathbf{u}} \left[1\{\mathbf{x}_{n-m} = \mathbf{v}\} \exp(H_{0,n-m}(\mathbf{x}_{\cdot})) \right] \\ =& 2^{n-m} E^{\mathbf{0}} \left[1\{\mathbf{x}_m = \mathbf{u},\, \mathbf{x}_n = \mathbf{v}\} \exp(H_{m,n}(\mathbf{x}_{\cdot})) \right] \end{aligned} \tag{3.11}$$

where $m = |\mathbf{u}|_1$ and $n = |\mathbf{v}|_1$.

Let $\tilde{Z}_{\mathbf{u},\mathbf{v}} = Z_{\mathbf{u},\mathbf{v}}/2^{|\mathbf{v}-\mathbf{u}|_1}$. $\log \tilde{Z}$ is also superadditive like $\log Z$: for $\mathbf{u} \leq \mathbf{v} \leq \mathbf{w} \in \mathbb{Z}_+^2$, $\log \tilde{Z}_{\mathbf{u},\mathbf{v}} + \log \tilde{Z}_{\mathbf{v},\mathbf{w}} \leq \log \tilde{Z}_{\mathbf{u},\mathbf{w}}$. For fixed $m, n \in \mathbb{N}$, define stopping times T_m and T^n by

$$T_m = \inf\{k \geq 0 : \mathbf{x}_k = (m, r) \quad \text{for some} \quad r \in \mathbb{Z}_+\} \tag{3.12}$$

and

$$T^n = \inf\{k \geq 0 : \mathbf{x}_k = (r, n) \quad \text{for some} \quad r \in \mathbb{Z}_+\}. \tag{3.13}$$

Lemma 3.10. *Let* $\mathbf{u} \leq \mathbf{v} \leq \mathbf{w} \in \mathbb{Z}_+^2$ *and* $\mathbf{v} = (m_1, n_1)$, $\mathbf{w} = (m, n)$. *Then we have*

$$\begin{aligned} \log Z_{\mathbf{u},\mathbf{w}} \leq& \log Z_{\mathbf{v},\mathbf{w}} + |\mathbf{v} - \mathbf{u}|_1 \log 2 \\ &+ \left[\Big(G_{\mathbf{u},(m_1,n)} - \sum_{j=n_1+1}^{n} \omega_{(m_1,j)} \Big) \vee \Big(G_{\mathbf{u},(m,n_1)} - \sum_{i=m_1+1}^{m} \omega_{(i,n_1)} \Big) \right], \end{aligned} \tag{3.14}$$

where G *is the last-passage time* (2.16).

Proof. Without loss of generality, we may assume $\mathbf{u} = \mathbf{0}$. Let $T = T_{m_1} \vee T^{n_1}$. Let $\mathbf{y} = (m_1, n)$ if $\mathbf{x}_T = (m_1, r)$ and $\mathbf{y} = (m, n_1)$ otherwise. By strong Markov property,

$$
\begin{aligned}
\tilde{Z}_{\mathbf{0},\mathbf{w}} &= E^{\mathbf{0}}\left[\exp(H_{0,T} + H_{T,m+n})1\{\mathbf{x}_{m+n} = \mathbf{w}\}\right] \\
&= E^{\mathbf{0}}\left[\exp(H_{0,T})\tilde{Z}_{\mathbf{x}_T,\mathbf{w}}\right] \\
&= E^{\mathbf{0}}\left[\exp(H_{0,T} - \log\tilde{Z}_{\mathbf{v},\mathbf{x}_T})\exp(\log\tilde{Z}_{\mathbf{v},\mathbf{x}_T} + \log\tilde{Z}_{\mathbf{x}_T,\mathbf{w}})\right] \\
&\le E^{\mathbf{0}}\left[\exp(H_{0,T} + H(\mathbf{x}_T \to \mathbf{y}) - \log\tilde{Z}_{\mathbf{v},\mathbf{x}_T} - H(\mathbf{x}_T \to \mathbf{y}))\exp(\log\tilde{Z}_{\mathbf{v},\mathbf{w}})\right] \\
&\le \tilde{Z}_{\mathbf{v},\mathbf{w}} E^{\mathbf{0}}\left[\exp(G_{\mathbf{0},\mathbf{y}} - H(\mathbf{v} \to \mathbf{y}))\right] \\
&\le \tilde{Z}_{\mathbf{v},\mathbf{w}} \cdot \exp(\text{the second line of } \quad (3.14)).
\end{aligned} \tag{3.15}
$$

Take the log then we obtain (3.14). □

Consider empirical measures on $\mathbf{S}$ for $n \in \mathbb{N}$, i, $j \in \mathbb{Z}_+$ and $\rho_{i,j} \in \mathbf{S}$

$$
\alpha_{n,j} = \frac{1}{n}\sum_{i=0}^{n-1}\delta_{\rho_{i,j}}, \quad \beta_{n,i} = \frac{1}{n}\sum_{j=0}^{n-1}\delta_{\rho_{i,j}}. \tag{3.16}
$$

For convenience let $\alpha_{0,i} = \beta_{0,j} = 0$. For any bounded measurable function $f : \mathbf{S} \to \mathbb{R}$, write

$$
\alpha_j(\rho, f) = \lim_{n\to\infty}\int f(r)\,\alpha_{n,j}(dr) = \lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1} f(\rho_{i,j}) \tag{3.17}
$$

and

$$
\beta_i(\rho, f) = \lim_{n\to\infty}\int f(r)\,\beta_{n,i}(dr) = \lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1} f(\rho_{i,j}) \tag{3.18}
$$

provided these limits exist.

For $y > 0$ and $r \in \mathbf{S}$, let

$$
F_2^y(r) = \int_{\mathbb{R}} \omega^y\, F_2(r, d\omega), \tag{3.19}
$$

where ω^y is the y-truncated weight in (2.29). Then from Assumption 2.1

$$
\left|F_2^y(r) - F_2(r)\right| \le \int |\omega| 1\{|\omega| > y\}\, F_2(r, d\omega) \le h(y)
$$

for some nonincreasing function h with $\lim_{y\to\infty} h(y) = 0$.

Lemma 3.11. *Suppose* (3.17) *and* (3.18) *hold for* F_2. *If* ω *has the distribution* $\mathbb{P}^\rho$ *in* (2.2), *then for* $(m_0, n_0) \le (m_1, n_1) \in \mathbb{Z}_+^2$ *we have*

$$\lim_{n\to\infty} \frac{1}{n} G^{\omega}_{(m_0,n_0),(m_1,n)} = \max_{m_0\le i\le m_1} \beta_i(\rho, F_2) \tag{3.20}$$

and

$$\lim_{m\to\infty} \frac{1}{m} G^{\omega}_{(m_0,n_0),(m,n_1)} = \max_{n_0\le j\le n_1} \alpha_j(\rho, F_2) \tag{3.21}$$

$\mathbb{P}^\rho$*-a.s. and in* $L^1(\mathbb{P}^\rho)$.

Proof. Without loss of generality, we may assume $(m_0, n_0) = (0, 0)$. For fixed $i \le m_1$, from the law of large numbers or Corollary 2.26 with shifting the origin, we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \big(\omega_{i,j} - F_2(\rho_{i,j})\big) = 0.$$

Our assumption that F_2 satisfies (3.18) gives

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} F_2(\rho_{i,j}) = \beta_i(\rho, F_2).$$

Therefore we have

$$\liminf_{n\to\infty} \frac{1}{n} G^{\omega}_{\mathbf{0},(m_1,n)} \ge \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \omega_{i,j} = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} F_2(\rho_{i,j}) = \beta_i(\rho, F_2)$$

$\mathbb{P}^\rho$-a.s. and hence,

$$\liminf_{n\to\infty} \frac{1}{n} G^{\omega}_{\mathbf{0},(m_1,n)} \ge \max_{0\le i\le m_1} \beta_i(\rho, F_2).$$

Consider ω^y for some fixed $y > 0$. Then by Theorem 2.19, there exists positive constants $C_1(y)$ and $C_2(y)$ such that for any path $\mathbf{x}_\cdot$ in $\Pi_{\mathbf{0},(m_1,n)}$,

$$\mathbb{P}^\rho\left[H_y(\mathbf{x}_\cdot) \ge \mathbb{E}^\rho H_y(\mathbf{x}_\cdot) + n\epsilon\right] \le C_1 e^{-C_2 n\epsilon^2},$$

where $H_y(\mathbf{x}_\cdot) = H^{\omega^y}(\mathbf{x}_\cdot)$. Since there are at most $(n+1)^{m_1+1}$ paths

$$\mathbb{P}^\rho \left[G^{\omega^y}_{\mathbf{0},(m_1,n)} \geq \max_{\mathbf{x}_\cdot} \mathbb{E}^\rho H_y(\mathbf{x}_\cdot) + n\epsilon \right] \leq C_1 (n+1)^{m_1+1} e^{-C_2 n \epsilon^2}. \tag{3.22}$$

Therefore from Borel-Cantelli lemma, $\mathbb{P}^\rho$ a.s.,

$$\limsup_{n\to\infty} \frac{1}{n} G^{\omega^y}_{\mathbf{0},(m_1,n)} \leq \limsup_{n\to\infty} \frac{1}{n} \max\{\mathbb{E}^\rho H_y(\mathbf{x}_\cdot) : \mathbf{x}_\cdot \in \Pi_{\mathbf{0},(m_1,n)}\}.$$

Any path in $\Pi_{\mathbf{0},(m_1,n)}$ can be decomposed into a disjoint union of paths from (i, J_i) to (i, J_{i+1}), $i = 0, 1, \ldots, m_1$, where $j_i \in \mathbb{Z}_+$ for each i and

$$0 = J_0 \leq J_1 \leq \cdots \leq J_{m_1+1} = n. \tag{3.23}$$

Note that $F_2(r)$ is bounded by some constant $M > 0$ from Assumptions 2.1. For $N \in \mathbb{N}$ define

$$\epsilon(N) = \max_{0\leq i\leq m_1} \sup\{\Big| \int F_2(r)\, \beta_{n,i}(dr) - \beta_i(\rho, F_2) \Big| : n \geq N\}.$$

Then from the assumption that F_2 satisfies (3.18), we have $\lim_{N\to\infty} \epsilon(N) = 0$. Note that

$J \cdot \epsilon(J) \le 2NM + n \cdot \epsilon(N)$ for any $J \le n$ and N. Since $\left|F_2^y(r) - F_2(r)\right| \le h(y)$, we have

$$
\begin{aligned}
&F_2^y(\rho_{0,0}) + \mathbb{E}^\rho H_y(\mathbf{x}_\cdot) \\
=&\sum_{i=0}^{m_1}\sum_{j=J_i}^{J_{i+1}} F_2^y(\rho_{i,j}) \\
\le&(n+m_1+1)h(y) + \sum_{i=0}^{m_1}\sum_{j=J_i}^{J_{i+1}} F_2(\rho_{i,j}) \\
=&(n+m_1+1)h(y) \\
+&\sum_{i=0}^{m_1}\left[(1+J_{i+1})\int F_2(r)\,\beta_{1+J_{i+1},i}(dr) - J_i\int F_2(r)\,\beta_{J_i,i}(dr)\right] \\
\le&(n+m_1+1)h(y) + (m_1+1)M \\
+&\sum_{i=0}^{m_1}\left[(J_{i+1}-J_i)\beta_i(\rho,F_2) + J_{i+1}\epsilon(J_{i+1}) + J_i\epsilon(J_i)\right] \\
\le&n\cdot\max_{0\le i\le m_1}\beta_i(\rho,F_2) + (n+m_1+1)h(y) \\
&+(m_1+1)M + 2n(m_1+1)\epsilon(N) + 4(m_1+1)NM.
\end{aligned}
\tag{3.24}
$$

Therefore

$$\limsup_{n\to\infty}\frac{1}{n}\max_{\mathbf{x}_\cdot}\mathbb{E}^\rho H_y(\mathbf{x}_\cdot) \le \max_{0\le i\le m_1}\beta_i(\rho,F_2) + 2(m_1+1)\epsilon(N) + h(y).$$

Letting $N\to\infty$, we have $\mathbb{P}^\rho$-a.s.,

$$\limsup_{n\to\infty}\frac{1}{n}G^{\omega^y}_{\mathbf{0},(m_1,n)} \le \max_{0\le i\le m_1}\beta_i(\rho,F_2) + h(y).$$

For general weights, from Theorem 2.22,

$$
\begin{aligned}
\limsup_{n\to\infty}\frac{1}{n}G^{\omega}_{\mathbf{0},(m_1,n)} &= \lim_{y\to\infty}\limsup_{n\to\infty}\frac{1}{n}G^{\omega^y}_{\mathbf{0},(m_1,n)} \\
&\le \max_{0\le i\le m_1}\beta_i(\rho,F_2).
\end{aligned}
\tag{3.25}
$$

Therefore (3.20) holds $\mathbb{P}^\rho$-a.s. Since G is dominated by weights of greedy lattice animal, from Theorem 2.20 and dominated convergence theorem, $L^1(\mathbb{P}^\rho)$ convergence also holds. (3.21) is proved similarly. □

Proof of Theorem 3.5. Note that when we proved Theorem 2.9, we used Assumption 2.6 in Theorem 2.31 to invoke Theorem 2.17(b) or (c), a nonstationary subadditive ergodic theorem. Therefore it is enough to show that the sequence $f_n(\rho) = \mathbb{E}^\rho \log Z_{\mathbf{0},n\mathbf{u}}$ for fixed $\mathbf{u} = (u_1, u_2) \in \mathbb{N}^2$ satisfies the condition (a) in Theorem 2.17.

For any $m \in \mathbb{Z}_+$, from superadditivity of f_n, we have $\overline{f} \geq \overline{f} \circ T^m$, where $T = T_{\mathbf{u}}$. Hence we need to show that $\overline{f} \leq \overline{f} \circ T^m$ for any m, $\mathcal{Q}$-a.s. Suppose that for some $(m_1, n_1) \in \mathbb{Z}_+^2$

$$\max_{0 \leq i \leq m_1} \beta_i(\rho, F_2) = \beta_{m_1}(\rho, F_2) \quad \text{and} \quad \max_{0 \leq j \leq n_1} \alpha_j(\rho, F_2) = \alpha_{n_1}(\rho, F_2). \tag{3.26}$$

Then for m and n with $m\mathbf{u} \leq (m_1, n_1) \leq n\mathbf{u}$, from Lemma 3.10,

$$\begin{aligned}
\log Z_{\mathbf{0},n\mathbf{u}} &\leq \log Z_{(m_1,n_1),n\mathbf{u}} + (m_1+n_1)\log 2 \\
&\quad + \left[\Big(G_{\mathbf{0},(m_1,nu_2)} - \sum_{j=n_1+1}^{nu_2} \omega_{(m_1,j)}\Big) \vee \Big(G_{\mathbf{0},(nu_1,n_1)} - \sum_{i=m_1+1}^{nu_1} \omega_{(i,n_1)}\Big) \right] \\
&\leq \log Z_{m\mathbf{u},n\mathbf{u}} - \log Z_{m\mathbf{u},(m_1,n_1)} + (m_1+n_1)\log 2 \\
&\quad + \left[\Big(G_{\mathbf{0},(m_1,nu_2)} - \sum_{j=n_1+1}^{nu_2} \omega_{(m_1,j)}\Big) \vee \Big(G_{\mathbf{0},(nu_1,n_1)} - \sum_{i=m_1+1}^{nu_1} \omega_{(i,n_1)}\Big) \right].
\end{aligned} \tag{3.27}$$

Hence from (3.2) and Lemma 3.11,

$$\limsup_{n\to\infty} \frac{1}{n}\mathbb{E}^\rho \log Z_{\mathbf{0},n\mathbf{u}} \leq \limsup_{n\to\infty} \frac{1}{n}\mathbb{E}^\rho \log Z_{m\mathbf{u},n\mathbf{u}}.$$

Therefore $\overline{f}(\rho) \leq \overline{f} \circ T^m(\rho)$ for this m and ρ. Thus if we can show that for $\mathcal{Q}$-a.e. ρ, there exists a sequence $\{(m_1(k), n_1(k)\}_{k=0}^\infty$ such that $m_1(k), n_1(k) \to \infty$ and (3.26) is satisfied, then $\overline{f}$ is $\mathcal{Q}$-almost surely invariant and this completes the proof. However we can find such a sequence from (3.3) and (3.4). $\square$

Proof of Theorem 3.6. Define $\Gamma : S_0 \to \Omega_0 = \mathbb{R}_+^{\mathbb{N}^2}$ by $\rho_{i,j} = \Gamma(\Lambda, \Theta)_{i,j} = \gamma(\lambda_i, \theta_j)$ for $i,\ j \in \mathbb{N}$, where γ is as in Assumption 3.3(b). Then Γ intertwines translation maps: $\Gamma \circ \tau_{k,l} = T_{k,l} \circ \Gamma$. Let $\mathcal{Q}' = \Gamma_\#(\mathcal{Q})$ and $\mathcal{Q}'_0 = \Gamma_\#(\mathcal{Q}_0)$ be pushforward measures on Ω_0 of $\mathcal{Q}$ and $\mathcal{Q}_0$, respectively. By Lemma 2.14, $\mathcal{Q}'$ and $\mathcal{Q}'_0$ satisfy Assumption 2.4.

Let q_1 and q_2 denote the coordinate projections from $\mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$ onto $\mathbb{R}^{\mathbb{N}}$ defined by $q_1(\Lambda, \Theta) = \Lambda$ and $q_2(\Lambda, \Theta) = \Theta$, respectively. Let $\tau : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ be the shift map defined by $\tau(\mathbf{x})_k = \mathbf{x}_{k+1}$ for $k \in \mathbb{N}$. Let $f : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ be a bounded measurable map. Since $\mathcal{Q}_0$ is $\tau_{1,1}$-ergodic, we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f \circ \tau^{k-1}(\Lambda) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f \circ q_1 \circ \tau_{1,1}^{k-1}(\Lambda, \Theta) = E^{\mathcal{Q}_0}[f \circ q_1] \tag{3.28}$$

and

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f \circ \tau^{k-1}(\Theta) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f \circ q_2 \circ \tau_{1,1}^{k-1}(\Lambda, \Theta) = E^{\mathcal{Q}_0}[f \circ q_2] \quad \mathcal{Q}\text{-a.s.} \tag{3.29}$$

Therefore Λ and Θ are separately ergodic. If we write $\mathcal{Q}_1$ and $\mathcal{Q}_2$ for the distributions of Λ and Θ under $\mathcal{Q}_0$, respectively, then we conclude that Λ and Θ are τ-AMS with stationary means with $\mathcal{Q}_1$ and $\mathcal{Q}_2$, respectively.

Now suppose Assumption 3.3(d)' holds. Then $F_2(r)$ is a nondecreasing function. Let

$$B_\lambda = \{s : F_2(\gamma(\lambda, s)) \text{ is not continuous at } s\} \quad \text{and}$$

$$B = \bigcup \{B_\lambda : \alpha(\{\lambda\}) > 0\}.$$

Then B is countable. Therefore for $\mathcal{Q}$-a.e. Λ, from (3.28)

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} F_2(\gamma(\lambda_k, s)) = \int F_2(\gamma(\lambda, s))\, \alpha(d\lambda) \tag{3.30}$$

for all $s \in B' = (\mathbb{Q} \cap [b_0, b_1]) \cup \{b_0, b_1\} \cup B$. Let Ω_0' be an event that (3.30) holds and $\mathcal{Q}(\Omega_0') = 1$. If $\theta(\Lambda, \Theta)$ is a random variable taking values in B', then we have for $(\Lambda, \Theta) \in \Omega_0'$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} F_2(\gamma(\lambda_k, \theta)) = \int F_2(\gamma(\lambda, \theta))\, \alpha(d\lambda)$$

For general θ, let $\theta_{n,1} = n^{-1}\lfloor n\theta \rfloor$ and $\theta_{n,2} = n^{-1}\lceil n\theta \rceil$. Then $\theta_{n,1} \le \theta \le \theta_{n,2}$. We have

$$\limsup_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} F_2(\gamma(\lambda_k, \theta)) \le \int F_2(\gamma(\lambda, \theta_{n,2}))\, \alpha(d\lambda)$$

and

$$\liminf_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} F_2(\gamma(\lambda_k, \theta)) \ge \int F_2(\gamma(\lambda, \theta_{n,1}))\, \alpha(d\lambda)$$

if $(\Lambda, \Theta) \in \Omega_0'$. Suppose $\theta \notin B'$. Then by letting $n \to \infty$, from the definition of B',

$$\begin{aligned} &\limsup_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} F_2(\gamma(\lambda_k, \theta)) - \liminf_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} F_2(\gamma(\lambda_k, \theta)) \\ &\le \int F_2(\gamma(\lambda, \theta)+) - F_2(\gamma(\lambda, \theta)-)\, \alpha(d\lambda) = 0. \end{aligned} \tag{3.31}$$

Therefore for $\mathcal{Q}$-a.e. Λ and Θ,

$$\alpha_j(\rho, F_2) = \lim_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} F_2(\gamma(\lambda_k, \theta_j)) = \int F_2(\gamma(\lambda, \theta_j))\, \alpha(d\lambda). \tag{3.32}$$

Similarly,

$$\beta_i(\rho, F_2) = \lim_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} F_2(\gamma(\lambda_i, \theta_k)) = \int F_2(\gamma(\lambda_i, \theta))\, \beta(d\theta). \tag{3.33}$$

From (3.32), if $\beta(\{b_1\}) = 0$, then $\alpha_j(\rho, F_2) \le \lim_{b\uparrow b_1} \int F_2(\gamma(\lambda, b))\, \alpha(d\lambda)$. We claim that

$$\limsup_{m\to\infty} \alpha_m(\rho, F_2) = \lim_{b\uparrow b_1} \int F_2(\gamma(\lambda, b))\, \alpha(d\lambda).$$

Let $f = 1_{[b_1-\epsilon, b_1]}$ be an indicator function for $\epsilon > 0$. Since Θ is AMS with stationary mean Q_2,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(\theta_k) = \int 1_{[b_1-\epsilon, b_1]}(\theta)\, \beta(d\theta) > 0.$$

Hence $\theta_j > b_1 - \epsilon$ infinitely often and the claim is justified. Therefore

$$\alpha_j(\rho, F_2) \le \limsup_{m\to\infty} \alpha_m(\rho, F_2)$$

If $\beta(\{b_1\}) > 0$, considering $f = 1_{\{b_1\}}$, we have the same result. Similarly, we have

$$\beta_i(\rho, F_2) \le \limsup_{m\to\infty} \beta_m(\rho, F_2).$$

Therefore $\mathcal{Q}'$ and $\mathcal{Q}'_0$ satisfy Assumption 3.1. The boundary conditions of $\bar{\phi}$ are proved in the following Lemma. □

Lemma 3.12. *Let $\epsilon > 0$. There exists $\delta > 0$ such that if $0 < x < \delta$, then*

$$\left|\bar{\phi}(x,1) - \alpha\text{-}\operatorname*{ess\,sup}_{\lambda_1} \int F_2(\gamma(\lambda_1,\theta))\,\beta(d\theta)\right| < \epsilon \tag{3.34}$$

and

$$\left|\bar{\phi}(1,x) - \beta\text{-}\operatorname*{ess\,sup}_{\theta_1} \int F_2(\gamma(\lambda,\theta_1))\,\alpha(d\lambda)\right| < \epsilon. \tag{3.35}$$

Proof. Since the scaling limit $\bar{\phi}$ is completely determined by $\mathcal{Q}_0$, we assume $\mathcal{Q} = \mathcal{Q}_0$. The proof is similar to that of Lemma 3.11. Let $\mathbf{u} = (u_1, u_2) \in \mathbb{N}^2$. We have

$$\log Z^{\omega}_{\mathbf{1},\mathbf{1}+n\mathbf{u}} \ge \sum_{j=1}^{nu_2} \omega_{1,j} + \sum_{i=1}^{nu_1} \omega_{i,nu_2}.$$

Hence $\mathcal{Q}_0$-a.s.,

$$\bar{\phi}(u_1,u_2) = \lim_{n\to\infty} \frac{1}{n} \log Z^{\omega}_{\mathbf{1},\mathbf{1}+n\mathbf{u}} \ge u_2\beta_1(\rho, F_2) - Mu_1 = u_2 \int F_2(\gamma(\lambda_1,\theta))\,\beta(d\theta) - Mu_1,$$

where M is an upper bound of $F_2(r)$. Since $\bar{\phi}$ is a deterministic function,

$$\bar{\phi}(u_1/u_2, 1) \ge \alpha\text{-}\operatorname*{ess\,sup}_{\lambda_1} \int F_2(\gamma(\lambda_1,\theta))\,\beta(d\theta) - Mu_1/u_2.$$

Since $\bar{\phi}$ is continuous, for all $x > 0$,

$$\bar{\phi}(x,1) \ge \alpha\text{-}\operatorname*{ess\,sup}_{\lambda_1} \int F_2(\gamma(\lambda_1,\theta))\,\beta(d\theta) - Mx. \tag{3.36}$$

Let $K_n = \left|\Pi_{\mathbf{1},\mathbf{1}+u\mathbf{u}}\right|$. By comparing $\int_1^n \log x\, dx$ and $\log(n!)$, one has

$$\log \binom{n+m}{n} \le (n+m)(-p\log p - q\log q) + \log(n+m)$$

where $p = n/(n+m)$ and $q = m/(n+m)$. Hence with $p = u_1/(u_1+u_2)$,

$$\log K_n \le n(u_1+u_2)(-p\log p - (1-p)\log(1-p)) + \log(nu_1 + nu_2)$$

By considering paths in $\Pi_{\mathbf{1},\mathbf{1}+u\mathbf{u}}$ and y-truncated weights, we have

$$\begin{aligned} &\mathbb{P}^\rho\left[\log Z^{\omega^y}_{\mathbf{1},\mathbf{1}+n\mathbf{u}} - \log K_n \ge \max_{\mathbf{x}_\cdot} \mathbb{E}^\rho H_y(\mathbf{x}_\cdot) + n\epsilon\right] \\ \le&\mathbb{P}^\rho\left[G^{\omega^y}_{\mathbf{1},\mathbf{1}+u\mathbf{u}} \ge \max_{\mathbf{x}_\cdot} \mathbb{E}^\rho H_y(\mathbf{x}_\cdot) + n\epsilon\right] \le C_1 K_n e^{-C_2 n(u_1+u_2)\epsilon^2} \end{aligned} \tag{3.37}$$

(see (3.22)).

If u_1/u_2 is sufficiently small, say $< \delta$, then the upper bound in (3.37) is summable over n. Therefore from Borel-Cantelli lemma, if $u_1/u_2 < \delta$,

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{n} \log Z^{\omega^y}_{\mathbf{1},\mathbf{1}+n\mathbf{u}} \le{}& (u_1+u_2)(-p\log p - (1-p)\log(1-p)) \\ &+ \limsup_{n\to\infty} \frac{1}{n} \max\{\mathbb{E}^\rho H_y(\mathbf{x}_\cdot) : \mathbf{x}_\cdot \in \Pi_{\mathbf{1},\mathbf{1}+n\mathbf{u}}\} + \epsilon. \end{aligned} \tag{3.38}$$

By choosing smaller δ, we have

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{n} \log Z^{\omega^y}_{\mathbf{1},\mathbf{1}+n\mathbf{u}} \le{}& \limsup_{n\to\infty} \frac{1}{n} \max\{\mathbb{E}^\rho H_y(\mathbf{x}_\cdot) : \mathbf{x}_\cdot \in \Pi_{\mathbf{1},\mathbf{1}+n\mathbf{u}}\} \\ &+ (u_1+u_2+1)\epsilon \end{aligned} \tag{3.39}$$

if $u_1/u_2 < \delta$. Using notations in (3.24), if $\lambda = \max_{1\le i\le nu_1} \lambda_i$ (we use Assumption 3.3(d')),

$$\begin{aligned}
&F_2^y(\rho_{0,0}) + \mathbb{E}^\rho H_y(\mathbf{x}_\cdot)\\
=&\sum_{i=1}^{nu_1}\sum_{j=J_i}^{J_{i+1}} F_2^y(\rho_{i,j})\\
\le&(nu_1+nu_2+1)h(y) + \sum_{i=1}^{nu_1}\sum_{j=J_i}^{J_{i+1}} F_2(\gamma(\lambda_i,\theta_j))\\
\le&(nu_1+nu_2+1)h(y) + \sum_{i=1}^{nu_1}\sum_{j=J_i}^{J_{i+1}} F_2(\gamma(\lambda,\theta_j))\\
\le&(nu_1+nu_2+1)h(y) + Mnu_1 + \sum_{j=1}^{nu_2} F_2(\gamma(\lambda,\theta_j))\\
\le&(nu_1+nu_2+1)h(y) + Mnu_1 + \sum_{j=1}^{nu_2} F_2(\gamma(b_2,\theta_j)),
\end{aligned} \tag{3.40}$$

where $b_2 = b_1$ if $\alpha(\{b_1\}) > 0$ and $b_2 = b_1-$ otherwise. For the latter case, $F_2(\gamma(b_1-,\theta)) = \lim_{b\uparrow b_1} F_2(\gamma(b,\theta))$. By ergodic theorem, we have

$$\limsup_{n\to\infty} \frac{1}{n}\max_{\mathbf{x}_\cdot} \mathbb{E}^\rho H_y(\mathbf{x}_\cdot) \le u_2\cdot \int F_2(\gamma(b_2,\theta))\,\beta(d\theta) + Mu_1 + (u_1+u_2)h(y). \tag{3.41}$$

Note that

$$\alpha\text{- ess sup}_{\lambda_1} \int F_2(\gamma(\lambda_1,\theta))\,\beta(d\theta) = \int F_2(\gamma(b_2,\theta))\,\beta(d\theta).$$

Therefore we have, $\mathbb{P}^\rho$-a.s.,

$$\begin{aligned}
\limsup_{n\to\infty}\frac{1}{n} Z^{\omega^y}_{\mathbf{1},\mathbf{1}+n\mathbf{u}} \le& u_2\cdot \alpha\text{- ess sup}_{\lambda_1}\int F_2(\gamma(\lambda_1,\theta))\,\beta(d\theta)\\
&+ Mu_1 + (u_1+u_2)h(y) + (u_1+u_2+1)\epsilon.
\end{aligned} \tag{3.42}$$

For general weights, from Theorem 2.22,

$$\begin{aligned}
&\limsup_{n\to\infty}\frac{1}{n} Z^{\omega}_{\mathbf{1},\mathbf{1}+n\mathbf{u}} = \lim_{y\to\infty}\limsup_{n\to\infty}\frac{1}{n} Z^{\omega^y}_{\mathbf{1},\mathbf{1}+n\mathbf{u}}\\
\le& u_2\cdot \alpha\text{- ess sup}_{\lambda_1}\int F_2(\gamma(\lambda_1,\theta))\,\beta(d\theta) + (M+2\epsilon)u_1 + 2\epsilon u_2.
\end{aligned} \tag{3.43}$$

Hence if $u_1/u_2 < \delta$ then

$$\bar{\phi}(u_1/u_2, 1) \leq \alpha\text{-}\operatorname*{ess\,sup}_{\lambda_1} \int F_2(\gamma(\lambda_1, \theta))\, \beta(d\theta) + (M + 2\epsilon)u_1/u_2 + 2\epsilon.$$

From continuity of $\bar{\phi}$ we have

$$\bar{\phi}(x, 1) \leq \alpha\text{-}\operatorname*{ess\,sup}_{\lambda_1} \int F_2(\gamma(\lambda_1, \theta))\, \beta(d\theta) + (M + 2\epsilon)x + 2\epsilon. \tag{3.44}$$

for all $0 < x < \delta$. (3.36) and (3.44) prove (3.34). We omit the proof of (3.35). $\square$

3.4 The log-gamma polymer model

3.4.1 Definitions and Conventions

In this section, we study an explicitly solvable log-gamma polymer model and mostly use results and conventions from previous work. Our model lives in $\mathbb{N}^2$.

First, we describe the CDF of our model precisely. Let $G_2 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a function defined by $G_2(r, x) = \frac{1}{\Gamma(r)} \int_{-\infty}^{x} \exp[ry - e^y]\, dy$ for $x \in \mathbb{R}$ and $r > 0$. $G_2(r, \cdot)$ is the CDF of a log-gamma(r) random variable (see (A.13)). We write $G_r(x)$ for $G_2(r, x)$. $H(r, u) = G_r^{-1}(u) = (0 < u < 1)$ is the inverse of $G_2(r, \cdot)$ and increasing in r by (A.19) and (A.22). If $-\omega$ is a log-gamma(r) random variable, the CDF of ω is given by

$$F_r(x) = F_2(r, x) = \frac{1}{\Gamma(r)} \int_{-\infty}^{x} \exp[-ry - e^{-y}]\, dy \tag{3.45}$$

and $F_r^{-1}(u) = -H(r, 1 - u)$. $F_r^{-1}(u)$ is decreasing in r. Note that $F_2(r) = -\mathbf{E}X = -\Psi_0(r)$, where X is a log-gamma(r) random variable.

Now we explain how this model can be handled in the framework we developed.

Proof of Theorem 3.7. We provide functions in Assumption 3.3 for this model. γ in (b) is given by $\gamma(\lambda,\theta) = \lambda + \theta$. We showed that F_2 is increasing function in the first variable (we restrict the domain of F_2 to $\mathbb{R}_{>0}$). Therefore F_2 satisfies (d). The weights in (1.5) are precisely as explained in (c). One only need to show that F_2 satisfies Assumption 2.1. One can easily construct F in (2.1). See Remark 3.14. Therefore we can apply Theorem 3.6. For the boundary values of $\bar{\phi}$, we have

$$\bar{\phi}(1,0) = \beta\text{- ess}\sup_{\theta_1} \int -\Psi_0(\lambda+\theta_1)\,\alpha(d\lambda) = A(b_0) \tag{3.46}$$

and

$$\bar{\phi}(0,1) = \alpha\text{- ess}\sup_{\lambda_1} \int -\Psi_0(\lambda_1+\theta)\,\beta(d\theta) = B(-a_0), \tag{3.47}$$

where A and B are functions in (3.8) □

To prove the variational formula (3.7) and to obtain some explicit formulas, we utilize *stationary processes with boundary conditions* and couple these processes with our original process without boundary conditions. The precise setting is as follows. To couple these stationary processes with the original model, we need to extend sites. Let $S_0 = \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$, $\Omega_0' = \mathbb{R}^{\mathbb{N}^2}$, and $\Omega_0 = \mathbb{R}^{\mathbb{Z}_+^2}$. These spaces are equipped with the product σ-algebras. We extend weight parameters ρ to sites in $\mathbb{Z}_+^2 \setminus \mathbb{N}^2$. For $-a_0 < z < b_0$,

$$\rho_{i,0} = \lambda_i + z, \quad \rho_{0,j} = \theta_j - z$$

for boundary points ($i, j \in \mathbb{N}$). We set $\rho_{0,0} = 0$. We write ρ_z for these weight parameters. Original parameters on $\mathbb{N}^2$ is denoted by ρ

Definition 3.13 (Inhomogeneous log-gamma polymer)**.** *Assume $-a_0 < z < b_0$. All polymer models are defined on $\mathbb{L} = \mathbb{N}^2$ or $\mathbb{Z}_+^2$. For given parameters $\tilde{\rho} \in \mathbb{R}^{\mathbb{L}}$, $\mathbb{P}^{\tilde{\rho}}$ is* *given by* (2.2) *(we use* $\mathbb{L}$ *instead of* $\mathbb{Z}_+^2$*) with* F_2 *in* (3.45)*.* $\mathcal{Q}$ *satisfies conditions in Theorem 3.7.*

(1) Let (Λ, Θ) *be fixed. The* (Λ, Θ)*-polymer is the up-right directed polymer model started at* $(1, 1) \in \mathbb{N}^2$*:* $\mathbb{L} = \mathbb{N}^2$ *and the distribution of* ω *is* $\mathbb{P}^{\rho}$*.*

(2) The (Λ, Θ, z)*-stationary polymer is a polymer started at the origin:* $\mathbb{L} = \mathbb{Z}_+^2$ *and the distribution of* ω *is* $\mathbb{P}^{\rho_z}$*. Note that the bulk weights are the same as in part (1) but we give* z*-dependent weights for boundaries.*

(3) The $\mathcal{Q}$*-polymer is a polymer started at* $(1, 1) \in \mathbb{N}^2$*:* $\mathbb{L} = \mathbb{N}^2$*. The distribution for parameters* (Λ, Θ) *is* $\mathcal{Q}$*. The conditional law of the weights* ω *given the parameters* (Λ, Θ) *is* $\mathbb{P}^{\rho}$*.*

(4) $(\mathcal{Q}, z)$*-stationary polymer is a polymer model started at the origin:* $\mathbb{L} = \mathbb{Z}_+^2$*. The distribution for parameters* (Λ, Θ) *is* $\mathcal{Q}$*. The conditional law of the weights* ω *given the parameters* (Λ, Θ) *is* $\mathbb{P}^{\rho_z}$*.*

Remark 3.14. *As in* (2.15) *we use couplings to realize various weights simultaneously.* $U = \{U_{\mathbf{x}}\}_{\mathbf{x} \in \mathbb{Z}_+^2}$ *are i.i.d. Uniform-(0,1) random variables.* (Λ, Θ) *is independent of* U*.* $\omega_{\mathbf{x}} = F_{\rho_{\mathbf{x}}}^{-1}(U_{\mathbf{x}})$ *for* $\mathbf{x} \in \mathbb{L}$*. By monotonicity of* F_r^{-1} *in* r *we have*

$$F_{a_1+b_1}^{-1}(U_{\mathbf{x}}) \leq \omega_{\mathbf{x}} \leq F_{(a_0+z)\wedge(b_0-z)}^{-1}(U_{\mathbf{x}})$$

for $\mathbf{x} \in \mathbb{Z}_+^2 \setminus \{(0, 0)\}$*. Let* F^z *be the CDF of*

$$\max\{|F_{a_1+b_1}^{-1}(U_{\mathbf{0}})|,\ |F_{(a_0+z)\wedge(b_0-z)}^{-1}(U_{\mathbf{0}})|,\ 1\}.$$

We write F *for* F^0*.* F^z *has an exponential tail. In particular*

$$k^z \triangleq \int_0^\infty (1 - F^z(t))^{1/2}\, dt < \infty.$$

η variables are given by $\eta_{\mathbf{x}} = (F^z)^{-1}(U_{\mathbf{x}})$ for $\mathbf{x} \in \mathbb{Z}_+^2$.

Remark 3.15 (Notations)**.** *E^μ is used to denote the expectation for a general probability measure μ. $(\Omega, \mathcal{F}, \mathbf{P})$ is a generic probability space that is not part of the polymer model (see Section 2.4.1).*

The distribution of ω is denoted by $\mathbb{P}^\rho$ in (1) and $\mathbb{P}$ in (3), respectively. For z-stationary models, the distribution is denoted by $\mathbb{P}^{\rho,z}$ in (2) and $\mathbb{P}^z$ in (4), respectively. Note that marginals of $\mathbb{P}^{\rho,z}$ and $\mathbb{P}^z$ on Ω_0' are $\mathbb{P}^\rho$ and $\mathbb{P}$, respectively.

Under $\mathbb{P}$ the expectation of X is $\mathbb{E}X$ and variance $\mathbb{V}\mathrm{ar}(X)$. Overline means centering: $\overline{X} = X - \mathbb{E}X$. Q^ω is the quenched polymer measure. The annealed measure is $P(\cdot) = \mathbb{E}Q^\omega(\cdot)$ with expectation E. Under $\mathbb{P}^\rho$ or $\mathbb{P}^{\rho,z}$ we use similar conventions. For example, the expectation of X is $\mathbb{E}^\rho X$ or $\mathbb{E}^{\rho,z}X$.

3.4.2 Stationary processes with boundary conditions

In this section we consider the inhomogeneous log-gamma polymer with boundary conditions, processes in a *stationary* ([21], not to be confused with $\hat{T}$-stationarity in this paper) situation as explained below. Working with these models is crucial for explicit computations. Note that stationary polymers are defined by altering the distribution of the weights on the boundaries of $\mathbb{Z}_+^2$, maintaining the same distribution on the bulk.

A remarkable feature of the stationary processes is that the horizontal and vertical increments of the free energy (or the ratio of partition functions) are *stationary.* Recall the convention in (2.17). Variables indexed by a single point do not have the parentheses, for example, $Z_{m,n} = Z_{(m,n)} = Z_{(0,0),(m,n)}$. Define

$$\begin{aligned} I_{m,n} &= \log Z_{m,n} - \log Z_{m-1,n} \quad \text{for } m \geq 1 \text{ and } n \geq 0 \\ J_{m,n} &= \log Z_{m,n} - \log Z_{m,n-1} \quad \text{for } m \geq 0 \text{ and } n \geq 1. \end{aligned} \tag{3.48}$$

The partition function satisfies

$$Z_{m,n} = e^{\omega_{m,n}}(Z_{m-1,n} + Z_{m,n-1}) \qquad \text{for } (m,n) \in \mathbb{N}^2 \tag{3.49}$$

and one can verify that for $(m,n) \in \mathbb{N}^2$

$$\begin{aligned} e^{-I_{m,n}} &= e^{-\omega_{m,n}} \frac{e^{-I_{m,n-1}}}{e^{-I_{m,n-1}} + e^{-J_{m-1,n}}} \\ e^{-J_{m,n}} &= e^{-\omega_{m,n}} \frac{e^{-J_{m-1,n}}}{e^{-I_{m,n-1}} + e^{-J_{m-1,n}}}. \end{aligned} \tag{3.50}$$

The following Proposition is a key to explicit computations and explains why we call *stationary polymers* stationary.

Proposition 3.16. *Consider the (Λ, Θ, z)-stationary polymer and the $(\mathcal{Q}, z)$-stationary polymer in Definition 3.13 for $-a_0 < z < b_0$. Let $k,\, l \in \mathbb{Z}_+$.*

(a) *$I_{i,l}$ has the same distribution as $\omega_{i,0}$ for any $i \in \mathbb{N}$.*

(b) *$J_{k,j}$ has the same distribution as $\omega_{0,j}$ for any $j \in \mathbb{N}$.*

(c) *For the (Λ, Θ, z)-stationary polymer we have that for any fixed $l \in \mathbb{Z}_+$, the random variables $\{I_{i,l} \,:\, i \in \mathbb{N}\}$ are independent, and for any fixed $k \in \mathbb{Z}_+$, the variables $\{J_{k,j} \,:\, j \in \mathbb{N}\}$ are independent.*

(d) *For the $(\mathcal{Q}, z)$-stationary polymer, part (c) holds if $\mathcal{Q}$ is given by a product measure on S_0.*

Proof. For an i.i.d. environment in the bulk ($\lambda_i = \lambda$, $\theta_j = \theta$ are constants) this is proved in Theorem 3.3 [33] with slightly different formulations. Reversibility or Burke property is given in Lemma 3.2 [33] where a special property of the Gamma distribution is used for the proof, and then by induction argument (a), (b), and (c) are proved. For the (Λ, Θ, z)-stationary polymer, we have the same proof so we omit it. See [33] for details.

For the $(\mathcal{Q}, z)$-stationary polymer, the conditional law of $I(i, l)$ conditioned on (Λ, Θ) is $F_{\lambda_i+z}(\cdot)$ by the result for the (Λ, Θ, z)-stationary polymer. In particular, the conditional distributions do not depend on Θ. Therefore the distribution of $I_{i,l}$ does not depend on l. This observation also indicates that $\{I_{i,l} : i \in \mathbb{N}\}$ are independent if $\mathcal{Q}$ is a product measure. Same proof holds for $J_{k,j}$. $\square$

Recall definitions of A and B in (3.8). We can compute the limiting free energy of z-stationary polymers. For $-a_0 < z < b_0$, let $\bar{\phi}_z$ denote the function

$$\bar{\phi}_z(x, y) = xA(z) + yB(z). \tag{3.51}$$

Theorem 3.17. *We have*

$$\lim_{n\to\infty} \frac{\log Z^{\omega}_{\lfloor nx\rfloor, \lfloor ny\rfloor}}{n} = \bar{\phi}_z(x, y) \quad \textit{for all } x,\, y \geq 0 \textit{ in } \mathbb{R} \tag{3.52}$$

for the following cases.

(a) *For the (Λ, Θ, z)-stationary polymer (3.52) holds $\mathbb{P}^{\rho,z}$-a.s. if empirical measures $\frac{1}{n}\sum_{i=1}^{n}\delta_{\lambda_i}$ and $\frac{1}{n}\sum_{j=1}^{n}\delta_{\theta_j}$ converge weakly to α and β, respectively.*

(b) *For the $(\mathcal{Q}, z)$-stationary polymer (3.52) holds $\mathbb{P}^z$-a.s.*

Proof. We prove this theorem for x, $y > 0$. The cases $x = 0$ and $y = 0$ can be handled in the same way. From (3.48), we have

$$\frac{\log Z^{\omega}_{\lfloor nx\rfloor,\lfloor ny\rfloor}}{n} = \frac{1}{n}\sum_{i=1}^{\lfloor nx\rfloor} I_{i,0} + \frac{1}{n}\sum_{j=1}^{\lfloor ny\rfloor} J_{\lfloor nx\rfloor,j}.$$

Averaging with respect to $\mathbb{P}^{\rho,z}$, from Proposition 3.16 we have

$$\frac{\mathbb{E}^{\rho,z}\log Z_{\lfloor nx\rfloor,\lfloor ny\rfloor}}{n} = -\frac{1}{n}\sum_{i=1}^{\lfloor nx\rfloor}\Psi_0(z+\lambda_i) - \frac{1}{n}\sum_{j=1}^{\lfloor ny\rfloor}\Psi_0(-z+\theta_j). \tag{3.53}$$

(a) Since Ψ_0 is continuous and

$$\begin{gathered}\Psi_0(z+a_0) \le \Psi_0(z+\lambda_i) \le \Psi_0(z+a_1) \\ \Psi_0(-z+b_0) \le \Psi_0(-z+\theta_j) \le \Psi_0(-z+b_1)\end{gathered} \tag{3.54}$$

we have for any x, y

$$\begin{gathered}-\frac{1}{n}\sum_{i=1}^{\lfloor nx\rfloor}\Psi_0(z+\lambda_i) \to xA(z) \\ -\frac{1}{n}\sum_{j=1}^{\lfloor ny\rfloor}\Psi_0(-z+\theta_j) \to yB(z)\end{gathered} \tag{3.55}$$

from the weak convergence. Corollary 2.26 gives the convergence in (3.52) for $\mathbb{P}^{\rho,z}$-a.s. ω.

(b) Note that for fixed x and y, (3.55) holds for $\mathcal{Q}$-a.e. (Λ,Θ) by (3.28) and (3.29). Therefore (3.54) and Proposition 2.30 give simultaneous convergence in (3.55) for all x and y, $\mathcal{Q}$-a.s. Finally, Corollary 2.27 implies the convergence in (3.52) for $\mathbb{P}^{z}$-a.s. ω. □

Theorem 3.18.

$$\bar\phi_z(1,1) = \sup_{0\le t\le 1}\{\bar\phi_z(1-t,0)+\bar\phi(t,1)\} \vee \sup_{0\le t\le 1}\{\bar\phi_z(0,1-t)+\bar\phi(1,t)\} \tag{3.56}$$

This theorem is stated and proved for some inhomogeneous corner growth models in Proposition 4.4 [17] and for i.i.d. log-gamma polymers in Lemma 4.1 [21]. In the following proof we adapt their argument.

Proof. Note the inequality

$$\bar{\phi}(x,y) \le \bar{\phi}_z(x,y) \tag{3.57}$$

for any x, $y \ge 0$. This follows from the inequality

$$\log Z_{(0,0),(1,1)} + \log Z_{(1,1),(m,n)} \le \log Z_{(0,0),(m,n)}$$

and from $\mathbb{P}$ being the projection of $\mathbb{P}^z$ onto Ω_0'. Since $\bar{\phi}_z$ is linear,

$$\bar{\phi}_z(1,1) = \bar{\phi}_z(1-t,0) + \bar{\phi}_z(t,1) \ge \bar{\phi}_z(1-t,0) + \bar{\phi}(t,1)$$

$$\bar{\phi}_z(1,1) = \bar{\phi}_z(1-t,0) + \bar{\phi}_z(1,t) \ge \bar{\phi}_z(0,1-t) + \bar{\phi}(1,t)$$

for any $0 \le t \le 1$. Taking the supremum of the right-hand sides over t gives (3.56) with $\ge$ in place of $=$.

We can decompose the partition function $Z_{m,n}$ according to the exit point of the path from the boundary:

$$Z^{\omega}_{m,n} = \sum_{k=1}^{m}\Big(\exp(\sum_{i=1}^{k}\omega_{i,0})\Big)\cdot Z^{\square}_{(k,1),(m,n)} + \sum_{\ell=1}^{n}\Big(\exp(\sum_{j=1}^{\ell}\omega_{0,j})\Big)\cdot Z^{\square}_{(1,\ell),(m,n)} \tag{3.58}$$

where $Z^{\square}_{u,v}$ is the partition function including the weight of the starting point:

$$Z^{\square}_{\mathbf{u},\mathbf{v}} = e^{\omega_{\mathbf{u}}} Z_{\mathbf{u},\mathbf{v}}. \tag{3.59}$$

Proof for the $\le$ half of (3.56) is similar to that of Corollary 2.10. Take $m = n$ to compute $\bar{\phi}_z(1,1)$. Let $L \in \mathbb{N}$, $u_i = \lfloor in/L \rfloor$ for $0 \le i \le L$ and consider $n > L$ large enough to ensure that $u_i < u_{i+1}$. For any $1 \le k \le n$ there exists some $0 \le i < L$ such that $u_i < k \le u_{i+1}$. A summand in (3.58) satisfies

$$\begin{aligned}\log Z^{\omega}_{k,0} + \log Z^{\square}_{(k,1),(n,n)} \le \log Z^{\omega}_{u_i,0} + \log Z^{|\omega|}_{(u_i,0),(u_{i+1},0)} \\ + \log Z^{|\omega|}_{(u_i,1),(u_{i+1},1)} + \log Z^{\omega}_{(1+u_i,1),(n,n)}.\end{aligned} \tag{3.60}$$

It follows that

$$
\begin{aligned}
\log Z^{\omega}_{n,n} \le \log(4n) + &\max_{0\le i<L}\{\log Z^{\omega}_{u_i,0} + \log Z^{\omega}_{(1+u_i,1),(n,n)}\}\vee \\
&\max_{0\le i<L}\{\log Z^{|\omega|}_{(u_i,0),(u_{i+1},0)} + \log Z^{|\omega|}_{(u_i,1),(u_{i+1},1)}\}\vee \\
&\max_{0\le i<L}\{\log Z^{\omega}_{0,u_i} + \log Z^{\omega}_{(1,1+u_i),(n,n)}\}\vee \\
&\max_{0\le i<L}\{\log Z^{|\omega|}_{(0,u_i),(0,u_{i+1})} + \log Z^{|\omega|}_{(1,u_i),(1,u_{i+1})}\}.
\end{aligned}
\tag{3.61}
$$

Since we already established the existence of limits we only need to identify the limits. All limit formulas are completely determined by the expectations under $\mathcal{Q}_0$. Hence we may assume $\mathcal{Q} = \mathcal{Q}_0$ and so our model is stationary relative to shift maps. We have the following equalities in distribution under $\mathbb{P}$.

$$
\begin{aligned}
Z^{\omega}_{(1+u_i,1),(n,n)} &= Z^{\omega}_{(1,1),(n-u_i,n)} \\
Z^{\omega}_{(1,1+u_i),(n,n)} &= Z^{\omega}_{(1,1),(n,n-u_i)}\,,
\end{aligned}
$$

which imply

$$
\begin{aligned}
&\frac{1}{n}\log Z^{\omega}_{(1+u_i,1),(n,n)} \to \bar{\phi}(1-i/L,1) \\
&\frac{1}{n}\log Z^{\omega}_{(1,1+u_i),(n,n)} \to \bar{\phi}(1,1-i/L)
\end{aligned}
\tag{3.62}
$$

in probability as $n \to \infty$. Hence these limits are a.s. if n tends to ∞ along suitable subsequences.

Divide through by n in (3.61), let $n \to \infty$ along suitable subsequences and consider the limit of each term. By Theorem 3.17, (3.62), and ergodic theorem for error terms

$$
\begin{aligned}
\bar{\phi}_z(1,1) &\le \max_{0\le i<L}\max\{\bar{\phi}_z(i/L,0) + \bar{\phi}(1-i/L,1), \\
&\qquad\qquad \bar{\phi}_z(0,i/L) + \bar{\phi}(1,1-i/L)\} + \frac{C(z)+D(z)}{L} \\
&\le \sup_{0\le t\le 1}\max\{\bar{\phi}_z(t,0) + \bar{\phi}(1-t,1), \\
&\qquad\qquad \bar{\phi}_z(0,t) + \bar{\phi}(1,1-t)\} + \frac{C(z)+D(z)}{L}
\end{aligned}
$$

where $C(z) = \mathbb{E}_z\big(|\omega_{1,0}| + |\omega_{1,1}|\big)$ and $D(z) = \mathbb{E}_z\big(|\omega_{0,1}| + |\omega_{1,1}|\big)$. Finally, let $L \to \infty$. $\square$

To complete the proof of Theorem 3.7, we state an analytic theorem. Suppose $A(z)$, $B(z)$ are real valued functions defined on $(-a_0, b_0)$. We assume that A and B are C^2 functions and A'', $B'' > 0$ on $(-a_0, b_0)$. We also assume that $A' < 0$ and $B' > 0$ on $(-a_0, b_0)$. For $-a_0 < z < b_0$, g_z is a linear function on $\mathbb{R}^2$ given by

$$g_z(x, y) = xA(z) + yB(z). \tag{3.63}$$

Let $g : \mathbb{R}^2_+ \to \mathbb{R}$ be a continuous, concave, and positive homogeneous function.

Theorem 3.19. *Suppose the identity*

$$g_z(1, 1) = \sup_{0 \le t \le 1} \{\max\{g(t, 1) + g_z(1 - t, 0),\, g(1, t) + g_z(0, 1 - t)\}\} \tag{3.64}$$

holds for all $-a_0 < z < b_0$ *and assume*

$$g(1, 0) = \lim_{z \uparrow b_0} A(z) \triangleq A(b_0), \quad g(0, 1) = \lim_{z \downarrow -a_0} B(z) \triangleq B(-a_0). \tag{3.65}$$

Then

$$g(x, y) = \inf_{-a_0 < z < b_0} \{xA(z) + yB(z)\} \tag{3.66}$$

for all x, $y \ge 0$.

Proof. This theorem is implied in the proof of Proposition 4.4 [17]. From (3.64) and linearity of g_z we have $0 = \sup\{g(x, y) - g_z(x, y) \,:\, |(x, y)|_\infty = 1,\, (x, y) \in \mathbb{R}^2_+\}$. Since g and g_z are continuous this supremum is achieved at some point (x_0, y_0). By (3.65) and strict monotonicity of A and B, (x_0, y_0) must be in the interior of $\mathbb{R}^2_+$. We claim that

$$0 = \sup\{g(t, 1) - g_z(t, 1) \,:\, 0 \le t < \infty\}$$

for all $-a_0 < z < b_0$. For $t \geq 0$ let $c_t = |(t,1)|_\infty$. Then from positive homogeneity of g, we get $g(t,1) - g_z(t,1) = c_t\big(g(t/c_t, 1/c_t) - g_z(t/c_t, 1/c_t)\big) \leq 0$. If $y_0 = 1$ take $t = x_0$ and we are done. Suppose $x_0 = 1$ and $0 < y_0 < 1$. In this case take $t = x_0/y_0$ and we are done. It follows immediately that

$$B(z) = \sup_{0 \leq t < \infty} \{-tA(z) + g(t,1)\}.$$

Define $h : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ by $h(t) = -g(t,1)$ for $t \geq 0$ and $h(t) = \infty$ for $t < 0$. Then h is lower semi-continuous and proper convex on $\mathbb{R}$. Let k be the function defined on $(-A(-a_0), -A(b_0))$ and given by $k(x) = B \circ A^{-1}(-x)$. Hence

$$k(x) = \sup_{0 \leq t < \infty} \{tx - h(x)\} \tag{3.67}$$

for any $x \in (-A(-a_0), -A(b_0))$. Let h^* denote the convex conjugate of h, that is,

$$h^*(x) = \sup_{t \in \mathbb{R}} \{tx - h(t)\} = \sup_{t \geq 0} \{tx - h(t)\} \tag{3.68}$$

for $x \in \mathbb{R}$. Comparison of (3.67) and (3.68) shows that h^* agrees with k on $(-A(-a_0), -A(b_0))$. Now we compute h^* on the complement of $(-A(-a_0), -A(b_0))$. From the second equality of (3.68), h^* is nondecreasing and is bounded below by $-h(0) = g(0,1) = B(-a_0)$. Since k is strictly increasing function,

$$\begin{aligned} h^*(-A(-a_0)) &\leq \lim_{x \downarrow -A(-a_0)} h^*(x) = \lim_{x \downarrow -A(-a_0)} k(x) \\ &= \lim_{x \downarrow -A(-a_0)} B \circ A^{-1}(-x) = \lim_{z \to -a_0} B(z) = B(-a_0). \end{aligned} \tag{3.69}$$

Hence $h^*(x) = B(-a_0)$ for $x \leq -A(-a_0)$. On the other hand, if $x > -A(b_0) = -g(1,0)$ then $h^*(x) = \infty$ since

$$\lim_{t \to \infty} tx - h(t) = \lim_{t \to \infty} t(x + g(1, 1/t)) = \infty.$$

For $x = -A(b_0)$, note that h^* is lower semi-continuous and nondecreasing. Therefore

$$h^*(-A(b_0)) = \lim_{x \uparrow -A(b_0)} h^*(x) = \lim_{x \uparrow -A(b_0)} k(x) = B(b_0)$$

and h^* is continuous on $(-\infty, -A(b_0)]$.

Since h is a lower semi-continuous and proper convex function, by the Fenchel-Moreau theorem, h equals the convex conjugate of h^*, hence,

$$h(t) = \sup_{x \in \mathbb{R}} \{tx - h^*(x)\} \tag{3.70}$$

for all $t \in \mathbb{R}$. We claim that, for $t \geq 0$, the supremum could be taken over the interval $(-A(-a_0), -A(b_0))$. For $t \geq 0$, the function $x \mapsto tx - h^*(x)$ is nondecreasing on $(-\infty, -A(-a_0)]$ and is $-\infty$ for $x > -A(b_0)$. Hence, from the continuity of h^* on $(-\infty, -A(b_0)]$,

$$\begin{aligned} h(t) &= \sup_{x \in [-A(-a_0), -A(b_0)]} \{tx - h^*(x)\} \\ &= \sup_{x \in (-A(-a_0), -A(b_0))} \{tx - h^*(x)\} \\ &= \sup_{z \in (-A(-a_0), -A(b_0))} \{-tA(z) - B(z)\} \\ &= -\inf_{z \in (-A(-a_0), -A(b_0))} \{tA(z) + B(z)\} \end{aligned} \tag{3.71}$$

which implies (3.66) from the positive homogeneity of g. □

Corollary 3.20. *Recall the definition of the sectors S, S_1, and S_2 in (3.9) and below. We have the following.*

(a) $g(x, y) = xA(-a_0) + yB(-a_0)$ *for* $(x, y) \in S_1$.

(b) $g(x, y) = xA(b_0) + yB(b_0)$ *for* $(x, y) \in S_2$.

(c) *$g(cx_1 + (1-c)x_2, cy_1 + (1-c)y_2) > cg(x_1, y_1) + (1-c)g(x_2, y_2)$ for $0 < c < 1$ and (x_1, y_1), $(x_2, y_2) \in S$ that are nonparallel.*

(d) *g is continuously differentiable.*

(e) $S_1 \neq \emptyset \Leftrightarrow A'(-a_0) < \infty$, $S_2 \neq \emptyset \Leftrightarrow B'(b_0) < \infty$.

Proof. This corollary is proved in Corollary 2.3 [17] with specific forms of A and B. Our proof is exactly same therein but for completeness we give details. Since A' and B' are increasing, for any fixed x, $y > 0$, the derivative $z \mapsto xA'(z) + yB'(z)$ is also increasing and continuous. If $(x, y) \in S_1$,

$$xA'(z) + yB'(z) \geq xA'(-a_0) + yB'(-a_0) \geq 0$$

so that the infimum of $g(x, y)$ is achieved at $z = -a_0$. If $(x, y) \in S_2$,

$$xA'(z) + yB'(z) \leq xA'(b_0) + yB'(b_0) \leq 0$$

and hence (b) follows.

If $(x, y) \in S$, the derivative has a unique zero. Suppose

$$g(x, y) = xA(z) + yB(z) \tag{3.72}$$

where $-a_0 < z < b_0$ is the unique solution of the equation

$$-\frac{B'(z)}{A'(z)} = \frac{x}{y}. \tag{3.73}$$

Since $-B'/A'$ is increasing and continuous, it has a strictly increasing and continuous inverse ζ defined on $\{r > 0 : (r, 1) \in S\}$. Also $-B'/A'$ is continuously differentiable with derivative $(A''B' - A'B'')/(A')^2 > 0$. Hence by the inverse function theorem, ζ is C^1 as well. Hence, for $(x, y) \in S$,

$$g(x, y) = xA(\zeta(x/y)) + yB(\zeta(x/y)) \tag{3.74}$$

and using (3.73), (3.74), we compute the gradient of g on S as

$$\nabla g(x, y) = (A(\zeta(x/y)), B(\zeta(x/y))). \tag{3.75}$$

This gradient tends to $(A(-a_0), B(-a_0))$ as (x, y) approaches S_1 and to $(A(b_0), B(b_0))$ as (x, y) approaches S_2. Hence we have (d).

If $(x_1, y_1), (x_2, y_2) \in S$ are nonparallel then $\zeta(x_1/y_1) \neq \zeta(x_2/y_2)$, which gives the strict inequality

$$\begin{aligned} g(x_1, y_1) + g(x_2, y_2) &= x_1 A(\zeta(x_1/y_1)) + y_1 B(\zeta(x_1/y_1)) \\ &\quad + x_2 A(\zeta(x_2/y_2)) + y_2 B(\zeta(x_2/y_2)) \\ &< (x_1 + x_2)A(z) + (y_1 + y_2)B(z) \end{aligned} \tag{3.76}$$

for any $-a_0 < z < b_0$. Setting $z = \zeta((x_1 + x_2)/(y_1 + y_2))$ gives $g(x_1 + x_2, y_1 + y_2) > g(x_1, y_1) + g(x_2, y_2)$, and (c) is proved from this and positive homogeneity of g. (e) is immediate from computations above. □

Proof of Theorem 3.7 completed. Note that A and B are infinitely differentiable and the derivatives are given by

$$\begin{aligned} \frac{d^k}{dz^k} A(z) &= -\int_{(0,\infty)} \Psi_k(z + \lambda)\, \alpha(d\lambda) \\ \frac{d^k}{dz^k} B(z) &= (-1)^{k+1} \int_{(0,\infty)} \Psi_k(-z + \theta)\, \beta(d\theta) \end{aligned} \tag{3.77}$$

for any $k \geq 1$. In particular, A is a strictly decreasing convex function and B is a strictly increasing convex function. The variational formula for the log-gamma polymer is immediate from Theorem 3.18 and Theorem 3.19. Note that condition (3.65) is satisfied by (3.46) and (3.47). □

Proof of Corollary 3.8. We obtain the result except for part (f) by Corollary 3.20. Since $S_1 \neq \emptyset$ if and only if $-B'(-a_0)/A'(-a_0) > 0$, we need to estimate $B'(-a_0)$ and $A'(-a_0)$. From (3.77) and (A.2), we have $B'(-a_0) > 0$. Therefore $S_1 \neq \emptyset$ if and only if $A'(-a_0) < \infty$, and this is equivalent to $\int \frac{1}{(\lambda - a_0)^2}\, \alpha(d\lambda) < \infty$ from (A.6). The condition for S_2 can be derived similarly. □

Chapter 4

Scaling exponents for the log-gamma polymer

4.1 Introduction

This chapter continues to study the log-gamma polymer with emphasis on the fluctuation exponents with slightly different weight assumptions. Recall that our parameters satisfy (1.5) and

$$a_0 \le \lambda_i \le a_1 \quad \text{and} \quad b_0 \le \theta_j \le b_1 \tag{4.1}$$

for $i \ge 1$ and $j \ge 1$. We refer the reader to Section 3.4 for the definitions of various polymer models (Definition 3.13) and notation conventions (Remark 3.15).

In previous chapters, we imposed the AMS conditions to the parameters (Λ, Θ) and expressed the limiting free energy in terms of marginal distributions of λ_1 and θ_1 under the stationary mean $\mathcal{Q}_0$. See Theorem 3.7 and Corollary 3.8. In this chapter, we leave the restrictive AMS settings but give weaker conditions to obtain the same results for the law of large numbers. Detailed analysis of stationary polymer models gives results for the scaling exponents also.

We assume the following. Suppose α and β are supported on $[a_0, a_1]$ and $[b_0, b_1]$, respectively. Suppose α and β satisfy

$$a_0 = \inf \operatorname{supp} \alpha \quad \text{and} \quad b_0 = \inf \operatorname{supp} \beta. \tag{4.2}$$

Consider the empirical measures for m, $n \in \mathbb{N}$

$$\alpha_m = \frac{1}{m}\sum_{i=1}^{m} \delta_{\lambda_i}, \quad \beta_n = \frac{1}{n}\sum_{j=1}^{n} \delta_{\theta_j} \tag{4.3}$$

where δ_x is a Dirac measure at the point $x \in \mathbb{R}$.

Assumption 4.1. *Parameters Λ, Θ satisfy the following conditions for measures α and β.*

(a) (Deterministic case) α_n and β_n converge weakly to α and β, respectively.

(b) (Random case) The distribution of (Λ, Θ) is $\mathcal{Q}$. For a.e. realization of (Λ, Θ), condition (a) is achieved.

The fluctuation of $\log Z$ is governed by the extremal statistics of parameters λ_i and θ_j. In particular, the case of linear sector S_1 (see (3.9) and the paragraph there) needs a careful analysis of these statistics. In this chapter, we assume that S_1 is nonempty. We introduce the following notations. For $m, n \geq 1$, rearrange the parameters $\lambda_1, \dots, \lambda_m$ and $\theta_1 \dots, \theta_n$ into a nondecreasing sequences

$$x_{m:1} \leq x_{m:2} \leq \cdots \leq x_{m:m} \tag{4.4}$$

and

$$y_{n:1} \leq y_{n:2} \leq \cdots \leq y_{n:n}. \tag{4.5}$$

Set

$$x_m = x_{m:1} = \min_{1 \leq i \leq m} \lambda_i \quad \text{and} \quad y_n = y_{n:1} = \min_{1 \leq j \leq n} \theta_j. \tag{4.6}$$

The behavior of $\log Z$ in S_1 is diffusive. To get concrete results, we further impose assumptions. We only state assumptions for the case $S_1 \neq \emptyset$. One can easily obtain a similar result in the case $S_2 \neq \emptyset$. Note that we have $\int \frac{1}{(\lambda - a_0)^2} \alpha(d\lambda) < \infty$ from Corollary 3.8.

Assumption 4.2. *We add the following conditions to Assumption 4.1. There are positive constants p_1, q_1, d_1 with $2 < p_1 \leq 3$ and $d_1 < (p_1 - 2)q_1/3$ for which the following hold.*

(a) (Convergence condition)

$$\lim_{m\to\infty} \int \frac{1}{(\lambda - a_0)^{p_1}} \alpha_m(d\lambda) = \int \frac{1}{(\lambda - a_0)^{p_1}} \alpha(d\lambda) < \infty.$$

(b) (Separability condition) There are constants $C_{q_1} > 0$ and $D_1 > 0$ such that

$$(x_{m:1} - a_0) \leq \frac{C_{q_1}}{m^{q_1}}$$

for $m \geq 1$ and

$$\frac{x_{m:1} - a_0}{x_{m:2} - a_0} \leq 1 - \frac{D_1}{m^{d_1}}$$

for all sufficiently large m.

(c) In the case of Assumption 4.1 (b), conditions (a) and (b) with random constants C_{q_1}, D_1 are satisfied for a.e. realization of (Λ, Θ).

Note that condition (a) enforces $\lambda_i > a_0$ for all sufficiently large i. Hence we assume $\lambda_i > a_0$ for all i. If λ_is are chosen randomly according to α and α satisfies appropriate moment conditions, condition (a) is achieved a.s. Condition (b) is also satisfied for a wide class of distributions. See conditions above Theorem 1 of [23] and Section 6 therein.

4.2 Results

In this section, we present our results. In Theorem 3.7, the limiting free energy is given by

$$\bar{\phi}(x,y) = \inf_{-a_0<z<b_0}\{xA(z)+yB(z)\},$$

where A and B are defined by (3.8). Recall that S denotes the sector of the first quadrant on which

$$-B'(-a_0)/A'(-a_0) < x/y < -B'(b_0)/A'(b_0)$$

and S_1, S_2 denote the sectors defined by the inequalities $x/y \le -B'(-a_0)/A'(-a_0)$ and $x/y \ge -B'(b_0)/A'(b_0)$, respectively. The boundary of $\bar{S}$ consists of two lines. Write $S_1 \cap \bar{S}$ and $S_2 \cap \bar{S}$ by lines

$$x/y = s_1 = -\frac{B'(-a_0)}{A'(-a_0)} \quad \text{and} \quad x/y = s_2 = -\frac{B'(b_0)}{A'(b_0)}, \tag{4.7}$$

respectively. For $(x,y) \in S$, there exists a unique minimizer $\zeta \in (-a_0, b_0)$ given by (3.10). ζ satisfies

$$xA'(\zeta) + yB'(\zeta) = 0.$$

The statements of our results involve discrete version of quantities A, B and ζ computed with α_m and β_n in place of α and β, respectively.

We focus on (Λ,Θ)-polymer and (Λ,Θ,z)-polymer. In the following we understand (Λ,Θ) are given and fixed. For a probability measure μ on $[a,b]$ define

$$A_\mu(z) = -\int \Psi_0(\lambda+z)\,\mu(d\lambda) \quad \text{for} \quad z > -a \tag{4.8}$$

and $B_\mu(z) = A_\mu(-z)$ for $z < a$. $A_\mu(z)$ and $B_\mu(z)$ are monotone functions because polygamma functions are monotone (see (A.1)). Note that $A(z) = A_\alpha(z)$ and $B(z) =$

$B_\beta(z)$. We write $A_m(z) = A_{\alpha_m}(z)$ and $B_n(z) = B_{\beta_n}(z)$ for α_m and β_n. Therefore

$$A_m(z) = -\int \Psi_0(\lambda + z)\,\alpha_m(d\lambda) = -\frac{1}{m}\sum_{i=1}^{m}\Psi_0(\lambda_i + z) \tag{4.9}$$

and

$$B_n(z) = -\int \Psi_0(\theta - z)\,\beta_n(d\theta) = -\frac{1}{n}\sum_{j=1}^{m}\Psi_0(\theta_j - z). \tag{4.10}$$

We interpret $A_0(z) = 0$. Note that

$$A'_m(z) < 0\,, A''_m(z) > 0 \quad \text{and} \quad B'_n(z) > 0\,, B''_n(z) > 0. \tag{4.11}$$

For fixed m, n, A_m and B_n are defined for $-x_m < z < y_n$, where x_m and y_n are given by (4.6). Note that $(-a_0, b_0) \subseteq (-x_m, y_n)$ and this inclusion could be proper. Define

$$G_{m,n}(z) = mA_m(z) + nB_n(z) \tag{4.12}$$

and

$$M_{m,n}(z) = mA''_m(z) + nB''_n(z) > 0. \tag{4.13}$$

From Proposition 3.16 we have

$$G_{m,n}(z) = \mathbb{E}^{\rho,z}\log Z_{m,n} \tag{4.14}$$

and

$$M_{m,n}(z) = \frac{d^2}{dz^2}\mathbb{E}^{\rho,z}\log Z_{m,n} \tag{4.15}$$

when we consider $\mathbb{E}^{\rho,z}\log Z_{m,n}$ as a function of z. Recall our notation conventions in Remark 3.15. Here ρ refers to the bulk parameters and z to boundary parameters.

For m, $n \geq 1$,

$$\lim_{s\to y_n} mA'_m(s) + nB'_n(s) = \infty \tag{4.16}$$

and

$$\lim_{s\to -x_m} mA_m'(s) + nB_n'(s) = -\infty. \tag{4.17}$$

Therefore there exists a unique $\zeta_{m,n} \in (-x_m, y_n)$ with

$$G_{m,n}'(\zeta_{m,n}) = 0 \tag{4.18}$$

since $G_{m,n}(z)$ is a smooth convex function of z. Hence $\zeta_{m,n}$ is the unique minimizer of $G_{m,n}(z)$. Note that since it is possible to get $x_m > a_0$, we may have $\zeta_{m,n} \in (-x_m, -a_0)$. Theorem 4.4 below will give conditions for the range of $\zeta_{m,n}$. Define

$$\phi_{m,n} = \inf_{-x_m<z<y_n} \{mA_m(z) + nB_n(z)\} = G_{m,n}(\zeta_{m,n}) \tag{4.19}$$

and

$$\sigma_{m,n} = \left(M_{m,n}(\zeta_{m,n})\right)^{1/3}. \tag{4.20}$$

We will show that $\mathbb{E}^{\rho} \log Z_{(1,1),(m,n)}$ is close to $\phi_{m,n}$ and fluctuation of $\log Z_{(1,1),(m,n)}$ around $\phi_{m,n}$ is controlled by $\sigma_{m,n}$. It turns out that $\sigma_{m,n}$ grows in the order of $N^{1/3}$ in S and $N^{1/2}$ in S_1, where $N = m + n$. We obtain these results first for some stationary polymer models and then, by coupling these models with the original model, for the (Λ, Θ)-polymer. Note that $\phi_{m,n} = \mathbb{E}^{\rho,\zeta_{m,n}} \log Z_{m,n}$.

Theorem 4.3. *For given $m, n \geq 1$, consider the $(\Lambda, \Theta, \zeta_{m,n})$-stationary polymer in Definition 3.13. Then there exist positive constants C_0, C and N_0 such that*

$$\mathbb{V}\mathrm{ar}^{\rho,\zeta_{m,n}}(\log Z_{m,n}) \leq C\big(\sigma_{m,n}\big)^2 \tag{4.21}$$

for all $N \geq 1$ and

$$\mathbb{V}\mathrm{ar}^{\rho,\zeta_{m,n}}(\log Z_{m,n}) \geq C_0\big(\sigma_{m,n}\big)^2 \tag{4.22}$$

for all $N \geq N_0$.

The constants C_0, C and N_0 do not depend on m, n. They only depend on a_0+b_0 and a_1+b_1. We have explicit formulas for these constants. See (4.117) for C. We can take $C_0=e^{-C^2}$ and $N_0=C^2$. See Lemma 4.28. This theorem reveals scaling exponents in the KPZ universality class. A key point in KPZ class is that the fluctuation exponents should be connected to curvature. This is explicit in the bounds (4.21) and (4.22) with (4.15) and in particular the quadratic provides the $2/3$ exponent.

This theorem does not use Assumptions 4.1 and 4.2. If we use Assumptions 4.1 and 4.2, then we can quantify $\sigma_{m,n}$ and find a connection to Theorem 3.7 and Corollary 3.8. In the remainder of this section, all constants implicitly depend on a_0, b_0 and a_1, b_1. Also under Assumptions 4.1 and 4.2, these constants depend on the convergence rate of α_n and β_n to α and β, respectively. If we give some conditions for the rate of convergence in terms of Wasserstein distance W_1 in (4.60), we can obtain explicit formulas. However, we do not pursue such details.

For the (Λ,Θ,z)-stationary polymer, *characteristic direction* is $v(z)=(v_1(z),v_2(z))=(B'(z),-A'(z))/(B'(z)-A'(z))$. For (Λ,Θ)-polymer without boundary, we say $\zeta(x/y)$ is the *characteristic value* of (x,y) if $(x,y)\in S$. Let $N=m+n$ denote the scaling parameter we take to ∞. Fix $(x,y)\in\mathbb{R}^2_{>0}$ with $x+y=1$. We take (m,n) along the direction (x,y). More precisely we assume that the coordinates (m,n) of the endpoint of the polymer satisfy

$$|m-Nx|\vee|n-Ny|\leq K \tag{4.23}$$

for some fixed constant $K>0$. In light of Theorem 3.7 and Corollary 3.8, we expect that $\zeta_{m,n}$ is close to the characteristic value $\zeta(x/y)$ for $(x,y)\in S$ and $-a_0$ for $(x,y)\in S_1$.

Theorem 4.4. *Suppose Assumptions 4.1 and 4.2 hold. Let* $(x,y)\in\mathbb{R}^2_{>0}$ *with* $x+y=1$

and $\epsilon > 0$ *be given. Assume* (m, n) *satisfy* (4.23)*. Let* $s = x/y$*. There exist positive constants* $N_0(s, K, \epsilon)$*,* $C_0(s)$*,* $C_1(s)$ *and* $C_2(s)$ *such that whenever* $N \geq N_0$ *the following hold.*

(1) For $(x, y) \in S$*, that is,* $s_1 < s < s_2$*,*

$$C_1 N^{1/3} \leq \sigma_{m,n} \leq C_2 N^{1/3} \tag{4.24}$$

and

$$|\zeta_{m,n} - \zeta(s)| \leq \epsilon. \tag{4.25}$$

(2) For $(x, y) \in S_1$ *and* $s < s_1$*,*

$$C_1 N^{1/2} \leq \sigma_{m,n} \leq C_2 N^{1/2} \tag{4.26}$$

and

$$|\zeta_{m,n} + x_m| \leq \frac{C_0}{\sqrt{m}}. \tag{4.27}$$

We also have

$$\lim_{m\to\infty} |\zeta_{m,n} + a_0| = 0, \quad \lim_{m\to\infty} \sqrt{m}|\zeta_{m,n} + a_0| = \infty. \tag{4.28}$$

(3) We have

$$\lim_{N\to\infty} \frac{\phi_{m,n}}{N} = \bar{\phi}(x, y). \tag{4.29}$$

The constants in the above Theorem continuously depend on s. Part (3) shows that the quenched shape converges to the annealed shape as N grows.

Next, we study the model without boundaries. First, we state a general fluctuation result. Define

$$\Delta\zeta_{m,n} = (x_m + \zeta_{m,n}) \wedge (y_n - \zeta_{m,n}) \tag{4.30}$$

and

$$K_{m,n} = \frac{\Delta\zeta_{m,n}(\sigma_{m,n})^3}{m|A'_m(\zeta_{m,n})|} = \frac{\Delta\zeta_{m,n}(\sigma_{m,n})^3}{nB'_n(\zeta_{m,n})}. \tag{4.31}$$

Theorem 4.5. *Consider the* (Λ, Θ)*-polymer. Then there exist constants* C*,* C_1 *and* N_0 *that depend on* a_0+b_0 *and* a_1+b_1 *such that whenever* $m+n \geq N_0$*, we have the following.*

$$\begin{aligned}
&\mathbb{P}^\rho\left\{\left|\log Z_{(1,1),(m,n)} - \phi_{m,n}\right| \geq t\sigma_{m,n}\right\} \\
&\leq \begin{cases} \frac{C}{t^2}, & (0 < t \leq (\Delta\zeta_{m,n}\sigma_{m,n})^2/4) \\ C\left(\frac{1}{t} \wedge \exp\left[-K_{m,n}\sqrt{t}\right]\right), & ((\Delta\zeta_{m,n}\sigma_{m,n})^2/4) \leq t \leq 4(a_1+b_1+1)^2\sigma^2_{m,n} \\ C\exp[-t], & (t \geq 4(a_1+b_1+1)^2\sigma^2_{m,n}) \end{cases} \qquad (4.32)\\
&\leq \frac{C_1}{t^2}\left(1 \vee \frac{1}{(K_{m,n})^4}\right).
\end{aligned}$$

If we use Assumptions 4.1 and 4.2, by estimating $K_{m,n}$, then we obtain the following results. Here we have a result for the limiting point-to-point free energy of the polymer without boundary.

Theorem 4.6. *Consider the* (Λ, Θ)*-polymer. Suppose Assumptions 4.1 and 4.2 hold and let* x*,* $y > 0$*. Then we have the same result as in Theorem 3.7.*

$$\lim_{L\to\infty} L^{-1}\log Z^\omega_{(1,1),(\lfloor Lx\rfloor,\lfloor Ly\rfloor)} = \bar{\phi}(x,y). \tag{4.33}$$

For the fluctuation results, we have the following.

Theorem 4.7. *Consider the* (Λ, Θ)*-polymer. Let* $(x,y) \in S$ *with* $x+y=1$*. Let* $s = x/y$*. If Assumption 4.1(a) holds then there exist positive constants* L_0*,* C_0*,* C_1*, and* C_2 *that depend on* s *such that, for* $t > 0$ *and* $L \geq L_0$*,*

$$\mathbb{P}^\rho\left[|\log Z_{(1,1),(\lfloor Lx\rfloor,\lfloor Ly\rfloor)} - \phi_{\lfloor Lx\rfloor,\lfloor Ly\rfloor}| \geq tL^{1/3}\right] \leq \frac{C_0}{t^2}, \tag{4.34}$$

and

$$\mathbb{E}^{\rho}\Big|\log Z_{(1,1),(\lfloor Lx\rfloor,\lfloor Ly\rfloor)} - \phi_{\lfloor Lx\rfloor,\lfloor Ly\rfloor}\Big|^2 \le C_1 L^{2/3}\log L. \tag{4.35}$$

For the lower bound, we have

$$C_2 L^{1/3} \le \mathbb{E}^{\rho}\Big|\log Z_{(1,1),(\lfloor Lx\rfloor,\lfloor Ly\rfloor)} - \phi_{\lfloor Lx\rfloor,\lfloor Ly\rfloor}\Big|. \tag{4.36}$$

Therefore in the region S, the expected KPZ behavior is proved. (4.34) and (4.35) are our improvement compared to Theorem 2.4 of [33]. There, they proved similar probability bounds with $C/t^{3/2}$. Next theorem shows that the fluctuation in S_1 is of order $L^{1/2}$.

Theorem 4.8. *Consider the* (Λ,Θ)*-polymer. Let* $(x,y)\in S_1$ *with* $x+y=1$*. Suppose* $s=x/y<s_1$*. If Assumptions 4.1 and 4.2 hold then there exist positive constants* L_0*,* C_0*,* C_1*, and* C_2 *that depend on* s *such that, for* $t>0$ *and* $L\ge L_0$*,*

$$\mathbb{P}^{\rho}\left[|\log Z_{(1,1),(\lfloor Lx\rfloor,\lfloor Ly\rfloor)} - \phi_{\lfloor Lx\rfloor,\lfloor Ly\rfloor}| \ge tL^{1/2}\right] \le C_0\exp[-C_1\sqrt{t}]. \tag{4.37}$$

For the lower bound, we have

$$C_2 L^{1/2} \le \mathbb{E}^{\rho}\Big|\log Z_{(1,1),(\lfloor Lx\rfloor,\lfloor Ly\rfloor)} - \phi_{\lfloor Lx\rfloor,\lfloor Ly\rfloor}\Big|. \tag{4.38}$$

The behavior of the free energy on the boundary of S is subtle to analysis. We do not have precise scaling exponent. We only prove the following Theorem, which shows that the fluctuation is at least subdiffusive.

Theorem 4.9. *Consider the* (Λ,Θ)*-polymer. Suppose Assumptions 4.1 and 4.2 hold and let* $x/y=s_1$*. Then we have*

$$\frac{\log Z_{(1,1),(\lfloor Lx\rfloor,\lfloor Ly\rfloor)} - \phi_{\lfloor Lx\rfloor,\lfloor Ly\rfloor}}{\sqrt{L}} \to 0 \tag{4.39}$$

in probability under $\mathbb{P}^{\rho}$*.*

Organization of Chapter 4. Before we prove the main results in Section 4.3, we collect basic properties of the model. From the Burke-type property, we obtain a formula for the variance of stationary polymers. The upper and lower bounds of Theorem 4.3 are proved in Sections 4.4 and 4.6. Theorem 4.4 is proved in Section 4.7. The results for the polymer without boundary conditions are proved in Section 4.8. Theorem 4.5 is the most important result. It is obtained by coupling the original polymer model with an appropriate stationary polymer. Then the remaining theorems are easily proved from these results.

4.3 Basic properties and technical results

We follow [33] for notations and quote some theorems therein. Sometimes it is convenient to use multiplicative weights: $Y_{i,j} = e^{\omega(i,j)}$, $(i,j) \in \mathbb{Z}_+^2$. Then the partition function is given by

$$Z_{m,n} = \sum_{\mathbf{x}_\cdot \in \Pi_{m,n}} \prod_{k=1}^{m+n} Y_{x_k} \tag{4.40}$$

where $\Pi_{m,n}$ denotes the set of admissible lattice paths $\mathbf{x}_\cdot = (\mathbf{x}_i)_{0\le i\le m+n}$ that satisfy $\mathbf{x}_0 = (0,0)$, $\mathbf{x}_i - \mathbf{x}_{i-1} \in \{(1,0),(0,1)\}$, $\mathbf{x}_{m+n} = (m,n)$. Symbols U and V will denote weights on the horizontal and vertical boundaries:

$$U_{i,0} = Y_{i,0} \quad \text{and} \quad V_{0,j} = Y_{0,j} \quad \text{for } i,j \in \mathbb{N}. \tag{4.41}$$

The partition function that includes the weight at the starting point is written as

$$Z^{\square}_{(i,j),(k,l)} = \sum_{\mathbf{x}_\cdot \in \Pi_{(i,j),(k,l)}} \prod_{r=0}^{k-i+l-j} Y_{x_r} = Y_{i,j} Z_{(i,j),(k,l)} \tag{4.42}$$

where $\Pi_{(i,j),(k,l)}$ is the set of up-right paths $\mathbf{x}_\cdot = (\mathbf{x}_r)_{0\le r\le k-i+l-j}$ from $\mathbf{x}_0 = (i,j)$ to $\mathbf{x}_{k-i+l-j} = (k,l)$.

Recall definitions of $I_{m,n}$ and $J_{m,n}$ in (3.48). Define for $(i,j) \in \mathbb{N}^2$

$$\begin{aligned} U_{i,j} = \frac{Z_{i,j}}{Z_{i-1,j}} = e^I_{i,j}, \quad V_{i,j} = \frac{Z_{i,j}}{Z_{i,j-1}} = e^J_{i,j} \\ \text{and} \quad X_{i-1,j-1} = \left(\frac{1}{U_{i,j-1}} + \frac{1}{V_{i-1,j}}\right)^{-1}. \end{aligned} \tag{4.43}$$

The partition function satisfies

$$Z_{m,n} = Y_{m,n}(Z_{m-1,n} + Z_{m,n-1}) \quad \text{and } (m,n) \in \mathbb{N}^2. \tag{4.44}$$

Lemma 4.10 (Lemma 3.1 of [33]). *Consider two sets of positive initial values* $\{U_{i,0}, V_{0,j}, Y_{i,j} : i,j \in \mathbb{N}\}$ *and* $\{\tilde{U}_{i,0}, \tilde{V}_{0,j}, \tilde{Y}_{i,j} : i,j \in \mathbb{N}\}$ *and satisfy* $U_{i,0} \geq \tilde{U}_{i,0}$, $V_{0,j} \leq \tilde{V}_{0,j}$, *and* $Y_{i,j} = \tilde{Y}_{i,j}$. *Then* $U_{i,j} \geq \tilde{U}_{i,j}$, $V_{i,j} \leq \tilde{V}_{i,j}$ *for all* $(i,j) \in \mathbb{N}^2$.

Proposition 4.11 (Theorem 3.3 of [33] and Proposition 3.16). *Consider the* (Λ, Θ, z)*-stationary polymer. Let* $(m,n) \in \mathbb{N}^2$. *The variables* $\{U_{i,n}, V_{m,j}, X_{k-1,l-1} : 1 \leq i,k \leq m, 1 \leq j,l \leq n\}$ *are mutually independent with marginal distributions*

$$\begin{aligned} U_{i,n}^{-1} \sim \mathit{Gamma}(\lambda_i + z),\ V_{m,j}^{-1} \sim \mathit{Gamma}(\theta_j - z), \\ \text{and } X_{k-1,l-1}^{-1} \sim \mathit{Gamma}(\lambda_k + \theta_l). \end{aligned} \tag{4.45}$$

Let $\sigma : \mathbb{N} \to \mathbb{N}$ be a finite permutation that $\sigma(i) \neq i$ for only finitely many i. Consider a sequence $\Lambda = (\lambda_1, \lambda_2, \dots) \in \mathbb{R}^{\mathbb{N}}$ and empirical measures associated with this sequence $\alpha_n = \frac{1}{n}\sum_{i=1}^n \delta_{\lambda_i}$. σ acts naturally on parameters Λ and α_n, that is, $(\sigma\Lambda)_i = \lambda_{\sigma(i)}$ and $(\sigma\alpha)_n = \frac{1}{n}\sum_{i=1}^n \delta_{\lambda_{\sigma(i)}}$.

$R_n : \mathbb{Z}_+ \to \mathbb{Z}_+$ is a reflection defined by $R_n(i) = n+1-i$ for $1 \leq i \leq n$ and $R_n(i) = i$ for $i = 0$ and $i > n$. The restriction of R_n onto $\mathbb{N}$ is also denoted by R_n. Define $R_{m,n} : \mathbb{Z}_+^2 \to \mathbb{Z}_+^2$ by

$$R_{m,n}(i,j) = (R_m(i), R_n(j)) \tag{4.46}$$

for $(i,j) \in \mathbb{Z}_+^2$. $R_{m,n}$ acts naturally on $\omega \in \mathbb{R}^{\mathbb{Z}_+^2}$ by

$$(R_{m,n}\omega)_{i,j} = \omega_{R_{m,n}(i,j)}.$$

For a fixed rectangle $B_{m,n} = \{0,\dots,m\} \times \{0,\dots,n\}$, we define the reversed environment by

$$\tilde{\omega} = R_{m,n}\omega. \tag{4.47}$$

We have the following lemma whose proof is elementary.

Lemma 4.12. *If ω defines a (Λ,Θ,z)-stationary polymer, then $\tilde{\omega}$ defines a $(R_m\Lambda, R_n\Theta, z)$-stationary polymer and $Z^{\square,\omega}_{(1,1),(m,n)} = Z^{\square,\tilde{\omega}}_{(1,1),(m,n)}$. Therefore $Z^{\square,\omega_1}_{(1,1),(m,n)} \overset{d}{=} Z^{\square,\omega_2}_{(1,1),(m,n)}$ if ω_1 and ω_2 are constructed by (Λ,Θ) and $(R_m\Lambda, R_n\Theta)$, respectively.*

Let

$$\xi_x = \max\{k \geq 0 : x_i = (i,0) \text{ for } 0 \leq i \leq k\} \tag{4.48}$$

and

$$\xi_y = \max\{k \geq 0 : x_j = (0,j) \text{ for } 0 \leq j \leq k\} \tag{4.49}$$

denote the exit points of a path from the x- and y-axes. Recall that the annealed measure is $P^{\rho,z}(\cdot) = \mathbb{E}^{\rho,z}Q^{\omega}(\cdot)$ with expectation $E^{\rho,z}(\cdot)$ (see Remark 3.15). The function $L(r,x)$ in the following Theorem is introduced in (A.18). We refer to Lemma A.2 for the properties of L.

Theorem 4.13 (Theorem 3.7 of [33])**.** *Consider (Λ,Θ,z)-stationary polymer. For $m,n \in \mathbb{Z}_+$ we have these identities:*

$$\mathbb{V}\mathrm{ar}^{\rho,z}[\log Z_{m,n}] = \sum_{j=1}^{n} \Psi_1(\theta_j - z) - \sum_{i=1}^{m} \Psi_1(\lambda_i + z) + 2E^{\rho,z}_{m,n}\left[\sum_{i=1}^{\xi_x} L(\lambda_i + z, -\omega_{i,0})\right] \tag{4.50}$$

and

$$\mathbb{V}\mathrm{ar}^{\rho,z}[\log Z_{m,n}] = -\sum_{j=1}^{n} \Psi_1(\theta_j - z) + \sum_{i=1}^{m} \Psi_1(\lambda_i + z) + 2E^{\rho,z}_{m,n}\left[\sum_{j=1}^{\xi_y} L(\theta_j - z, -\omega_{0,j})\right]. \tag{4.51}$$

When $\xi_x = 0$ *or* $\xi_y = 0$ *the sum is interpreted as* 0*.*

Proof. We follow [33] with appropriate modifications. We prove (4.50). Let us abbreviate temporarily, according to the compass directions of the rectangle $B_{m,n}$,

$$S_{\mathcal{N}} = \log Z_{m,n} - \log Z_{0,n},\ S_{\mathcal{S}} = \log Z_{m,0},\ S_{\mathcal{E}} = \log Z_{m,n} - \log Z_{m,0},\ S_{\mathcal{W}} = \log Z_{0,n}.$$

Then

$$\begin{aligned}
\mathbb{V}\mathrm{ar}^{\rho,z}[\log Z_{m,n}] &= \mathbb{V}\mathrm{ar}^{\rho,z}(S_{\mathcal{W}} + S_{\mathcal{N}}) = \mathbb{V}\mathrm{ar}^{\rho,z}(S_{\mathcal{W}}) + \mathbb{V}\mathrm{ar}^{\rho,z}(S_{\mathcal{N}}) + 2\,\mathbb{C}\mathrm{ov}^{\rho,z}(S_{\mathcal{W}}, S_{\mathcal{N}}) \\
&= \mathbb{V}\mathrm{ar}^{\rho,z}(S_{\mathcal{W}}) + \mathbb{V}\mathrm{ar}^{\rho,z}(S_{\mathcal{N}}) + 2\,\mathbb{C}\mathrm{ov}^{\rho,z}(S_{\mathcal{S}} + S_{\mathcal{E}} - S_{\mathcal{N}}, S_{\mathcal{N}}) \\
&= \mathbb{V}\mathrm{ar}^{\rho,z}(S_{\mathcal{W}}) - \mathbb{V}\mathrm{ar}^{\rho,z}(S_{\mathcal{N}}) + 2\,\mathbb{C}\mathrm{ov}^{\rho,z}(S_{\mathcal{S}}, S_{\mathcal{N}}).
\end{aligned} \tag{4.52}$$

The last equality is from Proposition 4.11. By assumption $\mathbb{V}\mathrm{ar}^{\rho,z}(S_{\mathcal{W}}) = \sum_{j=1}^{n} \Psi_1(\theta_j - z)$, and $\mathbb{V}\mathrm{ar}^{\rho,z}(S_{\mathcal{N}}) = \sum_{i=1}^{m} \Psi_1(\lambda_i + z)$ by Proposition 4.11.

It remains to work on $\mathbb{C}\mathrm{ov}^{\rho,z}(S_{\mathcal{S}}, S_{\mathcal{N}})$. Now consider a system with two independent parameters λ and θ with weight distributions (for $i, j \in \mathbb{N}$, $\lambda > -a_0, \theta > -b_0$)

$$-\omega_{i,0} \sim \text{log-gamma}(\lambda_i + \lambda), \quad -\omega_{0,j} \sim \text{log-gamma}(\theta_j + \theta)$$

and

$$-\omega_{i,j} \sim \text{log-gamma}(\lambda_i + \theta_j).$$

We show that

$$\mathbb{C}\mathrm{ov}^{\rho,z}(S_{\mathcal{S}}, S_{\mathcal{N}}) = -\frac{\partial}{\partial \lambda}\mathbb{E}^{\lambda,\theta}(S_{\mathcal{N}})\big|_{(\lambda,\theta)=(z,-z)}. \tag{4.53}$$

The joint p.d.f. of $\{-\omega_{i,j}\}$ is $(x = \{x_{i,j}\})$

$$\begin{aligned} g_{\lambda,\theta}(x) = &\prod_{i=1}^{m} \frac{1}{\Gamma(\lambda_i+\lambda)} \exp[(\lambda_i+\lambda)x_{i,0} - e^{x_{i,0}}] \cdot \prod_{j=1}^{n} \frac{1}{\Gamma(\theta_j+\theta)} \exp[(\theta_j+\theta)x_{0,j} - e^{x_{0,j}}] \\ &\cdot \prod_{\substack{1\le i\le m\\ 1\le j\le n}} \frac{1}{\Gamma(\lambda_i+\theta_j)} \exp[(\lambda_i+\theta_j)x_{i,j} - e^{-x_{i,j}}] \\ &= \prod_{i=1}^{m} \frac{1}{\Gamma(\lambda_i+\lambda)} \prod_{j=1}^{n} \frac{1}{\Gamma(\theta_j+\theta)} \cdot \exp(\lambda \sum_{i=1}^{m} x_{i,0}) \\ &\cdot \exp(\theta \sum_{j=1}^{n} x_{0,j}) \cdot G(\Lambda, \Theta, x). \end{aligned}$$

Therefore if we only focus on λ, we can write $g_{\lambda,\theta}(x)dx = \dfrac{e^{\lambda T(x)}\mu(dx)}{A(\lambda)}$ where $T(x) = \sum_{i=1}^{m} x_{i,0}$, $\mu(dx) = \exp(\theta \sum_{j=1}^{n} x_{0,j}) \cdot G(\Lambda, \Theta, x)dx$, and $A(\lambda)$ is a normalizing factor. Hence we can apply Lemma A.1 (3). Since $x_{i,j} = -\omega_{i,j}$, we have $T = -S_{\mathcal{S}}$ and (4.53) follows.

Next, we calculate $(\partial/\partial\lambda)\mathbb{E}^{\lambda,\theta}(S_{\mathcal{N}})$. We also utilize a direct functional dependence on λ in $Z_{m,n}$ by realizing the weights $\omega_{i,0}$ as functions of uniform random variables. Then if $\eta = (\eta_1, \ldots, \eta_m)$ is a vector of Uniform(0,1) random variables, $\omega_{i,0} = -H_{\lambda_i+\lambda}(\eta_i)$ where H is a function introduced in (A.16) . Note that

$$\frac{\partial}{\partial\lambda}\mathbb{E}^{\lambda,\theta}(S_{\mathcal{N}}) = \frac{\partial}{\partial\lambda}\mathbb{E}^{\lambda,\theta}(\log Z_{m,n} - S_{\mathcal{W}}) = \frac{\partial}{\partial\lambda}\mathbb{E}^{\lambda,\theta}\log Z_{m,n} = \mathbb{E}^{\lambda,\theta}\frac{\partial}{\partial\lambda}\log Z_{m,n}. \quad (4.54)$$

We justify the last equality soon. First, we compute $\frac{\partial}{\partial\lambda}\log Z_{m,n}$. A quenched measure can be written as

$$Q^{\omega}(d\mathbf{x}_\cdot) = \frac{e^{\mathcal{H}^{\omega}(\mathbf{x}_\cdot)}\mu_c(d\mathbf{x}_\cdot)}{Z_{m,n}} \quad (4.55)$$

where μ_c is the counting measure on $\Pi_{m,n}$ and

$$\mathcal{H}^{\omega}(\mathbf{x}_\cdot) = \sum_{k=1}^{m+n} \omega_{\mathbf{x}_k} = \sum_{i=1}^{\xi_x}(-H_{\lambda_i+\lambda}(\eta_i)) + \sum_{k=\xi_x+1}^{m+n} \omega_{\mathbf{x}_k}, \quad (4.56)$$

$$Z_{m,n} = Z_{m,n}(\lambda) = \sum_{\mathbf{x}_\bullet \in \Pi_{m,n}} e^{\mathcal{H}^\omega(\mathbf{x}_\bullet)}. \tag{4.57}$$

Recall that $\sum_{i=1}^{\xi_x}$ is interpreted as 0 when $\xi_x = 0$. Lemma A.1 and (A.19) give

$$\frac{\partial}{\partial \lambda} \log Z_{m,n} = E^{Q^\omega} \frac{\partial \mathcal{H}^\omega}{\partial \lambda} = -E^{Q^\omega} \sum_{i=1}^{\xi_x} L(\lambda_i + \lambda, H_{\lambda_i+\lambda}(\eta_i)). \tag{4.58}$$

Taking expectation, we obtain (4.50). Finally, we justify the interchange of expectation and differentiation in (4.54). Using (A.22) and strict monotonicity of $H(r,\eta)$ in the first coordinate, we can invoke the dominated convergence theorem.

We finish the proof by recording some useful identities. We replace the weights on the y-axis with functions of uniform random variables. Let $\eta' = (\eta'_1, \ldots, \eta'_n)$ be a vector of Uniform(0,1) random variables independent of η and the bulk weights ω. Let $\omega_{0,j} = -H_{\theta_j+\theta}(\eta'_j)$. Write $\tilde{\mathbb{E}}^\rho$ for the expectation over the uniform variables η, η' and the bulk weights $\{\omega_{i,j} : i, j \geq 1\}$. Then we have

$$\begin{aligned} E^{\rho,z}_{m,n}\left[\sum_{i=1}^{\xi_x} L(\lambda_i + z, -\omega_{i,0})\right] &= -\,\tilde{\mathbb{E}}^\rho\big(E^{Q^\omega} \frac{\partial \mathcal{H}^\omega}{\partial \lambda}\big)\big|_{(\lambda,\theta)=(z,-z)} \\ E^{\rho,z}_{m,n}\left[\sum_{j=1}^{\xi_y} L(\theta_i - z, -\omega_{0,j})\right] &= -\,\tilde{\mathbb{E}}^\rho\big(E^{Q^\omega} \frac{\partial \mathcal{H}^\omega}{\partial \theta}\big)\big|_{(\lambda,\theta)=(z,-z)}. \end{aligned} \tag{4.59}$$

□

In the remainder of this section, we collect facts for the Wasserstein distance W_1. For the proofs, we refer the reader to [7, 15]. Given a separable metric space (E,d), let $Z(E,d)$ denote the space of all Borel probability measures μ on E. For $p \geq 1$, denote by $Z_p(E,d)$, or just $Z_p(E)$ the collection of all probability measures μ in $Z(E,d)$ such that

$$\int_E d(x,x_0)^p \, \mu(dx) < \infty$$

for some, or equivalently all, $x_0 \in E$. The Wasserstein distance W_p between two probability measures μ, ν on E is defined by

$$W_p = \left(\inf\left\{\int_{E\times E} d(x,y)^p \,\xi(dx,dy) : \xi \in M(\mu,\nu)\right\}\right)^{1/p} \tag{4.60}$$

where $M(\mu,\nu)$ is the set of all probability measures on $E \times E$ with marginals μ and ν. Note that for $p < q$, we have $W_p \leq W_q$.

Theorem 4.14 (Convergence in W_p)**.** *Let $1 \leq p < \infty$. Given $\mu \in Z_p(E,d)$ and a sequence $\{\mu_n\}$ in $Z_p(E,d)$, the following properties are equivalent:*

(1) $W_p(\mu_n,\mu) \to 0$ as $n \to \infty$.

(2) $\mu_n \to \mu$ weakly and for some, or equivalently all $x_0 \in E$,

$$\lim_{n\to\infty} \int_E d(x,x_0)^p \,\mu_n(dx) = \int_E d(x,x_0)^p \,\mu(dx).$$

The Lipschitz semi-norm for suitable real-valued functions f on E is defined by

$$\|f\|_{Lip} = \sup\{\frac{|f(x)-f(y)|}{d(x,y)} \,:\, x \neq y \,\text{in}\, E\}.$$

Theorem 4.15 (Kantorovich-Rubinstein)**.** *Given a separable metric space (E,d), for all $\mu,\nu \in Z_1(E,d)$,*

$$W_1(\mu,\nu) = \sup_{\|u\|_{Lip}\leq 1} \left|\int_E u\,d\mu - \int_E u\,d\nu\right|$$

where the supremum is taken over all Lipschitz functions $u : E \to \mathbb{R}$ with Lipschitz semi-norm $\|u\|_{Lip} \leq 1$.

We apply these Theorems to our model. For a given constant $2 < p_1 \leq 3$, let

$$\hat{\lambda}_i = \frac{1}{(\lambda_i - a_0)^{p_1}}, \quad \hat{\alpha}_n = \frac{1}{n}\sum_{i=1}^{n} \delta_{\hat{\lambda}_i}. \tag{4.61}$$

Let $\hat{\alpha}$ denotes the distribution of $(\lambda - a_0)^{-p_1}$ under α. Then from Assumptions 4.1, 4.2 and Theorem 4.14, we have

$$\lim_{n\to\infty} W_1(\alpha_n, \alpha) = \lim_{n\to\infty} W_1(\beta_n, \beta) = 0 \tag{4.62}$$

and

$$\lim_{n\to\infty} W_1(\hat{\alpha}_n, \hat{\alpha}) = 0. \tag{4.63}$$

For $k \geq 1$ and $a \geq 0$, define a function $\varphi_{k,a} : (0, \infty) \to \mathbb{R}$ by

$$\varphi_{k,a}(x) = \Psi_k(a + x^{-1/p_1}). \tag{4.64}$$

Then from (A.6) we have

$$\begin{aligned}|\varphi'_{k,a}(x)| =&(1/p_1)\left|x^{-1-1/p_1}\Psi_{k+1}(a + x^{-1/p_1})\right| \leq (1/p_1)\left|x^{-1-1/p_1}\Psi_{k+1}(x^{-1/p_1})\right| \\ \leq& \frac{k!}{p_1}\left[(k+1)x^{(k+1)/p_1-1} + x^{k/p_1-1}\right].\end{aligned}$$

In particular, for $x \geq b > 0$,

$$|\varphi'_{1,a}(x)| \leq |\varphi'_{1,0}(x)| \leq \kappa_1(b) = \frac{1}{p_1}\left[2b^{2/p_1-1} + b^{1/p_1-1}\right]. \tag{4.65}$$

If $p_1 = 3$, we have

$$|\varphi'_{2,a}(x)| \leq |\varphi'_{2,0}(x)| \leq \kappa_2(b) = \frac{2}{3}\left[3 + b^{-1/3}\right]. \tag{4.66}$$

Recall definition (4.8) for a probability measure μ on $[a, b]$ and $z > -a$. Note that the k-th derivative of A_μ is well-defined and given by

$$A_\mu^{(k)}(z) = -\int \Psi_k(\lambda + z)\,\mu(d\lambda). \tag{4.67}$$

Since polygamma functions are monotonic, $A_\mu^{(k)}$ are also monotonic. Let

$$A_\mu^{(k)}(-a) = \lim_{z\downarrow -a} A_\mu^{(k)}(z). \tag{4.68}$$

Lemma 4.16. *For $k \geq 1$, $A_\mu^{(k)}(z)$ satisfies the following.*

(1) *If $-a < z_1 < z_2$ and $c = (a+z_1)/(a+z_2)$, then*

$$\left|A_\mu^{(k)}(z_1)\right| \leq \frac{1}{c^{k+1}}\left|A_\mu^{(k)}(z_2)\right|$$

and

$$\max\{\left|A_\mu^{(k)}(z)\right| : z_1 \leq z \leq z_2\} \leq \frac{1}{c^{k+1}}\left|A_\mu^{(k)}(z_2)\right|.$$

(2) $\left|A_\mu^{(k)}(-a)\right| < \infty$ *if and only if* $\int \frac{1}{(\lambda-a)^{k+1}}\,\mu(d\lambda) < \infty$

(3) *Let $1 < p \leq k+1$. If $\int \frac{1}{(\lambda-a)^p}\,\mu(d\lambda) < \infty$ then*

$$\left|A_\mu^{(k)}(z)\right| \leq \frac{k!}{(z+a)^{k+1-p}}\left[\int \frac{1}{(\lambda-a)^p}\,\mu(d\lambda) + \frac{1}{p-1}\int \frac{1}{(\lambda-a)^{p-1}}\,\mu(d\lambda)\right].$$

(4) *Under Assumption 4.1(a), for $z > -a_0$,*

$$\left|A_m^{(k)}(z) - A^{(k)}(z)\right| \leq \left|\Psi_{k+1}(a_0+z)\right| W_1(\alpha_m, \alpha)$$

and for $z < b_0$

$$\left|B_n^{(k)}(z) - B^{(k)}(z)\right| \leq \left|\Psi_{k+1}(b_0-z)\right| W_1(\beta_n, \beta).$$

(5) *Under Assumption 4.2, for $z \geq -a_0$,*

$$\left|A'_m(z) - A'(z)\right| \leq \frac{1}{p_1}\left[2(a_1-a_0)^{p_1-2} + (a_1-a_0)^{p_1-1}\right] W_1(\hat{\alpha}_m, \hat{\alpha}).$$

(6) *Under Assumption 4.2, if $p_1 = 3$, then for $z \geq -a_0$,*

$$\left|A''_m(z) - A''(z)\right| \leq \frac{2}{3}\left[3 + (a_1-a_0)^{-1/3}\right] W_1(\hat{\alpha}_m, \hat{\alpha}).$$

Proof. (1) From (A.7) we have

$$
\begin{aligned}
&\big|A_\mu^{(k)}(z_1)\big| = \big|A_\mu^{(k)}\big(-a + c(z_2+a)\big)\big| \\
=& \int \big|\Psi_k\big((\lambda - a) + c(z_2+a)\big)\big|\,\mu(d\lambda) \le \int \big|\Psi_k\big(c(\lambda - a) + c(z_2+a)\big)\big|\,\mu(d\lambda) \\
\le& \frac{1}{c^{k+1}} \int \big|\Psi_k\big((\lambda - a) + (z_2+a)\big)\big|\,\mu(d\lambda) = \frac{1}{c^{k+1}}\big|A_\mu^{(k)}(z_2)\big|.
\end{aligned}
\tag{4.69}
$$

Monotonicity of polygamma functions (see (A.1)) gives the proof of the second part.

(2) follows from (A.6).

(3) Note that for $\lambda > a$ and $z > -a$ from (A.5)

$$
\begin{aligned}
|\Psi_k(\lambda + z)| =& \sum_{i=0}^{\infty} \frac{k!}{(\lambda + z + i)^{k+1}} = \sum_{i=0}^{\infty} \frac{k!}{(\lambda + z + i)^{k+1-p}(\lambda + z + i)^p} \\
\le& \frac{k!}{(a+z)^{k+1-p}} \sum_{i=0}^{\infty} \frac{1}{(\lambda - a + i)^p} \\
\le& \frac{k!}{(a+z)^{k+1-p}} \left[\frac{1}{(\lambda - a)^p} + \frac{1}{(p-1)(\lambda - a)^{p-1}}\right].
\end{aligned}
\tag{4.70}
$$

Therefore (4.67) and (4.70) give the proof of part (3).

(4) From Theorem 4.15 we have

$$
\begin{aligned}
&\big|A_m^{(k)}(z) - A^{(k)}(z)\big| = \Big|\int \Psi_k(\lambda + z)\,\alpha_m(d\lambda) - \int \Psi_k(\lambda + z)\,\alpha(d\lambda)\Big| \\
\le& \big|\Psi_{k+1}(a_0 + z)\big| W_1(\alpha_m, \alpha)
\end{aligned}
$$

since

$$
\sup\{\big|\Psi_k'(\lambda + z)\big| : \lambda \ge a_0\} = \big|\Psi_{k+1}(a_0 + z)\big|.
$$

Note that α_m is supported on $[a_0, \infty)$ since $\lambda_i \ge a_0$ by assumption (4.1). Proof for B_n is similar.

(5) From (4.65) and Theorem 4.15 we have

$$\begin{aligned}
&\big|A'_m(z) - A'(z)\big| = \Big|\int \Psi_1(\lambda + z)\,\alpha_m(d\lambda) - \int \Psi_1(\lambda + z)\,\alpha(d\lambda)\Big| \\
=&\Big|\int \varphi_{1,z+a_0}(\hat\lambda)\,\hat\alpha_m(d\hat\lambda) - \int \varphi_{1,z+a_0}(\hat\lambda)\,\hat\alpha(d\hat\lambda)\Big| \\
\le&\kappa_1((a_1 - a_0)^{-p_1})W_1(\hat\alpha_m, \hat\alpha)
\end{aligned}$$

since $\hat\lambda \ge (a_1 - a_0)^{-p_1}$.

(6) From (4.66) and Theorem 4.15 we have

$$\begin{aligned}
&\big|A''_m(z) - A''(z)\big| = \Big|\int \Psi_2(\lambda + z)\,\alpha_m(d\lambda) - \int \Psi_2(\lambda + z)\,\alpha(d\lambda)\Big| \\
=&\Big|\int \varphi_{2,z+a_0}(\hat\lambda)\,\hat\alpha_m(d\hat\lambda) - \int \varphi_{2,z+a_0}(\hat\lambda)\,\hat\alpha(d\hat\lambda)\Big| \\
\le&\kappa_2((a_1 - a_0)^{-p_1})W_1(\hat\alpha_m, \hat\alpha).
\end{aligned}$$

□

4.4 Upper bound for the fluctuation

As in the proof of Theorem 4.13 we replace weights on the x- and y-axes with functions of uniform random variables: $-\omega_{i,j} = H(r_{i,j}, u_{i,j})$ with appropriate parameter $r_{i,j}$ and uniform random variable $u_{i,j}$. Recall that H is defined by (A.16). See the comments above (4.59) for more details. In particular, $\tilde{\mathbb{E}}^\rho$ refers to the expectation over uniform random variables and bulk weights. Hence

$$\mathbb{E}^{\rho,z} X = \tilde{\mathbb{E}}^\rho X(z) \tag{4.71}$$

if a random variable X is suitably realized by $X(z)$. For the annealed measure, we write $\tilde{E}^\rho_{m,n} X(z)$ for $E^{\rho,z}_{m,n} X$. In the remainder of this paper, we continue to use this realization of weights. Recall that $x_m = \min_{1\le i\le m} \lambda_i$ and $y_n = \min_{1\le j\le n} \theta_j$.

Lemma 4.17 (Lemma 4.1 of [33])**.** *Consider the (Λ, Θ, z)-stationary polymer. For all m, n and all $z \in (-x_m, y_n)$, we have*

$$\big|\frac{d}{dz}\,\mathbb{V}\mathrm{ar}^{\rho,z}[\log Z_{m,n}]\big| \leq C_1 \cdot (mA''_m(z) + nB''_n(z)) = C_1 \cdot \left(\frac{d^2}{dz^2}\mathbb{E}^{\rho,z}\log Z_{m,n}\right), \tag{4.72}$$

where

$$C_1 = 2 + 20e^2(1 + a_1 + b_1). \tag{4.73}$$

The last equality is from (4.15).

Proof. Identity (4.59) is convenient. Recall the definition of $\mathcal{H}^\omega$ in (4.56). Temporarily assume we can exchange expectation and differentiation. This will be justified later. By the chain rule and Lemma A.1 we have

$$\begin{aligned}
&\frac{d}{dz}\big[-\tilde{\mathbb{E}}^\rho E^{Q^\omega}\frac{\partial\mathcal{H}^\omega}{\partial\lambda}\big|_{(\lambda,\theta)=(z,-z)}\big] = -\tilde{\mathbb{E}}^\rho\frac{d}{dz}\big[E^{Q^\omega}\frac{\partial\mathcal{H}^\omega}{\partial\lambda}\big|_{(\lambda,\theta)=(z,-z)}\big]\\
&= -\tilde{\mathbb{E}}^\rho\big[\frac{\partial}{\partial\lambda}E^{Q^\omega}\frac{\partial\mathcal{H}^\omega}{\partial\lambda} - \frac{\partial}{\partial\theta}E^{Q^\omega}\frac{\partial\mathcal{H}^\omega}{\partial\lambda}\big]\big|_{(\lambda,\theta)=(z,-z)}\\
&= -\tilde{\mathbb{E}}^\rho\big[E^{Q^\omega}\frac{\partial^2\mathcal{H}^\omega}{\partial\lambda^2} + \mathrm{Cov}^{Q^\omega}(\frac{\partial\mathcal{H}^\omega}{\partial\lambda}, \frac{\partial\mathcal{H}^\omega}{\partial\lambda})\\
&\qquad - E^{Q^\omega}\frac{\partial^2\mathcal{H}^\omega}{\partial\theta\partial\lambda} - \mathrm{Cov}^{Q^\omega}(\frac{\partial\mathcal{H}^\omega}{\partial\lambda}, \frac{\partial\mathcal{H}^\omega}{\partial\theta})\big]\big|_{(\lambda,\theta)=(\mathrm{z},-\mathrm{z})}.
\end{aligned} \tag{4.74}$$

Similarly

$$\begin{aligned}
&\frac{d}{dz}\big[-\tilde{\mathbb{E}}^\rho E^{Q^\omega}\frac{\partial\mathcal{H}^\omega}{\partial\theta}\big|_{(\lambda,\theta)=(z,-z)}\big] = -\tilde{\mathbb{E}}^\rho\frac{d}{dz}\big[E^{Q^\omega}\frac{\partial\mathcal{H}^\omega}{\partial\theta}\big|_{(\lambda,\theta)=(z,-z)}\big]\\
&= -\tilde{\mathbb{E}}^\rho\big[\frac{\partial}{\partial\lambda}E^{Q^\omega}\frac{\partial\mathcal{H}^\omega}{\partial\theta} - \frac{\partial}{\partial\theta}E^{Q^\omega}\frac{\partial\mathcal{H}^\omega}{\partial\theta}\big]\big|_{(\lambda,\theta)=(z,-z)}\\
&= -\tilde{\mathbb{E}}^\rho\big[E^{Q^\omega}\frac{\partial^2\mathcal{H}^\omega}{\partial\lambda\partial\theta} + \mathrm{Cov}^{Q^\omega}(\frac{\partial\mathcal{H}^\omega}{\partial\theta}, \frac{\partial\mathcal{H}^\omega}{\partial\lambda})\\
&\qquad - E^{Q^\omega}\frac{\partial^2\mathcal{H}^\omega}{\partial\theta^2} - \mathrm{Cov}^{Q^\omega}(\frac{\partial\mathcal{H}^\omega}{\partial\theta}, \frac{\partial\mathcal{H}^\omega}{\partial\theta})\big]\big|_{(\lambda,\theta)=(\mathrm{z},-\mathrm{z})}.
\end{aligned} \tag{4.75}$$

Since

$$\begin{aligned}
\mathcal{H}^\omega(x_{\boldsymbol{.}}) &= \sum_{i=1}^{\xi_x}\omega_{\mathbf{x}_i} + \sum_{j=1}^{\xi_y}\omega_{\mathbf{x}_j} + \sum_{k=\xi_x\vee\xi_y+1}^{m+n}\omega_{\mathbf{x}_k}\\
&= -\sum_{i=1}^{\xi_x}H_{\lambda_i+\lambda}(\eta_i) - \sum_{j=1}^{\xi_y}H_{\theta_j+\theta}(\eta'_j) + \sum_{k=\xi_x\vee\xi_y+1}^{m+n}\omega_{\mathbf{x}_k},
\end{aligned} \tag{4.76}$$

we have, from (A.19),

$$\begin{aligned}\frac{\partial \mathcal{H}^\omega(x_\cdot)}{\partial \lambda} &= -\sum_{i=1}^{\xi_x} L(\lambda_i+\lambda, H_{\lambda_i+\lambda}(\eta_i)) \quad \text{and} \\ \frac{\partial \mathcal{H}^\omega(x_\cdot)}{\partial \theta} &= -\sum_{j=1}^{\xi_y} L(\theta_j+\theta, H_{\theta_j+\theta}(\eta_j')).\end{aligned} \tag{4.77}$$

Since $\xi_x\xi_y = 0$ and $L(\rho, x) > 0$, from (4.77), we have

$$\frac{\partial^2 \mathcal{H}^\omega}{\partial\lambda\partial\theta} = 0 \quad \text{and}$$

$$\begin{aligned}\operatorname{Cov}^{Q^\omega}\Big(\frac{\partial \mathcal{H}^\omega}{\partial \lambda}, \frac{\partial \mathcal{H}^\omega}{\partial \theta}\Big) &= E^{Q^\omega}\left(\frac{\partial \mathcal{H}^\omega}{\partial \lambda}\frac{\partial \mathcal{H}^\omega}{\partial \theta}\right) - \left(E^{Q^\omega}\frac{\partial \mathcal{H}^\omega}{\partial \lambda}\right)\left(E^{Q^\omega}\frac{\partial \mathcal{H}^\omega}{\partial \theta}\right) \\ &= -\left(E^{Q^\omega}\frac{\partial \mathcal{H}^\omega}{\partial \lambda}\right)\left(E^{Q^\omega}\frac{\partial \mathcal{H}^\omega}{\partial \theta}\right) < 0.\end{aligned}$$

Therefore, from (4.50), (4.59), and (4.74), we have

$$\begin{aligned}\frac{d}{dz}\mathbb{V}\mathrm{ar}^{\rho,z}[\log Z_{m,n}] \leq &-\sum_{j=1}^{n}\Psi_2(\theta_j - z) - \sum_{i=1}^{m}\Psi_2(\lambda_i + z) \\ &- \tilde{\mathbb{E}}^\rho\big(E^{Q^\omega}\frac{\partial^2\mathcal{H}^\omega}{\partial\lambda^2}\big)\big|_{(\lambda,\theta)=(z,-z)} \\ =& mA_m''(z) + nB_n''(z) - \tilde{\mathbb{E}}^\rho\big(E^{Q^\omega}\frac{\partial^2\mathcal{H}^\omega}{\partial\lambda^2}\big)\big|_{(\lambda,\theta)=(z,-z)}\end{aligned} \tag{4.78}$$

and from (4.51), (4.59), and (4.75),

$$\begin{aligned}\frac{d}{dz}\mathbb{V}\mathrm{ar}^{\rho,z}[\log Z_{m,n}] \geq &\sum_{j=1}^{n}\Psi_2(\theta_j - z) + \sum_{i=1}^{m}\Psi_2(\lambda_i + z) \\ &+ \tilde{\mathbb{E}}^\rho\big(E^{Q^\omega}\frac{\partial^2\mathcal{H}^\omega}{\partial\theta^2}\big)\big|_{(\lambda,\theta)=(z,-z)} \\ =& -mA_m''(z) - nB_n''(z) + \tilde{\mathbb{E}}^\rho\big(E^{Q^\omega}\frac{\partial^2\mathcal{H}^\omega}{\partial\theta^2}\big)\big|_{(\lambda,\theta)=(z,-z)}.\end{aligned} \tag{4.79}$$

To complete the proof, we have to estimate $\big|\tilde{\mathbb{E}}^\rho\big(E^{Q^\omega}\frac{\partial^2\mathcal{H}^\omega}{\partial\lambda^2}\big)\big|$ and $\big|\tilde{\mathbb{E}}^\rho\big(E^{Q^\omega}\frac{\partial^2\mathcal{H}^\omega}{\partial\theta^2}\big)\big|$.

From (4.77) and (A.20)

$$\frac{\partial^2 \mathcal{H}^\omega(x_\cdot)}{\partial \lambda^2} = -\sum_{i=1}^{\xi_x} L_1(\lambda_i+\lambda, H_{\lambda_i+\lambda}(\eta_i)).$$

From (A.27) we obtain

$$\begin{aligned}\big|\tilde{\mathbb{E}}^{\rho}\big(E^{Q^\omega}\frac{\partial^2\mathcal{H}^\omega}{\partial\lambda^2}\big)\big| &\leq \sum_{i=1}^{m}\tilde{\mathbb{E}}^{\rho}|L_1(r,H_r(\eta_i)|\big|_{r=\lambda_i+\lambda} \\ &\leq \big(1+20e^2(1+a_1+b_1)\big)\sum_{i=1}^{m}\big|\Psi_2(\lambda_i+\lambda)\big| \\ &= \big(1+20e^2(1+a_1+b_1)\big)mA''_m(\lambda).\end{aligned} \tag{4.80}$$

We have a similar inequality for $\big|\tilde{\mathbb{E}}^{\rho}\big(E^{Q^\omega}\frac{\partial^2\mathcal{H}^\omega}{\partial\theta^2}\big)\big|$.

Hence (4.78), (4.79) and (4.80) give

$$|\frac{d}{dz}\mathbb{V}\mathrm{ar}^{\rho,z}[\log Z_{m,n}]| \leq \big(2+20e^2(1+a_1+b_1)\big)(mA''_m(z)+nB''_n(z)). \tag{4.81}$$

Therefore it remains to justify the first equalities of (4.74) and (4.75). The integrands of the last expressions of (4.74) and (4.75) can be written in terms of L, L_1, λ_is, θ_js and ω. As in the proof of Theorem 4.13, (A.22), (A.23), and monotonicity of $H(r,\eta)$ in r are used to show that these integrands are dominated by some z-independent integrable functions. □

Corollary 4.18. *Consider the (Λ,Θ,z)-stationary polymer. For all m, n and all $z_1 < z_2$ in $(-x_m, y_n)$, we have*

$$\begin{aligned}&\big|\mathbb{V}\mathrm{ar}^{\rho,z_2}[\log Z_{m,n}] - \mathbb{V}\mathrm{ar}^{\rho,z_1}[\log Z_{m,n}]\big| \\ \leq &C_1\cdot\big|\big(mA'_m(z_2)+nB'_n(z_2)\big)-\big(mA'_m(z_1)+nB'_n(z_1)\big)\big|,\end{aligned} \tag{4.82}$$

where C_1 is given by (4.73).

Proof. We have

$$
\begin{aligned}
&\left|\mathbb{V}\mathrm{ar}^{\rho,z_2}[\log Z_{m,n}] - \mathbb{V}\mathrm{ar}^{\rho,z_1}[\log Z_{m,n}]\right| \\
=&\Big|\int_{z_1}^{z_2} \frac{d}{dz}\,\mathbb{V}\mathrm{ar}^{\rho,z}[\log Z_{m,n}]\,dz\Big| \\
\leq& C_1 \int_{z_1}^{z_2} mA''_m(z) + nB''_n(z)\,dz \\
=& C_1 \cdot \Big|\big(mA'_m(z_2) + nB'_n(z_2)\big) - \big(mA'_m(z_1) + nB'_n(z_1)\big)\Big|.
\end{aligned}
$$

□

We prove the upper bound of Theorem 4.3. We fix m, n and consider $z \in (-x_m, y_n)$. For $-x_m < z < y_n$, $0 \leq k \leq m$, and $0 \leq l \leq n$ define

$$
\begin{aligned}
f_z(k) = k\big|A'_k(z)\big| &= \sum_{i=1}^{k} \Psi_1(\lambda_i + z) \\
\text{and} \quad g_z(l) = lB'_l(z) &= \sum_{j=1}^{l} \Psi_1(\theta_j - z).
\end{aligned}
\tag{4.83}
$$

Note that f_z and g_z are strictly increasing functions in k and l, respectively.

Lemma 4.19. *For any $c_1 > 0$ we have the following.*

$$
E^{\rho,z}_{m,n}\left[\sum_{i=1}^{\xi_x} L(\lambda_i + z, -\omega_{i,0})\right] \leq E^{\rho,z}_{m,n} f_z(\xi_x) + c_1 E^{\rho,z}_{m,n}\xi_x + \frac{2^{75}}{c_1^4}\left[\frac{2}{(x_m+z)^{10}} + 1\right], \tag{4.84}
$$

$$
E^{\rho,z}_{m,n}\left[\sum_{j=1}^{\xi_y} L(\theta_j - z, -\omega_{0,j})\right] \leq E^{\rho,z}_{m,n} g_z(\xi_y) + c_1 E^{\rho,z}_{m,n}\xi_y + \frac{2^{75}}{c_1^4}\left[\frac{2}{(y_n-z)^{10}} + 1\right]. \tag{4.85}
$$

Proof. This lemma is a generalization of Lemma 4.2 in [33] which is proved for the homogeneous model. We give details for (4.84) to get a precise bound. Define $L_i = L(\lambda_i + z, -\omega_{i,0})$, $\bar{L}_i = L_i - \mathbb{E}^{\rho,z} L_i$ and $S_k = \sum_{i=1}^{k} \bar{L}_i$.

$$
E^{\rho,z}_{m,n}\left[\sum_{i=1}^{\xi_x} L_i\right] = E^{\rho,z}_{m,n}\sum_{i=1}^{\xi_x}\mathbb{E}^{\rho,z}L_i + E^{\rho,z}_{m,n}\left[\sum_{i=1}^{\xi_x}\bar{L}_i\right] = E^{\rho,z}_{m,n}\sum_{i=1}^{\xi_x}\mathbb{E}^{\rho,z}L_i + \sum_{k=1}^{m}\mathbb{E}^{\rho,z}\left[Q^{\omega}_{m,n}\{\xi_x = k\}S_k\right].
$$

The first term can be computed using (A.24):

$$E^{\rho,z}_{m,n}\sum_{i=1}^{\xi_x}\mathbb{E}^{\rho,z}L_i = E^{\rho,z}_{m,n}\sum_{i=1}^{\xi_x}\Psi_1(\lambda_i+z) = E^{\rho,z}_{m,n}[\xi_x\big|A'_{\xi_x}(z)\big|]$$

$$= E^{\rho,z}_{m,n}f_z(\xi_x).$$

For the second term, we have

$$\sum_{k=1}^{m}\mathbb{E}^{\rho,z}\left[Q^{\omega}_{m,n}\{\xi_x=k\}S_k\right] \le c_1E^{\rho,z}_{m,n}\xi_x + \sum_{k=1}^{m}\mathbb{E}^{\rho,z}\left[1\{S_k\ge c_1k\}S_k\right]$$

and

$$\mathbb{E}^{\rho,z}\left[1\{S_k\ge c_1k\}S_k\right] = c_1k\mathbb{P}^{\rho,z}(S_k\ge c_1k) + \int_{c_1k}^{\infty}\mathbb{P}^{\rho,z}(S_k\ge t)\,dt$$

$$\le \frac{\mathbb{E}^{\rho,z}|S_k|^{\ell}}{(c_1k)^{\ell-1}}(1+\frac{1}{\ell-1}) \le \frac{C_\ell}{c_1^{\ell-1}k^{\ell/2-1}}\cdot\left[\frac{2}{(x_m+z)^{2\ell}}+1\right]$$

for all $\ell\ge 2$. The last bound comes from (A.54). Taking $\ell=5$, we obtain the result. □

Theorem 4.13 and Lemma 4.19 say that the variance is controlled by the behavior of exit points. So we need to estimate $Ef_z(\xi_x)$ and $E\xi_x$.

Lemma 4.20. *Let $X(x_{\cdot},\omega)\ge 0$ be a random variable defined on $\Pi_{m,n}\times\Omega_0$. For $s>0$ let $q^{\omega}_s = Q^{\omega}_{m,n}(X\ge s)$. Suppose there are random variables $W_s(\omega)$ and functions $f(s)$, $g(s)$ such that $\log q_s\le W_s$, $\tilde{\mathbb{E}}^{\rho}W_s\le f(s)<0$ for $s\ge s_1>0$, and $\tilde{\mathbb{V}}\mathrm{ar}^{\rho}W_s\le g(s)$ for $s_1\le s\le s_2$. Then we have*

$$\tilde{E}^{\rho}_{m,n}X = \tilde{\mathbb{E}}^{\rho}E^{Q^{\omega}_{m,n}}X \le s_1 + \int_{s_1}^{s_2}\left(e^{f(s)/2}+\frac{4g(s)}{f(s)^2}\right)\,ds$$

$$+\int_{s_2}^{\infty}\left(e^{f(s)/2}+\tilde{\mathbb{P}}^{\rho}\{W_s-\tilde{\mathbb{E}}^{\rho}W_s\ge -\tfrac{1}{2}f(s)\}\right)\,ds.$$

Proof.

$$\begin{aligned}\tilde{E}^{\rho}_{m,n}X &= \int_0^\infty \tilde{\mathbb{E}}^\rho q_s\,ds = \int_0^\infty \int_0^1 \tilde{\mathbb{P}}^\rho(q_s \ge t)\,dt\,ds \\ &\le s_1 + \int_{s_1}^\infty \int_0^1 \tilde{\mathbb{P}}^\rho(q_s \ge t)\,dt\,ds \le s_1 + \int_{s_1}^\infty \int_0^1 \tilde{\mathbb{P}}^\rho(e^{W_s} \ge t)\,dt\,ds \\ &= s_1 + \int_{s_1}^\infty \left[\int_0^{e^{f(s)/2}} \tilde{\mathbb{P}}^\rho(e^{W_s} \ge t)\,dt + \int_{e^{f(s)/2}}^1 \tilde{\mathbb{P}}^\rho(e^{W_s} \ge t)\,dt\right] ds \\ &\le s_1 + \int_{s_1}^\infty [e^{f(s)/2} + \tilde{\mathbb{P}}^\rho(W_s - \tilde{\mathbb{E}}^\rho W_s \ge -\tfrac{1}{2}f(s))]\,ds.\end{aligned}$$

Now apply Chebyshev inequality for $s_1 \le s \le s_2$. □

For $-x_m < \lambda < z < y_n$ and $u \in [1,m] \cap \mathbb{N}$, we have

$$\begin{aligned}Q^{z,\omega}_{m,n}\{\xi_x \ge u\} &= \frac{1}{Z(z)} \sum_{\mathbf{x}_\cdot} 1\{\xi_x \ge u\} e^{\mathcal{H}^z(\mathbf{x}_\cdot)} \\ =&\frac{1}{Z(z)} \sum_{\mathbf{x}_\cdot} 1\{\xi_x \ge u\} e^{\mathcal{H}^\lambda(\mathbf{x}_\cdot)} \exp\left[\sum_{i=1}^{\xi_x} \big(H_{\lambda_i+\lambda}(\eta_i) - H_{\lambda_i+z}(\eta_i)\big)\right] \\ \le&\frac{Z(\lambda)}{Z(z)} \cdot \exp\left[\sum_{i=1}^{u} \big(H_{\lambda_i+\lambda}(\eta_i) - H_{\lambda_i+z}(\eta_i)\big)\right]\end{aligned} \tag{4.86}$$

where $Z = Z_{m,n}$ (see (4.57)) and $u \in \{1, 2, \ldots, m\}$. This is from

$$\mathcal{H}^z(\mathbf{x}_\cdot) = \mathcal{H}^\lambda(\mathbf{x}_\cdot) + \sum_{i=1}^{\xi_x} \big(H_{\lambda_i+\lambda}(\eta_i) - H_{\lambda_i+z}(\eta_i)\big)$$

for $\xi_x > 0$ and the fact that $H_r(\eta)$ is increasing in r. Now we estimate $E^{\rho,z}_{m,n} h(\xi_x)$ where $h : \mathbb{N} \to \mathbb{R}$ is a positive increasing function. We take $h(k) = f_z(k)$ and $h(k) = k$. For $s > 0$ let

$$v(s) = \inf\{k \in \mathbb{N} : \, h(k) \ge s\}. \tag{4.87}$$

We interpret $\inf \emptyset = \infty$. For our $h(k) = f_z(k) \ge k\Psi_1(a_1 + b_1)$ or $h(k) = k$, $v(s) < \infty$ for all $s > 0$. Note that for all $s > 0$, $h(v(s)) \ge s$ and

$$Q^{z,\omega}_{m,n}\{h(\xi_x) \ge s\} = Q^{z,\omega}_{m,n}\{\xi_x \ge v(s)\}.$$

If $h(k) = f_z(k)$, then $f_z(v(s)) = h(v(s)) \geq s$. If $h(k) = k$, then $v(s) = \lceil s \rceil \geq s$ and

$$f_z(v(s)) = v|A'_v(z)| \geq v(s)\Psi_1(a_1 + b_1) \geq s\Psi_1(a_1 + b_1).$$

Therefore we have

$$f_z(v(s)) \geq cs \tag{4.88}$$

for $c = 1$ if $h(k) = f_z(k)$ and for $c = \Psi_1(a_1 + b_1)$ if $h(k) = k$.

Now we start to prove the upper bound of variance. Recall definitions of A_m and B_n (see (4.8)). For $m, n \geq 1$ and $\zeta_{m,n}$ define

$$\Delta z^1_{m,n} = (x_m + \zeta_{m,n})/2. \tag{4.89}$$

For simplicity of notation, we set

$$z_0 = \zeta_{m,n}. \tag{4.90}$$

For $u \geq 1$ and $z_0 - \Delta z^1_{m,n} \leq \lambda < z \leq z_0$, let

$$Y_{u,z}(\lambda) = \left(\log Z_{m,n}(\lambda) + \sum_{i=1}^{u} H_{\lambda_i + \lambda}(\eta_i) \right) - \left(\log Z_{m,n}(z) + \sum_{i=1}^{u} H_{\lambda_i + z}(\eta_i) \right). \tag{4.91}$$

From (4.86), we have $\log Q^{z,\omega}_{m,n}\{\xi_x \geq u\} \leq Y_{u,z}(\lambda)$. Note that this inequality also holds for $u > m$ ($\log 0 = -\infty$). Here we consider not only $z = z_0$ but also $z < z_0$ for later use in Lemma 4.25. By Proposition 4.11 and (4.71),

$$\tilde{\mathbb{E}}^\rho \log Z(\lambda) = -\sum_{i=1}^{m} \Psi_0(\lambda_i + \lambda) - \sum_{j=1}^{n} \Psi_0(\theta_j - \lambda) = mA_m(\lambda) + nB_n(\lambda). \tag{4.92}$$

Hence by Taylor's theorem, for some $\lambda < z_1, z_2 < z$,

$$
\begin{aligned}
\tilde{\mathbb{E}}^\rho Y_{u,z}(\lambda) = &\left(mA_m(\lambda) + nB_n(\lambda) + \sum_{i=1}^{u} \Psi_0(\lambda_i + \lambda) \right) \\
&- \left(mA_m(z) + nB_n(z) + \sum_{i=1}^{u} \Psi_0(\lambda_i + z) \right) \\
=& \left(mA_m(\lambda) + nB_n(\lambda) - uA_u(\lambda)\right) - \left(mA_m(z) + nB_n(z) - uA_u(z)\right) \\
=& m[A_m(\lambda) - A_m(z)] + n[B_n(\lambda) - B_n(z)] - u[A_u(\lambda) - A_u(z)] \qquad (4.93) \\
=& \big(mA'_m(z) + nB'_n(z)\big) \cdot (\lambda - z) + \big(mA''_m(z_1) + nB''_n(z_1)\big) \cdot \frac{(\lambda - z)^2}{2} \\
&- uA'_u(z_2) \cdot (\lambda - z) \\
=& - \Delta z \cdot \Big[u\big|A'_u(z_2)\big| + \big(mA'_m(z) + nB'_n(z)\big) \\
&\qquad - \frac{\Delta z}{2}\big(mA''_m(z_1) + nB''_n(z_1)\big)\Big],
\end{aligned}
$$

where $\Delta z = z - \lambda \leq (z - z_0) + \Delta z^1_{m,n}$.

From Lemma 4.16 part (1), with $c = 1/2$, $z_2 = z_0$, $a = x_m$ and monotonicity of $B_k^{(l)}$ we have

$$
|A_k^{(l)}(\lambda_0)| \leq 2^{l+1} |A_k^{(l)}(z_0)| \quad \text{and} \quad |B_k^{(l)}(\lambda_0)| \leq |B_k^{(l)}(z_0)| \tag{4.94}
$$

for any $\lambda_0 \in [z_0 - \Delta z^1_{m,n}, z_0]$. In particular for $l = 2$ we have

$$
mA''_m(\lambda_0) + nB''_n(\lambda_0) \leq 8A''_m(z_0) + 8B''_n(z_0). \tag{4.95}
$$

Suppose Δz and u satisfy

$$
\Delta z \leq \frac{u\big|A'_u(z_0)\big|}{16 M_0}, \tag{4.96}
$$

where

$$
M_0 = \sigma^3_{m,n} = M_{m,n}(\zeta_{m,n}) = mA''_m(z_0) + nB''_n(z_0) \tag{4.97}
$$

is given by (4.20), and

$$
u\big|A'_u(z_0)\big| \geq 4\big|mA'_m(z) + nB'_n(z)\big|. \tag{4.98}
$$

Then from (4.93), (4.95) and monotonicity of $A'_u(z)$ in (4.11) we have

$$\begin{aligned}
\tilde{\mathbb{E}}^\rho Y_{u,z}(\lambda) = & -\Delta z \cdot \Big[\frac{1}{2}u\big|A'_u(z_2)\big| \\
& \qquad + \frac{1}{4}u\big|A'_u(z_2)\big| + \big(mA'_m(z) + nB'_n(z)\big) \\
& \qquad + \frac{1}{4}u\big|A'_u(z_2)\big| - \frac{\Delta z}{2}\big(mA''_m(z_1) + nB''_n(z_1)\big)\Big] \\
\le & -\Delta z \cdot \Big[\frac{1}{2}u\big|A'_u(z_0)\big|\Big] \qquad (4.99) \\
& -\Delta z \cdot \frac{1}{4}\Big[u\big|A'_u(z_0)\big| - 4\big|(mA'_m(z) + nB'_n(z)\big|\Big] \\
& -\Delta z \cdot \frac{1}{4}\Big[u\big|A'_u(z_0)\big| - 16\Delta z M_0\Big] \\
\le & -\Delta z \frac{u\big|A'_u(z_0)\big|}{2} = -\Delta z \frac{f_{z_0}(u)}{2},
\end{aligned}$$

where f_z is as defined in (4.83).

For given $s > 0$, we select u and Δz by $u = u(s) = v(s)$ and

$$\Delta z \le \frac{f_{z_0}(u)}{16M_0} \wedge (z - z_0 + \Delta z^1_{m,n}). \qquad (4.100)$$

So Δz depends on s. Below $u = u(s)$ throughout. Suppose (4.98) is satisfied. Then we have

$$
\begin{aligned}
&\tilde{\mathbb{P}}^\rho\left[Q^{z,\omega}_{m,n}\{h(\xi_x)\geq s\}\geq \exp[-\frac{f_{z_0}(u)\Delta z}{4}]\right]\\
&=\tilde{\mathbb{P}}^\rho\left[Q^{z,\omega}_{m,n}\{\xi_x\geq u\}\geq \exp[-\frac{f_{z_0}(u)\Delta z}{4}]\right]\\
&\leq \tilde{\mathbb{P}}^\rho\left[Y_{u,z}(\lambda)\geq -\frac{f_{z_0}(u)\Delta z}{4}\right]\\
&\leq \tilde{\mathbb{P}}^\rho\left[Y_{u,z}(\lambda)-\tilde{\mathbb{E}}^\rho Y_{u,z}(\lambda)\geq \frac{f_{z_0}(u)\Delta z}{4}\right]\\
&\leq \tilde{\mathbb{P}}^\rho\left[\left|\log Z(\lambda)-\tilde{\mathbb{E}}^\rho\log Z(\lambda)\right|\geq \frac{f_{z_0}(u)\Delta z}{16}\right]\\
&+\tilde{\mathbb{P}}^\rho\left[\left|\log Z(z)-\tilde{\mathbb{E}}^\rho\log Z(z)\right|\geq \frac{f_{z_0}(u)\Delta z}{16}\right]\\
&+\tilde{\mathbb{P}}^\rho\left[\sum_{i=1}^u H_{\lambda_i+\lambda}(\eta_i)-\tilde{\mathbb{E}}^\rho\sum_{i=1}^u H_{\lambda_i+\lambda}(\eta_i)\geq \frac{f_{z_0}(u)\Delta z}{16}\right]\\
&+\tilde{\mathbb{P}}^\rho\left[\sum_{i=1}^u H_{\lambda_i+z}(\eta_i)-\tilde{\mathbb{E}}^\rho\sum_{i=1}^u H_{\lambda_i+z}(\eta_i)\leq -\frac{f_{z_0}(u)\Delta z}{16}\right].
\end{aligned}
\tag{4.101}
$$

From (4.94), we have

$$\tilde{\mathbb{V}}\text{ar}^\rho[\sum_{i=1}^u H_{\lambda_i+z}(\eta_i)]=u|A'_u(z)|\leq 4u|A'_u(z_0)|$$

and

$$\tilde{\mathbb{V}}\text{ar}^\rho[\sum_{i=1}^u H_{\lambda_i+\lambda}(\eta_i)]=u|A'_u(\lambda)|\leq 4u|A'_u(z_0)|.$$

Apply Corollary A.5 part (3) with $A=4\left|A'_u(z_0)\right|$ and $r_0=\Delta z^1_{m,n}$:

$$x=\frac{f_{z_0}(u)\Delta z}{16},\quad h(x)=\left[\frac{x^2}{32f_{z_0}(u)}\wedge\frac{\Delta z^1_{m,n}x}{4}\right]=\frac{(\Delta z)^2 f_{z_0}(u)}{2^{13}}$$

Hence we have

$$
\begin{aligned}
&\tilde{\mathbb{P}}^\rho\left[\sum_{i=1}^u H_{\lambda_i+\lambda}(\eta_i)-\tilde{\mathbb{E}}^\rho\sum_{i=1}^u H_{\lambda_i+\lambda}(\eta_i)\geq \frac{f_{z_0}(u)\Delta z}{16}\right]+\\
&\tilde{\mathbb{P}}^\rho\left[\sum_{i=1}^u H_{\lambda_i+z}(\eta_i)-\tilde{\mathbb{E}}^\rho\sum_{i=1}^u H_{\lambda_i+z}(\eta_i)\leq -\frac{f_{z_0}(u)\Delta z}{16}\right]\\
&\leq 2e^{-h(x)}=2\exp[-\frac{(\Delta z)^2 f_{z_0}(u)}{2^{13}}].
\end{aligned}
$$

Therefore for $s > 0$, $u = u(s)$ with (4.98) and $\Delta z \le \frac{f_{z_0}(u)}{16M_0} \wedge (z - z_0 + \Delta z^1_{m,n})$, we have

$$\begin{aligned}
&\tilde{\mathbb{P}}^\rho \left[Q^{z,\omega}_{m,n}\{h(\xi_x) \ge s\} \ge \exp[-\frac{f_{z_0}(u)\Delta z}{4}] \right] \\
&\le \tilde{\mathbb{P}}^\rho \left[Y_{u,z}(\lambda) - \tilde{\mathbb{E}}^\rho Y_{u,z}(\lambda) \ge \frac{f_{z_0}(u)\Delta z}{4} \right] \\
&\le \tilde{\mathbb{P}}^\rho \left[\left|\log Z(\lambda) - \tilde{\mathbb{E}}^\rho \log Z(\lambda)\right| \ge \frac{f_{z_0}(u)\Delta z}{16} \right] \\
&+ \tilde{\mathbb{P}}^\rho \left[\left|\log Z(z) - \tilde{\mathbb{E}}^\rho \log Z(z)\right| \ge \frac{f_{z_0}(u)\Delta z}{16} \right] \\
&+ 2\exp[-\frac{(\Delta z)^2 f_{z_0}(u)}{2^{13}}].
\end{aligned} \tag{4.102}$$

Recall that our goal is to estimate $E^{\rho,z_0}_{m,n} h(\xi_x)$ and to do that we use Lemma 4.20 with appropriate choices of W_s. Suppose that, for some $c > 0$, $v(s)$ satisfies $f_{z_0}(v(s)) \ge cs$ for all $s > 0$. $h(k) = f_{z_0}(k)$ and $h(k) = k$ satisfy this condition by (4.88). Let

$$q_s = Q^{z_0,\omega}_{m,n}\{h(\xi_x) \ge s\}, \quad \Delta z(s) = \frac{cs}{16M_0} \wedge \Delta z^1_{m,n} \tag{4.103}$$

and

$$W_s = Y_{u(s),z_0}(\lambda), \quad f(s) = -\frac{\Delta z(s) f_{z_0}(u(s))}{2}. \tag{4.104}$$

We have $\tilde{\mathbb{E}}^\rho W_s \le f(s) \le 0$ by (4.99). Note that (4.98) is satisfied for any $s > 0$ in the case $z = z_0$. We will estimate probabilities in (4.102) with $z = z_0$ by dividing the range of s. Let

$$s_2 = 16\Delta z^1_{m,n} M_0 / c.$$

Case 1. $s > s_2$.

In this case $\Delta z = \Delta z^1_{m,n}$ and $f(s) \le -c\Delta z^1_{m,n} s/2$. Therefore from (4.102) we have

$$\begin{aligned}
&\int_{s_2}^{\infty} e^{f(s)/2} + \tilde{\mathbb{P}}^\rho(W_s - \tilde{\mathbb{E}}^\rho W_s \ge -\tfrac{1}{2}f(s))\,ds \\
\le& \frac{4}{c\Delta z^1_{m,n}} + \frac{2^{14}}{c(\Delta z^1_{m,n})^2} \\
&+ \frac{16}{c\Delta z^1_{m,n}}\Big[\tilde{\mathbb{E}}^\rho\big|\log Z(\lambda) - \tilde{\mathbb{E}}^\rho \log Z(\lambda)\big| + \tilde{\mathbb{E}}^\rho\big|\log Z(z_0) - \tilde{\mathbb{E}}^\rho \log Z(z_0)\big|\Big] \qquad (4.105)\\
\le& \frac{16}{c\Delta z^1_{m,n}}\left[1 + \sqrt{\tilde{\mathbb{V}}\mathrm{ar}^\rho \log Z(z)} + \sqrt{\tilde{\mathbb{V}}\mathrm{ar}^\rho \log Z(\lambda)} + \frac{2^{10}}{\Delta z^1_{m,n}}\right] \\
\le& \frac{16}{c\Delta z^1_{m,n}}\left[1 + 3\sqrt{m|A'_m(z_0)|} + 3\sqrt{nB'_n(z_0)} + \frac{2^{10}}{\Delta z^1_{m,n}}\right].
\end{aligned}$$

The last line comes from the fact that $\log Z(\lambda) = X + Y$ where X and Y are sums of independent random variables with $\tilde{\mathbb{V}}\mathrm{ar}^\rho(X) = \big|mA'_m(\lambda)\big|$ and $\tilde{\mathbb{V}}\mathrm{ar}^\rho(Y) = nB'_n(\lambda)$. We used (4.94) to bound $|A'_m(\lambda)|$ and $B'_n(\lambda)$ in terms of z_0.

Case 2. $s_1 \le s \le s_2$ for $s_1 = rM_0^{2/3}$ with $r > 0$ to be determined later.

In this case $\Delta z = \frac{cs}{16M_0}$ and $f(s) \le -\frac{c^2s^2}{32M_0}$. Now we compute the variance of W_s.

$$\tilde{\mathbb{V}}\mathrm{ar}^\rho\, W_s \le 2\,\tilde{\mathbb{V}}\mathrm{ar}^\rho\big[\sum_{i=1}^{u} H_{\lambda_i+\lambda}(\eta_i) - H_{\lambda_i+z_0}(\eta_i)\big] + 2\,\tilde{\mathbb{V}}\mathrm{ar}^\rho\big[\log Z(\lambda) - \log Z(z_0)\big].$$

The first term is $\le 4u(|A'_u(\lambda)| + |A'_u(z_0)|) \le 20 f_{z_0}(u)$ by (4.94). For the second term, apply (4.72) and (4.95) with $C_1 = 2 + 20e^2(1 + a_1 + b_1)$: For some $\lambda^* \in [\lambda, z_0]$,

$$\begin{aligned}
\tilde{\mathbb{V}}\mathrm{ar}^\rho\big[\log Z(\lambda) - \log Z(z_0)\big] &\le 2\,\tilde{\mathbb{V}}\mathrm{ar}^\rho \log Z(\lambda) + 2\,\tilde{\mathbb{V}}\mathrm{ar}^\rho \log Z(z_0) \\
&\le 4\,\tilde{\mathbb{V}}\mathrm{ar}^\rho \log Z(z_0) + 2C_1(mA''_m(\lambda^*) + nB''_n(\lambda^*))\Delta z \\
&\le 4\,\tilde{\mathbb{V}}\mathrm{ar}^\rho \log Z(z_0) + 16C_1 M_0 \Delta z \\
&= 4\,\tilde{\mathbb{V}}\mathrm{ar}^\rho \log Z(z_0) + cC_1 s.
\end{aligned}$$

Therefore

$$\tilde{\mathbb{V}}\mathrm{ar}^\rho\, W_s \le g(s) = 40 f_{z_0}(u(s)) + 8\,\tilde{\mathbb{V}}\mathrm{ar}^\rho \log Z(z_0) + 2cC_1 s.$$

and

$$\frac{4g(s)}{f(s)^2} \leq \frac{640}{(\Delta z)^2 f_{z_0}(u(s))} + \frac{2^{15} M_0^2 \, \tilde{\mathbb{V}}\mathrm{ar}^{\rho} \log Z(z_0)}{c^4 s^4} + \frac{2^{13} C_1 M_0^2}{c^3 s^3}$$
$$\leq \frac{5 \cdot 2^{15} M_0^2}{c^3 s^3} + \frac{2^{15} M_0^2 \, \tilde{\mathbb{V}}\mathrm{ar}^{\rho} \log Z(z_0)}{c^4 s^4} + \frac{2^{13} C_1 M_0^2}{c^3 s^3} \tag{4.106}$$
$$\leq \frac{2^{15} M_0^2 \, \tilde{\mathbb{V}}\mathrm{ar}^{\rho} \log Z(z_0)}{c^4 s^4} + \frac{2^{14} C_1 M_0^2}{c^3 s^3}.$$

We use Lemma 4.20 with (4.106) to compute $E^{\rho, z_0}_{m,n} h(\xi_x)$.

$$E^{\rho, z_0}_{m,n} h(\xi_x) \leq r M_0^{2/3} + (4.105)$$
$$+ \int_{r M_0^{2/3}}^{s_2} \left[\exp(-c^2 s^2/(64 M_0)) + \frac{2^{15} M_0^2 \, \tilde{\mathbb{V}}\mathrm{ar}^{\rho} \log Z(z_0)}{c^4 s^4} + \frac{2^{14} C_1 M_0^2}{c^3 s^3} \right] ds$$
$$\leq r M_0^{2/3} + \frac{16}{c \Delta z^1_{m,n}} \left[1 + 3\sqrt{m |A'_m(z_0)|} + 3\sqrt{n B'_n(z_0)} + \frac{2^{10}}{\Delta z^1_{m,n}} \right]$$
$$+ \frac{32 M_0^{1/3}}{c^2 r} \exp(-c^2 r^2 M_0^{1/3}/64) + 2^{13} \frac{\tilde{\mathbb{V}}\mathrm{ar}^{\rho} \log Z(z_0)}{c^4 r^3} + 2^{13} \frac{C_1 M_0^{2/3}}{c^3 r^2}. \tag{4.107}$$

Take $r = 2^6/c$. Then we obtain

$$c E^{\rho, z_0}_{m,n} h(\xi_x) \leq \frac{\tilde{\mathbb{V}}\mathrm{ar}^{\rho} \log Z(z_0)}{32} + (64 + 2C_1) M_0^{2/3} + 1/128$$
$$+ \frac{16}{\Delta z^1_{m,n}} \left[1 + 3\sqrt{m |A'_m(z_0)|} + 3\sqrt{n B'_n(z_0)} + \frac{2^{10}}{\Delta z^1_{m,n}} \right]. \tag{4.108}$$

Combining (4.50), (4.84) with $c_1 = \Psi(a_1 + b_1)$ and (4.108) gives the upper variance bound for the free energy:

$$\begin{aligned}\tilde{\mathbb{V}}\mathrm{ar}^{\rho}[\log Z_{m,n}(z_0)] \le\ & 1 + (2^9 + 16C_1)\cdot M_0^{2/3} \\ & + \frac{2^{10}}{x_m + z_0}\left(\sqrt{|mA_m'(z_0)|} + \sqrt{nB_n'(z_0)}\right) + \frac{2^8}{x_m + z_0} + \frac{2^{19}}{(x_m + z_0)^2} \\ & + \frac{2^{77}}{(\Psi_1(a_1 + b_1))^4}\left[\frac{2}{(x_m + z_0)^{10}} + 1\right] \\ & \le (2^9 + 16C_1)\cdot M_0^{2/3} + \frac{2^{79}}{(1 \wedge \Psi_1(a_1 + b_1))^4}\left[\frac{2}{(x_m + z_0)^{10}} + 1\right] \\ & + \frac{2^{10}}{x_m + z_0}\left(\sqrt{|mA_m'(z_0)|} + \sqrt{nB_n'(z_0)}\right).\end{aligned} \tag{4.109}$$

Using (4.51) we also have a upper bound :

$$\begin{aligned}\tilde{\mathbb{V}}\mathrm{ar}^{\rho}[\log Z_{m,n}(z_0)] \le\ & (2^9 + 16C_1)\cdot M_0^{2/3} + \frac{2^{79}}{(1 \wedge \Psi_1(a_1 + b_1))^4}\left[\frac{2}{(y_n - z_0)^{10}} + 1\right] \\ & + \frac{2^{10}}{y_n - z_0}\left(\sqrt{|mA_m'(z_0)|} + \sqrt{nB_n'(z_0)}\right).\end{aligned} \tag{4.110}$$

For $z_0 \ge (b_0 - a_0)/2$, we have $(x_m + z_0) \ge (a_0 + b_0)/2$ and for $z_0 \le (b_0 - a_0)/2$, we have $(y_n - z_0) \ge (a_0 + b_0)/2$. Therefore we have

$$\begin{aligned}\tilde{\mathbb{V}}\mathrm{ar}^{\rho}[\log Z_{m,n}(z_0)] \le\ & (2^9 + 16C_1)\cdot M_0^{2/3} + \frac{2^{79}}{(1 \wedge \Psi_1(a_1 + b_1))^4}\left[\frac{2^{11}}{(a_0 + b_0)^{10}} + 1\right] \\ & + \frac{2^{11}}{a_0 + b_0}\left(\sqrt{|mA_m'(z_0)|} + \sqrt{nB_n'(z_0)}\right).\end{aligned} \tag{4.111}$$

Note that from (A.6)

$$\begin{aligned}|mA_m'(z_0)|^{3/4} \le & \sum_{i=1}^{m}(\Psi_1(\lambda_i + z_0))^{3/4} \le \sum_{i=1}^{m}\left(\frac{1}{(\lambda_i + z_0)^{3/2}} + \frac{1}{(\lambda_i + z_0)^{3/4}}\right) \\ \le & 2(1 + a_1 + b_1)^{5/4}\sum_{i=1}^{m}\frac{1}{(\lambda_i + z_0)^2} \le 2(1 + a_1 + b_1)^{5/4} mA_m''(z_0) \\ \le & 2(1 + a_1 + b_1)^{5/4} M_0.\end{aligned}$$

A similar calculation for $nB'_n(z_0)$ shows that

$$\sqrt{|mA'_m(z_0)|} + \sqrt{nB'_n(z_0)} \le 4(1 + a_1 + b_1)M_0^{2/3}. \tag{4.112}$$

From (4.111) and (4.112) we have

$$\tilde{\mathbb{V}}\mathrm{ar}^{\rho}[\log Z_{m,n}(z_0)] \le C_2 M_0^{2/3} + C_3, \tag{4.113}$$

where

$$C_2 = 2^{10} + 320e^2(1 + a_1 + b_1) + \frac{2^{13}(1 + a_1 + b_1)}{a_0 + b_0} \tag{4.114}$$

and

$$C_3 = 2^{79}(1 + a_1 + b_1)^4 \left[\frac{2^{11}}{(a_0 + b_0)^{10}} + 1\right]. \tag{4.115}$$

Since $M_0 \ge |\Psi_2(\lambda_1 + z_0)| \ge 1/(1 + a_1 + b_1)^3$, from (4.113), (4.114) and (4.115) we have

$$\tilde{\mathbb{V}}\mathrm{ar}^{\rho}[\log Z_{m,n}(z_0)] \le C M_0^{2/3}, \tag{4.116}$$

where

$$C = 2^{91}(1 + a_1 + b_1)^6 \left[\frac{1}{(a_0 + b_0)^{10}} + 1\right]. \tag{4.117}$$

This completes the proof of (4.21).

4.5 Useful estimates

In this section, we collect technical lemmas for the remaining sections. All polymer models are (Λ, Θ, z)-stationary polymers. Weights on the axes are given by functions of uniform random variables as in section 4.4. z-dependence of models is achieved by the function $H(\rho, \eta)$ in (A.16). See the first paragraph in Section 4.4. Thus we use the notation $\tilde{\mathbb{P}}^{\rho}(X(z) \in \cdot)$ instead of $\mathbb{P}^{\rho,z}(X \in \cdot)$ for a random variable X. Recall the definitions of x_m and y_n in (4.6). z satisfies $-x_m < z < y_n$.

Recall definitions of A_m and B_n in (4.9) and (4.10). For fixed m, $n \geq 1$, let

$$\begin{aligned} M(z) = M_{m,n}(z) &= mA''_m(z) + nB''_n(z) \\ &= -\sum_{i=1}^{m} \Psi_2(\lambda_i + z) - \sum_{j=1}^{n} \Psi_2(\theta_j - z) > 0 \end{aligned} \tag{4.118}$$

be as in (4.13). Suppose $-x_m < z_1^* \leq z_1 < z_2 \leq z_2^* < y_n$. Let

$$M_1 = M(z_1) \quad \text{and} \quad M_2 = M(z_2).$$

Since $B'''_n(z) = \int \Psi_3(\theta - z)\,\beta_n(d\theta) > 0$, $B''_n(z_1) \leq B''_n(z_2)$. By Lemma 4.16 part (1), $0 < A''_m(z_1) \leq A''_m(z_2) \cdot \left(\frac{x_m+z_2}{x_m+z_1}\right)^3$. Hence we have $M_1/M_2 \leq \left(\frac{x_m+z_2^*}{x_m+z_1^*}\right)^3$. In a similar way, we obtain $M_2/M_1 \leq \left(\frac{y_n-z_1^*}{y_n-z_2^*}\right)^3$. Therefore we have

$$\left(\frac{x_m + z_1^*}{x_m + z_2^*}\right)^3 \leq M_2/M_1 \leq \left(\frac{y_n - z_1^*}{y_n - z_2^*}\right)^3. \tag{4.119}$$

Since

$$x_m + z \leq \lambda_i + z \leq a_1 + b_1 \quad \text{and} \quad y_n - z \leq \theta_j - z \leq a_1 + b_1, \tag{4.120}$$

we have, from (A.6),

$$M(z) \geq \left|\Psi_2(x_m + z)\right| + \left|\Psi_2(y_n - z)\right| \geq \frac{2}{(x_m + z)^3} + \frac{2}{(y_n - z)^3} \geq \frac{1}{(\Delta z_{m,n})^3}, \tag{4.121}$$

where

$$\Delta z_{m,n} = (x_m + z) \wedge (y_n - z). \tag{4.122}$$

Throughout this section, we assume the following. Recall the definitions of $\zeta_{m,n}$ in (4.18), $\sigma_{m,n}$ in (4.20), $\Delta\zeta_{m,n}$ in (4.30) and $K_{m,n}$ in (4.31). As in the previous section, for simplicity of notations, we sometimes write

$$z_0 = \zeta_{m,n}, \quad M_0 = \sigma_{m,n}^3$$

(see (4.90) and (4.97)).

Let

$$\Delta z^1_{m,n} = (x_m + \zeta_{m,n})/2, \quad \Delta z^2_{m,n} = (y_n - \zeta_{m,n})/2. \tag{4.123}$$

Note that M_0 and $K_{m,n}$ only depend on $\lambda_1, \dots, \lambda_m$ and $\theta_1, \dots, \theta_n$. M_0 and $K_{m,n}$ are symmetric functions of these parameters. Hence

$$M_0(\lambda_1, \dots, \lambda_m, \theta_1, \dots, \theta_n) = M_0(x_{m:1}, \dots, x_{m:m}, y_{n:1}, \dots, y_{n:n}) \tag{4.124}$$

and

$$K_{m,n}(\lambda_1, \dots, \lambda_m, \theta_1, \dots, \theta_n) = K_{m,n}(x_{m:1}, \dots, x_{m:m}, y_{n:1}, \dots, y_{n:n}), \tag{4.125}$$

where $x_{m:1}, \dots, x_{m:m}, y_{n:1}, \dots, y_{n:n}$ are rearranged parameters in (4.4) and (4.5). From (A.8),

$$\begin{aligned} A''_m(z_0) = & -\frac{1}{m}\sum_{i=1}^{m} \Psi_2(\lambda_i + z_0) \\ & \leq \frac{2}{m}\sum_{i=1}^{m} \frac{1}{\lambda_i + z_0}\Psi_1(\lambda_i + z_0) \leq \frac{2}{\Delta\zeta_{m,n}}|A'_m(z_0)|. \end{aligned}$$

Similarly, $B''_n(z_0) \leq \frac{2}{\Delta\zeta_{m,n}} B'_n(z_0)$. Hence we have

$$K_{m,n} \leq 4. \tag{4.126}$$

We consider parameters z with

$$\zeta_{m,n} - \Delta z^1_{m,n} \leq z \leq \zeta_{m,n} + \Delta z^2_{m,n}. \tag{4.127}$$

Lemma 4.21. *Suppose z satisfies (4.127). Let C be the constant in (4.117). Then we have the following.*

(1) For $t > 0$,

$$\begin{aligned} &\tilde{\mathbb{P}}^\rho \left[\left| \log Z_{m,n}(z) - \tilde{\mathbb{E}}^\rho \log Z_{m,n}(z) \right| \geq t \right] \\ \leq \quad & 4 \left(\exp\big[-\frac{t^2}{128m|A'_m(\zeta_{m,n})|} \big] \vee \exp\big[-\frac{(\Delta\zeta_{m,n})t}{16} \big] \right). \end{aligned} \tag{4.128}$$

(2) For $t > 0$,

$$\begin{aligned} &\tilde{\mathbb{P}}^\rho \left[\left| \log Z_{m,n}(z) - \tilde{\mathbb{E}}^\rho \log Z_{m,n}(z) \right| \geq t \right] \\ \leq \quad & \frac{20C\sigma_{m,n}^2}{t^2} \\ & + 2 \left(\exp\big[-\frac{t^2}{32|mA'_m(z) + nB'_n(z)|} \big] \vee \exp\big[-\frac{(\Delta\zeta_{m,n})t}{16} \big] \right). \end{aligned} \tag{4.129}$$

We interpret $1/0 = \infty$.

(3) Suppose $\left| mA'_m(z) + nB'_n(z) \right| \leq r\sigma_{m,n}^2$. *For* $t > 0$,

$$\begin{aligned} &\tilde{\mathbb{P}}^\rho \left[\left| \log Z_{m,n}(z) - \tilde{\mathbb{E}}^\rho \log Z_{m,n}(z) \right| \geq t\sigma_{m,n} \right] \\ \leq \quad & \frac{20C}{t^2} + 2 \left(\exp\left[-t^2/(32r) \right] \vee \exp[-t/16] \right). \end{aligned} \tag{4.130}$$

(4) For fixed $0 < c \leq 2$, *if* $t \geq c\Delta\zeta_{m,n}\sigma_{m,n}$, *then*

$$\begin{aligned} &\tilde{\mathbb{P}}^\rho \left[\left| \log Z_{m,n}(z) - \tilde{\mathbb{E}}^\rho \log Z_{m,n}(z) \right| \geq t\sigma_{m,n} \right] \\ \leq \quad & 4 \exp\left[-\frac{cK_{m,n}}{128} t \right]. \end{aligned} \tag{4.131}$$

Proof. Part (1). Write $z_0 = \zeta_{m,n}$ and $M_0 = \sigma_{m,n}^3$.

Recall that

$$\log Z_{m,n}(z) = \sum_{i=1}^m I_{i,0}(z) + \sum_{j=1}^n J_{m,j}(z) = X_m(z) + Y_n(z).$$

We apply (A.53) and Lemma 4.16(1). Since $\tilde{\mathbb{V}}\mathrm{ar}^\rho X(z) = m\left|A'_m(z)\right| \leq 4m\left|A'_m(z_0)\right|$ and $\tilde{\mathbb{V}}\mathrm{ar}^\rho Y(z) = nB'_n(z) \leq 4nB'_n(z_0)$ and $\lambda_i + z \geq \Delta z_{m,n}^1$, we have

$$\tilde{\mathbb{P}}^\rho \left[\left| X_m(z) - \tilde{\mathbb{E}}^\rho X_m(z) \right| \geq t \right] \leq 2 \left(\exp\big[-\frac{t^2}{32m|A'_m(z_0)|} \big] \vee \exp\big[-\frac{(x_m + z_0)t}{8} \big] \right). \tag{4.132}$$

Similarly,

$$\tilde{\mathbb{P}}^\rho\left[\left|Y_n(z)-\tilde{\mathbb{E}}^\rho Y_n(z)\right|\geq t\right]\leq 2\left(\exp[-\frac{t^2}{32nB_n'(z_0)}]\vee\exp[-\frac{(y_n-z_0)t}{8}]\right).$$

Since $|mA_m'(z_0)| = nB_n(z_0)$, substituting $t/2$ into above inequalities, we get (4.128).

Part (2).

First, we assume $mA_m'(z)+nB_n'(z)\leq 0$. Hence $z\leq z_0$ since $mA_m(z)+nB_n(z)$ is a smooth convex function of z. Define

$$g(k,z)=kA_k'(z)+nB_n'(z)\quad\text{for } 0\leq k\leq m.$$

Then $g(0,z)>0$ and $g(m,z)\leq 0$. Note that g is strictly decreasing in k for fixed z (see (4.11)). Define

$$u=u(z)=\min\{k:g(k,z)\leq 0\}\tag{4.133}$$

Note that $u\geq 1$. Write $\log Z_{m,n}(z)$ as

$$\log Z_{m,n}(z)=\log Z_{u,n}(z)+\sum_{i=u+1}^{m}I_{i,n}(z).$$

Then

$$\frac{d}{dz}\tilde{\mathbb{E}}^\rho\log Z_{u,n}(z)=g(u,z)=g(u-1,z)-\Psi_1(\lambda_u+z)$$

and $\left|g(u,z)\right|\leq\Psi_1(\lambda_u+z)$. Let λ_0 be the unique constant such that $uA_u'(\lambda_0)+nB_n'(\lambda_0)=0$. Since $mA_m'(z_0)+nB_n'(z_0)=0$, we have $uA_u'(z_0)+nB_n'(z_0)\geq 0$. Hence we have $z\leq\lambda_0\leq z_0$. From Theorem 4.3 and (4.82),

$$\tilde{\mathbb{V}}\mathrm{ar}^\rho\log Z_{u,n}(z)\leq C_1|g(u,z)|+C\left[uA_u''(\lambda_0)+nB_n''(\lambda_0)\right]^{2/3}.$$

From (A.6) and (A.7)

$$\begin{aligned}\left|g(u,z)\right| &\le \Psi_1(\lambda_u+z) \le 4\Psi_1(\lambda_u+z_0) \le \left(\frac{4}{(\lambda_u+z_0)^2}+\frac{4}{\lambda_u+z_0}\right)\\ &\le 8(1+a_1+b_1)\left[\frac{2}{(\lambda_u+z_0)^3}\vee\frac{1}{(\lambda_u+z_0)^2}\right]^{2/3}\\ &\le 8(1+a_1+b_1)|\Psi_2(\lambda_u+z_0)|^{2/3} \le CM_0^{2/3}.\end{aligned}$$

From Lemma 4.16(1) we also have

$$uA_u''(\lambda_0)+nB_n''(\lambda_0) \le 8uA_u''(z_0)+8nB_n''(z_0) \le 8M(z).$$

Therefore we have

$$\tilde{\mathbb{V}}\mathrm{ar}^\rho \log Z_{u,n}(z) \le 5CM_0^{2/3}. \tag{4.134}$$

Chebyshev's inequality with $t/2$ gives the first term of bound (4.129).

Now we consider

$$\sum_{i=u+1}^{m} I_{i,n}(z).$$

We have, since $g(u,z)\le 0$,

$$\begin{aligned}\tilde{\mathbb{V}}\mathrm{ar}^\rho \sum_{i=u+1}^{m} I_{i,n}(z) &= -\left[mA_m'(z)+nB_n'(z)\right]+\left[uA_u'(z)+nB_n'(z)\right]\\ &\le \left|mA_m'(z)+nB_n'(z)\right|.\end{aligned}$$

From (A.53),

$$\begin{aligned}&\tilde{\mathbb{P}}^\rho\left[\Big|\sum_{i=u+1}^{m} I_{i,n}(z)-\tilde{\mathbb{E}}^\rho\sum_{i=u+1}^{m} I_{i,n}(z)\Big|\ge t/2\right]\\ \le\quad & 2\left(\exp\Big[-\frac{t^2}{32|mA_m'(z)+nB_n'(z)|}\Big]\vee e^{-t\Delta z^0_{m,n}/16}\right).\end{aligned} \tag{4.135}$$

Collecting all the results we have (4.129).

If $mA_m'(z)+nB_n'(z)\ge 0$, then write $\log Z_{m,n}(z)$ as

$$\log Z_{m,n}(z)=\log Z_{m,v}(z)+\sum_{j=v+1}^{n} J_{m,j}(z)$$

. The same reasoning as above can be employed to prove (4.129).

Part (3). This is a consequence of (4.129). Note that we used (4.121) so that $\sigma_{m,n}\Delta\zeta_{m,n} \geq 1$.

Part (4). This is a consequence of (4.128). Here we also used (4.126) and (4.121). □

Next lemma estimates the distance between z and $\zeta_{m,n}$.

Lemma 4.22. *Suppose* $\left|mA'_m(z)+nB'_n(z)\right| \leq c\Delta z_{m,n}M(z)/8$ *for some* $0 < c < 1$*, where* $\Delta z_{m,n}$ *is given by* (4.122)*. Then*

$$|z - \zeta_{m,n}| \leq (c\Delta z_{m,n}) \wedge \frac{(1+c)^3|mA'_m(z) + nB'_n(z)|}{M(z)}.$$

We also have

$$|z - \zeta_{m,n}| \leq \frac{c}{1-c}\Delta z^0_{m,n}.$$

Proof. Let $z_1 = z - c\Delta z_{m,n}$ and $z_2 = z + c\Delta z_{m,n}$. Note that

$$\frac{x_m + z}{x_m + z_2} \geq \frac{1}{1+c} \quad \text{and} \quad \frac{y_n - z_1}{y_n - z} \leq 1 + c.$$

Then for some z^* with $z_1 < z^* < z$, from (4.119) with $z^*_1 = z_1$ and $z^*_2 = z$,

$$\begin{aligned} &[mA'_m(z_1) + nB'_n(z_1)] - [mA'_m(z) + nB'_n(z)] \\ =& - c\Delta z_{m,n}[mA''_m(z^*) + nB''_n(z^*)] \leq -c\Delta z_{m,n}M(z)/(1+c)^3. \end{aligned} \tag{4.136}$$

Hence $mA'_m(z_1) + nB'_n(z_1) \leq 0$ since $\left|mA'_m(z) + nB'_n(z)\right| \leq c\Delta z_{m,n}M(z)/8$. Similarly, $mA'_m(z_2) + nB'_n(z_2) \geq 0$. Therefore $z_1 \leq \zeta_{m,n} \leq z_2$. Since for some z^{**} between z and $\zeta_{m,n}$

$$\left|mA'_m(z) + nB'_n(z)\right| = \left|z - \zeta_{m,n}\right|[mA''_m(z^{**}) + nB''_n(z^{**})] \geq \frac{\left|z - \zeta_{m,n}\right|}{(1+c)^3}M(z),$$

we have

$$\left|z - \zeta_{m,n}\right| \le \frac{(1+c)^3 |mA'_m(z) + nB'_n(z)|}{M(z)}.$$

Since $|z - \zeta_{m,n}| \le c\Delta z_{m,n}$ we have

$$z - c\Delta z_{m,n} \le \zeta_{m,n} \le z + c\Delta z_{m,n}.$$

Hence

$$(1-c)(x_m + z) \le x_m + z - c\Delta z_{m,n} \le x_m + \zeta_{m,n}$$

and

$$(1-c)(y_n - z) \le y_n - z - c\Delta z_{m,n} \le y_n - \zeta_{m,n}.$$

Therefore $\Delta\zeta_{m,n} \ge (1-c)\Delta z_{m,n}$ so that

$$\frac{|z - \zeta_{m,n}|}{\Delta\zeta_{m,n}} \le \frac{|z - \zeta_{m,n}|}{(1-c)\Delta z_{m,n}} \le \frac{c}{1-c}.$$

□

Lemma 4.23. *Suppose z satisfies (4.127). Let C be the constant in (4.117). Write $z - \zeta_{m,n} = r/\sigma_{m,n}$ then*

$$P(z) = \mathbb{P}^{\rho,z}\left\{\left|\log Z_{m,n} - \mathbb{E}^{\rho,z}\log Z_{m,n}\right| \ge \left|\mathbb{E}^{\rho,z}\log Z_{m,n} - \mathbb{E}^{\rho,\zeta_{m,n}}\log Z_{m,n}\right|/4\right\}$$

and

$$P(\zeta_{m,n}) = \mathbb{P}^{\rho,\zeta_{m,n}}\left\{\left|\log Z_{m,n} - \mathbb{E}^{\rho,\zeta_{m,n}}\log Z_{m,n}\right| \ge \left|\mathbb{E}^{\rho,z}\log Z_{m,n} - \mathbb{E}^{\rho,\zeta_{m,n}}\log Z_{m,n}\right|/4\right\}$$

satisfy the following.

(1) For $0 < r \le \Delta\zeta_{m,n}\sigma_{m,n}/2$,

$$P(z) \vee P(\zeta_{m,n}) \le \frac{2^{17}C}{r^4}$$

and for $\Delta\zeta_{m,n}\sigma_{m,n}/2 \leq r \leq \Delta z^2_{m,n}\sigma_{m,n}$

$$P(z) \vee P(\zeta_{m,n}) \leq \frac{2^{19}C}{r^2} \wedge \left(4\exp\left[-\frac{K_{m,n}}{2^{22}}r\right]\right).$$

(2) For $-\Delta\zeta_{m,n}\sigma_{m,n}/2 \leq r < 0$,

$$P(z) \vee P(\zeta_{m,n}) \leq \frac{2^{17}C}{r^4}$$

and for $-\Delta z^1_{m,n}\sigma_{m,n} \leq r \leq -\Delta\zeta_{m,n}\sigma_{m,n}/2$

$$P(z) \vee P(\zeta_{m,n}) \leq \frac{2^{19}C}{r^2} \wedge \left(4\exp\left[-\frac{K_{m,n}}{2^{22}}r\right]\right).$$

Proof. We do the case $r > 0$. Write $z_0 = \zeta_{m,n}$ and $M_0 = \sigma^3_{m,n}$ as before. Let

$$f(z) = \tilde{\mathbb{E}}^\rho \log Z_{m,n}(z_0) - \tilde{\mathbb{E}}^\rho \log Z_{m,n}(z). \tag{4.137}$$

We estimate $f(z)$. From (4.119),

$$M(z) \leq 8M_0 \tag{4.138}$$

for $z_0 \leq z \leq z_0 + \Delta z^2_{m,n}$ and

$$M(z) \geq \frac{1}{8}M_0 \tag{4.139}$$

for $z_0 \leq z \leq z_0 + \Delta\zeta_{m,n}/2$.

Let $\Delta z = z - z_0$. For some z_1 with $z_0 < z_1 < z$,

$$\begin{aligned} f(z) = &\quad [mA_m(z_0) + nB_n(z_0)] - [mA_m(z) + nB_n(z)] \\ = &\quad -\Delta z[mA'_m(z_0) + nB'_n(z_0)] - \frac{(\Delta z)^2}{2}[mA''_m(z_1) + nB''_n(z_1)] \\ = &\quad -\frac{(\Delta z)^2}{2}[mA''_m(z_1) + nB''_n(z_1)]. \end{aligned}$$

Suppose $\left|\Delta z\right| \leq \Delta\zeta_{m,n}/2$. Then from (4.119), we have

$$f(z) \leq -\frac{(\Delta z)^2 M_0}{16} = -\frac{r^2 M_0^{1/3}}{16}.$$

Since $f(z)$ is concave and $f(z_0)=0$ we also have

$$\begin{aligned} f(z) &\leq \frac{f(z_0+\Delta z^0_{m,n}/2)}{\Delta z^0_{m,n}/2}\Delta z \\ &\leq -\frac{\Delta z^0_{m,n} M_0}{32}\Delta z = -\frac{\Delta z^0_{m,n} M_0^{2/3} r}{32} \leq -\frac{r M_0^{1/3}}{32} \end{aligned}$$

for $\Delta z \geq \Delta\zeta_{m,n}/2$. The last inequality is from (4.121).

Therefore we have

$$f(z) \leq \begin{cases} -\frac{r^2}{16} M_0^{1/3}, & (0 < r \leq \frac{\Delta\zeta_{m,n}}{2} M_0^{1/3}) \\ -\frac{r}{32} M_0^{1/3}, & (\frac{\Delta\zeta_{m,n}}{2} M_0^{1/3} \leq r \leq \Delta z^2_{m,n} M_0^{1/3}). \end{cases} \tag{4.140}$$

Also note that, from (4.138),

$$\left| m A'_m(z) + n B'_n(z) \right| \leq 8 M_0 \left|\Delta z\right| \leq 8 r M_0^{2/3}.$$

Therefore for $1/2 \leq r \leq \frac{\Delta\zeta_{m,n}}{2} M_0^{1/3}$, from (4.130) with $t = r^2/2^6$ we have

$$\begin{aligned} & P(z) \vee P(z_0) \\ \leq \quad & \frac{20 \cdot 2^{12} C}{r^4} + 2\left(\exp\left[-r^3/2^{20}\right] \vee \exp[-r^2/2^{10}]\right) \\ \leq \quad & \frac{20 \cdot 2^{12} C}{r^4} + 2\exp\left[-r^2/2^{21}\right] \leq \frac{5 \cdot 2^{14} C}{r^4} + \frac{2^{43}}{r^4} \leq \frac{2^{17} C}{r^4}. \end{aligned} \tag{4.141}$$

Note that this inequality holds also for $0 < r \leq 1/2$ since our C is sufficiently large. For $\frac{\Delta\zeta_{m,n}}{2} M_0^{1/3} \leq r \leq \Delta z^2_{m,n} M_0^{1/3}$, from (4.130) with $t = r/2^7$ we have

$$\begin{aligned} & P(z) \vee P(z_0) \\ \leq \quad & \frac{20 \cdot 2^{14} C}{r^2} + 2\left(\exp\left[-r/2^{22}\right] \vee \exp[-r/2^{11}]\right) \\ = \quad & \frac{20 \cdot 2^{14} C}{r^2} + 2\exp\left[-r/2^{22}\right] \leq \frac{20 \cdot 2^{14} C}{r^2} + \frac{2^{45}}{r^2} \leq \frac{2^{19} C}{r^2}. \end{aligned} \tag{4.142}$$

For this range, from (4.131) with $c = 1/2^8$ and $t = r/2^7$ we also have

$$\begin{aligned} & P(z) \vee P(z_0) \\ \leq \quad & 4\exp\left[-\frac{K_{m,n}}{2^{22}} r\right]. \end{aligned} \tag{4.143}$$

Collecting all the results, we complete the proof of part (1). The case $r < 0$ is handled similarly. $\square$

Lemma 4.24. *Suppose z satisfies (4.127). Let C be the constant in (4.117). Write $z - \zeta_{m,n} = r/\sigma_{m,n}$. If $m + n \geq (128(a_1 + b_1)\log 2)^3$ then we have the following.*

(1) For $0 < r \leq \Delta\zeta_{m,n}\sigma_{m,n}/2$,

$$\tilde{\mathbb{P}}^{\rho}\left[Q^{z,\omega}_{m,n}\{\xi_x > 0\} \geq \frac{1}{2}\right] \leq \frac{2^{18}C}{r^4}$$

and for $\Delta\zeta_{m,n}\sigma_{m,n}/2 \leq r \leq \Delta z^2_{m,n}\sigma_{m,n}$

$$\tilde{\mathbb{P}}^{\rho}\left[Q^{z,\omega}_{m,n}\{\xi_x > 0\} \geq \frac{1}{2}\right] \leq \frac{2^{20}C}{r^2} \wedge \left(8\exp\left[-\frac{K_{m,n}}{2^{22}}r\right]\right).$$

(2) For $-\Delta\zeta_{m,n}\sigma_{m,n}/2 \leq r < 0$,

$$\tilde{\mathbb{P}}^{\rho}\left[Q^{z,\omega}_{m,n}\{\xi_y > 0\} \geq \frac{1}{2}\right] \leq \frac{2^{18}C}{r^4}$$

and for $-\Delta z^1_{m,n}\sigma_{m,n} \leq r \leq -\Delta\zeta_{m,n}\sigma_{m,n}/2$

$$\tilde{\mathbb{P}}^{\rho}\left[Q^{z,\omega}_{m,n}\{\xi_y > 0\} \geq \frac{1}{2}\right] \leq \frac{2^{20}C}{r^2} \wedge \left(8\exp\left[-\frac{K_{m,n}}{2^{22}}r\right]\right).$$

Proof. We do the case $r > 0$. Write $z_0 = \zeta_{m,n}$ and $M_0 = \sigma^3_{m,n}$ as before. Note that $1/2 \leq \Delta\zeta_{m,n}\sigma_{m,n}/2$ by (4.121). Since our constant C is sufficiently large, the first inequality in Part (1) is trivially satisfied for $0 < r \leq 1/2$. Therefore we may assume $r \geq 1/2$. From the definition of the quenched measure and (4.56), we have

$$Q^{z,\omega}\{\xi_x > 0\} \leq \frac{Z_{m,n}(z_0)}{Z_{m,n}(z)}.$$

Let $f(z)$ be as in (4.137). Suppose $f(z) \leq -2\log 2$. Then we have

$$\begin{aligned} & \tilde{\mathbb{P}}^\rho\left[Q^{z,\omega}\{\xi_x > 0\} \geq \frac{1}{2}\right] \leq \tilde{\mathbb{P}}^\rho\left[\log Z_{m,n}(z_0) - \log Z_{m,n}(z) \geq -\log 2\right] \\ \leq\ & \tilde{\mathbb{P}}^\rho\left[\left|\log Z_{m,n}(z_0) - \log Z_{m,n}(z) - f(z)\right| \geq \frac{-f(z)}{2}\right] \\ \leq\ & \tilde{\mathbb{P}}^\rho\left[\left|\log Z_{m,n}(z_0) - \tilde{\mathbb{E}}^\rho \log Z_{m,n}(z_0)\right| \geq \frac{|f(z)|}{4}\right] \\ & + \tilde{\mathbb{P}}^\rho\left[\left|\log Z_{m,n}(z) - \tilde{\mathbb{E}}^\rho \log Z_{m,n}(z)\right| \geq \frac{|f(z)|}{4}\right] \end{aligned} \tag{4.144}$$

and, from Lemma 4.23(1), we have bounds in part (1).

Now we seek conditions for $f(z) \leq -2\log 2$. From (4.140), for $r \geq 1/2$, we have $f(z) \leq -M_0^{1/3}/64$. If $M_0^{1/3} \geq 128\log 2$, we have $f(z) \leq -2\log 2$. From (A.6),

$$M_0 \geq (m+n)|\Psi_2(a_1+b_1)| \geq (m+n)\frac{1}{(a_1+b_1)^3}.$$

Hence under our assumption $m+n \geq (128(a_1+b_1)\log 2)^3$, we have

$$M_0^{1/3} \geq 128\log 2 \tag{4.145}$$

and $f(z) \leq -2\log 2$ for $r \geq 1/2$. This completes the proof of part (1). The case $r < 0$ is handled similarly. $\square$

Lemma 4.25. *Consider z with*

$$\zeta_{m,n} - \Delta z^1_{m,n}/2 \leq z \leq \zeta_{m,n} + \Delta z^2_{m,n}/2$$

and write $z - \zeta_{m,n} = r/\sigma_{m,n}$. Let C be the constant in (4.117). Then we have the following.

(1) Suppose $z \leq \zeta_{m,n}$. For $2|r| < t \leq 4\Delta\zeta_{m,n}\sigma_{m,n}$

$$\tilde{\mathbb{P}}^\rho\left[Q^{z,\omega}_{m,n}\{f_{\zeta_{m,n}}(\xi_x) \geq t\sigma^2_{m,n}\} \geq \exp[-\frac{t^2\sigma_{m,n}}{64}]\right] \leq \frac{2^{22}C}{t^4} \tag{4.146}$$

and for $t \geq (4\Delta\zeta_{m,n}\sigma_{m,n}) \vee (2|r|)$,

$$\begin{aligned} &\tilde{\mathbb{P}}^{\rho}\left[Q^{z,\omega}_{m,n}\{f_{\zeta_{m,n}}(\xi_x) \geq t\sigma^2_{m,n}\} \geq \exp[-\frac{t\sigma_{m,n}}{16}]\right] \\ \leq &\frac{2^{18}C}{t^2} \wedge \left(10\exp\left[-\frac{K_{m,n}}{2^{19}}t\right]\right). \end{aligned} \tag{4.147}$$

(2) Suppose $z \geq \zeta_{m,n}$. *For* $2|r| < t \leq 4\Delta\zeta_{m,n}\sigma_{m,n}$

$$\tilde{\mathbb{P}}^{\rho}\left[Q^{z,\omega}_{m,n}\{g_{\zeta_{m,n}}(\xi_y) \geq t\sigma^2_{m,n}\} \geq \exp[-\frac{t^2\sigma_{m,n}}{64}]\right] \leq \frac{2^{22}C}{t^4} \tag{4.148}$$

and for $t \geq (4\Delta\zeta_{m,n}\sigma_{m,n}) \vee (2|r|)$,

$$\begin{aligned} &\tilde{\mathbb{P}}^{\rho}\left[Q^{z,\omega}_{m,n}\{g_{\zeta_{m,n}}(\xi_y) \geq t\sigma^2_{m,n}\} \geq \exp[-\frac{t\sigma_{m,n}}{16}]\right] \\ \leq &\frac{2^{18}C}{t^2} \wedge \left(10\exp\left[-\frac{K_{m,n}}{2^{19}}t\right]\right). \end{aligned} \tag{4.149}$$

Proof. We prove part (1). Part (2) can be proven similarly. Write $z_0 = \zeta_{m,n}$ and $M_0 = \sigma^3_{m,n}$ as before. We use (4.86) and (4.102) with $h(k) = f_{z_0}(k)$. Since $f_{z_0}(u(s)) \geq s$ by (4.88), if conditions for (4.102) are satisfied, then we have

$$\begin{aligned} &\tilde{\mathbb{P}}^{\rho}\left[Q^{z,\omega}_{m,n}\{f_{z_0}(\xi_x) \geq s\} \geq \exp[-\frac{s\Delta z}{4}]\right] \\ &\leq \tilde{\mathbb{P}}^{\rho}\left[\left|\log Z(\lambda) - \tilde{\mathbb{E}}^{\rho}\log Z(\lambda)\right| \geq \frac{s\Delta z}{16}\right] \\ &+ \tilde{\mathbb{P}}^{\rho}\left[\left|\log Z(z) - \tilde{\mathbb{E}}^{\rho}\log Z(z)\right| \geq \frac{s\Delta z}{16}\right] \\ &+ 2\exp[-\frac{(\Delta z)^2 s}{2^{13}}], \end{aligned} \tag{4.150}$$

where $\Delta z = z - \lambda > 0$ and Δz satisfies (4.100). To use (4.102), we need to check if $u(s)$ satisfies (4.98). Since

$$|mA'_m(z) + nB'_n(z)| = (z_0 - z)M(z^*)$$

for some $z \leq z^* \leq z_0$, from (4.119), we have

$$|mA'_m(z) + nB'_n(z)| \leq 8|r|M_0^{2/3}. \tag{4.151}$$

Therefore if $s \geq 2|r|M_0^{2/3}$, then

$$u|A_u'(z_0)| = f_{z_0}(u(s)) \geq s \geq \left|mA_m'(z) + nB_n'(z)\right|/4$$

and (4.98) is satisfied. Hence we can use (4.150) for $s > 2|r|M_0^{2/3}$. Substitute $s = tM_0^{2/3}$ into (4.150).

For $2|r| < t \leq 4\Delta\zeta_{m,n}M_0^{1/3}$ (if this range is nonempty), set $\Delta z = s/(16M_0)$. Then for some $\lambda^* \in [\lambda, z_0]$ we have

$$|mA_m'(\lambda) + nB_n'(\lambda)| = (z_0 - \lambda)M(\lambda^*) \leq 8(t + |r|)M_0^{2/3} \leq 16tM_0^{2/3}. \tag{4.152}$$

From (4.130) with $t^2/2^8$, (4.150), (4.151), and (4.152) we have

$$\begin{aligned}
&\tilde{\mathbb{P}}^\rho\left[Q_{m,n}^{z,\omega}\{f_{z_0}(\xi_x) \geq tM_0^{2/3}\} \geq \exp[-\frac{t^2M_0^{1/3}}{64}]\right] \\
\leq &\frac{40 \cdot 2^{16}C}{t^4} + 2\left(\exp[-\frac{t^4}{2^{24}|r|}] \vee \exp[-\frac{t^2}{2^{12}}]\right) \\
&+ 2\left(\exp[-\frac{t^4}{2^{24}(t + |r|)}] \vee \exp[-\frac{t^2}{2^{12}}]\right) + 2\exp[-\frac{t^3}{2^{21}}] \\
\leq &\frac{40 \cdot 2^{16}C}{t^4} + 4\left(\exp[-\frac{t^3}{2^{25}}] \vee \exp[-\frac{t^2}{2^{12}}]\right) + 2\exp[-\frac{t^3}{2^{21}}] \\
\leq &\frac{2^{22}C}{t^4}.
\end{aligned}$$

For $t \geq \left(4\Delta\zeta_{m,n}M_0^{1/3}\right) \vee (2|r|)$, set $\Delta z = \Delta\zeta_{m,n}/4$. Then for some $\lambda^* \in [\lambda, z_0]$ we have

$$\begin{aligned}
&|mA_m'(\lambda) + nB_n'(\lambda)| = (z_0 - \lambda)M(\lambda^*) \\
\leq &8\left(|r|M_0^{-1/3} + \Delta z_{m,n}^0/4\right)M_0 \leq 8(|r| + t/16)M_0^{2/3} \leq 8tM_0^{2/3}.
\end{aligned} \tag{4.153}$$

From (4.121), (4.130) with $t/2^6$, (4.150), (4.151), and (4.153) we have

$$\begin{aligned}
&\tilde{\mathbb{P}}^\rho\left[Q^{z,\omega}_{m,n}\{f_{z_0}(\xi_x) \geq tM_0^{2/3}\} \geq \exp[-\frac{tM_0^{1/3}}{16}]\right] \\
\leq &\frac{40 \cdot 2^{12}C}{t^2} + 2\left(\exp[-\frac{t^2}{2^{20}|r|}] \vee \exp[-\frac{t}{2^{10}}]\right) \\
&+ 2\left(\exp[-\frac{t}{2^{20}}] \vee \exp[-\frac{t}{2^{10}}]\right) + 2\exp[-\frac{t}{2^{17}}] \\
\leq &\frac{40 \cdot 2^{12}C}{t^2} + 6\exp[-\frac{t}{2^{20}}] \leq \frac{2^{18}C}{t^2}.
\end{aligned}$$

For this range, from (4.131) with $c = 1/2^4$ and $t/2^6$ we also have

$$\begin{aligned}
&\tilde{\mathbb{P}}^\rho\left[Q^{z,\omega}_{m,n}\{f_{z_0}(\xi_x) \geq tM_0^{2/3}\} \geq \exp[-\frac{tM_0^{1/3}}{16}]\right] \\
\leq &8\exp\left[-\frac{K_{m,n}}{2^{17}}t\right] + 2\exp[-\frac{t}{2^{17}}] \\
\leq &10\exp\left[-\frac{K_{m,n}}{2^{19}}t\right].
\end{aligned}$$

□

4.6 Lower bound for the fluctuation

In this section we finish the proof of Theorem 4.3. We continue to use our notation $\tilde{\mathbb{P}}^\rho$ as explained in the beginning of Section 4.4. However at some places we go back to notation $\mathbb{P}^\rho$ in Remark 3.15. We also refer readers to page 115 for various definitions and notations. We construct our environment as follows:

Let $\{\eta_{i,j} : i \geq 0, j \geq 0\}$ be i.i.d. uniform random variables on $(0, 1)$. Let $\omega_{i,j}(z) = -H(\rho_{i,j}, \eta_{i,j})$ with $\rho_{i,j} = \lambda_i + \theta_j$ for $i, j \geq 1$ and $\rho_{i,0} = \lambda_i + z$, $\rho_{0,j} = \theta_j - z$. For subsets $A \subseteq \Pi_{(i,j),(k,l)}$, let

$$Z^z_{(i,j),(k,l)}(A) = \sum_{\mathbf{x}_\cdot \in A} \exp\left[\sum_{r=1}^{k-i+l-j} \omega_{\mathbf{x}_k}(z)\right].$$

Lemma 4.26 (Lemma 5.1. of [33]). *For $m \geq 2$ and $n \geq 1$ we have this comparison of partition functions:*

$$\frac{Z_{m,n}(\xi_y > 0)}{Z_{m-1,n}(\xi_y > 0)} \leq \frac{Z_{(1,1),(m,n)}}{Z_{(1,1),(m-1,n)}} \leq \frac{Z_{m,n}(\xi_x > 0)}{Z_{m-1,n}(\xi_x > 0)}. \tag{4.154}$$

Lemma 4.27 (Lemma 5.4 of [33]). *For each fixed ω, $Q^{\omega}_{m_1,n}(\xi_x > 0) \leq Q^{\omega}_{m_2,n}(\xi_x > 0)$ for all $0 < m_1 < m_2$ and $n \geq 0$.*

The proof of the following lemma is adapted from [18].

Lemma 4.28. *Let C be the constant in (4.117). Then whenever $m + n \geq C^2$, we have*

$$\tilde{\mathbb{P}}^{\rho}\left[\log Z_{m,n}(\zeta_{m,n}) - \tilde{\mathbb{E}}^{\rho} \log Z_{m,n}(\zeta_{m,n}) \geq \sigma_{m,n}\right] \geq e^{-C^2}. \tag{4.155}$$

Proof. Write $z_0 = \zeta_{m,n}$ and $M_0 = (\sigma_{m,n})^3$ as before. First assume $\Delta\zeta_{m,n}$ in (4.30) satisfies $\Delta\zeta_{m,n} = y_n - z_0$. So we have $\Delta z^1_{m,n} = \frac{x_m + z_0}{2} \geq \frac{a_0 + b_0}{4}$. Consider λ with

$$0 < z_0 - \lambda \leq \Delta z^1_{m,n} \quad \text{and} \quad \lambda = z_0 - r/M_0^{1/3}.$$

We can use Lemma 4.24: Suppose $m+n \geq (128(a_1+b_1)\log 2)^3$. We have $M_0^{1/3} \geq 128 \log 2$ by (4.145).

For $0 < r \leq \Delta z^1_{m,n} M_0^{1/3}$,

$$\tilde{\mathbb{P}}^{\rho}\left[Q^{\lambda,\omega}_{m,n}\{\xi_x > 0\} \geq \frac{1}{2}\right] = 1 - \tilde{\mathbb{P}}^{\rho}\left[Q^{\lambda,\omega}_{m,n}\{\xi_y > 0\} \geq \frac{1}{2}\right] \geq 1 - \frac{2^{20}C}{r^2}. \tag{4.156}$$

If $r \leq \Delta z^1_{m,n} M_0^{1/3}/2$, from Lemma 4.25(1), we have

$$\tilde{\mathbb{P}}^{\rho}\left[Q^{\lambda,\omega}\{f_{z_0}(\xi_x) \geq tM_0^{2/3}\} \geq \tfrac{1}{4}\right] \leq \frac{2^{22}C}{t^2} \tag{4.157}$$

for $t \geq (2r) \vee 1$. Choose such r and t. From (4.156) and (4.157) we have

$$\tilde{\mathbb{P}}^{\rho}\left[Q^{\lambda,\omega}\{0 < f_{z_0}(\xi_x) < tM_0^{2/3}\} \geq 1/4\right] \geq 1 - \frac{2^{20}C}{r^2} - \frac{2^{22}C}{t^2}. \tag{4.158}$$

From (4.158)

$$
\begin{aligned}
&1-\frac{2^{20}C}{r^2}-\frac{2^{22}C}{t^2}\\
\leq&\tilde{\mathbb{P}}^{\rho}\left[Q^{\lambda,\omega}\{0<f_{z_0}(\xi_x)<tM_0^{2/3}\}\geq 1/4\right]\\
=&\tilde{\mathbb{P}}^{\rho}\left[\frac{1}{Z_{m,n}(\lambda)}Z^{\lambda}_{m,n}\{0<f_{z_0}(\xi_x)<tM_0^{2/3}\}\geq 1/4\right]\\
\leq&\tilde{\mathbb{P}}^{\rho}\left[Z^{\lambda}_{m,n}\{0<f_{z_0}(\xi_x)<tM_0^{2/3}\}\geq \frac{1}{4}\exp\frac{1}{2}\big(\tilde{\mathbb{E}}^{\rho}\log Z_{m,n}(\lambda)+\tilde{\mathbb{E}}^{\rho}\log Z_{m,n}(z_0)\big)\right] \qquad (4.159)\\
&+\tilde{\mathbb{P}}^{\rho}\left[Z_{m,n}(\lambda)\leq \exp\frac{1}{2}\big(\tilde{\mathbb{E}}^{\rho}\log Z_{m,n}(\lambda)+\tilde{\mathbb{E}}^{\rho}\log Z_{m,n}(z_0)\big)\right]. \qquad (4.160)
\end{aligned}
$$

We treat (4.160). From Lemma 4.23(2), we have

$$
(4.160)=\tilde{\mathbb{P}}^{\rho}\left[\log Z_{m,n}(\lambda)-\tilde{\mathbb{E}}^{\rho}\log Z_{m,n}(\lambda)\leq -\big|\tilde{\mathbb{E}}^{\rho}\log Z_{m,n}(z_0)-\tilde{\mathbb{E}}^{\rho}\log Z_{m,n}(\lambda)\big|/2\right]\leq\frac{2^{19}C}{r^2}.
$$

Note that if we do not use our standard realization of weights by uniform random variables, we can write

$$
(4.159)=\mathbb{P}^{\rho,\lambda}\left[\log Z^{\omega}_{m,n}\{0<f_{z_0}(\xi_x)<tM_0^{2/3}\}\geq 2^{-1}\left(\mathbb{E}^{\rho,z_0}\log Z_{m,n}+\mathbb{E}^{\rho,\lambda}\log Z_{m,n}\right)-\log 4\right].
$$

Let $S=S(\omega)$ denote the event in the above probability. Then we have

$$
(1-\frac{2^{21}C}{r^2}-\frac{2^{22}C}{t^2})\leq\mathbb{P}^{\rho,\lambda}(S).
$$

We construct a new environment to handle $\mathbb{P}^{\rho,\lambda}(S)$ and $\mathbb{P}^{\rho,z_0}(S)$. Let

$$
u=u(t)=\max\{k\geq 0: f_{z_0}(k)<tM_0^{2/3}\}.
$$

If $u>0$, then we use the same realization as the beginning of this section for bulk weights and modify boundary weights only:

$$
\omega_{i,0}=\omega_{i,0}(\lambda)\quad\text{for}\quad 1\leq i\leq u
$$

and

$$\omega_{i,j} = \omega_{i,j}(z_0) \quad \text{for remaining weights.}$$

Let $\hat{\omega}$ denote this new environment. If $u = 0$, then set $\hat{\omega} = \omega$. We have

$$\mathbb{P}^{\rho,\lambda}[S] = \tilde{\mathbb{P}}^{\rho}[S(\hat{\omega})].$$

Let ν^{z_0} be the distribution of $\omega(z_0)$, $\hat{\nu}$ the distribution of $\hat{\omega}$ and $h = d\hat{\nu}/d\nu^{z_0}$ the Radon-Nikodym derivative. Here we consider random environment ω as a collection of log-gamma random variables. Since only boundary weights are different, we have

$$h(\omega) = \prod_{i=1}^{u} \frac{\Gamma(\lambda_i + z_0)}{\Gamma(\lambda_i + \lambda)} \exp[(\lambda - z_0)\omega_{i,0}].$$

When $u = 0$, the product is interpreted as 1. Schwartz inequality gives

$$\tilde{\mathbb{P}}^{\rho}[S(\hat{\omega})] = \mathbb{E}^{\rho,z_0}(h1_S) \le (\mathbb{E}^{\rho,z_0}h^2)^{1/2}\mathbb{P}^{\rho,z_0}[S]^{1/2}.$$

Now we compute $\mathbb{E}^{\rho,z_0}h^2$: Introduce a function $H(\lambda)$ then we can write

$$\begin{aligned} \mathbb{E}^{z}h^2 &= \prod_{i=1}^{u} \frac{\Gamma(\lambda_i + z_0)\Gamma(\lambda_i + 2\lambda - z_0)}{\Gamma(\lambda_i + \lambda)^2} \\ &= \exp\left[\sum_{i=1}^{u} \log\Gamma(\lambda_i + z_0) + \log\Gamma(\lambda_i + 2\lambda - z_0) - 2\log\Gamma(\lambda_i + \lambda)\right] \\ &= \exp H(\lambda). \end{aligned} \tag{4.161}$$

We have

$$H''(\lambda) = \sum_{i=1}^{u}\big(4\Psi_1(\lambda_i + 2\lambda - z_0) - 2\Psi_1(\lambda_i + \lambda)\big) = -4uA'_u(z_0 + 2(\lambda - z_0)) + 2uA'_u(\lambda)$$

and by substituting $\lambda = z_0$,

$$H'(z_0) = 0 \quad \text{and} \quad H''(z_0) = 2u\big|A'_u(z_0)\big| = 2f_{z_0}(u).$$

Also note that if $z_0 - \lambda \leq (x_m + z_0)/4$, which is satisfied in our case (recall we assume $r \leq \Delta z^1_{m,n} M_0^{1/3}/2$), from Lemma 4.16 part (1),

$$H''(\lambda) \leq 16u\left|A'_u(z_0)\right| = 16 f_{z_0}(u).$$

Hence from Taylor's theorem and the definition of u, we have

$$H(\lambda) \leq 8 f_{z_0}(u) \cdot (\lambda - z_0)^2 \leq 8tr^2.$$

Collecting all the results, we have

$$\begin{aligned}(1 - \frac{2^{21}C}{r^2} - \frac{2^{22}C}{t^2})^2 \exp(-8tr^2) \leq \mathbb{P}^{\rho, z_0}[S] \\ \leq \mathbb{P}^{\rho, z_0}\left[\log Z^\omega_{m,n} - \mathbb{E}^{\rho, z_0} \log Z_{m,n} \geq 2^{-1}\left(\mathbb{E}^{\rho,\lambda} \log Z_{m,n} - \mathbb{E}^{\rho, z_0} \log Z_{m,n}\right) - \log 4\right].\end{aligned} \tag{4.162}$$

provided $1 - \frac{2^{21}C}{r^2} - \frac{2^{22}C}{t^2} > 0$. From (4.137) and (4.140), if $r \geq 64$, then

$$2^{-1}\left(\mathbb{E}^{\rho,\lambda} \log Z_{m,n} - \mathbb{E}^{\rho, z_0} \log Z_{m,n}\right) - \log 4 \geq M_0^{1/3}.$$

Take $r = t = (2^{12})\sqrt{C}$. Then we have

$$\mathbb{P}^{\rho, z_0}\left[\log Z^\omega_{m,n} \geq \mathbb{E}^{\rho, z_0} \log Z_{m,n} + M_0^{1/3}\right] \geq \delta,$$

where

$$\delta = \exp[-C^2].$$

Note that this is only possible when $\Delta z^1_{m,n} M_0^{1/3}/2$ is sufficiently large because we need $r \leq \Delta z^1_{m,n} M_0^{1/3}/2$. Since

$$M_0 \geq (m+n)|\Psi_2(a_1 + b_1)| \geq \frac{m+n}{(a_1+b_1)^3},$$

we have

$$\Delta z^1_{m,n} M_0^{1/3}/2 \geq \frac{(a_0+b_0)}{8(a_1+b_1)}(m+n)^{1/3}.$$

Hence if we choose $(m+n) \geq C^2$, then we have the desired result.

For the case $\Delta\zeta_{m,n} = x_m + z_0$, use $\lambda > z_0$ and $g_{z_0}(\xi_y)$. The proof is the same with appropriate modifications. □

Proof of the lower bound in (4.22). This is an immediate consequence of Lemma 4.28. For $N \geq C^2$,

$$\mathbb{V}\mathrm{ar}^{\rho,z_0} \log Z_{m,n} = \mathbb{E}^{\rho,z_0}\big(\log Z_{m,n} - \mathbb{E}^{\rho,z_0}\log Z_{m,n}\big)^2 \geq e^{-C^2}(\sigma_{m,n})^2.$$

□

Lemma 4.29. *Let C be the constant in* (4.117)*. Then for $N_0 = e^{4C^2}$, whenever $m+n \geq N_0$ we have*

$$\mathbb{E}^{\rho}\Big[\big(\log Z_{(1,1),(m,n)} - \phi_{m,n}\big)^-\Big] \geq 2^{-1}e^{-C^2}\sigma_{m,n}. \tag{4.163}$$

Proof. Write $z_0 = \zeta_{m,n}$ and $(\sigma_{m,n})^3 = M_0$ as before. Recall that $\phi_{m,n} = \mathbb{E}^{\rho,z_0}\log Z_{m,n}$ (see (4.14) and (4.19)). Let S be the event

$$S = \{\log Z_{m,n} \geq \mathbb{E}^{\rho,z_0}\log Z_{m,n} + M_0^{1/3}\}.$$

By Lemma 4.28 there exist positive $\delta = e^{-C^2}$ and $N_0 = C^2$ such that $\mathbb{P}^{\rho,z_0}[S] \geq \delta$ for $(m+n) \geq N_0$. Without loss of generality, assume $z_0 \geq (b_0-a_0)/2$. We use the simple upper bound

$$\log Z_{m,n}(z_0) \geq \log Z_{(1,1),(m,n)} + \omega_{1,0}(z_0) + \omega_{1,1}. \tag{4.164}$$

Let $\epsilon(z_0) = \omega_{1,0}(z_0) + \omega_{1,1}$. Note that

$$\begin{aligned}\tilde{\mathbb{E}}^{\rho}[\epsilon(z_0)^2] \leq\ & 2\tilde{\mathbb{E}}^{\rho}[(\omega_{1,0}(z_0))^2] + 2\tilde{\mathbb{E}}^{\rho}[(\omega_{1,1})^2] \\ \leq\ & 4\Big[\Psi_1(\frac{a_0+b_0}{2}) + \big(\Psi_0(\frac{a_0+b_0}{2})\big)^2 + \big(\Psi_0(a_1+b_1)\big)^2\Big].\end{aligned} \tag{4.165}$$

From these we have

$$\begin{aligned} 0 = \quad & \tilde{\mathbb{E}}^\rho[\big(\log Z_{m,n}(z_0) - \tilde{\mathbb{E}}^\rho \log Z_{m,n}(z_0)\big)1_S] \\ & + \tilde{\mathbb{E}}^\rho[\big(\log Z_{m,n}(z_0) - \tilde{\mathbb{E}}^\rho \log Z_{m,n}(z_0)\big)1_{S^c}] \\ \geq \quad & \mathbb{P}^{\rho,z_0}[S] M_0^{1/3} + \tilde{\mathbb{E}}^\rho[\big(\log Z_{(1,1),(m,n)} - \tilde{\mathbb{E}}^\rho \log Z_{m,n}(z_0)\big)1_{S^c}] + \tilde{\mathbb{E}}^\rho[\epsilon(z_0)1_{S^c}]. \end{aligned}$$

Therefore

$$\begin{aligned} \delta M_0^{1/3} \leq \quad & \tilde{\mathbb{E}}^\rho\left[\big(\tilde{\mathbb{E}}^\rho \log Z_{m,n}(z_0) - \log Z_{(1,1),(m,n)}\big)1_{S^c}\right] + \tilde{\mathbb{E}}^\rho[\big|\epsilon(z_0)\big|] \\ \leq \quad & \tilde{\mathbb{E}}^\rho\left[\big(\tilde{\mathbb{E}}^\rho \log Z_{m,n}(z_0) - \log Z_{(1,1),(m,n)}\big)^+\right] + \sqrt{\tilde{\mathbb{E}}^\rho[\epsilon(z_0)]^2}. \end{aligned}$$

If $\sqrt{\tilde{\mathbb{E}}^\rho[\epsilon(z_0)]^2} \leq \delta M_0^{1/3}/2$, we have

$$\tilde{\mathbb{E}}^\rho\left[\big(\tilde{\mathbb{E}}^\rho \log Z_{m,n}(z_0) - \log Z_{(1,1),(m,n)}\big)^+\right] \geq \frac{\delta M_0^{1/3}}{2}.$$

Direct calculation using (4.165) shows that we may take $N_0 = e^{4C^2}$. □

4.7 Identification of scaling exponents

In this section, we prove Theorem 4.4. We start from points in S.

Proof of Theorem 4.4 (1). As before, $M_0 = (\sigma_{m,n})^3$ and $z = \zeta(s)$. Note that $-a_0 < z < b_0$. Define

$$\delta z_1 = (a_0 + z)/2, \quad \delta z_2 = (b_0 - z)/2$$

and

$$\delta z = \delta z_1 \wedge \delta z_2.$$

Suppose $\zeta_{m,n}$ satisfies $\zeta_{m,n} \in [z - \delta z_1, z + \delta z_2]$. Then from (4.118) and monotonicity of Ψ_2 explained in (A.2) we have

$$(m+n)|\Psi_2(a_1+b_1)| \le M_0 \le (m+n)|\Psi_2(\delta z)|. \tag{4.166}$$

Therefore in this case (4.24) holds with

$$C_1 = |\Psi_2(a_1+b_1)|^{1/3} \tag{4.167}$$

and

$$C_2 = |\Psi_2(\delta z)|^{1/3}. \tag{4.168}$$

Hence we need to prove (4.25) to complete the proof.

From Lemma 4.16 part (4) and (4.23), we have

$$\begin{aligned}\left|mA_m'(z) + nB_n'(z)\right| &\le \left|mA'(z) + nB'(z)\right| + \left|\Psi_2(\delta z)\right|(mW_1(\alpha_m,\alpha) + nW_1(\beta_n,\beta)) \\ &\le \left|NxA'(z) + NyB'(z)\right| + K\left|A'(z)\right| + KB'(z) \\ &\quad + \left|\Psi_2(\delta z)\right|(mW_1(\alpha_m,\alpha) + nW_1(\beta_n,\beta)) \\ &\le \quad \left|\Psi_2(\delta z)\right|(2K + mW_1(\alpha_m,\alpha) + nW_1(\beta_n,\beta)).\end{aligned} \tag{4.169}$$

Since

$$M_{m,n}(z) \ge (m+n)|\Psi_2(a_1+b_1)|$$

for any $z \in (-a_0, b_0)$, from (4.62) we have

$$\frac{\left|mA_m'(z) + nB_n'(z)\right|}{M_{m,n}(z)} \to 0 \tag{4.170}$$

as $N \to \infty$. Use Lemma 4.22 with $\Delta z_{m,n} = 2\delta z$. Then we obtain (4.25). □

Next, we treat the case of S_1.

Lemma 4.30. *Suppose Assumption 4.2 holds. We have*

$$\lim_{m\to\infty} \frac{m^{3d_1}(x_m - a_0)}{m} \sum_{i=1}^{m} \frac{1}{(x_{m:i} - a_0)^3} = 0 \tag{4.171}$$

and

$$\lim_{m\to\infty} \frac{x_m - a_0}{m} \sum_{i=2}^{m} \frac{1}{(x_{m:i} - x_m)^3} = 0. \tag{4.172}$$

We also have

$$\lim_{m\to\infty} \frac{1}{m(x_m - a_0)^2} = \lim_{m\to\infty} \frac{\Psi_1(x_m - a_0)}{m} = 0. \tag{4.173}$$

Proof. For $2 < p_1 \le 3$, we have

$$\begin{aligned} &\frac{m^{3d_1}(x_m - a_0)}{m} \sum_{i=1}^{m} \frac{1}{(x_{m:i} - a_0)^3} \\ \le\ &\frac{m^{3d_1}(x_m - a_0)^{p_1-2}}{m} \sum_{i=1}^{m} \frac{1}{(x_{m:i} - a_0)^{p_1}} \\ \le\ &m^{3d_1}(x_m - a_0)^{p_1-2} C_{p_1} \end{aligned}$$

for some $C_{p_1} > 0$. From Assumption 4.2 part (b), we have

$$m^{3d_1}(x_m - a_0)^{p_1-2} C_{p_1} \le \frac{C_{p_1} C_{q_1}}{m^{(p_1-2)q_1 - 3d_1}} \to 0.$$

For (4.172), note that for all sufficiently large m,

$$\frac{x_{m:i} - x_m}{x_{m:i} - a_0} = 1 - \frac{x_m - a_0}{x_{m:i} - a_0} \ge 1 - \frac{x_m - a_0}{x_{m:2} - a_0} \ge \frac{D_1}{m^{d_1}}.$$

Application of (4.171) gives (4.172).

For (4.173), from (4.171) we have

$$\lim_{m\to\infty} \frac{1}{m(x_m - a_0)^2} = 0.$$

This result and (A.6) gives

$$\lim_{m\to\infty} \frac{\Psi_1(x_m - a_0)}{m} = 0.$$

□

Recall that our parameter $\zeta_{m,n}$ is in $(-x_m, y_n)$ and possibly $x_m = \min_{1\le i\le m} \lambda_i > a_0$. In particular, it is possible to have $-x_m < \zeta_{m,n} < -a_0$. In the flat region S_1, this is the case as the following lemma shows.

Lemma 4.31. *Let $y/x = v_1 = 1/s_1$ be the line $S_1 \cap \bar{S}$. Fix $v > v_1$ and assume $n/m \ge v$ for all but finitely many m and n.*

(1) There exists a positive constant m_v that depend on a_0, a_1, b_0 and b_1 such that for $m \ge m_v$,

$$\zeta_{m,n} < -a_0, \quad (x_m + \zeta_{m,n}) \le (y_n - \zeta_{m,n}).$$

(2) Let $n = n(m)$ and assume $\lim_{m\to\infty} \frac{n}{m} = w \ge v$. Then we have

$$\lim_{m\to\infty} \frac{\Psi_1(x_m + \zeta_{m,n})}{m} = (w - v_1)B'(-a_0) = -\frac{w - v_1}{v_1} A'(-a_0) \tag{4.174}$$

and

$$\lim_{m\to\infty} A'_m(\zeta_{m,n}) = -wB'(-a_0) = \frac{w}{v_1} A'(-a_0). \tag{4.175}$$

(3) There exists an another constant m_v such that for $m \ge m_v$,

$$\frac{\Psi_1(x_m + \zeta_{m,n})}{m|A'_m(\zeta_{m,n})|} \ge \frac{(v - v_1)}{2v}. \tag{4.176}$$

(4) There exist a positive constant $N_0(v, a_0, a_1, b_0, b_1)$ such that for $N = m + n \ge N_0$, we have

$$C_1 N \le (\sigma_{m,n})^2 \le C_2 N \tag{4.177}$$

and

$$\big((x_m + \zeta_{m,n})\sigma_{m,n}\big)^2 \le C_0, \tag{4.178}$$

where

$$C_1 = \frac{(v - v_1)\Psi_1(a_1 + b_1)}{2v(1 + a_1 + b_1)},$$

$$C_2 = (2 + a_1 + b_1)\Psi_1((a_0 + b_0)/2)$$

and

$$C_0 = (2 + a_1 + b_1)^2 \frac{2v}{v - v_1}.$$

(5) $K_{m,n}$ *in* (4.31) *satisfies*

$$K_{m,n} \geq \frac{v - v_1}{4v}. \tag{4.179}$$

Proof. (1) Note that

$$A'_m(-a_0) + \frac{n}{m} B'_n(-a_0) \geq (v - v_1)B'_n(-a_0) + \left[A'_m(-a_0) + v_1 B'_n(-a_0)\right].$$

From (4.62), (4.63), Lemma 4.16 part (4) and (5), we have

$$\lim_{m\to\infty} (v - v_1)B'_n(-a_0) = (v - v_1)B'(-a_0) > 0 \tag{4.180}$$

and

$$\lim_{m\to\infty} A'_m(-a_0) + v_1 B'_n(-a_0) = A'(-a_0) + v_1 B'(-a_0) = 0. \tag{4.181}$$

The last equality is from (4.7). Therefore for all sufficiently large m,

$$mA'_m(-a_0) + nB'_n(-a_0) > 0.$$

Since $G_{m,n}(z) = mA_m(z) + nB_n(z)$ is a convex function of z and $G'_{m,n}(\zeta_{m,n}) = 0$, we have $\zeta_{m,n} < -a_0$. Since $x_m \to a_0$,

$$(x_m + \zeta_{m,n}) \leq (b_0 + a_0) \leq (y_n - \zeta_{m,n})$$

for all large m. Assume these properties hold for $m \geq m_v$.

(2) From the definition of $\zeta_{m,n}$ we have

$$\frac{\Psi_1(x_m+\zeta_{m,n})}{m} = -\frac{1}{m}\sum_{i=2}^{m}\Psi_1(x_{m:i}+\zeta_{m,n}) + \frac{n}{m}B_n'(\zeta_{m,n}). \tag{4.182}$$

We claim that

$$\lim_{m\to\infty}\sup_{n\geq mv}\left|-\frac{1}{m}\sum_{i=2}^{m}\Psi_1(x_{m:i}+\zeta_{m,n}) - A'(-a_0)\right| = 0 \tag{4.183}$$

and

$$\lim_{m\to\infty}\sup_{n\geq mv}\left|B_n'(\zeta_{m,n}) - B'(-a_0)\right| = 0. \tag{4.184}$$

Suppose these hold. Write $z_0 = \zeta_{m,n}$. Then from (4.181) and (4.182) we have

$$\lim_{m\to\infty}\frac{\Psi_1(x_m+z_0)}{m} = A'(-a_0) + wB'(-a_0) = (w-v_1)B'(-a_0)$$

and

$$\lim_{m\to\infty}A_m'(z_0) = -\lim_{m\to\infty}\frac{n}{m}B_n'(z_0) = -wB'(-a_0).$$

This proves (4.174) and (4.175). Hence it remains to prove (4.183) and (4.184).

Suppose $m \geq m_v$. Hence $-x_m < z_0 < -a_0$. From Taylor's Theorem, for some z^* with $z_0 < z^* < -a_0$,

$$\begin{aligned}
& \left|-\frac{1}{m}\sum_{i=2}^{m}\Psi_1(x_{m:i}+z_0) - A'(-a_0)\right| \\
\leq\ & \left|-\frac{1}{m}\sum_{i=2}^{m}\Psi_1(x_{m:i}+z_0) - A_m'(-a_0)\right| + \left|A_m'(-a_0) - A'(-a_0)\right| \\
\leq\ & \left|-\frac{1}{m}\sum_{i=2}^{m}\Psi_1(x_{m:i}+z_0) + \frac{1}{m}\sum_{i=2}^{m}\Psi_1(x_{m:i}-a_0)\right| \\
& + \left|A_m'(-a_0) - A'(-a_0)\right| + \left|\frac{\Psi_1(x_m-a_0)}{m}\right| \\
=\ & \left|\frac{z_0+a_0}{m}\sum_{i=2}^{m}\Psi_2(x_{m:i}+z^*)\right| + \left|A_m'(-a_0) - A'(-a_0)\right| + \left|\frac{\Psi_1(x_m-a_0)}{m}\right| \\
\leq\ & \left|\frac{x_m-a_0}{m}\sum_{i=2}^{m}\Psi_2(x_{m:i}-x_m)\right| + \left|A_m'(-a_0) - A'(-a_0)\right| + \left|\frac{\Psi_1(x_m-a_0)}{m}\right|.
\end{aligned} \tag{4.185}$$

The last line of (4.185) goes to 0 as $m \to \infty$ by Lemma 4.30 and (A.6). This proves (4.183).

Similarly, for some other z^*,

$$\begin{aligned} & \left|B_n'(z_0) - B'(-a_0)\right| \le \left|B_n'(z_0) - B_n'(-a_0)\right| + \left|B_n'(-a_0) - B'(-a_0)\right| \\ \le \quad & B_n''(z^*) \cdot \left|z_0 + a_0\right| + \left|B_n'(-a_0) - B'(-a_0)\right| \\ \le \quad & B_n''(-a_0) \cdot (x_m - a_0) + \left|B_n'(-a_0) - B'(-a_0)\right| \\ \le \quad & \left|\Psi_2(a_0 + b_0)\right| \cdot (x_m - a_0) + \left|B_n'(-a_0) - B'(-a_0)\right|. \end{aligned} \tag{4.186}$$

Hence

$$\sup_{n \ge mv} \left|B_n'(z_0) - B'(-a_0)\right| \le \left|\Psi_2(a_0 + b_0)\right| \cdot (x_m - a_0) + \sup_{n \ge mv} \left|B_n'(-a_0) - B'(-a_0)\right|$$

and we have (4.184) by (4.62), Lemma 4.16 part (4) and the fact that $x_m \to a_0$.

(3) We need (4.183) and (4.184). Proof for part (3) is similar to that of part (2). We skip the details.

(4) Write $M_0 = (\sigma_{m,n})^3$. From (A.6), we have

$$\begin{aligned} M_0^{2/3} &\le \sum_{i=1}^{m} \left|\Psi_2(\lambda_i + z_0)\right|^{2/3} + \sum_{j=1}^{n} \left|\Psi_2(\theta_j - z_0)\right|^{2/3} \\ &\le (2 + a_1 + b_1)^{2/3} \left[\sum_{i=1}^{m} \frac{1}{(\lambda_i + z_0)^2} + \sum_{j=1}^{n} \frac{1}{(\theta_j - z_0)^2}\right] \\ &\le (2 + a_1 + b_1)^{2/3} \left[\sum_{i=1}^{m} \Psi_1(\lambda_i + z_0) + \sum_{j=1}^{n} \Psi_1(\theta_j - z_0)\right] \\ &= (2 + a_1 + b_1)^{2/3} \left[\left|mA_m'(z_0)\right| + nB_n'(z_0)\right]. \end{aligned} \tag{4.187}$$

On the other hand,

$$\left|mA_m'(z_0)\right| \le m\Psi_1(\frac{a_0 + b_0}{2}) \quad \text{if} \quad z_0 \ge (b_0 - a_0)/2$$

and

$$nB_n'(z_0) \leq n\Psi_1(\frac{a_0+b_0}{2}) \quad \text{if} \quad z_0 \leq (b_0-a_0)/2.$$

Therefore

$$\left|mA_m'(z_0)\right| + nB_n'(z_0) \leq 2\Psi_1(\frac{a_0+b_0}{2})N$$

since $\left|mA_m'(z_0)\right| = nB_n'(z_0)$. From (4.187), we have

$$M_0^{2/3} \leq C_2 N,$$

where $C_2 = (2+a_1+b_1)\Psi_1((a_0+b_0)/2)$.

Now we estimate a lower bound of $M_0^{2/3}$. For any z,

$$\left|mA_m'(z)\right| + nB_n'(z) \geq \Psi_1(a_1+b_1)N.$$

From (A.6), we have

$$M_0^{2/3} \geq \left|\Psi_2(x_m+z_0)\right|^{2/3} \geq \frac{1}{1+a_1+b_1}\Psi_1(x_m+z_0). \tag{4.188}$$

For $m \geq m_v$,

$$\frac{\Psi_1(x_m+z_0)}{\left|mA_m'(z_0)\right| + nB_n'(z_0)} = \frac{\Psi_1(x_m+z_0)}{\left|2mA_m'(z_0)\right|} \geq \frac{v-v_1}{2v} > 0$$

from (4.176). Combining these inequalities, we obtain a lower bound for $M_0^{2/3}$.

$$M_0^{2/3} \geq \frac{(v-v_1)\Psi_1(a_1+b_1)}{2v(1+a_1+b_1)}N.$$

From (A.6),

$$(x_m+z_0)^2 M_0^{2/3} \leq \frac{1+a_1+b_1}{\Psi_1(x_m+z_0)}M_0^{2/3}.$$

Hence by (4.187) we have

$$(x_m+z_0)^2 M_0^{2/3} \leq (2+a_1+b_1)^2 \frac{|mA_m'(z_0)| + nB_n'(z_0)}{\Psi_1(x_m+z_0)} \leq (2+a_1+b_1)^2 \frac{2v}{v-v_1}.$$

(5) If $m \geq m_v$, then from (A.8) and (4.31), we have

$$K_{m,n} \geq \frac{\Delta\zeta_{m,n}|\Psi_2(x_m + z_0)|}{m|A'_m(z_0)|} = \frac{(x_m + z_0)|\Psi_2(x_m + z_0)|}{m|A'_m(z_0)|} \geq \frac{\Psi_1(x_m + z_0)}{2m|A'_m(z_0)|}.$$

(4.176) gives (4.179). □

Proof of Theorem 4.4 (2). (4.28) is an immediate consequence of (4.173) and Lemma 4.31 (1). (4.27) is from (4.174) and (A.6). (4.26) is from (4.178). □

Proof of Theorem 4.4 (3). First, consider $(x, y) \in S$. Let $z = \zeta(s)$. The computation of (4.169) shows that

$$\lim_{N\to\infty} \frac{mA'_m(z) + nB'_n(z)}{N} = 0$$

For some z^* between z and $\zeta_{m,n}$ we have

$$\begin{aligned} \left|\phi_{m,n} - G_{m,n}(z)\right| &= |\Delta z_{m,n}| \cdot \left|mA'_m(z^*) + nB'_n(z^*)\right| \\ &\leq |\Delta z_{m,n}| \cdot \left|mA'_m(z) + nB'_n(z)\right|, \end{aligned}$$

where $\Delta z_{m,n} = \zeta_{m,n} - z$. The last inequality is from the convexity of $G_{m,n}(z)$ and the definition of $\zeta_{m,n}$. Since $\Delta z_{m,n}$ goes to 0 as $N \to \infty$ by (4.25), we have from Assumption 4.1(a)

$$\lim_{N\to\infty} \frac{\phi_{m,n}}{N} = \lim_{N\to\infty} \frac{G_{m,n}(z)}{N} = xA(z) + yB(z) = \bar{\phi}(x, y).$$

Next, consider $(x, y) \in S_1$ with $s < s_1$. For some z^* between $-a_0$ and $\zeta_{m,n}$ we have

$$\begin{aligned} \left|\phi_{m,n} - G_{m,n}(-a_0)\right| &= |\Delta z_{m,n}| \cdot \left|mA'_m(z^*) + nB'_n(z^*)\right| \\ &\leq |\Delta z_{m,n}| \cdot \left|mA'_m(-a_0) + nB'_n(-a_0)\right|, \end{aligned} \tag{4.189}$$

where $\Delta z_{m,n} = \zeta_{m,n} + a_0$. From (4.28), we have

$$\lim_{N\to\infty} \frac{\left|\phi_{m,n} - G_{m,n}(-a_0)\right|}{N} \leq 0 \cdot \left|xA'(-a_0) + yB'(-a_0)\right| = 0.$$

Therefore

$$\lim_{N\to\infty}\frac{\phi_{m,n}}{N}=\lim_{N\to\infty}\frac{G_{m,n}(-a_0)}{N}=xA(-a_0)+yB(-a_0)=\bar{\phi}(x,y).$$

Finally consider (x, y) with $s = s_1$. Let $z = -a_0 + \epsilon$ for some small $\epsilon > 0$. Then

$$\lim_{N\to\infty}\Big(\frac{m}{N}A'_m(z)+\frac{n}{N}B'_n(z)\Big)=xA'(z)+yB'(z)>0.$$

Hence $\zeta_{m,n} \le -a_0 + \epsilon$ for all sufficiently large N and

$$|\zeta_{m,n}+a_0|\le(x_m-a_0)+\epsilon.$$

Therefore from (4.189), we have

$$\lim_{N\to\infty}\frac{\big|\phi_{m,n}-G_{m,n}(-a_0)\big|}{N}=0$$

and

$$\lim_{N\to\infty}\frac{\phi_{m,n}}{N}=\lim_{N\to\infty}\frac{G_{m,n}(-a_0)}{N}=xA(-a_0)+yB(-a_0)=\bar{\phi}(x,y).$$

□

4.8 Fluctuation of the free energy

In this section, we obtain fluctuation results of the (Λ, Θ)-polymer. For the (Λ, Θ)-polymer, there is an explicit integral formula for the Laplace transform of the law of the partition function. The following result is taken from Theorem 3.8.ii of [14]. For our models, without loss of generality we may assume $m \ge n$ and $a_0 > 0$, $b_1 < 0$.

Proposition 4.32. *Fix $m \ge n$, and assume $a_0 > 0$ and $b_1 < 0$. For all $s > 0$*

$$\mathbb{E}^{\rho}\left[e^{-sZ^{\square}_{(1,1),(m,n)}}\right]=\int_{i\mathbb{R}^n}dw\, s^{\sum_{l=1}^{n}(\theta_l-w_l)}\prod_{k,l=1}^{n}\Gamma(w_k-\theta_l)\prod_{k=1}^{m}\prod_{l=1}^{n}\frac{\Gamma(w_l+\lambda_k)}{\Gamma(\lambda_k+\theta_l)}s_n(w),\quad(4.190)$$

where

$$s_n(w) = \frac{1}{(2\pi i)^n n!} \prod_{\substack{k,l=1\\k\neq l}}^{n} \frac{1}{\Gamma(w_k - w_l)} \tag{4.191}$$

and dw refers to multiple contour integrals in $\mathbb{C}^n$.

By the above Proposition, the law of $Z^{\square}_{(1,1),(m,n)}$ is a symmetric function of $(\lambda_1, \dots, \lambda_m)$ and $(\theta_1, \dots, \theta_n)$. Therefore we can assume

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m \quad \text{and} \quad \theta_1 \leq \theta_2 \leq \cdots \leq \theta_n. \tag{4.192}$$

In this section we couple (Λ, Θ)-polymer with (Λ, Θ, z_0)-polymer and obtain some fluctuation bounds of $Z^{\square}_{(1,1),(m,n)}$. For $0 \leq u_1 < u_2 \leq m$, decompose according to the value of ξ_x:

$$\frac{Z_{m,n}(u_1 < \xi_x \leq u_2)}{Z^{\square}_{(1,1),(m,n)}} = \sum_{k=u_1+1}^{u_2} \exp\left(\sum_{i=1}^{k} \omega_{i,0}\right) \cdot \frac{Z^{\square}_{(k,1),(m,n)}}{Z^{\square}_{(1,1),(m,n)}}.$$

Consider a new environment $\tilde{\omega}(\lambda)$ with a new parameter λ for the rectangle $B_{m,n} = \{0, \dots, m\} \times \{0, \dots, n\}$ (see (4.47)). By (4.154) and Lemma 4.27,

$$\begin{aligned}
\frac{Z^{\square}_{(k,1),(m,n)}}{Z^{\square}_{(1,1),(m,n)}} &= \frac{Z^{\square,\tilde{\omega}}_{(1,1),(m+1-k,n)}}{Z^{\square,\tilde{\omega}}_{(1,1),(m,n)}} \leq \frac{Z^{\tilde{\omega}}_{m+1-k,n}(\xi_y > 0)}{Z^{\tilde{\omega}}_{m,n}(\xi_y > 0)} \\
&= \frac{Q^{\tilde{\omega}}_{m+1-k,n}(\xi_y > 0) Z^{\tilde{\omega}}_{m+1-k,n}}{Q^{\tilde{\omega}}_{m,n}(\xi_y > 0) Z^{\tilde{\omega}}_{m,n}} \leq \frac{1}{Q^{\tilde{\omega}}_{m,n}(\xi_y > 0)} \exp\left(-\sum_{i=1}^{k-1} I^{\tilde{\omega}}_{m+1-i,n}\right).
\end{aligned}$$

Recall that $-\omega_{i,0} \sim \text{log-gamma}(\lambda_i + z_0)$ and $-I^{\tilde{\omega}}_{m+1-i,n} \sim \text{log-gamma}(\lambda_i + \lambda)$. Let

$$D_k = \omega_{k,0} + \sum_{i=1}^{k-1} (\omega_{i,0} - I^{\tilde{\omega}}_{m+1-i,n}), \quad T_k = \tilde{\mathbb{E}}^{\rho} D_k, \quad \text{and} \quad S_k = D_k - T_k.$$

Thus we have

$$\frac{Z_{m,n}(u_1 < \xi_x \leq u_2)}{Z^{\square}_{(1,1),(m,n)}} \leq \frac{u_2 - u_1}{Q^{\tilde{\omega}}_{m,n}(\xi_y > 0)} \exp\left(\sup_{u_1 < k \leq u_2} S_k + \sup_{u_1 < k \leq u_2} T_k\right). \tag{4.193}$$

Recall definitions of f_z and g_z in (4.83).

Lemma 4.33. *Let C be the constant in (4.117) and $s,\ t,\ r > 0$. Then whenever $m+n \geq 2^{21}(a_1+b_1+1)^6$, we have the following.*

(1) If $r \geq 1/2$ and $t \geq 2(sr+4)$, then

$$\mathbb{P}^{\rho,\zeta_{m,n}}\left[\frac{Z_{m,n}(0<f_{\zeta_{m,n}}(\xi_x)\leq s(\sigma_{m,n})^2)}{Z^{\square}_{(1,1),(m,n)}} \geq 4^{-1}e^{t\sigma_{m,n}}\right] \leq \quad 12\exp[-\frac{t}{2^{11}}] + \begin{cases} \frac{2^{18}C}{r^4}, & (1/2 \leq r \leq \Delta\zeta_{m,n}\sigma_{m,n}/2) \\ \frac{2^{20}C}{r^2} \wedge \left(8\exp\left[-\frac{K_{m,n}}{2^{22}}r\right]\right), & (\Delta\zeta_{m,n}\sigma_{m,n}/2 \leq r \leq \Delta z^2_{m,n}\sigma_{m,n}) \end{cases} \tag{4.194}$$

and

$$\mathbb{P}^{\rho,\zeta_{m,n}}\left[\frac{Z_{m,n}(0<g_{\zeta_{m,n}}(\xi_y)\leq s(\sigma_{m,n})^2)}{Z^{\square}_{(1,1),(m,n)}} \geq 4^{-1}e^{t\sigma_{m,n}}\right] \leq \quad 12\exp[-\frac{t}{2^{11}}] + \begin{cases} \frac{2^{18}C}{r^4}, & (1/2 \leq r \leq \Delta\zeta_{m,n}\sigma_{m,n}/2) \\ \frac{2^{20}C}{r^2} \wedge \left(8\exp\left[-\frac{K_{m,n}}{2^{22}}r\right]\right), & (\Delta\zeta_{m,n}\sigma_{m,n}/2 \leq r \leq \Delta z^1_{m,n}\sigma_{m,n}). \end{cases} \tag{4.195}$$

(2) If $t \geq 2(2s\Delta z^2_{m,n}\sigma_{m,n}+4)$, then

$$\mathbb{P}^{\rho,\zeta_{m,n}}\left[\frac{Z_{m,n}(0<f_{\zeta_{m,n}}(\xi_x)\leq s(\sigma_{m,n})^2)}{Z^{\square}_{(1,1),(m,n)}} \geq 4^{-1}e^{t\sigma_{m,n}}\right] \leq 12\exp[-\frac{t}{2^{11}}] \tag{4.196}$$

and if $t \geq 2(2s\Delta z^1_{m,n}\sigma_{m,n}+4)$, then

$$\mathbb{P}^{\rho,\zeta_{m,n}}\left[\frac{Z_{m,n}(0<g_{\zeta_{m,n}}(\xi_y)\leq s(\sigma_{m,n})^2)}{Z^{\square}_{(1,1),(m,n)}} \geq 4^{-1}e^{t\sigma_{m,n}}\right] \leq 12\exp[-\frac{t}{2^{11}}]. \tag{4.197}$$

(3) If $t \geq 4(a_1+b_1+1)(\sigma_{m,n})^2$ then

$$\mathbb{P}^{\rho,\zeta_{m,n}}\left[\log Z_{m,n} - \log Z^{\square}_{(1,1),(m,n)} \geq t\sigma_{m,n}\right] \leq 24\exp[-\frac{t}{2^{11}}]. \tag{4.198}$$

Proof. (1) We only give the proof for ξ_x. Write $z_0 = \zeta_{m,n}$ and $M_0 = (\sigma_{m,n})^3$. Let $u = \max\{k : f_{z_0}(k) \leq sM_0^{2/3}\}$ and $\lambda = z_0 + rM_0^{-1/3}$ with $r > 0$. We assume $u \geq 1$ since $u = 0$ case is trivial.

From (4.193), probability in (4.194)

$$\leq \tilde{\mathbb{P}}^{\rho}\left[Q^{\tilde{\omega}}_{m,n}(\xi_y > 0) \leq \frac{1}{2}\right] \tag{4.199}$$

$$+ \tilde{\mathbb{P}}^{\rho}\left[\sup_{0<k\leq u} S_k \geq tM_0^{1/3} - \sup_{0<k\leq u} T_k - \log(8u)\right]. \tag{4.200}$$

We estimate (4.199) first. By Lemma 4.24, if $m+n \geq (128(a_1+b_1)\log 2)^3$, then we have

$$(4.199) = \tilde{\mathbb{P}}^{\rho}\left[Q^{\tilde{\omega}}_{m,n}(\xi_x > 0) \geq \frac{1}{2}\right]$$

$$\leq \begin{cases} \frac{2^{18}C}{r^4}, & (1/2 \leq r \leq \Delta z^0_{m,n} M_0^{1/3}/2) \\ \frac{2^{20}C}{r^2} \wedge \left(8\exp\left[-\frac{K_{m,n}}{2^{22}} r\right]\right), & (\Delta z^0_{m,n} M_0^{1/3}/2 \leq r \leq \Delta z^2_{m,n} M_0^{1/3}). \end{cases} \tag{4.201}$$

Here we used the symmetry of $M(z)$ with respect to permutation of parameters (Λ, Θ).

Next, we treat probability (4.200). We use the following Lemma.

Lemma 4.34 (Etemadi's Inequality, M19 of [5]). *If $S_1, \ldots, S_n$ are sums of independent random variables, then for $t \geq 0$*

$$\mathbf{P}\{\max_{1\leq k\leq n} |S_k| \geq 3t\} \leq 3 \max_{1\leq k\leq n} \mathbf{P}\{|S_k| \geq t\}.$$

From Taylor's theorem and monotonicity of $A'_k(z)$ explained in (4.11),

$$T_k = kA_k(z_0) - kA_k(\lambda) - \Psi_0(\lambda_k + \lambda)$$

$$\leq k\left|A'_k(z_0)\right| \cdot (\lambda - z_0) - \Psi_0(\lambda_k + z_0) \leq srM_0^{1/3} + 2M_0^{1/3} = (sr+2)M_0^{1/3}$$

since

$$-\Psi_0(\lambda_k + z_0) \leq \frac{2}{\lambda_k + z_0} \quad \text{and} \quad \frac{1}{(\lambda_k + z_0)^3} \leq M_0$$

by (4.121) and (A.3).

Since

$$u \leq m + n \leq \frac{M_0}{|\Psi_2(a_1 + b_1)|} \leq (a_1 + b_1)^3 M_0$$

by (A.6), if $m+n \geq 6^3(a_1+b_1+1)^6$ then

$$\log(8u) \leq 3[\log(2(a_1+b_1)) + \log(M_0^{1/3})] \leq 6(a_1+b_1) + M_0^{1/3} \leq 2M_0^{1/3}. \tag{4.202}$$

Therefore we have

$$\sup_{0<k\leq u} T_k + \log(8u) \leq [sr+4]M_0^{1/3}.$$

Thus if $t \geq 2[sr+4]$, then

$$(4.200) \leq \tilde{\mathbb{P}}^\rho \left[\sup_{0<k\leq u} S_k \geq tM_0^{1/3}/2 \right]. \tag{4.203}$$

S_k can be written as

$$\begin{aligned} S_k = S_k^{(1)} + S_k^{(2)} = & \sum_{i=1}^{k} \left(\omega_{i,0} - \tilde{\mathbb{E}}^\rho \omega_{i,0} \right) \\ & - \sum_{i=1}^{k-1} \left(I^{\tilde{\omega}}_{m+1-i,n} - \tilde{\mathbb{E}}^\rho I^{\tilde{\omega}}_{m+1-i,n} \right). \end{aligned}$$

We have

$$\tilde{\mathbb{V}}\text{ar}^\rho \, S_k^{(2)} = f_{z_0}(k)$$

and

$$\tilde{\mathbb{V}}\text{ar}^\rho \, S_k^{(1)} = f_\lambda(k-1) \leq f_{z_0}(k).$$

Apply corollary A.5 part (3) with $A = |A_k'(z_0)|$ and $r_0 = x_m + z_0$:

$$\begin{aligned} & \tilde{\mathbb{P}}^\rho \left[|S_k| \geq \frac{tM_0^{1/3}}{8} \right] \leq \tilde{\mathbb{P}}^\rho \left[|S_k^{(1)}| \geq \frac{tM_0^{1/3}}{16} \right] + \tilde{\mathbb{P}}^\rho \left[|S_k^{(2)}| \geq \frac{tM_0^{1/3}}{16} \right] \\ \leq \;& 4 \left(\exp[-\frac{t^2 M_0^{2/3}}{2^{11} f_{z_0}(k)}] \vee \exp[-\frac{(x_m+z_0)tM_0^{1/3}}{64}] \right) \\ \leq \;& 4 \left(\exp[-\frac{t^2}{2^{11}s}] \vee \exp[-\frac{t}{64}] \right) \leq 4\exp[-\frac{t}{2^{11}}]. \end{aligned} \tag{4.204}$$

The last line is from the definition of u, (4.121) and the inequality $t \geq 2sr \geq s$. Lemma 4.34 gives

$$(4.200) \leq 12 \exp[-\frac{t}{2^{11}}]. \tag{4.205}$$

(2) We derive a different bound for the probability in (4.196). From the superadditivity of $\log Z$, we have

$$\sum_{i=1}^{k-1} \omega_{i,1} + \log Z^{\square}_{(k,1),(m,n)} \leq \log Z^{\square}_{(1,1),(m,n)}.$$

Hence we get

$$\begin{aligned}\frac{Z_{m,n}(0 < \xi_x \leq u)}{Z^{\square}_{(1,1),(m,n)}} &\leq \sum_{k=1}^{u} \exp\left[\omega_{k,1} + \sum_{i=1}^{k}(\omega_{i,0} - \omega_{i,1})\right] \\ &\leq u \exp\left[\sup_{0<k\leq u} S_k + \sup_{0<k\leq u} T_k\right]\end{aligned} \tag{4.206}$$

where S_k and T_k are defined by

$$C_k = \omega_{k,1} + \sum_{i=1}^{k}(\omega_{i,0} - \omega_{i,1}), \quad T_k = \tilde{\mathbb{E}}^{\rho} C_k \quad \text{and } S_k = C_k - T_k.$$

Therefore

$$\text{probability in (4.196)} \leq \tilde{\mathbb{P}}^{\rho}\left[\sup_{0<k\leq u} S_k \geq t M_0^{1/3} - \left(\sup_{0<k\leq u} T_k + \log(8u)\right)\right]. \tag{4.207}$$

Since $-\omega_{i,0} \sim$ log-gamma$(\lambda_i + z_0)$, $-\omega_{i,1} \sim$ log-gamma$(\lambda_i + \theta_1)$ and all $\omega_{i,0}$, $\omega_{i,1}$ are independent, from Taylor's theorem and monotonicity of $A_k'(z)$ (see (4.11)), we have

$$\begin{aligned}T_k &= kA_k(z_0) - kA_k(\theta_1) - \Psi_0(\lambda_k + \theta_1) \\ &\leq k\left|A_k'(z_0)\right| \cdot (\theta_1 - z_0) - \Psi_0(\lambda_k + z) \leq s(\theta_1 - z_0)M_0^{2/3} + 2M_0^{1/3}.\end{aligned}$$

Therefore from (4.202) and (4.192),

$$\sup_{0<k\leq u} T_k + \log(8u) \leq 2s\Delta z_{m,n}^2 M_0^{2/3} + 4M_0^{1/3}.$$

Thus

$$\text{probability in (4.196)} \leq \tilde{\mathbb{P}}^\rho \left[\sup_{0<k\leq u} S_k \geq tM_0^{1/3}/2 \right], \tag{4.208}$$

if

$$t \geq 2[2s\Delta z_{m,n}^2 M_0^{1/3} + 4].$$

S_k can be written as

$$\begin{aligned} S_k = S_k^{(1)} + S_k^{(2)} = & \sum_{i=1}^{k} \left(\omega_{i,0} - \tilde{\mathbb{E}}^\rho \log \omega_{i,0} \right) \\ & - \sum_{i=1}^{k-1} \left(\omega_{i,1} - \tilde{\mathbb{E}}^\rho \omega_{i,1} \right). \end{aligned}$$

We have

$$\tilde{\mathbb{V}}\mathrm{ar}^\rho\, S_k^{(1)} = f_{z_0}(k)$$

and

$$\tilde{\mathbb{V}}\mathrm{ar}^\rho\, S_k^{(2)} = f_{\theta_1}(k-1) \leq f_{z_0}(k)$$

since $z_0 < y_n = \theta_1$. Apply corollary A.5 part (3) with $A = \left|A_k'(z_0)\right|$ and $r_0 = x_m + z_0$:

$$\tilde{\mathbb{P}}^\rho \left[\left|S_k\right| \geq \frac{tM_0^{1/3}}{8} \right] \leq 4\exp[-\frac{t}{2^{11}}]. \tag{4.209}$$

Lemma 4.34 gives

$$\text{probability in (4.196)} \leq 12\exp[-\frac{t}{2^{11}}]. \tag{4.210}$$

(3) We use part (2). Set $s = \frac{m|A'_m(z_0)|}{M_0^{2/3}} = \frac{nB'_n(z_0)}{M_0^{2/3}}$. From (A.8),

$$m\big|A'_m(z_0)\big| \le 2(a_1+b_1)mA''_m(z_0) \le 2(a_1+b_1)M_0.$$

Hence $s \le 2(a_1+b_1)M_0^{1/3}$. Since $\Delta z^1_{m,n} \vee \Delta z^2_{m,n} \le (a_1+b_1)/2$, from (4.121), if

$$t \ge 4(a_1+b_1+1)^2 M_0^{2/3}$$

then t satisfies conditions in part (2). Therefore

$$\begin{aligned}
&\mathbb{P}^{\rho,z_0}\left[\log Z_{m,n} - \log Z^{\square}_{(1,1),(m,n)} \ge tM_0^{1/3}\right]\\
\le &\,\mathbb{P}^{\rho,z_0}\left[\frac{Z_{m,n}(\xi_x>0)}{Z^{\square}_{(1,1),(m,n)}} \ge 4^{-1}e^{tM_0^{1/3}}\right] + \mathbb{P}^{\rho,z_0}\left[\frac{Z_{m,n}(\xi_y>0)}{Z^{\square}_{(1,1),(m,n)}} \ge 4^{-1}e^{tM_0^{1/3}}\right]\\
\le &\,24\exp[-\frac{t}{2^{11}}].
\end{aligned}$$

□

Theorem 4.35. *Let C be the constant in (4.117). Then whenever*

$$m+n \ge 2^{21}(a_1+b_1+1)^6,$$

we have the following.

$$\begin{aligned}
&\mathbb{P}^{\rho,\zeta_{m,n}}\left[\left|\log Z_{m,n} - \log Z^{\square}_{(1,1),(m,n)}\right| \ge t\sigma_{m,n}\right]\\
\le &\begin{cases}
\frac{2^{32}C}{t^2}, & (0 < t \le (\Delta\zeta_{m,n})^2(\sigma_{m,n})^2/4)\\
\frac{2^{26}C}{t} \wedge \left(40\exp\left[-\frac{K_{m,n}}{2^{22}}\sqrt{t}\right]\right), & ((\Delta\zeta_{m,n})^2(\sigma_{m,n})/4) \le t \le 4(a_1+b_1+1)^2(\sigma_{m,n})^2\\
28\exp[-\frac{t}{2^{11}}], & (t \ge 4(a_1+b_1+1)^2(\sigma_{m,n})^2)
\end{cases}\\
\le &\frac{2^{32}C}{t^2}\left(1 \vee \frac{1}{(K_{m,n})^4}\right).
\end{aligned} \tag{4.211}$$

Proof. Without loss of generality, we may assume $\Delta\zeta_{m,n} = (x_m - \zeta_{m,n})$. Then $\Delta z^2_{m,n} = (\theta_1 - \zeta_{m,n})/2 \geq (a_0 + b_0)/2$ by (4.192). Write $z_0 = \zeta_{m,n}$ and $M_0 = (\sigma_{m,n})^3$. Supperadditivity of $\log Z$ gives

$$\log Z_{m,n} \geq \log Z^{\square}_{(1,1),(m,n)} + \omega_{0,1}.$$

Hence

$$\begin{aligned} &\mathbb{P}^{\rho,z_0}\left[\log Z_{m,n} - \log Z^{\square}_{(1,1),(m,n)} \leq -tM_0^{1/3}\right] \leq \mathbb{P}^{\rho,z_0}\left[\omega_{0,1} \leq -tM_0^{1/3}\right] \\ \leq &\frac{\mathbb{E}^{\rho,z_0} e^{-\omega_{0,1}}}{e^{tM_0^{1/3}}} = (\theta_1 - z_0)e^{-tM_0^{1/3}} \leq (a_0 + b_0)e^{-tM_0^{1/3}}. \end{aligned}$$

For the other direction let $s_1, s_2 > 0$ and $u = s_1 M_0^{2/3}$, $v = s_2 M_0^{2/3}$.

$$\begin{aligned} &\mathbb{P}^{\rho,z_0}\left[\frac{Z_{m,n}}{Z^{\square}_{(1,1),(m,n)}} \geq e^{tM_0^{1/3}}\right] \\ &= \mathbb{P}^{\rho,z_0}\left[\frac{Z_{m,n}(\{0 < f_{z_0}(\xi_x) \leq u\} \cup \{0 < g_{z_0}(\xi_y) \leq v\})}{Z^{\square}_{(1,1),(m,n)} Q_{m,n}(\{0 < f_{z_0}(\xi_x) \leq u\} \cup \{0 < g_{z_0}(\xi_y) \leq v\})} \geq e^{tM_0^{1/3}}\right] \\ &\leq \mathbb{P}^{\rho,z_0}\left[\frac{Z_{m,n}(0 < f_{z_0}(\xi_x) \leq u)}{Z^{\square}_{(1,1),(m,n)}} \geq \frac{1}{4}e^{tM_0^{1/3}}\right] + \mathbb{P}^{\rho,z_0}\left[\frac{Z_{m,n}(0 < g_{z_0}(\xi_y) \leq v)}{Z^{\square}_{(1,1),(m,n)}} \geq \frac{1}{4}e^{tM_0^{1/3}}\right] \end{aligned} \tag{4.212}$$

$$+ \mathbb{P}^{\rho,z_0}\left[Q_{m,n}(\{0 < f_{z_0}(\xi_x) \leq u\} \cup \{0 < g_{z_0}(\xi_y) \leq v\}) \leq \frac{1}{2}\right] \tag{4.213}$$

$$\leq \mathbb{P}^{\rho,z_0}\left[\frac{Z_{m,n}(0 < f_{z_0}(\xi_x) \leq u)}{Z^{\square}_{(1,1),(m,n)}} \geq \frac{1}{4}e^{tM_0^{1/3}}\right] + \mathbb{P}^{\rho,z_0}\left[Q_{m,n}(\{f_{z_0}(\xi_x) \geq u\}) \geq \frac{1}{4}\right] \tag{4.214}$$

$$+ \mathbb{P}^{\rho,z_0}\left[\frac{Z_{m,n}(0 < g_{z_0}(\xi_y) \leq v)}{Z^{\square}_{(1,1),(m,n)}} \geq \frac{1}{4}e^{tM_0^{1/3}}\right] + \mathbb{P}^{\rho,z_0}\left[Q_{m,n}(\{g_{z_0}(\xi_y) \geq v\}) \geq \frac{1}{4}\right]. \tag{4.215}$$

Set $r = \sqrt{t}$. First, assume $t \geq 16$.

Case 1. $0 < t \leq (\Delta z^0_{m,n})^2 M_0^{2/3}/4$.

We use Lemma 4.33(1). Set $s_1 = s_2 = s = \sqrt{t}/4$. Then we have $t \geq 2(sr+4)$. Therefore line (4.212) is bounded by $2^{20}C/t^2$. By Lemma 4.25,

$$\text{line (4.213)} \leq \mathbb{P}^{\rho,z_0}\left[Q_{m,n}(\{f_{z_0}(\xi_x) \geq u\}) \geq \frac{1}{4}\right] + \mathbb{P}^{\rho,z_0}\left[Q_{m,n}(\{g_{z_0}(\xi_y) \geq u\}) \geq \frac{1}{4}\right] \leq \frac{2^{31}C}{t^2}.$$

Case 2. $(\Delta z^0_{m,n})^2 M_0^{2/3}/4 \leq t \leq (\Delta z^2_{m,n})^2 M_0^{2/3}$.

Let

$$P(t) = \frac{2^{24}C}{t} \wedge \left(10 \exp\left[-\frac{K_{m,n}}{2^{22}}\sqrt{t}\right]\right).$$

Set $s_1 = \sqrt{t}/4$. From (4.194) and Lemma 4.25, we have

$$\text{line (4.214)} \leq 2P(t).$$

Set $s_2 = \sqrt{t}/8$. From (4.197) and Lemma 4.25,

$$\text{line (4.215)} \leq 2P(t).$$

Case 3. $(\Delta z^2_{m,n})^2 M_0^{2/3} \leq t \leq 4(a_1 + b_1 + 1)M_0^{2/3}$

Set $s_1 = s_2 = \sqrt{t}/8$. From (4.196), (4.197) and Lemma 4.25, we have

$$\text{line (4.214)} \quad \text{and} \quad \text{(4.215)} \leq 2P(t).$$

Case 4. $t \geq 4(a_1 + b_1 + 1)M_0^{2/3}$. Use (4.198).

These bounds hold even for $0 < t < 16$. □

Proof of Theorem 4.5. In Theorem 4.35, we can replace $\log Z_{m,n}$ with $\mathbb{E}^{\rho,\zeta_{m,n}} \log Z_{m,n} = \phi_{m,n}$ using Lemma 4.21. Since our constants are sufficiently large, direct computation shows that same upper bounds can be used. Since

$$\log Z^{\square}_{(1,1),(m,n)} = \log Z_{(1,1),(m,n)} + \omega_{1,1}$$

and $\omega_{1,1} \sim$ log-gamma$(\lambda_1+\theta_1)$, by changing C if necessary, we have upper bounds in (4.32). □

Corollary 4.36. *Under the same assumptions as in Theorem 4.5, there exists a positive constant C_1 that depends on a_0+b_0 and a_1+b_1 such that*

$$\mathbb{E}^\rho\left(\left|\log Z_{(1,1),(m,n)} - \phi_{m,n}\right|^2\right) \le C_1(\sigma_{m,n})^2\left[\log(\Delta\zeta_{m,n}\sigma_{m,n}) + \frac{1}{(K_{m,n})^4}\right]. \qquad (4.216)$$

Proof. From (4.32), with a new constant C_1, we have

$$\begin{aligned}&\frac{1}{(\sigma_{m,n})^2}\mathbb{E}^\rho\left|\log Z_{(1,1),(m,n)} - \phi_{m,n}\right|^2\\ =&2\int_0^\infty t\,\mathbb{P}^\rho\left[\left|\log Z_{(1,1),(m,n)} - \phi_{m,n}\right| \ge t\sigma_{m,n}\right]\,dt\\ \le&1+2C\int_1^{(\Delta\zeta_{m,n}\sigma_{m,n})^2}\frac{1}{t}\,dt + 2C\int_0^\infty te^{-K_{m,n}\sqrt{t}}\,dt\\ \le&C_1\left[\log(\Delta\zeta_{m,n}\sigma_{m,n}) + \frac{1}{(K_{m,n})^4}\right].\end{aligned} \qquad (4.217)$$

□

Proof of Theorem 4.7. Set $m=\lfloor Lx\rfloor$, $n=\lfloor Ly\rfloor$. From (A.8),

$$A''_m(\zeta_{m,n}) \ge \frac{1}{2(a_1+b_1)}|A'_m(\zeta_{m,n})|, \quad B''_n(\zeta_{m,n}) \ge \frac{1}{2(a_1+b_1)}B'_n(\zeta_{m,n}). \qquad (4.218)$$

Hence from the definition of $K_{m,n}$ (4.31),

$$K_{m,n} \ge \frac{\Delta\zeta_{m,n}}{a_1+b_1}. \qquad (4.219)$$

Let $z=\zeta(s)$. From (4.25), for all sufficiently large $L\ge L_0$,

$$\Delta\zeta_{m,n} \ge \delta z = \frac{1}{2}\left[(a_0+z)\wedge(b_0-z)\right].$$

Therefore we get (4.34) from (4.24) and (4.32). (4.35) is a consequence of Corollary 4.36. We have (4.36) from (4.163). □

Proof of Theorem 4.8. Set $m = \lfloor Lx \rfloor$, $n = \lfloor Ly \rfloor$ and $N = m + n$. From Lemma 4.31, we have

$$K_{m,n} \geq \frac{1}{4}(1 - s/s_1), \quad (\Delta\zeta_{m,n}\sigma_{m,n})^2 \leq C_0$$

for some constant.

Therefore we obtain (4.37) from (4.26) and (4.32). (4.38) is from (4.163). □

Proof of Theorem 4.9. Let $m = \lfloor Lx \rfloor$, $n = \lfloor Ly \rfloor$, and $M_0 = \sigma_{m,n}^3$. From Theorem 4.5, it is enough to show that

$$\lim_{L\to\infty} \frac{\sigma_{m,n}}{\sqrt{L}} = 0.$$

From Assumption 4.2(a),

$$\frac{1}{m(x_m - a_0)^{p_1}} \leq \int \frac{1}{(\lambda - a_0)^{p_1}}\,\alpha_m(d\lambda) \leq C_{p_1}$$

for some $C_{p_1} > 0$. Hence from (4.172) and (A.6)

$$-\sum_{i=2}^{m} \Psi_2(x_{m:i} + z_0) \leq -\sum_{i=2}^{m} \Psi_2(x_{m:i} - x_m) \leq \frac{C_0 m}{(x_m - a_0)} \leq C_1 m^{1+1/p_1} \tag{4.220}$$

for some C_0, $C_1 > 0$. For all large enough L, we have $\zeta_{m,n} \leq (b_0 - a_0)/2$ and

$$nB_n''(\zeta_{m,n}) \leq n\Big|\Psi_2(\frac{a_0 + b_0}{2})\Big|. \tag{4.221}$$

Therefore it is enough to show that

$$\lim_{L\to\infty} \frac{1}{(x_m + \zeta_{m,n})\sqrt{L}} = 0.$$

From the definition of $\zeta_{m,n}$, we have

$$\frac{\Psi_1(x_m + \zeta_{m,n})}{m} = -\frac{1}{m}\sum_{i=2}^{m} \Psi_1(x_{m:i} + \zeta_{m,n}) + \frac{n}{m}B_n'(\zeta_{m,n}). \tag{4.222}$$

If $\zeta_{m,n} \le -a_0$, by the computation in (4.185) and (4.186),

$$\begin{aligned} &\frac{\Psi_1(x_m+z_0)}{m} \\ \le\ & \Big|\frac{x_m-a_0}{m}\sum_{i=2}^{m}\Psi_2(x_{m:i}-x_m)\Big| + \big|A'_m(-a_0)-A'(-a_0)\big| + \Big|\frac{\Psi_1(x_m-a_0)}{m}\Big| \\ & + \big|n/m-1/s_1\big|\cdot\Big[\big|\Psi_2(a_0+b_0)\big|\cdot(x_m-a_0)+\big|B'_n(-a_0)-B'(-a_0)\big|\Big]. \end{aligned} \tag{4.223}$$

If $\zeta_{m,n} > -a_0$,

$$\frac{\Psi_1(x_m+\zeta_{m,n})}{m} \le \frac{\Psi_1(x_m-a_0)}{m}.$$

Therefore from Lemma 4.30

$$\lim_{L\to\infty}\frac{\Psi_1(x_m+\zeta_{m,n})}{L}=0$$

and hence

$$\lim_{L\to\infty}\frac{1}{(x_m+\zeta_{m,n})\sqrt{L}}=0.$$

□

Proof of Theorem 4.6. First consider the case $(x,y)\in S$ or $(x,y)\in S_1$ with $x/y<s_1$. By Theorems 4.7 and 4.8 we have

$$\lim_{L\to\infty}\frac{\mathbb{E}^\rho\log Z_{(1,1),(\lfloor Lx\rfloor,\lfloor Ly\rfloor)}-\phi_{\lfloor Lx\rfloor,\lfloor Ly\rfloor}}{L}=0.$$

On the other hand, we have

$$\lim_{L\to\infty}\frac{\phi_{\lfloor Lx\rfloor,\lfloor Ly\rfloor}}{L}=\bar{\phi}(x,y).$$

by (4.29). Therefore $\lim_{L\to\infty}\frac{\mathbb{E}^\rho\log Z_{(1,1),(\lfloor Lx\rfloor,\lfloor Ly\rfloor)}}{L}=\bar{\phi}(x,y)$. One can apply Theorem 2.23 to have a.s. convergence.

Convergence for boundary points can be proved approximation from S and S_1 and continuity of $\bar{\phi}$. If S_2 is nonempty, then we extend Assumption 4.2 to cover S_2 and obtain the same results. □

Appendix A

Appendix

In this section, we summarize basic facts for polygamma functions and log-gamma distributions.

A.1 Polygamma functions

Here we collect basic facts about polygamma functions.

The logarithm of Gamma function $\log\Gamma(s)$ is convex and (real) analytic on $(0,\infty)$. The derivatives are the polygamma funtions $\Psi_k(s) = (d^{k+1}/ds^{k+1})\log\Gamma(s)$, $k \geq 0$. We use the identities from [3] :

$$\Psi_0(x) = -\gamma + \sum_{i=0}^{\infty}\left(\frac{1}{i+1} - \frac{1}{i+x}\right),$$
$$\Psi_k(x) = (-1)^{k+1}k!\sum_{i=0}^{\infty}\frac{1}{(x+i)^{k+1}} \quad k \geq 1 \tag{A.1}$$

where γ is the Euler constant. From (A.1), all polygamma functions are monotonic functions. In this thesis, we use these properties a great deal. In particular, we have

$$\begin{aligned} &\Psi_1(x) > 0, \quad \Psi_1 \text{ is decreasing and} \\ &\Psi_2(x) < 0, \quad \Psi_2 \text{ is increasing.} \end{aligned} \tag{A.2}$$

From these series representations we have

$$\log x - \frac{1}{x} \leq \Psi_0(x) \leq \log(x+1) - \frac{1}{x} \tag{A.3}$$

for all $x > 0$ by Corollary 2.3 in [32]. We also have the inequality

$$\log x - \frac{1}{x} \leq \Psi_0(x) \leq \log x - \frac{1}{2x} \tag{A.4}$$

from Theorem 3.1 in [2].

For $p > 1$ and $x > 0$ we have

$$\frac{1}{x^p} \vee \frac{1}{(p-1)x^{p-1}} \leq \sum_{i=0}^{\infty} \frac{1}{(x+i)^p} \leq \frac{1}{x^p} + \frac{1}{(p-1)x^{p-1}}. \tag{A.5}$$

To prove (A.5), use the following inequalities:

$$\sum_{i=1}^{\infty} f(i) \leq \int_0^{\infty} f(t)\,dt = \frac{1}{(p-1)x^{p-1}} \leq \sum_{i=0}^{\infty} f(i),$$

where $f(t) = 1/(x+t)^p$. In particular we have

$$\frac{k!}{x^{k+1}} \vee \frac{(k-1)!}{x^k} \leq |\Psi_k(x)| \leq \frac{k!}{x^{k+1}} + \frac{(k-1)!}{x^k} \tag{A.6}$$

for all $x > 0$ and $k \geq 1$.

From (A.1) we have

$$1 \leq \frac{\left|\Psi_k(cx)\right|}{\left|\Psi_k(x)\right|} \leq \frac{1}{c^{k+1}} \tag{A.7}$$

for $0 < c < 1$ and $k \geq 1$.

We also have

$$\frac{k}{2x} \leq \frac{|\Psi_{k+1}(x)|}{|\Psi_k(x)|} \leq \frac{k+1}{x} \tag{A.8}$$

for $x > 0$ and $k \geq 1$. The left inequality is from (A.6) and the right is from (A.1).

A.2 Log-gamma distribution

In this section, we collect some facts about the log-gamma distribution. We need a technical lemma whose proof is given by the dominated convergence theorem. Part (4) of the following lemma is a well-known result for sufficient statistics (Fisher-Neyman factorization theorem).

Lemma A.1. *Let* Ω *be a measurable space with a* σ*-finite measure* μ *and* $H, Y : \Omega \times I \to \mathbb{R}$ *are measurable functions where* $I \subseteq \mathbb{R}$ *is an interval. Assume* H, Y *are differentiable in the second variable and their derivatives are dominated by a function* g*:* $|\frac{\partial H}{\partial r}(\omega, r)|, |\frac{\partial Y}{\partial r}(\omega, r)| \leq g(\omega)$ *for all* $(\omega, r) \in \Omega \times I$*. Also assume* H *is dominated by a function* h*:* $H(\omega, r) \leq h(\omega)$ *for all* $(\omega, r) \in \Omega \times I$*. Finally assume* e^h *and* ge^h *are* μ *integrable. Define a probability measure* Q_r *by* $dQ_r(\omega) = \dfrac{e^{H(\omega,r)}}{Z(r)} d\mu(\omega)$ *where* $Z(r) = \int_\Omega e^{H(\omega,r)} \mu(d\omega)$ *is a normalizing factor. Then we have*

(1) $\dfrac{\partial}{\partial r} \log Z(r) = E^{Q_r} \dfrac{\partial H}{\partial r}$

(2) $\dfrac{\partial}{\partial r} E^{Q_r} Y(r) = E^{Q_r} \dfrac{\partial Y}{\partial r} + \mathbb{C}\mathrm{ov}^{Q_r}(Y, \dfrac{\partial H}{\partial r})$

(3) In particular, if $H(\omega, r) = rT(\omega)$ *and* Y *does not depend on* r *then*

$$\frac{\partial \log Z}{\partial r} = E^{Q_r}(T) \tag{A.9}$$

$$\frac{\partial^2 \log Z}{\partial r^2} = \mathbb{V}\mathrm{ar}^{Q_r}(T) \tag{A.10}$$

$$\frac{\partial}{\partial r} E^{Q_r} Y = \mathbb{C}\mathrm{ov}^{Q^r}(Y, T) = E^{Q_r}\left[Y(T - E^{Q_r}T)\right] \tag{A.11}$$

$$\begin{aligned} \frac{\partial^2}{\partial r^2} E^{Q_r} Y &= \mathbb{C}\mathrm{ov}^{Q^r}\left(Y, (T - E^{Q_r}T)^2\right) \\ &= E^{Q_r}\left[Y\left((T - E^{Q_r}T)^2 - \mathbb{V}\mathrm{ar}^{Q_r}(T)\right)\right]. \end{aligned} \tag{A.12}$$

(4) If $H(\omega, r) = rT(\omega)$ *then the conditional law of* ω *under* Q_r *given* T *is independent of* r*:* $Q_r(\omega \in \cdot\ | T) = Q_{r_0}(\omega \in \cdot\ | T)$ *for all* $r, r_0 \in I$*.*

For $\alpha,\ \beta > 0$ we define the log-gamma(α,β) distribution by the distribution of ω if $e^\omega \sim$ Gamma(α,β). The CDF of the log-gamma(α, β) distribution is given by

$$F(\alpha, \beta, x) = \frac{\beta^\alpha}{\Gamma(\alpha)} \int_{-\infty}^{x} \exp\left[\alpha y - \beta e^y\right] dy \tag{A.13}$$

for $x \in \mathbb{R}$. Note that if $A \sim$ log-gamma(α,β) then

$$\mathbb{E}A = \Psi_0(\alpha) - \log\beta, \quad \mathbb{V}\text{ar}\, A = \Psi_1(\alpha) \tag{A.14}$$

and for $t > -\alpha$

$$\log \mathbb{E}e^{tA} = \log\Gamma(\alpha + t) - \log\Gamma(\alpha) - t\log\beta. \tag{A.15}$$

For $r > 0$ we write log-gamma(r) and $F_r(x) = F(r, x)$ for log-gamma(r,1) and $F(r, 1, x)$, respectively. Let $H_r : (0, 1) \to \mathbb{R}$ be the inverse of F_r satisfying

$$F(r, H_r(u)) = u \tag{A.16}$$

for all $u \in (0, 1)$. We write $H_r(u) = H(r, u)$ when subscripts are not convenient. Then if η is a Uniform$(0, 1)$ random variable, $H_r(\eta)$ is a log-gamma(r) random variable. From (A.16),

$$\frac{\partial H}{\partial r}(r, u) = -\frac{\partial F/\partial r}{\partial F/\partial x}(r, H_r(u)). \tag{A.17}$$

For $r > 0$ and $x \in \mathbb{R}$ define the function

$$\begin{aligned} L(r, x) &= -\frac{\partial F/\partial r}{\partial F/\partial x}(r, x) \\ &= \int_{-\infty}^{x} (\Psi_0(r) - y) e^{(y-x)r} e^{-e^y + e^x}\, dy. \end{aligned} \tag{A.18}$$

Then

$$\frac{\partial H}{\partial r}(r, u) = L(r, H_r(u)) \tag{A.19}$$

and

$$\frac{\partial^2 H}{\partial r^2}(r,u) = \left[\frac{\partial L}{\partial r} + L\frac{\partial L}{\partial x}\right](r, H_r(u)) = L_1(r, H_r(u)), \tag{A.20}$$

where $L_1(r,x)$ is defined by

$$L_1(r,x) = \frac{\partial L}{\partial r}(r,x) + L(r,x)\frac{\partial L}{\partial x}(r,x). \tag{A.21}$$

In next lemma, we collect some properties of H, L and L_1.

Lemma A.2. *L and L_1 satisfy*

$$0 < L(r,x) \leq \frac{e^2(1 \vee r)}{r}\left[|x - \Psi_0(r)| + \sqrt{\Psi_1(r)}\right] \tag{A.22}$$

and

$$\begin{aligned} \left|L_1(r,x)\right| \leq \left(1 + e^2(1 \vee r)\right)\frac{e^2(1 \vee r)}{r}\left[|x - \Psi_0(\rho)| + \sqrt{\Psi_1(r)}\right]^2 \\ + \frac{e^4(1 \vee r^2)}{r^2}\left[|x - \Psi_0(r)| + \sqrt{\Psi_1(r)}\right]. \end{aligned} \tag{A.23}$$

If $A \sim$ log-gamma(r) and $\eta \sim$ Uniform$(0,1)$, then

$$\mathbb{E}L(r,A) = \mathbb{E}\frac{\partial}{\partial r}H(r,\eta) = \frac{\partial}{\partial r}\mathbb{E}H(r,\eta) = \Psi_1(r) \tag{A.24}$$

and

$$\mathbb{E}L_1(r,A) = \mathbb{E}\frac{\partial^2}{\partial r^2}H(r,\eta) = \frac{\partial^2}{\partial r^2}\mathbb{E}H(r,\eta) = \Psi_2(r). \tag{A.25}$$

We also have

$$\mathbb{E}\left([L(r,A)]^2\right) \leq \frac{4e^4(1 \vee r^2)}{r^2}\Psi_1(r) \tag{A.26}$$

and

$$\mathbb{E}\left|L_1(r,A)\right| \leq \left(1 + 20e^2(1 \vee r)\right)|\Psi_2(r)|. \tag{A.27}$$

Finally, we have

$$\mathbb{E}e^{tL(r,A)} < \infty \quad \text{for} \quad |t| < \frac{r^2}{e^2(1 \vee r)}. \tag{A.28}$$

Proof. Let A be a random variable with the distribution log-gamma(r). For a function f, we denote the expectation of $f(A)$ by $E^r f(A)$. From (A.11),

$$\frac{\partial F}{\partial r}(r,x) = E^r[1\{A \le x\}(A - \Psi_0(r))].$$

Hence for $x_1 < x_2$,

$$\frac{\partial F}{\partial r}(r,x_2) - \frac{\partial F}{\partial r}(r,x_1) = E^r[1\{x_1 < A \le x_2\}(A - \Psi_0(r))].$$

Therefore $\frac{\partial}{\partial r}F(r,\cdot)$ is decreasing on $(-\infty, \Psi_0(r)]$ and increasing on $[\Psi_0(r), \infty)$. Since $\frac{\partial}{\partial r}F(r,-\infty) = \frac{\partial}{\partial r}F(r,\infty) = 0$, $\frac{\partial}{\partial r}F(r,x) < 0$ for all $x \in \mathbb{R}$. Thus we have $L > 0$ from (A.18).

We record some useful integral representations of L: From (A.18) and $E^r A = \Psi_0(r)$, we have

$$L(r,x) = \int_x^\infty (y - \Psi_0(r)) e^{(y-x)r} e^{-e^y + e^x}\, dy. \tag{A.29}$$

By a change of variables, substituting $t = y - x$, we get

$$L(r,x) = \int_{-\infty}^0 (\Psi_0(r) - x - t) e^{rt} \exp[-e^{x+t} + e^x]\, dt \tag{A.30}$$

and

$$L(r,x) = \int_0^\infty (x - \Psi_0(r) + t) e^{rt} \exp[-e^{x+t} + e^x]\, dt. \tag{A.31}$$

Now we estimate L: For $r > 0$ let

$$C(r) = \frac{\Gamma(r) \exp[e^{\Psi_0(r)}]}{e^{r\Psi_0(r)}}. \tag{A.32}$$

From (A.13), (A.14), (A.30) and (A.31), we have, for $x \le \Psi_0(r)$,

$$\begin{aligned} L(r,x) \le & \big(\Psi_0(r) - x\big) \int_{-\infty}^0 e^{rt} \exp[-e^{\Psi_0(r)+t} + e^{\Psi_0(r)}]\, dt \\ & + \int_{-\infty}^0 |t| e^{rt} \exp[-e^{\Psi_0(r)+t} + e^{\Psi_0(r)}]\, dt \\ \le & C(r) \left[|x - \Psi_0(r)| + \sqrt{\Psi_1(r)} \right] \end{aligned}$$

and for $x \geq \Psi_0(r)$

$$\begin{aligned} L(r,x) \leq & \big(x-\Psi_0(r)\big) \int_0^\infty e^{rt} \exp[-e^{\Psi_0(r)+t} + e^{\Psi_0(r)}]\, dt \\ & + \int_0^\infty |t| e^{rt} \exp[-e^{\Psi_0(r)+t} + e^{\Psi_0(r)}]\, dt \\ \leq & C(r) \left[|x-\Psi_0(r)| + \sqrt{\Psi_1(r)} \right]. \end{aligned}$$

Hence we have

$$L(r,x) \leq C(r) \left[|x-\Psi_0(r)| + \sqrt{\Psi_1(r)} \right]. \tag{A.33}$$

Therefore we obtain (A.22) from Lemma A.3 below. Next, we prove (A.23). First, we estimate $\partial L/\partial r$. From (A.30) and (A.31) we have

$$\begin{aligned} \frac{\partial L}{\partial r} = & \Psi_1(r) \int_{-\infty}^0 e^{rt} \exp[-e^{x+t} + e^x]\, dt \\ & + \int_{-\infty}^0 (\Psi_0(r) - x - t) t e^{rt} \exp[-e^{x+t} + e^x]\, dt \end{aligned} \tag{A.34}$$

and

$$\begin{aligned} \frac{\partial L}{\partial r} = & -\Psi_1(r) \int_0^\infty e^{rt} \exp[-e^{x+t} + e^x]\, dt \\ & + \int_0^\infty (x - \Psi_0(r) + t) t e^{rt} \exp[-e^{x+t} + e^x]\, dt. \end{aligned} \tag{A.35}$$

For $x \leq \Psi_0(r)$, from (A.34),

$$\begin{aligned} \frac{\partial L}{\partial r} \leq & \Psi_1(r) \int_{-\infty}^0 e^{rt} \exp[-e^{\Psi_0(r)+t} + e^{\Psi_0(r)}]\, dt \\ \leq & C(r)\Psi_1(r) \end{aligned}$$

and

$$\begin{aligned} \frac{\partial L}{\partial r} \geq & \int_{-\infty}^0 (\Psi_0(r) - x - t) t e^{rt} e^{rt} \exp[-e^{x+t} + e^x]\, dt \\ \geq & -|x-\Psi_0(r)| \int_{-\infty}^0 |t| e^{rt} \exp[-e^{\Psi_0(r)+t} + e^{\Psi_0(r)}]\, dt \\ & - \int_{-\infty}^0 t^2 e^{rt} \exp[-e^{\Psi_0(r)+t} + e^{\Psi_0(r)}]\, dt \\ \geq & -C(r) \left[|x-\Psi_0(r)| \sqrt{\Psi_1(r)} + \Psi_1(r) \right]. \end{aligned}$$

Hence we have

$$\left|\frac{\partial L}{\partial r}\right| \leq C(r)\left[|x-\Psi_0(r)|\sqrt{\Psi_1(r)}+\Psi_1(r)\right].$$

For $x \geq \Psi_0(r)$, we obtain the same inequality using (A.35). Therefore

$$\left|\frac{\partial L}{\partial r}\right| \leq C(r)\left[|x-\Psi_0(r)|\sqrt{\Psi_1(r)}+\Psi_1(r)\right]. \tag{A.36}$$

Second, we estimate $\partial L/\partial x$. From (A.18) and (A.31), we have

$$\frac{\partial L}{\partial x}(r,x) = \big(\Psi_0(r)-x\big)+(e^x-r)L(r,x) \tag{A.37}$$

and

$$\begin{aligned}\frac{\partial L}{\partial x} &= \int_0^\infty e^{rt}\exp[-e^{x+t}+e^x]\,dt \\ &\quad - e^x\int_0^\infty \big(x-\Psi_0(r)+t\big)e^{rt}(e^t-1)\exp[-e^{x+t}+e^x]\,dt.\end{aligned} \tag{A.38}$$

From (A.33) and (A.37) we get

$$\begin{aligned}L\frac{\partial L}{\partial x} \geq& -|x-\Psi_0(r)|L-\rho L^2 \\ \geq& -C(r)\left[\big(x-\Psi_0(r)\big)^2+|x-\Psi_0(r)|\sqrt{\Psi_1(r)}\right] \\ &- rC(r)^2\left[\big(x-\Psi_0(r)\big)^2+2|x-\Psi_0(r)|\sqrt{\Psi_1(r)}+\Psi_1(r)\right].\end{aligned} \tag{A.39}$$

For $x \leq \Psi_0(r)$, from (A.37),

$$\begin{aligned}L\frac{\partial L}{\partial x} \leq& |x-\Psi_0(r)|L \\ \leq& C(r)\left[\big(x-\Psi_0(r)\big)^2+|x-\Psi_0(r)|\sqrt{\Psi_1(r)}\right].\end{aligned} \tag{A.40}$$

since $e^x \leq e^{\Psi_0(r)} < r$. For $x \geq \Psi_0(r)$ we use (A.38) :

$$\begin{aligned}\frac{\partial L}{\partial x} &\leq \int_0^\infty e^{rt}\exp[-e^{x+t}+e^x]\,dt \\ &\leq \int_0^\infty e^{rt}\exp[-e^{\Psi_0(r)+t}+e^{\Psi_0(r)}]\,dt \leq C(r).\end{aligned}$$

Hence

$$L\frac{\partial L}{\partial x} \le C(r)L \le C(r)^2\left[|x-\Psi_0(r)|+\sqrt{\Psi_1(r)}\right]. \tag{A.41}$$

Now we can estimate L_1:

From (A.36), (A.40) and (A.41) we have

$$\begin{aligned} L_1(r,x) \le & C(r)\left[\left(x-\Psi_0(r)\right)^2+2|x-\Psi_0(r)|\sqrt{\Psi_1(r)}+\Psi_1(r)\right]+C(r)L \\ \le & C(r)\left[|x-\Psi_0(r)|+\sqrt{\Psi_1(r)}\right]^2+C(r)^2\left[|x-\Psi_0(r)|+\sqrt{\Psi_1(r)}\right] \end{aligned} \tag{A.42}$$

and from (A.36) and (A.39) we get

$$L_1(r,x) \ge -\left(1+rC(r)\right)C(r)\left[|x-\Psi_0(r)|+\sqrt{\Psi_1(r)}\right]^2. \tag{A.43}$$

Therefore we have

$$\begin{aligned} \left|L_1(r,x)\right| \le & \left(1+rC(r)\right)C(r)\left[|x-\Psi_0(r)|+\sqrt{\Psi_1(r)}\right]^2 \\ & + C(r)^2\left[|x-\Psi_0(r)|+\sqrt{\Psi_1(r)}\right]. \end{aligned} \tag{A.44}$$

(A.44) and (A.45) give (A.23).

We obtain (A.24) and (A.25) from (A.22), (A.23) and monotonicity of $H(\rho,u)$ in ρ using the dominated convergence theorem. (A.26) and (A.28) are from (A.22).

From the first line of (A.42) and (A.24), the positive part of L_1 satisfies

$$\mathbb{E}\left(L_1(r,A)\right)^+ \le 5C(r)\Psi_1(r).$$

Hence by (A.8) and (A.25),

$$\begin{aligned} \mathbb{E}|L_1(r,A)| &= 2\mathbb{E}\left(L_1(r,A)\right)^+ - \mathbb{E}L_1(r,A) \\ &\le |\Psi_2(r)| + 10C(r)\Psi_1(r) \\ &\le |\Psi_2(r)| + 20rC(r)|\Psi_2(r)| \\ &\le \left(1+20e^2(1\vee r)\right)|\Psi_2(r)|. \end{aligned}$$

This completes the proof of the Lemma. □

Lemma A.3. *C in (A.32) is a strictly decreasing function and satisfies*

$$0 < C(r) \leq e^2\big(\frac{1}{r} \vee 1\big). \tag{A.45}$$

Proof. Note that C satisfies

$$\log C(r) = \log \Gamma(r) + e^{\Psi_0(r)} - \rho\Psi_0(r)$$

and

$$\big(\log C(r)\big)' = (e^{\Psi_0(r)} - r)\Psi_1(r) < 0.$$

The last inequality is from the Jensen's inequality :

$$e^{\Psi_0(r)} = e^{E^r A} < E^r e^A = r.$$

Hence C is strictly decreasing.

Consider

$$C_1(r) = \frac{r}{e^r} C(r).$$

Then from (A.3)

$$\begin{aligned}\big(\log C_1(r)\big)' =& \frac{1}{r} - 1 + (e^{\Psi_0(r)} - r)\Psi_1(r)\\ =& [\frac{1}{r} - \rho\Psi_1(r)] + [e^{\Psi_0(r)}\Psi_1(r) - 1]\\ \leq& [e^{\Psi_0(r)}\Psi_1(r) - 1] < 0.\end{aligned}$$

The last inequality is from Lemma 1.2 of [4].

From (A.3), $C_1(0) = \lim_{r\to 0} C_1(r) = e$. Therefore

$$C(r) \leq \frac{e^{r+1}}{r}.$$

Since C is strictly decreasing, $C(r) \leq C(1) \leq e^2$ for $r \geq 1$. This proves (A.45). □

Next, we estimate some moments of log-gamma distribution related random variables. We need the following lemma.

Lemma A.4. *Let Z_1, $Z_2, \dots$ be independent random variables that satisfy for all $i = 1, 2, \dots$ $\mathbf{E}(Z_i) = 0$ and $\mathbf{E}(|Z_i|^p) < \infty$ for some $p \geq 2$. Set $S_k = Z_1 + \cdots + Z_k$. Below $C = \big(18(1-p^{-1})^{-1/2}p\big)^p$ is a constant that depends only on p. The following hold.*

(1)

$$\mathbf{E}|S_k|^p \leq C \frac{\sum_{i=1}^k \mathbf{E}|Z_i|^p}{k} k^{p/2}. \tag{A.46}$$

(2) Let $a_i > 0$ and $T > 0$ be constants such that for all $1 \leq i \leq k$ and $t \in [-T, T]$, $\log \mathbf{E} e^{tZ_i} \leq a_i t^2/2$. Then for $k \geq 1$ and $A \geq \frac{1}{k}\sum_{i=1}^k a_i$, we have for all $x > 0$

$$\mathbf{P}\left[\frac{S_k}{k} > x\right] \vee \mathbf{P}\left[\frac{S_k}{k} < -x\right] \leq e^{-kg(x)}, \tag{A.47}$$

where

$$g(x) = \begin{cases} \frac{1}{2A}x^2 \quad , & 0 \leq x \leq AT \\ Tx - \frac{AT^2}{2}, & x \geq AT. \end{cases}$$

Since $g(x) \geq \frac{1}{2A}x^2 \wedge \frac{1}{2}Tx$, we also have for all $x > 0$

$$\begin{aligned} &\mathbf{P}\left[\frac{S_k}{k} > x\right] \vee \mathbf{P}\left[\frac{S_k}{k} < -x\right] \\ \leq &\exp\big[-k(\tfrac{1}{2A}x^2 \wedge \tfrac{1}{2}Tx)\big] \\ = &\exp[-\tfrac{kx^2}{2A}] \vee \exp[-\tfrac{kTx}{2}]. \end{aligned} \tag{A.48}$$

In this case we have

$$\mathbf{E}|S_k|^p \leq p(2kA)^{p/2}\Gamma(p/2) + \frac{2^{p+1}}{T^p}\Gamma(p+1). \tag{A.49}$$

Proof. Part (1). This is a simple consequence of the Burkholder-Davis-Gundy inequality [11] for $p > 1$ and $q^{-1} = 1 - p^{-1}$:

$$\frac{1}{18p^{1/2}q} \left(\mathbf{E}Y_k^p\right)^{1/p} \le \left(\mathbf{E}|S_k|^p\right)^{1/p} \le 18q^{1/2}p \left(\mathbf{E}Y_k^p\right)^{1/p} \tag{A.50}$$

where

$$Y_k = \Big(\sum_{i=1}^{k} Z_i^2\Big)^{1/2}.$$

Jensen's inequality gives (1).

Part (2). (A.47) is a simple consequence of large deviation theory. For (A.49), we have

$$\begin{aligned}\mathbf{E}|S_k|^p &= p\int_0^\infty y^{p-1}\mathbf{P}\left[|S_k| \ge y\right]\, dy \\ &\le 2p\int_0^\infty y^{p-1}\left(\exp[-\frac{y^2}{2kA}] + \exp[-Ty/2]\right)\, dy \\ &= p(2kA)^{p/2}\Gamma(p/2) + \frac{2^{p+1}}{T^p}\Gamma(p+1).\end{aligned} \tag{A.51}$$

□

Corollary A.5. *Let Y_i be independent random variables with the distribution log-gamma(r_i) for $r_i > 0$. Set $Z_i = Y_i - \mathbf{E}Y_i$ and $S_k = Z_1 + \ldots + Z_k$. Let $\mu_k = \frac{1}{k}\sum_{i=1}^{k}\delta_{r_i}$. Then for*

$$C = 2^{2p+2}\Gamma(p+1)\left(18(1-p^{-1})^{-1/2}p\right)^p, \quad (p \ge 2)$$

the following hold.

(1)

$$\mathbf{E}|S_k|^p \le Ck^{p/2}\int\left(\frac{1}{r^p} + \frac{1}{r^{p/2}}\right)\mu_k(dr). \tag{A.52}$$

(2) For fixed $r_0 > 0$, if $r_i \ge r_0$ for all $i \ge 1$, for $kA \ge \mathbf{V}\mathrm{ar}\, S_k$ and $x > 0$, we have

$$\begin{aligned}\mathbf{P}\left[S_k > x\right] \vee \mathbf{P}\left[S_k < -x\right] \le& e^{-h(x)} = \exp\big(-[\frac{x^2}{8kA} \wedge \frac{r_0 x}{4}]\big) \\ \le& \exp(-\frac{x^2}{8\,\mathbf{V}\mathrm{ar}\, S_k}) \vee \exp(-\frac{r_0 x}{4}).\end{aligned} \tag{A.53}$$

(3) Set $W_i = L(r_i, Y_i) - \Psi_1(r_i)$ *and* $T_k = W_1 + \cdots + W_k$*, where* L *is given by* (A.18)*. Then*

$$\mathbf{E}|T_k|^p \le C_1 k^{p/2} \int \left(\frac{1}{r^{2p}} + \frac{1}{r^{p/2}} \right) \mu_k(dr), \tag{A.54}$$

where

$$C_1 = 2^{3p+3} e^{2p} \Gamma(p+1) \left(18(1-p^{-1})^{-1/2} p \right)^p, \quad (p \ge 2).$$

Proof. We need to estimate some moments of a log-gamma random variable Y with a parameter $r > 0$. The logarithmic moment generating function of $Z = Y - \mathbf{E}Y$ is

$$f(s) = \log \mathbf{E} e^{sZ} = \log \Gamma(r+s) - \log \Gamma(r) - s\Psi_0(r), \quad s > -r.$$

Note that $f(0) = f'(0) = 0$ and $f^{(k+1)}(0) = \Psi_k(r)$ for $k \ge 1$. Taylor's theorem gives

$$f(s) = f(0) + sf'(0) + \frac{1}{2} s^2 f''(s^*) = \frac{1}{2} s^2 \Psi_1(r + s^*),$$

for some s^* with $0 < |s^*| < |s|$. Therefore, for $|s| \le r/2$ by (A.7),

$$0 \le f(s) \le \frac{1}{2} s^2 \Psi_1(r/2) \le 2 s^2 \Psi_1(r).$$

From (A.6) and (A.49) with $k = 1$ we have

$$\begin{aligned} \mathbf{E}|Z|^p &\le p(2\Psi_1(r/2))^{p/2} \Gamma(p/2) + \frac{2^{2p+1}}{r^p} \Gamma(p+1) \\ &\le 2^{2p+2} \Gamma(p+1) \left(\frac{1}{r^p} + \frac{1}{r^{p/2}} \right). \end{aligned} \tag{A.55}$$

Let $W = L(r, Y) - \Psi_1(r)$. Then $\mathbf{E}W = 0$ and

$$|W|^p \le L(r, Y)^p \vee (\Psi_1(r))^p$$

since $L > 0$. By (A.22), we have

$$L(r, Y)^p \le \frac{2^{p-1} e^{2p} (1 \vee r^p)}{r^p} \left(Z^p + (\Psi_1(r))^{p/2} \right).$$

Since $\sqrt{\Psi_1(r)} \le e^2(1 \vee r)/r$ by (A.6) we have

$$|W|^p \le \frac{2^{p-1}e^{2p}(1 \vee r^p)}{r^p}\left(Z^p + (\Psi_1(r))^{p/2}\right).$$

Therefore (A.55) gives

$$\mathbf{E}|W|^p \le 2^{3p+3}e^{2p}\Gamma(p+1)\left(\frac{1}{r^{2p}} + \frac{1}{r^{p/2}}\right). \tag{A.56}$$

We can use (A.55) together with (A.46) to prove part (1) of this corollary. For part (2), we use (A.48). Set $a_i = 4\Psi_1(r_i)$, $T = r_0/2$ then $\sum_{i=1}^{k} a_i = 4\,\mathbf{V}\mathrm{ar}(S_k)$. Substitute kx for x, then we have the result. For part (3), use (A.56) and (A.46). $\square$

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