

Q-POLYNOMIAL DISTANCE-REGULAR GRAPHS AND A DOUBLE AFFINE HECKE ALGEBRA OF RANK ONE

By

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Abstract

We study a relationship between Q -polynomial distance-regular graphs and the double affine Hecke algebra of type (C_1^\vee, C_1) . Let Γ denote a Q -polynomial distance-regular graph with vertex set X . We assume that Γ has q -Racah type and contains a Delsarte clique C . Fix a vertex $x \in C$. We partition X according to the path-length distance to both x and C . This is an equitable partition. For each cell in this partition, consider the corresponding characteristic vector. These characteristic vectors form a basis for a \mathbb{C} -vector space \mathbf{W} .

The universal double affine Hecke algebra of type (C_1^\vee, C_1) is the \mathbb{C} -algebra \hat{H}_q defined by generators $\{t_n^{\pm 1}\}_{n=0}^3$ and relations (i) $t_n t_n^{-1} = t_n^{-1} t_n = 1$; (ii) $t_n + t_n^{-1}$ is central; (iii) $t_0 t_1 t_2 t_3 = q^{-1/2}$. In this thesis, we display an \hat{H}_q -module structure for \mathbf{W} . For this module and up to affine transformation,

- $t_0 t_1 + (t_0 t_1)^{-1}$ acts as the adjacency matrix of Γ ;
- $t_3 t_0 + (t_3 t_0)^{-1}$ acts as the dual adjacency matrix of Γ with respect to C ;
- $t_1 t_2 + (t_1 t_2)^{-1}$ acts as the dual adjacency matrix of Γ with respect to x .

To obtain our results we use the theory of Leonard systems.

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Chapter 1

Introduction

This thesis is about three classes of objects: (i) Leonard systems; (ii) Q -polynomial distance-regular graphs; (iii) double affine Hecke algebras. To motivate our results we will provide some background on each of these topics.

The concept of a Leonard system was introduced by Terwilliger [26, Definition 1.4]. To explain what this concept is, we begin with a more basic concept called a Leonard pair [26, Definition 1.1]. Roughly speaking, a Leonard pair consists of two diagonalizable linear transformations on a finite-dimensional vector space, each of which acts in an irreducible tridiagonal fashion on an eigenbasis for the other one. A Leonard system is a Leonard pair together with an appropriate ordering of their primitive idempotents. Leonard systems are classified up to isomorphism [26, Theorem 1.9]. This classification yields a bijection between the Leonard systems and a family of orthogonal polynomials consisting of the q -Racah polynomials and their relatives in the Askey scheme [29]. The Leonard systems that correspond to the q -Racah polynomials are said to have q -Racah type. This is the most general type of Leonard system. We will focus on the Leonard systems of q -Racah type.

In [8], Delsarte introduced the Q -polynomial property for distance-regular graphs. Since then, this property has been investigated by many authors [2, 3, 4, 7, 14, 15, 16, 22, 23]. A Q -polynomial distance-regular graph can be regarded as a discrete analogue of a rank 1

symmetric space [2, p. 311, 312]. Let Γ denote a Q -polynomial distance-regular graph with vertex set X . Let $\mathbf{M}_X(\mathbb{C})$ denote the \mathbb{C} -algebra consisting of the matrices with entries in \mathbb{C} whose rows and columns are indexed by X . Let \mathbf{V} denote the \mathbb{C} -vector space consisting of column vectors with rows indexed by X . We view \mathbf{V} as a left module for $\mathbf{M}_X(\mathbb{C})$. Fix $x \in X$. In [24] Terwilliger introduced the subconstituent algebra $T = T(x)$ (or Terwilliger algebra). The algebra T is the subalgebra of $\mathbf{M}_X(\mathbb{C})$ generated by the adjacency matrix A of Γ and a certain diagonal matrix $A^* = A^*(x)$, called the dual adjacency matrix of Γ with respect to x . We now discuss the T -modules. By a T -module we mean a T -submodule of \mathbf{V} . The algebra T is semisimple [24, Lemma 3.4], so \mathbf{V} decomposes into a direct sum of irreducible T -modules. Let W denote an irreducible T -module. Then W is called thin whenever the intersection of W with each eigenspace of A and A^* has dimension at most 1. It is known that the matrices A, A^* act as a Leonard pair on each thin irreducible T -module [24]. We now recall the primary T -module [24, Lemma 3.6]. Consider the T -module generated by the characteristic vector of x . This module is thin and irreducible. We call this the primary T -module. We say that Γ has q -Racah type whenever the Leonard system induced by the primary T -module has q -Racah type.

In [21] Suzuki extended the Terwilliger algebra concept by defining the Terwilliger algebra with respect to a set of vertices. For our purpose the set of vertices will be a Delsarte clique. A Delsarte clique C of Γ is a nonempty set of mutually adjacency vertices of Γ that has cardinality $1 - k/\theta_{\min}$, where k is the valency of Γ and θ_{\min} is the minimum eigenvalue of A . The Terwilliger algebra $\tilde{T} = \tilde{T}(C)$ is the subalgebra of $\mathbf{M}_X(\mathbb{C})$ generated by A and a certain diagonal matrix $\tilde{A}^* = \tilde{A}^*(C)$, called the dual adjacency matrix of Γ with respect to C . By a \tilde{T} -module we mean a \tilde{T} -submodule of \mathbf{V} . As we will

see in Section 4, the algebra \tilde{T} is semisimple. So \mathbf{V} decomposes into a direct sum of irreducible \tilde{T} -modules. Let W denote an irreducible \tilde{T} -module. Then W is called thin whenever the intersection of W with each eigenspace of A and \tilde{A}^* has dimension at most 1. By [24, Lemma 3.2] the matrices A, \tilde{A}^* act as a Leonard pair on each thin irreducible \tilde{T} -module. We now recall the primary \tilde{T} -module [21, Section 7]. Consider the \tilde{T} -module generated by the characteristic vector of C . It turns out that this module is thin and irreducible. We call this the primary \tilde{T} -module.

In this thesis we introduce an algebra \mathbf{T} . To define \mathbf{T} we assume that Γ contains a Delsarte clique C . Fix a vertex $x \in C$. The algebra $\mathbf{T} = \mathbf{T}(x, C)$ is generated by $T = T(x)$ and $\tilde{T} = \tilde{T}(C)$. The algebra \mathbf{T} is semisimple. It turns out that the \mathbf{T} -module generated by the characteristic vector of x is equal to the \mathbf{T} -module generated by the characteristic vector of C . We denote this module by \mathbf{W} . As we will see, the \mathbf{T} -module \mathbf{W} is irreducible. Moreover, the T -module \mathbf{W} decomposes into the direct sum of two irreducible T -modules, one of which is the primary T -module. Also, the \tilde{T} -module \mathbf{W} decomposes into the direct sum of two irreducible \tilde{T} -modules, one of which is the primary \tilde{T} -module. The \mathbf{T} -module \mathbf{W} will play a role in our main result.

In [6], Cherednik introduced the double affine Hecke algebra (or DAHA) for a reduced root system. In [20], Sahi extended this definition to include non-reduced root systems. In this thesis we consider the DAHA of type (C_1^\vee, C_1) . This is the most general DAHA of rank 1 [18]. In [17], Noumi and Stokman treated this algebra in detail, in order to study the Askey-Wilson polynomials. We now define the DAHA of type (C_1^\vee, C_1) .

Definition 1.0.1. [11] Fix nonzero scalars k_0, k_1, k_2, k_3, q in \mathbb{C} . Let $H = H(k_0, k_1, k_2, k_3; q)$

denote the \mathbb{C} -algebra defined by generators $\{t_n\}_{n=0}^3$ and relations

$$(t_n - k_n)(t_n - k_n^{-1}) = 0, \quad t_0 t_1 t_2 t_3 = q^{-1/2}.$$

This algebra is called the DAHA of type (C_1^\vee, C_1) .

In [30] Terwilliger defined a central extension \hat{H}_q of H . By definition \hat{H}_q has generators $\{t_n^{\pm 1}\}_{n=0}^3$ and relations (i) $t_n t_n^{-1} = t_n^{-1} t_n = 1$; (ii) $t_n + t_n^{-1}$ is central; (iii) $t_0 t_1 t_2 t_3 = q^{-1/2}$. We call \hat{H}_q the universal DAHA of type (C_1^\vee, C_1) . In the present thesis we will focus on \hat{H}_q .

We now summarize our main results. Let Γ denote a Q -polynomial distance-regular graph with vertex set X and diameter $d \geq 3$. We assume that Γ has q -Racah type and contains a Delsarte clique C . Fix a vertex $x \in C$. Consider the module \mathbf{W} for $\mathbf{T} = \mathbf{T}(x, C)$ which was discussed earlier. We show that \mathbf{W} is an irreducible \hat{H}_q -module as well as an irreducible \mathbf{T} -module. We show how the \hat{H}_q -action on \mathbf{W} is related to the \mathbf{T} -action on \mathbf{W} . Our central result of this thesis is as follows. Define

$$\mathbf{A} = t_0 t_1 + (t_0 t_1)^{-1}, \quad \mathbf{B} = t_3 t_0 + (t_3 t_0)^{-1}, \quad \mathbf{B}^\dagger = t_1 t_2 + (t_1 t_2)^{-1}.$$

On \mathbf{W} and up to affine transformation,

- (i) \mathbf{A} acts as the adjacency matrix of Γ ;
- (ii) \mathbf{B} acts as the dual adjacency matrix of Γ with respect to C ;
- (iii) \mathbf{B}^\dagger acts as the dual adjacency matrix of Γ with respect to x .

Moreover on \mathbf{W} and for appropriate $k_0, k_1 \in \mathbb{C}$,

- (iv) $\frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}}$ acts as the projection from \mathbf{W} onto the primary \tilde{T} -module;

- (v) $\frac{t_1 - k_1^{-1}}{k_1 - k_1^{-1}}$ acts as the projection from \mathbf{W} onto the primary T -module.

The thesis is organized as follows. It has two main chapters. In Chapter 2, we discuss Leonard systems and Q -polynomial distance-regular graphs. In Chapter 3, we discuss the DAHA of type (C_1^\vee, C_1) . Chapter 2 is organized as follows. In Section 2.1 we provide some background regarding Leonard systems. In Section 2.2 we recall some background concerning Q -polynomial distance-regular graphs and the Terwilliger algebra. In Section 2.3 we review the basic properties of a Delsarte clique C and study the Terwilliger algebra with respect to C . In Section 2.4 we define the algebra \mathbf{T} and construct a \mathbf{T} -module \mathbf{W} . In Sections 2.5–2.7 we study the Leonard systems associated with \mathbf{W} . In Section 2.8 we identify five bases for \mathbf{W} , and display the transition matrices between certain pairs of bases among the five. We also display the matrices representing various linear maps in $\text{End}(\mathbf{W})$ with respect to the five bases. These matrices will be used to prove the main results of the thesis. Chapter 3 is organized as follows. In Section 3.1 we define the algebra \hat{H}_q and the elements $\mathbf{A}, \mathbf{B}, \mathbf{B}^\dagger$ in \hat{H}_q . In Section 3.2 we turn \mathbf{W} into an \hat{H}_q -module. In Section 3.3 we display how the \hat{H}_q -action on \mathbf{W} is related to the \mathbf{T} -action on \mathbf{W} . Theorem 3.3.1 is the main result of the thesis. This thesis ends with an Appendix in which many details are made explicit for the case of diameter 4.

Chapter 2

Leonard Systems and Q -polynomial Distance-Regular Graphs

2.1 Leonard systems and parameter arrays

Throughout the thesis, let d denote a positive integer and let $q \in \mathbb{C}$ be a nonzero scalar such that $q^2 \neq 1$. Let $\mathbf{M}_{d+1}(\mathbb{C})$ denote the \mathbb{C} -algebra consisting of all $(d+1) \times (d+1)$ matrices that have entries in \mathbb{C} . We index the rows and columns by $0, 1, \dots, d$. Throughout the thesis V denotes a vector space over \mathbb{C} with dimension $d+1$. Let $\text{End}(V)$ denote the \mathbb{C} -algebra consisting of all \mathbb{C} -linear transformations from V to V . Let I denote the identity of $\text{End}(V)$. For $A \in \text{End}(V)$, A is called *multiplicity-free* whenever A has $d+1$ mutually distinct eigenvalues. Assume A is multiplicity-free. Let $\{\theta_i\}_{i=0}^d$ denote an ordering of the eigenvalues of A . For $0 \leq i \leq d$ let V_i denote the eigenspace of A associated with θ_i . Define $E_i \in \text{End}(V)$ such that $(E_i - I)V_i = 0$ and $E_i V_j = 0$ for $j \neq i$ ($0 \leq j \leq d$). We call E_i the *primitive idempotent* of A corresponding to V_i (or θ_i). Observe that (i) $E_i E_j = \delta_{ij} E_i$ ($0 \leq i, j \leq d$); (ii) $I = \sum_{i=0}^d E_i$; (iii) $A = \sum_{i=0}^d \theta_i E_i$. Moreover,

$$E_i = \prod_{\substack{0 \leq j \leq d \\ j \neq i}} \frac{A - \theta_j I}{\theta_i - \theta_j} \quad (0 \leq i \leq d).$$

Let \mathfrak{A} denote the \mathbb{C} -subalgebra of $\text{End}(V)$ generated by A . Observe that each of $\{A^i\}_{i=0}^d$ and $\{E_i\}_{i=0}^d$ is a basis for \mathfrak{A} . We now define a Leonard system on V .

Definition 2.1.1. [26, Definition 1.4] By a *Leonard system* on V we mean a sequence

$$\Phi := (A; A^*; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$$

that satisfies (i)–(v) below.

- (i) Each of A, A^* is a multiplicity-free element in $\text{End}(V)$.
- (ii) $\{E_i\}_{i=0}^d$ is an ordering of the primitive idempotents of A .
- (iii) $\{E_i^*\}_{i=0}^d$ is an ordering of the primitive idempotents of A^* .
- (iv) For $0 \leq i, j \leq d$,

$$E_i A^* E_j = \begin{cases} 0 & \text{if } |i - j| > 1, \\ \neq 0 & \text{if } |i - j| = 1. \end{cases}$$

- (v) For $0 \leq i, j \leq d$,

$$E_i^* A E_j^* = \begin{cases} 0 & \text{if } |i - j| > 1, \\ \neq 0 & \text{if } |i - j| = 1. \end{cases}$$

We call d the *diameter* of Φ , and say Φ is *over* \mathbb{C} .

There exists a natural action of the dihedral group D_4 on the set of all Leonard systems.

The action is described as follows. Let Φ denote the Leonard system from Definition 2.1.1. Then each of the following is a Leonard system on V :

$$\begin{aligned} \Phi^* &:= (A^*; A; \{E_i^*\}_{i=0}^d; \{E_i\}_{i=0}^d), \\ \Phi^\downarrow &:= (A; A^*; \{E_i\}_{i=0}^d; \{E_{d-i}^*\}_{i=0}^d), \\ \Phi^\updownarrow &:= (A; A^*; \{E_{d-i}\}_{i=0}^d; \{E_i^*\}_{i=0}^d). \end{aligned} \tag{2.1}$$

Viewing $*$, \downarrow , \Downarrow as permutations on the set of all Leonard systems, we have

$$*^2 = \downarrow^2 = \Downarrow^2 = 1, \quad \Downarrow * = * \downarrow, \quad \downarrow * = * \Downarrow, \quad \Downarrow \downarrow = \downarrow \Downarrow. \quad (2.2)$$

The group generated by the symbols $*$, \downarrow , \Downarrow subject to the relations (2.2) is the dihedral group D_4 of order 8. The permutations $*$, \downarrow , \Downarrow induce an action of D_4 on the set of all Leonard systems.

We recall the notion of isomorphism for Leonard systems. Let Φ denote the Leonard system from Definition 2.1.1. Let V' denote a vector space over \mathbb{C} with dimension $d + 1$. Let $f : \text{End}(V) \rightarrow \text{End}(V')$ denote an isomorphism of \mathbb{C} -algebras. Write $\Phi^f = (A^f; A^{*f}; \{E_i^f\}_{i=0}^d; \{E_i^{*f}\}_{i=0}^d)$ and observe Φ^f is a Leonard system on V' . Let Φ and Φ' denote any Leonard systems over \mathbb{C} . By an *isomorphism of Leonard systems* from Φ to Φ' we mean an isomorphism of \mathbb{C} -algebras $f : \text{End}(V) \rightarrow \text{End}(V')$ such that $\Phi^f = \Phi'$. We say that Φ and Φ' are *isomorphic* whenever there exists an isomorphism of Leonard systems from Φ to Φ' .

Let Φ denote the Leonard system from Definition 2.1.1. For $0 \leq i \leq d$ let θ_i (resp. θ_i^*) denote the eigenvalue of A (resp. A^*) associated with E_i (resp. E_i^*). We call $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^*\}_{i=0}^d$) the *eigenvalue sequence* (resp. *dual eigenvalue sequence*) of Φ . By [26, Theorem 3.2] there exist nonzero scalars $\{\varphi_i\}_{i=1}^d$ and a \mathbb{C} -algebra isomorphism

$\natural : \text{End}(V) \rightarrow \mathbf{M}_{d+1}(\mathbb{C})$ such that

$$A^\natural = \begin{bmatrix} \theta_0 & & & & & \mathbf{0} \\ & 1 & \theta_1 & & & \\ & & 1 & \theta_2 & & \\ & & & \cdot & \cdot & \\ & & & & \cdot & \cdot \\ \mathbf{0} & & & & & 1 & \theta_d \end{bmatrix}, \quad A^{*\natural} = \begin{bmatrix} \theta_0^* & \varphi_1 & & & & \mathbf{0} \\ & \theta_1^* & \varphi_2 & & & \\ & & \theta_2^* & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & \cdot & \varphi_d \\ \mathbf{0} & & & & & & \theta_d^* \end{bmatrix}. \quad (2.3)$$

We call the sequence $\{\varphi_i\}_{i=1}^d$ the *first split sequence* of Φ . We let $\{\phi_i\}_{i=0}^d$ denote the first split sequence of Φ^\downarrow and call this the *second split sequence* of Φ . By the *parameter array* of Φ we mean the sequence

$$(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d, \{\varphi_i\}_{i=1}^d, \{\phi_i\}_{i=1}^d). \quad (2.4)$$

We denote this parameter array by $p(\Phi)$. The following theorem shows that the isomorphism class of Φ is determined by $p(\Phi)$.

Theorem 2.1.2. [26, Theorem 1.9] *Let*

$$(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d, \{\varphi_i\}_{i=1}^d, \{\phi_i\}_{i=1}^d) \quad (2.5)$$

denote a sequence of scalars in \mathbb{C} . Then there exists a Leonard system Φ over \mathbb{C} with the parameter array (2.5) if and only if the following conditions (PA1)–(PA5) hold:

$$(PA1) \quad \theta_i \neq \theta_j, \quad \theta_i^* \neq \theta_j^* \quad \text{if} \quad i \neq j \quad (0 \leq i, j \leq d).$$

$$(PA2) \quad \varphi_i \neq 0, \quad \phi_i \neq 0 \quad (1 \leq i \leq d).$$

$$(PA3) \quad \varphi_i = \phi_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} + (\theta_i^* - \theta_0^*)(\theta_{i-1} - \theta_d) \quad (1 \leq i \leq d).$$

$$(PA4) \quad \phi_i = \varphi_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} + (\theta_i^* - \theta_0^*)(\theta_{d-i+1} - \theta_0) \quad (1 \leq i \leq d).$$

(PA5) *The expressions*

$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}, \quad \frac{\theta_{i-2}^* - \theta_{i+1}^*}{\theta_{i-1}^* - \theta_i^*}$$

are equal and independent of i for $2 \leq i \leq d-1$.

Moreover if (PA1)–(PA5) hold then Φ is unique up to isomorphism of Leonard systems.

By a *parameter array over \mathbb{C} of diameter d* we mean a sequence of complex scalars $(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d, \{\varphi_i\}_{i=1}^d, \{\phi_i\}_{i=1}^d)$ which satisfies the conditions (PA1)–(PA5). By Theorem 2.1.2, the map which sends a given Leonard system to its parameter array induces a bijection from the set of isomorphism classes of Leonard systems over \mathbb{C} to the set of parameter arrays over \mathbb{C} . In [27] Terwilliger displayed all the parameter arrays over \mathbb{C} .

Let Φ denote the Leonard system from Definition 2.1.1. We now recall the Φ -standard basis. Let \mathbf{v} be a nonzero vector in E_0V . By [28, Lemma 10.2], the sequence $\{E_i^*\mathbf{v}\}_{i=0}^d$ is a basis for V . This basis is said to be Φ -*standard*. The following is a characterization of the Φ -standard basis.

Lemma 2.1.3. [28, Lemma 10.4] *Let $\{\mathbf{v}_i\}_{i=0}^d$ denote a sequence of vectors in V , not all 0. Then this sequence is a Φ -standard basis for V if and only if both*

$$(i) \quad \mathbf{v}_i \in E_i^*V \text{ for } 0 \leq i \leq d;$$

$$(ii) \quad \sum_{i=0}^d \mathbf{v}_i \in E_0V.$$

Let Φ denote the Leonard system from Definition 2.1.1 and $(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d, \{\varphi_i\}_{i=1}^d, \{\phi_i\}_{i=1}^d)$ denote the parameter array of Φ . For all $X \in \text{End}(V)$ let X^b denote the matrix in

$\mathbf{M}_{d+1}(\mathbb{C})$ that represents X relative to a Φ -standard basis for V . By Definition 2.1.1 and construction we have

$$A^{*\flat} = \text{diag}(\theta_0^*, \theta_1^*, \theta_2^*, \dots, \theta_d^*). \quad (2.6)$$

Moreover, there exist scalars $\{a_i\}_{i=0}^d, \{b_i\}_{i=0}^{d-1}, \{c_i\}_{i=1}^d$ in \mathbb{C} such that

$$A^\flat = \begin{bmatrix} a_0 & b_0 & & & \mathbf{0} \\ c_1 & a_1 & b_1 & & \\ & c_2 & a_2 & \ddots & \\ & & \ddots & \ddots & b_{d-1} \\ \mathbf{0} & & & c_d & a_d \end{bmatrix}. \quad (2.7)$$

We call a_i, b_i, c_i the *intersection numbers of Φ* . For notational convenience define $b_d = 0$ and $c_0 = 0$. By [28, Lemma 10.5], A^\flat has constant row sum θ_0 . Therefore $a_i = \theta_0 - b_i - c_i$ ($0 \leq i \leq d$). In the following two lemmas, we give the intersection numbers of Φ in terms of the parameter array of Φ .

Lemma 2.1.4. [28, Theorem 23.5] *The intersection numbers b_i, c_i of Φ are*

$$b_i = \varphi_{i+1} \frac{(\theta_i^* - \theta_0^*)(\theta_i^* - \theta_1^*) \cdots (\theta_i^* - \theta_{i-1}^*)}{(\theta_{i+1}^* - \theta_0^*)(\theta_{i+1}^* - \theta_1^*) \cdots (\theta_{i+1}^* - \theta_i^*)} \quad (0 \leq i \leq d-1), \quad (2.8)$$

$$c_i = \phi_i \frac{(\theta_i^* - \theta_d^*)(\theta_i^* - \theta_{d-1}^*) \cdots (\theta_i^* - \theta_{i+1}^*)}{(\theta_{i-1}^* - \theta_d^*)(\theta_{i-1}^* - \theta_{d-1}^*) \cdots (\theta_{i-1}^* - \theta_i^*)} \quad (1 \leq i \leq d). \quad (2.9)$$

Lemma 2.1.5. [28, Theorem 23.6] *The intersection numbers a_i of Φ are*

$$a_0 = \theta_0 + \frac{\varphi_1}{\theta_0^* - \theta_1^*}, \quad (2.10)$$

$$a_i = \theta_i + \frac{\varphi_i}{\theta_i^* - \theta_{i-1}^*} + \frac{\varphi_{i+1}}{\theta_i^* - \theta_{i+1}^*} \quad (1 \leq i \leq d-1),$$

$$a_d = \theta_d + \frac{\varphi_d}{\theta_d^* - \theta_{d-1}^*}.$$

We recall the Leonard system Φ^* from (2.1). By *the dual intersection numbers of Φ* we mean the intersection numbers of Φ^* . These are denoted by a_i^*, b_i^*, c_i^* .

Lemma 2.1.6. [26, Theorem 1.11], [28, Theorem 23.5] *The dual intersection numbers b_i^*, c_i^* of Φ are*

$$\begin{aligned} b_i^* &= \varphi_{i+1} \frac{(\theta_i - \theta_0)(\theta_i - \theta_1) \cdots (\theta_i - \theta_{i-1})}{(\theta_{i+1} - \theta_0)(\theta_{i+1} - \theta_1) \cdots (\theta_{i+1} - \theta_i)} & (0 \leq i \leq d-1), \\ c_i^* &= \phi_{d-i+1} \frac{(\theta_i - \theta_d)(\theta_i - \theta_{d-1}) \cdots (\theta_i - \theta_{i+1})}{(\theta_{i-1} - \theta_d)(\theta_{i-1} - \theta_{d-1}) \cdots (\theta_{i-1} - \theta_i)} & (1 \leq i \leq d). \end{aligned}$$

Lemma 2.1.7. [26, Theorem 1.11], [28, Theorem 23.6] *The dual intersection numbers a_i^* of Φ are*

$$\begin{aligned} a_0^* &= \theta_0^* + \frac{\varphi_1}{\theta_0 - \theta_1}, \\ a_i^* &= \theta_i^* + \frac{\varphi_i}{\theta_i - \theta_{i-1}} + \frac{\varphi_{i+1}}{\theta_i - \theta_{i+1}} & (1 \leq i \leq d-1), \\ a_d^* &= \theta_d^* + \frac{\varphi_d}{\theta_d - \theta_{d-1}}. \end{aligned}$$

Let Φ denote the Leonard system from Definition 2.1.1. Recall the parameter array $p(\Phi)$ from (2.4). Let λ denote an indeterminate and let $\mathbb{C}[\lambda]$ denote the \mathbb{C} -algebra consisting of the polynomials in λ that have all coefficients in \mathbb{C} . For $0 \leq i \leq d$ define a polynomial $\mathbf{u}_i \in \mathbb{C}[\lambda]$ by

$$\mathbf{u}_i = \sum_{n=0}^i \frac{(\theta_i^* - \theta_0^*)(\theta_i^* - \theta_1^*) \cdots (\theta_i^* - \theta_{n-1}^*)(\lambda - \theta_0)(\lambda - \theta_1) \cdots (\lambda - \theta_{n-1})}{\varphi_1 \varphi_2 \cdots \varphi_n}. \quad (2.11)$$

By [27, Theorem 7.2],

$$\theta_j \mathbf{u}_i(\theta_j) = c_i \mathbf{u}_{i-1}(\theta_j) + a_i \mathbf{u}_i(\theta_j) + b_i \mathbf{u}_{i+1}(\theta_j) \quad (0 \leq i, j \leq d),$$

where a_i, b_i, c_i are the intersection numbers of Φ and \mathbf{u}_{-1} and \mathbf{u}_{d+1} are indeterminates.

By [27, Section 10],

$$\mathbf{u}_i(\theta_d) = \frac{\phi_1 \phi_2 \cdots \phi_i}{\varphi_1 \varphi_2 \cdots \varphi_i} \quad (0 \leq i \leq d). \quad (2.12)$$

We now recall the q -Racah family of parameter arrays [27]. This is the most general family.

Example 2.1.8. [27, Example 5.3] For $0 \leq i \leq d$ define

$$\theta_i = \theta_0 + h(1 - q^i)(1 - sq^{i+1})q^{-i}, \quad (2.13)$$

$$\theta_i^* = \theta_0^* + h^*(1 - q^i)(1 - s^*q^{i+1})q^{-i}, \quad (2.14)$$

and for $1 \leq i \leq d$ define

$$\varphi_i = hh^*q^{1-2i}(1 - q^i)(1 - q^{i-d-1})(1 - r_1q^i)(1 - r_2q^i), \quad (2.15)$$

$$\phi_i = hh^*q^{1-2i}(1 - q^i)(1 - q^{i-d-1})(r_1 - s^*q^i)(r_2 - s^*q^i)/s^*, \quad (2.16)$$

where θ_0 and θ_0^* are scalars in \mathbb{C} , and where h, h^*, s, s^*, r_1, r_2 are nonzero scalars in \mathbb{C} such that $r_1r_2 = ss^*q^{d+1}$. To avoid degenerate situations assume that none of $q^i, r_1q^i, r_2q^i, s^*q^i/r_1, s^*q^i/r_2$ is equal to 1 for $1 \leq i \leq d$ and that neither of sq^i, s^*q^i is equal to 1 for $2 \leq i \leq 2d$. Then the sequence $(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d, \{\varphi_i\}_{i=1}^d, \{\phi_i\}_{i=1}^d)$ is a parameter array over \mathbb{C} . This parameter array is said to have q -Racah type.

Let Φ denote the Leonard system from Definition 2.1.1. We say that Φ has q -Racah type whenever its parameter array has q -Racah type. Assume that Φ has q -Racah type with the parameter array as in Example 2.1.8. Recall the intersection numbers $\{b_i\}_{i=0}^{d-1}, \{c_i\}_{i=1}^d$ of Φ from Lemma 2.1.4. Evaluating (2.8) and (2.9) using (2.13)–(2.16) we find

$$b_0 = \frac{h(1 - q^{-d})(1 - r_1q)(1 - r_2q)}{1 - s^*q^2}, \quad (2.17)$$

$$b_i = \frac{h(1 - q^{i-d})(1 - s^*q^{i+1})(1 - r_1q^{i+1})(1 - r_2q^{i+1})}{(1 - s^*q^{2i+1})(1 - s^*q^{2i+2})} \quad (1 \leq i \leq d-1), \quad (2.18)$$

$$c_i = \frac{h(1 - q^i)(1 - s^*q^{i+d+1})(r_1 - s^*q^i)(r_2 - s^*q^i)}{s^*q^d(1 - s^*q^{2i})(1 - s^*q^{2i+1})} \quad (1 \leq i \leq d-1), \quad (2.19)$$

$$c_d = \frac{h(1 - q^d)(r_1 - s^*q^d)(r_2 - s^*q^d)}{s^*q^d(1 - s^*q^{2d})}; \quad (2.20)$$

see [28, Section 24]. To obtain the dual intersection numbers $\{b_i^*\}_{i=0}^{d-1}, \{c_i^*\}_{i=1}^d$ of Φ , replace (h, s^*) by (h^*, s) in (2.17)–(2.20).

Now consider (2.11). Pick integers i, j ($0 \leq i, j \leq d$). Evaluating (2.11) at $\lambda = \theta_j$ and simplifying the result using (2.13)–(2.16) we get

$$\mathbf{u}_i(\theta_j) = \sum_{n=0}^i \frac{(q^{-i}; q)_n (s^* q^{i+1}; q)_n (q^{-j}; q)_n (s q^{j+1}; q)_n q^n}{(r_1 q; q)_n (r_2 q; q)_n (q^{-d}; q)_n (q; q)_n}, \quad (2.21)$$

where

$$(a; q)_n := (1 - a)(1 - aq)(1 - aq^2) \cdots (1 - aq^{n-1}) \quad n = 0, 1, 2, \dots$$

From the definition of basic hypergeometric series [9, p. 4], the sum on the right in (2.21) is

$${}_4\phi_3 \left(\begin{matrix} q^{-i}, s^* q^{i+1}, q^{-j}, s q^{j+1} \\ r_1 q, r_2 q, q^{-d} \end{matrix} \middle| q, q \right). \quad (2.22)$$

The q -Racah polynomials are defined in [1]. By (2.21) and (2.22), the $\{\mathbf{u}_i\}_{i=0}^d$ are q -Racah polynomials.

2.2 Q -polynomial distance-regular graphs

We now turn our attention to distance-regular graphs. In this section we review those aspects of Q -polynomial distance-regular graphs that we will need later in the thesis. For more background information we refer the reader to Brouwer, Cohen and Neumaier [3] and Terwilliger [24].

Let X denote a nonempty finite set. Let $\mathbf{M}_X(\mathbb{C})$ denote the \mathbb{C} -algebra consisting of the matrices with entries in \mathbb{C} whose rows and columns are indexed by X . Let $\mathbf{V} = \mathbb{C}^X$

denote the \mathbb{C} -vector space consisting of column vectors with entries in \mathbb{C} and rows indexed by X . We view \mathbf{V} as a left module for $\mathbf{M}_X(\mathbb{C})$ and call this the *standard module*. We endow \mathbf{V} with the Hermitean inner product $\langle \cdot, \cdot \rangle$ that satisfies $\langle u, v \rangle = u^t \bar{v}$ for $u, v \in \mathbf{V}$, where t denotes transpose and $\bar{\cdot}$ denotes complex conjugate. For all $x \in X$ let \hat{x} denote the vector in \mathbf{V} with a 1 in the x coordinate and 0 in all other coordinates. Note that the set $\{\hat{x} \mid x \in X\}$ is an orthonormal basis for \mathbf{V} . For a subset $Y \subseteq X$, define $\hat{Y} = \sum_{y \in Y} \hat{y}$ and call this the *characteristic vector of Y* . We abbreviate $\mathbf{j} = \hat{X}$. The vector \mathbf{j} has x -coordinate 1 for all $x \in X$.

Let Γ denote an undirected, connected graph, without loops or multiple edges, with vertex set X and diameter $d \geq 3$. For $x \in X$ define

$$\Gamma_i(x) = \{y \in X \mid \partial(x, y) = i\} \quad (0 \leq i \leq d), \quad (2.23)$$

where ∂ denotes the path-length distance function. We abbreviate $\Gamma(x) = \Gamma_1(x)$. For an integer $k \geq 0$, Γ is said to be *regular with valency k* whenever $|\Gamma(x)| = k$ for all $x \in X$. We say that Γ is *distance-regular* whenever for all integers h, i, j ($0 \leq h, i, j \leq d$) and all vertices $x, y \in X$ with $\partial(x, y) = h$, the number $p_{i,j}^h = |\Gamma_i(x) \cap \Gamma_j(y)|$ is independent of x and y . The numbers $p_{i,j}^h$ are called the *intersection numbers of Γ* . Abbreviate

$$a_i(\Gamma) = p_{1,i}^i \quad (0 \leq i \leq d), \quad b_i(\Gamma) = p_{1,i+1}^i \quad (0 \leq i \leq d-1), \quad c_i(\Gamma) = p_{1,i-1}^i \quad (1 \leq i \leq d),$$

and define $b_d(\Gamma) = 0$ and $c_0(\Gamma) = 0$. By construction $a_0(\Gamma) = 0$ and $c_1(\Gamma) = 1$. For the rest of the thesis, assume that Γ is distance-regular. Observe that Γ is regular with valency $k = b_0(\Gamma)$.

We recall the Bose-Mesner algebra of Γ . For $0 \leq i \leq d$ let A_i denote the matrix in

$\mathbf{M}_X(\mathbb{C})$ with (x, y) -entry

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } \partial(x, y) = i \\ 0 & \text{if } \partial(x, y) \neq i \end{cases} \quad (x, y \in X).$$

We call A_i the i -th distance matrix of Γ . Observe that $\sum_{i=0}^d A_i = \mathbf{J}$, the all-ones matrix, and $A_i A_j = \sum_{h=0}^d p_{i,j}^h A_h$ for $0 \leq i, j \leq d$. We abbreviate $A = A_1$ and call this the *adjacency matrix* of Γ . Let \mathcal{M} denote the subalgebra of $\mathbf{M}_X(\mathbb{C})$ generated by A . The algebra \mathcal{M} is commutative. We call \mathcal{M} the *Bose-Mesner algebra* of Γ . By [3, p. 127] the set $\{A_i\}_{i=0}^d$ is a basis for \mathcal{M} . The algebra \mathcal{M} is semisimple since it is closed under the conjugate-transpose map. By [3, p. 45] \mathcal{M} has a basis $\{E_i\}_{i=0}^d$ such that (i) $E_0 = |X|^{-1} \mathbf{J}$; (ii) $I = \sum_{i=0}^d E_i$; (iii) $\bar{E}_i = E_i$ ($0 \leq i \leq d$); (iv) $E_i^t = E_i$ ($0 \leq i \leq d$); (v) $E_i E_j = \delta_{ij} E_i$ ($0 \leq i, j \leq d$). We call $\{E_i\}_{i=0}^d$ the *primitive idempotents* of Γ . Since $\{E_i\}_{i=0}^d$ is a basis for \mathcal{M} there exist complex scalars $\{\theta_i\}_{i=0}^d$ such that $A = \sum_{i=0}^d \theta_i E_i$. Observe that $A E_i = E_i A = \theta_i E_i$ for $0 \leq i \leq d$. By [2, p. 197] the scalars $\{\theta_i\}_{i=0}^d$ are real. The scalars $\{\theta_i\}_{i=0}^d$ are mutually distinct since A generates \mathcal{M} . We call θ_i the *eigenvalue* of Γ associated with E_i ($0 \leq i \leq d$). By (i) we have $k = \theta_0$. Note that $\mathbf{V} = \sum_{i=0}^d E_i \mathbf{V}$ (orthogonal direct sum). For $0 \leq i \leq d$, $E_i \mathbf{V}$ is the eigenspace of A associated with θ_i . Let m_i denote the rank of E_i and note that m_i is the dimension of $E_i \mathbf{V}$. We call m_i the *multiplicity* of E_i (or θ_i). Note that the vector \mathbf{j} is a basis for $E_0 \mathbf{V}$.

We recall the Q -polynomial property. Let \circ denote the entrywise product in $\mathbf{M}_X(\mathbb{C})$. Observe that $A_i \circ A_j = \delta_{ij} A_i$ for $0 \leq i, j \leq d$, so \mathcal{M} is closed under \circ . Therefore, there exist complex scalars $q_{i,j}^h$ ($0 \leq h, i, j \leq d$) such that $E_i \circ E_j = |X|^{-1} \sum_{h=0}^d q_{i,j}^h E_h$ for $0 \leq i, j \leq d$. By [3, p. 48, 49] each $q_{i,j}^h$ is real and nonnegative. The $q_{i,j}^h$ are called the

dual intersection numbers (or *Krein parameters*) of Γ . We abbreviate

$$a_i^*(\Gamma) = q_{1,i}^i \quad (0 \leq i \leq d), \quad b_i^*(\Gamma) = q_{1,i+1}^i \quad (0 \leq i \leq d-1), \quad c_i^*(\Gamma) = q_{1,i-1}^i \quad (1 \leq i \leq d),$$

and define $b_d^*(\Gamma) = 0$ and $c_0^*(\Gamma) = 0$. By [2, p. 193] $c_1^*(\Gamma) = 1$. The graph Γ is said to be *Q-polynomial* (with respect to the ordering $\{E_i\}_{i=0}^d$ of the primitive idempotents) whenever for $0 \leq h, i, j \leq d$, $q_{i,j}^h = 0$ (resp. $q_{i,j}^h \neq 0$) if one of h, i, j is greater than (resp. equal to) the sum of the other two [3, p. 235]. For the rest of this thesis, assume that Γ is *Q-polynomial* with respect to the ordering $\{E_i\}_{i=0}^d$. It is not necessarily the case that $\theta_0 > \theta_1 > \dots > \theta_d$. However, this ordering occurs in many examples [3, Chapter 8]. To keep our discussion simple, we always assume that our *Q-polynomial* structure satisfies $\theta_0 > \theta_1 > \dots > \theta_d$.

We recall the dual Bose-Mesner algebra of Γ . For the rest of this section, fix $x \in X$. We view x as a base vertex. For $0 \leq i \leq d$ let $E_i^* = E_i^*(x)$ denote the diagonal matrix in $\mathbf{M}_X(\mathbb{C})$ with (y, y) -entry

$$(E_i^*)_{yy} = \begin{cases} 1 & \text{if } \partial(x, y) = i \\ 0 & \text{if } \partial(x, y) \neq i \end{cases} \quad (y \in X). \quad (2.24)$$

We call E_i^* the *i-th dual primitive idempotent* of Γ with respect to x . Observe that (i) $I = \sum_{i=0}^d E_i^*$; (ii) $\overline{E_i^*} = E_i^*$ ($0 \leq i \leq d$); (iii) $(E_i^*)^t = E_i^*$ ($0 \leq i \leq d$); (iv) $E_i^* E_j^* = \delta_{ij} E_i^*$ ($0 \leq i, j \leq d$). By these facts, $\{E_i^*\}_{i=0}^d$ forms a basis for a commutative subalgebra $\mathcal{M}^* = \mathcal{M}^*(x)$ of $\mathbf{M}_X(\mathbb{C})$. We call \mathcal{M}^* the *dual Bose-Mesner algebra* of Γ with respect to x . The algebra \mathcal{M}^* is semisimple since it is closed under the conjugate-transpose map. For $0 \leq i \leq d$ let $A_i^* = A_i^*(x)$ denote the diagonal matrix in $\mathbf{M}_X(\mathbb{C})$ with (y, y) -entry

$$(A_i^*)_{yy} = |X|(E_i)_{xy} \quad (y \in X).$$

We call A_i^* the i -th dual distance matrix of Γ with respect to x . By [24, p. 379], $\{A_i^*\}_{i=0}^d$ is a basis for \mathcal{M}^* . We abbreviate $A^* = A_1^*$ and call this the *dual adjacency matrix* of Γ with respect to x . By [24, Lemma 3.11], A^* generates \mathcal{M}^* . Since $\{E_i^*\}_{i=0}^d$ is a basis for \mathcal{M}^* , there exist complex scalars $\{\theta_i^*\}_{i=0}^d$ such that $A^* = \sum_{i=0}^d \theta_i^* E_i^*$. Observe that $A^* E_i^* = E_i^* A^* = \theta_i^* E_i^*$ for $0 \leq i \leq d$. By [24, Lemma 3.11], the $\{\theta_i^*\}_{i=0}^d$ are real. These scalars are mutually distinct since A^* generates \mathcal{M}^* . We call θ_i^* the *dual eigenvalue* of Γ associated with E_i^* ($0 \leq i \leq d$). Note that $\mathbf{V} = \sum_{i=0}^d E_i^* \mathbf{V}$ (orthogonal direct sum). From (2.24), for $0 \leq i \leq d$ we find $E_i^* \mathbf{V} = \text{Span}\{\hat{y} \mid y \in X, \partial(x, y) = i\}$. $E_i^* \mathbf{V}$ is the eigenspace of A^* associated with θ_i^* . We call $E_i^* \mathbf{V}$ the i -th *subconstituent* of Γ with respect to x .

We now recall the Terwilliger algebra. Let $T = T(x)$ denote the subalgebra of $\mathbf{M}_X(\mathbb{C})$ generated by \mathcal{M} and \mathcal{M}^* . T is called the *Terwilliger algebra* (or *subconstituent algebra*) of Γ with respect to x ; see [24]. Note that A, A^* generate T . T is finite-dimensional and noncommutative. Moreover, T is semisimple since it is closed under the conjugate-transpose map. By [24, Lemma 3.2], the following are relations in T . For $0 \leq h, i, j \leq d$,

$$E_i^* A_h E_j^* = 0 \quad \text{if and only if} \quad p_{ij}^h = 0;$$

$$E_i A_h^* E_j = 0 \quad \text{if and only if} \quad q_{ij}^h = 0.$$

Setting $h = 1$ we find that for $0 \leq i, j \leq d$,

$$\begin{aligned} E_i^* A E_j^* &= 0 & \text{if} & \quad |i - j| > 1; \\ E_i A^* E_j &= 0 & \text{if} & \quad |i - j| > 1. \end{aligned} \tag{2.25}$$

By a T -module, we mean a subspace $W \subseteq \mathbf{V}$ such that $BW \subseteq W$ for all $B \in T$. Let W denote a T -module and let U be a T -submodule of W . Since T is closed under the

conjugate-transpose map, the orthogonal complement of U in W is a T -module. Hence W decomposes into an orthogonal direct sum of irreducible T -modules. In particular, \mathbf{V} is an orthogonal direct sum of irreducible T -modules.

Let W denote an irreducible T -module. Then W is a direct sum of the nonzero spaces among $\{E_i^*W\}_{i=0}^d$, and also a direct sum of the nonzero spaces among $\{E_iW\}_{i=0}^d$. By the *endpoint* of W we mean $\min\{i \mid 0 \leq i \leq d, E_i^*W \neq 0\}$. By the *dual endpoint* of W we mean $\min\{i \mid 0 \leq i \leq d, E_iW \neq 0\}$. By the *diameter* of W we mean $|\{i \mid 0 \leq i \leq d, E_i^*W \neq 0\}| - 1$. By the *dual diameter* of W we mean $|\{i \mid 0 \leq i \leq d, E_iW \neq 0\}| - 1$. By [19, Corollary 3.3] the diameter of W is equal to the dual diameter of W . By [24, Lemma 3.9, Lemma 3.12] $\dim E_i^*W \leq 1$ ($0 \leq i \leq d$) if and only if $\dim E_iW \leq 1$ ($0 \leq i \leq d$); in this case W is said to be *thin*.

Lemma 2.2.1. [31, Lemma 14.8] *Let W denote a thin irreducible T -module with endpoint σ , dual endpoint σ^* and diameter ρ . Then the elements*

$$(A; A^*; \{E_i\}_{i=\sigma^*}^{\rho+\sigma^*}; \{E_i^*\}_{i=\sigma}^{\rho+\sigma})$$

act on W as a Leonard system.

We give an example of a thin irreducible T -module.

Example 2.2.2. [24, Lemma 3.6] The all-ones vector \mathbf{j} satisfies $A_i\hat{x} = E_i^*\mathbf{j}$ and $A_i^*\mathbf{j} = |X|E_i\hat{x}$ for $0 \leq i \leq d$. Therefore $\mathcal{M}\hat{x} = \mathcal{M}^*\mathbf{j}$. The space $\mathcal{M}\hat{x}$ is a thin irreducible T -module with endpoint 0, dual endpoint 0, and diameter d . The space $\mathcal{M}\hat{x}$ has bases $\{A_i\hat{x}\}_{i=0}^d$ and $\{E_i\hat{x}\}_{i=0}^d$. We call $\mathcal{M}\hat{x}$ the *primary* T -module.

Lemma 2.2.3. [25, Theorem 4.1] *Let Φ denote the Leonard system on $\mathcal{M}\hat{x}$ from Lemma 2.2.1.*

(i) For $0 \leq i \leq d$, the intersection numbers a_i, b_i, c_i of Φ satisfy

$$a_i = a_i(\Gamma), \quad b_i = b_i(\Gamma), \quad c_i = c_i(\Gamma).$$

(ii) For $0 \leq i \leq d$, the dual intersection numbers a_i^*, b_i^*, c_i^* of Φ satisfy

$$a_i^* = a_i^*(\Gamma), \quad b_i^* = b_i^*(\Gamma), \quad c_i^* = c_i^*(\Gamma).$$

The Leonard system Φ in Lemma 2.2.3 will be called *primary*, and its parameter array will also be called *primary*. Moreover, the intersection numbers and dual intersection numbers of Φ will be called *primary*. From now on, the notation

$$p(\Phi) = (\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d, \{\varphi_i\}_{i=1}^d, \{\phi_i\}_{i=1}^d)$$

refers to the primary parameter array. Furthermore, the notation a_i, b_i, c_i (resp. a_i^*, b_i^*, c_i^*) refer to the primary intersection numbers (resp. dual intersection numbers) of Γ which are also the intersection numbers (resp. dual intersection numbers) of Γ .

We say that Γ has *q-Racah type* whenever the primary Leonard system has *q-Racah type*. For the rest of the thesis, assume that Γ has *q-Racah type*. Since the primary parameter array has *q-Racah type*, it satisfies (2.13)–(2.16) for some scalars h, h^*, s, s^*, r_1, r_2 . We fix this notation for the rest of the thesis.

Note 2.2.4. We mentioned earlier that the intersection number $c_1 = 1$. In (2.19) we set $i = 1$ and use $c_1 = 1$ to get

$$h = \frac{s^* q^d (1 - s^* q^2) (1 - s^* q^3)}{(1 - q) (1 - s^* q^{d+2}) (r_1 - s^* q) (r_2 - s^* q)}. \quad (2.26)$$

To get h^* replace s^* by s in (2.26). In the resulting formula, eliminate s using $r_1 r_2 = s s^* q^{d+1}$ to get

$$h^* = \frac{q^{2d-1} (s^* - r_1 r_2 q^{1-d}) (s^* - r_1 r_2 q^{2-d})}{(1 - q) (s^* - r_1 r_2 q) (r_1 - s^* q^d) (r_2 - s^* q^d)}. \quad (2.27)$$

2.3 Delsarte cliques in Γ

In this section we discuss the basic properties of a Delsarte clique C of Γ . We also discuss the Terwilliger algebra associated with C .

We recall the definition of a Delsarte clique. By a *clique* of Γ we mean a nonempty subset of X such that any two distinct vertices of this subset are adjacent. Let C denote a clique of Γ . We mention an upper bound on $|C|$. Recall that k is the valency of Γ . By [3, Corollary 3.5.4] $\theta_d \leq -1$. By [3, Proposition 4.4.6] $|C| \leq 1 - k/\theta_d$. We say that C is *Delsarte* whenever $|C| = 1 - k/\theta_d$. Throughout the rest of this thesis we will assume that Γ contains a Delsarte clique C . In our study of C , we will use the following fact.

Lemma 2.3.1. [13, Lemma 2.1] *For $0 \leq j \leq d$ and for $y, z \in X$,*

$$\langle E_j \hat{y}, E_j \hat{z} \rangle = |X|^{-1} m_j \mathbf{u}_i(\theta_j), \quad (2.28)$$

where $i = \partial(y, z)$ and \mathbf{u}_i is the polynomial from (2.11) attached to the primary parameter array. Recall that m_j is the multiplicity of θ_j .

Lemma 2.3.2. $E_j \hat{C} \neq 0$ for $0 \leq j \leq d - 1$. Moreover, $E_d \hat{C} = 0$.

Proof. We evaluate $\|E_j \hat{C}\|^2$ for $0 \leq j \leq d$. We find

$$\|E_j \hat{C}\|^2 = \sum_{y, z \in C} \langle E_j \hat{y}, E_j \hat{z} \rangle. \quad (2.29)$$

For $y, z \in C$ consider the corresponding summand in (2.29). First assume that $y = z$. In (2.28) set $i = 0$ and $\mathbf{u}_0 = 1$ to find $\|E_j \hat{y}\|^2 = |X|^{-1} m_j$. Next assume that $y \neq z$. Then y, z are adjacent. In (2.28) set $i = 1$ and $\mathbf{u}_1(\theta_j) = \theta_j/k$ to find $\langle E_j \hat{y}, E_j \hat{z} \rangle = |X|^{-1} m_j \theta_j/k$. Evaluate (2.29) using these comments to get

$$\|E_j \hat{C}\|^2 = |C| \frac{m_j}{|X|} + |C|(|C| - 1) \frac{m_j \theta_j}{|X|k}. \quad (2.30)$$

In (2.30) divide both sides by $|C|$ and use $|C| = 1 - k/\theta_d$ to get

$$\frac{\|E_j \hat{C}\|^2}{|C|} = \frac{m_j(\theta_d - \theta_j)}{|X|\theta_d}.$$

The factor $\theta_d - \theta_j$ is nonzero for $0 \leq j \leq d-1$, and zero for $j = d$. The result follows. ■

By a *partition* of X , we mean a set of mutually disjoint non-empty subsets of X whose union is X . For $y \in X$, define $\partial(y, C) = \min\{\partial(y, z) \mid z \in C\}$. By the *covering radius* of C we mean $\max\{\partial(y, C) \mid y \in X\}$. By [10, p. 277] the covering radius of C is $d - 1$. For $0 \leq i \leq d - 1$ define

$$C_i := \{y \in X \mid \partial(y, C) = i\}. \quad (2.31)$$

Observe that $C_0 = C$. Note that $\{C_i\}_{i=0}^{d-1}$ is the partition of X , and hence $\mathbf{j} = \sum_{i=0}^{d-1} \hat{C}_i$. Shortly we will show that this partition is equitable in the sense of [10, p. 75].

Lemma 2.3.3. [3, Corollary 4.1.2] *Let $\{\mathbf{u}_i\}_{i=0}^d$ be the polynomials from (2.11) attached to the primary parameter array. For $0 \leq i \leq d - 1$, we have $(-1)^i \mathbf{u}_i(\theta_d) > 0$.*

For $0 \leq i \leq d - 1$ and $z \in C_i$, define

$$N_i(z) = |\Gamma_i(z) \cap C|. \quad (2.32)$$

The following lemma shows that $N_i(z)$ is independent of the choice of z .

Lemma 2.3.4. *For $0 \leq i \leq d - 1$ and $z \in C_i$,*

$$N_i(z) = |C| \frac{\mathbf{u}_{i+1}(\theta_d)}{\mathbf{u}_{i+1}(\theta_d) - \mathbf{u}_i(\theta_d)}, \quad (2.33)$$

where $\{\mathbf{u}_j\}_{j=0}^d$ are the polynomials from (2.11) attached to the primary parameter array.

In (2.33) the denominator is nonzero by Lemma 2.3.3.

Proof. For notational convenience abbreviate $N_i = N_i(z)$. Observe that

$$\langle E_d \hat{z}, E_d \hat{C} \rangle = \sum_{y \in C} \langle E_d \hat{z}, E_d \hat{y} \rangle. \quad (2.34)$$

The left-hand side of (2.34) is zero by Lemma 2.3.2. Concerning the right-hand side of (2.34), among $y \in C$ exactly N_i are contained in $\Gamma_i(z)$ by (2.32), and for such y the summand in (2.34) becomes $|X|^{-1} m_d \mathbf{u}_i(\theta_d)$ by (2.28). The remaining $y \in C$ are contained in $\Gamma_{i+1}(z)$, and for such y the summand in (2.34) becomes $|X|^{-1} m_d \mathbf{u}_{i+1}(\theta_d)$ by (2.28). By these comments, line (2.34) becomes

$$0 = N_i |X|^{-1} m_d \mathbf{u}_i(\theta_d) + (|C| - N_i) |X|^{-1} m_d \mathbf{u}_{i+1}(\theta_d).$$

Solving the above equation for N_i , we obtain (2.33). ■

Referring to Lemma 2.3.4, for $0 \leq i \leq d-1$ the scalar $N_i(z)$ is independent of z . Therefore we define $N_i = N_i(z)$. By construction $N_0 = 1$. For notational convenience, define $N_{-1} = 0$. Recall the scalars h, s, s^*, r_1, r_2 from above Note 2.2.4.

Lemma 2.3.5. *For $0 \leq i \leq d-1$ both*

$$N_i = \frac{h(q^d - 1)(r_1 - s^* q^{i+1})(r_2 - s^* q^{i+1})}{\theta_d s^* q^d (1 - s^* q^{2i+2})}, \quad (2.35)$$

$$|C| - N_i = -\frac{h(q^d - 1)(1 - r_1 q^{i+1})(1 - r_2 q^{i+1})}{\theta_d q^d (1 - s^* q^{2i+2})}. \quad (2.36)$$

Proof. We first show (2.35). To do this, evaluate (2.33) using $|C| = 1 - k/\theta_d$ and (2.12).

Simplify the result to find

$$N_i = \frac{k - \theta_d}{\theta_d} \frac{\phi_{i+1}}{\varphi_{i+1} - \phi_{i+1}}. \quad (2.37)$$

By [26, Lemma 6.5],

$$\varphi_{i+1} - \phi_{i+1} = (\theta_{i+1}^* - \theta_i^*) \sum_{h=0}^i (\theta_h - \theta_{d-h}).$$

Also by [26, Lemma 10.2],

$$\sum_{h=0}^i \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} = \frac{(q^{i+1} - 1)(q^{d-i} - 1)}{(q - 1)(q^d - 1)}.$$

Recall $k = \theta_0$. Evaluate (2.37) using these comments and simplify the result to find

$$N_i = \frac{\phi_{i+1}(q - 1)(q^d - 1)}{\theta_d(\theta_{i+1}^* - \theta_i^*)(q^{i+1} - 1)(q^{d-i} - 1)}. \quad (2.38)$$

In (2.38), evaluate $\theta_{i+1}^* - \theta_i^*$ using (2.14) and evaluate ϕ_{i+1} using (2.16). Simplify the result to get (2.35).

We now verify (2.36). In the equation $|C| = 1 - k/\theta_d$, evaluate the right-hand side using (2.13) and $k = \theta_0$ to get

$$|C| = h(1 - q^d)(1 - sq^{d+1})/\theta_d q^d. \quad (2.39)$$

Combine this with (2.35) and eliminate s using $r_1 r_2 = ss^* q^{d+1}$. Simplify the result to get (2.36). ■

Corollary 2.3.6. *We have $0 < N_i < |C|$ for $0 \leq i \leq d - 1$.*

Proof. By (2.32), $0 \leq N_i \leq |C|$. We show $N_i \neq 0$ and $N_i \neq |C|$. To do this we use Lemma 2.3.5. In lines (2.35), (2.36) each factor is nonzero by the inequalities in Example 2.1.8. The result follows. ■

For $0 \leq i \leq d - 1$ and $z \in C_i$, define

$$\tilde{c}_i(z) = |\Gamma(z) \cap C_{i-1}|, \quad \tilde{a}_i(z) = |\Gamma(z) \cap C_i|, \quad \tilde{b}_i(z) = |\Gamma(z) \cap C_{i+1}|, \quad (2.40)$$

where C_{-1} and C_d are empty sets. Observe

$$\tilde{c}_i(z) + \tilde{a}_i(z) + \tilde{b}_i(z) = k. \quad (2.41)$$

The following theorem shows that $\tilde{a}_i(z), \tilde{b}_i(z), \tilde{c}_i(z)$ are independent of z .

Theorem 2.3.7. *The following (i), (ii) hold:*

(i) *For $1 \leq i \leq d-1$ and $z \in C_i$,*

$$\tilde{c}_i(z) = \frac{N_i}{N_{i-1}}c_i; \quad (2.42)$$

(ii) *For $0 \leq i \leq d-2$ and $z \in C_i$,*

$$\tilde{b}_i(z) = \frac{|C| - N_i}{|C| - N_{i+1}}b_{i+1}. \quad (2.43)$$

Proof. (i) Let m denote the number of ordered pairs (y, w) such that $y \in \Gamma_i(z) \cap C$ and $w \in \Gamma(z) \cap C_{i-1}$ and $\partial(y, w) = i-1$. We compute m in two ways. First, there are N_i choices for y since $|\Gamma_i(z) \cap C| = N_i$. For $y \in \Gamma_i(z) \cap C$ there are exactly c_i vertices $w \in \Gamma(z) \cap C_{i-1}$ such that $\partial(y, w) = i-1$. Therefore $m = N_i c_i$. Secondly, there are $\tilde{c}_i(z)$ choices for w since $|\Gamma(z) \cap C_{i-1}| = \tilde{c}_i(z)$. For $w \in \Gamma(z) \cap C_{i-1}$ there are exactly N_{i-1} vertices $y \in \Gamma_i(z) \cap C$ such that $\partial(y, w) = i-1$. Therefore $m = \tilde{c}_i(z) N_{i-1}$. By these comments $N_i c_i = \tilde{c}_i(z) N_{i-1}$. The result follows.

(ii) In a similar manner to (i), compute the number of ordered pairs (y, w) such that $y \in \Gamma_{i+1}(z) \cap C$ and $w \in \Gamma(z) \cap C_{i+1}$ and $\partial(y, w) = i+2$. Use $|\Gamma_{i+1}(z) \cap C| = |C| - N_i$ and $|\Gamma(z) \cap C_{i+1}| = \tilde{b}_i(z)$ to get the result. \blacksquare

By (2.41) and Theorem 2.3.7, for $0 \leq i \leq d-1$ and $z \in C_i$ the scalars $\tilde{a}_i(z)$, $\tilde{b}_i(z)$, $\tilde{c}_i(z)$ are independent of z . We define

$$\tilde{a}_i = \tilde{a}_i(z), \quad \tilde{b}_i = \tilde{b}_i(z), \quad \tilde{c}_i = \tilde{c}_i(z). \quad (2.44)$$

Note that

$$\tilde{c}_0 = 0, \quad \tilde{a}_0 = |C| - 1, \quad \tilde{b}_0 = k - |C| + 1,$$

and $\tilde{b}_{d-1} = 0$. For notational convenience, define $\tilde{b}_{-1} = 0$ and $\tilde{c}_d = 0$.

Corollary 2.3.8. [3, Section 11.1] *The partition $\{C_i\}_{i=0}^{d-1}$ is equitable.*

Proof. Immediate from (2.41) and Theorem 2.3.7. ■

Corollary 2.3.9. *The following hold.*

$$\tilde{b}_0 = \frac{h(1 - q^{-d+1})(1 - r_1q)(1 - r_2q)}{(1 - s^*q^3)}, \quad (2.45)$$

$$\tilde{b}_i = \frac{h(1 - q^{i-d+1})(1 - s^*q^{i+2})(1 - r_1q^{i+1})(1 - r_2q^{i+1})}{(1 - s^*q^{2i+2})(1 - s^*q^{2i+3})} \quad (1 \leq i \leq d-2), \quad (2.46)$$

$$\tilde{c}_i = \frac{h(1 - q^i)(1 - s^*q^{i+d+1})(r_1 - s^*q^{i+1})(r_2 - s^*q^{i+1})}{s^*q^d(1 - s^*q^{2i+1})(1 - s^*q^{2i+2})} \quad (1 \leq i \leq d-2), \quad (2.47)$$

$$\tilde{c}_{d-1} = \frac{h(1 - q^{d-1})(r_1 - s^*q^d)(r_2 - s^*q^d)}{s^*q^d(1 - s^*q^{2d-1})}. \quad (2.48)$$

Proof. To get (2.45), (2.46) evaluate (2.43) using (2.18), (2.36) and simplify the result.

Lines (2.47), (2.48) are similarly obtained using (2.19), (2.35). ■

Lemma 2.3.10. *We have $\tilde{b}_i \neq 0$ for $0 \leq i \leq d-2$ and $\tilde{c}_i \neq 0$ for $1 \leq i \leq d-1$.*

Proof. Either use Corollary 2.3.6 and Theorem 2.3.7 or use Example 2.1.8 and Corollary 2.3.9. ■

We now recall the Terwilliger algebra associated with C [21]. For $0 \leq i \leq d-1$, let \tilde{E}_i^* denote the diagonal matrix in $\mathbf{M}_X(\mathbb{C})$ with (y, y) -entry

$$(\tilde{E}_i^*)_{yy} = \begin{cases} 1 & \text{if } y \in C_i \\ 0 & \text{if } y \notin C_i \end{cases} \quad (y \in X). \quad (2.49)$$

We call \tilde{E}_i^* the i -th *dual primitive idempotent* of Γ with respect to C . Observe that (i) $I = \sum_{i=0}^{d-1} \tilde{E}_i^*$; (ii) $\overline{\tilde{E}_i^*} = \tilde{E}_i^*$ ($0 \leq i \leq d-1$); (iii) $(\tilde{E}_i^*)^t = \tilde{E}_i^*$ ($0 \leq i \leq d-1$); (iv) $\tilde{E}_i^* \tilde{E}_j^* = \delta_{ij} \tilde{E}_i^*$ ($0 \leq i, j \leq d-1$). By these facts, $\{\tilde{E}_i^*\}_{i=0}^{d-1}$ forms a basis for a commutative

subalgebra $\tilde{\mathcal{M}}^*$ of $\mathbf{M}_X(\mathbb{C})$. The algebra $\tilde{\mathcal{M}}^*$ is semisimple since it is closed under the conjugate-transpose map. We comment on how \mathcal{M}^* and $\tilde{\mathcal{M}}^*$ are related. For these subalgebras each element is a diagonal matrix. Therefore any element of \mathcal{M}^* commutes with any element of $\tilde{\mathcal{M}}^*$.

Define the diagonal matrix $\tilde{A}^* \in \mathbf{M}_X(\mathbb{C})$ by

$$\tilde{A}^* = |C|^{-1} \sum_{y \in C} A^*(y). \quad (2.50)$$

We call \tilde{A}^* the *dual adjacency matrix of Γ with respect to C* .

Lemma 2.3.11. *With the above notation,*

$$\tilde{A}^* = \sum_{i=0}^{d-1} \tilde{\theta}_i^* \tilde{E}_i^*, \quad (2.51)$$

where

$$\tilde{\theta}_i^* = \frac{N_i \theta_i^* + (|C| - N_i) \theta_{i+1}^*}{|C|} \quad (0 \leq i \leq d-1). \quad (2.52)$$

Proof. For $0 \leq i \leq d-1$ and $z \in C_i$ it suffices to show that $\tilde{\theta}_i^*$ is the (z, z) -entry of \tilde{A}^* . Consider the right-hand side in (2.50). Among $y \in C$ there are exactly N_i with $\partial(y, z) = i$, and for such y the (z, z) -entry of $A^*(y)$ is θ_i^* . The remaining $y \in C$ satisfy $\partial(y, z) = i+1$, and for such y the (z, z) -entry of $A^*(y)$ is θ_{i+1}^* . The result follows. ■

Corollary 2.3.12. $\tilde{A}^* \in \tilde{\mathcal{M}}^*$. Moreover $\{\tilde{\theta}_i^*\}_{i=0}^{d-1}$ are the eigenvalues of \tilde{A}^* .

Proof. Immediate from Lemma 2.3.11. ■

Lemma 2.3.13. *Referring to (2.52),*

$$\tilde{\theta}_i^* = \tilde{\theta}_0^* + \tilde{h}^*(1 - q^i)(1 - \tilde{s}^* q^{i+1}) q^{-i} \quad (0 \leq i \leq d-1), \quad (2.53)$$

where

$$\tilde{s}^* = s^*q, \quad \tilde{h}^* = \frac{s^*q^{-1}-r_1r_2}{s^*-r_1r_2}h^*, \quad (2.54)$$

$$\tilde{\theta}_0^* = \theta_0^* + h^* \left(\frac{s^*(q-1)(r_1+r_2)}{s^*-r_1r_2} + \frac{(s^*q^{-1}-r_1r_2)(1+s^*q^2)}{s^*-r_1r_2} - 1 - s^*q \right). \quad (2.55)$$

Moreover \tilde{h}^* is nonzero.

Proof. To get (2.53) evaluate (2.52) using (2.14) and Lemma 2.3.5 together with (2.39), and simplify. We now show $\tilde{h}^* \neq 0$. Since $h^* \neq 0$, it suffices to show that $r_1r_2 \neq s^*q^{-1}$. By Example 2.1.8 $r_1r_2 = ss^*q^{d+1}$ and $sq^i \neq 1$ for $2 \leq i \leq 2d$. Recall $d \geq 3$. The result follows. \blacksquare

Lemma 2.3.14. *Referring to Lemma 2.3.13,*

$$\tilde{\theta}_i^* - \tilde{\theta}_j^* = \tilde{h}^*(1 - q^{i-j})(1 - s^*q^{i+j+2})q^{-i} \quad (0 \leq i, j \leq d-1).$$

Proof. Routine using (2.53) and the first equation in (2.54). \blacksquare

Corollary 2.3.15. *The scalars $\{\tilde{\theta}_i^*\}_{i=0}^{d-1}$ are mutually distinct. Moreover \tilde{A}^* generates $\tilde{\mathcal{M}}^*$.*

Proof. To obtain the first assertion, use Lemma 2.3.14. By Lemma 2.3.13 $\tilde{h}^* \neq 0$. By Example 2.1.8 $q^i \neq 1$ for $1 \leq i \leq d$ and $s^*q^i \neq 1$ for $2 \leq i \leq 2d$. The second assertion follows from the first assertion. \blacksquare

Let \tilde{T} denote the subalgebra of $\mathbf{M}_X(\mathbb{C})$ generated by \mathcal{M} and $\tilde{\mathcal{M}}^*$. \tilde{T} is called the *Terwilliger algebra of Γ with respect to C* [21]. The algebra \tilde{T} is finite-dimensional and noncommutative. Observe that \tilde{T} is generated by A, \tilde{A}^* . \tilde{T} is semisimple since it is closed under the conjugate-transpose map. By a \tilde{T} -module we mean a subspace $W \subseteq \mathbf{V}$

such that $BW \subseteq W$ for all $B \in \tilde{T}$. Let W denote a \tilde{T} -module. Since \tilde{T} is closed under the conjugate-transpose map, W is an orthogonal direct sum of irreducible \tilde{T} -modules. In particular, \mathbf{V} is an orthogonal direct sum of irreducible \tilde{T} -modules. For more background information on \tilde{T} we refer the reader to [21].

Lemma 2.3.16. *The following hold:*

- (i) $\hat{C}_i = \tilde{E}_i^* \mathbf{j}$ ($0 \leq i \leq d-1$).
- (ii) $A_i \hat{C} = (|C| - N_{i-1}) \hat{C}_{i-1} + N_i \hat{C}_i$ ($0 \leq i \leq d-1$).
- (iii) $A_d \hat{C} = (|C| - N_{d-1}) \hat{C}_{d-1}$.
- (iv) $\mathcal{M} \hat{C} = \tilde{\mathcal{M}}^* \mathbf{j}$.
- (v) $\mathcal{M} \hat{C}$ has bases $\{\hat{C}_i\}_{i=0}^{d-1}$, $\{E_i \hat{C}\}_{i=0}^{d-1}$, and $\{A_i \hat{C}\}_{i=0}^{d-1}$.
- (vi) $\mathcal{M} \hat{C}$ is an irreducible \tilde{T} -module.

We call $\mathcal{M} \hat{C}$ the primary \tilde{T} -module.

Proof. (i) Use (2.49).

(ii) Assume $i \neq 0$; otherwise the result is trivial. For $z \in C_{i-1}$, there are precisely $|C| - N_{i-1}$ vertices $y \in C$ with $\partial(y, z) = i$. For $z \in C_i$, there are precisely N_i vertices $y \in C$ with $\partial(y, z) = i$. The equation follows.

(iii) Similar to (ii).

(iv) By construction $\{\tilde{E}_i^* \mathbf{j}\}_{i=0}^{d-1}$ forms a basis for $\tilde{\mathcal{M}}^* \mathbf{j}$, so $\tilde{\mathcal{M}}^* \mathbf{j}$ has dimension d . Recall $\{A_i\}_{i=0}^d$ spans \mathcal{M} so $\{A_i \hat{C}\}_{i=0}^d$ spans $\mathcal{M} \hat{C}$. By (i)–(iii) $A_i \hat{C} \in \tilde{\mathcal{M}}^* \mathbf{j}$ for $0 \leq i \leq d$. Therefore $\mathcal{M} \hat{C} \subseteq \tilde{\mathcal{M}}^* \mathbf{j}$. By Lemma 2.3.2 $\{E_i \hat{C}\}_{i=0}^{d-1}$ forms a basis for $\mathcal{M} \hat{C}$. Therefore $\mathcal{M} \hat{C}$ has dimension d . By these comments, $\mathcal{M} \hat{C} = \tilde{\mathcal{M}}^* \mathbf{j}$.

(v) In the proof of (iv), we saw that $\{\tilde{E}_i^* \mathbf{j}\}_{i=0}^{d-1}$ is a basis for $\mathcal{M}\hat{C}$. By this and (i) the set $\{\hat{C}_i\}_{i=0}^{d-1}$ forms a basis for $\mathcal{M}\hat{C}$. Also, in the proof of (iv) we saw that $\{E_i \hat{C}\}_{i=0}^{d-1}$ is a basis for $\mathcal{M}\hat{C}$. We now show that $\{A_i \hat{C}\}_{i=0}^{d-1}$ is a basis for $\mathcal{M}\hat{C}$. By construction $\text{Span}\{A_i \hat{C}\}_{i=0}^{d-1}$ is a subspace of $\mathcal{M}\hat{C}$. Since $\mathcal{M}\hat{C}$ has dimension d , it suffices to show that the vectors $\{A_i \hat{C}\}_{i=0}^{d-1}$ are linearly independent. Note that the $\{N_i\}_{i=0}^{d-1}$ are nonzero by Corollary 2.3.6 and $\{\hat{C}_i\}_{i=0}^{d-1}$ are linearly independent. By these comments and (ii), the vector $\{A_i \hat{C}\}_{i=0}^{d-1}$ are linearly independent.

(vi) By (iv), $\mathcal{M}\hat{C}$ is a \tilde{T} -module. We show the \tilde{T} -module $\mathcal{M}\hat{C}$ is irreducible. Express $\mathcal{M}\hat{C}$ as the orthogonal direct sum of irreducible \tilde{T} -modules. Since $\hat{C} \in \mathcal{M}\hat{C}$, among these irreducible \tilde{T} -modules there exists one, denoted W , that is not orthogonal to \hat{C} . Observe $\tilde{E}_0^* W \neq 0$. Also $\tilde{E}_0^* W \subseteq \tilde{E}_0^* \mathcal{M}\hat{C} = \text{Span}\{\hat{C}\}$, so $\tilde{E}_0^* W = \text{Span}\{\hat{C}\}$. By this and since W is a \tilde{T} -module, we have $\hat{C} \in \tilde{E}_0^* W \subseteq W$. But then $\mathcal{M}\hat{C} \subseteq W$ and thus $\mathcal{M}\hat{C} = W$ by the irreducibility of W . ■

Corollary 2.3.17.

- (i) A is multiplicity-free on $\mathcal{M}\hat{C}$ with eigenvalues $\{\theta_i\}_{i=0}^{d-1}$.
- (ii) \tilde{A}^* is multiplicity-free on $\mathcal{M}\hat{C}$ with eigenvalues $\{\tilde{\theta}_i^*\}_{i=0}^{d-1}$.

Proof. (i) Follows from Lemma 2.3.16(v).

(ii) Follows from Lemma 2.3.16(i), (v). ■

Lemma 2.3.18. *Consider the matrices*

$$(A; \tilde{A}^*; \{E_i\}_{i=0}^{d-1}; \{\tilde{E}_i^*\}_{i=0}^{d-1}). \tag{2.56}$$

Then the elements (2.56) act on $\mathcal{M}\hat{C}$ as a Leonard system.

Proof. We show that (2.56) satisfies conditions (i)–(v) in Definition 2.1.1. Conditions (i)–(iii) follow from Corollary 2.3.17, condition (iv) is from (2.25) and (2.50), and condition (v) follows from [21, Lemma 4.1]. \blacksquare

We let $\tilde{\Phi}$ denote the Leonard system on $\mathcal{M}\hat{C}$ from Lemma 2.3.18. We call $\tilde{\Phi}$ the *primary Leonard system with respect to C* . Recall the basis $\{\hat{C}_i\}_{i=0}^{d-1}$ for $\mathcal{M}\hat{C}$ from Lemma 2.3.16(v).

Lemma 2.3.19. *The vectors $\{\hat{C}_i\}_{i=0}^{d-1}$ form a $\tilde{\Phi}$ -standard basis for $\mathcal{M}\hat{C}$.*

Proof. We invoke Lemma 2.1.3. Abbreviate $W = \mathcal{M}\hat{C}$. By Lemma 2.3.16(i) we have $\hat{C}_i \in \tilde{E}_i^*W$ for $0 \leq i \leq d-1$. By Lemma 2.3.16(i) and the construction, $\sum_{j=0}^{d-1} \hat{C}_j = \mathbf{j} \in E_0W$. The result follows in view of Lemma 2.1.3. \blacksquare

By Lemma 2.3.11, the matrix representing \tilde{A}^* relative to the basis $\{\hat{C}_i\}_{i=0}^{d-1}$ is

$$\text{diag}(\tilde{\theta}_0^*, \tilde{\theta}_1^*, \tilde{\theta}_2^*, \dots, \tilde{\theta}_{d-1}^*). \quad (2.57)$$

By (2.40) and (2.44), the matrix representing A relative to $\{\hat{C}_i\}_{i=0}^{d-1}$ is

$$\begin{bmatrix} \tilde{a}_0 & \tilde{b}_0 & & & \mathbf{0} \\ \tilde{c}_1 & \tilde{a}_1 & \tilde{b}_1 & & \\ & \tilde{c}_2 & \tilde{a}_2 & \ddots & \\ & & \ddots & \ddots & \tilde{b}_{d-2} \\ \mathbf{0} & & & \tilde{c}_{d-1} & \tilde{a}_{d-1} \end{bmatrix}. \quad (2.58)$$

Comparing (2.7) and (2.58) we see that the $\tilde{a}_i, \tilde{b}_i, \tilde{c}_i$ are the intersection numbers of $\tilde{\Phi}$.

We now consider the parameter array of $\tilde{\Phi}$. We denote this parameter array by

$$p(\tilde{\Phi}) = (\{\tilde{\theta}_i\}_{i=0}^{d-1}; \{\tilde{\theta}_i^*\}_{i=0}^{d-1}; \{\tilde{\varphi}_i\}_{i=1}^{d-1}; \{\tilde{\phi}_i\}_{i=1}^{d-1}).$$

The scalars $\{\tilde{\theta}_i^*\}_{i=0}^{d-1}$ were evaluated in Lemma 2.3.13. We now consider $\{\tilde{\theta}_i\}_{i=0}^{d-1}$.

Lemma 2.3.20. *We have*

$$\tilde{\theta}_i = \theta_i \quad (0 \leq i \leq d-1).$$

Proof. By construction, $AE_i = \tilde{\theta}_i E_i$ on $\mathcal{M}\hat{C}$. But $AE_i = \theta_i E_i$, so $\tilde{\theta}_i = \theta_i$. ■

Recall the scalars h, h^*, s, s^*, r_1, r_2 from above Note 2.2.4

Theorem 2.3.21. *For $0 \leq i \leq d-1$,*

$$\tilde{\theta}_i = \tilde{\theta}_0 + \tilde{h}(1-q^i)(1-\tilde{s}q^{i+1})q^{-i}, \quad (2.59)$$

$$\tilde{\theta}_i^* = \tilde{\theta}_0^* + \tilde{h}^*(1-q^i)(1-\tilde{s}^*q^{i+1})q^{-i}, \quad (2.60)$$

and for $1 \leq i \leq d-1$,

$$\tilde{\varphi}_i = \tilde{h}\tilde{h}^*q^{1-2i}(1-q^i)(1-q^{i-d})(1-\tilde{r}_1q^i)(1-\tilde{r}_2q^i), \quad (2.61)$$

$$\tilde{\phi}_i = \tilde{h}\tilde{h}^*q^{1-2i}(1-q^i)(1-q^{i-d})(\tilde{r}_1 - \tilde{s}^*q^i)(\tilde{r}_2 - \tilde{s}^*q^i)/\tilde{s}^*, \quad (2.62)$$

where

$$\tilde{h} = h, \quad \tilde{s} = s, \quad \tilde{r}_1 = r_1, \quad (2.63)$$

$$\tilde{h}^* = \frac{s^*q^{-1}-r_1r_2}{s^*-r_1r_2}h^*, \quad \tilde{s}^* = s^*q, \quad \tilde{r}_2 = r_2, \quad (2.64)$$

and where $\tilde{\theta}_0^*$ is from (2.55).

Proof. The Leonard system $\tilde{\Phi}$ has diameter $d-1$. To verify (2.59), evaluate each term using (2.13) and Lemma 2.3.20. Line (2.60) is from Lemma 2.3.13. To verify (2.61), (2.62) use Lemma 2.1.4, Lemma 2.3.14 and (2.45)–(2.48) along with (2.63), (2.64). ■

Corollary 2.3.22. *The Leonard system $\tilde{\Phi}$ has q -Racah type.*

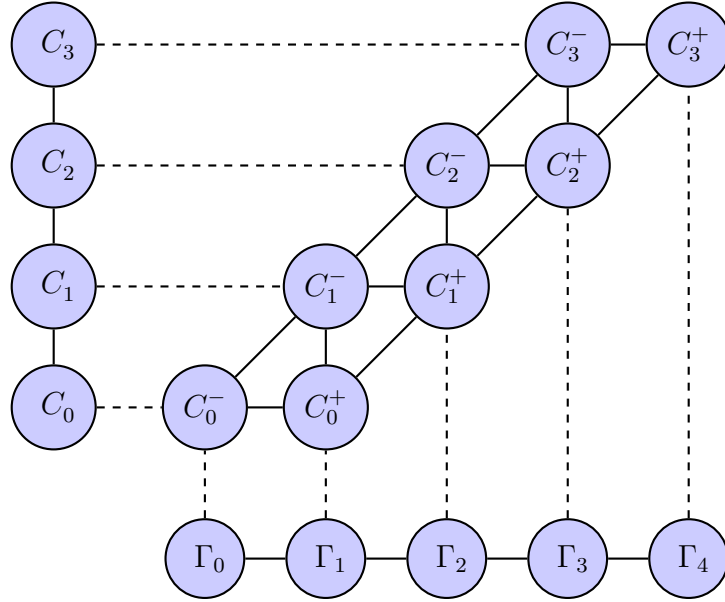
Proof. Compare Example 2.1.8 and Theorem 2.3.21. ■

2.4 The subspace \mathbf{W}

We continue to discuss the Delsarte clique C of Γ . For the rest of the thesis fix a vertex $x \in C$. In this section, using x and C we will construct a certain partition of X . Using this partition we construct a subspace \mathbf{W} of \mathbf{V} which has a module structure for both $T = T(x)$ and \tilde{T} . Recall the partition $\{C_i\}_{i=0}^{d-1}$ of X from (2.31). For $0 \leq i \leq d-1$ define

$$C_i^- = C_i \cap \Gamma_i, \quad C_i^+ = C_i \cap \Gamma_{i+1}, \quad (2.65)$$

where $\Gamma_j = \Gamma_j(x)$ for $0 \leq j \leq d$. For notational convenience, define $C_{-1}^\pm = \emptyset$ and $C_d^\pm = \emptyset$. Observe that $C_i = C_i^- \cup C_i^+$ for $0 \leq i \leq d-1$. Also $\Gamma_i = C_{i-1}^+ \cup C_i^-$ for $1 \leq i \leq d-1$ and $\Gamma_0 = C_0^- = \{x\}, \Gamma_d = C_{d-1}^+$. We visualize the $\{C_i^\pm\}_{i=0}^{d-1}$ as follows:



Example: The sets $\{C_i^\pm\}_{i=0}^{d-1}$ of X when $d = 4$.

Lemma 2.4.1. *The following (i)–(iii) hold.*

- (i) For $z \in C_0^-$,

- z is adjacent to precisely 0 vertices in C_0^- ,
 - z is adjacent to precisely $b_0 - \tilde{b}_0$ vertices in C_0^+ ,
 - z is adjacent to precisely \tilde{b}_0 vertices in C_1^- .
- (ii) For $1 \leq i \leq d-2$ and $z \in C_i^-$,
- z is adjacent to precisely c_i vertices in C_{i-1}^- ,
 - z is adjacent to precisely $\tilde{c}_i - c_i$ vertices in C_{i-1}^+ ,
 - z is adjacent to precisely $a_i - \tilde{c}_i + c_i$ vertices in C_i^- ,
 - z is adjacent to precisely $b_i - \tilde{b}_i$ vertices in C_i^+ ,
 - z is adjacent to precisely \tilde{b}_i vertices in C_{i+1}^- .
- (iii) For $z \in C_{d-1}^-$,
- z is adjacent to precisely c_{d-1} vertices in C_{d-2}^- ,
 - z is adjacent to precisely $\tilde{c}_{d-1} - c_{d-1}$ vertices in C_{d-2}^+ ,
 - z is adjacent to precisely $a_{d-1} - \tilde{c}_{d-1} + c_{d-1}$ vertices in C_{d-1}^- ,
 - z is adjacent to precisely b_{d-1} vertices in C_{d-1}^+ .

Proof. Routine using (2.65). ■

Lemma 2.4.2. *The following (i)–(iii) hold.*

- (i) For $z \in C_0^+$,
- z is adjacent to precisely c_1 vertex in C_0^- ,
 - z is adjacent to precisely $\tilde{a}_0 - c_1$ vertices in C_0^+ ,
 - z is adjacent to precisely $\tilde{b}_0 - b_1$ vertices in C_1^- ,
 - z is adjacent to precisely b_1 vertices in C_1^+ .
- (ii) For $1 \leq i \leq d-2$ and $z \in C_i^+$,

- z is adjacent to precisely \tilde{c}_i vertices in C_{i-1}^+ ,
- z is adjacent to precisely $c_{i+1} - \tilde{c}_i$ vertices in C_i^- ,
- z is adjacent to precisely $\tilde{a}_i - c_{i+1} + \tilde{c}_i$ vertices in C_i^+ ,
- z is adjacent to precisely $\tilde{b}_i - b_{i+1}$ vertices in C_{i+1}^- ,
- z is adjacent to precisely b_{i+1} vertices in C_{i+1}^+ .

(iii) For $z \in C_{d-1}^+$,

- z is adjacent to precisely \tilde{c}_{d-1} vertices in C_{d-2}^+ ,
- z is adjacent to precisely $c_d - \tilde{c}_{d-1}$ vertices in C_{d-1}^- ,
- z is adjacent to precisely $\tilde{a}_{d-1} - c_d + \tilde{c}_{d-1}$ vertices in C_{d-1}^+ .

Proof. Routine using (2.65). ■

Corollary 2.4.3. *The following (i)–(iv) hold.*

- (i) $\tilde{b}_i |C_i^-| = c_{i+1} |C_{i+1}^-|$ for $0 \leq i \leq d-2$.
- (ii) $b_{i+1} |C_i^+| = \tilde{c}_{i+1} |C_{i+1}^+|$ for $0 \leq i \leq d-2$.
- (iii) $(b_i - \tilde{b}_i) |C_i^-| = (c_{i+1} - \tilde{c}_i) |C_i^+|$ for $0 \leq i \leq d-1$.
- (iv) $(\tilde{b}_i - b_{i+1}) |C_i^+| = (\tilde{c}_{i+1} - c_{i+1}) |C_{i+1}^-|$ for $0 \leq i \leq d-2$.

Proof. (i) By the data of Lemma 2.4.1, every vertex in C_i^- is adjacent to precisely \tilde{b}_i vertices in C_{i+1}^- and every vertex in C_{i+1}^- is adjacent to precisely c_{i+1} vertices in C_i^- . The result follows.

(ii)–(iv) Similar to (i). ■

We now find the cardinality for each of $\{C_i^\pm\}_{i=0}^{d-1}$.

Lemma 2.4.4. For $0 \leq i \leq d-1$,

$$|C_i^-| = \frac{\tilde{b}_0 \tilde{b}_1 \cdots \tilde{b}_{i-1}}{c_1 c_2 \cdots c_i}, \quad |C_i^+| = \frac{b_1 b_2 \cdots b_i}{\tilde{c}_1 \tilde{c}_2 \cdots \tilde{c}_i} (|C| - 1).$$

Proof. To get the equation on the left, use Corollary 2.4.3(i). To get the equation on the right, use Corollary 2.4.3(ii). ■

Corollary 2.4.5. For $0 \leq i \leq d-1$, each of C_i^\pm is nonempty.

Proof. By Lemma 2.3.10 and Lemma 2.4.4. ■

By Corollary 2.4.5 and the construction, the $\{C_i^\pm\}_{i=0}^{d-1}$ is a partition of X .

Proposition 2.4.6. The partition $\{C_i^\pm\}_{i=0}^{d-1}$ of X is equitable.

Proof. By Lemma 2.4.1 and Lemma 2.4.2. ■

Recall the standard module \mathbf{V} from above line (2.23). Using the equitable partition $\{C_i^\pm\}_{i=0}^{d-1}$ we get a subspace \mathbf{W} of \mathbf{V} as follows. For $0 \leq i \leq d-1$, recall the characteristic vector \hat{C}_i^\pm of C_i^\pm . Let \mathbf{W} denote the subspace of \mathbf{V} spanned by $\{\hat{C}_i^\pm\}_{i=0}^{d-1}$. Note that \mathbf{W} contains the vectors \hat{x}, \hat{C} . Recall the all 1's vector $\mathbf{j} = \sum_{i=0}^{d-1} \hat{C}_i$ from below (2.31). The subspace \mathbf{W} contains \mathbf{j} since

$$\hat{C}_i = \hat{C}_i^- + \hat{C}_i^+ \quad (0 \leq i \leq d-1). \quad (2.66)$$

Lemma 2.4.7. The vectors $\{\hat{C}_i^\pm\}_{i=0}^{d-1}$ form an orthogonal basis for \mathbf{W} . Moreover,

$$\|\hat{C}_i^-\|^2 = \frac{\tilde{b}_0 \tilde{b}_1 \cdots \tilde{b}_{i-1}}{c_1 c_2 \cdots c_i}, \quad \|\hat{C}_i^+\|^2 = \frac{b_1 b_2 \cdots b_i}{\tilde{c}_1 \tilde{c}_2 \cdots \tilde{c}_i} (|C| - 1).$$

Proof. By construction the $\{C_i^\pm\}_{i=0}^{d-1}$ are mutually disjoint, so the $\{\hat{C}_i^\pm\}_{i=0}^{d-1}$ are mutually orthogonal. By Corollary 2.4.5, each of $\{C_i^\pm\}_{i=0}^{d-1}$ is nonzero. The first assertion follows from these comments. The second assertion follows from Lemma 2.4.4. ■

Corollary 2.4.8. *The dimension of \mathbf{W} is $2d$.*

Proof. Immediate from Lemma 2.4.7. ■

In Lemma 2.4.7 we gave a basis for \mathbf{W} . We now give the action of A on this basis.

Recall $C_{-1}^{\pm} = \emptyset$ and $C_d^{\pm} = \emptyset$, so

$$\hat{C}_{-1}^{-} = 0, \quad \hat{C}_{-1}^{+} = 0, \quad \hat{C}_d^{-} = 0, \quad \hat{C}_d^{+} = 0.$$

Lemma 2.4.9. *The element A acts on $\{\hat{C}_i^{\pm}\}_{i=0}^{d-1}$ as follows: for $0 \leq i \leq d-1$ both*

$$A.\hat{C}_i^{-} = \tilde{b}_{i-1}\hat{C}_{i-1}^{-} + (\tilde{b}_{i-1} - b_i)\hat{C}_{i-1}^{+} + (\tilde{a}_i - b_i + \tilde{b}_i)\hat{C}_i^{-} + (c_{i+1} - \tilde{c}_i)\hat{C}_i^{+} + c_{i+1}\hat{C}_{i+1}^{-}, \quad (2.67)$$

$$A.\hat{C}_i^{+} = b_i\hat{C}_{i-1}^{+} + (b_i - \tilde{b}_i)\hat{C}_i^{-} + (\tilde{a}_i - c_{i+1} + \tilde{c}_i)\hat{C}_i^{+} + (\tilde{c}_{i+1} - c_{i+1})\hat{C}_{i+1}^{-} + \tilde{c}_{i+1}\hat{C}_{i+1}^{+}. \quad (2.68)$$

Proof. Use Lemma 2.4.1 and Lemma 2.4.2. ■

Evaluating (2.67), (2.68) using (2.17)–(2.20) and (2.45)–(2.48), we obtain the following.

For $0 \leq i \leq d-1$,

$$A.\hat{C}_i^{-} =$$

term	coefficient
\hat{C}_{i-1}^{-}	$\frac{h(1-q^{i-d})(1-s^*q^{i+1})(1-r_1q^i)(1-r_2q^i)}{(1-s^*q^{2i})(1-s^*q^{2i+1})}$
\hat{C}_{i-1}^{+}	$\frac{h(1-q^{i-d})(1-s^*q^{i+1})}{1-s^*q^{2i+1}} \left(\frac{(1-r_1q^i)(1-r_2q^i)}{1-s^*q^{2i}} - \frac{(1-r_1q^{i+1})(1-r_2q^{i+1})}{1-s^*q^{2i+2}} \right)$
\hat{C}_i^{-}	$b_0 - h \left(\frac{(1-q^i)(1-s^*q^{i+d+1})(r_1-s^*q^{i+1})(r_2-s^*q^{i+1})}{s^*q^d(1-s^*q^{2i+1})(1-s^*q^{2i+2})} + \frac{(1-q^{i-d})(1-s^*q^{i+1})(1-r_1q^{i+1})(1-r_2q^{i+1})}{(1-s^*q^{2i+1})(1-s^*q^{2i+2})} \right)$
\hat{C}_i^{+}	$\frac{h(r_1-s^*q^{i+1})(r_2-s^*q^{i+1})}{s^*q^d(1-s^*q^{2i+2})} \left(\frac{(1-q^{i+1})(1-s^*q^{i+d+2})}{1-s^*q^{2i+3}} - \frac{(1-q^i)(1-s^*q^{i+d+1})}{1-s^*q^{2i+1}} \right)$
\hat{C}_{i+1}^{-}	$\frac{h(1-q^{i+1})(1-s^*q^{i+d+2})(r_1-s^*q^{i+1})(r_2-s^*q^{i+1})}{s^*q^d(1-s^*q^{2i+2})(1-s^*q^{2i+3})}$

(2.69)

and also

$$A.\hat{C}_i^+ =$$

term	coefficient
\hat{C}_{i-1}^+	$\frac{h(1-q^{i-d})(1-s^*q^{i+1})(1-r_1q^{i+1})(1-r_2q^{i+1})}{(1-s^*q^{2i+1})(1-s^*q^{2i+2})}$
\hat{C}_i^-	$\frac{h(1-r_1q^{i+1})(1-r_2q^{i+1})}{1-s^*q^{2i+2}} \left(\frac{(1-q^{i-d})(1-s^*q^{i+1})}{1-s^*q^{2i+1}} - \frac{(1-q^{i-d+1})(1-s^*q^{i+2})}{1-s^*q^{2i+3}} \right)$
\hat{C}_i^+	$b_0 - h \left(\frac{(1-q^{i-d+1})(1-s^*q^{i+2})(1-r_1q^{i+1})(1-r_2q^{i+1})}{(1-s^*q^{2i+2})(1-s^*q^{2i+3})} + \frac{(1-q^{i+1})(1-s^*q^{i+d+2})(r_1-s^*q^{i+1})(r_2-s^*q^{i+1})}{s^*q^d(1-s^*q^{2i+2})(1-s^*q^{2i+3})} \right)$
\hat{C}_{i+1}^-	$\frac{h(1-q^{i+1})(1-s^*q^{i+d+2})}{s^*q^d(1-s^*q^{2i+3})} \left(\frac{(r_1-s^*q^{i+2})(r_2-s^*q^{i+2})}{1-s^*q^{2i+4}} - \frac{(r_1-s^*q^{i+1})(r_2-s^*q^{i+1})}{1-s^*q^{2i+2}} \right)$
\hat{C}_{i+1}^+	$\frac{h(1-q^{i+1})(1-s^*q^{i+d+2})(r_1-s^*q^{i+2})(r_2-s^*q^{i+2})}{s^*q^d(1-s^*q^{2i+3})(1-s^*q^{2i+4})}$

(2.70)

The scalars b_0 and h in the above tables are given in (2.17) and (2.26), respectively. For $0 \leq r \leq d$ we now give the action of E_r^* on $\{\hat{C}_i^\pm\}_{i=0}^{d-1}$.

Lemma 2.4.10. *For $0 \leq r \leq d$ and $0 \leq i \leq d-1$,*

$$E_r^*.\hat{C}_i^- = \delta_{ri}\hat{C}_i^-, \quad E_r^*.\hat{C}_i^+ = \delta_{r,i+1}\hat{C}_i^+.$$

Proof. Use (2.65). ■

Corollary 2.4.11. *The matrix A^* acts on $\{\hat{C}_i^\pm\}_{i=0}^{d-1}$ as follows. For $0 \leq i \leq d-1$,*

$$A^*.\hat{C}_i^- = \theta_i^*.\hat{C}_i^-, \quad A^*.\hat{C}_i^+ = \theta_{i+1}^*.\hat{C}_i^+.$$

Proof. By Lemma 2.4.10 and since $A^* = \sum_{r=0}^d \theta_r^* E_r^*$. ■

Lemma 2.4.12. *We have*

$$\mathbf{W} = \sum_{i=0}^d E_i^* \mathbf{W} \quad (\text{orthogonal direct sum}).$$

Moreover,

(i) \hat{C}_0^- is a basis for $E_0^*\mathbf{W}$.

(ii) For $1 \leq i \leq d-1$ the vectors $\hat{C}_{i-1}^+, \hat{C}_i^-$ form a basis for $E_i^*\mathbf{W}$.

(iii) \hat{C}_{d-1}^+ is a basis for $E_d^*\mathbf{W}$.

Proof. Use (2.65) and Lemma 2.4.10. ■

Corollary 2.4.13. For $0 \leq i \leq d$,

$$\dim E_i^*\mathbf{W} = \begin{cases} 2 & \text{if } 1 \leq i \leq d-1, \\ 1 & \text{if } i \in \{0, d\}. \end{cases}$$

Proof. Immediate from Lemma 2.4.12. ■

Lemma 2.4.14. The subspace \mathbf{W} is a T -module.

Proof. Follows from Lemma 2.4.9 and Lemma 2.4.10. ■

For $0 \leq r \leq d-1$ we now give the action of \tilde{E}_r^* on $\{\hat{C}_i^\pm\}_{i=0}^{d-1}$.

Lemma 2.4.15. For $0 \leq i, r \leq d-1$,

$$\tilde{E}_r^*.\hat{C}_i^- = \delta_{ri}\hat{C}_i^-, \quad \tilde{E}_r^*.\hat{C}_i^+ = \delta_{ri}\hat{C}_i^+.$$

Proof. Use (2.65). ■

Corollary 2.4.16. The matrix \tilde{A}^* acts on $\{\hat{C}_i^\pm\}_{i=0}^{d-1}$ as follows. For $0 \leq i \leq d-1$,

$$\tilde{A}^*.\hat{C}_i^- = \tilde{\theta}_i^*.\hat{C}_i^-, \quad \tilde{A}^*.\hat{C}_i^+ = \tilde{\theta}_i^*.\hat{C}_i^+.$$

Proof. By (2.51) and Lemma 2.4.15. ■

Lemma 2.4.17. *We have*

$$\mathbf{W} = \sum_{i=0}^{d-1} \tilde{E}_i^* \mathbf{W} \quad (\text{orthogonal direct sum}).$$

Moreover, for $0 \leq i \leq d-1$ the vectors \hat{C}_i^+, \hat{C}_i^- form a basis for $\tilde{E}_i^* \mathbf{W}$.

Proof. Use (2.65) and Lemma 2.4.15. ■

Corollary 2.4.18. *The dimension of $\tilde{E}_i^* \mathbf{W}$ is 2 for $0 \leq i \leq d-1$.*

Proof. Immediate from Lemma 2.4.17. ■

Lemma 2.4.19. *The subspace \mathbf{W} is a \tilde{T} -module.*

Proof. Follows from Lemma 2.4.9 and Lemma 2.4.15. ■

Motivated by Lemma 2.4.14 and Lemma 2.4.19, we make a definition.

Definition 2.4.20. Let \mathbf{T} denote the subalgebra of $\mathbf{M}_X(\mathbb{C})$ generated by T and \tilde{T} . The algebra \mathbf{T} is finite-dimensional and noncommutative. Observe that A, A^*, \tilde{A}^* generate \mathbf{T} . The algebra \mathbf{T} is semisimple since it is closed under the conjugate-transpose map.

By a \mathbf{T} -module we mean a subspace $W \subseteq \mathbf{V}$ such that $BW \subseteq W$ for all $B \in \mathbf{T}$. By Lemma 2.4.14 and Lemma 2.4.19 \mathbf{W} is a \mathbf{T} -module. We call \mathbf{W} the *primary \mathbf{T} -module*.

We now describe the \mathbf{T} -module \mathbf{W} .

Lemma 2.4.21. *We have*

$$\mathbf{W} = \sum_{i=0}^{d-1} \sum_{j=i}^{i+1} \tilde{E}_i^* E_j^* \mathbf{W} \quad (\text{orthogonal direct sum}).$$

Moreover, for $0 \leq i \leq d-1$ both

- (i) *The vector \hat{C}_i^- is a basis for $\tilde{E}_i^* E_i^* \mathbf{W}$,*

(ii) The vector \hat{C}_i^+ is a basis for $\tilde{E}_i^* E_{i+1}^* \mathbf{W}$.

Proof. Use Lemma 2.4.12 and Lemma 2.4.17. ■

Lemma 2.4.22. For $0 \leq i \leq d-1$,

$$\tilde{E}_i^* E_i^* \mathbf{j} = \hat{C}_i^-, \quad \tilde{E}_i^* E_{i+1}^* \mathbf{j} = \hat{C}_i^+.$$

Proof. Use Lemma 2.4.10 and Lemma 2.4.15 along with $\mathbf{j} = \sum_{i=0}^{d-1} (\hat{C}_i^- + \hat{C}_i^+)$. ■

Lemma 2.4.23. For $0 \leq i \leq d-1$,

$$\hat{C}_i^- = \sum_{j=0}^i A_j \hat{x} - \sum_{j=0}^{i-1} \hat{C}_j, \quad \hat{C}_i^+ = \sum_{j=0}^i \hat{C}_j - \sum_{j=0}^i A_j \hat{x}.$$

Proof. Use (2.65) and induction on i . ■

Corollary 2.4.24. The following hold.

(i) $\mathcal{M}\hat{x} + \mathcal{M}\hat{C} = \mathbf{W}$.

(ii) $\mathcal{M}\hat{x} \cap \mathcal{M}\hat{C} = \mathbb{C}\mathbf{j}$.

Proof. (i) By Lemma 2.4.9 \mathbf{W} is invariant under \mathcal{M} . Also $\hat{x}, \hat{C} \in \mathbf{W}$. Thus $\mathcal{M}\hat{x} + \mathcal{M}\hat{C} \subseteq \mathbf{W}$. Concerning the reverse inclusion, recall by Lemma 2.3.16(v) that $\{\hat{C}_j\}_{j=0}^{d-1}$ forms a basis for $\mathcal{M}\hat{C}$. By Lemma 2.4.23, the set $\{\hat{C}_i^\pm\}_{i=0}^{d-1}$ is contained in $\mathcal{M}\hat{x} + \mathcal{M}\hat{C}$. The $\{\hat{C}_i^\pm\}_{i=0}^{d-1}$ span \mathbf{W} and therefore \mathbf{W} is contained in $\mathcal{M}\hat{x} + \mathcal{M}\hat{C}$. The result follows.

(ii) We saw earlier that $\mathbf{j} \in \mathcal{M}\hat{x}$ and $\mathbf{j} \in \mathcal{M}\hat{C}$, so $\mathbb{C}\mathbf{j} \subseteq \mathcal{M}\hat{x} \cap \mathcal{M}\hat{C}$. To finish the proof we show that the dimension of $\mathcal{M}\hat{x} \cap \mathcal{M}\hat{C}$ is 1. Using Example 2.2.2, Lemma 2.3.16(v), and part (i) along with Corollary 2.4.8,

$$\begin{aligned} \dim(\mathcal{M}\hat{x} \cap \mathcal{M}\hat{C}) &= \dim \mathcal{M}\hat{x} + \dim \mathcal{M}\hat{C} - \dim(\mathcal{M}\hat{x} + \mathcal{M}\hat{C}) \\ &= d + 1 + d - 2d \\ &= 1. \end{aligned}$$

The result follows. ■

Proposition 2.4.25. *The \mathbf{T} -module \mathbf{W} is irreducible.*

Proof. Since \mathbf{T} is closed under the conjugate-transpose map, \mathbf{W} is an orthogonal direct sum of irreducible \mathbf{T} -modules. Among these \mathbf{T} -modules, there exists one, denoted W , that is not orthogonal to \hat{x} . Now $E_0^*W \neq 0$. Also $E_0^*W \subseteq E_0^*\mathbf{W} = \text{Span}\{\hat{x}\}$. So $\hat{x} \in E_0^*W \subseteq W$ and hence $\mathcal{M}\hat{x} \subseteq W$. Thus $\mathbf{j} \in \mathcal{M}\hat{x} \subseteq W$, and so $\mathbf{T}\mathbf{j} \subseteq W$. Now $\{\hat{C}_i^\pm\}_{i=0}^{d-1} \subseteq W$ by Lemma 2.4.22. The $\{\hat{C}_i^\pm\}_{i=0}^{d-1}$ span \mathbf{W} and therefore $\mathbf{W} \subseteq W$. Consequently $\mathbf{W} = W$. The result follows. ■

We finish this section with a comment.

Lemma 2.4.26. *For $0 \leq i \leq d$,*

$$\dim E_i\mathbf{W} = \begin{cases} 2 & \text{if } 1 \leq i \leq d-1, \\ 1 & \text{if } i \in \{0, d\}. \end{cases}$$

Proof. Using Corollary 2.4.24(i), $E_i\mathbf{W} = E_i(\mathcal{M}\hat{x} + \mathcal{M}\hat{C}) = E_i\mathcal{M}\hat{x} + E_i\mathcal{M}\hat{C}$. By linear algebra,

$$\dim E_i\mathbf{W} = \dim E_i\mathcal{M}\hat{x} + \dim E_i\mathcal{M}\hat{C} - \dim (E_i\mathcal{M}\hat{x} \cap E_i\mathcal{M}\hat{C}). \quad (2.71)$$

Observe that $E_i(\mathcal{M}\hat{x} \cap \mathcal{M}\hat{C})$ is contained in $E_i\mathcal{M}\hat{x} \cap E_i\mathcal{M}\hat{C}$. By this and Corollary 2.4.24(ii),

$$\dim E_i(\mathbf{C}\mathbf{j}) \leq \dim (E_i\mathcal{M}\hat{x} \cap E_i\mathcal{M}\hat{C}). \quad (2.72)$$

Combining (2.71) and (2.72),

$$\dim E_i\mathbf{W} \leq \dim E_i\mathcal{M}\hat{x} + \dim E_i\mathcal{M}\hat{C} - \dim E_i(\mathbf{C}\mathbf{j}).$$

By Example 2.2.2, $\dim E_r \mathcal{M}\hat{x} = 1$ for $0 \leq r \leq d$. By Lemma 2.3.16(v), $\dim E_r \mathcal{M}\hat{C} = 1$ for $0 \leq r \leq d - 1$. Also $E_d \mathcal{M}\hat{C} = 0$ by Lemma 2.3.2. Moreover, $\dim E_0(\mathbb{C}\mathbf{j}) = 1$ and E_r vanishes on $\mathbb{C}\mathbf{j}$ for $1 \leq r \leq d$. By these comments the dimension of $E_i \mathbf{W}$ is at most 2 for $1 \leq i \leq d - 1$ and at most 1 for $i \in \{0, d\}$. The sum of these upper bounds is $2d$. Also $\sum_{i=0}^d \dim E_i \mathbf{W} = \dim \mathbf{W} = 2d$. By these comments the dimension of $E_i \mathbf{W}$ equals 2 for $1 \leq i \leq d - 1$ and equals 1 for $i \in \{0, d\}$. The result follows. \blacksquare

2.5 \mathbf{W} as a T -module

Recall the subspace \mathbf{W} from above line (2.66). In Lemma 2.4.14 we saw that \mathbf{W} is a T -module. Throughout this section we adopt this point of view. Recall the primary T -module $\mathcal{M}\hat{x}$ from Example 2.2.2. Observe that $\mathcal{M}\hat{x}$ is an irreducible T -submodule of \mathbf{W} . Let $\mathcal{M}\hat{x}^\perp$ denote the orthogonal complement of $\mathcal{M}\hat{x}$ in \mathbf{W} . Observe that $\dim \mathcal{M}\hat{x}^\perp = d - 1$, since $\dim \mathbf{W} = 2d$ by Corollary 2.4.8 and $\dim \mathcal{M}\hat{x} = d + 1$ by Example 2.2.2. The subspace $\mathcal{M}\hat{x}^\perp$ is a T -submodule of \mathbf{W} since T is closed under the conjugate-transpose map. By construction

$$\mathbf{W} = \mathcal{M}\hat{x} + \mathcal{M}\hat{x}^\perp \quad (\text{orthogonal direct sum of } T\text{-modules}). \quad (2.73)$$

Lemma 2.5.1. *The T -module $\mathcal{M}\hat{x}^\perp$ is irreducible and thin, with endpoint 1, dual endpoint 1, and diameter $d - 2$.*

Proof. Abbreviate $W = \mathcal{M}\hat{x}^\perp$. Using Example 2.2.2 and Corollary 2.4.13,

$$\dim E_i^* W = \begin{cases} 1 & \text{if } 1 \leq i \leq d - 1, \\ 0 & \text{if } i \in \{0, d\}. \end{cases} \quad (2.74)$$

Using Example 2.2.2 and Lemma 2.4.26,

$$\dim E_i W = \begin{cases} 1 & \text{if } 1 \leq i \leq d-1, \\ 0 & \text{if } i \in \{0, d\}. \end{cases}$$

To finish the proof it suffices to show that the T -module W is irreducible. Write W as a direct sum of irreducible T -modules. Among these T -modules, there exists a module U with endpoint 1 since $E_0^* W = 0$ and $E_1^* W \neq 0$ by (2.74). By [5, Lemma 5.1], the dimension of U is at least $d-1$. But $U \subseteq W$ and the dimension of W is $d-1$, so the dimension of U is at most $d-1$. Therefore $U = W$, which means W is irreducible. The result follows. \blacksquare

Recall the primary Leonard system Φ on $\mathcal{M}\hat{x}$ from Lemma 2.2.3, with its parameter array

$$p(\Phi) = (\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d).$$

We now consider the Leonard system on $\mathcal{M}\hat{x}^\perp$. By Lemma 2.2.1 and Lemma 2.5.1, the elements

$$(A; A^*; \{E_i\}_{i=1}^{d-1}; \{E_i^*\}_{i=1}^{d-1})$$

act on $\mathcal{M}\hat{x}^\perp$ as a Leonard system; we denote this Leonard system by Φ^\perp . We denote the parameter array of Φ^\perp by

$$p(\Phi^\perp) = (\{\theta_i^\perp\}_{i=0}^{d-2}; \{\theta_i^{*\perp}\}_{i=0}^{d-2}; \{\varphi_i^\perp\}_{i=1}^{d-2}; \{\phi_i^\perp\}_{i=1}^{d-2}). \quad (2.75)$$

Lemma 2.5.2. *With the above notation,*

$$\theta_i^\perp = \theta_{i+1}, \quad \theta_i^{*\perp} = \theta_{i+1}^*, \quad (0 \leq i \leq d-2). \quad (2.76)$$

Proof. The endpoint and dual endpoint of $\mathcal{M}\hat{x}^\perp$ are both 1. \blacksquare

Recall from Example 2.2.2 that the $\{A_i \hat{x}\}_{i=0}^d$ form a Φ -standard basis for $\mathcal{M}\hat{x}$. For notational convenience define $v_i = A_i \hat{x}$ for $0 \leq i \leq d$. Observe that

$$v_0 = \hat{C}_0^-, \quad v_i = \hat{C}_{i-1}^+ + \hat{C}_i^- \quad (1 \leq i \leq d-1), \quad v_d = \hat{C}_{d-1}^+. \quad (2.77)$$

We now discuss a Φ^\perp -standard basis for $\mathcal{M}\hat{x}^\perp$. Define the scalars

$$\epsilon_i := -\frac{|C_{i-1}^+|}{|C_i^-|} \quad (1 \leq i \leq d-1). \quad (2.78)$$

By Corollary 2.4.5, ϵ_i is well-defined and nonzero for $1 \leq i \leq d-1$.

Lemma 2.5.3. *For $1 \leq i \leq d-1$,*

$$\epsilon_i = \frac{(1-q^i)(1-s^*q^{i+d+1})}{q^d(1-q^{i-d})(1-s^*q^{i+1})}. \quad (2.79)$$

Proof. Using Theorem 2.3.7, Corollary 2.4.3(iv), and (2.78),

$$\epsilon_i = \frac{\tilde{c}_i - c_i}{b_i - \tilde{b}_{i-1}} = \frac{N_i - |C|}{N_{i-1}} \frac{c_i}{b_i}.$$

In the above line simplify the expression on the right using (2.18), (2.19), and Lemma 2.3.5 to get (2.79). ■

Lemma 2.5.4. *For $1 \leq i \leq d-1$ the vector*

$$\hat{C}_{i-1}^+ + \epsilon_i \hat{C}_i^-$$

is a basis for $E_i^(\mathcal{M}\hat{x}^\perp)$.*

Proof. By Lemma 2.4.12(ii), $0 \neq \hat{C}_{i-1}^+ + \epsilon_i \hat{C}_i^- \in E_i^* \mathbf{W}$. By (2.73) the subspace $E_i^*(\mathcal{M}\hat{x}^\perp)$ is the orthogonal complement of $E_i^* \mathcal{M}\hat{x}$ in $E_i^* \mathbf{W}$. Recall v_i is a basis for $E_i^* \mathcal{M}\hat{x}$. Using (2.77),

$$\langle v_i, \hat{C}_{i-1}^+ + \epsilon_i \hat{C}_i^- \rangle = \|\hat{C}_{i-1}^+\|^2 + \epsilon_i \|\hat{C}_i^-\|^2 = |C_{i-1}^+| + \epsilon_i |C_i^-| = 0.$$

Therefore $\hat{C}_{i-1}^+ + \epsilon_i \hat{C}_i^- \in E_i^*(\mathcal{M}\hat{x}^\perp)$. The result follows since $E_i^*(\mathcal{M}\hat{x}^\perp)$ has dimension 1 by (2.74). \blacksquare

Corollary 2.5.5. *The vectors*

$$\hat{C}_{i-1}^+ + \epsilon_i \hat{C}_i^- \quad (1 \leq i \leq d-1)$$

form an orthogonal basis for $\mathcal{M}\hat{x}^\perp$.

Proof. Follows from Lemma 2.5.4. \blacksquare

In Lemma 2.5.5 we found a basis $\{\hat{C}_{i-1}^+ + \epsilon_i \hat{C}_i^-\}_{i=1}^{d-1}$ for $\mathcal{M}\hat{x}^\perp$. As we will see, this basis is not a Φ^\perp -standard basis for $\mathcal{M}\hat{x}^\perp$. In order to turn it into a Φ^\perp -standard basis we make an adjustment. Pick a nonzero $w \in E_1(\mathcal{M}\hat{x}^\perp)$. Define the complex scalars $\{\xi_i\}_{i=1}^{d-1}$ such that

$$w = \sum_{i=1}^{d-1} \xi_i (\hat{C}_{i-1}^+ + \epsilon_i \hat{C}_i^-). \quad (2.80)$$

For notational convenience we rewrite (2.80) as

$$w = \sum_{i=0}^{d-2} \xi_{i+1} (\hat{C}_i^+ + \epsilon_{i+1} \hat{C}_{i+1}^-).$$

Define

$$v_i^\perp = \xi_{i+1} (\hat{C}_i^+ + \epsilon_{i+1} \hat{C}_{i+1}^-) \quad (0 \leq i \leq d-2). \quad (2.81)$$

For notational convenience define $v_{-1}^\perp = 0$ and $v_{d-1}^\perp = 0$.

Lemma 2.5.6. *The vectors $\{v_i^\perp\}_{i=0}^{d-2}$ form a Φ^\perp -standard basis for $\mathcal{M}\hat{x}^\perp$.*

Proof. By Lemma 2.5.4, $v_i^\perp \in E_{i+1}^*(\mathcal{M}\hat{x}^\perp)$ for $0 \leq i \leq d-2$. Also, $\sum_{i=0}^{d-2} v_i^\perp = w \in E_1(\mathcal{M}\hat{x}^\perp)$. The result follows from these comments and Lemma 2.1.3. \blacksquare

Corollary 2.5.7. *The scalars $\{\xi_i\}_{i=1}^{d-1}$ from (2.80) are all nonzero.*

Proof. By Lemma 2.5.6 $\{v_i^\perp\}_{i=0}^{d-2}$ are all nonzero. By this and (2.81) the result follows. ■

By the comments below Lemma 2.1.3, the matrix representing A^* relative to the basis $\{v_i^\perp\}_{i=0}^{d-2}$ is

$$\text{diag}(\theta_0^{*\perp}, \theta_1^{*\perp}, \theta_2^{*\perp}, \dots, \theta_{d-2}^{*\perp}). \quad (2.82)$$

Let $\{a_i^\perp\}_{i=0}^{d-2}$, $\{b_i^\perp\}_{i=0}^{d-3}$, $\{c_i^\perp\}_{i=1}^{d-2}$ denote the intersection numbers of the Leonard system Φ^\perp . By construction the matrix representing A relative to $\{v_i^\perp\}_{i=0}^{d-2}$ is

$$\begin{bmatrix} a_0^\perp & b_0^\perp & & & \mathbf{0} \\ c_1^\perp & a_1^\perp & b_1^\perp & & \\ & c_2^\perp & a_2^\perp & \ddots & \\ & & & \ddots & \ddots & b_{d-3}^\perp \\ \mathbf{0} & & & & c_{d-2}^\perp & a_{d-2}^\perp \end{bmatrix} \quad (2.83)$$

For convenience, define $b_{-1}^\perp = 0$ and $c_{d-1}^\perp = 0$. Recall the scalars a_i, b_i, c_i from below Lemma 2.2.3 and the $\tilde{a}_i, \tilde{b}_i, \tilde{c}_i$ from (2.44).

Lemma 2.5.8. *With the above notation the following (i)–(vi) hold.*

- (i) $b_{i-1}^\perp \xi_i = \xi_{i+1} b_i \quad (1 \leq i \leq d-2).$
- (ii) $b_{i-1}^\perp \xi_i \epsilon_i = \xi_{i+1} (b_i - \tilde{b}_i) + \xi_{i+1} \epsilon_{i+1} \tilde{b}_i \quad (1 \leq i \leq d-2).$
- (iii) $a_i^\perp = \tilde{a}_i - c_{i+1} + \tilde{c}_i + \epsilon_{i+1} (\tilde{b}_i - b_{i+1}) \quad (0 \leq i \leq d-2).$
- (iv) $a_i^\perp \epsilon_{i+1} = \tilde{c}_{i+1} - c_{i+1} + \epsilon_{i+1} (\tilde{a}_{i+1} - b_{i+1} + \tilde{b}_{i+1}) \quad (0 \leq i \leq d-2).$
- (v) $c_{i+1}^\perp \xi_{i+2} = \xi_{i+1} \tilde{c}_{i+1} + \xi_{i+1} \epsilon_{i+1} (c_{i+2} - \tilde{c}_{i+1}) \quad (0 \leq i \leq d-3).$

$$(vi) \quad c_{i+1}^\perp \xi_{i+2} \epsilon_{i+2} = \xi_{i+1} \epsilon_{i+1} c_{i+2} \quad (0 \leq i \leq d-3).$$

Proof. For $0 \leq i \leq d-2$ we evaluate Av_i^\perp in two ways. First, using (2.83),

$$Av_i^\perp = b_{i-1}^\perp v_{i-1}^\perp + a_i^\perp v_i^\perp + c_{i+1}^\perp v_{i+1}^\perp.$$

In this equation evaluate the right-hand side using (2.81). Secondly, in Av_i^\perp eliminate v_i^\perp using (2.81) and evaluate the result using Lemma 2.4.9. We have just evaluated Av_i^\perp in two ways. Compare the results using the linear independence of $\{\hat{C}_j^\pm\}_{j=0}^{d-1}$. The result follows. ■

Corollary 2.5.9. *Referring to Lemma 2.5.8, the following hold.*

$$(i) \quad b_i^\perp = \frac{\xi_{i+2}}{\xi_{i+1}} b_{i+1} \quad (0 \leq i \leq d-3).$$

$$(ii) \quad a_i^\perp = b_0 - \tilde{b}_i - \tilde{c}_{i+1} \quad (0 \leq i \leq d-2).$$

$$(iii) \quad c_i^\perp = \frac{\xi_i \epsilon_i}{\xi_{i+1} \epsilon_{i+1}} c_{i+1} \quad (1 \leq i \leq d-2).$$

Proof. (i) Immediate from Lemma 2.5.8(i).

(ii) In the equation from Lemma 2.5.8(iii), eliminate ϵ_{i+1} using Corollary 2.4.3(iv) along with (2.78) and simplify the result using $\tilde{a}_i + \tilde{b}_i + \tilde{c}_i = b_0$.

(iii) Immediate from Lemma 2.5.8(vi). ■

We now compare the parameter arrays $p(\Phi)$ and $p(\Phi^\perp)$. Recall that Φ has q -Racah type. Recall the scalars h, h^*, s, s^*, r_1, r_2 from above Note 2.2.4. Recall the parameter array of Φ^\perp from (2.75).

Theorem 2.5.10. *With the above notation, for $0 \leq i \leq d-2$,*

$$\theta_i^\perp = \theta_0^\perp + h^\perp (1 - q^i)(1 - s^\perp q^{i+1})q^{-i}, \quad (2.84)$$

$$\theta_i^{*\perp} = \theta_0^{*\perp} + h^{*\perp} (1 - q^i)(1 - s^{*\perp} q^{i+1})q^{-i}, \quad (2.85)$$

and for $1 \leq i \leq d - 2$,

$$\varphi_i^\perp = h^\perp h^{*\perp} q^{1-2i} (1 - q^i)(1 - q^{i-d+1})(1 - r_1^\perp q^i)(1 - r_2^\perp q^i), \quad (2.86)$$

$$\phi_i^\perp = h^\perp h^{*\perp} q^{1-2i} (1 - q^i)(1 - q^{i-d+1})(r_1^\perp - s^{*\perp} q^i)(r_2^\perp - s^{*\perp} q^i)/s^{*\perp}, \quad (2.87)$$

where $\theta_0^\perp, \theta_0^{*\perp}$ are from (2.76) and

$$h^\perp = hq^{-1}, \quad s^\perp = sq^2, \quad r_1^\perp = r_1q, \quad (2.88)$$

$$h^{*\perp} = h^*q^{-1}, \quad s^{*\perp} = s^*q^2, \quad r_2^\perp = r_2q. \quad (2.89)$$

Proof. Recall that Φ^\perp has diameter $d - 2$. To get (2.84), (2.85) use (2.13), (2.14) and Lemma 2.5.2. We now verify (2.86), (2.87). To this end, recall the intersection numbers $\{a_j^\perp\}_{j=0}^{d-2}$ of Φ^\perp . Using (2.10), $\varphi_1^\perp = (a_0^\perp - \theta_0^\perp)(\theta_0^{*\perp} - \theta_1^{*\perp})$. Evaluate this using Lemma 2.5.2 and Corollary 2.5.9(ii) to get

$$\varphi_1^\perp = (b_0 - \tilde{b}_0 - \tilde{c}_1 - \theta_1)(\theta_1^* - \theta_2^*). \quad (2.90)$$

Evaluate the right-hand side of (2.90) using $b_0 = \theta_0$, (2.13), (2.14), and Corollary 2.3.9 and simplify the result using (2.88), (2.89). This yields (2.86) for $i = 1$. Using this together with (2.84), (2.85) and the condition (PA4) of Theorem 2.1.2 we obtain (2.87) for $1 \leq i \leq d - 2$. We now use (2.87) at $i = 1$ together with (2.84), (2.85) and the condition (PA3) of Theorem 2.1.2 to obtain (2.86) for $1 \leq i \leq d - 2$. The result follows. ■

Corollary 2.5.11. *The Leonard system Φ^\perp has q -Racah type.*

Proof. Compare Example 2.1.8 and Theorem 2.5.10. ■

Recall the intersection numbers $\{b_i^\perp\}_{i=0}^{d-3}, \{c_i^\perp\}_{i=1}^{d-2}$ of the Leonard system Φ^\perp .

Corollary 2.5.12. *With the above notation,*

$$b_0^\perp = \frac{h(1 - q^{-d+2})(1 - r_1 q^2)(1 - r_2 q^2)}{q(1 - s^* q^4)}, \quad (2.91)$$

$$b_i^\perp = \frac{h(1 - q^{i-d+2})(1 - s^* q^{i+3})(1 - r_1 q^{i+2})(1 - r_2 q^{i+2})}{q(1 - s^* q^{2i+3})(1 - s^* q^{2i+4})} \quad (1 \leq i \leq d-3), \quad (2.92)$$

$$c_i^\perp = \frac{h(1 - q^i)(1 - s^* q^{i+d+1})(r_1 - s^* q^{i+1})(r_2 - s^* q^{i+1})}{s^* q^{d-1}(1 - s^* q^{2i+2})(1 - s^* q^{2i+3})} \quad (1 \leq i \leq d-3), \quad (2.93)$$

$$c_{d-2}^\perp = \frac{h(1 - q^{d-2})(r_1 - s^* q^{d-1})(r_2 - s^* q^{d-1})}{s^* q^{d-1}(1 - s^* q^{2d-2})}. \quad (2.94)$$

Proof. Use (2.17)–(2.20) and (2.88), (2.89). ■

We finish this section with some comments about the scalars $\{\xi_i\}_{i=1}^{d-1}$ from (2.80).

Lemma 2.5.13. *The vector w in line (2.80) can be chosen such that*

$$\xi_i = q^{1-i}(1 - q^{i-d})(1 - s^* q^{i+1}) \quad (1 \leq i \leq d-1). \quad (2.95)$$

Proof. Observe that the vector w is defined up to multiplication by a nonzero scalar in \mathbb{C} . Multiplying w by a nonzero scalar if necessary, we may assume that

$$\xi_1 = (1 - q^{1-d})(1 - s^* q^2). \quad (2.96)$$

Using Lemma 2.5.8(i) and induction on i ,

$$\xi_i = \frac{b_0^\perp b_1^\perp \cdots b_{i-2}^\perp}{b_1 b_2 \cdots b_{i-1}} \xi_1 \quad (1 \leq i \leq d-1). \quad (2.97)$$

Evaluate (2.97) using (2.18), (2.91), (2.92) and (2.96) to get (2.95). The result follows. ■

From now on we assume that the vector w in line (2.80) has been chosen such that (2.95) holds. Recall the vectors $\{v_i^\perp\}_{i=0}^{d-2}$ from (2.81).

Lemma 2.5.14. For $0 \leq i \leq d-2$,

$$v_i^\perp = q^{-i}(1 - q^{1+i-d})(1 - s^*q^{i+2})\hat{C}_i^+ + q^{-i-d}(1 - q^{i+1})(1 - s^*q^{i+d+2})\hat{C}_{i+1}^-.$$

Proof. Evaluate (2.81) using (2.79) and (2.95). ■

2.6 \mathbf{W} as a \tilde{T} -module

In the previous section we discussed \mathbf{W} as a T -module. Recall the algebra \tilde{T} from above Lemma 2.3.16. In Lemma 2.4.19 we saw that \mathbf{W} is a \tilde{T} -module. Throughout this section we adopt this point of view. Recall the primary \tilde{T} -module $\mathcal{M}\hat{C}$ from Lemma 2.3.16. Observe that $\mathcal{M}\hat{C}$ is an irreducible \tilde{T} -submodule of \mathbf{W} . Let $\mathcal{M}\hat{C}^\perp$ denote the orthogonal complement of $\mathcal{M}\hat{C}$ in \mathbf{W} . Observe that $\dim \mathcal{M}\hat{C}^\perp = d$, since $\dim \mathbf{W} = 2d$ and $\dim \mathcal{M}\hat{C} = d$. The subspace $\mathcal{M}\hat{C}^\perp$ is a \tilde{T} -submodule of \mathbf{W} since \tilde{T} is closed under the conjugate-transpose map. By construction

$$\mathbf{W} = \mathcal{M}\hat{C} + \mathcal{M}\hat{C}^\perp \quad (\text{orthogonal direct sum of } \tilde{T}\text{-modules}). \quad (2.98)$$

Our next goal is to show that the \tilde{T} -module $\mathcal{M}\hat{C}^\perp$ is irreducible.

Lemma 2.6.1. Let W denote the \tilde{T} -module $\mathcal{M}\hat{C}^\perp$. The following hold.

- (i) The dimension of $E_i W$ is 1 for $1 \leq i \leq d$. Moreover, $E_0 W = 0$.
- (ii) The dimension of $\tilde{E}_i^* W$ is 1 for $0 \leq i \leq d-1$.

Proof. (i) Use Corollary 2.3.17(i), Lemma 2.4.26, and (2.98).

(ii) Use Corollary 2.3.17(ii), Corollary 2.4.18, and (2.98). ■

Lemma 2.6.2. *The vector $\hat{C} - |C|\hat{x}$ is a basis for $\tilde{E}_0^*(\mathcal{M}\hat{C}^\perp)$.*

Proof. By (2.98) the subspace $\tilde{E}_0^*(\mathcal{M}\hat{C}^\perp)$ is the orthogonal complement of $\tilde{E}_0^*\mathcal{M}\hat{C}$ in $\tilde{E}_0^*\mathbf{W}$. By construction $\hat{C} - |C|\hat{x}$ is nonzero and contained in $\tilde{E}_0^*\mathbf{W}$. Recall \hat{C} is a basis for $\tilde{E}_0^*\mathcal{M}\hat{C}$. The vector $\hat{C} - |C|\hat{x}$ is orthogonal to \hat{C} since

$$\langle \hat{C} - |C|\hat{x}, \hat{C} \rangle = \|\hat{C}\|^2 - |C| = 0.$$

Therefore $\hat{C} - |C|\hat{x} \in \tilde{E}_0^*(\mathcal{M}\hat{C}^\perp)$. The result follows since $\tilde{E}_0^*(\mathcal{M}\hat{C}^\perp)$ has dimension 1 by Lemma 2.6.1(ii). ■

Lemma 2.6.3. *Abbreviate $u = \hat{C} - |C|\hat{x}$.*

(i) *The vectors $\{A_i u\}_{i=0}^{d-1}$ form a basis for $\mathcal{M}\hat{C}^\perp$.*

(ii) *The \mathcal{M} -module $\mathcal{M}\hat{C}^\perp$ is generated by u .*

Proof. (i) For $0 \leq i \leq d-1$, evaluate $A_i u$ using Lemma 2.3.16(ii). Express the result as a linear combination of $\{\hat{C}_j^\pm\}_{j=0}^{d-1}$ using (2.66) and (2.77). We find

$$A_i u = (|C| - N_{i-1})\hat{C}_{i-1}^- + (-N_{i-1})\hat{C}_{i-1}^+ + (N_i - |C|)\hat{C}_i^- + N_i\hat{C}_i^+. \quad (2.99)$$

Consider the last term on the right-hand side of (2.99). Recall N_i is nonzero by Corollary 2.3.6. Therefore the $\{A_i u\}_{i=0}^{d-1}$ are linearly independent. The result follows since $\dim \mathcal{M}\hat{C}^\perp = d$.

(ii) Consider $\mathcal{M}u = \text{Span}\{A_i u\}_{i=0}^d$. Since $u \in \mathcal{M}\hat{C}^\perp$, we have $\mathcal{M}u \subseteq \mathcal{M}\hat{C}^\perp$. But $\mathcal{M}u$ contains $\text{Span}\{A_i u\}_{i=0}^{d-1}$, which is equal to $\mathcal{M}\hat{C}^\perp$ by part (i). By these comments $\mathcal{M}u = \mathcal{M}\hat{C}^\perp$.

The result follows. ■

Proposition 2.6.4. *The \tilde{T} -module $\mathcal{M}\hat{C}^\perp$ is irreducible.*

Proof. Abbreviate $u = \hat{C} - |C|\hat{x}$. Express $\mathcal{M}\hat{C}^\perp$ as the orthogonal direct sum of irreducible \tilde{T} -modules. Since $u \in \mathcal{M}\hat{C}^\perp$, among these irreducible \tilde{T} -modules there exists one, denoted U , that is not orthogonal to u . Observe $\tilde{E}_0^*U \neq 0$. Also $\tilde{E}_0^*U \subseteq \tilde{E}_0^*(\mathcal{M}\hat{C}^\perp) = \text{Span}\{u\}$, so \tilde{E}_0^*U is spanned by u . By this and since U is \tilde{T} -module, we get $u \in \tilde{E}_0^*U \subseteq U$. But then $\mathcal{M}u \subseteq U$. By this and Lemma 2.6.3(ii) $\mathcal{M}\hat{C}^\perp \subseteq U$ and therefore $\mathcal{M}\hat{C}^\perp = U$, as desired. ■

We now show that the irreducible \tilde{T} -module $\mathcal{M}\hat{C}^\perp$ supports a Leonard system.

Lemma 2.6.5. *The following matrices*

$$(A; \tilde{A}^*; \{E_i\}_{i=1}^d; \{\tilde{E}_i^*\}_{i=0}^{d-1}) \quad (2.100)$$

act on $\mathcal{M}\hat{C}^\perp$ as a Leonard system.

Proof. We show that (2.100) satisfies conditions (i)–(v) in Definition 2.1.1. Conditions (i)–(iii) follow from (2.51), Corollary 2.3.15 and Lemma 2.6.1, condition (iv) follows from (2.25) and (2.50), and condition (v) follows from [21, Lemma 4.1]. ■

We let $\tilde{\Phi}^\perp$ denote the Leonard system on $\mathcal{M}\hat{C}^\perp$ from Lemma 2.6.5. We denote the parameter array of $\tilde{\Phi}^\perp$ by

$$p(\tilde{\Phi}^\perp) = (\{\tilde{\theta}_i^\perp\}_{i=0}^{d-1}; \{\tilde{\theta}_i^{*\perp}\}_{i=0}^{d-1}; \{\tilde{\varphi}_i^\perp\}_{i=1}^{d-1}; \{\tilde{\phi}_i^\perp\}_{i=1}^{d-1}). \quad (2.101)$$

Lemma 2.6.6. *With the above notation,*

$$\tilde{\theta}_i^\perp = \theta_{i+1}, \quad \tilde{\theta}_i^{*\perp} = \tilde{\theta}_i^* \quad (0 \leq i \leq d-1). \quad (2.102)$$

Proof. By construction, $AE_{i+1} = \tilde{\theta}_i^\perp E_{i+1}$ on $\mathcal{M}\hat{C}^\perp$. But $AE_{i+1} = \theta_{i+1}E_{i+1}$, so $\tilde{\theta}_i^\perp = \theta_{i+1}$. Similarly we get $\tilde{\theta}_i^{*\perp} = \tilde{\theta}_i^*$. ■

In Lemma 2.3.19, we saw that the $\{\hat{C}_i\}_{i=0}^{d-1}$ form a $\tilde{\Phi}$ -standard basis for $\mathcal{M}\hat{C}$. To keep our notation consistent, define $\tilde{v}_i = \hat{C}_i$ for $0 \leq i \leq d-1$. By (2.66),

$$\tilde{v}_i = \hat{C}_i^- + \hat{C}_i^+ \quad (0 \leq i \leq d-1). \quad (2.103)$$

Our next goal is to find a $\tilde{\Phi}^\perp$ -standard basis for $\mathcal{M}\hat{C}^\perp$. Define the scalars

$$\tau_i := -\frac{|C_i^+|}{|C_i^-|} \quad (0 \leq i \leq d-1). \quad (2.104)$$

By Corollary 2.4.5, τ_i is well-defined and nonzero for $0 \leq i \leq d-1$.

Lemma 2.6.7. *For $0 \leq i \leq d-1$,*

$$\tau_i = \frac{s^*(1-r_1q^{i+1})(1-r_2q^{i+1})}{(r_1-s^*q^{i+1})(r_2-s^*q^{i+1})}. \quad (2.105)$$

Proof. Evaluate (2.104) using Lemma 2.4.4 and eliminate the scalars $\{\tilde{b}_j\}_{j=0}^{i-1}, \{\tilde{c}_j\}_{j=1}^i$ using Theorem 2.3.7. Simplify the result to find

$$\tau_i = \frac{N_i - |C|}{N_i}.$$

Evaluate this equation using Lemma 2.3.5 to obtain the result. ■

Lemma 2.6.8. *For $0 \leq i \leq d-1$ the vector*

$$\tau_i \hat{C}_i^- + \hat{C}_i^+$$

is a basis for $\tilde{E}_i^(\mathcal{M}\hat{C}^\perp)$.*

Proof. By Lemma 2.4.17, $0 \neq \tau_i \hat{C}_i^- + \hat{C}_i^+ \in \tilde{E}_i^* \mathbf{W}$. By (2.98) the subspace $\tilde{E}_i^*(\mathcal{M}\hat{C}^\perp)$ is the orthogonal complement of $\tilde{E}_i^* \mathcal{M}\hat{C}$ in $\tilde{E}_i^* \mathbf{W}$. Recall \tilde{v}_i is a basis for $\tilde{E}_i^* \mathcal{M}\hat{C}$. Using (2.103),

$$\langle \tilde{v}_i, \tau_i \hat{C}_i^- + \hat{C}_i^+ \rangle = \tau_i \|\hat{C}_i^-\|^2 + \|\hat{C}_i^+\|^2 = \tau_i |C_i^-| + |C_i^+| = 0.$$

Therefore $\tau_i \hat{C}_i^- + \hat{C}_i^+ \in \tilde{E}_i^*(\mathcal{M}\hat{C}^\perp)$. The result follows since $\tilde{E}_i^*(\mathcal{M}\hat{C}^\perp)$ has dimension 1 by Lemma 2.6.1(ii). ■

Corollary 2.6.9. *The vectors*

$$\tau_i \hat{C}_i^- + \hat{C}_i^+ \quad (0 \leq i \leq d-1)$$

form an orthogonal basis for $\mathcal{M}\hat{C}^\perp$.

Proof. Follows from Lemma 2.6.8. ■

As we will see, the vectors $\{\tau_i \hat{C}_i^- + \hat{C}_i^+\}_{i=0}^{d-1}$ do not form a $\tilde{\Phi}^\perp$ -standard basis for $\mathcal{M}\hat{C}^\perp$. In order to turn it into a $\tilde{\Phi}^\perp$ -standard basis we make an adjustment. Pick a nonzero $\tilde{w} \in E_1(\mathcal{M}\hat{C}^\perp)$. Define the complex scalars $\{\zeta_i\}_{i=0}^{d-1}$ such that

$$\tilde{w} = \sum_{i=0}^{d-1} \zeta_i (\tau_i \hat{C}_i^- + \hat{C}_i^+). \quad (2.106)$$

Define

$$\tilde{v}_i^\perp = \zeta_i (\tau_i \hat{C}_i^- + \hat{C}_i^+) \quad (0 \leq i \leq d-1). \quad (2.107)$$

For notational convenience define $\tilde{v}_{-1}^\perp = 0$ and $\tilde{v}_d^\perp = 0$.

Lemma 2.6.10. *The vectors $\{\tilde{v}_i^\perp\}_{i=0}^{d-1}$ form a $\tilde{\Phi}^\perp$ -standard basis for $\mathcal{M}\hat{C}^\perp$.*

Proof. By Lemma 2.6.8, $\tilde{v}_i^\perp \in \tilde{E}_i^*(\mathcal{M}\hat{C}^\perp)$ for $0 \leq i \leq d-1$. Also, $\sum_{i=0}^{d-1} \tilde{v}_i^\perp = \tilde{w} \in E_1(\mathcal{M}\hat{C}^\perp)$. The result follows from these comments and Lemma 2.1.3. ■

Corollary 2.6.11. *The scalars $\{\zeta_i\}_{i=0}^{d-1}$ from (2.106) are all nonzero.*

Proof. By Lemma 2.6.10 $\{\tilde{v}_i^\perp\}_{i=0}^{d-1}$ are all nonzero. By this and (2.107) the result follows. ■

The matrix representing \tilde{A}^* relative to the basis $\{\tilde{v}_i^\perp\}_{i=0}^{d-1}$ is

$$\text{diag}(\tilde{\theta}_0^{*\perp}, \tilde{\theta}_1^{*\perp}, \tilde{\theta}_2^{*\perp}, \dots, \tilde{\theta}_{d-1}^{*\perp}). \quad (2.108)$$

Let $\{\tilde{a}_i^\perp\}_{i=0}^{d-1}$, $\{\tilde{b}_i^\perp\}_{i=0}^{d-2}$, $\{\tilde{c}_i^\perp\}_{i=1}^{d-1}$ denote the intersection numbers of the Leonard system $\tilde{\Phi}^\perp$. By construction the matrix representing A relative to $\{\tilde{v}_i^\perp\}_{i=0}^{d-1}$ is

$$\begin{bmatrix} \tilde{a}_0^\perp & \tilde{b}_0^\perp & & & \mathbf{0} \\ \tilde{c}_1^\perp & \tilde{a}_1^\perp & \tilde{b}_1^\perp & & \\ & \tilde{c}_2^\perp & \tilde{a}_2^\perp & \ddots & \\ & & \ddots & \ddots & \tilde{b}_{d-2}^\perp \\ \mathbf{0} & & & \tilde{c}_{d-1}^\perp & \tilde{a}_{d-1}^\perp \end{bmatrix}. \quad (2.109)$$

For convenience define $\tilde{b}_{-1}^\perp = 0$ and $\tilde{c}_d^\perp = 0$. Recall the scalars a_i, b_i, c_i from below Lemma 2.2.3 and the $\tilde{a}_i, \tilde{b}_i, \tilde{c}_i$ from (2.44).

Lemma 2.6.12. *With the above notation the following (i)–(vi) hold.*

- (i) $\zeta_{i-1}\tau_{i-1}\tilde{b}_{i-1}^\perp = \zeta_i\tau_i\tilde{b}_{i-1} \quad (1 \leq i \leq d-1),$
- (ii) $\zeta_{i-1}\tilde{b}_{i-1}^\perp = \zeta_i\tau_i(\tilde{b}_{i-1} - b_i) + \zeta_i b_i \quad (1 \leq i \leq d-1),$
- (iii) $\tilde{a}_i^\perp\tau_i = \tau_i(\tilde{a}_i - b_i + \tilde{b}_i) + (b_i - \tilde{b}_i) \quad (0 \leq i \leq d-1),$
- (iv) $\tilde{a}_i^\perp = \tau_i(c_{i+1} - \tilde{c}_i) + \tilde{a}_i - c_{i+1} + \tilde{c}_i \quad (0 \leq i \leq d-1),$
- (v) $\zeta_{i+1}\tau_{i+1}\tilde{c}_{i+1}^\perp = \zeta_i\tau_i c_{i+1} + \zeta_i(\tilde{c}_{i+1} - c_{i+1}) \quad (0 \leq i \leq d-2),$
- (vi) $\zeta_{i+1}\tilde{c}_{i+1}^\perp = \zeta_i\tilde{c}_{i+1} \quad (0 \leq i \leq d-2).$

Proof. For $0 \leq i \leq d-1$ we evaluate $A\tilde{v}_i^\perp$ in two ways. First, using (2.109),

$$A\tilde{v}_i^\perp = \tilde{b}_{i-1}^\perp \tilde{v}_{i-1}^\perp + \tilde{a}_i^\perp \tilde{v}_i^\perp + \tilde{c}_{i+1}^\perp \tilde{v}_{i+1}^\perp.$$

In this equation evaluate the right-hand side using (2.107). Secondly, in $A\tilde{v}_i^\perp$ eliminate \tilde{v}_i^\perp using (2.107) and evaluate the result using Lemma 2.4.9. We have evaluated $A\tilde{v}_i^\perp$ in two ways. Compare the results using the linear independence of $\{\hat{C}_j^\pm\}_{j=0}^{d-1}$. The result follows. ■

Corollary 2.6.13. *Referring to Lemma 2.6.12, the following holds.*

$$(i) \quad \tilde{b}_i^\perp = \frac{\zeta_{i+1}\tau_{i+1}}{\zeta_i\tau_i} \tilde{b}_i \quad (0 \leq i \leq d-2).$$

$$(ii) \quad \tilde{a}_i^\perp = b_0 - b_i - c_{i+1} \quad (0 \leq i \leq d-1).$$

$$(iii) \quad \tilde{c}_i^\perp = \frac{\zeta_{i-1}}{\zeta_i} \tilde{c}_i \quad (1 \leq i \leq d-1).$$

Proof. (i) Follows from Lemma 2.6.12(i).

(ii) In the equation from Lemma 2.6.12(iv), eliminate τ_i using Corollary 2.4.3(iii) along with (2.104) and simplify the result using $\tilde{a}_i + \tilde{b}_i + \tilde{c}_i = b_0$.

(iii) Follows from Lemma 2.6.12(vi). ■

Our next goal is to show that the Leonard system $\tilde{\Phi}^\perp$ has q -Racah type. To this end we compare the parameter arrays $p(\Phi)$ and $p(\tilde{\Phi}^\perp)$. Recall the scalars h, h^*, s, s^*, r_1, r_2 from above Note 2.2.4. Recall the parameter array $p(\tilde{\Phi}^\perp)$ from (2.101).

Theorem 2.6.14. *With the above notation, for $0 \leq i \leq d-1$,*

$$\tilde{\theta}_i^\perp = \tilde{\theta}_0^\perp + \tilde{h}^\perp (1-q^i)(1-\tilde{s}^\perp q^{i+1})q^{-i}, \quad (2.110)$$

$$\tilde{\theta}_i^{*\perp} = \tilde{\theta}_0^{*\perp} + \tilde{h}^{*\perp} (1-q^i)(1-\tilde{s}^{*\perp} q^{i+1})q^{-i}, \quad (2.111)$$

and for $1 \leq i \leq d-1$,

$$\tilde{\varphi}_i^\perp = \tilde{h}^\perp \tilde{h}^{*\perp} q^{1-2i} (1-q^i)(1-q^{i-d})(1-\tilde{r}_1^\perp q^i)(1-\tilde{r}_2^\perp q^i), \quad (2.112)$$

$$\tilde{\phi}_i^\perp = \tilde{h}^\perp \tilde{h}^{*\perp} q^{1-2i} (1-q^i)(1-q^{i-d})(\tilde{r}_1^\perp - \tilde{s}^{*\perp} q^i)(\tilde{r}_2^\perp - \tilde{s}^{*\perp} q^i) / \tilde{s}^{*\perp}, \quad (2.113)$$

where $\tilde{\theta}_0^\perp, \tilde{\theta}_0^{*\perp}$ are from (2.102) and

$$\tilde{h}^\perp = hq^{-1}, \quad \tilde{s}^\perp = sq^2, \quad \tilde{r}_1^\perp = r_1q, \quad (2.114)$$

$$\tilde{h}^{*\perp} = \frac{s^*q^{-1}-r_1r_2}{s^*-r_1r_2} h^*, \quad \tilde{s}^{*\perp} = s^*q, \quad \tilde{r}_2^\perp = r_2q. \quad (2.115)$$

Proof. Recall that the Leonard system $\tilde{\Phi}^\perp$ has diameter $d-1$. To get (2.110), (2.111) use (2.13), (2.53) and Lemma 2.6.6. We now verify (2.112), (2.113). To this end, recall the intersection numbers $\{\tilde{a}_j^\perp\}_{j=0}^{d-1}$ of $\tilde{\Phi}^\perp$. Using (2.10), $\tilde{\varphi}_1^\perp = (\tilde{a}_0^\perp - \tilde{\theta}_0^\perp)(\tilde{\theta}_0^{*\perp} - \tilde{\theta}_1^{*\perp})$. Evaluate this using Lemma 2.6.6 and Corollary 2.6.13(ii) to get

$$\tilde{\varphi}_1^\perp = (c_1 + \theta_1)(\tilde{\theta}_1^* - \tilde{\theta}_0^*). \quad (2.116)$$

Evaluate the right-hand side of (2.116) using (2.13), (2.19) and Lemma 2.3.14, and simplify the result using (2.114), (2.115). This yields (2.112) for $i=1$. Using this together with (2.110), (2.111) and the condition (PA4) of Theorem 2.1.2 we obtain (2.113) for $1 \leq i \leq d-1$. We now use (2.113) at $i=1$ together with (2.110), (2.111) and the condition (PA3) of Theorem 2.1.2 to obtain (2.112) for $1 \leq i \leq d-1$. The result follows. ■

Corollary 2.6.15. *The Leonard system $\tilde{\Phi}^\perp$ has q -Racah type.*

Proof. Compare Example 2.1.8 and Theorem 2.6.14. ■

Corollary 2.6.16. *With the above notation,*

$$\tilde{b}_0^\perp = \frac{h(1 - q^{-d+1})(1 - r_1 q^2)(1 - r_2 q^2)}{q(1 - s^* q^3)}, \quad (2.117)$$

$$\tilde{b}_i^\perp = \frac{h(1 - q^{i-d+1})(1 - s^* q^{i+2})(1 - r_1 q^{i+2})(1 - r_2 q^{i+2})}{q(1 - s^* q^{2i+2})(1 - s^* q^{2i+3})} \quad (1 \leq i \leq d-2), \quad (2.118)$$

$$\tilde{c}_i^\perp = \frac{h(1 - q^i)(1 - s^* q^{i+d+1})(r_1 - s^* q^i)(r_2 - s^* q^i)}{s^* q^{d-1}(1 - s^* q^{2i+1})(1 - s^* q^{2i+2})} \quad (1 \leq i \leq d-2), \quad (2.119)$$

$$\tilde{c}_{d-1}^\perp = \frac{h(1 - q^{d-1})(r_1 - s^* q^{d-1})(r_2 - s^* q^{d-1})}{s^* q^{d-1}(1 - s^* q^{2d-1})}. \quad (2.120)$$

Proof. Use (2.17)–(2.20) and (2.114), (2.115). ■

We finish this section with some comments about the scalars $\{\zeta_i\}_{i=0}^{d-1}$ from (2.106).

Lemma 2.6.17. *The vector \tilde{w} in line (2.106) can be chosen such that*

$$\zeta_i = q^{-i}(r_1 - s^* q^{i+1})(r_2 - s^* q^{i+1}) \quad (0 \leq i \leq d-1). \quad (2.121)$$

Proof. Observe that the vector \tilde{w} is defined up to multiplication by a nonzero scalar in \mathbb{C} . Multiplying \tilde{w} by a nonzero scalar if necessary, we may assume that

$$\zeta_0 = (r_1 - s^* q)(r_2 - s^* q). \quad (2.122)$$

Using Lemma 2.6.12(vi) and induction on i ,

$$\zeta_i = \frac{\tilde{c}_1 \tilde{c}_2 \cdots \tilde{c}_i}{\tilde{c}_1^\perp \tilde{c}_2^\perp \cdots \tilde{c}_i^\perp} \zeta_0 \quad (0 \leq i \leq d-1). \quad (2.123)$$

Evaluate (2.123) using (2.47), (2.48), (2.119), (2.120) and (2.122) to get (2.121). The result follows. ■

From now on we assume that the vector \tilde{w} in line (2.106) has been chosen such that (2.121) holds. Recall the $\tilde{\Phi}^\perp$ -standard basis $\{\tilde{v}_i^\perp\}_{i=0}^{d-1}$ for $\mathcal{M}\hat{C}^\perp$ from (2.107).

Lemma 2.6.18. For $0 \leq i \leq d-1$,

$$\tilde{v}_i^\perp = q^{-i}s^*(1-r_1q^{i+1})(1-r_2q^{i+1})\hat{C}_i^- + q^{-i}(r_1-s^*q^{i+1})(r_2-s^*q^{i+1})\hat{C}_i^+.$$

Proof. Evaluate (2.107) using (2.105) and (2.121). ■

2.7 The maps \mathfrak{p} and $\tilde{\mathfrak{p}}$

Recall the subspace \mathbf{W} from above line (2.66). In this section we introduce two \mathbb{C} -linear maps $\mathfrak{p} : \mathbf{W} \rightarrow \mathbf{W}$ and $\tilde{\mathfrak{p}} : \mathbf{W} \rightarrow \mathbf{W}$. These maps are defined as follows. In (2.73) and (2.98) we obtained the following direct sum decompositions of \mathbf{W} :

$$\mathbf{W} = \mathcal{M}\hat{x} + \mathcal{M}\hat{x}^\perp, \quad \mathbf{W} = \mathcal{M}\hat{C} + \mathcal{M}\hat{C}^\perp.$$

Define a \mathbb{C} -linear map $\mathfrak{p} : \mathbf{W} \rightarrow \mathbf{W}$ such that $(\mathfrak{p} - 1)\mathcal{M}\hat{x} = 0$ and $\mathfrak{p}(\mathcal{M}\hat{x}^\perp) = 0$. Thus \mathfrak{p} is the projection from \mathbf{W} onto $\mathcal{M}\hat{x}$. Define a \mathbb{C} -linear map $\tilde{\mathfrak{p}} : \mathbf{W} \rightarrow \mathbf{W}$ such that $(\tilde{\mathfrak{p}} - 1)\mathcal{M}\hat{C} = 0$ and $\tilde{\mathfrak{p}}(\mathcal{M}\hat{C}^\perp) = 0$. Thus $\tilde{\mathfrak{p}}$ is the projection from \mathbf{W} onto $\mathcal{M}\hat{C}$.

Lemma 2.7.1. For all $B \in T$ the action of B on \mathbf{W} commutes with \mathfrak{p} . In particular, on \mathbf{W} each of A, A^* commutes with \mathfrak{p} .

Proof. For $w \in \mathbf{W}$ we show $B\mathfrak{p}w = \mathfrak{p}Bw$. Write $w = u + v$, where $u \in \mathcal{M}\hat{x}$ and $v \in \mathcal{M}\hat{x}^\perp$. Observe $\mathfrak{p}u = u$ and $\mathfrak{p}v = 0$, so $B\mathfrak{p}w = Bu$. Each of $\mathcal{M}\hat{x}, \mathcal{M}\hat{x}^\perp$ is a T -module, so $Bu \in \mathcal{M}\hat{x}$ and $Bv \in \mathcal{M}\hat{x}^\perp$. Now $\mathfrak{p}Bu = Bu$ and $\mathfrak{p}Bv = 0$, so $\mathfrak{p}Bw = Bu$. The result follows. ■

Lemma 2.7.2. For all $B \in \tilde{T}$ the action of B on \mathbf{W} commutes with $\tilde{\mathfrak{p}}$. In particular, on \mathbf{W} each of A, \tilde{A}^* commutes with $\tilde{\mathfrak{p}}$.

Proof. Similar to the proof of Lemma 2.7.1. ■

Our next goal is to show how \mathfrak{p} acts on the basis $\{\hat{C}_i^\pm\}_{i=0}^{d-1}$ for \mathbf{W} . Recall from (2.77) the Φ -standard basis $\{v_i\}_{i=0}^d$ for $\mathcal{M}\hat{x}$, and from (2.81) the Φ^\perp -standard basis $\{v_i^\perp\}_{i=0}^{d-2}$ for $\mathcal{M}\hat{x}^\perp$. By construction,

$$\mathfrak{p}v_i = v_i \quad (0 \leq i \leq d), \quad \mathfrak{p}v_i^\perp = 0 \quad (0 \leq i \leq d-2). \quad (2.124)$$

Lemma 2.7.3. *For $1 \leq i \leq d-1$,*

$$\begin{aligned} \hat{C}_{i-1}^+ &= \frac{\epsilon_i}{\epsilon_i-1}v_i + \frac{1}{\xi_i(1-\epsilon_i)}v_{i-1}^\perp, \\ \hat{C}_i^- &= \frac{1}{1-\epsilon_i}v_i + \frac{1}{\xi_i(\epsilon_i-1)}v_{i-1}^\perp. \end{aligned}$$

Moreover

$$\hat{C}_0^- = v_0, \quad \hat{C}_{d-1}^+ = v_d.$$

Proof. Use (2.77) and (2.81). ■

Lemma 2.7.4. *The map \mathfrak{p} acts on $\{\hat{C}_i^\pm\}_{i=0}^{d-1}$ as follows. For $1 \leq i \leq d-1$,*

$$\mathfrak{p}\hat{C}_i^- = \frac{1}{1-\epsilon_i}(\hat{C}_{i-1}^+ + \hat{C}_i^-), \quad \mathfrak{p}\hat{C}_{i-1}^+ = \frac{\epsilon_i}{\epsilon_i-1}(\hat{C}_{i-1}^+ + \hat{C}_i^-).$$

Moreover $\mathfrak{p}\hat{C}_0^- = \hat{C}_0^-$ and $\mathfrak{p}\hat{C}_{d-1}^+ = \hat{C}_{d-1}^+$.

Proof. By (2.124) and Lemma 2.7.3. ■

We now show how $\tilde{\mathfrak{p}}$ acts on the basis $\{\hat{C}_i^\pm\}_{i=0}^{d-1}$ for \mathbf{W} . Recall from (2.103) the $\tilde{\Phi}$ -standard basis $\{\tilde{v}_i\}_{i=0}^{d-1}$ for $\mathcal{M}\hat{C}$, and from (2.107) the $\tilde{\Phi}^\perp$ -standard basis $\{\tilde{v}_i^\perp\}_{i=0}^{d-1}$ for $\mathcal{M}\hat{C}^\perp$. By construction,

$$\tilde{\mathfrak{p}}\tilde{v}_i = \tilde{v}_i, \quad \tilde{\mathfrak{p}}\tilde{v}_i^\perp = 0 \quad (0 \leq i \leq d-1). \quad (2.125)$$

Lemma 2.7.5. For $0 \leq i \leq d-1$,

$$\begin{aligned}\hat{C}_i^- &= \frac{1}{1-\tau_i} \tilde{v}_i + \frac{1}{\zeta_i(\tau_i-1)} \tilde{v}_i^\perp, \\ \hat{C}_i^+ &= \frac{\tau_i}{\tau_i-1} \tilde{v}_i + \frac{1}{\zeta_i(1-\tau_i)} \tilde{v}_i^\perp.\end{aligned}$$

Proof. Use (2.103) and (2.107). ■

Lemma 2.7.6. The map $\tilde{\mathfrak{p}}$ acts on $\{\hat{C}_i^\pm\}_{i=0}^{d-1}$ as follows. For $0 \leq i \leq d-1$,

$$\tilde{\mathfrak{p}}\hat{C}_i^- = \frac{1}{1-\tau_i}(\hat{C}_i^- + \hat{C}_i^+), \quad \tilde{\mathfrak{p}}\hat{C}_i^+ = \frac{\tau_i}{\tau_i-1}(\hat{C}_i^- + \hat{C}_i^+).$$

Proof. By (2.125) and Lemma 2.7.5. ■

2.8 Five bases and five linear maps for \mathbf{W}

Recall the subspace \mathbf{W} from above line (2.66). In this section we display five bases for \mathbf{W} . We then display the transition matrices between certain pairs of bases among the five. We also display the matrix representations of $A, A^*, \tilde{A}^*, \mathfrak{p}, \tilde{\mathfrak{p}}$ relative to these five bases.

We now define our five bases for \mathbf{W} . Recall the vectors $\{\hat{C}_i^\pm\}_{i=0}^{d-1}$ from Lemma 2.4.7.

The first basis for \mathbf{W} is

$$\mathfrak{e} := \{\hat{C}_0^-, \hat{C}_0^+, \hat{C}_1^-, \hat{C}_1^+, \hat{C}_2^-, \hat{C}_2^+, \dots, \hat{C}_{d-1}^-, \hat{C}_{d-1}^+\}. \quad (2.126)$$

In Section 2.5 we saw the Φ -standard basis $\{v_i\}_{i=0}^d$ for $\mathcal{M}\hat{x}$ and the Φ^\perp -standard basis $\{v_i^\perp\}_{i=0}^{d-2}$ for $\mathcal{M}\hat{x}^\perp$. The second basis for \mathbf{W} is

$$\mathfrak{B} := \{v_0, v_1, v_2, \dots, v_d, v_0^\perp, v_1^\perp, v_2^\perp, \dots, v_{d-2}^\perp\}. \quad (2.127)$$

The third basis for \mathbf{W} is

$$\mathcal{B}_{alt} := \{v_0, v_1, v_0^\perp, v_2, v_1^\perp, v_3, v_2^\perp, \dots, v_{d-1}, v_{d-2}^\perp, v_d\}. \quad (2.128)$$

In Section 2.6 we saw the $\tilde{\Phi}$ -standard basis $\{\tilde{v}_i\}_{i=0}^{d-1}$ for $\mathcal{M}\hat{C}$ and the $\tilde{\Phi}^\perp$ -standard basis $\{\tilde{v}_i^\perp\}_{i=0}^{d-1}$ for $\mathcal{M}\hat{C}^\perp$. The fourth basis for \mathbf{W} is

$$\tilde{\mathcal{B}} := \{\tilde{v}_0, \tilde{v}_1, \tilde{v}_2, \dots, \tilde{v}_{d-1}, \tilde{v}_0^\perp, \tilde{v}_1^\perp, \tilde{v}_2^\perp, \dots, \tilde{v}_{d-1}^\perp\}. \quad (2.129)$$

The fifth basis for \mathbf{W} is

$$\tilde{\mathcal{B}}_{alt} := \{\tilde{v}_0, \tilde{v}_0^\perp, \tilde{v}_1, \tilde{v}_1^\perp, \tilde{v}_2, \tilde{v}_2^\perp, \dots, \tilde{v}_{d-1}, \tilde{v}_{d-1}^\perp\}. \quad (2.130)$$

We now describe the transition matrices between certain pairs of bases among the five. A pair of bases will be considered whenever they are adjacent in the following diagram:

$$\mathcal{B} \quad \text{---} \quad \mathcal{B}_{alt} \quad \text{---} \quad \mathcal{C} \quad \text{---} \quad \tilde{\mathcal{B}}_{alt} \quad \text{---} \quad \tilde{\mathcal{B}} \quad (2.131)$$

Lemma 2.8.1. *The transition matrix from \mathcal{C} to \mathcal{B}_{alt} is*

$$\text{blockdiag}[\mathbf{T}_0, \mathbf{T}_1, \mathbf{T}_2 \cdots, \mathbf{T}_{d-1}, \mathbf{T}_d], \quad (2.132)$$

where

$$\mathbf{T}_0 = [1], \quad \mathbf{T}_d = [1],$$

and for $1 \leq i \leq d-1$,

$$\mathbf{T}_i = \begin{bmatrix} 1 & \xi_i \\ 1 & \xi_i \epsilon_i \end{bmatrix},$$

where

$$\xi_i = q^{1-i}(1 - q^{i-d})(1 - s^*q^{i+1}), \quad \xi_i \epsilon_i = q^{1-i-d}(1 - q^i)(1 - s^*q^{i+d+1}).$$

The transition matrix from \mathcal{B}_{alt} to \mathcal{C} is

$$\text{blockdiag}[\mathbf{S}_0, \mathbf{S}_1, \mathbf{S}_2, \dots, \mathbf{S}_{d-1}, \mathbf{S}_d], \quad (2.133)$$

where

$$\mathbf{S}_0 = [1], \quad \mathbf{S}_d = [1],$$

and for $1 \leq i \leq d-1$,

$$\mathbf{S}_i = \begin{bmatrix} \frac{\epsilon_i}{\epsilon_i - 1} & \frac{1}{1 - \epsilon_i} \\ \frac{1}{\xi_i(1 - \epsilon_i)} & \frac{1}{\xi_i(\epsilon_i - 1)} \end{bmatrix},$$

where

$$\begin{aligned} \frac{\epsilon_i}{\epsilon_i - 1} &= \frac{(1 - q^i)(1 - s^* q^{i+d+1})}{(1 - q^d)(1 - s^* q^{2i+1})}, & \frac{1}{1 - \epsilon_i} &= \frac{q^d(1 - q^{i-d})(1 - s^* q^{i+1})}{(q^d - 1)(1 - s^* q^{2i+1})}, \\ \frac{1}{\xi_i(\epsilon_i - 1)} &= \frac{q^{d+i-1}}{(1 - q^d)(1 - s^* q^{2i+1})}, & \frac{1}{\xi_i(1 - \epsilon_i)} &= \frac{q^{d+i-1}}{(q^d - 1)(1 - s^* q^{2i+1})}. \end{aligned}$$

Proof. Use (2.77) and (2.81) to obtain the matrix (2.132). Use Lemma 2.7.3 to obtain the matrix (2.133). ■

Lemma 2.8.2. The transition matrix from \mathcal{C} to $\tilde{\mathcal{B}}_{alt}$ is

$$\text{blockdiag}[\mathbf{R}_0, \mathbf{R}_1, \mathbf{R}_2, \dots, \mathbf{R}_{d-1}], \quad (2.134)$$

where for $0 \leq i \leq d-1$,

$$\mathbf{R}_i = \begin{bmatrix} 1 & \zeta_i \tau_i \\ 1 & \zeta_i \end{bmatrix},$$

and

$$\zeta_i \tau_i = q^{-i} s^* (1 - r_1 q^{i+1})(1 - r_2 q^{i+1}), \quad \zeta_i = q^{-i} (r_1 - s^* q^{i+1})(r_2 - s^* q^{i+1}).$$

The transition matrix from $\tilde{\mathcal{B}}_{alt}$ to \mathcal{C} is

$$\text{blockdiag}[\mathbf{Q}_0, \mathbf{Q}_1, \mathbf{Q}_2, \dots, \mathbf{Q}_{d-1}], \quad (2.135)$$

where for $0 \leq i \leq d-1$,

$$\mathbf{Q}_i = \begin{bmatrix} \frac{1}{1-\tau_i} & \frac{\tau_i}{\tau_i-1} \\ \frac{1}{\zeta_i(\tau_i-1)} & \frac{1}{\zeta_i(1-\tau_i)} \end{bmatrix},$$

and

$$\begin{aligned} \frac{1}{1-\tau_i} &= \frac{(r_1-s^*q^{i+1})(r_2-s^*q^{i+1})}{(r_1r_2-s^*)(1-s^*q^{2i+2})}, & \frac{\tau_i}{\tau_i-1} &= \frac{s^*(1-r_1q^{i+1})(1-r_2q^{i+1})}{(s^*-r_1r_2)(1-s^*q^{2i+2})}, \\ \frac{1}{\zeta_i(\tau_i-1)} &= \frac{q^i}{(s^*-r_1r_2)(1-s^*q^{2i+2})}, & \frac{1}{\zeta_i(1-\tau_i)} &= \frac{q^i}{(r_1r_2-s^*)(1-s^*q^{2i+2})}. \end{aligned}$$

Proof. Use (2.103) and (2.107) to obtain the matrix (2.134). Use Lemma 2.7.5 to obtain the matrix (2.135). ■

For $0 \leq j \leq d-1$ let \mathbf{e}_j denote the column vector with rows indexed by $0, 1, 2, \dots, 2d-1$, and with a 1 in the j -th coordinate and 0 in every other coordinate.

Lemma 2.8.3. *The transition matrix from \mathcal{B} to \mathcal{B}_{alt} is the permutation matrix*

$$[\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_{d+1}, \mathbf{e}_2, \mathbf{e}_{d+2}, \mathbf{e}_3, \mathbf{e}_{d+3}, \dots, \mathbf{e}_{d-1}, \mathbf{e}_{2d-1}, \mathbf{e}_d].$$

The transition matrix from \mathcal{B}_{alt} to \mathcal{B} is the permutation matrix

$$[\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_3, \mathbf{e}_5, \mathbf{e}_7, \dots, \mathbf{e}_{2d-1}, \mathbf{e}_2, \mathbf{e}_4, \mathbf{e}_6, \dots, \mathbf{e}_{2d-2}].$$

Proof. Compare (2.127) and (2.128). ■

Lemma 2.8.4. *The transition matrix from $\tilde{\mathcal{B}}$ to $\tilde{\mathcal{B}}_{alt}$ is the permutation matrix*

$$[\mathbf{e}_0, \mathbf{e}_d, \mathbf{e}_1, \mathbf{e}_{d+1}, \mathbf{e}_2, \mathbf{e}_{d+2}, \mathbf{e}_3, \mathbf{e}_{d+3}, \dots, \mathbf{e}_{d-1}, \mathbf{e}_{2d-1}].$$

The transition matrix from $\tilde{\mathcal{B}}_{alt}$ to $\tilde{\mathcal{B}}$ is the permutation matrix

$$[\mathbf{e}_0, \mathbf{e}_2, \mathbf{e}_4, \mathbf{e}_6, \dots, \mathbf{e}_{2d-2}, \mathbf{e}_1, \mathbf{e}_3, \mathbf{e}_5, \mathbf{e}_7, \dots, \mathbf{e}_{2d-1}].$$

Proof. Compare (2.129) and (2.130). ■

We have now found the transition matrices between every adjacent pair of bases in (2.131). For any pair of bases in (2.131) the transition matrix can be computed from the above lemmas and linear algebra. For example, the transition matrix from \mathcal{B}_{alt} to $\tilde{\mathcal{B}}_{alt}$ is the product of the transition matrix from \mathcal{B}_{alt} to \mathcal{C} given by (2.133), and the transition matrix from \mathcal{C} to $\tilde{\mathcal{B}}_{alt}$ given by (2.134).

We now display the matrices that represent

$$A, \quad A^*, \quad \tilde{A}^*, \quad \mathfrak{p}, \quad \tilde{\mathfrak{p}}$$

relative to the five bases (2.131). We first consider the matrices representing A relative to the five bases.

Lemma 2.8.5. *The matrix representing A relative to the basis \mathcal{C} is block tridiagonal:*

$$\begin{bmatrix} \mathbf{A}_0 & \mathbf{B}_0 & & & \mathbf{0} \\ \mathbf{C}_1 & \mathbf{A}_1 & \mathbf{B}_1 & & \\ & \mathbf{C}_2 & \mathbf{A}_2 & \ddots & \\ & & \ddots & \ddots & \mathbf{B}_{d-2} \\ \mathbf{0} & & & \mathbf{C}_{d-1} & \mathbf{A}_{d-1} \end{bmatrix}, \quad (2.136)$$

where

$$\begin{aligned} \mathbf{B}_i &= \begin{bmatrix} \tilde{b}_i & 0 \\ \tilde{b}_i - b_{i+1} & b_{i+1} \end{bmatrix} & (0 \leq i \leq d-2), \\ \mathbf{A}_i &= \begin{bmatrix} \tilde{a}_i - b_i + \tilde{b}_i & b_i - \tilde{b}_i \\ c_{i+1} - \tilde{c}_i & \tilde{a}_i - c_{i+1} + \tilde{c}_i \end{bmatrix} & (0 \leq i \leq d-1), \\ \mathbf{C}_i &= \begin{bmatrix} c_i & \tilde{c}_i - c_i \\ 0 & \tilde{c}_i \end{bmatrix} & (1 \leq i \leq d-1). \end{aligned}$$

Proof. Use Lemma 2.4.9. ■

Note that the matrix (2.136) is five-diagonal.

Lemma 2.8.6. *The matrix representing A relative to the basis \mathcal{B} is*

$$\begin{bmatrix} \mathbf{M}_0 & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_1 \end{bmatrix},$$

where

$$\mathbf{M}_0 = \begin{bmatrix} a_0 & b_0 & & & \\ c_1 & a_1 & b_1 & & \\ & c_2 & a_2 & \cdots & \\ & & \cdots & \cdots & b_{d-1} \\ & & & c_d & a_d \end{bmatrix}, \quad \mathbf{M}_1 = \begin{bmatrix} a_0^\perp & b_0^\perp & & & \\ c_1^\perp & a_1^\perp & b_1^\perp & & \\ & c_2^\perp & a_2^\perp & \cdots & \\ & & \cdots & \cdots & b_{d-3}^\perp \\ & & & c_{d-2}^\perp & a_{d-2}^\perp \end{bmatrix}.$$

Proof. By (2.7), (2.83) and (2.127). ■

Lemma 2.8.7. *The matrix representing A relative to the basis \mathcal{B}_{alt} is*

$$\begin{bmatrix} \mathbf{x}_0 & \mathbf{y}_0 & & & \mathbf{0} \\ & \mathbf{z}_1 & \mathbf{x}_1 & \mathbf{y}_1 & \\ & & \mathbf{z}_2 & \mathbf{x}_2 & \ddots \\ & & & \ddots & \ddots & \mathbf{y}_{d-1} \\ \mathbf{0} & & & & & \mathbf{z}_d & \mathbf{x}_d \end{bmatrix},$$

where

$$\begin{aligned} \mathbf{y}_0 &= \begin{bmatrix} b_0 & 0 \end{bmatrix}, & \mathbf{y}_i &= \begin{bmatrix} b_i & 0 \\ 0 & b_{i-1}^\perp \end{bmatrix} & (1 \leq i \leq d-2), & \mathbf{y}_{d-1} &= \begin{bmatrix} b_{d-1} \\ 0 \end{bmatrix}, \\ \mathbf{x}_0 &= \begin{bmatrix} a_0 \end{bmatrix}, & \mathbf{x}_i &= \begin{bmatrix} a_i & 0 \\ 0 & a_{i-1}^\perp \end{bmatrix} & (1 \leq i \leq d-1), & \mathbf{x}_d &= \begin{bmatrix} a_d \end{bmatrix}, \\ \mathbf{z}_1 &= \begin{bmatrix} c_1 \\ 0 \end{bmatrix}, & \mathbf{z}_i &= \begin{bmatrix} c_i & 0 \\ 0 & c_{i-1}^\perp \end{bmatrix} & (2 \leq i \leq d-1), & \mathbf{z}_d &= \begin{bmatrix} c_d & 0 \end{bmatrix}. \end{aligned}$$

Proof. Let $[A]_{\mathcal{B}}$ denote the matrix representing A relative to \mathcal{B} , and let L denote the transition matrix from \mathcal{B}_{alt} to \mathcal{B} . By linear algebra the matrix representing A relative to \mathcal{B}_{alt} is $L[A]_{\mathcal{B}}L^{-1}$. Evaluate this matrix using Lemma 2.8.3 and Lemma 2.8.6 to obtain the result. ■

Lemma 2.8.8. *The matrix representing A relative to the basis $\tilde{\mathcal{B}}$ is*

$$\begin{bmatrix} \tilde{\mathbf{M}}_0 & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{M}}_1 \end{bmatrix},$$

where

$$\widetilde{\mathbf{M}}_0 = \begin{bmatrix} \widetilde{a}_0 & \widetilde{b}_0 & & & & \\ \widetilde{c}_1 & \widetilde{a}_1 & \widetilde{b}_1 & & & \\ & \widetilde{c}_2 & \widetilde{a}_2 & \cdots & & \\ & & \cdots & \cdots & \widetilde{b}_{d-2} & \\ & & & \widetilde{c}_{d-1} & \widetilde{a}_{d-1} & \end{bmatrix}, \quad \widetilde{\mathbf{M}}_1 = \begin{bmatrix} \widetilde{a}_0^\perp & \widetilde{b}_0^\perp & & & & \\ \widetilde{c}_1^\perp & \widetilde{a}_1^\perp & \widetilde{b}_1^\perp & & & \\ & \widetilde{c}_2^\perp & \widetilde{a}_2^\perp & \cdots & & \\ & & \cdots & \cdots & \widetilde{b}_{d-2}^\perp & \\ & & & \widetilde{c}_{d-1}^\perp & \widetilde{a}_{d-1}^\perp & \end{bmatrix}.$$

Proof. By (2.58), (2.109) and (2.129). ■

Lemma 2.8.9. *The matrix representing A relative to the basis $\widetilde{\mathcal{B}}_{alt}$ is*

$$\begin{bmatrix} \widetilde{\mathbf{x}}_0 & \widetilde{\mathbf{y}}_0 & & & \mathbf{0} \\ \widetilde{\mathbf{z}}_1 & \widetilde{\mathbf{x}}_1 & \widetilde{\mathbf{y}}_1 & & \\ & \widetilde{\mathbf{z}}_2 & \widetilde{\mathbf{x}}_2 & \cdots & \\ & & \cdots & \cdots & \widetilde{\mathbf{y}}_{d-2} \\ \mathbf{0} & & & \widetilde{\mathbf{z}}_{d-1} & \widetilde{\mathbf{x}}_{d-1} \end{bmatrix},$$

where

$$\begin{aligned} \widetilde{\mathbf{y}}_i &= \begin{bmatrix} \widetilde{b}_i & 0 \\ 0 & \widetilde{b}_i^\perp \end{bmatrix} & (0 \leq i \leq d-2), \\ \widetilde{\mathbf{x}}_i &= \begin{bmatrix} \widetilde{a}_i & 0 \\ 0 & \widetilde{a}_i^\perp \end{bmatrix} & (0 \leq i \leq d-1), \\ \widetilde{\mathbf{z}}_i &= \begin{bmatrix} \widetilde{c}_i & 0 \\ 0 & \widetilde{c}_i^\perp \end{bmatrix} & (1 \leq i \leq d-1). \end{aligned}$$

Proof. Let $[A]_{\widetilde{\mathcal{B}}}$ denote the matrix representing A relative to $\widetilde{\mathcal{B}}$, and let L denote the transition matrix from $\widetilde{\mathcal{B}}_{alt}$ to $\widetilde{\mathcal{B}}$. By linear algebra the matrix representing A relative to

\mathcal{B}_{alt} is $L[A]_{\tilde{\mathcal{B}}}L^{-1}$. Evaluate this matrix using Lemma 2.8.4 and Lemma 2.8.8 to obtain the result. ■

We are done with A . We now consider the matrices representing A^* relative to the five bases.

Lemma 2.8.10. *The matrix representing A^* relative to the basis \mathcal{C} is*

$$\text{diag}(\theta_0^*, \theta_1^*, \theta_1^*, \theta_2^*, \theta_2^*, \dots, \theta_{d-1}^*, \theta_{d-1}^*, \theta_d^*).$$

Proof. From Corollary 2.4.11 and (2.126). ■

Lemma 2.8.11. *The matrix representing A^* relative to the basis \mathcal{B} is*

$$\text{diag}(\theta_0^*, \theta_1^*, \theta_2^*, \theta_3^*, \dots, \theta_d^*, \theta_1^*, \theta_2^*, \dots, \theta_{d-1}^*).$$

Proof. Use (2.6), (2.82), (2.127) along with (2.76). ■

Lemma 2.8.12. *The matrix representing A^* relative to the basis \mathcal{B}_{alt} is*

$$\text{diag}(\theta_0^*, \theta_1^*, \theta_1^*, \theta_2^*, \theta_2^*, \dots, \theta_{d-1}^*, \theta_{d-1}^*, \theta_d^*).$$

Proof. Use (2.6), (2.82), (2.128) along with (2.76). ■

Lemma 2.8.13. *The matrix representing A^* relative to the basis $\tilde{\mathcal{B}}$ is*

$$\begin{bmatrix} \mathbf{C} & \mathbf{D} \\ \mathbf{E} & \mathbf{F} \end{bmatrix},$$

where

$$\begin{aligned} \mathbf{C} &= \text{diag} \left[\frac{\theta_0^* - \tau_0 \theta_1^*}{1 - \tau_0}, \frac{\theta_1^* - \tau_1 \theta_2^*}{1 - \tau_1}, \frac{\theta_2^* - \tau_2 \theta_3^*}{1 - \tau_2}, \dots, \frac{\theta_{d-1}^* - \tau_{d-1} \theta_d^*}{1 - \tau_{d-1}} \right], \\ \mathbf{D} &= \text{diag} \left[\frac{\zeta_0 \tau_0 (\theta_0^* - \theta_1^*)}{1 - \tau_0}, \frac{\zeta_1 \tau_1 (\theta_1^* - \theta_2^*)}{1 - \tau_1}, \frac{\zeta_2 \tau_2 (\theta_2^* - \theta_3^*)}{1 - \tau_2}, \dots, \frac{\zeta_{d-1} \tau_{d-1} (\theta_{d-1}^* - \theta_d^*)}{1 - \tau_{d-1}} \right], \\ \mathbf{E} &= \text{diag} \left[\frac{\theta_0^* - \theta_1^*}{\zeta_0 (\tau_0 - 1)}, \frac{\theta_1^* - \theta_2^*}{\zeta_1 (\tau_1 - 1)}, \frac{\theta_2^* - \theta_3^*}{\zeta_2 (\tau_2 - 1)}, \dots, \frac{\theta_{d-1}^* - \theta_d^*}{\zeta_{d-1} (\tau_{d-1} - 1)} \right], \\ \mathbf{F} &= \text{diag} \left[\frac{\tau_0 \theta_0^* - \theta_1^*}{\tau_0 - 1}, \frac{\tau_1 \theta_1^* - \theta_2^*}{\tau_1 - 1}, \frac{\tau_2 \theta_2^* - \theta_3^*}{\tau_2 - 1}, \dots, \frac{\tau_{d-1} \theta_{d-1}^* - \theta_d^*}{\tau_{d-1} - 1} \right], \end{aligned}$$

and for $0 \leq i \leq d-1$

$$\begin{aligned} \frac{\theta_i^* - \tau_i \theta_{i+1}^*}{1 - \tau_i} &= \theta_i^* + \frac{h^* s^* q^{-i-1} (1-q)(1-r_1 q^{i+1})(1-r_2 q^{i+1})}{s^* - r_1 r_2}, \\ \frac{\zeta_i \tau_i (\theta_i^* - \theta_{i+1}^*)}{1 - \tau_i} &= \frac{h^* s^* q^{-2i-1} (1-q)(r_1 - s^* q^{i+1})(r_2 - s^* q^{i+1})(1-r_1 q^{i+1})(1-r_2 q^{i+1})}{s^* - r_1 r_2}, \\ \frac{\theta_i^* - \theta_{i+1}^*}{\zeta_i (\tau_i - 1)} &= \frac{h^* q^{-1} (q-1)}{s^* - r_1 r_2}, \\ \frac{\tau_i \theta_i^* - \theta_{i+1}^*}{\tau_i - 1} &= \theta_{i+1}^* + \frac{h^* s^* q^{-i-1} (q-1)(1-r_1 q^{i+1})(1-r_2 q^{i+1})}{s^* - r_1 r_2}. \end{aligned}$$

Proof. Let $[A^*]_{\mathcal{B}}$ denote the matrix representing A^* relative to \mathcal{B} , and let L denote the transition matrix from $\tilde{\mathcal{B}}$ to \mathcal{B} . By linear algebra the matrix representing A^* relative to $\tilde{\mathcal{B}}$ is $L[A^*]_{\mathcal{B}}L^{-1}$. The result follows. \blacksquare

Lemma 2.8.14. *The matrix representing A^* relative to the basis $\tilde{\mathcal{B}}_{alt}$ is*

$$\text{blockdiag}[\mathbf{X}_0, \mathbf{X}_1, \dots, \mathbf{X}_{d-1}],$$

where for $0 \leq i \leq d-1$,

$$\mathbf{X}_i = \begin{bmatrix} \frac{\theta_i^* - \tau_i \theta_{i+1}^*}{1 - \tau_i} & \frac{\zeta_i \tau_i (\theta_i^* - \theta_{i+1}^*)}{1 - \tau_i} \\ \frac{\theta_i^* - \theta_{i+1}^*}{\zeta_i (\tau_i - 1)} & \frac{\tau_i \theta_i^* - \theta_{i+1}^*}{\tau_i - 1} \end{bmatrix}.$$

Proof. Let $[A^*]_{\mathcal{B}_{alt}}$ denote the matrix representing A^* relative to \mathcal{B}_{alt} , and let L denote the transition matrix from $\tilde{\mathcal{B}}_{alt}$ to \mathcal{B}_{alt} . By linear algebra the matrix representing A^* relative to $\tilde{\mathcal{B}}_{alt}$ is $L[A^*]_{\mathcal{B}_{alt}}L^{-1}$. The result follows. \blacksquare

We are done with A^* . We now consider the matrices representing \tilde{A}^* relative to the five bases.

Lemma 2.8.15. *The matrix representing \tilde{A}^* relative to the basis \mathcal{C} is*

$$\text{diag}(\tilde{\theta}_0^*, \tilde{\theta}_0^*, \tilde{\theta}_1^*, \tilde{\theta}_1^*, \tilde{\theta}_2^*, \tilde{\theta}_2^*, \dots, \tilde{\theta}_{d-1}^*, \tilde{\theta}_{d-1}^*).$$

Proof. From Lemma 2.4.16 and (2.126). ■

Lemma 2.8.16. *The matrix representing \tilde{A}^* relative to the basis $\tilde{\mathcal{B}}$ is*

$$\text{diag}(\tilde{\theta}_0^*, \tilde{\theta}_1^*, \tilde{\theta}_2^*, \dots, \tilde{\theta}_{d-1}^*, \tilde{\theta}_0^*, \tilde{\theta}_1^*, \tilde{\theta}_2^*, \dots, \tilde{\theta}_{d-1}^*).$$

Proof. Use (2.57), (2.108), (2.129) along with Lemma 2.6.6. ■

Lemma 2.8.17. *The matrix representing \tilde{A}^* relative to the basis $\tilde{\mathcal{B}}_{alt}$ is*

$$\text{diag}(\tilde{\theta}_0^*, \tilde{\theta}_0^*, \tilde{\theta}_1^*, \tilde{\theta}_1^*, \tilde{\theta}_2^*, \tilde{\theta}_2^*, \dots, \tilde{\theta}_{d-1}^*, \tilde{\theta}_{d-1}^*).$$

Proof. Use (2.57), (2.108), (2.130) along with Lemma 2.6.6. ■

Lemma 2.8.18. *The matrix representing \tilde{A}^* relative to the basis \mathcal{B} is*

$$\begin{bmatrix} \tilde{\mathbf{C}} & \tilde{\mathbf{D}} \\ \tilde{\mathbf{E}} & \tilde{\mathbf{F}} \end{bmatrix},$$

where

$$\tilde{\mathbf{C}} = \text{diag} \left[\tilde{\theta}_0^*, \frac{\epsilon_1 \tilde{\theta}_0^* - \tilde{\theta}_1^*}{\epsilon_1 - 1}, \frac{\epsilon_2 \tilde{\theta}_1^* - \tilde{\theta}_2^*}{\epsilon_2 - 1}, \frac{\epsilon_3 \tilde{\theta}_2^* - \tilde{\theta}_3^*}{\epsilon_3 - 1}, \dots, \frac{\epsilon_{d-1} \tilde{\theta}_{d-2}^* - \tilde{\theta}_{d-1}^*}{\epsilon_{d-1} - 1}, \tilde{\theta}_{d-1}^* \right],$$

$$\tilde{\mathbf{D}} = \begin{bmatrix} 0 & 0 & \dots & 0 \\ \hline \frac{\epsilon_1 \xi_1 (\tilde{\theta}_0^* - \tilde{\theta}_1^*)}{\epsilon_1 - 1} & & & \mathbf{0} \\ & \frac{\epsilon_2 \xi_2 (\tilde{\theta}_1^* - \tilde{\theta}_2^*)}{\epsilon_2 - 1} & & \\ & & \ddots & \\ \mathbf{0} & & & \frac{\epsilon_{d-1} \xi_{d-1} (\tilde{\theta}_{d-2}^* - \tilde{\theta}_{d-1}^*)}{\epsilon_{d-1} - 1} \\ \hline 0 & 0 & \dots & 0 \end{bmatrix},$$

(in the above matrix the entries in the first row and the last row are all zero)

$$\tilde{\mathbf{E}} = \begin{bmatrix} 0 & \left| \frac{\tilde{\theta}_0^* - \tilde{\theta}_1^*}{\xi_1(1-\epsilon_1)} \right. & & \mathbf{0} & \left| 0 \right. \\ 0 & & \frac{\tilde{\theta}_1^* - \tilde{\theta}_2^*}{\xi_2(1-\epsilon_2)} & & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & \mathbf{0} & & & \frac{\tilde{\theta}_{d-2}^* - \tilde{\theta}_{d-1}^*}{\xi_{d-1}(1-\epsilon_{d-1})} \left| 0 \right. \end{bmatrix},$$

(in the above matrix the entries in the first column and the last column are all zero)

$$\tilde{\mathbf{F}} = \text{diag} \left[\frac{\tilde{\theta}_0^* - \epsilon_1 \tilde{\theta}_1^*}{1-\epsilon_1}, \frac{\tilde{\theta}_1^* - \epsilon_2 \tilde{\theta}_2^*}{1-\epsilon_2}, \frac{\tilde{\theta}_2^* - \epsilon_3 \tilde{\theta}_3^*}{1-\epsilon_3}, \dots, \frac{\tilde{\theta}_{d-2}^* - \epsilon_{d-1} \tilde{\theta}_{d-1}^*}{1-\epsilon_{d-1}} \right],$$

and for $1 \leq i \leq d-1$

$$\begin{aligned} \frac{\epsilon_i \tilde{\theta}_{i-1}^* - \tilde{\theta}_i^*}{\epsilon_i - 1} &= \tilde{\theta}_{i-1}^* + \frac{\tilde{h}^*(1-q)(1-q^{d-i})(1-s^*q^{i+1})}{1-q^d}, \\ \frac{\epsilon_i \xi_i (\tilde{\theta}_{i-1}^* - \tilde{\theta}_i^*)}{\epsilon_i - 1} &= \frac{\tilde{h}^* q^{1-2i} (1-q)(1-q^i)(1-q^{i-d})(1-s^*q^{i+1})(1-s^*q^{d+i+1})}{q^d - 1}, \\ \frac{\tilde{\theta}_{i-1}^* - \tilde{\theta}_i^*}{\xi_i(1-\epsilon_i)} &= \frac{\tilde{h}^* q^{d-1} (1-q)}{1-q^d}, \\ \frac{\tilde{\theta}_{i-1}^* - \epsilon_i \tilde{\theta}_i^*}{1-\epsilon_i} &= \tilde{\theta}_i^* + \frac{\tilde{h}^*(1-q)(1-q^{d-i})(1-s^*q^{i+1})}{q^d - 1}. \end{aligned}$$

Proof. Let $[\tilde{A}^*]_{\tilde{\mathcal{B}}}$ denote the matrix representing \tilde{A}^* relative to $\tilde{\mathcal{B}}$, and let L denote the transition matrix from \mathcal{B} to $\tilde{\mathcal{B}}$. By linear algebra the matrix representing \tilde{A}^* relative to \mathcal{B} is $L[\tilde{A}^*]_{\tilde{\mathcal{B}}}L^{-1}$. The result follows. \blacksquare

Lemma 2.8.19. *The matrix representing \tilde{A}^* relative to the basis \mathcal{B}_{att} is*

$$\text{blockdiag}[\tilde{\mathbf{X}}_0, \tilde{\mathbf{X}}_1, \tilde{\mathbf{X}}_2, \dots, \tilde{\mathbf{X}}_{d-1}, \tilde{\mathbf{X}}_d],$$

where

$$\tilde{\mathbf{X}}_0 = [\tilde{\theta}_0^*], \quad \tilde{\mathbf{X}}_d = [\tilde{\theta}_{d-1}^*]$$

and for $1 \leq i \leq d-1$

$$\tilde{\mathbf{X}}_i = \begin{bmatrix} \frac{\epsilon_i \tilde{\theta}_{i-1}^* - \tilde{\theta}_i^*}{\epsilon_i - 1} & \frac{\epsilon_i \xi_i (\tilde{\theta}_{i-1}^* - \tilde{\theta}_i^*)}{\epsilon_i - 1} \\ \frac{\tilde{\theta}_{i-1}^* - \tilde{\theta}_i^*}{\xi_i(1-\epsilon_i)} & \frac{\tilde{\theta}_{i-1}^* - \epsilon_i \tilde{\theta}_i^*}{1-\epsilon_i} \end{bmatrix}.$$

Proof. Let $[\tilde{A}^*]_{\tilde{\mathcal{B}}_{alt}}$ denote the matrix representing \tilde{A}^* relative to $\tilde{\mathcal{B}}_{alt}$, and let L denote the transition matrix from \mathcal{B}_{alt} to $\tilde{\mathcal{B}}_{alt}$. By linear algebra the matrix representing \tilde{A}^* relative to \mathcal{B}_{alt} is $L[\tilde{A}^*]_{\tilde{\mathcal{B}}_{alt}} L^{-1}$. The result follows. ■

We are done with \tilde{A}^* . We now consider the matrices representing \mathfrak{p} relative to the five bases.

Lemma 2.8.20. *The matrix representing \mathfrak{p} relative to the basis \mathcal{C} is*

$$\text{blockdiag}[\mathbf{Y}_0, \mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_{d-1}, \mathbf{Y}_d], \quad (2.137)$$

where

$$\mathbf{Y}_0 = [1], \quad \mathbf{Y}_d = [1]$$

and for $1 \leq i \leq d-1$,

$$\mathbf{Y}_i = \begin{bmatrix} \frac{\epsilon_i}{\epsilon_i-1} & \frac{1}{1-\epsilon_i} \\ \frac{\epsilon_i}{\epsilon_i-1} & \frac{1}{1-\epsilon_i} \end{bmatrix}.$$

Proof. By Lemma 2.7.4. ■

Lemma 2.8.21. *The matrix representing \mathfrak{p} relative to the basis \mathcal{B} is*

$$\left[\begin{array}{c|c} \mathbf{I}_{d+1} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} \end{array} \right],$$

where \mathbf{I}_{d+1} is the $(d+1) \times (d+1)$ identity matrix.

Proof. By (2.124) and (2.127). ■

Lemma 2.8.22. *The matrix representing \mathfrak{p} relative to the basis \mathcal{B}_{alt} is*

$$\text{diag}(1, 1, 0, 1, 0, 1, 0, \dots, 1, 0, 1).$$

Proof. By (2.124) and (2.128). ■

Lemma 2.8.23. *The matrix representing \mathfrak{p} relative to the basis $\tilde{\mathcal{B}}$ is*

$$\begin{bmatrix} \mathbf{P} & \mathbf{Q} \\ \mathbf{R} & \mathbf{S} \end{bmatrix},$$

where each matrix \mathbf{P} , \mathbf{Q} , \mathbf{R} , \mathbf{S} is $d \times d$ tridiagonal with rows and columns indexed by $0, 1, 2, \dots, d-1$ such that

$$\begin{aligned} \mathbf{P}_{00} &= \frac{1-\epsilon_1(1-\tau_0)}{(1-\tau_0)(1-\epsilon_1)}, & \mathbf{P}_{d-1,d-1} &= \frac{1+\tau_{d-1}(\epsilon_{d-1}-1)}{(1-\tau_{d-1})(1-\epsilon_{d-1})}, \\ \mathbf{P}_{ij} &= \begin{cases} \frac{-\epsilon_i}{(1-\tau_i)(1-\epsilon_i)} & \text{if } j = i-1 \quad (1 \leq i \leq d-1) \\ \frac{1-\epsilon_{i+1}+\epsilon_{i+1}\tau_i(1-\epsilon_i)}{(1-\tau_i)(1-\epsilon_i)(1-\epsilon_{i+1})} & \text{if } j = i \quad (1 \leq i \leq d-2) \\ \frac{-\tau_i}{(1-\tau_i)(1-\epsilon_{i+1})} & \text{if } j = i+1 \quad (0 \leq i \leq d-2) \end{cases} \end{aligned}$$

and

$$\begin{aligned} \mathbf{Q}_{00} &= \frac{\zeta_0\tau_0}{(1-\tau_0)(1-\epsilon_1)}, & \mathbf{Q}_{d-1,d-1} &= \frac{\zeta_{d-1}\tau_{d-1}\epsilon_{d-1}}{(1-\tau_{d-1})(1-\epsilon_{d-1})}, \\ \mathbf{Q}_{ij} &= \begin{cases} \frac{-\zeta_{i-1}\epsilon_i}{(1-\tau_i)(1-\epsilon_i)} & \text{if } j = i-1 \quad (1 \leq i \leq d-1) \\ \frac{\zeta_i\tau_i(1-\epsilon_i\epsilon_{i+1})}{(1-\tau_i)(1-\epsilon_i)(1-\epsilon_{i+1})} & \text{if } j = i \quad (1 \leq i \leq d-2) \\ \frac{-\tau_i\tau_{i+1}\zeta_{i+1}}{(1-\tau_i)(1-\epsilon_{i+1})} & \text{if } j = i+1 \quad (0 \leq i \leq d-2) \end{cases} \end{aligned}$$

and

$$\begin{aligned} \mathbf{R}_{00} &= \frac{-1}{\zeta_0(1-\tau_0)(1-\epsilon_1)}, & \mathbf{R}_{d-1,d-1} &= \frac{-\epsilon_{d-1}}{\zeta_{d-1}(1-\tau_{d-1})(1-\epsilon_{d-1})}, \\ \mathbf{R}_{ij} &= \begin{cases} \frac{\epsilon_i}{\zeta_i(1-\tau_i)(1-\epsilon_i)} & \text{if } j = i-1 \quad (1 \leq i \leq d-1) \\ \frac{-1+\epsilon_i\epsilon_{i+1}}{\zeta_i(1-\tau_i)(1-\epsilon_i)(1-\epsilon_{i+1})} & \text{if } j = i \quad (1 \leq i \leq d-2) \\ \frac{1}{\zeta_i(1-\tau_i)(1-\epsilon_{i+1})} & \text{if } j = i+1 \quad (0 \leq i \leq d-2) \end{cases} \end{aligned}$$

and

$$\mathbf{S}_{00} = \frac{\tau_0(\epsilon_1-1)-\epsilon_1}{(1-\tau_0)(1-\epsilon_1)}, \quad \mathbf{S}_{d-1,d-1} = \frac{1-\tau_{d-1}-\epsilon_{d-1}}{(1-\tau_{d-1})(1-\epsilon_{d-1})},$$

$$\mathbf{S}_{ij} = \begin{cases} \frac{\zeta_{i-1}\epsilon_i}{\zeta_i(1-\tau_i)(1-\epsilon_i)} & \text{if } j = i-1 \quad (1 \leq i \leq d-1) \\ \frac{\tau_i(\epsilon_{i+1}-1)+\epsilon_{i+1}(\epsilon_i-1)}{(1-\tau_i)(1-\epsilon_i)(1-\epsilon_{i+1})} & \text{if } j = i \quad (1 \leq i \leq d-2) \\ \frac{\zeta_{i+1}\tau_{i+1}}{\zeta_i(1-\tau_i)(1-\epsilon_{i+1})} & \text{if } j = i+1 \quad (0 \leq i \leq d-2). \end{cases}$$

Proof. Let $[\mathbf{p}]_{\mathcal{B}}$ denote the matrix representing \mathbf{p} relative to \mathcal{B} , and let L denote the transition matrix from $\tilde{\mathcal{B}}$ to \mathcal{B} . By linear algebra the matrix representing \mathbf{p} relative to $\tilde{\mathcal{B}}$ is $L[\mathbf{p}]_{\mathcal{B}}L^{-1}$. The result follows. \blacksquare

Lemma 2.8.24. *The matrix representing \mathbf{p} relative to the basis $\tilde{\mathcal{B}}_{alt}$ is*

$$\begin{bmatrix} \mathbf{a}_0 & \mathbf{b}_0 & & & \mathbf{0} \\ \mathbf{c}_1 & \mathbf{a}_1 & \mathbf{b}_1 & & \\ & \mathbf{c}_2 & \mathbf{a}_2 & \ddots & \\ & & \ddots & \ddots & \mathbf{b}_{d-2} \\ \mathbf{0} & & & \mathbf{c}_{d-1} & \mathbf{a}_{d-1} \end{bmatrix},$$

where

$$\begin{aligned}
\mathbf{b}_i &= \begin{bmatrix} \frac{-\tau_i}{(1-\tau_i)(1-\epsilon_{i+1})} & \frac{-\tau_i\tau_{i+1}\zeta_{i+1}}{(1-\tau_i)(1-\epsilon_{i+1})} \\ \frac{1}{\zeta_i(1-\tau_i)(1-\epsilon_{i+1})} & \frac{\tau_{i+1}\zeta_{i+1}}{\zeta_i(1-\tau_i)(1-\epsilon_{i+1})} \end{bmatrix} & (0 \leq i \leq d-2), \\
\mathbf{a}_0 &= \begin{bmatrix} \frac{1-\epsilon_1(1-\tau_0)}{(1-\tau_0)(1-\epsilon_1)} & \frac{\zeta_0\tau_0}{(1-\tau_0)(1-\epsilon_1)} \\ \frac{-1}{\zeta_0(1-\tau_0)(1-\epsilon_1)} & \frac{\tau_0(\epsilon_1-1)-\epsilon_1}{(1-\tau_0)(1-\epsilon_1)} \end{bmatrix}, \\
\mathbf{a}_i &= \begin{bmatrix} \frac{1-\epsilon_{i+1}+\tau_i\epsilon_{i+1}(1-\epsilon_i)}{(1-\tau_i)(1-\epsilon_i)(1-\epsilon_{i+1})} & \frac{\zeta_i\tau_i(1-\epsilon_i\epsilon_{i+1})}{(1-\tau_i)(1-\epsilon_i)(1-\epsilon_{i+1})} \\ \frac{-1+\epsilon_i\epsilon_{i+1}}{\zeta_i(1-\tau_i)(1-\epsilon_i)(1-\epsilon_{i+1})} & \frac{\tau_i(\epsilon_{i+1}-1)+\epsilon_{i+1}(\epsilon_i-1)}{(1-\tau_i)(1-\epsilon_i)(1-\epsilon_{i+1})} \end{bmatrix} & (1 \leq i \leq d-2), \\
\mathbf{a}_{d-1} &= \begin{bmatrix} \frac{1+\tau_{d-1}(\epsilon_{d-1}-1)}{(1-\tau_{d-1})(1-\epsilon_{d-1})} & \frac{\zeta_{d-1}\tau_{d-1}\epsilon_{d-1}}{(1-\tau_{d-1})(1-\epsilon_{d-1})} \\ \frac{-\epsilon_{d-1}}{\zeta_{d-1}(1-\tau_{d-1})(1-\epsilon_{d-1})} & \frac{1-\tau_{d-1}-\epsilon_{d-1}}{(1-\tau_{d-1})(1-\epsilon_{d-1})} \end{bmatrix}, \\
\mathbf{c}_i &= \begin{bmatrix} \frac{-\epsilon_i}{(1-\tau_i)(1-\epsilon_i)} & \frac{-\zeta_{i-1}\epsilon_i}{(1-\tau_i)(1-\epsilon_i)} \\ \frac{\epsilon_i}{\zeta_i(1-\tau_i)(1-\epsilon_i)} & \frac{\zeta_{i-1}\epsilon_i}{\zeta_i(1-\tau_i)(1-\epsilon_i)} \end{bmatrix} & (1 \leq i \leq d-1).
\end{aligned}$$

Proof. Let $[\mathbf{p}]_{\mathcal{B}_{alt}}$ denote the matrix representing \mathbf{p} relative to \mathcal{B}_{alt} , and let L denote the transition matrix from $\tilde{\mathcal{B}}_{alt}$ to \mathcal{B}_{alt} . By linear algebra the matrix representing \mathbf{p} relative to $\tilde{\mathcal{B}}_{alt}$ is $L[\mathbf{p}]_{\mathcal{B}_{alt}}L^{-1}$. The result follows. \blacksquare

We are done with \mathbf{p} . We now consider the matrices representing $\tilde{\mathbf{p}}$ relative to the five bases.

Lemma 2.8.25. *The matrix representing $\tilde{\mathbf{p}}$ relative to the basis \mathcal{C} is*

$$\text{blockdiag}[\tilde{\mathbf{Y}}_0, \tilde{\mathbf{Y}}_1, \tilde{\mathbf{Y}}_2, \dots, \tilde{\mathbf{Y}}_{d-1}], \quad (2.138)$$

where for $0 \leq i \leq d-1$,

$$\tilde{\mathbf{Y}}_i = \begin{bmatrix} \frac{1}{1-\tau_i} & \frac{\tau_i}{\tau_i-1} \\ \frac{1}{1-\tau_i} & \frac{\tau_i}{\tau_i-1} \end{bmatrix}.$$

Proof. By Lemma 2.7.6. ■

Lemma 2.8.26. *The matrix representing $\tilde{\mathfrak{p}}$ relative to the basis $\tilde{\mathcal{B}}$ is*

$$\left[\begin{array}{c|c} \mathbf{I}_d & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} \end{array} \right],$$

where \mathbf{I}_d is the $d \times d$ identity matrix.

Proof. By (2.125) and (2.129). ■

Lemma 2.8.27. *The matrix representing $\tilde{\mathfrak{p}}$ relative to the basis $\tilde{\mathcal{B}}_{alt}$ is*

$$\text{diag}(1, 0, 1, 0, 1, 0, \dots, 1, 0).$$

Proof. By (2.125) and (2.130). ■

Lemma 2.8.28. *The matrix representing $\tilde{\mathfrak{p}}$ relative to the basis \mathcal{B} is*

$$\left[\begin{array}{cc} \tilde{\mathbf{P}} & \tilde{\mathbf{Q}} \\ \tilde{\mathbf{R}} & \tilde{\mathbf{S}} \end{array} \right],$$

where $\tilde{\mathbf{P}}, \tilde{\mathbf{Q}}, \tilde{\mathbf{R}}, \tilde{\mathbf{S}}$ are described as follows.

The matrix $\tilde{\mathbf{P}}$ is $(d+1) \times (d+1)$ tridiagonal with rows and columns indexed by $0, 1, 2, \dots, d$ such that

$$\begin{aligned} \tilde{\mathbf{P}}_{00} &= \frac{1}{1-\tau_0}, & \tilde{\mathbf{P}}_{01} &= \frac{\tau_0}{\tau_0-1}, & \tilde{\mathbf{P}}_{d,d-1} &= \frac{1}{1-\tau_{d-1}}, & \tilde{\mathbf{P}}_{d,d} &= \frac{\tau_{d-1}}{\tau_{d-1}-1}, \\ \tilde{\mathbf{P}}_{ij} &= \begin{cases} \frac{-\epsilon_i}{(1-\tau_{i-1})(1-\epsilon_i)} & \text{if } j = i-1 \quad (1 \leq i \leq d-1) \\ \frac{\epsilon_i \tau_{i-1}(\tau_i-1) + \tau_{i-1}-1}{(\tau_{i-1}-1)(\tau_i-1)(\epsilon_i-1)} & \text{if } j = i \quad (1 \leq i \leq d-1) \\ \frac{-\tau_i}{(1-\epsilon_i)(1-\tau_i)} & \text{if } j = i+1 \quad (1 \leq i \leq d-1). \end{cases} \end{aligned}$$

The matrix $\tilde{\mathbf{Q}}$ is $(d+1) \times (d-1)$ with rows indexed by $0, 1, 2, \dots, d$ and columns indexed by $0, 1, 2, \dots, d-2$ such that

$$\tilde{\mathbf{Q}}_{00} = \frac{\tau_0 \xi_1}{\tau_0 - 1}, \quad \tilde{\mathbf{Q}}_{d,d-2} = \frac{\xi_{d-1} \epsilon_{d-1}}{1 - \tau_{d-1}},$$

$$\tilde{\mathbf{Q}}_{ij} = \begin{cases} \frac{-\xi_{i-1} \epsilon_{i-1} \epsilon_i}{(1 - \tau_{i-1})(1 - \epsilon_i)} & \text{if } j = i - 2 \quad (2 \leq i \leq d - 1) \\ \frac{\xi_i \epsilon_i (1 - \tau_{i-1} \tau_i)}{(1 - \tau_{i-1})(1 - \tau_i)(1 - \epsilon_i)} & \text{if } j = i - 1 \quad (1 \leq i \leq d - 1) \\ \frac{-\tau_i \xi_{i+1}}{(1 - \epsilon_i)(1 - \tau_i)} & \text{if } j = i \quad (1 \leq i \leq d - 2) \\ 0 & \text{otherwise.} \end{cases}$$

The matrix $\tilde{\mathbf{R}}$ is $(d-1) \times (d+1)$ with rows indexed by $0, 1, 2, \dots, d-2$ and columns indexed by $0, 1, 2, \dots, d$ such that

$$\tilde{\mathbf{R}}_{ij} = \begin{cases} \frac{1}{\xi_{i+1}(\epsilon_{i+1} - 1)(\tau_i - 1)} & \text{if } j = i \quad (0 \leq i \leq d - 2) \\ \frac{\tau_i \tau_{i+1} - 1}{\xi_{i+1}(1 - \epsilon_{i+1})(1 - \tau_{i+1})(1 - \tau_i)} & \text{if } j = i + 1 \quad (0 \leq i \leq d - 2) \\ \frac{\tau_{i+1}}{\xi_{i+1}(1 - \epsilon_{i+1})(1 - \tau_{i+1})} & \text{if } j = i + 2 \quad (0 \leq i \leq d - 2) \\ 0 & \text{otherwise.} \end{cases}$$

The matrix $\tilde{\mathbf{S}}$ is $(d-1) \times (d-1)$ tridiagonal with rows and columns indexed by $0, 1, 2, \dots, d-2$ such that

$$\tilde{\mathbf{S}}_{ij} = \begin{cases} \frac{\xi_i \epsilon_i}{\xi_{i+1}(\epsilon_{i+1} - 1)(\tau_i - 1)} & \text{if } j = i - 1 \quad (1 \leq i \leq d - 2) \\ \frac{\tau_i(\tau_{i+1} - 1) + \epsilon_{i+1}(\tau_i - 1)}{(1 - \tau_i)(1 - \tau_{i+1})(1 - \epsilon_{i+1})} & \text{if } j = i \quad (0 \leq i \leq d - 2) \\ \frac{\tau_{i+1} \xi_{i+2}}{\xi_{i+1}(1 - \epsilon_{i+1})(1 - \tau_{i+1})} & \text{if } j = i + 1 \quad (0 \leq i \leq d - 3). \end{cases}$$

Proof. Let $[\tilde{\mathbf{p}}]_{\tilde{\mathcal{B}}}$ denote the matrix representing $\tilde{\mathbf{p}}$ relative to $\tilde{\mathcal{B}}$, and let L denote the transition matrix from \mathcal{B} to $\tilde{\mathcal{B}}$. By linear algebra the matrix representing $\tilde{\mathbf{p}}$ relative to \mathcal{B} is $L[\tilde{\mathbf{p}}]_{\tilde{\mathcal{B}}}L^{-1}$. The result follows. \blacksquare

Lemma 2.8.29. *The matrix representing $\tilde{\mathfrak{p}}$ relative to the basis \mathcal{B}_{att} is*

$$\begin{bmatrix} \tilde{\mathbf{a}}_0 & \tilde{\mathbf{b}}_0 & & & \mathbf{0} \\ \tilde{\mathbf{c}}_1 & \tilde{\mathbf{a}}_1 & \tilde{\mathbf{b}}_1 & & \\ & \tilde{\mathbf{c}}_2 & \tilde{\mathbf{a}}_2 & \ddots & \\ & & & \ddots & \tilde{\mathbf{b}}_{d-1} \\ \mathbf{0} & & & \tilde{\mathbf{c}}_d & \tilde{\mathbf{a}}_d \end{bmatrix},$$

where

$$\begin{aligned} \tilde{\mathbf{b}}_0 &= \begin{bmatrix} \frac{\tau_0}{\tau_0-1} & \frac{\tau_0\xi_1}{\tau_0-1} \end{bmatrix}, \\ \tilde{\mathbf{b}}_i &= \begin{bmatrix} \frac{-\tau_i}{(1-\epsilon_i)(1-\tau_i)} & \frac{-\tau_i\xi_{i+1}}{(1-\epsilon_i)(1-\tau_i)} \\ \frac{\tau_i}{\xi_i(1-\epsilon_i)(1-\tau_i)} & \frac{\tau_i\xi_{i+1}}{\xi_i(1-\epsilon_i)(1-\tau_i)} \end{bmatrix} & (1 \leq i \leq d-2), \\ \tilde{\mathbf{b}}_{d-1} &= \begin{bmatrix} \frac{-\tau_{d-1}}{(1-\epsilon_{d-1})(1-\tau_{d-1})} \\ \frac{\tau_{d-1}}{\xi_{d-1}(1-\epsilon_{d-1})(1-\tau_{d-1})} \end{bmatrix}, \\ \tilde{\mathbf{a}}_0 &= \begin{bmatrix} 1 \\ 1-\tau_0 \end{bmatrix}, \\ \tilde{\mathbf{a}}_i &= \begin{bmatrix} \frac{\epsilon_i\tau_{i-1}(\tau_i-1)+\tau_{i-1}-1}{(\tau_{i-1}-1)(\tau_i-1)(\epsilon_i-1)} & \frac{\xi_i\epsilon_i(1-\tau_{i-1}\tau_i)}{(1-\tau_{i-1})(1-\tau_i)(1-\epsilon_i)} \\ \frac{\tau_{i-1}\tau_i-1}{\xi_i(1-\epsilon_i)(1-\tau_i)(1-\tau_{i-1})} & \frac{\tau_{i-1}(\tau_i-1)+\epsilon_i(\tau_{i-1}-1)}{(1-\epsilon_i)(1-\tau_i)(1-\tau_{i-1})} \end{bmatrix} & (1 \leq i \leq d-1), \\ \tilde{\mathbf{a}}_d &= \begin{bmatrix} \tau_{d-1} \\ \tau_{d-1}-1 \end{bmatrix}, \\ \tilde{\mathbf{c}}_1 &= \begin{bmatrix} \frac{-\epsilon_1}{(1-\epsilon_1)(1-\tau_0)} \\ \frac{1}{\xi_1(1-\epsilon_1)(1-\tau_0)} \end{bmatrix}, \\ \tilde{\mathbf{c}}_i &= \begin{bmatrix} \frac{-\epsilon_i}{(1-\tau_{i-1})(1-\epsilon_i)} & \frac{-\xi_{i-1}\epsilon_{i-1}\epsilon_i}{(1-\tau_{i-1})(1-\epsilon_i)} \\ \frac{1}{\xi_i(1-\epsilon_i)(1-\tau_{i-1})} & \frac{\xi_{i-1}\epsilon_{i-1}}{\xi_i(1-\epsilon_i)(1-\tau_{i-1})} \end{bmatrix} & (2 \leq i \leq d-1), \\ \tilde{\mathbf{c}}_d &= \begin{bmatrix} \frac{1}{1-\tau_{d-1}} & \frac{\xi_{d-1}\epsilon_{d-1}}{1-\tau_{d-1}} \end{bmatrix}. \end{aligned}$$

Proof. Let $[\tilde{\mathfrak{p}}]_{\tilde{\mathcal{B}}_{alt}}$ denote the matrix representing $\tilde{\mathfrak{p}}$ relative to $\tilde{\mathcal{B}}_{alt}$, and let L denote the transition matrix from \mathcal{B}_{alt} to $\tilde{\mathcal{B}}_{alt}$. By linear algebra the matrix representing $\tilde{\mathfrak{p}}$ relative to \mathcal{B}_{alt} is $L[\tilde{\mathfrak{p}}]_{\tilde{\mathcal{B}}_{alt}}L^{-1}$. The result follows. ■

Chapter 3

The Universal Double Affine Hecke Algebra of Type (C_1^\vee, C_1)

3.1 The Universal DAHA \hat{H}_q

The double affine Hecke algebra (DAHA) associated with the root system (C_1^\vee, C_1) was discovered by Sahi [20]. This algebra has rank one. It involves q and four additional nonzero parameters. It is known that the algebra controls the algebraic structure of the Askey-Wilson polynomials [17]. In [30] Terwilliger introduced a central extension of that algebra, denoted \hat{H}_q and called the universal DAHA of type (C_1^\vee, C_1) . \hat{H}_q has no parameter besides q and has a larger automorphism group. Our goal for the rest of this thesis is to display a representation of \hat{H}_q using a Delsarte clique of a Q -polynomial distance-regular graph, and to show how this representation is related to the algebra \mathbf{T} from Definition 2.4.20.

We now define the algebra \hat{H}_q . For notational convenience define a four element set

$$\mathbb{I} = \{0, 1, 2, 3\}.$$

For the rest of the thesis we fix a square root $q^{1/2}$ of q .

Definition 3.1.1. [30, Definition 3.1] Let \hat{H}_q denote the \mathbb{C} -algebra defined by generators

$\{t_n^{\pm 1}\}_{n \in \mathbb{I}}$ and relations

$$t_n t_n^{-1} = t_n^{-1} t_n = 1 \quad n \in \mathbb{I}; \quad (3.1)$$

$$t_n + t_n^{-1} \text{ is central} \quad n \in \mathbb{I}; \quad (3.2)$$

$$t_0 t_1 t_2 t_3 = q^{-1/2}. \quad (3.3)$$

We call \hat{H}_q the *universal DAHA of type (C_1^\vee, C_1)* . We note that in [30, Definition 3.1] Terwilliger defined \hat{H}_q in a slightly different way. To go from his definition to ours, replace q by $q^{1/2}$.

Definition 3.1.2. We define elements \mathbf{X} and \mathbf{Y} in \hat{H}_q as follows:

$$\mathbf{X} = t_3 t_0, \quad \mathbf{Y} = t_0 t_1.$$

Note that each of \mathbf{X} , \mathbf{Y} is invertible.

Definition 3.1.3. We define elements \mathbf{A} , \mathbf{B} , \mathbf{B}^\dagger in \hat{H}_q as follows:

$$\mathbf{A} = \mathbf{Y} + \mathbf{Y}^{-1} = t_0 t_1 + (t_0 t_1)^{-1},$$

$$\mathbf{B} = \mathbf{X} + \mathbf{X}^{-1} = t_3 t_0 + (t_3 t_0)^{-1},$$

$$\mathbf{B}^\dagger = q^{1/2} \mathbf{X} + q^{-1/2} \mathbf{X}^{-1} = t_1 t_2 + (t_1 t_2)^{-1}.$$

Note 3.1.4. By [12, Lemma 3.8] each of \mathbf{A} , \mathbf{B} commutes with t_0 , and each of \mathbf{A} , \mathbf{B}^\dagger commutes with t_1 . Moreover \mathbf{B} , \mathbf{B}^\dagger commute.

3.2 An action of \hat{H}_q on the \mathbf{T} -module \mathbf{W}

We now bring in the situation of Chapter 2. Recall the primary \mathbf{T} -module \mathbf{W} from below Definition 2.4.20. In this section we will show that \mathbf{W} has the structure of a

\hat{H}_q -module. To this end we define some block diagonal matrices whose blocks are 1×1 or 2×2 . Recall the scalars h, h^*, s, s^*, r_1, r_2 from above Note 2.2.4. In what follows, we will encounter square roots. These are interpreted as follows. For the rest of the thesis fix square roots

$$s^{1/2}, \quad s^{*1/2}, \quad r_1^{1/2}, \quad r_2^{1/2}$$

such that $r_1^{1/2}r_2^{1/2} = s^{1/2}s^{*1/2}q^{(d+1)/2}$.

Definition 3.2.1. We define some matrices as follows:

(a) For $0 \leq i \leq d-1$ define

$$\mathbf{t}_0(i) = \begin{bmatrix} \frac{1}{\sqrt{s^*r_1r_2}} \left(\frac{(r_1-s^*q^{i+1})(r_2-s^*q^{i+1})}{1-s^*q^{2i+2}} + s^* \right) & -\sqrt{\frac{s^*}{r_1r_2}} \frac{(1-r_1q^{i+1})(1-r_2q^{i+1})}{1-s^*q^{2i+2}} \\ \frac{1}{\sqrt{s^*r_1r_2}} \frac{(r_1-s^*q^{i+1})(r_2-s^*q^{i+1})}{1-s^*q^{2i+2}} & \sqrt{\frac{s^*}{r_1r_2}} \left(1 - \frac{(1-r_1q^{i+1})(1-r_2q^{i+1})}{1-s^*q^{2i+2}} \right) \end{bmatrix}.$$

(b) For $1 \leq i \leq d-1$ define

$$\mathbf{t}_1(i) = \begin{bmatrix} \frac{q^{d/2}(1-q^{i-d})(1-s^*q^{i+1})}{1-s^*q^{2i+1}} + \frac{1}{q^{d/2}} & \frac{q^{d/2}(q^{i-d}-1)(1-s^*q^{i+1})}{1-s^*q^{2i+1}} \\ \frac{(1-q^i)(1-s^*q^{d+i+1})}{q^{d/2}(1-s^*q^{2i+1})} & \frac{(q^i-1)(1-s^*q^{d+i+1})}{q^{d/2}(1-s^*q^{2i+1})} + \frac{1}{q^{d/2}} \end{bmatrix}.$$

Define

$$\mathbf{t}_1(0) = \left[\frac{1}{q^{d/2}} \right], \quad \mathbf{t}_1(d) = \left[\frac{1}{q^{d/2}} \right].$$

(c) For $1 \leq i \leq d-1$ define

$$\mathbf{t}_2(i) = \begin{bmatrix} \frac{1}{\sqrt{s^*q^{d+1}}} \left(\frac{s^*q^{d+1}(1-q^i)(1-q^{i-d})}{1-s^*q^{2i+1}} + 1 \right) & \frac{q^i\sqrt{s^*q^{d+1}}(1-q^{i-d})(1-s^*q^{i+1})}{1-s^*q^{2i+1}} \\ \frac{1}{q^i\sqrt{s^*q^{d+1}}} \frac{(q^i-1)(1-s^*q^{d+i+1})}{1-s^*q^{2i+1}} & \sqrt{s^*q^{d+1}} \left(\frac{(q^i-1)(1-q^{i-d})}{1-s^*q^{2i+1}} + 1 \right) \end{bmatrix}.$$

Define

$$\mathbf{t}_2(0) = \left[\sqrt{s^* q^{d+1}} \right], \quad \mathbf{t}_2(d) = \left[\frac{1}{\sqrt{s^* q^{d+1}}} \right].$$

(d) For $0 \leq i \leq d-1$ define

$$\mathbf{t}_3(i) = \begin{bmatrix} \frac{1}{q^{i+1} \sqrt{r_1 r_2}} \left(1 - \frac{(1-r_1 q^{i+1})(1-r_2 q^{i+1})}{1-s^* q^{2i+2}} \right) & \frac{1}{q^{i+1} \sqrt{r_1 r_2}} \frac{(1-r_1 q^{i+1})(1-r_2 q^{i+1})}{1-s^* q^{2i+2}} \\ -\frac{q^{i+1}}{\sqrt{r_1 r_2}} \frac{(r_1 - s^* q^{i+1})(r_2 - s^* q^{i+1})}{1-s^* q^{2i+2}} & \frac{q^{i+1}}{\sqrt{r_1 r_2}} \left(\frac{(r_1 - s^* q^{i+1})(r_2 - s^* q^{i+1})}{1-s^* q^{2i+2}} + s^* \right) \end{bmatrix}.$$

Definition 3.2.2. We define complex scalars $\{k_n\}_{n \in \mathbb{I}}$ as follows:

$$k_0 = \sqrt{\frac{r_1 r_2}{s^*}}, \quad k_1 = \frac{1}{\sqrt{q^d}}, \quad k_2 = \sqrt{s^* q^{d+1}}, \quad k_3 = \sqrt{\frac{r_2}{r_1}}. \quad (3.4)$$

Observe that k_0, k_1, k_2, k_3 are nonzero.

We discuss some properties of the matrices from Definition 3.2.1. We begin with the 2×2 matrices.

Lemma 3.2.3. For $n \in \mathbb{I}$ define $\varepsilon = 0$ if $n \in \{0, 3\}$ and $\varepsilon = 1$ if $n \in \{1, 2\}$. Referring to Definition 3.2.1, for $\varepsilon \leq i \leq d-1$ both

(i) $\det(\mathbf{t}_n(i)) = 1;$

(ii) $\text{trace}(\mathbf{t}_n(i)) = k_n + k_n^{-1}.$

Proof. Routine. ■

Lemma 3.2.4. Referring to Lemma 3.2.3, the matrix $\mathbf{t}_n(i)$ is invertible. Moreover

$$\mathbf{t}_n(i) + (\mathbf{t}_n(i))^{-1} = (k_n + k_n^{-1})I. \quad (3.5)$$

Proof. Use Lemma 3.2.3. ■

Lemma 3.2.5. *With reference to Definition 3.2.1 the following (i), (ii) hold:*

$$(i) \quad \mathbf{t}_3(i)\mathbf{t}_0(i) = \begin{bmatrix} \frac{1}{q^{i+1}\sqrt{s^*}} & 0 \\ 0 & q^{i+1}\sqrt{s^*} \end{bmatrix} \quad (0 \leq i \leq d-1).$$

$$(ii) \quad \mathbf{t}_1(i)\mathbf{t}_2(i) = \begin{bmatrix} \frac{1}{q^i\sqrt{s^*q}} & 0 \\ 0 & q^i\sqrt{s^*q} \end{bmatrix} \quad (1 \leq i \leq d-1).$$

Proof. Routine. ■

We have discussed the 2×2 matrices from Definition 3.2.1. We now consider the 1×1 matrices from Definition 3.2.1.

Lemma 3.2.6. *The following (i), (ii) hold:*

(i) *For $n \in \{1, 2\}$ and $i \in \{0, d\}$,*

$$\mathbf{t}_n(i) + (\mathbf{t}_n(i))^{-1} = k_n + k_n^{-1}.$$

(ii) *Both*

$$\mathbf{t}_1(0)\mathbf{t}_2(0) = \sqrt{s^*q}, \quad \mathbf{t}_1(d)\mathbf{t}_2(d) = \frac{1}{q^d\sqrt{s^*q}}.$$

Proof. Immediate from Definition 3.2.1. ■

Definition 3.2.7. *With reference to Definition 3.2.1, for $n \in \mathbb{I}$ we define a $2d \times 2d$ block diagonal matrix \mathcal{T}_n as follows:*

$$\mathcal{T}_0 = \text{blockdiag}[\mathbf{t}_0(0), \mathbf{t}_0(1), \dots, \mathbf{t}_0(d-1)],$$

$$\mathcal{T}_1 = \text{blockdiag}[\mathbf{t}_1(0), \mathbf{t}_1(1), \dots, \mathbf{t}_1(d-1), \mathbf{t}_1(d)],$$

$$\mathcal{T}_2 = \text{blockdiag}[\mathbf{t}_2(0), \mathbf{t}_2(1), \dots, \mathbf{t}_2(d-1), \mathbf{t}_2(d)],$$

$$\mathcal{T}_3 = \text{blockdiag}[\mathbf{t}_3(0), \mathbf{t}_3(1), \dots, \mathbf{t}_3(d-1)].$$

Lemma 3.2.9. *With reference to Definition 3.2.7, the following (i)–(iii) hold:*

- (i) \mathcal{T}_n is invertible $(n \in \mathbb{I})$;
- (ii) $\mathcal{T}_n + \mathcal{T}_n^{-1} = (k_n + k_n^{-1})I$ $(n \in \mathbb{I})$;
- (iii) $\mathcal{T}_0\mathcal{T}_1\mathcal{T}_2\mathcal{T}_3 = q^{-1/2}I$.

Proof: (i) Use Lemma 3.2.4.

(ii) Immediate from (3.5) and Lemma 3.2.6(i).

(iii) It suffices to show that $\mathcal{T}_3\mathcal{T}_0\mathcal{T}_1\mathcal{T}_2 = q^{-1/2}I$. This is routinely verified using Lemma 3.2.8. ■

We now describe an action of \hat{H}_q on \mathbf{W} . Recall the basis \mathcal{C} for \mathbf{W} from (2.126).

Proposition 3.2.10. *Let \mathbf{W} be as below Definition 2.4.20. Then there exists an \hat{H}_q -module structure on \mathbf{W} such that for $n \in \mathbb{I}$ the matrix \mathcal{T}_n represents the generator t_n relative to \mathcal{C} .*

Proof. Follows from Lemma 3.2.9. ■

It turns out that the \hat{H}_q -module \mathbf{W} is irreducible. We will show this in Section 3.3.

Corollary 3.2.11. *Referring to the \hat{H}_q -module \mathbf{W} from Proposition 3.2.10, the elements of \mathcal{C} are eigenvectors for the action of \mathbf{X} on \mathbf{W} . Moreover the action of \mathbf{X} on \mathbf{W} is multiplicity free.*

Proof. Recall $\mathbf{X} = t_3t_0$. By this and Proposition 3.2.10 the matrix $\mathcal{T}_3\mathcal{T}_0$ represents \mathbf{X} relative to \mathcal{C} . The result follows in view of Lemma 3.2.8. ■

Remark 3.2.12. In Proposition 3.2.10 we obtained an \hat{H}_q -action on \mathbf{W} . Let $H = H(k_0, k_1, k_2, k_3; q)$ be a double affine Hecke algebra of type (C_1^\vee, C_1) , where k_0, k_1, k_2, k_3 are from (3.4). Then there exists a surjective \mathbb{C} -algebra homomorphism $\hat{H}_q \rightarrow H$ that sends $t_n \mapsto t_n$ for all $n \in \mathbb{I}$. Taking the quotient of \hat{H}_q by the kernel of this homomorphism we obtain an H -module structure on \mathbf{W} .

We mention a result for future use.

Lemma 3.2.13. *With reference to Definition 3.2.2 neither of k_0, k_1 is equal to ± 1 .*

Proof. By Example 2.1.8 $\frac{r_1 r_2}{s^*} = sq^{d+1}$ and $sq^{d+1} \neq 1$, so $k_0^2 \neq 1$ by (3.4). Also by Example 2.1.8 $q^d \neq 1$, so $k_1^2 \neq 1$ by (3.4). ■

3.3 How the \hat{H}_q -action on \mathbf{W} is related to \mathbf{T}

We continue to discuss the \mathbf{T} -module \mathbf{W} . In the previous section we saw how the algebra \hat{H}_q acts on \mathbf{W} . In this section we will see how the \hat{H}_q -action on \mathbf{W} is related to the algebra \mathbf{T} . Recall from Definition 2.4.20 that the algebra \mathbf{T} is generated by A, A^*, \tilde{A}^* . Recall the projections $\mathfrak{p}, \tilde{\mathfrak{p}}$ from Section 2.7. The following theorem is the main result of the thesis.

Theorem 3.3.1. *Referring to the \hat{H}_q -module \mathbf{W} from Proposition 3.2.10, for each row*

of the table below the two displayed elements coincide on \mathbf{W} .

Element in \hat{H}_q	Element in \mathbf{T}
A	$\frac{1}{h\sqrt{s^*q}}(A - (\theta_0 - h - hsq)I)$
B	$\frac{1}{\tilde{h}^*\sqrt{\tilde{s}^*q}}(\tilde{A}^* - (\tilde{\theta}_0^* - \tilde{h}^* - \tilde{h}^*\tilde{s}^*q)I)$
B[†]	$\frac{1}{h^*\sqrt{s^*q}}(A^* - (\theta_0^* - h^* - h^*s^*q)I)$
$\frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}}$	$\tilde{\mathbf{p}}$
$\frac{t_1 - k_1^{-1}}{k_1 - k_1^{-1}}$	\mathbf{p}

The remainder of this section is devoted to the proof of Theorem 3.3.1. Recall from (2.126) that the basis \mathcal{C} consists of the vectors \hat{C}_i^-, \hat{C}_i^+ ($0 \leq i \leq d-1$). Recall from above Lemma 2.4.9 that

$$\hat{C}_{-1}^- = 0, \quad \hat{C}_{-1}^+ = 0, \quad \hat{C}_d^- = 0, \quad \hat{C}_d^+ = 0.$$

Lemma 3.3.2. *Let \mathbf{Y} be as in Definition 3.1.2. Then for $0 \leq i \leq d-1$, $\mathbf{Y}.\hat{C}_i^-$ and $\mathbf{Y}.\hat{C}_i^+$ are given as a linear combination with the following terms and coefficients:*

$$\mathbf{Y}.\hat{C}_i^- =$$

term	coefficient
\hat{C}_{i-1}^-	$\sqrt{\frac{s^*q^d}{r_1r_2}} \left(\frac{(1-q^{i-d})(1-s^*q^{i+1})(1-r_1q^i)(1-r_2q^i)}{(1-s^*q^{2i})(1-s^*q^{2i+1})} \right)$
\hat{C}_{i-1}^+	$\sqrt{\frac{s^*q^d}{r_1r_2}} \left(\frac{(1-q^{i-d})(1-s^*q^{i+1})}{1-s^*q^{2i+1}} \right) \left(\frac{(1-r_1q^i)(1-r_2q^i)}{1-s^*q^{2i}} - 1 \right)$
\hat{C}_i^-	$\frac{1}{\sqrt{s^*r_1r_2q^d}} \left(\frac{(q^i-1)(1-s^*q^{d+i+1})}{1-s^*q^{2i+1}} + 1 \right) \left(\frac{(r_1-s^*q^{i+1})(r_2-s^*q^{i+1})}{1-s^*q^{2i+2}} + s^* \right)$
\hat{C}_i^+	$\frac{1}{\sqrt{s^*r_1r_2q^d}} \left(\frac{(r_1-s^*q^{i+1})(r_2-s^*q^{i+1})}{1-s^*q^{2i+2}} \right) \left(\frac{(q^i-1)(1-s^*q^{d+i+1})}{1-s^*q^{2i+1}} + 1 \right)$

(3.8)

and

$$\mathbf{Y} \cdot \hat{C}_i^+ =$$

term	coefficient
\hat{C}_i^-	$-\sqrt{\frac{s^* q^d}{r_1 r_2}} \left(\frac{(1-r_1 q^{i+1})(1-r_2 q^{i+1})}{1-s^* q^{2i+2}} \right) \left(\frac{(1-q^{i-d+1})(1-s^* q^{i+2})}{1-s^* q^{2i+3}} + \frac{1}{q^d} \right)$
\hat{C}_i^+	$\sqrt{\frac{s^* q^d}{r_1 r_2}} \left(1 - \frac{(1-r_1 q^{i+1})(1-r_2 q^{i+1})}{1-s^* q^{2i+2}} \right) \left(\frac{(1-q^{i-d+1})(1-s^* q^{i+2})}{1-s^* q^{2i+3}} + \frac{1}{q^d} \right)$
\hat{C}_{i+1}^-	$\frac{1}{\sqrt{s^* r_1 r_2 q^d}} \left(\frac{(1-q^{i+1})(1-s^* q^{d+i+2})}{1-s^* q^{2i+3}} \right) \left(\frac{(r_1-s^* q^{i+2})(r_2-s^* q^{i+2})}{1-s^* q^{2i+4}} + S^* \right)$
\hat{C}_{i+1}^+	$\frac{1}{\sqrt{s^* r_1 r_2 q^d}} \left(\frac{(1-q^{i+1})(1-s^* q^{d+i+2})(r_1-s^* q^{i+2})(r_2-s^* q^{i+2})}{(1-s^* q^{2i+3})(1-s^* q^{2i+4})} \right)$

Proof. Compute $\mathcal{T}_0 \mathcal{T}_1$ using Definition 3.2.7 and the data in Definition 3.2.1. \blacksquare

Lemma 3.3.3. *Let \mathbf{Y} be as in Definition 3.1.2. For $0 \leq i \leq d-1$, $\mathbf{Y}^{-1} \cdot \hat{C}_i^-$ and $\mathbf{Y}^{-1} \cdot \hat{C}_i^+$ are given as a linear combination with the following terms and coefficients:*

$$\mathbf{Y}^{-1} \cdot \hat{C}_i^- =$$

term	coefficient
\hat{C}_{i-1}^+	$\sqrt{\frac{s^* q^d}{r_1 r_2}} \left(\frac{(1-q^{i-d})(1-s^* q^{i+1})}{1-s^* q^{2i+1}} \right) \left(1 - \frac{(1-r_1 q^{i+1})(1-r_2 q^{i+1})}{1-s^* q^{2i+2}} \right)$
\hat{C}_i^-	$\sqrt{\frac{s^* q^d}{r_1 r_2}} \left(1 - \frac{(1-r_1 q^{i+1})(1-r_2 q^{i+1})}{1-s^* q^{2i+2}} \right) \left(\frac{(1-q^{i-d})(1-s^* q^{i+1})}{1-s^* q^{2i+1}} + \frac{1}{q^d} \right)$
\hat{C}_i^+	$\frac{1}{\sqrt{s^* r_1 r_2 q^d}} \left(\frac{(r_1-s^* q^{i+1})(r_2-s^* q^{i+1})}{1-s^* q^{2i+2}} \right) \left(\frac{(1-q^{i+1})(1-s^* q^{d+i+2})}{1-s^* q^{2i+3}} - 1 \right)$
\hat{C}_{i+1}^-	$\frac{1}{\sqrt{s^* r_1 r_2 q^d}} \left(\frac{(1-q^{i+1})(1-s^* q^{d+i+2})(r_1-s^* q^{i+1})(r_2-s^* q^{i+1})}{(1-s^* q^{2i+2})(1-s^* q^{2i+3})} \right)$

and

$$\mathbf{Y}^{-1} \cdot \hat{C}_i^+ =$$

term	coefficient
\hat{C}_{i-1}^+	$\sqrt{\frac{s^* q^d}{r_1 r_2}} \left(\frac{(1-q^{i-d})(1-s^* q^{i+1})(1-r_1 q^{i+1})(1-r_2 q^{i+1})}{(1-s^* q^{2i+1})(1-s^* q^{2i+2})} \right)$
\hat{C}_i^-	$\sqrt{\frac{s^* q^d}{r_1 r_2}} \left(\frac{(1-r_1 q^{i+1})(1-r_2 q^{i+1})}{1-s^* q^{2i+2}} \right) \left(\frac{(1-q^{i-d})(1-s^* q^{i+1})}{1-s^* q^{2i+1}} + \frac{1}{q^d} \right)$
\hat{C}_i^+	$\frac{1}{\sqrt{s^* r_1 r_2 q^d}} \left(\frac{(r_1-s^* q^{i+1})(r_2-s^* q^{i+1})}{1-s^* q^{2i+2}} + S^* \right) \left(\frac{(q^{i+1}-1)(1-s^* q^{d+i+2})}{1-s^* q^{2i+3}} + 1 \right)$
\hat{C}_{i+1}^-	$\frac{1}{\sqrt{s^* r_1 r_2 q^d}} \left(\frac{(q^{i+1}-1)(1-s^* q^{d+i+2})}{1-s^* q^{2i+3}} \right) \left(\frac{(r_1-s^* q^{i+1})(r_2-s^* q^{i+1})}{1-s^* q^{2i+2}} + S^* \right)$

and

$$\mathbf{A} \cdot \hat{C}_i^+ =$$

term	coefficient
\hat{C}_{i-1}^+	$\sqrt{\frac{s^* q^d}{r_1 r_2}} \left(\frac{(1-q^{i-d})(1-s^* q^{i+1})(1-r_1 q^{i+1})(1-r_2 q^{i+1})}{(1-s^* q^{2i+1})(1-s^* q^{2i+2})} \right)$
\hat{C}_i^-	$\sqrt{\frac{s^* q^d}{r_1 r_2}} \left(\frac{(1-r_1 q^{i+1})(1-r_2 q^{i+1})}{1-s^* q^{2i+2}} \right) \left(\frac{(1-q^{i-d})(1-s^* q^{i+1})}{1-s^* q^{2i+1}} - \frac{(1-q^{i-d+1})(1-s^* q^{i+2})}{1-s^* q^{2i+3}} \right)$
\hat{C}_i^+	$\left(\sqrt{\frac{s^* q^d}{r_1 r_2}} + \sqrt{\frac{r_1 r_2}{s^* q^d}} \right) - \sqrt{\frac{s^* q^d}{r_1 r_2}} \times \left(\frac{(1-q^{i+1})(1-s^* q^{i+d+2})(r_1-s^* q^{i+1})(r_2-s^* q^{i+1})}{s^* q^d (1-s^* q^{2i+2})(1-s^* q^{2i+3})} \right. \\ \left. + \frac{(1-q^{i-d+1})(1-s^* q^{i+2})(1-r_1 q^{i+1})(1-r_2 q^{i+1})}{(1-s^* q^{2i+2})(1-s^* q^{2i+3})} \right)$
\hat{C}_{i+1}^-	$\sqrt{\frac{s^* q^d}{r_1 r_2}} \left(\frac{(1-q^{i+1})(1-s^* q^{i+d+2})}{s^* q^d (1-s^* q^{2i+3})} \right) \left(\frac{(r_1-s^* q^{i+2})(r_2-s^* q^{i+2})}{1-s^* q^{2i+4}} - \frac{(r_1-s^* q^{i+1})(r_2-s^* q^{i+1})}{1-s^* q^{2i+2}} \right)$
\hat{C}_{i+1}^+	$\sqrt{\frac{s^* q^d}{r_1 r_2}} \left(\frac{(1-q^{i+1})(1-s^* q^{i+d+2})(r_1-s^* q^{i+2})(r_2-s^* q^{i+2})}{s^* q^d (1-s^* q^{2i+3})(1-s^* q^{2i+4})} \right)$

(3.13)

Proof. Combine Lemma 3.3.2 and Lemma 3.3.3. ■

Proposition 3.3.5. *Let \mathbf{B} be as in Definition 3.1.3. Then \mathbf{B} acts on the basis \mathcal{C} as follows:*

- (i) $\mathbf{B} \cdot \hat{C}_i^- = \left(\frac{1}{q^{i+1} \sqrt{s^*}} + q^{i+1} \sqrt{s^*} \right) \hat{C}_i^- \quad (0 \leq i \leq d-1).$
- (ii) $\mathbf{B} \cdot \hat{C}_i^+ = \left(\frac{1}{q^{i+1} \sqrt{s^*}} + q^{i+1} \sqrt{s^*} \right) \hat{C}_i^+ \quad (0 \leq i \leq d-1).$

Proof. Use (3.6). ■

Lemma 3.3.6. *Referring to Proposition 3.3.5, the following scalars are mutually distinct:*

$$\frac{1}{q^{i+1} \sqrt{s^*}} + q^{i+1} \sqrt{s^*} \quad (0 \leq i \leq d-1).$$

Proof. By Example 2.1.8, $s^* q^i \neq 1$ for $2 \leq i \leq 2d$. ■

Corollary 3.3.7. *For $0 \leq i \leq d-1$, the vectors \hat{C}_i^-, \hat{C}_i^+ form a basis for an eigenspace of \mathbf{B} with eigenvalue $\frac{1}{q^{i+1} \sqrt{s^*}} + q^{i+1} \sqrt{s^*}$.*

Proof. Use Proposition 3.3.5 and Lemma 3.3.6. ■

Proposition 3.3.8. *Let \mathbf{B}^\dagger be as in Definition 3.1.3. Then \mathbf{B}^\dagger acts on the basis \mathcal{C} as follows:*

$$\begin{aligned} \text{(i)} \quad \mathbf{B}^\dagger \cdot \hat{C}_i^- &= \left(\frac{1}{q^i \sqrt{s^* q}} + q^i \sqrt{s^* q} \right) \hat{C}_i^- & (0 \leq i \leq d-1), \\ \text{(ii)} \quad \mathbf{B}^\dagger \cdot \hat{C}_i^+ &= \left(\frac{1}{q^{i+1} \sqrt{s^* q}} + q^{i+1} \sqrt{s^* q} \right) \hat{C}_i^+ & (0 \leq i \leq d-1). \end{aligned}$$

Proof. Use (3.7). ■

Lemma 3.3.9. *Referring to Proposition 3.3.8, the following scalars are mutually distinct:*

$$\frac{1}{q^i \sqrt{s^* q}} + q^i \sqrt{s^* q} \quad (0 \leq i \leq d-1).$$

Proof. By Example 2.1.8, $s^* q^i \neq 1$ for $2 \leq i \leq 2d$. ■

Corollary 3.3.10. *For $1 \leq i \leq d-1$ the vectors $\hat{C}_{i-1}^+, \hat{C}_i^-$ form a basis for an eigenspace of \mathbf{B}^\dagger with eigenvalue $\frac{1}{q^i \sqrt{s^* q}} + q^i \sqrt{s^* q}$.*

Proof. Use Proposition 3.3.8 and Lemma 3.3.9. ■

Recall the scalars k_0 and k_1 from (3.4). Recall by Lemma 3.2.13 that $k_0 \neq \pm 1$ and $k_1 \neq \pm 1$. We now describe the action of $\frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}}$ and $\frac{t_1 - k_1^{-1}}{k_1 - k_1^{-1}}$ on the basis \mathcal{C} .

Proposition 3.3.11. *The element $\frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}}$ acts on the basis \mathcal{C} as follows:*

$$\begin{aligned} \text{(i)} \quad \left(\frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} \right) \cdot \hat{C}_i^- &= \frac{(r_1 - s^* q^{i+1})(r_2 - s^* q^{i+1})}{(r_1 r_2 - s^*)(1 - s^* q^{2i+2})} (\hat{C}_i^- + \hat{C}_i^+) & (0 \leq i \leq d-1), \\ \text{(ii)} \quad \left(\frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}} \right) \cdot \hat{C}_i^+ &= \frac{s^*(1 - r_1 q^{i+1})(1 - r_2 q^{i+1})}{(s^* - r_1 r_2)(1 - s^* q^{2i+2})} (\hat{C}_i^- + \hat{C}_i^+) & (0 \leq i \leq d-1). \end{aligned}$$

Proof. Use the t_0 -action on the basis \mathcal{C} from Proposition 3.2.10. ■

Proposition 3.3.12. *The element $\frac{t_1 - k_1^{-1}}{k_1 - k_1^{-1}}$ acts on the basis \mathcal{C} as follows:*

- (i) $\left(\frac{t_1 - k_1^{-1}}{k_1 - k_1^{-1}}\right) \cdot \hat{C}_0^- = \hat{C}_0^-$,
- (ii) $\left(\frac{t_1 - k_1^{-1}}{k_1 - k_1^{-1}}\right) \cdot \hat{C}_i^- = \frac{q^d(1 - q^{i-d})(1 - s^*q^{i+1})}{(q^d - 1)(1 - s^*q^{2i+1})} (\hat{C}_{i-1}^+ + \hat{C}_i^-) \quad (1 \leq i \leq d - 1)$,
- (iii) $\left(\frac{t_1 - k_1^{-1}}{k_1 - k_1^{-1}}\right) \cdot \hat{C}_{i-1}^+ = \frac{(1 - q^i)(1 - s^*q^{i+d+1})}{(1 - q^d)(1 - s^*q^{2i+1})} (\hat{C}_{i-1}^+ + \hat{C}_i^-) \quad (1 \leq i \leq d - 1)$,
- (iv) $\left(\frac{t_1 - k_1^{-1}}{k_1 - k_1^{-1}}\right) \cdot \hat{C}_{d-1}^+ = \hat{C}_{d-1}^+$.

Proof. Use the t_1 -action on the basis \mathcal{C} from Proposition 3.2.10. ■

We are ready to prove Theorem 3.3.1.

Proof of Theorem 3.3.1: We refer to the table in the theorem statement. For each row we compare the matrices representing the two displayed elements relative to the basis \mathcal{C} . In each case we show that these matrices coincide.

A: From Proposition 3.3.4 we find the matrix representation of **A**. From (2.69) and (2.70) we obtain the matrix representation of A . From this we get the matrix representation of $\frac{1}{h\sqrt{sq}}(A - (\theta_0 - h - hsq)I)$. In this representation, eliminate s using $r_1r_2 = ss^*q^{d+1}$. The result coincides with the matrix representation of **A**.

B: From Proposition 3.3.5 we find the matrix representation of **B**. From Lemma 2.8.15 we obtain the matrix representation of \tilde{A}^* . From this we get the matrix representation of $\frac{1}{\tilde{h}^*\sqrt{\tilde{s}^*q}}(\tilde{A}^* - (\tilde{\theta}_0^* - \tilde{h}^* - \tilde{h}^*\tilde{s}^*q)I)$. Evaluate this representation using (2.60). The result coincides with the matrix representation of **B**.

B[†]: From Proposition 3.3.8 we find the matrix representation of **B[†]**. From Lemma 2.8.10 we obtain the matrix representation of A^* . From this we get the matrix representation

of $\frac{1}{h^*\sqrt{s^*q}}(A^* - (\theta_0^* - h^* - h^*s^*q)I)$. Evaluate this representation using (2.14). The result coincides with the matrix representation of \mathbf{B}^\dagger .

$\frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}}$: From Proposition 3.3.11 we find the matrix representation of $\frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}}$. The representation coincides with the matrix (2.138), which is the representation of $\tilde{\mathfrak{p}}$.

$\frac{t_1 - k_1^{-1}}{k_1 - k_1^{-1}}$: From Proposition 3.3.12 we find the matrix representation of $\frac{t_1 - k_1^{-1}}{k_1 - k_1^{-1}}$. The representation coincides with the matrix (2.137), which is the representation of \mathfrak{p} . ■

We finish this thesis with a comment.

Corollary 3.3.13. *Referring to Theorem 3.3.1, the \hat{H}_q -module \mathbf{W} is irreducible.*

Proof. By definition 2.4.20 the algebra \mathbf{T} is generated by A, A^*, \tilde{A}^* . By Proposition 2.4.25 the \mathbf{T} -module \mathbf{W} is irreducible. The result follows in view of Theorem 3.3.1. ■

Appendix A

An Example worked out in detail

In this Appendix we consider the case of diameter $d = 4$ in great detail. We briefly review and summarize the results of Chapter 2 and Chapter 3. Concerning Chapter 2, recall that Γ is a Q -polynomial distance-regular graph of q -Racah type that contains a Delsarte clique C . Recall the semisimple algebra \mathbf{T} from Definition 2.4.20, whose generators are A, A^*, \tilde{A}^* . Recall the primary \mathbf{T} -module \mathbf{W} from below Definition 2.4.20. The \mathbf{T} -module \mathbf{W} is decomposed in two ways:

$$\mathbf{W} = \mathcal{M}\hat{x} + \mathcal{M}\hat{x}^\perp, \quad \mathbf{W} = \mathcal{M}\hat{C} + \mathcal{M}\hat{C}^\perp. \quad (\text{orthogonal direct sum})$$

On each of the four summands, we found a natural Leonard system. For each Leonard system we found the parameter array, and we described how these parameter arrays are related. We recall our five linear maps in $\text{End}(\mathbf{W})$:

$$A, \quad A^*, \quad \tilde{A}^*, \quad \mathfrak{p}, \quad \tilde{\mathfrak{p}}. \quad (\text{A.1})$$

Recall from (2.131) our five bases for \mathbf{W} :

$$\mathcal{C}, \quad \mathcal{B}, \quad \mathcal{B}_{alt}, \quad \tilde{\mathcal{B}}, \quad \tilde{\mathcal{B}}_{alt}. \quad (\text{A.2})$$

In Section 2.8, we displayed the matrix representing each map in (A.1) relative to each basis in (A.2). We also displayed the transition matrices between certain pairs of bases among (A.2).

Concerning Chapter 3, recall the algebra \hat{H}_q from Definition 3.1.1 whose generators are $\{t_n^{\pm 1}\}_{n=0}^3$ and relations (3.1)–(3.3). Recall $\mathbf{X} = t_3 t_0$, $\mathbf{Y} = t_0 t_1$ from Definition 3.1.2, and $\mathbf{A}, \mathbf{B}, \mathbf{B}^\dagger$ from Definition 3.1.3. In Section 3.2 we constructed a \hat{H}_q -module on \mathbf{W} . For this module and up to affine transformation we showed that \mathbf{A} acts as A and \mathbf{B} (resp. \mathbf{B}^\dagger) acts as \tilde{A}^* (resp. A^*). Moreover, up to affine transformation, t_0 (resp. t_1) acts as $\tilde{\mathfrak{p}}$ (resp. \mathfrak{p}).

To give a concrete example we now take $d = 4$. In Section A.1 below, we display the matrices representing each map in (A.1) relative to each basis in (A.2) and the transition matrices between all pairs of bases among (A.2). In Section A.2, we display the matrices representing each generator $\{t_n\}_{n=0}^3$ of \hat{H}_q and some related elements of \hat{H}_q with respect to the basis \mathcal{C} from (A.2).

We recall the scalars h, h^*, s, s^*, r_1, r_2 from above Note 2.2.4. The scalars h, h^* satisfy (2.26), (2.27), respectively. Recall the formulae ϵ_i from (2.79), ξ_i from (2.95), τ_i from (2.105), and ζ_i from (2.121). For $d = 4$, these become

$$\epsilon_i = \frac{(1 - q^i)(1 - s^* q^{i+5})}{q^4(1 - q^{i-4})(1 - s^* q^{i+1})}, \quad \xi_i = q^{1-i}(1 - q^{i-4})(1 - s^* q^{i+1}), \quad (\text{A.3})$$

for $1 \leq i \leq 3$ and

$$\tau_i = \frac{s^*(1 - r_1 q^{i+1})(1 - r_2 q^{i+1})}{(r_1 - s^* q^{i+1})(r_2 - s^* q^{i+1})}, \quad \zeta_i = q^{-i}(r_1 - s^* q^{i+1})(r_2 - s^* q^{i+1}), \quad (\text{A.4})$$

for $0 \leq i \leq 3$. All entries of each matrix in this Appendix will be expressed in terms of q, s, s^*, r_1, r_2 and their square roots.

A.1 The matrices from Chapter 2

Consider the five bases in (A.2). For $d = 4$ these bases are

$$\begin{aligned}\mathcal{C} &= \{\hat{C}_0^-, \hat{C}_0^+, \hat{C}_1^-, \hat{C}_1^+, \hat{C}_2^-, \hat{C}_2^+, \hat{C}_3^-, \hat{C}_3^+\}, \\ \mathcal{B} &= \{v_0, v_1, v_2, v_3, v_4, v_0^\perp, v_1^\perp, v_2^\perp\}, \\ \mathcal{B}_{alt} &= \{v_0, v_1, v_0^\perp, v_2, v_1^\perp, v_3, v_2^\perp, v_4\}, \\ \tilde{\mathcal{B}} &= \{\tilde{v}_0, \tilde{v}_1, \tilde{v}_2, \tilde{v}_3, \tilde{v}_0^\perp, \tilde{v}_1^\perp, \tilde{v}_2^\perp, \tilde{v}_3^\perp\}, \\ \tilde{\mathcal{B}}_{alt} &= \{\tilde{v}_0, \tilde{v}_0^\perp, \tilde{v}_1, \tilde{v}_1^\perp, \tilde{v}_2, \tilde{v}_2^\perp, \tilde{v}_3, \tilde{v}_3^\perp\}.\end{aligned}$$

For each of the 20 ordered pairs of bases from above, we now display the corresponding transition matrix. This will be done in Example A.1.1 through Example A.1.10 below.

Example A.1.1. The transition matrix from \mathcal{C} to \mathcal{B}_{alt} is

$$\left[\begin{array}{c|ccc|cc|cc|c} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & \xi_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & \xi_1\epsilon_1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & \xi_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \xi_2\epsilon_2 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & \xi_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \xi_3\epsilon_3 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right],$$

where $\{\xi_i\}_{i=1}^3$ and $\{\epsilon_i\}_{i=1}^3$ are from (A.3). Moreover,

$$\xi_i\epsilon_i = q^{-i-3}(1 - q^i)(1 - s^*q^{i+5}) \quad (1 \leq i \leq 3). \quad (\text{A.5})$$

The transition matrix from \mathcal{B}_{alt} to \mathcal{C} is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\epsilon_1}{\epsilon_1-1} & \frac{1}{1-\epsilon_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\xi_1(1-\epsilon_1)} & \frac{1}{\xi_1(\epsilon_1-1)} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\epsilon_2}{\epsilon_2-1} & \frac{1}{1-\epsilon_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\xi_2(1-\epsilon_2)} & \frac{1}{\xi_2(\epsilon_2-1)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\epsilon_3}{\epsilon_3-1} & \frac{1}{1-\epsilon_3} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\xi_3(1-\epsilon_3)} & \frac{1}{\xi_3(\epsilon_3-1)} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Moreover for $1 \leq i \leq 3$,

$$\begin{aligned} \frac{\epsilon_i}{\epsilon_i-1} &= \frac{(1-q^i)(1-s^*q^{i+5})}{(1-q^4)(1-s^*q^{2i+1})}, & \frac{1}{1-\epsilon_i} &= \frac{q^4(1-q^{i-4})(1-s^*q^{i+1})}{(q^4-1)(1-s^*q^{2i+1})}, \\ \frac{1}{\xi_i(1-\epsilon_i)} &= \frac{q^{3+i}}{(q^4-1)(1-s^*q^{2i+1})}, & \frac{1}{\xi_i(\epsilon_i-1)} &= \frac{q^{3+i}}{(1-q^4)(1-s^*q^{2i+1})}. \end{aligned} \tag{A.6}$$

Example A.1.2. The transition matrix from \mathcal{C} to $\tilde{\mathcal{B}}_{alt}$ is

$$\begin{bmatrix} 1 & \zeta_0\tau_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & \zeta_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & \zeta_1\tau_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \zeta_1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & \zeta_2\tau_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \zeta_2 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & \zeta_3\tau_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \zeta_3 \end{bmatrix},$$

where $\{\zeta_i\}_{i=0}^3$ and $\{\tau_i\}_{i=0}^3$ are from (A.4). Moreover,

$$\zeta_i\tau_i = q^{-i}s^*(1-r_1q^{i+1})(1-r_2q^{i+1}), \quad (0 \leq i \leq 3) \tag{A.7}$$

The transition matrix from $\tilde{\mathcal{B}}_{alt}$ to \mathcal{C} is

$$\begin{bmatrix} \frac{1}{1-\tau_0} & \frac{\tau_0}{\tau_0-1} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{\zeta_0(\tau_0-1)} & \frac{1}{\zeta_0(1-\tau_0)} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{1-\tau_1} & \frac{\tau_1}{\tau_1-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\zeta_1(\tau_1-1)} & \frac{1}{\zeta_1(1-\tau_1)} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{1-\tau_2} & \frac{\tau_2}{\tau_2-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\zeta_2(\tau_2-1)} & \frac{1}{\zeta_2(1-\tau_2)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{1-\tau_3} & \frac{\tau_3}{\tau_3-1} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\zeta_3(\tau_3-1)} & \frac{1}{\zeta_3(1-\tau_3)} \end{bmatrix}.$$

Moreover for $0 \leq i \leq 3$,

$$\begin{aligned} \frac{1}{1-\tau_i} &= \frac{(r_1-s^*q^{i+1})(r_2-s^*q^{i+1})}{(r_1r_2-s^*)(1-s^*q^{2i+2})}, & \frac{\tau_i}{\tau_i-1} &= \frac{s^*(1-r_1q^{i+1})(1-r_2q^{i+1})}{(s^*-r_1r_2)(1-s^*q^{2i+2})}, \\ \frac{1}{\zeta_i(\tau_i-1)} &= \frac{q^i}{(s^*-r_1r_2)(1-s^*q^{2i+2})}, & \frac{1}{\zeta_i(1-\tau_i)} &= \frac{q^i}{(r_1r_2-s^*)(1-s^*q^{2i+2})}. \end{aligned} \quad (\text{A.8})$$

Example A.1.3. The transition matrix from \mathcal{B}_{alt} to $\tilde{\mathcal{B}}_{alt}$ is

$$\begin{bmatrix} 1 & \zeta_0\tau_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\epsilon_1}{\epsilon_1-1} & \frac{\epsilon_1\zeta_0}{\epsilon_1-1} & \frac{1}{1-\epsilon_1} & \frac{\zeta_1\tau_1}{1-\epsilon_1} & 0 & 0 & 0 & 0 \\ \frac{1}{\xi_1(1-\epsilon_1)} & \frac{\zeta_0}{\xi_1(1-\epsilon_1)} & \frac{1}{\xi_1(\epsilon_1-1)} & \frac{\zeta_1\tau_1}{\xi_1(\epsilon_1-1)} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\epsilon_2}{\epsilon_2-1} & \frac{\epsilon_2\zeta_1}{\epsilon_2-1} & \frac{1}{1-\epsilon_2} & \frac{\zeta_2\tau_2}{1-\epsilon_2} & 0 & 0 \\ 0 & 0 & \frac{1}{\xi_2(1-\epsilon_2)} & \frac{\zeta_1}{\xi_2(1-\epsilon_2)} & \frac{1}{\xi_2(\epsilon_2-1)} & \frac{\zeta_2\tau_2}{\xi_2(\epsilon_2-1)} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\epsilon_3}{\epsilon_3-1} & \frac{\epsilon_3\zeta_2}{\epsilon_3-1} & \frac{1}{1-\epsilon_3} & \frac{\zeta_3\tau_3}{1-\epsilon_3} \\ 0 & 0 & 0 & 0 & \frac{1}{\xi_3(1-\epsilon_3)} & \frac{\zeta_2}{\xi_3(1-\epsilon_3)} & \frac{1}{\xi_3(\epsilon_3-1)} & \frac{\zeta_3\tau_3}{\xi_3(\epsilon_3-1)} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \zeta_3 \end{bmatrix},$$

where $\{\xi_i\}_{i=1}^3, \{\epsilon_i\}_{i=1}^3$ are from (A.3) and $\{\zeta_i\}_{i=0}^3, \{\tau_i\}_{i=0}^3$ are from (A.4). Moreover

$$\left. \begin{aligned} \frac{\epsilon_i}{\epsilon_i-1} &= \frac{(1-q^i)(1-s^*q^{i+5})}{(1-q^4)(1-s^*q^{2i+1})}, & \frac{\epsilon_i \zeta_{i-1}}{\epsilon_i-1} &= \frac{q^{-i+1}(1-q^i)(1-s^*q^{i+5})(r_1-s^*q^i)(r_2-s^*q^i)}{(1-q^4)(1-s^*q^{2i+1})}, \\ \frac{1}{\xi_i(1-\epsilon_i)} &= \frac{q^{3+i}}{(q^4-1)(1-s^*q^{2i+1})}, & \frac{\zeta_{i-1}}{\xi_i(1-\epsilon_i)} &= \frac{q^4(r_1-s^*q^i)(r_2-s^*q^i)}{(q^4-1)(1-s^*q^{2i+1})}, \\ \frac{1}{1-\epsilon_i} &= \frac{q^4(1-q^{i-4})(1-s^*q^{i+1})}{(q^4-1)(1-s^*q^{2i+1})}, & \frac{\zeta_i \tau_i}{1-\epsilon_i} &= \frac{s^*q^{4-i}(1-q^{i-4})(1-s^*q^{i+1})(1-r_1q^{i+1})(1-r_2q^{i+1})}{(q^4-1)(1-s^*q^{2i+1})}, \\ \frac{1}{\xi_i(\epsilon_i-1)} &= \frac{q^{3+i}}{(1-q^4)(1-s^*q^{2i+1})}, & \frac{\zeta_i \tau_i}{\xi_i(\epsilon_i-1)} &= \frac{s^*q^3(1-r_1q^{i+1})(1-r_2q^{i+1})}{(1-q^4)(1-s^*q^{2i+1})}. \end{aligned} \right\} \quad (\text{A.9})$$

The transition matrix from $\tilde{\mathcal{B}}_{alt}$ to \mathcal{B}_{alt} is

$$\left[\begin{array}{ccc|cc|cc|c} \frac{1}{1-\tau_0} & \frac{\tau_0}{\tau_0-1} & \frac{\tau_0 \xi_1}{\tau_0-1} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{\zeta_0(\tau_0-1)} & \frac{1}{\zeta_0(1-\tau_0)} & \frac{\xi_1}{\zeta_0(1-\tau_0)} & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & \frac{1}{1-\tau_1} & \frac{\xi_1 \epsilon_1}{1-\tau_1} & \frac{\tau_1}{\tau_1-1} & \frac{\tau_1 \xi_2}{\tau_1-1} & 0 & 0 & 0 \\ 0 & \frac{1}{\zeta_1(\tau_1-1)} & \frac{\xi_1 \epsilon_1}{\zeta_1(\tau_1-1)} & \frac{1}{\zeta_1(1-\tau_1)} & \frac{\xi_2}{\zeta_1(1-\tau_1)} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \frac{1}{1-\tau_2} & \frac{\xi_2 \epsilon_2}{1-\tau_2} & \frac{\tau_2}{\tau_2-1} & \frac{\tau_2 \xi_3}{\tau_2-1} & 0 \\ 0 & 0 & 0 & \frac{1}{\zeta_2(\tau_2-1)} & \frac{\xi_2 \epsilon_2}{\zeta_2(\tau_2-1)} & \frac{1}{\zeta_2(1-\tau_2)} & \frac{\xi_3}{\zeta_2(1-\tau_2)} & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & \frac{1}{1-\tau_3} & \frac{\xi_3 \epsilon_3}{1-\tau_3} & \frac{\tau_3}{\tau_3-1} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\zeta_3(\tau_3-1)} & \frac{\xi_3 \epsilon_3}{\zeta_3(\tau_3-1)} & \frac{1}{\zeta_3(1-\tau_3)} \end{array} \right],$$

where $\{\xi_i\}_{i=1}^3, \{\epsilon_i\}_{i=1}^3$ are from (A.3) and $\{\zeta_i\}_{i=0}^3, \{\tau_i\}_{i=0}^3$ are from (A.4). Moreover

$$\left. \begin{aligned} \frac{1}{1-\tau_i} &= \frac{(r_1-s^*q^{i+1})(r_2-s^*q^{i+1})}{(r_1 r_2 - s^*)(1-s^*q^{2i+2})}, & \frac{\xi_i \epsilon_i}{1-\tau_i} &= \frac{q^{-i-3}(1-q^i)(1-s^*q^{i+5})(r_1-s^*q^{i+1})(r_2-s^*q^{i+1})}{(1-s^*q^{2i+2})(r_1 r_2 - s^*)}, \\ \frac{1}{\zeta_i(\tau_i-1)} &= \frac{q^i}{(s^*-r_1 r_2)(1-s^*q^{2i+2})}, & \frac{\xi_i \epsilon_i}{\zeta_i(\tau_i-1)} &= \frac{q^{-3}(1-q^i)(1-s^*q^{i+5})}{(1-s^*q^{2i+2})(s^*-r_1 r_2)}, \\ \frac{\tau_i}{\tau_i-1} &= \frac{s^*(1-r_1 q^{i+1})(1-r_2 q^{i+1})}{(s^*-r_1 r_2)(1-s^*q^{2i+2})}, & \frac{\tau_i \xi_{i+1}}{\tau_i-1} &= \frac{s^*q^{-i}(1-q^{i-3})(1-s^*q^{i+2})(1-r_1 q^{i+1})(1-r_2 q^{i+1})}{(1-s^*q^{2i+2})(s^*-r_1 r_2)}, \\ \frac{1}{\zeta_i(1-\tau_i)} &= \frac{q^i}{(r_1 r_2 - s^*)(1-s^*q^{2i+2})}, & \frac{\xi_{i+1}}{\zeta_i(1-\tau_i)} &= \frac{(1-q^{i-3})(1-s^*q^{i+2})}{(1-s^*q^{2i+2})(r_1 r_2 - s^*)}. \end{aligned} \right\} \quad (\text{A.10})$$

Example A.1.4. The transition matrix from \mathcal{B} to \mathcal{B}_{alt} is

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right].$$

The transition matrix from \mathcal{B}_{alt} to \mathcal{B} is

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \end{array} \right].$$

Example A.1.5. The transition matrix from $\tilde{\mathcal{B}}$ to $\tilde{\mathcal{B}}_{alt}$ is

$$\left[\begin{array}{cc|cc|cc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right].$$

The transition matrix from $\tilde{\mathcal{B}}_{alt}$ to $\tilde{\mathcal{B}}$ is

$$\left[\begin{array}{cc|cc|cc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right].$$

Example A.1.6. The transition matrix from \mathcal{C} to \mathcal{B} is

$$\left[\begin{array}{c|cc|cc|cc|c} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 & \xi_1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \xi_1\epsilon_1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 & \xi_2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \xi_2\epsilon_2 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 & 0 & \xi_3 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \xi_3\epsilon_3 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{array} \right],$$

where $\{\xi_i\}_{i=1}^3$ and $\{\xi_i\epsilon_i\}_{i=1}^3$ are from (A.3) and (A.5), respectively.

The transition matrix from \mathcal{B} to \mathcal{C} is

$$\left[\begin{array}{c|cc|cc|cc|c} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & \frac{\epsilon_1}{\epsilon_1-1} & \frac{1}{1-\epsilon_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\epsilon_2}{\epsilon_2-1} & \frac{1}{1-\epsilon_2} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & \frac{\epsilon_3}{\epsilon_3-1} & \frac{1}{1-\epsilon_3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & \frac{1}{\xi_1(1-\epsilon_1)} & \frac{1}{\xi_1(\epsilon_1-1)} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\xi_2(1-\epsilon_2)} & \frac{1}{\xi_2(\epsilon_2-1)} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & \frac{1}{\xi_3(1-\epsilon_3)} & \frac{1}{\xi_3(\epsilon_3-1)} & 0 \end{array} \right].$$

Each entry is from (A.6).

Example A.1.7. The transition matrix from \mathcal{C} to $\tilde{\mathcal{B}}$ is

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \zeta_0\tau_0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & \zeta_0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 & \zeta_1\tau_1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \zeta_1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 & \zeta_2\tau_2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \zeta_2 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 & 0 & \zeta_3\tau_3 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \zeta_3 \end{array} \right],$$

where $\{\zeta_i\}_{i=0}^3$ and $\{\zeta_i\tau_i\}_{i=0}^3$ are from (A.4) and (A.7), respectively.

The transition matrix from $\tilde{\mathcal{B}}$ to \mathcal{C} is

$$\left[\begin{array}{cc|cc|cc|cc} \frac{1}{1-\tau_0} & \frac{\tau_0}{\tau_0-1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{1-\tau_1} & \frac{\tau_1}{\tau_1-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{1-\tau_2} & \frac{\tau_2}{\tau_2-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{1-\tau_3} & \frac{\tau_3}{\tau_3-1} \\ \hline \frac{1}{\zeta_0(\tau_0-1)} & \frac{1}{\zeta_0(1-\tau_0)} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\zeta_1(\tau_1-1)} & \frac{1}{\zeta_1(1-\tau_1)} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\zeta_2(\tau_2-1)} & \frac{1}{\zeta_2(1-\tau_2)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\zeta_3(\tau_3-1)} & \frac{1}{\zeta_3(1-\tau_3)} \end{array} \right].$$

Each entry is from (A.8).

Example A.1.8. The transition matrix from \mathcal{B} to $\tilde{\mathcal{B}}_{alt}$ is

$$\left[\begin{array}{cc|cc|cc|cc} 1 & \zeta_0 \tau_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\epsilon_1}{\epsilon_1-1} & \frac{\epsilon_1 \zeta_0}{\epsilon_1-1} & \frac{1}{1-\epsilon_1} & \frac{\zeta_1 \tau_1}{1-\epsilon_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\epsilon_2}{\epsilon_2-1} & \frac{\epsilon_2 \zeta_1}{\epsilon_2-1} & \frac{1}{1-\epsilon_2} & \frac{\zeta_2 \tau_2}{1-\epsilon_2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\epsilon_3}{\epsilon_3-1} & \frac{\epsilon_3 \zeta_2}{\epsilon_3-1} & \frac{1}{1-\epsilon_3} & \frac{\zeta_3 \tau_3}{1-\epsilon_3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \zeta_3 \\ \hline \frac{1}{\xi_1(1-\epsilon_1)} & \frac{\zeta_0}{\xi_1(1-\epsilon_1)} & \frac{1}{\xi_1(\epsilon_1-1)} & \frac{\zeta_1 \tau_1}{\xi_1(\epsilon_1-1)} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\xi_2(1-\epsilon_2)} & \frac{\zeta_1}{\xi_2(1-\epsilon_2)} & \frac{1}{\xi_2(\epsilon_2-1)} & \frac{\zeta_2 \tau_2}{\xi_2(\epsilon_2-1)} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\xi_3(1-\epsilon_3)} & \frac{\zeta_2}{\xi_3(1-\epsilon_3)} & \frac{1}{\xi_3(\epsilon_3-1)} & \frac{\zeta_3 \tau_3}{\xi_3(\epsilon_3-1)} \end{array} \right].$$

Each entry is from (A.9).

The transition matrix from $\tilde{\mathcal{B}}_{alt}$ to \mathcal{B} is

$$\left[\begin{array}{cccc|ccc} \frac{1}{1-\tau_0} & \frac{\tau_0}{\tau_0-1} & 0 & 0 & 0 & \frac{\tau_0 \xi_1}{\tau_0-1} & 0 & 0 \\ \frac{1}{\zeta_0(\tau_0-1)} & \frac{1}{\zeta_0(1-\tau_0)} & 0 & 0 & 0 & \frac{\xi_1}{\zeta_0(1-\tau_0)} & 0 & 0 \\ \hline 0 & \frac{1}{1-\tau_1} & \frac{\tau_1}{\tau_1-1} & 0 & 0 & \frac{\xi_1 \epsilon_1}{1-\tau_1} & \frac{\tau_1 \xi_2}{\tau_1-1} & 0 \\ 0 & \frac{1}{\zeta_1(\tau_1-1)} & \frac{1}{\zeta_1(1-\tau_1)} & 0 & 0 & \frac{\xi_1 \epsilon_1}{\zeta_1(\tau_1-1)} & \frac{\xi_2}{\zeta_1(1-\tau_1)} & 0 \\ \hline 0 & 0 & \frac{1}{1-\tau_2} & \frac{\tau_2}{\tau_2-1} & 0 & 0 & \frac{\xi_2 \epsilon_2}{1-\tau_2} & \frac{\tau_2 \xi_3}{\tau_2-1} \\ 0 & 0 & \frac{1}{\zeta_2(\tau_2-1)} & \frac{1}{\zeta_2(1-\tau_2)} & 0 & 0 & \frac{\xi_2 \epsilon_2}{\zeta_2(\tau_2-1)} & \frac{\xi_3}{\zeta_2(1-\tau_2)} \\ \hline 0 & 0 & 0 & \frac{1}{1-\tau_3} & \frac{\tau_3}{\tau_3-1} & 0 & 0 & \frac{\xi_3 \epsilon_3}{1-\tau_3} \\ 0 & 0 & 0 & \frac{1}{\zeta_3(\tau_3-1)} & \frac{1}{\zeta_3(1-\tau_3)} & 0 & 0 & \frac{\xi_3 \epsilon_3}{\zeta_3(\tau_3-1)} \end{array} \right].$$

Each entry is from (A.10).

Example A.1.9. The transition from \mathcal{B}_{alt} to $\tilde{\mathcal{B}}$ is

1	0	0	0	$\zeta_0 \tau_0$	0	0	0
$\frac{\epsilon_1}{\epsilon_1-1}$	$\frac{1}{1-\epsilon_1}$	0	0	$\frac{\epsilon_1 \zeta_0}{\epsilon_1-1}$	$\frac{\zeta_1 \tau_1}{1-\epsilon_1}$	0	0
$\frac{1}{\xi_1(1-\epsilon_1)}$	$\frac{1}{\xi_1(\epsilon_1-1)}$	0	0	$\frac{\zeta_0}{\xi_1(1-\epsilon_1)}$	$\frac{\zeta_1 \tau_1}{\xi_1(\epsilon_1-1)}$	0	0
0	$\frac{\epsilon_2}{\epsilon_2-1}$	$\frac{1}{1-\epsilon_2}$	0	0	$\frac{\epsilon_2 \zeta_1}{\epsilon_2-1}$	$\frac{\zeta_2 \tau_2}{1-\epsilon_2}$	0
0	$\frac{1}{\xi_2(1-\epsilon_2)}$	$\frac{1}{\xi_2(\epsilon_2-1)}$	0	0	$\frac{\zeta_1}{\xi_2(1-\epsilon_2)}$	$\frac{\zeta_2 \tau_2}{\xi_2(\epsilon_2-1)}$	0
0	0	$\frac{\epsilon_3}{\epsilon_3-1}$	$\frac{1}{1-\epsilon_3}$	0	0	$\frac{\epsilon_3 \zeta_2}{\epsilon_3-1}$	$\frac{\zeta_3 \tau_3}{1-\epsilon_3}$
0	0	$\frac{1}{\xi_3(1-\epsilon_3)}$	$\frac{1}{\xi_3(\epsilon_3-1)}$	0	0	$\frac{\zeta_2}{\xi_3(1-\epsilon_3)}$	$\frac{\zeta_3 \tau_3}{\xi_3(\epsilon_3-1)}$
0	0	0	1	0	0	0	ζ_3

Each entry is from (A.9).

The transition matrix from $\tilde{\mathcal{B}}$ to \mathcal{B}_{alt} is

$\frac{1}{1-\tau_0}$	$\frac{\tau_0}{\tau_0-1}$	$\frac{\tau_0 \xi_1}{\tau_0-1}$	0	0	0	0	0
0	$\frac{1}{1-\tau_1}$	$\frac{\xi_1 \epsilon_1}{1-\tau_1}$	$\frac{\tau_1}{\tau_1-1}$	$\frac{\tau_1 \xi_2}{\tau_1-1}$	0	0	0
0	0	0	$\frac{1}{1-\tau_2}$	$\frac{\xi_2 \epsilon_2}{1-\tau_2}$	$\frac{\tau_2}{\tau_2-1}$	$\frac{\tau_2 \xi_3}{\tau_2-1}$	0
0	0	0	0	0	$\frac{1}{1-\tau_3}$	$\frac{\xi_3 \epsilon_3}{1-\tau_3}$	$\frac{\tau_3}{\tau_3-1}$
$\frac{1}{\zeta_0(\tau_0-1)}$	$\frac{1}{\zeta_0(1-\tau_0)}$	$\frac{\xi_1}{\zeta_0(1-\tau_0)}$	0	0	0	0	0
0	$\frac{1}{\zeta_1(\tau_1-1)}$	$\frac{\xi_1 \epsilon_1}{\zeta_1(\tau_1-1)}$	$\frac{1}{\zeta_1(1-\tau_1)}$	$\frac{\xi_2}{\zeta_1(1-\tau_1)}$	0	0	0
0	0	0	$\frac{1}{\zeta_2(\tau_2-1)}$	$\frac{\xi_2 \epsilon_2}{\zeta_2(\tau_2-1)}$	$\frac{1}{\zeta_2(1-\tau_2)}$	$\frac{\xi_3}{\zeta_2(1-\tau_2)}$	0
0	0	0	0	0	$\frac{1}{\zeta_3(\tau_3-1)}$	$\frac{\xi_3 \epsilon_3}{\zeta_3(\tau_3-1)}$	$\frac{1}{\zeta_3(1-\tau_3)}$

Each entry is from (A.10).

Example A.1.10. The transition matrix from \mathcal{B} and $\tilde{\mathcal{B}}$ is

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \zeta_0 \tau_0 & 0 & 0 & 0 \\ \frac{\epsilon_1}{\epsilon_1-1} & \frac{1}{1-\epsilon_1} & 0 & 0 & \frac{\epsilon_1 \zeta_0}{\epsilon_1-1} & \frac{\zeta_1 \tau_1}{1-\epsilon_1} & 0 & 0 \\ 0 & \frac{\epsilon_2}{\epsilon_2-1} & \frac{1}{1-\epsilon_2} & 0 & 0 & \frac{\epsilon_2 \zeta_1}{\epsilon_2-1} & \frac{\zeta_2 \tau_2}{1-\epsilon_2} & 0 \\ 0 & 0 & \frac{\epsilon_3}{\epsilon_3-1} & \frac{1}{1-\epsilon_3} & 0 & 0 & \frac{\epsilon_3 \zeta_2}{\epsilon_3-1} & \frac{\zeta_3 \tau_3}{1-\epsilon_3} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \zeta_3 \\ \hline \frac{1}{\xi_1(1-\epsilon_1)} & \frac{1}{\xi_1(\epsilon_1-1)} & 0 & 0 & \frac{\zeta_0}{\xi_1(1-\epsilon_1)} & \frac{\zeta_1 \tau_1}{\xi_1(\epsilon_1-1)} & 0 & 0 \\ 0 & \frac{1}{\xi_2(1-\epsilon_2)} & \frac{1}{\xi_2(\epsilon_2-1)} & 0 & 0 & \frac{\zeta_1}{\xi_2(1-\epsilon_2)} & \frac{\zeta_2 \tau_2}{\xi_2(\epsilon_2-1)} & 0 \\ 0 & 0 & \frac{1}{\xi_3(1-\epsilon_3)} & \frac{1}{\xi_3(\epsilon_3-1)} & 0 & 0 & \frac{\zeta_2}{\xi_3(1-\epsilon_3)} & \frac{\zeta_3 \tau_3}{\xi_3(\epsilon_3-1)} \end{array} \right].$$

Each entry is from (A.9).

The transition matrix from $\tilde{\mathcal{B}}$ to \mathcal{B} is

$$\left[\begin{array}{cccc|ccc} \frac{1}{1-\tau_0} & \frac{\tau_0}{\tau_0-1} & 0 & 0 & 0 & \frac{\tau_0 \xi_1}{\tau_0-1} & 0 & 0 \\ 0 & \frac{1}{1-\tau_1} & \frac{\tau_1}{\tau_1-1} & 0 & 0 & \frac{\xi_1 \epsilon_1}{1-\tau_1} & \frac{\tau_1 \xi_2}{\tau_1-1} & 0 \\ 0 & 0 & \frac{1}{1-\tau_2} & \frac{\tau_2}{\tau_2-1} & 0 & 0 & \frac{\xi_2 \epsilon_2}{1-\tau_2} & \frac{\tau_2 \xi_3}{\tau_2-1} \\ 0 & 0 & 0 & \frac{1}{1-\tau_3} & \frac{\tau_3}{\tau_3-1} & 0 & 0 & \frac{\xi_3 \epsilon_3}{1-\tau_3} \\ \hline \frac{1}{\zeta_0(\tau_0-1)} & \frac{1}{\zeta_0(1-\tau_0)} & 0 & 0 & 0 & \frac{\xi_1}{\zeta_0(1-\tau_0)} & 0 & 0 \\ 0 & \frac{1}{\zeta_1(\tau_1-1)} & \frac{1}{\zeta_1(1-\tau_1)} & 0 & 0 & \frac{\xi_1 \epsilon_1}{\zeta_1(\tau_1-1)} & \frac{\xi_2}{\zeta_1(1-\tau_1)} & 0 \\ 0 & 0 & \frac{1}{\zeta_2(\tau_2-1)} & \frac{1}{\zeta_2(1-\tau_2)} & 0 & 0 & \frac{\xi_2 \epsilon_2}{\zeta_2(\tau_2-1)} & \frac{\xi_3}{\zeta_2(1-\tau_2)} \\ 0 & 0 & 0 & \frac{1}{\zeta_3(\tau_3-1)} & \frac{1}{\zeta_3(1-\tau_3)} & 0 & 0 & \frac{\xi_3 \epsilon_3}{\zeta_3(\tau_3-1)} \end{array} \right].$$

Each entry is from (A.10).

We are done with display of transition matrices.

For the rest of this section, we do following. Referring to (A.1) and (A.2), we give the matrix representing each map in (A.1) relative to each basis in (A.2). We start with

A. Recall the formulae b_i, c_i from (2.17)–(2.20) and \tilde{b}_i, \tilde{c}_i from (2.45)–(2.48), and further b_i^\perp, c_i^\perp from (2.91)–(2.94) and $\tilde{b}_i^\perp, \tilde{c}_i^\perp$ from (2.117)–(2.120).

Example A.1.11. The matrix representing A relative to the basis \mathcal{C} is

$\tilde{a}_0 - b_0 + \tilde{b}_0$	$b_0 - \tilde{b}_0$	\tilde{b}_0	0	0	0	0	0
$c_1 - \tilde{c}_0$	$\tilde{a}_0 - c_1 + \tilde{c}_0$	$\tilde{b}_0 - b_1$	b_1	0	0	0	0
c_1	$\tilde{c}_1 - c_1$	$\tilde{a}_1 - b_1 + \tilde{b}_1$	$b_1 - \tilde{b}_1$	\tilde{b}_1	0	0	0
0	\tilde{c}_1	$c_2 - \tilde{c}_1$	$\tilde{a}_1 - c_2 + \tilde{c}_1$	$\tilde{b}_1 - b_2$	b_2	0	0
0	0	c_2	$\tilde{c}_2 - c_2$	$\tilde{a}_2 - b_2 + \tilde{b}_2$	$b_2 - \tilde{b}_2$	\tilde{b}_2	0
0	0	0	\tilde{c}_2	$c_3 - \tilde{c}_2$	$\tilde{a}_2 - c_3 + \tilde{c}_2$	$\tilde{b}_2 - b_3$	b_3
0	0	0	0	c_3	$\tilde{c}_3 - c_3$	$\tilde{a}_3 - b_3 + \tilde{b}_3$	$b_3 - \tilde{b}_3$
0	0	0	0	0	\tilde{c}_3	$c_4 - \tilde{c}_3$	$\tilde{a}_3 - c_4 + \tilde{c}_3$

The matrix representing A relative to the basis \mathcal{B} is

a_0	b_0	0	0	0	0	0	0
c_1	a_1	b_1	0	0	0	0	0
0	c_2	a_2	b_2	0	0	0	0
0	0	c_3	a_3	b_3	0	0	0
0	0	0	c_4	a_4	0	0	0
0	0	0	0	0	a_0^\perp	b_0^\perp	0
0	0	0	0	0	c_1^\perp	a_1^\perp	b_1^\perp
0	0	0	0	0	0	c_2^\perp	a_2^\perp

The matrix representing A relative to \mathcal{B}_{alt} is

$$\left[\begin{array}{ccc|cc|cc|c} a_0 & b_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline c_1 & a_1 & 0 & b_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_0^\perp & 0 & b_0^\perp & 0 & 0 & 0 \\ \hline 0 & c_2 & 0 & a_2 & 0 & b_2 & 0 & 0 \\ 0 & 0 & c_1^\perp & 0 & a_1^\perp & 0 & b_1^\perp & 0 \\ \hline 0 & 0 & 0 & c_3 & 0 & a_3 & 0 & b_3 \\ 0 & 0 & 0 & 0 & c_2^\perp & 0 & a_2^\perp & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & c_4 & 0 & a_4 \end{array} \right].$$

The matrix representing A relative to the basis $\tilde{\mathcal{B}}$ is

$$\left[\begin{array}{cccc|cccc} \tilde{a}_0 & \tilde{b}_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \tilde{c}_1 & \tilde{a}_1 & \tilde{b}_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \tilde{c}_2 & \tilde{a}_2 & \tilde{b}_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \tilde{c}_3 & \tilde{a}_3 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \tilde{a}_0^\perp & \tilde{b}_0^\perp & 0 & 0 \\ 0 & 0 & 0 & 0 & \tilde{c}_1^\perp & \tilde{a}_1^\perp & \tilde{b}_1^\perp & 0 \\ 0 & 0 & 0 & 0 & 0 & \tilde{c}_2^\perp & \tilde{a}_2^\perp & \tilde{b}_2^\perp \\ 0 & 0 & 0 & 0 & 0 & 0 & \tilde{c}_3^\perp & \tilde{a}_3^\perp \end{array} \right].$$

The matrix representing A relative to $\tilde{\mathcal{B}}_{alt}$ is

$$\left[\begin{array}{cc|cc|cc|cc} \tilde{a}_0 & 0 & \tilde{b}_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \tilde{a}_0^\perp & 0 & \tilde{b}_0^\perp & 0 & 0 & 0 & 0 \\ \hline \tilde{c}_1 & 0 & \tilde{a}_1 & 0 & \tilde{b}_1 & 0 & 0 & 0 \\ 0 & \tilde{c}_1^\perp & 0 & \tilde{a}_1^\perp & 0 & \tilde{b}_1^\perp & 0 & 0 \\ \hline 0 & 0 & \tilde{c}_2 & 0 & \tilde{a}_2 & 0 & \tilde{b}_2 & 0 \\ 0 & 0 & 0 & \tilde{c}_2^\perp & 0 & \tilde{a}_2^\perp & 0 & \tilde{b}_2^\perp \\ \hline 0 & 0 & 0 & 0 & \tilde{c}_3 & 0 & \tilde{a}_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & \tilde{c}_3^\perp & 0 & \tilde{a}_3^\perp \end{array} \right].$$

We are done with A . We now consider A^* . Recall from (2.14) the formula

$$\theta_i^* = \theta_0^* + h^*(1 - q^i)(1 - s^*q^{i+1})q^{-i}, \quad (\text{A.11})$$

for $0 \leq i \leq 4$.

Example A.1.12. The matrix representing A^* relative to the basis \mathcal{C} is

$$\text{diag}(\theta_0^*, \theta_1^*, \theta_1^*, \theta_2^*, \theta_2^*, \theta_3^*, \theta_3^*, \theta_4^*).$$

The matrix representing A^* relative to the basis \mathcal{B} is

$$\text{diag}(\theta_0^*, \theta_1^*, \theta_2^*, \theta_3^*, \theta_4^*, \theta_1^*, \theta_2^*, \theta_3^*).$$

The matrix representing A^* relative to the basis \mathcal{B}_{alt} is

$$\text{diag}(\theta_0^*, \theta_1^*, \theta_1^*, \theta_2^*, \theta_2^*, \theta_3^*, \theta_3^*, \theta_4^*).$$

The matrix representing A^* relative to the basis $\tilde{\mathcal{B}}$ is

$$\left[\begin{array}{cccc|cccc} \frac{\theta_0^* - \tau_0 \theta_1^*}{1 - \tau_0} & 0 & 0 & 0 & \frac{\zeta_0 \tau_0 (\theta_0^* - \theta_1^*)}{1 - \tau_0} & 0 & 0 & 0 \\ 0 & \frac{\theta_1^* - \tau_1 \theta_2^*}{1 - \tau_1} & 0 & 0 & 0 & \frac{\zeta_1 \tau_1 (\theta_1^* - \theta_2^*)}{1 - \tau_1} & 0 & 0 \\ 0 & 0 & \frac{\theta_2^* - \tau_2 \theta_3^*}{1 - \tau_2} & 0 & 0 & 0 & \frac{\zeta_2 \tau_2 (\theta_2^* - \theta_3^*)}{1 - \tau_2} & 0 \\ 0 & 0 & 0 & \frac{\theta_3^* - \tau_3 \theta_4^*}{1 - \tau_3} & 0 & 0 & 0 & \frac{\zeta_3 \tau_3 (\theta_3^* - \theta_4^*)}{1 - \tau_3} \\ \hline \frac{\theta_0^* - \theta_1^*}{\zeta_0 (\tau_0 - 1)} & 0 & 0 & 0 & \frac{\tau_0 \theta_0^* - \theta_1^*}{\tau_0 - 1} & 0 & 0 & 0 \\ 0 & \frac{\theta_1^* - \theta_2^*}{\zeta_1 (\tau_1 - 1)} & 0 & 0 & 0 & \frac{\tau_1 \theta_1^* - \theta_2^*}{\tau_1 - 1} & 0 & 0 \\ 0 & 0 & \frac{\theta_2^* - \theta_3^*}{\zeta_2 (\tau_2 - 1)} & 0 & 0 & 0 & \frac{\tau_2 \theta_2^* - \theta_3^*}{\tau_2 - 1} & 0 \\ 0 & 0 & 0 & \frac{\theta_3^* - \theta_4^*}{\zeta_3 (\tau_3 - 1)} & 0 & 0 & 0 & \frac{\tau_3 \theta_3^* - \theta_4^*}{\tau_3 - 1} \end{array} \right],$$

where $\{\theta_i^*\}_{i=0}^3$ are from (A.11) and $\{\zeta_i\}_{i=0}^3, \{\tau_i\}_{i=0}^3$ are from (A.4). Moreover,

$$\left. \begin{aligned} \frac{\theta_i^* - \tau_i \theta_{i+1}^*}{1 - \tau_i} &= \theta_i^* + \frac{h^* s^* q^{-i-1} (1-q) (1-r_1 q^{i+1}) (1-r_2 q^{i+1})}{s^* - r_1 r_2}, \\ \frac{\zeta_i \tau_i (\theta_i^* - \theta_{i+1}^*)}{1 - \tau_i} &= \frac{h^* s^* q^{-2i-1} (1-q) (r_1 - s^* q^{i+1}) (r_2 - s^* q^{i+1}) (1-r_1 q^{i+1}) (1-r_2 q^{i+1})}{s^* - r_1 r_2}, \\ \frac{\theta_i^* - \theta_{i+1}^*}{\zeta_i (\tau_i - 1)} &= \frac{h^* q^{-1} (1-q)}{r_1 r_2 - s^*}, \\ \frac{\tau_i \theta_i^* - \theta_{i+1}^*}{\tau_i - 1} &= \theta_{i+1}^* + \frac{h^* s^* q^{-i-1} (1-q) (1-r_1 q^{i+1}) (1-r_2 q^{i+1})}{r_1 r_2 - s^*}. \end{aligned} \right\} \quad (\text{A.12})$$

The matrix representing A^* relative to the basis $\tilde{\mathcal{B}}_{alt}$ is

$$\left[\begin{array}{cc|cc|cc|cc} \frac{\theta_0^* - \tau_0 \theta_1^*}{1 - \tau_0} & \frac{\zeta_0 \tau_0 (\theta_0^* - \theta_1^*)}{1 - \tau_0} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\theta_0^* - \theta_1^*}{\zeta_0 (\tau_0 - 1)} & \frac{\tau_0 \theta_0^* - \theta_1^*}{\tau_0 - 1} & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \frac{\theta_1^* - \tau_1 \theta_2^*}{1 - \tau_1} & \frac{\zeta_1 \tau_1 (\theta_1^* - \theta_2^*)}{1 - \tau_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\theta_1^* - \theta_2^*}{\zeta_1 (\tau_1 - 1)} & \frac{\tau_1 \theta_1^* - \theta_2^*}{\tau_1 - 1} & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \frac{\theta_2^* - \tau_2 \theta_3^*}{1 - \tau_2} & \frac{\zeta_2 \tau_2 (\theta_2^* - \theta_3^*)}{1 - \tau_2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\theta_2^* - \theta_3^*}{\zeta_2 (\tau_2 - 1)} & \frac{\tau_2 \theta_2^* - \theta_3^*}{\tau_2 - 1} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & \frac{\theta_3^* - \tau_3 \theta_4^*}{1 - \tau_3} & \frac{\zeta_3 \tau_3 (\theta_3^* - \theta_4^*)}{1 - \tau_3} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\theta_3^* - \theta_4^*}{\zeta_3 (\tau_3 - 1)} & \frac{\tau_3 \theta_3^* - \theta_4^*}{\tau_3 - 1} \end{array} \right].$$

Each entry is from (A.12).

We are done with A^* . We now consider \tilde{A}^* . Recall from (2.53) the formula

$$\tilde{\theta}_i^* = \tilde{\theta}_0^* + \tilde{h}^*(1 - q^i)(1 - s^*q^{i+2})q^{-i}, \quad (\text{A.13})$$

for $0 \leq i \leq 4$.

Example A.1.13. The matrix representing \tilde{A}^* relative to the basis \mathcal{C} is

$$\text{diag}(\tilde{\theta}_0^*, \tilde{\theta}_0^*, \tilde{\theta}_1^*, \tilde{\theta}_1^*, \tilde{\theta}_2^*, \tilde{\theta}_2^*, \tilde{\theta}_3^*, \tilde{\theta}_3^*).$$

The matrix representing \tilde{A}^* relative to the basis $\tilde{\mathcal{B}}$ is

$$\text{diag}(\tilde{\theta}_0^*, \tilde{\theta}_1^*, \tilde{\theta}_2^*, \tilde{\theta}_3^*, \tilde{\theta}_0^*, \tilde{\theta}_1^*, \tilde{\theta}_2^*, \tilde{\theta}_3^*).$$

The matrix representing \tilde{A}^* relative to the basis $\tilde{\mathcal{B}}_{alt}$ is

$$\text{diag}(\tilde{\theta}_0^*, \tilde{\theta}_0^*, \tilde{\theta}_1^*, \tilde{\theta}_1^*, \tilde{\theta}_2^*, \tilde{\theta}_2^*, \tilde{\theta}_3^*, \tilde{\theta}_3^*).$$

The matrix representing \tilde{A}^* relative to the basis \mathcal{B} is

$$\left[\begin{array}{cccc|ccc} \tilde{\theta}_0^* & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\epsilon_1 \tilde{\theta}_0^* - \tilde{\theta}_1^*}{\epsilon_1 - 1} & 0 & 0 & 0 & \frac{\epsilon_1 \xi_1 (\tilde{\theta}_0^* - \tilde{\theta}_1^*)}{\epsilon_1 - 1} & 0 \\ 0 & 0 & \frac{\epsilon_2 \tilde{\theta}_1^* - \tilde{\theta}_2^*}{\epsilon_2 - 1} & 0 & 0 & 0 & \frac{\epsilon_2 \xi_2 (\tilde{\theta}_1^* - \tilde{\theta}_2^*)}{\epsilon_2 - 1} \\ 0 & 0 & 0 & \frac{\epsilon_3 \tilde{\theta}_2^* - \tilde{\theta}_3^*}{\epsilon_3 - 1} & 0 & 0 & \frac{\epsilon_3 \xi_3 (\tilde{\theta}_2^* - \tilde{\theta}_3^*)}{\epsilon_3 - 1} \\ 0 & 0 & 0 & 0 & \tilde{\theta}_3^* & 0 & 0 \\ \hline 0 & \frac{\tilde{\theta}_0^* - \tilde{\theta}_1^*}{\xi_1 (1 - \epsilon_1)} & 0 & 0 & 0 & \frac{\tilde{\theta}_0^* - \epsilon_1 \tilde{\theta}_1^*}{1 - \epsilon_1} & 0 \\ 0 & 0 & \frac{\tilde{\theta}_1^* - \tilde{\theta}_2^*}{\xi_2 (1 - \epsilon_2)} & 0 & 0 & 0 & \frac{\tilde{\theta}_1^* - \epsilon_2 \tilde{\theta}_2^*}{1 - \epsilon_2} \\ 0 & 0 & 0 & \frac{\tilde{\theta}_2^* - \tilde{\theta}_3^*}{\xi_3 (1 - \epsilon_3)} & 0 & 0 & \frac{\tilde{\theta}_2^* - \epsilon_3 \tilde{\theta}_3^*}{1 - \epsilon_3} \end{array} \right],$$

where $\{\tilde{\theta}_i^*\}_{i=0}^3$ are from (A.13) and $\{\xi_i\}_{i=1}^3, \{\epsilon_i\}_{i=1}^3$ are from (A.3). Moreover,

$$\left. \begin{aligned} \frac{\epsilon_i \tilde{\theta}_{i-1}^* - \tilde{\theta}_i^*}{\epsilon_i - 1} &= \tilde{\theta}_{i-1}^* + \frac{\tilde{h}^* (1-q)(1-q^{4-i})(1-s^* q^{i+1})}{1-q^4}, \\ \frac{\epsilon_i \xi_i (\tilde{\theta}_{i-1}^* - \tilde{\theta}_i^*)}{\epsilon_i - 1} &= \frac{\tilde{h}^* q^{1-2i} (1-q)(1-q^i)(1-q^{i-4})(1-s^* q^{i+1})(1-s^* q^{i+5})}{q^4 - 1}, \\ \frac{\tilde{\theta}_{i-1}^* - \tilde{\theta}_i^*}{\xi_i (1 - \epsilon_i)} &= \frac{\tilde{h}^* q^3 (1-q)}{1-q^4}, \\ \frac{\tilde{\theta}_{i-1}^* - \epsilon_i \tilde{\theta}_i^*}{1 - \epsilon_i} &= \tilde{\theta}_i^* + \frac{\tilde{h}^* (q-1)(1-q^{4-i})(1-s^* q^{i+1})}{1-q^4}. \end{aligned} \right\} \quad (\text{A.14})$$

The matrix representing \tilde{A}^* relative to the basis \mathcal{B}_{alt} is

$$\begin{bmatrix} \tilde{\theta}_0^* & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\epsilon_1 \tilde{\theta}_0^* - \tilde{\theta}_1^*}{\epsilon_1 - 1} & \frac{\epsilon_1 \xi_1 (\tilde{\theta}_0^* - \tilde{\theta}_1^*)}{\epsilon_1 - 1} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\tilde{\theta}_0^* - \tilde{\theta}_1^*}{\xi_1 (1 - \epsilon_1)} & \frac{\tilde{\theta}_0^* - \epsilon_1 \tilde{\theta}_1^*}{1 - \epsilon_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\epsilon_2 \tilde{\theta}_1^* - \tilde{\theta}_2^*}{\epsilon_2 - 1} & \frac{\epsilon_2 \xi_2 (\tilde{\theta}_1^* - \tilde{\theta}_2^*)}{\epsilon_2 - 1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\tilde{\theta}_1^* - \tilde{\theta}_2^*}{\xi_2 (1 - \epsilon_2)} & \frac{\tilde{\theta}_1^* - \epsilon_2 \tilde{\theta}_2^*}{1 - \epsilon_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\epsilon_3 \tilde{\theta}_2^* - \tilde{\theta}_3^*}{\epsilon_3 - 1} & \frac{\epsilon_3 \xi_3 (\tilde{\theta}_2^* - \tilde{\theta}_3^*)}{\epsilon_3 - 1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\tilde{\theta}_2^* - \tilde{\theta}_3^*}{\xi_3 (1 - \epsilon_3)} & \frac{\tilde{\theta}_2^* - \epsilon_3 \tilde{\theta}_3^*}{1 - \epsilon_3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \tilde{\theta}_3^* \end{bmatrix}.$$

Each entry is from (A.14).

We are done with \tilde{A}^* . We now consider the map \mathfrak{p} .

The matrix representing \mathfrak{p} relative to the basis \mathcal{B}_{alt} is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The matrix representing \mathfrak{p} relative to $\tilde{\mathcal{B}}$ is

$$\begin{bmatrix} \frac{1-\epsilon_1(1-\tau_0)}{(1-\tau_0)(1-\epsilon_1)} & \frac{-\tau_0}{(1-\tau_0)(1-\epsilon_1)} & 0 & 0 & \frac{\zeta_0\tau_0}{(1-\tau_0)(1-\epsilon_1)} & \frac{-\tau_0\tau_1\zeta_1}{(1-\tau_0)(1-\epsilon_1)} & 0 & 0 \\ \frac{-\epsilon_1}{(1-\tau_1)(1-\epsilon_1)} & \frac{1-\epsilon_2+\tau_1\epsilon_2(1-\epsilon_1)}{(1-\tau_1)(1-\epsilon_1)(1-\epsilon_2)} & \frac{-\tau_1}{(1-\tau_1)(1-\epsilon_2)} & 0 & \frac{-\zeta_0\epsilon_1}{(1-\tau_1)(1-\epsilon_1)} & \frac{\zeta_1\tau_1(1-\epsilon_1\epsilon_2)}{(1-\tau_1)(1-\epsilon_1)(1-\epsilon_2)} & \frac{-\tau_1\tau_2\zeta_2}{(1-\tau_1)(1-\epsilon_2)} & 0 \\ 0 & \frac{-\epsilon_2}{(1-\tau_2)(1-\epsilon_2)} & \frac{1-\epsilon_3+\tau_2\epsilon_3(1-\epsilon_2)}{(1-\tau_2)(1-\epsilon_2)(1-\epsilon_3)} & \frac{-\tau_2}{(1-\tau_2)(1-\epsilon_3)} & 0 & \frac{-\zeta_1\epsilon_2}{(1-\tau_2)(1-\epsilon_2)} & \frac{\zeta_2\tau_2(1-\epsilon_2\epsilon_3)}{(1-\tau_2)(1-\epsilon_2)(1-\epsilon_3)} & \frac{-\tau_2\tau_3\zeta_3}{(1-\tau_2)(1-\epsilon_3)} \\ 0 & 0 & \frac{-\epsilon_3}{(1-\tau_3)(1-\epsilon_3)} & \frac{1+\tau_3(\epsilon_3-1)}{(1-\tau_3)(1-\epsilon_3)} & 0 & 0 & \frac{-\zeta_2\epsilon_3}{(1-\tau_3)(1-\epsilon_3)} & \frac{\zeta_3\tau_3\epsilon_3}{(1-\tau_3)(1-\epsilon_3)} \\ \hline \frac{-1}{\zeta_0(1-\tau_0)(1-\epsilon_1)} & \frac{1}{\zeta_0(1-\tau_0)(1-\epsilon_1)} & 0 & 0 & \frac{\tau_0(\epsilon_1-1)-\epsilon_1}{(1-\tau_0)(1-\epsilon_1)} & \frac{\zeta_1\tau_1}{\zeta_0(1-\tau_0)(1-\epsilon_1)} & 0 & 0 \\ \frac{\epsilon_1}{\zeta_1(1-\tau_1)(1-\epsilon_1)} & \frac{-1+\epsilon_1\epsilon_2}{\zeta_1(1-\tau_1)(1-\epsilon_1)(1-\epsilon_2)} & \frac{1}{\zeta_1(1-\tau_1)(1-\epsilon_2)} & 0 & \frac{\zeta_0\epsilon_1}{\zeta_1(1-\tau_1)(1-\epsilon_1)} & \frac{\tau_1(\epsilon_2-1)+\epsilon_2(\epsilon_1-1)}{(1-\tau_1)(1-\epsilon_1)(1-\epsilon_2)} & \frac{\zeta_2\tau_2}{\zeta_1(1-\tau_1)(1-\epsilon_2)} & 0 \\ 0 & \frac{\epsilon_2}{\zeta_2(1-\tau_2)(1-\epsilon_2)} & \frac{-1+\epsilon_2\epsilon_3}{\zeta_2(1-\tau_2)(1-\epsilon_2)(1-\epsilon_3)} & \frac{1}{\zeta_2(1-\tau_2)(1-\epsilon_3)} & 0 & \frac{\zeta_1\epsilon_2}{\zeta_2(1-\tau_2)(1-\epsilon_2)} & \frac{\tau_2(\epsilon_3-1)+\epsilon_3(\epsilon_2-1)}{(1-\tau_2)(1-\epsilon_2)(1-\epsilon_3)} & \frac{\zeta_3\tau_3}{\zeta_2(1-\tau_2)(1-\epsilon_3)} \\ 0 & 0 & \frac{\epsilon_3}{\zeta_3(1-\tau_3)(1-\epsilon_3)} & \frac{-\epsilon_3}{\zeta_3(1-\tau_3)(1-\epsilon_3)} & 0 & 0 & \frac{\zeta_2\epsilon_3}{\zeta_3(1-\tau_3)(1-\epsilon_3)} & \frac{1-\tau_3-\epsilon_3}{(1-\tau_3)(1-\epsilon_3)} \end{bmatrix},$$

where $\{\epsilon_i\}_{i=1}^3$ are from (A.3) and $\{\zeta_i\}_{i=0}^3, \{\tau_i\}_{i=0}^3$ are from (A.4).

The matrix representing \mathfrak{p} relative to $\tilde{\mathcal{B}}_{alt}$ is

$$\left[\begin{array}{cc|cc|cc|cc} \frac{1-\epsilon_1(1-\tau_0)}{(1-\tau_0)(1-\epsilon_1)} & \frac{\zeta_0\tau_0}{(1-\tau_0)(1-\epsilon_1)} & \frac{-\tau_0}{(1-\tau_0)(1-\epsilon_1)} & \frac{-\tau_0\tau_1\zeta_1}{(1-\tau_0)(1-\epsilon_1)} & 0 & 0 & 0 & 0 \\ \frac{-1}{\zeta_0(1-\tau_0)(1-\epsilon_1)} & \frac{\tau_0(\epsilon_1-1)-\epsilon_1}{(1-\tau_0)(1-\epsilon_1)} & \frac{1}{\zeta_0(1-\tau_0)(1-\epsilon_1)} & \frac{\zeta_1\tau_1}{\zeta_0(1-\tau_0)(1-\epsilon_1)} & 0 & 0 & 0 & 0 \\ \hline \frac{-\epsilon_1}{(1-\tau_1)(1-\epsilon_1)} & \frac{-\zeta_0\epsilon_1}{(1-\tau_1)(1-\epsilon_1)} & \frac{1-\epsilon_2+\tau_1\epsilon_2(1-\epsilon_1)}{(1-\tau_1)(1-\epsilon_1)(1-\epsilon_2)} & \frac{\zeta_1\tau_1(1-\epsilon_1\epsilon_2)}{(1-\tau_1)(1-\epsilon_1)(1-\epsilon_2)} & \frac{-\tau_1}{(1-\tau_1)(1-\epsilon_2)} & \frac{-\tau_1\tau_2\zeta_2}{(1-\tau_1)(1-\epsilon_2)} & 0 & 0 \\ \frac{\epsilon_1}{\zeta_1(1-\tau_1)(1-\epsilon_1)} & \frac{\zeta_0\epsilon_1}{\zeta_1(1-\tau_1)(1-\epsilon_1)} & \frac{-1+\epsilon_1\epsilon_2}{\zeta_1(1-\tau_1)(1-\epsilon_1)(1-\epsilon_2)} & \frac{\tau_1(\epsilon_2-1)+\epsilon_2(\epsilon_1-1)}{(1-\tau_1)(1-\epsilon_1)(1-\epsilon_2)} & \frac{1}{\zeta_1(1-\tau_1)(1-\epsilon_2)} & \frac{\zeta_2\tau_2}{\zeta_1(1-\tau_1)(1-\epsilon_2)} & 0 & 0 \\ \hline 0 & 0 & \frac{-\epsilon_2}{(1-\tau_2)(1-\epsilon_2)} & \frac{-\zeta_1\epsilon_2}{(1-\tau_2)(1-\epsilon_2)} & \frac{1-\epsilon_3+\tau_2\epsilon_3(1-\epsilon_2)}{(1-\tau_2)(1-\epsilon_2)(1-\epsilon_3)} & \frac{\zeta_2\tau_2(1-\epsilon_2\epsilon_3)}{(1-\tau_2)(1-\epsilon_2)(1-\epsilon_3)} & \frac{-\tau_2}{(1-\tau_2)(1-\epsilon_3)} & \frac{-\tau_2\tau_3\zeta_3}{(1-\tau_2)(1-\epsilon_3)} \\ 0 & 0 & \frac{\epsilon_2}{\zeta_2(1-\tau_2)(1-\epsilon_2)} & \frac{\zeta_1\epsilon_2}{\zeta_2(1-\tau_2)(1-\epsilon_2)} & \frac{-1+\epsilon_2\epsilon_3}{\zeta_2(1-\tau_2)(1-\epsilon_2)(1-\epsilon_3)} & \frac{\tau_2(\epsilon_3-1)+\epsilon_3(\epsilon_2-1)}{(1-\tau_2)(1-\epsilon_2)(1-\epsilon_3)} & \frac{1}{\zeta_2(1-\tau_2)(1-\epsilon_3)} & \frac{\zeta_3\tau_3}{\zeta_2(1-\tau_2)(1-\epsilon_3)} \\ \hline 0 & 0 & 0 & 0 & \frac{-\epsilon_3}{(1-\tau_3)(1-\epsilon_3)} & \frac{-\zeta_2\epsilon_3}{(1-\tau_3)(1-\epsilon_3)} & \frac{1+\tau_3(\epsilon_3-1)}{(1-\tau_3)(1-\epsilon_3)} & \frac{\zeta_3\tau_3\epsilon_3}{(1-\tau_3)(1-\epsilon_3)} \\ 0 & 0 & 0 & 0 & \frac{\epsilon_3}{\zeta_3(1-\tau_3)(1-\epsilon_3)} & \frac{\zeta_2\epsilon_3}{\zeta_3(1-\tau_3)(1-\epsilon_3)} & \frac{-\epsilon_3}{\zeta_3(1-\tau_3)(1-\epsilon_3)} & \frac{1-\tau_3-\epsilon_3}{(1-\tau_3)(1-\epsilon_3)} \end{array} \right],$$

where $\{\epsilon_i\}_{i=1}^3$ are from (A.3) and $\{\zeta_i\}_{i=0}^3, \{\tau_i\}_{i=0}^3$ are from (A.4).

We are done with \mathfrak{p} . Finally we consider the map $\tilde{\mathfrak{p}}$.

Example A.1.15. The matrix representing $\tilde{\mathfrak{p}}$ relative to \mathcal{C} is

$$\left[\begin{array}{cc|cc|cc|cc} \frac{1}{1-\tau_0} & \frac{\tau_0}{\tau_0-1} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{1-\tau_0} & \frac{\tau_0}{\tau_0-1} & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \frac{1}{1-\tau_1} & \frac{\tau_1}{\tau_1-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{1-\tau_1} & \frac{\tau_1}{\tau_1-1} & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \frac{1}{1-\tau_2} & \frac{\tau_2}{\tau_2-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{1-\tau_2} & \frac{\tau_2}{\tau_2-1} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{1-\tau_3} & \frac{\tau_3}{\tau_3-1} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{1-\tau_3} & \frac{\tau_3}{\tau_3-1} \end{array} \right],$$

where $\{\tau_i\}_{i=0}^3$ are from (A.4). Moreover,

$$\frac{1}{1-\tau_i} = \frac{(r_1-s^*q^{i+1})(r_2-s^*q^{i+1})}{(r_1r_2-s^*)(1-s^*q^{2i+2})}, \quad \frac{\tau_i}{\tau_i-1} = \frac{s^*(1-r_1q^{i+1})(1-r_2q^{i+1})}{(s^*-r_1r_2)(1-s^*q^{2i+2})}, \quad (\text{A.16})$$

for $0 \leq i \leq 3$.

The matrix representing $\tilde{\mathfrak{p}}$ relative to $\tilde{\mathcal{B}}$ is

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right].$$

The matrix representing $\tilde{\mathfrak{p}}$ relative to $\tilde{\mathcal{B}}_{alt}$ is

$$\left[\begin{array}{cc|cc|cc|cc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right].$$

The matrix representing $\tilde{\mathfrak{p}}$ relative to \mathcal{B} is

$$\left[\begin{array}{cccc|ccc} \frac{1}{1-\tau_0} & \frac{\tau_0}{\tau_0-1} & 0 & 0 & 0 & \frac{\tau_0 \xi_1}{\tau_0-1} & 0 & 0 \\ \frac{-\epsilon_1}{(1-\tau_0)(1-\epsilon_1)} & \frac{\epsilon_1 \tau_0 (\tau_1-1) + \tau_0 - 1}{(\tau_0-1)(\tau_1-1)(\epsilon_1-1)} & \frac{-\tau_1}{(1-\epsilon_1)(1-\tau_1)} & 0 & 0 & \frac{\xi_1 \epsilon_1 (1-\tau_0 \tau_1)}{(1-\tau_0)(1-\tau_1)(1-\epsilon_1)} & \frac{-\tau_1 \xi_2}{(1-\epsilon_1)(1-\tau_1)} & 0 \\ 0 & \frac{-\epsilon_2}{(1-\tau_1)(1-\epsilon_2)} & \frac{\epsilon_2 \tau_1 (\tau_2-1) + \tau_1 - 1}{(\tau_1-1)(\tau_2-1)(\epsilon_2-1)} & \frac{-\tau_2}{(1-\epsilon_2)(1-\tau_2)} & 0 & \frac{-\xi_1 \epsilon_1 \epsilon_2}{(1-\tau_1)(1-\epsilon_2)} & \frac{\xi_2 \epsilon_2 (1-\tau_1 \tau_2)}{(1-\tau_1)(1-\tau_2)(1-\epsilon_2)} & \frac{-\tau_2 \xi_3}{(1-\epsilon_2)(1-\tau_2)} \\ 0 & 0 & \frac{-\epsilon_3}{(1-\tau_2)(1-\epsilon_3)} & \frac{\epsilon_3 \tau_2 (\tau_3-1) + \tau_2 - 1}{(\tau_2-1)(\tau_3-1)(\epsilon_3-1)} & \frac{-\tau_3}{(1-\epsilon_3)(1-\tau_3)} & 0 & \frac{-\xi_2 \epsilon_2 \epsilon_3}{(1-\tau_2)(1-\epsilon_3)} & \frac{\xi_3 \epsilon_3 (1-\tau_2 \tau_3)}{(1-\tau_2)(1-\tau_3)(1-\epsilon_3)} \\ 0 & 0 & 0 & \frac{1}{1-\tau_3} & \frac{\tau_3}{\tau_3-1} & 0 & 0 & \frac{\xi_3 \epsilon_3}{1-\tau_3} \\ \hline \frac{1}{\xi_1(\epsilon_1-1)(\tau_0-1)} & \frac{\tau_0 \tau_1 - 1}{\xi_1(1-\epsilon_1)(1-\tau_1)(1-\tau_0)} & \frac{\tau_1}{\xi_1(1-\epsilon_1)(1-\tau_1)} & 0 & 0 & \frac{\tau_0(\tau_1-1) + \epsilon_1(\tau_0-1)}{(1-\epsilon_1)(1-\tau_1)(1-\tau_0)} & \frac{\tau_1 \xi_2}{\xi_1(1-\epsilon_1)(1-\tau_1)} & 0 \\ 0 & \frac{1}{\xi_2(\epsilon_2-1)(\tau_1-1)} & \frac{\tau_1 \tau_2 - 1}{\xi_2(1-\epsilon_2)(1-\tau_2)(1-\tau_1)} & \frac{\tau_2}{\xi_2(1-\epsilon_2)(1-\tau_2)} & 0 & \frac{\xi_1 \epsilon_1}{\xi_2(1-\epsilon_2)(1-\tau_1)} & \frac{\tau_1(\tau_2-1) + \epsilon_2(\tau_1-1)}{(1-\epsilon_2)(1-\tau_2)(1-\tau_1)} & \frac{\tau_2 \xi_3}{\xi_2(1-\epsilon_2)(1-\tau_2)} \\ 0 & 0 & \frac{1}{\xi_3(\epsilon_3-1)(\tau_2-1)} & \frac{\tau_2 \tau_3 - 1}{\xi_3(1-\epsilon_3)(1-\tau_3)(1-\tau_2)} & \frac{\tau_3}{\xi_3(1-\epsilon_3)(1-\tau_3)} & 0 & \frac{\xi_2 \epsilon_2}{\xi_3(1-\epsilon_3)(1-\tau_2)} & \frac{\tau_2(\tau_3-1) + \epsilon_3(\tau_2-1)}{(1-\epsilon_3)(1-\tau_3)(1-\tau_2)} \end{array} \right],$$

where $\{\xi_i\}_{i=1}^3$, $\{\epsilon_i\}_{i=1}^3$ are from (A.3) and $\{\tau_i\}_{i=0}^3$ are from (A.4).

The matrix representing $\tilde{\mathfrak{p}}$ relative to \mathcal{B}_{alt} is

$$\left[\begin{array}{ccc|cc|cc|c} \frac{1}{1-\tau_0} & \frac{\tau_0}{\tau_0-1} & \frac{\tau_0 \xi_1}{\tau_0-1} & 0 & 0 & 0 & 0 & 0 \\ \frac{-\epsilon_1}{(1-\tau_0)(1-\epsilon_1)} & \frac{\epsilon_1 \tau_0 (\tau_1-1) + \tau_0 - 1}{(\tau_0-1)(\tau_1-1)(\epsilon_1-1)} & \frac{\xi_1 \epsilon_1 (1-\tau_0 \tau_1)}{(1-\tau_0)(1-\tau_1)(1-\epsilon_1)} & \frac{-\tau_1}{(1-\epsilon_1)(1-\tau_1)} & \frac{-\tau_1 \xi_2}{(1-\epsilon_1)(1-\tau_1)} & 0 & 0 & 0 \\ \frac{1}{\xi_1(\epsilon_1-1)(\tau_0-1)} & \frac{\tau_0 \tau_1 - 1}{\xi_1(1-\epsilon_1)(1-\tau_1)(1-\tau_0)} & \frac{\tau_0(\tau_1-1) + \epsilon_1(\tau_0-1)}{(1-\epsilon_1)(1-\tau_1)(1-\tau_0)} & \frac{\tau_1}{\xi_1(1-\epsilon_1)(1-\tau_1)} & \frac{\tau_1 \xi_2}{\xi_1(1-\epsilon_1)(1-\tau_1)} & 0 & 0 & 0 \\ 0 & \frac{-\epsilon_2}{(1-\tau_1)(1-\epsilon_2)} & \frac{-\xi_1 \epsilon_1 \epsilon_2}{(1-\tau_1)(1-\epsilon_2)} & \frac{\epsilon_2 \tau_1 (\tau_2-1) + \tau_1 - 1}{(\tau_1-1)(\tau_2-1)(\epsilon_2-1)} & \frac{\xi_2 \epsilon_2 (1-\tau_1 \tau_2)}{(1-\tau_1)(1-\tau_2)(1-\epsilon_2)} & \frac{-\tau_2}{(1-\epsilon_2)(1-\tau_2)} & \frac{-\tau_2 \xi_3}{(1-\epsilon_2)(1-\tau_2)} & 0 \\ 0 & \frac{1}{\xi_2(\epsilon_2-1)(\tau_1-1)} & \frac{\xi_1 \epsilon_1}{\xi_2(1-\epsilon_2)(1-\tau_1)} & \frac{\tau_1 \tau_2 - 1}{\xi_2(1-\epsilon_2)(1-\tau_2)(1-\tau_1)} & \frac{\tau_1(\tau_2-1) + \epsilon_2(\tau_1-1)}{(1-\epsilon_2)(1-\tau_2)(1-\tau_1)} & \frac{\tau_2}{\xi_2(1-\epsilon_2)(1-\tau_2)} & \frac{\tau_2 \xi_3}{\xi_2(1-\epsilon_2)(1-\tau_2)} & 0 \\ 0 & 0 & 0 & \frac{-\epsilon_3}{(1-\tau_2)(1-\epsilon_3)} & \frac{-\xi_2 \epsilon_2 \epsilon_3}{(1-\tau_2)(1-\epsilon_3)} & \frac{\epsilon_3 \tau_2 (\tau_3-1) + \tau_2 - 1}{(\tau_2-1)(\tau_3-1)(\epsilon_3-1)} & \frac{\xi_3 \epsilon_3 (1-\tau_2 \tau_3)}{(1-\tau_2)(1-\tau_3)(1-\epsilon_3)} & \frac{-\tau_3}{(1-\epsilon_3)(1-\tau_3)} \\ 0 & 0 & 0 & \frac{1}{\xi_3(\epsilon_3-1)(\tau_2-1)} & \frac{\xi_2 \epsilon_2}{\xi_3(1-\epsilon_3)(1-\tau_2)} & \frac{\tau_2 \tau_3 - 1}{\xi_3(1-\epsilon_3)(1-\tau_3)(1-\tau_2)} & \frac{\tau_2(\tau_3-1) + \epsilon_3(\tau_2-1)}{(1-\epsilon_3)(1-\tau_3)(1-\tau_2)} & \frac{\tau_3}{\xi_3(1-\epsilon_3)(1-\tau_3)} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{1-\tau_3} & \frac{\xi_3 \epsilon_3}{1-\tau_3} & \frac{\tau_3}{\tau_3-1} \end{array} \right],$$

where $\{\xi_i\}_{i=1}^3$, $\{\epsilon_i\}_{i=1}^3$ are from (A.3) and $\{\tau_i\}_{i=0}^3$ are from (A.4).

A.2 The matrices from Chapter 3

Consider the basis \mathcal{C} from (A.2). Recall that for $d = 4$ this basis consists of

$$\hat{C}_0^-, \hat{C}_0^+, \hat{C}_1^-, \hat{C}_1^+, \hat{C}_2^-, \hat{C}_2^+, \hat{C}_3^-, \hat{C}_3^+. \quad (\text{A.17})$$

In this section, for each element

$$\{t_n\}_{n=0}^3, \quad \mathbf{X}^{\pm 1}, \quad \mathbf{Y}^{\pm 1}, \quad \mathbf{A}, \quad \mathbf{B}, \quad \mathbf{B}^\dagger, \quad \frac{t_0 - k_0^{-1}}{k_0 - k_0^{-1}}, \quad \frac{t_1 - k_1^{-1}}{k_1 - k_1^{-1}}$$

of \hat{H}_q we give the matrix representing that element relative to the basis (A.17).

The matrix representing t_0 relative to (A.17) is block diagonal:

$$\begin{bmatrix} \mathbf{t}_0(0) & & & \mathbf{0} \\ & \mathbf{t}_0(1) & & \\ & & \mathbf{t}_0(2) & \\ \mathbf{0} & & & \mathbf{t}_0(3) \end{bmatrix},$$

where for $0 \leq i \leq 3$,

$$\mathbf{t}_0(i) = \begin{bmatrix} \frac{1}{\sqrt{s^* r_1 r_2}} \left(\frac{(r_1 - s^* q^{i+1})(r_2 - s^* q^{i+1})}{1 - s^* q^{2i+2}} + s^* \right) & -\sqrt{\frac{s^*}{r_1 r_2}} \frac{(1 - r_1 q^{i+1})(1 - r_2 q^{i+1})}{1 - s^* q^{2i+2}} \\ \frac{1}{\sqrt{s^* r_1 r_2}} \frac{(r_1 - s^* q^{i+1})(r_2 - s^* q^{i+1})}{1 - s^* q^{2i+2}} & \sqrt{\frac{s^*}{r_1 r_2}} \left(1 - \frac{(1 - r_1 q^{i+1})(1 - r_2 q^{i+1})}{1 - s^* q^{2i+2}} \right) \end{bmatrix}.$$

The matrix representing t_1 relative to (A.17) is block diagonal:

$$\begin{bmatrix} \mathbf{t}_1(0) & & & \mathbf{0} \\ & \mathbf{t}_1(1) & & \\ & & \mathbf{t}_1(2) & \\ & & & \mathbf{t}_1(3) \\ \mathbf{0} & & & & \mathbf{t}_1(4) \end{bmatrix},$$

where for $1 \leq i \leq 3$,

$$\mathbf{t}_1(i) = \begin{bmatrix} \frac{q^2(1 - q^{i-4})(1 - s^* q^{i+1})}{1 - s^* q^{2i+1}} + \frac{1}{q^2} & \frac{q^2(q^{i-4} - 1)(1 - s^* q^{i+1})}{1 - s^* q^{2i+1}} \\ \frac{(1 - q^i)(1 - s^* q^{i+5})}{q^2(1 - s^* q^{2i+1})} & \frac{(q^i - 1)(1 - s^* q^{i+5})}{q^2(1 - s^* q^{2i+1})} + \frac{1}{q^2} \end{bmatrix}$$

and

$$\mathbf{t}_1(0) = [q^{-2}], \quad \mathbf{t}_1(4) = [q^{-2}].$$

The matrix representing t_2 relative to (A.17) is block diagonal:

$$\begin{bmatrix} \mathbf{t}_2(0) & & & & \mathbf{0} \\ & \mathbf{t}_2(1) & & & \\ & & \mathbf{t}_2(2) & & \\ & & & \mathbf{t}_2(3) & \\ \mathbf{0} & & & & \mathbf{t}_2(4) \end{bmatrix},$$

where for $1 \leq i \leq 3$,

$$\mathbf{t}_2(i) = \begin{bmatrix} \frac{1}{\sqrt{s^*q^5}} \left(\frac{s^*q^5(1-q^i)(1-q^{i-4})}{1-s^*q^{2i+1}} + 1 \right) & \frac{q^i \sqrt{s^*q^5} (1-q^{i-4})(1-s^*q^{i+1})}{1-s^*q^{2i+1}} \\ \frac{1}{q^i \sqrt{s^*q^5}} \frac{(q^i-1)(1-s^*q^{i+5})}{1-s^*q^{2i+1}} & \sqrt{s^*q^5} \left(\frac{(q^i-1)(1-q^{i-4})}{1-s^*q^{2i+1}} + 1 \right) \end{bmatrix}$$

and

$$\mathbf{t}_2(0) = [\sqrt{s^*q^5}], \quad \mathbf{t}_2(4) = \left[\frac{1}{\sqrt{s^*q^5}} \right].$$

The matrix representing t_3 relative to (A.17) is block diagonal:

$$\begin{bmatrix} \mathbf{t}_3(0) & & & \mathbf{0} \\ & \mathbf{t}_3(1) & & \\ & & \mathbf{t}_3(2) & \\ \mathbf{0} & & & \mathbf{t}_3(3) \end{bmatrix},$$

where for $0 \leq i \leq 3$,

$$\mathbf{t}_3(i) := \begin{bmatrix} \frac{1}{q^{i+1} \sqrt{r_1 r_2}} \left(1 - \frac{(1-r_1 q^{i+1})(1-r_2 q^{i+1})}{1-s^*q^{2i+2}} \right) & \frac{1}{q^{i+1} \sqrt{r_1 r_2}} \frac{(1-r_1 q^{i+1})(1-r_2 q^{i+1})}{1-s^*q^{2i+2}} \\ -\frac{q^{i+1}}{\sqrt{r_1 r_2}} \frac{(r_1-s^*q^{i+1})(r_2-s^*q^{i+1})}{1-s^*q^{2i+2}} & \frac{q^{i+1}}{\sqrt{r_1 r_2}} \left(\frac{(r_1-s^*q^{i+1})(r_2-s^*q^{i+1})}{1-s^*q^{2i+2}} + s^* \right) \end{bmatrix}.$$

So the matrix representations of t_0, t_3 and the matrix representations of t_1, t_2 take the form

$$\begin{pmatrix} * & * & & & & & & & & & \\ & * & * & & & & & & & & \\ & & & * & * & & & & & & \\ & & & & * & * & & & & & \\ & & & & & * & * & & & & \\ & & & & & & * & * & & & \\ & & & & & & & * & * & & \\ & & & & & & & & * & * & \\ & & & & & & & & & * & * \\ & & & & & & & & & & * \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} * & & & & & & & & & & \\ & * & * & & & & & & & & \\ & & * & * & & & & & & & \\ & & & * & * & & & & & & \\ & & & & * & * & & & & & \\ & & & & & * & * & & & & \\ & & & & & & * & * & & & \\ & & & & & & & * & * & & \\ & & & & & & & & * & * & \\ & & & & & & & & & * & * \end{pmatrix},$$

respectively.

The matrix representing \mathbf{X} relative to (A.17) is diagonal:

$$\text{diag} \left[\frac{1}{q\sqrt{s^*}}, q\sqrt{s^*}, \frac{1}{q^2\sqrt{s^*}}, q^2\sqrt{s^*}, \frac{1}{q^3\sqrt{s^*}}, q^3\sqrt{s^*}, \frac{1}{q^4\sqrt{s^*}}, q^4\sqrt{s^*} \right].$$

The matrix representing \mathbf{X}^{-1} relative to (A.17) is diagonal:

$$\text{diag} \left[q\sqrt{s^*}, \frac{1}{q\sqrt{s^*}}, q^2\sqrt{s^*}, \frac{1}{q^2\sqrt{s^*}}, q^3\sqrt{s^*}, \frac{1}{q^3\sqrt{s^*}}, q^4\sqrt{s^*}, \frac{1}{q^4\sqrt{s^*}} \right].$$

The matrix representing \mathbf{Y} relative to (A.17) is $\sqrt{\frac{s^*q^4}{r_1r_2}}$ times

$$\begin{bmatrix} \mathbf{a}_0 & \mathbf{b}_0 & \mathbf{0} \\ & \mathbf{a}_1 & \mathbf{b}_1 \\ & & \mathbf{a}_2 & \mathbf{b}_2 \\ \mathbf{0} & & & \mathbf{a}_3 & \mathbf{b}_3 \end{bmatrix},$$

where

$$\mathbf{a}_0 = \frac{1}{s^*q^4} \begin{bmatrix} \frac{(r_1-s^*q)(r_2-s^*q)}{1-s^*q^2} + s^* \\ \frac{(r_1-s^*q)(r_2-s^*q)}{1-s^*q^2} \end{bmatrix}, \quad \mathbf{b}_3 = \begin{bmatrix} -\frac{(1-r_1q^4)(1-r_2q^4)}{q^4(1-s^*q^8)} \\ \frac{1}{q^4} - \frac{(1-r_1q^4)(1-r_2q^4)}{q^4(1-s^*q^8)} \end{bmatrix},$$

and for $1 \leq i \leq 3$,

$\mathbf{a}_i =$

$$\frac{1}{s^*q^4} \begin{bmatrix} \frac{(1-q^i)(1-s^*q^{i+5})}{1-s^*q^{2i+1}} \left(\frac{(r_1-s^*q^{i+1})(r_2-s^*q^{i+1})}{1-s^*q^{2i+2}} + s^* \right) & \left(1 - \frac{(1-q^i)(1-s^*q^{i+5})}{1-s^*q^{2i+1}} \right) \left(\frac{(r_1-s^*q^{i+1})(r_2-s^*q^{i+1})}{1-s^*q^{2i+2}} + s^* \right) \\ \frac{(1-q^i)(1-s^*q^{i+5})(r_1-s^*q^{i+1})(r_2-s^*q^{i+1})}{(1-s^*q^{2i+1})(1-s^*q^{2i+2})} & \left(1 - \frac{(1-q^i)(1-s^*q^{i+5})}{1-s^*q^{2i+1}} \right) \frac{(r_1-s^*q^{i+1})(r_2-s^*q^{i+1})}{1-s^*q^{2i+2}} \end{bmatrix},$$

and for $0 \leq i \leq 2$,

$\mathbf{b}_i =$

$$\begin{bmatrix} -\frac{(1-r_1q^{i+1})(1-r_2q^{i+1})}{1-s^*q^{2i+2}} \left(\frac{(1-q^{i-3})(1-s^*q^{i+2})}{1-s^*q^{2i+3}} + q^{-4} \right) & \frac{(1-q^{i-3})(1-s^*q^{i+2})(1-r_1q^{i+1})(1-r_2q^{i+1})}{(1-s^*q^{2i+3})(1-s^*q^{2i+2})} \\ \left(1 - \frac{(1-r_1q^{i+1})(1-r_2q^{i+1})}{1-s^*q^{2i+2}} \right) \left(\frac{(1-q^{i-3})(1-s^*q^{i+2})}{1-s^*q^{2i+3}} + q^{-4} \right) & \frac{(1-q^{i-3})(1-s^*q^{i+2})}{1-s^*q^{2i+3}} \left(\frac{(1-r_1q^{i+1})(1-r_2q^{i+1})}{1-s^*q^{2i+2}} - 1 \right) \end{bmatrix}.$$

The matrix representing \mathbf{Y}^{-1} relative to (A.17) is $\sqrt{\frac{s^*q^4}{r_1r_2}}$ times

$$\begin{bmatrix} \mathbf{a}_0 & \mathbf{0} \\ \mathbf{c}_1 & \mathbf{a}_1 \\ & \mathbf{c}_2 & \mathbf{a}_2 \\ & & \mathbf{c}_3 & \mathbf{a}_3 \\ \mathbf{0} & & & \mathbf{c}_4 \end{bmatrix},$$

where

$$\mathbf{a}_0 = \begin{bmatrix} \left(1 - \frac{(1-r_1q)(1-r_2q)}{1-s^*q^2} \right) \left(\frac{(1-q^{-4})(1-s^*q)}{1-s^*q} + \frac{1}{q^4} \right) & \frac{(1-r_1q)(1-r_2q)}{1-s^*q^2} \left(\frac{(1-q^{-4})(1-s^*q)}{1-s^*q} + \frac{1}{q^4} \right) \end{bmatrix},$$

$$\mathbf{c}_4 = \frac{1}{s^*q^4} \begin{bmatrix} \frac{(r_1-s^*q^4)(r_2-s^*q^4)}{1-s^*q^8} \left(\frac{(1-q^4)(1-s^*q^9)}{1-s^*q^9} - 1 \right) & \left(\frac{(r_1-s^*q^4)(r_2-s^*q^4)}{1-s^*q^8} + s^* \right) \left(\frac{(q^4-1)(1-s^*q^9)}{1-s^*q^9} + 1 \right) \end{bmatrix},$$

where for $0 \leq i \leq 3$,

$\mathbf{a}_i =$

$$\left[\begin{array}{c|c} 1 + \frac{r_1 r_2}{s^* q^4} - \frac{(1-q^i)(1-s^* q^{i+5})(r_1-s^* q^{i+1})(r_2-s^* q^{i+1})}{s^* q^4(1-s^* q^{2i+1})(1-s^* q^{2i+2})} & \frac{(1-r_1 q^{i+1})(1-r_2 q^{i+1})}{1-s^* q^{2i+2}} \\ & \times \left(\frac{(1-q^{i-4})(1-s^* q^{i+1})}{1-s^* q^{2i+1}} - \frac{(1-q^{i-3})(1-s^* q^{i+2})}{1-s^* q^{2i+3}} \right) \\ \hline \frac{(r_1-s^* q^{i+1})(r_2-s^* q^{i+1})}{s^* q^4(1-s^* q^{2i+2})} & 1 + \frac{r_1 r_2}{s^* q^4} - \frac{(1-q^{i+1})(1-s^* q^{i+6})(r_1-s^* q^{i+1})(r_2-s^* q^{i+1})}{s^* q^4(1-s^* q^{2i+2})(1-s^* q^{2i+3})} \\ \times \left(\frac{(1-q^{i+1})(1-s^* q^{i+6})}{1-s^* q^{2i+3}} - \frac{(1-q^i)(1-s^* q^{i+5})}{1-s^* q^{2i+1}} \right) & - \frac{(1-q^{i-3})(1-s^* q^{i+2})(1-r_1 q^{i+1})(1-r_2 q^{i+1})}{(1-s^* q^{2i+2})(1-s^* q^{2i+3})} \end{array} \right],$$

and for $0 \leq i \leq 2$,

$$\mathbf{b}_i = \left[\begin{array}{c|c} \frac{(1-q^{i-3})(1-s^* q^{i+2})(1-r_1 q^{i+1})(1-r_2 q^{i+1})}{(1-s^* q^{2i+2})(1-s^* q^{2i+3})} & 0 \\ \hline \frac{(1-q^{i-3})(1-s^* q^{i+2})}{1-s^* q^{2i+3}} \left(\frac{(1-r_1 q^{i+1})(1-r_2 q^{i+1})}{1-s^* q^{2i+2}} - \frac{(1-r_1 q^{i+2})(1-r_2 q^{i+2})}{1-s^* q^{2i+4}} \right) & \frac{(1-q^{i-3})(1-s^* q^{i+2})(1-r_1 q^{i+2})(1-r_2 q^{i+2})}{(1-s^* q^{2i+3})(1-s^* q^{2i+4})} \end{array} \right],$$

and for $1 \leq i \leq 3$,

$$\mathbf{c}_i = \left[\begin{array}{c|c} \frac{(1-q^i)(1-s^* q^{i+5})(r_1-s^* q^i)(r_2-s^* q^i)}{s^* q^4(1-s^* q^{2i})(1-s^* q^{2i+1})} & \frac{(1-q^i)(1-s^* q^{i+5})}{s^* q^4(1-s^* q^{2i+1})} \left(\frac{(r_1-s^* q^{i+1})(r_2-s^* q^{i+1})}{1-s^* q^{2i+2}} - \frac{(r_1-s^* q^i)(r_2-s^* q^i)}{1-s^* q^{2i}} \right) \\ \hline 0 & \frac{(1-q^i)(1-s^* q^{i+5})(r_1-s^* q^{i+1})(r_2-s^* q^{i+1})}{s^* q^4(1-s^* q^{2i+1})(1-s^* q^{2i+2})} \end{array} \right].$$

Each entry is shown in (A.16).

The matrix representing $\frac{t_1 - k_1^{-1}}{k_1 - k_1^{-1}}$ relative to (A.17) is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\epsilon_1}{\epsilon_1 - 1} & \frac{1}{1 - \epsilon_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\epsilon_1}{\epsilon_1 - 1} & \frac{1}{1 - \epsilon_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\epsilon_2}{\epsilon_2 - 1} & \frac{1}{1 - \epsilon_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\epsilon_2}{\epsilon_2 - 1} & \frac{1}{1 - \epsilon_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\epsilon_3}{\epsilon_3 - 1} & \frac{1}{1 - \epsilon_3} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\epsilon_3}{\epsilon_3 - 1} & \frac{1}{1 - \epsilon_3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Each entry is shown in (A.15).

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