LIE STRUCTURES IN DERIVED CATEGORIES AND THEIR APPLICATIONS

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Abstract

For a closed embedding $X \hookrightarrow S$ of smooth schemes with a first order splitting, the derived self-intersection $X \times^R_S X$ and the shifted normal bundle $N_{X/S}[-1]$ carry Lie structures, a fact which we review. A fundamental construction due to Arinkin-Căldăraru, Arinkin-Căldăraru-Hablicsek, and Grivaux is that of an analogue of the exponential map in this context, the HKR isomorphisms from the shifted normal bundle to the self-intersection.

We study functoriality property of the HKR isomorphism defined by Arinkin-Căldăraru for a sequence of closed embeddings $X \hookrightarrow Y \hookrightarrow S$. We show that the HKR isomorphism is functorial when a certain cohomology class, which we call the Bass-Quillen class, vanishes. We obtain Lie theoretic interpretations as well.

We apply this functoriality result to study the multiplicative structure of orbifold Hochschild cohomology in an attempt to generalize the results of Kontsevich and Calaque-Van den Bergh relating the Hochschild and polyvector field cohomology rings of a smooth variety. We define a product on the cohomology of polyvector fields of a global quotient orbifold and we prove that the product is associative when certain Bass-Quillen classes vanish.

In the case of the diagonal embedding $X \hookrightarrow X \times X$, the study of the Lie structure on the shifted tangent bundle $T_X[-1]$ is applied to obtain a result in deformation theory and Hochschild cohomology, which solves a question of Eyal Markman.

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Chapter 1

Introduction

1.1 Lie structures in algebraic geometry

The study of Lie algebra objects in derived categories started with the work of Kapranov and Kontsevich [25]. Consider the shifted tangent bundle $T_X[-1]$ of a smooth algebraic variety X. It can be thought as a chain complex which has the tangent bundle of X in degree one, and is zero in all other degrees. Kapranov [25] showed that it carries an anti-symmetric bracket satisfying the Jacobi identity in the derived category of X. Therefore $T_X[-1]$ has a Lie algebra structure. Since then this structure has proven useful in a wide variety of fields of mathematics. To mention a few examples, Roberts-Willerton [30] proved that the category of representations of the Lie algebra $T_X[-1]$ is the derived category of X, Calaque-Rossi-Van den Bergh [9] related it with formal geometry and deformation quantization, and Arinkin-Căldăraru-Hablicsek [3] used a generalized version of the Lie algebra interpretation above to study the orbifold Hochschild cohomology.

The study of $T_X[-1]$ originated from the study of Hochschild cohomology. Let X be an algebraic variety. The Hochschild cohomology $\mathrm{HH}^*(X)$ of X is defined as $\mathrm{Ext}^*(\Delta_*\mathscr{O}_X, \Delta_*\mathscr{O}_X)$, where $\Delta : X \hookrightarrow X \times X$ is the diagonal embedding. If X is smooth over a field of characteristic zero, Swan [31] gave an explicit formula computing the

Hochschild cohomology in more familiar terms:

$$\mathrm{HH}^*(X) \cong \mathrm{HT}^*(X) \overset{\mathrm{def}}{=} \bigoplus_{p+q=*} H^p(X, \wedge^q T_X).$$

The isomorphism above is called the HKR isomorphism, and the right hand side is called the polyvector field cohomology of X. Kapranov and Kontsevich observed that there is a Lie theoretic interpretation of the HKR isomorphism. The derived self-intersection $X \times^R_{X\times X} X$ has a natural group structure in the derived category of differential graded (dg) schemes, arising from its interpretation as the derived loop space of X. Its structure complex as an $\mathscr{O}_X$-module is $\Delta^*\Delta_*\mathscr{O}_X$. The Lie algebra corresponding to this group is the shifted tangent bundle $T_X[-1]$. One can consider the total space $\mathbb{T}_X[-1]$ of the shifted tangent bundle as a dg scheme. Its structure complex as an $\mathscr{O}_X$-module is $\mathrm{Sym}(\Omega_X[1])$. There is an analogue of the exponential map, which is an isomorphism of derived schemes

$$\exp : \mathbb{T}_X[-1] \cong X \times^R_{X\times X} X.$$

One can restate the isomorphism above in terms of structure complexes as $\Delta^*\Delta_*\mathscr{O}_X \cong \mathrm{Sym}(\Omega_X[1])$. Applying the functor $\mathrm{Hom}(-, \mathscr{O}_X)$ on both sides and taking cohomology of the chain complexes, we get the original HKR isomorphism constructed by Swan.

1.2 Generalized HKR isomorphism

In [2] Arinkin and Căldăraru replaced the diagonal embedding $\Delta : X \hookrightarrow X \times X$ by an arbitrary closed embedding $i : X \hookrightarrow S$ of smooth schemes. One can still get a generalized HKR isomorphism

$$\mathbb{N}_{X/S}[-1] \cong X \times^R_S X$$

from the choice of a fixed first order splitting (when one exists). Here $\mathbb{N}_{X/S}[-1]$ is the total space of the shifted normal bundle. Calaque-Căldăraru-Tu and Calaque-Grivaux [5, 7] established the existence of a Lie structure on $N_{X/S}[-1]$ later, in the presence of additional conditions. In particular, they showed that the shifted normal bundle has a natural Lie algebra structure if a certain condition which they call *tameness* is satisfied. When the embedding splits to infinite order, the derived self-intersection $X \times_S^R X$ has a group structure. Therefore in this case the generalized HKR isomorphism can be viewed as the exponential map from the Lie algebra $N_{X/S}[-1]$ to the group $X \times_S^R X$.

In Chapter 2 we study the functoriality of the generalized HKR isomorphisms and the corresponding Lie theoretic interpretation. Consider a sequence of closed embeddings $X \hookrightarrow Y \hookrightarrow S$ with first order splittings. There are natural maps

$$X \times_Y^R X \to X \times_S^R X \to Y \times_S^R Y|_X$$

between derived schemes and natural maps

$$N_{X/Y}[-1] \to N_{X/S}[-1] \to N_{Y/S}|_X[-1]$$

between vector bundles. We want to understand whether the maps are compatible with the HKR isomorphisms induced by the first order splittings. Theorems 2.2 and 2.3 provide an answer to this question; the answer depends on the vanishing of a certain cohomology class which we call the Bass-Quillen class (see Definition 2.1).

1.3 Applications to orbifold Hochschild cohomology

In Chapter 3 we apply the functoriality of HKR isomorphisms to obtain a result on the product structure of orbifold Hochschild cohomology. The motivation comes from

Kontsevich's work on deformation quantization of Poisson manifolds.

Deformation Quantization. Consider the HKR isomorphism

$$I^{\mathsf{HKR}} : \mathrm{HT}^*(X) = \bigoplus_{p+q=*} H^p(X, \wedge^q T_X) \xrightarrow{\cong} \mathrm{HH}^*(X) = \mathrm{Ext}^*(\Delta_*\mathscr{O}, \Delta_*\mathscr{O})$$

for diagonal embeddings. There are natural algebra structures on both sides above. We have the wedge product on $\mathrm{HT}^*(X)$ and the Yoneda product on the Ext algebra. However, the HKR isomorphism is not an isomorphism of algebras in general. Kontsevich [27] claimed that the HKR isomorphism with a correction term

$$\mathrm{HT}^*(X) \xrightarrow{\lrcorner\,\mathrm{td}^{-\frac{1}{2}}} \mathrm{HT}^*(X) \xrightarrow{I^{\mathsf{HKR}}} \mathrm{HH}^*(X)$$

is an isomorphism of algebras. This result was proved by Calaque and Van den Bergh [6]. We hope to generalize the isomorphism above to orbifolds.

Orbifold Hochschild cohomology. Consider an orbifold $[S/G]$, where S is a smooth scheme with the action of a finite group G. There is an orbifold version of the HKR isomorphism [3]

$$\mathrm{HT}^*([S/G]) \overset{\mathrm{def}}{=} (\mathrm{HT}^*(S;G))^G = \left(\bigoplus_{g\in G} \bigoplus_{p+q=*} H^{p-c_g}(S^g, \wedge^q T_{S^g} \otimes \omega_g) \right)^G \xrightarrow{\simeq} \mathrm{HH}^*([S/G]),$$

where S^g is the fixed locus of $g \in G$, c_g is the codimension of S^g in S, and ω_g is the dualizing sheaf of the embedding $S^g \hookrightarrow S$.

One may hope to generalize the isomorphism of algebras for smooth schemes to orbifolds. However, it is not even clear what the correct product structure on $\mathrm{HT}^*(S;G)$ should be. This product structure should generalize the wedge product on $\mathrm{HT}^*(X)$ for a scheme X. In Chapter 3 we define a product on $\mathrm{HT}^*(S;G)$ when G is abelian. We prove its associativity in some special cases. In particular, our product is associative when S is affine or when S is an abelian variety with $\mathbb{Z}/2\mathbb{Z}$ action.

Our results have further possible applications in the following topics.

Homological mirror symmetry. Chen and Ruan [11] defined orbifold singular cohomology for orbifolds. In homological mirror symmetry, the orbifold Hochschild cohomology of an orbifold should match with the orbifold singular cohomology of its mirror as algebras.

Cohomological hyperkähler resolution conjecture. Let $[S/G]$ be a global holomorphic symplectic quotient. If Z is a hyperkähler resolution of the singular space S/G, then the conjecture predicts that the singular cohomology of Z should be isomorphic to the orbifold singular cohomology of $[S/G]$ as rings. If all the work in the items listed above is done, then it would give a line of attack on a proof of the cohomological hyperkähler resolution conjecture.

1.4 Applications in deformation theory and hyperkähler manifolds

In Chapter 4 we discuss further applications to deformation theory and especially to the study of hyperkähler manifolds.

Let X be a smooth complex scheme. Consider the first order deformation $\tilde{X}$ of X associated to a class $\tilde{\alpha} \in H^1(X, T_X)$. In general, a vector bundle $\mathscr{F}$ on X may not deform to a bundle $\tilde{\mathscr{F}}$ on $\tilde{X}$. The obstruction $\alpha_{\mathscr{F}} \in \operatorname{Ext}^2(\mathscr{F}, \mathscr{F})$ to the existence of a vector bundle $\tilde{\mathscr{F}}$ on $\tilde{X}$ such that $\tilde{\mathscr{F}}|_X \cong \mathscr{F}$ was described in [4, 32] as the contraction

$$\alpha_{\mathscr{F}} = \tilde{\alpha} \lrcorner \operatorname{at}_{\mathscr{F}} \in \operatorname{Ext}^2(\mathscr{F}, \mathscr{F}).$$

Here $\operatorname{at}_{\mathscr{F}} \in \operatorname{Ext}^1(\mathscr{F}, \mathscr{F} \otimes \Omega_X)$ is the Atiyah class of $\mathscr{F}$. Moreover, if $\mathscr{F}$ does deform, then its Chern classes, and hence its Mukai vector, all stay of Hodge type on the deformed space $\tilde{X}$. This implies that the class

$$\tilde{\alpha}\lrcorner\, v(\mathscr{F}) \in \mathrm{H}\Omega_*(X) \overset{\text{def}}{=} \bigoplus_{q-p=*} H^p(X, \wedge^q \Omega_X)$$

vanishes, where $v(\mathscr{F})$ is the Mukai vector of $\mathscr{F}$.

Thus, in the simple case where $\tilde{\alpha} \in H^1(X, T_X)$ we conclude that if $\tilde{\alpha}\lrcorner\, \mathrm{at}_{\mathscr{F}}$ is zero, then $\tilde{\alpha}\lrcorner\, v(\mathscr{F})$ is zero.

In email correspondence Eyal Markman asked if the above statement can be generalized to the case where $\tilde{\alpha}$ is an arbitrary polyvector field in $\mathrm{HT}^*(X)$. According to Markman, this question is central to his study of the deformations of hyperkähler manifolds.

We argue in Theorem 4.1 that there is a commutative diagram

$$\begin{tikzcd}
\mathrm{HH}^*(X) \arrow[r] & \mathrm{Ext}^*(\mathscr{F}, \mathscr{F}) \\
\mathrm{HT}^*(X) \arrow[u, "I^{\mathsf{HKR}}"] \arrow[ur, "(-)\lrcorner\exp(\mathrm{at}_{\mathscr{F}})"'] &
\end{tikzcd}$$

for any smooth variety X and any coherent sheaf $\mathscr{F}$ on X. More details are in Chapter 4.

Let $\tilde{\alpha}$ be a class in $\mathrm{HT}^*(X)$. The commutativity of the diagram above implies Theorem 4.5 which shows

$$D(\tilde{\alpha})\lrcorner\, v(\mathscr{F}) = 0$$

if $\tilde{\alpha}\lrcorner \exp(\mathrm{at}_{\mathscr{F}}) = 0$. Here D is the Duflo operator,

$$D(\tilde{\alpha}) = \mathrm{td}^{\frac{1}{2}} \lrcorner\, \tilde{\alpha},$$

where td is the Todd class of X. The original statement for a class $\tilde{\alpha} \in H^1(X, T_X)$ follows easily from the one above as shown in Chapter 4.

The inspiration of the commutative diagram above comes from a similar statement in Lie theory. Let V be a finite dimensional representation of a finite dimensional Lie algebra $\mathfrak{g}$. Rewrite the representation map $\mathfrak{g} \otimes V \to V$ as a map $\Lambda : V \to V \otimes \mathfrak{g}^*$. One can draw a similar commutative diagram

$$\begin{tikzcd}
(U\mathfrak{g})^{\mathfrak{g}} \arrow[r] & \mathrm{Hom}(V, V) \\
(S\mathfrak{g})^{\mathfrak{g}}. \arrow[u, "\mathsf{PBW}"] \arrow[ru, "(-)\lrcorner\exp(\Lambda)"']
\end{tikzcd}$$

As we explained above, the shifted tangent bundle $T_X[-1]$ is a Lie algebra in the derived category of X. The category of representations of $T_X[-1]$ is the derived category of X and the universal enveloping algebra of $T_X[-1]$ is the Hochschild cochain complex $\underline{\mathsf{Hom}}(\Delta_*\mathscr{O}_X, \Delta_*\mathscr{O}_X)$ [30]. The functor $(-)^{\mathfrak{g}}$ is the 0-th Lie algebra cohomology which is similar to $H^*(X, -)$. Setting $\mathfrak{g}$ to be equal to $T_X[-1]$, we see that the diagram in Theorem 4.1 is a complete analogue of the Lie algebraic statement. The Hochschild cohomology $\mathrm{HH}^*(X)$ plays the role of $(U\mathfrak{g})^{\mathfrak{g}}$, $\mathrm{HT}^*(X)$ plays the role of $(S\mathfrak{g})^{\mathfrak{g}}$, and the HKR map is precisely the PBW map.

The results in this thesis have been divided into three papers [15, 21, 22] for publication.

1.5 Conventions

Throughout this work all the schemes are smooth over a field of characteristic zero. The schemes are over the field of complex numbers in the last chapter.

Chapter 2

Functoriality of HKR isomorphisms

In this chapter we study functoriality of HKR isomorphisms and its Lie theoretic interpretations.

2.1 Background

We recall results known about HKR isomorphisms in this section.

2.1.1 The diagonal embedding

The HKR isomorphism

$$\mathrm{HH}^*(X) \cong \bigoplus_{p+q=*} H^p(X, \wedge^q T_X)$$

identifies Hochschild cohomology of a smooth scheme X with polyvector fields as vector spaces. More precisely, we have an HKR isomorphism at the level of sheaves in the derived category of X

$$\Delta^*\Delta_*\mathscr{O}_X \cong \mathrm{Sym}_{\mathscr{O}_X}(\Omega_X[1]),$$

where $\Delta : X \hookrightarrow X \times X$ is the diagonal embedding. The sheaf version of the HKR isomorphism can be rewritten as an isomorphism of derived schemes over X

$$\mathbb{T}_X[-1] \xrightarrow{\cong} LX = X \times^R_{X\times X} X.$$

We get the original isomorphism on cohomology by applying $\mathrm{Hom}(-, \mathscr{O}_X)$ to the isomorphism of sheaves.

2.1.2 General embeddings

There exist generalized HKR isomorphisms if we replace the diagonal embedding by an arbitrary closed embedding $i : X \hookrightarrow S$ of smooth schemes.

The embedding i factors as

$$\begin{tikzcd} X \arrow[r, hook, "\mu"] & X_S^{(1)} \arrow[r, hook, "\nu"] & S, \end{tikzcd}$$

where $X_S^{(1)}$ is the first order neighborhood of X in S. We say i splits to first order if and only if the map μ is split, i.e., there exists a map of schemes $\varphi : X_S^{(1)} \to X$ such that $\varphi \circ \mu = id$. There is a bijection between first order splittings of i and splittings of the short exact sequence below [20, 20.5.12 (iv)]

$$\begin{tikzcd} 0 \arrow[r] & T_X \arrow[r] & T_S|_X \arrow[r] \arrow[l, dashed, bend right] & N_{X/S} \arrow[r] & 0. \end{tikzcd}$$

Arinkin and Căldăraru [2] provided a necessary and sufficient condition for $i^* i_* \mathscr{O}_X$ to be isomorphic to $\mathrm{Sym}(N_{X/S}^{\vee}[1])$. In [3] Arinkin, Căldăraru, and Hablicsek proved that the derived intersection $X \times_S^R X$ is isomorphic to $\mathbb{N}_{X/S}[-1]$ over $X \times X$ if and only if the embedding i splits to first order. Grivaux independently proved a similar result for complex manifolds in [19].

2.1.3 The first definition of the general HKR isomorphism

Let us briefly recall how the HKR isomorphism $i^*i_*\mathscr{O}_X \cong \mathrm{Sym}(N^\vee_{X/S}[1])$ was constructed in [2]. It is defined as the composite map

$$\mu^*\nu^*\nu_*\mu_*\mathscr{O}_X \longrightarrow \mu^*\mu_*\mathscr{O}_X \xrightarrow{\cong} \mathrm{T}^c(N^\vee_{X/S}[1]) \xrightarrow{\exp} \mathrm{T}(N^\vee_{X/S}[1]) \longrightarrow \mathrm{Sym}(N^\vee_{X/S}[1]).$$

The left most map is given by the counit of the adjunction $\nu^* \dashv \nu_*$. The map exp is multiplying by $1/k!$ on the degree k piece, and the last one is the natural projection map. The $\mathrm{T}^c(N^\vee_{X/S}[1])$ is the free coalgebra on $N^\vee_{X/S}$ with the shuffle product structure, and $\mathrm{T}(N^\vee_{X/S}[1])$ is the tensor algebra on $N^\vee_{X/S}$. The isomorphism $\mu^*\mu_*\mathscr{O}_X \cong \mathrm{T}^c(N^\vee_{X/S}[1])$ in the middle is non-trivial and needs more explanation. With the splitting φ one can build an explicit resolution of $\mu_*\mathscr{O}_X$ as an $\mathscr{O}_{X^{(1)}_S}$-algebra

$$(\mathrm{T}^c(\varphi^*N^\vee_{X/S}[1]), d) \longrightarrow \mu_*\mathscr{O}_X,$$

where $(\mathrm{T}^c(\varphi^*N^\vee_{X/S}[1]), d)$ is the free coalgebra on $\varphi^*N^\vee_{X/S}$ with the shuffle product structure and a differential d. The differential is defined as follows. There is a short exact sequence on $X^{(1)}_S$

$$0 \to \mu_*N^\vee_{X/S} \to \mathscr{O}_{X^{(1)}_S} \to \mu_*\mathscr{O}_X \to 0.$$

Consider the composite map

$$\varphi^*N^\vee_{X/S} \to \mu_*\mu^*\varphi^*N^\vee_{X/S} = \mu_*N^\vee_{X/S} \to \mathscr{O}_{X^{(1)}_S},$$

whose cokernel is $\mu_*\mathscr{O}_X$. Tensor the morphism above with $(\varphi^*N^\vee_{X/S})^{\otimes(k-1)}$. We get the degree k-th piece of the differential $d_k : (\varphi^*N^\vee_{X/S})^{\otimes k} \to (\varphi^*N^\vee_{X/S})^{\otimes(k-1)}$. The differential vanishes once we pull this resolution back on X via μ, so we get the desired isomorphism.

For any vector bundle $\mathscr{E}$ on X, we tensor the resolution above by $\varphi^*\mathscr{E}$. Using the projection formula and $\varphi \circ \mu = id$, one can show that we get a resolution of $\mu_*\mathscr{E}$

$$(\mathrm{T}^c(\varphi^* N^\vee_{X/S}[1]) \otimes \varphi^*\mathscr{E}, d) \to \mu_*\mathscr{E}.$$

The same argument shows that $i^*i_*(\mathscr{E}) \cong \mathscr{E} \otimes \mathrm{Sym}(N^\vee_{X/S}[1])$, i.e., that $i^*i_*(-) \cong (-) \otimes \mathrm{Sym}(N^\vee_{X/S}[1])$ as dg functors. This shows that $X \times^R_S X \cong \mathbb{N}_{X/S}[-1]$ over $X \times X$.

2.1.4 The second definition of the general HKR isomorphism

Let us recall how $I^{\mathsf{HKR}} : \mathbb{N}_{X/S}[-1] \cong X \times^R_S X$ was constructed in [3] and [19]. It is defined as the composite map

$$\begin{tikzcd}
\mathbb{N}_{X/S}[-1] \arrow[r, dashed] & \mathbb{T}_S|_X[-1] \arrow[r, "\cong"] & S \times^R_{S \times S} S|_X \arrow[dll, "="] \\
S \times^R_{S \times S} X \arrow[r, "id_S \times \Delta"'] & S \times^R_{S \times S} (X \times X) \arrow[r, "\cong"] & X \times^R_S X.
\end{tikzcd}$$

The dotted arrow is the splitting we fixed. The isomorphism in the middle $\mathbb{T}_S[-1] \cong S \times^R_{S \times S} S$ is the HKR isomorphism of diagonal embeddings $S \hookrightarrow S \times S$ discussed in (2.1.3). There are two splittings to define HKR for the diagonal embeddings. We always choose p_1, i.e., the projection onto the left factor

$$\begin{tikzcd}
\Delta_S : S \arrow[r] & S \times S. \arrow[l, bend right, dashed, "p_1"'] \arrow[l, bend left, dashed, "p_2"]
\end{tikzcd}$$

We do not know whether the constructions in (2.1.3) and (2.1.4) define the same isomorphism or not. We only consider the one in (2.1.3) in this chapter.

2.1.5 Lie theoretic interpretations for self-intersections

It was observed by Kapranov and Kontsevich that there is a Lie theoretic interpretation of the HKR isomorphism. The derived loop space $LX = X \times^R_{X\times X} X$ has the structure of a derived group scheme over X and the shifted tangent bundle $T_X[-1]$ is its Lie algebra [25]. The HKR isomorphism $\mathbb{T}_X[-1] \to LX$ can be thought of as a version of the exponential map [8].

Consider a closed embedding $i : X \hookrightarrow S$ of smooth schemes. The derived self-intersection $X \times^R_S X$ has an ∞-groupoid structure in the $(\infty, 1)$-category of dg schemes over X. The associated L_∞ algebroid is $N_{X/S}[-1]$. Passing to the derived category, we get a groupoid in the derived category of dg schemes having X as the space of objects. The target and source maps are the two projections π_1 and $\pi_2 : X \times^R_S X \to X$. See [5] for more details. When $S = X \times X$ and i is the diagonal embedding $\Delta : X \to X \times X$, there are two projections p_1 and $p_2 : X \times X \to X$ such that $p_i \circ \Delta = id$. This implies that the source map π_1 and the target map π_2 are equal in the derived category in this case, so $X \times^R_{X\times X} X$ becomes a group over X [3]. A similar argument works if the inclusion from X to its formal neighborhood in S splits.

Generally speaking, $N_{X/S}[-1]$ has an L_∞ algebroid structure in the $(\infty, 1)$-category of dg quasi-coherent sheaves on X. However, the Lie bracket may not be $\mathscr{O}_X$-linear, and it may not satisfy the Jacobi identity when we pass to the derived category of X. Calaque, Căldăraru, and Tu proved that the induced bracket in the derived category is $\mathscr{O}_X$-linear if i splits to first order, and it satisfies the Jacobi identity if i splits to second order [5]. As a consequence $N_{X/S}[-1]$ admits a natural Lie algebra structure in the derived category if i splits to second order. Later, Calaque and Grivaux showed that

$N_{X/S}[-1]$ has a natural Lie algebra structure if $X \hookrightarrow S$ is a tame quantized cycle [7], a weaker condition than splitting to second order. More precisely, for an embedding $i : X \hookrightarrow S$ with a chosen first order splitting, they described the $\mathscr{O}_X$-linear bracket $N_{X/S} \otimes N_{X/S} \to \operatorname{Sym}^2 N_{X/S} \to N_{X/S}[1]$ explicitly as the extension class of the short exact sequence of vector bundles on X

$$0 \to \operatorname{Sym}^2 N_{X/S}^{\vee} \cong \frac{I_X^2}{I_X^3} \to \varphi_* \frac{I_X}{I_X^3} \to \frac{I_X}{I_X^2} \cong N_{X/S}^{\vee} \to 0,$$

where I_X is the ideal sheaf of X in S. This bracket satisfies the Jacobi identity under the tameness assumption. In the rest of the proofs and theorems, we only use embeddings which split to first order without requiring the Jacobi identity to hold for this specific bracket. We will call this type of bracket to be a pre-Lie bracket.

In the case of classical Lie groups we have a commutative diagram

$$\begin{tikzcd} G \arrow[r, "f"] & H \\ \mathfrak{g} \arrow[u, "\exp"] \arrow[r, "df"] & \mathfrak{h}. \arrow[u, "\exp"'] \end{tikzcd}$$

A map of schemes $X \to Y$ induces a map of derived group schemes $LX \to LY|_X$ over X. The analogous statement for derived schemes to the above Lie theoretic statement is that the diagram

$$\begin{tikzcd} LX \arrow[r] & LY|_X \\ \mathbb{T}_X[-1] \arrow[u, "I^{\mathsf{HKR}}"] \arrow[r] & \mathbb{T}_Y|_X[-1] \arrow[u, "I^{\mathsf{HKR}}"'] \end{tikzcd}$$

commutes. One can prove the commutativity of the diagram above easily using the methods in this chapter.

We would like to consider the commutativity of this diagram in a more general setting. We will be in the following setting from now on. Let $X \hookrightarrow Y \hookrightarrow S$ be a sequence of closed embeddings of smooth schemes. Assume that X is split to first order in Y, and similarly for X in Y and Y in S. We want to understand if the two diagrams

$$\begin{tikzcd}
X \times^R_Y X \arrow[r] & X \times^R_S X \arrow[r] & Y \times^R_S X = Y \times^R_S Y|_X \\
\mathbb{N}_{X/Y}[-1] \arrow[r] \arrow[u, "\cong"] & \mathbb{N}_{X/S}[-1] \arrow[r] \arrow[u, "\cong"] & \mathbb{N}_{Y/S}|_X[-1] \arrow[u, "\cong"]
\end{tikzcd}$$

are commutative.

2.2 The proof of the functoriality

Before we study the commutativity of the diagram at the end of the previous section, we need to consider the following question and define an important cohomology class.

The restriction of the conormal bundle $N^\vee_{Y/S}$ to the first order neighborhood $X^{(1)}_Y$ is a vector bundle on $X^{(1)}_Y$. One can ask whether this bundle is isomorphic to $s^*(N^\vee_{Y/S}|_X)$, where s is the chosen first order splitting of X in Y. If the answer to this question is positive, i.e., there is an isomorphism of vector bundles on $X^{(1)}_Y$

$$N^\vee_{Y/S}|_{X^{(1)}_Y} \cong s^*(N^\vee_{Y/S}|_X), \quad (*)$$

we will say condition $(*)$ is satisfied.

The condition $(*)$ is equivalent to the vanishing of a certain cohomology class in $\mathrm{Ext}^1(N_{X/Y} \otimes N_{Y/S}|_X, N_{Y/S}|_X)$ associated to $N^\vee_{Y/S}|_{X^{(1)}_Y}$. One can construct a similar cohomology class $\alpha_{s,\mathscr{M}} \in \mathrm{Ext}^1(N_{X/Y} \otimes \mathscr{M}^\vee|_X, \mathscr{M}^\vee|_X)$ for any vector bundle $\mathscr{M}$ on $X^{(1)}_Y$ and the fixed splitting s.

Definition 2.1 *Consider the short exact sequence of $\mathscr{O}_{X^{(1)}_Y}$-modules*

$$0 \to t_* N^\vee_{X/Y} \to \mathscr{O}_{X^{(1)}_Y} \to t_*\mathscr{O}_X \to 0,$$

where t is the inclusion $X \hookrightarrow X_Y^{(1)}$.

For any vector bundle $\mathscr{M}$ on $X_Y^{(1)}$, tensor it with this short exact sequence. Then push-forward the sequence onto X via s. Using the fact that $s \circ t = id$ and the projection formula, we get a short exact sequence of vector bundles on X

$$0 \to N_{X/Y}^{\vee} \otimes \mathscr{M}|_X \to s_* \mathscr{M} \to \mathscr{M}|_X \to 0.$$

Dualizing, we get an extension class $\alpha_{s,\mathscr{M}} : N_{X/Y} \otimes \mathscr{M}^{\vee}|_X \to \mathscr{M}^{\vee}|_X[1]$. We call $\alpha_{s,\mathscr{M}}$ the Bass-Quillen class for the pair $(s, \mathscr{M})$.

The class $\alpha_{s,\mathscr{M}}$ vanishes if and only if $\mathscr{M}$ is isomorphic to $s^*\mathscr{M}|_X$ [7]. We call this class the Bass-Quillen class since it is related to the Bass-Quillen conjecture as we explain below. Suppose Y is the total space of a vector bundle $\mathscr{G}$ on a smooth scheme X. Let $\mathscr{F}$ be a vector bundle on Y. Then one can ask if $\mathscr{F}$ is isomorphic to the pull back of some vector bundle on X. When X is affine this is known as the Bass-Quillen problem and was answered affirmatively in [28]. However, in the global case the answer to this question is negative. In particular, there can be no vector bundle on X whose pull back is $\mathscr{F}$ if the Bass-Quillen class in $\mathrm{Ext}^1(N_{X/Y} \otimes \mathscr{F}^{\vee}|_X, \mathscr{F}^{\vee}|_X)$ associated to $\mathscr{F}|_{X_Y^{(1)}}$ is not zero.

We are ready to state the result about functoriality of HKR isomorphisms.

Theorem 2.2 *Let $X \hookrightarrow Y \hookrightarrow S$ be a sequence of closed embeddings of smooth schemes. Further assume that there are compatible splittings on the tangent bundles*

$$\begin{tikzcd}
T_X \arrow[r] & T_Y|_X \arrow[l, dashed, bend right, "p"'] \arrow[r] & T_S|_X . \arrow[l, dashed, bend right, "q|_X"'] \arrow[ll, dashed, bend left, "\rho"]
\end{tikzcd}$$

Compatibility means that $p \circ q|_X = \rho$.

The left square below is commutative. If condition $(*)$ *is satisfied, i.e., the Bass-Quillen class of* $N^{\vee}_{Y/S}|_{X^{(1)}_Y}$ *is zero, then the right square below is also commutative.*

$$\begin{tikzcd}
X \times^R_Y X \arrow[r] & X \times^R_S X \arrow[r] & Y \times^R_S X = Y \times^R_S Y|_X \\
\mathbb{N}_{X/Y}[-1] \arrow[r] \arrow[u, "\cong"] & \mathbb{N}_{X/S}[-1] \arrow[r] \arrow[u, "\cong"] & \mathbb{N}_{Y/S}|_X[-1]. \arrow[u, "\cong"]
\end{tikzcd}$$

The vertical maps are the HKR isomorphisms defined in [2]. The horizontal maps between normal bundles are the linear ones, i.e., they are vector bundle maps.

Proof. We have a commutative diagram

$$\begin{tikzcd}
X \arrow[dr, "t"] \arrow[drr, "f"] \arrow[ddr, "\mu"'] & & & \\
& X^{(1)}_Y \arrow[r, "a"] \arrow[d, "g"] \arrow[ul, dashed, bend right, "s"'] & Y \arrow[d, "b"] & \\
& X^{(1)}_S \arrow[r, "f^{(1)}"] \arrow[uul, dashed, bend left, "\varphi"] \arrow[u, dashed, bend right, "\sigma"'] & Y^{(1)}_S \arrow[r] \arrow[u, dashed, bend right, "\pi"'] & S
\end{tikzcd}$$

under the assumptions in Theorem 2.2. The solid arrows are the obvious ones. The dotted arrows π, s, and φ are the first order splittings of the closed embeddings $j : Y \to S$, $f : X \to Y$, and $i : X \to S$ respectively. Notice that $X^{(1)}_Y$ is the fiber product of $X^{(1)}_S$ and Y over $Y^{(1)}_S$, so we can pull π back along the morphism $f^{(1)}$ to define σ. The compatibility condition on the splittings means that $s \circ \sigma = \varphi$.

The HKR isomorphism $i^* i_* \mathscr{O}_X \cong \mathrm{Sym}(N^{\vee}_{X/S}[1])$ in[2] is defined as the composite map

$$\mu^*\nu^*\nu_*\mu_*\mathscr{O}_X \longrightarrow \mu^*\mu_*\mathscr{O}_X \xrightarrow{\cong} \mathrm{T}^c(N^{\vee}_{X/S}[1]) \xrightarrow{\exp} \mathrm{T}(N^{\vee}_{X/S}[1]) \longrightarrow \mathrm{Sym}(N^{\vee}_{X/S}[1]).$$

It is easy to see that all the constructions are canonical except for the isomorphism $\mu^*\mu_*\mathscr{O}_X \cong \mathrm{T}^c(N^{\vee}_{X/S}[1])$ which depends on the choice of the splitting φ.

To check the commutativity of the left square in Theorem 2.2, it suffices to show that the diagram

$$\begin{tikzcd}
\mu^*\mu_*\mathscr{O}_X \arrow[r, "\cong"] \arrow[d] & \mathrm{T}^c(N^\vee_{X/S}[1]) \arrow[d] \\
t^*t_*\mathscr{O}_X \arrow[r, "\cong"] & \mathrm{T}^c(N^\vee_{X/Y}[1])
\end{tikzcd}$$

is commutative since all the other constructions are canonical. The right vertical map is obtained from the natural vector bundle map $N^\vee_{X/S} \to N^\vee_{X/Y}$. The horizontal isomorphisms are constructed using the splittings, from explicit resolutions of $\mu_*\mathscr{O}_X$ and $t_*\mathscr{O}_X$ on $X^{(1)}_S$ and $X^{(1)}_Y$ respectively. These resolutions are of the form $(\mathrm{T}^c(\varphi^*N^\vee_{X/S}[1]), d)$ and $(\mathrm{T}^c(s^*N^\vee_{X/Y}[1]), d)$ as explained in (2.1.3).

We have $g^*\varphi^*N^\vee_{X/S} = s^*N^\vee_{X/S}$ using the fact that $\varphi = s \circ \sigma$ and $\sigma \circ g = id$. There is a natural map of vector bundles $g^*\varphi^*N^\vee_{X/S} = s^*N^\vee_{X/S} \to s^*N^\vee_{X/Y}$ which induces a map of complexes $g^*(\mathrm{T}^c(\varphi^*N^\vee_{X/S}[1]), d) \to (\mathrm{T}^c(s^*N^\vee_{X/Y}[1]), d)$. One can check carefully the induced map is indeed a map of complexes, i.e., the differentials are preserved. This proves that the diagram

$$\begin{tikzcd}
g^*(\mathrm{T}^c(\varphi^*N^\vee_{X/S}[1]), d) \arrow[r] \arrow[d] & g^*\mu_*\mathscr{O}_X \arrow[r] \arrow[d] & 0 \\
(\mathrm{T}^c(s^*N^\vee_{X/Y}[1]), d) \arrow[r] & t_*\mathscr{O}_X \arrow[r] & 0
\end{tikzcd}$$

which relates the two explicit resolutions of $\mathscr{O}_X$ as an $\mathscr{O}_{X^{(1)}_S}$-algebra and as an $\mathscr{O}_{X^{(1)}_Y}$-algebra is commutative. If we pull the natural map $g^*\varphi^*N^\vee_{X/S} = s^*N^\vee_{X/S} \to s^*N^\vee_{X/Y}$ back to X, we get the natural vector bundle map $N^\vee_{X/S} \to N^\vee_{X/Y}$. This proves that we get our desired commutative diagram at the beginning of the proof of Theorem 2.2 once we pull the commutative diagram above back to X.

Similarly, to prove the commutativity of the right square of Theorem 2.2, it suffices to show that the diagram

$$\begin{tikzcd}
\mu^*\mu_*\mathscr{O}_X \arrow[r, "\cong"] & \mathrm{T}^c(N^\vee_{X/S}[1]) \\
f^*b^*b_*\mathscr{O}_Y \arrow[r, "\cong"] \arrow[u] & f^*\mathrm{T}^c(N^\vee_{Y/S}[1]) \arrow[u]
\end{tikzcd}$$

is commutative. The right vertical map is induced by the natural map of vector bundles $N^\vee_{Y/S}|_X \to N^\vee_{X/S}$.

If the Bass-Quillen class $\alpha_{s,N^\vee_{Y/S}|_{X^{(1)}_Y}}$ vanishes, then we have an isomorphism between $a^*N^\vee_{Y/S}$ and $s^*(N^\vee_{Y/S}|_X)$. The latter maps to $s^*N^\vee_{X/S}$ naturally. Therefore, we get a map $\sigma^*a^*N^\vee_{Y/S} \cong \sigma^*s^*N^\vee_{Y/S}|_X \to \sigma^*s^*N^\vee_{X/S} = \varphi^*N^\vee_{X/S}$. Notice that $a \circ \sigma = \pi \circ f^{(1)}$ by the definition of σ, so we get a map $f^{(1)*}\pi^*N^\vee_{Y/S} = \sigma^*a^*N^\vee_{Y/S} \to \sigma^*s^*N^\vee_{X/S} = \varphi^*N^\vee_{X/S}$. This map induces a map of complexes $f^{(1)*}(\mathrm{T}^c(\pi^*N^\vee_{Y/S}[1]), d) \to (\mathrm{T}^c(\varphi^*N^\vee_{X/S}[1]), d)$. As a consequence the diagram of resolutions of $\mu_*\mathscr{O}_X$ and $b_*\mathscr{O}_Y$

$$\begin{tikzcd}
(\mathrm{T}^c(\varphi^*N^\vee_{X/S}[1]), d) \arrow[r] & \mu_*\mathscr{O}_X \arrow[r] & 0 \\
f^{(1)*}(\mathrm{T}^c(\pi^*N^\vee_{Y/S}[1]), d) \arrow[r] \arrow[u] & f^{(1)*}b_*\mathscr{O}_Y \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

is commutative. We recover the map of vector bundles $N^\vee_{Y/S}|_X \to N^\vee_{X/S}$ if we pull the natural map $\sigma^*a^*N^\vee_{Y/S} = f^{(1)*}\pi^*N^\vee_{Y/S} \to \varphi^*N^\vee_{X/S} = \sigma^*s^*N^\vee_{X/S}$ back to X. This proves that we get our desired commutative diagram once we pull the commutative diagram above back to X. $\square$

We apply this theorem in Chapter 3 to study the Hochschild cohomology of an orbifold. The main example we consider is the setting where S admits an action of a finite group G and X is the fixed locus of G and Y is the fixed locus of a subgroup $H \leq G$. It is easy to see that the splittings obtained from the averaging maps are compatible in this case.

2.3 Lie theoretic interpretations

All normal bundles in Theorem 2.2 carry pre-Lie brackets as explain before. The map $N_{X/Y}[-1] \to N_{Y/S}|_X[-1]$ respects the brackets in general. However, $N_{X/S}[-1] \to N_{Y/S}|_X[-1]$ may not preserve the brackets. The Bass-Quillen class $\alpha_{s,N^\vee_{Y/S}|_{X^{(1)}_Y}}$ is precisely the obstruction to this map preserving the brackets.

Theorem 2.3 *In the same setting as Theorem 2.2, the vector bundle map* $N_{X/S}[-1] \to N_{Y/S}|_X[-1]$ *preserves the pre-Lie brackets if and only if the Bass-Quillen class*

$$\alpha_{s,N^\vee_{Y/S}|_{X^{(1)}_Y}} : N_{X/Y} \otimes N_{Y/S}|_X \to N_{Y/S}|_X[1]$$

is zero.

We prove Theorem 2.3 in Section 2.6.

Lie theoretic interpretations for Theorem 2.3 Under certain extra assumptions, all derived self-intersections in Theorem 2.2 are groups in the derived category of dg schemes. The shifted normal bundles are their Lie algebras.

To make our Lie theoretic interpretation clearer, let us assume that all the three derived self-intersections $X \times^R_Y X$, $X \times^R_S X$, and $Y \times^R_S Y$ are groups in this subsection. However, we will state and explain theorems and propositions later in Sections 2.5 and 2.6 which only assume the existence of first order splittings. One can check that the natural maps

$$X \times^R_Y X \longrightarrow X \times^R_S X \longrightarrow X \times^R_S Y = Y \times^R_S Y|_X$$

are maps of groups. All the shifted normal bundles are Lie algebras under this assumption. In this section we denote $N_{X/Y}[-1]$, $N_{X/S}[-1]$, and $N_{Y/S}|_X[-1]$ by $\mathfrak{h}$, $\mathfrak{g}$, and $\mathfrak{n}$

respectively. The functoriality of HKR isomorphisms can be viewed as the functoriality of the exponential maps from Lie algebras to Lie groups.

The map $\mathfrak{h} = N_{X/Y}[-1] \hookrightarrow \mathfrak{g} = N_{X/S}[-1]$ preserves the Lie brackets, so we are able to prove the commutativity of the left square in Theorem 2.2 with no difficulty. Moreover, the compatibility of the Lie brackets implies that $\mathfrak{g}$ is an $\mathfrak{h}$-module, and $\mathfrak{h} \hookrightarrow \mathfrak{g}$ is a map of $\mathfrak{h}$-modules. Therefore, $\mathfrak{n} = \mathfrak{g}/\mathfrak{h} = N_{Y/S}|_X[-1]$ has a natural $\mathfrak{h}$-module structure. We get a short exact sequence of $\mathfrak{h}$-modules

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{h} = N_{X/Y}[-1] \arrow[r] & \mathfrak{g} = N_{X/S}[-1] \arrow[r] & \mathfrak{n} = \mathfrak{g}/\mathfrak{h} = N_{Y/S}|_X[-1] \arrow[r] & 0. \end{tikzcd}$$

The $\mathfrak{h}$-module structure on $\mathfrak{g}/\mathfrak{h}$: $N_{X/Y} \otimes N_{Y/S}|_X \to N_{Y/S}|_X[1]$ is exactly the Bass-Quillen class $\alpha_{s, N^\vee_{Y/S}|_{X^{(1)}_Y}}$. We prove this statement in Section 2.5.

On the other hand, the map $\mathfrak{g} = N_{X/S}[-1] \to \mathfrak{n} = N_{Y/S}|_X[-1]$ may not in general preserve the Lie brackets even if we assume that all the derived self-intersections are groups. This explains the difficulty for proving the functoriality of the exponential maps in the right square of Theorem 2.2. In Section 2.6 we will show that $\mathfrak{h} = N_{X/Y}[-1]$ acts on its module $\mathfrak{g}/\mathfrak{h} = N_{Y/S}|_X[-1]$ trivially if and only if the Lie brackets are preserved, i.e., $\mathfrak{g} \to \mathfrak{g}/\mathfrak{h} = \mathfrak{n}$ is a map of Lie algebras. This is Theorem 2.3. As a consequence the right square in Theorem 2.2 commutes when the Lie brackets are preserved. Thus Theorems 2.2 and 2.3 provide a generalization of the original result for Lie groups to the setting of groups obtained as self-intersections.

2.4 Theorem 2.3 in classical Lie theory

As a warm-up to proving Theorem 2.3, we present here an analogous result in Lie theory. We give a proof of this result using techniques that can be adapted to the derived setting of Theorem 2.3.

Consider an injective map of Lie algebras in vector spaces $\alpha : \mathfrak{h} \hookrightarrow \mathfrak{g}$. There is a short exact sequence of $\mathfrak{h}$-modules

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{h} \arrow[r, "\alpha"] & \mathfrak{g} \arrow[r, "\beta"] & \mathfrak{n} = \mathfrak{g}/\mathfrak{h} \arrow[r] & 0. \end{tikzcd}$$

Given a vector space map $\gamma : \mathfrak{n} \dashrightarrow \mathfrak{g}$ splitting β we define a pre-Lie bracket on $\mathfrak{n}$ by the formula $[x, y]_{\mathfrak{n}} = \beta([\gamma(x), \gamma(y)]_{\mathfrak{g}})$ for any $x, y \in \mathfrak{n}$. This bracket becomes a Lie bracket under the tameness assumption [7], but we do not need this pre-Lie bracket to be a Lie bracket throughout our discussion. In general, the map β may not respect the pre-Lie brackets.

We define a map $\mathfrak{g} \otimes \mathfrak{n} \to \mathfrak{n}$

$$\sum_i x_i \otimes y_i \to \sum_i \beta([x_i, \gamma(y_i)]),$$

for $x_i \in \mathfrak{g}$ and $y_i \in \mathfrak{n}$. This map may not define a $\mathfrak{g}$-module structure on $\mathfrak{n}$ if β is not a morphism of Lie algebras. We state a proposition the analogous statement of which is important in Sections 2.4 and 2.6. Its proof is left to the reader.

Proposition 2.4 *In general, we do not expect*

$$\begin{tikzcd} \mathfrak{g} \otimes \mathfrak{g} \arrow[r] \arrow[d, "id\otimes\beta"] & \mathfrak{g} \arrow[d, "\beta"] \\ \mathfrak{g} \otimes \mathfrak{n} \arrow[r] & \mathfrak{n} \end{tikzcd}$$

to be commutative. However, the diagram

$$\begin{tikzcd} \wedge^2\mathfrak{g} = \wedge^2\mathfrak{h} \oplus \wedge^2\mathfrak{n} \oplus (\mathfrak{h} \otimes \mathfrak{n}) \arrow[r] \arrow[d] & \mathfrak{g} \arrow[d, "\beta"] \\ (\mathfrak{h} \otimes \mathfrak{n}) \oplus \wedge^2\mathfrak{n} \arrow[r] & \mathfrak{n} \end{tikzcd}$$

is commutative if we identify $\mathfrak{g}$ *with* $\mathfrak{h} \oplus \mathfrak{n}$ *as a direct sum of vector spaces via* γ*.*

Therefore the right hand side square of the diagram

$$\begin{tikzcd}
\mathfrak{g} \otimes \mathfrak{g} \arrow[r] \arrow[d, "id\otimes\beta"] & \wedge^2\mathfrak{g} = \wedge^2\mathfrak{h} \oplus \wedge^2\mathfrak{n} \oplus (\mathfrak{h} \otimes \mathfrak{n}) \arrow[r] \arrow[d] & \mathfrak{g} \arrow[d, "\beta"] \\
\mathfrak{g} \otimes \mathfrak{n} \arrow[r] & (\mathfrak{h} \otimes \mathfrak{n}) \oplus \wedge^2\mathfrak{n} \arrow[r] & \mathfrak{n}
\end{tikzcd}$$

is commutative, but the square on the left is not.

Here is the analogous theorem to Theorem 2.3.

Theorem 2.5 *The map* β *preserves the pre-Lie brackets of* $\mathfrak{g}$ *and* $\mathfrak{n}$ *if and only if* $\mathfrak{h}$ *acts trivially on* $\mathfrak{n}$*. This is also equivalent to saying that* β *is a morphism of Lie algebras.*

Proof. It is easy to see that the map $\mathfrak{g} \otimes \mathfrak{n} \to \mathfrak{n}$ has the following properties.

(I) It is compatible with the $\mathfrak{h}$-module structure on $\mathfrak{n}$. Equivalently, the diagram

$$\begin{tikzcd}
\mathfrak{h} \otimes \mathfrak{n} \arrow[r] \arrow[d, "\alpha\otimes id"] & \mathfrak{n} \arrow[d, "id"] \\
\mathfrak{g} \otimes \mathfrak{n} \arrow[r] & \mathfrak{n}
\end{tikzcd}$$

is commutative. This follows from the observation that the $\mathfrak{h}$-module structure on $\mathfrak{n}$ can be defined as

$$x \otimes y \to \beta([\alpha(x), \gamma(y)]),$$

for any $x \in \mathfrak{h}$ and $y \in \mathfrak{n}$.

(II) It defines a $\mathfrak{g}$-module structure on $\mathfrak{n}$ if β is a morphism of Lie algebras. Equivalently, the diagram

$$\begin{tikzcd}
\mathfrak{g} \otimes \mathfrak{n} \arrow[r] \arrow[d, "\beta\otimes id"] & \mathfrak{n} \arrow[d, "id"] \\
\mathfrak{n} \otimes \mathfrak{n} \arrow[r] & \mathfrak{n}
\end{tikzcd}$$

is commutative if β is a morphism of Lie algebras.

(III) The diagram

$$\begin{tikzcd}
\mathfrak{n}\otimes\mathfrak{n} \arrow[r] \arrow[d, dashed, "\gamma\otimes id"] & \wedge^2\mathfrak{n} \arrow[r] \arrow[d] & \mathfrak{n} \\
\mathfrak{g}\otimes\mathfrak{n} \arrow[r] & (\mathfrak{h}\otimes\mathfrak{n})\oplus\wedge^2\mathfrak{n} \arrow[r] & \mathfrak{n} \arrow[u]
\end{tikzcd}$$

is commutative.

We first prove that $\mathfrak{h}$ acts on $\mathfrak{n}$ trivially if β preserves the pre-Lie brackets. Compose the two commutative diagrams in (I) and (II) together

$$\begin{tikzcd}
\mathfrak{h}\otimes\mathfrak{n} \arrow[r] \arrow[d, "\alpha\otimes id"] & \mathfrak{n} \arrow[d, "id"] \\
\mathfrak{g}\otimes\mathfrak{n} \arrow[r] \arrow[d, "\beta\otimes id"] & \mathfrak{n} \arrow[d, "id"] \\
\mathfrak{n}\otimes\mathfrak{n} \arrow[r] & \mathfrak{n}.
\end{tikzcd}$$

The map $\mathfrak{h}\otimes\mathfrak{n}\to\mathfrak{n}$ is zero because $\beta\circ\alpha=0$.

Let us turn to prove that β preserves the pre-Lie brackets if the Lie module structure map $\mathfrak{h}\otimes\mathfrak{n}\to\mathfrak{n}$ is zero. Identify $\mathfrak{g}$ with $\mathfrak{h}\oplus\mathfrak{n}$ via γ. With property (I) and (III) one can conclude that the map $\mathfrak{g}\otimes\mathfrak{n}=(\mathfrak{h}\otimes\mathfrak{n})\oplus(\mathfrak{n}\otimes\mathfrak{n})\to\mathfrak{n}$ that we defined at the beginning of this section is the Lie module structure map $\mathfrak{h}\otimes\mathfrak{n}\to\mathfrak{n}$ plus the pre-Lie bracket on $\mathfrak{n}$: $\mathfrak{n}\otimes\mathfrak{n}\to\mathfrak{n}$.

If $\mathfrak{h}\otimes\mathfrak{n}\to\mathfrak{n}$ is zero, then the diagram

$$\begin{tikzcd}
\mathfrak{g}\otimes\mathfrak{n} \arrow[r] \arrow[d] & (\mathfrak{h}\otimes\mathfrak{n})\oplus\wedge^2\mathfrak{n} \arrow[r] \arrow[d] & \mathfrak{n} \arrow[d] \\
\mathfrak{n}\otimes\mathfrak{n} \arrow[r] & \wedge^2\mathfrak{n} \arrow[r] & \mathfrak{n}
\end{tikzcd}$$

is commutative. Put the commutative diagram in Proposition 2.4 and the diagram above together

$$\begin{tikzcd}
\wedge^2\mathfrak{g} \arrow[r] \arrow[d] & \mathfrak{g} \arrow[d] \\
(\mathfrak{h}\otimes\mathfrak{n})\oplus\wedge^2\mathfrak{n} \arrow[r] \arrow[d] & \mathfrak{n} \arrow[d] \\
\wedge^2\mathfrak{n} \arrow[r] & \mathfrak{n}.
\end{tikzcd}$$

We conclude that β is a map of Lie algebras. □

2.5 The Bass-Quillen class as a Lie module structure map

In this section we prove the Bass-Quillen class can be viewed as a Lie module structure map as explained in Section 2.3. We begin by stating the result under the assumption that closed embeddings split to first order only. Then we provide explanations in Lie theoretic terms. We turn to the proof at last.

Lemma 2.6 *The following short exact sequence splits*

$$\begin{tikzcd}
0 \arrow[r] & N_{X/Y} \arrow[r] & N_{X/S} \arrow[r] & N_{Y/S}|_X \arrow[l, dashed, bend right, "\Psi"'] \arrow[r] & 0.
\end{tikzcd}$$

Proof. We have a map $\Psi : N_{Y/S}|_X \dashrightarrow T_S|_X \to N_{X/S}$ which splits the short exact sequence because Y splits to first order in S. □

The short exact sequence above shifted by negative one is analogous to the sequence

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{h} \arrow[r, "\alpha"] & \mathfrak{g} \arrow[r, "\beta"] & \mathfrak{n} = \mathfrak{g}/\mathfrak{h} \arrow[r] & 0.
\end{tikzcd}$$

in Section 2.4.

Most of this section will be devoted to constructing a map $\kappa : N_{X/Y} \otimes N_{X/S} \to N_{X/S}[1]$. This map will be given by the extension class of an explicit short exact sequence. Using this map we will prove the following proposition.

Proposition 2.7 *In the same setting as Theorem 2.2, the vector bundle map $N_{X/Y}[-1] \to N_{X/S}[-1]$ preserves the pre-Lie brackets constructed by Calaque and Grivaux [7]. There exists a map $\kappa : N_{X/Y} \otimes N_{X/S} \to N_{X/S}[1]$ defined explicitly by the extension class of a short exact sequence. The diagram*

$$\begin{tikzcd}
N_{X/Y} \otimes N_{Y/S}|_X \arrow[r] & N_{Y/S}|_X[1] \\
N_{X/Y} \otimes N_{X/S} \arrow[r, "\kappa"] \arrow[u] \arrow[d] & N_{X/S}[1] \arrow[u] \arrow[d] \\
N_{X/S} \otimes N_{X/S} \arrow[r] & N_{X/S}[1]
\end{tikzcd}$$

is commutative. Here all the vertical maps are the obvious maps of vector bundles. The top horizontal map is the Bass-Quillen class $\alpha_{s,N^\vee_{Y/S}|_{X^{(1)}_Y}}$, and the bottom horizontal map is the pre-Lie bracket.

To explain in Lie theoretic terms the meaning of Proposition 2.7, assume for simplicity that the embedding $X \hookrightarrow S$ satisfies the additional conditions that make $\mathfrak{g} = N_{X/S}[-1]$ a Lie algebra. Then denote by $\mathfrak{h} = N_{X/Y}[-1]$. It is a subalgebra of $\mathfrak{g}$ because the map $\mathfrak{h} \hookrightarrow \mathfrak{g}$ preserves the brackets by Proposition 2.7. The bundle $N_{Y/S}|_X[-1]$ can be identified with $\mathfrak{g}/\mathfrak{h}$. Then the diagram in Proposition 2.7 becomes

$$\begin{tikzcd}
\mathfrak{h} \otimes \mathfrak{g}/\mathfrak{h} \arrow[r] & \mathfrak{g}/\mathfrak{h} \\
\mathfrak{h} \otimes \mathfrak{g} \arrow[r, "\kappa"] \arrow[u] \arrow[d] & \mathfrak{g} \arrow[u] \arrow[d] \\
\mathfrak{g} \otimes \mathfrak{g} \arrow[r] & \mathfrak{g}.
\end{tikzcd}$$

The commutativity of

$$\begin{tikzcd}
N_{X/Y} \otimes N_{X/S} \arrow[r, "\kappa"] \arrow[d] & N_{X/S}[1] \arrow[d] & \mathfrak{h} \otimes \mathfrak{g} \arrow[r, "\kappa"] \arrow[d] & \mathfrak{g} \arrow[d] \\
N_{X/S} \otimes N_{X/S} \arrow[r] & N_{X/S}[1] & \mathfrak{g} \otimes \mathfrak{g} \arrow[r] & \mathfrak{g}
\end{tikzcd}$$

says that the morphism κ is the structure map of the natural $\mathfrak{h}$-module structure on $\mathfrak{g} = N_{X/S}[-1]$, where $\mathfrak{h} = N_{X/Y}[-1]$ is the Lie algebra.

The commutativity of the diagram

$$\begin{tikzcd}
N_{X/Y} \otimes N_{Y/S}|_X \arrow[r] & N_{Y/S}|_X[1] & \mathfrak{h} \otimes \mathfrak{g}/\mathfrak{h} \arrow[r] & \mathfrak{g}/\mathfrak{h} \\
N_{X/Y} \otimes N_{X/S} \arrow[r, "\kappa"] \arrow[u] & N_{X/S}[1] \arrow[u] & \mathfrak{h} \otimes \mathfrak{g} \arrow[r, "\kappa"] \arrow[u] & \mathfrak{g} \arrow[u]
\end{tikzcd}$$

says that the Bass-Quillen class $\alpha_{s,N^{\vee}_{Y/S}|_{X^{(1)}_{Y}}}$ at the top of the diagram is the structure map of the $\mathfrak{h}$-module structure on $\mathfrak{g}/\mathfrak{h} = N_{Y/S}|_X[-1]$.

Before we prove Proposition 2.7, we have to define the morphism $\kappa : N_{X/Y} \otimes N_{X/S} \to N_{X/S}[1]$ that appears in the middle of the diagram in Proposition 2.7. We hope to describe the morphism κ explicitly as the extension class of a short exact sequence. There is a technical detail we need to deal with. As we can see, the pre-Lie bracket is defined as $\operatorname{Sym}^2 N_{X/S} \to N_{X/S}[1]$ instead of $N_{X/S} \otimes N_{X/S} \to N_{X/S}[1]$. It is easy to describe the short exact sequence corresponding to $\operatorname{Sym}^2 N_{X/S} \to N_{X/S}[1]$. However, it is hard to describe explicitly what short exact sequence the morphism $N_{X/S} \otimes N_{X/S} \to N_{X/S}[1]$ corresponds to. The same phenomenon appears when we try to define $N_{X/Y} \otimes N_{X/S} \to N_{X/S}[1]$. There is an anti-symmetric part $\wedge^2 N_{X/Y} \hookrightarrow N_{X/Y} \otimes N_{X/S}$. We can only define our desired Lie module structure map κ via the extension class of a short exact sequence after we kill this anti-symmetric part. Lemma 2.8 below describes how to kill the anti-symmetric part of $N_{X/Y} \otimes N_{X/S} \cong (N_{X/Y} \otimes N_{X/Y}) \oplus (N_{X/Y} \otimes N_{Y/S}|_X)$ canonically.

Lemma 2.8 *The vector bundle* $\frac{I_X^2}{I_X^3+I_Y^2}$ *on* X *is isomorphic to* $\operatorname{Sym}^2 N_{X/Y}^\vee \oplus (N_{X/Y}^\vee \otimes N_{Y/S}^\vee|_X)$*, where* I_X *and* I_Y *are the ideal sheaves of* X *and* Y *in* S*.*

Proof. The cokernel of $\operatorname{Sym}^2 N_{Y/S}^\vee|_X \hookrightarrow \operatorname{Sym}^2 N_{X/S}^\vee$ is isomorphic to $\operatorname{Sym}^2 N_{X/Y}^\vee \oplus (N_{X/Y}^\vee \otimes N_{Y/S}^\vee|_X)$ using the splitting in Lemma 2.6.

There is a commutative diagram on X

$$\begin{tikzcd}
0 \arrow[r] & \frac{I_Y^2}{I_Y^2 I_X} \arrow[r] \arrow[d, "\cong"] & \frac{I_X^2}{I_X^3} \arrow[r] \arrow[d, "\cong"] & \frac{I_X^2}{I_X^3+I_Y^2} \arrow[r] & 0 \\
0 \arrow[r] & \operatorname{Sym}^2 N_{Y/S}^\vee|_X \arrow[r] & \operatorname{Sym}^2 N_{X/S}^\vee \arrow[r] & \operatorname{Sym}^2 N_{X/Y}^\vee \oplus (N_{X/Y}^\vee \otimes N_{Y/S}^\vee|_X) \arrow[r] & 0.
\end{tikzcd}$$

The two vertical maps above are isomorphisms, so we can complete the diagram above as an isomorphism of short exact sequences. This implies our desired isomorphism. □

Definition 2.9 *Define the morphism* $\kappa : N_{X/Y} \otimes N_{X/S} \to N_{X/S}[1]$ *as follows*

$$N_{X/Y} \otimes N_{X/S} \to \operatorname{Sym}^2 N_{X/Y} \oplus (N_{X/Y} \otimes N_{Y/S}|_X) \cong (\frac{I_X^2}{I_X^3+I_Y^2})^\vee \to N_{X/S}[1],$$

where the map on the left is the obvious map under the identification $N_{X/S} \cong N_{X/Y} \oplus N_{Y/S}|_X$ *in Lemma 2.6, and the map on the right is given by the extension class of the short exact sequence*

$$0 \to \frac{I_X^2}{I_X^3+I_Y^2} \to \varphi_* \frac{I_X}{I_X^3+I_Y^2} \to \frac{I_X}{I_X^2} \to 0.$$

We will focus on the proof of Proposition 2.7. The result will follow from Lemma 2.11 and Propositions 2.12 and 2.13 below.

Lemma 2.10 $\operatorname{Sym}^2 N_{X/Y}^\vee$ *is isomorphic to* $\frac{I_X^2}{I_X^3+I_X I_Y}$.

Proof. The ideal sheaf of X in Y is $\frac{I_X}{I_Y} \subset \mathcal{O}_Y = \mathcal{O}_S/I_Y$. Note that $\frac{I_X^n}{I_Y^n} \neq (\frac{I_X}{I_Y})^n \subset \mathcal{O}_S/I_Y$. It is easy to show that $(\frac{I_X}{I_Y})^n \cong \frac{I_X^n+I_Y}{I_Y} \subset \mathcal{O}_Y = \mathcal{O}_S/I_Y$. Therefore,

$$\operatorname{Sym}^2 N_{X/Y}^\vee \cong \frac{(\frac{I_X}{I_Y})^2}{(\frac{I_X}{I_Y})^3} \cong \frac{I_X^2+I_Y}{I_X^3+I_Y} \cong \frac{I_X^2}{I_X^2 \cap (I_X^3+I_Y)}.$$

We have $I_X^2 \cap (I_X^3 + I_Y) = (I_X^2 \cap I_X^3) + (I_X^2 \cap I_Y)$ because $I_X^3 \subset I_X^2$. The equality $I_X^2 \cap I_Y = I_X I_Y$ is due to the injective map below

$$N_{Y/S}^\vee|_X = \frac{I_Y}{I_Y I_X} \hookrightarrow N_{X/S}^\vee = \frac{I_X}{I_X^2}. \qquad \square$$

Lemma 2.11 *The map of vector bundles* $N_{X/Y}[-1] \to N_{X/S}[-1]$ *preserves the pre-Lie brackets.*

Proof. One can check that the two short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \dfrac{I_X^2}{I_X^3} \arrow[r] \arrow[d] & \varphi_*\dfrac{I_X}{I_X^3} \arrow[r] \arrow[d] & \dfrac{I_X}{I_X^2} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \dfrac{I_X^2}{I_X^3+I_XI_Y} \arrow[r] & s_*\dfrac{I_X}{I_X^3+I_Y} \arrow[r] & \dfrac{I_X}{I_X^2+I_Y} \arrow[r] & 0
\end{tikzcd}$$

are compatible. □

On the other hand, the map $N_{X/S}[-1] \to N_{Y/S}|_X[-1]$ may not preserve the pre-Lie brackets because there is no map $(\pi_* \frac{I_Y}{I_Y^3})|_X \to \varphi_* \frac{I_X}{I_X^3}$ generally.

We prove the commutativity of the two diagrams in Proposition 2.7. It is divided into two propositions below.

Proposition 2.12 *The map in Definition 2.9 is compatible with the pre-Lie bracket of* $N_{X/S}[-1]$. *It is equivalent to saying that the diagram*

$$\begin{tikzcd}
N_{X/Y} \otimes N_{X/S} \arrow[r] \arrow[d] & N_{X/S}[1] \arrow[d] \\
N_{X/S} \otimes N_{X/S} \arrow[r] & N_{X/S}[1]
\end{tikzcd}$$

is commutative.

Proof. We need to show that the three small diagrams

$$\begin{tikzcd}
N_{X/Y} \otimes N_{X/S} \arrow[r] \arrow[d] & \operatorname{Sym}^2 N_{X/Y} \oplus (N_{X/Y} \otimes N_{Y/S}|_X) \arrow[r, "\cong"] \arrow[d] & \left(\frac{I_X^2}{I_X^3 + I_Y^2}\right)^\vee \arrow[r] \arrow[d] & N_{X/S}[1] \arrow[d] \\
N_{X/S} \otimes N_{X/S} \arrow[r] & \operatorname{Sym}^2 N_{X/S} \arrow[r, "\cong"] & \left(\frac{I_X^2}{I_X^3}\right)^\vee \arrow[r] & N_{X/S}[1]
\end{tikzcd}$$

are commutative. Clearly the one on the left is commutative. The commutativity of the isomorphism in the middle follows from the compatibility of the two short exact sequences in Lemma 2.8. The diagram on the right commutes because the two short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \frac{I_X^2}{I_X^3 + I_Y^2} \arrow[r] & \varphi_* \frac{I_X}{I_X^3 + I_Y^2} \arrow[r] & \frac{I_X}{I_X^2} \arrow[r] & 0 \\
0 \arrow[r] & \frac{I_X^2}{I_X^3} \arrow[r] \arrow[u] & \varphi_* \frac{I_X}{I_X^3} \arrow[r] \arrow[u] & \frac{I_X}{I_X^2} \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

are compatible. ☐

Proposition 2.13 *There is a commutative diagram*

$$\begin{tikzcd}
N_{X/Y} \otimes N_{X/S} \arrow[r] \arrow[d] & N_{X/Y} \otimes N_{Y/S}|_X \arrow[d] \\
N_{X/S}[1] \arrow[r] & N_{Y/S}[1],
\end{tikzcd}$$

where the left vertical map is in Definition 2.9, and the right vertical map is the Bass-Quillen class $\alpha_{s,N^\vee_{Y/S}|_{X^{(1)}_Y}}$.

Proof. We need to prove that the three small diagrams in the diagram (1)

$$\begin{tikzcd}
N_{X/Y} \otimes N_{X/S} \arrow[r] \arrow[d] & N_{X/Y} \otimes N_{Y/S}|_X \arrow[d] \\
\operatorname{Sym}^2 N_{X/Y} \oplus (N_{X/Y} \otimes N_{Y/S}|_X) \arrow[r] \arrow[d, "\cong"] & N_{X/Y} \otimes N_{Y/S}|_X \arrow[d, "\cong"] \\
(\dfrac{I_X^2}{I_X^3 + I_Y^2})^\vee \arrow[r] \arrow[d] & (\dfrac{I_X}{I_X^2 + I_Y})^\vee \otimes (\dfrac{I_Y}{I_Y I_X})^\vee \arrow[d] \\
N_{X/S}[1] \arrow[r] & N_{Y/S}[1]
\end{tikzcd} \tag{1}$$

are commutative. Obviously the one on the top is commutative.

To prove the isomorphism in the middle is compatible, we construct a commutative diagram

$$\begin{tikzcd}
N^\vee_{X/Y} \otimes N^\vee_{Y/S}|_X \arrow[r, dashed] \arrow[d, "\cong"] & N^\vee_{X/S} \otimes N^\vee_{X/S} \arrow[r] \arrow[d, "\cong"] & \operatorname{Sym}^2 N^\vee_{X/S} \arrow[r] \arrow[d, "\cong"] & \operatorname{Sym}^2 N^\vee_{X/Y} \oplus (N^\vee_{X/Y} \otimes N^\vee_{Y/S}|_X) \arrow[d, "\cong"] \\
\dfrac{I_X}{I_X^2 + I_Y} \otimes \dfrac{I_Y}{I_Y I_X} \arrow[r, dashed] & \dfrac{I_X}{I_X^2} \otimes \dfrac{I_X}{I_X^2} \arrow[r] & \dfrac{I_X^2}{I_X^3} \arrow[r] & \dfrac{I_X^2}{I_X^3 + I_Y^2},
\end{tikzcd}$$

where the dotted arrows are defined by the splitting in Lemma 2.6, and the right square commutes as mentioned in the proof of Proposition 2.12. Clearly, the left and middle squares are commutative. We hope to prove that this big commutative diagram is exactly dual to the one in the middle of (1). It suffices to show that the map

$$\begin{tikzcd}
\dfrac{I_X}{I_X^2 + I_Y} \otimes \dfrac{I_Y}{I_Y I_X} \arrow[r, dashed] & \dfrac{I_X}{I_X^2} \otimes \dfrac{I_X}{I_X^2} \arrow[r] & \dfrac{I_X^2}{I_X^3} \arrow[r] & \dfrac{I_X^2}{I_X^3 + I_Y^2}
\end{tikzcd}$$

defined using the splitting is equal to the natural map

$$\frac{I_X}{I_X^2 + I_Y} \otimes \frac{I_Y}{I_Y I_X} \to \frac{I_X^2}{I_X^3 + I_Y^2},$$

$$(a \otimes b) \to ab, \text{ for } a \in \frac{I_X}{I_X^2 + I_Y}, \text{ and } b \in \frac{I_Y}{I_Y I_X}.$$

One can check this easily.

Let us focus on the commutativity of the bottom square in (1)

$$\begin{tikzcd}
(\frac{I_X^2}{I_X^3 + I_Y^2})^\vee \arrow[r] \arrow[d] & N_{X/S}[1] \arrow[d] \\
(\frac{I_X}{I_X^2 + I_Y})^\vee \otimes (\frac{I_Y}{I_Y I_X})^\vee \arrow[r] & N_{Y/S}[1].
\end{tikzcd}$$

The bottom horizontal map is defined by a short exact sequence

$$0 \to N_{X/Y}^\vee \otimes N_{Y/S}^\vee|_X \to s_* a^* N_{Y/S}^\vee \to N_{Y/S}^\vee \to 0.$$

Moreover, we have a morphism of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & N_{X/Y}^\vee \otimes N_{Y/S}^\vee|_X \arrow[r] \arrow[d] & s_* a^* N_{Y/S}^\vee \arrow[r] \arrow[d, "\cong"] & N_{Y/S}^\vee|_X \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \frac{I_X I_Y}{I_Y(I_X^2 + I_Y)} \arrow[r] & s_* \frac{I_Y}{I_Y(I_X^2 + I_Y)} \arrow[r] & \frac{I_Y}{I_Y I_X} \arrow[r] & 0.
\end{tikzcd}$$

This implies that $N_{X/Y}^\vee \otimes N_{Y/S}^\vee \cong \frac{I_X I_Y}{I_Y(I_X^2+I_Y)}$. It suffices to show that the short exact sequence $0 \to \frac{I_X I_Y}{I_Y(I_X^2+I_Y)} \to s_* \frac{I_Y}{I_Y(I_X^2+I_Y)} \to \frac{I_Y}{I_Y I_X} \to 0$ is compatible with the short exact sequence $0 \to \frac{I_X^2}{I_X^3+I_Y^2} \to \varphi_* \frac{I_X}{I_X^3+I_Y^2} \to \frac{I_X}{I_X^2} \to 0$.

There is a natural map of sheaves $g_* \frac{I_Y}{I_Y(I_X^2+I_Y)} \to \frac{I_X}{I_X^3+I_Y^2}$. We get $s_* \frac{I_Y}{I_Y(I_X^2+I_Y)} \to \varphi_* \frac{I_X}{I_X^3+I_Y^2}$ by applying φ_* on both sides, so the two short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \frac{I_X I_Y}{I_Y(I_X^2 + I_Y)} \arrow[r] \arrow[d] & s_* \frac{I_Y}{I_Y(I_X^2 + I_Y)} \arrow[r] \arrow[d] & \frac{I_Y}{I_Y I_X} \arrow[r] \arrow[d] & 0. \\
0 \arrow[r] & \frac{I_X^2}{I_X^3 + I_Y^2} \arrow[r] & \varphi_* \frac{I_X}{I_X^3 + I_Y^2} \arrow[r] & \frac{I_X}{I_X^2} \arrow[r] & 0
\end{tikzcd}$$

are compatible. □

2.6 The proof of Theorem 2.3

We generalize the proof in Section 2.4 to prove Theorem 2.3. We first define a morphism $N_{X/S} \otimes N_{Y/S}|_X \to N_{Y/S}|_X[1]$ which is analogous to the map $\mathfrak{g} \otimes \mathfrak{n} \to \mathfrak{n}$ in Section 2.4. Then we prove similar statements to properties (I) and (II), and Proposition 2.4 in Section 2.4. We prove Theorem 2.3 at last.

The first thing that we need is the map $N_{X/S} \otimes N_{Y/S}|_X \to N_{Y/S}|_X[1]$ which is the analogue of the map $\mathfrak{g} \otimes \mathfrak{n} \to \mathfrak{n}$ from Section 2.4. We need to deal with the same technical issue that appears in Section 2.5. Using the splitting in Lemma 2.6 we see that there is an anti-symmetric part $\wedge^2 N_{Y/S}|_X$ in $N_{X/S} \otimes N_{Y/S}|_X = (N_{X/Y} \oplus N_{Y/S}|_X) \otimes N_{Y/S}|_X$. We need to kill this anti-symmetric part canonically. Lemma 2.14 and 2.15 describe how to do this.

Lemma 2.14 $I_Y^2 \cap I_X^2 I_Y = I_Y^2 I_X$.

Proof. We have $I_Y^2 \cap I_X^3 = I_Y^2 I_X$ because the map

$$\operatorname{Sym}^2 N_{Y/S}^\vee|_X = \frac{I_Y^2}{I_Y^2 I_X} \to \operatorname{Sym}^2 N_{X/S}^\vee = \frac{I_X^2}{I_X^3}$$

is injective. Then we have

$$I_Y^2 I_X \subset I_Y^2 \cap I_X^2 I_Y \subset I_Y^2 \cap I_X^3 = I_Y^2 I_X. \qquad \square$$

Lemma 2.15 *There is an isomorphism of vector bundles* $(\frac{I_X I_Y}{I_X^2 I_Y})^\vee \cong \operatorname{Sym}^2 N_{Y/S}|_X \oplus (N_{X/Y} \otimes N_{Y/S}|_X)$ *on* X.

Proof. There is a morphism of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \frac{I_Y}{I_YI_X} \otimes \frac{I_Y}{I_YI_X} \arrow[r] \arrow[d] & \frac{I_X}{I_X^2} \otimes \frac{I_Y}{I_YI_X} \arrow[r] \arrow[d, "v"] & \frac{I_X}{I_X^2+I_Y} \otimes \frac{I_Y}{I_YI_X} \arrow[r] \arrow[d, "u"] \arrow[l, bend right, dashed, "\tau"'] & 0 \\
0 \arrow[r] & \frac{I_Y^2}{I_Y^2I_X} \arrow[r] & \frac{I_XI_Y}{I_X^2I_Y} \arrow[r] & \frac{I_XI_Y}{I_Y(I_X^2+I_Y)} \arrow[r] & 0.
\end{tikzcd}$$

Everything above is clear except for the injectivity of $\frac{I_Y^2}{I_Y^2I_X} \to \frac{I_XI_Y}{I_X^2I_Y}$. This is due to Lemma 2.14. $\square$

The short exact sequence on the top is the dual of the sequence of the normal bundles tensored with $N_{Y/S}^\vee|_X$, so it splits naturally. The map u is an isomorphism as mentioned in the proof of Proposition 2.13. One can construct a splitting $v \circ \tau \circ u^{-1}$ for the short exact sequence on the bottom. Therefore $\frac{I_XI_Y}{I_X^2I_Y} \cong \frac{I_Y^2}{I_Y^2I_X} \oplus \frac{I_XI_Y}{I_Y(I_X^2+I_Y)} \cong \operatorname{Sym}^2 N_{Y/S}^\vee|_X \oplus (N_{X/Y}^\vee \otimes N_{Y/S}^\vee|_X)$. The diagram of two short exact sequences above says that there is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & N_{Y/S}^\vee|_X \otimes N_{Y/S}^\vee|_X \arrow[r] \arrow[d] & N_{X/S}^\vee \otimes N_{Y/S}^\vee|_X \arrow[r] \arrow[d, "v"] & N_{X/Y}^\vee \otimes N_{Y/S}^\vee|_X \arrow[r] \arrow[d, "id"] \arrow[l, bend right, dashed, "\tau"'] & 0 \\
0 \arrow[r] & \operatorname{Sym}^2 N_{Y/S}^\vee|_X \arrow[r] \arrow[d, "\cong"] & \operatorname{Sym}^2 N_{Y/S}^\vee|_X \oplus (N_{Y/S}^\vee|_X \otimes N_{X/Y}^\vee) \arrow[r] \arrow[d, "\cong"] & N_{X/Y}^\vee \otimes N_{Y/S}^\vee|_X \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \frac{I_Y^2}{I_Y^2I_X} \arrow[r] & \frac{I_XI_Y}{I_X^2I_Y} \arrow[r] & \frac{I_XI_Y}{(I_X^2+I_Y)I_Y} \arrow[r] & 0.
\end{tikzcd}$$

Definition 2.16 *Define the morphism $N_{X/S} \otimes N_{Y/S}|_X \to N_{Y/S}[1]$ which is analogous to $\mathfrak{g} \otimes \mathfrak{n} \to \mathfrak{n}$ in Section 2.4 as follows*

$$N_{X/S} \otimes N_{Y/S}|_X \to \operatorname{Sym}^2(N_{Y/S}|_X) \oplus (N_{X/Y} \otimes N_{Y/S}|_X) \cong (\frac{I_XI_Y}{I_X^2I_Y})^\vee \to N_{Y/S}|_X[1],$$

where the map on the left hand side is the obvious one under the identification $N_{X/S} \cong N_{X/Y} \oplus N_{Y/S}|_X$, and the map on the right hand side is given by the extension class of *the short exact sequence*

$$0 \to \frac{I_X I_Y}{I_X^2 I_Y} \to \varphi_* \frac{I_Y}{I_Y I_X^2} \to \frac{I_Y}{I_Y I_X} \to 0.$$

The following proposition is analogous to property (I) in Section 2.4.

Proposition 2.17 *There is a commutative diagram*

$$\begin{tikzcd}
N_{X/S} \otimes N_{Y/S}|_X \arrow[r] & N_{Y/S}|_X[1] \\
N_{X/Y} \otimes N_{Y/S}|_X \arrow[r] \arrow[u] & N_{Y/S}|_X[1], \arrow[u]
\end{tikzcd}$$

where the horizontal map at top is in Definition 2.16, and the horizontal map at bottom is the Bass-Quillen class $\alpha_{s,N^\vee_{Y/S}|_{X^{(1)}_Y}}$.

Proof. The isomorphism $N^\vee_{X/Y} \otimes N^\vee_{Y/S}|_X \cong \frac{I_X I_Y}{(I_X^2+I_Y)I_Y}$ is mentioned in the proof of Proposition 2.13. It suffices to prove that the three small diagrams

$$\begin{tikzcd}
N_{X/S} \otimes N_{Y/S}|_X \arrow[r] & \operatorname{Sym}^2 N_{Y/S}|_X \oplus (N_{X/Y} \otimes N_{Y/S}|_X) \arrow[r, "\cong"] & (\frac{I_X I_Y}{I_X^2 I_Y})^\vee \arrow[r] & N_{Y/S}|_X[1] \\
N_{X/Y} \otimes N_{Y/S}|_X \arrow[r] \arrow[u] & N_{X/Y} \otimes N_{Y/S}|_X \arrow[r, "\cong"] \arrow[u] & (\frac{I_X I_Y}{(I_X^2+I_Y)I_Y})^\vee \arrow[r] \arrow[u] & N_{Y/S}|_X[1] \arrow[u]
\end{tikzcd}$$

are commutative. Obviously, the left one is commutative. Commutativity of the one in the middle is due to the compatibility of the short exact sequence in Lemma 2.15. The rest of our proof is devoted to the commutativity of the diagram on the right.

There is a natural map $\frac{I_Y}{I_Y I_X^2} \to g_* \frac{I_Y}{(I_X^2+I_Y)I_Y}$. We get $\varphi_* \frac{I_Y}{I_Y I_X^2} \to s_* \frac{I_Y}{(I_X^2+I_Y)I_Y}$ by applying φ_* on both sides. This gives the two compatible short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \frac{I_X I_Y}{I_X^2 I_Y} \arrow[r] \arrow[d] & \varphi_* \frac{I_Y}{I_Y I_X^2} \arrow[r] \arrow[d] & \frac{I_Y}{I_Y I_X} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \frac{I_X I_Y}{(I_X^2+I_Y)I_Y} \arrow[r] & s_* \frac{I_Y}{(I_X^2+I_Y)I_Y} \arrow[r] & \frac{I_Y}{I_Y I_X} \arrow[r] & 0,
\end{tikzcd}$$

which proves that the diagram on the right is commutative. □

The following proposition is similar to Proposition 2.4.

Proposition 2.18 *There is a commutative diagram*

$$\begin{tikzcd}
 & \mathrm{Sym}^2 N_{X/S} \arrow[r] \arrow[d] & N_{X/S}[1] \arrow[d] \\
 & \mathrm{Sym}^2 N_{Y/S}|_X \oplus (N_{X/Y} \otimes N_{Y/S}|_X) \cong (\frac{I_X I_Y}{I_X^2 I_Y})^\vee \arrow[r] & N_{Y/S}|_X[1],
\end{tikzcd}$$

where the bottom horizontal map is in Definition 2.16.

However, we do not expect the following big diagram

$$\begin{tikzcd}
N_{X/S} \otimes N_{X/S} \arrow[r] \arrow[d] & \mathrm{Sym}^2 N_{X/S} \arrow[r] \arrow[d] & N_{X/S}[1] \arrow[d] \\
N_{X/S} \otimes N_{Y/S}|_X \arrow[r] & \mathrm{Sym}^2 N_{Y/S}|_X \oplus (N_{X/Y} \otimes N_{Y/S}|_X) \cong (\frac{I_X I_Y}{I_X^2 I_Y})^\vee \arrow[r] & N_{Y/S}|_X[1]
\end{tikzcd}$$

is commutative generally. One can check that the left square is not commutative as mentioned in Proposition 2.4.

Proof. We need to prove that the two small diagrams in diagram (2)

$$\begin{tikzcd}
\mathrm{Sym}^2 N_{X/S} \arrow[r, "\cong"] \arrow[d] & (\frac{I_X^2}{I_X^3})^\vee \arrow[r] \arrow[d] & N_{X/S}[1] \arrow[d] \\
\mathrm{Sym}^2 N_{Y/S}|_X \oplus (N_{X/Y} \otimes N_{Y/S}|_X) \arrow[r, "\cong"] & (\frac{I_X I_Y}{I_X^2 I_Y})^\vee \arrow[r] & N_{Y/S}|_X[1]
\end{tikzcd} \qquad (2)$$

are commutative. The diagram on the right of (2) commutes because the two short exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \frac{I_XI_Y}{I_X^2I_Y} \arrow[r] \arrow[d] & \varphi_*\frac{I_Y}{I_YI_X^2} \arrow[r] \arrow[d] & \frac{I_Y}{I_YI_X} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \frac{I_X^2}{I_X^3} \arrow[r] & \varphi_*\frac{I_X}{I_X^3} \arrow[r] & \frac{I_X}{I_X^2} \arrow[r] & 0.
\end{tikzcd}$$

are compatible. Let us prove that the left diagram in (2) commutes. We construct a diagram below

$$\begin{tikzcd}
\operatorname{Sym}^2 N^\vee_{Y/S} \oplus (N^\vee_{X/Y} \otimes N^\vee_{Y/S}|_X) \arrow[rr, dashed, "\epsilon"] \arrow[drr, "\zeta"] & & \operatorname{Sym}^2 N^\vee_{Y/S}|_X \arrow[rr, "\epsilon'"] & & \operatorname{Sym}^2 N^\vee_{X/S} \\
& & N^\vee_{X/Y} \otimes N^\vee_{Y/S}|_X \arrow[rr, dashed, "\zeta'"] & & N^\vee_{X/S} \otimes N^\vee_{X/S} \arrow[u, "\vartheta"] \\
\frac{I_XI_Y}{I_X^2I_Y} \arrow[uu] \arrow[rr, dashed, "\delta"] \arrow[drr, "\xi"] & & \frac{I_Y^2}{I_Y^2I_X} \arrow[rr, "\delta'"] \arrow[uu, bend right] & & \frac{I_X^2}{I_X^3} \arrow[uu, bend right] \\
& & \frac{I_XI_Y}{(I_X^2+I_Y)I_Y} \cong \frac{I_X}{I_X^2+I_Y} \otimes \frac{I_Y}{I_YI_X} \arrow[uu, bend left] \arrow[rr, dashed, "\xi'"] & & \frac{I_X}{I_X^2} \otimes \frac{I_X}{I_X^2}, \arrow[u, "\vartheta'"] \arrow[uu, bend right]
\end{tikzcd}$$

where all the vertical maps are natural isomorphisms. The dotted arrows are constructed by splittings in the proof of Lemma 2.15, and the solid arrows are the obvious ones which also appear in the proof of Lemma 2.15. The two short exact sequences and their splittings in the proof of Lemma 2.15 are compatible, so the diagram above is commutative. Taking the direct sum and direct product of the maps above, we get a commutative diagram

$$\begin{tikzcd}
\operatorname{Sym}^2 N^\vee_{Y/S} \oplus (N^\vee_{X/Y} \otimes N^\vee_{Y/S}|_X) \arrow[r, "\epsilon\times\zeta"] & \operatorname{Sym}^2 N^\vee_{Y/S} \oplus (N^\vee_{X/Y} \otimes N^\vee_{Y/S}|_X) \arrow[r, "\epsilon'\oplus(\vartheta\circ\zeta')"] & \operatorname{Sym}^2 N^\vee_{X/S} \\
\frac{I_XI_Y}{I_X^2I_Y} \arrow[u, "\cong"] \arrow[r, "\xi\times\delta"] & \frac{I_Y^2}{I_Y^2I_X} \oplus \frac{I_XI_Y}{I_X^2+I_Y} \arrow[u, "\cong"] \arrow[r, "(\vartheta'\circ\xi')\oplus\delta'"] & \frac{I_X^2}{I_X^3}. \arrow[u, "\cong"]
\end{tikzcd}$$

We hope to prove that the diagram above is dual to the left square in (2). This says that we need to prove the following statement (3):

The map we constructed above $((\vartheta' \circ \xi') \oplus \delta') \circ (\xi \times \delta)$ is equal to the natural map $\frac{I_X I_Y}{I_X^2 I_Y} \to \frac{I_X^2}{I_X^3}$.

Consider the following diagram

$$\begin{tikzcd}
\frac{I_X I_Y}{I_X^2 I_Y} \arrow[rr, dashed, "\delta"] \arrow[drr, "\xi"] & & \frac{I_Y^2}{I_Y^2 I_X} \arrow[rr, "\delta'"] & & \frac{I_X^2}{I_X^3} \\
& & \frac{I_X I_Y}{(I_X^2 + I_Y) I_Y} \cong \frac{I_X}{I_X^2 + I_Y} \otimes \frac{I_Y}{I_Y I_X} \arrow[rr, dashed, "\xi'"'] & & \frac{I_X}{I_X^2} \otimes \frac{I_X}{I_X^2} \arrow[u, "\vartheta'"] \\
N^\vee_{X/S} \otimes N^\vee_{Y/S}|_X \arrow[uu] \arrow[rr, dashed, "\theta"] \arrow[drr, "\lambda"] & & N^\vee_{Y/S}|_X \otimes N^\vee_{Y/S}|_X \arrow[uu, bend right] \arrow[uurr, "\theta'"] & & \\
& & N^\vee_{X/Y} \otimes N^\vee_{Y/S}|_X, \arrow[uu, bend left] \arrow[uurr, "\lambda'"] & &
\end{tikzcd}$$

where the dotted arrows are the splittings in the proof of Lemma 2.15. The diagram above is commutative because the two short exact sequences and their splittings in the proof of Lemma 2.15 are compatible.

Notice that $N^\vee_{X/S} \otimes N^\vee_{Y/S}|_X \to \frac{I_X I_Y}{I_X^2 I_Y}$ is surjective. To prove the statement (3), it suffices to show that the map $(\theta' \oplus (\vartheta' \circ \lambda')) \circ (\theta \times \lambda) : N^\vee_{X/S} \otimes N^\vee_{Y/S}|_X = \frac{I_X}{I_X^2} \otimes \frac{I_Y}{I_Y I_X} \to \operatorname{Sym}^2 N^\vee_{X/S} = \frac{I_X^2}{I_X^3}$ constructed via the splittings is equal to the natural map

$$\frac{I_X}{I_X^2} \otimes \frac{I_Y}{I_Y I_X} \to \frac{I_X^2}{I_X^3},$$

$$(a \otimes b) \to ab, \text{ for } a \in \frac{I_X}{I_X^2}, \text{ and } b \in \frac{I_Y}{I_Y I_X}.$$

One can verify it easily. □

The following lemma is analogous to property (II) in Section 2.4.

Lemma 2.19 *If* $N_{X/S}[-1] \to N_{Y/S}|_X[-1]$ *preserves the pre-Lie brackets, then there is a commutative diagram*

$$\begin{tikzcd}
\mathrm{Sym}^2 N_{Y/S}|_X \arrow[r] & N_{Y/S}|_X[1] \\
\mathrm{Sym}^2 N_{Y/S} \oplus (N_{X/Y} \otimes N_{Y/S}|_X) \arrow[u] \arrow[r] & N_{Y/S}|_X[1], \arrow[u, "id"']
\end{tikzcd}$$

where the bottom horizontal map is in Definition 2.16.

Proof. Put what we want to prove into a larger diagram

$$\begin{tikzcd}
\mathrm{Sym}^2 N_{Y/S}|_X \arrow[rr, "\chi"] & & N_{Y/S}|_X[1] \\
& \mathrm{Sym}^2 N_{Y/S}|_X \oplus (N_{X/Y} \otimes N_{Y/S}|_X) \arrow[ul, "\psi"'] \arrow[ur, "\phi"] & \\
\mathrm{Sym}^2 N_{X/S} \arrow[uu] \arrow[ur, "\iota"] \arrow[rr] & & N_{Y/S}|_X[1]. \arrow[uu]
\end{tikzcd}$$

The outer square commutes because we assume that the brackets are preserved. We want to show that $\chi \circ \psi = \phi$. The commutativity of the outer square and Proposition 2.18 show that $\chi \circ \psi \circ \iota = \phi \circ \iota$. The map ι splits, so we get our desired result. $\square$

There should be a statement analogous to property (III). It will appear in the proof of Theorem 2.3 below.

Proof of Theorem 2.3. We first prove that the Bass-Quillen Lie module map

$\alpha_{s,N^\vee_{Y/S}|_{X^{(1)}_Y}}$ is zero if the pre-Lie brackets are preserved. There is a commutative diagram

$$\begin{tikzcd}
N_{Y/S}|X \otimes N_{Y/S}|_X \arrow[r] & \operatorname{Sym}^2 N_{Y/S}|_X \arrow[r] & N_{Y/S}|_X[1] \\
 & \operatorname{Sym}^2 N_{Y/S}|_X \oplus (N_{X/Y} \otimes N_{Y/S}|_X) \arrow[u] \arrow[r] & N_{Y/S}|_X[1] \arrow[u, "id"'] \\
 & N_{X/S} \otimes N_{Y/S}|_X \arrow[u] \arrow[r] \arrow[uul] & N_{Y/S}|_X[1] \arrow[u, "id"'] \\
 & N_{X/Y} \otimes N_{Y/S}|_X \arrow[u] \arrow[r] & N_{Y/S}|_X[1] \arrow[u, "id"']
\end{tikzcd}$$

due to Lemma 2.19 and Proposition 2.17. The Bass-Quillen class $\alpha_{s,N^\vee_{Y/S}|_{X^{(1)}_Y}}$ that appears at the bottom of the diagram above vanishes because the composition $N_{X/Y} \to N_{X/S} \to N_{Y/S}|_X$ is zero.

Let us turn to prove that the vector bundle map $N_{X/S}[-1] \to N_{Y/S}|_X[-1]$ preserves the pre-Lie brackets if the Bass-Quillen class $\alpha_{s,N^\vee_{Y/S}|_{X^{(1)}_Y}}$ is zero. The proof is similar to the one in Section 2.4. In [7] Calaque and Grivaux showed that the pre-Lie brackets on $N_{X/S}[-1]$ and $N_{Y/S}|_X[-1]$ defined by the extension classes of short exact sequences can be also defined as follows

$$\begin{tikzcd}
\operatorname{Sym}^2 N_{X/S} \arrow[r, dashed] & \operatorname{Sym}^2 T_S|_X \arrow[r] & T_S|_X[1] \arrow[r] & N_{X/S}[1],
\end{tikzcd}$$

$$\begin{tikzcd}
\operatorname{Sym}^2 N_{Y/S}|_X \arrow[r, dashed] & \operatorname{Sym}^2 T_S|_X \arrow[r] & T_S|_X[1] \arrow[r] & N_{Y/S}|_X[1].
\end{tikzcd}$$

The dotted arrow is due to the fact that $f : X \hookrightarrow S$ and $j : Y \hookrightarrow S$ split to first order. The map in the middle is the Atiyah class.

Using the compatibility condition on splittings of tangent bundles and the fact above, we conclude that the diagram

$$\begin{tikzcd}
\operatorname{Sym}^2 N_{Y/S}|_X \arrow[r, dashed] \arrow[d, dashed] & \operatorname{Sym}^2 T_S|_X \arrow[r] \arrow[d, "id"] & T_S|_X[1] \arrow[r] \arrow[d, "id"] & N_{Y/S}|_X[1] \\
\operatorname{Sym}^2 N_{X/S} \arrow[r, dashed] & \operatorname{Sym}^2 T_S|_X \arrow[r] & T_S|_X[1] \arrow[r] & N_{X/S}[1] \arrow[u]
\end{tikzcd}$$

is commutative. The diagram above and Proposition 2.18 say that we have a commutative diagram

$$\begin{tikzcd}
\operatorname{Sym}^2 N_{Y/S}|_X \arrow[r] \arrow[d] & N_{Y/S}|_X[1] \arrow[d, "id"] \\
\operatorname{Sym}^2 N_{Y/S}|_X \oplus (N_{Y/S}|_X \otimes N_{X/Y}) \arrow[r] & N_{Y/S}|_X[1]
\end{tikzcd}$$

which is analogous to property (III) in Section 2.4. Based on the diagram above and Proposition 2.17 it is clear that the diagram

$$\begin{tikzcd}
\operatorname{Sym}^2 N_{Y/S}|_X \oplus (N_{Y/S}|_X \otimes N_{X/Y}) \arrow[r] \arrow[d] & N_{Y/S}|_X[1] \arrow[d, "id"] \\
\operatorname{Sym}^2 N_{Y/S}|_X \arrow[r] & N_{Y/S}|_X[1]
\end{tikzcd}$$

is commutative if the Bass-Quillen class $\alpha_{s,N^\vee_{Y/S}|_{X_Y^{(1)}}}$ is zero. Compose the diagram above with the one in Proposition 2.18. We get a commutative diagram

$$\begin{tikzcd}
\operatorname{Sym}^2 N_{X/S} \arrow[r] \arrow[d] & N_{X/S}[1] \arrow[d] \\
\operatorname{Sym}^2 N_{Y/S}|_X \oplus (N_{Y/S}|_X \otimes N_{X/Y}) \arrow[r] \arrow[d] & N_{Y/S}|_X[1] \arrow[d, "id"] \\
\operatorname{Sym}^2 N_{Y/S}|_X \arrow[r] & N_{Y/S}|_X[1],
\end{tikzcd}$$

so we conclude that the brackets are preserved if the Bass-Quillen class $\alpha_{s,N^\vee_{Y/S}|_{X_Y^{(1)}}}$ is zero. □

We end this section by providing an example where the Bass-Quillen class is not zero. Consider the embeddings

$$\begin{tikzcd} X \arrow[r, "\Delta_X"] & X \times X = Y \arrow[l, bend right, dashed, "p_1"'] \arrow[r, "\Delta_{X \times X}"] & S = X \times X \times X \times X \arrow[l, bend right, dashed, "\pi_1"'] \end{tikzcd}$$

for a smooth scheme X, where all the inclusions are the diagonal embeddings, and all the splittings are the projections to the first factor. The normal bundle $N_{Y/S}$ is $p_1^*T_X \oplus p_2^*T_X$ in this case, where p_1 and p_2 are the two projections: $X \times X \to X$. It is clear that the Bass-Quillen class $\alpha_{p_1, N^\vee_{Y/S}|_{X^{(1)}_Y}}$: $T_X \otimes (T_X \oplus T_X) \to T_X[1]$ is zero plus the Atiyah class: $T_X[-1] \otimes T_X[-1] \to T_X[-1]$. It is not zero in general.

Chapter 3

Orbifold Hochschild cohomology

We study the multiplicative structure of orbifold Hochschild cohomology in an attempt to generalize the results of Kontsevich and Calaque-Van den Bergh relating the Hochschild and polyvector field cohomology rings of a smooth variety.

We introduce the concept of linearized derived scheme, and we argue that when X is a smooth algebraic variety and G is a finite abelian group acting on X, the derived fixed locus $\widetilde{X}^G$ admits an HKR linearization. This allows us to define a product on the cohomology of polyvector fields of the orbifold $[X/G]$. We analyze the obstructions to the associativity of this product and show that they vanish in certain special cases. We conjecture that in these cases the resulting polyvector field cohomology ring is isomorphic to the Hochschild cohomology of $[X/G]$.

Inspired by mirror symmetry we introduce a bigrading on the Hochschild homology of Calabi-Yau orbifolds. We propose a conjectural product which respects this bigrading and simplifies the previously introduced product.

3.1 Background

In this section we review the theorem of Kontsevich and Calaque-Van den Bergh about the multiplicative structure on Hochschild cohomology of smooth algebraic varieties and a similar theorem in Lie theory. We recall Kontsevich's proof based on deformation quantization of Poisson manifolds. Then we move to the orbifold case and review the construction of the orbifold HKR isomorphism. A few new definitions and interpretations in this section will be important throughout this chapter.

3.1.1 Multiplicative structure on Hochschild cohomology

Both sides in the HKR isomorphism below

$$I^{\mathsf{HKR}} : \mathrm{HT}^*(X) \xrightarrow{\sim} \mathrm{HH}^*(X)$$

are graded commutative rings: polyvector field cohomology classes can be multiplied using the wedge product on $\wedge^* TX$ and cup product on cohomology, while Hochschild cohomology classes can be composed using the Yoneda product. However, the HKR isomorphism is not a ring map in general.

Kontsevich [27] claimed that the rings $\mathrm{HT}^*(X)$ and $\mathrm{HH}^*(X)$ *are* in fact isomorphic, via a modification of the HKR isomorphism. This result was later proved by Calaque and Van den Bergh [6].

Theorem 3.1 (Calaque-Van den Bergh, Kontsevich) *The map*

$$\mathrm{HT}^*(X) \xrightarrow{\lrcorner \mathrm{td}^{-\frac{1}{2}}} \mathrm{HT}^*(X) \xrightarrow{\mathsf{HKR}} \mathrm{HH}^*(X)$$

is a ring *isomorphism. Here* td *is the Todd class of* X*.*

The ring $\mathrm{HT}^*(X)$ is bigraded, and the product respects this bigrading. Moreover, the Hochschild cochain complex carries a filtration given by order of polydifferential operators, which in turn induces a filtration on Hochschild cohomology. Kontsevich's claim (along with the explicit formula for the corrected ring isomorphism) can be interpreted as saying that this filtration admits a multiplicative splitting, yielding a bigrading on $\mathrm{HH}^*(X)$ which refines the usual grading.

Before we move on to the orbifold case, we recall a similar theorem in Lie theory and Kontsevich's proof. If we replace the Lie algebra $T_X[-1]$ in the derived category by a Lie algebra $\mathfrak{g}$ in vector spaces, we get a similar theorem first proved by Duflo [17].

Theorem 3.2 (Duflo, Kontsevich) *The map*

$$H^*(\mathfrak{g}, \mathrm{Sym}(\mathfrak{g})) \xrightarrow{J^{\frac{1}{2}}} H^*(\mathfrak{g}, \mathrm{Sym}(\mathfrak{g})) \xrightarrow{\mathsf{PBW}} H^*(\mathfrak{g}, U\mathfrak{g})$$

is an isomorphism of algebras, where J is considered as an element in the completed symmetric algebra $\widehat{\mathrm{Sym}(\mathfrak{g}^*)}$ and given by the formula

$$J(x) = \det(\frac{1 - \exp^{-ad_x}}{ad_x})$$

for $x \in \mathfrak{g}$.

The theorem above is a consequence of Kontsevich's main theorem on deformation quantization of Poisson manifolds [27] which we recall in the next subsection.

3.1.2 Deformation quantization of Poisson manifolds

All the contents in this subsection are from [27]. Let M be a real smooth manifold and $\Pi \in \Gamma(\wedge^2 TM)$ be a smooth bi-vector field. The formula $\{f, g\} = \Pi(df \wedge dg)$ defines a bilinear operator for smooth functions f and g. We say that the bi-vector field defines a Poisson structure on M if $\{-, -\}$ satisfies the Jacobi identity.

Kontsevich relates the Poisson structure on M with the deformation of the algebra of smooth functions on M. Let A be the algebra of smooth functions on the manifold

M. The star-product on A is an associative $\mathbb{R}[[\hbar]]$-linear product on $A[[\hbar]] = A \otimes_{\mathbb{R}} \mathbb{R}[[\hbar]]$ given by the formula for $f, g \in A$

$$f \star g = fg + \hbar B_1(f,g) + \hbar^2 B_2(f,g) + \cdots \in A[[\hbar]],$$

where B_i are bidifferential operators. We extend the product to $A[[\hbar]]$ by linearity. We are only interested in the star products up to the following natural isomorphisms.

The map

$$D : f \to f + \hbar D_1(f) + \hbar^2 D_2(f) + \cdots$$

extended by $\hbar$-linearity defines an automorphism $D(\hbar)$ of $A[[\hbar]]$. It acts on the set of star-products as

$$\star \to \star', x \star' y = D(\hbar)(D(\hbar)^{-1} x \star D(\hbar)^{-1} y),$$

for $x, y \in A[[\hbar]]$.

Theorem 3.3 (Kontsevich) *The set of equivalence classes of star-products on a smooth manifold M can be naturally with the set of equivalence classes of Poisson structures on M depending formally on $\hbar$*

$$\Pi = \Pi_1 \hbar + \Pi_2 \hbar^2 + \cdots \in \Gamma(\wedge^2 TM)[[\hbar]]$$

modulo diffeomorphism.

It is observed by Deligne is that every deformation problem is governed by some differential graded Lie algebra. The theorem above is a consequence of a more general statement about differential graded Lie algebras.

Definition 3.4 *A differential graded Lie algebra (DGLA) is a graded vector space L with $[-,-]$ and d, where $[-,-]$ is graded antisymmetric and satisfies graded Jacobi identity, and d is a differential satisfying the Leibniz rule*

$$d[x,y]=[dx,y]+(-1)^{|x|}[x,dy].$$

We define two DGLAs associated with a smooth manifold M. Let A be the algebra of smooth functions on M the Hochschild cochain complex is

$$C^*(A,A)=\bigoplus_{k\geq 0} C^k(A,A), C^k(A,A)=\mathrm{Hom}_k(A^{\otimes k},A).$$

The shifted complex $C^*(A,A)[1]$ is a DGLA. The differential is

$$d\Phi(a_0\otimes\cdots\otimes a_{k+1})=a_0\Phi(a_1\otimes\cdots\otimes a_{k+1})-\sum_{i=0}^{k}\Phi(a_0\otimes\cdots\otimes(a_ia_{i+1})\otimes\cdots\otimes a_{k+1})$$
$$+(-1)^k\Phi(a_0\otimes\cdots\otimes a_k)a_{k+1}, \Phi\in\mathrm{Hom}_k(A^{\otimes k+1},A),$$

and the Lie bracket is

$$[\Phi_1,\Phi_2]=\Phi_1\circ\Phi_2-(-1)^{k_1k_2}\Phi_2\circ\Phi_1$$

for $\Phi_i\in\mathrm{Hom}_k(A^{\otimes k_i+1},A)$, where the operation $\circ$ is defined as

$$(\Phi_1\circ\Phi_2)(a_0\otimes\cdots\otimes a_{k_1+k_2})$$
$$=\sum_{i=0}^{k_1}(-1)^{ik_2}\Phi_1(a_0\otimes\cdots\otimes a_{i-1}\otimes(\Phi_2(a_i\otimes\cdots\otimes a_{i+k_2}))\otimes a_{i+k_2+1}\otimes\cdots\otimes a_{k_1+k_2}).$$

The first DGLA $D_{\mathsf{poly}}(M)$ we hope to study is the subalgebra of the shifted Hochschild cochain complex defined as follows. The space $D^n_{\mathsf{poly}}(M)$ consists of Hochschild cochains $A^{\otimes n+1}\to A$ given by polydifferential operators. In local coordinates (x^i) an element in $D^n_{\mathsf{poly}}(M)$ can be written as

$$f_0\otimes\cdots\otimes f_n\to\sum_{(I_0,\ldots,I_n)}C^{I_0,\ldots,I_n}(x_1,\cdots,x_d)\partial_{I_0}(f_0)\ldots\partial_{I_n}(f_n),$$

where the sum is finite over the multi-indices I_k.

The second DGLA is the polyvector fields $T_{\mathsf{poly}}(M)$ defined as

$$T^n_{\mathsf{poly}}(M) = \Gamma(M, \wedge^{n+1}TM).$$

The differential is zero and the bracket is

$$[\xi_0 \wedge \cdots \wedge \xi_k, \eta_0 \wedge \cdots \wedge \eta_l] =$$

$$\sum_{i=0}^{k}\sum_{j=0}^{l}(-1)^{i+j+k}[\xi_i, \eta_j] \wedge \xi_0 \wedge \cdots \wedge \xi_{i-1} \wedge \xi_{i+1} \wedge \eta_0 \wedge \cdots \wedge \eta_{j-1} \wedge \eta_{j+1} \wedge \cdots \wedge \eta_l,$$

for $k, l \geq 0$ and $\xi_i, \eta_i \in \Gamma(M, TM)$, and

$$[\xi_0 \wedge \cdots \wedge \xi_k, h] = \sum_{i=0}^{k}(-1)^i \xi_i(h)(\xi_0 \wedge \cdots \wedge \xi_{i-1} \wedge \xi_{i+1} \wedge \cdots \wedge \xi_k),$$

for a smooth function h.

We discuss the relation between DGLAs and deformations. Let $\mathfrak{g}$ be a nilpotent DGLA. We define the deformation functor $\mathsf{Def}_{\mathfrak{g}}$ as follows. We define $\mathcal{MC}(\mathfrak{g})$ to be the set of Maurer-Cartan equation modulo the gauge equivalence

$$\mathcal{MC}(\mathfrak{g}) = \{x \in \mathfrak{g}^1 | dx + \frac{1}{2}[x, x] = 0\}/\Gamma^0$$

where Γ^0 is the nilpotent group associated with the Lie algebra g^0. The functor $\mathsf{Def}_{\mathfrak{g}}$ is from the category of finite-dimensional nilpotent commutative algebras without unit to the category of sets

$$\mathsf{Def}_{\mathfrak{g}}(m) = \mathcal{MC}(\mathfrak{g} \otimes m).$$

It can be extended to $\hbar\mathbb{R}[[\hbar]] = \varprojlim(\hbar\mathbb{R}[[\hbar]]/\hbar^k\mathbb{R}[\hbar])$ by taking limits.

Let A be the algebra of smooth functions on M. Suppose there is a deformation of the associative algebra A

$$x * y = a \cdot b + \hbar B_1(x, y) + \hbar^2 B_2(x, y) + \cdots .$$

One can check that $*$ is associative if and only if $B = \hbar B_1 + \hbar^2 B_2 + \cdots$ is an element in $\mathsf{Def}_{\mathfrak{g}}(\hbar\mathbb{R}[[\hbar]] \otimes \mathfrak{g})$, i.e.,

$$dB + \frac{1}{2}[B, B] = 0,$$

where the DGLA $\mathfrak{g}$ is the shifted Hochschild cochain complex.

The deformation functor associated with $D_{\mathsf{poly}}(M)$ classifies the star-product on $A[[\hbar]]$ and the deformation functor associated with $T_{\mathsf{poly}}(M)$ classifies the Poisson structures on M modulo diffeomorphism.

The morphisms between DGLAs are called L_∞-morphisms introduced in [27]. The L_∞-morphisms induce maps on homology of DGLAs. We say the L_∞-morphism is a quasi-isomorphism if the induced map on homology is an isomorphism.

Proposition 3.5 (Kontsevich) *An L_∞-morphism from $\mathfrak{g}_1$ to $\mathfrak{g}_2$ induce natural transformation of the deformation functors, where $\mathfrak{g}_i$ are DGLAs. If an L_∞-morphism is a quasi-isomorphism, then it induces an isomorphism of deformation functors.*

Theorem 3.6 (Kontsevich) *There is a quasi-isomorphism $\mathcal{U} : T_{\mathsf{poly}}(M) \to D_{\mathsf{poly}}(M)$. The formula of the morphism $\mathcal{U}$ is written down explicitly in [27].*

Proposition 3.5 and the theorem above imply Theorem 3.3 immediately. To prove Theorem 3.2, we need to study the tangent spaces and tangent maps associated with the deformation functor of a DGLA $\mathfrak{g}$. If $x \in (\mathfrak{g} \otimes m)^1$ satisfies the Maurer-Cartan equation $dx + \frac{1}{2}[x, x] = 0$ where m is a finite dimensional nilpotent without unit, the tangent space T_x at x is defined as the complex $\mathfrak{g}[1] \otimes m$ with the differential $d + [x, -]$. When $\mathfrak{g}$ is the Hochschild cochain complex of an algebra A and $m = \hbar\mathbb{R}[[\hbar]]$, the tangent space at x is the Hochschild cochain complex of the deformed algebra $A[[\hbar]]$ corresponding to

$x \in \mathsf{Def}_{\mathfrak{g}}(m)$. Kontsevich [27] defines the cup product on the tangent space $\mathfrak{g}[1] \otimes m$ when $\mathfrak{g}$ is $T_{\mathsf{poly}}(M)$ or $D_{\mathsf{poly}}(M)$.

Theorem 3.7 (Kontsevich) *The L_∞-morphism $\mathcal{U}$ in Theorem 3.6 maps the cup product for $T_{\mathsf{poly}}(M)$ to the cup product for $D_{\mathsf{poly}}(M)$.*

Kontsevich applied Theorem 3.7 to prove Theorem 3.2. Let $\mathfrak{g}$ be a finite dimensional Lie algebra over $\mathbb{R}$. The dual space $\mathfrak{g}^*$ is naturally a Poisson manifold with the Kirillov-Poisson bracket [26]. The space $\mathfrak{g}^*$ can be considered as an algebraic Poisson manifold because the coefficients of the bracket are linear functions on $\mathfrak{g}^*$. We consider the algebra of algebraic functions on $\mathfrak{g}^*$ instead of the algebra of smooth functions. The algebra of algebraic functions on $\mathfrak{g}^*$ is the symmetric algebra $\mathrm{Sym}(\mathfrak{g})$.

Theorem 3.8 (Kontsevich) *The canonical quantization of the algebra of algebraic functions on the Poisson manifold $\mathfrak{g}^*$ is isomorphic to the family of algebras $U_\hbar(\mathfrak{g})$ defined as the universal enveloping algebras of $\mathfrak{g}$ with the bracket $\hbar[-,-]$.*

The Kirillov-Poisson bracket and the deformed product satisfy Maurer-Cartan equation for $T_{\mathsf{poly}}(M)$ and $D_{\mathsf{poly}}(M)$ respectively. The corresponding tangent spaces are the Chevalley-Eilenberg complex of the $\mathfrak{g}$-module $\mathrm{Sym}(\mathfrak{g})$ and the Hochschild cochain complex of $U\mathfrak{g}$ respectively. Applying Theorem 3.7 and passing to the cohomology, we obtain Theorem 3.2.

3.1.3 Formality of derived schemes

We go back to the case of algebraic geometry. The problem of understanding an analogue of Kontsevich's claim for orbifolds has been open for at least 20 years. The most recent (negative) progress is due to Negron-Schedler [29] who argue that the Hochschild cochain complex of an orbifold does not satisfy a formality result similar to the one Kontsevich used. However, this does not rule out the possibility that an analogue of the Kontsevich claim holds for a *corrected* filtration from the one they study. This problem is particularly interesting in view of its connections to Ruan's crepant resolution conjecture. For example, getting a good understanding of the orbifold Hochschild cohomology product would explain the matching between the cohomology ring of the Hilbert scheme of n-points on a K3 surface S and the Chen-Ruan orbifold cohomology ring of $[S^n/\Sigma_n]$, as observed by Fantechi-Göttsche [18].

The HKR map relates the derived Lie group LX with its Lie algebra $N_{X/LX} = \mathbb{T}_X[-1]$. In general, suppose $\widetilde{X}$ is an arbitrary derived scheme which is not necessary a derived group. We can still consider the total space $\mathbb{N}_{X/\widetilde{X}}$ of the normal bundle of X in $\widetilde{X}$, where X is the underlying classical scheme $X \hookrightarrow \widetilde{X}$. This leads to the following definition.

Definition 3.9 *For a derived scheme $\widetilde{X}$, the* linearization $\mathbb{L}_{\widetilde{X}}$ *of $\widetilde{X}$ is defined to be the total space of the normal bundle $N_{X/\widetilde{X}}$, where X is the underlying classical scheme $X \hookrightarrow \widetilde{X}$. A choice of isomorphism $\mathbb{L}_{\widetilde{X}} \cong \widetilde{X}$ (if one exists) will be called a* linearization *of $\widetilde{X}$.*

For example, consider $X \hookrightarrow LX$. Then

$$\mathbb{L}_{LX} = \mathbb{N}_{X/LX} = \mathbb{T}_X[-1],$$

and the HKR isomorphism provides a linearization of LX.

We need to address a technical detail about the above isomorphism. The linearization $\mathbb{L}_{\widetilde{X}}$ of a derived scheme $\widetilde{X}$ is by definition the total space of the normal bundle $N_{X/\widetilde{X}}$, hence it comes with a natural projection which makes it a scheme over X. Moreover, this projection splits the inclusion $X \hookrightarrow \widetilde{X}$. However, in general $\widetilde{X}$ may not admit such a projection. This explains why in general here is no way to define an isomorphism $\mathbb{L}_{\widetilde{X}} \cong \widetilde{X}$ over X. If we hope to define an isomorphism $\mathbb{L}_{\widetilde{X}} \cong \widetilde{X}$, we usually need to find a natural base scheme Y such that both $\widetilde{X}$ and $\mathbb{L}_{\widetilde{X}}$ are affine over Y. Then we can consider the structure complex of $\mathbb{L}_{\widetilde{X}}$ and $\widetilde{X}$ as $\mathscr{O}_Y$-algebras. There is a bijection between the set of isomorphisms $\mathscr{O}_{\mathbb{L}_{\widetilde{X}}} \cong \mathscr{O}_{\widetilde{X}}$ in the derived category of Y and the set of isomorphisms $\mathbb{L}_{\widetilde{X}} \cong \widetilde{X}$ over Y. A choice of such an isomorphism will be called a linearization of $\widetilde{X}$ over Y.

The HKR isomorphism

$$\exp : \mathbb{T}_X[-1] = \mathbb{L}_{LX} \xrightarrow{\cong} LX$$

linearizes the derived loop space LX over X. Here $LX = X \times^R_{X \times X} X$ is to be viewed as a scheme over X via one of the two projection maps onto the left or right factors $X \times^R_{X\times X} X \to X$.

For most derived schemes $\widetilde{X}$, even if they are affine over a scheme Y, it is not true that $\widetilde{X}$ is isomorphic to $\mathbb{L}_{\widetilde{X}}$ over Y. If one thinks about the structure complexes as $\mathscr{O}_Y$-algebras, the existence of such an isomorphism would say that the structure complex $\mathscr{O}_{\widetilde{X}}$ of $\widetilde{X}$ would be quasi-isomorphic to its cohomology. This is equivalent to saying that the derived scheme $\widetilde{X}$ is formal over Y in the sense of [16]. See [3] for more discussions.

The HKR isomorphism on Hochschild cohomology is obtained by dualizing the structure complexes of $\mathbb{T}_X[-1]$ and LX with respect to X, and taking global sections:

$$\begin{aligned}\mathrm{HH}^*(X) &= \mathrm{Hom}_{X\times X}(\Delta_*\mathscr{O}_X, \Delta_*\mathscr{O}_X) = \mathrm{Hom}_X(\Delta^*\Delta_*\mathscr{O}_X, \mathscr{O}_X)\\ &= \Gamma(X, (\mathscr{O}_L X)^\vee) \cong \Gamma(X, \operatorname{Sym} T_X[-1]) = \bigoplus H^p(X, \wedge^q T_X)\\ &= \mathrm{HT}^*(X).\end{aligned}$$

Duals of functions are distributions. We will need a relative version of this concept, made precise in the following definition.

Definition 3.10 *For a map of spaces $f : Y \to X$ the space of relative distributions is defined by*

$$\mathbb{D}(Y/X) = \mathrm{Hom}(f_*\mathscr{O}_Y, \mathscr{O}_X).$$

We will often omit the space X when it is clear from context.

For example consider the map $LX \to X$, where X is a smooth algebraic variety. Then the Hochschild cohomology of X is naturally identified with the space of distributions on LX,

$$\mathbb{D}(LX/X) = \mathrm{Hom}(\Delta^*\mathscr{O}_\Delta, \mathscr{O}_X) = \mathrm{HH}^*(X),$$

while the polyvector field cohomology of X is naturally the space of distributions on $\mathbb{T}_X[-1]$, the linearization of LX,

$$\mathbb{D}(\mathbb{T}_X[-1]) = \mathrm{Hom}(\operatorname{Sym}\Omega_X[1], \mathscr{O}_X) = \mathrm{HT}^*(X).$$

Therefore we should think of polyvector fields as (invariant) distributions on the Lie algebra $\mathbb{T}_X[-1]$ and Hochschild cohomology as (invariant) distributions on the derived group LX. The HKR isomorphism is then interpreted as the isomorphism on distributions induced by the exponential map. The product structures on the two sides are given by convolution of distributions, where the group structure on $\mathbb{T}_X[-1]$ is given by addition in the fibers.

This interpretation was probably the basis for Kontsevich's claim: for ordinary Lie algebras a theorem of Duflo [17] asserts that the rings of invariant distributions on a Lie group and on its Lie algebra are isomorphic, after a correction to the exponential map by what is known as the Duflo element.

3.1.4 The orbifold HKR isomorphism

Before we begin we need an analogue of the HKR isomorphism for orbifolds. Let X be a smooth algebraic variety, let G be a finite group acting on X, and denote by $\mathfrak{X} = [X/G]$ the corresponding global quotient orbifold.

We have the following diagram

$$\begin{tikzcd}
L\mathfrak{X} \arrow[drr, "q"] \arrow[ddr, "p"'] & & \\
& I\mathfrak{X} \arrow[ul] \arrow[r] \arrow[d] & \mathfrak{X} \arrow[d, "\Delta"] \\
& \mathfrak{X} \arrow[r, "\Delta"] & \mathfrak{X} \times \mathfrak{X}.
\end{tikzcd}$$

Here $L\mathfrak{X}$ denotes the loop space of the stack $\mathfrak{X}$ defined by analogy with the case of ordinary spaces as the derived self-intersection

$$L\mathfrak{X} = \mathfrak{X} \times^R_{\mathfrak{X}\times\mathfrak{X}} \mathfrak{X}.$$

Its underlying underived stack $I\mathfrak{X}$ is the inertia stack of $\mathfrak{X}$,

$$I\mathfrak{X} = \mathfrak{X} \times_{\mathfrak{X}\times\mathfrak{X}} \mathfrak{X}.$$

Unlike the case where X is a smooth space with no group action, the two maps $I\mathfrak{X} \to \mathfrak{X}$ are no longer isomorphisms: it is not hard to see that

$$I\mathfrak{X} = \left[(\coprod_{g \in G} X^g)/G \right],$$

where X^g denotes the fixed locus of the action of $g \in G$ on X.

We can rewrite X^g as $\Delta \times_{X \times X} \Delta^g$, where $\Delta = \{(x,x)\} \subset X \times X$ and $\Delta^g = \{(x,gx)\} \subset X \times X$. We get an explicit formula for the derived loop space $L\mathfrak{X}$ if we replace the above intersection by the corresponding derived intersection:

$$L\mathfrak{X} = \left[(\coprod_{g \in G} \widetilde{X^g})/G \right],$$

where

$$\widetilde{X^g} = \Delta \times^R_{X \times X} \Delta^g.$$

We will call $\widetilde{X^g}$ the derived fixed locus of g.

The orbifold HKR isomorphism expresses the derived loop space $L\mathfrak{X}$ as the total space of a certain vector bundle over $I\mathfrak{X}$, as explained by Arinkin, Căldăraru and Hablicsek [3].

The derived loop space $L\mathfrak{X}$ decomposes naturally into connected components, so it is better to look at each component $\widetilde{X^g}$ of $L\mathfrak{X}$ individually. The orbifold HKR isomorphism identifies $\widetilde{X^g}$ with the total space of the tangent bundle of X^g. More precisely, for each $g \in G$ [3] construct a linearization isomorphism of derived schemes over X

$$\mathbb{T}_{X^g}[-1] = \mathbb{L}_{\widetilde{X^g}} \xrightarrow{\sim} \widetilde{X^g}.$$

In explicit terms this translates into an isomorphism of commutative $\mathscr{O}_X$-algebras

$$q_* \mathscr{O}_{\widetilde{X^g}} \xrightarrow{\sim} i_{g*} \operatorname{Sym}(\Omega_{X^g}[1]),$$

where $i_g : X^g \hookrightarrow X$ is the inclusion of the fixed locus. Applying $\mathrm{Hom}(-, \mathscr{O}_X)$ to this algebra isomorphism we get an induced isomorphism on distributions

$$\mathbb{D}(\mathbb{T}_{X^g}[-1]/X) \xrightarrow{\sim} \mathbb{D}(\widetilde{X^g}/X).$$

We will denote $\mathbb{D}(\mathbb{T}_{X^g}[-1]/X)$ by $\mathrm{HT}^*(X; g)$, and $\mathbb{D}(\widetilde{X^g}/X)$ by $\mathrm{HH}^*(X; g)$.

Grothendieck duality allows us to give an explicit form to the space $\mathbb{D}(\mathbb{T}_{X^g}[-1]/X)$:

$$\mathbb{D}(\mathbb{T}_{X^g}[-1]/X) = \mathrm{Hom}_X(i_{g*} \operatorname{Sym} \Omega_{X^g}[1], \mathscr{O}_X) = \bigoplus_{p+q=*} H^{p-c_g}(X^g, \wedge^q T_{X^g} \otimes \omega_g),$$

where c_g is the codimension of X_g/X and ω_g is the dualizing sheaf of the inclusion $X^g \subseteq X$. Taking G-invariants of the direct sum over $g \in G$ we get the final form of the orbifold HKR isomorphism for $\mathfrak{X}$:

$$\begin{aligned}\mathrm{HH}^*(\mathfrak{X}) &= \left(\bigoplus_{g\in G} \mathrm{HH}^*(X; g)\right)^G = \left(\bigoplus_{g\in G} \mathbb{D}(\widetilde{X^g}/X)\right)^G \\ &\cong \left(\bigoplus_{g\in G} \mathrm{HT}^*(X; g)\right)^G = \left(\bigoplus_{g\in G} \bigoplus_{p+q=*} H^{p-c_g}(X^g, \wedge^q T_{X^g} \otimes \omega_g)\right)^G.\end{aligned}$$

We think of the right hand side above as the definition of the space of polyvector fields on $\mathfrak{X}$,

$$\mathrm{HT}^*(\mathfrak{X}) = \left(\bigoplus_{g\in G} \mathrm{HT}^*(X; g)\right)^G.$$

We define

$$\mathrm{HT}^*(X; G) = \left(\bigoplus_{g\in G} \bigoplus_{p+q=*} H^{p-c_g}(X^g, \wedge^q T_{X^g} \otimes \omega_g)\right).$$

Note that $\mathrm{HT}^*(X; G)$ carries a natural G action, and we set

$$\mathrm{HT}^*([X/G]) = \mathrm{HT}^*(X; G)^G.$$

As stated above, for a smooth variety X there is an obvious associative product on $\mathrm{HT}^*(X)$. However, when G is non-trivial, it is not at all obvious what the analogous product structure should be on $\mathrm{HT}^*([X/G])$. Understanding candidates for such a product is the goal of this chapter.

We close this subsection by noting that the above HKR isomorphisms can be assembled to an analogue of the exponential map

$$\exp : \mathbb{T}_{I\mathfrak{X}}[-1] = \mathbb{L}_{L\mathfrak{X}} \xrightarrow{\sim} L\mathfrak{X}$$

from the Lie algebra $\mathbb{L}_{L\mathfrak{X}}$ to the derived group $L\mathfrak{X}$.

3.2 Definition of the product on orbifold polyvector fields

In this section we define the product on orbifold polyvector fields. The technical results used in the definition are introduced in this section, but will be proved in Section 3.3.

As we have explained previously, the Hochschild cohomology and the polyvector fields cohomology of a space X can be viewed as the distributions on the derived loop space (a derived Lie group) and on its Lie algebra, respectively. The product structures on these come from the convolution of distributions. We begin by recalling the definition of the convolution product of distributions on (classical) Lie groups and Lie algebras.

3.2.1 Distributions on Lie groups and Lie algebras

Let G be a Lie group with Lie algebra $\mathfrak{g}$. The convolution product of distributions $\mathbb{D}(G)$ on G is defined as follows

$$\begin{tikzcd}\mathbb{D}(G) \otimes \mathbb{D}(G) \arrow[r] & \mathbb{D}(G \times G) \arrow[r, "m_*"] & \mathbb{D}(G),\end{tikzcd}$$

where m is the multiplication map $G \times G \to G$, and m_* is the induced map on distributions. The Lie algebra $\mathfrak{g}$ of G is a vector space. It is considered as an abelian group under the addition operation of vectors. One can define the convolution product for $\mathbb{D}(\mathfrak{g})$ similarly.

In the derived setting, the convolution product on orbifold Hochschild cohomology is known as the composition of morphisms in the derived category. We hope to define the convolution product on the polyvector fields. Therefore, it is important to know how to recover the convolution product on $\mathbb{D}(\mathfrak{g})$ with the knowledge of the group G only. The following is how we do this.

First there is a multiplication map $m : G \times G \to G$. Taking derivative of this map, we get the induced map on tangent spaces $\mathbb{L}_m : \mathbb{L}_{G \times G} \to \mathbb{L}_G$, where $\mathbb{L}_G = \mathfrak{g}$ is the tangent space of G at origin. We use the same notation $\mathbb{L}$ as the notation for the linearization of derived schemes in the derived setting.

There is a natural isomorphism $\mathbb{L}_{G \times G} \cong \mathbb{L}_G \times \mathbb{L}_G$. Under this natural identification, the map $\mathbb{L}_m : \mathbb{L}_G \times \mathbb{L}_G \to \mathbb{L}_G$ is nothing but the addition law on the vector space $\mathbb{L}_G$. We can recover the convolution product of $\mathbb{D}(\mathbb{L}_G)$ now

$$\begin{tikzcd}\mathbb{D}(\mathbb{L}_G) \otimes \mathbb{D}(\mathbb{L}_G) \arrow[r] & \mathbb{D}(\mathbb{L}_G \times \mathbb{L}_G) \arrow[r, "\cong"] & \mathbb{D}(\mathbb{L}_{G \times G}) \arrow[r, "\mathbb{L}_{m*}"] & \mathbb{D}(\mathbb{L}_G).\end{tikzcd}$$

3.2.2 A non-trivial isomorphism in the derived setting

We can try to do exactly the same thing in the derived setting. However, there is a technical issue. The natural isomorphism $\mathbb{L}_{G\times G} \cong \mathbb{L}_G \times \mathbb{L}_G$ is not at all obvious for the derived loop space. The analogous statement would be

$$\mathbb{L}_{L\mathfrak{X}\times^R_{\mathfrak{X}}L\mathfrak{X}} \cong \mathbb{L}_{L\mathfrak{X}} \times^R_{\mathfrak{X}} \mathbb{L}_{L\mathfrak{X}}$$

for the derived loop space of an orbifold $\mathfrak{X}$. The left hand side is obviously linear: it is a total space of a vector bundle over the inertia stack $I\mathfrak{X}$. On the other hand, it is not at all obvious that the right hand side can be linearized.

The following two propositions will be proved in the next sections.

Proposition 3.11 *Let $\mathfrak{X} = [X/G]$ be a global quotient orbifold of a finite group G acting on a smooth algebraic variety. If we further assume G is abelian, then there is an isomorphism*

$$\mathbb{L}_{(L\mathfrak{X}\times^R_{\mathfrak{X}}L\mathfrak{X})} \cong \mathbb{L}_{L\mathfrak{X}} \times^R_{\mathfrak{X}} \mathbb{L}_{L\mathfrak{X}}.$$

The derived loop space $L\mathfrak{X}$ decomposes naturally into connected components, so we can restate the above proposition on components.

Proposition 3.12 *In the same setting as Proposition 3.11, there is an isomorphism*

$$\mathbb{L}_{\widetilde{X^g}\times^R_X\widetilde{X^h}} \cong \mathbb{L}_{\widetilde{X^g}} \times^R_X \mathbb{L}_{\widetilde{X^h}}$$

for any $g, h \in G$.

3.2.3 The definition of the convolution product in the derived setting

The multiplication map for Lie groups plays an important role in the case of Lie groups and Lie algebras. We need to know what the multiplication map is for the derived loop space $L\mathfrak{X} = \mathfrak{X} \times^R_{\mathfrak{X}\times\mathfrak{X}} \mathfrak{X}$ of $\mathfrak{X}$. It is the projection map $p_1 \times p_3$ onto the first and the third factors

$$\begin{tikzcd}
L\mathfrak{X} \times_{\mathfrak{X}} L\mathfrak{X} \arrow[r, "m"] \arrow[d, "="] & L\mathfrak{X}. \arrow[d, "="] \\
\mathfrak{X} \times^R_{\mathfrak{X}\times\mathfrak{X}} \mathfrak{X} \times^R_{\mathfrak{X}\times\mathfrak{X}} \mathfrak{X} \arrow[r, "p_1\times p_3"] & \mathfrak{X} \times^R_{\mathfrak{X}\times\mathfrak{X}} \mathfrak{X}.
\end{tikzcd}$$

We need three lemmas for derived groups which are generalizations of well-known results from classical Lie group theory.

Lemma 3.13 *A map $f : X \longrightarrow Y$ between derived schemes induces a map on linearlizations $\mathbb{L}_f : \mathbb{L}_X \longrightarrow \mathbb{L}_Y$.*

Proof. We have a commutative diagram

$$\begin{tikzcd}
X^0 \arrow[r, "g"] \arrow[d, "i"] & Y^0 \arrow[d, "j"] \\
X \arrow[r, "f"] & Y,
\end{tikzcd}$$

where X^0 and Y^0 are the classical schemes of X and Y respectively. Then there is a commutative diagram of derived tangent complexes

$$\begin{tikzcd}
T_{X^0} \arrow[r] \arrow[d] & g^*T_{Y^0} \arrow[d] \\
i^*T_X \arrow[r] & i^*f^*T_Y = g^*j^*T_Y.
\end{tikzcd}$$

Passing to the quotient, we get an induced map

$$N_{X^0/X} = i^*T_X/T_{X^0} \to g^*(j^*T_Y)/g^*T_{Y^0} = g^*(j^*T_Y/T_{Y^0}) = g^*N_{Y^0/Y}.$$

The map above is equivalent to a map $\mathbb{N}_{X^0/X} \to \mathbb{N}_{Y^0/Y} \times_{Y^0} X^0$ in terms of total spaces. □

Applying the above lemma to the multiplication map of derived loop space yields an induced map $\mathbb{L}_m : \mathbb{L}_{L\mathfrak{X}\times^R_{\mathfrak{X}} L\mathfrak{X}} \to \mathbb{L}_{L\mathfrak{X}}$.

Lemma 3.14 *Suppose there is a commutative diagram*

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "i"'] & & Y \arrow[dl, "j"] \\
& S &
\end{tikzcd}$$

of (derived) schemes. Then there is a pushforward map for relative distributions, i.e., there is a natural induced map $f_* : \mathbb{D}(X/S) \to \mathbb{D}(Y/S)$.

Proof. We have $\mathbb{D}(X/S) = \mathrm{Hom}(i_*\mathscr{O}_X, \mathscr{O}_S)$. Applying j_* to the map $\mathscr{O}_Y \to f_*\mathscr{O}_X$, we get a map $j_*\mathscr{O}_Y \to j_*f_*\mathscr{O}_X = i_*\mathscr{O}_X$. Composing it with $i_*\mathscr{O}_X \to \mathscr{O}_S$, we get the desired pushforward map. □

Lemma 3.15 *Suppose there is a commutative diagram*

$$\begin{tikzcd}
W = X \times^R_S Y \arrow[r] \arrow[d] \arrow[dr, "\pi"] & Y \arrow[d, "j"'] \\
X \arrow[r, "i"] & S
\end{tikzcd}$$

of (derived) schemes. Then there is a natural map $\mathbb{D}(X/S) \otimes \mathbb{D}(Y/S) \to \mathbb{D}(W/S)$.

Proof. We have

$$\begin{aligned}\mathrm{Hom}(i_*\mathscr{O}_X, \mathscr{O}_S) \otimes \mathrm{Hom}(j_*\mathscr{O}_Y, \mathscr{O}_S) &\to \mathrm{Hom}(i_*\mathscr{O}_X \otimes_{\mathscr{O}_S} j_*\mathscr{O}_Y, \mathscr{O}_S) \\ &= \mathrm{Hom}(\pi_*\mathscr{O}_W, \mathscr{O}_S). \qquad \square\end{aligned}$$

With the three lemmas above we are able to define our desired product.

Definition 3.16 *Under the assumptions in Proposition 3.11 we define the following binary operation on $\mathbb{D}(\mathbb{L}_{L\mathfrak{X}}/\mathfrak{X})$, which is our proposed definition for a product on orbifold polyvector fields:.*

$$\begin{aligned}\mathbb{D}(\mathbb{L}_{L\mathfrak{X}}/\mathfrak{X}) \otimes \mathbb{D}(\mathbb{L}_{L\mathfrak{X}}/\mathfrak{X}) \longrightarrow \mathbb{D}(\mathbb{L}_{L\mathfrak{X}} \times^R_{\mathfrak{X}} \mathbb{L}_{L\mathfrak{X}}/\mathfrak{X}) &\xrightarrow{\sim} \mathbb{D}(\mathbb{L}_{(L\mathfrak{X} \times_{\mathfrak{X}} L\mathfrak{X})}/\mathfrak{X}) \\ &\xrightarrow{\mathbb{L}_{m*}} \mathbb{D}(\mathbb{L}_{\mathfrak{X}}/\mathfrak{X}),\end{aligned}$$

where the first arrow is due to Lemma 3.15, the second arrow is the non-trivial isomorphism in Proposition 3.11, and the last map is due to Lemmas 3.13 and 3.14.

Looking at each connected component of $L\mathfrak{X}$ individually, the definition gives a map for every $g, h \in G$

$$\mathbb{D}(\mathbb{L}_{\widetilde{X^g}}/X) \otimes \mathbb{D}(\mathbb{L}_{\widetilde{X^h}}/X) \to \mathbb{D}(\mathbb{L}_{\widetilde{X^g}} \times^R_X \mathbb{L}_{\widetilde{X^h}}/X) \xrightarrow{\sim} \mathbb{D}(\mathbb{L}_{(\widetilde{X^g \times_X X^h})}/X) \xrightarrow{\mathbb{L}_{m*}} \mathbb{D}(\mathbb{L}_{\widetilde{X^{gh}}}/X).$$

3.3 The formality of double fixed loci

We begin by studying the cohomology sheaves of the structure complex of $\mathbb{L}_{\widetilde{X^g}} \times^R_X \mathbb{L}_{\widetilde{X^h}}$. Then we compute the linearization $\mathbb{L}_{\widetilde{X^g \times^R_X X^h}}$ explicitly. At last, we prove Propositions 3.11 and 3.12, in other words we construct a formality isomorphism

$$\mathbb{L}_{\widetilde{X^g}} \times^R_X \mathbb{L}_{\widetilde{X^h}} \cong \mathbb{L}_{\widetilde{X^g \times^R_X X^h}}.$$

The construction will be indirect: we will use known results to show that both sides are isomorphic to

$$\mathbb{T}_{X^g}[-1]|_{X^{g,h}} \oplus \mathbb{T}_{X^h}[-1]|_{X^{g,h}} \oplus \mathbb{E}[-1],$$

where $\mathbb{E}$ is the total space of the excess intersection bundle for the intersection of X^g and X^h in X.

Throughout this section X will be a smooth variety, and g, h will denote commuting elements of a group G which acts on X.

Before we begin we note that the derived fixed locus $\widetilde{X^g} = \Delta \times^R_{X \times X} \Delta^g$ is not directly a scheme over X. It is more naturally viewed as a scheme over Δ or Δ^g, both of which are isomorphic (in different ways) to X. We will use the latter when computing the fiber product $\widetilde{X^g} \times^R_X \widetilde{X^h}$.

Similarly, the derived fixed locus $\widetilde{X^h} = \Delta \times^R_{X \times X} \Delta^h$ is naturally isomorphic to $\Delta^g \times^R_{X \times X} \Delta^{gh}$, which is also a scheme over Δ^g. Therefore, while the notation $\widetilde{X^g} \times^R_X \widetilde{X^h}$ is imprecise, what we will really mean by it is

$$(\Delta \times^R_{X \times X} \Delta^g) \times^R_{\Delta^g} (\Delta^g \times^R_{X \times X} \Delta^{gh}) = \Delta \times^R_{X \times X} \Delta^g \times^R_{X \times X} \Delta^{gh}.$$

We think of this as the derived fixed locus of g and h, and denote it by $\widetilde{X^{g,h}}$.

3.3.1 The cohomology sheaves of the structure complex

It is difficult to compute $\mathscr{O}_{\mathbb{L}_{\widetilde{X^g}} \times^R_X \mathbb{L}_{\widetilde{X^h}}}$ directly, but we can compute its cohomology sheaves more easily, and we begin with this computation.

We hope to compute the cohomology sheaves of

$$\mathscr{O}_{\mathbb{L}_{\widetilde{X^g}} \times^R_X \mathbb{L}_{\widetilde{X^h}}} = \mathrm{Sym}(\Omega_{X^g}[1]) \otimes^L_{\mathscr{O}_X} \mathrm{Sym}(\Omega_{X^h}[1]).$$

The calculation becomes straightforward using the following lemma.

Lemma 3.17 *Suppose i, j are closed embeddings of classical schemes, and $\mathscr{E}$, $\mathscr{F}$ are vector bundles on X and on Y, respectively. Denote by W the fiber product of classical schemes below*

$$\begin{tikzcd}
X \times_S Y = W \arrow[r, "l"] \arrow[d, "k"'] & Y \arrow[d, "j"] \\
X \arrow[r, "i"] & S.
\end{tikzcd}$$

Then

$$\mathcal{H}_q(i_*\mathscr{E} \otimes^L_{\mathscr{O}_S} j_*\mathscr{F}) = j_*l_*(\mathscr{E}|_W \otimes \mathscr{F}|_W \otimes E^\vee),$$

where E is the excess intersection bundle,

$$E = \frac{T_S|_W}{T_X|_W + T_Y|_W}.$$

Proof. [13, Proposition A.6]. □

The lemma above shows that

$$\mathcal{H}^n(\mathscr{O}_{\mathbb{L}_{\widetilde{X^g}} \times^R_X \mathbb{L}_{\widetilde{X^h}}}) = \bigoplus_{p+q+i=n} (\wedge^p \Omega_{X^g}|_{X^{g,h}} \otimes \wedge^q \Omega_{X^h}|_{X^{g,h}} \otimes \wedge^i E^\vee).$$

In other words, if we knew that $\mathbb{L}_{\widetilde{X^g}} \times^R_X \mathbb{L}_{\widetilde{X^h}}$ is formal over its underlying classical scheme $X^{g,h}$, the above calculation would imply that

$$\mathbb{L}_{\widetilde{X^g}} \times^R_X \mathbb{L}_{\widetilde{X^h}} \cong \mathbb{T}_{X^g}[-1]|_{X^{g,h}} \oplus \mathbb{T}_{X^h}[-1]|_{X^{g,h}} \oplus \mathbb{E}[-1].$$

We will prove the formality statement in (3.3.3).

We will now argue that the linearization $\mathbb{L}_{\widetilde{X^{g,h}}}$ is precisely the space that appears on the right hand side of the equality above,

$$\mathbb{L}_{\widetilde{X^{g,h}}} = \mathbb{T}_{X^g}[-1]|_{X^{g,h}} \oplus \mathbb{T}_{X^h}[-1]|_{X^{g,h}} \oplus \mathbb{E}[-1].$$

By definition, $\mathbb{L}_{\widetilde{X^{g,h}}} = \mathbb{N}_{X^{g,h}/\widetilde{X^{g,h}}}$. To compute the normal bundle, we need to know what the derived tangent complex of $\widetilde{X^{g,h}}$ is.

3.3.2 The derived tangent complex

The standard reference for the derived tangent complex is [23]. Suppose X and Y are closed subschemes of a scheme S. Let $\widetilde{W} = X \times_S^R Y$ be the derived intersection and $W = X \times_S^R Y$ be the classical intersection. There is an exact sequence

$$0 \to T_X|_W \cap T_Y|_W = T_W \to T_X|_W \oplus T_Y|_W \to T_S|_W \to E \to 0.$$

The complex

$$T_X|_W \oplus T_Y|_W \to T_S|_W = \operatorname{Cone}(T_X|_W \oplus T_Y|_W \to T_S|_W)[-1]$$

is the restriction to W of the derived tangent complex $T_{\widetilde{W}}|_W$ of $\widetilde{W}$. Since only its $\mathcal{H}^0$ and $\mathcal{H}^1$ sheaves are non-zero, the information contained in it is equivalent to the data of the triple $(\mathcal{H}^0, \mathcal{H}^1, \eta)$, where $\mathcal{H}^0(T_{\widetilde{W}}|_W) = T_W$, $\mathcal{H}^1(T_{\widetilde{W}}|_W) = N_{W/\widetilde{W}} = E$, and the class η is an element in $\operatorname{Ext}^2_S(E, T_W)$.

For example, if we consider the situation where $S = X \times X$, $X = \Delta$, $Y = \Delta^g$, so that $W = X^g$, we have $\mathcal{H}^0(T_{\widetilde{X^g}}) = T_{X^g}$ and $\mathcal{H}^1(T_{\widetilde{X^g}}) = E$. Moreover, the excess bundle in this case equals the coinvariant bundle $(T_X|_{X^g})_g$, which in characteristic zero is canonically isomorphic to the invariant bundle T_{X^g}.

The linearization $\mathbb{L}_{\widetilde{W}}$ is by definition the total space of the normal bundle $N_{W/\widetilde{W}}$, the cone of the map $T_W \to T_{\widetilde{W}}|_W$. Since $\mathcal{H}^0(T_{\widetilde{W}}|_W) = T_W$, it follows that the normal bundle $N_{W/\widetilde{W}}$ is the first cohomology $\mathcal{H}^1(T_{\widetilde{W}}|_W)[-1]$ of $T_{\widetilde{W}}|_W$. In the example considered above this shows that $\mathbb{L}_{\widetilde{X^g}} = \mathbb{T}_{X^g}[-1]$.

The above discussion also works for derived schemes. If we replace X, Y, and S by derived schemes in the commutative diagram at the beginning of (3.3.2), we have the same formula

$$T_{\widetilde{W}}|_W = \operatorname{Cone}(T_X|_W \oplus T_Y|_W \longrightarrow T_S|_W)[-1],$$

where T_X, T_Y, and T_S are the derived tangent complexes of X, Y, and S. The scheme W is the underlying classical scheme of $\widetilde{W} = X \times_S^R Y$. Since all the complexes are restricted to W, we will omit the restrictions from X, Y, S, and $\widetilde{W}$ to W for simplicity from now on.

It helps us to compute the derived tangent complex $T_{\widetilde{X^{g,h}}}$ if we set $S = X$, $X = \widetilde{X^g}$, $Y = \widetilde{X^h}$, and $\widetilde{W} = \widetilde{X^{g,h}}$ respectively.

Lemma 3.18 *The derived tangent complex of $\widetilde{X^{g,h}}$ is quasi-isomorphic to*

$$T_\Delta \oplus T_{\Delta^g} \oplus T_{\Delta^h} \to T_{X\times X} \oplus T_{X\times X} \oplus T_{X\times X} \to T_{X\times X},$$

where the maps are of the form

$$(v_1, v_2, v_3) \to (v_1 - v_2, v_2 - v_3, v_3 - v_1),$$

and

$$(a, b, c) \to a + b + c.$$

We can compute the cohomology of the derived tangent complex of $\widetilde{X^{g,h}}$ using the above lemma. We have $\mathcal{H}^0(T_{\widetilde{X^{g,h}}}) = T_{X^{g,h}}$. To compute the first cohomology it suffices to compute the cokernel of the map below

$$V \oplus V \oplus V \to V \oplus V \oplus V \oplus V,$$

where ($V = T_X \cong T_\Delta \cong T_{\Delta^g} \cong T_{\Delta^h}$) and the maps are $(v, v', v'') \to (v - v', v - gv', v - v'', v - hv'')$. This is done in the lemma below.

Lemma 3.19 *Suppose V is a finite dimensional representation of a finite group G over a field of characteristic 0. Let g and h be two elements of G. Then the quotient of*

$V \oplus V \oplus V \oplus V$ by the relations (v, v, v, v), $(v, gv, 0, 0)$, and $(0, 0, v, hv)$ is isomorphic to

$$V_g \oplus V_h \oplus \frac{V}{V^g + V^h}.$$

Proof. Let L be the linear subspace (v, v, v, v), and note that $L = H_1 \cap H_2 \cap H_3$, where H_1 is defined by $v_1 = v_2$, H_2 is defined by $v_1 = v_3$, and H_3 is defined by $v_3 = v_4$. Then we have an isomorphism

$$\frac{V \oplus V \oplus V \oplus V}{L} \cong \frac{V^{\oplus 4}}{H_1} \oplus \frac{V^{\oplus 4}}{H_2} \oplus \frac{V^{\oplus 4}}{H_3} \cong V \oplus V \oplus V.$$

Under this identification, the second and third relations become $(v - gv, v, 0)$ and $(0, -v, v - hv)$. There is a natural projection to the first and third components,

$$\frac{V \oplus V \oplus V}{(v - gv, v, 0), (0, -v', v' - hv')} \to \frac{V \oplus V}{(v - gv, 0), (0, v' - hv')} = V_g \oplus V_h.$$

It is easy to show that the kernel is $\frac{V}{V^g + V^h}$. So we get a short exact sequence

$$0 \to \frac{V}{V^g + V^h} \to \frac{V \oplus V \oplus V}{(v - gv, v, 0), (0, -v', v' - hv')} \to V_g \oplus V_h \to 0.$$

By the averaging map $v \to \frac{1}{\mathrm{ord}(g)} \sum_{i=1}^{\mathrm{ord}(g)} g^i \cdot v$, the map $V \to V_g$ splits in characteristic 0. We can use the averaging map of g and h to get a canonical splitting of the short exact sequence above. □

The discussion above shows that the first cohomology of the tangent complex of $\widetilde{X^{g,h}}$ is $E \oplus T_{X^g} \oplus T_{X^h}$, where $E = \frac{T_X}{T_{X^g} + T_{X^h}}$. As a consequence we have an isomorphism

$$\mathbb{L}_{\widetilde{X^g} \times^R_X \widetilde{X^h}} = \mathbb{L}_{\widetilde{X^{g,h}}} \cong \mathbb{T}_{X^g}|_{X^{g,h}}[-1] \oplus \mathbb{T}_{X^h}|_{X^{g,h}}[-1] \oplus \mathbb{E}[-1].$$

3.3.3 Formality of $\mathbb{L}_{\widetilde{X^g}} \times^R_X \mathbb{L}_{\widetilde{X^h}}$

The linearizations $\mathbb{L}_{\widetilde{X^g}}$ and $\mathbb{L}_{\widetilde{X^h}}$ are by definition the total spaces of vector bundles over X^g and X^h respectively, so we study the formality of $X^g \times^R_X X^h$ first. The key tools are [3, Theorem 1.8 and Lemma 4.3].

Proof. [Proof of Proposition 3.12.] The inclusion $X^g \to X$ splits to first order. By [3, Lemma 4.3], the derived scheme $X^g \times^R_X X^h$ is formal over $X^g \times X^h$ if and only if the short exact sequence on $X^{g,h} = X^g \times_X X^h$

$$0 \to \frac{T_{X^h}}{T_{X^g} \cap T_{X^h}} \to \frac{T_X}{T_{X^g}} \to E = \frac{T_X}{T_{X^g} + T_{X^h}} \to 0$$

splits.

Define a map

$$\frac{T_X}{T_{X^g}} \to \frac{T_{X^h}}{T_{X^g} \cap T_{X^h}}$$

by the formula

$$v \mapsto \frac{1}{\mathrm{ord}(h)} \sum h^i \cdot v.$$

The map is well-defined because g and h commute under our initial assumptions (or the ones in Proposition 3.12). It splits the short exact sequence above. This shows $X^g \times^R_X X^h$ is formal over $X^g \times X^h$.

Consider the following commutative diagram

$$\begin{tikzcd}
\widehat{X^{g,h}} = X^g \times^R_X X^h \arrow[rrd] \arrow[rdd] & & \\
& X^{g,h} \arrow[lu] \arrow[r, "p"'] \arrow[d, "q"] & X^g \arrow[d, "i"'] \\
& X^h \arrow[r, "j"] & X
\end{tikzcd}$$

By [3, Theorem 1.8] we know that the dg functor $j^* i_*(-)$ is isomorphic to the dg functor $q_* \left(p^*(-) \otimes \mathrm{Sym}(E^\vee[1])\right)$.

The structure complex of $\mathbb{L}_{\widehat{X^g}} \times^R_X \mathbb{L}_{\widehat{X^h}}$ is

$$i_* \mathrm{Sym}(\Omega_{X^g}[1]) \otimes^L_{\mathscr{O}_X} j_* \mathrm{Sym}(\Omega_{X^h}[1]) = j_* \left(j^* i_* \mathrm{Sym}(\Omega_{X^g}[1]) \otimes \mathrm{Sym}(\Omega_{X^h}[1])\right).$$

Using the isomorphism of the two dg functors above, we see that $j^*i_*(\mathrm{Sym}(\Omega_{X^g}[1])) \cong q_*(p^*(\mathrm{Sym}(\Omega_{X^g}[1])) \otimes \mathrm{Sym}(E^\vee[1]))$. As a consequence

$$\begin{aligned} & i_* \,\mathrm{Sym}(\Omega_{X^g}[1]) \otimes^L_{\mathscr{O}_X} j_* \,\mathrm{Sym}(\Omega_{X^h}[1]) = \\ &\qquad\qquad = j_* q_* \left(\mathrm{Sym}(\Omega_{X^g}|_{X^{g,h}}[1]) \otimes \mathrm{Sym}(E^\vee[1]) \otimes \mathrm{Sym}(\Omega_{X^h}|_{X^{g,h}}[1])\right) \\ &\qquad\qquad = j_* q_* \,\mathrm{Sym}\left((\Omega_{X^g}|_{X^{g,h}} \oplus \Omega_{X^h}|_{X^{g,h}} \oplus E^\vee)[1]\right). \end{aligned}$$

Therefore $\mathbb{L}_{\widetilde{X^g}} \times^R_X \mathbb{L}_{\widetilde{X^h}}$ is formal over X (and it is isomorphic to $\mathbb{L}_{\widetilde{X^{g,h}}}$). ☐

3.4 Associativity of the product

In this section we explain the strategy for studying the associativity of the product. In other words we want to show that, under the assumption that certain Bass-Quillen cohomology classes vanish, the product defined in Section 3.2 is associative. The proof is reduced to Propositions 3.21 and 3.22, which will be proved in Section 3.5.

Formality of triple intersections. To prove the associativity it is natural to study the triple intersection $\widetilde{X^{g,h,k}} = \widetilde{X^g} \times^R_X \times \widetilde{X^h} \times^R_X \widetilde{X^k}$ for g, h, and $k \in G$. More precisely we define

$$\begin{aligned} \widetilde{X^{g,h,k}} &= (\Delta \times^R_{X\times X} \Delta^g) \times_{\Delta^g} (\Delta^g \times^R_{X\times X} \Delta^{gh}) \times_{\Delta^{gh}} (\Delta^{gh} \times^R_{X\times X} \Delta^{ghk}) \\ &= \Delta \times^R_{X\times X} \Delta^g \times^R_{X\times X} \Delta^{gh} \times^R_{X\times X} \Delta^{ghk}, \end{aligned}$$

as explained at the beginning of Section 3.3. Under the assumption that G is abelian it is not hard to see that $\widetilde{X^{g,h,k}}$ is formal over X, and it is isomorphic to $\mathbb{L}_{\widetilde{X^{g,h,k}}}$. The proof is essentially the same as the one in Section 3.3.

The diagram

$$\begin{tikzcd}
\widetilde{X^{g,h,k}} \arrow[r] \arrow[d] & \widetilde{X^{g,hk}} \arrow[d] \\
\widetilde{X^{gh,k}} \arrow[r] & \widetilde{X^{ghk}}
\end{tikzcd}$$

is commutative because it is the associativity of the group law of the loop space $L[X/G]$. Taking distributions over on the corresponding linearizations, we get the following commutative diagram

$$\begin{tikzcd}
\mathbb{D}(\mathbb{L}_{\widetilde{X^{g,h,k}}}) \arrow[r] \arrow[d] & \mathbb{D}(\mathbb{L}_{\widetilde{X^{g,hk}}}) \arrow[d] \\
\mathbb{D}(\mathbb{L}_{\widetilde{X^{gh,k}}}) \arrow[r] & \mathbb{D}(\mathbb{L}_{\widetilde{X^{ghk}}}).
\end{tikzcd}$$

For simplicity we have denoted the relative distributions with respect to X as $\mathbb{D}(-)$ instead of $\mathbb{D}(-/X)$.

Consider the following diagram $(**)$:

$$\begin{tikzcd}
\mathbb{D}(\mathbb{L}_{\widetilde{X^g}}) \otimes \mathbb{D}(\mathbb{L}_{\widetilde{X^h}}) \otimes \mathbb{D}(\mathbb{L}_{\widetilde{X^k}}) \arrow[rrr, "\mathrm{id} \otimes m"] \arrow[dr] \arrow[ddd, "m \otimes \mathrm{id}"'] & & & \mathbb{D}(\mathbb{L}_{\widetilde{X^g}}) \otimes \mathbb{D}(\mathbb{L}_{\widetilde{X^{hk}}}) \arrow[dl] \arrow[ddd, "m"] \\
& \mathbb{D}(\mathbb{L}_{\widetilde{X^{g,h,k}}}) \arrow[r] \arrow[d] & \mathbb{D}(\mathbb{L}_{\widetilde{X^{g,hk}}}) \arrow[d] & \\
& \mathbb{D}(\mathbb{L}_{\widetilde{X^{gh,k}}}) \arrow[r] & \mathbb{D}(\mathbb{L}_{\widetilde{X^{ghk}}}) & \\
\mathbb{D}(\mathbb{L}_{\widetilde{X^{gh}}}) \otimes \mathbb{D}(\mathbb{L}_{\widetilde{X^k}}) \arrow[ur] \arrow[rrr, "m"'] & & & \mathbb{D}(\mathbb{L}_{\widetilde{X^{ghk}}}), \arrow[ul, "="]
\end{tikzcd}$$

where m is the product on orbifold polyvector fields in Definition 3.16. Associativity of m is equivalent to commutativity of the outer part of the diagram.

The middle square is commutative by the discussion in the previous paragraph. The squares on the bottom and right are commutative because they are the definitions of our product.

We need to examine the ones on the top and left. The left one is the diagram

$$\begin{tikzcd}
\mathbb{D}(\mathbb{L}_{\widetilde{X^g}}) \otimes \mathbb{D}(\mathbb{L}_{\widetilde{X^h}}) \otimes \mathbb{D}(\mathbb{L}_{\widetilde{X^k}}) \arrow[r] \arrow[d, "m\otimes\mathrm{id}"'] & \mathbb{D}(\mathbb{L}_{\widetilde{X^{g,h,k}}}) \arrow[d] \\
\mathbb{D}(\mathbb{L}_{\widetilde{X^{gh}}}) \otimes \mathbb{D}(\mathbb{L}_{\widetilde{X^k}}) \arrow[r] & \mathbb{D}(\mathbb{L}_{\widetilde{X^{gh,k}}}).
\end{tikzcd}$$

Expand the diagram in detail

$$\begin{tikzcd}
\mathbb{D}(\mathbb{L}_{\widetilde{X^g}}) \otimes \mathbb{D}(\mathbb{L}_{\widetilde{X^h}}) \otimes \mathbb{D}(\mathbb{L}_{\widetilde{X^k}}) \arrow[r] \arrow[d, "m\otimes\mathrm{id}"'] & \mathbb{D}(\mathbb{L}_{\widetilde{X^{g,h}}}) \otimes \mathbb{D}(\mathbb{L}_{\widetilde{X^k}}) \arrow[r] \arrow[d] & \mathbb{D}(\mathbb{L}_{\widetilde{X^{g,h}}} \times^R_X \mathbb{L}_{\widetilde{X^k}}) \arrow[r, "\sim"] \arrow[d] & \mathbb{D}(\mathbb{L}_{\widetilde{X^{g,h,k}}}) \arrow[d] \\
\mathbb{D}(\mathbb{L}_{\widetilde{X^{gh}}}) \otimes \mathbb{D}(\mathbb{L}_{\widetilde{X^k}}) \arrow[r, "="] & \mathbb{D}(\mathbb{L}_{\widetilde{X^{gh}}}) \otimes \mathbb{D}(\mathbb{L}_{\widetilde{X^k}}) \arrow[r] & \mathbb{D}(\mathbb{L}_{\widetilde{X^{gh}}} \times^R_X \mathbb{L}_{\widetilde{X^k}}) \arrow[r, "\sim"] & \mathbb{D}(\mathbb{L}_{\widetilde{X^{gh,k}}}).
\end{tikzcd}$$

Clearly, the left and middle squares of the diagram above are commutative. We have to show commutativity of the square on the right. Note that the maps on distributions are induced from maps on spaces, so we only need to show the commutativity of the diagram below

$$\begin{tikzcd}
\mathbb{L}_{\widetilde{X^{g,h}}} \times^R_X \mathbb{L}_{\widetilde{X^k}} \arrow[r, "\sim"] \arrow[d] & \mathbb{L}_{\widetilde{X^{g,h,k}}} \arrow[d] \\
\mathbb{L}_{\widetilde{X^{gh}}} \times^R_X \mathbb{L}_{\widetilde{X^k}} \arrow[r, "\sim"] & \mathbb{L}_{\widetilde{X^{gh,k}}}.
\end{tikzcd}$$

Similarly, the commutativity of the top square in the big diagram (∗∗) reduces to the commutativity of the diagram below

$$\begin{tikzcd}
\mathbb{L}_{\widetilde{X^g}} \times^R_X \mathbb{L}_{\widetilde{X^{h,k}}} \arrow[r, "\sim"] \arrow[d] & \mathbb{L}_{\widetilde{X^{g,h,k}}} \arrow[d] \\
\mathbb{L}_{\widetilde{X^g}} \times^R_X \mathbb{L}_{\widetilde{X^{hk}}} \arrow[r, "\sim"] & \mathbb{L}_{\widetilde{X^{g,hk}}}.
\end{tikzcd}$$

We will only analyze the former diagram; the proof of the commutativity of the latter is entirely similar.

There is, however, one more compatibility that needs to be discussed. Even though we wrote the top left diagonal map in the big diagram (∗∗) as a single map, it is in fact clear from the discussion above that there are two maps here,

$$\mathbb{D}(\mathbb{L}_{\widetilde{X^g}}) \otimes \mathbb{D}(\mathbb{L}_{\widetilde{X^h}}) \otimes \mathbb{D}(\mathbb{L}_{\widetilde{X^k}}) \rightrightarrows \mathbb{D}(\mathbb{L}_{\widetilde{X^{g,h,k}}}).$$

One is the one that appears in the left square in the big diagram (∗∗), and the other one is the one that is in the top square of the big diagram (∗∗). We need to prove that these two maps are the same. This question is easily reduced to the following problem.

As mentioned in the previous section, the linearizations $\mathbb{L}_{\widetilde{X^g}}$, $\mathbb{L}_{\widetilde{X^h}}$, and $\mathbb{L}_{\widetilde{X^k}}$ are vector bundles over the underlying schemes X^g, X^h, and X^k. To prove the formality of derived intersections of linearizations, it suffices to prove the formality of the underlying schemes. There are two ways to define the isomorphism $\mathbb{L}_{\widetilde{X^g}} \times^R_X \mathbb{L}_{\widetilde{X^h}} \times^R_X \mathbb{L}_{\widetilde{X^k}} \cong \mathbb{L}_{\widetilde{X^{g,h,k}}}$. One uses the fact that $\widehat{X^{g,h}} = X^g \times^R_X X^h$ is formal and $\widehat{X^{(g,h),k}} = X^{g,h} \times^R_X X^k$ is formal. The other uses the fact that $\widehat{X^{h,k}} = X^h \times^R_X X^k$ is formal and $\widehat{X^{g,(h,k)}} = X^g \times^R_X X^{h,k}$ is formal. Therefore, we need to prove that the two isomorphisms agree, i.e., that the diagram below is commutative

$$\begin{tikzcd}
(X^g \times^R_X X^h) \times^R_X X^k \arrow[r, "\sim"] \arrow[d, "\mathrm{id}"] & \mathbb{L}_{\widehat{X^{g,h}}} \times^R_X X^k = \mathbb{L}_{\widehat{X^{g,h}}} \times^R_{X^{g,h}} (X^{g,h} \times^R_X X^k) \arrow[r, "\sim"] & \mathbb{L}_{\widehat{X^{g,h}}} \times^R_{X^{g,h}} \mathbb{L}_{\widehat{X^{(g,h),k}}} \arrow[r] & \mathbb{L}_{\widehat{X^{g,h,k}}} \arrow[d, "\mathrm{id}"] \\
X^g \times^R_X (X^h \times^R_X X^k) \arrow[r, "\sim"] & X^g \times^R_X \mathbb{L}_{\widehat{X^{h,k}}} = (X^g \times^R_X X^{h,k}) \times^R_{X^{h,k}} \mathbb{L}_{\widehat{X^{h,k}}} \arrow[r, "\sim"] & \mathbb{L}_{\widehat{X^{g,(h,k)}}} \times^R_{X^{h,k}} \mathbb{L}_{\widehat{X^{h,k}}} \arrow[r] & \mathbb{L}_{\widehat{X^{g,h,k}}}.
\end{tikzcd}$$

Unfortunately we can not prove the commutativity of the diagrams above without further assumptions. The Bass-Quillen class plays an important role in what follows. We review it below.

Let $X \hookrightarrow Y \hookrightarrow S$ be a sequence of closed embedding of smooth schemes, and assume that there is a fixed first order splitting of the map $X \hookrightarrow Y$.

The class we need is the Bass-Quillen class associated to the restriction $N_{Y/S}|_{X^{(1)}}$ of the normal bundle $N_{Y/S}$ to the first order neighborhood $X^{(1)}$. In this chapter we call this class the Bass-Quillen class associated to the sequence of embeddings $X \hookrightarrow Y \hookrightarrow S$.

The following two statements, which will be proven in the next section, imply the commutativity of the diagrams above, under the assumption that the Bass-Quillen classes associated to $X^{g,h} \hookrightarrow X^{gh} \hookrightarrow X$ and $X^{g,h} \hookrightarrow X^{g} \hookrightarrow X$ vanish for all $g, h \in G$. This will give the following Theorem 3.20.

Theorem 3.20 *Suppose* $[X/G]$ *is a global quotient orbifold, where* X *is a smooth algebraic variety and* G *is a finite* abelian *group acting on* X. *Then the construction in Section 3.2 defines an operation on* $\mathrm{HT}^*(X; G)$ *which recovers the wedge product on* $\mathrm{HT}^*(X)$ *when* G *is trivial.*

This operation is associative if the Bass-Quillen class associated to the sequence of closed embeddings $X^{g,h} \hookrightarrow X^{g} \hookrightarrow X$ *vanishes for all* $g, h \in G$.

Proposition 3.21 *Under the assumptions of Theorem 3.20, assume that the Bass-Quillen class associated to* $X^{g,h} \hookrightarrow X^{gh} \hookrightarrow X$ *vanishes. Then the diagram*

$$\begin{tikzcd}
\mathbb{L}_{\widetilde{X^{g,h}}} \times^R_X \mathbb{L}_{\widetilde{X^k}} \arrow[r, "\sim"] \arrow[d] & \mathbb{L}_{\widetilde{X^{g,h,k}}} \arrow[d] \\
\mathbb{L}_{\widetilde{X^{gh}}} \times^R_X \mathbb{L}_{\widetilde{X^k}} \arrow[r, "\sim"] & \mathbb{L}_{\widetilde{X^{gh,k}}}.
\end{tikzcd}$$

is commutative.

Proposition 3.22 *Under the assumptions of Theorem 3.20, assume that the Bass-Quillen class associated to* $X^{g,h} \hookrightarrow X^{g} \hookrightarrow X$ *and* $X^{g,h} \hookrightarrow X^{h} \hookrightarrow X$ *vanish. Then the diagram*

$$\begin{tikzcd}
(X^g \times^R_X X^h) \times^R_X X^k \arrow[r, "\sim"] \arrow[d, "\mathrm{id}"] & \mathbb{L}_{\widehat{X^{g,h}}} \times^R_X X^k \arrow[r, "\sim"] & \mathbb{L}_{\widehat{X^{g,h,k}}} \arrow[d, "\mathrm{id}"] \\
X^g \times^R_X (X^h \times^R_X X^k) \arrow[r, "\sim"] & X^g \times^R_X \mathbb{L}_{\widehat{X^{h,k}}} \arrow[r, "\sim"] & \mathbb{L}_{\widehat{X^{g,h,k}}}.
\end{tikzcd}$$

is commutative.

Examples.

- If X is affine, then all the Bass-Quillen classes above are zero.
- Consider the $G = \mathbb{Z}/2\mathbb{Z}$ action on an abelian variety X. We have either $X^{g,h} = X^{gh}$ or $X^{gh} = X$ in this case, so it is easy to show that all the Bass-Quillen classes are zero.

Therefore the product on $\mathrm{HT}^*(X;G)$ defined in Section 3.2 is associative in the cases above.

3.5 Consequences of vanishing of Bass-Quillen class

We prove Propositions 3.21 and 3.22 in this section.

All the linearizations are total spaces of vector bundles over the underlying schemes, so we can reduce the result of Proposition 3.21 to the commutativity of the following two formality isomorphisms

$$\begin{tikzcd}
X^{g,h} \arrow[d] & X^{g,h} \times^R_X X^k \arrow[l] \arrow[r, "\sim"] \arrow[d] & \mathbb{E}_{(g,h),k}[-1] = \mathbb{L}_{\widehat{X^{(g,h),k}}} \arrow[d] \\
X^{gh} \arrow[d] & X^{gh} \times^R_X X^k \arrow[l] \arrow[r, "\sim"] \arrow[d] & \mathbb{E}_{gh,k}[-1] = \mathbb{L}_{\widehat{X^{gh,k}}} \\
X & X^k \arrow[l] &
\end{tikzcd}$$

where $E_{(g,h),k} = \frac{T_X}{T_{X^{g,h}} + T_{X^k}}$ and $E_{gh,k} = \frac{T_X}{T_{X^{gh}} + T_{X^k}}$ are excess bundles supported on $X^{g,h,k}$ and $X^{gh,k}$ respectively.

To check the commutativity, we need to look at how the isomorphism is defined in [3]. For simplicity, denote T_X by V. The two isomorphisms are defined based on two splittings of the two short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \frac{V^k}{V^{g,h} \cap V^k} \arrow[r] \arrow[d] & \frac{V}{V^{g,h}} = N_{X^{g,h}/X} \arrow[r] \arrow[d] \arrow[l, dashed, bend right] & \frac{V}{V^{g,h} + V^k} = E_{(g,h),k} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \frac{V^k}{V^{gh} \cap V^k} \arrow[r] & \frac{V}{V^{gh}} = N_{X^{gh}/X} \arrow[r] \arrow[l, dashed, bend right] & \frac{V}{V^{gh} + V^k} = E_{gh,k} \arrow[r] & 0.
\end{tikzcd}$$

The two splittings are compatible in the sense that the diagram above commutes because the two splittings are the averaging map by the element $k \in G$

$$v \to \frac{1}{\text{ord}(k)} \sum_{i=1}^{\text{ord}(k)} k^i \cdot v.$$

Proposition 3.21 is a consequence of the more general result Proposition 3.23 below, by replacing X, Y, Z, and S in by $X^{g,h}$, X^{gh}, X^k, and X. Note that all the assumptions in Proposition 3.23 except for the last one hold trivially for $X^{g,h}$, X^{gh}, X^k, and X.

Proposition 3.23 *Consider a sequence of closed embeddings $X \hookrightarrow Y \hookrightarrow S$, and a separate closed embedding $Z \hookrightarrow S$.*

Assume that all the closed embeddings split to first order in the sense of [3], and that the first order splittings of $X \hookrightarrow Y \hookrightarrow S$ are compatible in the sense that was explained in Theorem 2.2. We further assume that the Bass-Quillen class associated to $X \hookrightarrow Y \hookrightarrow S$ is zero. Then the diagram

$$\begin{tikzcd}
X \times_S^R Z \arrow[r, "\sim"] \arrow[d] & \mathbb{E}_{X,Z}[-1] = \mathbb{E}_W[-1] = \mathbb{L}_{X \times_S^R Z} \arrow[d] \\
Y \times_S^R Z \arrow[r, "\sim"] & \mathbb{E}_{Y,Z}[-1] = \mathbb{E}_T[-1] = \mathbb{L}_{Y \times_S^R Z}
\end{tikzcd}$$

is commutative, where $E_{X,Z} = E_W = \frac{T_S}{T_X + T_Z}$ and $E_{Y,Z} = E_T = \frac{T_S}{T_Y + T_Z}$ are the excess bundles supported on

$$W = X \times_S Z \text{ and } T = Y \times_S Z.$$

The horizontal isomorphisms are defined in [3] and will be explained in the proof below. The map $\mathbb{E}_{X,Z}[-1] \to \mathbb{E}_{Y,Z}[-1]$ *is induced by the obvious map of vector bundles.*

Before we begin the proof we note that the setup of the above proposition gives rise to the following diagram of spaces, where W and T are the underived fiber products,

$$\begin{tikzcd}
X \arrow[dd] & & X \times^R_S Z \arrow[ll] \arrow[dd] \\
& W = X \times_S Z \arrow[ul] \arrow[ur] \arrow[dl] \arrow[dr] \arrow[dd] & \\
Y \arrow[dd] & & Y \times^R_S Z \arrow[ll] \arrow[dd] \\
& T = Y \times_S Z \arrow[ul] \arrow[ur] \arrow[dl] \arrow[dr] & \\
S & & Z. \arrow[ll]
\end{tikzcd}$$

Proof. The compatibility of first order splittings implies that the following two short exact sequences and their splittings are compatible

$$\begin{tikzcd}
0 \arrow[r] & N_{W/Z} \arrow[r] \arrow[d] & N_{X/S}|_W \arrow[r] \arrow[d] \arrow[l, dashed, bend right] & E_W \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & N_{T/Z}|_W \arrow[r] & N_{Y/S}|_W \arrow[r] \arrow[l, dashed, bend right] & E_T|_W \arrow[r] & 0.
\end{tikzcd}$$

The two isomorphisms $\mathbb{E}_W[-1] \cong X \times^R_S Z$, and $\mathbb{E}_T[-1] \cong Y \times^R_S Z$ are defined using the two splittings of short exact sequences above. The three horizontal maps on the left of the diagram below are the splittings of the short exact sequences. The composition of horizontal maps below are the desired isomorphisms $\mathbb{E}_W[-1] \cong X \times^R_S Z$ and $\mathbb{E}_T[-1] \cong$

$Y \times^R_S Z$,

$$\begin{tikzcd}
\mathbb{E}_W[-1] \arrow[r, dashed] \arrow[d] & \mathbb{N}_{X/S}[-1]|_W \arrow[r, "\sim"] \arrow[d] & X \times^R_S X|_W = X \times^R_S W \arrow[r] \arrow[d] & X \times^R_S Z \arrow[d] \\
\mathbb{E}_T[-1]|_W \arrow[r, dashed] \arrow[d] & \mathbb{N}_{Y/S}[-1]|_W \arrow[r, "\sim"] \arrow[d] & Y \times^R_S Y|_W = Y \times^R_S W \arrow[r] \arrow[d] & Y \times^R_S Z \arrow[d] \\
\mathbb{E}_T[-1] \arrow[r, dashed] & \mathbb{N}_{Y/S}[-1]|_T \arrow[r, "\sim"] & Y \times^R_S Y|_T = Y \times^R_S T \arrow[r] & Y \times^R_S Z.
\end{tikzcd}$$

We only need to prove the commutativity of the isomorphisms in the middle

$$\begin{tikzcd}
\mathbb{N}_{X/S}[-1]|_W \arrow[r, "\sim"] \arrow[d] & X \times^R_S X|_W = X \times^R_S W \arrow[d] \\
\mathbb{N}_{Y/S}[-1]|_W \arrow[r, "\sim"] & Y \times^R_S Y|_W = Y \times^R_S W
\end{tikzcd}$$

because all the others are commutative. We can restrict everything to X first, and then restrict to W. Therefore, it suffices to show the commutativity of

$$\begin{tikzcd}
\mathbb{N}_{X/S}[-1] \arrow[r, "\sim"] \arrow[d] & X \times^R_S X \arrow[d] \\
\mathbb{N}_{Y/S}[-1]|_X \arrow[r, "\sim"] & Y \times^R_S Y|_X = Y \times^R_S X.
\end{tikzcd}$$

This is Theorem 2.2 in Chapter 2. ☐

Proof of Proposition 3.22. For simplicity denote the space X^g, X^h, X^k, and X in Proposition 3.22 by X, Y, Z, and S.

Because of Proposition 3.23 we have the commutativity of

$$\begin{tikzcd}
X \times^R_S Y \arrow[r, "\sim"] & \mathbb{E}_{X,Y}[-1] = \mathbb{L}_{X \times^R_S Y} & W \times^R_S Z \arrow[r, "\sim"] \arrow[d] & \mathbb{E}_{W,Z}[-1] = \mathbb{L}_{W \times^R_S Z} \arrow[d] \\
X \times^R_S T \arrow[r, "\sim"] \arrow[u] & \mathbb{E}_{X,T}[-1] = \mathbb{L}_{X \times^R_S T} \arrow[u] & Y \times^R_S Z \arrow[r, "\sim"] & \mathbb{E}_{Y,Z}[-1] = \mathbb{L}_{Y \times^R_S Z},
\end{tikzcd}$$

where $T = Y \times_S Z$ and $W = X \times_S Y$. As a consequence we get the commutative diagram

$$\begin{tikzcd}
X \times^R_S Y \times^R_S Z \arrow[r] \arrow[d, "id"'] & \mathbb{E}_{X,Y}[-1] \times^R_S Z = \mathbb{E}_{X,Y}[-1] \times^R_W (W \times^R_S Z) \arrow[r] \arrow[d] & \mathbb{E}_{X,Y}[-1] \times^R_W \mathbb{E}_{W,Z}[-1] \arrow[r] \arrow[d] & \mathbb{E}_{X,Y,Z}[-1] \arrow[d, "id"'] \\
X \times^R_S Y \times^R_S Z \arrow[r] \arrow[d, leftrightarrow, "id"'] & \mathbb{E}_{X,Y}[-1] \times^R_S Z = \mathbb{E}_{X,Y}[-1] \times^R_Y (Y \times^R_S Z) \arrow[r] & \mathbb{E}_{X,Y}[-1] \times^R_Y \mathbb{E}_{Y,Z}[-1] \arrow[r] \arrow[d, leftrightarrow, "id"'] & \mathbb{E}_{X,Y,Z}[-1] \arrow[d, leftrightarrow, "id"'] \\
X \times^R_S Y \times^R_S Z \arrow[r] & X \times^R_S \mathbb{E}_{Y,Z}[-1] = (X \times^R_S Y) \times^R_Y \mathbb{E}_{Y,Z}[-1] \arrow[r] & \mathbb{E}_{X,Y}[-1] \times_Y \mathbb{E}_{Y,Z}[-1] \arrow[r] & \mathbb{E}_{X,Y,Z}[-1] \\
X \times^R_S Y \times^R_S Z \arrow[r] \arrow[u, "id"] & X \times^R_S \mathbb{E}_{Y,Z}[-1] = (X \times^R_S T) \times^R_T \mathbb{E}_{Y,Z}[-1] \arrow[r] \arrow[u] & \mathbb{E}_{X,T}[-1] \times^R_T \mathbb{E}_{Y,Z}[-1] \arrow[r] \arrow[u] & \mathbb{E}_{X,Y,Z}[-1], \arrow[u, "id"']
\end{tikzcd}$$

where all the arrows are isomorphisms, and $E_{X,Y,Z}$ is the excess bundle of the triple intersection $X \times^R_S Y \times^R_S Z$.

The two rightmost squares of the diagram above commute because there are natural isomorphisms $E_{X,Y,Z} \cong E_{X,Y}|_U \oplus E_{W,Z} \cong E_{X,T} \oplus E_{Y,Z}|_U$, where $U = X \cap Y \cap Z = X \times_S Y \times_S Z$. The commutativity of the outer square of the diagram above is the one that we needed to prove in Proposition 3.22. $\square$

3.6 A possible simplification

This section is more speculative. After we rewrite our product in Definition 3.16 in more concrete terms, we propose a way to simplify the formulas for Calabi-Yau global quotient orbifolds. The simplification is motivated by the definition of Chen-Ruan orbifold cohomology, so we need to first review this definition before proceeding. We then provide several examples comparing the simplified product with the Chen-Ruan orbifold cohomology, via homological mirror symmetry. We end this chapter by stating a number of questions that remain open for future research.

In Section 3.3 we computed the structure complexes of $\mathbb{L}_{\widetilde{X^{g,h}}}$ and $\mathbb{L}_{\widetilde{X^g}} \times^R_X \mathbb{L}_{\widetilde{X^h}}$, so we can write the two-step map

$$\mathbb{D}(\mathbb{L}_{\widetilde{X^g}}) \otimes \mathbb{D}(\mathbb{L}_{\widetilde{X^h}}) \to \mathbb{D}(\mathbb{L}_{\widetilde{X^g}} \times^R_X \mathbb{L}_{\widetilde{X^h}}) \cong \mathbb{D}(\mathbb{L}_{\widetilde{X^{g,h}}}) \to \mathbb{D}(\mathbb{L}_{\widetilde{X^{gh}}})$$

in Definition 3.16 in more concrete terms. Explicitly, for $g \in G$ define

$$\mathrm{HT}^{(p,q)}(X; g) = H^{p-c_g}(X^g, \wedge^q T_{X^g} \otimes \omega_g).$$

The two composition of the maps above can then be written as a direct sum over p, p', q, q' of maps

$$\mathrm{HT}^{(p,q)}(X; g) \otimes \mathrm{HT}^{(p',q')}(X; h) \to \bigoplus_{i=0}^{\operatorname{rk} E} \mathrm{HT}^{(p+p'-i,q+q'+i)}(X; gh)$$

factoring through the middle term (coming from $\mathbb{D}(\mathbb{L}_{\widetilde{X^{g,h}}})$)

$$\bigoplus_{i=0}^{\operatorname{rk} E} H^{p+p'-c_{g,h}-i}(X^{g,h}, \wedge^q T_{X^g}|_{X^{g,h}} \otimes \wedge^{q'} T_{X^h}|_{X^{g,h}} \otimes \wedge^i E \otimes \omega_{g,h}).$$

Here E is the excess bundle for the intersection of X^g and X^h in X. Note in particular that if G is trivial our definition recovers the classical product on polyvector fields.

Observe that the $\mathrm{HT}^{(p,q)}$ notation does *not* give a bigrading – *a priori* all the maps above, for $0 \leq i \leq \operatorname{rk} E$ could be non-zero. The simplification we propose, for the Calabi-Yau case, is to leave only one of these maps, for a specific i. (Conjecturally, all the other maps would be zero anyway.)

The formulas for Chen-Ruan orbifold cohomology. We now discuss some preparations for the motivation for the simplification of the product we defined. The idea is to draw inspiration from mirror symmetry, and to regard, in the Calabi-Yau case, Chen-Ruan orbifold cohomology as the mirror of orbifold Hochschild cohomology.

Let X be a complex manifold endowed with the action of a finite group G. Chen and Ruan [11] defined a version of the singular cohomology *ring* for the orbifold $[X/G]$. Fantechi and Göttsche [18] wrote down the formula for the product explicitly as follows. They first constructed an associative product on

$$H^*_{\mathsf{orb}}(X;G) = \bigoplus_{g\in G} H^{*-2\iota(g)}(X^g, \mathbb{C})$$

which maps $\alpha_g \in H^{*-2\iota(g)}(X^g, \mathbb{C})$ and $\beta_h \in H^{*-2\iota(h)}(X^h, \mathbb{C})$ to

$$(\alpha_g, \beta_h) \mapsto i^{gh}_{g,h*}(\alpha_g|_{X^{g,h}} \cdot \beta_h|_{X^{g,h}} \cdot \gamma_{g,h}).$$

Here $\gamma_{g,h}$ is the top Chern class of a certain twist bundle whose rank is $\iota(g) + \iota(h) - \iota(gh) - \operatorname{codim}(X^{g,h}, X^{gh})$, where $\iota(g)$ is the so-called *age* of g, see [18].

The Chen-Ruan orbifold singular cohomology ring is obtained by taking G-invariants:

$$H^*_{\mathsf{orb}}([X/G]) = H^*_{\mathsf{orb}}(X;G)^G.$$

Note that the above ring is *bigraded* with respect to the orbifold Hodge decomposition [1]

$$H^{n-2\iota(g)}(X^g, \mathbb{C}) = \bigoplus_{p+q=n} H^{p-\iota(g)}(X, \wedge^{q-\iota(g)}\Omega_{X^g}).$$

Hyperkähler conjecture. We discuss the formula of Fantechi-Göttsche and its relation with hyperkähler conjecture. Let S be a hyperkähler surface and S^n be the n copies of direct product of S. The symmetric group Σ_n acts naturally on S^n. One part of the conjecture states that the orbifold singular cohomology $H^*_{\mathsf{orb}}([S^n/\Sigma_n])$ is isomorphic to the singular cohomology of the Hilbert scheme of n points on S. Fantechi-Göttsche [18] proves the statement above holds up to a sign. We discuss the sign issue first. Let

$l(g)$ be the minimal number of transpositions whose product is g for $g \in G$. Define a modified product on $H^*_{\mathsf{orb}}([S^n/\Sigma_n])$ by

$$\alpha_g * \beta_h = (-1)^{\frac{l(g)+l(h)-l(gh)}{2}} \alpha_g \cdot \beta_h.$$

Denote this new ring by $(H^*_{\mathsf{orb}}([S^n/\Sigma_n], *)$.

Theorem 3.24 (Fantechi-Göttsche) *Let S be a complex projective surface with trivial canonical bundle. There is a canonical ring isomorphism between $(H^*_{\mathsf{orb}}([S^n/\Sigma_n], *)$ and $H^*(S^{[n]})$, where $S^{[n]}$ is the Hilbert scheme of n points on S.*

The formula of this canonical ring isomorphism is given in [18].

Orbifold Hochschild cohomology. Mirror symmetry associates two graded commutative rings to a Calabi-Yau space: the A- and the B-model state spaces, which are interchanged by the mirror operation. When the target space is a compact Calabi-Yau manifold X, the A-space is $H^*(X, \mathbb{C})$, while the B-space is $\mathrm{HH}^*(X)$. When it is an orbifold $[X/G]$, these spaces are naturally the Chen-Ruan orbifold cohomology and the orbifold Hochschild cohomology rings of $[X/G]$.

Since the product in the A-model preserves the p, q bidegree, the yoga of mirror symmetry suggests that the product in the B-model should also preserve *some* bidegree for Calabi-Yau orbifolds.

The proofs of the following two lemmas are left as exercises to the reader.

Lemma 3.25 *There is a natural isomorphism*

$$\omega_g|_{X^{g,h}}[-c_g] \otimes \omega_h|_{X^{g,h}}[-c_h] \cong \wedge^r E[r] \otimes \omega_{g,h}[-c_{g,h}],$$

where r is the rank of the excess bundle E.

Lemma 3.26 *The bundle* $T_{X^g}|_{X^{g,h}}$ *decomposes naturally into a direct sum as* $T_{X^{g,h}} \oplus N_{X^{g,h}/X^g}$*, and similarly for* $T_{X^h}|_{X^{g,h}}$*.*

The class $\gamma_{g,h} \in H^k(X^{g,h}, \wedge^k \Omega_{X^{g,h}})$ *in Fantechi and Göttsche's paper [18] acts naturally on*

$$\bigoplus_{p,q} H^p(X^{g,h}, \wedge^q T_{X^g}|_{X^{g,h}} \otimes \wedge^{q'} T_{X^h}|_{X^{g,h}} \otimes \wedge^r E^\vee \otimes \omega_{g,h}),$$

where

$$k = \iota(g) + \iota(h) - \iota(gh) - \operatorname{codim}(X^{g,h}, X^{gh})$$

and the action is given by the contraction of $\Omega_{X^{g,h}}$ *with* $T_{X^{g,h}}$*.*

We are now ready to give a new construction for an operation on $\mathrm{HT}^*(X;G)$ which mimics more closely the Fantechi-Göttsche product [18]. Define the bigraded piece $\mathrm{HT}^{p,q}(X;G)$ of bidegree p, q of $\mathrm{HT}(X;G)$ by

$$\mathrm{HT}^{p,q}(X;G) = H^{p-\iota(g)}(X, \wedge^{q+\iota(g)-c_g} T_{X^g} \otimes \omega_g).$$

The product will be bigraded, being given by maps

$$\mathrm{HT}^{p,q}(X;G) \otimes \mathrm{HT}^{p',q'}(X;G) \to \mathrm{HT}^{p+p',q+q'}(X;G).$$

Note that unlike the product in Definition 3.16, only one of the maps there is non-zero. We conjecture that in Calabi-Yau situations, the two products agree – in other words, all the maps in Definition 3.16 which do not preserve the bigrading are zero. This is the case in all the examples we study below.

Definition 3.27 *The new product is defined as the following composition:*

$$\begin{aligned}
&H^p(X^g, \wedge^q T_{X^g} \otimes \omega_g[-c_g]) \otimes H^{p'}(X^h, \wedge^{q'} T_{X^h} \otimes \omega_h[-c_h]) \\
&\qquad \to H^{p+p'}(X^{g,h}, \wedge^q T_{X^g}|_{X^{g,h}} \otimes \omega_g|_{X^{g,h}}[-c_g] \otimes \wedge^{q'} T_{X^h}|_{X^{g,h}} \otimes \omega_h|_{X^{g,h}}[-c_h]) \\
&\qquad \cong H^{p+p'-r}(X^{g,h}, \wedge^q T_{X^g}|_{X^{g,h}} \otimes \wedge^{q'} T_{X^h}|_{X^{g,h}} \otimes \omega_{g,h}[-c_{g,h}] \otimes \wedge^r E) \\
&\qquad \to \bigoplus_{i+j=k} H^{p+p'-r+k}(X^{g,h}, \wedge^{q-i} T_{X^g}|_{X^{g,h}} \otimes \wedge^{q'-j} T_{X^h}|_{X^{g,h}} \otimes \omega_{g,h}[-c_{g,h}] \otimes \wedge^r E) \\
&\qquad \to H^{p+p'-r+k}(X^{gh}, \wedge^{q+q'+r-k} T_{X^{gh}} \otimes \omega_{gh}[-c_{gh}]).
\end{aligned}$$

The first arrow is the naive restriction from X^g and X^h to $X^{g,h}$. The isomorphisms in the middle are due to Lemma 3.25. The last arrow is the map $\mathbb{L}_{m*}$ in Definition 3.16. The second arrow in the middle involving k is the action of $\gamma_{g,h}$ in Lemma 3.26. One does indeed verify that this map respects the bigrading defined above.

Examples. For a first example consider an abelian surface A endowed with the action of $\mathbb{Z}/2\mathbb{Z}$, acting by negation in the group law of A. The tangent bundle of A is trivial, so there are no Duflo correction terms. The mirror of the orbifold $[A/G]$ is expected to be $[A/G]$ itself in this case. This suggests that the product we defined should match with the one on the orbifold cohomology of $[A/G]$, so we expect to find isomorphisms

$$\mathrm{HT}^*([A/G]) \cong \mathrm{HH}^*([A/G]) \cong H^*_{\mathsf{orb}}([A/G], \mathbb{C}),$$

It is known [18] that in this case the classes $\gamma_{g,h}$ are trivial. Write $G = \{e, \tau\}$ where e is the identity element. Then we have

$$\mathrm{HH}^*([A/G]) = (\mathrm{HH}^*(A, e) \oplus \mathrm{HH}^*(A, \tau))^G = \mathrm{HH}^*(A, e) \oplus \mathrm{HH}^*(A, \tau)^\tau,$$

where for $g \in G$ the notation $\mathrm{HH}^*(A, g)$ was explained in Section 3.1. The space $\mathrm{HH}^*(A, e)$ is the Hochschild cohomology of A, and its product is well-understood from the Kontsevich and Calaque-Van den Bergh theorem. The only non-trivial product we need to understand is

$$\mathrm{HH}^*(A,\tau)\otimes \mathrm{HH}^*(A,\tau)\to \mathrm{HH}^*(A,e).$$

Note that the space

$$\mathrm{HH}^*(A,\tau)=H^0(A^\tau,\wedge^0 T_{A^\tau}\otimes\omega_\tau)=H^0(A^\tau,\mathbb{C}),$$

is a 16-dimensional vector space in cohomological degree 2. It is of bidegree $(1,1)$ under the new bigrading we defined in Definition 3.27. By the definition of our product, it is also clear that the product of two $(1,1)$-form gives a $(2,2)$-form which lands in $H^2(A,\wedge^2 T_A)$. This matches perfectly with the product on orbifold cohomology [18].

For another example, consider a holomorphic symplectic orbifold $[X/G]$. Again, the mirror of $[X/G]$ is expected to be $[X/G]$, so we expect to get

$$\mathrm{HT}^*([X/G])\cong \mathrm{HH}^*(X/G)\cong H^*_{\mathsf{orb}}([X/G],\mathbb{C}).$$

The right hand side decomposes into

$$H^{*-2\iota(g)}(X^g,\mathbb{C})=\bigoplus_{p+q=*}H^{p-\iota(g)}(X^g,\wedge^{q-\iota(g)}\Omega_{X^g})$$

by the Hodge decomposition. The left hand side is

$$\bigoplus_{g\in G}H^{p-\iota(g)}(X^g,\wedge^{q+\iota(g)-c_g}T_{X^g}\otimes\omega_g).$$

Moreover, ω_g is trivial and $\Omega_{X^g}\cong T_{X^g}$ because of the holomorphic symplectic condition. There is a canonical identification between the two sides as vector spaces, and we believe the two products should agree. The bigradings of the two sides match completely because $2\iota(g)=c_g$.

One important example we have in mind is when $X = K^n$ consists of n copies of a K3 surface K and $G = \Sigma_n$ is the symmetric group acting on K^n by permutation. The group is not abelian in this case, but the constructions and results in Sections 3.2 and 3.3 still work because one can check directly that Proposition 3.12 holds in this situation. The key point is that in this situation all the tangent bundles and normal bundles involved are copies of direct sums of T_K, so the short exact sequence in the proof of Proposition 3.12 splits naturally.

Open questions. (1) We can not prove that the simplified product in Definition 3.27 agrees with the one in Definition 3.16. We conjecture that they agree under the Calabi-Yau assumption.

(2) For any Calabi-Yau orbifold, we believe that our product on orbifold polyvector fields should match with the Chen-Ruan orbifold cohomology of the mirror.

(3) This is our main conjecture. For any orbifold $[X/G]$ with an abelian group action, we believe that Kontsevich's Theorem holds, i.e., the orbifold Hochschild cohomology should be isomorphic to the orbifold polyvector fields. More precisely, we conjecture that the diagram

$$\begin{tikzcd}
\mathbb{D}(\mathbb{L}_{\widetilde{X^g}}) \otimes \mathbb{D}(\mathbb{L}_{\widetilde{X^h}}) \arrow[r, "HKR\circ(-\lrcorner\mathrm{td}(T_{X^g})^{-\frac{1}{2}})\otimes HKR\circ(-\lrcorner\mathrm{td}(T_{X^h})^{-\frac{1}{2}})"] \arrow[d] & \mathbb{D}(\widetilde{X^g}) \otimes \mathbb{D}(\widetilde{X^h}) \arrow[d] \\
\mathbb{D}(\mathbb{L}_{\widetilde{X^g \times^R_X X^h}}) \cong \mathbb{D}(\mathbb{L}_{\widetilde{X^g}} \times^R_X \mathbb{L}_{\widetilde{X^h}}) \arrow[r, "HKR\circ(-\lrcorner\mathrm{td}(T_{X^{g,h}})^{-\frac{1}{2}})"] \arrow[d, "\mathbb{L}_{m*}"] & \mathbb{D}(\widetilde{X^g} \times^R_X \widetilde{X^h}) = \mathbb{D}(\widetilde{X^{g,h}}) \arrow[d, "m_*"] \\
\mathbb{D}(\mathbb{L}_{\widetilde{X^{gh}}}) \arrow[r, "HKR\circ(-\lrcorner\mathrm{td}(T_{X^{gh}})^{-\frac{1}{2}})"] & \mathbb{D}(\widetilde{X^{gh}})
\end{tikzcd}$$

is commutative, where the horizontal maps are isomorphisms. All the HKR maps that appear in the horizontal isomorphisms are the formality isomorphisms in Sections 3.1-3.3. They generalize the classical HKR isomorphism as explained in [3]. As mentioned at the very beginning of this paper, HKR can not be an isomorphism of rings, so we need to add the Duflo correction term in the horizontal isomorphisms.

Chapter 4

Applications in deformation theory and hyperkähler manifolds

We discuss applications in deformation theory and hyperkähler manifolds in Chapter 4. In particular, we solve the question in Section 1.4 raised by Markman.

4.1 A commutative diagram of representations of the shifted bundle

We need the following contraction operation due to Toda [32] before we state the results. Consider the exponential Atiyah class

$$\exp(\mathrm{at}_{\mathscr{F}}) = 1 + \mathrm{at}_{\mathscr{F}} + \cdots + \frac{(\mathrm{at}_{\mathscr{F}})^k}{k!} + \cdots ,$$

where $(\mathrm{at}_{\mathscr{F}})^k \in \mathrm{Ext}^k(\mathscr{F}, \mathscr{F} \otimes \wedge^k \Omega_X)$.

Let $\tilde{\alpha}$ be a class in $\mathrm{HT}^*(X)$ and $\tilde{\alpha}^{p,k} \in H^p(X, \wedge^k T_X)$ be the homogenous degree (p, k) part of $\tilde{\alpha}$. We can contract $\tilde{\alpha}^{p,k}$ with $\frac{(\mathrm{at}_{\mathscr{F}})^k}{k!}$ to get an element in $\mathrm{Ext}^{p+k}(\mathscr{F}, \mathscr{F})$. Taking the sum over all (p, k), we get the desired class which will be denoted by $\alpha \lrcorner \, \tilde{\exp}(\mathrm{at}_{\mathscr{F}}) \in \mathrm{Ext}^*(\mathscr{F}, \mathscr{F})$. When $\tilde{\alpha}$ is a class in $H^1(X, T_X)$, we recover the previous contraction $\tilde{\alpha} \lrcorner \, \mathrm{at}_{\mathscr{F}}$ in Section 1.4.

There is another way to produce a class in $\mathrm{Ext}^*(\mathscr{F},\mathscr{F})$ with a given class $\tilde{\alpha} \in \mathrm{HT}^*(X)$. It uses the HKR isomorphism

$$I^{\mathsf{HKR}} : \mathrm{HT}^*(X) \to \mathrm{HH}^*(X).$$

Since $\mathrm{HH}^*(X)$ can be interpreted as natural transformations of the identity functor at the dg level, this yields a natural map $\mathrm{HH}^*(X) \to \mathrm{Ext}^*(\mathscr{F},\mathscr{F})$.

Theorem 4.1 *The two classes defined above are the same. In other words the diagram*

$$\begin{tikzcd}
\mathrm{HH}^*(X) \arrow[r] & \mathrm{Ext}^*(\mathscr{F},\mathscr{F}) \\
\mathrm{HT}^*(X) \arrow[u, "I^{\mathsf{HKR}}"] \arrow[ur, "(-)\lrcorner\exp(\mathrm{at}_{\mathscr{F}})"'] &
\end{tikzcd}$$

is commutative.

There is an analogous result for Hopf algebras. See Theorem 2.7 in [10] and see [24] for more details. We prove Theorem 4.1 in Subsection 4.1.2.

The inspiration for Theorem 4.1 comes from a similar statement in Lie theory.

4.1.1 A similar diagram for Lie algebras.

Let $\mathfrak{g}$ be a finite dimensional Lie algebra over a field of characteristic zero and let V be a finite dimensional representation of $\mathfrak{g}$. There is a diagram

$$\begin{tikzcd}
(U\mathfrak{g})^{\mathfrak{g}} \arrow[r] & \mathrm{Hom}(V,V) \\
(S\mathfrak{g})^{\mathfrak{g}}. \arrow[u, "\mathsf{PBW}"] \arrow[ur] &
\end{tikzcd}$$

The PBW map from the symmetric algebra $S\mathfrak{g}$ to the universal enveloping algebra $U\mathfrak{g}$ is defined on the degree n-th component of $S\mathfrak{g}$ as follows

$$x_1 \cdots x_n \to \frac{1}{n!} \sum_{\sigma \in \Sigma_n} x_{\sigma(1)} \cdots x_{\sigma(n)}.$$

Here Σ_n is the symmetric group on a finite set of n symbols. The universal enveloping algebra $U\mathfrak{g}$ acts naturally on V. This natural action defines the map $(\mathrm{U}\mathfrak{g})^{\mathfrak{g}} \to \mathrm{Hom}(V, V)$ on the top of the diagram above. The map $(\mathrm{S}\mathfrak{g})^{\mathfrak{g}} \to \mathrm{Hom}(V, V)$ is defined as follows. We can rewrite the representation map $\mathfrak{g} \otimes V \to V$ as a map $\Lambda : V \to V \otimes \mathfrak{g}^*$. Take the exponent

$$\exp(\Lambda) = id_V + \Lambda + \cdots + \frac{\Lambda^k}{k!} + \cdots$$

of the map Λ. Then we can contract $\exp(\Lambda)$ with $S\mathfrak{g}$. The similarity between this diagram and the diagram in Theorem 4.1 was explained in Section 1.4.

Proposition 4.2 *The diagram of the Lie algebra* $\mathfrak{g}$ *above is commutative.*

Proof. We can prove that the Lie algebra diagram is commutative even before taking $\mathfrak{g}$-invariants, i.e., the diagram

$$\begin{tikzcd} U\mathfrak{g} \arrow[r] & \mathrm{Hom}(V, V) \\ S\mathfrak{g} \arrow[u, "\mathsf{PBW}"] \arrow[ur] & \end{tikzcd}$$

is commutative. The map PBW factors through the tensor algebra $T\mathfrak{g}$

$$\mathsf{PBW} : S\mathfrak{g} \xrightarrow{\psi} T\mathfrak{g} \longrightarrow U\mathfrak{g},$$

so we can replace $U\mathfrak{g}$ at the top left corner of the diagram by $T\mathfrak{g}$. It is easy to check that the map $S\mathfrak{g} \to \mathrm{Hom}(V, V)$ is equal to the following map

$$S\mathfrak{g} \xrightarrow{\psi} T\mathfrak{g} \xrightarrow{\varphi} \mathrm{Hom}(V, V),$$

where the map $\varphi : T\mathfrak{g} \to \mathrm{Hom}(V, V)$ is defined as follows. Rewrite the representation map $\mathfrak{g} \otimes V \to V$ as a map $\Lambda : V \to V \otimes \mathfrak{g}^*$. Instead of taking the exponential of the map Λ, we compose the map Λ with itself k times. We get a map $\Lambda^{\otimes k} : V \to V \otimes (\mathfrak{g}^*)^{\otimes k}$ in this way. Contract $\Lambda^{\otimes k}$ with $\mathfrak{g}^{\otimes k}$ and get a map $\mathfrak{g}^{\otimes k} \to \mathrm{Hom}(V, V)$. Adding the k-th components for all $k \in \mathbb{N}$, we obtain the desired map $\varphi : T\mathfrak{g} \to \mathrm{Hom}(V, V)$.

Now we have two maps $T\mathfrak{g} \to \mathrm{Hom}(V, V)$. One of them is the map φ, and the other one is $\Theta : T\mathfrak{g} \to U\mathfrak{g} \to \mathrm{Hom}(V, V)$. We want to show that they agree. This follows from Lemma 4.3 below by setting W_1 to be V and W_2 to be $\mathfrak{g}^{\otimes k}$. □

Lemma 4.3 *Let W_1 and W_2 be finite dimensional vector spaces over a field k and f be a map $W_2 \otimes W_1 \to W_1$. Rewrite the map as $g : W_1 \to W_2^* \otimes W_1$ by the adjunction formula* $\mathrm{Hom}(W_2 \otimes_k W_1, W_1) = \mathrm{Hom}(W_1, W_2^* \otimes_k W_1)$. *Fix an element $x \in W_2$. Then $f(x \otimes -)$ is a map from W_1 to W_1. This map is precisely g followed by the contraction with x.*

Proof. This is due to the adjunction property

$$\mathrm{Hom}(W_2 \otimes_k W_1, W_1) = \mathrm{Hom}(W_1, W_2^* \otimes_k W_1). \qquad \square$$

4.1.2 Proof of Theorem 4.1

The proof for the Lie algebra diagram reduces the commutativity of the Lie algebra diagram to a statement about tensor algebras. The statement about tensor algebras remains valid in the case of derived categories.

Proof of Theorem 4.1. One can define a map $\mathrm{Sym}(T_X[-1]) \to T(T_X[-1])$ given by the formula

$$x_1 \wedge \cdots \wedge x_n \to \frac{1}{n!} \sum_{\sigma \in \Sigma_n} (-1)^{sgn(\sigma)} x_{\sigma(1)} \cdots x_{\sigma(n)},$$

where $T(T_X[-1])$ is the tensor algebra on $T_X[-1]$.

The map above is a differential graded version of the map ψ in (4.1.1). Let $X^{(1)}$ be the first order neighborhood of X in $X \times X$. There are embeddings $i : X \hookrightarrow X^{(1)}$ and $j : X^{(1)} \hookrightarrow X \times X$. Arinkin and Căldăraru [2] showed that $T(T_X[-1])$ is isomorphic to $(i^*i_*\mathscr{O}_X)^\vee$, where $(-)^\vee$ is the dual. The map

$$(i^*i_*\mathscr{O}_X)^\vee \to (i^*j^*j_*i_*\mathscr{O}_X)^\vee = (\Delta^*\Delta_*\mathscr{O}_X)^\vee = \underline{\mathsf{Hom}}(\Delta_*\mathscr{O}_X, \Delta_*\mathscr{O}_X)$$

is defined by the adjunction $j^* \dashv j_*$. The composite map

$$\mathrm{Sym}(T_X[-1]) \to T(T_X[-1]) \cong (i^*i_*\mathscr{O}_X)^\vee \to (i^*j^*j_*i_*\mathscr{O}_X)^\vee$$

$$= (\Delta^*\Delta_*\mathscr{O}_X)^\vee = \underline{\mathsf{Hom}}(\Delta_*\mathscr{O}_X, \Delta_*\mathscr{O}_X)$$

is the sheaf version HKR isomorphism as showed in [2]. Taking cohomology on both sides of the equality above, we get the HKR isomorphism

$$I^{\mathsf{HKR}} : \mathrm{HT}^*(X) = \bigoplus_{p+q=*} H^p(X, \wedge^q T_X) \to \mathrm{HH}^*(X).$$

Now it is clear that we have a commutative diagram

$$\begin{tikzcd}
\underline{\mathsf{Hom}}(\Delta_*\mathscr{O}_X, \Delta_*\mathscr{O}_X) \arrow[r] & \underline{\mathsf{Hom}}(\mathscr{F}, \mathscr{F}) \\
T(T_X[-1]) \arrow[u] \arrow[ur] & \\
\mathrm{Sym}(T_X[-1]), \arrow[u] \arrow[uur] &
\end{tikzcd}$$

which is similar to the Lie algebra diagram in (4.1.1). Taking cohomology on the diagram above, we get the diagram

$$\begin{tikzcd}
\mathrm{HH}^*(X) \arrow[r] & \mathrm{Ext}^*(\mathscr{F}, \mathscr{F}) \\
\mathrm{HT}^*(X) = \bigoplus_{p+q=*} H^p(X, \wedge^q T_X) \arrow[u, "I^{\mathsf{HKR}}"] \arrow[ur, "(-)\lrcorner\exp(at_{\mathscr{F}})"'] &
\end{tikzcd}$$

that we start with in Theorem 4.1. $\square$

4.2 Applications of Theorem 4.1

We apply Theorem 4.1 to obtain Theorem 4.5 below. Then we explain Theorem 4.5 answers the question of Markman in Section 1.4.

We first introduce a few notations. Denote $I^{\mathsf{HKR}}(\tilde{\alpha})$ by

$$\alpha \in \mathrm{HH}^*(X) = \mathrm{Ext}^*_{X\times X}(\mathscr{O}_\Delta, \mathscr{O}_\Delta),$$

where $\mathscr{O}_\Delta = \Delta_*\mathscr{O}_X$. Denote the image of α in $\mathrm{Ext}^*(\mathscr{F}, \mathscr{F})$ by $\alpha_{\mathscr{F}}$. For any vector bundle $\mathscr{F}$ on X, Căldăraru and Willerton [14] defined an abstract Chern character $\mathrm{ch}(\mathscr{F})$ which lies in the degree zero part of the Hochschild homology $\mathrm{HH}_*(X) = \mathrm{Ext}^*_{X\times X}(S_\Delta^{-1}, \mathscr{O}_\Delta)$, where $S_\Delta^{-1} = \Delta_*(\omega_X^\vee[-\dim X])$. There is an HKR isomorphism for Hochschild homology

$$I_{\mathsf{HKR}} : \mathrm{HH}_*(X) \to \mathrm{H}\Omega_*(X) = \bigoplus_{q-p=*} H^p(X, \wedge^q\Omega_X).$$

The image of the abstract Chern character under the map I_{HKR} is the usual Chern character of $\mathscr{F}$ [12].

Lemma 4.4 *If $\alpha_{\mathscr{F}}$ is zero, then $\alpha \circ \mathrm{ch}(\mathscr{F})$ is zero. Here $\circ$ is the composition of morphisms in $\mathbf{D}^b(X \times X)$ and $\mathrm{ch}(\mathscr{F})$ is the abstract Chern character.*

Proof. The proof is known in an email correspondence with Eyal Markman. Let β be any class in $\mathrm{Ext}^*_{X\times X}(\mathscr{O}_\Delta, S_\Delta)$, where $S_\Delta = \Delta_*(\omega_X[\dim X])$. Similar to the definition of the class $\alpha_{\mathscr{F}}$ associated to $\alpha \in \mathrm{Ext}^*_{X\times X}(\mathscr{O}_\Delta, \mathscr{O}_\Delta)$, we get a class $\beta_{\mathscr{F}} \in \mathrm{Ext}^*_X(\mathscr{F}, S_X\mathscr{F})$, where $S_X(-) = \omega_X[\dim X] \otimes -$. It is shown in [12] that the class $\mathrm{ch}(\mathscr{F})$ is characterized by the identity

$$\mathrm{Tr}_{X\times X}(\beta \circ \mathrm{ch}(\mathscr{F})) = \mathrm{Tr}_X(\beta_{\mathscr{F}}).$$

Due to the equality above, we have

$$\mathrm{Tr}_{X\times X}(\gamma\circ\alpha\circ\mathrm{ch}(\mathscr{F}))=\mathrm{Tr}_X((\gamma\circ\alpha)_{\mathscr{F}})=\mathrm{Tr}_X(\gamma_{\mathscr{F}}\circ\alpha_{\mathscr{F}})$$

for any $\gamma\in\mathrm{Ext}^*_{X\times X}(\mathscr{O}_\Delta,S_\Delta)$. The right hand side is zero since we assume that $\alpha_{\mathscr{F}}$ is zero. We can conclude that $\alpha\circ\mathrm{ch}(\mathscr{F})$ is zero because the equality $\mathrm{Tr}_{X\times X}(\gamma\circ\alpha\circ\mathrm{ch}(\mathscr{F}))=0$ holds for any γ and $\mathrm{Tr}(-)$ is non-degenerate. □

We discuss the action of $\mathrm{HT}^*(X)$ on $\mathrm{H}\Omega_*(X)$. The space $\mathrm{H}\Omega_*(X)$ is naturally a module over $\mathrm{HT}^*(X)$, mimicking the module structure of Hochschild homology over cohomology. For an object $\mathscr{F}$ in the derived category of X, its Mukai vector $v(\mathscr{F})$ lies in $\mathrm{H}\Omega_*(X)$. Thus we can act with the class $\tilde{\alpha}$ to obtain $\tilde{\alpha}\lrcorner\, v(\mathscr{F})\in\mathrm{H}\Omega_*(X)$.

The two HKR isomorphisms I_{HKR} and I^{HKR} can be twisted by the Todd class. We denote the resulting twisted isomorphisms by I_K and I^K

$$I_K:\mathrm{HH}_*(X)\to\mathrm{H}\Omega_*(X)=\bigoplus_{q-p=*}H^p(X,\wedge^q\Omega_X),$$

$$I^K:\mathrm{HT}^*(X)=\bigoplus_{p+q=*}H^p(X,\wedge^qT_X)\to\mathrm{HH}^*(X).$$

They are given by the formula $I_K=(-\wedge\mathrm{td}^{\frac{1}{2}})\circ I_{\mathsf{HKR}}$ and $I^K=I^{\mathsf{HKR}}\circ D^{-1}$, where D^{-1} is the inverse of the Duflo operator.

The Mukai vector $v(\mathscr{F})$ of $\mathscr{F}$ is $I_K(\mathrm{ch}(\mathscr{F}))$ by definition. There are natural ring structures on $\mathrm{HH}^*(X)$ and $\mathrm{HT}^*(X)$: the product on $\mathrm{HH}^*(X)$ is the Yoneda product, and the product on $\mathrm{HT}^*(X)$ is the wedge product. Kontsevich [27] claimed that the map I^K is a ring isomorphism. This statement was proved by Calaque and Van den Bergh [6]. The Hochschild homology is a module over the Hochschild cohomology and similarly $\mathrm{H}\Omega_*(X)$ is a module over $\mathrm{HT}^*(X)$. Calaque, Rossi, and Van den Bergh [9] proved that the maps I_K and I^K respect the module structures.

With the help of Theorem 4.1, we obtain the following result.

Theorem 4.5 *If* $\tilde{\alpha} \lrcorner \exp(at_{\mathscr{F}}) = 0$*, then we have*

$$D(\tilde{\alpha}) \lrcorner v(\mathscr{F}) = 0.$$

Here D *is the Duflo operator,*

$$D(\tilde{\alpha}) = \mathrm{td}^{\frac{1}{2}} \lrcorner \tilde{\alpha},$$

where td *is the Todd class of* X*.*

Remark 4.6 *We are using the contraction symbol* $\lrcorner$ *in three different ways in this paper.*

- *A polyvector field* $\tilde{\alpha} \in \mathrm{HT}^*(X)$ *acts on a class* $v \in \mathrm{H}\Omega_*(X)$*. This action is denoted by* $\tilde{\alpha} \lrcorner v \in \mathrm{H}\Omega_*(X)$*.*
- *A class* $v \in \mathrm{H}\Omega_*(X)$ *acts on a polyvector field* $\tilde{\alpha} \in \mathrm{HT}^*(X)$*. This action yields an element* $v \lrcorner \tilde{\alpha} \in \mathrm{HT}^*(X)$*. We only use the second contraction in the Duflo operator* $D(\tilde{\alpha}) = \mathrm{td}^{\frac{1}{2}} \lrcorner \tilde{\alpha}$ *in this paper. Note that* D *is an automorphism of* $\mathrm{HT}^*(X)$*. The inverse operator is* $D^{-1}(\tilde{\alpha}) = \mathrm{td}^{-\frac{1}{2}} \lrcorner \tilde{\alpha}$*.*
- *The third contraction map is* $\beta \lrcorner \exp(\mathrm{at}_{\mathscr{F}}) \in \mathrm{Ext}^*(\mathscr{F}, \mathscr{F})$ *for* $\beta \in \mathrm{HT}^*(X)$*. An element* β *in* $H^p(X, \wedge^k T_X)$ *can only contract with the term* $\frac{(\mathrm{at}_{\mathscr{F}})^k}{k!}$ *in the Taylor expansion of* $\exp(\mathrm{at}_{\mathscr{F}})$*. It is easy to distinguish this map from the previous two maps.*

Proof of Theorem 4.5. The commutative diagram in Theorem 4.1 shows that

$$\tilde{\alpha} \lrcorner \exp(\mathrm{at}_{\mathscr{F}}) = \alpha_{\mathscr{F}},$$

which is zero under the assumption of Theorem 4.5. We conclude that $\alpha \circ \mathrm{ch}(\mathscr{F})$ is zero by Lemma 4.4. Since I_K and I^K respect the module structures, we have

$$0 = I_K(\alpha \circ \mathrm{ch}(\mathscr{F})) = (I^K)^{-1}(\alpha) \lrcorner\, I_K(\mathrm{ch}(\mathscr{F})) = (I^K)^{-1}(\alpha) \lrcorner\, v(\mathscr{F}).$$

The inverse map of I^K is the composite map

$$(I^K)^{-1} : \mathrm{HH}^*(X) \xrightarrow{(I^{\mathsf{HKR}})^{-1}} \mathrm{HT}^*(X) \xrightarrow{\quad D \quad} \mathrm{HT}^*(X).$$

As a consequence

$$0 = I_K(\alpha \circ \mathrm{ch}(\mathscr{F})) = (I^K)^{-1}(\alpha) \lrcorner\, v(\mathscr{F}) = D(\tilde{\alpha}) \lrcorner\, v(\mathscr{F}).$$

The special case when $\tilde{\alpha} \in H^1(X, T_X)$. Note that our statement in Theorem 4.5 appears to be different from the original one in Section 1.4, which did not have the Duflo operator D. We prove that the original statement follows easily from ours.

The result in Section 1.4 says that $\tilde{\alpha} \lrcorner\, v(\mathscr{F})$ is zero if $\tilde{\alpha} \lrcorner \exp(\mathrm{at}_{\mathscr{F}})$ is zero for any $\tilde{\alpha} \in H^1(X, T_X)$. From now on let $\tilde{\alpha}$ be an element in $H^1(X, T_X)$. The only term in $\exp(\mathrm{at}_{\mathscr{F}}) = 1 + \mathrm{at}_{\mathscr{F}} + \frac{(\mathrm{at}_{\mathscr{F}})^2}{2!} + \cdots$ that can contract with $\tilde{\alpha}$ is $\mathrm{at}_{\mathscr{F}}$, so $\tilde{\alpha} \lrcorner \exp(\mathrm{at}_{\mathscr{F}}) = \tilde{\alpha} \lrcorner\, \mathrm{at}_{\mathscr{F}}$ in this case.

Choose $\mathscr{F} = \mathscr{O}_X$. We have $\tilde{\alpha} \lrcorner \exp(\mathrm{at}_{\mathscr{O}_X}) = 0$. Therefore

$$D(\tilde{\alpha}) \lrcorner\, v(\mathscr{O}_X) = (\mathrm{td}^{\frac{1}{2}} \lrcorner\, \tilde{\alpha}) \lrcorner\, \mathrm{td}^{\frac{1}{2}} = 0$$

according to Theorem 4.5.

Expand the Todd class td as $1 + \frac{c_1}{2} + \frac{c_1^2 + c_2}{12} + \cdots$, and note that the only term of $(\mathrm{td}^{\frac{1}{2}} \lrcorner\, \tilde{\alpha}) \lrcorner\, \mathrm{td}^{\frac{1}{2}}$ in $H^2(X, \mathscr{O}_X)$ is $\tilde{\alpha} \lrcorner \frac{c_1}{2}$. Since $(\mathrm{td}^{\frac{1}{2}} \lrcorner\, \tilde{\alpha}) \lrcorner\, \mathrm{td}^{\frac{1}{2}} = 0$, we can conclude that $\tilde{\alpha} \lrcorner\, c_1$ is zero for any $\tilde{\alpha} \in H^1(X, T_X)$. The fact that $\tilde{\alpha} \lrcorner\, c_1 = 0$ for $\tilde{\alpha} \in H^1(X, T_X)$ is also known due to Griffiths. Consider the first order deformation of X corresponding to $\tilde{\alpha}$. The vanishing of $\tilde{\alpha} \lrcorner c_1$ is equivalent to the class c_1 remaining of type (p, p).

The term $\tilde{\alpha} \lrcorner \frac{c_1}{4}$ is exactly the difference between $D(\tilde{\alpha})$ and $\tilde{\alpha}$ because

$$D(\tilde{\alpha}) = \mathrm{td}^{\frac{1}{2}} \lrcorner \tilde{\alpha} = (1 + \frac{c_1}{4} + \cdots) \lrcorner \tilde{\alpha} = \tilde{\alpha} + \frac{c_1}{4} \lrcorner \tilde{\alpha} + 0.$$

We conclude that $\tilde{\alpha} \lrcorner v(\mathscr{F})$ is zero if and only if $D(\tilde{\alpha}) \lrcorner v(\mathscr{F})$ is zero for $\tilde{\alpha} \in H^1(X, T_X)$.

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