Further properties of viscosity solutions to some fully nonlinear partial differential equations

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Abstract

I present two research directions in this dissertation. The first direction is on further properties of viscosity solutions to geometric equations, a part of which is about the mean curvature motion with the simultaneous effect of a forcing term and a contact angle condition. The interaction of the two effects is analyzed during the estimate with a new method. In conclusion, a sufficient condition on the forcing term to be coercive is provided [66] (joint work with Dohyun Kwon, Hiroyoshi Mitake, and Hung Vinh Tran) for the right-angle condition and [64] for the nearly right-angle condition.

The remaining part of the first direction is devoted to periodic homogenization of geometric equations with the theory of viscosity solutions. I present results on qualitative homogenization of general geometric equations and on quantitative homogenization of forced mean curvature flows [65], obtained by using perturbed approximate correctors. Also, an improved rate of convergence is given in the laminar setting by directly using correctors [63].

The second direction of this dissertation is concerned with the study on properties of viscosity solutions to Hamilton-Jacobi equations. Specifically, an improved homogenization rate of multi-scale first-order convex Hamilton-Jacobi equations, which models the motion of a particle following classical mechanics, in the periodic setting is obtained [57] (joint work with Yuxi Han). Also, I present results on basic properties of viscosity solutions that describe the minimum eradication time for time-varying SIR models [70] (joint work with Yeoneung Kim).

Dedication

To my parents, Suig Jang and Eunhee Heo.

Declaration

I declare that this thesis has been composed solely by myself and that it has not been submitted, in whole or in part, in any previous application for a degree. Except where states otherwise by reference or acknowledgment, the work presented is entirely my own.

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Chapter 1

Viscosity solutions to geometric equations

The very first development of the viscosity solution theory originates from [26, 27], where the theory is considered for first-order Hamilton-Jacobi equations. In these pioneering works, a given equation is approximated with an additional regularization term with a coefficient $\varepsilon > 0$. The number $\varepsilon > 0$ represents the *viscosity*, and the idea of finding the solution from the approximated problems by letting $\varepsilon \to 0$ is called the *vanishing viscosity technique*. The term *viscosity solutions* is coined for this reason.

In this chapter, we give a minimal introduction to the theory of viscosity solutions with emphasis on the existence and the uniqueness. Although the development for first-order Hamilton-Jacobi equations comes before the case of *geometric equations*, we consider only the latter throughout this chapter. This is because this dissertation is primarily concerned with viscosity solutions to geometric equations. We instead leave [26, 27, 28, 36, 98] and the references therein for first-order Hamilton-Jacobi equations.

1.1 Definitions

Let $n \geq 1$. Let S^n denote the set of $n \times n$ symmetric matrices. Let $F : S^n \times (\mathbb{R}^n \setminus \{0\}) \to \mathbb{R}$ be an operator which is continuous on its domain of definition. We consider the following

Cauchy problem with a uniformly continuous datum u_0 on $\mathbb{R}^n$:

$$\begin{cases} u_t + F\left(D^2u, Du\right) = 0, & \text{in} \quad \mathbb{R}^n \times (0,\infty), \\ u(\cdot,0) = u_0, & \text{on} \quad \mathbb{R}^n. \end{cases} \tag{1.1}$$

Throughout this chapter, the functions F^*, F_* are defined by, for $(X,p) \in S^n \times \mathbb{R}^n$,

$$F^*(X,p) := \limsup_{\eta\to 0} \left\{F(Y,q) \ : \ (Y,q) \in S^n \times (\mathbb{R}^n \setminus \{0\}), \ \|Y-X\|, |q-p| \le \eta\right\},$$

$$F_*(X,p) := \liminf_{\eta\to 0} \left\{F(Y,q) \ : \ (Y,q) \in S^n \times (\mathbb{R}^n \setminus \{0\}), \ \|Y-X\|, |q-p| \le \eta\right\},$$

respectively. We similarly define the functions u^*, u_*.

Definition 1.1.1 (Viscosity sub/super/solutions). *An upper semicontinuous (lower semicontinuous, resp.) function $u : \mathbb{R}^n \times [0,\infty) \to [-\infty,+\infty]$ is a viscosity subsolution (a viscosity supersolution, resp.) to* (1.1) *if $u < +\infty$ ($u > -\infty$, resp.), $u(\cdot,0) \le u_0$ ($u(\cdot,0) \ge u_0$, resp.), and if $u - \varphi$ attains a local maximum (a local minimum, resp.) at $P_0 = (x_0,t_0)$ with $t_0 > 0$ for some smooth function $\varphi = \varphi(x,t)$ defined near P_0, then we have*

$$\varphi_t + F_*(D^2\varphi, D\varphi) \le 0 \ \ (\varphi_t + F^*(D^2\varphi, D\varphi) \ge 0, \ \textit{resp.}) \ \textit{at } P_0.$$

We say that u is a viscosity solution to (1.1) *if u^* is a subsolution to* (1.1) *and if u_* is a supersolution to* (1.1).

What we are interested in is equations with *degenerate elliptic* and *geometric* operators:

Definition 1.1.2 (Degenerate ellipticity, geometricity). *A continuous operator $F : S^n \times (\mathbb{R}^n \setminus \{0\}) \to \mathbb{R}$ is called degenerate elliptic if*

$$F(Y,p) \le F(X,p)$$

for all $X, Y \in S^n$ *with* $X \leq Y$ *and all* $p \in (\mathbb{R}^n \setminus \{0\})$, *and is called geometric if*

$$F(\lambda X + \mu p \otimes p, \lambda p) = \lambda F(X, p)$$

for all $(X, p) \in S^n \times (\mathbb{R}^n \setminus \{0\})$ *and all* $\lambda > 0$, $\mu \in \mathbb{R}$.

1.2 Existence and Uniqueness

We present the existence and the uniqueness results regarding viscosity solutions of geometric equations. We remark that [24] and [35], conducted independently and simultaneously, are considered as the first level-set approach for mean curvature motions using the viscosity solution theory. The contents of this subsection are mostly taken from [24]. We also leave [51] for more comprehensive theory.

Theorem 1.2.1. *Suppose that a continuous operator* $F : S^n \times (\mathbb{R}^n \setminus \{0\}) \to \mathbb{R}$ *is degenerate elliptic and geometric. Suppose that* $(\mp F)_*(\pm I, p) \leq c_\pm(|p|)$ *for some* $c_\pm \in C^1([0, \infty))$ *and* $c_\pm(\sigma) \geq c_0 > 0$ *with some constant* $c_0 > 0$. *Suppose that*

$$-\infty < F_*(O, 0) = F^*(O, 0) < +\infty.$$

Then, for any continuous initial datum u_0 *with compact support, there exists a unique viscosity* $u \in C(\mathbb{R}^n \times [0, \infty))$ *solution to* (1.1) *such that its restriction on* $\mathbb{R}^n \times [0, T]$ *is compactly supported for any* $T > 0$.

Depending on the subject, we can think of operators F with the spatial dependence. Also, we can also of boundary conditions of various types. In all cases, the existence and the uniqueness are the first issues to be addressed.

In the following chapters of this dissertation, which are considered to be independent to each other, the definition, the existence, and the uniqueness of viscosity solutions will be provided chapter by chapter. We also explore the further properties of viscosity solutions, such as gradient estimates, the large-time behavior, homogenization, and so on, according to the subject of the chapter.

Before we move on the next, we remark that the large-time behavior is one of the most important subjects of study that are beyond the well-posedness. In this direction for first-order and second-order Hamilton-Jacobi equations, [17] provides the most general results that can handle the degenerate case.

1.3 The contents of the subsequent chapters

In the following, solutions are understood as viscosity solutions. The characters κ, c, Ω denote the mean curvature, a given forcing term, a given domain, respectively.

Chapters 2, 3, 4, 5 are concerned with geometric equations.

Chapter 2 studies the propagation of a surface $\Gamma \subset \Omega$ with the normal velocity $V = \kappa + c$ with the right-angle condition $\Gamma \perp \partial\Omega$. The convexity/concavity of $\partial\Omega$ affects the behavior of the surface, which is analyzed by finding a role of c. The main reference is [66].

Chapter 3 extends the study in Chapter 2 to an angle condition which is not necessarily to be the right-angle. Gradient estimates, large-time behaviors are analyzed in Chapter 3, as well as in Chapter 2. The main reference is [64].

Chapter 4 provides a convergence rate of periodic homogenization of forced mean curvature flows in laminated media. The method is based on the regular selection of correctors, in the framework of the doubling variable method and the perturbed test function method. The main reference is [63].

Chapter 5 proves the homogenization of general geometric equations in the periodic setting assuming the solvability of cell problems. A quantitative homogenization of forced mean curvature flows in general media is also presented. Perturbed approximate correctors are used for the both cases. The main reference is [65].

Chapters 6, 7 are concerned with first-order Hamilton-Jacobi equations.

Chapter 6 provides a convergence rate of periodic homogenization of first-order convex Hamilton-Jacobi equations in multiscales. The idea can be seen as the combination of the separation of scales and the curve cutting technique [97]. The main reference is [57].

Chapter 7 suggests a viewpoint on the minimum eradication time for SIR (Susceptible-Infectious-Recovered) models with time-dependent coefficients. It turns out that there are two value functions both describing the minimum eradication time. A sufficient condition on the threshold for the two functions to agree is also given. The main reference is [70].

Chapter 2

Level-set forced mean curvature flow with the Neumann boundary condition

2.1 Introduction

In this chapter, we study the level-set equation for the forced mean curvature flow

$$\begin{cases} u_t = |Du| \operatorname{div}\left(\dfrac{Du}{|Du|}\right) + c(x)|Du| & \text{in } \Omega \times (0,\infty), \quad (2.1)\\ \dfrac{\partial u}{\partial \vec{\mathbf{n}}} = 0 & \text{on } \partial\Omega \times [0,\infty), \quad (2.2)\\ u(x,0) = u_0(x) & \text{on } \overline{\Omega}. \quad (2.3) \end{cases}$$

The domain $\Omega \subset \mathbb{R}^n$ with $n \geq 2$ is assumed to be bounded and $C^{2,\theta}$ for some $\theta \in (0,1)$. Here, $c = c(x)$ is a forcing function, which is in $C^1(\overline{\Omega})$, and $\vec{\mathbf{n}}$ is the outward unit normal vector to $\partial\Omega$. Throughout this chapter, we assume that $u_0 \in C^{2,\theta}(\overline{\Omega})$, and $\frac{\partial u_0}{\partial \vec{\mathbf{n}}} = 0$ on $\partial\Omega$ for compatibility.

We first notice that the well-posedness and the comparison principle for (2.1)–(2.3) are well established in the theory of viscosity solutions (see [24, 35, 48, 49] for instance). Our main interest in this chapter is to go beyond the well-posedness theory to understand the Lipschitz regularity and large time behavior of the solution. The Lipschitz regularity for the solution is rather subtle because of the competition between the forcing term and the mean curvature term together with the constraint on perpendicular intersections of the level sets of the solution with the boundary of Ω. It is worth emphasizing that the geometry of $\partial\Omega$ plays a crucial role in the analysis.

We now describe our main results. First of all, we show that u is Lipschitz in time and locally Lipschitz in space.

Theorem 2.1.1. *Let u be the unique viscosity solution u of (2.1)–(2.3). Then, there exists a constant $M > 0$ and for each $T > 0$, there exists a constant $C_T > 0$ depending on T such that*

$$\begin{cases} |u(x,t) - u(x,s)| \le M|t-s|, \\ |u(x,t) - u(y,t)| \le C_T|x-y|, \end{cases} \qquad \text{for all } x, y \in \overline{\Omega},\ t, s \in [0,T].$$

We next show that if we put some further conditions on the forcing term c, then we have the global Lipschitz estimate in x of the solution. Denote by

$$\begin{cases} C_0 := \max\{-\lambda : \lambda \text{ is a principal curvature of } \partial\Omega \text{ at } x_0 \text{ for } x_0 \in \partial\Omega\} \in \mathbb{R}, \\ K_0 := \min\{d : d \text{ is the diameter of an open ball inscribed in } \Omega\} > 0. \end{cases}$$

Theorem 2.1.2. *Assume that there exists $\delta > 0$ such that*

$$\frac{1}{n}c(x)^2 - |Dc(x)| - \delta > \max\left\{0,\ C_0|c(x)| + \frac{2nC_0}{K_0}\right\} \quad \text{for all } x \in \Omega. \tag{2.4}$$

Let u be the unique viscosity solution to (2.1)–(2.3). Then, there exist constants $M, L > 0$ depending only on the forcing term c and the initial data u_0 such that

$$\begin{cases} |u(x,t) - u(x,s)| \le M|t-s|, \\ |u(x,t) - u(y,t)| \le L|x-y|, \end{cases} \qquad \text{for all } x, y \in \overline{\Omega},\ t, s \in [0,\infty). \tag{2.5}$$

Let us now explain a bit the geometric meaning of K_0. For each $x \in \partial\Omega$, let

$$K_x = \max\{2r > 0 : B(x - r\vec{\mathbf{n}}(x), r) \subset \Omega\}.$$

Then, $K_0 = \min_{x\in\partial\Omega} K_x$. We notice next that if Ω is convex in Theorem 2.1.2, then we clearly have $C_0 \le 0$. In this case, (2.4) becomes $\frac{1}{n}c(x)^2 - |Dc(x)| - \delta > 0$, a kind of coercive assumption, which often appears in the usage of the classical Bernstein method to obtain Lipschitz regularity (see [78] for instance).

In the specific case where $c \equiv 0$ and Ω is convex and bounded, the global Lipschitz estimate of the solution was obtained in [47]. Moreover, a very interesting example was given in [47] to show that the solution is not globally Lipschitz continuous if Ω is not convex. Motivated by this example, we give two examples showing that u is not globally Lipschitz continuous if we do not impose (2.4). Furthermore, the examples demonstrate that condition (2.4) is sharp.

Let us note that the graph mean curvature flow with the Neumann boundary conditions has been studied much in the literature (see [54, 58, 80] and the references therein).

We next study the large time behavior of u under condition (2.4).

Theorem 2.1.3. *Assume* (2.4). *Let u be the unique viscosity solution to* (2.1)–(2.3). *Then,*

$$u(\cdot,t) \to v, \qquad \text{as } t \to \infty,$$

uniformly on $\overline{\Omega}$ for some Lipschitz function v, which is a viscosity solution to

$$\begin{cases} -\left(\operatorname{div}\left(\frac{Dv}{|Dv|}\right) + c(x)\right)|Dv| = 0 & \text{in } \Omega, \\ \dfrac{\partial v}{\partial \vec{\mathbf{n}}} = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.6}$$

We prove Theorem 2.1.3 by using a Lyapunov function, which is quite standard. We say that v is the large time profile of the solution u. It is important to note that the stationary problem (2.6) may have various different solutions, and thus, the question on how the large time profile v depends on the initial data u_0 is rather delicate and challenging. We are able to answer this question in the radially symmetric setting, and it is still widely open in the general settings.

Theorem 2.1.4. *Suppose that, by abuse of notions,*

$$\begin{cases} \Omega = B(0,R) \text{ for some } R > 0, \\ c(x) = c(r) \text{ for } |x| = r \in [0,R], \\ u_0(x) = u_0(r) \text{ for } |x| = r \in [0,R]. \end{cases} \tag{2.7}$$

Here, $c \in C^1([0,R],[0,\infty))$, *and* $u_0 \in C^2([0,R])$ *with* $u_0'(R) = 0$. *Denote by*

$$\mathcal{A} := \left\{ r \in (0,R] \,:\, c(r) = \frac{n-1}{r} \right\},$$
$$\mathcal{A}_+ := \left\{ r \in (0,R] \,:\, c(r) > \frac{n-1}{r} \right\},$$
$$\mathcal{A}_- := \left\{ r \in (0,R] \,:\, c(r) < \frac{n-1}{r} \right\}.$$

Define $d : (0,R] \to (0,R]$ *as*

$$d(r) = \begin{cases} r & \text{if } r \in \mathcal{A}, \\ \max\left(\mathcal{A} \cap (0,r)\right) & \text{if } r \in \mathcal{A}_+, \\ \min\left(\mathcal{A} \cap (r,R]\right) & \text{if } r \in \mathcal{A}_- \text{ and } \mathcal{A} \cap (r,R] \neq \emptyset, \\ R & \text{if } r \in \mathcal{A}_- \text{ and } \mathcal{A} \cap (r,R] = \emptyset. \end{cases}$$

Write $u(x,t) = \phi(|x|,t)$ *for* $x \in \Omega = B(0,R)$ *and* $t \geq 0$. *Then, the limiting profile* $\phi_\infty(r) = \lim_{t\to\infty} \phi(r,t)$ *can be written in terms of* u_0 *as: for each* $r_0 \in (0,R]$,

$$\phi_\infty(r_0) = \max\left\{u_0(r) \,:\, r \geq d(r_0)\right\}. \tag{2.8}$$

As a by-product, Theorem 2.1.4 shows that the solution to (2.1)–(2.3) is not globally Lipschitz continuous with an appropriate choice of initial data u_0.

Corollary 2.1.1. *Consider the setting in Theorem* 2.1.4. *Assume that there exist* $0 < a < b < R$ *such that* $a, b \in \mathcal{A}$ *and* $(a,b) \subset \mathcal{A}_-$. *Assume further that* u_0 *is a* C^2 *function* *on* $[0, R]$ *such that*

$$u_0(r) = \begin{cases} 1 & \text{for } r \le a, \\ \in (0,1) & \text{for } a < r \le b, \\ 0 & \text{for } b < r \le R. \end{cases}$$

Then, u is not globally Lipschitz, and

$$\phi_\infty(r) = \begin{cases} 1 & \text{for } r \le a, \\ 0 & \text{for } a < r \le R. \end{cases}$$

Lastly, we give another example to show the non global Lipschitz phenomenon in Theorem 2.6.1. Since we deal with the situation where Ω is unbounded there, we leave the precise statement of Theorem 2.6.1 and corresponding adjustments to Section 2.6.

Our problem (2.1)–(2.2) basically describes a level-set forced mean curvature flow with the homogeneous Neumann boundary condition. If a level set of the unknown u is a smooth enough surface, then it evolves with the normal velocity $V(x) = \kappa + c(x)$, where κ equals $(n-1)$ times the mean curvature of the surface at x, and it perpendicularly intersects $\partial\Omega$ (if ever). What is really interesting and delicate here is the competition between the forcing term $c(x)$ and the mean curvature term κ coupled with the constraint on perpendicular intersections of the level sets with the boundary. It is worth emphasizing that we do not assume Ω is convex, and the geometry of $\partial\Omega$ plays a crucial role in the behavior of the solution here. Indeed, analyzing the competition between the two constraints, the force and the boundary condition subjected to $\partial\Omega$, as time evolves in viscosity sense is the main topic of this chapter.

We now briefly describe our approaches to get the aforementioned results. We use the maximum principle and rely on the classical Bernstein method to establish *a priori* gradient estimates for the solution. The main difficulty is when a maximizer is located on the boundary, which we cannot apply the maximum principle directly. We deal with this difficulty by considering a multiplier that puts the maximizer, with the homogeneous Neumann boundary condition, inside the domain so that the maximum principle is applicable. To the best of our knowledge, the idea of handling a maximizer in the proof of Theorem 2.1.2 for the level-set equation for forced mean curvature flows under the Neumann boundary condition is new in the literature.

Once we get a global Lipschitz estimate for the solution, by using a standard Lyapunov function, we prove the convergence in Theorem 2.1.3. Next, the radially symmetric setting is considered, and (2.1)–(2.3) are reduced to a first-order singular Hamilton-Jacobi equation with the homogeneous Neumann boundary condition; see [46, 45] for a related problem on the whole space. By using the representation formula for the Neumann problem (see, e.g., [62]), we are able to obtain Theorem 2.1.4 and Corollary 2.1.1. The situation considered in Theorem 2.6.1 is related to that in [90, Section 4] with no forcing term. As we have a constant forcing c interacting with the boundary, the construction in the proof of Theorem 2.6.1 is rather delicate and involved. It is worth emphasizing that Corollary 2.1.1 and Theorem 2.6.1 demonstrate that condition (2.4), which is needed for the global Lipschitz regularity of u, is essentially optimal.

We conclude this introduction by giving a non exhaustive list of related works to this chapter. There are several asymptotic analysis results on the forced mean curvature flows with Neumann boundary conditions [54, 82, 86, 103] or with periodic boundary conditions [22], but they are all for graph-like surfaces. The volume preserving mean curvature flow, which is a different type of forced mean curvature flows, was studied in [68, 69]. Recently, the relation between the level set approach and the varifold approach for (2.1) with $c \equiv 0$ was investigated in [1]. We also refer to [45, 55] for some recent results on the asymptotic growth speed of solutions to forced mean curvature flows with discontinuous source terms in the whole space.

Organization of the chapter

The chapter is organized as follows. In Section 2.2, we give the notion of viscosity solutions to the problem and some basic results. In Section 2.3, we prove the local and global gradient estimates. Section 2.4 is devoted to the study on large time behavior of the solution and its large time profile. We give two examples that the spatial gradient of the solution grows to infinity as time tends to infinity in Sections 2.5 and 2.6 if we do not impose assumption (2.4) on the force c.

2.2 Preliminaries

In this section, we recall the notion of viscosity solutions to the Neumann boundary problem (2.1)–(2.3) and give some related results.

Let $\mathcal{S}^n$ be the set of symmetric matrices of size n. Define $F : \overline{\Omega} \times (\mathbb{R}^n \setminus \{0\}) \times \mathcal{S}^n \to \mathbb{R}$ by

$$F(x, p, X) = \operatorname{trace}\left(\left(I - \frac{p \otimes p}{|p|^2}\right) X\right) + c(x)|p|.$$

We denote the semicontinuous envelopes of F by, for $(x, p, X) \in \overline{\Omega} \times \mathbb{R}^n \times \mathcal{S}^n$,

$$F_*(x, p, X) = \liminf_{(y,q,Y)\to(x,p,X)} F(y, q, Y), \quad F^*(x, p, X) = \limsup_{(y,q,Y)\to(x,p,X)} F(y, q, Y).$$

Definition 2.2.1. *An upper semicontinuous function $u : \overline{\Omega} \times [0, \infty) \to \mathbb{R}$ is said to be a viscosity subsolution of* (2.1)–(2.3) *if $u(\cdot, 0) \le u_0$ on $\overline{\Omega}$, and, for any $\varphi \in C^2(\overline{\Omega} \times [0, \infty))$, if $(\hat{x}, \hat{t}) \in \overline{\Omega} \times (0, \infty)$ is a maximizer of $u - \varphi$, and if $\hat{x} \in \Omega$, then*

$$\varphi_t(\hat{x}, \hat{t}) - F^*(\hat{x}, D\varphi(\hat{x}, \hat{t}), D^2\varphi(\hat{x}, \hat{t})) \le 0;$$

if $\hat{x} \in \partial\Omega$, then

$$\min\left\{\varphi_t(\hat{x}, \hat{t}) - F^*(\hat{x}, D\varphi(\hat{x}, \hat{t}), D^2\varphi(\hat{x}, \hat{t})), \frac{\partial \varphi}{\partial \vec{\mathbf{n}}}(\hat{x}, \hat{t})\right\} \le 0.$$

Similarly, a lower semicontinuous function $u : \overline{\Omega} \times [0, \infty) \to \mathbb{R}$ is said to be a viscosity supersolution of (2.1)–(2.3) *if $u(\cdot, 0) \ge u_0$ on $\overline{\Omega}$, and, for any $\varphi \in C^2(\overline{\Omega} \times [0, \infty))$, if*

$(\hat{x},\hat{t}) \in \overline{\Omega} \times (0,\infty)$ *is a minimizer of* $u-\varphi$, *and if* $\hat{x} \in \Omega$, *then*

$$\varphi_t(\hat{x},\hat{t}) - F_*(\hat{x}, D\varphi(\hat{x},\hat{t}), D^2\varphi(\hat{x},\hat{t})) \geq 0;$$

if $\hat{x} \in \partial\Omega$, *then*

$$\max\left\{\varphi_t(\hat{x},\hat{t}) - F_*(\hat{x}, D\varphi(\hat{x},\hat{t}), D^2\varphi(\hat{x},\hat{t})), \frac{\partial \varphi}{\partial \vec{\mathbf{n}}}(\hat{x},\hat{t})\right\} \geq 0.$$

Finally, a continuous function u *is said to be a viscosity solution of* (2.1)–(2.3) *if* u *is both its viscosity subsolution and its viscosity supersolution.*

Henceforth, since we are always concerned with viscosity solutions, the adjective "viscosity" is omitted. The following comparison principle for solutions to (2.1)–(2.3) in a bounded domain is well known (see, e.g., [48]).

Proposition 2.2.1 (Comparison principle for (2.1)–(2.3))**.** *Let* u *and* v *be a subsolution and a supersolution of* (2.1)–(2.3), *respectively. Then,* $u \leq v$ *in* $\overline{\Omega} \times [0,\infty)$.

To obtain Lipschitz estimates, it is convenient to consider an approximate problem of (2.1)–(2.3) by considering, for $\varepsilon > 0$, $T > 0$,

$$\begin{cases} u^\varepsilon_t = \sqrt{\varepsilon^2 + |Du^\varepsilon|^2}\,\mathrm{div}\left(\frac{Du^\varepsilon}{\sqrt{\varepsilon^2+|Du^\varepsilon|^2}}\right) + c(x)\sqrt{\varepsilon^2 + |Du^\varepsilon|^2} & \text{in } \Omega \times (0,T], \\ \dfrac{\partial u^\varepsilon}{\partial \vec{\mathbf{n}}} = 0 & \text{on } \partial\Omega \times [0,T], \\ u^\varepsilon(x,0) = u_0(x) & \text{on } \overline{\Omega}. \end{cases} \tag{2.9}$$

Equation (2.9) describes the motion of the graph of $\frac{u^\varepsilon}{\varepsilon}$ under the forced mean curvature flow $V = \kappa + c$ in Ω with right contact angle condition on $\partial\Omega$. The following result on a priori estimates on the gradient of u^ε plays a crucial role in our analysis.

Theorem 2.2.1 (A priori estimates)**.** *Assume that* $\partial\Omega$ *is smooth and* $c \in C^\infty(\overline{\Omega})$. *For each* $\varepsilon \in (0,1)$ *and* $T > 0$, *assume that* $u^\varepsilon \in C^\infty(\overline{\Omega} \times (0,T]) \cap C^1(\overline{\Omega} \times [0,T])$ *is the unique* *solution of* (2.9). *Then, there exist a constant* $M > 0$ *and a constant* $C_T > 0$ *depending on* T *such that*

$$\|u_t^\varepsilon\|_{L^\infty(\overline{\Omega}\times[0,T])} \le M \quad \text{and} \quad \|Du^\varepsilon\|_{L^\infty(\overline{\Omega}\times[0,T])} \le C_T. \tag{2.10}$$

Here, M *and* C_T *are independent of* $\varepsilon \in (0,1)$.

The proof of Theorem 2.2.1 is given in the next section. The a priori estimates then allow us to get the existence and uniqueness of solutions to (2.9).

Proposition 2.2.2. *For each* $\varepsilon \in (0,1)$ *and* $T > 0$, *equation* (2.9) *has a unique continuous solution* u^ε. *Furthermore,* $u^\varepsilon \in C^{2,1}(\Omega \times (0,T]) \cap C^1(\overline{\Omega} \times [0,T])$ *and* (2.10) *holds.*

Proposition 2.2.2 can be obtained by the classical parabolic PDE theory. For instance, we refer to [86] for a similar form of Proposition 2.2.2. The proof of this proposition is quite standard, and hence, is omitted here.

Once we get (2.10), by the standard stability result of viscosity solutions, and the uniqueness of viscosity solutions to (2.1)–(2.3), we imply that

$$u^\varepsilon \to u \quad \text{as } \varepsilon \to 0 \quad \text{uniformly on } \overline{\Omega} \times [0,T]$$

for each $T > 0$. Moreover, Theorem 2.2.1 and Proposition 2.2.2 give us right away Theorem 2.1.1.

2.3 Lipschitz regularity

In this section, we prove Theorems 2.1.1, 2.1.2, and 2.2.1. As noted, it is actually enough to prove Theorems 2.1.2 and 2.2.1. First, we prove that the time derivative of u^ε is bounded.

Lemma 2.3.1. *Assume that* $\partial\Omega$ *is smooth and* $c \in C^\infty(\overline{\Omega})$. *Suppose that* $u^\varepsilon \in C^\infty(\overline{\Omega} \times (0,T]) \cap C^1(\overline{\Omega} \times [0,T])$ *is the unique solution of* (2.9) *for each* $\varepsilon \in (0,1)$ *and* $T > 0$. *Then, there exists* $M > 0$ *depending only on the forcing term* c *and the initial data* u_0 *such that,* *for* $\varepsilon \in (0,1)$,

$$\|u_t^\varepsilon\|_{L^\infty(\overline{\Omega}\times[0,T])} \le \|u_t^\varepsilon(\cdot,0)\|_{L^\infty(\overline{\Omega})} \le M.$$

Proof. Set $b(p) = I_n - p\otimes p/(\varepsilon^2+|p|^2)$. Then (2.9) is expressed as

$$u_t^\varepsilon - b^{ij}(Du^\varepsilon)u_{ij}^\varepsilon - c(x)\sqrt{\varepsilon^2+|Du^\varepsilon|^2} = 0 \quad \text{in } \Omega\times(0,T]. \tag{2.11}$$

Here, we use the Einstein summation convention, and we write $f_i = \frac{\partial f}{\partial x_i}$ and $f_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$ for $i,j = 1,\ldots,n$, where $f = f(x,t)$ is a given function. We now show that

$$\|u_t^\varepsilon\|_{L^\infty(\overline{\Omega}\times[0,T])} \le \|u_t^\varepsilon(\cdot,0)\|_{L^\infty(\overline{\Omega})}. \tag{2.12}$$

To prove (2.12), it is enough to obtain the upper bound

$$\max_{\overline{\Omega}\times[0,T]} u_t^\varepsilon = \max_{\overline{\Omega}} u_t^\varepsilon(\cdot,0)$$

as the lower bound can be obtained analogously.

Differentiating (2.11) with respect to t yields

$$(u_t^\varepsilon)_t - b^{ij}(u_t^\varepsilon)_{ij} - (b^{ij})_t u_{ij}^\varepsilon - c(x)\frac{(u_t^\varepsilon)_l u_l^\varepsilon}{\sqrt{\varepsilon^2+|Du^\varepsilon|^2}} = 0,$$

where

$$(b^{ij})_t = -\frac{(u_t^\varepsilon)_i u_j^\varepsilon}{\varepsilon^2+|Du^\varepsilon|^2} - \frac{u_i^\varepsilon (u_t^\varepsilon)_j}{\varepsilon^2+|Du^\varepsilon|^2} + \frac{2u_i^\varepsilon u_j^\varepsilon u_l^\varepsilon (u_t^\varepsilon)_l}{(\varepsilon^2+|Du^\varepsilon|^2)^2}.$$

Suppose, on the contrary, that $u_t^\varepsilon(x,t) > \max_{\overline{\Omega}} u_t^\varepsilon(\cdot,0)$ for some $(x,t) \in \overline{\Omega}\times(0,T]$. Then, there exist a small number $\delta > 0$ and $(x_0,t_0) \in \overline{\Omega}\times(0,T]$ such that $(x_0,t_0) \in \operatorname{argmax}_{\overline{\Omega}\times(0,T]}(u_t^\varepsilon - \delta t)$.

At (x_0,t_0), we have $Du_t^\varepsilon = 0$, and note that the boundary case $x_0 \in \partial\Omega$ is included due to the homogeneous Neumann boundary condition. Thus,

$$(u_t^\varepsilon)_t - b^{ij}(u_t^\varepsilon)_{ij} = 0, \quad \text{at } (x_0,t_0). \tag{2.13}$$

On the other hand, $(u_t^\varepsilon - \delta t)_t \geq 0$, $-b^{ij}(u_t^\varepsilon)_{ij} \geq 0$ at (x_0, t_0). Note that the Neumann boundary condition is used for $D^2 u_t^\varepsilon \leq 0$ at (x_0, t_0) as well. Since $(u_t^\varepsilon)_t \geq \delta > 0$, we arrive at a contradiction in (2.13). Thus, (2.12) holds. Choose

$$M = n^2 \|D^2 u_0\|_{L^\infty(\overline{\Omega})} + \|c\sqrt{1 + |Du_0|^2}\|_{L^\infty(\overline{\Omega})}$$

to complete the proof. □

We are now ready to prove Theorems 2.1.2 and 2.2.1 using the classical Bernstein method. It is important emphasizing that the boundary behavior needs to be handled rather carefully. We first give a proof of Theorem 2.1.2.

Proof of Theorem 2.1.2. Assume first that $\partial\Omega$ is smooth and $c \in C^\infty(\overline{\Omega})$. For each $\varepsilon \in (0, 1)$ and $T > 0$, let $u^\varepsilon \in C^\infty(\overline{\Omega} \times (0, T]) \cap C^1(\overline{\Omega} \times [0, T])$ be the unique solution of (2.9).

Let $w^\varepsilon = \sqrt{\varepsilon^2 + |Du^\varepsilon|^2}$. In view of Lemma 2.3.1, we only need to show that

$$\max_{\overline{\Omega} \times [0,T]} w^\varepsilon \leq C \tag{2.14}$$

for some positive constant C depending only on $\|u_0\|_{C^2(\overline{\Omega})}$, $\|c\|_{C^1(\overline{\Omega})}$, the constants n, C_0, K_0, and δ from (2.4). The crucial point here is C does not depend on T and ε. Fix $(x_0, t_0) \in \operatorname{argmax}_{\overline{\Omega} \times [0,T]} w^\varepsilon$. If $t_0 = 0$, then

$$\max_{\overline{\Omega} \times [0,T]} w^\varepsilon \leq w^\varepsilon(x_0, 0) \leq \|Du_0\|_{L^\infty(\overline{\Omega})} + 1,$$

and (2.14) is valid. We next consider the case $t_0 > 0$.

We write $u = u^\varepsilon$, $w = w^\varepsilon$ in this proof for brevity. Differentiate (2.11) in x_k and multiply the result by u_k to get

$$u_k u_{kt} - (D_p b^{ij} \cdot Du_k) u_k u_{ij} - b^{ij} u_k u_{kij} - u_k c_k w - c\frac{u_k u_{lk} u_l}{w} = 0.$$

Substituting $ww_t = u_k u_{kt},\ ww_k = u_l u_{kl}$ and $ww_{ij} = u_{kij}u_k + b^{kl}u_{ki}u_{lj}$, we get

$$ww_t - w(D_p b^{ij} \cdot Dw)u_{ij} - wb^{ij}w_{ij} + b^{ij}b^{kl}u_{ki}u_{lj} - wDu \cdot Dc - cDu \cdot Dw = 0. \qquad (2.15)$$

We divide the proof into two cases: $x_0 \in \Omega$ and $x_0 \in \partial\Omega$.

Case 1: the interior case $x_0 \in \Omega$. We follow the computations of [44, Lemma 4.1]. At (x_0, t_0), we have $w_t \geq 0$, $Dw = 0$, $D^2 w \leq 0$, and thus

$$wDu \cdot Dc \geq b^{ij}b^{kl}u_{ki}u_{lj}.$$

We then use the Cauchy-Schwarz inequality

$$(\mathrm{tr}\alpha\beta)^2 \leq \mathrm{tr}(\alpha^2)\mathrm{tr}(\beta^2)$$

for all $\alpha, \beta \in \mathcal{S}^n$, and put $\alpha = A^{\frac{1}{2}}BA^{\frac{1}{2}}$, $\beta = I_n$, where $A = (b^{ij})$, $B = (u_{kl})$, I_n the n by n identity matrix to get $\mathrm{tr}(AB)^2 \geq (\mathrm{tr}AB)^2/\mathrm{tr}(I_n)$.

Therefore, at (x_0, t_0),

$$|Dc(x_0)|\, w^2 \geq wDu \cdot Dc \geq b^{ij}b^{kl}u_{ki}u_{lj} = \mathrm{tr}(AB)^2 \geq \frac{(\mathrm{tr}AB)^2}{\mathrm{tr}(I_n)} = \frac{1}{n}\left(u_t - c(x_0)w\right)^2$$

Since $\frac{1}{n}c(x)^2 - |Dc(x)| \geq \delta > 0$ by (2.4), we imply that at (x_0, t_0),

$$\delta w^2 \leq \frac{2u_t c(x_0)}{n} w \quad \Longrightarrow \quad w \leq \frac{2M\|c\|_{L^\infty(\overline{\Omega})}}{n\delta},$$

which confirms (2.14).

Case 2: the boundary case $x_0 \in \partial\Omega$. As $\partial\Omega$ is $C^{2,\theta}$, we assume that $\mathbf{n}$ is defined as a C^1 function in a neighborhood of $\partial\Omega$. Note that the Neumann boundary condition $Du \cdot \vec{\mathbf{n}} = 0$ gives $\big(D^2u\,\vec{\mathbf{n}} + D\vec{\mathbf{n}}Du\big) \cdot v = 0$ for all $v \in \mathbb{R}^n$ perpendicular to $\vec{\mathbf{n}}$ on $\partial\Omega \times [0, T]$.

Thus, on $\partial\Omega \times [0,T]$,

$$\frac{\partial w}{\partial \vec{\mathbf{n}}} = \frac{D^2uDu}{w} \cdot \vec{\mathbf{n}} = -\frac{D\vec{\mathbf{n}}Du \cdot Du}{w} \leq C_0 \frac{|Du|^2}{w},$$

where $C_0 = \sup\{-\lambda : \lambda \text{ is a principal curvature of } \partial\Omega \text{ at } x_0 \text{ for } x_0 \in \partial\Omega\}$.

If $C_0 < 0$, then $\frac{\partial w}{\partial \vec{\mathbf{n}}} < 0$ on $\partial\Omega \times [0,T]$, and hence w cannot attain its maximum on $\partial\Omega \times [0,T]$. Therefore, $C_0 \geq 0$. We consider the case when $C_0 > 0$ first, and deal with the case when $C_0 = 0$ later. We note that if $C_0 > 0$, then

$$\frac{\partial w}{\partial \vec{\mathbf{n}}} \leq C_0 \frac{|Du|^2}{w} < C_0 w.$$

Take $x_c \in \Omega$ so that $B := B(x_c, K_0/2)$ is inside Ω and tangent to the boundary $\partial\Omega$ at x_0. Consider a multiplier

$$\rho(x) = -\frac{C_0}{K_0}|x - x_c|^2 + \frac{C_0 K_0}{4} + 1 \quad \text{for } x \in \overline{\Omega}.$$

Then, $\rho > 1$ in B, $\rho = 1$ on ∂B, and $\rho \leq 1$ on $\overline{\Omega} \setminus B$. Besides, $C_0\rho(x_0) + \frac{\partial \rho}{\partial \vec{\mathbf{n}}}(x_0) = 0$.

Denote by $\psi = \rho w$. Then, at (x_0, t_0),

$$\frac{\partial \psi}{\partial \vec{\mathbf{n}}} = \frac{\partial(\rho w)}{\partial \vec{\mathbf{n}}} = \rho \frac{\partial w}{\partial \vec{\mathbf{n}}} + w \frac{\partial \rho}{\partial \vec{\mathbf{n}}} < w \left(C_0 \rho + \frac{\partial \rho}{\partial \vec{\mathbf{n}}} \right) = 0. \tag{2.16}$$

By the choice of ρ, it is clear that

$$\psi(z,t) \leq w(z,t) \leq w(x_0,t_0) = \psi(x_0,t_0) \quad \text{for } (z,t) \in \left(\overline{\Omega} \setminus B\right) \times [0,T],$$

and, by (2.16),

$$\max_{\overline{\Omega}\times[0,T]} \rho w = \max_{\overline{B}\times[0,T]} \rho w > \psi(x_0,t_0) = w(x_0,t_0). \tag{2.17}$$

Let $(x_1,t_1)\in \operatorname{argmax}_{\overline{\Omega}\times[0,T]}\rho w$. If $t_1=0$, then for all $(x,t)\in\overline{\Omega}\times[0,T]$,

$$\begin{aligned} w(x,t) &\le w(x_0,t_0)=\rho(x_0)w(x_0,t_0)\le \rho(x_1)w(x_1,0)\\ &\le \left(\frac{C_0K_0}{4}+1\right)\left(\|Du_0\|_{L^\infty(\overline{\Omega})}+1\right), \end{aligned}$$

and we are done. Thus, we may assume that $t_1>0$. In light of (2.16)–(2.17), we yield that $x_1\in B\subset\Omega$. At this point (x_1,t_1), we have $\psi_t\ge 0$, $D\psi=0$, $D^2\psi\le 0$. Consequently, as $\psi_t=\rho_t w+\rho w_t$, $D\psi = wD\rho+\rho Dw$, and $\psi_{ij}=w_{ij}\rho+w_i\rho_j+w_j\rho_i+w\rho_{ij}$, we have at (x_1,t_1),

$$\begin{cases} w_t\ge -\frac{\rho_t}{\rho}w=0,\\ Dw=-\frac{w}{\rho}D\rho,\\ w_{ij}=\frac{1}{\rho}(\psi_{ij}-w_i\rho_j-w_j\rho_i-w\rho_{ij}). \end{cases}$$

Therefore, at (x_1,t_1), by (2.15)

$$\begin{aligned} -\frac{\rho_t}{\rho}w^2+\frac{w^2}{\rho}(D_pb^{ij}\cdot D\rho)u_{ij}+\frac{w}{\rho}b^{ij}(w_i\rho_j+w_j\rho_i+w\rho_{ij})&\\ +b^{ij}b^{kl}u_{ki}u_{lj}-wDu\cdot Dc+\frac{cw}{\rho}Du\cdot D\rho\le 0.& \end{aligned}$$

Now,

$$b^{ij}_{p_l}=-\frac{\delta_{il}u_j}{\varepsilon^2+|Du|^2}-\frac{\delta_{jl}u_i}{\varepsilon^2+|Du|^2}+\frac{2u_iu_ju_l}{(\varepsilon^2+|Du|^2)^2},$$

and thus,

$$\begin{aligned} w(D_pb^{ij}\cdot D\rho)u_{ij} &= w\left(-\frac{\rho_iu_ju_{ij}}{\varepsilon^2+|Du|^2}-\frac{\rho_ju_iu_{ij}}{\varepsilon^2+|Du|^2}+\frac{2u_iu_ju_l\rho_lu_{ij}}{(\varepsilon^2+|Du|^2)^2}\right)\\ &= -2Dw\cdot D\rho+\frac{2(Du\cdot D\rho)(Du\cdot Dw)}{w^2}. \end{aligned}$$

Hence,

$$\begin{aligned} & w(D_p b^{ij} \cdot D\rho) u_{ij} + b^{ij} w_i \rho_j + b^{ij} w_j \rho_i \\ = & \frac{2(Du \cdot D\rho)(Du \cdot Dw)}{w^2} - \frac{u_i u_j w_i \rho_j}{w^2} - \frac{u_i u_j w_j \rho_i}{w^2} = 0. \end{aligned}$$

All in all, at $(x_1, t_1) \in \operatorname{argmax}_{\overline{\Omega} \times (0,T]} \rho w$ with $x_1 \in B \subset \Omega$, the inequality

$$-\frac{\rho_t}{\rho} w^2 + \frac{\rho_{ij}}{\rho} b^{ij} w^2 + b^{ij} b^{kl} u_{ki} u_{lj} - w Du \cdot Dc + \frac{cw}{\rho} Du \cdot D\rho \leq 0 \tag{2.18}$$

holds. Note that $\rho_t = 0$ here, but we keep this term in the above formula for the usage in the proof of Theorem 2.2.1 later.

Using the Cauchy-Schwarz type inequality as in the above, we obtain

$$\begin{aligned} \frac{1}{n}(u_t - c(x_1)w)^2 \leq b^{ij} b^{kl} u_{il} u_{kj} & \leq -\frac{w^2}{\rho} b^{ij} \rho_{ij} + w Du \cdot Dc - \frac{cw}{\rho} Du \cdot D\rho \\ & \leq \frac{2C_0}{K_0} \frac{w^2}{\rho} \left(n - \frac{|Du|^2}{\varepsilon^2 + |Du|^2} \right) + |Dc| w^2 + C_0 |c| w^2 \\ & \leq \left(\frac{2nC_0}{K_0} + |Dc(x_1)| + C_0 |c(x_1)| \right) w^2. \end{aligned}$$

By (2.4),

$$\frac{1}{n} c(x)^2 - |Dc(x)| - C_0 |c(x)| - \frac{2nC_0}{K_0} \geq \delta > 0 \qquad \text{for all } x \in \overline{\Omega}$$

for some $\delta > 0$, we see that $w(x_1, t_1) \leq \frac{2M \|c\|_{L^\infty(\overline{\Omega})}}{n\delta}$. Thus,

$$w(x_0, t_0) \leq \rho(x_1) w(x_1, t_1) \leq \left(\frac{C_0 K_0}{4} + 1 \right) \frac{2M \|c\|_{L^\infty(\overline{\Omega})}}{n\delta}.$$

Now, we handle the case when $C_0 = 0$. We consider a multiplier

$$\rho(x) = -\frac{\delta_1}{K_0} |x - x_c|^2 + \frac{\delta_1 K_0}{4} + 1 \quad \text{ for } x \in \overline{\Omega},$$

where

$$\delta_1 = \frac{\delta}{2(\|c\|_{L^\infty} + \frac{2n}{K_0})} > 0.$$

Then, at (x_0, t_0),

$$\frac{\partial w}{\partial \vec{\mathbf{n}}} \le C_0 \frac{|Du|^2}{w} = 0,$$

and

$$\frac{\partial \psi}{\partial \vec{\mathbf{n}}} = \frac{\partial(\rho w)}{\partial \vec{\mathbf{n}}} = \rho \frac{\partial w}{\partial \vec{\mathbf{n}}} + w\frac{\partial \rho}{\partial \vec{\mathbf{n}}} \le w \frac{\partial \rho}{\partial \vec{\mathbf{n}}} < 0.$$

Following the same argument as above with δ_1 in place of C_0, we see that

$$\frac{1}{n}(u_t - c(x_1)w)^2 \le \left(\frac{2n\delta_1}{K_0} + |Dc(x_1)| + C_0|c(x_1)|\right) w^2.$$

This inequality, together with the fact that

$$\frac{1}{n}c(x)^2 - |Dc(x)| - \delta_1|c(x)| - \frac{2n\delta_1}{K_0} \ge \delta - \frac{1}{2}\delta = \frac{1}{2}\delta > 0 \qquad \text{for all } x \in \overline{\Omega},$$

implies (2.14).

By (2.14) and Lemma 2.3.1, Du^ε and u_t^ε are uniformly bounded in $\Omega \times [0,T]$ for all $\varepsilon \in (0,1)$ and $T > 0$. Note that the bound depends only on $\|u_0\|_{C^2(\overline{\Omega})}$, $\|c\|_{C^1(\overline{\Omega})}$, the constants n, C_0, K_0, and δ from (2.4). By approximations, we see that the same result holds true in the case that $\partial\Omega \in C^{2,\theta}$ and $c \in C^1(\overline{\Omega})$. From the uniform convergence of u^ε to the unique viscosity solution u of (2.1)–(2.3), we conclude that u satisfies (2.5). □

We remark for later usage that for any smooth function $\rho > 0$, (2.18) is valid at $(x_1, t_1) \in \operatorname{argmax}(\rho w) \cap (\Omega \times (0,T])$.

Remark 2.3.1. *Let us discuss a bit the case where $c \equiv 0$ and Ω is convex and bounded. Then, w satisfies*

$$ww_t - w(D_p b^{ij} \cdot Dw)u_{ij} - wb^{ij}w_{ij} + b^{ij}b^{kl}u_{ki}u_{lj} = 0.$$

And, on $\partial\Omega \times [0,T]$,

$$\frac{\partial w}{\partial \vec{\mathbf{n}}} = \frac{D^2uDu}{w} \cdot \vec{\mathbf{n}} = -\frac{D\vec{\mathbf{n}}Du \cdot Du}{w} \leq 0.$$

By the usual maximum principle, we yield that

$$\max_{\overline{\Omega}\times[0,T]} w = \max_{\overline{\Omega}} w(\cdot, 0) \leq C.$$

We thus recover the gradient bound in [47]. It is worth to note that in this specific situation, condition (2.4) *is not needed.*

Proof of Theorem 2.2.1. Let $u = u^\varepsilon$ and $w = \sqrt{\varepsilon^2 + |Du^\varepsilon|^2}$ as in the proof of Theorem 2.1.2. As above, we may assume $\partial\Omega$ is smooth and $c \in C^\infty(\overline{\Omega})$. Pick

$$M > \frac{2n(|C_0|+1)}{K_0} + \|Dc\|_{L^\infty(\overline{\Omega})} + (|C_0|+1)\|c\|_{L^\infty(\overline{\Omega})}$$

and $(x_0, t_0) \in \operatorname{argmax}_{\overline{\Omega}\times[0,T]} e^{-Mt}w(x,t)$. If $t_0 = 0$, then we have that for $(x,t) \in \overline{\Omega}\times[0,T]$,

$$w(x,t) \leq e^{MT}\left(\|Du_0\|_{L^\infty(\overline{\Omega})} + 1\right).$$

Consider next the case that $t_0 > 0$. If $x_0 \in \Omega$, then by (2.18) with $\rho = e^{-Mt}$, at (x_0, t_0),

$$Mw^2 + b^{ij}b^{kl}u_{il}u_{kj} - wDu \cdot Dc \leq 0.$$

As $Mw^2 - wDu \cdot Dc > 0$ by the choice of M and $b^{ij}b^{kl}u_{il}u_{kj} \geq 0$, we arrive at a contradiction. Thus, $x_0 \in \partial\Omega$.

We repeat the proof of Theorem 2.1.2. Since $x_0 \in \operatorname{argmax}_{\overline{\Omega}} w(\cdot, t_0) \cap \partial\Omega$, we see as before that $C_0 \geq 0$. We use a new multiplier

$$\rho(x,t) = e^{-Mt}\left(-\frac{C_0+1}{K_0}|x - x_c|^2 + \frac{(C_0+1)K_0}{4} + 1\right) \quad \text{for } (x,t) \in \overline{\Omega} \times [0,\infty).$$

Here, $B = B(x_c, K_0/2)$ is inside Ω and tangent to the boundary $\partial\Omega$ at x_0.

Put $w_M = e^{-Mt}w$ and note that $w_M(x_0,t_0) = \max_{\overline{\Omega}} w_M$, $\frac{\partial w_M}{\partial \vec{\mathbf{n}}} \le C_0 w_M$ on $\partial\Omega \times [0,T]$, and

$$\rho w = \left(-\frac{C_0+1}{K_0}|x-x_c|^2 + \frac{(C_0+1)K_0}{4} + 1 \right) w_M.$$

Observe as in the proof of Theorem 2.1.2 that $\frac{\partial(\rho w)}{\partial \vec{\mathbf{n}}}(x_0,t_0) < 0$, $\rho w \le w_M$ on $\left(\overline{\Omega} \setminus B\right) \times [0,T]$, and therefore, $\operatorname{argmax}(\rho w) \subset B \times [0,T]$. Then, there is a point $(x_1,t_1) \in \operatorname{argmax}_{\overline{\Omega}\times[0,T]} \rho w$ with $(x_1,t_1) \in B \times [0,T]$. Consider the case $t_1 = 0$. For all $(x,t) \in \overline{\Omega} \times [0,T]$,

$$\begin{aligned} w_M(x,t) &\le w_M(x_0,t_0) = (\rho w)(x_0,t_0) \le (\rho w)(x_1,0) \\ &\le \left(\frac{(C_0+1)K_0}{4} + 1 \right) \left(\|Du_0\|_{L^\infty(\overline{\Omega})} + 1 \right). \end{aligned}$$

Thus, for $(x,t) \in \overline{\Omega} \times [0,T]$,

$$w(x,t) \le e^{MT} \left(\frac{(C_0+1)K_0}{4} + 1 \right) \left(\|Du_0\|_{L^\infty(\overline{\Omega})} + 1 \right). \tag{2.19}$$

Next, we consider the case $t_1 > 0$. At (x_1,t_1), thanks to (2.18), we have

$$Mw^2 + \frac{\rho_{ij}}{\rho} b^{ij} w^2 + b^{ij} b^{kl} u_{ki} u_{lj} - wDu \cdot Dc + \frac{cw}{\rho} Du \cdot D\rho \le 0.$$

From this, recalling the choice of M, we obtain, as before,

$$0 \le b^{ij} b^{kl} u_{ki} u_{lj} \le \left(-M + \frac{2n(C_0+1)}{K_0} + |Dc(x_1)| + (C_0+1)|c(x_1)| \right) w^2 < 0,$$

which is absurd. Thus, the case $t_1 > 0$ does not occur, and (2.19) holds true. Lemma 2.3.1 and (2.19) then complete the proof.

□

Remark 2.3.2. *We note that Theorems 2.1.1 and 2.1.2 are still valid when* $\partial\Omega \in C^2$, $c \in C^1(\overline{\Omega})$, *and* $u_0 \in C^2(\overline{\Omega})$ *by approximations as the Lipschitz bounds depend only on* $\|u_0\|_{C^2(\overline{\Omega})}$, $\|c\|_{C^1(\overline{\Omega})}$, *the constants* n, C_0, K_0, *and* $T > 0$ *in case of Theorem 2.1.1, and* δ

from (2.4) *in case of Theorem 2.1.2 .*

2.4 Large time behavior of the solution

In this section, we prove the large time behavior of u, which is globally Lipschitz continuous thanks to Theorem 2.1.2. Let L be the spatial Lipschitz constant of u^ε for $\varepsilon \in (0,1)$ given by the proof of Theorem 2.1.2.

Proof of Theorem 2.1.3. Although the proof is almost same as that of [50, Theorem 1.2], we give it for completeness.

We consider the following Lyapunov function

$$I^\varepsilon(t) = \int_\Omega \sqrt{\varepsilon^2 + |Du^\varepsilon|^2}\, dx.$$

By calculation,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \sqrt{\varepsilon^2 + |Du^\varepsilon|^2}\, dx = \int_\Omega \frac{Du^\varepsilon \cdot Du^\varepsilon_t}{\sqrt{\varepsilon^2 + |Du^\varepsilon|^2}}\, dx = -\int_\Omega u^\varepsilon_t \,\mathrm{div}\left(\frac{Du^\varepsilon}{\sqrt{\varepsilon^2 + |Du^\varepsilon|^2}} \right) dx,$$

and thus,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \sqrt{\varepsilon^2 + |Du^\varepsilon|^2}\, dx &= -\int_\Omega u^\varepsilon_t \left(\frac{u^\varepsilon_t}{\sqrt{\varepsilon^2 + |Du^\varepsilon|^2}} - c(x) \right) dx \\
&= -\int_\Omega \left(\frac{(u^\varepsilon_t)^2}{\sqrt{\varepsilon^2 + |Du^\varepsilon|^2}} - c(x) u^\varepsilon_t \right) dx \\
&\le -\frac{1}{\sqrt{\varepsilon^2 + L^2}} \int_\Omega (u^\varepsilon_t)^2 dx + \int_\Omega c(x) u^\varepsilon_t\, dx.
\end{aligned}$$

Rearranging the terms,

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\int_\Omega \sqrt{\varepsilon^2 + |Du^\varepsilon|^2}\, dx - \int_\Omega c(x) u^\varepsilon\, dx \right) \le -\frac{1}{\sqrt{\varepsilon^2 + L^2}} \int_\Omega (u^\varepsilon_t)^2\, dx.$$

Integrating the inequality above, we have

$$\begin{aligned}\int_0^T\int_\Omega (u_t^\varepsilon)^2\,dxdt &\le \sqrt{\varepsilon^2+L^2}\int_\Omega c(x)(u^\varepsilon(x,T)-u^\varepsilon(x,0))\,dx\\ &+ \sqrt{\varepsilon^2+L^2}\int_\Omega\left(\sqrt{\varepsilon^2+|Du^\varepsilon|^2(x,0)}-\sqrt{\varepsilon^2+|Du^\varepsilon|^2(x,T)}\right)\,dx.\end{aligned}$$

Note that $\|u\|_{L^\infty(\overline{\Omega}\times[0,\infty))}\le\|u_0\|_{L^\infty(\overline{\Omega})}$. Therefore,

$$\limsup_{\varepsilon\to 0}\int_0^T\int_\Omega (u_t^\varepsilon)^2\,dxdt\le C,$$

where C is a constant independent of $\varepsilon\in(0,1)$ and $T>0$. Hence, we get that $u_t^\varepsilon\rightharpoonup u_t$ weakly in $L^2(\overline{\Omega}\times[0,T])$ as $\varepsilon\to 0$ for each $T>0$.

By weakly lower semi-continuity,

$$\int_0^T\int_\Omega (u_t)^2\,dxdt\le\liminf_{\varepsilon\to 0}\int_0^T\int_\Omega (u_t^\varepsilon)^2\,dxdt\le C.$$

Since the constant C is independent of $\varepsilon,\ T$, we see that

$$\int_0^\infty\int_\Omega (u_t)^2\,dxdt\le C. \tag{2.20}$$

For every $\{t_k\}\to\infty$, by the Arzelà-Ascoli theorem, there exist a subsequence $\{t_{k_j}\}$ and a Lipschitz continuous function v such that

$$u_{k_j}(x,t)=u(x,t+t_{k_j})\to v(x,t),$$

locally uniformly on $\overline{\Omega}\times[0,\infty)$. In particular,

$$u_{k_j}(x,t)=u(x,t+t_{k_j})\to v(x,t), \tag{2.21}$$

uniformly on $\overline{\Omega}\times[0,T]$, for every $T>0$. By stability results of viscosity solutions, v

satisfies

$$\begin{cases} v_t = |Dv|\text{div}\left(\frac{Dv}{|Dv|}\right) + c|Dv| & \text{in } \Omega \times (0,\infty), \\ \dfrac{\partial v}{\partial \vec{\mathbf{n}}} = 0 & \text{on } \partial\Omega \times [0,\infty). \end{cases}$$

Thanks to (2.20), we have

$$\int_0^1 \int_\Omega (u_{k_j})_t^2 \, dxdt = \int_{t_{k_j}}^{1+t_{k_j}} \int_\Omega (u_t)^2 \, dxdt \to 0,$$

as $j \to \infty$. This shows that

$$(u_{k_j})_t \rightharpoonup 0,$$

weakly in $L^2(\overline{\Omega} \times [0,1])$ as $j \to \infty$. On the other hand, (2.21) implies that

$$(u_{k_j})_t \rightharpoonup v_t,$$

weakly in $L^2(\overline{\Omega} \times [0,1])$ as $j \to \infty$. Consequently, $v_t = 0$ weakly, and v is constant in t. Thus, v is a solution of (2.6), that is, v solves

$$\begin{cases} |Dv| \,\text{div}\left(\frac{Dv}{|Dv|}\right) + c(x)|Dv| = 0 & \text{in } \Omega, \\ \dfrac{\partial v}{\partial \vec{\mathbf{n}}} = 0 & \text{on } \partial\Omega. \end{cases}$$

Equation (2.6) has many viscosity solutions in general. For example, as v is a solution, $v + C$ is also a solution for any $C \in \mathbb{R}$. Therefore, v may depend on the choice of subsequence $\{t_k\}_k$.

At last, we prove that v is independent of the choice of subsequence $\{t_k\}_k$. Since u_{k_j} converges uniformly to v on $\overline{\Omega} \times [0,1]$, for every $\varepsilon > 0$ there exists j large enough such that

$$|u_{k_j}(x,t) - v(x)| < \varepsilon, \qquad \text{for all } (x,t) \in \overline{\Omega} \times [0,1].$$

In particular, $v(x) - \varepsilon < u_{k_j}(x,0) = u(x,t_{k_j}) < v(x) + \varepsilon$ for all $x \in \overline{\Omega}$. By the comparison principle,

$$v(x) - \varepsilon \le u(x,t) \le v(x) + \varepsilon \quad \text{for } (x,t) \in \overline{\Omega} \times [t_{k_j}, \infty).$$

This implies that $u(\cdot, t)$ converges uniformly to v on $\overline{\Omega}$ without taking a subsequence. $\square$

2.5 The large time profile in the radially symmetric setting

In this section, we study the radially symmetric setting and illustrate some examples of multiplicity of solutions to the stationary problem (2.6). We always assume here (2.7), that is,

$$\begin{cases} \Omega = B(0,R) \text{ for some } R > 0, \\ c(x) = c(r) \text{ for } |x| = r \in [0,R], \\ u_0(x) = u_0(r) \text{ for } |x| = r \in [0,R]. \end{cases}$$

Here, $c \in C^1([0,R],[0,\infty))$, and $u_0 \in C^2([0,R])$ with $u_0'(R) = 0$ are given. In this setting, (2.6) reduces to the following Hamilton-Jacobi equation with Neumann boundary condition

$$\begin{cases} -\frac{n-1}{r}\phi_r - c(r)|\phi_r| = 0, \quad \text{in } (0,R), \\ \phi_r(R) = 0. \end{cases} \tag{2.22}$$

It is worth noting that no boundary condition is needed at $r = 0$, and that the Hamiltonian is concave and maybe noncoercive. Clearly, every constant is a solution to (2.22). Also, if ϕ is a solution to (2.22), then so is $C\phi$ for any given constant $C \ge 0$.

We have the following proposition.

Proposition 2.5.1. *Let* $\mathcal{A} = \left\{ r \in (0,R] \,:\, c(r) = \frac{n-1}{r} \right\}$. *Denote by*

$$r_{\min} = \begin{cases} \min\{r \,:\, r \in \mathcal{A}\} > 0 & \text{if } \mathcal{A} \ne \emptyset, \\ R & \text{if } \mathcal{A} = \emptyset. \end{cases}$$

Let ϕ *be a Lipschitz solution to* (2.22). *Then,* ϕ *is constant on each connected component of* $(0,R) \setminus \operatorname{int}(\mathcal{A})$. *In particular,* ϕ *is constant on* $[0, r_{\min}]$.

Proof. Factoring (2.22) into $\left(-\frac{n-1}{r} \pm c(r)\right)\phi_r(r) = 0$, we see that either $-\frac{n-1}{r} \pm c(r) = 0$ or $\phi_r(r) = 0$ at each point of differentiability of ϕ.

Take $(a, b) \subset ((0, R) \setminus \operatorname{int}(\mathcal{A}))$ for some $a < b$. By the above, we have that $\phi_r(r) = 0$ for a.e. $r \in (a, b)$, and thus, ϕ is constant on $[a, b]$. □

Example 2.5.1 (A toy model)**.** *We consider the case that $c(r)$ is of the form*

$$c(r) = \begin{cases} \frac{n-1}{a}, & 0 \le r < a, \\ \frac{n-1}{r}, & a \le r \le b, \\ \frac{n-1}{b}, & b < r \le R, \end{cases}$$

for some $0 < a < b < R$, then the stationary problem (2.22) admits multiple solutions of the form

$$\phi(r) = \begin{cases} c_1, & 0 \le r \le a, \\ g(r), & a \le r \le b, \\ c_2, & b \le r \le R, \end{cases}$$

where $c_1 \ge c_2$ are constants, $g(r)$ is any nonincreasing function on $[a, b]$ with $g(a) = c_1,\ g(b) = c_2$. Here, the function g can be discontinuous if we extend the definition of viscosity solutions to discontinuous functions (see [51] for instance).

Example 2.5.1 shows further the multiplicity of solutions to (2.22) besides the constant functions noted above. Thus, it is important to address how the large-time limit ϕ_∞ depends on the initial data u_0. In this radially symmetric setting, we are able to characterize the limiting profile and specify its dependence on the initial data.

Equations (2.1)–(2.3) become

$$\begin{cases} \phi_t - \frac{n-1}{r}\phi_r - c(r)|\phi_r| = 0 & \text{in } (0, R) \times (0, \infty), \\ \phi_r(R, t) = 0 & \text{for } t \ge 0, \\ \phi(r, 0) = u_0(r) & \text{for } r \in [0, R]. \end{cases}$$

Here, $u(x, t) = \phi(|x|, t)$ for $(x, t) \in B(0, R) \times [0, \infty)$. Note that this is a first-order

Hamilton-Jacobi equation with a concave Hamiltonian. The associated Lagrangian $L = L(r,q)$ to the Hamiltonian $H(r,p) = -\frac{n-1}{r}p - c(r)|p|$ is

$$\begin{aligned} L(r,q) &= \inf_{p\in\mathbb{R}} \left\{ p\cdot q - \left(-\frac{n-1}{r}p - c(r)|p| \right) \right\} \\ &= \inf_{p\in\mathbb{R}} \left\{ \left(q + \frac{n-1}{r} \right) p + c(r)|p| \right\} \\ &= \begin{cases} 0, & \text{if } \left| q + \frac{n-1}{r} \right| \le c(r), \\ -\infty, & \text{otherwise.} \end{cases} \end{aligned}$$

Therefore, we have the following representation formula for $\phi = \phi(r,t)$

$$\phi(r,t) = \sup \left\{ u_0(\gamma(0)) \,:\, (\gamma, v, l) \in \mathrm{SP}(r,t) \right\},$$

where we denote by $\mathrm{SP}(r,t)$ the Skorokhod problem. For a given $r \in (0,R],\ v \in L^\infty([0,t])$, the Skorokhod problem seeks to find a solution $(\gamma, l) \in \mathrm{Lip}((0,t)) \times L^\infty((0,t))$ such that

$$\begin{cases} \gamma(t) = r, & \gamma([0,t]) \subset (0,R], \\ l(s) \ge 0 & \text{for almost every } s > 0, \\ l(s) = 0 & \text{if } \gamma(s) \ne R, \\ \left| -v(s) + \frac{n-1}{\gamma(s)} \right| \le c(\gamma(s)), \\ v(s) = -\dot\gamma(s) + l(s) n(\gamma(s)), \end{cases}$$

and the set $\mathrm{SP}(r,t)$ collects all the associated triples (γ, v, l). Here, $n(R) = 1$ is the outward normal vector to $(0,R)$ at R. See [62, Theorem 4.2] for the existence of solutions of the Skorokhod problem and [62, Theorem 5.1] for the representation formula. See [46] for a related problem on large time behavior and large time profile.

Example 2.5.2. *Consider Example 2.5.1. To recall, $c(r)$ is defined in the following way*

$$c(r) = \begin{cases} \frac{n-1}{a}, & 0 \le r < a, \\ \frac{n-1}{r}, & a \le r \le b, \\ \frac{n-1}{b}, & b < r \le R. \end{cases}$$

for some $0 < a < b < R$. We analyze the velocity condition $\left|\dot{\gamma}(s) + \frac{n-1}{\gamma(s)}\right| \le c(\gamma(s))$. Note that $c(r)$ is less than $\frac{n-1}{r}$, equal to $\frac{n-1}{r}$, and greater than $\frac{n-1}{r}$ in the written order, respectively. In each case, then, the velocity condition becomes

$$\begin{cases} -\frac{n-1}{a} - \frac{n-1}{\gamma(s)} \le \dot{\gamma}(s) \le \frac{n-1}{a} - \frac{n-1}{\gamma(s)} < 0, & 0 < \gamma(s) < a, \\ -\frac{2(n-1)}{\gamma(s)} \le \dot{\gamma}(s) \le 0, & a \le \gamma(s) \le b, \\ -\frac{n-1}{b} - \frac{n-1}{\gamma(s)} \le \dot{\gamma}(s) \le \frac{n-1}{b} - \frac{n-1}{\gamma(s)}, & b \le \gamma(s) < R. \end{cases}$$

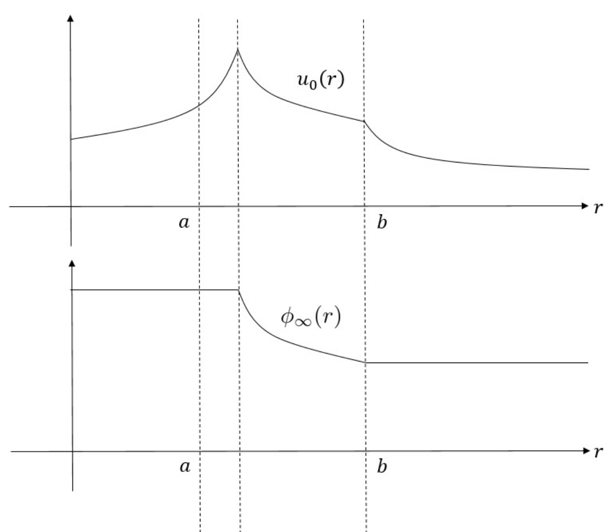

Figure 2.1: Stationary solution of (2.22)

Focusing the right hand side in each case, we see that the point $\gamma(s)$ must move left as time s increases, can stay still, and can go right in the written order, respectively. This point of view in terms of the Lagrangian $L(r, q)$ and Proposition 2.5.1 explain the limit $\phi_\infty(r)$ of $\phi(r, t)$ as $t \to \infty$ in the above illustration of Figure 2.1.

The description in Example 2.5.2 shows how to formulate and write the limit ϕ_∞ in terms of the initial data u_0 in full generality. We note one more thing on the boundary. If $c(h) < \frac{n-1}{h}$ for all $h \in (0, R]$, then the reversed curve $\eta(s) := \gamma(t-s)$ of an admissible curve γ must go right, and it stays on the boundary $r = R$ once it reaches there. This is where the effect of the Skorokhod problem comes in, and it means that the solution $\phi(r,t)$ needs to be understood in the sense of viscosity solutions. We also note that in this setting, we can prove that ϕ is same as the value function of the state constraint problem. Together with this observation on the boundary, analyzing curves $\gamma(s)$ explains how the limit ϕ_∞ depends on the initial data u_0, and indeed the analysis of admissible curves yields the proof of Theorem 2.1.4.

We now give some preparation steps in order to prove Theorem 2.1.4. Let $\eta(s) := \gamma(t-s)$, $s \in [0,t]$, be the reversed curve of a curve $\gamma \in \mathrm{AC}([0,t],(0,R])$ with $(\gamma, v, l) \in \mathrm{SP}(r,t)$. Then, we have the following velocity condition for η

$$-c(\eta(s)) + \frac{n-1}{\eta(s)} \le \dot{\eta}(s) \le c(\eta(s)) + \frac{n-1}{\eta(s)} \quad \text{for a.e. } s \in [0,t] \text{ with } \eta(s) \neq R. \tag{2.23}$$

The following lemma is a direct consequence of the comparison principle.

Lemma 2.5.1. *Let $r_0 \in (0,R)$. Let $\eta_1 \in \mathrm{AC}([0,\infty),(0,R])$ be a curve satisfying*

$$\begin{cases} \dot{\eta}_1(s) = -c(\eta_1(s)) + \frac{n-1}{\eta_1(s)}, & \text{for } s > 0 \text{ provided that } \eta_1(s) < R, \\ \eta_1(0) = r_0. \end{cases}$$

If $\eta_1(s_0) = R$ for some $s_0 > 0$, then we set $\eta_1(s) = R$ for all $s \ge s_0$.

For each $t > 0$, let $\eta \in \mathrm{AC}([0,t],(0,R])$ be the reversed curve given above with $\eta(0) \ge r_0$. Then, $\eta_1(s) \le \eta(s)$ for all $s \in [0,t]$.

Lemma 2.5.2. *Assume the settings of Theorem 2.1.4 and Lemma 2.5.1. Then,*

$$\lim_{s\to\infty} \eta_1(s) = d(r_0). \tag{2.24}$$

Proof. If $r_0 \in \mathcal{A}$, then $\eta_1(s) = r_0$ for all $s \geq 0$, and hence (2.24) holds.

Next, we only need to consider the case that $r_0 \in \mathcal{A}_+$ as the proof of the case that $r_0 \in \mathcal{A}_-$ follows analogously. It is clear that η_1 is decreasing, and by Lemma 2.5.1, $\eta_1(s) \geq d(r_0)$ for all $s \geq 0$. Therefore, $\lim_{s\to\infty} \eta_1(s)$ exists, and

$$\lim_{s\to\infty} \eta_1(s) = r_1 \geq d(r_0).$$

This yields further that

$$\limsup_{s\to\infty} \dot{\eta}_1(s) = 0.$$

Hence,

$$-c(r_1) + \frac{n-1}{r_1} = 0,$$

which implies that $r_1 = d(r_0)$.

□

Proof of Theorem 2.1.4. For $(r_0, t) \in (0, R) \times [0, \infty)$, we have

$$\phi(r_0, t) = \sup\{u_0(\eta(t)) : (\gamma, v, l) \in \mathrm{SP}(r_0, t),\ \eta(s) = \gamma(t-s),\ s \in [0, t]\}.$$

We say that $\eta \in \mathrm{AC}\,([0, t], (0, R])$ is admissible if $\eta(s) = \gamma(t-s),\ s \in [0, t]$ for some γ with $(\gamma, v, l) \in \mathrm{SP}(r_0, t)$. Let η_1 be the curve given in the statement of Lemma 2.5.1. By Lemma 2.5.1, $\eta(s) \geq \eta_1(s)$ for $s \in [0, t]$ for any admissible curve η. From this fact, we see that

$$\phi(r_0, t) \leq \sup\{u_0(r) : r \geq \eta_1(t)\},$$

and therefore, by Lemma 2.5.2,

$$\limsup_{t\to\infty} \phi(r_0, t) \leq \max\{u_0(r) : r \geq d(r_0)\}.$$

In order to complete the proof, it suffices to show the other direction

$$\liminf_{t\to\infty} \phi(r_0,t) \geq \max\{u_0(r) : r \geq d(r_0)\}. \tag{2.25}$$

To show this, let $r_1 \in [d(r_0), R]$ be such that

$$u_0(r_1) = \max\{u_0(r) : r \geq d(r_0)\}.$$

We consider first the case $r_0 \in \mathcal{A}$. Then, $r_1 \geq r_0$. Let η_2 solve

$$\begin{cases} \dot{\eta}_2(s) = c(\eta_2(s)) + \frac{n-1}{\eta_2(s)}, & \text{for } s > 0, \\ \eta_2(0) = r_0. \end{cases}$$

Note that $c(r) + (n-1)/r \geq (n-1)/R > 0$ for all $r \in (0, R]$. Then, there is a unique number $t_2 \geq 0$ such that $\eta_2(t_2) = r_1$. Now, for $t \geq t_2$, let η be defined as

$$\eta(s) = \begin{cases} r_0, & \text{if } s \leq t - t_2, \\ \eta_2(s-(t-t_2)), & \text{if } s \geq t - t_2. \end{cases}$$

Then, η is admissible, and $\phi(r_0,t) \geq u_0(\eta(t)) = u_0(r_1)$. Thus, (2.25) holds.

Next, we consider the case $r_0 \in \mathcal{A}_+$. If $r_1 \geq r_0$, then we repeat the above process to conclude. If $r_1 < r_0$, then $r_1 \in [d(r_0), r_0)$ necessarily, and in this case, we use the curve η_1. We note that if $r_1 > d(r_0)$, then there is a unique number $t_1 \geq 0$ such that $\eta_1(t_1) = r_1$. Now, for $t \geq t_1$, let η be defined as

$$\eta(s) = \begin{cases} r_0, & \text{if } s \leq t - t_1, \\ \eta_1(s-(t-t_1)), & \text{if } s \geq t - t_1. \end{cases}$$

Then, the curve η is admissible, and $\phi(r_0,t) \geq u_0(\eta(t)) = u_0(r_1)$. If $r_1 = d(r_0)$, we take $\eta = \eta_1$ and recall that $\lim_{t\to\infty} \eta_1(t) = d(r_0)$, which gives $\phi(r_0,t) \geq u_0(\eta(t)) \to u_0(r_1)$ as

$t \to \infty$. Therefore, (2.25) holds.

Finally, we study the case $r_0 \in \mathcal{A}_-$. Let η_2, t_2 be defined as above. There exists a unique $t_3 > 0$ such that $\eta_2(t_3) = d(r_0)$. In this case, $r_1 \geq d(r_0)$ and $t_2 \geq t_3$. For $t \geq t_2$, define

$$\eta(s) = \begin{cases} \eta_2(s), & \text{if } 0 \leq s \leq t_3, \\ d(r_0), & \text{if } t_3 \leq s \leq t - (t_2 - t_3), \\ \eta_2(s - (t - t_2)), & \text{if } t - (t_2 - t_3) \leq s \leq t. \end{cases}$$

Then, η is admissible, and $\eta(t) = r_1$, which yields (2.25). □

Next, we prove Corollary 2.1.1, and discuss the sharpness of condition (2.4).

Proof of Corollary 2.1.1. The values of ϕ_∞ are computed directly from Theorem 2.1.4. This tells us the fact that the solution $u = u(r,t)$ is not globally Lipschitz because if it were globally Lipschitz, then the limit ϕ_∞ would be as well. □

Corollary 2.1.1 realizes a jump discontinuity in the limit, which indicates that condition (2.4), which is needed for the globally Lipschitz continuity of u, is almost optimal. As the domain $\Omega = B(0,R)$ is convex, $C_0 \leq 0$, and (2.4) becomes $\frac{1}{n}c(x)^2 - |Dc(x)| - \delta > 0$. Let us now assume that $c(r)$ touches $\frac{n-1}{r}$ from below at a. Then,

$$c(a) = \frac{n-1}{a} \quad \text{and} \quad c'(a) = -\frac{n-1}{a^2}.$$

At $r = a$, we see that

$$\frac{1}{n}c(a)^2 - |c'(a)| = \frac{(n-1)^2}{na^2} - \frac{n-1}{a^2} = -\frac{n-1}{na^2} < 0.$$

Moreover, we see that condition (2.4) is essentially optimal if we seek to find sufficient conditions on the force c that are uniform in dimensions n and in R because the left hand side of the above goes to zero as $a \to \infty$.

2.6 The gradient growth as time tends to infinity in two dimensions

Let $n = 2$. Let the forcing term c be a positive constant in Ω, that is, $c(x) = c$ for all $x \in \overline{\Omega}$ for some $c > 0$. Consider the following nonconvex domain,

$$\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : |x_2| < f(x_1)\}, \tag{2.26}$$

where $f(x) = \frac{m}{2}x^2 + k$ for fixed $m > 0$ and $k > 0$. Here, Ω is unbounded.

In this unbounded setting, let $R_0 > 0$ be a sufficiently large constant. Let $\widetilde{\Omega} \subset \mathbb{R}^n$ be a bounded $C^{2,\theta}$ domain such that

$$\Omega \cap B(0, R_0) \subset \widetilde{\Omega} \subset \Omega.$$

We say that u is a solution (resp., subsolution, supersolution) of (2.1)–(2.3) on $\overline{\Omega} \times [0, \infty)$ if there exists $\alpha \in \mathbb{R}$ such that

$$u - \alpha = u_0 - \alpha = 0 \quad \text{ on } (\overline{\Omega} \setminus B(0, R_0)) \times [0, \infty), \tag{2.27}$$

and u is a solution (resp., subsolution, supersolution) of (2.1)–(2.3) with $\widetilde{\Omega}$ in place of Ω.

Let u be the solution to (2.1)–(2.3). If a level set of u is a smooth curve, then it is evolved by the forced curvature flow equation $V = \kappa + c$, where V is the normal velocity and κ is the curvature in the direction of the normal. Then, the classical Neumann boundary condition becomes the right angle condition for the level-set curves with respect to $\partial\Omega$, that is, if a smooth level curve and $\partial\Omega$ intersect, then their normal vectors are perpendicular at the points of intersections.

We show that if c is too small and fails to satisfy (2.4), then there exist discontinuous viscosity solutions to (2.6). In particular, we find that one such discontinuous solution of (2.6) is stable in the sense that the solution of (2.1)–(2.3) with a suitable choice of initial data converges to this discontinuous stationary solution as time goes to infinity. This implies that the global Lipschitz estimate for the solution of (2.1)–(2.3) does not hold. The following is the main result of this section.

Theorem 2.6.1. *Let* Ω *be the set given by* (2.26)*, and* $c(x) = c$ *for all* $x \in \overline{\Omega}$ *for* $c \in (0, r_{\text{min}}^{-1})$*, where* r_{min} *is defined by* (2.32)*. Let* $u \in C(\overline{\Omega} \times [0,\infty))$ *be the solution of* (2.1)*–*(2.3) *with the given initial data* $u_0 \in C^{2,\theta}(\overline{\Omega})$ *satisfying that* $\frac{\partial u_0}{\partial \vec{\mathbf{n}}} = 0$ *on* $\partial\Omega$ *and there exist constants* l_1, l_2, α *and* β *such that* $l_1 \in (0, a_1)$*,* $l_2 \in (0, a_2 - a_1)$*,* $\alpha < \beta$*,*

$$u_0(x) = \begin{cases} \beta & \text{for } x = (x_1, x_2) \in U(a_1 - l_1), \\ \alpha & \text{for } x = (x_1, x_2) \in \overline{\Omega} \setminus \overline{U(a_1 + l_2)}, \end{cases} \tag{2.28}$$

and $\alpha \le u_0 \le \beta$*, where* $U(a)$ *is defined by* (2.31) *for* $a > 0$*, and* $0 < a_1 < a_2$ *is given in Theorem 2.6.2. Then,*

$$\lim_{t\to\infty} u(x,t) = \begin{cases} \beta & \text{if } x \in U(a_1), \\ \alpha & \text{if } x \in \overline{\Omega} \setminus \overline{U(a_1)}. \end{cases}$$

2.6.1 Set-theoretic stationary solutions

For $a > 0$, consider a family of curves with constant curvature in Ω,

$$X(a,\theta) = (X_1(a,\theta), X_2(a,\theta)) = p(a) + r(a)(\cos\theta, \sin\theta), \quad |\theta| < \arctan(ma), \tag{2.29}$$

where we choose $p(a)$, $r(a)$ so that the curve

$$\Gamma := \{(X_1(a,\theta), X_2(a,\theta)) \,:\, |\theta| < \arctan(ma)\} \cup \{(-X_1(a,\theta), X_2(a,\theta)) \,:\, |\theta| < \arctan(ma)\}$$

has a constant curvature, and is perpendicular to the boundary $\partial\Omega$. Indeed, set

$$p(a) := \left(\frac{a}{2} - \frac{k}{ma}, 0\right).$$

Then, we see that the tangent line for $\{(x_1, x_2) \mid x_2 = f(x_1)\}$ at $x_1 = a$ goes through $p(a)$. Moreover, setting

$$r(a) := \left| \left(a, \frac{ma^2}{2} + k\right) - p(a) \right| = \left(\frac{a}{2} + \frac{k}{ma}\right) \sqrt{m^2a^2 + 1},$$

by elementary geometry, we can check that

$$\Gamma \perp \partial\Omega.$$

The parameter a will be specified so that

$$c = \frac{1}{r(a)}$$

in Lemma 2.6.1.

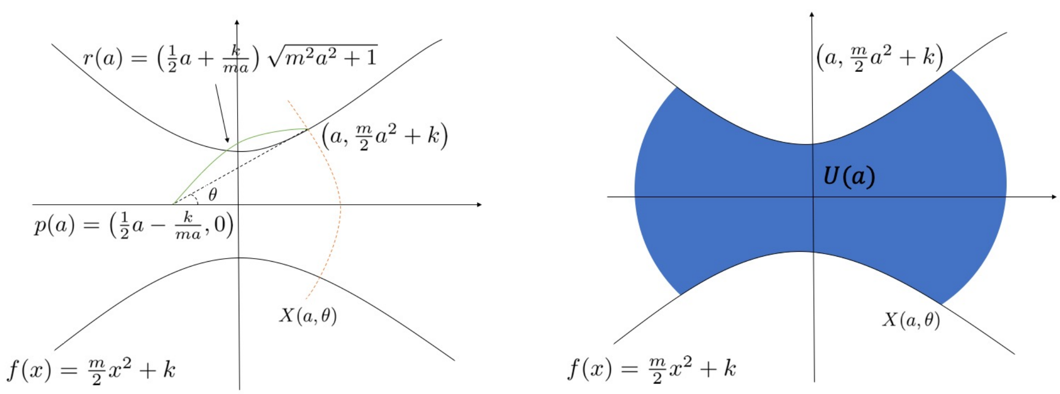

Figure 2.2: Illustrations of (2.29) and (2.31)

The following definition is taken from [51, Definition 5.1.1].

Definition 2.6.1. *Let G be a set in $\mathbb{R}^n \times J$, where J is an open interval in $(0, T)$. We say that G is a set-theoretic subsolution (resp., supersolution) of*

$$V = \kappa + c \quad \text{on } \Gamma_t \quad \text{with} \quad \Gamma_t \perp \partial\Omega \tag{2.30}$$

if χ_G^* *is a viscosity subsolution (resp.,* $(\chi_G)_*$ *is a viscosity supersolution) of* (2.1)–(2.2) *in* $\mathbb{R}^n \times J$*, where* $\chi_G(x,t) = 1$ *if* $(x,t) \in G$*, and* $\chi_G(x,t) = 0$ *if* $(x,t) \notin G$*, and* χ_G^* *and* $(\chi_G)_*$ *denote the upper semicontinuous envelope and the lower semicontinuous envelope of* χ_G*, respectively. If* G *is both a set-theoretic subsolution and supersolution of* (2.30)*,* G *is called a set-theoretic solution of* (2.30)*.*

Set

$$U(a) := \{(x_1, x_2) \in \Omega : |x_1| < X_1(a,\theta), |x_2| < X_2(a,\theta), |\theta| < \arctan(ma)\}, \tag{2.31}$$

and

$$r_{\min} := \inf\{r(a) : a > 0\}. \tag{2.32}$$

Then, $r_{\min}$ is positive since r is a continuous positive function in $(0,\infty)$ and

$$\lim_{a\to 0} r(a) = \lim_{a\to\infty} r(a) = \infty. \tag{2.33}$$

Moreover, by direct computation, we have

$$r'(a) = \frac{1}{\sqrt{m^2a^2+1}}\left(m^2a^2 + \frac{1}{2} - \frac{k}{ma^2}\right).$$

Therefore, r has only one critical point $a_* = \frac{1}{2m}\sqrt{-1+\sqrt{1+16mk}}$ in $(0,\infty)$ and $r_{\min} = r(a_*)$. In addition,

$$r'(a) < 0 \text{ if } a < a_*, \text{ and } r'(a) > 0 \text{ if } a > a_*. \tag{2.34}$$

Lemma 2.6.1. *If* $c = \frac{1}{r(a)}$ *for some* $a > 0$*, then* $U(a)$ *is a set-theoretic stationary solution of* (2.1)–(2.2)*.*

Proof. As a consequence of the nice characterization of set-theoretic solutions in [51, Theorem 5.1.2], $U(a)$ is a set-theoretic stationary solution of (2.30) if and only if $0 = \kappa + c$ on $\partial U(a) \cap \Omega$ and the right angle condition holds. The equality follows from the fact that $\partial U(a) \cap \Omega$ contains two arcs of two circles of the same radius $r(a)$ and curvature $\kappa = -r(a)^{-1} = -c$.

On the other hand, these arcs intersect with $\partial\Omega$ at four points $(a, \pm f(a))$, $(-a, \pm f(a))$. By symmetry, it suffices to prove the right angle condition at $(a, f(a))$. Notice that

$$(a, f(a)) = (X_1(a, \arctan(ma)), X_2(a, \arctan(ma))) = p(a) + \frac{r(a)}{\sqrt{m^2a^2+1}} \cdot (1, ma).$$

Therefore, the line joining $(a, f(a))$ and $p(a)$, the center of the arc, is tangent to $\partial\Omega$ at $(a, f(a))$. Thus, $\partial U(a) \cap \Omega$ satisfies the right angle condition at $(a, f(a))$. $\square$

Theorem 2.6.2. *If $c \in (0, \frac{1}{r_{\min}})$, then there exist two positive constants $a_1 < a_2$ such that $U(a_i)$ is a set-theoretic stationary solution of* (2.30) *for $i = 1, 2$.*

Proof. Thanks to (2.32)–(2.34), there exist two positive constants a_1, a_2 with $a_1 < a_* < a_2$ such that

$$r(a_1) = r(a_2) = \frac{1}{c}. \tag{2.35}$$

By Lemma 2.6.1, $U(a_i)$ is a set-theoretic stationary solution of (2.30) for $i = 1, 2$. $\square$

2.6.2 Stability

Let a_i be the constants given by Theorem 2.6.2 for $i = 1, 2$. In this section, we prove that $U(a_1)$ given by (2.31) is a set-theoretic solution which is stable in the sense of Theorem 2.6.1.

Lemma 2.6.2. *Let $l_1 \in (0, a_1)$, $l_2 \in (0, a_2 - a_1)$ and $\delta > 0$. Set $\underline{a}(t) := a_1 - l_1 e^{-\delta t}$ and $\overline{a}(t) := a_1 + l_2 e^{-\delta t}$. There exists $\delta_0 = \delta_0(m, k, l_1, l_2)$ such that $U(\underline{a}(t))$ and $U(\overline{a}(t))$ are a set-theoretic subsolution and supersolution to* (2.30) *for all $\delta \in (0, \delta_0)$, respectively.*

Proof. We only prove that $U(\underline{a}(t))$ is a set-theoretic subsolution, since we can similarly prove that $U(\overline{a}(t))$ is a set-theoretic supersolution. Let $\tilde{X}(t) := X(\underline{a}(t), \theta)$. From the characterization of set-theoretic solutions in [51, Theorem 5.1.2], it suffices to show that for $t \geq 0$,

$$\frac{d\tilde{X}}{dt} \cdot \vec{\mathbf{n}} \leq -\frac{1}{r(\underline{a}(t))} + c \quad \text{for all } t > 0, \tag{2.36}$$

where $\vec{\mathbf{n}}$ is the outward normal vector $\vec{\mathbf{n}}$ of $U(\underline{a}(t))$, that is, $\vec{\mathbf{n}} = (\cos\theta, \sin\theta)$.

Note that

$$\frac{d\tilde{X}}{dt} \cdot \vec{\mathbf{n}} = \frac{\partial \underline{a}}{\partial t}\frac{\partial X}{\partial a} \cdot \vec{\mathbf{n}} = \delta l_1 e^{-\delta t}\frac{\partial X}{\partial a} \cdot \vec{\mathbf{n}} = \delta(a_1 - \underline{a}(t))\frac{\partial X}{\partial a} \cdot \vec{\mathbf{n}}.$$

Also, for any constant $L > 0$, there exists $C = C(m, k, L) > 0$ such that

$$\begin{aligned} &\frac{\partial X}{\partial a}(a,\theta) \cdot \vec{\mathbf{n}} = p'(a) \cdot \vec{\mathbf{n}} + r'(a) \leq |p'(a)| + r'(a) \\ =\ &\frac{1}{2} + \frac{m^2a^2 + \frac{1}{2}}{\sqrt{m^2a^2+1}} + \frac{mk}{m^2a^2 + 1 + \sqrt{m^2a^2+1}} \leq C \end{aligned}$$

for all $a \in (0, L)$ and $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Therefore,

$$\frac{d\tilde{X}}{dt} \cdot \vec{\mathbf{n}} = \delta(a_1 - \underline{a}(t))\frac{\partial X}{\partial a} \cdot \vec{\mathbf{n}} \leq C\delta(a_1 - \underline{a}(t)).$$

The observation (2.34) implies that $r(\underline{a}(t)) > r(a_1) = c^{-1}$ for all $t \geq 0$, and thus we get

$$\left(\frac{d\tilde{X}}{dt} \cdot \vec{\mathbf{n}}\right)\left(-\frac{1}{r(\underline{a}(t))} + c\right)^{-1} \leq \delta C \frac{a_1 - \underline{a}(t)}{\frac{1}{r(a_1)} - \frac{1}{r(\underline{a}(t))}}.$$

Thus, (2.36) holds for $\delta \in (0, \delta_0)$, where

$$\delta_0 := \left(C \sup_{a \in [a_1 - l_1, a_1 + l_2]} h(a)\right)^{-1}.$$

Here the function $h : [a_1 - l_1, a_1 + l_2] \to \mathbb{R}$ is given by

$$h(a) := \begin{cases} \dfrac{a_1 - a}{\frac{1}{r(a_1)} - \frac{1}{r(a)}} & \text{for } a \in [a_1 - l_1, a_1 + l_2] \setminus \{a_1\}, \\ \\ \dfrac{-r^2(a_1)}{r'(a_1)} & \text{for } a = a_1. \end{cases}$$

Since $a_1 + l_2 < a_2$, by (2.34) we have $r(a) \neq r(a_1)$ in $[a_1 - l_1, a_1 + l_2] \setminus \{a_1\}$ and $r'(a_1) < 0$. Therefore, h is well-defined and continuous in $[a_1 - l_1, a_1 + l_2]$. Thus, h is bounded in $[a_1 - l_1, a_1 + l_2]$, and hence, $\delta_0 > 0$ is well-defined, which implies that (2.36) holds for all $\delta \in (0, \delta_0)$. □

Proof of Theorem 2.6.1. We let $\alpha = 0$ and $\beta = 1$ for simplicity. Set

$$\underline{u}(x,t) := \chi_{\overline{U(\underline{a}(t))}}(x) \quad \text{and} \quad \overline{u}(x,t) := \chi_{U(\overline{a}(t))}(x)$$

for $(x,t) \in \overline{\Omega} \times [0, \infty)$, where $\underline{a}$ and $\overline{a}$ are the functions defined in Lemma 2.6.2. By Lemma 2.6.2, we see that $\underline{u}$ and $\overline{u}$ are a subsolution and a supersolution of (2.1)–(2.2), respectively. Due to (2.28), we get

$$\underline{u}(\cdot, 0) = \chi_{\overline{U(\underline{a}(0))}} \leq u_0 \leq \chi_{U(\overline{a}(0))} = \overline{u}(\cdot, 0) \text{ on } \overline{\Omega}.$$

In addition, since

$$U(a) \subset V(a) := [-(|p(a)| + r(a)), |p(a)| + r(a)] \times [-f(a), f(a)]$$

by construction for $p(a)$ and $r(a)$ given in (2.29) and $f(a) = \frac{m}{2}a^2 + k$, we obtain

$$\mathrm{supp}(\underline{u}) \subset \bigcup_{a \in [a_1 - l_1, a_1]} V(a) \times [0, \infty) \text{ and } \mathrm{supp}(\overline{u}) \subset \bigcup_{a \in [a_1, a_1 + l_2]} V(a) \times [0, \infty).$$

As $|p(\cdot)| + r(\cdot)$ and f are continuous on $[a_1 - l_1, a_1 + l_2]$, there exists a constant $R_0 > 0$

satisfying (2.27).

By the comparison principle for (2.1)–(2.3), Proposition 2.2.1, we get

$$\underline{u}(\cdot,t) \leq u(\cdot,t) \leq \overline{u}(\cdot,t) \quad \text{on } \overline{\Omega} \quad \text{for all } t > 0.$$

On the other hand, since both $a_1 - l_1 e^{-\delta t}$ and $a_1 + l_2 e^{-\delta t}$ converge to a_1 as t goes to infinity,

$$\lim_{t\to\infty} \underline{u}(x,t) = \lim_{t\to\infty} \overline{u}(x,t) = 1 \quad \text{for } x \in U(a_1),$$

and

$$\lim_{t\to\infty} \underline{u}(x,t) = \lim_{t\to\infty} \overline{u}(x,t) = 0 \quad \text{for } x \in \overline{\Omega} \setminus \overline{U(a_1)},$$

which finish the proof. $\square$

Chapter 3

Capillary-type boundary value problems of mean curvature flows with force and transport terms on a bounded domain

3.1 Introduction

In this chapter, we study the following two problems

$$\begin{cases} u_t = \sqrt{1+|Du|^2}\,\mathrm{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) + c(x,u)\sqrt{1+|Du|^2} - f(x,u) & \text{in } \Omega\times(0,T), \\ \dfrac{\partial u}{\partial \vec{\mathbf{n}}} = \phi(x)(\sqrt{1+|Du|^2})^{1-q} & \text{on } \partial\Omega\times[0,T), \\ u(x,0) = u_0(x) & \text{on } \overline{\Omega}, \end{cases} \tag{3.1}$$

and

$$\begin{cases} u_t = |Du|\,\mathrm{div}\left(\frac{Du}{|Du|}\right) + c(x,u)|Du| - f(x,u) & \text{in } \Omega\times(0,T), \\ \dfrac{\partial u}{\partial \vec{\mathbf{n}}} = \phi(x) & \text{on } \partial\Omega\times[0,T), \\ u(x,0) = u_0(x) & \text{on } \overline{\Omega}, \end{cases} \tag{3.2}$$

where $q > 0$ in (3.1) is a fixed positive number, and $T > 0$ denotes values in $(0,\infty]$. Solutions of (3.2) are understood in the viscosity sense. A forcing term $c = c(x,z)$ and a transport term $f = f(x,z)$ depend on the spatial position $x \in \overline{\Omega}$ and the value $z \in \mathbb{R}$, and they are functions in $C^{1,\alpha}(\overline{\Omega}\times\mathbb{R})$ for a fixed $\alpha\in(0,1)$. The functions c and f of $(x,z)\in\overline{\Omega}\times\mathbb{R}$ are assumed, throughout this chapter, to be $C^{1,\alpha}$ functions and to satisfy, for some constant C,

$$|c|\le C,\quad |D_x c|\le C,\quad c_z\le 0, \tag{3.3}$$

and

$$|f|\le C,\quad |D_x f|\le C,\quad f_z\ge 0, \tag{3.4}$$

for all arguments $(x,z)\in\overline{\Omega}\times\mathbb{R}$. The vector $\vec{\mathbf{n}}$ denotes the outward unit normal vector to $\partial\Omega$, and $\phi=\phi(x)\in C^3(\overline{\Omega})$. Throughout this chapter, we assume that the domain $\Omega\subset\mathbb{R}^n$ is bounded and C^3-regular. We also assume that $u_0\in C^{2,\alpha}(\overline{\Omega})$ with the same $\alpha\in(0,1)$ as above, and we say the initial condition u_0 is compatible with the boundary condition if

$$\frac{\partial u_0}{\partial\vec{\mathbf{n}}}=\phi(x)(\sqrt{1+|Du_0|^2})^{1-q} \text{ on } \partial\Omega$$

in (3.1) and

$$\frac{\partial u_0}{\partial\vec{\mathbf{n}}}=\phi(x) \text{ on } \partial\Omega$$

in (3.2), and we always assume the compatibility in this chapter. Next, we consider the following forced mean curvature equations

$$\begin{cases} -\sum_{i,j=1}^n\left(\delta^{ij}-\frac{w_iw_j}{1+|Dw|^2}\right)w_{ij}-c(x)\sqrt{1+|Dw|^2}+f(x)=-\lambda & \text{in } \Omega,\\ \dfrac{\partial w}{\partial\vec{\mathbf{n}}}=\phi(x)(\sqrt{1+|Dw|^2})^{1-q} & \text{on } \partial\Omega \end{cases} \tag{3.5}$$

with general capillary-type boundary conditions and

$$\begin{cases} -\sum_{i,j=1}^n\left(\delta^{ij}-\frac{w_iw_j}{|Dw|^2}\right)w_{ij}-c(x)|Dw|+f(x)=-\lambda & \text{in } \Omega,\\ \dfrac{\partial w}{\partial\vec{\mathbf{n}}}=\phi(x) & \text{on } \partial\Omega, \end{cases} \tag{3.6}$$

with Neumann boundary conditions. Here, $u_i = u_{x_i},\ u_{ij} = u_{x_i x_j}$ (and the same for w) denote the partial derivatives of u in x_i, x_i and x_j in order, respectively. The term δ^{ij} is the (i,j)-entry of the n by n identity matrix for $i,j = 1,\cdots,n$. Equation (3.6) is understood in the viscosity sense. Equations (3.5) and (3.6) correspond to (3.1) and (3.2), respectively. λ is a real number, and it is called an eigenvalue. The stationary problems (3.5) and (3.6) are also considered as additive eigenvalue problems.

The four equations above, (3.1), (3.2), (3.5) and (3.6), will be studied by obtaining *a priori* C^1 estimates for

$$\begin{cases} u_t = \sqrt{\eta^2+|Du|^2}\,\mathrm{div}\left(\frac{Du}{\sqrt{\eta^2+|Du|^2}}\right) + c(x,u)\sqrt{\eta^2+|Du|^2} - f(x,u) & \text{in } \Omega\times(0,T), \\ \dfrac{\partial u}{\partial \vec{\mathbf{n}}} = \phi(x)v^{1-q} & \text{on } \partial\Omega\times[0,T), \\ u(x,0) = u_0(x) & \text{on } \overline{\Omega}, \end{cases} \tag{3.7}$$

and *a priori* C^0, C^1 estimates for

$$\begin{cases} -\sum_{i,j=1}^n \left(\delta^{ij} - \frac{u_i u_j}{\eta^2+|Du|^2}\right) u_{ij} - c(x)\sqrt{\eta^2+|Du|^2} + f(x) = -ku & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \vec{\mathbf{n}}} = \phi(x)v^{1-q} & \text{on } \partial\Omega \end{cases} \tag{3.8}$$

where $v = \sqrt{\eta^2+|Du|^2}$ and $k > 0$. The choices $\eta = 1,\ q > 0$ and $\eta = 0,\ q = 1$ in (3.7) yield (3.1) and (3.2), respectively. The same choices in (3.8) correspond to (3.5) and (3.6), respectively after letting $k \to 0$. In the choice of $\eta = 0,\ q = 1$, we first take $\eta \in (0,1]$, and then we let $\eta \to 0$, considered as a vanishing viscosity parameter. Whenever we discuss the vanishing viscosity parameter $\eta \in (0,1]$, especially obtaining estimates uniform in $\eta \in (0,1]$, we refer to the case $q = 1$.

We note that if $q = 0$, (3.1) is the capillary problem, and (3.2) is the capillary problem formulated as the level-set equation. If $q = 1$, (3.1) and (3.2) are Neumann boundary value problems. We investigate the well-posedness and the large time behavior of the forced mean curvature flow on a C^3 bounded domain with general capillary-type boundary conditions, i.e., $q > 0$.

The novelty of this chapter is threefold; first of all, the multiplier method in [66] can be combined with the method in [101] in order to get *a priori* gradient estimates of (3.7) uniform in $\eta \in (0,1]$. The combination of the methods allows us to handle the difficulties coming from the nonconvexity of Ω, a forcing term c, a transport term f, a nonzero boundary condition with $\phi \not\equiv 0$ at the same time. By using the two methods simultaneously, we get a uniform *a priori* gradient estimate, and therefore, we get quite general results. This is the main contribution of this chapter. In the gradient estimate, we derive a sufficient condition on a forcing term c to ensure the global Lipschitz regularity, which we call the *coercivity* assumption on c. Second of all, we keep the force term c coercive during the interpolation, while we apply the Leray-Schauder fixed point theorem, so that a uniform gradient estimate is maintained. This extra care on the force c is a new step, not arising in [101], and it is necessary and natural since we observe that the coercivity condition is crucial to study the large time behavior. We accordingly are able to study the mean curvature equations (3.5) and (3.6). Finally, by adopting the approaches in [66, 46], we discuss the optimality of the coercive condition on c, and compute the eigenvalue, the large time profile based on the optimal control formula in the radially symmetric setting of (3.2). We also give a dynamics proof in order to deal with the boundary, which does not appear in [46], when we study the asymptotic behavior.

The multiplier method in [66] has been considered new and devised only recently, and it successfully treats the homogeneous Neumann boundary condition. The method is natural, and it explains how the geometry of $\partial\Omega$ affects gradient estimates, which turn out to be sharp. This chapter presents as a new contribution that the multiplier method can be generalized to deal with general capillary-type boundary conditions by combining with the method that has been established in [101]. The result is general because (3.1) and (3.2) cover a wide range of equations on a general bounded domain. The process of combining is linear and natural, which justifies that each of the methods is natural. Moreover, the multiplier method highlights the coercivity assumption on the force c with the right angle condition. Another observation of this chapter is that we can study the additive eigenvalue problem with this coercivity condition.

We first discuss the literature, which is not an exhaustive list at all, on the capillary problem and the Neumann boundary value problem of mean curvature flows in Subsection 3.1.1. Next, we provide the main results in Subsection 3.1.2, and we outline the approaches of this chapter in Subsection 3.1.3.

3.1.1 Literature

The capillary problem has been an important subject for decades because of motivations and applications in physics, such as wetting phenomena [18, 42], behaviors of droplets [2, 15, 30, 93]. It also has been investigated with emphasis on obtaining gradient estimates. For instance, [100, 95, 43, 75] study gradient estimates of the mean curvature equation with test function technique. In 1975, the maximum principle was first used to get gradient estimates [96], and [72, 75] are based on the maximum principle. Paper [75] also deals with boundary conditions $q = 0$ and $q > 1$, and in these cases, boundary gradient estimates have been shown [102] recently with a new proof using the maximum principle. The results when $0 < q < 1$ have been obtained in [101]. For the mean curvature flow, the well-posedness and the large time behavior of solutions has been studied in [3, 54]. In particular, [3] deals with the case when $q = 0$ in the dimension $n = 2$, and the questions about the well-posedness and the large time behavior in higher dimensions are still open. The vertical capillary problem, i.e., when $\phi(x) = 0$ and thus when the problem is also the homogeneous Neumann boundary problem, has been investigated [58].

The mean curvature flow with Neumann boundary conditions has been of significance on its own. Paper [4] investigates the mean curvature equation with the homogeneous Neumann condition on a convex domain in the graph case. Recently, the mean curvature flow with general Neumann boundary conditions has been studied [103], and a uniform gradient estimate has been obtained for Neumann boundary conditions on a strictly convex domain [80]. Also, [86] studies gradient estimates with Neumann boundary conditions.

The level-set formulation of the mean curvature flow with the homogeneous Neumann boundary condition, understood in the viscosity sense, has been studied [47] on a smoothly bounded convex domain, based on the maximum principle. Paper [47] also contains an illustration where we lose a global gradient estimate on a nonconvex domain. Note that the illustration justifies the necessity of a nonzero force term in order to have a global gradient estimate on a nonconvex domain. In this context, the results on the forced mean curvature flow with the right angle condition have been obtained [66] recently, which explains the effect of the constraints by the forcing term and by the geometry of the boundary. However, there are no results on the forced mean curvature flow and the forced mean curvature equation with more general boundary conditions on a general bounded domain, for neither the graph case nor the level-set case.

In the context of the above, the main goal of this chapter is to study the well-posedness and the large time behavior of solutions of capillary-type boundary value problems, i.e., $q > 0$, of the mean curvature flow with a forcing term and a transport term for the graph case, and to study Neumann boundary problems, $q = 1$, for the level-set case, on a bounded domain with C^3 boundary, which is not necessarily convex. It generalizes [101] to capillary-type boundary value problems on a nonconvex domain with a force, and generalizes [66] to nonzero Neumann boundary value problems with a transport term.

3.1.2 Main results

We first list the main results of this chapter, and then discuss the main difficulties and the approaches to overcome.

We start with a local gradient estimate.

Theorem 3.1.1. *Let Ω be a C^3 bounded domain in $\mathbb{R}^n$, $n \geq 2$. Suppose that c and f satisfy* (3.3) *and* (3.4)*. Then, for each $T \in (0, \infty)$, there exists a unique solution $u \in C^{2,\sigma}(\overline{\Omega} \times [0, T]) \cap C^{3,\sigma}(\Omega \times (0, T])$ of* (3.1) *for some $\sigma \in (0, 1)$, and there exists a unique viscosity solution u of* (3.2)*. For both* (3.1) *and* (3.2)*, moreover, there exists a constant $M > 0$ such that and for each $T \in (0, \infty)$, there exists a constant $R_T > 0$*

depending only on T, Ω, c, f, ϕ, q, u_0 such that

$$\begin{cases} |u(x,t) - u(x,s)| \leq M|t-s|, \\ |u(x,t) - u(y,t)| \leq R_T|x-y|, \end{cases} \qquad \textit{for all } x, y \in \overline{\Omega},\ t, s \in [0,T].$$

For each $x \in \mathbb{R}^n,\ r > 0$, we let $B(x,r)$ denote the open ball centered at x with a radius r. We recall that for $y \in \partial\Omega$, $\vec{\mathbf{n}}(y)$ is defined to be the outward unit normal vector to $\partial\Omega$ at y. For each $y \in \partial\Omega$, we define the number $K_0(y)$ by

$$K_0(y) = \sup\{r > 0 : B(y - r\vec{\mathbf{n}}(y), r) \subseteq \Omega\}.$$

Note that the domain Ω satisfies the uniform interior ball condition since Ω is a C^3 bounded domain. Therefore, there exists a number $\hat{r} > 0$ such that $B(y - \hat{r}\vec{\mathbf{n}}(y), \hat{r}) \subseteq \Omega$ for all $y \in \partial\Omega$, which implies $K_0(y) \geq \hat{r}$ for all $y \in \partial\Omega$. We also note that for each $y \in \partial\Omega$, $B(y - K_0(y)\vec{\mathbf{n}}(y), K_0(y)) \subseteq \Omega$, and $B(y - (K_0(y)+\varepsilon)\vec{\mathbf{n}}(y), K_0(y)+\varepsilon) \nsubseteq \Omega$ for any $\varepsilon > 0$.

For each $y \in \partial\Omega$, we define the number $C_0(y)$ by

$$C_0(y) = \max\{\lambda : \lambda \text{ is an eigenvalue of } -\kappa\},$$

where $\kappa := \left(\kappa^{\ell j}\right)_{\ell,j=1}^{n-1}$ is the curvature matrix of $\partial\Omega$ at y.

Next we show that a solution u is globally Lipschitz under further conditions on the forcing term c.

Theorem 3.1.2. *Let Ω be a C^3 bounded domain in $\mathbb{R}^n,\ n \geq 2$. Let*

$$\begin{cases} C_0 = \sup\{C_0(y) : y \in \partial\Omega\}, \\ K_0 = \inf\{K_0(y) : y \in \partial\Omega\}. \end{cases}$$

Suppose that c and f satisfy (3.3) *and* (3.4)*. Suppose that there exists $\delta > 0$ such that*

$$\frac{1}{n-1}c(x,z)^2 - |Dc(x,z)| - \delta > \max\left\{0,\ C_0|c(x,z)| + \frac{(n-1)C_0}{K_0} + (1+q)\mathrm{sgn}(C_0)C_0^2\right\} \tag{3.9}$$

for all $(x,z) \in \overline{\Omega}\times\mathbb{R}$, where $\mathrm{sgn}(C_0)$ *is the sign of the real number C_0. Let $T \in (0,\infty)$, and let $u \in C^{2,\sigma}(\overline{\Omega}\times[0,T]) \cap C^{3,\sigma}(\Omega\times(0,T])$ be the unique solution of* (3.1)*, $\sigma \in (0,1)$, and with abuse of notations, let u be the unique viscosity solution u of* (3.2)*. In both cases, there exist constants $M, L > 0$, depending only on Ω, c, f, ϕ, q, u_0 such that*

$$\begin{cases} |u(x,t) - u(x,s)| \le M|t-s|, \\ |u(x,t) - u(y,t)| \le L|x-y|, \end{cases} \qquad \text{for all } x,y \in \overline{\Omega},\ t,s \in [0,T].$$

We can relax the conditions (3.3) and (3.4) quite a bit if we have *a priori* C^0 estimate on u. For instance, $\widetilde{f}(x,z) = f(x) + kz$, $k > 0$, is not bounded as z runs over $\mathbb{R}$. However, if we know that a solution u is bounded *a priori*, then $\widetilde{f}(x,u) = f(x) + ku$ is bounded as well. Therefore, once we get *a priori* C^0 estimate on u, we can drop the assumptions $|c| \le C,\ |f| \le C$ in (3.3), (3.4), respectively, for Theorem 3.1.1 and Theorem 3.1.2.

The condition (3.9) serves as a coercivity assumption, which appears in the classical Bernstein method. In this sense, we sometimes call the forcing term c *coercive* if c satisfies (3.9). One more remark is that the coercivity condition (3.9) is an open condition, in the sense that it remains true even if we perturb the force c a little bit.

When the domain Ω is convex so that $C_0 \le 0$, the condition (3.9) is equivalent to taking only zero on the right hand of (3.9) into account. On the other hand, if the domain Ω is nonconvex so that $C_0 > 0$, the condition (3.9) considers only $C_0|c(x,z)| + \frac{(n-1)C_0}{K_0} + (1+q)\mathrm{sgn}(C_0)C_0^2$, and moreover, this condition is stronger than the convex case. In other words, we require a stronger coercivity condition on the force to deal with the nonconvex boundary $\partial\Omega$. We may refer to the example on a nonconvex domain suggested in [66, Section 6].

The condition (3.9) is slightly better than the one given in [66, Theorem 1.2] in the case when $\phi \equiv 0$ so that the boundary condition is the homogeneous Neumann boundary condition, or the right angle condition equivalently. More precisely, when $\phi \equiv 0$, one can see easily that the condition (3.9) with $q = 0$ follows from the condition in [66, Theorem 1.2]. Thus, the condition in [66, Theorem 1.2] is assuming more. We also note that the condition (3.9) with $q = 0$ works as a sufficient condition by following the proof of Theorem 3.1.2.

We note that C_0 measures the curvature on the boundary $\partial\Omega$, and K_0 measures the width of the domain Ω with inscribed balls. The appearance of the fraction $\frac{C_0}{K_0}$ in (3.9) reflects the battle of the two constraints, namely, from the normal velocity $V = k_1 + c$ and from the boundary condition $\frac{\partial u}{\partial \vec{\mathbf{n}}} = \phi(x)v^{1-q}$, where k_1 is $(n-1)$ times of the mean curvature of a level-set of u.

We also note that if Ω is strictly convex, then $C_0 < 0$ so that $C_0|c(x,z)| + \frac{(n-1)C_0}{K_0} - 2(1+q)C_0^2 < 0$. This implies that there is a room for improvement of estimates if Ω is strictly convex, and indeed it turns out that we can recover a global gradient estimate if $c(x,z) \equiv 0$. We state the following corollary for $c \equiv 0$, which is [101, Theorem 1.1] for (3.1), together with the corresponding conclusion for (3.2).

Corollary 3.1.1. *Let Ω be a strictly convex C^3 bounded domain in $\mathbb{R}^n$, $n \geq 2$. Let $c \equiv 0$. Suppose that the term f satisfies* (3.4). *Then, for each $T \in (0,\infty)$, there exists a unique solution $u \in C^{2,\sigma}(\overline{\Omega} \times [0,T]) \cap C^{3,\sigma}(\Omega \times (0,T])$ of* (3.1) *for some $\sigma \in (0,1)$, and there exists a unique viscosity solution u of* (3.2)*, with abuse of notations. In both cases, moreover, there exist constants $M, L > 0$ depending only on Ω, c, f, ϕ, q, u_0 such that $|u(x,t) - u(x,s)| \leq M|t-s|$, $|u(x,t) - u(y,t)| \leq L|x-y|$ for all $x, y \in \overline{\Omega}$, $t, s \in [0,T]$.*

As we have obtained gradient estimates, we next study the additive eigenvalue problems (3.5) and (3.6) under the assumption (3.9) on the forcing term c. In the additive eigenvalue problems, we will consider the terms $c = c(x)$ and $f = f(x)$ that depend only on $x \in \overline{\Omega}$. That being said, the z-dependence in the estimates obtained so far plays a role in the additive eigenvalue problems.

Before we introduce the next results, we explain how the additive eigenvalue problem is approached briefly. First of all, we get uniform C^0 *a priori* estimates of $|ku|$ in (3.8) by the maximum principle. Then, we establish uniform C^1 *a priori* estimates of (3.8). Applying Leray-Schauder fixed point theorem (see [74]), we get the existence of solutions of (3.8). Finding a pair of an eigenvalue and an eigenfunction of (3.5) and (3.6) is called additive eigenvalue problems, which have been extensively studied. The problems naturally appear in ergodic optimal control theory, in the homogenization of Hamilton-Jacobi equations, in the large time behavior of the Cauchy problem of Hamilton-Jacobi equations and in weak KAM theory. See [8, 39, 98, 77] and the references therein. We also leave the references [29, 39, 38, 62] for the *Aubry* set, as it is treated separately as an important set in this chapter.

Theorem 3.1.3. *Let Ω be a C^∞ bounded domain in $\mathbb{R}^n$, $n \geq 2$, and let $q > 0$. Suppose that $c = c(x)$ satisfies* (3.9)*. For $\phi \in C^\infty(\overline{\Omega})$, there exists a unique $\lambda \in \mathbb{R}$ such that there exists a solution $u \in C^\infty(\overline{\Omega})$ of* (3.5)*. Moreover, a solution u is unique upto an additive constant.*

Moreover, we get the following result on the large time behavior of solutions of (3.1) by following the argument in [80, 101, 94].

Theorem 3.1.4. *Let Ω be a C^∞ bounded domain in $\mathbb{R}^n$, $n \geq 2$, and let $q > 0$. Suppose that $c, f, \phi \in C^\infty(\overline{\Omega})$, and that c satisfies* (3.9)*. Let u^i, $i = 1, 2$, be the solution of*

$$\begin{cases} u_t = \sqrt{1+|Du|^2}\,\mathrm{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) + c(x)\sqrt{1+|Du|^2} - f(x) & \text{in } \Omega \times (0,\infty), \\ \dfrac{\partial u}{\partial \vec{\mathbf{n}}} = \phi(x)(\sqrt{1+|Du|^2})^{1-q} & \text{on } \partial\Omega \times [0,\infty), \\ u(x,0) = u_0^i(x) & \text{on } \overline{\Omega}, \end{cases} \tag{3.10}$$

with initial data u_0^i compatible with the boundary condition, respectively for $i = 1, 2$. Then $\lim_{t\to\infty} |u^1 - u^2|_{C^\infty(\overline{\Omega})} = 0$. In particular, for the solution u of (3.1) *and the solution (λ, w) of* (3.5)*, it holds that $\lim_{t\to\infty} |u(x,t) - \lambda t - w(x)|_{C^\infty(\overline{\Omega})} = 0$.*

We also study the large time behavior of solutions of (3.2). We go though the same procedure as we do in Theorem 3.1.3. During the limit process in which we send k to 0, the gradient estimates remain uniform in the viscosity parameter $\eta \in (0,1]$, which allows us to find a viscosity solution of the stationary problem (3.6).

Theorem 3.1.5. *Let Ω be a C^∞ bounded domain in $\mathbb{R}^n$, $n \geq 2$. Suppose that c satisfies* (3.9). *For $\phi \in C^\infty(\overline{\Omega})$, there exists a unique $\lambda \in \mathbb{R}$ such that there exists a viscosity solution w of* (3.6). *Moreover, $\lambda = \lim_{t\to\infty} \frac{u(x,t)}{t}$ and the convergence as $t \to \infty$ is uniform in $x \in \overline{\Omega}$, where u is the unique viscosity solution of* (3.2) *with $T = \infty$.*

The questions on classifying viscosity solutions w of (3.6), and on whether or not $u(x,t) - \lambda t$ converges to a stationary solution w as $t \to \infty$ are challenging, and they are still widely open. For partial resolutions, we refer to [47, 66], where a Lyapunov function is used.

In the radially symmetric setting, we can prove the convergence of $u(x,t) - \lambda t$ to a stationary solution w as $t \to \infty$. Moreover, we are able to compute the eigenvalue λ and the large time profile w of the solution u based on the optimal control formula. We will see in Chapter 3.4 that the curves $c(r)$ and $\frac{n-1}{r}$ meet at most one point on $[0,R]$ because of the coercivity assumption (3.9) on c. This fact allows us to follow the argument in [46] overall, with the dynamics suggested in [62], called the Skorokhod problem.

We also note that the eigenvalue $\lambda = \lim_{t\to\infty} \frac{u(x,t)}{t}$ is constant in $x \in \overline{\Omega}$, but this is under the condition (3.9). We will find an example in the radially symmetric setting, where the limit $\lim_{t\to\infty} \frac{u(x,t)}{t}$ is not constant, which thus disobeys (3.9). It turns out this example demonstrates that the condition (3.9) is optimal, which we will discuss in Section 3.4.

Theorem 3.1.6. *Assume the radially symmetric setting* (3.66). *Assume* (3.9). *Let $u = u(r,t)$ be the unique radial viscosity solution of* (3.2), *and let (λ, w) be a pair of a real number and a Lipschitz continuous function satisfying* (3.6) *in the sense of viscosity solutions. Then,*

(i) $u(r,t) - \lambda t \to w(r)$ as $t \to \infty$ uniformly in $r \in [0,R]$, and

(ii) the asymptotic speed λ and the asymptotic profile w are described as follows; if the curves $r \mapsto c(r)$ and $r \mapsto \frac{n-1}{r}$ cross at $r \in [0,R]$, then such numbers r are unique, which

we call r_{cr}. If the curves do not cross on the interval $[0, R]$, we let $r_{cr} := \infty$. Then,

$$\lambda = \sup\left\{-f(r) + \delta(r-R)\phi(R)\left(\frac{n-1}{R} + \operatorname{sgn}(\phi(R))c(R)\right) :\ r \geq r_{cr}\ \textit{or}\ r = R\right\}, \tag{3.11}$$

where δ is the function on $\mathbb{R}$ having its value 1 at the origin, 0 elsewhere, and the asymptotic profile w is given by

$$w(r) = \max\left\{d(r,s) + w_0(s) :\ s \in \widetilde{\mathcal{A}}\right\}. \tag{3.12}$$

Here,

$$d(r_0, r_1) := \sup\left\{\int_0^t -f(\eta(s)) - \phi(\eta(s))l(s)ds :\ t \geq 0,\ (\eta, l) \in \mathcal{C}(0, t; r_0, r_1)\right\} \tag{3.13}$$

for any $r_0, r_1 \in [0, R]$, where we set

$$\mathcal{C}(0, t; r_0, r_1) := \{(\eta, l) \in \mathrm{AC}([0,t]; (0,R]) \times L^\infty([0,t]) : \\ \eta(0) = r_0,\ \eta(t) = r_1,\ (\eta, v, l) \in \mathrm{SP}(r_0)\},$$

and $\mathrm{SP}(r)$ denotes the Skorokhod problem, and

$$w_0(r) := \max\{d(r,\rho) + u_0(\rho) :\ \rho \in [0, R]\},$$

$$\widetilde{\mathcal{A}} := \{r \geq r_{cr} :\ \textit{the supremum of (3.11) is attained}\} \qquad \textit{if } r_{cr} < \infty.$$

If $r_{cr} = \infty$, we let $\widetilde{\mathcal{A}} := \{R\}$.

3.1.3 Discussions and our main ideas

In the following, we first discuss the necessity of a nonzero force in order to get a global gradient estimate and its geometric interpretation. Next, we outline the approaches taken to obtain the results of this chapter.

We start with the special case of (3.2) when $c(x, z) \equiv 0$, $f(x, z) \equiv 0$, $\phi(x) \equiv 0$, which corresponds to the homogeneous Neumann boundary problem with zero force. Paper [47] obtains a global gradient estimate for the problem on a convex domain, and additionally, [47] describes an example, which is constructed rigorously in [90] as well, on a nonconvex domain where the global gradient estimate fails. In this context, [66] provides the computation realizing the description, which means we need a nonzero force on a nonconvex domain to get a global estimate. Also, [66] studies the problem with a nonzero force $c = c(x)$, and it generally investigates the competition between the two geometric constraints, one from the normal velocity $V = k_1 + c$ where k_1 is $(n-1)$ times of the mean curvature, the other from the right angle condition of surfaces and $\partial\Omega$ given by the boundary condition.

We now describe the approaches of this chapter. We overall rely on the maximum principle to get *a priori* gradient estimates. The difficult case is when a maximizer is on the boundary, where we cannot expect the maximum principle to hold as it is inside the domain. In [101], the difficulty is overcome by considering a slanted gradient in order to get rid of u_{nn}, the second derivative of a solution in the normal direction, which is hard to know from the maximum principle. In [66], the difficulty is handled by considering a multiplier which allows us to put the maximizer inside, so that we can apply the maximum principle. This idea is the crux of the multiplier method, which plays a main role in the estimates in [66]. Moreover, the multiplier method explains how the geometry of the domain affects the estimates, which is natural and geometric. It ultimately enables us to generalize the results of [101] on nonconvex domains in a natural way for a wide class of equations (3.1) and (3.2). This is how we overcome the difficulty, and it is the main novelty of this chapter.

To outline the structure of gradient estimates, we start by observing that both of the methods are relying on the same major term coming from the square norm of the second fundamental form. This is the reason why it is possible to apply the two methods at the same time, and why the process of mix is linear and natural. The whole chain of inequalities starts with applying the maximum principle, and is basically an expansion of a polynomial in $v = \sqrt{\eta^2 + |Du|^2}$. Finally, we focus on the coefficient of the highest power of v, which yields the coercivity condition (3.9) on c. We also note that we can get rid of bad terms in the linearized equation.

After we get a global gradient estimate, we next study the mean curvature equations and the large time behavior, as suggested in [101]. The part different from [101] is where we apply Leray-Schauder fixed point theorem for the mean curvature equations. As we deal with the additional term concerning a nonzero force, we interpolate (3.8) with a carefully chosen equation so that we keep the force c coercive during the interpolation. A force that is being kept coercive yields a uniform C^1 estimate by the gradient estimate obtained above. As an exchange for keeping coercivity in the interpolation, we change the transport term f, and this is allowed as long as it is *a priori* bounded. We then follow [101] to verify the asymptotic behavior for the graph case, and go through vanishing viscosity process as $\eta \to 0$ for the level-set case.

For the level-set mean curvature flow, we compute the eigenvalue and the large time profile, and prove the asymptotic behavior in the radially setting. Equation (3.2) is reduced to a first-order singular Hamilton-Jacobi equation with Neumann boundary conditions. Based on the optimal control formula [62], we are able to compute the eigenvalue. By providing an example where the eigenvalue is not constant, we discuss the optimality of the condition (3.9), which serves as the most important condition to ensure global gradient estimates. The use of the optimal control formula for computing the limit and for an example in this way follows [66], and it is extended to an equation with a transport term and nonzero boundary conditions. Then, by observing the monotonicity on the Aubry set as in [46], we prove the asymptotic behavior. To deal with the boundary, which does not appear in [46], we instead give a dynamics proof for the monotonicity, written in the style of [29].

Organization of the chapter

In Section 3.2, we prove the existence of solutions of (3.1) and (3.2) by giving *a priori* local and global gradient estimates. We also recover [101, Theorem 1.1] and the corresponding result for (3.2) when the domain Ω is strictly convex. In Section 3.3, we prove the existence of solutions of (3.5) and (3.6) through homogenization. In Section 3.4, we compute the eigenvalue and the large time profile, and prove the asymptotic behavior of the solution of (3.2) in the radially symmetric setting. In Appendix, we provide the definitions and the results on the comparison principle and on the stability of viscosity solutions of (3.2).

3.2 Gradient estimates

In this section, we give *a priori* local gradient estimates of (3.7), and under the condition (3.9) on the forcing term c, we prove *a priori* global gradient estimates. Throughout this section, we assume that the conditions (3.3) and (3.4) hold, and that Ω is bounded with C^3 boundary.

We leave a remark that for the choice $\eta = 1,\ q > 0$ in (3.1), the function u_0 serves as an initial data that is compatible with the boundary condition. In (3.2), by setting $q = 1$, we see that the function u_0, which is independent of $\eta \in (0, 1]$, serves as an initial data that is compatible with the boundary condition even if $\eta \in (0, 1]$ varies. We understand its viscosity solution as the limit of solutions of (3.7) as $\eta \to 0$. We also note from the compatibility condition that $|\phi v^{-q}| < 1$ on the boundary $\partial\Omega$.

The following lemma states that the time derivative of a solution of (3.7) is bounded.

Lemma 3.2.1. *Suppose that* u^η *is the unique solution of* (3.7) *for each* $\eta \in (0, 1]$*. Suppose* (3.4) *and* (3.3)*. Fix* $T \in (0, \infty)$*. Then, there exists* $M > 0$ *depending only on* Ω*,* c*,* f*,* ϕ*,* q*,* u_0 *such that*

$$\|u_t^\eta\|_{L^\infty(\overline{\Omega}\times[0,T])} \leq \|u_t^\eta(\cdot, 0)\|_{L^\infty(\overline{\Omega})} \leq M.$$

Proof. The proof follows the argument in [101, Lemma 2.1]. □

Now we state *a priori* gradient estimates.

Proposition 3.2.1. *Let $T \in (0,\infty)$, $\eta \in (0,1]$. Suppose that a solution u^η of* (3.7) *exists and it is of class $C^{2,\sigma}(\overline{\Omega}\times[0,T]) \cap C^{3,\sigma}(\Omega\times(0,T])$ for some $\sigma \in (0,1)$. Suppose that the force c satisfies* (3.9)*. Then u^η satisfies that*

$$\|Du^\eta\|_{L^\infty(\overline{\Omega}\times[0,T])} \le R,$$

where $R > 1$ is a constant depending only on $\Omega, c, f, \phi, q, u_0$.

Once we prove Proposition 3.2.1 (and Proposition 3.2.2 introduced later), we obtain the existence of solutions $u = u^\eta$ to (3.7) with the bound $\|Du^\eta\|_{L^\infty(\overline{\Omega}\times[0,T])} \le R$ (and therefore prove Theorem 3.1.2), due to the standard theory of quasilinear uniformly parabolic equations, for which we refer to [73]. See [86, Section 5] for the usage of [73], [76, Theorem 8.8]. We also briefly describe the existence from *a priori* estimates in Appendix for completeness.

Before getting into the proof of Proposition 3.2.1, we introduce the notations for scalars, vectors, and matrices. After that, we state Lemma 3.2.2 and Lemma 3.2.3 for later use, whose proofs are provided in Appendix.

We set notations. Let $p, q \in \mathbb{R}^n$ be column vectors and M be a symmetric n by n matrix. A real number $p \cdot q$ is the scalar obtained from the standard inner product of $\mathbb{R}^n$, and we let $|p| = \sqrt{p\cdot p}$. A vector Mp is the vector obtained from the standard matrix product. Let $\alpha = \left(\alpha^{ij}\right)_{i,j=1}^n, \beta = \left(\beta^{ij}\right)_{i,j=1}^n$ be two n by n matrices that are not necessarily symmetric. We let $\alpha\beta$ denote the matrix obtained from the standard matrix multiplication of α in the left and β in the right. We write $\mathrm{tr}\{\alpha\beta^{\mathrm{Tr}}\} = \sum_{i,j=1}^n \alpha^{ij}\beta^{ij}$, where $\mathrm{tr}\{\cdot\}$ denotes the trace, and Tr denotes the transpose. We let $\|\alpha\| = \sqrt{\mathrm{tr}\{\alpha\alpha^{\mathrm{Tr}}\}}$.

For a C^1 function μ in $x = (x_1, \cdots, x_n)$, we let μ_i denote the partial derivative μ_{x_i} of μ in x_i for each $i = 1, \cdots, n$, and we let $D\mu = (\mu_1, \cdots, \mu_n)^{\mathrm{Tr}}$ be the gradient of μ. For a C^2 function, say μ again, in $x = (x_1, \cdots, x_n)$, we let μ_{ij} denote the second order partial derivative $\mu_{x_i x_j}$ of μ in x_i and x_j in order for each $i, j = 1, \cdots, n$, and we let $D^2\mu = (\mu_{ij})_{i,j=1}^n$ be the Hessian of μ. For a C^3 function μ and a vector $\xi = (\xi^1, \cdots, \xi^n)^{\mathrm{Tr}}$, we let $\mu_{\ell ij}$ denote the third order partial derivative $\mu_{x_\ell x_i x_j}$ of μ in x_ℓ, x_i and x_j in order for each $\ell, i, j = 1, \cdots, n$, and we let $D^3\mu \odot \xi$ denote the matrix $\left(\sum_{\ell=1}^n \mu_{\ell ij}\xi^\ell\right)_{i,j=1}^n$. For $\nu = (\nu^1, \cdots, \nu^n)^{\mathrm{Tr}}$, ν^i a C^1 function for each $i = 1, \cdots, n$, we let $D\nu$ denote the matrix $\left(\nu^i_{x_j}\right)_{i,j=1}^n$. Then, for a C^2 function μ, we check that $D^2\mu = D(D\mu)$.

We define the matrix $a = a(p)$ by $a(p) = I_n - \frac{p\otimes p}{\eta^2+|p|^2}$, where $p \otimes p$ denotes the matrix $\left(p^i p^j\right)_{i,j=1}^n$ for $p = \left(p^1, \cdots, p^n\right)^{\mathrm{Tr}}$, and I_n denotes the n by n identity matrix. We let $p \otimes q$ denotes the matrix $\left(p^i q^j\right)_{i,j=1}^n$ for $p = \left(p^1, \cdots, p^n\right)^{\mathrm{Tr}}, q = \left(q^1, \cdots, q^n\right)^{\mathrm{Tr}} \in \mathbb{R}^n$. For a vector $\xi = (\xi^1, \cdots, \xi^n)^{\mathrm{Tr}}$, we let $D_p a \odot \xi$ denote the matrix

$$D_p a \odot \xi = \left(\sum_{\ell=1}^n a^{ij}_{p^\ell}\xi^\ell\right)_{i,j=1}^n,$$

where $a^{ij}_{p^\ell} = a^{ij}_{p^\ell}(p)$ is the partial derivative of a^{ij}, the (i,j)-entry of the matrix a for $i, j = 1, \cdots, n$, in its ℓ-th variable p^ℓ of $p = (p^1, \cdots, p^n)^{\mathrm{Tr}}$.

Now, we give the setup for Lemma 3.2.2. Suppose that $x_0 = (0, \cdots, 0) \in \partial\Omega$, and that $\vec{\mathbf{n}}(x_0) = (0, \cdots, 0, -1)$. Then, there exist an open neighborhood U_1 of x_0 in $\mathbb{R}^n$ and a C^3 function φ defined on $\{x' = (x_1, \cdots, x_{n-1}) : (x', 0) \in U_1\}$ such that $x = (x', x_n) \in \partial\Omega$ if and only if $x_n = \varphi(x')$. The eigenvalues $\kappa_1, \cdots, \kappa_{n-1}$ of the matrix $D^2\varphi(x'_0)$ are called the principal curvatures of $\partial\Omega$ at x_0, where $x'_0 = (0, \cdots, 0) \in \mathbb{R}^{n-1}$, and the corresponding eigenvectors are called the principal directions of $\partial\Omega$ at x_0.

By applying a rotation of coordinates to $x' = (x_1, \cdots, x_{n-1})$, we may assume that the $x_\ell-$axis lies along a principal direction corresponding to κ_ℓ, $\ell = 1, \cdots, n-1$, respectively. We call such a coordinate system a principal coordinate system of $\partial\Omega$ at x_0. The Hessian matrix $D^2\varphi(x_0)$ with respect to a principal coordinate system of $\partial\Omega$ at x_0 is given by the diagonal matrix, as

$$D^2\varphi(x_0) = \begin{bmatrix} \kappa_1 & & 0 \\ & \ddots & \\ 0 & & \kappa_{n-1} \end{bmatrix}.$$

We state Lemma 3.2.2, which provides a local parametrization $y' = (y_1, \cdots, y_{n-1})$ of the surface $\partial\Omega$ around $(0, \cdots, 0)$ and the derivatives of C^1 (or C^2) functions in $y =$

$(y_1, \cdots, y_n)$. See [52, Lemma 14.16] for the reference of Lemma 3.2.2.

Lemma 3.2.2. *Let $x_0 \in \partial\Omega$. For a coordinate $x = (x_1, \cdots, x_n)$ of $\mathbb{R}^n$, suppose that $x_0 = (0, \cdots, 0)$, and that $\vec{\mathbf{n}}(x_0) = (0, \cdots, 0, -1)$. Suppose also that $x' = (x_1, \cdots, x_{n-1})$ is a principal coordinate system of $\partial\Omega$ at x_0, i.e., the $x_\ell-$axis lies along a principal direction corresponding to a principal curvature κ_ℓ of $\partial\Omega$ at x_0, $\ell = 1, \cdots, n-1$, respectively.*

Then, there are open neighborhoods U, V of $(0, \cdots, 0)$ in $\mathbb{R}^n$ and a C^2 diffeomorphism $g : U \to V$, and there is a number $\sigma > 0$ satisfying the following properties;

(i) It holds that $g(0, \cdots, 0) = (0, \cdots, 0)$, and that

$$\{g(y', 0) : |y'| < \sigma\} \subseteq \partial\Omega \quad \text{and} \quad \{g(y', y_n) : |y'| + |y_n| < \sigma,\ y_n > 0\} \subseteq \Omega.$$

where $y' = (y_1, \cdots, y_{n-1}) \in \mathbb{R}^{n-1}$, and

(ii) g is the identity function on the line $\{(0, \cdots, 0, y_n) : |y_n| < \sigma\}$.

If we write $x = g(y),\ y \in U,\ x \in V$, then

(iii)

$$\frac{\partial \overline{\zeta}}{\partial y_\ell} = (1 - \kappa_\ell y_n)\frac{\partial \zeta}{\partial x_\ell} \qquad \text{for } \ell = 1, \cdots, n,$$

on the line $\{(0, \cdots, 0, y_n) : |y_n| < \sigma\}$, which is a subset of U. Here, $\zeta = \zeta(x)$ is a C^1 function defined on V, $\overline{\zeta}(y)$ is the C^1 function defined by $\zeta(g(y))$ on U, and κ_n is set to be 0. The number $\sigma > 0$ satisfies $\sigma^{-1} > \max\{|\kappa_1|, \cdots, |\kappa_{n-1}|\}$.

(iv)

$$\frac{\partial}{\partial y_n}\left(\frac{\partial \overline{\zeta}}{\partial y_\ell}\right) = (1 - \kappa_\ell y_n)\frac{\partial}{\partial y_n}\left(\frac{\partial \zeta}{\partial x_\ell}\right) - \frac{\kappa_\ell}{1 - \kappa_\ell y_n}\frac{\partial \overline{\zeta}}{\partial y_\ell} \qquad \text{for } \ell = 1, \cdots, n,$$

on the line $\{(0, \cdots, 0, y_n) : |y_n| < \sigma\}$ if the functions $\zeta, \overline{\zeta}$ given as above are C^2 functions.

We introduce the following lemma in advance, which will be used in the proof of Proposition 3.2.1.

Lemma 3.2.3. *Let $u \in C^{2,\sigma}(\overline{\Omega} \times [0, T]) \cap C^{3,\sigma}(\Omega \times (0, T])$, and let $v = \sqrt{\eta^2 + |Du|^2}$ for*

$T \in (0,\infty),\ \eta \in (0,1]$. Let $\xi \in \mathbb{R}^n$. Then,

$$v tr\{(D_p a(Du) \odot \xi) D^2 u\} + 2tr\{a(Du)(\xi \otimes Dv)\} = 0. \tag{3.14}$$

Proof of Proposition 3.2.1. The proof of Proposition 3.2.1 follows the classical Bernstein method by applying the maximum principle to the function $w := v^{q+1} - (q+1)\phi Du \cdot Dh$, where $v := \sqrt{\eta^2 + |Du|^2}$.

Let $T \in (0,\infty)$, $\eta \in (0,1]$. Let $u = u^\eta \in C^{2,\sigma}(\overline{\Omega} \times [0,T]) \cap C^{3,\sigma}(\Omega \times (0,T])$ be a solution to (3.7) for some $\sigma \in (0,1)$. We need to show that $\|v\|_{L^\infty(\overline{\Omega}\times[0,T])} \le R$ for some constant $R > 1$ independent of $T \in (0,\infty)$ and of $\eta \in (0,1]$. Throughout the proof, $R > 1$ will denote constants which vary line by line and which do not depend on $T \in (0,\infty)$ and also on $\eta \in (0,1]$. Note that η is fixed to be 1 when $q > 0$, and $\eta \in (0,1]$ when $q = 1$. Accordingly, $\eta \in (0,1]$ in all cases. Also, $C > 0$ will denote constants which vary line by line throughout the proof and also which do not depend on $T \in (0,\infty)$ and also on $\eta \in (0,1]$.

We drop the super and subscript regarding η, but we are still dealing with (3.7) together with the η-dependence when $q = 1$, which is of importance for (3.2). Once we obtain bounds uniform in $\eta \in (0,1]$, we also drop the η-dependence throughout the estimate.

Let h be a function in $C^3(\overline{\Omega})$ such that $h \equiv C,\ Dh = \vec{\mathbf{n}}$ on the boundary $\partial\Omega$ for some constant C. Let

$$w = v^{q+1} - (q+1)\phi Du \cdot Dh$$

on $\overline{\Omega} \times [0,T]$. The reason why we choose this w instead of $v = \sqrt{\eta^2 + |Du|^2}$ is that we want to cancel out terms involving $\frac{\partial^2 u}{\partial \vec{\mathbf{n}}^2}$, the second derivative of u in the normal direction on the boundary. The reason will be explained with more details when the cancellation occurs.

Fix $(x_0,t_0) \in \operatorname{argmax}_{\overline{\Omega}\times[0,T]} w$. The goal is to show that $v(x_0,t_0) \le R$ for some constant $R > 1$ independent of $T \in (0,\infty)$, $\eta \in (0,1]$. Once it is shown, then we obtain

$\|v\|_{L^\infty(\overline{\Omega}\times[0,T])} \le R$, which completes the proof. This is seen by the fact that

$$w \le v^{q+1} + (q+1)\|\phi\|_{C^0(\overline{\Omega})}\|h\|_{C^1(\overline{\Omega})}$$

at (x_0, t_0), and by the fact that

$$v^{q+1} - (q+1)\|\phi\|_{C^0(\overline{\Omega})}\|h\|_{C^1(\overline{\Omega})} \le w \le w(x_0,t_0) \le R$$

at $(x,t) \in \overline{\Omega} \times [0,T]$.

If $t_0 = 0$, we get a uniform bound $v(x_0,t_0) \le R$, so we are done. It remains the case when $t_0 > 0$, and we divide the proof into two cases: $x_0 \in \Omega$ and $x_0 \in \partial\Omega$.

Case 1: $x_0 \in \Omega$.

Step 1. We apply the maximum principle at (x_0,t_0) and simplify the resulting inequality.

As $x_0 \in \Omega,\ t_0 > 0$, the maximum principle yields $D^2 w \le 0,\ w_t \ge 0$ at (x_0,t_0). Therefore, together with the fact that $a(p) \ge 0$ as a matrix, we obtain

$$0 \ge \frac{1}{q+1}\big(\mathrm{tr}\{a(Du)D^2 w\} - w_t\big) \qquad \text{at } (x_0,t_0). \tag{3.15}$$

This is the point where we start a chain of inequalities.

Write $u_t = G + cv - f$, where $G := \mathrm{tr}\{a(Du)D^2u\}$, so that (3.15) becomes

$$\begin{aligned} 0 &\ge \frac{1}{q+1}\big(\mathrm{tr}\{a(Du)D^2 w\} - w_t\big) \\ &= \mathrm{tr}\{a(Du)D(v^q Dv)\} - \mathrm{tr}\{a(Du)D^2(\phi Du\cdot Dh)\} - (v^q v_t - \phi Du_t \cdot Dh) \\ &= \mathrm{tr}\{a(Du)D(v^q Dv)\} - \mathrm{tr}\{a(Du)D^2(\phi Du\cdot Dh)\} \\ &\qquad + (-v^{q-1}Du + \phi Dh)\cdot DG + (-v^{q-1}Du + \phi Dh)\cdot D(cv - f) \end{aligned} \tag{3.16}$$

at (x_0,t_0). Here, we have used the fact that $vv_t = Du \cdot Du_t$.

For the first term $\mathrm{tr}\{a(Du)D(v^q Dv)\}$ of (3.16), we substitute $D(v^q Dv) = qv^{q-1}Dv \otimes$

$Dv + v^q D^2 v$ to get

$$\mathrm{tr}\{a(Du)D(v^q Dv)\} = qv^{q-1}\mathrm{tr}\{a(Du)Dv \otimes Dv\} + v^q \mathrm{tr}\{a(Du)D^2 v\}.$$

We first check that $vD^2v = Qa(Du)D^2u + D^3u \odot Du$ with $Q = D^2u$. Differentiating $vDv = D^2uDu$, and using the fact that $p \otimes q = pq^{\mathrm{Tr}}$ for two vectors p, q, we get

$$\begin{aligned} vD^2v &= D^3u \odot Du + (D^2u)^2 - Dv \otimes Dv \\ &= (D^2u)^2 - \frac{D^2uDu}{v} \otimes \frac{D^2uDu}{v} + D^3u \odot Du \\ &= QI_nQ - Q\left(\frac{Du}{v} \otimes \frac{Du}{v}\right)Q + D^3u \odot Du \\ &= Qa(Du)Q + D^3u \odot Du. \end{aligned}$$

Therefore,

$$\mathrm{tr}\{a(Du)D(v^q Dv)\} = v^{q-1}\mathrm{tr}\{(a(Du)D^2u)^2\} + qV + X_1, \tag{3.17}$$

where $V := v^{q-1}\mathrm{tr}\{a(Du)Dv \otimes Dv\}$ and $X_1 := v^{q-1}\mathrm{tr}\{a(Du)(D^3u \odot Du)\}$.

To compute the second term of (3.16), we expand $D^2(\phi Du \cdot Dh)$ so that

$$\begin{aligned} D^2(\phi Du \cdot Dh) = (Du \cdot Dh)D^2\phi + (D^2uDh + D^2hDu) \otimes D\phi + D\phi \otimes (D^2uDh + D^2hDu) \\ + \phi(D^3u \odot Dh + D^2uD^2h + D^3h \odot Du + D^2hD^2u). \end{aligned}$$

Since $\mathrm{tr}\{a(p)(q \otimes r)\} = \mathrm{tr}\{a(p)(r \otimes q)\}$, $\mathrm{tr}\{a(p)AB\} = \mathrm{tr}\{a(p)BA\}$ for $p, q, r \in \mathbb{R}^n$, symmetric matrices A, B, we obtain

$$\mathrm{tr}\{a(Du)D^2(\phi Du \cdot Dh)\} = 2\mathrm{tr}\{a(Du)(D\phi \otimes (D^2uDh))\} + 2\phi\mathrm{tr}\{a(Du)D^2uD^2h\} + X_2 + J_1,$$

where $X_2 := \phi\mathrm{tr}\{a(Du)(D^3u \odot Dh)\}$ and

$$J_0 := (Du \cdot Dh)\mathrm{tr}\{a(Du)D^2\phi\} + 2\mathrm{tr}\{a(Du)(D\phi \otimes (D^2hDu))\} + \phi\mathrm{tr}\{a(Du)(D^3h \odot Du)\}.$$

Applying Cauchy-Schwarz inequality to the terms of J_0, we see that there exists a constant $C > 0$ independent of $T \in (0, \infty),\ \eta \in (0, 1]$ such that

$$\begin{aligned} J_0 &= (Du \cdot Dh)\mathrm{tr}\{a(Du)D^2\phi\} + 2\mathrm{tr}\{a(Du)(D\phi \otimes (D^2hDu))\} + \phi\mathrm{tr}\{a(Du)(D^3h \odot Du)\} \\ &\le |Du||Dh|\|a\|\|D^2\phi\| + 2\|a\||D\phi||D^2hDu| + |\phi|\|a\|\|D^3h \odot Du\| \\ &\le Cv\|a\| \\ &\le C\left(\frac{\eta^2}{v} + v\right). \end{aligned}$$

We have used the fact that $\|a\| = \left(\frac{\eta^4}{v^4} + n - 1\right)^{1/2} \le \frac{\eta^2}{v^2} + n - 1$, that $\|p \otimes q\| = |p||q|$ for $p, q \in \mathbb{R}^n$. We also have used the fact that, seen again by Cauchy-Schwarz inequality,

$$\begin{aligned} |D^2hDu| &= \sqrt{\|(D^2hDu) \otimes (D^2hDu)\|} = \sqrt{\|D^2hDuDu^{\mathrm{Tr}}D^2h^{\mathrm{Tr}}\|} \\ &\le \sqrt{\|D^2h\|\|DuDu^{\mathrm{Tr}}\|\|D^2h^{\mathrm{Tr}}\|} = \|D^2h\||Du| \le \|D^2h\|v, \end{aligned}$$

and

$$\|D^3h \odot Du\| = \sqrt{\sum_{i,j=1}^{n}\left(\sum_{\ell=1}^{n} h_{ij\ell}u_\ell\right)^2} \le \sqrt{\sum_{i,j=1}^{n}\left(\sum_{\ell=1}^{n} h_{ij\ell}^2\right)\left(\sum_{\ell=1}^{n} u_\ell^2\right)} \le C|Du| \le Cv,$$

where $C > 0$ is a constant depending on $\|h\|_{C^3(\overline{\Omega})}$. Since $\eta \in (0, 1]$, we see that there exist constants $R > 1,\ C > 0$ independent of $T \in (0, \infty),\ \eta \in (0, 1]$ such that

$$J_0 \le Cv$$

whenever $v > R$, and therefore that

$$\begin{aligned} -\mathrm{tr}\{a(Du)D^2(\phi Du \cdot Dh)\} \ge -2\mathrm{tr}\{a(Du)(D\phi \otimes (D^2uDh))\} - 2\phi\mathrm{tr}\{a(Du)D^2uD^2h\} \\ - X_2 - Cv. \qquad (3.18) \end{aligned}$$

whenever $v > R$.

We compute the third term $(-v^{q-1}Du+\phi Dh)\cdot DG$ of (3.16). By differentiating $G=\mathrm{tr}\{a(Du)D^2u\}$ and taking inner product, we obtain

$$\begin{aligned} Du\cdot DG &= \mathrm{tr}\{(D_pa(Du)\odot(D^2uDu))D^2u\}+\mathrm{tr}\{a(Du)(D^3u\odot Du)\} \\ &= v\mathrm{tr}\{(D_pa(Du)\odot Dv)D^2u\}+\mathrm{tr}\{a(Du)(D^3u\odot Du)\} \end{aligned}$$

and

$$Dh\cdot DG=\mathrm{tr}\{(D_pa(Du)\odot(D^2uDh))D^2u\}+\mathrm{tr}\{a(Du)(D^3u\odot Dh)\}.$$

Therefore,

$$\begin{aligned} (-v^{q-1}Du+\phi Dh)\cdot DG = &-v^q\mathrm{tr}\{(D_pa(Du)\odot Dv)D^2u\} \\ &+\phi\mathrm{tr}\{(D_pa(Du)\odot(D^2uDh))D^2u\}-X_1+X_2. \end{aligned} \tag{3.19}$$

Recall that $X_1=v^{q-1}\mathrm{tr}\{a(Du)(D^3u\odot Du)\}$ and $X_2=\phi\mathrm{tr}\{a(Du)(D^3u\odot Dh)\}$.

Now, we compute and estimate the fourth term $(-v^{q-1}Du+\phi Dh)\cdot D(cv-f)$ of (3.16). By expansion,

$$\begin{aligned} (-v^{q-1}Du+\phi Dh)\cdot D(cv-f) = &(-c_zv+f_z)(v^{q-1}|Du|^2-\phi Du\cdot Dh) \\ &+(-v^{q-1}Du+\phi Dh)\cdot(vDc-Df)+cDv\cdot(-v^{q-1}Du+Dh). \end{aligned}$$

Since $\eta\in(0,1]$, there exist constants $R>1,\ C>0$ independent of $T\in(0,\infty),\ \eta\in(0,1]$ such that

$$v^{q-1}|Du|^2-\phi Du\cdot Dh\geq v^{q+1}-\eta^2v^{q-1}-\|\phi\|_{C^0(\overline{\Omega})}\|h\|_{C^1(\overline{\Omega})}v\geq 0$$

if $v>R$, and therefore that

$$(-c_zv+f_z)(v^{q-1}|Du|^2-\phi Du\cdot Dh)\geq 0$$

if $v > R$. Here, we have used the assumption that $c_z \le 0,\ f_z \ge 0$ from (3.3), (3.4). Also, again by (3.3), (3.4), there exists a constant $C > 0$ independent of $T \in (0,\infty),\ \eta \in (0,1]$ such that

$$\begin{aligned}(-v^{q-1}Du + \phi Dh)\cdot(vDc - Df) &\ge -v^q|Du||Dc| - v^{q-1}|Du|\|Df\|_{C^0(\overline{\Omega}\times\mathbb{R})}\\ &\quad - \|h\|_{C^1(\overline{\Omega})}|Dc|v - \|h\|_{C^1(\overline{\Omega})}\|Df\|_{C^0(\overline{\Omega}\times\mathbb{R})}\\ &\ge -|Dc|v^{q+1} - C(v+v^q).\end{aligned}$$

Therefore, there exist constants $R > 1,\ C > 0$ independent of $T \in (0,\infty),\ \eta \in (0,1]$ such that at (x_0,t_0)

$$\begin{aligned}(-v^{q-1}Du + \phi Dh)\cdot D(cv - f) &\ge -|Dc|v^{q+1} - C(v+v^q)\\ &\quad + cDv\cdot(-v^{q-1}Du + \phi Dh)\end{aligned} \tag{3.20}$$

whenever $v > R$. We will give a bound of the term $cDv\cdot(-v^{q-1}Du + \phi Dh)$ at (x_0,t_0) later.

All in all, by the estimates (3.17), (3.18), (3.19), (3.20), we obtain that there exist constants $R > 1,\ C > 0$ independent of $T \in (0,\infty),\ \eta \in (0,1]$ such that at (x_0,t_0),

$$\begin{aligned}0 &\ge \frac{1}{q+1}\big(\mathrm{tr}\{a(Du)D^2w\} - w_t\big)\\ &\ge J_1 + J_2 - |Dc|v^{q+1} + (q+1-\varepsilon)V - C(v+v^q)\end{aligned} \tag{3.21}$$

if $v > R$, where

$$J_1 := (1-\varepsilon)v^{q-1}\mathrm{tr}\{(a(Du)D^2u)^2\} - \frac{1}{2}v^q\mathrm{tr}\{(D_pa(Du)\odot Dv)D^2u\} + cDv\cdot(-v^{q-1}Du + \phi Dh)$$

$$J_2 := \varepsilon v^{q-1}\mathrm{tr}\{(a(Du)D^2u)^2\} - \frac{1}{2}\varepsilon v^q\mathrm{tr}\{(D_pa(Du)\odot Dv)D^2u\} - 2\mathrm{tr}\{a(Du)(D\phi\otimes(D^2uDh))\} - 2\phi\mathrm{tr}\{a(Du)D^2uD^2h\} + \phi\mathrm{tr}\{(D_pa(Du)\odot(D^2uDh))D^2u\}.$$

Here, $\varepsilon \in (0,1)$ is a number to be determined, and we have used the fact, from Lemma 3.2.3 with $\xi = Dv$, that

$$-\frac{1}{2}v^q\mathrm{tr}\{(D_pa(Du)\odot Dv)D^2u\} = v^{q-1}\mathrm{tr}\{a(Du)Dv\otimes Dv = V.$$

Step 2. We estimate J_1.

We first write, with $Q = D^2u$,

$$\begin{aligned}\mathrm{tr}\{(a(Du)D^2u)^2\} &= \mathrm{tr}\left\{\left(I_n - \frac{Du\otimes Du}{v^2}\right)Qa(Du)Q\right\}\\ &= \mathrm{tr}\{a(Du)Q^2\} - \mathrm{tr}\left\{a(Du)\left(\frac{D^2uDu}{v}\otimes\frac{D^2uDu}{v}\right)\right\}\\ &= \mathrm{tr}\{a(Du)(D^2u)^2\} - \mathrm{tr}\{a(Du)Dv\otimes Dv\}.\end{aligned}$$

Apply Cauchy-Schwarz inequality $\|\alpha\|^2\|\beta\|^2 \geq \mathrm{tr}\{\alpha\beta^{\mathrm{Tr}}\}^2$ for $\mathrm{tr}\{a(Du)(D^2u)^2\}$ with $\alpha = \sqrt{a}D^2u$, $\beta = \sqrt{a}$ to obtain

$$\begin{aligned}\mathrm{tr}\{a(Du)(D^2u)^2\} = \|\alpha\|^2 &\geq \frac{\mathrm{tr}\{\alpha\beta^{\mathrm{Tr}}\}^2}{\|\beta\|^2} = \frac{G^2}{n-1+\frac{\eta^2}{v^2}}\\ &= \left(\frac{1}{n-1} - \frac{\eta^2}{v^2(n-1)\left(n-1+\frac{\eta^2}{v^2}\right)}\right)(u_t - cv + f)^2\\ &\geq \frac{1}{n-1}c^2v^2 - Cv\end{aligned}$$

for some constant $C > 0$ depending only on $\|f\|_{C^0(\overline{\Omega}\times\mathbb{R})}, \|c\|_{C^0(\overline{\Omega}\times\mathbb{R})}$ and $M > 0$ in Lemma 3.2.1. We have used Lemma 3.2.1, the assumptions (3.3), (3.4) and the fact that $\eta \in (0, 1]$. Therefore, there exist constants $R > 1,\ C > 0$ independent of $T \in (0, \infty),\ \eta \in (0, 1]$ such that

$$\mathrm{tr}\{a(Du)(D^2u)^2\} \geq \frac{1}{n-1}c^2v^2 - Cv$$

if $v > R$, and thus such that

$$(1-\varepsilon)v^{q-1}\mathrm{tr}\{(a(Du)D^2u)^2\} \geq \frac{1-\varepsilon}{n-1}c^2v^{q+1} - (1-\varepsilon)V - Cv^q \tag{3.22}$$

if $v > R$, $\varepsilon \in (0, 1)$. The number $\varepsilon \in (0, 1)$ will be explicitly chosen later. We note that the term $\mathrm{tr}\{(a(Du)D^2u)^2\}$ is used to derive the term $\frac{1}{n-1}c^2v^{q+1}$ as a lower bound, which is crucial to obtain the bound $v \leq R$.

For the third term of J_1, we claim that at (x_0, t_0), it holds that

$$|cDv \cdot (-v^{q-1}Du + \phi Dh)| \leq Cv \tag{3.23}$$

for some constant $C > 0$ independent of $T \in (0, \infty),\ \eta \in (0, 1]$. Note that $Dw = 0$ at (x_0, t_0), so that

$$\begin{aligned} 0 &= \frac{1}{q+1}Dw \cdot Du \\ &= v^qDu \cdot Dv - (Du \cdot D\phi)(Du \cdot Dh) - \phi(D^2uDu) \cdot Dh - \phi(D^2hDu) \cdot Du. \end{aligned}$$

This implies that at (x_0, t_0),

$$cDv \cdot (-v^{q-1}Du + \phi Dh) = -\frac{c}{v}\left((Du \cdot D\phi)(Du \cdot Dh) + \phi(D^2hDu) \cdot Du\right),$$

and thus that at (x_0, t_0),

$$|cDv \cdot (-v^{q-1}Du + \phi Dh)| \le \|c\|_{C^0(\overline{\Omega}\times\mathbb{R})} \left(\|\phi\|_{C^1(\overline{\Omega})} \|h\|_{C^1(\overline{\Omega})} + \|h\|_{C^2(\overline{\Omega})} \|\phi\|_{C^0(\overline{\Omega})} \right) \frac{1}{v} |Du|^2$$
$$\le Cv$$

for some constant $C > 0$ independent of $T \in (0, \infty)$, $\eta \in (0, 1]$. We have used the fact that $|Du| \le v$ and the assumptions (3.3), (3.4).

Together with the fact that

$$-\frac{1}{2} v^q \mathrm{tr}\{(D_p a(Du) \odot Dv) D^2 u\} = v^{q-1} \mathrm{tr}\{a(Du) Dv \otimes Dv\} = V,$$

and with (3.22), (3.23), we conclude that there exist constants $R > 1$, $C > 0$ independent of $T \in (0, \infty)$, $\eta \in (0, 1]$ such that

$$J_1 \ge \frac{1-\varepsilon}{n-1} c^2 v^{q+1} + \varepsilon V - C(v + v^q) \tag{3.24}$$

if $v > R$.

Step 3. We estimate J_2.

Before we start the estimate of J_2, we rotate the axes at x_0 and compute the second derivatives of u with respect to these axes. Take axes at x_0 such that

$$u_1 = |Du|, \qquad u_i = 0,\ i = 2, \cdots, n, \qquad (u_{ij})_{2 \le i,j \le n} \text{ is diagonal.} \tag{3.25}$$

Then, $a^{ij} = a^{ij}(Du)$ is simplified as

$$a^{11} = \frac{\eta^2}{v^2}, \qquad a^{ii} = 1,\ i = 2, \cdots n, \qquad a^{ij} = 0,\ i \ne j. \tag{3.26}$$

Using $Dw = 0$ at (x_0, t_0), we obtain, at (x_0, t_0),

$$v^{q-1}u_1u_{1i} - \phi \sum_{\ell=1}^{n} u_{\ell i}h_\ell = (\phi_i h_1 + \phi h_{1i})u_1, \quad i = 1, \cdots, n.$$

For $i \geq 2$,

$$v^{q-1}u_1u_{1i} - \phi u_{1i}h_1 - \phi u_{ii}h_i = (\phi_i h_1 + \phi h_{1i})u_1, \quad i = 2, \cdots, n,$$

and thus,

$$u_{1i} = E_i u_1 + F_i u_{ii}, \qquad i = 2, \cdots, n, \tag{3.27}$$

where

$$E_i := \frac{\phi_i h_1 + \phi h_{1i}}{v^{q-1}u_1 - \phi h_1}, \quad F_i := \frac{\phi h_i}{v^{q-1}u_1 - \phi h_1}, \qquad i = 2, \cdots, n. \tag{3.28}$$

For $i = 1$,

$$v^{q-1}u_1u_{11} - \phi h_1 u_{11} - \phi \sum_{\ell=2}^{n} h_\ell u_{1\ell} = (\phi_1 h_1 + \phi h_{11})u_1.$$

As above, we get

$$u_{11} = E_1 u_1 + \sum_{\ell=2}^{n} F_\ell^2 u_{\ell\ell}, \tag{3.29}$$

where

$$E_1 := \frac{\phi_1 h_1 + \phi h_{11}}{v^{q-1}u_1 - \phi h_1} + \frac{\phi}{v^{q-1}u_1 - \phi h_1} \sum_{\ell=2}^{n} h_\ell E_\ell. \tag{3.30}$$

Now, we write $J_2 = \varepsilon v^{q-1}\mathrm{tr}\{(a(Du)D^2u)^2\} + S_1 + S_2$, where

$$S_1 := -2\mathrm{tr}\{a(Du)(D\phi \otimes (D^2uDh))\} - 2\phi\mathrm{tr}\{a(Du)D^2uD^2h\},$$
$$S_2 := -\frac{1}{2}\varepsilon v^q \mathrm{tr}\{(D_p a(Du) \odot Dv)D^2u\} + \phi\mathrm{tr}\{(D_p a(Du) \odot (D^2uDh))D^2u\},$$

and we bound $S_1,\ S_2$.

We start with S_1. By expansion,

$$\begin{aligned}S_1 &= -2\left(\frac{\eta^2}{v^2}\phi_1 Du_1\cdot Dh + \sum_{\ell=2}^n \phi_\ell Du_\ell\cdot Dh + \frac{\eta^2}{v^2}\phi Du_1\cdot Dh_1 + \phi\sum_{\ell=2}^n Du_\ell\cdot Dh_\ell\right)\\ &= -2\left(\frac{\eta^2}{v^2}Du_1\cdot(\phi_1 Dh + \phi Dh_1) + \sum_{\ell=2}^n Du_\ell\cdot(\phi_\ell Dh + \phi Dh_\ell)\right).\end{aligned}$$

Let $H_{\ell i} := \phi_\ell h_i + \phi h_{\ell i}$ for each $\ell, i = 1,\cdots,n$. Then,

$$\begin{aligned}S_1 &= -2\left(\frac{\eta^2}{v^2}\sum_{\ell=1}^n u_{1\ell}H_{1\ell} + \sum_{\ell=2}^n(u_{1\ell}H_{\ell 1} + u_{\ell\ell}H_{\ell\ell})\right)\\ &= -2\left(\frac{\eta^2}{v^2}u_{11}H_{11} + \sum_{\ell=2}^n u_{1\ell}\left(\frac{\eta^2}{v^2}H_{1\ell} + H_{\ell 1}\right) + \sum_{\ell=2}^n u_{\ell\ell}H_{\ell\ell}\right).\end{aligned}$$

Using (3.27), (3.29), we get

$$\begin{aligned}S_1 = -2\Bigg(&\left(\frac{\eta^2}{v^2}H_{11}E_1 + \sum_{\ell=2}^n\left(\frac{\eta^2}{v^2}H_{1\ell} + H_{\ell 1}\right)E_\ell\right)u_1\\ &+ \sum_{\ell=2}^n\left(\frac{\eta^2}{v^2}H_{11}F_\ell^2 + \left(\frac{\eta^2}{v^2}H_{1\ell} + H_{\ell 1}\right)F_\ell + H_{\ell\ell}\right)u_{\ell\ell}\Bigg)\end{aligned}$$

Note that since $\eta \in (0,1]$,

$$\left|\frac{\eta^2}{v^2}H_{11}E_1 + \sum_{\ell=2}^n\left(\frac{\eta^2}{v^2}H_{1\ell} + H_{\ell 1}\right)E_\ell\right| \le Cv^{-q},$$

$$\left|\frac{\eta^2}{v^2}H_{11}F_\ell^2 + \left(\frac{\eta^2}{v^2}H_{1\ell} + H_{\ell 1}\right)F_\ell + H_{\ell\ell}\right| \le C$$

for $v > 1$, for some constant $C > 0$ that depends only on $\|\phi\|_{C^1(\overline{\Omega})},\ \|h\|_{C^2(\overline{\Omega})}$. Therefore, there exist constants $R > 1,\ C > 0$ independent of $T \in (0,\infty),\ \eta \in (0,1]$ such that

$$S_1 \ge -C\left(v^{1-q} + \sum_{\ell=2}^n |u_{\ell\ell}|\right) \tag{3.31}$$

for $v > R$.

Now, we estimate S_2. Applying Lemma 3.2.3 with $\xi = Dv$ and with $\xi = D^2uDh$, and by expansion, we see that

$$S_2 = \varepsilon v^{q-1}\mathrm{tr}\{a(Du)Dv \otimes Dv\} - \frac{2\phi}{v}\mathrm{tr}\{a(Du)(D^2uDh \otimes Dv)\}$$
$$= \varepsilon v^{q-1}\left(\frac{\eta^2}{v^2}v_1^2 + \sum_{\ell=2}^{n} v_\ell^2\right) - \frac{2\phi}{v}\left(\frac{\eta^2}{v^2}v_1(Du_1 \cdot Dh) + \sum_{\ell=2}^{n} v_\ell(Du_\ell \cdot Dh)\right).$$

Let $K_\ell := \phi v^{-1}(Du_\ell \cdot Dh)$ for each $\ell = 1, \cdots, n$. Then,

$$S_2 = \frac{\eta^2}{v^2}(\varepsilon v^{q-1}v_1^2 - 2K_1v_1) + \sum_{\ell=2}^{n}(\varepsilon v^{q-1}v_\ell^2 - 2K_\ell v_\ell)$$
$$= \frac{\eta^2}{v^2}\left(\varepsilon v^{q-1}\left(v_1 - \frac{K_1}{\varepsilon v^{q-1}}\right)^2 - \frac{K_1^2}{\varepsilon v^{q-1}}\right) + \sum_{\ell=2}^{n}\left(\varepsilon v^{q-1}\left(v_\ell - \frac{K_\ell}{\varepsilon v^{q-1}}\right)^2 - \frac{K_\ell^2}{\varepsilon v^{q-1}}\right)$$
$$\geq -\varepsilon^{-1}v^{-1-q}K_1^2 - \varepsilon^{-1}v^{1-q}\sum_{\ell=2}^{n} K_\ell^2.$$

In the last inequality, we have used the fact that $\eta \in (0, 1]$. By expansion and (3.27), (3.29), we have

$$K_1 = \phi v^{-1}\left(h_1u_{11} + \sum_{\ell=2}^{n} h_\ell u_{1\ell}\right) = K_{11}u_1 + \sum_{\ell=2}^{n} K_{1\ell}u_{\ell\ell},$$

where

$$K_{11} := \phi v^{-1}\sum_{\ell=1}^{n} h_\ell E_\ell, \quad K_{1\ell} := \phi v^{-1}(h_1F_\ell^2 + h_\ell F_\ell), \quad \text{for } \ell = 2, \cdots, n. \tag{3.32}$$

Then, by Cauchy-Schwarz inequality,

$$-K_1^2 = -\left(K_{11}u_1 + \sum_{\ell=2}^{n} K_{1\ell}u_{\ell\ell}\right)^2 \geq -nK_{11}^2u_1^2 - \sum_{\ell=2}^{n} nK_{1\ell}^2u_{\ell\ell}^2.$$

Similarly, it holds that, for $\ell = 2, \cdots, n$,

$$K_\ell = \phi v^{-1}\left(h_1u_{1\ell} + h_\ell u_{\ell\ell}\right) = K_{\ell 1}u_1 + K_{\ell\ell}u_{\ell\ell},$$

where

$$K_{\ell 1} := \phi v^{-1} h_1 E_\ell, \quad K_{\ell\ell} := \phi v^{-1}(h_1 F_\ell + h_\ell), \quad \text{for } \ell = 2, \cdots, n, \tag{3.33}$$

and, by Cauchy-Schwarz inequality, for $\ell = 2, \cdots, n$,

$$-K_\ell^2 = -\left(K_{\ell 1} u_1 + K_{\ell\ell} u_{\ell\ell}\right)^2 \geq -2K_{\ell 1}^2 u_1^2 - 2K_{\ell\ell}^2 u_{\ell\ell}^2.$$

Therefore,

$$\begin{aligned} S_2 &\geq -\varepsilon^{-1} v^{-1-q} K_1^2 - \varepsilon^{-1} v^{1-q} \sum_{\ell=2}^{n} K_\ell^2 \\ &\geq -\varepsilon^{-1} S_{21} u_1^2 - \varepsilon^{-1} \sum_{\ell=2}^{n} S_{2\ell} u_{\ell\ell}^2, \end{aligned} \tag{3.34}$$

where

$$S_{21} := n v^{-1-q} K_{11}^2 + 2v^{1-q} \sum_{\ell=2}^{n} K_{\ell 1}^2, \; S_{2\ell} := n v^{-1-q} K_{1\ell}^2 + 2v^{1-q} K_{\ell\ell}^2, \;\; \text{for } \ell = 2, \cdots, n. \tag{3.35}$$

By (3.31), (3.34), we see that there exist constants $R > 1,\ C > 0$ independent of $T \in (0, \infty),\ \eta \in (0, 1]$ such that

$$J_2 \geq \varepsilon v^{q-1} \sum_{\ell=2}^{n} u_{\ell\ell}^2 - C(v^{1-q} + \sum_{\ell=2}^{n} |u_{\ell\ell}|) - \varepsilon^{-1} S_{21} u_1^2 - \varepsilon^{-1} \sum_{\ell=2}^{n} S_{2\ell} u_{\ell\ell}^2$$

for $v > R$. Note that there exists a constant $C > 0$ that depends only on $\|\phi\|_{C^1(\overline{\Omega})},\ \|h\|_{C^2(\overline{\Omega})}$ such that, by (3.28), (3.30),

$$|E_1| + \sum_{\ell=2}^{n} |E_\ell| + \sum_{\ell=2}^{n} |F_\ell| \leq C v^{-q},$$

for $v > 1$, and in turn, by (3.32), (3.33)

$$|K_{11}| + \sum_{\ell=2}^{n} |K_{1\ell}| + \sum_{\ell=2}^{n} |K_{\ell 1}| + \sum_{\ell=2}^{n} |K_{\ell\ell}| \leq Cv^{-1-q},$$

for $v > 1$, and thus such that, by (3.35)

$$|S_{21}| + \sum_{\ell=2}^{n} |S_{2\ell}| \leq Cv^{-1-3q}.$$

Therefore, there exist constants $R > 1,\ C > 0$ independent of $T \in (0, \infty),\ \eta \in (0, 1]$ such that

$$\begin{aligned} J_2 &\geq \varepsilon v^{q-1} \sum_{\ell=2}^{n} u_{\ell\ell}^2 - C(v^{1-q} + \sum_{\ell=2}^{n} |u_{\ell\ell}|) - C\varepsilon^{-1} v^{1-3q} - C\varepsilon^{-1} \sum_{\ell=2}^{n} v^{-1-3q} u_{\ell\ell}^2 \\ &\geq -Cv^{1-q} - C\varepsilon^{-1} v^{1-3q} + \sum_{\ell=2}^{n} \left(v^{q-1} \left(\varepsilon - C\varepsilon^{-1} v^{-4q} \right) u_{\ell\ell}^2 - C|u_{\ell\ell}| \right) \end{aligned}$$

for $v > R$.

For a given $\varepsilon \in (0, 1)$, choose $R_\varepsilon > 1$ that may depend on $\varepsilon \in (0, 1)$ but not on $T \in (0, \infty),\ \eta \in (0, 1]$ such that $C\varepsilon^{-1} v^{-4q} < \frac{\varepsilon}{2}$ if $v > R_\varepsilon$. Then, for $v > R_\varepsilon$,

$$\begin{aligned} J_2 &\geq -Cv^{1-q} - C\varepsilon^{-1} v^{1-3q} + \sum_{\ell=2}^{n} \left(\frac{\varepsilon}{2} v^{q-1} u_{\ell\ell}^2 - C|u_{\ell\ell}| \right) \\ &= -Cv^{1-q} - C\varepsilon^{-1} v^{1-3q} + \sum_{\ell=2}^{n} \left(\frac{\varepsilon}{2} v^{q-1} \left(|u_{\ell\ell}| - \frac{C}{\varepsilon v^{q-1}} \right)^2 - \frac{C^2}{2\varepsilon v^{q-1}} \right). \end{aligned}$$

All in all, for each $\varepsilon \in (0, 1)$, there exist a constant $C > 0$ independent of $T \in (0, \infty),\ \eta \in (0, 1]$ (also of $\varepsilon \in (0, 1)$) and a constant $R_\varepsilon > 1$ that may depend on $\varepsilon \in (0, 1)$ but not on $T \in (0, \infty),\ \eta \in (0, 1]$ such that

$$J_2 \geq -Cv^{1-q}(1 + \varepsilon^{-1}) \tag{3.36}$$

for $v > R_\varepsilon$.

Step 4. We finish Case 1.

We come back to the maximum principle (3.21) applied at (x_0, t_0). By (3.24), (3.36), we see that there exist a constant $C > 0$ independent of $T \in (0, \infty)$, $\eta \in (0, 1]$ (also of $\varepsilon \in (0, 1)$) and a constant $R_\varepsilon > 1$ that may depend on $\varepsilon \in (0, 1)$ but not on $T \in (0, \infty)$, $\eta \in (0, 1]$ such that

$$0 \geq \left(\frac{1-\varepsilon}{n-1} c^2 - |Dc| \right) v^{q+1} + (q+1)V - C(v + v^q) - Cv^{1-q}\varepsilon^{-1}$$

for $v > R_\varepsilon$. Now, we apply the condition (3.9); take

$$\varepsilon = \frac{1}{2} \min \left\{ 1, \frac{(n-1)\delta}{\|c\|^2_{L^\infty(\overline{\Omega} \times \mathbb{R})}} \right\}.$$

For this choice of $\varepsilon \in (0, 1)$, it holds that $\frac{1-\varepsilon}{n-1} c^2 - |Dc| \geq \frac{1}{2}\delta$, and R_ε, ε^{-1} are fixed. Therefore, by taking this choice of $\varepsilon \in (0, 1)$, we see that there exist constants $R > 1$, $C > 0$ independent of $T \in (0, \infty)$, $\eta \in (0, 1]$ such that at (x_0, t_0),

$$0 \geq \frac{\delta}{2} v^{q+1} - C(v + v^q)$$

if $v > R$. Here, we have used the fact that $V \geq 0$. On the other hand, there is also a constant $R_0 > R$ independent of $T \in (0, \infty)$, $\eta \in (0, 1]$ such that

$$0 < \frac{\delta}{2} v^{q+1} - C(v + v^q)$$

if $v > R_0$. Therefore, it must hold that $v = v(x_0, t_0) \leq R_0$, which completes Case 1.

Case 2: $x_0 \in \partial\Omega$.

Step 5. We bound the normal derivative of w at (x_0, t_0) with a geometric constant.

Recall that $C_0(x_0) = \max\{\lambda : \lambda \text{ is an eigenvalue of } -\kappa\}$, where $\kappa := \left(\kappa^{\ell j}\right)_{\ell,j=1}^{n-1}$ is the curvature matrix of $\partial\Omega$ at x_0, and that $C_0 = \sup\{C_0(y) : y \in \partial\Omega\}$. For $\varepsilon_0 \in (0, 1)$, we let $L = (q+1)\,(C_0 + \varepsilon_0)$. The goal of this step is to prove that for any given number $\varepsilon_0 \in (0, 1)$, there exists a constant $R_{\varepsilon_0} > 0$ which depends on ε_0 but not on $T \in (0,\infty)$, $\eta \in (0,1]$ (also not on $x_0 \in \partial\Omega$) such that $w > 0$ and $\frac{\partial w}{\partial \vec{\mathbf{n}}} < Lw$ at (x_0, t_0) whenever $v > R_{\varepsilon_0}$.

Changing a coordinate on $\mathbb{R}^n$, we may assume without loss of generality that $x_0 = (0,\cdots,0)$, $\vec{\mathbf{n}}(x_0) = (0,\cdots,0,-1)$, and that $x' = (x_1,\cdots,x_{n-1})$ is a principal coordinate system of $\partial\Omega$ at x_0. We may assume that the $x_\ell-$axis lies along a principal direction corresponding to κ_ℓ, $\ell = 1,\cdots,n-1$, respectively. By Lemma 3.2.2, there are open neighborhoods U, V of $(0,\cdots,0)$ in $\mathbb{R}^n$ and a C^2 diffeomorphism $g : U \to V$, and there is a number $\sigma > 0$ satisfying the properties (i), $\cdots$, (iv) of Lemma 3.2.2. For each function $\zeta = u, v, w, \phi, h$ on $V \cap \overline{\Omega}$, we define the function $\overline{\zeta}$ on $U \cap g^{-1}(\overline{\Omega}) = \{y = (y_1,\cdots,y_n) : y \in U,\ y_n \geq 0\}$ by $\overline{\zeta} = \zeta \circ g$. We let $y_0 = g^{-1}(x_0)$. The different characters x_0, y_0 are used to distinguish where they belong to, i.e., the domains V, U of definitions, respectively, though the both are the origin.

We introduce notations to denote vectors and derivatives in $y = (y_1,\cdots,y_n)$. For a C^1 function $\overline{\zeta}$ defined on U, let

$$\nabla\overline{\zeta} := \left(\frac{\partial\overline{\zeta}}{\partial y_1},\cdots,\frac{\partial\overline{\zeta}}{\partial y_n}\right)^{\mathrm{Tr}}, \quad \nabla'\overline{\zeta} := \left(\frac{\partial\overline{\zeta}}{\partial y_1},\cdots,\frac{\partial\overline{\zeta}}{\partial y_{n-1}}\right)^{\mathrm{Tr}},$$

and for the C^1 function $\zeta := \overline{\zeta} \circ g^{-1}$ on V, let

$$D\zeta := \left(\frac{\partial\zeta}{\partial x_1},\cdots,\frac{\partial\zeta}{\partial x_n}\right)^{\mathrm{Tr}}, \quad D'\zeta := \left(\frac{\partial\zeta}{\partial x_1},\cdots,\frac{\partial\zeta}{\partial x_{n-1}}\right)^{\mathrm{Tr}}.$$

If ζ is a C^2 function on V, we let

$$\frac{\partial}{\partial y_n}\left(\nabla\overline{\zeta}\right) := \left(\frac{\partial}{\partial y_n}\left(\frac{\partial\overline{\zeta}}{\partial y_1}\right),\cdots,\frac{\partial}{\partial y_n}\left(\frac{\partial\overline{\zeta}}{\partial y_n}\right)\right)^{\mathrm{Tr}},$$
$$\frac{\partial}{\partial y_n}\left(\nabla'\overline{\zeta}\right) := \left(\frac{\partial}{\partial y_n}\left(\frac{\partial\overline{\zeta}}{\partial y_1}\right),\cdots,\frac{\partial}{\partial y_n}\left(\frac{\partial\overline{\zeta}}{\partial y_{n-1}}\right)\right)^{\mathrm{Tr}},$$
$$\frac{\partial}{\partial y_n}\left(D\zeta\right) := \left(\frac{\partial}{\partial y_n}\left(\frac{\partial\zeta}{\partial x_1}\right),\cdots,\frac{\partial}{\partial y_n}\left(\frac{\partial\zeta}{\partial x_n}\right)\right)^{\mathrm{Tr}},$$
$$\frac{\partial}{\partial y_n}\left(D'\zeta\right) := \left(\frac{\partial}{\partial y_n}\left(\frac{\partial\zeta}{\partial x_1}\right),\cdots,\frac{\partial}{\partial y_n}\left(\frac{\partial\zeta}{\partial x_{n-1}}\right)\right)^{\mathrm{Tr}}.$$

We use the same notation, $\cdot$, for the inner product in $\mathbb{R}^n$, now including the vectors in $\mathbb{R}^n$ just introduced above. By abuse of notations, we use the notation, $\cdot$, for the inner product in $\mathbb{R}^{n-1}$, also including the above vectors in $\mathbb{R}^{n-1}$ just introduced. We write the curvature κ as

$$\kappa = \begin{bmatrix} \kappa_1 & & 0 \\ & \ddots & \\ 0 & & \kappa_{n-1} \end{bmatrix},$$

and we let

$$\widetilde{\kappa} = \begin{bmatrix} \kappa_1 & & & 0 \\ & \ddots & & \\ & & \kappa_{n-1} & \\ 0 & & & \kappa_n \end{bmatrix},$$

with $\kappa_n = 0$ for convenience for later.

With the above notations, Lemma 3.2.2 states that

$$\nabla\overline{\zeta} = (I_n - y_n\widetilde{\kappa})D\zeta,$$

and

$$\frac{\partial}{\partial y_n}(\nabla\overline{\zeta}) = (I_n - y_n\widetilde{\kappa})\frac{\partial}{\partial y_n}(D\zeta) - (I_n - y_n\widetilde{\kappa})^{-1}\widetilde{\kappa}\nabla\overline{\zeta}$$

on the line $\{(0, \cdots, 0, y_n) \in U : 0 \leq y_n < \sigma\}$, in the setting of Lemma 3.2.2.

We start the estimate of $\frac{\partial w}{\partial \vec{\mathbf{n}}}(x_0, t_0)$. In order to estimate $\frac{\partial w}{\partial \vec{\mathbf{n}}}(x_0, t_0)(= -\frac{\partial w}{\partial x_n}(x_0, t_0) = -\frac{\partial \overline{w}}{\partial y_n}(y_0, t_0))$, we first compute $\frac{\partial \overline{v}}{\partial y_n}$, $\nabla'\overline{v}$, $\nabla'\overline{u} \cdot \frac{\partial}{\partial y_n}(\nabla'\overline{u})$ in turn. Note that for the normal derivatives, we have the additional negative sign, since $\vec{\mathbf{n}}(x_0)$ denotes the outward unit normal vector at x_0, while the inward unit normal vector at x_0 and the inward unit normal vector at y_0 lie on the positive x_n-axis and the positive y_n-axis, respectively.

To compute $\frac{\partial \overline{v}}{\partial y_n}$, we differentiate $\overline{v}^2 = \eta^2 + |Du|^2$ on the line $\{(0, \cdots, 0, y_n) : 0 \leq y_n <$

$\sigma\}$ in y_n to obtain

$$2\overline{v}\frac{\partial\overline{v}}{\partial y_n} = 2\frac{\partial}{\partial y_n}\left((I_n - y_n\widetilde{\kappa})^{-1}\nabla\overline{u}\right)\cdot(I_n - y_n\widetilde{\kappa})^{-1}\nabla\overline{u}$$
$$= 2\left((I_n - y_n\widetilde{\kappa})^{-3}\widetilde{\kappa}\nabla\overline{u}\right)\cdot\nabla\overline{u} + 2\left((I_n - y_n\widetilde{\kappa})^{-2}\frac{\partial}{\partial y_n}(\nabla\overline{u})\right)\cdot\nabla\overline{u}$$

on the line $\{(0,\cdots,0,y_n) : 0 \le y_n < \sigma\}$. Since $\frac{\partial\overline{u}}{\partial y_n} = -\overline{\phi}\overline{v}^{1-q}$ at (y_0,t_0) and $\kappa_n = 0$, we obtain

$$\frac{\partial\overline{v}}{\partial y_n} = \frac{1}{\overline{v}}\frac{\partial}{\partial y_n}(\nabla'\overline{u})\cdot\nabla'\overline{u} - \overline{\phi}\overline{v}^{-q}\frac{\partial^2\overline{u}}{\partial y_n^2} + \frac{1}{\overline{v}}(\kappa\nabla'\overline{u})\cdot\nabla'\overline{u}. \tag{3.37}$$

at (y_0,t_0).

We compute $\nabla'\overline{v}$ at (y_0,t_0). Since y_0 is a maximizer of $\overline{w}(\cdot,t_0)$ on $U\cap g^{-1}(\partial\Omega) = \{y = (y',0)\in U : y' = (y_1,\cdots,y_{n-1})\}$, it holds that $\nabla'\overline{w}(y_0,t_0) = 0$. Note also that $\overline{w} = \overline{v}^{q+1} - (q+1)\overline{\phi}^2\overline{v}^{1-q}$ on $\left(U\cap g^{-1}(\partial\Omega)\right)\times\{t_0\}$. Hence, at (y_0,t_0),

$$0 = \frac{1}{q+1}\nabla'\overline{w} = \overline{v}^q\nabla'\overline{v} - 2\overline{\phi}\overline{v}^{1-q}\nabla'\overline{\phi} - (1-q)\overline{\phi}^2\overline{v}^{-q}\nabla'\overline{v},$$

which gives

$$\nabla'\overline{v} = \frac{2\overline{\phi}\overline{v}^{1-q}}{\overline{v}^q - (1-q)\overline{\phi}^2\overline{v}^{-q}}\nabla'\overline{\phi} \tag{3.38}$$

at (y_0,t_0). Here, we are assuming $(\overline{v}(y_0,t_0) =)v(x_0,t_0) > \left(|1-q|\|\phi\|^2_{C^0(\partial\Omega)}\right)^{1/2q}$ so that $\overline{v}^q - (1-q)\overline{\phi}^2\overline{v}^{-q} > 0$. In the other case when $v = v(x_0,t_0) \le \left(|1-q|\|\phi\|^2_{C^0(\partial\Omega)}\right)^{1/2q}$, we already achieve our goal.

We compute $\nabla'\overline{u}\cdot\frac{\partial}{\partial y_n}(\nabla'\overline{u})$ before getting into the estimate of $\frac{\partial w}{\partial\vec{\mathbf{n}}}$ at (x_0,t_0). We differentiate $\frac{\partial\overline{u}}{\partial y_n} = -\overline{\phi}\overline{v}^{1-q}$ on $\left(U\cap g^{-1}(\partial\Omega)\right)\times\{t_0\}$ in y_ℓ, $\ell = 1,\cdots,n-1$, to have

$$\frac{\partial}{\partial y_n}\left(\nabla'\overline{u}\right) = \nabla'\left(\frac{\partial\overline{u}}{\partial y_n}\right) = -\overline{v}^{1-q}\nabla'\overline{\phi} - (1-q)\overline{\phi}\overline{v}^{-q}\nabla'\overline{v}.$$

By (3.38), we obtain

$$\begin{aligned}\nabla'\overline{u}\cdot\frac{\partial}{\partial y_n}\left(\nabla'\overline{u}\right) &= -\overline{v}^{1-q}\nabla'\overline{u}\cdot\nabla'\overline{\phi} - \frac{2(1-q)\overline{\phi}^2\overline{v}^{1-2q}}{\overline{v}^q-(1-q)\overline{\phi}^2\overline{v}^{-q}}\nabla'\overline{u}\cdot\nabla'\overline{\phi}\\ &= -\frac{\overline{v}+(1-q)\overline{\phi}^2\overline{v}^{1-2q}}{\overline{v}^q-(1-q)\overline{\phi}^2\overline{v}^{-q}}\nabla'\overline{u}\cdot\nabla'\overline{\phi}.\end{aligned} \tag{3.39}$$

We now estimate $\frac{\partial w}{\partial \mathbf{n}}$ at (x_0,t_0). On the line $\{(0,\cdots,0,y_n):0\le y_n<\sigma\}$, we have

$$\begin{aligned}\frac{1}{q+1}\frac{\partial\overline{w}}{\partial y_n} &= \frac{1}{q+1}\frac{\partial}{\partial y_n}\left(\overline{v}^{q+1}-(q+1)\overline{\phi}(I_n-y_n\widetilde{\kappa})^{-2}\nabla\overline{u}\cdot\nabla\overline{h}\right)\\ &= \overline{v}^q\frac{\partial\overline{v}}{\partial y_n}-\frac{\partial\overline{\phi}}{\partial y_n}(I_n-y_n\widetilde{\kappa})^{-2}\nabla\overline{u}\cdot\nabla\overline{h}-2\overline{\phi}(I_n-y_n\widetilde{\kappa})^{-3}\widetilde{\kappa}\nabla\overline{u}\cdot\nabla\overline{h}\\ &\quad -\overline{\phi}(I_n-y_n\widetilde{\kappa})^{-2}\frac{\partial}{\partial y_n}(\nabla\overline{u})\cdot\nabla\overline{h}-\overline{\phi}(I_n-y_n\widetilde{\kappa})^{-2}\nabla\overline{u}\cdot\frac{\partial}{\partial y_n}(\nabla\overline{h}).\end{aligned}$$

Note that $\kappa_n=0$ and that $\nabla'\overline{h}=0$, $\frac{\partial\overline{h}}{\partial y_n}=-1$ at (y_0,t_0). Also, $\nabla'\left(\frac{\partial\overline{h}}{\partial y_n}\right)=0$ on $U\cap g^{-1}(\partial\Omega)$ since $\frac{\partial\overline{h}}{\partial y_n}=-1$ on $U\cap g^{-1}(\partial\Omega)$. Therefore, at (y_0,t_0), we get

$$\frac{1}{q+1}\frac{\partial\overline{w}}{\partial y_n}=\overline{v}^q\frac{\partial\overline{v}}{\partial y_n}+\frac{\partial\overline{\phi}}{\partial y_n}\frac{\partial\overline{u}}{\partial y_n}+\overline{\phi}\frac{\partial^2\overline{u}^2}{\partial y_n^2}-\overline{\phi}\frac{\partial\overline{u}}{\partial y_n}\frac{\partial^2\overline{h}}{\partial y_n^2}.$$

By (3.37), (3.39) and the boundary condition that $\frac{\partial\overline{u}}{\partial y_n}=-\overline{\phi}\overline{v}^{1-q}$ on $\left(U\cap g^{-1}(\partial\Omega)\right)\times\{t_0\}$, we obtain, at (y_0,t_0),

$$\begin{aligned}\frac{1}{q+1}\frac{\partial\overline{w}}{\partial y_n} &= \overline{v}^{q-1}\left(\nabla'\overline{u}\cdot\frac{\partial}{\partial y_n}(\nabla'\overline{u})+(\kappa\nabla'\overline{u})\cdot\nabla'\overline{u}\right)-\overline{\phi}\frac{\partial^2\overline{u}}{\partial y_n^2}\\ &\qquad +\frac{\partial\overline{\phi}}{\partial y_n}\frac{\partial\overline{u}}{\partial y_n}+\overline{\phi}\frac{\partial^2\overline{u}^2}{\partial y_n^2}-\overline{\phi}\frac{\partial\overline{u}}{\partial y_n}\frac{\partial^2\overline{h}}{\partial y_n^2}\\ &= -\frac{\overline{v}^q+(1-q)\overline{\phi}^2\overline{v}^{-q}}{\overline{v}^q-(1-q)\overline{\phi}^2\overline{v}^{-q}}\nabla'\overline{u}\cdot\nabla'\overline{\phi}+\overline{v}^{q-1}(\kappa\nabla'\overline{u})\cdot\nabla'\overline{u}\\ &\qquad -\overline{\phi}\frac{\partial\overline{\phi}}{\partial y_n}\overline{v}^{1-q}-\overline{\phi}\frac{\partial\overline{u}}{\partial y_n}\frac{\partial^2\overline{h}}{\partial y_n^2}.\end{aligned} \tag{3.40}$$

At this point, we emphasize the cancellation of the terms $\pm\overline{\phi}\frac{\partial^2\overline{u}}{\partial y_n^2}$ while we compute the normal derivative $\frac{\partial\overline{w}}{\partial y_n}$ at (y_0,t_0). The term $\frac{\partial^2\overline{u}}{\partial y_n^2}$ is the hardest term to get information among the terms in the Hessian $D^2\overline{u}$ of $\overline{u}$.

We recall the definitions of $C_0(x_0),\ C_0$;

$$C_0(x_0) = \max\{-\lambda : \lambda \text{ is an eigenvalue of } \kappa \text{ at } x_0\},$$
$$C_0 = \sup\{C_0(x_0) : x_0 \in \partial\Omega\}.$$

Also, if $v > \left(2|1-q|\|\phi\|^2_{C^0(\partial\Omega)}\right)^{1/2q} =: R_0$, then $|(1-q)\phi^2 v^{-2q}| < \frac{1}{2}$ at (x_0, t_0), and thus,

$$\frac{1}{3} < \frac{v^q + (1-q)\phi^2 v^{-q}}{v^q - (1-q)\phi^2 v^{-q}}(x_0, t_0) < 3.$$

Note that R_0 is independent of $T \in (0, \infty),\ \eta \in (0, 1],\ x_0 \in \partial\Omega$. Lastly, we check that $\frac{\partial^2 \overline{h}}{\partial y_n^2}(y_0) = \frac{\partial^2 h}{\partial x_n^2}(x_0)$ since the coordinate change $g : U \to V$ is the identity on the line $\{(0, \cdots, 0, y_n) : |y_n| < \sigma\}$.

Finally, if $v = v(x_0, t_0) > R_0$, and also if $\eta \in (0, 1]$, then

$$\begin{aligned}
&\frac{1}{q+1}\frac{\partial w}{\partial \vec{\mathbf{n}}}(x_0, t_0) \\
&\le 3\,|Du(x_0, t_0)|\,|D\phi(x_0)| + C_0 v(x_0, t_0)^{q-1}|D'u(x_0, t_0)|^2 \\
&\qquad + |\phi(x_0)||D\phi(x_0)|v(x_0, t_0)^{1-q} + |\phi(x_0)|Du(x_0, t_0)||D^2 h(x_0)||,
\end{aligned}$$

from the fact that $\frac{1}{q+1}\frac{\partial w}{\partial \vec{\mathbf{n}}}(x_0, t_0) = -\frac{1}{q+1}\frac{\partial \overline{w}}{\partial y_n}(y_0, t_0)$ and (3.40). By the boundary condition $\frac{\partial u}{\partial x_n} = -\phi v^{1-q}$ at (x_0, t_0), we see that

$$|D'u(x_0, t_0)|^2 = v(x_0, t_0)^2 - \left(\frac{\partial u}{\partial x_n}(x_0, t_0)\right)^2 - \eta^2 = v(x_0, t_0)^2 - \phi(x_0)^2 v(x_0, t_0)^{2-2q} - \eta^2.$$

Together with the fact that $\eta \in (0, 1]$ and that

$$C_0 v(x_0, t_0)^{q-1}(-\phi(x_0)^2 v(x_0, t_0)^{2-2q} - \eta^2) \le |C_0|\|\phi\|^2_{C^0(\partial\Omega)} v(x_0, t_0)^{1-q} + |C_0|v(x_0, t_0)^{q-1},$$

we obtain that

$$
\begin{aligned}
&\frac{1}{q+1}\frac{\partial w}{\partial \vec{\mathbf{n}}}(x_0,t_0)\\
&\leq 3\|D\phi\|_{C^0(\partial\Omega)} v(x_0,t_0) + C_0 v(x_0,t_0)^{q+1} + |C_0|\|\phi\|^2_{C^0(\partial\Omega)} v(x_0,t_0)^{1-q} + |C_0| v(x_0,t_0)^{q-1}\\
&\qquad + \|\phi\|_{C^0(\partial\Omega)}\|D\phi\|_{C^0(\partial\Omega)} v(x_0,t_0)^{1-q} + \|\phi\|_{C^0(\partial\Omega)}\|h\|_{C^2(\partial\Omega)} v(x_0,t_0)
\end{aligned}
$$

Therefore, if $v = v(x_0,t_0) > R_0$, and also if $\eta \in (0,1]$, then

$$
\frac{1}{q+1}\frac{\partial w}{\partial \vec{\mathbf{n}}} \leq L_1 v^{q+1}
$$

at (x_0,t_0), where

$$
\begin{aligned}
L_1 := C_0 &+ 3\|D\phi\|_{C^0(\partial\Omega)} v^{-q} + |C_0|\|\phi\|^2_{C^0(\partial\Omega)} v^{-2q} + |C_0| v^{-2}\\
&+ \|\phi\|_{C^0(\partial\Omega)}\|D\phi\|_{C^0(\partial\Omega)} v^{-2q} + \|\phi\|_{C^0(\partial\Omega)}\|h\|_{C^2(\partial\Omega)} v^{-q},
\end{aligned}
$$

with $v = v(x_0,t_0)$.

Note that for a given $\varepsilon_0' \in (0,1)$, it holds that $1-\varepsilon_0' < 1-(q+1)\phi^2 v^{-2q} < 1+\varepsilon_0'$ when $v > \max\left\{1, R_0, \left((q+1)\|\phi\|^2_{C^0(\partial\Omega)}(\varepsilon_0')^{-1}\right)^{1/2q}\right\}$. Thus, for a given $\varepsilon_0' \in (0,1)$, there exists $R_{\varepsilon_0'} > 1$ that may depend on ε_0' but not on $T \in (0,\infty)$, $\eta \in (0,1]$, $x_0 \in \partial\Omega$ such that $C_0 \leq L_1 < C_0 + \varepsilon_0'$, and $1-\varepsilon_0' < 1-(q+1)\phi^2 v^{-2q} < 1+\varepsilon_0'$ and that $\frac{1}{q+1}\frac{\partial w}{\partial \vec{\mathbf{n}}} \leq L_1 v^{q+1}$ whenever $v > R_{\varepsilon_0'}$. Also, $w = v^{q+1} - (q+1)\phi^2 v^{1-q} > (1-\varepsilon_0') v^{q+1} > 0$ on $\partial\Omega \times \{t_0\}$ whenever $v > R_{\varepsilon_0'}$.

For a given $\varepsilon_0' \in (0,1)$ and for $v = v(x_0,t_0) > R_{\varepsilon_0'}$, $\eta \in (0,1]$, we have

$$
\begin{aligned}
\frac{1}{q+1}\frac{\partial w}{\partial \vec{\mathbf{n}}} &\leq L_1 v^{q+1}\\
&= L_1 \frac{v^{q+1}}{v^{q+1} - (q+1)\phi^2 v^{1-q}} w\\
&= \frac{L_1}{1-(q+1)\phi^2 v^{-2q}} w.
\end{aligned}
$$

at (x_0, t_0). If $C_0 + \varepsilon_0' \geq 0$,

$$\frac{L_1}{1-(q+1)\phi^2 v^{-2q}} < \frac{C_0+\varepsilon_0'}{1-\varepsilon_0'},$$

and if $C_0 + \varepsilon_0' < 0$,

$$\frac{L_1}{1-(q+1)\phi^2 v^{-2q}} < \frac{C_0+\varepsilon_0'}{1+\varepsilon_0'}.$$

For a given $\varepsilon_0 \in (0,1)$, there exists $\varepsilon_0' \in (0,1)$ that depends only on ε_0 such that

$$\frac{C_0+\varepsilon_0'}{1-\varepsilon_0'} < C_0 + \varepsilon_0 \quad \text{and} \quad \frac{C_0+\varepsilon_0'}{1+\varepsilon_0'} < C_0 + \varepsilon_0.$$

Therefore, for a given $\varepsilon_0 \in (0,1)$, there exists a constant $R_{\varepsilon_0} > 1$ that may depend on ε_0 but not on $T \in (0,\infty)$, $\eta \in (0,1]$ and also not on $x_0 \in \partial\Omega$ such that at (x_0, t_0), $w > 0$ for $v > R_{\varepsilon_0}$, and

$$\frac{1}{q+1}\frac{\partial w}{\partial \vec{\mathbf{n}}} < (C_0 + \varepsilon_0)\, w,$$

or

$$\frac{\partial w}{\partial \vec{\mathbf{n}}} < Lw \tag{3.41}$$

for $v > R_{\varepsilon_0}$, where $L = (q+1)(C_0 + \varepsilon_0)$. Note that we relied on the fact that x_0 is a maximizer of w on $\partial\Omega \times \{t_0\}$, and this condition will be emphasized in future applications in the estimate on the boundary.

We claim that if $C_0 < 0$, then $v(x_0, t_0) \leq R$ for some constant $R > 1$ that does not depend on $T \in (0,\infty)$, $\eta \in (0,1]$ and also not on $x_0 \in \partial\Omega$. This is because if we choose $\varepsilon_0 = \frac{1}{2}\min\{\frac{1}{2}, -\frac{1}{2}C_0\}$, then there is a constant $R = R_{\varepsilon_0}$, which is now fixed by the choice of ε_0, such that $w > 0$ and

$$\frac{1}{q+1}\frac{\partial w}{\partial \vec{\mathbf{n}}} < (C_0 + \varepsilon_0)\, w$$

if $v(x_0,t_0) = v > R = R_{\varepsilon_0}$. If it really were that $v(x_0,t_0) > R = R_{\varepsilon_0}$, then we would have

$$\frac{1}{q+1}\frac{\partial w}{\partial \vec{\mathbf{n}}} < (C_0 + \varepsilon_0)\, w < 0.$$

However, this is a contradiction, since x_0 is a maximizer of w on $\overline{\Omega} \times \{t_0\}$, it must hold that $\frac{\partial w}{\partial \vec{\mathbf{n}}} \geq 0$ at (x_0,t_0). Therefore, $v(x_0,t_0) \leq R$ for some constant $R > 0$ that does not depend on $T \in (0,\infty)$, $\eta \in (0,1]$ (also not on $x_0 \in \partial\Omega$). Since our goal is to prove the bound $v(x_0,t_0) \leq R$, we are done in the case when $C_0 < 0$, and this argument verifies Theorem 3.1.2 in the case when $C_0 < 0$ under the assumption (3.9) with $C_0 < 0$.

It remains the case when $C_0 \geq 0$. From now on, we assume that $C_0 \geq 0$, and thus that $L \geq 0$.

Step 6. For a new function $\psi := \rho w$, we get a new maximizer (x_1,t_1) of ψ with $x_1 \in \Omega, t_1 > 0$ by choosing a specific multiplier ρ. We apply the maximum principle to ψ at (x_1,t_1) in order to bound $v(x_1,t_1)$.

Let $\psi := \rho w$ with a multiplier $\rho = \rho(x)$ that is smooth on $\mathbb{R}^n$. We require that $\rho(x_0) = 1$, $\frac{\partial \rho}{\partial \vec{\mathbf{n}}}(x_0) = -L$. Let $B = B(x_c, K_0)$ be the open ball with the center $x_c := x_0 - K_0\vec{\mathbf{n}}(x_0)$ so that $B \subseteq \Omega$ and $\overline{B} \cap (\mathbb{R}^n \setminus \Omega) = \{x_0\}$. Choose

$$\rho(x) := -\frac{L}{2K_0}|x - x_c|^2 + \frac{LK_0}{2} + 1.$$

Since we assume $L \geq 0$, it holds that $\rho \geq 1$ in B. Also, ρ is a quadratic function in $|x-x_c|$, and $\rho(x_0) = 1$, $\frac{\partial \rho}{\partial \vec{\mathbf{n}}}(x_0) = -L$. Then, by (3.41),

$$\frac{\partial \psi}{\partial \vec{\mathbf{n}}} = \rho\frac{\partial w}{\partial \vec{\mathbf{n}}} + w\frac{\partial \rho}{\partial \vec{\mathbf{n}}} = \frac{\partial w}{\partial \vec{\mathbf{n}}} + (-L)w < 0, \qquad \text{at } (x_0,t_0),$$

if $v(x_0,t_0) > R_{\varepsilon_0}$ for a given $\varepsilon_0 \in (0,1)$.

For a given $\varepsilon_0 \in (0,1)$, assume $v(x_0,t_0) > R_{\varepsilon_0}$. Say the maximum of $\psi = \rho w$ on

$\overline{B} \times [0,T]$ occurs at $(x_1, t_1) \in \overline{B} \times [0,T]$. If $t_1 = 0$, then

$$w(x_0, t_0) = \rho(x_0)w(x_0, t_0) \le \rho(x_1)w(x_1, t_1) = \rho(x_1)w(x_1, 0) \le R,$$

for some constant $R > 1$ independent of $T \in (0, \infty)$, $\eta \in (0, 1]$. Thus, it proves that $w(x_0, t_0) \le R$ in this case. Using the fact that $v^{q+1} - Cv \le w$ for some constant $C > 0$ depending only on $\|\phi\|_{C^0(\overline{\Omega})}$, $\|h\|_{C^1(\overline{\Omega})}$, we see that $v(x_0, t_0) \le R$, and we reach our goal. Therefore, we now consider the case when $t_1 > 0$, and we assume $t_1 > 0$ from now on. If $x_1 \in \partial B$, then $\rho(x_1) = \rho(x_0)$, and thus,

$$\rho(x_1)w(x_1, t_1) \le \rho(x_0)w(x_0, t_0).$$

However, $\rho(x_0)w(x_0, t_0) < \rho(x)w(x, t_0)$ for some $x \in B$ since $\frac{\partial(\rho w)}{\partial \vec{\mathbf{n}}}(x_0, t_0) < 0$. It contradicts with the choice of $(x_1, t_1) \in \operatorname{argmax}_{\overline{B} \times [0,T]} \psi$. Therefore, $x_1 \in B$, and it suffices to consider the case $(x_1, t_1) \in B \times (0, T]$.

For a given $\varepsilon_0 \in (0, 1)$, we always assume from now on that $v(x_0, t_0) > R_{\varepsilon_0}$ so that $w > 0 and (3.41)$ are valid. Also, we assume that a maximizer $(x_1, t_1) \in \operatorname{argmax}_{\overline{B} \times [0,T]} \psi$ happens in $B \times (0, T]$, since we achieve the goal, i.e., to prove $v(x_0, t_0) \le R$, in the other cases from the above argument. Fix $(x_1, t_1) \in \operatorname{argmax}_{\overline{B} \times [0,T]} \psi \cap (B \times (0, T])$.

Before we move on the next step, we check that there exists a constant $C > 0$ depending only on $\|\phi\|_{C^0(\overline{\Omega})}$, $\|h\|_{C^1(\overline{\Omega})}$ such that the condition $v(x_0, t_0) > R_{\varepsilon_0}$ with $R_{\varepsilon_0} > (8C)^{\frac{1}{q+1}}$ implies the condition $v(x_1, t_1) > \left(\frac{1}{4C}\right)^{\frac{1}{q+1}} R_{\varepsilon_0} =: R'_{\varepsilon_0}$. This is because there exists a constant $C > 0$ depending only on $\|\phi\|_{C^0(\overline{\Omega})}$, $\|h\|_{C^1(\overline{\Omega})}$ such that

$$v(x_0, t_0)^{q+1} - Cv(x_0, t_0) \le w(x_0, t_0) \le \rho(x_1, t_1)w(x_1, t_1) \le C(v(x_1, t_1)^{q+1} + v(x_1, t_1)).$$

Moreover, if $v(x_0, t_0) > R_{\varepsilon_0}$ with $R_{\varepsilon_0} > (8C)^{\frac{1}{q+1}}$, then

$$\frac{1}{2}R_{\varepsilon_0}^{q+1} < \frac{1}{2}v(x_0, t_0)^{q+1} \le v(x_0, t_0)^{q+1} - Cv(x_0, t_0) \le C(v(x_1, t_1)^{q+1} + v(x_1, t_1)).$$

If $v(x_1,t_1) \le 1$, then we would have $\frac{1}{2}R_{\varepsilon_0}^{q+1} < 2C$, which contradicts to $R_{\varepsilon_0} > (8C)^{\frac{1}{q+1}}$. Thus, $v(x_1,t_1) > 1$, which gives $\frac{1}{2}R_{\varepsilon_0}^{q+1} < 2Cv(x_1,t_1)^{q+1}$ and the conclusion that $v(x_1,t_1) > R'_{\varepsilon_0}$. We note that this is true whenever we replace the constant $C > 0$ by a larger one.

Writing $R'_{\varepsilon_0} = \left(\frac{1}{4C}\right)^{\frac{1}{q+1}} R_{\varepsilon_0}$, $R_{\varepsilon_0} = (4C)^{\frac{1}{q+1}} R'_{\varepsilon_0}$ (and also for $R,\ R'$ similarly), we can state the above equivalently that if $v(x_1,t_1) \le R'_{\varepsilon_0}$, then $v(x_0,t_0) \le \max\left\{R_{\varepsilon_0}, (8C)^{\frac{1}{q+1}}\right\}$. Accordingly, we change our goal from verifying $v(x_0,t_0) \le R$ to proving $v(x_1,t_1) \le R'$.

By the maximum principle, $D^2\psi \le 0,\ \psi_t \ge 0$ at (x_1,t_1), and thus,

$$0 \ge \frac{1}{(q+1)\rho}\left(\mathrm{tr}\{a(Du)D^2\psi\} - \psi_t\right) \tag{3.42}$$

at (x_1,t_1). Substituting the derivatives of ψ with those of ρ and w, we obtain, at (x_1,t_1),

$$0 \ge \frac{w}{(q+1)\rho}\mathrm{tr}\{a(Du)D^2\rho\} + \frac{2}{(q+1)\rho}\mathrm{tr}\{a(Du)Dw\otimes D\rho\} + \frac{1}{q+1}(\mathrm{tr}\{a(Du)D^2w\} - w_t). \tag{3.43}$$

Following the computations up to (3.21) in Step 1, we see that there exist a constant $C > 0$ independent of $T \in (0,\infty),\ \eta \in (0,1],\ \varepsilon_0 \in (0,1)$ and a constant $R'_{\varepsilon_0} > 1$ that may depend on $\varepsilon_0 \in (0,1)$ but not on $T \in (0,\infty),\ \eta \in (0,1]$ such that, at (x_1,t_1),

$$\begin{aligned} 0 &\ge \frac{1}{q+1}\left(\mathrm{tr}\{a(Du)D^2w\} - w_t\right) \\ &\ge J_1 + J_2 - |Dc|v^{q+1} + (q+1-\varepsilon_0)V - C(v+v^q) \end{aligned} \tag{3.44}$$

if $v = v(x_1,t_1) > R'_{\varepsilon_0}$, with the same definitions of $J_1,\ J_2$ (ε replaced by ε_0).

We check for a moment that, at (x_1,t_1),

$$V \ge V_1 + V_2, \tag{3.45}$$

where

$$V_1 := v^{-q-1}\mathrm{tr}\left\{a(Du)\left(\frac{w}{(q+1)\rho}D\rho\right)\otimes\left(\frac{w}{(q+1)\rho}D\rho\right)\right\},$$
$$V_2 := -2v^{-q-1}\mathrm{tr}\left\{a(Du)\left(\frac{w}{(q+1)\rho}D\rho\right)\otimes\big((Du\cdot Dh)D\phi+\phi D^2uDh+\phi D^2hDu\big)\right\}.$$

At (x_1,t_1), we have that $D\psi = wD\rho+\rho Dw = 0$ so that

$$-\frac{w}{\rho}D\rho = (q+1)(v^qDv-(Du\cdot Dh)D\phi-\phi D^2uDh-\phi D^2hDu).$$

By putting

$$v^qDv = -\frac{w}{(q+1)\rho}D\rho+(Du\cdot Dh)D\phi+\phi D^2uDh+\phi D^2hDu$$

into $V = v^{q-1}\mathrm{tr}\{a(Du)Dv\otimes Dv\} = v^{-q-1}\mathrm{tr}\{a(Du)(v^qDv)\otimes(v^qDv)\}$, we obtain (3.45).

By (3.43), (3.44), (3.45), there exist a constant $C>0$ independent of $T\in(0,\infty)$, $\eta\in(0,1]$, $\varepsilon_0\in(0,1)$ and a constant $R'_{\varepsilon_0}>1$ that may depend on $\varepsilon_0\in(0,1)$ but not on $T\in(0,\infty)$, $\eta\in(0,1]$ such that, at (x_1,t_1),

$$0 \geq \frac{w}{(q+1)\rho}\mathrm{tr}\{a(Du)D^2\rho\}+\left(\frac{2}{(q+1)\rho}\mathrm{tr}\{a(Du)Dw\otimes D\rho\}+(q+1)V_1\right)$$
$$+J_1'+J_2'-|Dc|v^{q+1}-C(v+v^q) \qquad (3.46)$$

if $v = v(x_1, t_1) > R'_{\varepsilon_0}$, where

$$\begin{aligned} J_1' :&= J_1 - \varepsilon_0 V \\ &= (1-\varepsilon_0)v^{q-1}\mathrm{tr}\{(a(Du)D^2u)^2\} - \frac{1}{2}v^q \mathrm{tr}\{(D_p a(Du) \odot Dv)D^2u\} \\ &\qquad + cDv \cdot (-v^{q-1}Du + \phi Dh) - \varepsilon_0 V, \\ J_2' :&= J_2 + (q+1)V_2 \\ &= \varepsilon_0 v^{q-1}\mathrm{tr}\{(a(Du)D^2u)^2\} - \frac{1}{2}\varepsilon_0 v^q \mathrm{tr}\{(D_p a(Du) \odot Dv)D^2u\} \\ &\qquad - 2\mathrm{tr}\{a(Du)(D\phi \otimes (D^2uDh))\} - 2\phi \mathrm{tr}\{a(Du)D^2uD^2h\} \\ &\qquad + \phi \mathrm{tr}\{(D_p a(Du) \odot (D^2uDh))D^2u\} + (q+1)V_2. \end{aligned}$$

Step 7. We estimate the terms of (3.46).

We start with the first term of (3.46). By the fact that $D^2\rho = -\frac{L}{K_0}I_n$ and $\rho \geq 1$ in B, we see that

$$\begin{aligned} &\frac{w}{(q+1)\rho}\mathrm{tr}\{a(Du)D^2\rho\} \\ &= \frac{w}{(q+1)\rho}\left(-\frac{L}{K_0}\right)\left(\frac{\eta^2}{v^2} + n - 1\right) \\ &\geq -\frac{L}{(q+1)K_0}(v^{q+1} + (q+1)\|\phi\|_{C^0(\overline{\Omega})}\|h\|_{C^1(\overline{\Omega})}v)\left(\frac{\eta^2}{v^2} + n - 1\right). \end{aligned}$$

Therefore, there exists a constant $C > 0$ independent of $T \in (0, \infty)$, $\eta \in (0, 1]$, $\varepsilon_0 \in (0, 1)$ such that, at (x_1, t_1),

$$\frac{w}{(q+1)\rho}\mathrm{tr}\{a(Du)D^2\rho\} \geq -\frac{(n-1)(C_0+\varepsilon_0)}{K_0}v^{q+1} - C(v + v^q) \tag{3.47}$$

if $v = v(x_1, t_1) > 1$. Here, we have used the fact that $\eta \in (0, 1]$.

We bound the second term of (3.46). Since $Dw = -\frac{w}{\rho}D\rho$ at (x_1, t_1), we obtain

$$\frac{2}{(q+1)\rho}\mathrm{tr}\{a(Du)Dw \otimes D\rho\} + (q+1)V_1 = \frac{(wv^{-1-q} - 2)w}{q+1}\mathrm{tr}\left\{a(Du)\frac{D\rho}{\rho} \otimes \frac{D\rho}{\rho}\right\}$$

at (x_1, t_1). From

$$v^{q+1} - (q+1)\|\phi\|_{C^0(\overline{\Omega})}\|h\|_{C^1(\overline{\Omega})}v \leq w \leq v^{q+1} + (q+1)\|\phi\|_{C^0(\overline{\Omega})}\|h\|_{C^1(\overline{\Omega})}v,$$

we see that there exists a constant $R'_{\varepsilon_0} > 1$ that may depend on $\varepsilon_0 \in (0,1)$ but not on $T \in (0,\infty)$, $\eta \in (0,1]$ such that $|wv^{-q-1} - 1| < \varepsilon_0$ for $v > R'_{\varepsilon_0}$. Using the fact that

$$0 \leq \operatorname{tr}\left\{a(Du)\frac{D\rho}{\rho} \otimes \frac{D\rho}{\rho}\right\} \leq \left|\frac{D\rho}{\rho}\right|^2 = \frac{L^2}{K_0^2}|x_1 - x_c|^2 \leq (C_0 + \varepsilon_0)^2,$$

and the fact that

$$w \leq v^{q+1} + (q+1)\|\phi\|_{C^0(\overline{\Omega})}\|h\|_{C^1(\overline{\Omega})}v$$

once again, we see that there exist a constant $C > 0$ independent of $T \in (0,\infty)$, $\eta \in (0,1]$, $\varepsilon_0 \in (0,1)$ and a constant $R'_{\varepsilon_0} > 1$ that may depend on $\varepsilon_0 \in (0,1)$ but not on $T \in (0,\infty)$, $\eta \in (0,1]$ such that, at (x_1, t_1),

$$\frac{2}{(q+1)\rho}\operatorname{tr}\{a(Du)Dw \otimes D\rho\} + (q+1)V_1 \geq -(q+1)(C_0+\varepsilon_0)^2(1+\varepsilon_0)v^{q+1} - Cv. \quad (3.48)$$

if $v = v(x_1, t_1) > R'_{\varepsilon_0}$.

We give an estimate of the term J_1' of (3.46). Following the same computation of J_1, we have (3.22) with ε_0 instead of ε, and thus, we see that there exist a constant $C > 0$ independent of $T \in (0,\infty)$, $\eta \in (0,1]$, $\varepsilon_0 \in (0,1)$ and a constant $R'_{\varepsilon_0} > 1$ that may depend on $\varepsilon_0 \in (0,1)$ but not on $T \in (0,\infty)$, $\eta \in (0,1]$ such that

$$(1-\varepsilon_0)v^{q-1}\operatorname{tr}\{(a(Du)D^2u)^2\} - \frac{1}{2}v^q\operatorname{tr}\{(D_pa(Du) \odot Dv)D^2u\} \geq \frac{1-\varepsilon_0}{n-1}c^2v^{q+1} + \varepsilon_0 V - Cv^q. \quad (3.49)$$

if $v > R'_{\varepsilon_0}$.

We claim that at (x_1, t_1), it holds that, for $v > 1$,

$$|cDv \cdot (-v^{q-1}Du + \phi Dh)| \leq Cv + (C_0 + \varepsilon_0)|c|v^{q+1} \tag{3.50}$$

for some constant $C > 0$ independent of $T \in (0, \infty)$, $\eta \in (0, 1]$, $\varepsilon_0 \in (0, 1)$. Since $D\psi = 0$ at (x_1, t_1),

$$\begin{aligned} 0 &= \frac{1}{(q+1)\rho} D\psi \cdot Du \\ &= v^q Du \cdot Dv - (Du \cdot D\phi)(Du \cdot Dh) - \phi(D^2uDu) \cdot Dh - \phi(D^2hDu) \cdot Du \\ &\qquad + \frac{w}{(q+1)\rho} D\rho \cdot Du. \end{aligned}$$

This implies that at (x_1, t_1),

$$cDv \cdot (-v^{q-1}Du + \phi Dh) = -\frac{c}{v}\left((Du \cdot D\phi)(Du \cdot Dh) + \phi(D^2hDu) \cdot Du - \frac{w}{(q+1)\rho} D\rho \cdot Du\right),$$

and thus that at (x_1, t_1),

$$\begin{aligned} |cDv \cdot (-v^{q-1}Du + \phi Dh)| &\leq Cv + \frac{L|c|}{K_0(q+1)}|x_1 - x_c|(v^{q+1} + (q+1)\|\phi\|_{C^0(\overline{\Omega})}\|h\|_{C^1(\overline{\Omega})}v) \\ &\leq Cv + (C_0 + \varepsilon_0)|c|v^{q+1} \end{aligned}$$

for some constant $C > 0$ independent of $T \in (0, \infty)$, $\eta \in (0, 1]$, $\varepsilon_0 \in (0, 1)$.

By (3.49), (3.50), we conclude that there exist a constant $C > 0$ independent of $T \in (0, \infty)$, $\eta \in (0, 1]$, $\varepsilon_0 \in (0, 1)$ and a constant $R'_{\varepsilon_0} > 1$ that may depend on $\varepsilon_0 \in (0, 1)$ but not on $T \in (0, \infty)$, $\eta \in (0, 1]$ such that

$$J_1' \geq \left(\frac{1 - \varepsilon_0}{n-1}c^2 - (C_0 + \varepsilon_0)|c|\right) v^{q+1} - C(v + v^q) \tag{3.51}$$

if $v > R'_{\varepsilon_0}$.

Now, we bound the term J_2' of (3.46). Taking the axes at x_1 so that (3.25) holds, and calculating $u_{1i},\ i=2,\cdots,n,\ u_{11}$ using $\rho Dw + wD\rho = 0$ at (x_1,t_1), we obtain

$$u_{1i} = E_i u_1 + F_i u_{ii} + G_i w, \qquad i=2,\cdots,n, \tag{3.52}$$

where

$$E_i := \frac{\phi_i h_1 + \phi h_{1i}}{v^{q-1}u_1 - \phi h_1}, \quad F_i := \frac{\phi h_i}{v^{q-1}u_1 - \phi h_1}, \qquad i=2,\cdots,n$$

and

$$G_i := -\frac{\rho_i}{(q+1)\rho(v^{q-1}u_1 - \phi h_1)}, \qquad i=2,\cdots,n.$$

For $i=1$, we get

$$u_{11} = E_1 u_1 + \sum_{\ell=2}^{n} F_\ell^2 u_{\ell\ell} + G_1 w, \tag{3.53}$$

where

$$E_1 := \frac{\phi_1 h_1 + \phi h_{11}}{v^{q-1}u_1 - \phi h_1} + \frac{\phi}{v^{q-1}u_1 - \phi h_1}\sum_{\ell=2}^{n} h_\ell E_\ell,$$

and

$$G_1 := -\frac{\rho_1}{(q+1)\rho(v^{q-1}u_1 - \phi h_1)} + \frac{\phi}{v^{q-1}u_1 - \phi h_1}\sum_{\ell=2}^{n} h_\ell G_\ell.$$

The definitions of E_i's and F_i's are the same as before, and we display them to recall. Note that the denominator $v^{q-1}u_1 - \phi h_1$ is nonzero for $v > R'$ for some constant $R' > 1$ independent of $T \in (0,\infty),\ \eta \in (0,1],\ \varepsilon_0 \in (0,1)$.

Write

$$J_2' = \varepsilon_0 v^{q-1}\mathrm{tr}\{(a(Du)D^2u)^2\} + S_1' + S_2' - \frac{2wv^{-q-1}}{\rho}\mathrm{tr}\{a(Du)D\rho \otimes ((Du\cdot Dh)D\phi + \phi D^2 h Du)\},$$

where

$$
\begin{aligned}
S_1' :&= S_1 - \frac{2wv^{-q-1}\phi}{\rho}\mathrm{tr}\{a(Du)D\rho\otimes(D^2uDh)\} \\
&= -2\mathrm{tr}\{a(Du)(D\phi\otimes(D^2uDh))\} - 2\phi\mathrm{tr}\{a(Du)D^2uD^2h\} \\
&\qquad\qquad\qquad\qquad - \frac{2wv^{-q-1}\phi}{\rho}\mathrm{tr}\{a(Du)D\rho\otimes(D^2uDh)\}, \\
S_2' :&= S_2 \\
&= -\frac{1}{2}\varepsilon_0 v^q\mathrm{tr}\{(D_pa(Du)\odot Dv)D^2u\} + \phi\mathrm{tr}\{(D_pa(Du)\odot(D^2uDh))D^2u\},
\end{aligned}
$$

with $S_1,\ S_2$ defined as in Case 1 (ε replaced by ε_0).

Computing S_1' in a similar manner as before, we get

$$
\begin{aligned}
S_1' = -2\Bigg(\left(\frac{\eta^2}{v^2}H'_{11}E_1 + \sum_{\ell=2}^n\left(\frac{\eta^2}{v^2}H'_{1\ell}+H'_{\ell 1}\right)E_\ell\right)u_1 \\
+\sum_{\ell=2}^n\left(\frac{\eta^2}{v^2}H'_{11}F_\ell^2 + \left(\frac{\eta^2}{v^2}H'_{1\ell}+H'_{\ell 1}\right)F_\ell + H'_{\ell\ell}\right)u_{\ell\ell}\Bigg) \\
-2\left(\frac{\eta^2}{v^2}H'_{11}G_1 + \sum_{\ell=2}^n G_\ell\left(\frac{\eta^2}{v^2}H'_{1\ell}+H'_{\ell 1}\right)\right)w,
\end{aligned}
$$

where $H'_{\ell i} := H_{\ell i} + \frac{wv^{-q-1}\phi}{\rho}\rho_\ell h_i = \phi_\ell h_i + \phi h_{\ell i} + \frac{wv^{-q-1}\phi}{\rho}\rho_\ell h_i$ for each $\ell, i = 1,\cdots,n$. Note that since $\eta\in(0,1]$,

$$
\left|\frac{\eta^2}{v^2}H'_{11}E_1 + \sum_{\ell=2}^n\left(\frac{\eta^2}{v^2}H'_{1\ell}+H'_{\ell 1}\right)E_\ell\right| \le Cv^{-q},
$$

$$
\left|\frac{\eta^2}{v^2}H'_{11}F_\ell^2 + \left(\frac{\eta^2}{v^2}H'_{1\ell}+H'_{\ell 1}\right)F_\ell + H'_{\ell\ell}\right| \le C,
$$

$$
\left|\frac{\eta^2}{v^2}H'_{11}G_1 + \sum_{\ell=2}^n G_\ell\left(\frac{\eta^2}{v^2}H'_{1\ell}+H'_{\ell 1}\right)\right| \le Cv^{-q}
$$

for $v > R'_{\varepsilon_0}$. Here, $R'_{\varepsilon_0} > 1$ is some constant that may depend on $\varepsilon_0\in(0,1)$ but not on $T\in(0,\infty),\ \eta\in(0,1]$, and $C>0$ is another constant independent of $T\in(0,\infty),\ \eta\in(0,1],\ \varepsilon_0\in(0,1)$. Using the fact that $|w|\le v^{q+1}+(q+1)\|\phi\|_{C^0(\overline{\Omega})}\|h\|_{C^1(\overline{\Omega})}v$, we see that there exist a constant $C > 0$ independent of $T \in (0, \infty)$, $\eta \in (0, 1]$, $\varepsilon_0 \in (0, 1)$ and a constant $R'_{\varepsilon_0} > 1$ that may depend on $\varepsilon_0 \in (0, 1)$ but not on $T \in (0, \infty)$, $\eta \in (0, 1]$ such that

$$S'_1 \geq -C \left(v + \sum_{\ell=2}^{n} |u_{\ell\ell}| \right) \tag{3.54}$$

for $v > R'_{\varepsilon_0}$.

Following the same computation of S_2, we have

$$S'_2 \geq -\varepsilon_0^{-1} v^{-1-q} K_1^2 - \varepsilon_0^{-1} v^{1-q} \sum_{\ell=2}^{n} K_\ell^2,$$

where $K_\ell := \phi v^{-1}(Du_\ell \cdot Dh)$ for each $\ell = 1, \cdots, n$. By expansion and (3.52), (3.53), we have

$$K_1 = K_{11} u_1 + \sum_{\ell=2}^{n} K_{1\ell} u_{\ell\ell} + M_1 w,$$

where

$$K_{11} := \phi v^{-1} \sum_{\ell=1}^{n} h_\ell E_\ell, \quad K_{1\ell} := \phi v^{-1}(h_1 F_\ell^2 + h_\ell F_\ell), \quad \text{for } \ell = 2, \cdots, n.$$

and

$$M_1 := \phi v^{-1} \sum_{\ell=1}^{n} h_\ell G_\ell.$$

For $\ell = 2, \cdots, n$,

$$K_\ell = K_{\ell 1} u_1 + K_{\ell\ell} u_{\ell\ell},$$

where

$$K_{\ell 1} := \phi v^{-1} h_1 E_\ell, \quad K_{\ell\ell} := \phi v^{-1}(h_1 F_\ell + h_\ell), \quad \text{for } \ell = 2, \cdots, n,$$

and

$$M_\ell := \phi v^{-1} G_\ell, \quad \text{for } \ell = 2, \cdots, n.$$

Applying Cauchy-Schwarz inequality as before in S_2, we obtain

$$S_2' \geq -\varepsilon_0^{-1} S_{21} u_1^2 - \varepsilon_0^{-1} \sum_{\ell=2}^{n} S_{2\ell} u_{\ell\ell}^2 - \varepsilon_0^{-1} M w^2, \tag{3.55}$$

where

$$S_{21} := n v^{-1-q} K_{11}^2 + 2 v^{1-q} \sum_{\ell=2}^{n} K_{\ell 1}^2, \ S_{2\ell} := n v^{-1-q} K_{1\ell}^2 + 2 v^{1-q} K_{\ell\ell}^2, \ \text{ for } \ell = 2, \cdots, n,$$

and

$$M := (n+1) M_1^2 v^{-1-q} + 3 v^{1-q} \sum_{\ell=2}^{n} M_\ell^2.$$

We note that there exist a constant $C > 0$ independent of $T \in (0, \infty)$, $\eta \in (0, 1]$, $\varepsilon_0 \in (0, 1)$ and a constant $R'_{\varepsilon_0} > 1$ that may depend on $\varepsilon_0 \in (0, 1)$ but not on $T \in (0, \infty)$, $\eta \in (0, 1]$ such that, at (x_1, t_1),

$$\begin{aligned} &\left| -\frac{2wv^{-q-1}}{\rho} \operatorname{tr}\{a(Du) D\rho \otimes ((Du \cdot Dh) D\phi + \phi D^2 h Du)\} \right| \\ &\leq C \|a\| |D\rho| (|D\phi||Dh||Du| + |\phi| \|D^2 h\| |Du|) \\ &\leq Cv. \end{aligned} \tag{3.56}$$

Here, we have used the fact that $|wv^{-q-1} - 1| < \varepsilon_0$ for $v > R'_{\varepsilon_0}$ (making $R'_{\varepsilon_0} > 1$ larger if necessary), that $\rho \geq 1$ at $x_1 \in B$ and that $\|a\| = \left(\frac{\eta^4}{v^4} + n - 1\right)^{1/2} \leq \frac{\eta^2}{v^2} + n - 1 \leq C$ for

$v > 1,\ \eta \in (0,1]$. Also, the constants $C > 0,\ R'_{\varepsilon_0} > 1$ can be taken in a way that they may depend on $\|\rho\|_{C^1(\overline{\Omega})},\ \|\phi\|_{C^1(\overline{\Omega})},\ \|h\|_{C^2(\overline{\Omega})}$, but not on a specific position $x_1 \in \overline{\Omega}$.

By (3.54), (3.55), (3.56), we see that there exist a constant $C > 0$ independent of $T \in (0,\infty),\ \eta \in (0,1],\ \varepsilon_0 \in (0,1)$ and a constant $R'_{\varepsilon_0} > 1$ that may depend on $\varepsilon_0 \in (0,1)$ but not on $T \in (0,\infty),\ \eta \in (0,1]$ such that

$$J_2' \geq \varepsilon_0 v^{q-1} \sum_{\ell=2}^{n} u_{\ell\ell}^2 - C(v + \sum_{\ell=2}^{n} |u_{\ell\ell}|) - \varepsilon_0^{-1} S_{21} u_1^2 - \varepsilon_0^{-1} \sum_{\ell=2}^{n} S_{2\ell} u_{\ell\ell}^2$$

for $v > R'_{\varepsilon_0}$. As before, there exists a constant $C > 0$ that depends only on $\|\phi\|_{C^1(\overline{\Omega})},\ \|h\|_{C^2(\overline{\Omega})}$ such that, for $v > R'_{\varepsilon_0}$

$$|S_{21}| + \sum_{\ell=2}^{n} |S_{2\ell}| + |M| \leq C v^{-1-3q}.$$

Using the fact that $|w| \leq v^{q+1} + (q+1)\|\phi\|_{C^0(\overline{\Omega})}\|h\|_{C^1(\overline{\Omega})} v$, we see that there exist a constant $C > 0$ independent of $T \in (0,\infty),\ \eta \in (0,1],\ \varepsilon_0 \in (0,1)$ and a constant $R'_{\varepsilon_0} > 1$ that may depend on $\varepsilon_0 \in (0,1)$ but not on $T \in (0,\infty),\ \eta \in (0,1]$ such that

$$J_2' \geq -Cv - C\varepsilon_0^{-1} v^{1-q} + \sum_{\ell=2}^{n} \left(v^{q-1} \left(\varepsilon_0 - C\varepsilon_0^{-1} v^{-4q} \right) u_{\ell\ell}^2 - C|u_{\ell\ell}| \right)$$

for $v > R'_{\varepsilon_0}$.

As before, by choosing $R'_{\varepsilon_0} > 1$ that may depend on $\varepsilon_0 \in (0,1)$ but not on $T \in (0,\infty),\ \eta \in (0,1]$ such that $C\varepsilon_0^{-1} v^{-4q} < \frac{\varepsilon_0}{2}$ if $v > R'_{\varepsilon_0}$. Then, for $v > R'_{\varepsilon_0}$,

$$J_2' \geq -Cv - C\varepsilon_0^{-1} v^{1-q} + \sum_{\ell=2}^{n} \left(\frac{\varepsilon_0}{2} v^{q-1} \left(|u_{\ell\ell}| - \frac{C}{\varepsilon_0 v^{q-1}} \right)^2 - \frac{C^2}{2\varepsilon_0 v^{q-1}} \right).$$

All in all, for each $\varepsilon_0 \in (0,1)$, there exist a constant $C > 0$ independent of $T \in (0,\infty),\ \eta \in (0,1],\ \varepsilon_0 \in (0,1)$ and a constant $R'_{\varepsilon_0} > 1$ that may depend on $\varepsilon_0 \in (0,1)$ but not on $T \in (0,\infty),\ \eta \in (0,1]$ such that

$$J_2' \geq -Cv - C\varepsilon_0^{-1} v^{1-q}. \tag{3.57}$$

for $v > R'_{\varepsilon_0}$.

Step 8. We finish Case 2.

All in all, by (3.46), (3.47), (3.48), (3.51), (3.57), we see that there exist a constant $C > 0$ independent of $T \in (0, \infty)$, $\eta \in (0, 1]$, $\varepsilon_0 \in (0, 1)$ and a constant $R'_{\varepsilon_0} > 1$ that may depend on $\varepsilon_0 \in (0, 1)$ but not on $T \in (0, \infty)$, $\eta \in (0, 1]$ such that, at (x_1, t_1),

$$0 \geq \Big(\frac{1 - \varepsilon_0}{n - 1} c^2 - |Dc| - (C_0 + \varepsilon_0)|c| - \frac{(n-1)(C_0 + \varepsilon_0)}{K_0} \\ - (q+1)(C_0 + \varepsilon_0)^2 (1 + \varepsilon_0) \Big) v^{q+1} - C(v + v^q) - C\varepsilon_0^{-1} v^{1-q}.$$

if $v > R'_{\varepsilon_0}$. From the condition (3.9) and the assumption (3.3), we see that there exists $\varepsilon_0 \in (0, 1)$ such that the coefficient of v^{q+1} satisfies

$$\frac{1 - \varepsilon_0}{n - 1} c^2 - |Dc| - (C_0 + \varepsilon_0)|c| - \frac{(n-1)(C_0 + \varepsilon_0)}{K_0} - (q+1)(C_0 + \varepsilon_0)^2 (1 + \varepsilon_0) \geq \frac{\delta}{2}.$$

Fix such $\varepsilon_0 \in (0, 1)$. Then, R'_{ε_0}, ε_0^{-1} are fixed as well. Therefore, with this fixed $\varepsilon_0 \in (0, 1)$, there exist constants $R' > 1$, $C > 0$ independent of $T \in (0, \infty)$, $\eta \in (0, 1]$ such that at (x_1, t_1),

$$0 \geq \frac{\delta}{2} v^{q+1} - C(v + v^q)$$

if $v > R'$. There is, on the other hand, also a constant $R'_0 > R'$ independent of $T \in (0, \infty)$, $\eta \in (0, 1]$ such that

$$0 < \frac{\delta}{2} v^{q+1} - C(v + v^q)$$

if $v > R'_0$. Therefore, it must hold that $v = v(x_1, t_1) \leq R'_0$, which completes Case 2. □

Next, in order to prove Theorem 3.1.1, we prove *a priori* local gradient estimates, namely the following proposition 3.2.2.

Proposition 3.2.2. *Let $T \in (0, \infty)$, $\eta \in (0, 1]$. Suppose that a solution u^η of (3.7) exists and it is of class $C^{2,\sigma}(\overline{\Omega} \times [0, T]) \cap C^{3,\sigma}(\Omega \times (0, T])$ for some $\sigma \in (0, 1)$. Then u^η satisfies* *that*

$$\|Du^\eta\|_{L^\infty(\overline{\Omega}\times[0,T])} \le R_T,$$

where $R_T > 1$ *is a constant depending only on* $T, \Omega, c, f, \phi, q, u_0$.

Note that no assumption on the forcing term c is made, except for being $C^{1,\alpha}$. In the following proof of Proposition 3.2.2, we introduce a time-dependent multiplier.

Proof of Proposition 3.2.2. Now we only assume (3.3) and (3.4). Let $T \in (0,\infty)$, $\eta \in (0,1]$. Let $u = u^\eta \in C^{2,\sigma}(\overline{\Omega}\times[0,T]) \cap C^{3,\sigma}(\Omega\times(0,T])$ be a solution to (3.7) for some $\sigma \in (0,1)$. Let $w := v^{q+1} - (q+1)\phi Du \cdot Dh$ on $\overline{\Omega}\times[0,T]$. Let $R_T > 1$ denote a constant that may depend on $T \in (0,\infty)$ but not on $\eta \in (0,1)$. As before, $R_T > 1$ may vary line by line.

The goal is to prove that $w(x,t) \le R_T$ for all $(x,t) \in \overline{\Omega}\times[0,T]$. Once we achieve this goal, we complete the proof of Proposition 3.2.2 by using the fact that $v^{q+1} - Cv \le w$ for some constant $C > 0$ depending only on $\|h\|_{C^1(\overline{\Omega})}$, $\|\phi\|_{C^0(\overline{\Omega})}$ (and $q > 0$).

Let $M > 1$ be a constant to be determined. Let $(x_0,t_0) \in \operatorname{argmax}_{\overline{\Omega}\times[0,T]} e^{-Mt}w(x,t)$. We claim that in both cases of $t_0 = 0$ and $t_0 > 0$, $v(x_0,t_0)$ is bounded by a constant R_T that may depend on $T \in (0,\infty)$ but not on $\eta \in (0,1]$. In the case of $t_0 = 0$, we readily get a local gradient estimate. Indeed,

$$e^{-Mt}w(x,t) \le w(x_0,0) \le R \qquad \text{for all } (x,t) \in \overline{\Omega}\times[0,T]$$

for some constant $R > 1$ depending only on $\|u_0\|_{C^1(\overline{\Omega})}$, $\|h\|_{C^1(\overline{\Omega})}$, $\|\phi\|_{C^0(\overline{\Omega})}$, which proves our goal. Here, we have used the fact that $\eta \in (0,1]$.

It remains the case of $t_0 > 0$. Let $\rho(x,t) = e^{-Mt}\rho^0(x)$, where $\rho^0(x)$ will be chosen again according to the following cases; again divide into $x_0 \in \Omega$ and $x_0 \in \partial\Omega$.

Case 1: $x_0 \in \Omega$.

Take $\rho^0 \equiv 1$. Since $x_0 \in \Omega,\ t_0 > 0$, and $D\rho = 0,\ D^2\rho = 0$, we have that

$$\begin{aligned} 0 &\geq \frac{1}{(q+1)\rho}\left(\mathrm{tr}\{a(Du)D^2\psi\} - \psi_t\right) \\ &\geq \frac{1}{q+1}(\mathrm{tr}\{a(Du)D^2 w\} - w_t) - \frac{\rho_t w}{(q+1)\rho} \end{aligned}$$

at (x_0, t_0), where $\psi := \rho w$ as before.

Following the same argument in Step 1 of the proof of Proposition 3.2.1, we see that there exist constants $R > 1,\ C > 0$ independent of $T \in (0, \infty),\ \eta \in (0, 1]$ such that (3.21) holds true at (x_0, t_0) for $v > R$. Moreover, since $x_0 \in \mathrm{argmax}_{\overline{\Omega}}\, w(\cdot, t_0) \cap \Omega$ so that $Dw = 0$ at (x_0, t_0), (3.23) (for some constant $C > 0$ independent of $T \in (0, \infty),\ \eta \in (0, 1]$), (3.27), (3.28), (3.29), (3.30) are valid at (x_0, t_0). Therefore, we can follow the estimates in Step 3, Step 4 of the proof of Proposition 3.2.1 to conclude that for a given $\varepsilon \in (0, 1)$, there exists a constant $R_\varepsilon > 1$ that may depend on $\varepsilon \in (0, 1)$ but not on $T \in (0, \infty),\ \eta \in (0, 1]$ and a constant $C > 0$ independent of $T \in (0, \infty),\ \eta \in (0, 1],\ \varepsilon \in (0, 1)$ such that (3.24), (3.36) are valid at (x_0, t_0) for $v > R_\varepsilon$. We take $\varepsilon = \frac{1}{2}$, and we note that $-\frac{\rho_t w}{(q+1)\rho} = \frac{Mw}{q+1}$. Together with the fact that $w \geq v^{q+1} - (q+1)\|\phi\|_{C^0(\overline{\Omega})}\|h\|_{C^1(\overline{\Omega})} v$, we see that there exists a constant $R > 1,\ C > 0$ independent of $T \in (0, \infty),\ \eta \in (0, 1]$ such that

$$0 \geq \left(\frac{c^2}{2(n-1)} - |Dc| + \frac{M}{q+1}\right) v^{q+1} - C(v + v^q)$$

at (x_0, t_0) for $v > R$. From the assumption (3.3), we can choose a constant $M > 1$ independent of $T \in (0, \infty),\ \eta \in (0, 1]$ such that

$$\frac{c(x, z)^2}{2(n-1)} - |Dc(x, z)| + \frac{M}{q+1} > 1$$

for all $(x, z) \in \overline{\Omega} \times \mathbb{R}$. Since there exists a constant $R_0 > R$ independent of $T \in (0, \infty),\ \eta \in (0, 1]$ such that

$$0 < v^{q+1} - C(v + v^q)$$

for $v > R_0$, it must hold true that $v(x_0, t_0) \leq R_0$ with the above choice of $M > 1$. Using once again the fact that $w \leq v^{q+1} + (q+1)\|\phi\|_{C^0(\overline{\Omega})}\|h\|_{C^1(\overline{\Omega})}v$, we get

$$e^{-Mt}w(x,t) \leq e^{-Mt_0}w(x_0,t_0) \leq R, \qquad \text{for all } (x,t) \in \overline{\Omega} \times [0,T]$$

for some constant $R > 1$ independent of $T \in (0,\infty)$, $\eta \in (0,1]$, which proves our goal in Case 1.

Case 2: $x_0 \in \partial\Omega$.

Since $x_0 \in \operatorname{argmax}_{\overline{\Omega}} w(\cdot,t_0)$, we have both $\frac{\partial w}{\partial \vec{\mathbf{n}}}(x_0,t_0) \geq 0$ and $x_0 \in \operatorname{argmax}_{\partial\Omega} w(\cdot,t_0)$. From the latter, we see that for a given $\varepsilon_0 \in (0,1)$, there exists a constant $R_{\varepsilon_0} > 1$ that may depend on $\varepsilon_0 \in (0,1)$ but not on $T \in (0,\infty)$, $\eta \in (0,1]$ (also not on $x_0 \in \partial\Omega$) such that $w > 0$ and (3.41) holds at (x_0,t_0) for $v > R_{\varepsilon_0}$, where $L := (q+1)(C_0 + \varepsilon_0)$.

As in Step 5 of the proof of Theorem 3.1.2, if $C_0 < 0$, we see that, by taking $\varepsilon_0 = \frac{1}{2}\min\{\frac{1}{2}, -\frac{1}{2}C_0\}$, there exists a constant $R > 1$ independent of $T \in (0,\infty)$, $\eta \in (0,1]$ such that $v(x_0,t_0) \leq R$. Here, we have used the fact that $\frac{\partial w}{\partial \vec{\mathbf{n}}}(x_0,t_0) \geq 0$, as in Step 5 of the proof of Theorem 3.1.2. Using the fact that $w \leq v^{q+1} + Cv$ for some constant $C > 0$ depending only on $\|h\|_{C^1(\overline{\Omega})}$, $\|\phi\|_{C^0(\overline{\Omega})}$, we consequently see that

$$e^{-Mt}w(x,t) \leq e^{-Mt_0}w(x_0,t_0) \leq R \qquad \text{for all } (x,t) \in \overline{\Omega} \times [0,T],$$

and thus,

$$w(x,t) \leq Re^{MT} \qquad \text{for all } (x,t) \in \overline{\Omega} \times [0,T].$$

We achieved our goal accordingly when $C_0 < 0$. Now we assume the other case when $C_0 \geq 0$.

Let $B = B(x_c, K_0)$ be the open ball with the center $x_c := x_0 - K_0\vec{\mathbf{n}}(x_0)$ so that $B \subseteq \Omega$ and $\overline{B} \cap (\mathbb{R}^n \setminus \Omega) = \{x_0\}$. For $x \in \overline{B}$, we let

$$\rho^0(x) = -\frac{L}{2K_0}|x - x_c|^2 + \frac{LK_0}{2} + 1.$$

We then extend the function ρ^0 on $\overline{B}$ to a function (keeping the same notation ρ^0) on

$\mathbb{R}^n$ satisfying the requirement that $\rho^0(x) \geq \frac{1}{2}$ for all $x \in \mathbb{R}^n$, and that $\rho^0(x)$ is C^∞ on $\mathbb{R}^n$, a nondecreasing function in $|x - x_c|$. Then, $\rho^0(x_0) = 1$, $\frac{\partial \rho^0}{\partial \vec{\mathbf{n}}}(x_0) = -L$. Hence, for $\varepsilon_0 \in (0,1)$, there exists a constant $R_{\varepsilon_0} > 1$ that may depend on $\varepsilon_0 \in (0,1)$ but not on $T \in (0,\infty)$, $\eta \in (0,1]$ such that $w > 0$ and (3.41) at (x_0,t_0) are valid if $v > R_{\varepsilon_0}$, and thus that $\frac{\partial(\rho^0 w)}{\partial \vec{\mathbf{n}}} < 0$ at (x_0,t_0) if $v > R_{\varepsilon_0}$.

Since $\rho^0(z) = \rho^0(x_0) = 1$ for all $z \in \partial B$, and by the choice of (x_0,t_0), we have

$$e^{-Mt}\rho^0(z)w(z,t) \leq e^{-Mt_0}\rho^0(x_0)w(x_0,t_0), \qquad \text{for all } (z,t) \in \partial B \times [0,T].$$

Since $\frac{\partial(\rho^0 w)}{\partial \vec{\mathbf{n}}}(x_0,t_0) < 0$, we also have

$$e^{-Mt_0}\rho^0(x_0)w(x_0,t_0) < e^{-Mt_0}\rho^0(x)w(x,t_0)$$

for some $x \in B$. Combining these two points, we conclude that a maximizer (x_1,t_1) of $e^{-Mt}\rho^0(x)w(x,t)$ on $\overline{B} \times [0,T]$ occurs only inside B, i.e., x_1 must be inside B.

Let $(x_1,t_1) \in \operatorname{argmax}_{\overline{B}\times[0,T]} e^{-Mt}\rho^0(x)w(x,t)$ with $x_1 \in B$. If $t_1 = 0$, then

$$\max_{\overline{B}\times[0,T]} e^{-Mt}\rho^0(x)w(x,t) = \rho^0(x_1)w(x_1,0) \leq R$$

for some constant $R > 0$ depending only on $\|u_0\|_{C^1(\overline{\Omega})}$, $\|h\|_{C^1(\overline{\Omega})}$, $\|\phi\|_{C^0(\overline{\Omega})}$, Ω. Here, we have used the fact that $\eta \in (0,1]$, $\varepsilon_0 \in (0,1)$. It consequently yields that for all $(x,t) \in \overline{\Omega} \times [0,T]$,

$$\begin{aligned} e^{-Mt}\rho^0(x)w(x,t) = \rho^0(x)(e^{-Mt}w(x,t)) &\leq \left(\frac{\rho^0(x)}{\rho^0(x_0)}\right)\rho^0(x_0)(e^{-Mt_0}w(x_0,t_0)) \\ &\leq R \max_{\overline{B}\times[0,T]} e^{-Mt}\rho^0(x)w(x,t) \leq R \end{aligned}$$

since $(x_0,t_0) \in \overline{B} \times [0,T]$ and $\frac{\rho^0(x)}{\rho^0(x_0)} \leq R$ for all $x \in \overline{\Omega}$. Here, constants $R > 1$ change side by side. Then, for all $(x,t) \in \overline{\Omega} \times [0,T]$,

$$w(x,t) \leq \frac{R}{\rho^0(x)} e^{Mt} \leq Re^{MT}$$

since $\rho^0(x) \geq \frac{1}{2}$ for all $x \in \mathbb{R}^n$, which proves our goal. Now it remains the case when $t_1 > 0$.

We fix $(x_1,t_1) \in \mathrm{argmax}_{\overline{B}\times[0,T]}\, e^{-Mt}\rho^0(x)w(x,t)$ with $x_1 \in B$, $t_1 > 0$. Applying the maximum principle to $\psi = \rho w$ at (x_1,t_1), we obtain

$$\begin{aligned} 0 &\geq \frac{1}{(q+1)\rho}\left(\mathrm{tr}\{a(Du)D^2\psi\} - \psi_t\right) \\ &= \frac{w}{(q+1)\rho}\mathrm{tr}\{a(Du)D^2\rho\} + \frac{2}{(q+1)\rho}\mathrm{tr}\{a(Du)Dw \otimes D\rho\} \\ &\qquad\qquad + \frac{1}{q+1}(\mathrm{tr}\{a(Du)D^2 w\} - w_t) - \frac{\rho_t w}{(q+1)\rho} \end{aligned}$$

at (x_1,t_1). Following the same computations up to (3.46) in Step 6 of the proof of Proposition 3.2.1, we see that there exist a constant $C > 0$ independent of $T \in (0,\infty)$, $\eta \in (0,1]$, $\varepsilon_0 \in (0,1)$ and a constant $R_{\varepsilon_0} > 1$ that may depend on $\varepsilon_0 \in (0,1)$ but not on $T \in (0,\infty)$, $\eta \in (0,1]$ such that, at (x_1,t_1),

$$\begin{aligned} 0 \geq \frac{w}{(q+1)\rho}\mathrm{tr}\{a(Du)D^2\rho\} &+ \left(\frac{2}{(q+1)\rho}\mathrm{tr}\{a(Du)Dw \otimes D\rho\} + (q+1)V_1\right) \\ &\qquad + J_1' + J_2' - |Dc|v^{q+1} + \frac{M}{q+1}v^{q+1} - C(v + v^q) \end{aligned}$$

if $v = v(x_1,t_1) > R_{\varepsilon_0}$, with the same definitions of J_1', J_2' as in Step 6 of the proof of Proposition 3.2.1. Here, we have used the fact that $-\frac{\rho_t w}{(q+1)\rho} = \frac{Mw}{q+1}$ and that $w \geq v^{q+1} - (q+1)\|\phi\|_{C^0(\overline{\Omega})}\|h\|_{C^1(\overline{\Omega})}v$. Note that $\frac{D\rho}{\rho} = \frac{D\rho^0}{\rho^0}$, $\frac{D^2\rho}{\rho} = \frac{D^2\rho^0}{\rho^0}$, and that $x_1 \in \mathrm{argmax}_{\overline{\Omega}}\, \rho^0(\cdot)w(\cdot,t_1) \cap \Omega$. Therefore, we have $wD\rho^0 + \rho^0 Dw = 0$ at (x_1,t_1). Consequently, there exist a constant $C > 0$ independent of $T \in (0,\infty)$, $\eta \in (0,1]$, $\varepsilon_0 \in (0,1)$ and a constant $R_{\varepsilon_0} > 1$ that may depend on $\varepsilon_0 \in (0,1)$ but not on $T \in (0,\infty)$, $\eta \in (0,1]$ such that (3.47), (3.48), (3.50), (3.52), (3.53) hold true at (x_1,t_1) if $v > R_{\varepsilon_0}$, and thus that

(3.51), (3.57) hold true at (x_1,t_1) if $v > R_{\varepsilon_0}$.

Hence, there exist a constant $C>0$ independent of $T\in(0,\infty)$, $\eta\in(0,1]$, $\varepsilon_0\in(0,1)$ and a constant $R_{\varepsilon_0}>1$ that may depend on $\varepsilon_0\in(0,1)$ but not on $T\in(0,\infty)$, $\eta\in(0,1]$ such that, at (x_1,t_1),

$$
\begin{aligned}
0 \geq \Bigg(\frac{1-\varepsilon_0}{n-1}c^2 - |Dc| - (C_0+\varepsilon_0)|c| - \frac{(n-1)(C_0+\varepsilon_0)}{K_0}& \\
-(q+1)(C_0+\varepsilon_0)^2(1+\varepsilon_0) + \frac{M}{q+1}\Bigg) v^{q+1} - C(v+v^q) - C\varepsilon_0^{-1}v^{1-q}.&
\end{aligned}
$$

if $v > R_{\varepsilon_0}$. Now, take $\varepsilon_0 = \frac{1}{2}$, and take $M>1$ large enough, possible due to the assumption (3.3), that

$$
\frac{M}{q+1} - |Dc| - (C_0+\frac{1}{2})|c| - \frac{(n-1)(C_0+\frac{1}{2})}{K_0} - \frac{3}{2}(q+1)(C_0+\frac{1}{2})^2 > 1,
$$

where $c = c(x,z)$, for all $(x,z)\in\overline{\Omega}\times\mathbb{R}$. Since there exists a constant $R_0 > R$ independent of $T\in(0,\infty)$, $\eta\in(0,1]$ such that

$$
0 < v^{q+1} - C(v+v^q)
$$

for $v > R_0$, it must hold true that $v(x_1,t_1)\leq R_0$. Consequently, for all $(x,t)\in\overline{\Omega}\times[0,T]$,

$$
\begin{aligned}
e^{-Mt}w(x,t) \leq e^{-Mt_0}w(x_0,t_0) &= e^{-Mt_0}\rho^0(x_0)w(x_0,t_0)\frac{1}{\rho^0(x_0)} \\
&\leq e^{-Mt_1}\rho^0(x_1)w(x_1,t_1)\frac{1}{\rho^0(x_0)} \leq R.
\end{aligned}
$$

Here, we have used the fact that $w \leq v^{q+1} + (q+1)\|\phi\|_{C^0(\overline{\Omega})}\|h\|_{C^1(\overline{\Omega})}v$ so that $w(x_1,t_1)\leq R$. Therefore,

$$
w(x,t) \leq Re^{MT}
$$

for all $(x,t)\in\overline{\Omega}\times[0,T]$, which proves our goal in Case 2. This completes the proof. □

Finally, when Ω is strictly convex, we can recover gradient estimates in [101]. The following proof uses a strictly convex C^2 defining function of Ω when we choose a multiplier.

Proof of Corollary 3.1.1. In order to prove Corollary 3.1.1, it suffices to verify following, which is a similar statement to Proposition 3.2.1; let Ω be a C^3 strictly convex domain. Let $T \in (0,\infty)$, $\eta \in (0,1]$, and let $u = u^\eta \in C^{2,\sigma}(\overline{\Omega}\times[0,T]) \cap C^{3,\sigma}(\Omega\times(0,T])$ be a solution to (3.7) for some $\sigma \in (0,1)$, now with $c \equiv 0$. Then, it holds that

$$\|Du^\eta\|_{L^\infty(\overline{\Omega}\times[0,T])} \leq R,$$

where $R > 1$ is a constant independent of $T \in (0,\infty)$ and of $\eta \in (0,1]$.

Let g be a C^2 defining function of Ω such that $g < 0$ in Ω, $g = 0$ on $\partial\Omega$, $D^2 g \geq k_0 I_n$ on $\overline{\Omega}$ for some $k_0 > 0$, $\sup_\Omega |Dg| \leq 1$, $\frac{\partial g}{\partial \mathbf{n}} = 1$ on $\partial\Omega$. Let $\rho = \gamma g + 1$, where $\gamma \in \left(0, \frac{1}{2}\min\{1, \|g\|^{-1}_{C^0(\overline{\Omega})}\}\right)$ so that $\frac{1}{2} \leq \rho \leq 1$ on $\overline{\Omega}$.

Let $(x_0, t_0) \in \operatorname{argmax}_{\overline{\Omega}\times[0,T]} \rho w$, where w is defined as in the proof of Proposition 3.2.1. Again, our goal is to show $v(x_0,t_0) \leq R$, where $R > 1$ is a constant independent of $T \in (0,\infty)$, $\eta \in (0,1]$. Once it is shown, then we have a global gradient estimate, as

$$v(x,t) = \frac{1}{\rho(x)}\rho(x)v(x,t) \leq 2\rho(x_0)v(x_0,t_0) \leq R \qquad \text{for all } (x,t) \in \overline{\Omega}\times[0,T],$$

together with the fact that $\rho \geq \frac{1}{2}$ on $\overline{\Omega}$. In the case of $t_0 = 0$, we readily have that there exists a constant $R > 1$ depending only on $\|u_0\|_{C^1(\overline{\Omega})}$, $\|h\|_{C^1(\overline{\Omega})}$, $\|\phi\|_{C^0(\overline{\Omega})}$ such that $w(x_0,t_0) = w(x_0,0) \leq R$. Using the fact that $v^{q+1} - (q+1)\|\phi\|_{C^0(\overline{\Omega})}\|h\|_{C^1(\overline{\Omega})} v \leq w$, we see that there exists a constant $R > 1$ independent of $T \in (0,\infty)$, $\eta \in (0,1]$ such that $v(x_0,t_0) \leq R$, which proves our goal.

We assume the remaining case when $t_0 > 0$. We again divide the proof into two cases, but we consider the case $x_0 \in \partial\Omega$ first, and the case $x_0 \in \Omega$ next.

Case 1. $x_0 \in \partial\Omega$.

In this case, it holds that $x_0 \in \operatorname{argmax}_{\partial\Omega} w(\cdot, t_0)$ since $\rho \equiv 1$ on $\partial\Omega$. Therefore, by the argument of Step 5 of the proof of Proposition 3.2.1, we see that for a given

$\varepsilon_0 \in (0,1)$, there exists a constant $R_{\varepsilon_0} > 1$ that may depend on $\varepsilon_0 \in (0,1)$ but not on $T \in (0,\infty)$, $\eta \in (0,1]$ such that $w > 0$ and (3.41) hold true at (x_0, t_0) if $v > R_{\varepsilon_0}$. Since Ω is strictly convex, we have $C_0 < 0$. Take $\varepsilon_0 = \frac{1}{2}\min\{1, -C_0\} \in (0,1)$, and choose a constant $R = R_{\varepsilon_0} > 1$ accordingly. If $v(x_0,t_0) \le R$, we achieve our goal, and now we assume that $v(x_0,t_0) > R$ so that $w > 0$ and(3.41) are valid at (x_0,t_0). By replacing $R > 1$ by a larger one if necessary, we also have that $w(x_0,t_0) > 0$ if $v(x_0,t_0) > R$ (from the fact that $w \ge v^{q+1} - (q+1)\|\phi\|_{C^0(\overline{\Omega})}\|h\|_{C^1(\overline{\Omega})} v > 0$).

Note that $L = (q+1)(C_0 + \varepsilon_0) \le \frac{1}{2}(q+1)C_0 < 0$. Since $x_0 \in \operatorname{argmax}_{\overline{\Omega}} \rho(\cdot,t_0)w(\cdot,t_0)$, we have $\frac{\partial(\rho w)}{\partial \vec{\mathbf{n}}}(x_0,t_0) \ge 0$. However, if we choose $\gamma \in \left(0, \frac{1}{2}\min\{1, \|g\|^{-1}_{C^0(\overline{\Omega})}, -L\}\right)$ so that $\gamma < -L$, then, by (3.41),

$$\frac{\partial(\rho w)}{\partial \vec{\mathbf{n}}} = \rho \frac{\partial w}{\partial \vec{\mathbf{n}}} + w \frac{\partial \rho}{\partial \vec{\mathbf{n}}} < Lw + \gamma w < 0.$$

at (x_0,t_0), which contradicts to $\frac{\partial(\rho w)}{\partial \vec{\mathbf{n}}}(x_0,t_0) \ge 0$. Therefore, it must hold true that $v(x_0,t_0) \le R$, which proves our goal.

Case 2. $x_0 \in \Omega$.

In this case, a maximizer (x_0,t_0) of $\psi := \rho w$ happens in $B \times (0,T]$, and thus we can apply the maximum principle, which results in (3.42), (3.43) at (x_0,t_0). Following the same computations as in Step 6 of the proof of Proposition 3.2.1, we have (3.45) at (x_0,t_0). Fix $\varepsilon_0 = \frac{1}{2}$. Then, there exist constants $R > 1$, $C > 0$ independent of $T \in (0,\infty)$, $\eta \in (0,1]$ such that (3.44), (3.46) are true at (x_0,t_0) if $v > R$ with the same definitions of J_1', J_2' and $\varepsilon_0 = \frac{1}{2}$, $c \equiv 0$. Now that we have chosen a multiplier different from the one in the proof of Proposition 3.2.1, we estimate the first term and the second term of (3.46), which will replace (3.47) and (3.48), respectively.

We start with the first term of (3.46). Since $\rho = \gamma g + 1$ and $D^2\rho \ge \gamma k_0 I_n$, there exists a constant $C > 0$ independent of $T \in (0, \infty),\ \eta \in (0, 1]$ such that

$$\begin{aligned} &\frac{w}{(q+1)\rho}\mathrm{tr}\{a(Du)D^2\rho\} \\ &\geq \frac{1}{(q+1)\rho}\gamma k_0 \left(\frac{\eta^2}{v^2} + n - 1\right)(v^{q+1} - (q+1)\|\phi\|_{C^0(\overline{\Omega})}\|h\|_{C^1(\overline{\Omega})}v) \\ &\geq \frac{(n-1)\gamma k_0}{q+1}v^{q+1} - C(v + v^q) \end{aligned} \tag{3.58}$$

at (x_0, t_0) for $v > 1$. Here, we have used the fact that $\eta \in (0, 1]$ and that $\frac{1}{2} \leq \rho \leq 1$ on $\overline{\Omega}$.

We estimate the second term of (3.46). Since $D\rho = \gamma Dg$ and $|Dg| \leq 1,\ \rho \geq \frac{1}{2}$ on $\overline{\Omega}$, we have

$$0 \leq \mathrm{tr}\left\{a(Du)\frac{D\rho}{\rho} \otimes \frac{D\rho}{\rho}\right\} \leq \left|\frac{D\rho}{\rho}\right| \leq 4\gamma^2.$$

Choose a constant $R > 1$ independent of $T \in (0, \infty),\ \eta \in (0, 1]$ such that $|wv^{-q-1} - 1| < \frac{1}{2}$ for $v > R$. Then, we see that there exist a constant $R > 1,\ C > 0$ independent of $T \in (0, \infty),\ \eta \in (0, 1]$ such that

$$\begin{aligned} &\frac{2}{(q+1)\rho}\mathrm{tr}\{a(Du)Dw \otimes D\rho\} + (q+1)V_1 \\ &= \frac{(wv^{-1-q} - 2)w}{q+1}\mathrm{tr}\left\{a(Du)\frac{D\rho}{\rho} \otimes \frac{D\rho}{\rho}\right\} \\ &\geq -\frac{6\gamma^2}{q+1}v^{q+1} - Cv \end{aligned} \tag{3.59}$$

at (x_0, t_0) for $v > R$.

Following the computations of Step 7 of the proof of Proposition 3.2.1, we see that there exist constants $R > 1,\ C > 0$ independent of $T \in (0, \infty),\ \eta \in (0, 1]$ such that (3.49), (3.51) hold at (x_0, t_0) for $v > R$ with $\varepsilon_0 = \frac{1}{2},\ c = 0$. Note that the left hand side of (3.50) is zero, as $c = 0$. As $\rho Dw + wD\rho = 0$ at (x_0, t_0), (3.52), (3.53) are valid at (x_0, t_0), and therefore, there exists constants $R > 1,\ C > 0$ independent of $T \in (0, \infty),\ \eta \in (0, 1]$ such that (3.57) holds at (x_0, t_0) if $v > R$ with $\varepsilon_0 = \frac{1}{2}$.

All in all, by (3.46), (3.51), (3.57), (3.58), (3.59), there exist constants $R > 1,\ C > 0$

independent of $T \in (0,\infty),\ \eta \in (0,1]$ such that

$$0 \geq \frac{\gamma}{q+1}((n-1)k_0 - 6\gamma)v^{q+1} - C(v+v^q)$$

at (x_0,t_0) for $v > R$. Choose $\gamma = \frac{1}{4}\min\left\{1, \|g\|^{-1}_{C^0(\overline{\Omega})}, \frac{(n-1)k_0}{6}\right\} \in (0,1)$ so that $\frac{\gamma}{q+1}((n-1)k_0 - 6\gamma) \geq \frac{3(n-1)k_0}{16(q+1)} > 0$. Since there exists a constant $R_0 > R$ independent of $T \in (0,\infty),\ \eta \in (0,1]$ such that

$$0 < \frac{3(n-1)k_0}{16(q+1)}v^{q+1} - C(v+v^q)$$

for $v > R_0$, it must hold that $v = v(x_0,t_0) \leq R_0$, which proves our goal in Case 2. This completes the proof. □

3.3 The additive eigenvalue problem

In this section, we prove Theorem 3.1.3, Theorem 3.1.4 and Theorem 3.1.5. We leave the main reference [101], and we will highlight details that are different from [101]. We also refer to [98, Section 7] that go through the limit $k \to 0$ first and $\eta \to 0$ next.

We consider

$$\begin{cases} -\sum_{i,j=1}^n \left(\delta^{ij} - \frac{u_i u_j}{\eta^2+|Du|^2}\right) u_{ij} - c(x)\sqrt{\eta^2+|Du|^2} + f(x) = -ku & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \vec{\mathbf{n}}} = \phi(x)v^{1-q} & \text{on } \partial\Omega, \end{cases} \tag{3.60}$$

where $k \in (0,1),\ \eta \in (0,1]$ and $v = \sqrt{\eta^2+|Du|^2}$. Note that the choices $\eta = 1,\ q > 0$ and $\eta = 0,\ q = 1$ correspond to (3.5) and (3.6), respectively. The case $\eta = 0,\ q = 1$ will be studied by obtaining estimates uniform in $\eta \in (0,1]$ when $q = 1$.

First of all, we start with *a priori* C^0 and C^1 estimates and get the existence of solutions of (3.60) using the method of continuity with the estimates.

Proposition 3.3.1. *Let Ω be a C^∞ bounded domain in $\mathbb{R}^n$, $n \geq 2$. Assume that $c \in C^\infty(\overline{\Omega})$ satisfies (3.9). Then there exists a unique solution $u \in C^\infty(\overline{\Omega})$ of (3.60).*

Moreover, we have the following estimate uniform in $k \in (0,1)$ *and also in* $\eta \in (0,1]$ *when* $q = 1$*;*

$$\sup_{\overline{\Omega}} |ku| + \sup_{\overline{\Omega}} |Du| \le R,$$

where $R > 1$ *is a constant independent of* $k \in (0,1)$ *and also of* $\eta \in (0,1]$ *when* $q = 1$*.*

Proof. We apply Leray-Schauder fixed point theorem to the following family of boundary value problems, parametrized by $\tau \in [0,1]$,

$$\begin{cases} \tau\left(-\mathrm{tr}\{a(Du)D^2u\} - c(x)\sqrt{\eta^2 + |Du|^2} + f(x) + ku\right) & \\ \quad +(1-\tau)\left(-\mathrm{tr}\{a(Du)D^2u\} - c(x)\sqrt{\eta^2 + |Du|^2} + \eta c(x) + ku\right) = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \vec{\mathbf{n}}} = \tau\phi(x)v^{1-q} & \text{on } \partial\Omega, \end{cases} \tag{3.61}$$

where $a(p) := I_n - \frac{p\otimes p}{\eta^2+|p|^2}$ for $p \in \mathbb{R}^n$. When $\tau = 0$, $u \equiv 0$ is a solution, and we need to find a solution when $\tau = 1$. By Leray-Schauder fixed point theorem, the existence of a solution u when $\tau = 1$ can be shown by establishing *a priori* C^0 and C^1 estimates, uniform in $\tau \in [0,1]$,

$$\sup_{\overline{\Omega}} |ku| + \sup_{\overline{\Omega}} |Du| \le R,$$

which is also uniform in $k \in (0,1)$, and also in $\eta \in (0,1]$ when $q = 1$.

Let $u \in C^2(\overline{\Omega}) \cap C^3(\Omega)$ be a solution of (3.61). We first get *a priori* C^0 estimate, as it is used to obtain *a priori* C^1 estimate. A C^0 estimate can be obtained as before. Consider a smooth function g on $\overline{\Omega}$ that has a so large positive slope in the outward normal direction on the boundary that $\left(\sqrt{\eta^2 + |Dg|^2}\right)^{q-1} \frac{\partial g}{\partial \vec{\mathbf{n}}} > \sup_{\overline{\Omega}} |\phi|$ on $\partial\Omega$. Note again that we are dealing with $\eta = 1,\ q > 0$ for the graph case and with $\eta \in (0,1],\ q = 1$ for the level-set case.

We prove *a priori* C^0 estimate, and we first check that $g - u$ attains a minimum inside Ω for *a priori* C^0 estimate. Suppose not, and say $x_0 \in \partial\Omega$ is a minimizer of $g - u$. Then, at $x_0 \in \partial\Omega$, we have $0 < \frac{\partial g}{\partial \vec{\mathbf{n}}} \le \frac{\partial u}{\partial \vec{\mathbf{n}}}$ and $D'g = D'u$. The latter follows from $\nabla' g = \nabla' u$ at x_0 and Lemma 3.2.2, in the notations introduced in Step 5 of the proof of Proposition 3.2.1.

Using the fact that for a fixed $a \in \mathbb{R}$, the function $\left(\sqrt{\eta^2 + a^2 + b^2}\right)^{q-1} b$ is monotonically increasing in $b > 0$ when $\eta = 1,\ q > 0$ and also when $\eta \in (0,1],\ q = 1$, we see that

$$\left(\sqrt{\eta^2 + |Dg|^2}\right)^{q-1} \frac{\partial g}{\partial \vec{\mathbf{n}}} \leq \left(\sqrt{\eta^2 + |Du|^2}\right)^{q-1} \frac{\partial u}{\partial \vec{\mathbf{n}}} = \phi(x_0)$$

at $x_0 \in \partial\Omega$. This contradicts with the choice of a function g.

Let $x_0 \in \Omega$ be a minimizer of $g - u$. Applying the maximum principle at x_0 to $g - u$, i.e., $Dg(x_0) = Du(x_0),\ D^2 g(x_0) \geq D^2 u(x_0)$, we see that, at x_0,

$$\begin{aligned} C \geq \operatorname{tr}\{a(Dg)D^2 g\} &\geq \operatorname{tr}\{a(Du)D^2 u\} \\ &= ku - c\sqrt{\eta^2 + |Du|^2} + \tau f + (1-\tau)\eta c \\ &= ku - c\sqrt{\eta^2 + |Dg|^2} + \tau f + (1-\tau)\eta c \\ &\geq ku - C, \end{aligned}$$

for some constant $C > 0$ depending only on $\Omega,\ g,\ f,\ c$. Here, we have used the fact that $\tau, \eta \in [0,1]$ and the assumptions (3.3), (3.4). Therefore, for all $x \in \overline{\Omega}$,

$$ku(x) \leq kg(x) - kg(x_0) + ku(x_0) \leq R$$

for some constant $R > 1$ uniform in $\tau \in [0,1],\ k \in (0,1)$, and also in $\eta \in (0,1]$ when $q = 1$. Similarly, we can get a lower bound of $ku(x)$.

A C^1 estimate can be established similarly as in the proof of Proposition 3.2.1, but now with $\widetilde{c}(x,z) := c(x),\ \widetilde{f}(x,z) := \tau f(x) + (1-\tau)\eta c(x) + kz$ and $\widetilde{\phi}(x) := \tau\phi(x)$ for $x \in \overline{\Omega},\ z \in \mathbb{R}$. Equation (3.61) can be written as

$$\begin{cases} \operatorname{tr}\{a(Du)D^2 u\} + \widetilde{c}(x,u)v - \widetilde{f}(x,u) = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \vec{\mathbf{n}}} = \widetilde{\phi}(x)v^{1-q} & \text{on } \partial\Omega. \end{cases} \tag{3.62}$$

The force $\widetilde{c}(x,z) = c(x)$ is in $C^{1,\alpha}(\overline{\Omega})$ and satisfies (3.3), (3.9). Also, $\widetilde{\phi}(x)$ is in $C^3(\overline{\Omega})$

with a C^3 norm uniform in $\tau \in [0,1]$. Moreover, $\widetilde{f}(x,u) = \tau f(x) + (1-\tau)\eta c(x) + ku$ is *a priori* in $C^{1,\alpha}(\overline{\Omega} \times \mathbb{R})$ and *a priori* satisfies (3.4) with a constant $C > 0$ independent of $\tau \in [0,1],\ k \in (0,1)$ and of $\eta \in (0,1]$ when $q = 1$.

We now prove a *a priori* C^1 estimate. Throughout the remaining part of the proof, $R > 1,\ C > 0$ denote constants, which may vary from line to line, independent of $\tau \in [0,1],\ k \in (0,1)$ and also of $\eta \in (0,1]$ when $q = 1$. Let h be a function in $C^3(\overline{\Omega})$ such that $h \equiv C,\ Dh = \vec{\mathbf{n}}$ on the boundary $\partial\Omega$ for some constant C. Let $v = \sqrt{\eta^2 + |Du|^2}$ and let $w = v^{q+1} - (q+1)\widetilde{\phi} Du \cdot Dh$ on $\overline{\Omega}$.

The proof is similar to that of Proposition 3.2.1. We use the idea and the estimate from the proof of Proposition 3.2.1, and we highlight the difference coming from not having the time derivative involved.

Let $x_0 \in \operatorname{argmax}_{\overline{\Omega}} w$. The goal is to show that $v(x_0) \le R$ for some constant $R > 1$ independent of $\tau \in [0,1],\ k \in (0,1)$ and of $\eta \in (0,1]$ when $q = 1$. We again divide the proof into two cases when $x_0 \in \Omega$ and when $x_0 \in \partial\Omega$.

Case 1: $x_0 \in \Omega$.

At x_0, we apply the maximum principle to w to obtain

$$0 \ge \frac{1}{q+1} \operatorname{tr}\{a(Du)D^2 w\},$$

which leads to

$$0 \ge \operatorname{tr}\{a(Du)D(v^q Dv)\} - \operatorname{tr}\{a(Du)D^2(\widetilde{\phi} Du \cdot Dh)\}$$

at x_0. Write $0 = G + \widetilde{c}v - \widetilde{f}$, where $G := \operatorname{tr}\{a(Du)D^2 u\}$. Then,

$$(v^{q-1}Du - \widetilde{\phi} Dh) \cdot (DG + D(\widetilde{c}v - \widetilde{f})) = 0,$$

and thus, we have (3.16) at x_0 with $\widetilde{c}, \widetilde{f}, \widetilde{\phi}$ instead of c, f, ϕ.

We proceed the same estimate as in Case 1 of the proof of Proposition 3.2.1, except for the part we remark here that with $\alpha = \sqrt{a}D^2u$, $\beta = \sqrt{a}$,

$$\begin{aligned}
\mathrm{tr}\{a(Du)(D^2u)^2\} = \|\alpha\|^2 \geq \frac{\mathrm{tr}\{\alpha\beta^{\mathrm{Tr}}\}^2}{\|\beta\|^2} &= \frac{G^2}{n-1+\frac{\eta^2}{v^2}} \\
&= \left(\frac{1}{n-1} - \frac{\eta^2}{v^2(n-1)\left(n-1+\frac{\eta^2}{v^2}\right)}\right)(\widetilde{c}v - \widetilde{f})^2 \\
&\geq \frac{1}{n-1}\widetilde{c}^2v^2 - Cv
\end{aligned}$$

for some constant $C > 0$ depending only on $\|f\|_{C^0(\overline{\Omega}\times\mathbb{R})}, \|c\|_{C^0(\overline{\Omega}\times\mathbb{R})}$. We have used *a priori* C^0 estimate, the assumptions (3.3), (3.4) and the fact that $\tau \in [0,1]$, $\eta \in (0,1]$ when $q = 1$. Therefore, there exists a constant $R > 1$ independent of $\tau \in [0,1]$, $k \in (0,1)$ and of $\eta \in (0,1]$ when $q = 1$ such that whenever $v > R$, (3.22) holds.

Also, we have (3.23), (3.27), (3.29) at x_0 for some constant $C > 0$ independent of $\tau \in [0,1]$, $k \in (0,1)$ and of $\eta \in (0,1]$ when $q = 1$, since $Dw = 0$ at x_0. Following the same argument in Case 1 of the proof of Proposition 3.2.1, i.e., as in (3.24), (3.36), we see that for $\varepsilon \in (0,1)$ there exist constants $R > 1$, $C > 0$ independent of $\tau \in [0,1]$, $k \in (0,1)$ and of $\eta \in (0,1]$ when $q = 1$ such that

$$0 \geq \frac{\delta}{2}v^{q+1} - C(v + v^q)$$

at x_0 if $v > R_\varepsilon$. As there is a constant $R_0 > R$ independent of $\tau \in [0,1]$, $k \in (0,1)$ and not of $\eta \in (0,1]$ when $q = 1$ such that

$$0 < \frac{\delta}{2}v^{q+1} - C(v + v^q)$$

if $v > R_0$, it must hold that $v = v(x_0) \leq R_0$, which finishes Case 1.

Case 2: $x_0 \in \partial\Omega$.

We see that Step 5 of the proof of Proposition 3.2.1 carries over verbatim, since the time $t = t_0$ is fixed throughout the step, and since x_0 is a maximizer of w on $\overline{\Omega}$. Therefore, for each $\varepsilon_0 \in (0,1)$, there exists $R_{\varepsilon_0} > 1$ that may depend on ε_0 but not on $\tau \in [0,1]$, $k \in (0,1)$

and of $\eta \in (0,1]$ when $q=1$ such that $w>0$ and

$$\frac{\partial w}{\partial \vec{\mathbf{n}}} < Lw$$

at x_0 for $v > R_{\varepsilon_0}$, where $L := (q+1)(C_0+\varepsilon_0)$. We also see that if $C_0 < 0$, then $v(x_0) \le R$ for some constant $R > 1$ independent of $\tau \in [0,1]$, $k \in (0,1)$ and of $\eta \in (0,1]$ when $q=1$, by the argument at the end of Step 5 of the proof of Proposition 3.2.1, and thus, we achieve the goal in this case. Therefore, we assume that $C_0 \ge 0$, and thus that $L \ge 0$.

Let $B = B(x_c, K_0)$ be the open ball with the center $x_c := x_0 - K_0\vec{\mathbf{n}}(x_0)$ so that $B \subseteq \Omega$ and $\overline{B} \cap (\mathbb{R}^n \setminus \Omega) = \{x_0\}$. Let $\psi := \rho w$, with

$$\rho(x) := -\frac{L}{2K_0}|x-x_c|^2 + \frac{LK_0}{2} + 1$$

as before. Then, $\rho(x_0) = 1$, $\frac{\partial \rho}{\partial \vec{\mathbf{n}}}(x_0) = -L$, and thus,

$$\frac{\partial \psi}{\partial \vec{\mathbf{n}}} = \rho\frac{\partial w}{\partial \vec{\mathbf{n}}} + w\frac{\partial \rho}{\partial \vec{\mathbf{n}}} = \frac{\partial w}{\partial \vec{\mathbf{n}}} + (-L)w < 0, \qquad \text{at } x_0.$$

Since $\rho(z)w(z) \le \rho(x_0)w(x_0)$ for all $z \in \partial B$ from $\rho \equiv 1$ on ∂B, and since $\frac{\partial \psi}{\partial \vec{\mathbf{n}}}(x_0) < 0$, we derive that $x_1 \in B$ for $x_1 \in \operatorname{argmax}_{\overline{B}} \psi$. As in Step 5 of the proof of Proposition 3.2.1, we see that there exists a constant $C > 0$ depending only on $\|\phi\|_{C^0(\overline{\Omega})}$, $\|h\|_{C^1(\overline{\Omega})}$ such that the condition $v(x_0) > R_{\varepsilon_0}$ with $R_{\varepsilon_0} > (8C)^{\frac{1}{q+1}}$ implies the condition $v(x_1) > \left(\frac{1}{4C}\right)^{\frac{1}{q+1}} R_{\varepsilon_0} =: R'_{\varepsilon_0}$. Writing $R'_{\varepsilon_0} = \left(\frac{1}{4C}\right)^{\frac{1}{q+1}} R_{\varepsilon_0}$, $R_{\varepsilon_0} = (4C)^{\frac{1}{q+1}} R'_{\varepsilon_0}$ (and also for R, R' similarly), we can state equivalently that if $v(x_1) \le R'_{\varepsilon_0}$, then $v(x_0) \le \max\left\{R_{\varepsilon_0}, (8C)^{\frac{1}{q+1}}\right\}$. Accordingly, we change our goal from verifying $v(x_0) \le R$ to proving $v(x_1) \le R'$.

Fix $x_1 \in \operatorname{argmax}_{\overline{B}} \psi \cap B$. At x_1,

$$\begin{aligned} 0 &\ge \frac{1}{(q+1)\rho}\operatorname{tr}\{a(Du)D^2\psi\} \\ &= \frac{w}{(q+1)\rho}\operatorname{tr}\{a(Du)D^2\rho\} + \frac{2}{(q+1)\rho}\operatorname{tr}\{a(Du)Dw \otimes D\rho\} + \frac{1}{q+1}\operatorname{tr}\{a(Du)D^2 w\}, \end{aligned}$$

Since $D\psi = \rho Dw + wD\rho = 0$ at x_1, we have (3.45) at x_1. Also, since

$$(v^{q-1}Du - \widetilde{\phi}Dh) \cdot (DG + D(\widetilde{c}v - \widetilde{f})) = 0,$$

there exist a constant $R'_{\varepsilon_0} > 1$ that may depend on $\varepsilon_0 \in (0,1)$ but not on $\tau \in [0,1],\ k \in (0,1)$ and not on $\eta \in (0,1]$ when $q = 1$ and a constant $C > 0$ independent of $\tau \in [0,1],\ k \in (0,1)$ and of $\eta \in (0,1]$ when $q = 1$ such that (3.46) holds true at x_1 for $v > R'_{\varepsilon_0}$. Following the same computations in Step 7 of the proof of Proposition 3.2.1, we see that there exist a constant $R'_{\varepsilon_0} > 1$ that may depend on $\varepsilon_0 \in (0,1)$ but not on $\tau \in [0,1],\ k \in (0,1)$ and not on $\eta \in (0,1]$ when $q = 1$ and a constant $C > 0$ independent of $\tau \in [0,1],\ k \in (0,1)$ and of $\eta \in (0,1]$ when $q = 1$ such that (3.47), (3.48), (3.51), (3.52), (3.53), (3.57) at x_1 for $v > R'_{\varepsilon_0}$.

All in all, choosing $\varepsilon_0 \in (0,1)$ as in Step 8 of the proof of Proposition 3.2.1, we see that there exist constants $R' > 1,\ C > 0$ independent of $\tau \in [0,1],\ k \in (0,1)$ and of $\eta \in (0,1]$ when $q = 1$ such that

$$0 \geq \frac{\delta}{2} v^{q+1} - C(v + v^q)$$

at x_1 if $v > R'$. There is, on the other hand, also a constant $R'_0 > R'$ independent of $\tau \in [0,1],\ k \in (0,1)$ and of $\eta \in (0,1]$ when $q = 1$ such that

$$0 < \frac{\delta}{2} v^{q+1} - C(v + v^q)$$

if $v > R'_0$. Therefore, it must hold that $v = v(x_1) \leq R'_0$, which completes Case 2.

All in all, we have obtained *a priori* C^0 and C^1 estimates for $u \in C^2(\overline{\Omega}) \cap C^3(\Omega)$ solving (3.60), and thus the existence of a solution $u \in C^\infty(\overline{\Omega})$ of (3.60) by Leray-Schauder fixed point theorem. The higher regularity and the fixed point theorem are referred to [74].

For the rest of the proof, we refer to the proof of [101, Theorem 4.2] for more details and the uniqueness upto an additive constant. □

Take $\eta = 1,\ q > 0$ to prove Theorem 3.1.3 and Theorem 3.1.4.

Proof of Theorem 3.1.3 and Theorem 3.1.4. For each $k \in (0,1)$, let u_k be the solution of (3.8) with $\eta = 1,\ q > 0$. Then the function $w_k = u_k - \frac{\int_\Omega u_k}{|\Omega|}$ solves

$$\begin{cases} -a(Dw_k) : D^2 w_k - c(x)\sqrt{\eta^2 + |Dw_k|^2} + f(x) = -kw_k - k\frac{\int_\Omega u_k}{|\Omega|} & \text{in } \Omega, \\ \dfrac{\partial w_k}{\partial \vec{\mathbf{n}}} = \phi(x)\left(\sqrt{1 + |Dw_k|^2}\right)^{1-q} & \text{on } \partial\Omega. \end{cases} \tag{3.63}$$

Then we have that $\sup |w_k| + \sup |Dw_k| \leq R$. By Schauder theory, there is an exponent $\alpha \in (0,1)$ such that $\|w_k\|_{C^{2,\alpha}(\overline{\Omega})} \leq R$. Therefore, $w_k \to w$ in $C^{2,\alpha'}$ for some $\alpha' \in (0,\alpha)$, and $-kw_k - k\frac{\int_\Omega u_k}{|\Omega|} \to -\lambda$ where (λ, w) solves (3.5).

See the proof of [101, Theorem 4.2] for more details and the uniqueness upto an additive constant. The proof of Theorem 3.1.4 goes the same as that of [101, Theorem 5.1]. □

Now, we study (3.6) by vanishing viscosity procedure $\eta \to 0$ when $q = 1$.

Proposition 3.3.2. *Let Ω be a C^∞ bounded domain in $\mathbb{R}^n,\ n \geq 2$. Let $\eta \in (0,1]$. Assume $c \in C^\infty(\overline{\Omega})$ satisfies (3.9). Then, there exists a unique $\lambda_\eta \in \mathbb{R}$ such that there exists a solution $w \in C^\infty(\overline{\Omega})$ of*

$$\begin{cases} -\sum_{i,j=1}^n \left(\delta^{ij} - \frac{w_i w_j}{\eta^2 + |Dw|^2}\right) w_{ij} - c\sqrt{\eta^2 + |Dw|^2} + f = -\lambda_\eta & \textit{in } \Omega, \\ \dfrac{\partial w}{\partial \vec{\mathbf{n}}} = \phi(x) & \textit{on } \partial\Omega. \end{cases} \tag{3.64}$$

Moreover, a solution w is unique upto an additive constant, and we have the following estimate uniform in $\eta \in (0,1]$;

$$|\lambda_\eta| + \sup_{\overline{\Omega}} |Dw| \leq R, \tag{3.65}$$

where $R > 0$ is a constant not depending on $\eta \in (0,1]$.

Proof. We proceed the same limit process as $k \to 0$ as in the proof of Theorem 3.1.3. Note that the estimates are uniform in $\eta \in (0,1]$ when $q = 1$. □

Proof of Theorem 3.1.5. Fix $x_0 \in \Omega$. For each $\eta \in (0,1]$, let (λ_η, w_η) be a pair that solves

(3.64) with $w_\eta(x_0) = 0$. By (3.65) and Arzela-Ascoli Theorem, as $\eta \to 0$, we can find a subsequence of (λ_η, w_η) such that λ_η converges to $\lambda \in \mathbb{R}$, and w_η converges to a Lipschitz function w uniformly on $\overline{\Omega}$. By the stability of viscosity solutions, we see that (λ, w) solves (3.6).

Let u be the unique viscosity solution of (3.2). Then for some constant $C > 0$, $w(x) - C + \lambda t$ and $w(x) + C + \lambda t$ are a subsolution and supersolution of (3.2), respectively. By the comparison principle (Proposition 3.4.3) for (3.2), we have

$$w(x) - C + \lambda t \le u(x,t) \le w(x) + C + \lambda t.$$

Therefore, we can draw the conclusion that $\lambda = \lim_{t\to\infty} \frac{u(x,t)}{t}$ and that the convergence is uniform in $x \in \overline{\Omega}$. The uniqueness of such a number $\lambda \in \mathbb{R}$ follows from the uniqueness of a solution u of (3.2) and the limit $\lambda = \lim_{t\to\infty} \frac{u(x,t)}{t}$. □

3.4 Radially symmetric cases

In this section, we study the radially symmetric setting of (3.2). We find the Lagrangian, the optimal control formula and a counterexample of the condition (3.9) in Subsection 3.4.1, and we define the *Aubry set*, prove the comparison principle on the Aubry set and prove Theorem 3.1.6 in Subsection 3.4.2. We mention an example of nonuniqueness for (3.2) when $0 < q < 1$ at the end of this section. We leave the reference [46, 66] for the analysis of the radially symmetric setting, and [98] for Aubry sets.

We always assume here that, by abuse of notations,

$$\begin{cases} \Omega = B(0,R) & \text{for some } R > 0, \\ c(x) = c(r) & \text{for } |x| = r \in [0,R], \\ f(x) = f(r) & \text{for } |x| = r \in [0,R], \\ \phi(x) = \phi(r) & \text{for } |x| = r \in [0,R], \\ u_0(x) = u_0(r) & \text{for } |x| = r \in [0,R]. \end{cases} \tag{3.66}$$

Here, $R > 0$ is a fixed positive number, $c \in C^1([0,R],[0,\infty))$, $f \in C^1([0,R])$ and $u_0 \in C^2([0,R])$ with $u_0'(R) = \phi(R)$ are given. The function $\phi(x)$ can be understood as the constant $\phi(R)$.

3.4.1 The optimal control formula and a counterexample

Equation (3.2) becomes

$$\begin{cases} \varphi_t - \frac{n-1}{r}\varphi_r - c(r)|\varphi_r| + f(r) = 0 & \text{in } (0,R)\times(0,\infty), \\ \varphi_r(R) = \phi(R) & \\ \varphi(r,0) = u_0(r) & \text{for } r \in [0,R]. \end{cases} \tag{3.67}$$

Note that this is a first-order Hamilton-Jacobi equation with a concave Hamiltonian. The associated Lagrangian $L = L(r,q)$ to the Hamiltonian $H(r,p) = -\frac{n-1}{r}p - c(r)|p| + f(r)$ is

$$\begin{aligned} L(r,q) &= \inf_{p\in\mathbb{R}} \left\{ p\cdot q - \left(-\frac{n-1}{r}p - c(r)|p| + f(r) \right) \right\} \\ &= \inf_{p\in\mathbb{R}} \left\{ \left(q + \frac{n-1}{r} \right) p + c(r)|p| - f(r) \right\} \\ &= \begin{cases} -f(r), & \text{if } \left|q + \frac{n-1}{r}\right| \le c(r), \\ -\infty, & \text{otherwise.} \end{cases} \end{aligned}$$

Therefore, we have the following representation formula for $\phi = \phi(r,t)$

$$\varphi(r,t) = \sup\left\{ \int_0^t (-f(\eta(s)) + \phi(\eta(s))l(s))\,ds + u_0(\eta(t)) :\ (\eta, v, l) \in \mathrm{SP}(r) \right\}, \tag{3.68}$$

where we denote by $\mathrm{SP}(r)$ the Skorokhod problem. See [45, Section 4.5] for the derivation of the formula. For a given $r \in (0,R]$, $v \in L^\infty([0,t])$, the Skorokhod problem seeks to find a solution $(\eta, l) \in \mathrm{Lip}((0,t)) \times L^\infty((0,t))$ such that

$$\begin{cases} \eta(0) = r, & \eta([0,t]) \subset (0,R], \\ l(s) \geq 0 & \text{for almost every } s > 0, \\ l(s) = 0 & \text{if } \eta(s) \neq R, \\ \left| -v(s) + \frac{n-1}{\gamma(s)} \right| \leq c(\gamma(s)), \\ v(s) = \dot{\eta}(s) + l(s)n(\eta(s)), \end{cases}$$

and the set $\mathrm{SP}(r)$ collects all the associated triples (η, v, l). Here, $n(R) = 1$ is the outward normal vector to $(0,R)$ at R. See [62, Theorem 4.2] for the existence of solutions of the Skorokhod problem and [62, Theorem 5.1] for the representation formula. See [46] for a related problem on the large time behavior and the large time profile.

We remark that at $\eta(s) \neq R$, we have

$$\frac{n-1}{\eta(s)} - c(\eta(s)) \leq \dot{\eta}(s) \leq \frac{n-1}{\eta(s)} + c(\eta(s)), \tag{3.69}$$

and at $\eta(s) = R$,

$$\frac{n-1}{R} - c(R) \leq \dot{\eta}(s) + l(s)n(R) \leq \frac{n-1}{R} + c(R).$$

This implies that

$$\frac{n-1}{R} - c(R) \leq l(s) \leq \frac{n-1}{R} + c(R). \tag{3.70}$$

We will find the eigenvalue $\lambda = \lim_{t\to\infty} \frac{\varphi(r,t)}{t}$ in terms of given functions c, f and a constant $\phi(R)$ when the force c satisfies (3.9). Before that, let us see that $\lim_{t\to\infty} \frac{\varphi(r,t)}{t}$ is not constant in $r \in [0,R]$ when c does not satisfy (3.9) with the following example.

Example 3.4.1. *We consider a case when $c(r)$ is of the form*

$$c(r)\begin{cases} < \frac{n-1}{a}, & 0 \le r < a, \\ = \frac{n-1}{r}, & a \le r \le b, \\ > \frac{n-1}{b}, & b < r \le R, \end{cases}$$

for some $0 < a < b < R$. Let $u_0 \equiv 0$, $\phi(R) = 0$. By (3.69), a curve $\eta(s)$ with $(\eta, v, l) \in$ SP(r)

- *can stay still or go right when $a \le \eta(s) \le b$,*
- *must go right when $\eta(s) < a$*
- *can move both left and right when $\eta(s) > b$.*

Then, by (3.68),

$$\lim_{t\to\infty} \frac{\varphi(r,t)}{t} = \begin{cases} \sup\{-f(s): \ s \ge a\}, & r \le a, \\ \sup\{-f(s): \ s \ge r\}, & a \le r \le b, \\ \sup\{-f(s): \ s \ge b\}, & r \ge b. \end{cases}$$

We see that the limit is not constant in $r \in [0, R]$ for a suitable choice of f. For instance, take a smooth function $f(r)$ such that

$$f(r)\begin{cases} = 1, & 0 \le r < a, \\ \in (0,1), & a \le r \le b, \\ > 0, & b < r \le R. \end{cases}$$

In the above example, the force c does not satisfy (3.9); at $r \in (a, b)$,

$$\frac{1}{n-1}c(r)^2 - |Dc(r)| = \frac{1}{n-1}\left(\frac{n-1}{r}\right)^2 - \frac{n-1}{r^2} = 0.$$

Therefore, the condition (3.9) is sharp.

3.4.2 Aubry set, the comparison principle and the large-time behavior

From now on, we assume that c is coercive, i.e., c satisfies (3.9). Then there is at most one r, which we call r_{cr} if it exists, such that $c(r) = \frac{n-1}{r}$. Otherwise, there would exist two points $a < b$ where the curves $c(r)$ and $\frac{n-1}{r}$ cross. At $r = b$,

$$\frac{1}{n-1}c(b)^2 - |Dc(b)| \leq \frac{1}{n-1}\left(\frac{n-1}{b}\right)^2 - \frac{n-1}{b^2} = 0,$$

since $Dc(b) \leq \frac{d}{dr}\big|_b \left(\frac{n-1}{r}\right) = -\frac{n-1}{b^2} < 0$. If $c(r) < \frac{n-1}{r}$ for all $r \leq R$, we let $r_{cr} := \infty$.

In the both cases of $r_{cr} < \infty$ and $r_{cr} = \infty$, by (3.68) and (3.70), we obtain

$$\lambda = \sup\left\{-f(r) + \delta(r-R)\phi(R)\left(\frac{n-1}{R} + \operatorname{sgn}(\phi(R))c(R)\right) :\ r \geq r_{cr} \text{ or } r = R\right\},$$

which is (3.11).

We define the *Aubry set* $\widetilde{\mathcal{A}}$ by

$$\widetilde{\mathcal{A}} := \{r \geq r_{cr} :\ \text{the supremum of (3.11) is attained}\} \quad \text{if } r_{cr} < \infty.$$

Note that if $r_{cr} < \infty$, then the function $-f(r) + \delta(r-R)\phi(R)\left(\frac{n-1}{R} + \operatorname{sgn}(\phi(R))c(R)\right)$ is upper semicontinuous on the interval $[r_{cr}, R]$. Thus, $\widetilde{\mathcal{A}}$ is well-defined, and it is a nonempty closed subset of $[0, R]$. If $r_{cr} = \infty$, we let $\widetilde{\mathcal{A}} = \{R\}$.

Let

$$\begin{cases} \lambda - \frac{n-1}{r}w_r - c(r)|w_r| + f(r) = 0 & \text{in } (0,R)\times(0,\infty), \\ w_r(R) = \phi(R) \end{cases} \tag{3.71}$$

be the stationary problem of (3.67). Here, we are assuming that c satisfies (3.9), and thus, the eigenvalue λ is given as in (3.11).

The propositions in [46, Section 2] follow for (3.67) with little changes. Here, we state [46, Lemma 2.4] and [46, Theorem 2.5] for problem (3.71).

Proposition 3.4.1. *Let w^1, w^2 be two solutions of* (3.71)*. Assume that $w^1(r_0) = w^2(r_0)$ and $w^1(M) = w^2(M)$, where $r_0 := \min\{r : r \in \widetilde{\mathcal{A}}\}$ and $M := \max\{r : r \in \widetilde{\mathcal{A}}\}$. Then*

$w^1 = w^2$ *on* $[r_{cr}, r_0] \cup [M, R]$.

Proof. The only part that changes is where we prove $w^1 = w^2$ on $[M, R]$. To prove this, we may assume without loss of generality that $0 < r_{cr} < R$ and $M < R$. We claim that $w^1$and w^2 cannot have a corner from below in (M, R) so that they agree on $[M, R]$ by (3.71).

Suppose not, i.e., there would exist $i \in \{1, 2\}$ and $y \in [M, R)$ such that

$$(w^i)_r(r) = \frac{-r(-f(r) - \lambda)}{rc(r) + (n-1)} \qquad \text{for all } r \geq y.$$

At $r = R$,

$$\phi(R) = (w^i)_r(R) = \frac{-R(-f(R) - \lambda)}{Rc(R) + (n-1)}.$$

This means that $\phi(R) \geq 0$. However, from the assumption that $R \notin \widetilde{\mathcal{A}}$, we have

$$-f(R) + \phi(R)\left(\frac{n-1}{R} + c(R)\right) < \lambda,$$

or,

$$-f(R) - \lambda < -\phi(R)\left(\frac{n-1}{R} + c(R)\right) < 0.$$

This yields a contradiction, as

$$\phi(R) = \frac{-R(-f(R) - \lambda)}{Rc(R) + (n-1)} > \frac{-R}{Rc(R) + (n-1)} \cdot \left(-\phi(R)\left(\frac{n-1}{R} + c(R)\right)\right) = \phi(R).$$

□

This proposition implies the following proposition of the uniqueness set property of the Aubry set $\widetilde{\mathcal{A}}$.

Proposition 3.4.2. *The following hold;*

(i) If w^1, w^2 *are solutions of* (3.71) *such that* $w^1 = w^2$ *on* $\widetilde{\mathcal{A}}$*, then* $w^1 = w^2$ *on* $[0, R]$.

(ii) If w^1 *and* w^2 *are a subsolution and a supersolution of* (3.71)*, respectively, and if* $w^1 \leq w^2$ *on* $\widetilde{\mathcal{A}}$*, then* $w^1 \leq w^2$ *on* $[0, R]$.

Now we prove Theorem 3.1.6 based on the uniqueness set property of the Aubry set.

Proof of Theorem 3.1.6. Since we already found the eigenvalue λ, defined the Aubry set $\widetilde{\mathcal{A}}$ and the number r_{cr} in the preceding discussions, it suffices to prove the asymptotic behavior and to find the large time profile in this proof.

The proof follows almost the same as that of [46, Theorem 1.1], but we put a extra care on the boundary $r = R$. Following the proof of [46, Theorem 1.3], we can prove (ii) of Theorem 3.1.6 once we prove (i) of Theorem 3.1.6. Thus, it suffices to show that $\varphi(r,t) - \lambda t$ converges as $t \to \infty$ uniformly in $r \in [0, R]$.

The first case we consider is when $r_{cr} = \infty$. Note that by (3.69) every admissible curve $\eta = \eta(s)$, i.e., $(\eta, v, l) \in \mathrm{SP}(r)$ for some v, l, r, satisfies

$$\dot{\eta}(s) \geq \frac{n-1}{\eta(s)} - c(\eta(s)). \tag{3.72}$$

Then η always moves to the right with minimal speed $\delta > 0$ for some $\delta > 0$. Therefore, using the formula (3.68),

$$\varphi(r,t) - \lambda t = \sup \left\{ \int_0^t \left(-f(\eta(s)) + \phi(\eta(s))l(s) - \lambda\right) ds + u_0(\eta(t)) : \ (\eta, v, l) \in \mathrm{SP}(r) \right\}$$

does not change as t varies after $t > \frac{R}{\delta}$.

The second case is when $r_{cr} < \infty$. We claim that for any $r \in \widetilde{\mathcal{A}}$, and for any $t_1 \leq t_2$, we have

$$\varphi(r, t_1) - \lambda t_1 \leq \varphi(r, t_2) - \lambda t_2.$$

Let us write the Skorokhod problem in (3.72) as $\mathrm{SP}(r,t) = \mathrm{SP}(r)$ to show the dependence in t. Then a triple $(\eta, v, l) \in \mathrm{SP}(r, t_1)$ induces a triple $(\widetilde{\eta}, \widetilde{v}, \widetilde{l}) \in \mathrm{SP}(r, t_2)$ by means of

$$(\widetilde{\eta}, \widetilde{v}, \widetilde{l})(s) = \begin{cases} (\eta, v, l)(0), & \text{for } 0 \leq s \leq t_2 - t_1, \\ (\eta, v, l)(s - (t_2 - t_1)), & \text{for } t_2 - t_1 \leq s \leq t_2. \end{cases}$$

This yields

$$\int_0^{t_2} \left(-f(\widetilde{\eta}(s)) + \phi(\widetilde{\eta}(s))\widetilde{l}(s) - \lambda\right) ds + u_0(\widetilde{\eta}(t_2)) = \int_0^{t_1} (-f(\eta(s)) + \phi(\eta(s))l(s) - \lambda)\, ds + u_0(\eta(t_1)),$$

and this is because $r \in \widetilde{\mathcal{A}}$ so that the integrand above is zero while $(\widetilde{\eta}, \widetilde{v}, \widetilde{l}) \in \mathrm{SP}(r, t_2)$ stays still upto $s = t_2 - t_1$. This argument of embedding $\mathrm{SP}(r, t_1)$ into $\mathrm{SP}(r, t_2)$ gives, together with (3.72), that $\varphi(r, t_1) - \lambda t_1 \leq \varphi(r, t_2) - \lambda t_2$.

The rest proof follows the same as that of [46, Theorem 1.1]. We also refer to [29] □

We give an example of nonuniqueness of (3.2) when $0 < q < 1$ before we end the section.

Example 3.4.2. *Consider*

$$\begin{cases} \lambda - \frac{n-1}{r} w_r - c(r)|w_r| + f(r) = 0 & \text{in } (0, R) \times (0, \infty), \\ w_r(R) = \phi(R)|w_r(R)|^{1-q}, \end{cases} \tag{3.73}$$

where $0 < q < 1$. Let $\phi(R) = 1$. We also let $f \equiv 0$, $c \equiv 0$. Then c is coercive by Corollary 3.1.1.

By the definition of viscosity solutions, we see that the condition $w_r(R) = \phi(R)|w_r(R)|^{1-q}$ is satisfied if $w_r(R) = \mathrm{sgn}(\phi(R))|\phi(R)|^{\frac{1}{q}}$ in the classical sense. Then, one can check that $\lambda_1 = \frac{n-1}{R}$, $w^1(r) = \frac{r^2}{2R}$ solve (3.73).

Also, if the boundary condition $v_r(R) = 0$ is true in the classical sense, then the condition $w_r(R) = \phi(R)|w_r(R)|^{1-q}$ is satisfied in the viscosity sense. Then $\lambda_2 = 0$, $w^2 \equiv C$, where C is a constant, solve (3.73).

Therefore, we have two distinct eigenvalues admitting a solution, which result in two *different solutions* $\varphi^i(r,t) = \lambda_i t + w^i(x),\ i = 1, 2,$ *of*

$$\begin{cases} \varphi_t - \frac{n-1}{r}\varphi_r - c(r)|\varphi_r| + f(r) = 0 & \textit{in } (0,R)\times(0,\infty), \\ \qquad\qquad \varphi_r(R) = \phi(R)|\varphi_r(R)|^{1-q} & \\ \qquad\qquad \varphi(r,0) = u_0(r) & \textit{for } r \in [0,R]. \end{cases} \tag{3.74}$$

Appendix A

In this appendix, we provide the definition of viscosity solutions of (3.2) and give the results on the comparison principle and the stability under the conditions (3.3), (3.4) on c, f, respectively.

Let $F : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \setminus \{0\} \times \mathcal{S}_n \to \mathbb{R}$ be such that

$$F(x,z,p,X) = \text{trace}\left(\left(I - \frac{p\otimes p}{|p|^2}\right)X\right) + c(x,z)|p| - f(x,z),$$

where $\mathcal{S}_n$ is the set of square symmetric matrices of size n. Together with the assumption that $c_z \le 0,\ f_z \ge 0$, we see that $-F$ is degenerate elliptic and proper, i.e.,

$$-F(x,z,p,X) \le -F(x,w,p,Y) \quad \text{whenever } Y \le X,\ z \le w.$$

Define the lower and upper semicontinuous envelopes of F by, for $(x,z,p,X) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}_n$,

$$F_*(x,z,p,X) = \liminf_{(y,w,q,Y)\to(x,z,p,X)} F(y,w,q,Y),$$

and

$$F^*(x,z,p,X) = \limsup_{(y,w,q,Y)\to(x,z,p,X)} F(y,w,q,Y),$$

respectively.

Definition 3.4.1. *A function* $u : \overline{\Omega} \times [0,\infty) \to \mathbb{R}$ *is said to be a viscosity subsolution (a viscosity supersolution, resp.)of* (3.2) *if*

- *u is upper semicontinuous (lower semicontinuous, resp.);*

- *for all $x \in \overline{\Omega}$, $u^*(x,0) \le u_0(x)$ ($u_*(x,0) \ge u_0(x)$, resp.);*

- *for any function $\varphi \in C^2(\overline{\Omega} \times [0,\infty))$, if $(\hat{x},\hat{t}) \in \overline{\Omega} \times (0,\infty)$ is a maximizer (a minimizer, resp.) of $u - \varphi$, then, at $(\hat{x},\hat{t})$,*

$$\begin{cases} \varphi_t(\hat{x},\hat{t}) - F^*(\hat{x},u(\hat{x},\hat{t}),D\varphi(\hat{x},\hat{t}),D^2\varphi(\hat{x},\hat{t})) \le 0 & \text{if } \hat{x} \in \Omega, \\ \min\left\{\varphi_t(\hat{x},\hat{t}) - F^*(\hat{x},u(\hat{x},\hat{t}),D\varphi(\hat{x},\hat{t}),D^2\varphi(\hat{x},\hat{t})), \frac{\partial \varphi}{\partial \mathbf{n}}(\hat{x},\hat{t}) - \phi(\hat{x},\hat{t})\right\} \le 0 & \text{if } \hat{x} \in \partial\Omega. \end{cases}$$

$$\left(\begin{cases} \varphi_t(\hat{x},\hat{t}) - F_*(\hat{x},u(\hat{x},\hat{t}),D\varphi(\hat{x},\hat{t}),D^2\varphi(\hat{x},\hat{t})) \ge 0 & \text{if } \hat{x} \in \Omega, \\ \max\left\{\varphi_t(\hat{x},\hat{t}) - F_*(\hat{x},u(\hat{x},\hat{t}),D\varphi(\hat{x},\hat{t}),D^2\varphi(\hat{x},\hat{t})), \frac{\partial \varphi}{\partial \mathbf{n}}(\hat{x},\hat{t}) - \phi(\hat{x},\hat{t})\right\} \ge 0 & \text{if } \hat{x} \in \partial\Omega, \text{ resp.} \end{cases}\right)$$

A function $u : \overline{\Omega} \times [0,\infty) \to \mathbb{R}$ is a viscosity solution of (3.2) *if u is both its viscosity subsolution and its viscosity supersolution.*

Proposition 3.4.3 (Comparison principle for (3.2))**.** *Let Ω be a bounded domain in $\mathbb{R}^n$ with C^3 boundary $\partial\Omega$. Suppose that c, f satisfy* (3.3), (3.4), *respectively. Let u be a subsolution and v be a supersolution of* (3.2), *respectively. Then, $u^* \le v_*$ in $\overline{\Omega} \times [0,\infty)$.*

We can follow [6] with slight modifications for the comparison principle of viscosity solutions of (3.2). We also refer to [28, 48].

Lemma 3.4.1. *Suppose that u^η is the unique solution of* (3.7) *for each $\eta > 0$, and there exists $u \in C(\overline{\Omega} \times [0,\infty))$ such that*

$$u^\eta \to u, \quad \text{as } \eta \to 0,$$

uniformly on $\overline{\Omega} \times [0,T)$ for each $T > 0$. Then u is the unique viscosity solution of (3.2).

We refer to [28] for Lemma 3.4.1.

Appendix B

In this appendix, we provide a reason of why a priori gradient estimates (Propositions 3.2.1 and 3.2.2) yield the existence of solutions to (3.7). We leave [86] as the main reference.

Let $T \in (0,\infty)$, $X = C^{1,\alpha}(\Omega \times (0,T))$. For a given $w \in X$, we consider the following linear parabolic equation with a source term

$$\begin{cases} u_t = \operatorname{tr}\left\{a(Dw)D^2u\right\} + c(x,w)\sqrt{\eta^2 + |Dw|^2} - f(x,w) & \text{in } \Omega \times (0,T), \\ \dfrac{\partial u}{\partial \vec{\mathbf{n}}} = \phi(x)(\sqrt{\eta^2 + |Dw|^2})^{1-q} & \text{on } \partial\Omega \times [0,T), \\ u(x,0) = u_0(x) & \text{on } \overline{\Omega}. \end{cases} \tag{3.75}$$

Then, for any $w \in X$, there exists a unique solution $u_w \in C^{2,\alpha'}(\Omega \times (0,T)) \subseteq X$ to (3.75) for some $\alpha' \in (0,\alpha)$ with

$$\|u_w\|_{C^{2,\alpha'}(\Omega\times(0,T))} \leq C_1,$$

where $C_1 > 0$ is a constant depending only on $n, \alpha, \|w\|_X, \|u_0\|_{C^{2,\alpha}(\Omega)}$ and on the constants in (3.3), (3.4) (see [73, Theorem 4.5.2]).

Define a map $A : X \to X$ with $Aw = u_w$. Then A is a continuous and compact map. To apply Schauder fixed point theorem, it suffices to prove that the set

$$S = \{u \in X : u = \sigma A u \text{ for some } \sigma \in [0,1]\}$$

is bounded in X. Then, A admits a fixed point $u \in C^{2,\alpha'}(\Omega \times (0,T))$, and moreover, $u \in C^{1,\alpha'}(\overline{\Omega} \times [0,T])$ (see [73, 76]) since $c, f \in C^{1,\alpha}(\overline{\Omega} \times \mathbb{R})$ and are bounded. Therefore, u becomes a solution to (3.7), and the regularity of the solution u is improved so that $u \in C^{3,\alpha'}(\Omega \times (0,T)) \cap C^{2,\alpha'}(\overline{\Omega} \times [0,T])$ for some $\alpha' \in (0,\alpha)$ from the Schauder theory.

Let $u \in S$. Then, for some $\sigma \in [0,1]$, u solves

$$\begin{cases} u_t = \operatorname{tr}\left\{a(Du)D^2u\right\} + \sigma c(x,u)\sqrt{\eta^2 + |Du|^2} - \sigma f(x,u) & \text{in } \Omega \times (0,T), \\ \dfrac{\partial u}{\partial \vec{\mathbf{n}}} = \sigma\phi(x)(\sqrt{\eta^2 + |Du|^2})^{1-q} & \text{on } \partial\Omega \times [0,T), \\ u(x,0) = \sigma u_0(x) & \text{on } \overline{\Omega}. \end{cases} \tag{3.76}$$

By Proposition 3.2.2, we have that

$$\|Du\|_{L^\infty(\Omega\times[0,T))} \le C_2$$

where $C_2 > 0$ is a constant depending only on $T, \Omega, c, f, \phi, q, u_0$. Here, we have used the fact that $\sigma \in [0,1]$. By interior Schauder estimates, we also have that

$$\|Du\|_{C^\alpha(\Omega\times(0,T))} \le C_3$$

where $C_3 > 0$ is a constant depending only $T, \Omega, n, \alpha, c, f, \phi, q, u_0$. This yields that the set S is bounded in X, and therefore, we obtain the existence.

Now, we apply Proposition 3.2.1 to the obtained solution to conclude Theorem 3.1.2.

Appendix C

In this section, we provide the proof of Lemma 3.2.2 and that of 3.2.3.

Proof of Lemma 3.2.2. We take a copy of the space $(\mathbb{R}^n, x)$ with the coordinate x given in the hypothesis of this lemma, and relabel the coordinate x by y. We also relabel x_0 by y_0. We now construct a C^2 map g from $(\mathbb{R}^n, y)$ to $(\mathbb{R}^n, x)$ around y_0 as follows.

Take an open neighborhood U_1 of $y_0 = (0, \cdots, 0)$ in $\mathbb{R}^n$ and a C^3 function φ defined on $\{y' = (y_1, \cdots, y_{n-1}) : (y', 0) \in U_1\}$ such that $y = (y', y_n) \in \partial\Omega$ if and only if $y_n = \varphi(y')$. Then, the $y_\ell-$axis lies along an eigenvector corresponding to the eigenvalue κ_ℓ of the matrix $D^2\varphi(y_0)$, $\ell = 1, \cdots, n-1$, respectively. Define the map $g : U_1 \to \mathbb{R}^n$ by

$$g(y', y_n) = (y', \varphi(y')) - \vec{\mathbf{n}}(y', \varphi(y'))y_n.$$

Then, g is a C^2 function on U_1. Moreover, with respect to the coordinates y on the domain $U_1 \subseteq \mathbb{R}^n$ and x on the codomain $\mathbb{R}^n$, the Jacobian Jg at $(0, \cdots, 0, y_n)$, $|y_n| < \sigma$, is the diagonal matrix, as

$$Jg(0, \cdots, 0, y_n) = \begin{bmatrix} 1 - \kappa_1 y_n & & 0 \\ & \ddots & \\ 0 & & 1 - \kappa_n y_n \end{bmatrix},$$

where $\sigma > 0$ is a positive number such that $\{(0, \cdots, 0, y_n) : |y_n| < \sigma\} \subseteq U_1$ and that $\sigma^{-1} > \max\{|\kappa_1|, \cdots, |\kappa_{n-1}|\}$. In particular, $Jg(0, \cdots, 0)$ is the identity matrix, and therefore, by Inverse Function Theorem, there are an open neighborhood U of $(0, \cdots, 0)$ in $U_1(\subseteq \mathbb{R}^n)$ and an open neighborhood V of $(0, \cdots, 0)$ in $\mathbb{R}^n$ such that $g : U \to V$ is a C^2 diffeomorphism from U onto V. We take a smaller number $\sigma > 0$ if necessary so that $\{(0, \cdots, 0, y_n) : |y_n| < \sigma\} \subseteq U$ and that $\sigma^{-1} > \max\{|\kappa_1|, \cdots, |\kappa_{n-1}|\}$

By the chain rule, we obtain (iii), and then we obtain (iv) by differentiating (iii) in y_n when $\zeta, \overline{\zeta}$ are C^2 functions. For (i), (ii), we refer to [52, Lemma 14.16]. □

We next give the proof of Lemma 3.2.3.

Proof of Lemma 3.2.3. From $a(p) = I_n - \frac{p \otimes p}{\eta^2 + |p|^2}$, we see that, for each $\ell = 1, \cdots, n$,

$$a_{p^\ell}(Du) = -\frac{1}{\eta^2 + |Du|^2}\left(e_\ell \otimes Du + Du \otimes e_\ell\right) + \frac{2u_\ell}{(\eta^2 + |Du|^2)^2} Du \otimes Du,$$

where e_ℓ is the ℓ-th element of the standard basis of $\mathbb{R}^n$. Thus,

$$D_p a \odot \xi = -\frac{1}{\eta^2 + |Du|^2}\left(\xi \otimes Du + Du \otimes \xi\right) + \frac{2Du \cdot \xi}{(\eta^2 + |Du|^2)^2} Du \otimes Du.$$

Together with the fact that $\mathrm{tr}\{(p \otimes q)M\} = p \cdot (Mq) = q \cdot (Mp)$ for vectors $p, q \in \mathbb{R}^n$ and a symmetric matrix M, we obtain

$$
\begin{aligned}
v\mathrm{tr}\{(D_p(Du)\odot\xi)D^2u\} &= -\frac{2}{\eta^2+|Du|^2}\mathrm{tr}\{(\xi\otimes Du)vD^2u\} + \frac{2Du\cdot\xi}{(\eta^2+|Du|^2)^2}\mathrm{tr}\{(Du\otimes Du)vD^2u\}\\
&= -\frac{2}{\eta^2+|Du|^2}\xi\cdot(vD^2uDu) + \frac{2Du\cdot\xi}{(\eta^2+|Du|^2)^2}Du\cdot(vD^2uDu)\\
&= -\frac{2}{\eta^2+|Du|^2}\xi\cdot(v^2Dv) + \frac{2Du\cdot\xi}{(\eta^2+|Du|^2)^2}Du\cdot(v^2Dv)\\
&= -2\xi\cdot Dv + \frac{2Du\cdot\xi}{\eta^2+|Du|^2}Du\cdot Dv.
\end{aligned}
$$

We have used the fact that $vDv = D^2uDu$. Now use the fact that $(p_1\cdot p_2)(q_1\cdot q_2) = \mathrm{tr}\{(p_1\otimes q_1)(p_2\otimes q_2)\}$ for $p_1, p_2, q_1, q_2 \in \mathbb{R}^n$. Then,

$$
\begin{aligned}
v\mathrm{tr}\{(D_p(Du)\odot\xi)D^2u &= -2\left(\xi\cdot Dv - \frac{(Du\cdot\xi)(Du\cdot Dv)}{\eta^2+|Du|^2}\right)\\
&= -2\left(\mathrm{tr}\{I_n(\xi\otimes Dv)\} - \frac{\mathrm{tr}\{(Du\otimes Du)(\xi\otimes Dv)\}}{\eta^2+|Du|^2}\right)\\
&= -2\left(\mathrm{tr}\left\{\left(I_n - \frac{Du\otimes Du}{\eta^2+|Du|^2}\right)(\xi\otimes Dv)\right\}\right)\\
&= -2\mathrm{tr}\{a(Du)(\xi\otimes Dv)\},
\end{aligned}
$$

and therefore, (3.14) is proved. □

Chapter 4

A convergence rate of periodic homogenization for forced mean curvature flow of graphs in the laminar setting

4.1 Introduction

In this chapter, we are interested in the quantitative understanding of convergence of graphical hypersurfaces $\Gamma^\varepsilon(t)(\subseteq \mathbb{R}^{n+1})$ to $\Gamma(t)$ as $\varepsilon \to 0$ in a laminated environment, where the hypersurfaces $\Gamma^\varepsilon(t)$ evolve by the normal velocity

$$V = \varepsilon\kappa + c\left(\frac{x}{\varepsilon}\right).$$

Here, κ is the mean curvature of the hypersurface, and c is a given force depending on the spatial variable periodically. Fixing axes of $\mathbb{R}^{n+1}$, we write $c = c(x)$ with $x = (x_1, \cdots, x_n)$. The media is laminated so that c is independent of x_{n+1}.

If $\Gamma^\varepsilon(t)$ has a height function $u^\varepsilon(\cdot, t)$ so that $\Gamma^\varepsilon(t) = \{(x, u^\varepsilon(x,t)) : x \in \mathbb{R}^n\}$, then the evolution of hypersurfaces $\Gamma^\varepsilon(t)$, with an initial graph $\{(x, u_0(x)) : x \in \mathbb{R}^n\}$, is described by the equation

$$\begin{cases} u_t^\varepsilon + F\left(\varepsilon D^2 u^\varepsilon, Du^\varepsilon, \frac{x}{\varepsilon}\right) = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ u^\varepsilon(x,0) = u_0(x) & \text{on } \mathbb{R}^n \end{cases} \tag{4.1}$$

for $\varepsilon \in (0,1]$, where $F = F(X,p,y)$ is the mean curvature operator with a forcing term of graphs

$$F(X,p,y) = -\mathrm{tr}\left\{\left(I_n - \frac{p\otimes p}{1+|p|^2}\right)X\right\} - c\,(y)\,\sqrt{1+|p|^2},$$

for $(X,p,y) \in S^n \times \mathbb{R}^n \times \mathbb{R}^n$, $n \geq 1$. The precise meaning of notations will be introduced later.

Throughout this chapter, we impose the following assumptions on the forcing term c;

(A1) $c \in C^2(\mathbb{R}^n)$;

(A2) $c = c(y)$ is $\mathbb{Z}^n$-periodic in $y \in \mathbb{R}^n$, i.e., $c(y+k) = c(y)$ for $k \in \mathbb{Z}^n,\ y \in \mathbb{R}^n$;

(A3) $c(y)^2 - (n-1)|Dc(y)| > \delta \quad \text{for all } y \in \mathbb{R}^n,\ \text{for some } \delta > 0.$

We also assume that $u_0 \in \mathrm{Lip}(\mathbb{R}^n)$.

Under the assumptions (A1)–(A3), it is known (see [33, 78] for instance) that u^ε converges locally uniformly to u as $\varepsilon \to 0^+$ on $\mathbb{R}^n \times [0,\infty)$, which is a viscosity solution to the effective equation

$$\begin{cases} u_t + \overline{F}(Du) = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ u(x,0) = u_0(x) & \text{on } \mathbb{R}^n. \end{cases} \tag{4.2}$$

Here, $\overline{F}(p)$ is the unique real number such that the cell problem

$$F(D^2 v, p + Dv, y) = \overline{F}(p) \qquad \text{on } \mathbb{R}^n.$$

admits a $\mathbb{Z}^n$-periodic solution $v \in C^{2,\alpha}(\mathbb{R}^n)$ for some $\alpha \in (0,1)$. We refer to [33, 77] for the definition of $\overline{F}(p)$, or that of the effective Hamiltonian $\overline{H}(p)$.

The main goal of this chapter is to obtain a rate of convergence of u^ε to u as $\varepsilon \to 0^+$ by proving (i) that $\|u^\varepsilon - u\|_{L^\infty(\mathbb{R}^n\times[0,T])}$ is $O(\varepsilon^{1/2})$ for any given $T > 0$, and (ii) that $|u^\varepsilon(x_0,t_0) - u(x_0,t_0)|$ is $\Omega(\varepsilon^{1/2})$, i.e., $|u^\varepsilon(x_0,t_0) - u(x_0,t_0)| \geq C\varepsilon^{1/2}$ for some $C > 0$, $(x_0,t_0) \in \mathbb{R}^n \times (0,\infty)$ in certain cases.

4.1.1 Literature and main results

Periodic homogenization of geometric motions has been studied only recently. In [78], Lipschitz continuous correctors were found under the assumption (A3) in the periodic setting. When the gradient of the force c is large and $n \geq 3$, it is shown in [16] by an example in the laminar setting that homogenization may not occur. It is also shown in [16] that homogenization always takes place when $n = 2$ for the level-set fronts as long as the force is positive, whose argument is 2-d arguments, showing the front is trapped in two parallel translations of an initial front in a bounded distance. Without any sign condition on c, Lipschitz continuous correctors were found in [31] under the condition that $c \in C^2(\mathbb{T}^n)$ and that $\|c\|_{C^2(\mathbb{T}^n)}$ is small enough, whose part of the proof is based on [14]. A further analysis on asymptotic speeds is given in [40]. For more related works, we refer to [25, 21, 22]. See also the recent works [41, 83] on the curvature G-equation. To the best of our knowledge, quantitative homogenization of geometric motions in the periodic environment has not been treated.

Quantitative homogenization for Hamilton-Jacobi equations in the periodic setting has received a lot of attention. The rate $O(\varepsilon^{1/3})$ was obtained for first-order equations in [20]. For convex first-order Hamilton-Jacobi equations, the optimal rate of convergence $O(\varepsilon)$ was obtained very recently in [97]. We refer to [57, 85, 97, 99] and the references therein for earlier progress in this direction.

In this chapter, we obtain the rate $O(\varepsilon^{1/2})$ for periodic homogenization of forced mean curvature flow of graphs. We follow the framework of [20], and we utilize the additional fact that there is a regular selection of correctors (see Proposition 4.2.1). Based on this observation, we derive the improved rate $O(\varepsilon^{1/2})$. Also, we list an example that shows that we cannot expect a faster rate than $O(\varepsilon^{1/2})$ if we expect only the Lipschitz continuity of solutions and a regular selection of correctors. In the study of Hamilton-Jacobi equations, this improvement of rates is noted in [98, Theorem 4.40] and used to obtain the optimal rate of periodic homogenization of viscous Hamilton-Jacobi equations in [92]. Our work is closely related to [92], [98, Theorem 4.40].

We now give the precise statements of our main results. The rate $O(\varepsilon^{1/2})$ is obtained in the following theorem.

Theorem 4.1.1. *Assume* (A1)-(A3)*, and let u_0 be a globally Lipschitz function on $\mathbb{R}^n$ with $\|Du_0\|_{L^\infty(\mathbb{R}^n)} \le N_0 < +\infty$. For $\varepsilon \in (0,1]$, let u^ε be the unique classical solution to* (4.1)*, and let u be the unique viscosity solution to* (4.2)*. Fix $T > 0$. Then, there exists a constant $C > 0$ depending only on $n, \|c\|_{C^2(\mathbb{R}^n)}, N_0, \delta$ such that*

$$\|u^\varepsilon - u\|_{L^\infty(\mathbb{R}^n \times [0,T])} \le C(1+T)\varepsilon^{1/2}.$$

The next theorem shows that in the absence of a forcing term, i.e., $c \equiv 0$, one obtains the rate $\Omega(\varepsilon^{1/2})$. Also, one can check that the rate is $O(\varepsilon^{1/2})$ for general Lipschitz continuous, positively 1-homogeneous initial data when $c \equiv 0$.

Theorem 4.1.2. *Let $c \equiv 0$, and let $u_0(x) = |x|$. For $\varepsilon \in (0,1]$, let u^ε be the unique classical solution to* (4.1)*, and let u be the unique viscosity solution to* (4.2)*. Then, there exists an absolute constant $C > 0$ such that*

$$|u^\varepsilon(0,1) - u(0,1)| \ge C\varepsilon^{1/2}.$$

Organization of the chapter

In Section 4.2.1, we state propositions about the well-posedness of (4.1). In Section 4.2, we simplify the settings of the problem by using a priori estimates, and give a proof of Theorem 4.1.1. In Section 4.3, we obtain the optimality of the rate in Theorem 4.1.1 by proving Theorem 4.1.2.

Notations

The set of all n by n matrices is denoted by S^n. The matrix I_n denotes the n by n identity matrix, and $p \otimes p$ is the matrix $\left(p^i p^j\right)_{i,j=1}$ for $p = (p^1, \cdots, p^n)^{\mathrm{t}} \in \mathbb{R}^n$.

In the subsequent sections, $\langle x \rangle$ denotes the number $(1+|x|^2)^{1/2}$ for $x \in \mathbb{R}^n$. Note that

$D\langle x\rangle = \frac{x}{\langle x\rangle}$, and $D^2\langle x\rangle = \frac{1}{\langle x\rangle}\left(I_n - \frac{x}{\langle x\rangle}\otimes\frac{x}{\langle x\rangle}\right)$. We also let, for $p \in \mathbb{R}^n$, $a(p)$ denote the matrix $I_n - \frac{p}{\langle p\rangle}\otimes\frac{p}{\langle p\rangle}$. For a nonzero vector p in $\mathbb{R}^n$ (or in $\mathbb{R}^{n+1}$), we let $\widehat{p} = \frac{p}{|p|}$. For a square matrix α, we let $\|\alpha\| = \sqrt{\mathrm{tr}\{\alpha^t\alpha\}}$, where $\mathrm{tr}\{\cdot\}$ is the trace of a given argument square matrix. Numbers $C, M > 0$ denotes constants that may vary line by line, and their dependency on parameters will be specified in arguments.

For $p \in \mathbb{R}^n$, we let $a^{ij}(p)$ be the (i,j)-entry of the matrix $a(p)$. We define $D_p a(p) \odot q$ to be the matrix $\left(\sum_{k=1}^n \left(\frac{\partial}{\partial p^k}a^{ij}(p)\right) q^k\right)_{i,j=1,\cdots,n}$ for $q = (q^1, \cdots, q^n) \in \mathbb{R}^n$.

4.2 Proof of Theorem 4.1.1

4.2.1 Well-posedness of (4.1)

We consider the forced mean curvature flow of graphs

$$\begin{cases} w_t = \mathrm{tr}\left\{a(Dw)D^2w\right\} + c(x)\sqrt{1+|Dw|^2} & \text{in } \mathbb{R}^n \times (0,T), \\ w(x,0) = w_0(x) & \text{on } \mathbb{R}^n. \end{cases} \tag{4.3}$$

We note that by change of variables, namely $u^\varepsilon(x,t) = \varepsilon w\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}\right)$, or $w(x,t) = \frac{1}{\varepsilon}u^\varepsilon(\varepsilon x, \varepsilon t)$, we can go back and forth between (4.1) and (4.3) (when $T = +\infty$), with the change $w_0(x) = \frac{1}{\varepsilon}u_0(\varepsilon x),\ u_0(x) = \varepsilon w_0(\frac{1}{\varepsilon}x)$. We also note that Lipschitz constants on initial data are preserved through this change of variables.

We state the well-posedness of (4.3), which ensures that of (4.1).

Theorem 4.2.1. *Assume* (A1) and (A3). Let w_0 be a globally Lipschitz function on $\mathbb{R}^n$ with $\|Dw_0\|_{L^\infty(\mathbb{R}^n)} \leq N_0 < +\infty$. Then, (4.3) has a unique classical solution for all time ($T = +\infty$), and moreover, there exists $M = M(n, \|c\|_{C^2(\mathbb{R}^n)}, N_0, \delta) > 0$ such that

$$\|Dw\|_{L^\infty(\mathbb{R}^n\times[0,\infty))} \leq M,$$

and

$$\|w_t(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \le M\left(\frac{1}{\min\{\sqrt{t},1\}}+1\right) \quad \text{for all } t>0. \tag{4.4}$$

Here, $\delta>0$ is the number appearing in the condition (A3).

We outline a sketch of this theorem in Appendix 4.3. The references for the theorem we refer to are [32] (when $c\equiv 0$) and [31, Appendix A] (with a forcing term c).

4.2.2 Settings and simplifications

We assume the conditions (A1)-(A3) in the rest of this section. Let u_0 be a globally Lipschitz function on $\mathbb{R}^n$ with $\|Du_0\|_{L^\infty(\mathbb{R}^n)} \le N_0 < +\infty$, we consider (4.1). Then, $w(x,t) = \frac{1}{\varepsilon}u^\varepsilon(\varepsilon x,\varepsilon t)$ solves (4.3) with $w_0(x) = \frac{1}{\varepsilon}u_0(\varepsilon x)$ with the same Lipschitz constant N_0. Through this change of variables, we see that there exists $M = M(n,\|c\|_{C^2(\mathbb{R}^n)},N_0,\delta)>0$ such that

$$\|u^\varepsilon_t\|_{L^\infty(\mathbb{R}^n\times[\varepsilon,\infty))} + \|Du^\varepsilon\|_{L^\infty(\mathbb{R}^n\times[0,\infty))} \le M. \tag{4.5}$$

Also, by (4.4), we have

$$\|u^\varepsilon_t(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \le M\left(\left\|D^2w\left(\cdot,\frac{t}{\varepsilon}\right)\right\|_{L^\infty(\mathbb{R}^n)}+1\right) \le M\left(\max\left\{\sqrt{\frac{\varepsilon}{t}},1\right\}+1\right) \tag{4.6}$$

for $t>0$. Combining (4.6) with (4.5), we see that for each compact set $K\subseteq\mathbb{R}^n\times[0,\infty)$, there exists $M = M(n,\|c\|_{C^2(\mathbb{R}^n)},N_0,\delta,K)>0$ such that $\|u^\varepsilon\|_{L^\infty(K)}\le M$ for each $\varepsilon\in(0,1]$ by integration. By the Arzelà-Ascoli Theorem, u^ε converges to u locally uniformly on $\mathbb{R}^n\times[0,\infty)$ as $\varepsilon\to 0$, and u solves (4.2) (see [33]), and satisfies

$$\|u_t\|_{L^\infty(\mathbb{R}^n\times[0,\infty))} + \|Du\|_{L^\infty(\mathbb{R}^n\times[0,\infty))} \le M. \tag{4.7}$$

Therefore, changing the values of $F(X,p,x)$ for $|p|>M$ does not affect the equations (4.1) and (4.2).

Let $\xi \in C^\infty(\mathbb{R}^n, [0,1])$ be a cut-off function such that

$$\xi(r) = \begin{cases} 1 & \text{for } r \le \sqrt{1+M^2}+1, \\ 0 & \text{for } r \ge \sqrt{1+M^2}+2. \end{cases}$$

Let

$$\tilde{F}(X,p,y) = -\mathrm{tr}\left\{ \left(I_n - \frac{p \otimes p}{1+|p|^2} \right) X \right\} - \tilde{c}(y,p)\sqrt{1+|p|^2}$$

for $(X,p,x) \in S^n \times \mathbb{R}^n \times \mathbb{R}^n$, where

$$\tilde{c}(y,p) = \xi\left(\sqrt{1+|p|^2}\right) c(y) + \left(1 - \xi\left(\sqrt{1+|p|^2}\right)\right) c_0.$$

Here, $c_0 = \sup_{y \in \mathbb{R}^n}(c(y))$ if $c > 0$, and $c_0 = \inf_{y \in \mathbb{R}^n}(c(y))$ if $c < 0$. Note that $c = c(y)$ is either always positive or always negative due to the assumption (A3). From this choice of the constant c_0, $\tilde{c} = \tilde{c}(y,p)$ satisfies

(A1)' $\tilde{c}(y,p)$ is C^2 in $y \in \mathbb{R}^n$ and C^∞ in $p \in \mathbb{R}^n$;

(A2)' $\tilde{c}(y+k,p) = \tilde{c}(y,p)$ for all $y, p \in \mathbb{R}^n,\ k \in \mathbb{Z}^n$;

(A3)' $\tilde{c}(y,p)^2 - (n-1)|D_y\tilde{c}(y,p)| > \delta$ for $y, p \in \mathbb{R}^n,$ for the same $\delta > 0$ in (A3).

Also, it holds that $\tilde{c}(y,p) \in [\min_{\mathbb{R}^n}(c), \max_{\mathbb{R}^n}(c)]$ for all $(y,p) \in \mathbb{R}^n \times \mathbb{R}^n$. We note that u^ε solves (4.1) with $\tilde{F}$ in place of F as expected.

Since the modified force $\tilde{c} = \tilde{c}(y,p)$ satisfies the assumption (A3)', we have the following proposition (see [78, Proposition 3.1]).

Proposition 4.2.1. *For each $p \in \mathbb{R}^n$, there exists a unique real number, denoted by $\overline{\tilde{F}}(p)$, such that the cell problem*

$$\begin{cases} \tilde{F}(D^2\tilde{v}, p + D\tilde{v}, y) = \overline{\tilde{F}}(p) & \text{on } \mathbb{R}^n, \\ \tilde{v}(0,p) = 0. \end{cases} \tag{4.8}$$

has a unique $\mathbb{Z}^n$*-periodic solution* $\tilde{v} = \tilde{v}(\cdot, p) \in C^2(\mathbb{R}^n)$*. Moreover, for each* $y \in \mathbb{R}^n$*, the map* $p \mapsto \tilde{v}(y, p)$ *is well-defined and is a* C^2 *map from* $\mathbb{R}^n$ *to* $\mathbb{R}$*. Also, the map* $p \mapsto \overline{\tilde{F}}(p)$ *is a* C^2 *map from* $\mathbb{R}^n$ *to* $\mathbb{R}$*, and* $\overline{\tilde{F}}(p) \in [\min_{\mathbb{R}^n}(-\tilde{c}(\cdot, p))\sqrt{1+|p|^2}, \max_{\mathbb{R}^n}(-\tilde{c}(\cdot, p))\sqrt{1+|p|^2}]$ *for all* $p \in \mathbb{R}^n$*.*

Before we move on to the proof of Theorem 4.1.1, we explain the additional property coming from the modified operator $\tilde{F}$. For $|p| \geq M+10$, $\tilde{c}(y,p) = c_0$, and thus, $\tilde{v}(\cdot, p) \equiv 0$. Therefore, there exists a constant $C = C(n, \|c\|_{C^2(\mathbb{R}^n)}, N_0, \delta) > 0$ such that

$$\sup_{y,p\in\mathbb{R}^n} |\tilde{v}(y,p)| + \sup_{y,p\in\mathbb{R}^n} |D_{(y,p)}\tilde{v}(y,p)| + \sup_{y,p\in\mathbb{R}^n} \|D^2\tilde{v}(y,p)\| \leq C. \tag{4.9}$$

Here, the gradient $D_{(y,p)}\tilde{v}(y,p)$ is with respect to the variable $(y,p) \in \mathbb{R}^n \times \mathbb{R}^n$, and the Hessian $D^2\tilde{v}(y,p)$ is with respect to the variable $y \in \mathbb{R}^n$.

4.2.3 Proof

We prove that for a fixed $T > 0$,

$$\|u^\varepsilon - u\|_{L^\infty(\mathbb{R}^n\times[0,T])} \leq C(1+T)\varepsilon^{1/2} \tag{4.10}$$

for some constant $C = C(n, \|c\|_{C^2(\mathbb{R}^n)}, N_0, \delta) > 0$. From now on, we use F and v to denote $\tilde{F}$ and $\tilde{v}$, respectively, by abuse of notations.

Proof of Theorem 4.1.1. We first show that

$$u^\varepsilon(x,t) - u(x,t) \leq C(1+T)\varepsilon^{1/2}$$

for $(x,t) \in \mathbb{R}^n \times [0,T]$. We set the auxiliary function

$$\Phi(x,y,z,t,s) := u^\varepsilon(x,t) - u(y,s) - \varepsilon v\left(\frac{x}{\varepsilon}, \frac{z-y}{\varepsilon^{1/2}}\right) \\ - \frac{|x-y|^2 + |t-s|^2}{2\varepsilon^{1/2}} - \frac{|x-z|^2}{2\varepsilon^{1/2}} - K(t+s) - \gamma\langle x\rangle,$$

where $K, \gamma > 0$ are numbers that will be chosen later. Then, the global maximum of Φ on $\mathbb{R}^{3n} \times [0,T]^2$ is attained at a certain point $(\hat{x}, \hat{y}, \hat{z}, \hat{t}, \hat{s}) \in \mathbb{R}^{3n} \times [0,T]^2$.

From $\Phi(\hat{x}, \hat{y}, \hat{z}, \hat{t}, \hat{s}) \geq \Phi(\hat{x}, \hat{y}, \hat{x}, \hat{t}, \hat{s})$ with (4.9), we have

$$\frac{|\hat{x}-\hat{z}|^2}{2\varepsilon^{1/2}} \leq \varepsilon \left(v\left(\frac{\hat{x}}{\varepsilon}, \frac{\hat{x}-\hat{y}}{\varepsilon^{1/2}}\right) - v\left(\frac{\hat{x}}{\varepsilon}, \frac{\hat{z}-\hat{y}}{\varepsilon^{1/2}}\right)\right) \leq C\varepsilon^{1/2}|\hat{x}-\hat{z}|,$$

which gives $|\hat{x}-\hat{z}| \leq C\varepsilon$. Similarly, from $\Phi(\hat{x}, \hat{y}, \hat{z}, \hat{t}, \hat{s}) \geq \Phi(\hat{x}, \hat{x}, \hat{x}, \hat{t}, \hat{s})$, we get

$$\begin{aligned}
\frac{|\hat{x}-\hat{y}|^2 + |\hat{x}-\hat{z}|^2}{2\varepsilon^{1/2}} &\leq u(\hat{x}, \hat{s}) - u(\hat{y}, \hat{s}) + \varepsilon \left(v\left(\frac{\hat{x}}{\varepsilon}, 0\right) - v\left(\frac{\hat{x}}{\varepsilon}, \frac{\hat{z}-\hat{y}}{\varepsilon^{1/2}}\right)\right) \\
&\leq C|\hat{x}-\hat{y}| + C\varepsilon^{1/2}|\hat{y}-\hat{z}|,
\end{aligned}$$

which yields $|\hat{x}-\hat{y}|+|\hat{y}-\hat{z}| \leq C\varepsilon^{1/2}$. Lastly, $\Phi(\hat{x}, \hat{y}, \hat{z}, \hat{t}, \hat{s}) \geq \Phi(\hat{x}, \hat{y}, \hat{z}, \hat{t}, \hat{t})$ and $\Phi(\hat{x}, \hat{y}, \hat{z}, \hat{t}, \hat{s}) \geq \Phi(\hat{x}, \hat{y}, \hat{z}, \hat{s}, \hat{s})$ give $|\hat{t}-\hat{s}| \leq C\varepsilon^{1/2}$, which can be obtained by using (4.5), (4.6), (4.7).

Next, we show that the case $\hat{t}, \hat{s} > 0$ does not happen if $\gamma \leq \varepsilon^{1/2}$, $K = K_1\varepsilon^{1/2}$, $K_1 > 0$ a constant sufficiently large. Assume first that $\hat{t}, \hat{s} > 0$. Then, $(x,t) \mapsto \Phi(x, \hat{y}, \hat{z}, t, \hat{s})$ attains a maximum at $(\hat{x}, \hat{t})$, and thus, by the maximum principle,

$$\begin{aligned}
Du^\varepsilon(\hat{x}, \hat{t}) &= Dv\left(\frac{\hat{x}}{\varepsilon}, \frac{\hat{z}-\hat{y}}{\varepsilon^{1/2}}\right) + \frac{\hat{x}-\hat{y}}{\varepsilon^{1/2}} + \frac{\hat{x}-\hat{z}}{\varepsilon^{1/2}} + \gamma\frac{\hat{x}}{\langle\hat{x}\rangle}, \\
u_t^\varepsilon(\hat{x}, \hat{t}) &\geq K + \frac{\hat{t}-\hat{s}}{\varepsilon^{1/2}}, \\
D^2u^\varepsilon(\hat{x}, \hat{t}) &\leq \frac{1}{\varepsilon}D^2v\left(\frac{\hat{x}}{\varepsilon}, \frac{\hat{z}-\hat{y}}{\varepsilon^{1/2}}\right) + \frac{2}{\varepsilon^{1/2}}I_n + \frac{\gamma}{\langle\hat{x}\rangle}\left(I_n - \frac{\hat{x}}{\langle\hat{x}\rangle} \otimes \frac{\hat{x}}{\langle\hat{x}\rangle}\right).
\end{aligned}$$

From $u_t^\varepsilon(\hat{x}, \hat{t}) + F\left(\varepsilon D^2u^\varepsilon(\hat{x}, \hat{t}), Du^\varepsilon(\hat{x}, \hat{t}), \frac{\hat{x}}{\varepsilon}\right) = 0$, we obtain

$$\begin{aligned}
0 &\geq K + \frac{\hat{t}-\hat{s}}{\varepsilon^{1/2}} + F\left(D^2v\left(\frac{\hat{x}}{\varepsilon}, \frac{\hat{z}-\hat{y}}{\varepsilon^{1/2}}\right) + 2\varepsilon^{1/2}I_n\right. \\
&\qquad\qquad \left. + \frac{\varepsilon\gamma}{\langle\hat{x}\rangle}\left(I_n - \frac{\hat{x}}{\langle\hat{x}\rangle} \otimes \frac{\hat{x}}{\langle\hat{x}\rangle}\right), Du^\varepsilon(\hat{x}, \hat{t}), \frac{\hat{x}}{\varepsilon}\right) \\
&= K + \frac{\hat{t}-\hat{s}}{\varepsilon^{1/2}} + F\left(D^2v\left(\frac{\hat{x}}{\varepsilon}, \frac{\hat{z}-\hat{y}}{\varepsilon^{1/2}}\right), Du^\varepsilon(\hat{x}, \hat{t}), \frac{\hat{x}}{\varepsilon}\right) \\
&\qquad\qquad - \operatorname{tr}\left\{a\left(Du^\varepsilon(\hat{x}, \hat{t})\right)\left(2\varepsilon^{1/2}I_n + \frac{\varepsilon\gamma}{\langle\hat{x}\rangle}\left(I_n - \frac{\hat{x}}{\langle\hat{x}\rangle} \otimes \frac{\hat{x}}{\langle\hat{x}\rangle}\right)\right)\right\}.
\end{aligned}$$

Since $0 \le a(Du^\varepsilon(\hat{x},\hat{t})) \le I_n$ and $0 \le 2\varepsilon^{1/2} I_n + \frac{\varepsilon\gamma}{\langle \hat{x}\rangle}\left(I_n - \frac{\hat{x}}{\langle \hat{x}\rangle} \otimes \frac{\hat{x}}{\langle \hat{x}\rangle}\right) \le (2\varepsilon^{1/2} + \varepsilon\gamma) I_n$, we obtain

$$0 \ge K + \frac{\hat{t}-\hat{s}}{\varepsilon^{1/2}} + F\left(D^2 v\left(\frac{\hat{x}}{\varepsilon}, \frac{\hat{z}-\hat{y}}{\varepsilon^{1/2}}\right), Du^\varepsilon(\hat{x},\hat{t}), \frac{\hat{x}}{\varepsilon}\right) - n\varepsilon^{1/2}(2+\gamma\varepsilon^{1/2}). \tag{4.11}$$

Besides, $v = v\left(\cdot, \frac{\hat{z}-\hat{y}}{\varepsilon^{1/2}}\right)$ solves

$$F\left(D^2 v, \frac{\hat{z}-\hat{y}}{\varepsilon^{1/2}} + Dv, y\right) = \overline{F}\left(\frac{\hat{z}-\hat{y}}{\varepsilon^{1/2}}\right).$$

Since $\left|Du^\varepsilon(\hat{x},\hat{t}) - \left(\frac{\hat{z}-\hat{y}}{\varepsilon^{1/2}} + Dv\right)\right| = \left|\frac{2(\hat{x}-\hat{z})}{\varepsilon^{1/2}} + \gamma\frac{\hat{x}}{\langle \hat{x}\rangle}\right| \le C(\varepsilon^{1/2}+\gamma)$,

$$\begin{aligned}
&\left|F\left(D^2 v\left(\frac{\hat{x}}{\varepsilon}, \frac{\hat{z}-\hat{y}}{\varepsilon^{1/2}}\right), Du^\varepsilon(\hat{x},\hat{t}), \frac{\hat{x}}{\varepsilon}\right)\right. \\
&\qquad\qquad \left. - F\left(D^2 v\left(\frac{\hat{x}}{\varepsilon}, \frac{\hat{z}-\hat{y}}{\varepsilon^{1/2}}\right), \frac{\hat{z}-\hat{y}}{\varepsilon^{1/2}} + Dv\left(\frac{\hat{x}}{\varepsilon}, \frac{\hat{z}-\hat{y}}{\varepsilon^{1/2}}\right), \frac{\hat{x}}{\varepsilon}\right)\right| \\
&\le C(\varepsilon^{1/2}+\gamma)\left(\left\|D^2 v\left(\frac{\hat{x}}{\varepsilon}, \frac{\hat{z}-\hat{y}}{\varepsilon^{1/2}}\right)\right\| + \max_{y\in\mathbb{R}^n}|c(y)|\right) \\
&\le C(\varepsilon^{1/2}+\gamma).
\end{aligned} \tag{4.12}$$

Here, we are using the property of a that for $p, q \in \mathbb{R}^n$,

$$\|a(p) - a(q)\| \le C|p-q|,$$

and using Cauchy-Schwarz's inequality $\left|\mathrm{tr}\{\alpha\beta^t\}\right| \le \|\alpha\|\|\beta\|$ for two square matrices α, β of the same size. Therefore, combining (4.11) and (4.12), we have

$$0 \ge K + \frac{\hat{t}-\hat{s}}{\varepsilon^{1/2}} + \overline{F}\left(\frac{\hat{z}-\hat{y}}{\varepsilon^{1/2}}\right) - C(\varepsilon^{1/2}+\gamma). \tag{4.13}$$

Now, we fix $(\hat{x}, \hat{z}, \hat{t})$. For $\sigma > 0$, we let

$$\Psi(y,\xi,s) := u(y,s) + \varepsilon v\left(\frac{\hat{x}}{\varepsilon}, \frac{\hat{z}-\xi}{\varepsilon^{1/2}}\right) + \frac{|\hat{x}-y|^2 + |\hat{t}-s|^2}{2\varepsilon^{1/2}} + \frac{|y-\xi|^2}{2\sigma} + Ks.$$

Then, Ψ attains a minimum at $(y_\sigma, \xi_\sigma, s_\sigma) \in \mathbb{R}^{2n} \times [0, T]$, and $(y_\sigma, \xi_\sigma, s_\sigma) \to (\hat{y}, \hat{y}, \hat{t})$ as $\sigma \to 0$ upto a subsequence. From $\Psi(y_\sigma, \xi_\sigma, s_\sigma) \le \Psi(y_\sigma, y_\sigma, s_\sigma)$, we get

$$\frac{|y_\sigma - \xi_\sigma|^2}{2\sigma} \le \varepsilon \left(v\left(\frac{\hat{x}}{\varepsilon}, \frac{\hat{z} - y_\sigma}{\varepsilon^{1/2}}\right) - v\left(\frac{\hat{x}}{\varepsilon}, \frac{\hat{z} - \xi_\sigma}{\varepsilon^{1/2}}\right)\right) \le C\varepsilon^{1/2}|y_\sigma - \xi_\sigma|,$$

which implies $|y_\sigma - \xi_\sigma| \le C\varepsilon^{1/2}\sigma$. Now, $(y, s) \mapsto \Psi(y, \xi_\sigma, s)$ attains a minimum at (y_σ, s_σ), and by the viscosity supersolution test for u at (y_σ, s_σ), we obtain

$$-K - \frac{s_\sigma - \hat{t}}{\varepsilon^{1/2}} + \overline{F}\left(-\frac{y_\sigma - \hat{x}}{\varepsilon^{1/2}} - \frac{y_\sigma - \xi_\sigma}{\sigma}\right) \ge 0.$$

Letting $\sigma \to 0$, we get

$$-K + \frac{\hat{t} - \hat{s}}{\varepsilon^{1/2}} + \overline{F}\left(\frac{\hat{x} - \hat{y}}{\varepsilon^{1/2}}\right) \ge -C\varepsilon^{1/2}. \tag{4.14}$$

Combining (4.13) and (4.14), we obtain

$$2K \le C\varepsilon^{1/2} + C\gamma.$$

For the choices $\gamma \le \varepsilon^{1/2}$, $K = K_1\varepsilon^{1/2}$, $K_1 > 0$ a constant sufficiently large, we see that this is a contradiction.

Therefore, we have either $\hat{t} = 0$ or $\hat{s} = 0$. In case when $\hat{s} = 0$, we have $\hat{t} \le C\varepsilon^{1/2}$, and therefore, we obtain $u^\varepsilon(\hat{x}, \hat{t}) - u_0(\hat{x}) = \int_0^{\hat{t}} u_t^\varepsilon(\hat{x}, s)ds \le C\varepsilon^{1/2}$ by using (4.6). Consequently,

$$\Phi(\hat{x}, \hat{y}, \hat{z}, \hat{t}, \hat{s}) \le u^\varepsilon(\hat{x}, \hat{t}) - u(\hat{y}, \hat{s}) - \varepsilon v\left(\frac{\hat{x}}{\varepsilon}, \frac{\hat{z} - \hat{y}}{\varepsilon^{1/2}}\right) \le C\varepsilon^{1/2}.$$

In case when $\hat{t} = 0$, the above follows from the fact that $\|u_t\|_{L^\infty(\mathbb{R}^n \times [0,\infty))} \le C$.

Since $\Phi(x, x, x, t, t) \le C\varepsilon^{1/2}$ for all $(x, t) \in \mathbb{R}^n \times [0, T]$, it holds that

$$u^\varepsilon(x, t) - u(x, t) \le C\varepsilon^{1/2} + \varepsilon v\left(\frac{x}{\varepsilon}, 0\right) + 2K_1\varepsilon^{1/2}t + \gamma\langle x\rangle.$$

By letting $\gamma \to 0$, we obtain the upper bound

$$u^\varepsilon(x,t) - u(x,t) \le C(1+T)\varepsilon^{1/2}$$

for all $(x,t) \in \mathbb{R}^n \times [0,T]$.

To prove the lower bound

$$u^\varepsilon(x,t) - u(x,t) \ge -C(1+T)\varepsilon^{1/2}$$

for all $(x,t) \in \mathbb{R}^n \times [0,T]$, we alternatively consider another auxiliary function

$$\begin{aligned}\Phi_1(x,y,z,t,s) := u^\varepsilon(x,t) - u(y,s) - \varepsilon v\left(\frac{x}{\varepsilon}, \frac{z-y}{\varepsilon^{1/2}}\right)\\ + \frac{|x-y|^2+|t-s|^2}{2\varepsilon^{1/2}} + \frac{|x-z|^2}{2\varepsilon^{1/2}} + K(t+s) + \gamma\langle x\rangle.\end{aligned}$$

Then, we follow a similar argument as the above to obtain the lower bound. □

4.3 Proof of Theorem 1.2

In this section, we prove Theorem 4.1.2. We let $c(x) = 0$, $u_0(x) = |x|$ for $x \in \mathbb{R}^n$. Note that $u(x,t) = |x|$ is the unique viscosity solution to the effective equation (4.2) with $\overline{F} \equiv 0$.

Proof of Theorem 4.1.2. Let $\overline{w} = \overline{w}(x,t)$ be the unique Lipschitz classical solution to the mean curvature flow

$$\begin{cases} \overline{w}_t = \operatorname{tr}\left\{a(D\overline{w})D^2\overline{w}\right\} & \text{in } \mathbb{R}^n \times (0,\infty), \\ \overline{w}(x,0) = |x| & \text{on } \mathbb{R}^n. \end{cases} \tag{4.15}$$

As (4.15) enjoys the comparison principle among Lipschitz solutions, we have that $\frac{1}{\lambda}\overline{w}(\lambda x, \lambda^2 t) = \overline{w}(x,t)$ for any $\lambda > 0$. Therefore, $\overline{w}(0,t) = \sqrt{t}\overline{w}(0,1)$ for any $t \ge 0$, and the fact that $\overline{w}(0,1) > 0$ can be proved by taking a barrier function from below whose initial data is a smooth convex function that passes through the origin and is less or equal to the function

$w_0(x) = |x|$.

Since $u^\varepsilon(x,t) = \varepsilon \overline{w}\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}\right)$ and $u(x,t) = |x|$, we have

$$u^\varepsilon(0,t) - u(0,t) = \varepsilon \overline{w}\left(0, \frac{t}{\varepsilon}\right) = \overline{w}(0,1)\sqrt{t\varepsilon} > 0.$$

By taking $t = 1,\ C = \overline{w}(0,1) > 0$, we complete the proof of Theorem 4.1.2. □

A. Proof of Theorem 4.2.1

We prove Theorem 4.2.1 in this appendix. We will separate the steps into Propositions 4.3.1, 4.3.2, 4.3.3, whose statements are about the estimates of gradients, Hessians and time derivatives.

We state the short-time existence of classical solutions to (4.3). We skip the proof as known in the literature. The uniqueness follows from the standard comparison principle, for which we refer to [28]. See [7] for more general results in this direction. For the existence with gradient and Hessian estimates, we refer to [32] (in the absence of a forcing) and to [31, Appendix A] (when with a C^2 forcing term).

Proposition 4.3.1. *Let w_0 be a globally Lipschitz function on $\mathbb{R}^n$ with $\|Dw_0\|_{L^\infty(\mathbb{R}^n)} \leq N_0 < +\infty$. Then, there exists $T^* = T^*(\|c\|_{C^2(\mathbb{R}^n)}, N_0) > 0$ such that (4.3) with $T = T^*$ has a unique classical solution $w = w(x,t)$. Moreover, for each $T \in (0, T^*)$, there exist $N = N(\|c\|_{C^2(\mathbb{R}^n)}, N_0, T) > 0$ and $C = C(\|c\|_{C^2(\mathbb{R}^n)}, N_0, T) > 0$ such that*

$$\|Dw\|_{L^\infty(\mathbb{R}^n \times [0,T])} \leq N,$$

and

$$\|D^2 w(\cdot, t)\|_{L^\infty(\mathbb{R}^n)} \leq \frac{C}{\sqrt{t}}$$

for $t \in (0, T]$.

Next, we state and prove a priori time derivative estimates based on the maximum principle. See [66, Lemma 3.1].

Proposition 4.3.2. *Let w_0 be a globally Lipschitz function on $\mathbb{R}^n$ with $\|Dw_0\|_{L^\infty(\mathbb{R}^n)} \le N_0 < +\infty$. Let $T^* = T^*(\|c\|_{C^2(\mathbb{R}^n)}, N_0) > 0$ be chosen such that* (4.3) *with $T = T^*$ has the unique classical solution $w = w(x,t)$. Then, for any $\tau \in (0, T^*)$, it holds that*

$$\|w_t\|_{L^\infty(\mathbb{R}^n \times [\tau, T^*))} \le \|w_t(\cdot, \tau)\|_{L^\infty(\mathbb{R}^n)} < +\infty.$$

Proof of Proposition 4.3.2. Let $T \in (\tau, T^*)$. Then, by Proposition 4.3.1, there exist $N = N(\|c\|_{C^2(\mathbb{R}^n)}, N_0, T) > 0$ and $C = C(\|c\|_{C^2(\mathbb{R}^n)}, N_0, T) > 0$ such that

$$\|Dw\|_{L^\infty(\mathbb{R}^n \times [0,T])} \le N,$$

and

$$\|D^2 w(\cdot, t)\|_{L^\infty(\mathbb{R}^n)} \le \frac{C}{\sqrt{t}}$$

for $t \in (\tau, T]$. Therefore, by the equation (4.3), we see that $\|w_t\|_{L^\infty(\mathbb{R}^n \times [\tau, T])} \le K = K(\|c\|_{C^2(\mathbb{R}^n)}, N_0, \tau, T)$.

We aim to prove

$$\sup_{\mathbb{R}^n \times [\tau, T]} w_t \le \sup_{\mathbb{R}^n} w_t(\cdot, \tau).$$

Suppose for the contrary that there exists $(x_0, t_0) \in \mathbb{R}^n \times (\tau, T]$ such that

$$w_t(x_0, t_0) > \sup_{\mathbb{R}^n} w_t(\cdot, \tau).$$

Then there would exist a number $\lambda \in (0,1)$ such that

$$w_t(x_0, t_0) - \lambda t_0 > \sup_{\mathbb{R}^n} w_t(\cdot, \tau) - \lambda \tau.$$

We run Bernstein method now with $\Phi(x,t) := w_t(x,t) - \lambda t$. Let $\Phi^*(t) := \sup_{\mathbb{R}^n} \Phi(\cdot, t)$ for each $t \in [\tau, T]$. Then $\Phi^*(t_0) > \Phi^*(\tau)$. Fix a sequence $\{\varepsilon_j\}_j$ of positive numbers that converges to 0 as $j \to \infty$. For each $t \in [\tau, T]$, let $x_j(t)$ be a maximizer of $\Phi_j(x,t) = \Phi(x,t) - \varepsilon_j |x|^2$. Then, $\Phi(x_j(t), t) \to \Phi^*(t)$, $D\Phi(x_j(t), t) \to 0$ as $j \to \infty$, and

$\limsup_{j\to\infty} D^2\Phi(x_j(t),t) \le 0$ in the sense that $\limsup_{j\to\infty}(D^2\Phi(x_j(t),t)v)\cdot v \le 0$ for any $v \in \mathbb{R}^n$.

Note that $\{t \in [\tau,T] : \Phi^*(t) = \sup_{[\tau,T]}\Phi^*(\cdot)\}$ is a closed subinterval of $[\tau,T]$ not containing τ. Consequently, there exists $t^* \in (\tau,T]$ such that $\Phi^*(t^*) = \sup_{[\tau,T]}\Phi^*(\cdot)$, $\Phi^*(t) < \Phi^*(t^*)$ for all $t \in [\tau,t^*)$, and thus that $\liminf_{j\to\infty}\Phi_t(x_j(t^*),t^*) \ge 0$.

Differentiating the first line of (4.3) in t, we obtain

$$(w_t)_t - \mathrm{tr}\{a(Dw)D^2w_t\} = \mathrm{tr}\{a(Dw)_tD^2w\} + c\frac{Dw\cdot Dw_t}{\sqrt{1+|Dw|^2}}.$$

Also, at $(x,t) \in \mathbb{R}^n \times [\tau,T]$,

$$\begin{aligned}\mathrm{tr}\{a(Dw)_tD^2w\} &= \mathrm{tr}\{(D_pa(Dw)\odot Dw_t)D^2w\}\\ &\le \frac{4n^3}{\sqrt{1+|Dw|^2}}|Dw_t|\|D^2w\| \le C(n,\|c\|_{C^2(\mathbb{R}^n)},N_0,\tau)|Dw_t|\end{aligned}$$

for some constant $C = C(n,\|c\|_{C^2(\mathbb{R}^n)},N_0,\tau) > 0$ depending only on its argument. Here, we have used the fact that $\left|\frac{\partial}{\partial p_k}a^{ij}(p)\right| \le \frac{4}{\sqrt{1+|p|^2}}$ for all $p \in \mathbb{R}^n$. Also,

$$c\frac{Dw\cdot Dw_t}{\sqrt{1+|Dw|^2}} \le \|c\|_{L^\infty(\mathbb{R}^n)}|Dw_t|.$$

Therefore, evaluated at $(x_j(t^*),t^*)$ in the following limit,

$$\begin{aligned}0 &\le \liminf_{j\to\infty}\left(\Phi_t - \mathrm{tr}\left\{a(Dw)D^2\Phi\right\}\right)\\ &\le \liminf_{j\to\infty}\left(-\lambda + (w_t)_t - \mathrm{tr}\{a(Dw)D^2w_t\}\right)\\ &\le -\lambda + \liminf_{j\to\infty} C(n,\|c\|_{C^2(\mathbb{R}^n)},N_0,\tau)|Dw_t|\\ &= -\lambda,\end{aligned}$$

a contradiction.

The statement $\inf_{\mathbb{R}^n\times[\tau,T]} w_t \ge \inf_{\mathbb{R}^n} w_t(\cdot,\tau)$ can be verified similarly, and $T \in (\tau,T^*)$ can be chosen arbitrarily. Therefore, we complete the proof. □

We state and prove a priori gradient estimates. The point of the following proposition is to remove the dependency on $T \in (0, T^*)$ in the estimate of Proposition 4.3.1. We refer to [78, 66] regarding gradient estimates from the coercivity condition (A3).

Proposition 4.3.3. *Let w_0 be a globally Lipschitz function on $\mathbb{R}^n$ with $\|Dw_0\|_{L^\infty(\mathbb{R}^n)} \le N_0 < +\infty$. Let $T^* = T^*(\|c\|_{C^2(\mathbb{R}^n)}, N_0) > 0$ be chosen such that (4.3) with $T = T^*$ has the unique classical solution $w = w(x,t)$. Then, for any $\tau \in (0, T^*)$, there exists $M = M(n, \|c\|_{L^\infty(\mathbb{R}^n)}, \|w_t(\cdot,\tau)\|_{L^\infty(\mathbb{R}^n)}, \delta) > 0$ such that*

$$\|Dw\|_{L^\infty(\mathbb{R}^n \times [\tau, T^*))} \le \max\{\|Dw(\cdot,\tau)\|_{L^\infty(\mathbb{R}^n)}, M\}.$$

Here, $\delta > 0$ is the number appearing in the condition (A3).

Proof of Proposition 4.3.3. Let $T \in (\tau, T^*)$. By Proposition 4.3.1, there exists $N = N(\|c\|_{C^2(\mathbb{R}^n)}, N_0, T) > 0$ such that

$$\|Dw\|_{L^\infty(\mathbb{R}^n \times [\tau, T])} \le N. \tag{4.16}$$

The goal of this proof is to make this estimate independent of $T \in (\tau, T^*)$.

We now run Bernstein method with $\Phi(x,t) := z(x,t)$. Let $\Phi^*(t) := \sup_{\mathbb{R}^n} w(\cdot,t)$ for $t \in [\tau, T]$. Let $\{\varepsilon_j\}_j$ be a sequence of positive numbers that converges to 0 as $j \to \infty$. For each $t \in [\tau, T]$, a maximizer $\{x_j(t)\}_j$ of $\Phi_j(x,t) := \Phi(x,t) - \varepsilon_j |x|^2$ satisfies that $\Phi(x_j(t), t) \to \Phi^*(t)$, $D\Phi(x_j(t), t) \to 0$ as $j \to \infty$, and that $\limsup_{j\to\infty} D^2\Phi(x_j(t), t) \le 0$. Here, we are using the estimate (4.16).

If $\{t \in [\tau, T] : \Phi^*(t) = \sup_{[\tau,T]} \Phi^*(\cdot)\}$ contains τ, we obtain the conclusion. We assume the other case so that there exists $t_1 \in (\tau, T]$ such that $\Phi^*(t) < \Phi^*(t_1) = \sup_{[\tau,T]} \Phi^*(\cdot)$ for all $t \in [\tau, t_1)$. Then, it holds that $\liminf_{j\to\infty} \Phi_t(x_j(t_1), t_1) \ge 0$.

We differentiate the first line of (4.3) in x_k and multiply by w_{x_k} and then sum over

$k = 1, \cdots, n$. We get, as a result,

$$zz_t - z\mathrm{tr}\{a(Dw)D^2z\} = z\mathrm{tr}\{(D_pa(Dw) \odot Dz)D^2w\} - \mathrm{tr}\{(a(Dw)D^2w)^2\} \\ +zDc \cdot Dw + cDw \cdot Dz. \qquad (4.17)$$

We estimate the term $\mathrm{tr}\{(a(Dw)D^2w)^2\}$. Using the fact that $Dz = z^{-1}D^2wDw$, we see that

$$\begin{aligned} &\mathrm{tr}\{(a(Dw)D^2w)^2\} \\ &= \mathrm{tr}\{a(Dw)D^2wI_nD^2w\} - \mathrm{tr}\{a(Dw)D^2w\frac{Dw}{z} \otimes \frac{Dw}{z}D^2w\} \\ &= \mathrm{tr}\{a(Dw)(D^2w)^2\} - \mathrm{tr}\{a(Dw)Dz \otimes Dz\}. \qquad (4.18) \end{aligned}$$

Recall Cauchy-Schwarz's inequality; for two square matrices α, β of the same size, we have

$$\mathrm{tr}\{\alpha\beta^t\}^2 \leq \|\alpha\|^2\|\beta\|^2$$

Assume $n \geq 2$. We put $\alpha = (a(Dw))^{1/2}D^2w,\ \beta = (a(Dw))^{1/2}$ to obtain

$$\begin{aligned} &\mathrm{tr}\{a(Dw)(D^2w)^2\} \\ &\geq \frac{1}{n-1+\frac{1}{z^2}}\mathrm{tr}\{a(Dw)D^2w\}^2 \\ &\geq \frac{1}{n-1}(w_t - cz)^2 - \frac{1}{(n-1)^2z^2}(w_t - cz)^2 \\ &\geq \frac{c^2}{n-1}z^2 - \frac{2\|w_t(\cdot,\tau)\|_{L^\infty(\mathbb{R}^n)}\|c\|_{L^\infty(\mathbb{R}^n)}}{n-1}z - C \qquad (4.19) \end{aligned}$$

for some constant $C = C(n, \|c\|_{L^\infty(\mathbb{R}^n)}, \|w_t(\cdot,\tau)\|_{L^\infty(\mathbb{R}^n)}) > 0$. Here, we have used Proposition 4.3.2.

Note that

$$z\mathrm{tr}\{(D_pa(Dw) \odot Dz)D^2w\} \leq 4n^3|Dz|\|D^2w\|_{L^\infty(\mathbb{R}^n \times [\tau,T])} \leq C|Dz|$$

for some constant $C = C(n, \|c\|_{C^2(\mathbb{R}^n)}, N_0, \tau, T) > 0$ from the Hessian estimate in Proposition 4.3.1. We also have used the fact that $\left|\frac{\partial}{\partial p^k} a^{ij}(p)\right| \leq \frac{4}{\sqrt{1+|p|^2}}$ for $p \in \mathbb{R}^n$.

From (4.17), (4.18), (4.19), we have

$$
\begin{aligned}
& zz_t - z\mathrm{tr}\{a(Dw)D^2 w\} \\
& \leq C(n, \|c\|_{C^2(\mathbb{R}^n)}, N_0, \tau, T)|Dz| - \left(\frac{c^2}{n-1} - |Dc|\right) z^2 \\
& \quad + \frac{2\|w_t(\cdot,\tau)\|_{L^\infty(\mathbb{R}^n)}\|c\|_{L^\infty(\mathbb{R}^n)}}{n-1} z \ + |Dz|^2 + \|c\|_{L^\infty(\mathbb{R}^n)}|Dz|z + C
\end{aligned}
$$

for some constant $C = C(n, \|c\|_{L^\infty(\mathbb{R}^n)}, \|w_t(\cdot,\tau)\|_{L^\infty(\mathbb{R}^n)}) > 0$. Evaluate at $(x_j(t_1), t_1)$ and let $j \to \infty$ to obtain

$$0 \leq -\delta\Phi^*(t_1)^2 + C\Phi^*(t_1) + C$$

for some constant $C = C(n, \|c\|_{L^\infty(\mathbb{R}^n)}, \|w_t(\cdot,\tau)\|_{L^\infty(\mathbb{R}^n)}) > 0$ (taking a larger one than the previous lines if necessary) depending only on its arguments. This completes the proof when $n \geq 2$. In the case of $n = 1$, the estimate can be carried out similarly. □

Combining Propositions 4.3.1, 4.3.2, 4.3.3, we obtain the long-time existence, proved in the following.

Proof of Theorem 4.2.1. The uniqueness is standard [28, 7]. We prove the existence of classical solutions for all time.

Let $T^* = T^*(\|c\|_{C^2(\mathbb{R}^n)}, N_0) > 0$ be chosen as in Proposition 4.3.1. Fix $\tau_0 = \frac{1}{2}T^*$. Tracking the dependency on parameters using Propositions 4.3.1, 4.3.2, 4.3.3, we see that there exists $M = M(n, \|c\|_{C^2(\mathbb{R}^n)}, N_0, \delta) > 0$ such that

$$\|Dw\|_{L^\infty(\mathbb{R}^n \times [\tau_0, T^*))} \leq M.$$

Let $\tau_1 = T^* - \varepsilon \in (\tau_0, T^*)$. Starting from $w(\cdot, \tau_1)$ at $t = \tau_1$, seen as an initial data, we can extend the solution on time interval $[\tau_1, \tau_1 + \frac{1}{(C_1+1)\sqrt{1+M^2}})$ (see the proof of Proposition

4.3.1 for this explicit expression). As $\varepsilon > 0$ can be arbitrarily small, the solution exists on time interval $[0, T_1^*)$ with $T_1^* = T^* + \frac{1}{(C_1+1)\sqrt{1+M^2}}$. Not changing the choice $\tau_0 = \frac{1}{2}T^*$, we still have

$$\|Dw\|_{L^\infty(\mathbb{R}^n \times [\tau_0, T_1^*))} \leq M.$$

with the same constant $M = M(n, \|c\|_{C^2(\mathbb{R}^n)}, N_0, \delta) > 0$ by applying the proofs of Propositions 4.3.2, 4.3.3. Then, we can extend the solution on time interval $[0, T_2^*)$ with $T_2^* = T^* + \frac{2}{(C_1+1)\sqrt{1+M^2}}$ as we just did from $[0, T^*)$ to $[0, T_1^*)$. We inductively proceed to conclude the solution exists for all time.

The estimate (4.4) is a simple consequence of Propositions 4.3.1, 4.3.2. $\square$

Chapter 5

Periodic homogenization of geometric equations without perturbed correctors

5.1 Introduction

5.1.1 Settings and motivations

In this chapter, we are interested in homogenization of geometric equations in the periodic setting, i.e., the convergence of solutions $u^\varepsilon(x,t)$ to

$$\begin{cases} u_t^\varepsilon + F\left(\varepsilon D^2 u^\varepsilon, Du^\varepsilon, \frac{x}{\varepsilon}\right) = 0, & \text{in} \quad \mathbb{R}^n \times (0,\infty), \\ u^\varepsilon(\cdot,0) = u_0, & \text{on} \quad \mathbb{R}^n, \end{cases} \tag{5.1}$$

to the solution $u(x,t)$ to

$$\begin{cases} u_t + \overline{F}\left(\frac{Du}{|Du|}\right)|Du| = 0, & \text{in} \quad \mathbb{R}^n \times (0,\infty), \\ u(\cdot,0) = u_0, & \text{on} \quad \mathbb{R}^n. \end{cases} \tag{5.2}$$

as $\varepsilon \to 0^+$, where an operator F is periodic in the spatial variable. The equations (5.1), (5.2) describe the large-scale behavior of the level-sets $\{u^\varepsilon(\cdot,t) = 0\}$, understood as the fronts. The environment and the curvature determine the rule of evolution of the fronts as the normal velocity in typical models. As the curvature effect is now involved, we call the operator F *geometric*, of which we will provide the precise definition in the Subsection 5.1.3.

Our main motivations of this study are the following two geometric equations; one is the curvature G-equation [41] with the normal velocity $V = (1-\varepsilon d\kappa)_+ + W\left(\frac{x}{\varepsilon}\right)\cdot\vec{\mathrm{n}}$ with the microscale parameter $\varepsilon > 0$. Here, the normal vector is outward to the set $\{u^\varepsilon(\cdot,t) > 0\}$, and κ represents the mean curvature, which is nonpositive when $\{u^\varepsilon(\cdot,t) > 0\}$ is convex. The number $d > 0$ describes the flame thickness [81, 91], and W is a vector field modeling wind. The other is the mean curvature equation with the normal velocity $V = \varepsilon\kappa + c\left(\frac{x}{\varepsilon}\right)$ with a spatial forcing term c.

It was first pointed out in [16] that the classical perturbed test function method [34, 33] does not directly imply the full homogenization for geometric equations because of the discontinuity of geometric operators F at $p = 0$. Instead, the conditional homogenization [16, Theorem 1.5] is derived under a stronger condition, namely, that *perturbed correctors* exist. This condition is satisfied by coercive forced equations, as the perturbation respects the coercivity condition on c. However, this is not the case for the curvature G-equation due to the noncoercive nature, meaning that homogenization of the curvature G-equation has been unclear so far. Motivatived by this circumstance, we relax the condition on perturbed correctors in this chapter.

5.1.2 Literature overview

Homogenization of geometric equations in periodic media has received a lot of attentions. In [78], it is shown that mean curvature motions with a forcing term c admit Lipschitz continuous correctors under the coercivity condition that

$$\operatorname*{ess\,inf}_{\mathbb{R}^n}\left(c^2 - (n-1)|Dc|\right) \geq \delta > 0 \tag{5.3}$$

holds for some $\delta > 0$. This condition can be seen as a small oscillation condition on $|Dc|$, depending on the magnitude of a positive force. For a positive forcing term without assuming the condition (5.3), [16] shows by an example that homogenization does not happen in 3-d or higher. In contrast, also shown in [16], homogenization holds true in 2-d as long as the force is positive. In [40], a further asymptotic analysis on the head speed and the tail speed is given for a positive and Lipschitz continuous force. It is concluded in [40] that the head and tail speeds are continuous in the normal directions, and homogenization happens for any uniformly continuous initial data if and only if the two speeds agree.

The case of sign-changing forces also has been investigated with smallness conditions. Namely, [31] found Lipschitz continuous correctors under the condition that $c \in C^2(\mathbb{T}^n)$ and $\|c\|_{C^2(\mathbb{T}^n)}$ is small enough. A variational approach is taken in [22], showing that if $c(x) = g(x_1, \cdots, x_{n-1})$ (a laminated environment), and if

$$\int_{(0,1)^{n-1}} g > 0, \ \min g \leq 0 \text{ and } \max g - \min g < C_n 2^{1/n},$$

with the isoperimetric constant $C_n > 0$ (appearing in [7]), then there exists a Lipschitz continuous corrector for $p = e_n$. In general, moreover, [22] proved the existence of *generalized* traveling waves whose support is not necessarily a full cell. Also, interesting questions about homogenization with a sign-changing force, together with first-order Hamilton-Jacobi equations without curvature effect, are discussed in [21]. The paper [21] proved that when $n = 2, \ c(x) = g(x_1)$, there exists a Lipschitz continuous corrector under the condition

$$\int_0^1 g > 0 \text{ and } \int_0^1 g - \min g < 2.$$

Allowing a large oscillation, a counterexample to the homogenization with $\int_0^1 g = 0$ is also given in [21].

One of the main points of [16] is about the issue on deriving homogenization from the perturbed test function method [34, 33]. Unlike equations with continuous operators, the discontinuity at $p = 0$ of geometric operators causes a gap when applying the method. The paper [16] provided a sufficient condition to guarantee the full homogenization that the cell problems are solved for perturbed equations. Mean curvature motions with a coercive forcing term c, satisfying either (5.3) or $\inf c > 0$, meet this stronger condition, and consequently, homogenization is concluded. However, general curvature G-equations [41, 83] do not satisfy the condition since homogenization fails for 3-d shear flows when the flow intensity surpasses a bifurcation value [83].

The curvature G-equation has been introduced in [81] and been mathematically studied in [41, 83] recently. The curvature effect in $V = (1 - \varepsilon d\kappa)_+ + W\left(\frac{x}{\varepsilon}\right) \cdot \vec{\mathrm{n}}$ is considered to describe the physical phenomenon of the flame propagation that a concave part of the flame front propagates faster proportionally to the flame thickness, called the Markstein number [81]. The $()_+$-correction is considered in [41, 83] in order to ensure the physical validity, and this correction was first introduced in [104].

The difficulty when analyzing the curvature G-equation comes from the lack of coercivity, which necessitates a new idea. The paper [41] adopted a game theory analysis [71] (say, with Players I and II), and provided a uniform bound on the magnitude of approximate correctors. A natural strategy of Player I, indeed which serves as the key of the analysis, is to follow the flow-invariant curve of the two dimensional cellar flow. This strategy has two advantages, namely, that any choice of movement by Player II is nullified during the strategy, and that Player I can return back to the starting point of the curve, the latter of which is particularly important. This is because in general taking a certain strategy yields one direction of estimates, and the other direction is compensated by applying the minimum value principle with the closed boundary. However, when we consider perturbed equations, Player II has more options to move in each round, which hinders Player I from closing the flow-invariant curve.

Facing this difficulty, it is natural to ask whether or not having correctors, not perturbed ones, ensures homogenization. This is a nontrivial issue for geometric equations as pointed out in [16], although commonly believed to be true from the perturbed test function method [34, 33]. In this chapter, we confirm that this belief is true even for geometric equations. The idea is that we use perturbed approximate correctors instead of perturbed correctors, which is in line with the use of approximate correctors in [20].

Using the fact that mean curvature motions with a coercive forcing term enjoy Lipschitz estimates including their perturbed equations, we leave a rate of homogenization of the motions in the periodic setting in this note. See [5] in this direction for the random setting. A rate $O(\varepsilon^{1/2})$ for the case of periodic, laminated media can be obtained with a simpler estimate [64], and we refer to [92] for the case of viscous Hamilton-Jacobi equations. Obtaining an optimal rate is a different issue, and see [97] for the very recent development of the case of first-order convex Hamilton-Jacobi equations.

5.1.3 Main results

Let $n \geq 2$ throughout this chapter. Let S^n denote the set of $n \times n$ symmetric matrices. We consider operators $F : S^n \times (\mathbb{R}^n \setminus \{0\}) \times \mathbb{R}^n \to \mathbb{R}$ satisfying the following properties.

(I) F is continuous in $S^n \times (\mathbb{R}^n \setminus \{0\}) \times \mathbb{R}^n$.

(II) F is *degenerate elliptic*; $F(Y, p, y) \leq F(X, p, y)$ for all $X, Y \in S^n$ with $X \leq Y$ and all $(p, y) \in (\mathbb{R}^n \setminus \{0\}) \times \mathbb{R}^n$.

(III) F is *geometric*; $F(\lambda X + \mu p \otimes p, \lambda p, y) = \lambda F(X, p, y)$ for all $(X, p, y) \in S^n \times (\mathbb{R}^n \setminus \{0\}) \times \mathbb{R}^n$ and all $\lambda > 0,\ \mu \in \mathbb{R}$.

(IV) F is $\mathbb{Z}^n$-periodic; $F(X, p, y + k) = F(X, p, y)$ for all $(X, p, y) \in S^n \times (\mathbb{R}^n \setminus \{0\}) \times \mathbb{R}^n$ and $k \in \mathbb{Z}^n$.

(V) F is *regular*;

(i) For every $R > 0$, there exists $M > 0$ such that $|F(X, p, y)| \leq M$ for all $(X, p, y) \in S^n \times (\mathbb{R}^n \setminus \{0\}) \times \mathbb{R}^n$ with $\|X\| \leq R,\ 0 < |p| \leq R$.

(ii) There exist $K > 0$ and $\omega : [0, \infty) \to [0, \infty)$ such that $\omega(0^+) = 0$ and

$$F^*(X, \alpha(x - y), x) - F_*(Y, \alpha(x - y), y) \leq \omega\left(|x - y|(1 + \alpha|x - y|)\right)$$

for all $\alpha \geq 0,\ X, Y \in S^n,\ x, y \in \mathbb{R}^n$ satisfying

$$-K\alpha I_{2n} \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq K\alpha \begin{pmatrix} I_n & -I_n \\ -I_n & I_n \end{pmatrix}$$

with $\alpha = 0$ when $x = y$.

The notations appearing in the above conditions are explained before Section 5.2. Now we state the main theorem.

Theorem 5.1.1. *Suppose that a given operator $F : S^n \times (\mathbb{R}^n \setminus \{0\}) \times \mathbb{R}^n \to \mathbb{R}$ satisfies the conditions (I)–(V). Suppose that for each $p \in \mathbb{R}^n$, there exists a unique real number $\overline{F}(p)$ such that*

$$F(D^2 v, p + Dv, y) = \overline{F}(p) \qquad \text{on } \mathbb{R}^n \tag{5.4}$$

admits a $\mathbb{Z}^n$-periodic viscosity solution $v : \mathbb{R}^n \to \mathbb{R}$. Then, homogenization takes place, that is, u^ε converges to u locally uniformly on $\mathbb{R}^n \times [0, \infty)$ as $\varepsilon \to 0$, where u^ε and u are the unique viscosity solution to (5.1) *and to* (5.2)*, respectively, and u_0 represents a given uniformly continuous function on $\mathbb{R}^n$.*

As a corollary, we conclude homogenization of the curvature G-equation for the two dimensional cellular flow. Let us first state the main result of [41, Theorem 1.1] on the effective burning velocity and the uniform flatness.

Theorem 5.1.2. *[41, Theorem 1.1] For $\varepsilon > 0,\ p \in \mathbb{R}^2$, let u^ε denote the unique viscosity solution to*

$$u_t^\varepsilon + \left(1 - \varepsilon d \operatorname{div}\left(\frac{Du^\varepsilon}{|Du^\varepsilon|}\right)\right)_+ |Du^\varepsilon| + V\left(\frac{x}{\varepsilon}\right) \cdot Du^\varepsilon = 0, \quad \text{in} \quad \mathbb{R}^2 \times (0, \infty) \tag{5.5}$$

with $u^\varepsilon(x, 0) = p \cdot x,\ x \in \mathbb{R}^2$, where $d > 0$, and $V(x) = A(-\cos x_2 \sin x_1, \cos x_1 \sin x_2)$ is the two dimensional cellular flow with the flow intensity $A > 0$ for $x = (x_1, x_2) \in \mathbb{R}^2$.

Then, there exists a unique real number $\overline{H}(p)$ *such that*

$$|u^\varepsilon(x,t) - p\cdot x + \overline{H}(p)t| \le C\varepsilon \qquad \text{on } \mathbb{R}^2 \times [0,\infty),$$

for some constant $C > 0$ *depending only on* $d, A, |p|$*. Moreover, the map* $p \mapsto \overline{H}(p)$ *is a continuous, positive homogeneous function of degree one from* $\mathbb{R}^2 \setminus \{0\}$ *to* $(0,\infty)$*.*

As mentioned in [41, Section 4], the existence of the effective burning velocities $\overline{H}(p)$ such that the above conclusion holds can be extended to more general two dimensional incompressible flows.

Homogenization of the curvature G-equation with the two dimensional cellular flow (as well as general two dimensional incompressible flows) follows from the fact that this uniform flatness result, Theorem 5.1.2, implies the existence of correctors with $\overline{F}(p) = \overline{H}(p)$ in (5.4), which is proved in the Proposition 5.2.1.

Corollary 5.1.1. *For* $\varepsilon > 0$ *and a uniformly continuous initial function* u_0 *on* $\mathbb{R}^2$*, let* u^ε *be the unique viscosity solution to* (5.5) *with* $u^\varepsilon(\cdot,0) = u_0$*. Then, as* $\varepsilon \to 0$*, the solution* u^ε *converges locally uniformly on* $\mathbb{R}^2 \times [0,\infty)$ *to the unique viscosity solution* u *to*

$$\begin{cases} u_t + \overline{H}(Du) = 0, & \text{in} \quad \mathbb{R}^2 \times (0,\infty), \\ u(\cdot,0) = u_0, & \text{on} \quad \mathbb{R}^2. \end{cases} \tag{5.6}$$

We note that not only the case of the forced mean motion and the curvature G-equation that have effective velocities of certain signs, namely, $\overline{F}(p) < 0$ and $\overline{H}(p) > 0$ for $p \neq 0$, respectively, the statement of the Theorem 5.1.1 includes but also sign-changing cases, for which we refer to [31, 21, 22]. Technically speaking, the direction of the sup/inf-ball convolution [16, Lemma 13.1] should be carefully taken following the sign of $\overline{F}(p)$ in the proof of the Theorem 5.1.1.

We have stated the qualitative conclusion of homogenization deriving only from the the solvability of the cell problems. For mean curvature motions with a coercive forcing term satisfying (5.3), their perturbed forces also satisfy (5.3) as stated in the Proposition

5.3.1, which we cannot expect in general (the curvature G-equation, for instance). We utilize this fact to obtain a rate of periodic homogenization of mean curvature motions with a coercive forcing term. For a Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}^m$, we let $\|f\|_{C^{0,1}(\mathbb{R}^n)} := \|f\|_{L^\infty(\mathbb{R}^n)} + \|Df\|_{L^\infty(\mathbb{R}^n)}$, where Df is the Jacobian. Let $\hat{p} = \frac{p}{|p|}$ for $p \in \mathbb{R}^n \setminus \{0\}$.

Theorem 5.1.3. *Let*

$$F(X,p,y) = -\mathrm{tr}\,\{(I_n - \widehat{p} \otimes \widehat{p})\, X\} - c(y)|p|,$$

for $(X,p,y) \in S^n \times (\mathbb{R}^n \setminus \{0\}) \times \mathbb{R}^n$, *and suppose that a forcing term* c *is Lipschitz continuous,* $\mathbb{Z}^n$*-periodic and satisfies* (5.3)*. Let* u_0 *be a function on* $\mathbb{R}^n$ *with* $\|Du_0\|_{C^{0,1}(\mathbb{R}^n)} < \infty$*. For* $\varepsilon > 0$ *let* u^ε *denote the unique viscosity solution to* (5.1)*, and* u *denote the unique viscosity solution to* (5.2)*. Then, there exists a constant* $C > 0$ *depending only on* $n, \|c\|_{C^{0,1}(\mathbb{R}^n)}, \|Du_0\|_{C^{0,1}(\mathbb{R}^n)}, \delta$ *such that*

$$\|u^\varepsilon - u\|_{L^\infty(\mathbb{R}^n \times [0,T])} \leq C(1+T)\varepsilon^{1/8}$$

for all $T > 0$*,* $\varepsilon \in (0,1)$*.*

The optimal rate is of natural interest, and the following example shows that the optimal rate is slower than $O(\varepsilon)$.

Proposition 5.1.1. *Let* $n = 2$*,* $c(x) \equiv 1$*, and let* $u_0(x) = -|x|$*. Let* F, u^ε, u *be as in the statement of Theorem 5.1.3. Then, for any* $\varepsilon > 0$ *and for any* $(x,t) \in \mathbb{R}^2 \times [0,\infty)$ *with* $|x| = t > \varepsilon(1+e^{-1})$*, we have*

$$|u^\varepsilon(x,t) - u(x,t)| \geq \frac{1}{2}\varepsilon\left(\log\left(\frac{t}{\varepsilon} - 1\right) + 1\right). \tag{5.7}$$

In the next, we present examples of traveling graphs with prescribed asymptotics when a forcing term is a positive constant, as they demonstrate that homogenization rate is related to the stability of the traveling waves if we start with 1-positively homogeneous initial data. The traveling graphs are known as the "V-shaped traveling fronts" [89] in

2-d, and they are studied in [87] in arbitrary dimensions.

Proposition 5.1.2. *Let* $\alpha \in \left(0, \frac{\pi}{2}\right]$, $c(x) \equiv 1$ *and let*

$$u_0(x) = \sup_{\nu \in A} \left\{ (\cot \alpha) x \cdot (\nu, 0) \right\},$$

where A *is a given nonempty finite subset of the sphere* $\mathbb{S}^{n-2}$*. Let* F, u^ε, u *be as in the statement of Theorem 5.1.3. Then, there exists a constant* $C > 0$ *depending only on* $\alpha, |A|$ *such that*

$$\|u^\varepsilon - u\|_{L^\infty(\mathbb{R}^n \times [0,\infty))} \leq C\varepsilon$$

for all $\varepsilon > 0$.

In view of the metric problem [5], the fast rate $O(\varepsilon)$ can be seen as a result of the fact that the traveling waves stay similar to themselves as time changes, and we can ask whether or not this happens in laminated media with general uniformly continuous initial data, when a forcing term satisfies (5.3). A contrasting case in general media is the example in the Proposition 5.1.1, whose slowing effect is due to the accumulation of the curvature effect from the varying radii.

Organization of the chapter

We prove the Theorem 5.1.1 and the Corollary 5.1.1 in the Section 5.2, and we prove the Theorem 5.1.3 and the Propositions 5.1.1, 5.1.2 in the Section 5.3.

Notations and conventions

For each $n \geq 1$, we set and use the following notations throughout the chapter.

- $x_+ = \max\{x, 0\}$ for $x \in \mathbb{R}$.
- $\hat{p} = \frac{p}{|p|}$ for $p \in \mathbb{R}^n \setminus \{0\}$.
- $\langle p \rangle = \sqrt{1 + |p|^2}$ for $p \in \mathbb{R}^n$.

- S^n : the set of $n \times n$ symmetric matrices, for each $n \geq 1$.
- I_n : the $n \times n$ identity matrix.
- $p \otimes p$: the matrix $\left(p^i p^j\right)_{i,j=1}^n$ for $p = (p^1, \cdots, p^n) \in \mathbb{R}^n$.
- $\mathrm{tr}\,\{A\}$: the trace of a square matrix A.
- $\|A\| = \sup_{v \in \mathbb{R}^n : |v|=1} |(Av) \cdot v)|$ for each $n \times n$ matrix A.
- $B_r(x) = \{y \in \mathbb{R}^n : |y - x| < r\}$ for $x \in \mathbb{R}^n,\ r > 0$.
- $Q_r(P) = B_r(x) \times ((t - r, t + r) \cap [0, \infty))$ for $P = (x, t) \in \mathbb{R}^n \times [0, \infty),\ r > 0$.
- $\overline{Q}_r(P)$: the closure of $Q_r(P)$ in $\mathbb{R}^n \times [0, \infty)$ for $P \in \mathbb{R}^n \times [0, \infty),\ r > 0$.
- $\|f\|_{C^{0,1}(\mathbb{R}^n)} = \|f\|_{L^\infty(\mathbb{R}^n)} + \|Df\|_{L^\infty(\mathbb{R}^n)}$ for a Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}^m$ for $m \geq 1$, where Df denotes the Jacobian.
- F^* and F_* : the upper and lower-semicontinuous envelope of $F : S^n \times (\mathbb{R}^n \setminus \{0\}) \times \mathbb{R}^n \to \mathbb{R}$, respectively.

We follow the convention throughout the chapter that a number $C = C(\cdot) > 0$ denotes a positive constant that may vary line by line, and that its dependency on parameters (such as, $\varepsilon, \eta, \mu, r, \cdots$) is specified in its arguments. Specifying the dependency in the arguments is also applied to various parameters that appear in this chapter, not just to $C > 0$.

5.2 Proof of Theorem 5.1.1

This section is mainly devoted to the proof of the Theorem 5.1.1.

Let $F : S^n \times (\mathbb{R}^n \setminus \{0\}) \times \mathbb{R}^n \to \mathbb{R}$ be given. Throughout this section, we let

$$F^\eta(X, p, y) := \inf_{|e| \leq \eta} F(X, p, y + e),$$

$$F_\eta(X, p, y) := \sup_{|e| \leq \eta} F(X, p, y + e).$$

for $(X,p,y) \in S^n \times (\mathbb{R}^n \setminus \{0\}) \times \mathbb{R}^n$ and $\eta \geq 0$.

Proof of Theorem 5.1.1.

Step 0: Checking the initial condition with barrier functions.

Let $\overline{u} := \limsup^*_{\varepsilon\to 0} u^\varepsilon$ and $\underline{u} := \liminf_{\varepsilon\to 0}{}_* u^\varepsilon$. It suffices to prove that $\overline{u}$ is a viscosity subsolution to (5.2) and that $\underline{u}$ is a viscosity supersolution to (5.2). Then, the comparison principle for (5.2) implies that $\overline{u} \leq \underline{u}$, which then implies the local uniform convergence of u^ε to $\overline{u} = \underline{u}(=: u)$ on $\mathbb{R}^n \times [0,\infty)$.

We first of all note that there are sub/supersolutions $u^\mp$ to (5.2), which are independent of $\varepsilon \in (0,1]$ (see [51, Lemma 4.3.4, Theorem 4.3.1]) such that

$$u^- \leq u^+, \quad \text{and} \qquad \lim_{t\to 0} \sup_{x\in\mathbb{R}^2} |u^\mp(x,t) - u_0(x)| = 0,$$

which follows from the conditions (I), (II), (III) and (i) of (V). By the definition of $\underline{u}, \overline{u}$, we have that

$$u^- \leq \underline{u} \leq \overline{u} \leq u^+,$$

and thus that $\underline{u}(\cdot,0) = \overline{u}(\cdot,0) = u_0$.

Claim 1: The function $\underline{u}$ is a viscosity supersolution to (5.2).

Step 1.1: Parameters $r, \theta > 0$ from the assumption for the contrary.

Suppose the contrary for contradiction. Then, there exist $P_0 = (x_0,t_0) \in \mathbb{R}^2 \times (0,\infty)$, $r \in (0,t_0)$ and a C^2 function φ in $Q_r(P_0)$ such that

$$\begin{cases} \underline{u}(P_0) = \varphi(P_0), & \\ \underline{u} \geq \varphi & \text{on} \quad \overline{Q}_r(P_0), \\ \varphi_t(P_0) + \overline{F}(D\varphi(P_0)) =: -\theta < 0. & \end{cases}$$

Let $p = D\varphi(P_0),\ \overline{\lambda} = \overline{F}(p),\ \lambda_t = \varphi_t(P_0)$ so that we have

$$\overline{\lambda} + \theta = -\lambda_t. \tag{5.8}$$

By replacing φ by $-|(x,t) - P_0|^4 + \varphi$ if necessary, we can assume without loss of generality that there exists $\delta_1 = \delta_1(r) > 0$ such that

$$\begin{cases} \underline{u} - 2\delta_1 \geq \varphi & \text{on} \quad \overline{Q}_r(P_0) \setminus Q_{r/2}(P_0), \\ \underline{u} > \varphi & \text{on} \quad \overline{Q}_r(P_0) \setminus \{P_0\}. \end{cases}$$

We only cover the case $\overline{\lambda} = \overline{F}(p) \geq 0$ in the proof of the Claim 1 to avoid lengthiness. The other case $\overline{\lambda} = \overline{F}(p) < 0$ of the Claim 1, i.e., proving that $\underline{u}$ is a supersolution when $\overline{\lambda} = \overline{F}(p) < 0$ is omitted, as this case corresponds to the argument of [16, Section 8] that shows $\underline{u}$ is a supersolution. We use perturbed approximate correctors instead of perturbed correctors in both of the cases, and how we use perturbed correctors is demonstrated in detail from now on.

Step 1.2: Approximate correctors w_λ and $w_{\lambda,2\eta}$.

For $\lambda > 0$, we let w_λ be the solution to

$$\lambda w_\lambda + F\left(D^2 w_\lambda, p + Dw_\lambda, y\right) = 0 \quad \text{on} \quad \mathbb{R}^n.$$

By a simple comparison argument, we see that

$$-\sup v - \frac{\overline{F}(p)}{\lambda} + v \leq w_\lambda \leq -\inf v - \frac{\overline{F}(p)}{\lambda} + v,$$

where v is a $\mathbb{Z}^n$-periodic viscosity solution to (5.4) satisfying

$$\sup v - \inf v \leq \frac{1}{4}\overline{\kappa}_0$$

for some constant $\overline{\kappa}_0 = \overline{\kappa}_0(p) > 0$, from the hypothesis of the Theorem 5.1.1. Accordingly, we have

$$\sup w_\lambda - \inf w_\lambda \le \frac{1}{2}\overline{\kappa}_0,$$

and there exists $\lambda = \lambda(\theta, p) \in (0,1)$ such that

$$\|(-\lambda w_\lambda) - \overline{\lambda}\|_{L^\infty(\mathbb{R}^n)} \le \frac{1}{16}\theta.$$

We now consider an approximate corrector of the perturbed problem; for $\eta \ge 0$, let $w_{\lambda,2\eta}$ the solution to

$$\lambda w_{\lambda,2\eta} + F_{2\eta}\left(D^2 w_{\lambda,2\eta}, p + Dw_{\lambda,2\eta}, y\right) = 0 \qquad \text{on} \quad \mathbb{R}^n. \tag{5.9}$$

Choose $\eta = \eta(\lambda, \theta, \overline{\kappa}_0, p) = \eta(\theta, p) \in (0,1)$ such that

$$\|w_{\lambda,2\eta} - w_\lambda\|_{L^\infty(\mathbb{R}^n)} \le \min\left\{\frac{1}{16}\theta, \frac{1}{4}\overline{\kappa}_0\right\}$$

so that

$$\sup w_{\lambda,2\eta} - \inf w_{\lambda,2\eta} \le \overline{\kappa}_0 \tag{5.10}$$

and

$$\|(-\lambda w_{\lambda,2\eta}) - \overline{\lambda}\|_{L^\infty(\mathbb{R}^n)} \le \frac{1}{8}\theta. \tag{5.11}$$

Step 1.3: Extension and (sup-)convolution of the test function φ.

Write

$$\varphi(x,t) = \varphi(P_0) + p \cdot (x - x_0) + \lambda_t(t - t_0) + \psi(x,t) \tag{5.12}$$

with

$$|D\psi|, |\psi_t| \le \mu \quad \text{on} \quad \overline{Q}_r(P_0) \tag{5.13}$$

for some $\mu = \mu(r) > 0$ that goes to 0 as $r \to 0$. We extend ψ to a C^2 function on $\mathbb{R}^n \times [0, \infty)$, still denoted by ψ, that satisfies

$$|D\psi| + \|D^2\psi\| \le \mu_0 \quad \text{on} \quad \mathbb{R}^n \times [0, \infty) \tag{5.14}$$

for some $\mu_0 \le 1$ by replacing $r > 0$ by a smaller number if necessary. We also keep the notation for φ.

Let

$$\begin{cases} \widetilde{\varphi}^\varepsilon(x,t) := \varphi(x,t) + \varepsilon\left(w_{\lambda.2\eta}\left(\frac{x}{\varepsilon}\right) - w_{\lambda,2\eta}(0)\right), \\ \overline{\varphi}^\varepsilon(x,t) := \sup_{z \in \overline{B}_{\varepsilon\eta}(x)} \widetilde{\varphi}^\varepsilon(z,t), \\ \varphi^\varepsilon(x,t) := \sup_{z\in\mathbb{R}^n}\left(\overline{\varphi}^\varepsilon(z,t) - \frac{|x-z|^4}{4\varepsilon^3\rho}\right), \end{cases} \tag{5.15}$$

where $\rho \in (0,1)$ is to be determined later. By [16, Lemma 13.2.(B)], there exists $\varepsilon = \varepsilon(\delta_1, \overline{\kappa}_0, |p| + \mu_0) = \varepsilon(r,p) \in (0,1)$ such that

$$u^\varepsilon - \delta_1 \ge \varphi^\varepsilon \quad \text{on} \quad \overline{Q}_r(P_0) \setminus Q_{r/2}(P_0),$$

and that the infimum of $u^\varepsilon - \varphi^\varepsilon$ on $\overline{Q}_r(P_0)$ is attained in $Q_{r/2}(P_0)$, say at $P_\varepsilon = (x_\varepsilon, t_\varepsilon) \in Q_{r/2}(P_0)$. By [16, Lemma 13.2.(B)], and also by (5.10), there exist $\rho = \rho(\eta, \overline{\kappa}_0, |p| + \mu_0) = \rho(\theta, p) \in (0,1)$, $\overline{x}_\varepsilon \in \mathbb{R}^n$ such that

$$\varphi^\varepsilon(x_\varepsilon, t_\varepsilon) = \overline{\varphi}^\varepsilon(\overline{x}_\varepsilon, t_\varepsilon) - \frac{|x_\varepsilon - \overline{x}_\varepsilon|^4}{4\varepsilon^3\rho} \quad \text{and} \quad |\overline{x}_\varepsilon - x_\varepsilon| \le \varepsilon\eta. \tag{5.16}$$

Step 1.4: The viscosity inequalities from the Crandall-Ishii's Lemma

Unraveling the infimum in (5.15) and by the choice of $\overline{x}_\varepsilon \in \mathbb{R}^n$, we see that

$$(x,(y,t)) \in \mathbb{R}^n \times \overline{Q}_r(P_0) \longmapsto \overline{\varphi}^\varepsilon(x,t) - u^\varepsilon(y,t) - \frac{|x-y|^4}{4\varepsilon^3\rho}$$

attains a maximum at $(\overline{x}_\varepsilon,(x_\varepsilon,t_\varepsilon)) \in \mathbb{R}^n \times Q_{r/2}(P_0)$. Since [28, (8.5)] holds for our F (from the condition (i) of (V)) and for $\overline{\varphi}^\varepsilon, u^\varepsilon$, we can apply the Crandall-Ishii's Lemma [28, Theorem 8.3] to see that for every $\gamma > 0$, there exist $X, Y \in S^n$ such that

$$\begin{cases} (b_1, q, X) \in \overline{\mathcal{P}}^{2,+}\overline{\varphi}^\varepsilon(\overline{x}_\varepsilon, t_\varepsilon), \\ (b_2, q, Y) \in \overline{\mathcal{P}}^{2,-}u^\varepsilon(x_\varepsilon, t_\varepsilon), \\ b_1 - b_2 = 0 = \Phi_t(\overline{x}_\varepsilon, x_\varepsilon, t_\varepsilon), \\ -\left(\frac{1}{\gamma} + \|A\|\right) I_{2n} \le \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le A + \gamma A^2, \end{cases} \tag{5.17}$$

where $\Phi(x,y,t) := \frac{|x-y|^4}{4\varepsilon^3\rho}$ and

$$q := D_x\Phi(\overline{x}_\varepsilon, x_\varepsilon, t_\varepsilon) = -D_y\Phi(\overline{x}_\varepsilon, x_\varepsilon, t_\varepsilon) = \delta(\overline{x}_\varepsilon - x_\varepsilon) \quad \text{with} \quad \delta := \frac{|x_\varepsilon - \overline{x}_\varepsilon|^2}{\varepsilon^3\rho},$$

$$A := D^2_{(x,y)}\Phi(\overline{x}_\varepsilon, x_\varepsilon, t_\varepsilon) = \delta \begin{pmatrix} I_n + 2\widehat{q}\otimes\widehat{q} & -I_n - 2\widehat{q}\otimes\widehat{q} \\ -I_n - 2\widehat{q}\otimes\widehat{q} & I_n + 2\widehat{q}\otimes\widehat{q} \end{pmatrix} \text{ with } \widehat{q} := \frac{q}{|q|} \text{ (if } q \neq 0).$$

By [16, Lemma 13.1.(ii)], there exists $\widetilde{x}_\varepsilon \in \mathbb{R}^n$ such that

$$(b_1, q, X) \in \overline{\mathcal{P}}^{2,+}\widetilde{\varphi}^\varepsilon(\widetilde{x}_\varepsilon, t_\varepsilon), \quad \overline{\varphi}^\varepsilon(\overline{x}_\varepsilon, t_\varepsilon) = \widetilde{\varphi}^\varepsilon(\widetilde{x}_\varepsilon, t_\varepsilon), \quad |\widetilde{x}_\varepsilon - \overline{x}_\varepsilon| \le \varepsilon\eta. \tag{5.18}$$

Note that $v_{\lambda,2\eta}(x) := \varepsilon\left(w_{\lambda,2\eta}\left(\frac{x}{\varepsilon}\right) - w_{\lambda,2\eta}(0)\right)$ is a viscosity solution to

$$F_{2\eta}\left(\varepsilon D^2 v_{\lambda,2\eta}(x), p + Dv_{\lambda,2\eta}(x), \frac{x}{\varepsilon}\right) \le \overline{\lambda} + \frac{1}{8}\theta \quad \text{on} \quad \mathbb{R}^n,$$

from (5.9), (5.11). Therefore, from (5.18) and $\widetilde{\varphi}^{\varepsilon} = \varphi + v_{\lambda,2\eta}$, we have

$$b_1 = \varphi_t(\widetilde{P}_\varepsilon),\ (q - D\varphi(\widetilde{P}_\varepsilon), X - D^2\varphi(\widetilde{P}_\varepsilon)) \in \overline{\mathcal{J}}^{2,+} v_{\lambda,2\eta}(\widetilde{x}_\varepsilon) \text{ with } \widetilde{P}_\varepsilon = (\widetilde{x}_\varepsilon, t_\varepsilon) \tag{5.19}$$

and

$$(F_{2\eta})_* \left(\varepsilon X - \varepsilon D^2\varphi(\widetilde{P}_\varepsilon), p + q - D\varphi(\widetilde{P}_\varepsilon), \frac{\widetilde{x}_\varepsilon}{\varepsilon}\right) \leq \overline{\lambda} + \frac{1}{8}\theta.$$

Finally, by (5.12) and $(b_2, q, Y) \in \overline{\mathcal{P}}^{2,-} u^{\varepsilon}(x_\varepsilon, t_\varepsilon)$, we obtain the viscosity inequalities

$$\begin{cases} (F_{2\eta})_* \left(\varepsilon X - \varepsilon D^2\psi(\widetilde{P}_\varepsilon), q - D\psi(\widetilde{P}_\varepsilon), \frac{\widetilde{x}_\varepsilon}{\varepsilon}\right) \leq \overline{\lambda} + \frac{1}{8}\theta, \\ b_2 + F^* \left(\varepsilon Y, q, \frac{x_\varepsilon}{\varepsilon}\right) \geq 0. \end{cases} \tag{5.20}$$

Step 1.5: Bound of the gradient q and of the Hessians X, Y.

We can check that $\|A\| = 6\delta$ and

$$A^2 \leq 18\delta^2 \begin{pmatrix} I_n & -I_n \\ -I_n & I_n \end{pmatrix}.$$

Setting $\gamma = \frac{1}{3\delta}$ in (5.17), we obtain

$$-9\delta I_n \leq X \leq Y \leq 9\delta I_n$$

and in turn, with $\delta \leq \frac{\eta^2}{\varepsilon\rho}$ (from (5.16)),

$$|q| \leq \frac{\eta^3}{\rho} \quad \text{and} \quad -\frac{9\eta^2}{\rho} I_n \leq \varepsilon X \leq \varepsilon Y \leq \frac{9\eta^2}{\rho} I_n. \tag{5.21}$$

Now, as a crucial step, we separate the gradient q from the origin with a constant that depends only on θ, p. Choose $r = r(\theta) \in (0,1)$ small, $\varepsilon = \varepsilon(r,p) = \varepsilon(\theta,p) \in (0,1)$ small enough that $\mu = \mu(r) \leq \frac{\theta}{4}$ where $\mu > 0$ is given in (5.13), and that

$$P_\varepsilon = (x_\varepsilon, t_\varepsilon), \quad |\overline{x}_\varepsilon - x_\varepsilon| \leq \varepsilon\eta, \quad |\widetilde{x}_\varepsilon - \overline{x}_\varepsilon| \leq \varepsilon\eta$$

imply $\widetilde{P}_\varepsilon = (\widetilde{x}_\varepsilon, t_\varepsilon) \in Q_r(P_0)$, and thus that

$$|\psi_t(\widetilde{P}_\varepsilon)| \leq \mu \leq \frac{\theta}{4}. \tag{5.22}$$

From (5.12), (5.17), (5.19), we have $b_1 = b_2 = \lambda_t + \psi_t(\widetilde{P}_\varepsilon)$. Using the assumption $\overline{\lambda} = \overline{F}(p) \geq 0$, we link the identities/inequalities (5.8), (5.17), (5.20), (5.21), (5.22) to obtain a lower bound as follows:

$$\begin{aligned} \frac{\theta}{2} \leq \overline{\lambda} + \theta - \psi_t(\widetilde{P}_\varepsilon) = -\lambda_t - \psi_t(\widetilde{P}_\varepsilon) = -b_2 & \\ \leq F^*\left(\varepsilon Y, q, \frac{x_\varepsilon}{\varepsilon}\right) \leq F^*\left(\varepsilon X, q, \frac{x_\varepsilon}{\varepsilon}\right). \end{aligned}$$

Moreover, the function $\overline{\varphi}^\varepsilon$ is sup-convoluted by the definition (5.15), and therefore, its semijet $(b_1, q, X) \in \overline{\varphi}^\varepsilon$ enjoys a bound with the geometric operator F as follows ([16, Lemma 13.1.(ii)]):

$$\frac{\theta}{2} \leq F^*\left(\varepsilon X, q, \frac{x_\varepsilon}{\varepsilon}\right) \leq c|q|$$

for some $c = c(\eta) = c(\theta, p) > 0$. Therefore, we obtain

$$\frac{\theta}{2c} \leq |q|. \tag{5.23}$$

Step 1.6: Deriving a contradiction.

Note that the operator F is uniformly continuous on

$$\left\{(X', p', y') \in S^n \times \mathbb{R}^n \times \mathbb{R}^n : \|X'\| \leq 1 + \frac{9\eta^2}{\rho},\ \frac{\theta}{4c} \leq |p'| \leq 1 + \frac{\eta^3}{\rho}\right\}.$$

By this fact, we can choose $r = r(\theta, p) \in (0,1)$, $\varepsilon = \varepsilon(r, \theta, p) \in (0,1)$ such that

$$\varepsilon \| D^2 \psi(\widetilde{P}_\varepsilon)\|, |D\psi(\widetilde{P}_\varepsilon)| \le \mu(r) \le \min\left\{1, \frac{\theta}{2}, \frac{\theta}{4c}\right\},$$

and thus that, by (5.21), (5.23),

$$\|\varepsilon X\|, \|\varepsilon X - \varepsilon D^2\psi(\widetilde{P}_\varepsilon)\| \le 1 + \frac{9\eta^2}{\rho} \quad \text{and} \quad \frac{\theta}{4c} \le |q|, |q - D\psi(\widetilde{P}_\varepsilon)| \le 1 + \frac{\eta^3}{\rho},$$

and finally that

$$F\left(\varepsilon X - \varepsilon D^2\psi(\widetilde{P}_\varepsilon), q - D\psi(\widetilde{P}_\varepsilon), \frac{x_\varepsilon}{\varepsilon}\right) \ge -\frac{1}{4}\theta + F\left(\varepsilon X, q, \frac{x_\varepsilon}{\varepsilon}\right). \tag{5.24}$$

We derive a contradiction by linking the inequalities (5.8), (5.20), (5.21), (5.22), (5.24):

$$\begin{aligned}
\overline{\lambda} + \frac{1}{8}\theta &\ge (F_{2\eta})_* \left(\varepsilon X - \varepsilon D^2\psi(\widetilde{P}_\varepsilon), q - D\psi(\widetilde{P}_\varepsilon), \frac{\widetilde{x}_\varepsilon}{\varepsilon}\right) \\
&= F_{2\eta}\left(\varepsilon X - \varepsilon D^2\psi(\widetilde{P}_\varepsilon), q - D\psi(\widetilde{P}_\varepsilon), \frac{\widetilde{x}_\varepsilon}{\varepsilon}\right) \\
&\ge F\left(\varepsilon X - \varepsilon D^2\psi(\widetilde{P}_\varepsilon), q - D\psi(\widetilde{P}_\varepsilon), \frac{x_\varepsilon}{\varepsilon}\right) \\
&\ge -\frac{1}{4}\theta + F\left(\varepsilon X, q, \frac{x_\varepsilon}{\varepsilon}\right) \\
&\ge -\frac{1}{4}\theta + F\left(\varepsilon Y, q, \frac{x_\varepsilon}{\varepsilon}\right) \\
&\ge -\frac{1}{4}\theta - b_2 \\
&= -\frac{1}{4}\theta - \lambda_t - \psi_t(\widetilde{P}_\varepsilon) \\
&\ge \overline{\lambda} + \frac{1}{2}\theta,
\end{aligned}$$

which completes the proof of the Claim 1 in the case $\overline{\lambda} = \overline{F}(p) \ge 0$.

Claim 2: The function $\overline{u}$ is a viscosity subsolution to (5.2).

Step 2.1: Parameters $r, \theta > 0$ from the assumption for the contrary.

Suppose the contrary for contradiction. Then, there exist $P_0 = (x_0, t_0) \in \mathbb{R}^2 \times (0, \infty)$,

$r \in (0, t_0)$ and a C^2 function φ in $Q_r(P_0)$ such that

$$\begin{cases} \overline{u}(P_0) = \varphi(P_0), \\ \overline{u} \leq \varphi & \text{on} \quad \overline{Q}_r(P_0), \\ \varphi_t(P_0) + \overline{F}(D\varphi(P_0)) =: \theta > 0. \end{cases}$$

Let $p = D\varphi(P_0),\ \overline{\lambda} = \overline{F}(p),\ \lambda_t = \varphi_t(P_0)$ so that we have

$$\lambda_t + \overline{\lambda} = \theta. \tag{5.25}$$

By replacing φ by $|(x,t) - P_0|^4 + \varphi$ if necessary, we can assume without loss of generality that there exists $\delta_1 = \delta_1(r) > 0$ such that

$$\begin{cases} \overline{u} + 2\delta_1 \leq \varphi & \text{on} \quad \overline{Q}_r(P_0) \setminus Q_{r/2}(P_0), \\ \overline{u} < \varphi & \text{on} \quad \overline{Q}_r(P_0) \setminus \{P_0\}. \end{cases}$$

We only handle the case $\overline{\lambda} = \overline{F}(p) > 0$ in the proof of the Claim 2. The other case $\overline{\lambda} = \overline{F}(p) \leq 0$ of the Claim 2, i.e., proving that $\overline{u}$ is a subsolution when $\overline{\lambda} = \overline{F}(p) \leq 0$ is omitted, as this case corresponds to the argument of [16, Section 8] that shows $\overline{u}$ is a subsolution. We instead explain the use of perturbed approximate correctors in detail in the below (but in the opposite direction of perturbation to the proof of Claim 1).

Step 2.2: Approximate correctors w_λ and $w_\lambda^{2\eta}$.

Let w_λ be the solution to

$$\lambda w_\lambda + F\left(D^2 w_\lambda, p + Dw_\lambda, y\right) = 0 \quad \text{on} \quad \mathbb{R}^n.$$

Similarly as in the Step 1.1.1, namely by comparing w_λ with v (with additional constants), we see that there exist $\overline{\kappa}_0 = \overline{\kappa}_0(p) > 0$ and $\lambda = \lambda(\theta, p) \in (0, 1)$ such that

$$\sup w_\lambda - \inf w_\lambda \leq \frac{1}{2}\overline{\kappa}_0$$

and

$$\|(-\lambda w_\lambda) - \overline{\lambda}\|_{L^\infty(\mathbb{R}^n)} \leq \min\left\{\frac{1}{16}\theta, \frac{1}{16}\overline{\lambda}\right\}.$$

We consider an approximate corrector of the perturbed problem; for $\eta \geq 0$, let $w_\lambda^{2\eta}$ the solution to

$$\lambda w_\lambda^{2\eta} + F^{2\eta}\left(D^2 w_\lambda^{2\eta}, p + D w_\lambda^{2\eta}, y\right) = 0 \quad \text{on} \quad \mathbb{R}^n. \tag{5.26}$$

Choose $\eta = \eta(\lambda, \theta, \overline{\kappa}_0, p) = \eta(\theta, p) \in (0, 1)$ such that

$$\|w_\lambda^{2\eta} - w_\lambda\|_{L^\infty(\mathbb{R}^n)} \leq \min\left\{\frac{1}{16}\theta, \frac{1}{16}\overline{\lambda}, \frac{1}{4}\overline{\kappa}_0\right\}$$

so that

$$\sup w_\lambda^{2\eta} - \inf w_\lambda^{2\eta} \leq \overline{\kappa}_0 \tag{5.27}$$

and

$$\|(-\lambda w_\lambda^{2\eta}) - \overline{\lambda}\|_{L^\infty(\mathbb{R}^n)} \leq \min\left\{\frac{1}{8}\theta, \frac{1}{8}\overline{\lambda}\right\}. \tag{5.28}$$

Step 2.3: Extension of the test function φ, definition of a linear functional ℓ and the (sup-)convolution of u^ε.

Let ψ be a C^2 function on $\mathbb{R}^n \times [0, \infty)$ defined as in (5.12) (with abuse of notations for the extension) satisfying (5.13), (5.14). For $(x,t) \in \mathbb{R}^n \times [0,\infty)$, let

$$\begin{cases} \ell(x,t) := \varphi(P_0) + p \cdot (x - x_0) + \left(-\lambda w_\lambda^{2\eta}(0)\right)(t - t_0), \\ \widetilde{\ell}^\varepsilon(x,t) := \ell(x,t) + \varepsilon\left(w_\lambda^{2\eta}\left(\frac{x}{\varepsilon}\right) - w_\lambda^{2\eta}(0)\right), \\ \widehat{\ell}^\varepsilon(x,t) := \inf_{z\in\mathbb{R}^n}\left(\widetilde{\ell}^\varepsilon(z,t) + \frac{|x-z|^4}{4\varepsilon^3\rho}\right), \end{cases} \tag{5.29}$$

where $\rho \in (0,1)$ is to be determined later, and

$$\begin{cases} \overline{u}^\varepsilon(x,t) := u^\varepsilon(x,t) + \left((-\lambda w_\lambda^{2\eta}(0)) - \lambda_t\right)(t - t_0) - \psi(x,t), \\ \widehat{u}^\varepsilon(x,t) := \sup_{z\in\overline{B}_{\varepsilon\eta}(x)} \overline{u}^\varepsilon(z,t). \end{cases} \tag{5.30}$$

For $\varepsilon = \varepsilon(\delta_1) = \varepsilon(r) \in (0,1)$ small enough,

$$u^\varepsilon + \delta_1 \le \varphi \quad \text{on} \quad \overline{Q}_r(P_0) \setminus Q_{r/2}(P_0),$$

which in turn implies, by (5.27), (5.29), (5.30),

$$\overline{u}^\varepsilon + \delta_1 \le \widetilde{\ell}^\varepsilon + \varepsilon\overline{\kappa}_0 \quad \text{on} \quad \overline{Q}_r(P_0) \setminus Q_{r/2}(P_0).$$

By the fact that $\limsup^*_\varepsilon \widehat{u}^\varepsilon = \limsup^*_\varepsilon \overline{u}^\varepsilon$ and by [16, Lemma 13.2.(A).(vii)], we see that there exists $\varepsilon = \varepsilon(P_0, r, \delta_1, \overline{\kappa}_0, |p|) = \varepsilon(P_0, r, p) \in (0,1)$ small enough such that

$$\widehat{u}^\varepsilon + \frac{1}{2}\delta_1 \le \widehat{\ell}^\varepsilon \quad \text{on} \quad \overline{Q}_r(P_0) \setminus Q_{r/2}(P_0),$$

and that the supremum of $\widehat{u}^\varepsilon - \widehat{\ell}^\varepsilon$ on $\overline{Q}_r(P_0)$ is attained in $Q_{r/2}(P_0)$, say at $\widehat{P}_\varepsilon = (\widehat{x}_\varepsilon, t_\varepsilon) \in Q_{r/2}(P_0)$. Also, by [16, Lemma 13.2.(A).(vii)], there exist $\rho = \rho(\eta, \overline{\kappa}_0, |p|) = \rho(\theta, p) \in (0,1)$, $\widetilde{x}_\varepsilon \in \mathbb{R}^n$ such that

$$\widehat{\ell}^\varepsilon(\widehat{x}_\varepsilon, t_\varepsilon) = \widetilde{\ell}^\varepsilon(\widetilde{x}_\varepsilon, t_\varepsilon) + \frac{|\widehat{x}_\varepsilon - \widetilde{x}_\varepsilon|^4}{4\varepsilon^3\rho} \quad \text{and} \quad |\widehat{x}_\varepsilon - \widetilde{x}_\varepsilon| \le \varepsilon\eta. \tag{5.31}$$

Step 2.4: The viscosity inequalities from the Crandall-Ishii's Lemma.

Unraveling the infimum in (5.29), the supremum in (5.30) and by the choice of $\widetilde{x}_\varepsilon \in \mathbb{R}^n$, we see that

$$(x, y, t) \in \overline{B}_r(x_0) \times \mathbb{R}^n \times [t_0 - r, t_0 + r] \longmapsto \widehat{u}^\varepsilon(x, t) - \widetilde{\ell}^\varepsilon(y, t) - \frac{|x-y|^4}{4\varepsilon^3\rho}$$

attains a maximum at $(\widehat{x}_\varepsilon, \widetilde{x}_\varepsilon, t_\varepsilon) \in B_{r/2}(x_0) \times \mathbb{R}^n \times \left(t_0 - \frac{1}{2}r, t_0 + \frac{1}{2}r\right)$. Since [28, (8.5)] holds for our F (by the condition (i) of (V)) and for $\widehat{u}^\varepsilon, \widetilde{\ell}^\varepsilon$ (with the aid of [16, Lemma 13.1.(ii)] for $\widehat{u}^\varepsilon$), we can apply the Crandall-Ishii's Lemma [28, Theorem 8.3] to see that for every $\gamma > 0$, there exist $X, Y \in S^n$ such that

$$\begin{cases} (b_1, q, X) \in \overline{\mathcal{P}}^{2,+}\widehat{u}^\varepsilon(\widehat{x}_\varepsilon, t_\varepsilon), \\ (b_2, q, Y) \in \overline{\mathcal{P}}^{2,-}\widetilde{\ell}^\varepsilon(\widetilde{x}_\varepsilon, t_\varepsilon), \\ b_1 - b_2 = 0 = \Phi_t(\overline{x}_\varepsilon, x_\varepsilon, t_\varepsilon), \\ -\left(\frac{1}{\gamma} + \|A\|\right) I_{2n} \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq A + \gamma A^2, \end{cases} \tag{5.32}$$

where $\Phi(x, y, t) := \frac{|x-y|^4}{4\varepsilon^3\rho}$ and

$$q := D_x\Phi(\widehat{x}_\varepsilon, \widetilde{x}_\varepsilon, t_\varepsilon) = -D_y\Phi(\widehat{x}_\varepsilon, \widetilde{x}_\varepsilon, t_\varepsilon) = \delta(\widehat{x}_\varepsilon - \widetilde{x}_\varepsilon) \quad \text{with} \quad \delta := \frac{|\widehat{x}_\varepsilon - \widetilde{x}_\varepsilon|^2}{\varepsilon^3\rho},$$

$$A := D^2_{(x,y)}\Phi(\widehat{x}_\varepsilon, \widetilde{x}_\varepsilon, t_\varepsilon) = \delta \begin{pmatrix} I_n + 2\widehat{q} \otimes \widehat{q} & -I_n - 2\widehat{q} \otimes \widehat{q} \\ -I_n - 2\widehat{q} \otimes \widehat{q} & I_n + 2\widehat{q} \otimes \widehat{q} \end{pmatrix} \text{ with } \widehat{q} := \frac{q}{|q|} \text{ (if } q \neq 0).$$

By [16, Lemma 13.1.(ii)], there exists $\overline{x}_\varepsilon \in \mathbb{R}^n$ such that

$$(b_1, q, X) \in \overline{\mathcal{P}}^{2,+}\overline{u}^\varepsilon(\overline{x}_\varepsilon, t_\varepsilon), \quad \widehat{u}^\varepsilon(\widehat{x}_\varepsilon, t_\varepsilon) = \overline{u}^\varepsilon(\overline{x}_\varepsilon, t_\varepsilon), \quad |\widehat{x}_\varepsilon - \overline{x}_\varepsilon| \leq \varepsilon\eta, \tag{5.33}$$

and consequently, by (5.30), that

$$\Big(b_1 - (-\lambda w_\lambda^{2\eta}(0) - \lambda_t) + \psi_t(\overline{P}_\varepsilon), q + D\psi(\overline{P}_\varepsilon), X + D^2\psi(\overline{P}_\varepsilon)\Big) \in \overline{\mathcal{P}}^{2,+} u^\varepsilon(\overline{x}_\varepsilon, t_\varepsilon) \qquad (5.34)$$

with $\overline{P}_\varepsilon := (\overline{x}_\varepsilon, t_\varepsilon)$.

Also, we note that $v_\lambda^{2\eta}(x) := \varepsilon \left(w_\lambda^{2\eta}\left(\frac{x}{\varepsilon}\right) - w_\lambda^{2\eta}(0)\right)$ is a viscosity solution to

$$F^{2\eta}\left(\varepsilon D^2 v_\lambda^{2\eta}(x), p + Dv_\lambda^{2\eta}(x), \frac{x}{\varepsilon}\right) \geq \overline{\lambda} - \min\left\{\frac{1}{8}\theta, \frac{1}{8}\overline{\lambda}\right\} \quad \text{on} \quad \mathbb{R}^n,$$

from (5.26), (5.28). Therefore, from (5.29) and the second line of (5.32), we have

$$b_2 = -\lambda w_\lambda^{2\eta}(0), \ (q - p, Y) \in \overline{\mathcal{J}}^{2,-} v_\lambda^{2\eta}(\widetilde{x}_\varepsilon). \qquad (5.35)$$

Hence, by (5.34) and (5.35), we obtain the viscosity inequalities

$$\begin{cases} \lambda_t + \psi_t(\overline{P}_\varepsilon) + F_*\left(\varepsilon X + \varepsilon D^2\psi(\overline{P}_\varepsilon), q + D\psi(\overline{P}_\varepsilon), \frac{\overline{x}_\varepsilon}{\varepsilon}\right) \leq 0, \\ (F^{2\eta})^*\left(\varepsilon Y, q, \frac{\widetilde{x}_\varepsilon}{\varepsilon}\right) \geq \overline{\lambda} - \min\left\{\frac{1}{8}\theta, \frac{1}{8}\overline{\lambda}\right\}. \end{cases} \qquad (5.36)$$

Step 2.5: Bound of the gradient q and of the Hessians X, Y.

Similarly as before, we set $\gamma = \frac{1}{3\delta}$ in (5.32) and combine the fact that $\delta \leq \frac{\eta^2}{\varepsilon\rho}$ (from (5.33)) to obtain

$$|q| \leq \frac{\eta^3}{\rho} \quad \text{and} \quad -\frac{9\eta^2}{\rho} I_n \leq \varepsilon X \leq \varepsilon Y \leq \frac{9\eta^2}{\rho} I_n. \qquad (5.37)$$

We separate the gradient q from the origin with a constant that depends only on θ, p. Note that $(b_1, q, X) \in \overline{\mathcal{P}}^{2,+}\widehat{u}^\varepsilon(\widehat{x}_\varepsilon, t_\varepsilon)$ and that the function $\widehat{u}^\varepsilon$ is sup-convoluted by the definition (5.30). Therefore, [16, Lemma 13.1(ii)] implies, together with (5.36), (5.37), that

$$\frac{1}{2}\overline{\lambda} \le (F^{2\eta})^* \left(\varepsilon Y, q, \frac{\widetilde{x}_\varepsilon}{\varepsilon}\right) \le (F^{2\eta})^* \left(\varepsilon X, q, \frac{\widetilde{x}_\varepsilon}{\varepsilon}\right) \le c|q|$$

for some constant $c = c(\eta) = c(\theta, p) > 0$, which then yields

$$\frac{\overline{\lambda}}{2c} \le |q|. \tag{5.38}$$

Step 2.6: Deriving a contradiction.

Note that the operator F is uniformly continuous on

$$\left\{(X', p', y') \in S^n \times \mathbb{R}^n \times \mathbb{R}^n : \|X'\| \le 1 + \frac{9\eta^2}{\rho},\ \frac{\overline{\lambda}}{4c} \le |p'| \le 1 + \frac{\eta^3}{\rho}\right\}.$$

Combining the above uniform continuity with (5.31), (5.33), we can choose $r = r(\theta, p) \in (0,1)$, $\varepsilon = \varepsilon(P_0, r, \theta, p) \in (0,1)$ satisfying

$$\varepsilon\|D^2\psi(\overline{P}_\varepsilon)\|, |D\psi(\overline{P}_\varepsilon)|, |\psi_t(\overline{P}_\varepsilon)| \le \mu(r) \le \min\left\{1, \frac{\theta}{4}, \frac{\overline{\lambda}}{4c}\right\}, \tag{5.39}$$

so that, by (5.37), (5.38),

$$\|\varepsilon X\|, \|\varepsilon X + \varepsilon D^2\psi(\overline{P}_\varepsilon)\| \le 1 + \frac{9\eta^2}{\rho} \quad \text{and} \quad \frac{\overline{\lambda}}{4c} \le |q|, |q + D\psi(\overline{P}_\varepsilon)| \le 1 + \frac{\eta^3}{\rho},$$

and so that

$$F\left(\varepsilon X, q, \frac{\overline{x}_\varepsilon}{\varepsilon}\right) \le \frac{1}{4}\theta + F\left(\varepsilon X + \varepsilon D^2\psi(\overline{P}_\varepsilon), q + D\psi(\overline{P}_\varepsilon), \frac{\overline{x}_\varepsilon}{\varepsilon}\right). \tag{5.40}$$

We derive a contradiction by linking the inequalities (5.25), (5.36), (5.37), (5.39), (5.40):

$$\begin{aligned}
\overline{\lambda} - \frac{1}{8}\theta &\le \left(F^{2\eta}\right)^* \left(\varepsilon X, q, \frac{\widetilde{x}_\varepsilon}{\varepsilon}\right) \\
&= F^{2\eta}\left(\varepsilon X, q, \frac{\widetilde{x}_\varepsilon}{\varepsilon}\right) \\
&\le F\left(\varepsilon X, q, \frac{\overline{x}_\varepsilon}{\varepsilon}\right) \\
&\le \frac{1}{4}\theta + F\left(\varepsilon X + \varepsilon D^2\psi(\widetilde{P}_\varepsilon), q + D\psi(\widetilde{P}_\varepsilon), \frac{\overline{x}_\varepsilon}{\varepsilon}\right) \\
&\le \frac{1}{4}\theta - \lambda_t - \psi_t(\overline{P}_\varepsilon) \\
&\le \frac{1}{4}\theta + \overline{\lambda} - \theta - \psi_t(\overline{P}_\varepsilon) \\
&\le \overline{\lambda} - \frac{1}{2}\theta,
\end{aligned}$$

which completes the proof of the Claim 2 in the case $\overline{\lambda} = \overline{F}(p) > 0$.

□

We finish this section by proving the following proposition, which implies the Corollary 5.1.1 together with the Theorem 5.1.1. The proof is a simple argument using the Perron's method.

Proposition 5.2.1. *Let $d, A > 0$ and $V(x) = A(-\cos x_2 \sin x_1, \cos x_1 \sin x_2)$ for $x = (x_1, x_2) \in \mathbb{R}^2$. Let*

$$F(X, p, y) = (|p| - d\,tr\{(I_2 - \hat{p}\otimes\hat{p})\,X\})_+ + V(y)\cdot p$$

for $(X, p, y) \in S^2 \times (\mathbb{R}^2 \setminus \{0\}) \times \mathbb{R}^2$. For each $p \in \mathbb{R}^2$, let $\overline{H}(p)$ be the unique real number as in the statement of the Corollary 5.1.1. Then, (5.4) *with the right-hand side replaced by $\overline{H}(p)$ admits a $\mathbb{Z}^2$-periodic viscosity solution v.*

Proof. We skip tracking the dependency on d and V as they are fixed. From [41, Corollary 3.3], we see that there exist a viscosity subsolution $\overline{v}$ and a viscosity supersolution $\underline{v}$ to

(5.4) with the right-hand side replaced by $\overline{H}(p)$, and moreover,

$$\sup_{\mathbb{R}^2}\{|\overline{v}|, |\underline{v}|\} \le C_0$$

for some constant $C_0 > 0$ depending only on p, and the both are $\mathbb{Z}^2$-periodic. Then, $\overline{v} - C_0$ and $\underline{v} + C_0$ are also a viscosity sub and supersolution, respectively, and they satisfy

$$-2C_0 \le \overline{v} - C_0 \le \underline{v} + C_0 \le 2C_0.$$

By the Perron's method, namely by taking the supremum of $\mathbb{Z}^2$-periodic subsolutions between $\overline{v} - C_0$ and $\underline{v} + C_0$, we see that there exists a $\mathbb{Z}^2$-periodic solution v to (5.4) with the right-hand side replaced by $\overline{H}(p)$ satisfying

$$-2C_0 \le \overline{v} - C_0 \le v \le \underline{v} + C_0 \le 2C_0.$$

□

5.3 Quantitative homogenization of the forced mean curvature equation

We turn our attention to the forced mean curvature equation in this section. We are interested in a rate of periodic homogenization of the flow. We provide the rate $O\left(\varepsilon^{1/8}\right)$ by proving the Theorem 5.1.3 in this subsection.

5.3.1 Proof of Theorem 5.1.3

Throughout this subsection, we let

$$c^\eta := \sup_{z\in\mathbb{R}^n:|z|\le\eta} c(\cdot + z),$$

$$c_\eta := \inf_{z\in\mathbb{R}^n:|z|\le\eta} c(\cdot + z),$$

and

$$F(X,p,y) := -\mathrm{tr}\left\{(I_n - \widehat{p}\otimes\widehat{p})\,X\right\} - c(y)|p|,$$

$$F^{\eta}(X,p,y) := -\mathrm{tr}\left\{(I_n - \widehat{p}\otimes\widehat{p})\,X\right\} - c^{\eta}(y)|p|,$$

$$F_{\eta}(X,p,y) := -\mathrm{tr}\left\{(I_n - \widehat{p}\otimes\widehat{p})\,X\right\} - c^{\eta}(y)|p|$$

for $(X,p,y) \in S^n \times (\mathbb{R}^n \setminus \{0\}) \times \mathbb{R}^n$, $\eta \geq 0$.

We follow the framework of [20]. Before we get into the proof of the Theorem 5.1.3, we leave Lipschitz estimates of solutions u^{ε} and of perturbed approximate correctors $v^{\lambda,\eta}, v^{\lambda}_{\eta}$ in the following proposition.

Proposition 5.3.1. *Suppose that c is $\mathbb{Z}^n$-periodic, Lipschitz continuous and satisfies* (5.3).

(i) Then, c^{η}, c_{η} are $\mathbb{Z}^n$-periodic, Lipschitz continuous and satisfiy (5.3) *(with the same $\delta > 0$) as well for $\eta \geq 0$.*

(ii) Let u_0 be a function on $\mathbb{R}^n$ such that $\|Du_0\|_{C^{0,1}(\mathbb{R}^n)} < \infty$. For each $\varepsilon \in (0,1)$, there exists a unique viscosity solution u^{ε} to (5.1), *and u^{ε} enjoys Lipschitz estimates:*

$$\|u_t\|_{L^{\infty}(\mathbb{R}^n\times[0,\infty))} + \|Du\|_{L^{\infty}(\mathbb{R}^n\times[0,\infty))} \leq M \tag{5.41}$$

for some constant $M = M(n, \|c\|_{L^{\infty}(\mathbb{R}^n)}, \|Du_0\|_{C^{0,1}(\mathbb{R}^n)}, \delta) > 0$.

(iii) Let $\lambda > 0, \eta \geq 0$. For each $p \in \mathbb{R}^n$, let $v^{\lambda,\eta} = v^{\lambda,\eta}(\cdot,p)$ denote the unique viscosity solution to

$$\lambda v^{\lambda,\eta} + F^{\eta}\left(D^2 v^{\lambda,\eta}, p + Dv^{\lambda,\eta}, y\right) = 0 \quad \textit{on } \mathbb{R}^n.$$

Then, there exists a constant $C = C(n, \|c\|_{C^{0,1}(\mathbb{R}^n)}, \delta) > 0$ *that for* $x, y, p, q \in \mathbb{R}^n$,

$$\begin{cases} |v^{\lambda,\eta}(x,p) - v^{\lambda,\eta}(y,p)| \leq C|p||x-y|, \\ |v^{\lambda,\eta}(x,p) - v^{\lambda,\eta}(x,q)| \leq \frac{C}{\lambda}|p-q|, \\ \|\lambda v^{\lambda,\eta}(\cdot,p) + \overline{F}(p)\|_{L^\infty(\mathbb{R}^n)} \leq C|p|(\lambda+\eta). \end{cases} \tag{5.42}$$

Here, $\overline{F}(p)$ *denotes the unique real number such that*

$$F\left(D^2 v, p + Dv, y\right) = \overline{F}(p) \quad \text{on } \mathbb{R}^n.$$

admits a $\mathbb{Z}^n$*-periodic viscosity solution. Also, the similar holds for a solution* $v_\eta^\lambda = v_\eta^\lambda(\cdot, p)$ *to*

$$\lambda v_\eta^\lambda + F_\eta\left(D^2 v_\eta^\lambda, p + Dv_\eta^\lambda, y\right) = 0 \quad \text{on } \mathbb{R}^n.$$

We refer to [78, Lemma 3.2] for (iii). The statement (i) is an easy consequence of convolution, and (ii) can be shown by considering a vanishing viscosity parameter [98, Theorem 1.13]. During the derivation of (5.41), the time-derivative is bounded by Hessians, for which we leave [31, Appendix A.1], [64, Section 2] as references.

Now, we prove the Theorem 5.1.3.

Proof of Theorem 5.1.3. Throughout the proof, C denotes positive constants varying line by line, and they depend only on $n, \|c\|_{C^{0,1}(\mathbb{R}^n)}, \|Du_0\|_{C^{0,1}(\mathbb{R}^n)}, \delta$, which we call the data from now on.

We first show that

$$u^\varepsilon(x,t) - u(x,t) \leq C(1+T)\varepsilon^{1/8} \tag{5.43}$$

for $(x,t) \in \mathbb{R}^n \times [0,T]$.

Step 0: The framework of doubling variable method with approximate

correctors for quantification [20].

For a given $\varepsilon \in (0,1)$, we let $\eta = \varepsilon^\beta$ and let

$$\Phi(x,y,z,t,s) := u^\varepsilon(x,t) - u(y,s) - \varepsilon v^{\lambda,\eta}\left(\frac{x}{\varepsilon}, \frac{z-y}{\varepsilon^\beta}\right) - \frac{|x-y|^2 + |t-s|^2}{2\varepsilon^\beta} - \frac{|x-z|^2}{2\varepsilon^\beta} - K(t+s) - \gamma_1\langle y\rangle,$$

where $\lambda = \varepsilon^\theta$, $K = K_1\varepsilon^\beta$, $\gamma_1 \in (0,\varepsilon^\beta]$, and $\theta,\beta \in (0,1], K_1 > 0$ are constants to be determined. Then, the global maximum of Φ on $\mathbb{R}^{3n} \times [0,T]^2$ is attained at a certain point, say at $(\hat{x},\hat{y},\hat{z},\hat{t},\hat{s}) \in \mathbb{R}^{3n} \times [0,T]^2$ (abusing the notation $\hat{p} = \frac{p}{|p|}$ for $p \in \mathbb{R}^n$).

From $\Phi(\hat{x},\hat{y},\hat{z},\hat{t},\hat{s}) \geq \Phi(\hat{x},\hat{y},\hat{x},\hat{t},\hat{s})$ with (5.42), we have

$$\frac{|\hat{x}-\hat{z}|^2}{2\varepsilon^\beta} \leq \varepsilon\left(v^{\lambda,\eta}\left(\frac{\hat{x}}{\varepsilon}, \frac{\hat{x}-\hat{y}}{\varepsilon^\beta}\right) - v^{\lambda,\eta}\left(\frac{\hat{x}}{\varepsilon}, \frac{\hat{z}-\hat{y}}{\varepsilon^\beta}\right)\right) \leq C\varepsilon^{1-\theta-\beta}|\hat{x}-\hat{z}|,$$

which gives $|\hat{x}-\hat{z}| \leq C\varepsilon^{1-\theta}$. Also, from $\Phi(\hat{x},\hat{y},\hat{z},\hat{t},\hat{s}) \geq \Phi(\hat{x},\hat{x},\hat{x},\hat{t},\hat{s})$, we get

$$\begin{aligned}\frac{|\hat{x}-\hat{y}|^2 + |\hat{x}-\hat{z}|^2}{2\varepsilon^\beta} &\leq u(\hat{x},\hat{s}) - u(\hat{y},\hat{s}) + \varepsilon\left(v^{\lambda,\eta}\left(\frac{\hat{x}}{\varepsilon}, 0\right) - v^{\lambda,\eta}\left(\frac{\hat{x}}{\varepsilon}, \frac{\hat{z}-\hat{y}}{\varepsilon^\beta}\right)\right) \\ &\leq C\left(|\hat{x}-\hat{y}| + |\hat{y}-\hat{z}|\right),\end{aligned}$$

as long as $\theta+\beta \leq 1$. This yields $|\hat{x}-\hat{y}|+|\hat{y}-\hat{z}| \leq C\varepsilon^\beta$. Lastly, $\Phi(\hat{x},\hat{y},\hat{z},\hat{t},\hat{s}) \geq \Phi(\hat{x},\hat{y},\hat{z},\hat{t},\hat{t})$ and $\Phi(\hat{x},\hat{y},\hat{z},\hat{t},\hat{s}) \geq \Phi(\hat{x},\hat{y},\hat{z},\hat{s},\hat{s})$ give $|\hat{t}-\hat{s}| \leq C\varepsilon^\beta$. Thus, we have

$$\begin{cases} |\hat{x}-\hat{z}| \leq C\varepsilon^{1-\theta}, \\ |\hat{x}-\hat{y}| + |\hat{y}-\hat{z}| + |\hat{t}-\hat{s}| \leq C\varepsilon^\beta. \end{cases} \tag{5.44}$$

Claim. For $\beta = \frac{1}{8}$, $\theta \in \left[\frac{1}{8}, \frac{3}{8}\right]$, there exists $K_1 > 0$ depending only on the data such that either $\hat{t} = 0$ or $\hat{s} = 0$.

Once we establish this claim, we then obtain (5.43). Indeed, by (5.41), (5.44),

$$\Phi(x,x,x,t,t) \leq \Phi(\hat{x},\hat{y},\hat{z},\hat{t},\hat{s}) \leq C\varepsilon^{1/8}.$$

for all $(x,t) \in \mathbb{R}^n \times [0,T]$ in either case. Therefore, for all $(x,t) \in \mathbb{R}^n \times [0,T]$,

$$u^\varepsilon(x,t) - u(x,t) \le \varepsilon v^{\lambda,\eta}\left(\frac{\hat{x}}{\varepsilon}, 0\right) + K_1\varepsilon^{1/8}t + \gamma_1\langle x\rangle + C\varepsilon^{1/8}.$$

Since it holds for arbitrary $\gamma_1 \in (0, \varepsilon^{1/8}]$, we deduce (5.43).

From now on, we suppose that $\hat{t}, \hat{s} > 0$. We postpone the explicit choice of β, θ.

By the fact that $(y,s) \mapsto \Phi(\hat{x}, y, \hat{z}, \hat{t}, s)$ attains a maximum at $(\hat{y}, \hat{s})$ and by the super-solution test of u to (5.2), for some

$$-K + \frac{\hat{t} - \hat{s}}{\varepsilon^\beta} + \overline{F}\left(\frac{\hat{x} - \hat{y}}{\varepsilon^\beta} + \gamma_1 \frac{\hat{y}}{\langle \hat{y}\rangle} - q\right) \ge 0, \tag{5.45}$$

for some $q \in \overline{\mathcal{D}}^{1,-}\left(\varepsilon v^{\lambda,\eta}\left(\frac{\hat{x}}{\varepsilon}, \frac{\hat{z} - \cdot}{\varepsilon^\beta}\right)\right)$. See [20, Lemma 2.4] for the existence of q. Note that, by (5.42), we have $|q| \le C\varepsilon^{1-\theta-\beta}$.

As the direction of the sup/inf-involution [16, Lemma 13.1] follows the sign of $\hat{t} - \hat{s}$, which shall be explained, we divide the cases accordingly.

Case 1. $\hat{t} \le \hat{s}$**.**

Step 1.1: Sup-involutions of auxiliary functions and their maximizers.

Let

$$\begin{aligned}\Psi(x,\xi,z,t) := &\left(u^\varepsilon(x,t) - \frac{|x-\hat{y}|^2 + |x-z|^2 - 2x\cdot(z-\hat{y})}{2\varepsilon^\beta}\right)\\ &- \varepsilon\left(v^{\lambda,\eta}\left(\xi, \frac{z-\hat{y}}{\varepsilon^\beta}\right) + \frac{z-\hat{y}}{\varepsilon^\beta}\cdot\xi\right) - (x - \varepsilon\xi)\cdot\frac{z-\hat{y}}{\varepsilon^\beta}\\ &- \frac{|x-\varepsilon\xi|^2}{2\alpha} - \frac{|z-\hat{z}|^2}{4\varepsilon^\beta} - \frac{|t-\hat{s}|^2 + |t-\hat{t}|^2}{2\varepsilon^\beta} - Kt.\end{aligned}$$

Note that this auxiliary function is nothing but the terms of Φ involving (x,z,t), keeping $(\hat{y},\hat{s})$ fixed, if we ignore for the term $-\frac{|z-\hat{z}|^2}{4\varepsilon^\beta} - \frac{|t-\hat{t}|^2}{2\varepsilon^\beta}$. This additional term is attached to quantify the distance between maximizers $(\overline{z}_\alpha, \overline{t}_\alpha)$, which will be taken soon, and $(\hat{z}, \hat{t})$.

Let

$$\overline{\Psi}^{\mu}(x,\xi,z,t) := \sup_{w\in\overline{B}_{\varepsilon\mu}(x)}\left(u^{\varepsilon}(w,t) - \frac{|w-\hat{y}|^2+|w-z|^2-2w\cdot(z-\hat{y})}{2\varepsilon^{\beta}}\right)$$
$$- \varepsilon\left(v^{\lambda,\eta}\left(\xi,\frac{z-\hat{y}}{\varepsilon^{\beta}}\right) + \frac{z-\hat{y}}{\varepsilon^{\beta}}\cdot\xi\right) - (x-\varepsilon\xi)\cdot\frac{z-\hat{y}}{\varepsilon^{\beta}}$$
$$- \frac{|x-\varepsilon\xi|^2}{2\alpha} - \frac{|z-\hat{z}|^2}{4\varepsilon^{\beta}} - \frac{|t-\hat{s}|^2+|t-\hat{t}|^2}{2\varepsilon^{\beta}} - Kt,$$

where $\mu = \frac{1}{2}\eta = \frac{1}{2}\varepsilon^{\beta}$, and $\alpha > 0$ is to be determined later. Then, $\overline{\Psi}^{\mu}$ attains a global maximum, say at $(\overline{x}_{\alpha},\overline{\xi}_{\alpha},\overline{z}_{\alpha},\overline{t}_{\alpha}) \in \mathbb{R}^{3n}\times[0,T]$.

From $\overline{\Psi}^{\mu}(\overline{x}_{\alpha},\overline{\xi}_{\alpha},\overline{z}_{\alpha},\overline{t}_{\alpha}) \geq \overline{\Psi}^{\mu}(\overline{x}_{\alpha},\frac{\overline{x}_{\alpha}}{\varepsilon},\overline{z}_{\alpha},\overline{t}_{\alpha})$ with (5.42), (5.44), we have

$$\frac{|\overline{x}_{\alpha}-\varepsilon\overline{\xi}_{\alpha}|^2}{2\alpha} \leq \varepsilon\left(v^{\lambda,\eta}\left(\frac{\overline{x}_{\alpha}}{\varepsilon},\frac{\overline{z}_{\alpha}-\hat{y}}{\varepsilon^{\beta}}\right) - v^{\lambda,\eta}\left(\overline{\xi}_{\alpha},\frac{\overline{z}_{\alpha}-\hat{y}}{\varepsilon^{\beta}}\right)\right)$$
$$\leq C|\overline{x}_{\alpha}-\varepsilon\overline{\xi}_{\alpha}|\left|\frac{\overline{z}_{\alpha}-\hat{y}}{\varepsilon^{\beta}}\right|,$$

and in turn,

$$|\overline{x}_{\alpha}-\varepsilon\overline{\xi}_{\alpha}| \leq C\alpha\left|\frac{\overline{z}_{\alpha}-\hat{y}}{\varepsilon^{\beta}}\right|. \tag{5.46}$$

Now, we estimate $|\overline{z}_{\alpha}-\hat{z}|$ and $|\overline{t}_{\alpha}-\hat{t}|$ by using the term $-\frac{|z-\hat{z}|^2}{4\varepsilon^{\beta}} - \frac{|t-\hat{t}|^2}{2\varepsilon^{\beta}}$.

$$\overline{\Psi}^{\mu}(\overline{x}_{\alpha},\overline{\xi}_{\alpha},\overline{z}_{\alpha},\overline{t}_{\alpha})$$
$$\leq C\varepsilon\mu + u^{\varepsilon}(\overline{x}_{\alpha},\overline{t}_{\alpha}) + \sup_{w\in\overline{B}_{\varepsilon\mu}(\overline{x}_{\alpha})}\left(-\frac{|w-\hat{y}|^2+|w-\overline{z}_{\alpha}|^2-2(w-\overline{x}_{\alpha})\cdot(\overline{z}_{\alpha}-\hat{y})}{2\varepsilon^{\beta}}\right)$$
$$-\varepsilon v^{\varepsilon,\eta}\left(\overline{\xi}_{\alpha},\frac{\overline{z}_{\alpha}-\hat{y}}{\varepsilon^{\beta}}\right) - \frac{|\overline{x}_{\alpha}-\varepsilon\overline{\xi}_{\alpha}|^2}{2\alpha} - \frac{|\overline{z}_{\alpha}-\hat{z}|^2}{4\varepsilon^{\beta}} - \frac{|\overline{t}_{\alpha}-\hat{s}|^2+|\overline{t}_{\alpha}-\hat{t}|^2}{2\varepsilon^{\beta}} - K\overline{t}_{\alpha}$$
$$\leq -\frac{|\overline{z}_{\alpha}-\hat{z}|^2+|\overline{t}_{\alpha}-\hat{t}|^2}{4\varepsilon^{\beta}} + C\varepsilon\mu + \varepsilon\left(v^{\lambda,\eta}\left(\frac{\overline{x}_{\alpha}}{\varepsilon},\frac{\overline{z}_{\alpha}-\hat{y}}{\varepsilon^{\beta}}\right) - v^{\lambda,\eta}\left(\overline{\xi}_{\alpha},\frac{\overline{z}_{\alpha}-\hat{y}}{\varepsilon^{\beta}}\right)\right)$$
$$+ \sup_{w\in\overline{B}_{\varepsilon\mu}(\overline{x}_{\alpha})}\left(-\frac{2(w-\overline{x}_{\alpha})\cdot((w-\overline{x}_{\alpha})-2(\overline{z}_{\alpha}-\overline{x}_{\alpha}))}{2\varepsilon^{\beta}}\right)$$
$$+ u^{\varepsilon}(\overline{x}_{\alpha},\overline{t}_{\alpha}) - \frac{|\overline{x}_{\alpha}-\hat{y}|^2}{2\varepsilon^{\beta}} - \frac{|\overline{x}_{\alpha}-\overline{z}_{\alpha}|^2}{2\varepsilon^{\beta}} - \varepsilon v^{\varepsilon,\eta}\left(\frac{\overline{x}_{\alpha}}{\varepsilon},\frac{\overline{z}_{\alpha}-\hat{y}}{\varepsilon^{\beta}}\right) - \frac{|\overline{t}_{\alpha}-\hat{s}|^2}{2\varepsilon^{\beta}} - K\overline{t}_{\alpha}$$

Here, we used the fact that $u^\varepsilon(w,\overline{t}_\alpha) \le u^\varepsilon(\overline{x}_\alpha,\overline{t}_\alpha) + C\varepsilon\mu$ for $w \in \overline{B}_{\varepsilon\mu}(\overline{x}_\alpha)$ in the first inequality, and used the fact that

$$\begin{aligned}&(|w-\hat{y}|^2 - |\overline{x}_\alpha-\hat{y}|^2) + (|w-\overline{z}_\alpha|^2 - |\overline{x}_\alpha-\overline{z}_\alpha|^2) - 2(w-\overline{x}_\alpha)\cdot(\overline{z}_\alpha-\hat{y})\\&= 2(w-\overline{x}_\alpha)\cdot((w-\overline{x}_\alpha) - 2(\overline{z}_\alpha-\overline{x}_\alpha)),\end{aligned}$$

in the second inequality. The others are rearrangement of the terms. Now, from $\Phi(\hat{x},\hat{y},\hat{z},\hat{t},\hat{s}) \ge \Phi(\overline{x}_\alpha,\hat{y},\overline{z}_\alpha,\overline{t}_\alpha,\hat{s})$, we have

$$\begin{aligned}&\overline{\Psi}^\mu(\overline{x}_\alpha,\overline{\xi}_\alpha,\overline{z}_\alpha,\overline{t}_\alpha)\\&\le -\frac{|\overline{z}_\alpha-\hat{z}|^2 + |\overline{t}_\alpha-\hat{t}|^2}{4\varepsilon^\beta} + C\varepsilon\mu + C|\overline{x}_\alpha - \varepsilon\overline{\xi}_\alpha|\left|\frac{\overline{z}_\alpha-\hat{y}}{\varepsilon^\beta}\right| + \varepsilon^{1-\beta}\mu(\varepsilon\mu + 2\,|\overline{x}_\alpha-\overline{z}_\alpha|)\\&\qquad+\underbrace{u^\varepsilon(\hat{x},\hat{t}) - \varepsilon v^{\varepsilon,\eta}\left(\frac{\hat{x}}{\varepsilon},\frac{\hat{z}-\hat{y}}{\varepsilon^\beta}\right) - \frac{|\hat{x}-\hat{y}|^2}{2\varepsilon^\beta} - \frac{|\hat{x}-\hat{z}|^2}{2\varepsilon^\beta} - \frac{|\hat{t}-\hat{s}|^2}{2\varepsilon^\beta} - K\hat{t}}_{=\Psi\left(\hat{x},\frac{\hat{x}}{\varepsilon},\hat{z},\hat{t}\right)\le\overline{\Psi}^\mu\left(\hat{x},\frac{\hat{x}}{\varepsilon},\hat{z},\hat{t}\right)\le\overline{\Psi}^\mu(\overline{x}_\alpha,\overline{\xi}_\alpha,\overline{z}_\alpha,\overline{t}_\alpha)},\end{aligned}$$

and thus, by (5.46) and by the inequality $\left|\frac{\overline{z}_\alpha-\hat{y}}{\varepsilon^\beta}\right|^2 \le 2\left(C + \left|\frac{\overline{z}_\alpha-\hat{z}}{\varepsilon^\beta}\right|^2\right)$ from (5.44),

$$\frac{|\overline{z}_\alpha-\hat{z}|^2 + |\overline{t}_\alpha-\hat{t}|^2}{4\varepsilon^\beta} \le C\varepsilon\mu + C\alpha + C\alpha\left|\frac{\overline{z}_\alpha-\hat{z}}{\varepsilon^\beta}\right|^2 + \varepsilon^{2-\beta}\mu + 2\varepsilon^{1-\beta}\mu\,|\overline{x}_\alpha-\overline{z}_\alpha|\,. \quad (5.47)$$

Choose $\overline{x}_\alpha^1 \in \overline{B}_{\varepsilon\mu}(\overline{x}_\alpha)$ such that the supremum

$$\sup_{w\in\overline{B}_{\varepsilon\mu}(\overline{x}_\alpha)}\left(u^\varepsilon(w,\overline{t}_\alpha) - \frac{|w-\hat{y}|^2 + |w-\overline{z}_\alpha|^2 - 2(w-\overline{x}_\alpha)\cdot(\overline{z}_\alpha-\hat{y})}{2\varepsilon^\beta}\right)$$

is attained. From $\overline{\Psi}^\mu(\overline{x}_\alpha,\overline{\xi}_\alpha,\overline{z}_\alpha,\overline{t}_\alpha) \ge \overline{\Psi}^\mu(\overline{x}_\alpha,\overline{\xi}_\alpha,\overline{x}_\alpha,\overline{t}_\alpha)$, we get, by (5.42),

$$\begin{aligned}&2|\overline{x}_\alpha^1-\overline{z}_\alpha|^2 - 2|\overline{x}_\alpha^1-\overline{x}_\alpha|^2 + |\overline{z}_\alpha-\hat{z}|^2 - |\overline{x}_\alpha-\hat{z}|^2\\&\le 4(\overline{x}_\alpha^1-\overline{x}_\alpha)\cdot(\overline{z}_\alpha-\overline{x}_\alpha) + 4\varepsilon^{1+\beta}\left(v^{\lambda,\eta}\left(\overline{\xi}_\alpha,\frac{\overline{x}_\alpha-\hat{y}}{\varepsilon^\beta}\right) - v^{\lambda,\eta}\left(\overline{\xi}_\alpha,\frac{\overline{z}_\alpha-\hat{y}}{\varepsilon^\beta}\right)\right)\\&\le 4(\overline{x}_\alpha^1-\overline{x}_\alpha)\cdot(\overline{z}_\alpha-\overline{x}_\alpha) + C\varepsilon^{1-\theta}|\overline{x}_\alpha-\overline{z}_\alpha|.\end{aligned}$$

By elementary calculations using

$$\begin{cases} |\overline{x}^1_\alpha - \overline{z}_\alpha|^2 = |\overline{x}^1_\alpha - \overline{x}_\alpha|^2 + 2(\overline{x}^1_\alpha - \overline{x}_\alpha)\cdot(\overline{x}_\alpha - \overline{z}_\alpha) + |\overline{x}_\alpha - \overline{z}_\alpha|^2, \\ |\overline{x}_\alpha - \hat{z}|^2 = |\overline{x}_\alpha - \overline{z}_\alpha|^2 + 2(\overline{x}_\alpha - \overline{z}_\alpha)\cdot(\overline{z}_\alpha - \hat{z}) + |\overline{z}_\alpha - \hat{z}|^2, \end{cases}$$

we see that there exists a constant $C_0 > 0$ depending only on the data such that (5.46), (5.47) hold with C_0 in place of C,

$$|\overline{x}_\alpha - \overline{z}_\alpha| \leq C_0 \left(\varepsilon^{1-\theta} + |\overline{z}_\alpha - \hat{z}|\right). \tag{5.48}$$

Now, we consider $\alpha \in \left(0, \frac{1}{8C_0}\varepsilon^{1+\beta}\right)$ so that

$$\frac{|\overline{z}_\alpha - \hat{z}|^2}{8\varepsilon^\beta} \leq C\left(\varepsilon^{1+\beta} + \varepsilon\, |\overline{x}_\alpha - \overline{z}_\alpha|\right)$$

from (5.47) with another constant $C > 0$. Combining with (5.48), we obtain

$$\begin{cases} |\overline{z}_\alpha - \hat{z}| \leq C\varepsilon^{\frac{1}{2}+\beta}, \\ |\overline{x}_\alpha - \overline{z}_\alpha| \leq C\varepsilon^{\min\left\{\frac{1}{2}+\beta, 1-\theta\right\}}, \end{cases} \tag{5.49}$$

which in turn implies, again by (5.47),

$$|\overline{t}_\alpha - \hat{t}| \leq C\varepsilon^{\frac{1}{2}+\beta}. \tag{5.50}$$

Also, by (5.44), (5.47) and (5.49), there exists a constant $C_1 > 4C_0$ depending only on the data such that

$$|\overline{x}_\alpha - \varepsilon\overline{\xi}_\alpha| \leq C_0\alpha \left|\frac{\overline{z}_\alpha - \hat{y}}{\varepsilon^\beta}\right| \leq C\alpha\left(1 + \left|\frac{\overline{z}_\alpha - \hat{z}}{\varepsilon^\beta}\right|\right) \leq C_1\alpha.$$

Now, we take $\alpha = \frac{1}{2C_1}\varepsilon^{1+\beta} = \frac{1}{2C_1}\varepsilon\eta \in \left(0, \frac{1}{8C_0}\varepsilon^{1+\beta}\right)$ so that

$$\left|\frac{\overline{x}_\alpha}{\varepsilon} - \overline{\xi}_\alpha\right| \le \frac{1}{2}\eta. \tag{5.51}$$

Step 1.2: The viscosity inequalities from the Crandall-Ishii's Lemma.

If $\overline{t}_\alpha = 0$, we then necessarily have $\hat{t}, \hat{s} \le C\varepsilon^\beta$, which implies (5.43) as before. We assume the other case $\overline{t}_\alpha > 0$.

Let

$$\begin{cases} h(x,\xi,t) := \frac{|x-\varepsilon\xi|^2}{2\alpha} + (x-\varepsilon\xi)\cdot\frac{\overline{z}_\alpha - \hat{y}}{\varepsilon^\beta} + \frac{|t-\hat{s}|^2+|t-\hat{t}|^2}{2\varepsilon^\beta} + Kt + \frac{|\overline{z}_\alpha - \hat{z}|^2}{4\varepsilon^\beta}, \\ \overline{u}^{\varepsilon,\mu}(x,t) := \sup_{w\in\overline{B}_{\varepsilon\mu}(x)}\left(u^\varepsilon(w,t) - \frac{|w-\hat{y}|^2+|w-\overline{z}_\alpha|^2 - 2w\cdot(\overline{z}_\alpha-\hat{y})}{2\varepsilon^\beta}\right), \\ \widetilde{\ell}^\varepsilon(\xi) := v^{\lambda,\eta}\left(\xi, \frac{\overline{z}_\alpha-\hat{y}}{\varepsilon^\beta}\right) + \frac{\overline{z}_\alpha-\hat{y}}{\varepsilon^\beta}\cdot\xi \end{cases}$$

so that

$$(x,\xi,t) \longmapsto \overline{\Psi}^\mu(x,\xi,\overline{z}_\alpha,t) = \overline{u}^{\varepsilon,\mu}(x,t) - \varepsilon\widetilde{\ell}^\varepsilon(\xi) - h(x,\xi,t)$$

attains a global maximum at $(\overline{x}_\alpha, \overline{\xi}_\alpha, \overline{t}_\alpha) \in \mathbb{R}^{2n}\times(0,T]$.

Since [28, (8.5)] holds for our F and for $\overline{u}^{\varepsilon,\mu}$, $\varepsilon\widetilde{\ell}^\varepsilon$, we can apply the Crandall-Ishii's Lemma [28, Theorem 8.3] to see that for every $\gamma > 0$, there exist $X, Y \in S^n$ such that

$$\begin{cases} (b_1, p, X) \in \overline{\mathcal{P}}^{2,+}\overline{u}^{\varepsilon,\mu}(\overline{x}_\alpha, \overline{t}_\alpha), \\ (b_2, q, Y) \in \overline{\mathcal{P}}^{2,-}\varepsilon\widetilde{\ell}^\varepsilon(\overline{\xi}_\alpha), \\ b_1 = b_1 - b_2 = h_t(\overline{x}_\alpha, \overline{\xi}_\alpha, \overline{t}_\alpha) = K + \frac{\overline{t}_\alpha - \hat{s}}{\varepsilon^\beta} + \frac{\overline{t}_\alpha - \hat{t}}{\varepsilon^\beta}, \\ -\left(\frac{1}{\gamma} + \|A\|\right) I_{2n} \le \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le A + \gamma A^2, \end{cases} \tag{5.52}$$

where

$$p := D_x h(\overline{x}_\alpha, \overline{\xi}_\alpha, \overline{t}_\alpha) = \frac{\overline{x}_\alpha - \varepsilon\overline{\xi}_\alpha}{\alpha} + \frac{\overline{z}_\alpha - \hat{y}}{\varepsilon^\beta},$$
$$q := -D_\xi h(\overline{x}_\alpha, \overline{\xi}_\alpha, \overline{t}_\alpha) = \varepsilon p,$$
$$A := D^2_{(x,\xi)} h(\overline{x}_\alpha, \overline{\xi}_\alpha, \overline{t}_\alpha) = \frac{1}{\alpha}\begin{pmatrix} I_n & -\varepsilon I_n \\ -\varepsilon I_n & \varepsilon^2 I_n \end{pmatrix}.$$

As $\|A\|$ is comparable to $\frac{1}{\alpha}$, we take $\gamma = \alpha$ so that we can deduce from (5.52) that

$$\begin{cases} -\frac{C}{\varepsilon^\beta} I_n \le \varepsilon X \le \frac{C}{\varepsilon^\beta} I_n, \\ \quad \varepsilon X \le \frac{1}{\varepsilon} Y \end{cases} \tag{5.53}$$

with a constant $C > 0$ depending only on the data.

By the choice of $\overline{x}^1_\alpha \in \overline{B}_{\varepsilon\mu}(\overline{x}_\alpha)$ and by the definition of $\overline{u}^{\varepsilon,\mu}$, we can apply [16, Lemma 13.2] to obtain

$$\left(b_1, p + \frac{2(\overline{x}^1_\alpha - \overline{z}_\alpha)}{\varepsilon^\beta}, X + \frac{2}{\varepsilon^\beta} I_n\right) \in \overline{\mathcal{P}}^{2,+} u^\varepsilon(\overline{x}^1_\alpha, \overline{t}_\alpha),$$

which gives, from the subsolution test of u^ε and (5.52),

$$K + \frac{\overline{t}_\alpha - \hat{s}}{\varepsilon^\beta} + \frac{\overline{t}_\alpha - \hat{t}}{\varepsilon^\beta} + F_*\left(\varepsilon X + 2\varepsilon^{1-\beta} I_n, p + \frac{2(\overline{x}^1_\alpha - \overline{z}_\alpha)}{\varepsilon^\beta}, \frac{\overline{x}^1_\alpha}{\varepsilon}\right) \le 0. \tag{5.54}$$

Also, from the supersolution test of $\widetilde{\ell}^\varepsilon$ and (5.52), we have

$$\lambda v^{\lambda,\eta}\left(\overline{\xi}_\alpha, \frac{\overline{z}_\alpha - \hat{y}}{\varepsilon^\beta}\right) + (F^\eta)^*\left(\frac{1}{\varepsilon} Y, p, \overline{\xi}_\alpha\right) \ge 0. \tag{5.55}$$

Step 1.3: Separation of the gradient p from the origin.

By (5.42), (5.55) and by the fact that $\left|\frac{\overline{z}_\alpha - \hat{y}}{\varepsilon^\beta}\right| \leq C$, we have

$$C(\varepsilon^\theta + \varepsilon^\beta) - \overline{F}\left(\frac{\overline{z}_\alpha - \hat{y}}{\varepsilon^\beta}\right) \geq \lambda v^{\lambda,\eta}\left(\overline{\xi}_\alpha, \frac{\overline{z}_\alpha - \hat{y}}{\varepsilon^\beta}\right)$$
$$\geq (-F^\eta)_*\left(\frac{1}{\varepsilon}Y, p, \overline{\xi}_\alpha\right)$$
$$\geq (-F^\eta)_*\left(\varepsilon X, p, \overline{\xi}_\alpha\right)$$
$$\geq -\frac{C}{\mu}|p|,$$

where we used [16, Lemma 13.1] for $(b_1, p, X) \in \overline{\mathcal{P}}^{2,+}\overline{u}^{\varepsilon,\mu}(\overline{x}_\alpha, \overline{t}_\alpha)$ in the last inequality. On the other hand, by (5.42), (5.45), (5.44), (5.49), it holds that

$$-K \geq -K + \frac{\hat{t} - \hat{s}}{\varepsilon^\beta}$$
$$\geq -\overline{F}\left(\frac{\hat{x} - \hat{y}}{\varepsilon^\beta} + \gamma_1 \frac{\hat{y}}{\langle \hat{y} \rangle} - q\right)$$
$$\geq -\overline{F}\left(\frac{\overline{z}_\alpha - \hat{y}}{\varepsilon^\beta}\right) - C\varepsilon^{\min\left\{\frac{1}{2}, \beta, 1-\theta-\beta\right\}}.$$

Linking the two inequalities, we see that there exists a constant $C > 0$ depending only on the data such that

$$|p| \geq \varepsilon^\beta \left(C^{-1}K_1\varepsilon^\beta - C\varepsilon^{\min\left\{\frac{1}{2}, \theta, \beta, 1-\theta-\beta\right\}}\right).$$

We require that $K_1 > C^2$ and $\beta \leq \min\left\{\frac{1}{2}, \theta, 1-\theta-\beta\right\}$ so that

$$|p| \geq \left(C^{-1}K_1 - C\right)\varepsilon^{2\beta}. \tag{5.56}$$

Step 1.4: Deriving a contradiction for a large constant $K_1 > 0$.

Note that $\left|\frac{\overline{x}^1_\alpha}{\varepsilon} - \overline{\xi}_\alpha\right| \leq \left|\frac{\overline{x}^1_\alpha}{\varepsilon} - \frac{\overline{x}_\alpha}{\varepsilon}\right| + \left|\frac{\overline{x}_\alpha}{\varepsilon} - \overline{\xi}_\alpha\right| \leq \eta$ by (5.51) and the fact that $\overline{x}^1_\alpha \in \overline{B}_{\varepsilon\mu}(\overline{x}_\alpha)$. Therefore, by connecting the viscosity inequalities (5.54), (5.55), we obtain, by

(5.42), that

$$\begin{aligned} C(\varepsilon^\theta + \varepsilon^\beta) - \overline{F}\left(\frac{\overline{z}_\alpha - \hat{y}}{\varepsilon^\beta}\right) &\geq \lambda v^{\lambda,\eta}\left(\overline{\xi}_\alpha, \frac{\overline{z}_\alpha - \hat{y}}{\varepsilon^\beta}\right) \\ &\geq -F^\eta\left(\frac{1}{\varepsilon}Y, p, \overline{\xi}_\alpha\right) \\ &\geq -F\left(\varepsilon X, p, \frac{\overline{x}_\alpha^1}{\varepsilon}\right) \\ &\geq -F\left(\varepsilon X + 2\varepsilon^{1-\beta} I_n, p + \frac{2(\overline{x}_\alpha^1 - \overline{z}_\alpha)}{\varepsilon^\beta}, \frac{\overline{x}_\alpha^1}{\varepsilon}\right) + E_1 + E_2 \\ &\geq K + \frac{\overline{t}_\alpha^1 - \hat{s}}{\varepsilon^\beta} + \frac{\overline{t}_\alpha^1 - \hat{t}}{\varepsilon^\beta} + E_1 + E_2 \end{aligned} \tag{5.57}$$

where

$$\begin{aligned} E_1 &:= F\left(\varepsilon X, p + \frac{2(\overline{x}_\alpha^1 - \overline{z}_\alpha)}{\varepsilon^\beta}, \frac{\overline{x}_\alpha^1}{\varepsilon}\right) - F\left(\varepsilon X, p, \frac{\overline{x}_\alpha^1}{\varepsilon}\right), \\ E_2 &:= F\left(\varepsilon X + 2\varepsilon^{1-\beta} I_n, p + \frac{2(\overline{x}_\alpha^1 - \overline{z}_\alpha)}{\varepsilon^\beta}, \frac{\overline{x}_\alpha^1}{\varepsilon}\right) - F\left(\varepsilon X, p + \frac{2(\overline{x}_\alpha^1 - \overline{z}_\alpha)}{\varepsilon^\beta}, \frac{\overline{x}_\alpha^1}{\varepsilon}\right). \end{aligned}$$

Note that, from $|\overline{x}_\alpha^1 - \overline{z}_\alpha| \leq |\overline{x}_\alpha^1 - \overline{x}_\alpha| + |\overline{x}_\alpha - \overline{z}_\alpha| \leq C\varepsilon^{\min\left\{\frac{1}{2}+\beta, 1-\theta\right\}}$ by (5.49). Therefore, we have, by (5.56),

$$\left| p + \frac{2(\overline{x}_\alpha^1 - \overline{z}_\alpha)}{\varepsilon^\beta} \cdot \nu \right| \geq \left(C^{-1}K_1 - C\right)\varepsilon^{2\beta}$$

for $\nu \in [0,1]$ if we require $K_1 > C^2$, $2\beta \leq \min\left\{\frac{1}{2}, 1-\theta-\beta\right\}$ (and also $\beta \leq \theta$ from the previous requirement). This implies, with (5.53), that

$$|E_1| \leq C\varepsilon^{-\beta}\left((C^{-1}K_1 - C)\varepsilon^{2\beta}\right)^{-1}\varepsilon^{\min\left\{\frac{1}{2}, 1-\theta-\beta\right\}},$$

and therefore, we see that there exists a constant $C > 0$ depending only on the data such that if $K_1 > C$, then

$$\begin{cases} |E_1| \leq \frac{C}{K_1 - C}\varepsilon^{\min\left\{\frac{1}{2}, 1-\theta-\beta\right\} - 3\beta}, \\ |E_2| \leq 2n\varepsilon^{1-\beta}. \end{cases} \tag{5.58}$$

Therefore, by (5.45), (5.57), (5.58), we have

$$2K_1\varepsilon^\beta \le C\left(\varepsilon^{\min\{\theta,\beta\}} + \frac{1}{K_1 - C}\varepsilon^{\min\left\{\frac{1}{2},1-\theta-\beta\right\}-3\beta}\right)$$

for some constant $C > 0$ (with a larger one if necessary) depending only on the data. Now, we take $\beta = \frac{1}{8}$ and any $\theta \in \left[\frac{1}{8}, \frac{3}{8}\right]$ as an optimal choice. Then, taking $K_1 = C + 1$ yields a contradiction. Therefore, there exists a constant $K_1 > 0$ depending only on the data such that if $\hat{t} \le \hat{s}$, then $\hat{t} = 0$.

Case 2. $\hat{t} \ge \hat{s}$**.**

Step 2.1: Inf-involutions of auxiliary functions and their maximizers.

Let

$$\Psi_1(x,\xi,z,t) := u^\varepsilon(x,t) - \left(\varepsilon v^{\lambda,\eta}\left(\xi, \frac{z-\hat{y}}{\varepsilon^\beta}\right) + \frac{|\varepsilon\xi - \hat{y}|^2 + |\varepsilon\xi - z|^2}{2\varepsilon^\beta}\right) - \frac{|x-\varepsilon\xi|^2}{2\alpha} - \frac{|z-\hat{z}|^2}{4\varepsilon^\beta} - \frac{|t-\hat{s}|^2 + |t-\hat{t}|^2}{2\varepsilon^\beta} - Kt.$$

Also, we let

$$\overline{\Psi}_1^\mu(x,\xi,z,t) := u^\varepsilon(x,t) - \inf_{w\in\overline{B}_\mu(\xi)}\left(\varepsilon v^{\lambda,\eta}\left(w, \frac{z-\hat{y}}{\varepsilon^\beta}\right) + \frac{|\varepsilon w - \hat{y}|^2 + |\varepsilon w - z|^2}{2\varepsilon^\beta}\right) - \frac{|x-\varepsilon\xi|^2}{2\alpha} - \frac{|z-\hat{z}|^2}{4\varepsilon^\beta} - \frac{|t-\hat{s}|^2 + |t-\hat{t}|^2}{2\varepsilon^\beta} - Kt,$$

where $\mu = \frac{1}{2}\eta = \frac{1}{2}\varepsilon^\beta$, and $\alpha > 0$ is to be determined later. Then, $\overline{\Psi}_1^\mu$ attains a global maximum, say at $(\overline{x}_\alpha, \overline{\xi}_\alpha, \overline{z}_\alpha, \overline{t}_\alpha) \in \mathbb{R}^{3n} \times [0,T]$ (by abuse of notations).

From $\overline{\Psi}_1^\mu(\overline{x}_\alpha, \overline{\xi}_\alpha, \overline{z}_\alpha, \overline{t}_\alpha) \ge \overline{\Psi}_1^\mu(\varepsilon\overline{\xi}_\alpha, \overline{\xi}_\alpha, \overline{z}_\alpha, \overline{t}_\alpha)$, we have

$$u^\varepsilon(\overline{x}_\alpha, \overline{t}_\alpha) - \frac{|\overline{x}_\alpha - \varepsilon\overline{\xi}_\alpha|^2}{2\alpha} \ge u^\varepsilon(\varepsilon\overline{\xi}_\alpha, \overline{t}_\alpha),$$

which implies

$$\left|\overline{\xi}_\alpha - \frac{\overline{x}_\alpha}{\varepsilon}\right| \le \frac{1}{2}\eta \tag{5.59}$$

with $\alpha := \frac{1}{2(C+1)}\varepsilon\eta$. Here, $C > 0$ depends only on the Lipschitz constant of u^ε.

We estimate $|\overline{z}_\alpha - \hat{z}|$ and $|\overline{t}_\alpha - \hat{t}|$ by using the term $-\frac{|z-\hat{z}|^2}{4\varepsilon^\beta} - \frac{|t-\hat{t}|^2}{2\varepsilon^\beta}$ as in Case 1. First of all, it holds that

$$\begin{aligned}
&\overline{\Psi}_1^\mu(\overline{x}_\alpha, \overline{\xi}_\alpha, \overline{z}_\alpha, \overline{t}_\alpha) \\
&\le -\frac{|\overline{z}_\alpha - \hat{z}|^2 + |\overline{t}_\alpha - \hat{t}|^2}{4\varepsilon^\beta} + \left(u^\varepsilon(\overline{x}_\alpha, \overline{t}_\alpha) - u^\varepsilon(\varepsilon\overline{\xi}_\alpha, \overline{t}_\alpha)\right) \\
&\qquad - \inf_{w \in \overline{B}_\mu(\overline{\xi}_\alpha)} \left(\varepsilon v^{\lambda,\eta}\left(w, \frac{\overline{z}_\alpha - \hat{y}}{\varepsilon^\beta}\right) - \varepsilon v^{\lambda,\eta}\left(\overline{\xi}_\alpha, \frac{\overline{z}_\alpha - \hat{y}}{\varepsilon^\beta}\right)\right) \\
&\qquad\quad - \inf_{w \in \overline{B}_\mu(\overline{\xi}_\alpha)} \left(\frac{\left(|\varepsilon w - \hat{y}|^2 - |\varepsilon\overline{\xi}_\alpha - \hat{y}|^2\right) - \left(|\varepsilon w - \overline{z}_\alpha|^2 - |\varepsilon\overline{\xi}_\alpha - \overline{z}_\alpha|^2\right)}{2\varepsilon^\beta}\right) \\
&+ u^\varepsilon(\varepsilon\overline{\xi}_\alpha, \overline{t}_\alpha) - \varepsilon v^{\lambda,\eta}\left(\overline{\xi}_\alpha, \frac{\overline{z}_\alpha - \hat{y}}{\varepsilon^\beta}\right) - \frac{|\varepsilon\overline{\xi}_\alpha - \hat{y}|^2 + |\varepsilon\overline{\xi}_\alpha - \overline{z}_\alpha|^2}{2\varepsilon^\beta} - \frac{|\overline{t}_\alpha - \hat{s}|^2}{2\varepsilon^\beta} - K\overline{t}_\alpha.
\end{aligned}$$

We note that $\Phi(\hat{x}, \hat{y}, \hat{z}, \hat{t}, \hat{s}) \ge \Phi(\varepsilon\overline{\xi}_\alpha, \hat{y}, \overline{z}_\alpha, \overline{t}_\alpha, \hat{s})$ and

$$\begin{aligned}
&\left(|\varepsilon w - \hat{y}|^2 - |\varepsilon\overline{\xi}_\alpha - \hat{y}|^2\right) - \left(|\varepsilon w - \overline{z}_\alpha|^2 - |\varepsilon\overline{\xi}_\alpha - \overline{z}_\alpha|^2\right) \\
&= 2\varepsilon(w - \overline{\xi}_\alpha) \cdot \left(\varepsilon(w - \overline{\xi}_\alpha) + (\varepsilon\overline{\xi}_\alpha - \hat{y}) + (\varepsilon\overline{\xi}_\alpha - \overline{z}_\alpha)\right).
\end{aligned}$$

By these facts, together with (5.41), (5.42), we have

$$\begin{aligned}
&\overline{\Psi}_1^\mu(\overline{x}_\alpha, \overline{\xi}_\alpha, \overline{z}_\alpha, \overline{t}_\alpha) + \frac{|\overline{z}_\alpha - \hat{z}|^2 + |\overline{t}_\alpha - \hat{t}|^2}{4\varepsilon^\beta} \\
&\le C\varepsilon^{1+\beta} + C\varepsilon^{1+\beta}\left|\frac{\overline{z}_\alpha - \hat{y}}{\varepsilon^\beta}\right| + \varepsilon\left(\varepsilon^{1+\beta} + |\varepsilon\overline{\xi}_\alpha - \hat{y}| + |\varepsilon\overline{\xi}_\alpha - \overline{z}_\alpha|\right) \\
&+ \underbrace{u^\varepsilon(\hat{x}, \hat{t}) - \varepsilon v^{\lambda,\eta}\left(\frac{\hat{x}}{\varepsilon}, \frac{\hat{z} - \hat{y}}{\varepsilon^\beta}\right) - \frac{|\hat{x} - \hat{y}|^2 + |\hat{x} - \hat{z}|^2}{2\varepsilon^\beta} - \frac{|\hat{t} - \hat{s}|^2}{2\varepsilon^\beta} - K\hat{t}}_{=\Psi_1\left(\hat{x}, \frac{\hat{x}}{\varepsilon}, \hat{z}, \hat{t}\right) \le \overline{\Psi}_1^\mu\left(\hat{x}, \frac{\hat{x}}{\varepsilon}, \hat{z}, \hat{t}\right) \le \overline{\Psi}_1^\mu(\overline{x}_\alpha, \overline{\xi}_\alpha, \overline{z}_\alpha, \overline{t}_\alpha)},
\end{aligned}$$

which then yields, with (5.44),

$$\frac{|\overline{z}_\alpha - \hat{z}|^2 + |\overline{t}_\alpha - \hat{t}|^2}{4\varepsilon^\beta} \leq C(\varepsilon^{1+\beta} + \varepsilon|\overline{z}_\alpha - \hat{z}| + \varepsilon|\varepsilon\overline{\xi}_\alpha - \overline{z}_\alpha|). \tag{5.60}$$

Choose $\overline{\xi}^1_\alpha \in \overline{B}_\mu(\overline{\xi}_\alpha)$ such that the infimum

$$\inf_{w\in\overline{B}_\mu(\overline{\xi}_\alpha)} \left(\varepsilon v^{\lambda,\eta}\left(w, \frac{\overline{z}_\alpha - \hat{y}}{\varepsilon^\beta}\right) + \frac{|\varepsilon w - \hat{y}|^2 + |\varepsilon w - \overline{z}_\alpha|^2}{2\varepsilon^\beta}\right)$$

is attained. Then, $\overline{\Psi}^1_\mu(\overline{x}_\alpha, \overline{\xi}_\alpha, \overline{z}_\alpha, \overline{t}_\alpha) \geq \overline{\Psi}^1_\mu(\overline{x}_\alpha, \overline{\xi}_\alpha, \varepsilon\overline{\xi}_\alpha, \overline{t}_\alpha)$ yields, with (5.42),

$$2|\varepsilon\overline{\xi}^1_\alpha - \overline{z}_\alpha|^2 - 2|\varepsilon\overline{\xi}^1_\alpha - \varepsilon\overline{\xi}_\alpha|^2 + |\overline{z}_\alpha - \hat{z}|^2 - |\varepsilon\overline{\xi}_\alpha - \hat{z}|^2 \leq C\varepsilon^{1-\theta}|\varepsilon\overline{\xi}_\alpha - \overline{z}_\alpha|.$$

By elementary calculations using

$$\begin{cases} |\varepsilon\overline{\xi}^1_\alpha - \overline{z}_\alpha|^2 = |\varepsilon\overline{\xi}^1_\alpha - \varepsilon\overline{\xi}_\alpha|^2 + 2(\varepsilon\overline{\xi}^1_\alpha - \varepsilon\overline{\xi}_\alpha)\cdot(\varepsilon\overline{\xi}_\alpha - \overline{z}_\alpha) + |\varepsilon\overline{\xi}_\alpha - \overline{z}_\alpha|^2, \\ |\varepsilon\overline{\xi}_\alpha - \hat{z}|^2 = |\varepsilon\overline{\xi}_\alpha - \overline{z}_\alpha|^2 + 2(\varepsilon\overline{\xi}_\alpha - \overline{z}_\alpha)\cdot(\overline{z}_\alpha - \hat{z}) + |\overline{z}_\alpha - \hat{z}|^2, \end{cases}$$

we see that

$$|\varepsilon\overline{\xi}_\alpha - \overline{z}_\alpha| \leq C\left(\varepsilon^{1-\theta} + |\overline{z}_\alpha - \hat{z}|\right).$$

Combining this with (5.60), we obtain

$$\begin{cases} |\overline{z}_\alpha - \hat{z}| \leq C\varepsilon^{\frac{1}{2}+\beta}, \\ |\varepsilon\overline{\xi}_\alpha - \overline{z}_\alpha| \leq C\varepsilon^{\min\left\{\frac{1}{2}+\beta, 1-\theta\right\}}, \end{cases} \tag{5.61}$$

which in turn implies, again by (5.60),

$$|\overline{t}_\alpha - \hat{t}| \leq C\varepsilon^{\frac{1}{2}+\beta}. \tag{5.62}$$

Step 2.2: The viscosity inequalities from the Crandall-Ishii's Lemma.

If $\bar{t}_\alpha = 0$, we then necessarily have $\hat{t}, \hat{s} \le C\varepsilon^\beta$, which implies (5.43) as before. We assume the other case $\bar{t}_\alpha > 0$.

Let

$$\begin{cases} h(x,\xi,t) := \frac{|x-\varepsilon\xi|^2}{2\alpha} + \frac{|t-\hat{s}|^2+|t-\hat{t}|^2}{2\varepsilon^\beta} + Kt + \frac{|\overline{z}_\alpha-\hat{z}|^2}{4\varepsilon^\beta}, \\ \overline{\ell}^{\varepsilon,\mu}(\xi) := \inf_{w\in\overline{B}_\mu(\xi)} \left(\varepsilon v^{\lambda,\eta}\left(w, \frac{\overline{z}_\alpha-\hat{y}}{\varepsilon^\beta}\right) + \frac{|\varepsilon w-\hat{y}|^2+|\varepsilon w-\overline{z}_\alpha|^2}{2\varepsilon^\beta}\right), \end{cases}$$

so that

$$(x,\xi,t) \longmapsto \overline{\Psi}_1^\mu(x,\xi,\overline{z}_\alpha,t) = u^\varepsilon(x,t) - \overline{\ell}^{\varepsilon,\mu}(\xi) - h(x,\xi,t)$$

attains a global maximum at $(\overline{x}_\alpha, \overline{\xi}_\alpha, \overline{t}_\alpha) \in \mathbb{R}^{2n} \times (0,T]$.

Since [28, (8.5)] holds for our F and for u^ε, $\overline{\ell}^{\varepsilon,\mu}$, we can apply the Crandall-Ishii's Lemma [28, Theorem 8.3] to see that for every $\gamma > 0$, there exist $X, Y \in S^n$ such that

$$\begin{cases} (b_1, p, X) \in \overline{\mathcal{P}}^{2,+} u^\varepsilon(\overline{x}_\alpha, \overline{t}_\alpha), \\ (b_2, q, Y) \in \overline{\mathcal{P}}^{2,-} \overline{\ell}^{\varepsilon,\mu}(\overline{\xi}_\alpha), \\ b_1 = b_1 - b_2 = h_t(\overline{x}_\alpha, \overline{\xi}_\alpha, \overline{t}_\alpha) = K + \frac{\overline{t}_\alpha-\hat{s}}{\varepsilon^\beta} + \frac{\overline{t}_\alpha-\hat{t}}{\varepsilon^\beta}, \\ -\left(\frac{1}{\gamma} + \|A\|\right) I_{2n} \le \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le A + \gamma A^2, \end{cases} \tag{5.63}$$

where

$$\begin{aligned} p &:= D_x h(\overline{x}_\alpha, \overline{\xi}_\alpha, \overline{t}_\alpha) = \frac{\overline{x}_\alpha - \varepsilon\overline{\xi}_\alpha}{\alpha}, \\ q &:= -D_\xi h(\overline{x}_\alpha, \overline{\xi}_\alpha, \overline{t}_\alpha) = \varepsilon p, \\ A &:= D^2_{(x,\xi)} h(\overline{x}_\alpha, \overline{\xi}_\alpha, \overline{t}_\alpha) = \frac{1}{\alpha}\begin{pmatrix} I_n & -\varepsilon I_n \\ -\varepsilon I_n & \varepsilon^2 I_n \end{pmatrix}. \end{aligned}$$

Taking $\gamma = \alpha$, we have, from (5.63), that

$$\begin{cases} -\frac{C}{\varepsilon^\beta} I_n \le \varepsilon X \le \frac{C}{\varepsilon^\beta} I_n, \\ \quad \varepsilon X \le \frac{1}{\varepsilon} Y \end{cases} \tag{5.64}$$

as before.

From the viscosity subsolution test to u^ε at $(\overline{x}_\alpha, \overline{t}_\alpha)$,

$$K + \frac{\overline{t}_\alpha - \hat{t}}{\varepsilon^\beta} + \frac{\overline{t}_\alpha - \hat{s}}{\varepsilon^\beta} + F_* \left(\varepsilon X, p, \frac{\overline{x}_\alpha}{\varepsilon} \right) \le 0. \tag{5.65}$$

Also, by the choice of $\overline{\xi}^1_\alpha \in \overline{B}_\mu(\overline{\xi}_\alpha)$ and by the definition of $\overline{\ell}^{\varepsilon,\mu}$, we can apply [16, Lemma 13.2] to obtain

$$\left(b_2, q - \varepsilon^{1-\beta} \left((\varepsilon \overline{\xi}^1_\alpha - \hat{y}) + (\varepsilon \overline{\xi}^1_\alpha - \overline{z}_\alpha) \right), Y - 2\varepsilon^{2-\beta} I_n \right) \in \overline{\mathcal{P}}^{2,-} \varepsilon v^{\lambda,\eta} \left(\overline{\xi}^1_\alpha, \frac{\overline{z}_\alpha - \hat{y}}{\varepsilon^\beta} \right),$$

which gives, from the supersolution test to $v^{\lambda,\eta}$,

$$\lambda v^{\lambda,\eta} \left(\overline{\xi}^1_\alpha, \frac{\overline{z}_\alpha - \hat{y}}{\varepsilon^\beta} \right) + (F^\eta)^* \left(\frac{1}{\varepsilon} Y - 2\varepsilon^{1-\beta} I_n, p - \frac{2(\varepsilon \overline{\xi}^1_\alpha - \overline{z}_\alpha)}{\varepsilon^\beta}, \frac{\overline{x}^1_\alpha}{\varepsilon} \right) \ge 0. \tag{5.66}$$

Step 2.3: Separation of the gradient p from the origin.

From $\hat{t} \ge \hat{s}$ and (5.65), we have

$$\begin{aligned} K + \frac{2(\overline{t}_\alpha - \hat{t})}{\varepsilon^\beta} \le (-F)^* \left(\varepsilon X, p, \frac{\overline{x}_\alpha}{\varepsilon} \right) &\le (-F)^* \left(\frac{1}{\varepsilon} Y, p, \frac{\overline{x}_\alpha}{\varepsilon} \right) \\ &\le \frac{1}{\varepsilon} (-F)^* \left(Y, q, \frac{\overline{x}_\alpha}{\varepsilon} \right) \le \frac{C|q|}{\varepsilon \mu} \le \frac{C|p|}{\varepsilon^\beta} \end{aligned}$$

where we used [16, Lemma 13.1] for $(b_2, q, Y) \in \overline{\mathcal{P}}^{2,-} \overline{\ell}^{\varepsilon,\mu}(\overline{\xi}_\alpha)$ in the second-last inequality.

By (5.60), we see that there exists a constant $C > 0$ depending only on the data such that

$$|p| \geq \left(C^{-1}K_1 - C\right)\varepsilon^{2\beta}. \tag{5.67}$$

whenever $K_1 > C^2$. Here, we require $\beta \leq \frac{1}{2}$.

Step 2.4: Deriving a contradiction for a large constant $K_1 > 0$.

Note that $\left|\frac{\overline{x}_\alpha}{\varepsilon} - \overline{\xi}^1_\alpha\right| \leq \left|\frac{\overline{x}_\alpha}{\varepsilon} - \overline{\xi}_\alpha\right| + \left|\overline{\xi}_\alpha - \overline{\xi}^1_\alpha\right| \leq \eta$ by (5.59) and the fact that $\overline{\xi}^1_\alpha \in \overline{B}_\mu(\overline{\xi}_\alpha)$. Therefore, by connecting the viscosity inequalities (5.65), (5.66), we obtain, by (5.42), that

$$\begin{aligned}
C(\varepsilon^\theta + \varepsilon^\beta) - \overline{F}\left(\frac{\overline{z}_\alpha - \hat{y}}{\varepsilon^\beta}\right) &\geq \lambda v^{\lambda,\eta}\left(\overline{\xi}^1_\alpha, \frac{\overline{z}_\alpha - \hat{y}}{\varepsilon^\beta}\right) \\
&\geq -F^\eta\left(\frac{1}{\varepsilon}Y - 2\varepsilon^{1-\beta}I_n, p - \frac{2(\varepsilon\overline{\xi}^1_\alpha - \overline{z}_\alpha)}{\varepsilon^\beta}, \overline{\xi}^1_\alpha\right) \\
&\geq -F\left(\varepsilon X - 2\varepsilon^{1-\beta}I_n, p - \frac{2(\varepsilon\overline{\xi}^1_\alpha - \overline{z}_\alpha)}{\varepsilon^\beta}, \frac{\overline{x}_\alpha}{\varepsilon}\right) \\
&\geq -F\left(\varepsilon X, p, \frac{\overline{x}_\alpha}{\varepsilon}\right) + E_1 + E_2 \\
&\geq K + \frac{\overline{t}^1_\alpha - \hat{s}}{\varepsilon^\beta} + \frac{\overline{t}^1_\alpha - \hat{t}}{\varepsilon^\beta} + E_1 + E_2
\end{aligned} \tag{5.68}$$

where

$$\begin{aligned}
E_1 &:= F\left(\varepsilon X - 2\varepsilon^{1-\beta}I_n, p, \frac{\overline{x}_\alpha}{\varepsilon}\right) - F\left(\varepsilon X - 2\varepsilon^{1-\beta}I_n, p - \frac{2(\varepsilon\overline{\xi}^1_\alpha - \overline{z}_\alpha)}{\varepsilon}, \frac{\overline{x}_\alpha}{\varepsilon}\right), \\
E_2 &:= F\left(\varepsilon X - 2\varepsilon^{1-\beta}I_n, p, \frac{\overline{x}_\alpha}{\varepsilon}\right) - F\left(\varepsilon X, p, \frac{\overline{x}_\alpha}{\varepsilon}\right).
\end{aligned}$$

Note that, from $|\varepsilon\overline{\xi}^1_\alpha - \overline{z}_\alpha| \leq |\varepsilon\overline{\xi}^1_\alpha - \varepsilon\overline{\xi}_\alpha| + |\varepsilon\overline{\xi}_\alpha - \overline{z}_\alpha| \leq C\varepsilon^{\min\left\{\frac{1}{2}+\beta, 1-\theta\right\}}$ by (5.61). Therefore, we have, by (5.67),

$$\left|p + \frac{2(\varepsilon\overline{\xi}^1_\alpha - \overline{z}_\alpha)}{\varepsilon^\beta}\cdot\nu\right| \geq \left(C^{-1}K_1 - C\right)\varepsilon^{2\beta}$$

for $\nu \in [0,1]$ if we require $K_1 > C^2$, $2\beta \leq \min\left\{\frac{1}{2}, 1 - \theta - \beta\right\}$ with a larger constant $C > 0$.

This implies, with (5.53), that

$$|E_1| \le C\varepsilon^{-\beta}\left((C^{-1}K_1 - C)\varepsilon^{2\beta}\right)^{-1}\varepsilon^{\min\left\{\frac{1}{2},1-\theta-\beta\right\}},$$

and therefore, we see that there exists a constant $C > 0$ depending only on the data such that if $K_1 > C$, then

$$\begin{cases} |E_1| \le \frac{C}{K_1 - C}\varepsilon^{\min\left\{\frac{1}{2},1-\theta-\beta\right\}-3\beta}, \\ |E_2| \le 2n\varepsilon^{1-\beta}. \end{cases} \tag{5.69}$$

Therefore, by (5.45), (5.57), (5.58), we have

$$2K_1\varepsilon^{\beta} \le C\left(\varepsilon^{\min\{\theta,\beta\}} + \frac{1}{K_1 - C}\varepsilon^{\min\left\{\frac{1}{2},1-\theta-\beta\right\}-3\beta}\right)$$

for some constant $C > 0$ (with a larger one if necessary) depending only on the data. Now, we take $\beta = \frac{1}{8}$ and any $\theta \in \left[\frac{1}{8}, \frac{3}{8}\right]$ as an optimal choice. Then, taking $K_1 = C + 1$ yields a contradiction. Therefore, there exists a constant $K_1 > 0$ depending only on the data such that if $\hat{t} \ge \hat{s}$, then $\hat{s} = 0$.

To prove the lower bound

$$u^{\varepsilon}(x,t) - u(x,t) \ge -C(1+T)\varepsilon^{1/8}$$

for all $(x,t) \in \mathbb{R}^n \times [0,T]$, we alternatively consider another auxiliary function, for a given $\varepsilon \in (0,1)$,

$$\begin{aligned}\Phi_1(x,y,z,t,s) := u^{\varepsilon}(x,t) - u(y,s) - \varepsilon v_{\eta}^{\lambda}\left(\frac{x}{\varepsilon}, \frac{z-y}{\varepsilon^{\beta}}\right) \\ + \frac{|x-y|^2 + |t-s|^2}{2\varepsilon^{\beta}} + \frac{|x-z|^2}{2\varepsilon^{\beta}} + K(t+s) + \gamma_1\langle y\rangle,\end{aligned}$$

where $\lambda = \varepsilon^{\theta},\ \eta = \varepsilon^{\beta},\ K = K_1\varepsilon^{\beta},\ \gamma_1 \in (0,\varepsilon^{\beta}]$, and $\theta, \beta \in (0,1], K_1 > 0$ are constants to be determined. Then, the global minimum of Φ_1 on $\mathbb{R}^{3n} \times [0,T]^2$ is attained at a certain point $(\hat{x},\hat{y},\hat{z},\hat{t},\hat{s}) \in \mathbb{R}^{3n} \times [0,T]^2$, and we proceed estimates similarly as before. □

5.3.2 An example

We prove the Propositions 5.1.1, 5.1.2 in this subsection.

Proof of Proposition 5.1.1. As the forcing term and the initial data are radially symmetric, we have $u^\varepsilon(x,t) = \varphi^\varepsilon(r,t)$ for $r = |x| \geq 0, t \geq 0$, where φ^ε solves

$$\begin{cases} \varphi^\varepsilon_t - \frac{\varepsilon}{r}\varphi^\varepsilon_r - |\varphi^\varepsilon_r| = 0, & \text{in} \quad (0,\infty)\times(0,\infty), \\ \varphi^\varepsilon(r,0) = -r, & \text{on} \quad [0,\infty). \end{cases}$$

By the optimal control formula for solutions to first-order convex/concave Hamilton-Jacobi equations, we have

$$\begin{aligned} \varphi^\varepsilon(r,t) &= \sup\left\{ -|\eta(t)| : \eta(0) = r, \ \left|\dot\eta(s) - \frac{\varepsilon}{\eta(s)}\right| \leq 1, \ s \in [0,t] \right\} \\ &= \sup\left\{ -\left|\varepsilon\xi\left(\frac{t}{\varepsilon}\right)\right| : \xi(0) = \frac{r}{\varepsilon}, \ \left|\dot\xi(s_1) - \frac{1}{\xi(s_1)}\right| \leq 1, \ s_1 \in \left[0, \frac{t}{\varepsilon}\right] \right\} \end{aligned}$$

for $r, t \geq 0$. Here, we made the changes of variables $\eta(s) = \varepsilon\xi\left(\frac{s}{\varepsilon}\right)$. and $s_1 = \frac{s}{\varepsilon}$ for $s \in [0,t]$. We moreover have, for $r > t$,

$$\varphi^\varepsilon(r,t) = -\varepsilon\xi_1\left(\frac{t}{\varepsilon}\right),$$

where $\xi_1 : \left[0, \frac{t}{\varepsilon}\right] \to (0,\infty)$ is the solution to

$$\begin{cases} \dot\xi_1(s_1) = -1 + \frac{1}{\xi_1(s_1)}, & \text{in} \quad (0,\infty)\times(0,\infty), \\ \xi_1(0) = \frac{r}{\varepsilon}, & \text{on} \quad [0,\infty). \end{cases}$$

Then, ξ_1 can be expressed as

$$\xi_1(s_1) = W\left(\left(\frac{r}{\varepsilon} - 1\right)\exp\left(\frac{r}{\varepsilon} - s_1 - 1\right)\right) + 1,$$

where $W = W(z) : [0,\infty) \to [0,\infty)$ is the Lambert W function defined by

$$w = W(z) \quad \Longleftrightarrow \quad z = we^w$$

for $z \geq 0$. We can check easily that $W'(z) = \frac{W(z)}{z(1+W(z))}$ and

$$W(z) \geq \frac{1}{2}\log z \quad \text{for } z > 0.$$

Therefore, we immediately obtain, for $r = t > \varepsilon(1+e^{-1})$, that

$$-\varepsilon\xi_1\left(\frac{t}{\varepsilon}\right) \leq -\varepsilon\left(\frac{1}{2}\log\left(e^{-1}\left(\frac{t}{\varepsilon}-1\right)\right)+1\right) = -\frac{1}{2}\varepsilon\left(\log\left(\frac{t}{\varepsilon}-1\right)+1\right) < 0.$$

As $u(x,t) = 0$ whenever $|x| = t$, we complete the proof. □

Now, we prove the Proposition 5.1.2.

Proof of Proposition 5.1.2. By [87, Theorem 1.2], there exists a smooth convex function ϕ in the variable $x' = (x_1, \cdots, x_{n-1}) \in \mathbb{R}^{n-1}$ such that $\phi(x') + (\csc\alpha)t$ is a traveling wave solution to (5.1) with $\varepsilon = 1$ and with the initial datum ϕ, and ϕ satisfies

$$u_0 - C \leq \phi \leq u_0 + C \tag{5.70}$$

with $C = 2|A|\csc\alpha$, which we freeze in this proof.

Let u^1 be the solution to the unit scale problem (5.1) with $\varepsilon = 1$ and with the initial datum u_0. Applying the comparison principle to (5.70), we obtain

$$u^1 - C \leq \phi + (\csc\alpha)t \leq u^1 + C. \tag{5.71}$$

Note that $u(x,t) = u_0(x) + (\csc\alpha)t$. Together with this fact, we apply (5.70) once more to (5.71) to obtain

$$|u^1(x,t) - u(x,t)| \le 2C \quad \text{for all } (x,t) \in \mathbb{R}^n \times [0,\infty). \tag{5.72}$$

As u_0 is positively 1-homogeneous, we have $u^\varepsilon(x,t) = \varepsilon u^1\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}\right)$ and $\varepsilon u\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}\right) = u(x,t)$, which we apply to (5.72) to derive the conclusion. □

Chapter 6

Rate of convergence in periodic homogenization for convex Hamilton–Jacobi equations with multiscales

6.1 Introduction

We consider the periodic homogenization problem for convex Hamilton–Jacobi equations in the multiscale setting. For $\epsilon > 0$, let u^ϵ be the unique viscosity solution to

$$\begin{cases} u_t^\epsilon + H\left(x, \dfrac{x}{\epsilon}, Du^\epsilon\right) = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ u^\epsilon(x,0) = g(x) & \text{on } \mathbb{R}^n, \end{cases} \tag{6.1}$$

where g is a given function as the initial data and the Hamiltonian $H = H(x,y,p) : \mathbb{R}^n \times \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{R}$ is continuous and convex in p. Here, $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ is the n-dimensional flat torus. It is well known that under appropriate assumptions, u^ϵ converges uniformly to the unique viscosity solution u to

$$\begin{cases} u_t + \overline{H}\left(x, Du\right) = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ u(x,0) = g(x) & \text{on } \mathbb{R}^n, \end{cases} \tag{6.2}$$

on $\mathbb{R}^n \times [0,T]$ for any $T > 0$ as $\epsilon \to 0$, where $\overline{H}$ is the effective Hamiltonian of H (see [77]). However, the optimal rate of convergence of u^ϵ to u in this multiscale setting has not been studied thoroughly in the literature. In this chapter, we prove that the rate of convergence is $O(t\sqrt{\epsilon})$ for $t \geq \sqrt{\epsilon}$ and $O\left(\min\{t, \epsilon\}\right)$ for $t \in (0, \sqrt{\epsilon})$. Furthermore, examples are provided to demonstrate the optimality of this convergence rate for $0 < t < \sqrt{\epsilon}$ and $t \sim \sqrt{\epsilon}$.

6.1.1 Relevant Literature

Periodic homogenization for coercive Hamilton–Jacobi equations was first proved in [77]. Subsequently, numerous works in the literature have focused on determining the rate of convergence of the homogenization problem for Hamilton–Jacobi equations. For general nonconvex Hamiltonians with multiscales, the best known rate of convergence is $O(\epsilon^{1/3})$, which was obtained in [20] by the doubling variable method and the perturbed test function method (see [34, 33]). For convex Hamiltonians $H = H(y,p)$ that depend only on the oscillatory variable and the momentum, the optimal rate of convergence was first studied in [85] using weak KAM theory and Aubry-Mather theory. In particular, it was proved that the lower bound of $u^\epsilon - u \geq -C\epsilon$ is optimal and the upper bound holds with additional assumptions on H, u, g. Recently, the optimal rate of $O(\epsilon)$ was proved in [97] using a curve cutting lemma from metric geometry (see [13]), which concludes the study in the setting of convex Hamiltonians $H = H(y,p)$ that depend only on the oscillatory variable and the momentum. Additionally, the optimal rate of $O(\epsilon)$ was obtained in [88] for convex Hamiltonians $H = H(y,s,p)$ that also depend periodically on the time variable. For a recent study on the rate of convergence for time-fractional Hamilton–Jacobi equations with Caputo fractional derivatives, see [84]. We refer the reader to [20, 85, 97] for further references therein.

To our best knowledge, the most closely related previous research in this area is [99], where the approach in [85] was extended to attain the optimal rate of $O(\epsilon)$ in one dimension with further assumptions on H. In this study, we investigate this problem for dimensions

$n \geq 1$ and prove that the convergence rate, in general, is $O(t\sqrt{\epsilon})$ for $t \geq \sqrt{\epsilon}$.

6.1.2 Settings

Throughout this chapter, we will assume that the following conditions hold for the Hamiltonian $H : \mathbb{R}^n \times \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{R}$:

(H1) For each $R > 0$, $H \in \mathrm{BUC}\left(\mathbb{R}^n \times \mathbb{T}^n \times \mathrm{B}\left(0, R\right)\right)$, where $\mathrm{BUC}\left(\mathbb{R}^n \times \mathbb{T}^n \times \mathrm{B}\left(0, R\right)\right)$ stands for the set of bounded and uniformly continuous functions on $\mathbb{R}^n \times \mathbb{T}^n \times \mathrm{B}\left(0, R\right)$.

(H2) $\lim_{|p|\to\infty}\left(\inf_{x\in\mathbb{R}^n, y\in\mathbb{T}^n} H\left(x, y, p\right)\right) = +\infty$.

(H3) For each $x \in \mathbb{R}^n$ and $y \in \mathbb{T}^n$, the map $p \mapsto H(x, y, p)$ is convex.

(H4) There exists a constant $\mathrm{Lip}(H) > 0$ such that $|H(x_1, y, p) - H(x_2, y, p)| \leq \mathrm{Lip}(H)|x_1 - x_2|$, for any $x_1, x_2 \in \mathbb{R}^n$, $y \in \mathbb{T}^n$, and $p \in \mathbb{R}^n$.

We also assume $g \in \mathrm{BUC}(\mathbb{R}^n) \cap \mathrm{Lip}(\mathbb{R}^n)$.

We emphasize that condition (H4) is essential for the validity of our main result (as discussed in Remark 6.1.2). In Section 6.4, we present an example (refer to Proposition 6.4.2) to demonstrate that in the absence of this condition, the rate of convergence of u^ϵ to u as ϵ tends to zero cannot be bounded by $O(\sqrt{\epsilon})$.

The well-posedness of the equation (6.1) has already been extensively studied. The classical theory of viscosity solutions can be used to demonstrate the existence and uniqueness of solutions to (6.1) (see [98]). Moreover, the solution u^ϵ is uniformly bounded and Lipschitz, which can be expressed as follows:

$$\|u^\epsilon_t\|_{L^\infty(\mathbb{R}^n\times[0,\infty))} + \|Du^\epsilon\|_{L^\infty(\mathbb{R}\times[0,\infty))} \leq C_0, \quad \forall \epsilon > 0\,, \tag{6.3}$$

where $C_0 > 0$ is a constant that depends only on H and $\|Dg\|_{L^\infty(\mathbb{R}^n)}$. Based on (6.3), we can modify $H(x, y, p)$ for $|p| > 2C_0 + 1$ without changing the solutions to (6.1). This modification ensures that for all $x, p \in \mathbb{R}^n$ and $y \in \mathbb{T}^n$,

$$\frac{|p|^2}{2} - K_0 \leq H(x, y, p) \leq \frac{|p|^2}{2} + K_0 \tag{6.4}$$

for some constant $K_0 > 0$ that depends only on H and $\|Dg\|_{L^\infty(\mathbb{R}^n)}$. Consequently, for all $x, v \in \mathbb{R}^n$ and $y \in \mathbb{T}^n$,

$$\frac{|v|^2}{2} - K_0 \leq L(x,y,v) \leq \frac{|v|^2}{2} + K_0 \tag{6.5}$$

where $L : \mathbb{R}^n \times \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{R}$ is the Legendre transform of H.

Moreover, we have optimal control formulas for u^ϵ and u, that is,

$$u^\epsilon(x,t) = \inf \left\{ \int_0^t L\left(\gamma(s), \frac{\gamma(s)}{\epsilon}, -\dot{\gamma}(s)\right) ds + g\left(\gamma(t)\right) : \gamma \in \mathrm{AC}([0,t];\mathbb{R}^n),\ \gamma(0) = x \right\} \tag{6.6}$$

and

$$u(x,t) = \inf \left\{ \int_0^t \overline{L}\left(\overline{\gamma}(s), -\dot{\overline{\gamma}}(s)\right) ds + g\left(\overline{\gamma}(t)\right) : \overline{\gamma} \in \mathrm{AC}([0,t];\mathbb{R}^n),\ \overline{\gamma}(0) = x \right\}, \tag{6.7}$$

respectively. Here, AC denotes the class of absolutely continuous functions and $\overline{L}$ is the Legendre transform of $\overline{H} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$.

6.1.3 Main results and proof strategies

To establish our main result, we first introduce the following notation that can be viewed as a metric between any two points x and y in $\mathbb{R}^n$.

Definition 6.1.1. *Let $c, x, y \in \mathbb{R}^n$, $\epsilon > 0$, $0 \leq t_1 \leq t_2 < +\infty$. Define*

$$\begin{aligned}
\Gamma(t_1,t_2,x,y) &:= \left\{\gamma \in \mathrm{AC}([t_1,t_2],\mathbb{R}^n) :\ \gamma(t_1) = x,\ \gamma(t_2) = y\right\}, \\
m^\epsilon(t_1,t_2,x,y) &:= \inf\left\{ \int_{t_1}^{t_2} L\left(\gamma(s), \frac{\gamma(s)}{\epsilon}, -\dot{\gamma}(s)\right) ds \ :\ \gamma \in \Gamma(t_1,t_2,x,y) \right\}, \\
m_c^\epsilon(t_1,t_2,x,y) &:= \inf\left\{ \int_{t_1}^{t_2} L\left(c, \frac{\gamma(s)}{\epsilon}, -\dot{\gamma}(s)\right) ds \ :\ \gamma \in \Gamma(t_1,t_2,x,y) \right\}, \\
\overline{m}(t_1,t_2,x,y) &:= \inf\left\{ \int_{t_1}^{t_2} \overline{L}(\overline{\gamma}(s), -\dot{\overline{\gamma}}(s)) ds \ :\ \overline{\gamma} \in \Gamma(t_1,t_2,x,y) \right\}, \\
\overline{m}_c(t_1,t_2,x,y) &:= \inf\left\{ \int_{t_1}^{t_2} \overline{L}(c, -\dot{\overline{\gamma}}(s)) ds \ :\ \overline{\gamma} \in \Gamma(t_1,t_2,x,y) \right\}.
\end{aligned}$$

Although only the time difference $t_2 - t_1$ impacts the calculation of the cost in the above notations, we still specify the start and end time points to maintain consistency with the notation used for the discounted static problem.

We note that the optimal control formulas (6.6), (6.7) can be reformulated as

$$u^{\epsilon}(x,t) = \inf\left\{m^{\epsilon}(0,t,x,y) + g\left(y\right) : y \in \mathbb{R}^n\right\}$$

and

$$u(x,t) = \inf\left\{\overline{m}(0,t,x,y) + g\left(y\right) : y \in \mathbb{R}^n\right\},$$

respectively.

We now present our main result, which establishes a rate of $O(t\sqrt{\epsilon})$ for the multiscale setting. Our findings address the problem of the optimal rate of convergence for homogenization in the multiscale setting that Hitoshi Ishii initially proposed in 2018. (See [61])

Theorem 6.1.1. *Assume* (H1)-(H4) *and let* $g \in \mathrm{BUC}(\mathbb{R}^n) \cap \mathrm{Lip}(\mathbb{R}^n)$. *For* $\epsilon > 0$, *let* u^{ϵ} *be the unique viscosity solution to* (6.1) *and* u *be the unique viscosity solution to* (6.2). *Then there exists a constant* $C > 0$ *depending only on* n, H *and* $\|Dg\|_{L^{\infty}(\mathbb{R}^n)}$ *such that for any* $(x,t) \in \mathbb{R}^n \times (0,\infty)$ *and* $\epsilon \in (0,1)$, *we have*

$$\begin{aligned} |u^{\epsilon}(x,t) - u(x,t)| &\leq Ct\sqrt{\epsilon}, && \text{if } t \geq \sqrt{\epsilon}, \\ |u^{\epsilon}(x,t) - u(x,t)| &\leq C\min\left\{t,\epsilon\right\}, && \text{if } 0 < t < \sqrt{\epsilon}. \end{aligned} \tag{6.8}$$

We also state a similar result for the static problem.

Theorem 6.1.2. *Assume* (H1)-(H4). *For* $\lambda, \epsilon > 0$, *let* u^{ϵ} *be the unique viscosity solution to*

$$\lambda u^{\epsilon} + H\left(x, \frac{x}{\epsilon}, Du^{\epsilon}\right) = 0 \qquad \text{in } \mathbb{R}^n, \tag{6.9}$$

and let u *be the unique viscosity solution to*

$$\lambda u + \overline{H}\left(x, Du\right) = 0 \qquad \text{in } \mathbb{R}^n. \tag{6.10}$$

Then, there exists a constant $C > 0$ depending only on n and H such that for $\lambda, \epsilon \in (0,1)$, we have

$$\|u^\epsilon - u\|_{L^\infty(\mathbb{R}^n)} \leq \frac{C\sqrt{\epsilon}}{\lambda}. \tag{6.11}$$

A notable difference (for the Cauchy problems) between the multiscale setting and the case where Hamiltonians $H = H(y,p)$ (as studied in [97]) is that the rate of convergence in the former depends on time t, as opposed to being uniform in t for the latter. Specifically, in the multiscale setting, for t large, the rate of convergence is $O(\sqrt{\epsilon})$, with the power of ϵ being $\frac{1}{2}$. This power arises from balancing the macroscale and microscale variables, which is a key feature of the multiscale setting.

We now outline the proof strategy for the lower bound when $t \geq \sqrt{\epsilon}$ in the multiscale setting. As the proof of Theorem 6.1.2 is based on exactly the same idea, we focus on presenting the proof idea of Theorem 6.1.1.

First, we consider a minimizing curve $\gamma_0 : [0,t] \to \mathbb{R}^n$ for $u^\epsilon(x,t)$, i.e., $\gamma_0\,(0) = x$ and

$$u^\epsilon(x,t) = \int_0^t L\left(\gamma_0(s), \frac{\gamma_0(s)}{\epsilon}, -\dot{\gamma}_0(s)\right) ds + g(\gamma_0(t)). \tag{6.12}$$

The main idea is to break γ_0 into N evenly spaced pieces with respect to time, where N needs to be determined appropriately. For each piece, we approximate its cost by fixing the first argument of L in (6.12). More precisely, for the k-th piece where $k = 0, 1, \cdots, N-1$, the time runs from $t_k = k\sqrt{\epsilon}$ to $t_{k+1} = (k+1)\sqrt{\epsilon}$, and we estimate the running cost within this time with the first argument fixed in Lagrangian by the value of the curve at the beginning $x_k = \gamma_0(t_k)$ of this piece, that is,

$$\sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} L\left(x_k, \frac{\gamma_0(s)}{\epsilon}, -\dot{\gamma}_0(s)\right) ds \tag{6.13}$$

The error for fixing the first argument of L in the running cost for N pieces of shorter curves is $\frac{t^2}{N}$ (under condition (H4), see Lemma 6.2.2).

For each piece with the first argument of L fixed in the cost, we can use the definition of $m^{\epsilon}_{x_k}$ to obtain

$$\sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} L\left(x_k, \frac{\gamma_0(s)}{\epsilon}, -\dot{\gamma}_0(s)\right) ds \geq \sum_{k=0}^{N-1} m^{\epsilon}_{x_k}(t_k, t_{k+1}, \gamma_0(t_k), \gamma_0(t_{k+1})). \tag{6.14}$$

Further, we can use the following lemma to connect $m^{\epsilon}_{x_k}$ with $\overline{m}_{x_k}$ and hence $u(x,t)$.

Lemma 6.1.1. *Assume* (H1)-(H3). *Fix $c \in \mathbb{R}^n$. Let $x, y \in \mathbb{R}^n, \epsilon, t > 0$ and $M_0 > 0$ with $|y - x| \leq M_0 t$. Let $K_0 > 0$ be a constant that satisfies* (6.4), (6.5). *Then, there exists a constant $C = C(n, M_0, K_0) > 0$ such that*

$$|m^{\epsilon}_c(0, t, x, y) - \overline{m}_c(0, t, x, y)| \leq C\epsilon. \tag{6.15}$$

Remark 6.1.1. *This lemma is a generalization of [97, Lemmas 3.1, 3.2]. In [97], it is proved that for a fixed $c \in \mathbb{R}^n$, and for the Lagrangian $L^c(\cdot,\cdot) = L(c,\cdot,\cdot)$, there exists a constant $C = C(n, L^c, M_0) > 0$ such that for any $x, y \in \mathbb{R}^n$, $\epsilon, t > 0$ with $|y - x| \leq M_0 t$, we have the conclusion of Lemma 6.1.1 as above. Although the constant $C = C(n, L^c, M_0) > 0$ could potentially depend on $c \in \mathbb{R}^n$ due to the dependence of L^c on $c \in \mathbb{R}^n$, it can be shown, under the assumptions* (H1)-(H3), *that the constant $C > 0$ depends only on n, M_0, K_0, as presented in Appendix.*

Using Lemma 6.1.1, we can approximate each term on the right-hand side of (6.14) by $\overline{m}_{x_k}$ with the corresponding arguments, incurring an error of ϵN for the sum of N terms. Furthermore, by constructing an admissible path for $u(x,t)$, we can replace $\overline{m}_{x_k}$ with $u(x,t)$, introducing an additional error of $\frac{t^2}{N}$. Thus, we obtain the inequality

$$u^{\epsilon}(x,t) \geq u(x,t) - C\frac{t^2}{N} - CN\epsilon,$$

where $C > 0$ is a constant that depends only on n, M_0, K_0. In summary, we have one source of error coming from fixing the macroscale variable in approximating the running cost and the other source of error caused by handling the microscale variable with Lemma

6.1.1. To minimize the total error, that is, to balance between $\frac{t^2}{N}$ and $N\epsilon$, the best N we can choose is $N = \frac{t}{\sqrt{\epsilon}}$, which yields a bound of $Ct\sqrt{\epsilon}$ on the total error.

The balance between the spatial variable and the oscillatory variable in homogenization is a key feature of the multiscale setting, and it is the first work in the literature where scale separations occur at the level of optimal curves for the solutions. As we can see, it is crucial in the proof of Theorem 6.1.1 that the constant $C > 0$ in Lemma 6.1.1 is independent of $c \in \mathbb{R}^n$, as we freeze the spatial variable at various places along minimizing curves. Also, we will see that the involvement of time t in the bound $Ct\sqrt{\epsilon}$ is necessary by an example, which is also a feature distinguished from the case where Hamiltonians do not depend on the spatial variable.

Remark 6.1.2. *Condition* (H4) *is a necessary assumption for the approach of fixing the x-arguments to work. This condition enables us to bound the error caused by freezing the spatial variable. In Proposition 6.4.2, we provide an illustration of the case where* (H4) *is not satisfied, and the error cannot be controlled in this way.*

Organization of this chapter

In Section 6.2, we prove Theorem 6.1.1. In Section 6.3, we verify Theorem 6.1.2. In Section 6.4, we provide examples that demonstrate the optimality of the rate of convergence suggested in Theorem 6.1.1. In the Appendix, we show Lemma 6.1.1 in detail.

6.2 Proof of Theorem 6.1.1

6.2.1 Preliminaries

We begin by stating that throughout this chapter, we will use $C, C_0, K_0, M, M_0 > 0$ to denote positive constants, and their dependence on parameters will be specified as their arguments. The constants $C_0 = C_0\left(H, \|Dg\|_{L^\infty(\mathbb{R}^n)}\right)$, $K_0 = K_0\left(H, \|Dg\|_{L^\infty(\mathbb{R}^n)}\right)$, $M_0 = M_0\left(H, \|Dg\|_{L^\infty(\mathbb{R}^n)}\right) > 0$ will be fixed throughout this chapter, while $C, M > 0$ may vary line by line.

Prior to proving Theorem 6.1.1, we introduce two essential lemmas that will assist us in constraining the errors that arise when we freeze the first argument of L in the running cost.

We first state the lemma about the boundedness of velocities of minimizing curves.

Lemma 6.2.1. *Assume* (H1)-(H3)*. Let $x \in \mathbb{R}^n$, $t > 0$ and $\epsilon > 0$. Suppose that $\gamma : [0, t] \to \mathbb{R}^n$ is a minimizing curve of $u^\epsilon(x, t)$ in the sense that γ is absolutely continuous, and*

$$u^\epsilon(x,t) = \int_0^t L\left(\gamma(s), \frac{\gamma(s)}{\epsilon}, -\dot{\gamma}(s)\right) ds + g\left(\gamma(t)\right) \tag{6.16}$$

with $\gamma(0) = x$. Then, there exists a constant $M_0 = M_0\left(H, \|Dg\|_{L^\infty(\mathbb{R}^n)}\right) > 0$ such that $\|\dot{\gamma}\|_{L^\infty([0,t])} \leq M_0$. Similarly, if $\overline{\gamma} : [0, t] \to \mathbb{R}^n$ is a minimizing curve of $u(x, t)$ in the sense that $\overline{\gamma}$ is absolutely continuous, and

$$u(x,t) = \int_0^t \overline{L}\left(\overline{\gamma}(s), -\dot{\overline{\gamma}}(s)\right) ds + g\left(\overline{\gamma}(t)\right) \tag{6.17}$$

with $\overline{\gamma}(0) = x$, then there exists a constant $M_0 = M_0\left(H, \|Dg\|_{L^\infty(\mathbb{R}^n)}\right) > 0$ such that $\left\|\dot{\overline{\gamma}}\right\|_{L^\infty([0,t])} \leq M_0$.

The following lemma states that $L(\cdot, y, v)$ and $\overline{L}(\cdot, v)$ are Lipschitz uniformly in y and v under the condition (H4).

Lemma 6.2.2. *Assume* (H1)-(H4)*. Then,*

$$|L(x_1, y, v) - L(x_2, y, v)| \leq \mathrm{Lip}(H)|x_1 - x_2|,$$

and

$$\left|\overline{L}(x_1, v) - \overline{L}(x_2, v)\right| \leq \mathrm{Lip}(H)|x_1 - x_2|,$$

for any $x_1, x_2 \in \mathbb{R}^n$, $y \in \mathbb{T}^n$, and $v \in \mathbb{R}^n$.

The proofs of the above two lemmas are omitted here. See [98] for more details.

6.2.2 Proof

We are now ready to prove Theorem 6.1.1.

Proof of Theorem 6.1.1. Let $x \in \mathbb{R}^n$, $\epsilon, t > 0$. We first show that for some constant $C = C(n, H, \|Dg\|_{L^\infty(\mathbb{R}^n)}) > 0$, it holds that $u^\epsilon(x,t) - u(x,t) \geq -Ct\sqrt{\epsilon}$ for $t \geq \sqrt{\epsilon}$, and that $u^\epsilon(x,t) - u(x,t) \geq -C\epsilon$ for $t \in (0, \sqrt{\epsilon})$.

Let $\gamma_0 : [0,t] \to \mathbb{R}^n$ be an absolutely continuous curve with $\gamma_0(0) = x$ such that

$$u^\epsilon(x,t) = \int_0^t L\left(\gamma_0(s), \frac{\gamma_0(s)}{\epsilon}, -\dot{\gamma_0}(s)\right) ds + g(\gamma_0(t)) = m^\epsilon(0,t,x,y) + g(y),$$

where y denotes the point $\gamma_0(t) \in \mathbb{R}^n$. Then,

$$u(x,t) \leq \overline{m}(0,t,x,y) + g(y),$$

and thus,

$$u^\epsilon(x,t) - u(x,t) \geq m^\epsilon(0,t,x,y) - \overline{m}(0,t,x,y). \tag{6.18}$$

In order to give a lower bound of $m^\epsilon(0,t,x,y) - \overline{m}(0,t,x,y)$, we consider a partition

$$0 = t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_k \leq t_{k+1} \leq \cdots \leq t_N \leq t_{N+1} = t$$

of the interval $[0,t]$, where N is a nonnegative integer that will be determined later together with the division. On each interval $[t_k, t_{k+1}]$ for $k = 0, \cdots, N$, we freeze the spatial variable, homogenize in the oscillatory variable, and then unfreeze the spatial variable in divided steps as follows. We finally estimate the commutators arising from these steps.

Step 1: Freeze the spatial variable.

For each $k = 0, \cdots, N$, let $x_k := \gamma_0(t_k)$. Then, for each $k = 0, \cdots, N$,

$$m^\epsilon(t_k, t_{k+1}, \gamma_0(t_k), \gamma_0(t_{k+1})) = \int_{t_k}^{t_{k+1}} L\left(x_k, \frac{\gamma_0(s)}{\epsilon}, -\dot{\gamma}_0(s)\right) ds + E_k,$$

where

$$E_k := \int_{t_k}^{t_{k+1}} L\left(\gamma_0(s), \frac{\gamma_0(s)}{\epsilon}, -\dot{\gamma}_0(s)\right) ds - \int_{t_k}^{t_{k+1}} L\left(x_k, \frac{\gamma_0(s)}{\epsilon}, -\dot{\gamma}_0(s)\right) ds.$$

Step 2: Homogenize in the oscillatory variable.

We apply Lemma 6.1.1 and Lemma 6.2.1 to see that there exists a constant $C = C(n, H, \|Dg\|_{L^\infty(\mathbb{R}^n)}) > 0$ such that

$$\begin{aligned} \int_{t_k}^{t_{k+1}} L\left(x_k, \frac{\gamma_0(s)}{\epsilon}, -\dot{\gamma}_0(s)\right) ds &\geq m^\epsilon_{x_k}(t_k, t_{k+1}, \gamma_0(t_k), \gamma_0(t_{k+1})) \\ &\geq \overline{m}_{x_k}(t_k, t_{k+1}, \gamma_0(t_k), \gamma_0(t_{k+1})) - C\epsilon \end{aligned}$$

for each $k = 0, \cdots, N$. It is a crucial fact that the constant $C > 0$ is independent of $k = 0, \cdots, N$, i.e., independent of the spatial positions.

Step 3: Unfreeze the spatial variable.

For each $k = 0, \cdots, N$, let $\overline{\gamma}_k : [t_k, t_{k+1}] \to \mathbb{R}^n$ be an absolutely continuous curve with $\overline{\gamma}_k(t_k) = \gamma_0(t_k)$, $\overline{\gamma}_k(t_{k+1}) = \gamma_0(t_{k+1})$ such that

$$\overline{m}_{x_k}(t_k, t_{k+1}, \gamma_0(t_k), \gamma_0(t_{k+1})) = \int_{t_k}^{t_{k+1}} \overline{L}(x_k, -\dot{\overline{\gamma}}_k(s)) ds.$$

Then,

$$\begin{aligned} \overline{m}_{x_k}(t_k, t_{k+1}, \gamma_0(t_k), \gamma_0(t_{k+1})) &= \int_{t_k}^{t_{k+1}} \overline{L}(\overline{\gamma}_k(s), -\dot{\overline{\gamma}}_k(s)) ds - \overline{E}_k \\ &\geq \overline{m}(t_k, t_{k+1}, \gamma_0(t_k), \gamma_0(t_{k+1})) - \overline{E}_k, \end{aligned}$$

where

$$\overline{E}_k := \int_{t_k}^{t_{k+1}} \overline{L}(\overline{\gamma}_k(s), -\dot{\overline{\gamma}}_k(s))ds - \int_{t_k}^{t_{k+1}} \overline{L}(x_k, -\dot{\overline{\gamma}}_k(s))ds$$

for each $k = 0, \cdots, N$.

Step 4: Estimate the errors $E_k, \overline{E}_k$ and obtain a lower bound.

From Steps 1-3, we have that for each $k = 0, \cdots, N$,

$$m^\epsilon(t_k, t_{k+1}, \gamma_0(t_k), \gamma_0(t_{k+1})) \geq \overline{m}(t_k, t_{k+1}, \gamma_0(t_k), \gamma_0(t_{k+1})) + E_k - \overline{E}_k - C\epsilon.$$

Since

$$m^\epsilon(0, t, x, y) = \sum_{k=0}^{N} m^\epsilon(t_k, t_{k+1}, \gamma_0(t_k), \gamma_0(t_{k+1}))$$

and

$$\overline{m}(0, t, x, y) \leq \sum_{k=0}^{N} \overline{m}(t_k, t_{k+1}, \gamma_0(t_k), \gamma_0(t_{k+1})),$$

we obtain

$$m^\epsilon(0, t, x, y) \geq \overline{m}(0, t, x, y) + \sum_{k=0}^{N} \left(E_k - \overline{E}_k - C\epsilon\right). \tag{6.19}$$

Now, we estimate the errors $E_k, \overline{E}_k$. By Lemmas 6.2.1, 6.2.2, we get

$$\begin{aligned}
|E_k| &= \left| \int_{t_k}^{t_{k+1}} L\left(\gamma_0(s), \frac{\gamma_0(s)}{\epsilon}, -\dot{\gamma}_0(s)\right) ds - \int_{t_k}^{t_{k+1}} L\left(x_k, \frac{\gamma_0(s)}{\epsilon}, -\dot{\gamma}_0(s)\right) ds \right| \\
&\leq \int_{t_k}^{t_{k+1}} \mathrm{Lip}(H)|\gamma_0(s) - x_k| ds \\
&\leq \mathrm{Lip}(H) M_0 \int_{t_k}^{t_{k+1}} |s - t_k| ds \\
&\leq \mathrm{Lip}(H) M_0 (t_{k+1} - t_k)^2
\end{aligned}$$

for each $k = 0, \cdots, N$. By the same estimate, we also get $|\overline{E}_k| \le \mathrm{Lip}(H) M_0 (t_{k+1} - t_k)^2$ for each $k = 0, \cdots, N$.

Set $N = \left\lfloor \frac{t}{\sqrt{\epsilon}} \right\rfloor$ and $t_k = k\sqrt{\epsilon}$ for each $k = 0, \cdots, N$. With this choice of division, it holds that $t_{k+1} - t_k \le \sqrt{\epsilon}$ for all $k = 0, \cdots, N$. Note that $t_N = t_{N+1} = t$ when $\frac{t}{\sqrt{\epsilon}}$ is a positive integer. If $t \in (0, \sqrt{\epsilon})$, then $N = 0$, and thus,

$$\left| \sum_{k=0}^{N} \left(E_k - \overline{E}_k - C\epsilon \right) \right| \le 2\mathrm{Lip}(H) M_0 \epsilon + C\epsilon \le C\epsilon$$

with $C = C(n, H, \|Dg\|_{L^\infty(\mathbb{R}^n)}) > 0$ changed to a larger constant in the last inequality. If $t \ge \sqrt{\epsilon}$, then $N + 1 \le \frac{2t}{\sqrt{\epsilon}}$, and thus,

$$\begin{aligned} \left| \sum_{k=0}^{N} \left(E_k - \overline{E}_k - C\epsilon \right) \right| &\le 2(N+1)\mathrm{Lip}(H) M_0 \epsilon + (N+1)C\epsilon \\ &\le 4\mathrm{Lip}(H) M_0 t\sqrt{\epsilon} + 2Ct\sqrt{\epsilon} \\ &\le Ct\sqrt{\epsilon} \end{aligned}$$

with $C = C(n, H, \|Dg\|_{L^\infty(\mathbb{R}^n)}) > 0$ changed to a larger constant in the last inequality. In all cases, we obtain a desired lower bound by combining (6.18), (6.19).

To prove an upper bound of $u^\epsilon(x,t) - u(x,t)$, we instead obtain a lower bound of $u(x,t) - u^\epsilon(x,t)$. Since Lemmas 6.1.1, 6.2.1, 6.2.2 are written entirely symmetric in m^ϵ and $\overline{m}$, L and $\overline{L}$, the same arguments as the above (but swapping u^ϵ and u, m^ϵ and $\overline{m}$, L and $\overline{L}$, respectively) also prove lower bounds $u(x,t) - u^\epsilon(x,t) \ge -Ct\sqrt{\epsilon}$ for $t \ge \sqrt{\epsilon}$ and $u(x,t) - u^\epsilon(x,t) \ge -C\epsilon$ for $t \in (0, \sqrt{\epsilon})$.

Finally, for $t \in (0, \sqrt{\epsilon})$, we apply the comparison principle to see that there exists a constant $C = C(H, \|Dg\|_{L^\infty(\mathbb{R}^n)}) > 0$ such that

$$|u^\epsilon(x,t) - g(x)| \le Ct$$

and

$$|u(x,t) - g(x)| \leq Ct.$$

Therefore, there exists a constant $C = C(H, \|Dg\|_{L^\infty(\mathbb{R}^n)}) > 0$ such that

$$|u^\epsilon(x,t) - u(x,t)| \leq Ct,$$

which yields (6.8) together with the bounds $|u^\epsilon(x,t) - u(x,t)| \leq Ct\sqrt{\epsilon}$ for $t \geq \sqrt{\epsilon}$ and $|u^\epsilon(x,t) - u(x,t)| \leq C\epsilon$ for $t \in (0, \sqrt{\epsilon})$. This completes the proof. □

6.3 Proof of Theorem 6.1.2

6.3.1 Preliminaries

Let u^ϵ be the unique viscosity solution to (6.9), and let u be the unique viscosity solution to (6.10). Then, we have the optimal control formulas for u^ϵ and u, that is,

$$\begin{aligned} &u^\epsilon(x) \\ =&\inf\left\{\int_0^\infty e^{-\lambda s} L\left(\gamma(s), \frac{\gamma(s)}{\epsilon}, -\dot{\gamma}(s)\right) ds :\ \gamma(0) = x,\ \gamma \in \mathrm{AC}([0,T];\mathbb{R}^n),\ \text{for any } T > 0\right\}, \end{aligned} \tag{6.20}$$

and

$$\begin{aligned} &u(x) \\ =&\inf\left\{\int_0^\infty e^{-\lambda s} \overline{L}\left(\overline{\gamma}(s), -\dot{\overline{\gamma}}(s)\right) ds :\ \overline{\gamma}(0) = x,\ \overline{\gamma} \in \mathrm{AC}([0,T];\mathbb{R}^n),\ \text{for any } T > 0\right\} \end{aligned} \tag{6.21}$$

respectively.

We state the lemma about the boundedness of velocities of minizing curves for this problem, which corresponds to Lemma 6.2.1.

Lemma 6.3.1. *Assume* (H1)-(H3). *Let $x \in \mathbb{R}^n$, and $\lambda, \epsilon > 0$. Suppose that $\gamma : [0, \infty) \to$*

$\mathbb{R}^n$ is a minimizing curve of $u^\epsilon(x)$ in the sense that $\gamma \in \mathrm{AC}([0,T];\mathbb{R}^n)$ for any $T > 0$, and

$$u^\epsilon(x) = \int_0^\infty e^{-\lambda s} L\left(\gamma(s), \frac{\gamma(s)}{\epsilon}, -\dot{\gamma}(s)\right) ds \tag{6.22}$$

with $\gamma(0) = x$. Then, there exists a constant $M_0 = M_0(H) > 0$ such that $\|\dot{\gamma}\|_{L^\infty([0,\infty))} \leq M_0$. Similarly, if $\overline{\gamma} : [0,\infty) \to \mathbb{R}^n$ is a minimizing curve of $u(x)$ in the sense that $\overline{\gamma} \in \mathrm{AC}([0,T];\mathbb{R}^n)$ for any $T > 0$, and

$$u(x) = \int_0^\infty e^{-\lambda s} \overline{L}\left(\overline{\gamma}(s), -\dot{\overline{\gamma}}(s)\right) ds \tag{6.23}$$

with $\overline{\gamma}(0) = x$, then there exists a constant $M_0 = M_0(H) > 0$ such that $\left\|\dot{\overline{\gamma}}\right\|_{L^\infty([0,\infty))} \leq M_0$.

Also, by applying the comparison principle, we have the L^∞-bound of u^ϵ and u.

Lemma 6.3.2. *Assume* (H1)-(H3)*. Let u^ϵ be the unique viscosity solution to* (6.9)*, and let u be the unique viscosity solution to* (6.10)*. Let $M := \|H(\cdot,\cdot,0)\|_{L^\infty(\mathbb{R}^n\times\mathbb{T}^n)}$. Then,*

$$\|u^\epsilon\|_{L^\infty(\mathbb{R}^n)} \leq \frac{M}{\lambda}$$

and

$$\|u\|_{L^\infty(\mathbb{R}^n)} \leq \frac{M}{\lambda}.$$

6.3.2 Proof

We introduce the additional notations with the discount term for the proof of Theorem 6.1.2; for $x, y \in \mathbb{R}^n$, $\lambda, \epsilon > 0$, $0 \leq t_1 \leq t_2 < +\infty$, we let

$$m^{\epsilon,\lambda}(t_1,t_2,x,y) := \inf\left\{\int_{t_1}^{t_2} e^{-\lambda s} L\left(\gamma(s), \frac{\gamma(s)}{\epsilon}, -\dot{\gamma}(s)\right) ds \;:\; \gamma \in \Gamma(t_1,t_2,x,y)\right\},$$
$$\overline{m}^{\lambda}(t_1,t_2,x,y) := \inf\left\{\int_{t_1}^{t_2} e^{-\lambda s} \overline{L}(\overline{\gamma}(s), -\dot{\overline{\gamma}}(s)) ds \;:\; \overline{\gamma} \in \Gamma(t_1,t_2,x,y)\right\}.$$

Now we prove Theorem 6.1.2.

Proof of Theorem 6.1.2. Let $M := \|H(\cdot,\cdot,0)\|_{L^\infty(\mathbb{R}^n\times\mathbb{T}^n)}$. Let $H^M(x,y,p) := H(x,y,p) -$

M and $\overline{H}^M$ be its effective Hamiltonian, which coincides with $\overline{H}-M$. Then, $u^\epsilon_M := u^\epsilon + \frac{M}{\lambda}$ ($u_M := u + \frac{M}{\lambda}$, resp.) is the unique viscosity solution to

$$\lambda u^\epsilon_M + H^M\left(x, \frac{x}{\epsilon}, Du^\epsilon_M\right) = 0 \qquad \left(\lambda u_M + \overline{H}^M(x, Du_M) = 0, \text{ resp.}\right).$$

The additional property of the Hamiltonian H^M is that its Lagrangian and effective Lagrangian are nonnegative. Since $u^\epsilon_M - u_M = u^\epsilon - u$, it suffices to prove that $\|u^\epsilon_M - u_M\|_{L^\infty(\mathbb{R}^n)} \le \frac{C\sqrt{\epsilon}}{\lambda}$. Therefore, it suffices to prove that $\|u^\epsilon - u\|_{L^\infty(\mathbb{R}^n)} \le \frac{C\sqrt{\epsilon}}{\lambda}$ when $L, \overline{L} \ge 0$, which we assume from now on without loss of generality.

Let $x \in \mathbb{R}^n$, and let $\lambda, \epsilon \in (0,1)$. The goal is to prove $u^\epsilon(x) - u(x) \ge -\frac{C\sqrt{\epsilon}}{\lambda}$ for some constant $C = C(n, H) > 0$. Let γ_0 be a curve such that with $\gamma_0(0) = x$, $\gamma_0 \in \mathrm{AC}([0,T];\mathbb{R}^n)$ for any $T > 0$, and

$$u^\epsilon(x) = \int_0^\infty e^{-\lambda s} L\left(\gamma_0(s), \frac{\gamma_0(s)}{\epsilon}, -\dot{\gamma}_0(s)\right) ds.$$

Consider a partition

$$0 = t_0 < t_1 < t_2 < \cdots < t_k < t_{k+1} < \cdots$$

of the interval $[0, +\infty)$ with $t_k \to +\infty$ as $k \to +\infty$, which will be determined later.

Step 1: Freeze the spatial variable.

For each $k = 0, 1, 2, \cdots$, let $x_k := \gamma_0(t_k)$. Then, for each $k = 0, 1, 2, \cdots$,

$$m^{\epsilon,\lambda}(t_k, t_{k+1}, \gamma_0(t_k), \gamma_0(t_{k+1})) = \int_{t_k}^{t_{k+1}} e^{-\lambda s} L\left(x_k, \frac{\gamma_0(s)}{\epsilon}, -\dot{\gamma}_0(s)\right) ds + E_k,$$

where

$$E_k := \int_{t_k}^{t_{k+1}} e^{-\lambda s} L\left(\gamma_0(s), \frac{\gamma_0(s)}{\epsilon}, -\dot{\gamma}_0(s)\right) ds - \int_{t_k}^{t_{k+1}} e^{-\lambda s} L\left(x_k, \frac{\gamma_0(s)}{\epsilon}, -\dot{\gamma}_0(s)\right) ds.$$

Step 2: Homogenize in the oscillatory variable.

We apply Lemma 6.1.1 and Lemma 6.3.1 to see that there exists a constant $C = C(n, H) > 0$ such that

$$\begin{aligned}\int_{t_k}^{t_{k+1}} e^{-\lambda s} L\left(x_k, \frac{\gamma_0(s)}{\epsilon}, -\dot{\gamma}_0(s)\right) ds &\geq e^{-\lambda t_{k+1}} \int_{t_k}^{t_{k+1}} L\left(x_k, \frac{\gamma_0(s)}{\epsilon}, -\dot{\gamma}_0(s)\right) ds \\ &\geq e^{-\lambda t_{k+1}} m^{\epsilon}_{x_k}(t_k, t_{k+1}, \gamma_0(t_k), \gamma_0(t_{k+1})) \\ &\geq e^{-\lambda t_{k+1}} \overline{m}_{x_k}(t_k, t_{k+1}, \gamma_0(t_k), \gamma_0(t_{k+1})) - Ce^{-\lambda t_{k+1}}\epsilon.\end{aligned}$$

for each $k = 0, 1, 2, \cdots$.

Step 3: Unfreeze the spatial variable.

For each $k = 0, 1, 2, \cdots$, let $\overline{\gamma}_k : [t_k, t_{k+1}] \to \mathbb{R}^n$ be an absolutely continuous curve with $\overline{\gamma}_k(t_k) = \gamma_0(t_k),\ \overline{\gamma}_k(t_{k+1}) = \gamma_0(t_{k+1})$ such that

$$\overline{m}_{x_k}(t_k, t_{k+1}, \gamma_0(t_k), \gamma_0(t_{k+1})) = \int_{t_k}^{t_{k+1}} \overline{L}(x_k, -\dot{\overline{\gamma}}_k(s))ds.$$

Then,

$$\begin{aligned}&e^{-\lambda t_{k+1}} \overline{m}_{x_k}(t_k, t_{k+1}, \gamma_0(t_k), \gamma_0(t_{k+1})) \\ =&e^{-\lambda t_{k+1}} \int_{t_k}^{t_{k+1}} \overline{L}(\overline{\gamma}_k(s), -\dot{\overline{\gamma}}_k(s))ds - e^{-\lambda t_{k+1}} \overline{E}_k \\ \geq&e^{-\lambda(t_{k+1}-t_k)} \overline{m}^{\lambda}(t_k, t_{k+1}, \gamma_0(t_k), \gamma_0(t_{k+1})) - e^{-\lambda t_{k+1}} \overline{E}_k,\end{aligned}$$

where

$$\overline{E}_k := \int_{t_k}^{t_{k+1}} \overline{L}(\overline{\gamma}_k(s), -\dot{\overline{\gamma}}_k(s))ds - \int_{t_k}^{t_{k+1}} \overline{L}(x_k, -\dot{\overline{\gamma}}_k(s))ds$$

for each $k = 0, 1, 2, \cdots$.

Step 4: Estimate the errors $E_k, \overline{E}_k$ and obtain a lower bound.

From Steps 1-3, we have that for each $k = 0, 1, 2, \cdots$,

$$\begin{aligned} & m^{\epsilon,\lambda}(t_k, t_{k+1}, \gamma_0(t_k), \gamma_0(t_{k+1})) \\ \geq & e^{-\lambda(t_{k+1}-t_k)}\overline{m}^{\lambda}(t_k, t_{k+1}, \gamma_0(t_k), \gamma_0(t_{k+1})) + E_k - e^{-\lambda t_{k+1}}\overline{E}_k - Ce^{-\lambda t_{k+1}}\epsilon. \end{aligned}$$

Since

$$u^{\epsilon}(x) = \sum_{k=0}^{\infty} m^{\epsilon,\lambda}(t_k, t_{k+1}, \gamma_0(t_k), \gamma_0(t_{k+1}))$$

and

$$u(x) \leq \sum_{k=0}^{\infty} \overline{m}^{\lambda}(t_k, t_{k+1}, \gamma_0(t_k), \gamma_0(t_{k+1})),$$

we obtain

$$u^{\epsilon}(x) \geq e^{-\lambda \sup_{k\geq 0}(t_{k+1}-t_k)}u(x) + \sum_{k=0}^{\infty}\left(E_k - e^{-\lambda t_{k+1}}\overline{E}_k - Ce^{-\lambda t_{k+1}}\epsilon\right). \tag{6.24}$$

We now estimate the errors $E_k, \overline{E}_k$. By Lemmas 6.2.2, 6.3.1, we get

$$\begin{aligned} |E_k| &= \left|\int_{t_k}^{t_{k+1}} e^{-\lambda s}L\left(\gamma_0(s), \frac{\gamma_0(s)}{\epsilon}, -\dot{\gamma_0}(s)\right) ds - \int_{t_k}^{t_{k+1}} e^{-\lambda s}L\left(x_k, \frac{\gamma_0(s)}{\epsilon}, -\dot{\gamma_0}(s)\right) ds\right| \\ &\leq e^{-\lambda t_k}\int_{t_k}^{t_{k+1}} \mathrm{Lip}(H)|\gamma_0(s) - x_k| ds \\ &\leq e^{-\lambda t_k}\mathrm{Lip}(H)M_0(t_{k+1} - t_k)^2 \end{aligned}$$

for each $k = 0, 1, 2, \cdots$. Similarly, we also get $|\overline{E}_k| \leq \mathrm{Lip}(H)M_0(t_{k+1} - t_k)^2$ for each $k = 0, 1, 2, \cdots$.

Set $t_k = k\sqrt{\epsilon}$ for each $k = 0, 1, 2, \cdots$. Then, from (6.24), we have

$$u^{\epsilon}(x) \geq e^{-\lambda\sqrt{\epsilon}}u(x) - 2\mathrm{Lip}(H)M_0\frac{\sqrt{\epsilon}}{\lambda}\sum_{k=0}^{\infty}\lambda\sqrt{\epsilon}e^{-k\lambda\sqrt{\epsilon}} - C\frac{\sqrt{\epsilon}}{\lambda}\sum_{k=0}^{\infty}\lambda\sqrt{\epsilon}e^{-k\lambda\sqrt{\epsilon}}.$$

Also, by Lemma 6.3.2,

$$e^{-\lambda\sqrt{\epsilon}}u(x) - u(x) = -\left(1 - e^{-\lambda\sqrt{\epsilon}}\right)u(x) \geq -\lambda\sqrt{\epsilon}u(x) \geq -M\sqrt{\epsilon},$$

where $M := \|H(\cdot,\cdot,0)\|_{L^\infty(\mathbb{R}^n\times\mathbb{T}^n)}$. By the elementary fact that $\sum_{k=0}^{\infty}\lambda\sqrt{\epsilon}e^{-k\lambda\sqrt{\epsilon}}$ is bounded by a universal constant for $\lambda,\epsilon\in(0,1)$, we have

$$u^\epsilon(x) \geq u(x) - \frac{C\sqrt{\epsilon}}{\lambda}$$

for some constant $C = C(n,H) > 0$, as desired.

To prove an upper bound of $u^\epsilon(x)-u(x)$, we instead obtain a lower bound of $u(x)-u^\epsilon(x)$ by interchanging u^ϵ and u, m^ϵ and $\overline{m}$, L and $\overline{L}$, respectively, in the above arguments, which yields $u(x) \geq u^\epsilon(x) - \frac{C\sqrt{\epsilon}}{\lambda}$. □

6.4 Examples

In this section, we first establish the optimality of (6.8) in Theorem 6.1.1 for $0 < t < \sqrt{\epsilon}$ and $t \sim \sqrt{\epsilon}$. We then provide an example to illustrate the necessity of condition (H4) for Theorem 6.1.1. Finally, we present an example that demonstrates the necessity of involving the time variable t in (6.8).

The following proposition demonstrates the optimality of the bound (6.8). We consider two cases: when $0 < t < \sqrt{\epsilon}$ and $t \geq \sqrt{\epsilon}$. For $0 < t < \sqrt{\epsilon}$, the optimality of the rate of convergence in (6.8) is evident from (6.25). Moreover, (6.25) implies that the rate of convergence in (6.8) is also optimal for $t = C\sqrt{\epsilon}$, where $C > 1$. This example is discussed in [85, Proposition 4.3], and it shows that the rate $O(\epsilon)$ is optimal when the Hamiltonian $H = H(y,p)$ depends solely on the oscillatory variable and momentum. The optimality for the whole range $t \geq \sqrt{\epsilon}$ is still unclear because of the mixed involvement of both $\sqrt{\epsilon}$ and t.

Proposition 6.4.1. *Consider the case where $n = 1$, $H(y,p) = -V(y) + \frac{1}{2}p^2$ for a given continuous function $V \in C(\mathbb{T})$ with $\min_{\mathbb{T}} V = 0$ and $V \geq 1$ on $[-3^{-1}, 3^{-1}]$, and $g \equiv 0$. For $\epsilon > 0$, let u^ϵ be the solution to* (6.1)*, and let u be the solution to* (6.2)*. Then, for $\epsilon \in (0,1)$ and $t > 0$,*

$$u^\epsilon(0,t) - u(0,t) \geq \frac{\sqrt{2}}{3}\min\{t,\epsilon\}. \tag{6.25}$$

Proof. Due to the optimal control formula, we have

$$u^\epsilon(0,t) = \inf\left\{\epsilon\int_0^{\frac{t}{\epsilon}} V(\eta(s)) + \frac{1}{2}|\dot\eta(s)|^2 ds \;:\; \eta \in \mathrm{AC}\left(\left[0,\frac{t}{\epsilon}\right]\right),\ \eta(0)=0\right\}.$$

Let $\eta \in \mathrm{AC}\left(\left[0,\frac{t}{\epsilon}\right]\right)$ with $\eta(0)=0$. If $\eta\left(\left[0,\frac{t}{\epsilon}\right]\right) \subset \left[-\frac{1}{3},\frac{1}{3}\right]$, then

$$\epsilon\int_0^{\frac{t}{\epsilon}} V(\eta(s)) + \frac{1}{2}|\dot\eta(s)|^2 ds \geq \epsilon\int_0^{\frac{t}{\epsilon}} V(\eta(s))ds \geq t.$$

If not, without loss of generality, we may assume that there exists $s_1 \in \left(0,\frac{t}{\epsilon}\right)$ such that $\eta(s_1)=\frac{1}{3}$ and that $\eta([0,s_1)) \subset \left(-\frac{1}{3},\frac{1}{3}\right)$. Then,

$$\begin{aligned}\epsilon\int_0^{\frac{t}{\epsilon}} V(\eta(s)) + \frac{1}{2}|\dot\eta(s)|^2 ds &\geq \epsilon\left(\int_0^{s_1} V(\eta(s))ds + \frac{1}{2}\int_0^{s_1}|\dot\eta(s)|^2 ds\right)\\ &\geq \epsilon\left(s_1 + \frac{1}{2s_1}\left|\int_0^{s_1}\dot\eta(s)ds\right|^2\right) = \epsilon\left(s_1+\frac{1}{18s_1}\right) \geq \frac{\sqrt{2}}{3}\epsilon.\end{aligned}$$

Since u^ϵ converges to $u\equiv 0$ locally uniformly on $\mathbb{R}\times[0,\infty)$, we obtain (6.25). □

The following example explains why the assumption (H4) is needed.

Proposition 6.4.2. *Consider the Hamiltonian $H:\mathbb{R}\times\mathbb{T}\times\mathbb{R}\to\mathbb{R}$ defined by*

$$H(x,y,p) := -f(x) - W(y) + \frac{|p|^2}{2}$$

where $f:\mathbb{R}\to\mathbb{R}$ is defined by

$$f(x) := \begin{cases} |x|^{\frac{1}{4}}, & \text{if } |x|\leq 1,\\ 1, & \text{if } |x|>1,\end{cases}$$

and $W:\mathbb{T}\to\mathbb{R}$ is defined by

$$W(y) = \frac{1}{2} - |y|, \qquad \text{if } y \in \left[-\frac{1}{2},\frac{1}{2}\right].$$

Then, for $\epsilon \in (0,2^{-80})$, the corresponding solutions u^ϵ to (6.1) and u to (6.2) with

$g \equiv 0$ *satisfy*

$$u^{\epsilon}(0,1) - u(0,1) \geq \frac{1}{32}\epsilon^{\frac{1}{4}}.$$

Proof. Let $H_1 : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ be defined by $H_1(y,p) := -W(y) + \frac{|p|^2}{2}$. Then, the effective Hamiltonian $\overline{H}_1$ of H_1 is

$$\overline{H}_1(p) := \begin{cases} 0, & \text{if } |p| \leq \dfrac{2}{3} \\ \lambda, & \text{if } |p| \geq \dfrac{2}{3}, \text{ where } \lambda > 0 \text{ is a solution of } 2\sqrt{2}\displaystyle\int_0^{\frac{1}{2}} \sqrt{\lambda + \frac{1}{2} - y}\, dy = |p|. \end{cases}$$

In particular, for the Legendre transform of $\overline{H}_1$, denoted by $\overline{L}_1$, we know $\overline{L}_1(0) = 0$ (see [77], [98]). The effective Hamiltonian $\overline{H}$ of H is

$$\overline{H}(x,p) = -f(x) + \overline{H}_1(p).$$

Hence, the optimal control formula of u is

$$u(x,t) = \inf\left\{ \int_0^t f(\eta(s)) + \overline{L}_1(\dot{\eta}(s))\, ds : \eta \in \mathrm{AC}([0,t];\mathbb{R}),\ \eta(t) = x \right\},$$

which implies $u(0,1) = 0$.

Let $\gamma : \left[0, \frac{1}{\epsilon}\right] \to \mathbb{R}$ be a minimizing curve of $u^{\epsilon}(0,1)$ such that

$$u^{\epsilon}(0,1) = \epsilon \int_0^{\frac{1}{\epsilon}} \left(f(\epsilon\gamma(s)) + W(\gamma(s)) + \frac{|\dot{\gamma}(s)|^2}{2} \right) ds$$

with $\gamma(0) = 0$. If there exists a subinterval $[t_0, t_1] \subset \left[0, \frac{1}{\epsilon}\right]$ with $t_0 < t_1$ such that $\gamma(t_0) = \gamma(t_1) = \frac{1}{2}$, $\gamma((t_0,t_1)) \subset \left(\frac{1}{2}, +\infty\right)$, then the curve $\gamma_1 : \left[0, \frac{1}{\epsilon}\right] \to \mathbb{R}$ defined by $\gamma_1(t) = \frac{1}{2}$ for $t \in [t_0, t_1]$ and $\gamma_1(t) = \gamma(t)$ for $t \notin [t_0, t_1]$ would result in a lower value. The other cases, such as when $\gamma(t) > \frac{1}{2}$ for all $t > t_2$ for some $t_2 \in \left[0, \frac{1}{\epsilon}\right]$ and when with $-\frac{1}{2}$ instead of $\frac{1}{2}$, are similar. Therefore, we necessarily have $\gamma\left(\left[0, \frac{1}{\epsilon}\right]\right) \subset \left[-\frac{1}{2}, \frac{1}{2}\right]$. Consider the following two cases.

1. For any $s \in \left[0, \frac{1}{\epsilon}\right]$ such that $\gamma(s) \in \left[-\frac{1}{2} + \epsilon^{\frac{1}{4}}, \frac{1}{2} - \epsilon^{\frac{1}{4}}\right]$, there holds

$$f(\epsilon\gamma(s)) + W(\gamma(s)) \geq \frac{1}{2} - \left(\frac{1}{2} - \epsilon^{\frac{1}{4}}\right) = \epsilon^{\frac{1}{4}}.$$

2. For any $s \in \left[0, \frac{1}{\epsilon}\right]$ such that $\gamma(s) \in \left[-\frac{1}{2}, -\frac{1}{2} + \epsilon^{\frac{1}{4}}\right] \cup \left[\frac{1}{2} - \epsilon^{\frac{1}{4}}, \frac{1}{2}\right]$, there holds

$$f(\epsilon\gamma(s)) + W(\gamma(s)) \geq \left(\epsilon\left(\frac{1}{2} - \epsilon^{\frac{1}{4}}\right)\right)^{\frac{1}{4}} = \frac{1}{16}\epsilon^{\frac{1}{4}} - \epsilon^{\frac{5}{16}} \geq \frac{1}{32}\epsilon^{\frac{1}{4}},$$

   if $\epsilon \in (0, 2^{-80})$.

Therefore,

$$u^{\epsilon}(0,1) = \epsilon \int_0^{\frac{1}{\epsilon}} \left(f(\epsilon\gamma(s)) + W(\gamma(s)) + \frac{|\dot{\gamma}(s)|^2}{2}\right) ds \geq \frac{1}{32}\epsilon^{\frac{1}{4}},$$

that is,

$$u^{\epsilon}(0,1) - u(0,1) \geq \frac{1}{32}\epsilon^{\frac{1}{4}}.$$

□

Finally, we illustrate the necessity of involving the time variable t in (6.8) and the discount coefficient λ in (6.11) in the following example.

Proposition 6.4.3. *Let $n = 1$ and $H(x, y, p) = -f(x) - W(y) + \frac{1}{2}p^2$, where $f : \mathbb{R} \to \mathbb{R}$ is defined by*

$$f(x) := \begin{cases} |x|, & \text{if } |x| \leq 1, \\ 1, & \text{if } |x| > 1, \end{cases}$$

and $W : \mathbb{T} \to \mathbb{R}$ is defined by

$$W(y) = \frac{1}{2} - |y|, \quad \text{if } y \in \left[-\frac{1}{2}, \frac{1}{2}\right].$$

(i) Let u^{ϵ} be the unique solution to (6.1)*, and let u be the unique solution to* (6.2) *with*

$g \equiv 0$, respectively. Then, for $\epsilon \in (0,1)$,

$$u^{\epsilon}(0,t) - u(0,t) \geq \frac{1}{2}\epsilon t$$

for all $t > 0$.

(ii) Let u^{ϵ} be the unique solution to (6.9)*, and let u be the unique solution to* (6.10)*. Then, for $\lambda, \epsilon \in (0,1)$,*

$$u^{\epsilon}(0) - u(0) \geq \frac{\epsilon}{2\lambda}.$$

Proof. We adopt the notations introduced in the proof of Proposition 6.4.2. We give a proof of (i) and that of (ii) in order.

(i) By the fact that $\overline{L}_1(v) \geq 0$ for all $v \in \mathbb{R}$ and $\overline{L}_1(0) = 0$, and by the optimal control formula of u,

$$u(x,t) = \inf\left\{\int_0^t f(\eta(s)) + \overline{L}_1(\dot{\eta}(s))\, ds : \eta \in \mathrm{AC}([0,t];\mathbb{R}),\ \eta(t) = x\right\},$$

we conclude that $u(0,t) = 0$ for all $t > 0$.

On the other hand, note that the function $x \in \mathbb{R} \mapsto f(x) + W\left(\frac{x}{\varepsilon}\right)$ has minimum value $\frac{1}{2}\epsilon$. Therefore, by the optimal control formula of u^{ϵ},

$$u^{\epsilon}(x,t) = \inf\left\{\int_0^t f(\eta(s)) + W\left(\frac{\eta(s)}{\epsilon}\right) + \frac{1}{2}|\dot{\eta}(s)|^2 ds : \eta \in \mathrm{AC}([0,t];\mathbb{R}),\ \eta(t) = x\right\},$$

we see that $u^{\epsilon}(0,t) \geq \frac{1}{2}\epsilon t$.

(ii) Due to the same reason, we see that $u(0) = 0$. Also, by the optimal control formula of u^{ϵ},

$$\begin{aligned} &u^{\epsilon}(x) \\ =&\inf\left\{\int_0^{\infty} e^{-\lambda s}\left(f(\eta(s)) + W\left(\frac{\eta(s)}{\epsilon}\right) + \frac{1}{2}|\dot{\eta}(s)|^2\right) ds : \eta \in \mathrm{AC}([0,+\infty);\mathbb{R}^n),\ \eta(0) = x\right\} \\ \geq&\frac{1}{2}\epsilon\int_0^{\infty} e^{-\lambda s} ds = \frac{\epsilon}{2\lambda}. \end{aligned}$$

This completes the proof. □

Appendix

In this appendix, we prove Lemma 6.1.1 with emphasis on the dependence on parameters. The goal is to show the constant $C > 0$ appearing in the conclusion of the theorem is independent of the choice of $c \in \mathbb{R}^n$. Before we move into the proof, we set the following notation for convenience; for $c, x, y \in \mathbb{R}^n$ and $t > 0$, we let

$$m_c(t, x, y) := m_c^1(0, t, x, y)$$

where the right-hand side $m_c^1(0, t, x, y)$ is as defined in Definition 6.1.1 with $\epsilon = 1$.

First of all, from [97, Lemma 3.2], we see that for any $y \in \mathbb{R}^n,\ \epsilon, t > 0$ with $|y| \leq M_0 t$, there exists a constant $C = C(n, M_0, K_0) > 0$ such that

$$2m_c(t, 0, y) \leq m_c(2t, 0, 2y) + C,$$

which results in one direction of the conclusion, i.e.,

$$m_c^\epsilon(0, t, a, b) \leq \overline{m}_c(0, t, a, b) + C\epsilon,$$

for any $a, b \in \mathbb{R}^n$, $\epsilon, t > 0$ with $|b - a| \leq M_0 t$. The independence of $C > 0$ on $c \in \mathbb{R}^n$ is well shown by the argument of the proof of [97, Lemma 3.2] together with (6.4), (6.5), which hold with a constant K_0 uniform in $c \in \mathbb{R}^n$ under the assumption (H1) in this chapter. Thus, we skip the proof, and focus on the other direction instead.

Next, we show the other direction by verifying that for any $c \in \mathbb{R}^n$, and for any $y \in \mathbb{R}^n,\ \epsilon, t > 0$ with $|y| \leq M_0 t$, there exists a constant $C = C(n, M_0, K_0) > 0$ such that

$$m_c(2t, 0, 2y) \leq 2m_c(t, 0, y) + C,$$

which completes the proof of Lemma 6.1.1 and also shows the independence of the constant on $c \in \mathbb{R}^n$.

Lemma 6.4.1. *Assume* (H1)-(H3)*. Let* $M_0 > 0$*, and let* $K_0 > 0$ *be a constant that satisfies* (6.4), (6.5)*. Then, there exists a constant* $C = C(n, M_0, K_0) > 0$ *such that for any* $c \in \mathbb{R}^n$*,* $t > 0$ *and any* $y \in \mathbb{R}^n$ *such that* $|y| \leq M_0 t$*, we have*

$$m_c(2t, 0, 2y) \leq 2m_c(t, 0, y) + C.$$

Proof. Since $m_c(2t, 0, 2y) \leq m_c(t, 0, y) + m_c(t, y, 2y)$, it suffices to show that for some constant $C = C(n, M_0, K_0) > 0$, it holds that

$$m_c(t, y, 2y) \leq m_c(t, 0, y) + C.$$

1. If $t \leq 6$, then, by considering $\alpha : [0, t] \to \mathbb{R}^n$ defined by $\alpha(s) = y + \frac{s}{t} y$ for $s \in [0, t]$, we obtain, by (6.5),

   $$m_c(t, y, 2y) \leq \int_0^t \frac{1}{2} M_0^2 + K_0 ds \leq 3M_0^2 + 6K_0.$$

   Also,

   $$m_c(t, 0, y) \geq \int_0^t -K_0 ds \geq -6K_0.$$

   Hence, $m_c(t, y, 2y) \leq m_c(t, 0, y) + 3M_0^2 + 12K_0$.

2. If $t > 6$, let $\zeta : [0, t] \to \mathbb{R}^n$ be an absolutely continuous curve with $\zeta(0) = 0, \zeta(t) = y$ such that

   $$\int_0^t L(c, \zeta(s), -\dot{\zeta}(s)) ds \leq m_c(t, 0, y) + 1.$$

   By considering a straight line $\alpha : [0, t] \to \mathbb{R}^n$ defined by $\alpha(s) = \frac{s}{t} y$ for $s \in [0, t]$, we see that

   $$\int_0^t L(c, \zeta(s), -\dot{\zeta}(s)) ds \leq \int_0^t \frac{1}{2} M_0^2 + K_0 ds + 1 = \left(\frac{1}{2} M_0^2 + K_0 \right) t + 1.$$

We claim that there exists a number $d \in \left\{\frac{3}{2}k : 0 \le k < \lfloor \frac{2}{3}t \rfloor, k \in \mathbb{Z}\right\}$ such that

$$\int_d^{d+\frac{3}{2}} L(c, \zeta(s), -\dot{\zeta}(s))ds \le M_0^2 + 3K_0 + 1.$$

Otherwise, we would have

$$\int_0^{\frac{3}{2}\lfloor \frac{2}{3}t \rfloor} L(c, \zeta(s), -\dot{\zeta}(s))ds > \left\lfloor \frac{2}{3}t \right\rfloor \left(M_0^2 + 3K_0 + 1\right),$$

which then leads to

$$\begin{aligned}\left(\frac{1}{2}M_0^2 + K_0\right) t + 1 &\ge \int_0^{\frac{3}{2}\lfloor \frac{2}{3}t \rfloor} L(c, \zeta(s), -\dot{\zeta}(s))ds + \int_{\frac{3}{2}\lfloor \frac{2}{3}t \rfloor}^{t} L(c, \zeta(s), -\dot{\zeta}(s))ds \\ &> \left(\frac{2}{3}t - 1\right)\left(M_0^2 + 3K_0 + 1\right) - \left| t - \frac{3}{2}\left\lfloor \frac{2}{3}t \right\rfloor \right| K_0 \\ &> \left(\frac{2}{3}t - 1\right)\left(M_0^2 + 3K_0 + 1\right) - 3K_0.\end{aligned}$$

This is absurd for $t > 6$.

Let $w \in \mathbb{Z}^n$ such that $y - w \in [0, 1]^n$. Define a new curve $\tilde{\zeta} : [0, t] \to \mathbb{R}^n$ by

$$\tilde{\zeta}(s) := \begin{cases} w + \dfrac{\frac{1}{2} - s}{\frac{1}{2}}(y - w), & 0 \le s \le \dfrac{1}{2}, \\ \zeta\left(s - \dfrac{1}{2}\right) + w, & \dfrac{1}{2} \le s \le d + \dfrac{1}{2}, \\ \zeta\left(d + 3\left(s - \left(d + \dfrac{1}{2}\right)\right)\right) + w, & d + \dfrac{1}{2} \le s \le d + 1, \\ \zeta\left(s + \dfrac{1}{2}\right) + w, & d + 1 \le s \le t - \dfrac{1}{2}, \\ y + w + \dfrac{s - t + \frac{1}{2}}{\frac{1}{2}}(y - w), & t - \dfrac{1}{2} \le s \le t. \end{cases}$$

Then, $m_c(t, y, 2y) \le \int_0^t L(c, \tilde{\zeta}(s), -\dot{\tilde{\zeta}}(s))ds$ since $\tilde{\zeta}$ is an absolutely continuous curve from y to $2y$. From the definition of $\tilde{\zeta}$, we see that

$$\int_0^{\frac{1}{2}} L(c, \tilde{\zeta}(s), -\dot{\tilde{\zeta}}(s))ds + \int_{t-\frac{1}{2}}^{t} L(c, \tilde{\zeta}(s), -\dot{\tilde{\zeta}}(s))ds \le 2n + K_0$$

by (6.5), and that

$$\begin{aligned}
&\int_{\frac{1}{2}}^{d+\frac{1}{2}} L(c,\tilde{\zeta}(s),-\dot{\tilde{\zeta}}(s))ds + \int_{d+1}^{t-\frac{1}{2}} L(c,\tilde{\zeta}(s),-\dot{\tilde{\zeta}}(s))ds \\
=&\int_0^d L(c,\zeta(s),-\dot{\zeta}(s))ds + \int_{d+\frac{3}{2}}^t L(c,\zeta(s),-\dot{\zeta}(s))ds \\
\leq& m_c(t,0,y)+1-\int_d^{d+\frac{3}{2}} L(c,\zeta(s),-\dot{\zeta}(s))ds \\
\leq& m_c(t,0,y)+\frac{3}{2}K_0+1
\end{aligned}$$

by the fact that $L(x,y,v)$ is periodic in y and (6.5) again. Finally, by the change of variables and by the choice of the number d, we get

$$\begin{aligned}
\int_{d+\frac{1}{2}}^{d+1} L(c,\tilde{\zeta}(s),-\dot{\tilde{\zeta}}(s))ds &= \frac{1}{3}\int_d^{d+\frac{3}{2}} L(c,\zeta(s),-3\dot{\zeta}(s))ds \\
&\leq \frac{1}{2}K_0 + \frac{3}{2}\int_d^{d+\frac{3}{2}} \left|\dot{\zeta}(s)\right|^2 ds \\
&\leq 5K_0 + 3\int_d^{d+\frac{3}{2}} L(c,\zeta(s),-\dot{\zeta}(s))ds \leq 3M_0^2+14K_0+3.
\end{aligned}$$

All in all, in the case when $t > 6$, we see that there exists a constant $C = C(n,M_0,K_0) > 0$ such that

$$m_c(t,y,2y) \leq m_c(t,0,y)+C,$$

and we complete the proof.

□

Chapter 7

On a minimum eradication time for the SIR model with time-dependent coefficients

7.1 Introduction

We are interested in studying an eradication time for the controlled Susceptible-Infectious-Recovered (SIR in short henceforth) model with time-varying rates $\beta(t)$ and $\gamma(t)$:

$$\begin{cases} \dot{S} &= -\beta(t)SI - \alpha(t)S, \\ \dot{I} &= \beta(t)SI - \gamma(t)I, \end{cases}$$

where $\beta(t)$ and $\gamma(t)$ denote a time-dependent infected/recovery rate, respectively, and $\alpha(t)$ represents a vaccination control. The goal of this chapter is to study the mathematical properties of the value function that represents the minimum eradication time, defined by the first time at which the population I of infectious is less than or equal to μ and remains below afterward for a given small threshold $\mu > 0$. It turns out that the eradication time should be defined carefully, and its precise definition will be given in Subsection 7.1.2.

For time-independent rates $\beta, \gamma > 0$, the minimum eradication time is always well-defined as I shows a simple behavior, either decreasing or increasing first and decreasing afterward. However, for our case, more careful analysis should be carried out as the number of infectious individuals I can oscillate; for instance, even after I goes below a given threshold $\mu > 0$, it can bounce up and down several times.

In this regard, for time-varying rates $\beta(t)$ and $\gamma(t)$, given $\mu_0 > 0$, the selection of the threshold parameter, denoted as μ, plays a crucial role in accurately identifying the minimum time at which the variable I crosses μ and remains below this threshold for the duration of the observation as long as $I(0) \geq \mu_0$. More precisely, when I oscillates around the threshold, one can observe the ambiguity and the discontinuity of the eradication time as demonstrated in Figure 7.1. This chapter proves with the compactness argument that given any μ_0, we can select μ small enough so that the ambiguity and instability of the two types of eradication are avoided as long as $I(0) \geq \mu_0$. Furthermore, we also present the time-dependent Hamilton–Jacobi equation associated with as well as the local semiconcavity result.

7.1.1 Literature review

We introduce a list, but by no means complete, of the works on the vaccination strategy and the eradication time for SIR epidemic models. The SIR model is a classical model as studied in [67], and its variants have received a lot of attention particularly during and after the outbreak of COVID-19. The vaccination strategy as a control and the eradication time as a minimum cost function were investigated with optimal control theory [9, 11, 12, 53]. For numerical simulations of the eradication for the time-varying SIR model, we refer to [23]. The minimum eradication time problem in the aspect of free end-time optimal control problem was first studied by [12] where the authors claim that the optimal plan is to remain inactive and provide the maximum control after a certain point, which is called switching control.

In [53], various sufficient conditions to ensure the eradication of disease for a time-varying SIR model were provided under some structural assumptions on the dynamics and the transmission rate such as periodicity. Another interesting work related to our chapter is [79] where the authors study the eradication time for the Susceptible-Exposed-Infected-Susceptible compartmental model under the constraint of resources. In their paper, it was shown that the optimal vaccination control is indeed bang-bang control and there is a trade-off between the minimum eradication time and the total resources under the assumption that all parameters in the model are constants.

For mathematical treatments, the eradication time for controlled SIR models with constant infected and recovery rates was first studied as a viscosity solution to a static first-order Hamilton-Jacobi equation in [59]. Also, a critical time at which the infected population starts decreasing was analyzed in [60]. The works [59, 60] are for the SIR model with constant rates β and γ.

To the best of our knowledge, this is the first work that studies the minimum eradication time for time-dependent SIR epidemic models in the framework of the dynamic programming principle and viscosity solutions. Also, we observe that for an arbitrarily given threshold, denoted by μ below, we may not have a unique description of the eradication time in time-varying environments, and we show that for a suitable choice of μ, we necessarily have a unique definition of the eradication time and that this enjoys mathematical properties (such as the continuity and the semiconcavity). For this purpose, we separate the threshold μ from an initial population $I(0)$ of infectious, and this may suggest that with time-dependent rates, μ needs to be small enough compared to $I(0)$ for simulations, where the continuity is implicitly assumed.

7.1.2 Notations

We fix $\mu > 0$, $\overline{\beta} \geq \underline{\beta} > 0$, $\overline{\gamma} \geq \underline{\gamma} > 0$ and continuous functions $\beta : [0,\infty) \to [\underline{\beta}, \overline{\beta}]$, $\gamma : [0,\infty) \to [\underline{\gamma}, \overline{\gamma}]$ throughout this chapter. Let us define the set $\mathcal{A}$ of admissible controls and the data set $\mathcal{D}$ as follows:

$$\mathcal{A} := \{\alpha \in L^\infty([0,\infty)) : 0 \leq \alpha(t) \leq 1 \text{ a.e. } t \geq 0\},$$
$$\mathcal{D} := [0,\infty) \times [\mu,\infty) \times [0,\infty) \times \mathcal{A}.$$

The set $\mathcal{A}$ is endowed with the weak* topology inherited from that of $L^\infty([0,\infty))$. The intervals $[0,\infty), [\mu,\infty), [0,\infty)$ are endowed with their usual topologies, and the data set

$\mathcal{D}$ is endowed with their product topology.

For a given datum $d = (x, y, t, \alpha) \in \mathcal{D}$, we define (S^d, I^d) to be the flow of the following ODE:

$$\begin{cases} \dot{S}^d &= -\beta^t S^d I - \alpha^t S^d, \\ \dot{I}^d &= \beta^t S^d I^d - \gamma^t I^d, \\ S^d(0) &= x, \\ I^d(0) &= y, \end{cases} \tag{7.1}$$

where $\alpha^t = \alpha(\cdot + t)$, $\beta^t = \beta(\cdot + t)$, $\gamma^t = \gamma(\cdot + t)$. By (S, I) the flow associated with a datum d, we mean $(S, I) = (S^d, I^d)$ in this chapter. When the associated datum d is clear in the context, we abbreviate the superscript d in (S^d, I^d).

For a given datum $d = (x, y, t, \alpha) \in \mathcal{D}$, we define the upper (lower) value functions $\overline{u}^\alpha(x, y, t)$ $(\underline{u}^\alpha(x, y, t))$, respectively, by

$$\overline{u}^\alpha(x, y, t) := \sup\{s \geq 0 : I^d(s) \geq \mu\},$$
$$\underline{u}^\alpha(x, y, t) := \inf\{s \geq 0 : I^d(s + a) \leq \mu, \forall a \geq 0\}.$$

For $(x, y, t) \in [0, \infty) \times [\mu, \infty) \times [0, \infty)$, we let

$$\overline{u}(x, y, t) := \inf_{\alpha \in \mathcal{A}} \overline{u}^\alpha(x, y, t),$$
$$\underline{u}(x, y, t) := \inf_{\alpha \in \mathcal{A}} \underline{u}^\alpha(x, y, t).$$

It turns out that the value functions $\overline{u}, \underline{u}$ enjoy the following important properties, which are our main contributions and are stated in the following subsection.

7.1.3 Main results

Theorem 7.1.1. *The value function $\overline{u}$ ($\underline{u}$, resp.) is upper semicontinuous (lower semicontinuous, resp.) on $[0, \infty) \times [\mu, \infty) \times [0, \infty)$. Moreover, $\overline{u}$ ($\underline{u}$, resp.) is a viscosity* *subsolution (supersolution, resp.) to*

$$-\partial_t u + \beta(t)xy\partial_x u + x(\partial_x u)_+ + (\gamma(t) - \beta(t)x)y\partial_y u = 1 \tag{7.2}$$

in $(0,\infty)\times(\mu,\infty)\times(0,\infty)$. *Here,* $(\cdot)_+$ *denotes the positive part of the argument.*

The functions $\overline{u}, \underline{u}$ are natural in this aspect. However, Figure 7.1 indicates the discrepancy of $\overline{u}$ and $\underline{u}$, meaning we might not have $\overline{u} = \underline{u}$. This ambiguity is not observed in the time-independent SIR model studied in [59].

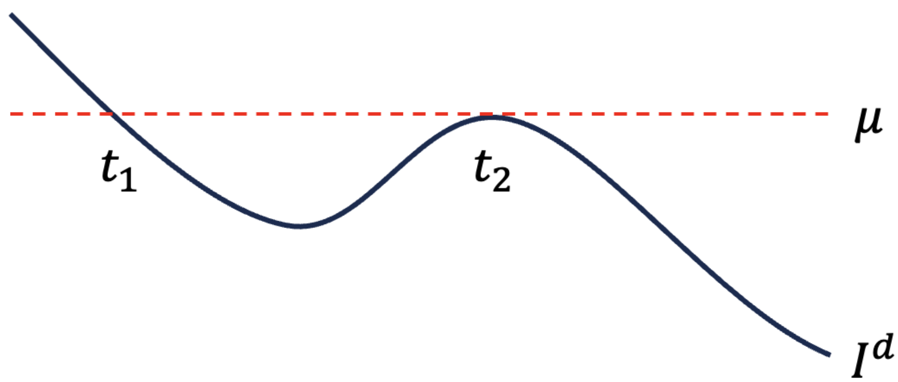

Figure 7.1: Two types of eradication time

However, we can resolve this ambiguity by taking the following viewpoint; for an initial infected population $I(0)$ that is noticeable, say greater than or equal to μ_0, we require the threshold μ be much smaller, depending on μ_0 (for instance, at least smaller than μ_0). This perspective aligns with the practical goal of vaccination intervention, aiming to control the spread of disease within the population, especially when controlling the number of infectious individuals under a small threshold.

We start with a fixed $\mu_0 > 0$. The next result states that for $\mu \in (0, \mu_0]$ small enough, we have $\overline{u} = \underline{u}$, which now becomes a viscosity solution to (7.2) in $(0,\infty)\times(\mu_0,\infty)\times(0,\infty)$. Also, the value function $u := \overline{u} = \underline{u}$ is characterized by its boundary value conditions. Note that we only assume $\underline{\beta} \le \beta(t) \le \overline{\beta}$, $\underline{\gamma} \le \gamma(t) \le \overline{\gamma}$ for $t \ge 0$, allowing an oscillatory behavior.

Theorem 7.1.2. *There exists* $\mu_1 \in (0, \mu_0]$ *depending only on* μ_0, β, γ *such that for every*

$\mu \in (0.\mu_1]$, it holds that $\overline{u} = \underline{u}$ on $[0,\infty) \times [\mu_0,\infty) \times [0,\infty)$, and therefore, $u := \overline{u} = \underline{u}$ is a viscosity solution to (7.2) in $(0,\infty) \times (\mu_0,\infty) \times (0,\infty)$. Moreover, if v is a viscosity solution to (7.2) and satisfies the boundary conditions

$$\begin{cases} v(x,\mu_0,t) = u(x,\mu_0,t) & \textit{for} \quad (x,t) \in \left[0, \overline{\gamma}/\underline{\beta}\right] \times [0,\infty), \\ v(0,y,t) = u(0,y,t) & \textit{for} \quad (y,t) \in [\mu_0,\infty) \times [0,\infty), \end{cases}$$

then we have $v = u$.

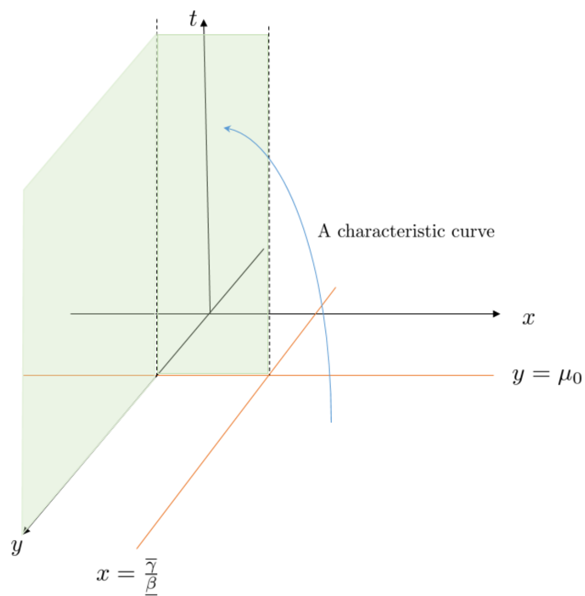

Figure 7.2: Effective boundary (the shaded region in light green color)

Although (7.2) has $-\partial_t u$ instead of $\partial_t u$, the notion of viscosity solutions to (7.2) is the same as the usual one to forward Cauchy problems, for which we refer to [98, Chapter 1].

It is worth noting that the Hamiltonian

$$H(t,x,y,p,q) = \beta(t)xyp + xp_+ + (\gamma(t) - \beta(t)x)yq$$

is positively homogeneous of degree 1, and thus, (7.2) has a hidden underlying front propagation structure [98]. Note that a growth condition is not necessary, as the equation basically is about the front propagation of level-sets, rather than the value function itself.

We compose with a bounded monotone function without loss of generality, which does not change the arrangement of level-sets in principle. Furthermore, only a part of the boundary is needed for the uniqueness result in Theorem 7.1.2. The front propagation nature and the boundary condition deserve further study.

We state a further regularity property when the transmission rate β and the recovery rate γ become constant in a small time.

Theorem 7.1.3. *Let $\mu \in (0, \mu_0]$ be chosen as in Theorem 7.1.2 so that $\overline{u} = \underline{u} (= u)$. Suppose that $\beta(t) \equiv \beta_0$, $\gamma(t) \equiv \gamma_0$ for all $t \geq T := \frac{1}{2\overline{\gamma}} \log\left(\frac{\mu_0}{\mu}\right)$ for some constants β_0, γ_0. Then, $u(x, y, t)$ is locally semiconcave in $(0, \infty) \times (\mu_0, \infty) \times (0, \infty)$.*

Organization of the chapter.

The chapter is organized as follows. In Section 7.2, we review the basic results of the flow (7.1) and prove Theorem 7.1.1. Section 7.3 is entirely devoted to the proof of the existence of $\mu \in (0, \mu_0]$ satisfying $\overline{u} = \underline{u}$ on $[0, \infty) \times [\mu_0, \infty) \times [0, \infty)$. In Section 7.4, we complete the proof of Theorem 7.1.2 by verifying the uniqueness (Theorem 7.4.1), and we also prove Theorem 7.1.3.

7.2 Properties of $\overline{u}$ and $\underline{u}$

In this section, we go over the basic properties of the flow of (7.1). Then, we investigate the semicontinuity of the value functions $\overline{u}$, $\underline{u}$. Finally, we check the dynamic programming principle and viscosity sub/supersolution tests. The main reference is [59], and we skip similar proofs. The properties coming from the split of $\overline{u}$ and $\underline{u}$ will be explained.

7.2.1 Flow of (7.1)

Lemma 7.2.1. *For any $d = (x, y, t, \alpha) \in \mathcal{D}$, there is a unique flow (S, I) associated with d. The flow (S, I) is Lipschitz continuous. Namely, we have*

$$|\dot{S}|, |\dot{I}| \leq \overline{\beta}(x + y)^2 + \max\{1, \overline{\gamma}\}(x + y).$$

Lemma 7.2.2. *For any $d = (x, y, t, \alpha) \in \mathcal{D}$, the associated flow (S, I) satisfies $\lim_{t\to\infty} I(t) = 0$.*

Proposition 7.2.1. *Let $d_k = (x_k, y_k, t_k, \alpha_k) \in \mathcal{D}$ and (S_k, I_k) be the associated flow for each $k = 0, 1, 2, \cdots$. Suppose that $d_k \to d_0$ as $k \to \infty$ in $\mathcal{D}$. Then, $(S_k(t), I_k(t)) \to (S_0(t), I_0(t))$ as $k \to \infty$ locally uniformly in $t \in [0, \infty)$.*

Now, we state the existence of an optimal control associated with the value function $\underline{u}$. As this property is expected for $\overline{u}$, we give a proof.

Proposition 7.2.2. *For any $(x, y, t) \in [0, \infty) \times [\mu, \infty) \times [0, \infty)$, there exists $\alpha^* \in \mathcal{A}$ such that*

$$\underline{u}^{\alpha^*}(x, y, t) \leq \underline{u}^{\alpha}(x, y, t)$$

for any $\alpha \in \mathcal{A}$.

Proof. Choose a sequence $\{\alpha_k\}_{k=1,2,\cdots}$ in $\mathcal{A}$ such that $\inf_{\alpha\in\mathcal{A}} \underline{u}^{\alpha}(x, y, t) = \lim_{k\to\infty} \underline{u}^{\alpha_k}(x, y, t)$. As the space $\mathcal{A}$ is (sequentially) weak* compact, there is a subsequence $\{\alpha_{k_j}\}_{j=1,2,\cdots}$ of $\{\alpha_k\}_{k=1,2,\cdots}$ such that $\alpha_{k_j} \to \alpha_0$ weak* for some $\alpha_0 \in \mathcal{A}$ as $j \to \infty$.

For each $j = 0, 1, 2, \cdots$, let (S_j, I_j) be the flow associated with the datum $(x, y, t, \alpha_{k_j}) \in \mathcal{D}$ with $k_0 := 0$ Let $a \geq 0$. Then, by the definition of $\underline{u}^{\alpha}$ with general control $\alpha \in \mathcal{A}$, we have $I_j(\underline{u}^{\alpha_{k_j}} + a) \leq \mu$ for all $j = 1, 2, \cdots$. Taking the limit $j \to \infty$, we obtain that $I_0(\inf_{\alpha\in\mathcal{A}} \underline{u}^{\alpha} + a) \leq \mu$. Since $a \in [0, \infty)$ was arbitrary, we conclude $\underline{u}^{\alpha_0}(x, y, t) \leq \inf_{\alpha\in\mathcal{A}} \underline{u}^{\alpha}(x, y, t)$ from the definition of $\underline{u}^{\alpha_0}(x, y, t)$. □

7.2.2 Semicontinuity

The following states the semicontinuity of $\underline{u}^{\alpha}(x, y, t), \overline{u}^{\alpha}(x, y, t)$.

Lemma 7.2.3. *Let $d_k = (x_k, y_k, t_k, \alpha_k) \in \mathcal{D}$ for $k = 1, 2, \cdots$. Suppose that $d_k \to d$ as $k \to \infty$ in $\mathcal{D}$ for some $d = (x, y, t, \alpha) \in \mathcal{D}$. Then,*

$$\underline{u}^{\alpha}(x, y, t) \leq \liminf_{k\to\infty} \underline{u}^{\alpha_k}(x_k, y_k, t_k)$$

and

$$\limsup_{k\to\infty} \overline{u}^{\alpha_k}(x_k, y_k, t_k) \le \overline{u}^{\alpha}(x, y, t).$$

We skip the proof, as it is identical to that of [59, Corollary 3.2]. Now, we prove the semicontinuity of $\underline{u}(x, y, t), \overline{u}(x, y, t)$.

Proposition 7.2.3. *The value function* $\underline{u}$ *is lower semicontinuous on* $[0, \infty) \times [\mu, \infty) \times [0, \infty)$*. Also, the value function* $\overline{u}$ *is upper semicontinuous on* $[0, \infty) \times [\mu, \infty) \times [0, \infty)$*.*

Proof. Say $(x_k, y_k, t_k) \to (x, y, t)$ as $k \to \infty$ in $[0, \infty) \times [\mu, \infty) \times [0, \infty)$. For each $k = 1, 2, \cdots$, choose $\alpha_k \in \mathcal{A}$ such that $\underline{u}(x_k, y_k, t_k) = \underline{u}^{\alpha_k}(x_k, y_k, t_k)$, which is possible due to Proposition 7.2.2. Select a subsequence $\{\alpha_{k_j}\}_j$ such that $\liminf_{k\to\infty} \underline{u}(x_k, y_k, t_k) = \lim_{j\to\infty} \underline{u}^{\alpha_{k_j}}(x_{k_j}, y_{k_j}, t_{k_j})$ and that $\alpha_{k_j} \to \alpha$ weak* as $j \to \infty$ for some $\alpha \in \mathcal{A}$. Then, by the above lemma, we get

$$\liminf_{k\to\infty} \underline{u}(x_k, y_k, t_k) = \lim_{j\to\infty} \underline{u}^{\alpha_{k_j}}(x_{k_j}, y_{k_j}, t_{k_j}) \ge \underline{u}^{\alpha}(x, y, t) \ge \underline{u}(x, y, t).$$

Let $\delta > 0$ be given. Then, we can choose $\alpha \in \mathcal{A}$ such that $\overline{u}^{\alpha}(x, y, t) < \overline{u}(x, y, t) + \delta$. For a sequence (x_k, y_k, t_k) that converges to (x, y, t) in $[0, \infty) \times [\mu, \infty) \times [0, \infty)$, we have

$$\limsup_{k\to\infty} \overline{u}(x_k, y_k, t_k) \le \limsup_{k\to\infty} \overline{u}^{\alpha}(x_k, y_k, t_k) \le \overline{u}^{\alpha}(x, y, t) < \overline{u}(x, y, t) + \delta.$$

Here we used the above lemma. Letting $\delta \to 0$ gives the upper semicontinuity. □

7.2.3 Dynamic programming principle and a viscosity sub/supersolution

For $d = (x, y, s, \alpha) \in \mathcal{D}$, we have the dynamic programming principle: for $t \in [0, \inf\{t_1 \ge 0 : I^d(t_1) = \mu\}]$, we have

$$\overline{u}^{\alpha}(x, y, s) = t + \overline{u}^{\alpha}(S^d(t), I^d(t), t + s),$$
$$\underline{u}^{\alpha}(x, y, s) = t + \underline{u}^{\alpha}(S^d(t), I^d(t), t + s).$$

This is also true for $\overline{u}, \underline{u}$, as stated in the next proposition.

Proposition 7.2.4. *Let* $x \geq 0, y \geq \mu, s \geq 0$*. If* $t \in [0, \inf_{\alpha \in \mathcal{A}} \inf\{t_1 \geq 0 : I^d(t_1) = \mu\}]$ *with* $d = (x, y, s, \alpha)$*,*

$$\overline{u}(x, y, s) = \inf_{\alpha \in \mathcal{A}} \left\{ t + \overline{u}(S^d(t), I^d(t), t + s) \right\},$$
$$\underline{u}(x, y, s) = \inf_{\alpha \in \mathcal{A}} \left\{ t + \underline{u}(S^d(t), I^d(t), t + s) \right\}.$$

Moreover, for any control $\alpha^* \in \mathcal{A}$ *such that* $\underline{u}(x, y, s) = \underline{u}^{\alpha^*}(x, y, s)$*, we have*

$$\underline{u}(x, y, s) = t + \underline{u}(S^*(t), I^*(t), t + s)$$

and

$$\underline{u}(S^*(t), I^*(t), t + s) = \underline{u}^{\alpha^*}(S^*(t), I^*(t), t + s).$$

for $t \in [0, \inf\{t_1 \geq 0 : I^*(t_1) = \mu\}]$*. Here, the flow* (S^*, I^*) *is associated with* $(x, y, s, \alpha^*) \in \mathcal{D}$*.*

We omit the proof as it is basically the same as that of [59, Proposition 3.4]. As a corollary from the dynamic programming principle, we see that the value functions $\overline{u}$ and $\underline{u}$ are a viscosity sub and supersolution, respectively. Once we have the dynamic programming principle, we are able to verify naturally that $\overline{u}$ ($\underline{u}$, resp.) is a viscosity subsolution (supersolution, resp.) to (7.2) (see [37, 98]).

Corollary 7.2.1 (Theorem 7.1.1)**.** *The value function* $\overline{u}$ *(*$\underline{u}$*, resp.) is a viscosity subsolution (supersolution, resp.) to*

$$-\partial_t u + \beta(t)xy\partial_x u + x(\partial_x u)_+ + (\gamma(t) - \beta(t)x)y\partial_y u = 1$$

in $(0, \infty) \times (\mu, \infty) \times (0, \infty)$*.*

Proof. Fix the ball $B_\delta(x_0, y_0, t_0) \subset (0, \infty) \times (\mu, \infty) \times (0, \infty)$ with a center (x_0, y_0, t_0) and a radius $\delta > 0$. Let ϕ be a C^1 function defined in $B_\delta(x_0, y_0, t_0)$ such that $\overline{u} - \phi$ attains a maximum at (x_0, y_0, t_0) in $B_\delta(x_0, y_0, t_0)$. Let $a \in [0, 1]$ and $\alpha \equiv a \in \mathcal{A}$. Let (S, I)

be the flow associated with $(x_0, y_0, t_0, \alpha) \in \mathcal{D}$. As $y_0 > \mu$, we have $\overline{u}(x_0, y_0, t_0) > 0$. As (S, I) is continuous in time, there exists $t_1 > 0$ depending also on $\delta > 0$ such that $(S(t), I(t)) \in B_\delta(x_0, y_0, t_0)$ for all $t \in [0, t_1]$.

From

$$(\overline{u} - \phi)(S(t), I(t), t_0 + t) \leq (\overline{u} - \phi)(x_0, y_0, t_0)$$

for $t \in [0, \min\{\overline{u}(x_0, y_0, t_0), t_1\}]$, we obtain

$$\begin{aligned} -t &\leq \overline{u}(S(t), I(t), t_0 + t) - \overline{u}(x_0, y_0, t_0) \\ &\leq \phi(S(t), I(t), t_0 + t) - \phi(x_0, y_0, t_0). \end{aligned}$$

Here, we used Proposition 7.2.4 in the first line. This yields

$$\begin{aligned} -1 &\leq \frac{d}{dt}\phi(S(t), I(t), t + t_0)|\Big|_{t=0} \\ &= \partial_t\phi(x_0, y_0, t_0) - (\beta(t_0)x_0y_0 + ax_0)\partial_x\phi(x_0, y_0, t_0) - (\gamma(t_0) - \beta(t_0)x_0)y_0\partial_y\phi(x_0, y_0, t_0). \end{aligned}$$

Taking the supremum over $a \in [0, 1]$, we obtain

$$-\partial_t\phi(x_0, y_0, t_0) + \beta(t_0)x_0y_0\partial_x\phi(x_0, y_0, t_0) + x_0(\partial_x\phi(x_0, y_0, t_0))_+ + (\gamma(t_0) - \beta(t_0)x_0)y_0\partial_y\phi(x_0, y_0, t_0) \leq 1.$$

Now, let ψ be a C^1 function defined in $B_\delta(x_0, y_0, t_0)$ such that $\underline{u} - \psi$ attains a minimum at (x_0, y_0, t_0) in $B_\delta(x_0, y_0, t_0)$. Take $\alpha^* \in \mathcal{A}$ such that $\underline{u}(x_0, y_0, t_0) = \underline{u}^{\alpha^*}(x_0, y_0, t_0)$. Let (S, I) be the flow associated with $(x_0, y_0, t_0, \alpha^*) \in \mathcal{D}$. As $y_0 > \mu$, we have $\underline{u}(x_0, y_0, t_0) > 0$. As (S, I) is continuous in time, there exists $t_1 > 0$ depending also on $\delta > 0$ such that $(S(t), I(t)) \in B_\delta(x_0, y_0, t_0)$ for all $t \in [0, t_1]$.

By Proposition 7.2.4, we have

$$\underline{u}(x_0, y_0, t_0) = t + \underline{u}(S(t), I(t), t + t_0)$$

for $t \in [0, \underline{u}(x_0, y_0, t_0)]$. From

$$(\underline{u} - \psi)(S(t), I(t), t_0 + t) \geq (\underline{u} - \psi)(x_0, y_0, t_0)$$

for $t \in [0, \min\{\underline{u}(x_0, y_0, t_0), t_1\}]$, we obtain

$$\begin{aligned} -t &= \underline{u}(S(t), I(t), t_0 + t) - \underline{u}(x_0, y_0, t_0) \\ &\geq \psi(S(t), I(t), t_0 + t) - \psi(x_0, y_0, t_0). \end{aligned}$$

Consequently, the fact that $\alpha^*(s) \in [0, 1]$ for almost every $s \geq 0$ yields that, for $t \in [0, \min\{\underline{u}(x_0, y_0, t_0), t_1\}]$,

$$\begin{aligned} -1 &\geq \frac{1}{t}\int_0^t \frac{d}{ds}\psi(S(s), I(s), s + t_0)ds \\ &= \frac{1}{t}\int_0^t (\partial_t\psi(S(s), I(s), s + t_0) + (-\beta(s + t_0)S(s)I(s) - \alpha^*(s + t_0)S(s))\partial_x\psi(S(s), I(s), s + t_0) \\ &\qquad + (\beta(s + t_0)S(s)I(s) - \gamma(s + t_0)I(s))\partial_y\psi(S(s), I(s), s + t_0))\, ds \\ &\geq \frac{1}{t}\int_0^t (\partial_t\psi(S(s), I(s), s + t_0) - \beta(s + t_0)S(s)I(s)\partial_x\psi(S(s), I(s), s + t_0) \\ &\qquad - S(s)(\partial_x\psi(S(s), I(s), s + t_0))_+ + (\beta(s + t_0)S(s)I(s) - \gamma(s + t_0)I(s))\partial_y\psi(S(s), I(s), s + t_0))\, ds. \end{aligned}$$

Sending $t \to 0^+$ and rearranging the terms gives

$$-\partial_t\psi(x_0, y_0, t_0) + \beta(t_0)x_0y_0\partial_x\psi(x_0, y_0, t_0) + x_0(\partial_x\psi(x_0, y_0, t_0))_+ + (\gamma(t_0) - \beta(t_0)x_0)y_0\partial_y\psi(x_0, y_0, t_0) \geq 1.$$

□

7.3 A viscosity solution $u = \overline{u} = \underline{u}$ from the choice of $\mu > 0$

This section is devoted to the proof of the following theorem.

Theorem 7.3.1. *There exists $\mu_1 \in (0, \mu_0]$ depending only on μ_0, β, γ such that for every $\mu \in (0, \mu_1]$, we have $\overline{u} = \underline{u}$ on $[0, \infty) \times [\mu_0, \infty) \times [0, \infty)$.*

The theorem means that we can avoid the splitting of the two eradication times by choosing $\mu \in (0, \mu_0]$ small enough that we obtain a viscosity solution $u := \overline{u} = \underline{u}$ to (7.2) in $(0, \infty) \times (\mu_0, \infty) \times (0, \infty)$.

Lemma 7.3.1. *Let (S, I) be the flow of (7.1) associated with a datum $d = (x_0, y_0, t_0, \alpha) \in \mathcal{D}$. If $\underline{u}^\alpha(x_0, y_0, t_0) < \overline{u}^\alpha(x_0, y_0, t_0)$, then $\dot{I}(s) = 0$ for $s = \overline{u}^\alpha(x_0, y_0, t_0)$.*

Proof. The proof of this lemma is straightforward; if $\underline{u}^\alpha(x_0, y_0, t_0) < \overline{u}^\alpha(x_0, y_0, t_0)$, then there exists $\delta \in (0, \overline{u}^\alpha(x_0, y_0, t_0) - \underline{u}^\alpha(x_0, y_0, t_0))$ such that $I(s) \leq \mu$ for all $s \in [\overline{u}^\alpha(x_0, y_0, t_0) - \delta, \overline{u}^\alpha(x_0, y_0, t_0) + \delta]$. Since $I(s) = \mu$ for $s = \overline{u}^\alpha(x_0, y_0, t_0)$, we get $\dot{I}(s) = 0$ for $\overline{u}^\alpha(x_0, y_0, t_0)$. □

Therefore, once we prove the following proposition, then we obtain Theorem 7.3.1.

Proposition 7.3.1. *For any $\mu_0 > 0$, there is $\mu_1 \in (0, \mu_0]$ depending only on μ_0, β, γ such that for every $d = (x_0, y_0, t_0, \alpha) \in \mathcal{D}$ with $y \geq \mu_0$, it holds that $\{I(s) : \dot{I}(s) = 0,\ s \geq 0\} \subset (\mu_1, \infty)$, where (S, I) is the flow of (7.1) associated with d.*

The rest of this section is devoted to the proof of this proposition.

Proof. Fix $d = (x_0, y_0, t_0, \alpha) \in \mathcal{D}$ with $y \geq \mu_0$. We divide the proof into three steps.

Step 1: Case reductions.

First of all, for the case $t_0 > 0$, proving the conclusion for $d = (x_0, y_0, t_0, \alpha)$ is the same as proving for $(x_0, y_0, 0, \alpha^{t_0})$ with the changed coefficients $\beta^{t_0}, \gamma^{t_0}$. Thus, it suffices to show the case when $t_0 = 0$.

When $x_0 \leq \underline{\gamma}/\overline{\beta}$, we can take $\mu_1 = \mu_0$, as I is strictly decreasing when this is the case, and from now on, we may assume $x_0 \geq \underline{\gamma}/\overline{\beta}, t_0 = 0$ without loss of generality.

If $x_0 > \overline{\gamma}/\underline{\beta}$, then we can find the minimal time $t_1 > 0$ such that $S(t_1) = \overline{\gamma}/\underline{\beta}$. The derivative vanishing $\dot{I}(t) = 0$ happens only for $t \geq t_1$, and therefore, it suffices to prove the proposition for $(S(t_1), I(t_1), 0, \alpha^{t_1})$ with the changed coefficients $\beta^{t_1}, \gamma^{t_1}$, which belongs to the case when $x_0 \in \left[\underline{\gamma}/\overline{\beta}, \overline{\gamma}/\underline{\beta}\right], t_0 = 0$. From now on, we may assume $x_0 \in \left[\underline{\gamma}/\overline{\beta}, \overline{\gamma}/\underline{\beta}\right], t_0 = 0$ without loss of generality.

If $y_0 > \mu_0$, then we can find the minimal time $t_2 > 0$ such that $I(t_2) = \mu_0$ since $\lim_{t\to\infty} I(t) = 0$. As $\dot{I}(t_2) \le 0$, we necessarily have $\beta(t_2)S(t_2) - \gamma(t_2) \le 0$, which implies $S(t_2) \le \overline{\gamma}/\underline{\beta}$. It may happen that $\dot{I}(t) = 0$ for some $t \in [0, t_2)$, but $I(t) > \mu_0$ for such t. Thus, as long as we require $\mu_1 \in (0, \mu_0]$, the case $y_0 > \mu_0$ is reduced to the case for $(S(t_2), \mu_0, 0, \alpha^{t_2})$ (with the changed coefficients $\beta^{t_2}, \gamma^{t_2}$). If $S(t_2) \le \underline{\gamma}/\overline{\beta}$, we can take $\mu_1 = \mu_0$. If not, then $S(t_2) \in \left[\underline{\gamma}/\overline{\beta}, \overline{\gamma}/\underline{\beta}\right]$, which belongs to the case $x_0 \in \left[\underline{\gamma}/\overline{\beta}, \overline{\gamma}/\underline{\beta}\right], y_0 = \mu_0, t_0 = 0$.

From now on, we may assume that $x_0 \in \left[\underline{\gamma}/\overline{\beta}, \overline{\gamma}/\underline{\beta}\right], y_0 = \mu_0, t_0 = 0$ without loss of generality. We also now fix $\alpha \in \mathcal{A}$.

Step 2: Definition of a mapping T.

Let $X := \left[\underline{\gamma}/\overline{\beta}, \overline{\gamma}/\underline{\beta}\right]$. Define the mapping T as follows:

$$\begin{aligned} T \ : \ X &\longrightarrow [0, \infty) \\ x_0 &\longmapsto T(x_0) \end{aligned}$$

where $T(x_0) := \inf\left\{t \ge 0 : S(t) \le \underline{\gamma}/\overline{\beta}\right\}$, (S, I) is the flow associated with $(x_0, \mu_0, 0, \alpha) \in \mathcal{D}$. By the definition of T, we have $S(T) = \underline{\gamma}/\overline{\beta}$.

If we can prove that T is bounded, say by $M > 0$, then we are done the proof of the proposition. This is because for any $x_0 \in X$, it holds that

$$\dot{I} = \beta SI - \gamma I \ge -\overline{\gamma} I \quad \Longrightarrow \quad I(t) \ge \mu_0 e^{-\overline{\gamma}M} \text{ for any } t \in [0, M].$$

The derivative vanishing $\dot{I}(t) = 0$ happens only if $S(t) \in \left[\underline{\gamma}/\overline{\beta}, \overline{\gamma}/\underline{\beta}\right]$, and this occurs only when $t \le M$. Consequently, taking $\mu_1 = \frac{1}{2}\mu_0 e^{-\overline{\gamma}M}$ completes the proof.

Now, since X is compact, it suffices to show the continuity of T.

Step 3: Continuity of the mapping T.

In this step, we prove that if $x_k \to x$ as $k \to \infty$ in X, then $T_k := T(x_k) \to T_0 := T(x)$ as $k \to \infty$.

Let (S_k, I_k) be the flow associated with $(x_k, \mu_0, 0, \alpha)$ for each $k = 1, 2, \cdots$, and let (S_0, I_0) be the flow associated with $(x, \mu_0, 0, \alpha)$. Then, by Proposition 7.2.1, (S_k, I_k) converges to (S_0, I_0) locally uniformly in $[0, \infty)$ as $k \to \infty$.

We first check that $T_k \leq T_0 + 1$ for all but finitely many k. If not, then there would be a subsequence $\left\{T_{k_j}\right\}_j$ such that

$$\underline{\gamma}/\overline{\beta} \leq S_{k_j}(T_{k_j}) \leq S_{k_j}(T_0 + 1).$$

Letting $j \to \infty$ gives

$$\underline{\gamma}/\overline{\beta} \leq S_0(T_0 + 1) < S_0(T_0) = \underline{\gamma}/\overline{\beta},$$

which is a contradiction.

Let

$$m_S := \inf_{k \geq 1, t \in [0, T_0+1]} \{S_k(t)\},$$
$$m_I := \inf_{k \geq 1, t \in [0, T_0+1]} \{I_k(t)\}.$$

Since (S_k, I_k) converges to (S_0, I_0) uniformly in $[0, T_0 + 1]$ as $k \to \infty$, we have $m_S, m_I > 0$.

Now, for $k \geq 1$,

$$\dot{S}_k = -\beta S_k I_k - \alpha_k S_k \leq -\underline{\beta} m_S m_I < 0,$$

on $[0, T_0 + 1]$. This implies, by the mean value theorem, that for $k \geq 1$ large enough,

$$\left| \frac{S_k(T_k) - S_k(T_0)}{T_k - T_0} \right| \geq \underline{\beta} m_S m_I > 0,$$

unless $T_k = T_0$. Therefore,

$$|T_k - T_0| \leq \frac{1}{\underline{\beta} m_S m_I} |S_k(T_k) - S_k(T_0)| = \frac{1}{\underline{\beta} m_S m_I} \left|\underline{\gamma}/\overline{\beta} - S_k(T_0)\right|.$$

As $S_k(T_0) \to S_0(T_0) = \underline{\gamma}/\overline{\beta}$, we see that $T_k \to T_0$ as $k \to \infty$.

Therefore, T is continuous, and this finishes the proof. □

Remark 7.3.1. *The argument is essentially the proof of the inverse function theorem, as T is the inverse function of S. The continuity of S yields that of T. We necessarily separate the derivative $\dot{S}$ from 0, which corresponds to the nonzero determinant assumption of the inverse function theorem.*

7.4 Further properties of $u = \overline{u} = \underline{u}$

7.4.1 Uniqueness

From now on, we assume that $\mu_1 \in (0, \mu_0]$ is chosen as in Theorem 7.3.1 and let $\mu \in (0, \mu_1]$ so that we have a viscosity solution $\overline{u} = \underline{u}$ to (7.2) in $(0, \infty) \times (\mu_0, \infty) \times (0, \infty)$. In this section, we discuss the uniqueness of the solution under prescribed boundary values with a boundedness assumption.

Let us denote $\Omega := (0, \infty) \times (\mu_0, \infty) \subset \mathbb{R}^2$. Let $u(x, y, t) \in C(\overline{\Omega} \times [0, \infty))$ be a viscosity solution to

$$\begin{cases} -\partial_t u + \beta(t) xy \partial_x u + x(\partial_x u)_+ + (\gamma(t) - \beta(t)x) y \partial_y u = 1 & \text{in} \quad \Omega \times (0, \infty), \\ u(x, \mu_0, t) = f(x, t) & \text{on} \quad \left[0, \overline{\gamma}/\underline{\beta}\right] \times [0, \infty), \\ u(0, y, t) = g(y, t) & \text{on} \quad [\mu_0, \infty) \times [0, \infty), \end{cases} \tag{7.3}$$

where f, g are given continuous functions on $\left[0, \overline{\gamma}/\underline{\beta}\right] \times [0, \infty)$, $[\mu_0, \infty) \times [0, \infty)$, respectively.

We now prove the uniqueness of viscosity solutions to (7.3).

Proposition 7.4.1. *There is at most one viscosity solution to* (7.3)*.*

Proof. Suppose that u^1, u^2 are two continuous viscosity solutions to (7.3). Fix a function smooth function $\phi : \mathbb{R} \to (-1, 1)$ that satisfies $\phi' > 0$ on $\mathbb{R}$. Then, $v^1(x, y, t) =$

$\phi(u^1(x,y,t)+t),\ v^2(x,y,t)=\phi(u^2(x,y,t)+t)$ are viscosity solutions to

$$\begin{cases} -\partial_t v+\beta(t)xy\partial_x v+x(\partial_x v)_+ +(\gamma(t)-\beta(t)x)y\partial_y v=0 & \text{in}\quad \Omega\times(0,\infty),\\ v(x,\mu_0,t)=\phi(f(x,t)+t) & \text{on}\quad \left[0,\overline{\gamma}/\underline{\beta}\right]\times[0,\infty),\\ v(0,y,t)=\phi(g(y,t)+t) & \text{on}\quad [\mu_0,\infty)\times[0,\infty), \end{cases} \tag{7.4}$$

and they satisfy $\|v^1\|_{L^\infty(\overline{\Omega}\times[0,\infty))}, \|v^2\|_{L^\infty(\overline{\Omega}\times[0,\infty))}\le 1$. Our goal is to prove $v^1\le v^2$, which then leads to $u^1\le u^2$. A symmetric argument switching u^1 and u^2 gives $u^1=u^2$, which proves the uniqueness.

Define

$$w(x,y,t)=\begin{cases} \frac{1}{2}\underline{\gamma}\mu_0 t+\frac{1}{t}+x+y+\frac{h(x)}{y-\mu_0}, & x\ge 0, y>\mu_0, t>0,\\ \frac{1}{2}\underline{\gamma}\mu_0 t+\frac{1}{t}+x+y, & 0\le x\le \overline{\gamma}/\underline{\beta}, y=\mu_0, t>0,\\ +\infty, & t=0,\ \text{or}\ x>\overline{\gamma}/\underline{\beta}, y=\mu_0, t>0, \end{cases}$$

where $h=h(x):[0,\infty)\to[0,\infty)$ is a nondecreasing function that vanishes on $\left[0,\overline{\gamma}/\underline{\beta}\right]$ and that is positive in $\left(\overline{\gamma}/\underline{\beta},\infty\right)$. Then, w is nonnegative, lower semicontinuous, and it satisfies $\partial_x w>0$, and

$$-\partial_t w+\beta xy\partial_x w+(\gamma-\beta x)y\partial_y w>0 \quad \text{in} \quad (0,\infty)\times(\mu_0,\infty)\times(0,\infty). \tag{7.5}$$

For given $\varepsilon,\alpha\in(0,1)$, we let

$$\Phi(X_1,X_2):=v^1(X_1)-v^2(X_2)-\frac{\varepsilon}{2}(w(X_1)+w(X_2))-\frac{1}{2\alpha}|X_1-X_2|^2,$$

for $X_1,X_2\in[0,\infty)\times[\mu_0,\infty)\times[0,\infty)$. Then, it suffices to prove $\Phi(X,X)\le 0$ for all $X\in[0,\infty)\times[\mu_0,\infty)\times[0,\infty)$. Indeed, this implies $v^1\le v^2$ as $\varepsilon\in(0,1)$ was arbitrary.

We suppose for the contrary that there exists $\delta>0$ independent of $\alpha\in(0,1)$ such that

$$\Phi(X,X) \geq \delta \quad \text{for some } X \in [0,\infty) \times [\mu_0,\infty) \times [0,\infty). \tag{7.6}$$

Our goal changes to deriving a contradiction from this. From this assumption, we see that the supremum

$$\sup\{\Phi(X_1,X_2) : X_1, X_2 \in [0,\infty) \times [\mu_0,\infty) \times [0,\infty)\}$$

is attained, say at $(X_1^\alpha, X_2^\alpha) \in \left[0, \frac{4}{\varepsilon}\right] \times \left[\mu_0, \frac{4}{\varepsilon}\right] \times \left[\frac{\varepsilon}{4}, \frac{8}{\mu_0 \varepsilon}\right]$.

Note that $\Phi(X_1^\alpha, X_2^\alpha) \geq \Phi((1,\mu_0,1),(1,\mu_0,1))$ yields

$$\frac{\varepsilon}{2}(w(X_1^\alpha) + w(X_2^\alpha)) + \frac{1}{2\alpha}|X_1^\alpha - X_2^\alpha|^2 \leq 6 + \frac{1}{2}\underline{\gamma}\mu_0 + \mu_0.$$

Therefore, passing to a subsequence of $\alpha \to 0$ if necessary (which we now follow and denote still by α by abuse of notations), there exists a limit $\hat{X} := (\hat{x}, \hat{y}, \hat{t}) = \lim_{\alpha\to 0} X_1^\alpha = \lim_{\alpha \to 0} X_2^\alpha \in \left[0, \frac{4}{\varepsilon}\right] \times \left[\mu_0, \frac{4}{\varepsilon}\right] \times \left[\frac{\varepsilon}{4}, \frac{8}{\mu_0\varepsilon}\right]$.

Also, $\Phi(X_1^\alpha, X_2^\alpha) \geq \Phi(X_1^\alpha, X_1^\alpha)$ implies

$$\frac{1}{2\alpha}|X_1^\alpha - X_2^\alpha|^2 \leq \frac{\varepsilon}{2}(w(X_1^\alpha) - w(X_2^\alpha)) + v^2(X_1^\alpha) - v^2(X_2^\alpha).$$

Since the function w is continuous on $\left\{(x,y,t) \ : \ w(x,y,t) \leq \frac{2}{\varepsilon}\left(6 + \frac{1}{2}\underline{\gamma}\mu_0 + \mu_0\right), \ x \geq 0, y \geq 0, t \geq 0\right\}$, this implies that

$$\lim_{\alpha \to 0} \frac{1}{2\alpha}|X_1^\alpha - X_2^\alpha|^2 = 0. \tag{7.7}$$

By the fact that $\Phi(X_1^\alpha, X_2^\alpha) \geq \delta$, which follows from (7.6), we have

$$\Phi(\hat{X}, \hat{X}) \geq \limsup_{\alpha \to 0} \Phi(X_1^\alpha, X_2^\alpha) \geq \delta > 0,$$

since $\delta > 0$ is independent of $\alpha \in (0,1)$ and the function w is lower semicontinuous.

Therefore, by the boundary conditions of (7.4) and the choice of the function w, we have $\hat{x} > 0, \hat{y} > \mu_0$.

For $X_1, X_2 \in [0,\infty) \times [\mu_0, \infty) \times [0,\infty)$, let

$$\begin{cases} \phi_1(X_1) := \frac{\varepsilon}{2} w(X_1) + \frac{1}{2\alpha}|X_1 - X_2^\alpha|^2, \\ \phi_2(X_2) := -\frac{\varepsilon}{2} w(X_2) - \frac{1}{2\alpha}|X_2 - X_1^\alpha|^2. \end{cases}$$

Then, $X_1 \mapsto v^1(X_1) - \phi^1(X_1)$ attains a local maximum at X_1^α, and $X_2 \mapsto v^2(X_2) - \phi^2(X_2)$ attains a local minimum at X_2^α, from which we have the following viscosity inequalities (for $\alpha \in (0,1)$ small enough):

$$\begin{cases} \left(-\partial_t \phi^1 + \beta x y \partial_x \phi^1 + x(\partial_x \phi^1)_+ + (\gamma - \beta x) y \partial_y \phi^1\right)\big|_{X_1^\alpha} \le 0, \\ \left(-\partial_t \phi^2 + \beta x y \partial_x \phi^2 + x(\partial_x \phi^2)_+ + (\gamma - \beta x) y \partial_y \phi^2\right)\big|_{X_2^\alpha} \ge 0. \end{cases}$$

We substract the two inequalities and let $\alpha \to 0$ to obtain

$$\varepsilon\left(-\partial_t w + \beta x y \partial_x w + (\gamma - \beta x) y \partial_y w\right)\big|_{\hat{X}} \le 0,$$

which contradicts to (7.5). □

7.4.2 Local semiconcavity

In this section, we establish the local semiconcavity of the value function $u(x, y, t)$ when $\beta(t) \equiv \beta_0, \gamma(t) \equiv \gamma_0$ for all $t \ge T$ for some $T > 0$ depending only on $\mu_0, \mu, \overline{\gamma}$. Here, we keep $\mu_1 \in (0, \mu_0]$ as in Theorem 7.3.1 and let $\mu \in (0, \mu_1]$ so that $\overline{u} = \underline{u}$ on $[0,\infty) \times [\mu_0, \infty) \times [0, \infty)$.

The observation is from the fact that the local semiconcavity property propagates from that of initial/terminal data along the time (see [10, 19]). Recently, global semiconcavity was proved in some situations in [56]. We first note the semiconcavity property of the terminal data $u(x, y, T)$ when $\beta(t) \equiv \beta_0, \gamma(t) \equiv \gamma_0$ for all $t \ge T$, which is the case studied in [59].

Theorem 7.4.1. *[59, Theorem 1.3] Suppose that $\beta(t) \equiv \beta_0, \gamma(t) \equiv \gamma_0$ for all $t \geq T$. Then, we have $\overline{u}(x,y,t) = \underline{u}(x,y,t)$ for all $x \geq 0, y \geq \mu, t \geq T$. Moreover, $u(x,y,T) := \overline{u}(x,y,T) = \underline{u}(x,y,T)$ is locally semiconcave in $(x,y) \in (0,\infty) \times (\mu,\infty)$.*

Now, we prove Theorem 7.1.3.

Proof of Theorem 7.1.3. When $\mu = \mu_0$, the theorem follows from Theorem 7.4.1 since $T = 0$. We assume the other case $\mu < \mu_0$ in the rest of the proof.

We start with the fact that for $x > 0, y > \mu_0$, we have $I^d(t) \geq ye^{-\overline{\gamma}t} > \mu_0 e^{-\overline{\gamma}t}$. Therefore, for $T := \frac{1}{2\overline{\gamma}} \log\left(\frac{\mu_0}{\mu}\right)$, it holds that $I^d(T) \geq y\sqrt{\frac{\mu}{\mu_0}} > \mu_0$ for every $d \in \mathcal{D}$, which also implies

$$\inf\left\{I^d(T) : d = (x,y,t,\alpha) \in \mathcal{D},\ y > \mu_0 + \delta\right\} > \mu \tag{7.8}$$

for every $\delta > 0$.

Let $v(x,y,t) = u(x,y,t) + t - T$ for $x \geq 0, y \geq \mu_0, t \geq 0$, and let $g(x,y) = u(x,y,T)$ for $x \geq 0, y \geq \mu$. Then, by Proposition 7.2.4, we have, for $(x,y,t) \in (0,\infty) \times (\mu_0,\infty) \times (0,T]$,

$$v(x,y,t) = \inf_{\alpha \in \mathcal{A}} g\left(S^d(T-t), I^d(T-t)\right),$$

where $d = (x,y,t,\alpha) \in \mathcal{D}$.

Our goal is to show the local semiconcavity of v, instead of that of u, which is sufficient to complete the proof. To this end, we fix $t \in (0,T]$, $K \subset\subset \Omega := (0,\infty) \times (\mu_0,\infty)$ and $(x,y) \in K$. By Propositions 7.2.2, 7.2.4, there exists an optimal control $\alpha^* \in \mathcal{A}$ such that $v(x,y,t) = g\left(S^{d^*}(T-t), I^{d^*}(T-t)\right)$ where $d^* = (x,y,t,\alpha^*) \in \mathcal{D}$.

Let $h = (h_1,h_2)$ be a vector in $\mathbb{R}^2$ whose magnitude is smaller than $\mathrm{dist}(K,\partial\Omega)$. Let $z(s) = \left(S^{d^*}(s), I^{d^*}(s)\right)$ and $z_\pm(s) = \left(S^{d^*_\pm}(s), I^{d^*_\pm}(s)\right)$ for $s \in [0,t]$, where $d^*_\pm = (x \pm h_1, y \pm h_2, t, \alpha^*)$. Then, it holds that

$$|z_+(t) - z_-(t)| \leq c|h| \quad \text{and} \quad |z_+(t) + z_-(t) - 2z(t)| \leq c|h|^2,$$

where c is a constant depending only on the compact set $K \subset\subset \Omega$ and on the rates β, γ. Therefore,

$$\begin{aligned}
&v(x+h_1, y+h_2, t) + v(x-h_1, y-h_2, t) - 2v(x,y,t) \\
&\leq g(z_+(t)) + g(z_-(t)) - 2g\left(\frac{z_+(t)+z_-(t)}{2}\right) + 2g\left(\frac{z_+(t)+z_-(t)}{2}\right) - 2g(z(t)) \\
&\leq c\left(\left|\frac{z_+(t)-z_-(t)}{2}\right|^2 + |z_+(t)+z_-(t)-2z(t)|\right) \\
&\leq c|h|^2.
\end{aligned}$$

Here, c denotes constants that may vary line by line, but all of which depend only on the compact set $K \subset\subset \Omega$ and on the rates β, γ. We used (7.8) and Theorem 7.4.1. Note also that the local semiconcavity of g in $(0,\infty) \times (\mu, \infty)$ implies the local Lipschitz regularity of g in $(0,\infty) \times (\mu,\infty)$. This proves the local semiconcavity of $v(\cdot,\cdot,t)$ (or, of $u(\cdot,\cdot,t)$) in Ω for each $t \in (0,T]$.

The local semiconcavity in time also follows from a similar argument and we refer the details to [19]. □

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