Structures of Free and Virtual Resolutions

by

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A dissertation submitted in partial fulfillment
of the requirements for the degree of

Doctor of Philosophy
(Mathematics)

at the
University of Wisconsin-Madison
2023

Date of Final Oral Exam: 06/16/2023
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Abstract

Free and virtual resolutions of modules are fundamental tools in commutative algebra that are used to study geometric objects in projective and toric varieties. We analyze the structure of such resolutions in a few particular settings. First, we investigate when the minimal free resolutions of random monomial ideals in a standard-graded polynomial ring are dependent on the characteristic of the underlying field. We use this result to develop a heuristic for the asymptotic characteristic dependence of the Betti numbers of algebraic varieties. Secondly, we add to the collection of known virtual resolutions by providing criteria for when members of the family of generalized Eagon–Northcott complexes associated to a map of finitely generated free modules are virtual resolutions. Thirdly, we explore the structure of virtual resolutions corresponding to finite sets of points in the product of two projective lines. In particular, we determine sufficient numerical conditions that describe when these virtual resolutions are of Hilbert–Burch type and when they are not.

Dedication

This thesis is dedicated to my father, Timothy Booms. Ever since I was a little girl, my dad encouraged me to pursue my dreams. An engineer by trade, he introduced me to math at a young age (he even used to give me math problems to work on in church), and he nurtured my analytical mind as I grew up.

In January 2021, in the middle of my graduate studies, my dad was unexpectedly diagnosed with grade IV Glioblastoma (GBM), an aggressive form of brain cancer. From the day that he had brain surgery, he wasn't the same. His tumor took away so much of the man that I knew my father to be, and watching him battle his disease for more than 28 months was gut-wrenching. My dad entered eternal life on May 27, 2023 after having watched my graduation ceremony virtually on May 11. I wouldn't be here today without the love, support, and guidance that he gave me during his life.

Furthermore, this and all of my works are dedicated to God, my source of peace, hope, joy, love, and strength. Without my faith and Your abundant grace, I would not have been able to persevere through the many trials I encountered in graduate school.

Acknowledgements

I would like to express my deepest gratitude to my advisor for his unwavering support throughout my graduate career. Not only has he taught me a tremendous amount, but I have sincerely enjoyed working with him. I am extremely grateful to the faculty and staff in the UW-Madison Mathematics Department, especially my committee members and the graduate program directors and coordinators.

I am very grateful to have been financially supported by the National Science Foundation Graduate Research Fellowship Program under Grant No. DGE-1747503. Note that any opinions, findings, and conclusions or recommendations expressed in this thesis are my own and do not necessarily reflect the views of the National Science Foundation. I am also indebted to the financial support of my advisor, the Elizabeth Hirschfelder Prize, the UW-Madison Mathematics Department, the AMS, and the AWM.

Many thanks to my peers and colleagues, especially my collaborators, who supported my learning and development as a mathematician.

Special thanks to my husband, son, parents, in-laws, and other family members for helping motivate and inspire me throughout my graduate program. I would be remiss in not mentioning St. Paul Catholic Student Center, where I rediscovered my faith and formed several wonderful friendships, as well as the Cathedral Parish community. Lastly, I want to thank my friends for always encouraging me and my counselors at UHS for providing emotional support and advice.

Contents

Chapter 1

Introduction

The research projects investigated in this thesis lie at the intersection of commutative algebra and algebraic geometry, and they utilize tools from other fields, such as combinatorics, topology, and probability. At the heart of algebraic geometry is the study of the zero sets of polynomials, called varieties, which correspond to ideals in polynomial rings. Commutative algebra provides techniques and invariants (Betti numbers, projective dimension, regularity, etc.) for studying varieties in order to shed light on their geometric properties (dimension, smoothness, irreducibility, reducedness, etc.). When working with varieties in projective space, a fundamental algebraic tool for understanding these invariants is the minimal free resolution of the corresponding ideal in the standard-graded polynomial ring, first introduced by Hilbert in the late 1800's [Hil90; Hil93]. When working with more general toric varieties, virtual resolutions, which generalize minimal free resolutions by allowing for a particular type of homology, of the corresponding ideal in the multigraded Cox ring are better suited for studying the geometric properties, as argued in [BES20].

Each chapter of this thesis is devoted to one of the research projects that I worked on during my graduate career. The first addresses a subtle but important question about the characteristic dependence of minimal free resolutions, and the latter two projects relate to virtual resolutions: in understanding how to produce examples of them and analyzing their properties in the particular setting of points in $\mathbb{P}^1 \times \mathbb{P}^1$. Below are brief introductions to each chapter.

Given a field k, we will be interested in studying objects in the polynomial ring in n variables over k, $S = k[x_1, \dots, x_n]$, which is a commutative ring that can be graded in different ways depending on the degrees of the variables. For example, the standard grading is where each x_i has degree 1, whereas in a multigrading each x_i has a degree given by an element in $\mathbb{N}^r$ for some r. Given a graded ideal $I \subseteq S$, we can obtain a unique complex of free graded S-modules $\mathcal{F}_\bullet$ called the **minimal free resolution** of S/I:

$$\mathcal{F}_\bullet : 0 \to F_n \xrightarrow{\varphi_n} F_{n-1} \to \cdots \to F_2 \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} S \to 0.$$

Each module F_i is a direct sum of twisted free modules of the form $S(-a)$ for $a \in \mathbb{Z}$ (assuming S has the standard grading), and this complex is exact except in homological degree 0, where $H_0(\mathcal{F}_\bullet) = S/I$. This resolution has length at most n by Hilbert's Syzygy Theorem, and it provides information about the ideal I. Notably, the **Betti table**, which is the collection of the ranks and degrees of each F_i, gives key information such as the projective dimension, regularity, and complexity of the ideal. The resolution $\mathcal{F}_\bullet$ acts as an extended generators and relations presentation of S/I: the image of φ_1 is I, and from there, φ_2 gives the relations (or syzygies) on the generators of I, φ_3 gives the relations on these relations (second syzygies), and so on.

Although it is not inherently obvious, if you start with an ideal in a polynomial ring with coefficients in $\mathbb{Z}$ where the quotient ring is flat over $\mathbb{Z}$, then specializing the ideal to different fields k can result in minimal free resolutions that depend on the characteristic of k. Known examples include certain monomial ideals [DK14; Kat06], Veronese embeddings of $\mathbb{P}^r$ [And92; Jon10], and determinantal ideals [Has90]. Yet, for many well-studied families of examples, very little is known about when to expect characteristic dependence. In Chapter 2, Daniel Erman, Jay Yang, and I investigate if dependence on the characteristic is a common or rare phenomenon. Specifically, for a family of random monomial ideals, namely the Stanley–Reisner ideals of random flag complexes, we prove that the Betti numbers asymptotically almost always depend on the characteristic. Using this result, we also develop a heuristic for characteristic dependence of asymptotic syzygies of algebraic varieties.

Chapter 3 focuses on producing more families of virtual resolutions. When working in projective varieties, minimal free resolutions capture geometric properties well because the maximal ideal, which plays a key role in the definition of such resolutions, is the irrelevant ideal. However, when working in toric varieties, the irrelevant ideal is strictly contained in the maximal ideal. For example, the Cox ring of $\mathbb{P}^1 \times \mathbb{P}^1$ is $S = k[x_0, x_1, y_0, y_1]$, which has irrelevant ideal $B = \langle x_0, x_1\rangle \cap \langle y_0, y_1\rangle \subsetneq \langle x_0, x_1, y_0, y_1\rangle$. This inequality results in the minimal free resolution containing algebraic structure that is geometrically irrelevant. By focusing on the irrelevant ideal, virtual resolutions are better able to represent important geometric information. Given any map of finitely generated free modules, Buchsbaum and Eisenbud define a family of generalized Eagon-Northcott complexes associated to it [BE75]. In Chapter 3, John Cobb and I give sufficient criterion for these complexes to be virtual resolutions, thus adding to the known examples of virtual resolutions, particularly those not coming from minimal free resolutions.

In Chapter 4, we direct our attention to virtual resolutions in the simplest nontrivial geometric setting: finite sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$. Building off of work of Harada, Nowroozi, and Van Tuyl which provided particular length two virtual resolutions for finite sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$, we prove that the vast majority of virtual resolutions of a pair for minimal elements of the multigraded regularity in this setting are of Hilbert–Burch type. We give explicit descriptions of these short virtual resolutions that depend only on the number of points (see Appendix A). Moreover, despite initial evidence, we show that these virtual resolutions are not always short, and we give sufficient conditions for when they are length three.

Chapter 2

Characteristic dependence of syzygies of random monomial ideals

Joint work with Daniel Erman and Jay Yang [BEY22]

2.1 Introduction

The minimal free resolution of an ideal can depend on the characteristic of the ground field. Known examples include certain monomial ideals [DK14; Kat06], Veronese embeddings of $\mathbb{P}^r$ [And92; Jon10], and determinantal ideals [Has90]. This paper is motivated by a desire to understand if dependence on the characteristic is a common or rare phenomenon. To make such a question precise, we can restrict to specific families, such as:

Question 2.1.1. *For which $d \geq 1$ does the minimal free resolution of the d-uple embedding of $\mathbb{P}^r$ depend on the characteristic? Does it happen for all $d \gg 0$? Or does it happen rarely?*

Question 2.1.2. *Let $\Delta \sim \Delta(n, p)$ be a random flag complex (see section 2.2.3). As $n \to \infty$, what is the probability that the minimal free resolution of the Stanley–Reisner ideal of Δ depends on the characteristic?*

We do not offer new results on question 2.1.1, though we discuss in section 2.1.1 how questions like this motivated our work. Our main result is theorem 2.1.3, which answers question 2.1.2 and shows that in this context, dependence on the characteristic is quite common.

To analyze dependence on characteristic, we will say that the Betti table of the Stanley–Reisner ideal of Δ **has ℓ-torsion** if this Betti table is different when defined over a field of characteristic ℓ than it is over $\mathbb{Q}$. See section 2.2 for further details on notation. We prove:

Theorem 2.1.3. *Let $\Delta \sim \Delta(n,p)$ be a random flag complex with $n^{-1/6} \ll p \leq 1-\epsilon$ for $\epsilon > 0$. If we fix any $m \geq 2$, then with high probability as $n \to \infty$, the Betti table of the Stanley–Reisner ideal of Δ has ℓ-torsion for every prime ℓ dividing m.*

Assuming the hypotheses of the theorem, this implies that with high probability as $n \to \infty$, the Betti table of the Stanley-Reisner ideal of Δ depends on the characteristic. The proof of theorem 2.1.3 proceeds as follows. By Hochster's formula [BH93, Theorem 5.5.1], it suffices to show that some induced subcomplex of Δ has m-torsion in its homology. For each m, we modify Newman's construction [New18, §3] to build a flag complex X_m with a small number of vertices and with m-torsion in $H_1(X_m)$. We then apply a variant of Bollobás's theorem on subgraphs of a random graph [Bol81, Theorem 8] to prove that X_m appears as an induced subcomplex of Δ with high probability as $n \to \infty$, yielding theorem 2.1.3.

The most common example of characteristic dependence is Reisner's example, coming from a triangulation of $\mathbb{RP}^2$ [BH93, §5.3]. Other previous research on characteristic *independence* of monomial ideals includes [TH96; Kat06; HKM10] for edge ideals and [DK14, Theorem 5.1] for monomial ideals with component-wise linear resolutions.

Theorem 2.1.3 also fits into an emerging literature on random monomial ideals. This began with [De +19b], which outlined an array of frameworks for random monomial ideals, including models related to random simplicial complexes such as [CF16; Kah14]. The average Betti table of a random monomial ideal is analyzed in [De +19a], while [SWY20] examines threshold phenomena in random models from [De +19b]. Banerjee and Yogeshwaran study homological properties of the edge ideals of Erdős–Rényi random graphs in [BY20]. There is also [EY18], which uses random monomial methods to demonstrate some asymptotic syzygy phenomena from [EL12; EEL15]. And finally, theorem 2.1.3 is thematically connected with [Kah+20], which analyzes torsion homology in random simplicial complexes (whereas theorem 2.1.3 analyzes the simpler question of finding m-torsion in the homology of *some* induced subcomplex of $\Delta(n, p)$).

2.1.1 Asymptotic syzygies and heuristics

One of our main motivations for studying question 2.1.2 is a belief that this will provide heuristic insights into more geometric questions like question 2.1.1. We now explain this connection in more detail.

The study of asymptotic syzygies, as introduced by Ein and Lazarsfeld in [EL12], examines the overarching behavior of syzygies of algebraic varieties under increasingly ample embeddings. Specifically, Ein and Lazarsfeld fixed a smooth variety X with a very ample line bundle A and considered the syzygies of X embedded by dA for $d \gg 0$. They proved an asymptotic nonvanishing result which showed that the limiting behavior essentially only depended on $\dim X$. Other researchers then found comparable limiting behavior for other families from geometry [Zho14; EEL16] and combinatorics [CJW18; EY18]. In a similar vein, [EEL15] conjectured that the syzygies of smooth varieties should asymptotically converge to a normal distribution, in an appropriate sense; that conjecture was verified for the combinatorial families in [EY18].

In short, work on asymptotic syzygies suggests that the overarching behavior will be similar across many geometric and combinatorial examples. This is the context in which question 2.1.1 and question 2.1.2 are connected. Whereas Ein and Lazarsfeld identified behavior in geometric settings which carried over to combinatorial settings, we look in the opposite direction: could a combinatorial result shed light on asymptotic syzygies in geometric examples?[1]

The study of ℓ-torsion is ripe for such a heuristic due to the lack of results and the difficulty of computing the Betti numbers of higher dimensional varieties. For instance,

[1]A similar idea appears in [EEL15], where a random model based on Boij-Söderberg theory is used to generate quantitative conjectures about the entries of Betti tables.

for Veronese embeddings of $\mathbb{P}^r$, the only results on ℓ-torsion are for the 2-uple embedding (exploiting the combinatorial description of [RR00]): Andersen's thesis [And92] shows that the Betti table of the 2-uple embedding of $\mathbb{P}^r$ has 5-torsion for any $r \geq 6$, and Jonsson generalized this to produce ℓ-torsion for $\ell = 3, 5, 7, 11$, and 13 and for various r [Jon10]. See [Bou92; Has90] for similar results. But even for d-uple embeddings of $\mathbb{P}^r$, there are no examples of torsion when $d > 2$ and no conjectures for any fixed $r \geq 2$.

The random flag complex model used in this paper was previously studied in work of Erman and Yang [EY18, Theorem 1.3], and they showed that if $n^{-1/(r-1)} \ll p \ll n^{-1/r}$, then the Betti table of the Stanley–Reisner ideal of $\Delta(n, p)$ exhibits some of the asymptotic behavior of r-dimensional varieties from [EL12]. We view theorem 2.1.3, which holds for $n^{-1/(r-1)} \ll p \ll n^{-1/r}$ when $r \geq 7$, as providing a heuristic for ℓ-torsion in the asymptotic syzygies of a smooth variety X of $\dim X \geq 7$. For concreteness, in the case of $\mathbb{P}^r$, we conjecture:

Conjecture 1. *Let $r \geq 7$. For any $d \gg 0$, the Betti table of $\mathbb{P}^r$ under the d-uple embedding depends on the characteristic.*

Conjecture 2. *Let $r \geq 7$. As $d \to \infty$, the number of primes ℓ such that the Betti table of $\mathbb{P}^r$ under the d-uple embedding has ℓ-torsion is unbounded.*

We will discuss some related conjectures and questions, in more detail, in section 2.7.

This paper is organized as follows. In section 2.2, we review notation and background, including on Betti numbers, Hochster's formula, and random flag complexes. section 2.3 contains our main construction in which we construct an explicit flag complex X_m with m-torsion in homology; see theorem 2.3.1. In section 2.4, we apply a minor variant of Bollobás's theorem on subgraphs of a random graph to show that, with high probability, X_m appears as an induced subcomplex of $\Delta(n, p)$ for any $n^{-1/6} \ll p \leq 1-\epsilon$ where $\epsilon > 0$ and $m \geq 2$. In section 2.5, we analyze the case of 2-torsion more closely, using the techniques from section 2.4 to expand known results from [CFH15]. In section 2.6, we combine results from section 2.4 with Hochster's formula to prove theorem 2.1.3. Finally, in section 2.7, we discuss questions about ℓ-torsion in asymptotic syzygies.

2.2 Background and Notation

2.2.1 Torsion in Betti tables

Throughout this paper we will analyze graded algebras, all of which have the following form: there is an ideal J in a polynomial ring T with coefficients in $\mathbb{Z}$, where T/J is flat over $\mathbb{Z}$, and we are interested in specializations $(T/J)\otimes_{\mathbb{Z}} k$ to various fields k. Our results focus on graded algebras that arise as the Stanley–Reisner rings of simplicial complexes. But there are many other potential examples, such as the coordinate rings of Veronese embeddings of projective space, Grassmanians, toric varieties, and so on. The central questions of this paper are concerned with when the Betti numbers of such algebras depend on the characteristic of k.

Let J be a monomial ideal in $T = \mathbb{Z}[x_1, \ldots, x_n]$. For a field k, the algebraic Betti numbers of $(T/J)\otimes_{\mathbb{Z}} k$ are given by

$$\beta_{i,j}((T/J)\otimes_{\mathbb{Z}} k) := \dim_k \operatorname{Tor}_i^{T\otimes_{\mathbb{Z}} k}((T/J)\otimes_{\mathbb{Z}} k, k)_j.$$

The collection of all of these Betti numbers is called the Betti table. Since field extensions are flat, Betti numbers are invariant under field extensions and will therefore be the same for any field of the same characteristic. Semicontinuity implies that $\beta_{i,j}((T/J)\otimes_{\mathbb{Z}} \mathbb{Q}) \leq \beta_{i,j}((T/J)\otimes_{\mathbb{Z}} \mathbb{F}_\ell)$. We say that the Betti table of J **has ℓ-torsion** if this inequality is strict for some i, j, and we say that the Betti table of J **depends on the characteristic** if it has ℓ-torsion for some prime ℓ.

Remark 2.2.1. Let J be an ideal in $T = \mathbb{Z}[x_1, \ldots, x_n]$ which is flat over $\mathbb{Z}$. Let $S = T\otimes_{\mathbb{Z}} \mathbb{F}_\ell = \mathbb{F}_\ell[x_1, \ldots, x_n]$ and $I = JS$. By a standard argument, it follows that

$$\dim_{\mathbb{F}_\ell} \operatorname{Tor}_i^S(S/I, \mathbb{F}_\ell)_j = \dim_{\mathbb{F}_\ell}(\operatorname{Tor}_i^T(T/J, \mathbb{Z})_j \otimes_{\mathbb{Z}} \mathbb{F}_\ell) + \dim_{\mathbb{F}_\ell}(\operatorname{Tor}_1^{\mathbb{Z}}(\operatorname{Tor}_{i+1}^T(T/J, \mathbb{Z})_j, \mathbb{F}_\ell)).$$

In particular, the Betti table of J has ℓ-torsion if and only if one of the $\operatorname{Tor}_{i+1}^T(T/J, \mathbb{Z})_j$ has ℓ-torsion as an abelian group.

Figure 2.1: In the graphs shown above, H is a subgraph of G, but it is not the induced subgraph on the vertex set $\{1, 2, 3\}$ since H is missing the diagonal edge connecting vertices 1 and 3.

2.2.2 Graphs and simplicial complexes

For a simplicial complex X, we write $V(X)$, $E(X)$, and $F(X)$ for the set of vertices, edges, and (2-dimensional) faces of X, respectively. We use $|*|$ to denote the number of elements in these sets. The degree of a vertex v, denoted $\deg(v)$, is the number of edges in X containing v. We write $\text{maxdeg}(X)$ for the maximum degree of any vertex of X, and we write $\text{avgdeg}(X)$ for the average degree of a vertex in X.

For a pair of graphs H, G, we write $H \subset G$ if H is a subgraph of G. We write $H \overset{ind}{\subset} G$ if H is an induced subgraph of G, that is, if the vertices of H are a subset of the vertices of G and the edges of H are precisely the edges connecting those vertices within G (see fig. 2.1). We use similar definitions and notations for a simplicial complex Δ' to be a subcomplex (or an induced subcomplex) of another complex Δ. If $\alpha \subset V(\Delta)$, then we let $\Delta|_\alpha$ denote the induced subcomplex of Δ on α.

The following definitions, adapted from [Bol81] and [Col+17], will be used in sections 2.4 to 2.6.

Definition 2.2.2. The **essential density** of a graph G is

$$m(G) := \max \left\{ \frac{|E(H)|}{|V(H)|} \; : \; H \subset G, \, |V(H)| > 0 \right\},$$

and G is **strictly balanced** if $m(H) < m(G)$ for all proper subgraphs $H \subset G$.

For a field k, a simplicial complex Δ on n vertices has a corresponding Stanley–Reisner ideal $I_\Delta \subset S = k[x_1, \ldots, x_n]$. Since these I_Δ are squarefree monomial ideals, Hochster's formula [BH93, Theorem 5.5.1] relates the Betti table of S/I_Δ to topological properties of

Δ, providing our key tool for studying this Betti table for various fields k. An immediate consequence of Hochster's formula is the following fact, which characterizes when these Betti tables are different over a field of characteristic ℓ than over $\mathbb{Q}$.

Fact 2.2.3. *For a simplicial complex Δ, the Betti table of the Stanley–Reisner ideal I_Δ has ℓ-torsion if and only if there exists a subset $\alpha \subset V(\Delta)$ such that $\Delta|_\alpha$ has ℓ-torsion in one of its homology groups.*

2.2.3 Monomial ideals from random flag complexes

Recall that a flag complex is a simplicial complex obtained from a graph by adjoining a k-simplex to every $(k+1)$-clique in the graph, which is called taking the clique complex. Therefore, a flag complex is entirely determined by its underlying graph. We write $\Delta \sim \Delta(n,p)$ to denote the flag complex which is the clique complex of an Erdős–Rényi random graph $G(n,p)$ on n vertices, where each edge is attached with probability p. If $\alpha \subset V(\Delta)$, then we note that $\Delta|_\alpha$ is also flag. The properties of random flag complexes have been analyzed extensively, with [Kah14] providing an overview. As discussed in the introduction, the syzygies of Stanley–Reisner ideals of random flag complexes were first studied in [EY18].

2.2.4 Probability

We use the notation $\mathbf{P}[*]$ for the probability of an event. If X_n is a sequence of random variables, then we say that the event $X_n = x_0$ occurs **with high probability as** $n \to \infty$ if $\mathbf{P}[X_n = x_0] \to 1$ as $n \to \infty$. For a random variable X, we use $\mathbb{E}[X]$ for the expected value of X and $\mathrm{Var}(X)$ for the variance of X.

For functions $f(x)$ and $g(x)$, we write $f \ll g$ if $\lim_{x\to\infty} f/g \to 0$. We use $f \in O(g)$ if there is a constant N where $|f(x)| \leq N|g(x)|$ for all sufficiently large values of x, and we use $f \in \Omega(g)$ if there is a constant N' where $|f(x)| \geq N'|g(x)|$ for all sufficiently large values of x.

2.3 Constructing a flag complex with m-torsion in homology

The goal of this section is to prove the following result:

Theorem 2.3.1. *For every $m \geq 2$, there exists a two-dimensional flag complex X_m such that* $\text{maxdeg}(X_m) \leq 12$ *and the torsion subgroup of $H_1(X_m)$ is isomorphic to $\mathbb{Z}/m\mathbb{Z}$.*

This result is the foundation of our proof of theorem 2.1.3 as we will show that this specific complex X_m appears as an induced subcomplex of $\Delta(n, p)$ with high probability as $n \to \infty$ under the hypotheses of that theorem.

Here is an overview of our proof of theorem 2.3.1, which is largely based on ideas from [New18]. Given an integer $m \geq 2$, we write its binary expansion as $m = 2^{n_1} + \cdots + 2^{n_k}$ with $0 \leq n_1 < \cdots < n_k$. Note that k is the Hamming weight of m and $n_k = \lfloor \log_2(m) \rfloor$. With this setup, the "repeated squares presentation" of $\mathbb{Z}/m\mathbb{Z}$ is given by

$$\mathbb{Z}/m\mathbb{Z} = \langle \gamma_0, \gamma_1, \ldots, \gamma_{n_k} \mid 2\gamma_0 = \gamma_1, 2\gamma_1 = \gamma_2, \ldots, 2\gamma_{n_k-1} = \gamma_{n_k}, \gamma_{n_1} + \cdots + \gamma_{n_k} = 0 \rangle.$$

We will construct a two-dimensional flag complex X_m such that the torsion subgroup of $H_1(X_m)$ has this presentation. To do so, we follow Newman's "telescope and sphere" construction in [New18], where Y_1 is the telescope satisfying

$$H_1(Y_1) \cong \langle \gamma_0, \gamma_1, \ldots, \gamma_{n_k} \mid 2\gamma_0 = \gamma_1, 2\gamma_1 = \gamma_2, \ldots, 2\gamma_{n_k-1} = \gamma_{n_k} \rangle,$$

Y_2 is the sphere satisfying

$$H_1(Y_2) \cong \langle \tau_1, \ldots, \tau_k \mid \tau_1 + \cdots + \tau_k = 0 \rangle,$$

and X_m is created by gluing Y_1 and Y_2 together (by identifying τ_i with γ_{n_i} for $i = 1, \ldots, k$) to yield a complex with the desired H_1-group. Because we want our construction to be a flag complex with $\text{maxdeg}(X_m) \leq 12$, we cannot simply quote Newman's results. Instead, we must alter the triangulations to ensure that Y_1, Y_2, and X_m are flag complexes. Then, we must further alter the construction to reduce $\text{maxdeg}(X_m)$. However, each of our

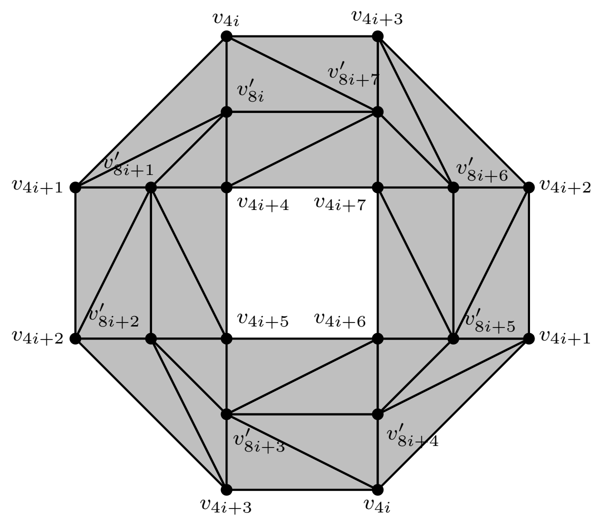

Figure 2.2: Building block for the telescope construction where $i \in \{0, 1, \ldots, (n_k - 1)\}$.

constructions is homeomorphic to each of Newman's constructions.

Notation 2.3.2. Throughout the remainder of this section we assume that $m \geq 2$ is given. We write $m = 2^{n_1} + \cdots + 2^{n_k}$ with $0 \leq n_1 < \cdots < n_k$. To simplify notation, we also denote X_m by X for the remainder of this section.

2.3.1 The telescope construction

The telescope Y_1 that we construct will be homeomorphic to the Y_1 that Newman constructs in [New18, Proof of Lemma 3.1] for the $d = 2$ case. We start with building blocks which are punctured projective planes; in contrast with [New18], our blocks are triangulated so that each is a flag complex. Explicitly, for each $i = 0, \ldots, (n_k - 1)$, we produce a building block which is a triangulated projective plane with a square face removed, with vertices, edges, and faces as illustrated in fig. 2.2. Our building blocks differ from Newman's in order to ensure that Y_1 and the final simplicial complex X are flag complexes; for instance, we need to add extra vertices $v'_{8i}, \ldots, v'_{8i+7}$.

We construct Y_1 by identifying edges and vertices of these n_k building blocks as labeled. The underlying vertex set is $V(Y_1) = \{v_0, v_1, v_2, \ldots, v_{4n_k+3}, v'_0, v'_1, \ldots, v'_{8n_k-1}\}$, so we have $|V(Y_1)| = 12n_k + 4$. Since each building block has 44 edges, 4 of which are identified with edges on the next building block, and 28 faces, we have $|E(Y_1)| = 40n_k + 4$ and $|F(Y_1)| = 28n_k$. In addition, observe that the vertices of highest degree are those in the squares in the "middle" of the telescope, such as vertex v_4 when $n_k \geq 2$. In this case, v_4 is adjacent to $v_5, v_7, v'_0, v'_1, v'_7, v'_8, v'_{15}, v'_{11}$, and v'_{12}, so $\deg(v_4) = 9$. By the symmetry of Y_1, we have that $\text{maxdeg}(Y_1) = 9$ when $n_k \geq 2$, and $\text{maxdeg}(Y_1) = 6$ when $n_k = 1$ (when $m = 2$ or 3).

To compute $H_1(Y_1)$, we simply apply the identical argument from [New18]. We order the vertices in the natural way, where $v_j > v_k$ if $j > k$, similarly for the v'_ℓ, and where $v'_\ell > v_j$ for all ℓ, j. We let these vertex orderings induce orientations on the edges and faces of Y_1. For each $i = 0, \dots, n_k$, denote by γ_i the 1-cycle of Y_1 represented by $[v_{4i}, v_{4i+1}] + [v_{4i+1}, v_{4i+2}] + [v_{4i+2}, v_{4i+3}] - [v_{4i}, v_{4i+3}]$. Then $2\gamma_i - \gamma_{i+1}$ is a 1-boundary of Y_1 for each $i = 0, \dots, (n_k - 1)$, and, as in Newman's construction, we have that $H_1(Y_1)$ can be presented as $\langle \gamma_0, \gamma_1, \dots, \gamma_{n_k} \mid 2\gamma_0 = \gamma_1, 2\gamma_1 = \gamma_2, \dots, 2\gamma_{n_k-1} = \gamma_{n_k} \rangle$.

2.3.2 The sphere construction

The sphere part Y_2 is a flag triangulation of the sphere S^2 that has k square holes such that the squares are all vertex disjoint and nonadjacent. Our Y_2 will be homeomorphic to the Y_2 that Newman constructs in [New18] for the $d = 2$ case, but our construction involves a few different steps. First, we will show that for any integer $k \geq 1$, there exists a flag triangulation T_i of S^2 (here $i = \lfloor \frac{k-1}{4} \rfloor$) with at least k faces such that $\text{maxdeg}(T_i) \leq 6$. Then, we will insert square holes on k of the faces of T_i, while subdividing the edges, and call the resulting flag complex $\widetilde{T}_i$. Finally, we describe a process to replace each vertex of degree 14 in $\widetilde{T}_i$ with two degree 9 vertices so that the resulting complex, Y_2, has $\text{maxdeg}(Y_2) \leq 12$. Throughout these constructions, we will have four cases corresponding to the value of $k \mod 4$, and we carefully keep track of the degrees of each vertex in T_i, $\widetilde{T}_i$, and Y_2 for each case.

T_i and flag bistellar 0-moves

We begin by constructing an infinite sequence $T_0, T_1, \ldots$ of flag triangulations of S^2 such that $\text{maxdeg}(T_i) \leq 6$ for all i. To do so, we adapt the bistellar 0-moves used in [New18, Lemma 5.6]. Let T_0 be the 3-simplex boundary on the vertex set $\{w_0, w_1, w_2, w_3\}$. Note that each vertex of T_0 has degree 3. We will construct the remaining T_i inductively. To build T_1, first remove the face $[w_1, w_2, w_3]$ and edge $[w_1, w_3]$. Then, add two new vertices w_4 and w_5 as well as new edges $[w_0, w_4], [w_1, w_4], [w_3, w_4], [w_1, w_5]$, $[w_2, w_5], [w_3, w_5]$, and $[w_4, w_5]$. Taking the clique complex will then give T_1. See fig. 2.3.

Essentially, this process is the same as making the face $[w_1, w_2, w_3]$ into a square face $[w_1, w_2, w_3, w_4]$, removing that square face, taking the cone over it, and then ensuring that the resulting complex is a flag triangulation of S^2. We will call such a move a **flag bistellar 0-move**. Each T_{i+1} for $i \geq 0$ will be obtained from T_i by performing a flag bistellar 0-move on the face $[w_{2i+1}, w_{2i+2}, w_{2i+3}]$ of T_i. Explicitly, to construct T_{i+1}, remove the face $[w_{2i+1}, w_{2i+2}, w_{2i+3}]$ and the edge $[w_{2i+1}, w_{2i+3}]$. Then, add new vertices w_{2i+4} and w_{2i+5} and new edges $[w_{2i}, w_{2i+4}], [w_{2i+1}, w_{2i+4}], [w_{2i+3}, w_{2i+4}]$, $[w_{2i+1}, w_{2i+5}]$, $[w_{2i+2}, w_{2i+5}], [w_{2i+3}, w_{2i+5}]$, $[w_{2i+4}, w_{2i+5}]$, and take the clique complex to get T_{i+1}. Note that each flag bistellar 0-move adds 2 vertices, 6 edges, and 4 faces. Since $|V(T_0)| = 4, |E(T_0)| = 6$, and $|F(T_0)| = 4$, this means that $|V(T_i)| = 2i + 4$, $|E(T_i)| = 6i + 6$, and $|F(T_i)| = 4i + 4$.

Further, table 2.1 summarizes the degrees of the vertices in each T_i. To compute the

Table 2.1: Degrees of the vertices in T_i.

T_i	Degree	Vertices
T_0	3	w_0, w_1, w_2, w_3
T_1	4	$w_0, w_1, w_2, w_3, w_5, w_6$
T_2	4	w_0, w_1, w_6, w_7
	5	w_2, w_3, w_4, w_5
T_i	4	$w_0, w_1, w_{2i+2}, w_{2i+3}$
$i \geq 3$	5	$w_2, w_3, w_{2i}, w_{2i+1}$
	6	$w_4, \ldots, w_{2i-1}$

degrees of vertices in T_i for $i \geq 3$, observe that when the new vertices w_{2i+2} and w_{2i+3} are

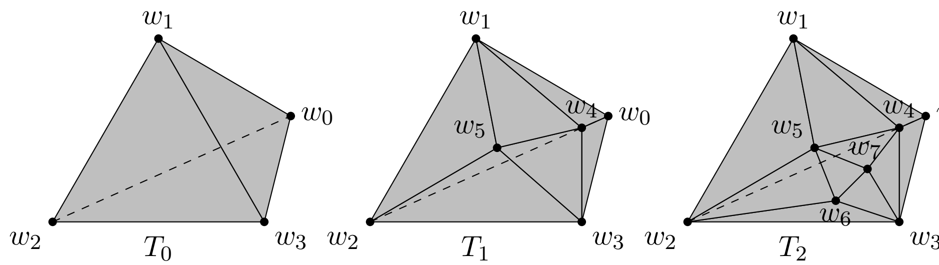

Figure 2.3: The first few flag triangulations of S^2 using flag bistellar 0-moves.

added, they have degree 4 in T_i. For each of the next two iterations of the flag bistellar-0 move, the degree of these vertices increases by one, resulting in degree 6 in T_{i+2}. In the remaining triangulations T_j with $j \geq i+3$, these vertices are not affected. Therefore, $\text{maxdeg}(T_i) \leq 6$ for each i.

From this infinite sequence of flag triangulations of S^2 with bounded degree, we are interested in the particular T_i with $i = \lfloor \frac{k-1}{4} \rfloor$ to use in our construction of Y_2, where k is the Hamming weight of m as in notation 2.3.2. Note that this T_i has vertex set $\{w_0, \dots, w_{2i+3}\}$ and has $4\lfloor \frac{k-1}{4} \rfloor + 4$ faces. Let δ be the integer $0 \leq \delta \leq 3$ where $\delta \equiv -k \mod 4$. Then T_i has exactly $k + \delta$ faces.

Constructing $\widetilde{T}_i$

Next, we insert square holes in the first k faces of T_i and subdivide the remaining faces in such a way that the squares will be vertex disjoint and nonadjacent.

First, we will insert square holes in k of the faces of T_i, making sure to triangulate the resulting faces and take the clique complex so that our simplicial complex remains flag. Let $[w_r, w_s, w_t]$ with $r < s < t$ be the jth of these k faces with respect to a fixed ordering of the faces (where j ranges from 1 to k). We remove this face and subdivide the edges by adding new vertices $w'_{r,s}$, $w'_{r,t}$, and $w'_{s,t}$ and new edges $[w_r, w'_{r,s}]$, $[w_s, w'_{r,s}]$, $[w_r, w'_{r,t}]$, $[w_t, w'_{r,t}]$, $[w_s, w'_{s,t}]$, and $[w_t, w'_{s,t}]$. Then, we add vertices $u_{4j-4}, u_{4j-3}, u_{4j-2}$, and u_{4j-1} to form a square inside

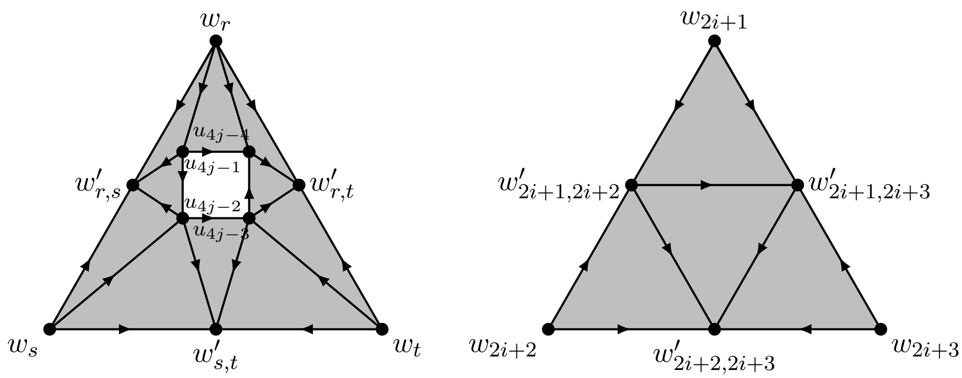

Figure 2.4: Example of square insertion done on k faces of T_i (left), and subdivided triangulation on remaining faces (right).

the original face with indices increasing counterclockwise. Moreover, we add edges

$$[w_r, u_{4j-4}], [w_r, u_{4j-1}], [u_{4j-4}, w'_{r,s}], [u_{4j-3}, w'_{r,s}], [w_s, u_{4j-3}]$$
$$[u_{4j-3}, w'_{s,t}], [u_{4j-2}, w'_{s,t}], [w_t, u_{4j-2}], [u_{4j-2}, w'_{r,t}], [u_{4j-1}, w'_{r,t}].$$

After applying this process, we take the clique complex. The result of this operation on face $[w_r, w_s, w_t]$ is depicted in fig. 2.4 (left).

The remaining δ faces of T_i will simply be subdivided and triangulated before taking the clique complex. Explicitly, this means that after removing the face $[w_{2i+1}, w_{2i+2}, w_{2i+3}]$ and its edges, we add vertices $w'_{2i+1,2i+2}, w'_{2i+1,2i+3}$, and
$w'_{2i+2,2i+3}$ and edges

$$[w_{2i+1}, w'_{2i+1,2i+2}], [w_{2i+2}, w'_{2i+1,2i+2}], [w_{2i+1}, w'_{2i+1,2i+3}],$$
$$[w_{2i+3}, w'_{2i+1,2i+3}], [w'_{2i+1,2i+2}, w'_{2i+1,2i+3}], [w_{2i+2}, w'_{2i+2,2i+3}],$$
$$[w_{2i+3}, w'_{2i+2,2i+3}], [w'_{2i+1,2i+2}, w'_{2i+2,2i+2}], [w'_{2i+1,2i+3}, w'_{2i+2,2i+3}].$$

This subdivision of face $[w_{2i+1}, w_{2i+2}, w_{2i+3}]$ is shown in fig. 2.4 (right). We do similarly for the faces $[w_{2i-1}, w_{2i+2}, w_{2i+3}]$ and $[w_{2i}, w_{2i+1}, w_{2i+3}]$, if necessary. The clique complex of this construction is a flag complex which is homeomorphic to S^2 with k distinct points removed. Call this complex $\widetilde{T}_i$.

Let's consider the degrees of the vertices of $\widetilde{T}_i$. We have that $\deg(w'_{s,t}) = 6$ for all s, t and $\deg(u_\ell) \in \{4, 5\}$ for all ℓ, where the "top" u_ℓ have degree 4 and the "bottom" u_ℓ have degree 5. To determine the degrees of the w_j vertices, we need to consider their degrees in T_i and how their degrees increase during the subdivision and square face removal processes. As we are interested in bounding the maximum degree of the vertices of $\widetilde{T}_i$, we need only consider the case when $\delta = 0$ and all k faces of T_i have a square hole. table 2.2 gives the

Table 2.2: Degrees of the vertices in $\widetilde{T}_i$ when $k \equiv 0 \mod 4$.

$\widetilde{T}_i$	Degree	Vertices
$\tilde{T}_0$ $(k = 4)$	6	w_2, w_3
	7	w_1
	9	w_0
$\tilde{T}_1$ $(k = 8)$	8	w_4, w_5
	9	w_2, w_3
	10	w_1
	12	w_0
$\tilde{T}_2$ $(k = 12)$	8	w_6, w_7
	10	w_1
	11	w_4, w_5
	12	w_0, w_2, w_3
$\widetilde{T}_i$ $i \geq 3$ $(k = 4i + 4)$	8	w_{2i+2}, w_{2i+3}
	10	w_1
	11	w_{2i}, w_{2i+1}
	12	w_0, w_2, w_3
	14	$w_4, \ldots, w_{2i-1}$

degrees of each of the w_j vertices in $\widetilde{T}_i$ when $\delta = 0$.

To verify the degrees of the w_j in $\widetilde{T}_i$ when $i \geq 3$, we consider how the degrees of the vertices change as i increases. Between $\tilde{T}_{i-1}$ and $\widetilde{T}_i$ (with $\delta = 0$ for both), the only vertices that change degree are $w_{2i-2}, w_{2i-1}, w_{2i}, w_{2i+1}$, each of which increase degree by 3. This is because they each get one new edge from the T_i flag bistellar 0-move and two new edges from the square removal triangulation process (since each vertex is the smallest indexed and hence the "top" vertex of one new triangular face). Further, the new vertices w_{2i+2}, w_{2i+3} in $\widetilde{T}_i$ have degree 8, and they increase degree by 3 in the next two iterations, resulting in degree 14 in $\tilde{T}_{i+2}$ and all future iterations.

The above argument shows that regardless of m and k, $\text{maxdeg}(\widetilde{T}_i) \leq 14$, where $i =$

$\lfloor \frac{k-1}{4} \rfloor$. Furthermore, the only vertices that could have degree 14 are $w_4, \ldots, w_{2i-1}$, each of which is separated from the others by a $w'_{s,t}$ vertex, which only has degree 6. We want to know exactly which vertices in $\widetilde{T}_i$ have degree 14, for all possible k with $i \geq 3$, because we plan to alter these vertices to decrease $\operatorname{maxdeg}(\widetilde{T}_i)$. Note that as δ increases from 0 to 3, the degree of each w_j vertex is nonincreasing. When $k = 4i + 4$ and $\delta = 0$, table 2.2 gives that $w_4, \ldots, w_{2i-1}$ have degree 14. When $k = 4i + 3$ and $\delta = 1$, the face $[w_{2i+1}, w_{2i+2}, w_{2i+3}]$ is subdivided instead of having a square removed, but this does not change the degrees of $w_4, \ldots, w_{2i-1}$, so these all still have degree 14. When $k = 4i + 2$ and $\delta = 2$, the faces $[w_{2i+1}, w_{2i+2}, w_{2i+3}]$ and $[w_{2i-1}, w_{2i+2}, w_{2i+3}]$ are subdivided. Therefore, w_{2i-1} has two fewer edges than in the previous case since w_{2i-1} is the smallest indexed vertex in $[w_{2i-1}, w_{2i+2}, w_{2i+3}]$ and so would have two "top" u_ℓ adjacent to it if this face had a square removed from it. So, in this case, $w_4, \ldots, w_{2i-2}$ have degree 14 and w_0, w_2, w_3, w_{2i-1} have degree 12 in $\widetilde{T}_i$. Finally, if $k = 4i+1$ and $\delta = 3$, then additionally the face $[w_{2i}, w_{2i+1}, w_{2i+3}]$ is subdivided, which means that the degree 12 and 14 vertices are the same as in the previous case.

Replacing degree 14 vertices to construct Y_2

Having identified the vertices of $\widetilde{T}_i$ of the highest degree, we now describe a process by which we will replace each vertex of degree 14 by two vertices of degree 9 in order to ensure that $\operatorname{maxdeg}(\widetilde{T}_i) \leq 12$ for all k (and i). The resulting flag complex, given by taking the clique complex of this construction, will be the final Y_2, and it will be homeomorphic to $\widetilde{T}_i$. The process is summarized by fig. 2.5 and described in detail in the following paragraphs.

Suppose w_j is a vertex of degree 14 in $\widetilde{T}_i$. Locally, on a small neighborhood of w_j, $\widetilde{T}_i$ is homeomorphic to a 2-manifold. Since $\deg(w_j) = 14$, w_j is surrounded by six triangular faces coming from T_i, all of which have had a square removed. By our construction, two of these squares (which are in adjacent triangular faces) have both of their "top" u_ℓ vertices connected to w_j, but the other four squares just have a single edge connecting one of their "bottom" u_ℓ vertices to w_j. So, w_j has six $w'_{s,t}$ neighbors and eight u_ℓ neighbors, which

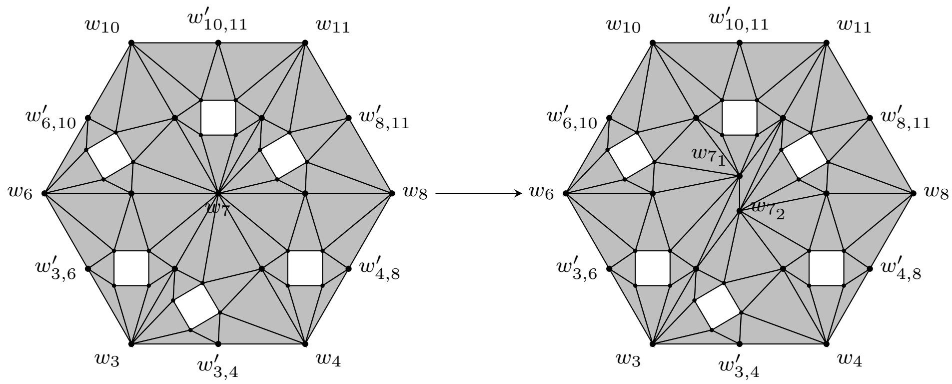

Figure 2.5: Replacing a degree 14 vertex in $\tilde{T}_4$ when $k = 20$.

form a 14-sided polygon with w_j as its "star" point. Choose two $w'_{s,t}$ vertices which are across from each other in this 14-sided polygon, say $w'_{a,b}$ and $w'_{c,d}$. Next, we will remove w_j and all of the 14 faces that it is contained in. Then, we add vertices w_{j_1} and w_{j_2} in place of w_j and add edges in such a way that $\deg(w_{j_1}) = \deg(w_{j_2}) = 9$, there are edges $[w_{j_1}, w_{j_2}], [w_{j_1}, w'_{a,b}], [w_{j_1}, w'_{c,d}], [w_{j_2}, w'_{a,b}]$, and $[w_{j_2}, w'_{c,d}]$, and the 14-sided polygon is triangulated with 16 triangles. This process only changes the degree of $w'_{a,b}$ and $w'_{c,d}$, each of which now have degree 7. Therefore, the maximum degree of w_{j_1}, w_{j_2}, and the 14 vertices in the polygon is 9 (since $\deg(u_\ell) \in \{4, 5\}$ and $\deg(w'_{s,t}) = 6$). To illustrate this construction, we consider the case when $k = 20$. Then $i = 4$, $\delta = 0$, and $\deg(w_7) = 14$ in $\tilde{T}_4$. fig. 2.5 depicts this process when $w'_{a,b} = w'_{3,7}$ and $w'_{c,d} = w'_{7,11}$.

After repeating the above process for each degree 14 vertex in $\widetilde{T}_i$, we take the clique complex and call the resulting flag complex Y_2. Observe that this process increases the number of vertices by 1, the number of edges by 3, and the number of faces by 2 each time a degree 14 vertex in $\widetilde{T}_i$ is replaced. Also, note that $\text{maxdeg}(Y_2) \leq 12$ for all m.

Now, we give the w_j, $w'_{s,t}$, and u_ℓ vertices their natural orderings and say that $w'_{s,t} > w_j$ and $w'_{s,t} > u_\ell$ for all ℓ, s, t, and j, and then let these vertex orderings induce orientations on the edges and faces of Y_2 (as shown in fig. 2.3). Counting the vertices, edges, and faces of Y_2 we have that if $0 \leq k \leq 12$, then there were no degree 14 vertices to remove, so $|V(Y_2)| = 6k + 2\delta + 2$, $|E(Y_2)| = 17k + 6\delta$, and $|F(Y_2)| = 10k + 4\delta$. If $k \geq 13$, then $i \geq 3$

and at least one degree 14 vertex was removed to construct Y_2 from $\widetilde{T}_i$. table 2.3 gives the number of vertices, edges, and faces of Y_2 for all values of $k \geq 13$.

Table 2.3: Number of vertices, edges, and faces in Y_2 when $k \geq 13$.

k	δ	$\lvert V(Y_2)\rvert$	$\lvert E(Y_2)\rvert$	$\lvert F(Y_2)\rvert$
$4i+4$	0	$\frac{13}{2}k-4$	$\frac{37}{2}k-18$	$11k-12$
$4i+3$	1	$\frac{13}{2}k-\frac{3}{2}$	$\frac{37}{2}k-\frac{21}{2}$	$11k-7$
$4i+2$	2	$\frac{13}{2}k$	$\frac{37}{2}k-6$	$11k-4$
$4i+1$	3	$\frac{13}{2}k+\frac{5}{2}$	$\frac{37}{2}k+\frac{3}{2}$	$11k+1$

Homology of Y_2

Since Y_2 is an oriented flag triangulation of S^2 with k square holes, each of which are vertex disjoint and nonadjacent, our Y_2 is homeomorphic to Newman's Y_2 in the $d = 2$ case of [New18, Lemma 5.7], and we can apply the same argument to compute the homology of Y_2. We denote the 1-cycles that are the boundaries of the k square holes by $\tau_1, \ldots, \tau_k$. Explicitly, for $j = 1, \ldots, k$, we define

$$\tau_j := [u_{4j-4}, u_{4j-3}] + [u_{4j-3}, u_{4j-2}] + [u_{4j-2}, u_{4j-1}] - [u_{4j-4}, u_{4j-1}].$$

Then, by our construction, each τ_j is a positively-oriented 1-cycle in $H_1(Y_2)$, and exactly as in [New18, Proof of Lemma 5.7], we have that

$$H_1(Y_2) = \langle \tau_1, \ldots, \tau_k | \tau_1 + \cdots + \tau_k = 0 \rangle.$$

2.3.3 Construction of X and proof of theorem 2.3.1

Now we attach Y_1 and Y_2 together to form the two-dimensional flag complex X such that the torsion subgroup of $H_1(X)$ is isomorphic to $\mathbb{Z}/m\mathbb{Z}$. This part essentially follows [New18, §3], though we must confirm that the resulting complex is flag and satisfies the desired bound of vertex degree.

Proof of theorem 2.3.1. For a given m, let Y_1 and Y_2 be the complexes constructed in the previous subsections. Let S denote the subcomplex of Y_2 induced by the $4k$ vertices $u_0, \ldots, u_{4k-1}$. Since the square holes in Y_2 are vertex-disjoint and have no edges between any two of them, S is a disjoint union of k square boundaries. Let $f : S \to Y_1$ be the simplicial map defined, for $j = 1, \ldots, k$, by

$$u_{4j-4} \mapsto v_{4n_j}, \qquad u_{4j-3} \mapsto v_{4n_j+1}, \qquad u_{4j-2} \mapsto v_{4n_j+2}, \qquad u_{4j-1} \mapsto v_{4n_j+3}.$$

Following [New18, §3], let $X = Y_1 \sqcup_f Y_2$ and observe that this is a simplicial complex by the same argument as Newman gives. In addition, X is a flag complex because Y_1 and Y_2 are flag, and we subdivided the edges of Y_1 and Y_2 to avoid the possibility that X might contain a 3-cycle which doesn't have a face. Furthermore, in X the squares τ_j and γ_{n_j} are identified by f for $j = 1, \ldots, k$, and, as in [New18],

$$H_1(X) \cong \mathbb{Z}^{k-1} \oplus \mathbb{Z}/m\mathbb{Z},$$

where $\mathbb{Z}/m\mathbb{Z}$ has the repeated squares representation given by

$$\langle \gamma_0, \gamma_1, \ldots, \gamma_{n_k} \mid 2\gamma_0 = \gamma_1, 2\gamma_1 = \gamma_2, \ldots, 2\gamma_{n_k-1} = \gamma_{n_k}, \gamma_{n_1} + \cdots + \gamma_{n_k} = 0 \rangle.$$

Finally, using our counts for the number of vertices, edges, and faces of Y_1 and Y_2 and with δ defined as above, if $0 \leq k \leq 12$, we have $|V(X)| = 2k + 12n_k + 6 + 2\delta$, $|E(X)| = 13k + 40n_k + 4 + 6\delta$, and $|F(X)| = 10k + 28n_k + 4\delta$. If $k \geq 13$, then table 2.4 gives the number of vertices, edges, and faces in X (where $i = \lfloor \frac{k-1}{4} \rfloor$).

Additionally, recall that $\text{maxdeg}(Y_1) \leq 9$ and $\text{maxdeg}(Y_2) \leq 12$. Since in X we are only identifying the squares of Y_2 with k of the squares of Y_1, to find the maximum degree of any vertex of X, we need only check the degrees of the identified vertices. In Y_1, we know that $\deg(v_j) \leq 9$ for each j, and in Y_2, we know that $\deg(u_\ell) \in \{4, 5\}$ for each ℓ. Let v_j and u_ℓ be vertices that are identified in X. Since two of their adjacent edges in the squares are identified as well, in X we see that $\deg(v_j) = \deg(u_\ell) \leq 12$. Thus, $\text{maxdeg}(X) \leq 12$. □

Table 2.4: Number of vertices, edges, and faces in X when $k \geq 13$.

k	δ	$\lvert V(X)\rvert$	$\lvert E(X)\rvert$	$\lvert F(X)\rvert$
$4i+4$	0	$\frac{5}{2}k + 12n_k$	$\frac{29}{2}k + 40n_k - 14$	$11k + 28n_k - 12$
$4i+3$	1	$\frac{5}{2}k + 12n_k + \frac{5}{2}$	$\frac{29}{2}k + 40n_k - \frac{13}{2}$	$11k + 28n_k - 7$
$4i+2$	2	$\frac{5}{2}k + 12n_k + 4$	$\frac{29}{2}k + 40n_k - 2$	$11k + 28n_k - 4$
$4i+1$	3	$\frac{5}{2}k + 12n_k + \frac{13}{2}$	$\frac{29}{2}k + 40n_k + \frac{11}{2}$	$11k + 28n_k + 1$

We also note the following corollary:

Corollary 2.3.3. *For every finite abelian group G there is a two-dimensional flag complex X such that the torsion subgroup of $H_1(X)$ is isomorphic to G and* $\operatorname{maxdeg}(X) \leq 12$.

Proof. Let $G = \mathbb{Z}/m_1\mathbb{Z} \oplus \mathbb{Z}/m_2\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/m_r\mathbb{Z}$ with $m_1 | m_2 | \cdots | m_r$ be an arbitrary finite abelian group. By theorem 2.3.1, there exist two-dimensional flag complexes X_{m_i} such that the torsion subgroup of $H_1(X_{m_i})$ is isomorphic to $\mathbb{Z}/m_i\mathbb{Z}$ and $\operatorname{maxdeg}(X_{m_i}) \leq 12$. If X is the disjoint union of all the X_{m_i}, then X satisfies the hypotheses of the corollary. □

2.4 Appearance of subcomplexes in $\Delta(n, p)$

The goal of this section is to show that, for attaching probabilities p in an appropriate range, the flag complex X_m from theorem 2.3.1 will appear with high probability as an induced subcomplex of $\Delta(n, p)$. See section 2.2 for the relevant definitions and notation used throughout this section. Here is our main result:

Proposition 2.4.1. *Let $m \geq 2$, and let X_m be as in theorem 2.3.1. If $\Delta \sim \Delta(n, p)$ is a random flag complex with $n^{-1/6} \ll p \leq 1 - \epsilon$ for some $\epsilon > 0$, then $\mathbf{P}\left[X_m \overset{ind}{\subset} \Delta(n,p)\right] \to 1$ as $n \to \infty$.*

Our proof of this result will rely on Bollobás's theorem on the appearance of subgraphs of a random graph, which we state here for reference.

Theorem 2.4.2 (Bollobás [Bol81])**.** *Let G' be a fixed graph, let $m(G')$ be the essential density of G' defined in definition 2.2.2, and let $G(n, p)$ be the Erdős-Rényi random graph* *on n vertices with attaching probability p. As $n \to \infty$, we have*

$$\mathbf{P}\left[G' \subset G(n,p)\right] \to \begin{cases} 0 & \textit{if } p \ll n^{-1/m(G')} \\ 1 & \textit{if } p \gg n^{-1/m(G')} \end{cases}.$$

Since any flag complex is determined by its underlying graph, we can almost apply this to prove proposition 2.4.1. However, proposition 2.4.1 (and our eventual application of it via Hochster's formula to theorem 2.1.3) requires X_m to appear as an induced subcomplex, whereas Bollobás's result is for not necessarily induced subgraphs. The following proposition, which is likely known to experts, shows that so long as p is bounded away from 1, this distinction is immaterial in the limit.

Proposition 2.4.3. *Let G' be a fixed graph, let $m(G')$ be the essential density of G' defined in definition 2.2.2, and let $G(n,p)$ be the Erdős-Rényi random graph on n vertices with attaching probability p. Suppose $p = p(n) \leq 1-\epsilon$ for some $\epsilon > 0$. Then as $n \to \infty$, we have*

$$\mathbf{P}\left[G' \overset{ind}{\subset} G(n,p)\right] \to \begin{cases} 0 & \textit{if } p \ll n^{-1/m(G')} \\ 1 & \textit{if } p \gg n^{-1/m(G')} \end{cases}.$$

Proof. Since an induced subgraph is a subgraph, if $\mathbf{P}[G' \subset G(n,p)] \to 0$, then $\mathbf{P}\left[G' \overset{ind}{\subset} G(n,p)\right] \to 0$. Thus, the first half of the threshold is a direct consequence of theorem 2.4.2, and all that needs to be shown is the second half of the threshold.

Suppose that $p \gg n^{-1/m(G')}$. We will mirror the proof of Bollobàs's theorem from [FK16, Theorem 5.3] (originally due to [RV86]), which relies on the second moment method. Let $\Lambda(G',n)$ be the set containing all of the possible ways that G' can appear as a induced subgraph of $G(n,p)$. Thus, an element $H \in \Lambda(G',n)$ corresponds to a subset of the n vertices and specified edges among those vertices such that the resulting graph is a copy of G'. We want to count the number of times G' appears as an induced subgraph of $G(n,p)$. For each $H \in \Lambda(G',n)$, we let $\mathbf{1}_H$ be the corresponding indicator random variable, where $\mathbf{1}_H = 1$ occurs in the event that restricting $G(n,p)$ to the vertices of H is precisely the copy of G' indicated by H. Note that the random variables $\mathbf{1}_H$ are not independent, as two distinct elements from $\Lambda(G', n)$ might have overlapping vertex sets. If we let $N_{G'}$ be the random variable for the number of copies of G' appearing as induced subgraphs in $G(n,p)$, then we have $N_{G'} = \sum_{H \in \Lambda(G',n)} \mathbf{1}_H$.

Our goal is to show that $\mathbf{P}[N_{G'} \geq 1] \to 1$, or equivalently that $\mathbf{P}[N_{G'} = 0] \to 0$. Since $N_{G'}$ is non-negative, the second moment method as seen in [AS16, Theorem 4.3.1] states that $\mathbf{P}[N_{G'} = 0] \leq \frac{\mathrm{Var}(N_{G'})}{\mathbb{E}[N_{G'}]^2}$, so it suffices to show that $\frac{\mathrm{Var}(N_{G'})}{\mathbb{E}[N_{G'}]^2} \to 0$. To start, we will bound the expected value. To simplify notation throughout the following computation, we let $v = |V(G')|$ and $e = |E(G')|$ denote the number of vertices and edges of G'.

$$\begin{aligned}
\mathbb{E}[N_{G'}] &= \sum_{H \in \Lambda(G',n)} \mathbb{E}[\mathbf{1}_H] \\
&= \sum_{H \in \Lambda(G',n)} p^e (1-p)^{\binom{v}{2} - e} \\
&= \Omega(n^v) \cdot p^e (1-p)^{\binom{v}{2} - e}.
\end{aligned}$$

Now let us repeat this with the variance instead.

$$\begin{aligned}
\mathrm{Var}(N_{G'}) &= \sum_{H,H' \in \Lambda(G',n)} \mathbb{E}[\mathbf{1}_H \mathbf{1}_{H'}] - \mathbb{E}[\mathbf{1}_H]\mathbb{E}[\mathbf{1}_{H'}] \\
&= \sum_{H,H' \in \Lambda(G',n)} \mathbf{P}[\mathbf{1}_H = 1 \text{ and } \mathbf{1}_{H'} = 1] - \mathbf{P}[\mathbf{1}_H = 1]\mathbf{P}[\mathbf{1}_{H'} = 1] \\
&= \sum_{H,H' \in \Lambda(G',n)} \mathbf{P}[\mathbf{1}_H = 1] \left(\mathbf{P}[\mathbf{1}_{H'} = 1 \mid \mathbf{1}_H = 1] - \mathbf{P}[\mathbf{1}_{H'} = 1] \right) \\
&= p^e (1-p)^{\binom{v}{2} - e} \sum_{H,H' \in \Lambda(G',n)} \mathbf{P}[\mathbf{1}_{H'} = 1 \mid \mathbf{1}_H = 1] - \mathbf{P}[\mathbf{1}_{H'} = 1]
\end{aligned}$$

If H and H' don't share at least two vertices, $\mathbf{1}_H$ and $\mathbf{1}_{H'}$ are independent of each other. So we can restrict to the case where they share at least two vertices, which gives

$$= p^e(1-p)^{\binom{v}{2}-e} \sum_{i=2}^{v} \sum_{\substack{H,H' \in \Lambda(G',n) \\ |V(H) \cap V(H')| = i}} \mathbf{P}[\mathbf{1}_{H'} = 1 \mid \mathbf{1}_H = 1] - \mathbf{P}[\mathbf{1}_{H'} = 1].$$

We now come to the key observation, which is also at the heart of the proof in [FK16, Theorem 5.3]: $\mathbf{P}[\mathbf{1}_{H'} = 1 \mid \mathbf{1}_H = 1]$ is maximized if those edges and non-edges in H are exactly those that are required by H'. Thus, by applying the fact that any subgraph of G' with i vertices, has at most $i \cdot m(G')$ edges and at most $\binom{i}{2}$ non-edges we get the following bound for $H, H' \in \Lambda(G', n)$ sharing i vertices:

$$\mathbf{P}[\mathbf{1}_{H'} = 1 \mid \mathbf{1}_H = 1] \leq \mathbf{P}[\mathbf{1}_{H'} = 1] \cdot p^{-i \cdot m(G')}(1-p)^{-\binom{i}{2}}$$

From here, it is a standard computation. Substituting this back into the previous equation and simplifying, we get

$$\begin{aligned}
\mathrm{Var}(N_{G'}) &\leq p^e(1-p)^{\binom{v}{2}-e} \sum_{i=2}^{v} \sum_{\substack{H,H' \in \Lambda(G',n) \\ |V(H) \cap V(H')| = i}} \mathbf{P}[\mathbf{1}_{H'} = 1] \left(p^{-i \cdot m(G')}(1-p)^{-\binom{i}{2}} - 1\right) \\
&\leq \left(p^e(1-p)^{\binom{v}{2}-e}\right)^2 \sum_{i=2}^{v} O\left(n^{2v-i}\right) \left(p^{-i \cdot m(G')}(1-p)^{-\binom{i}{2}} - 1\right).
\end{aligned}$$

And since p is bounded away from 1 and $1-p$ is bounded away from 0, we get

$$\leq \left(p^e(1-p)^{\binom{v}{2}-e}\right)^2 \sum_{i=2}^{v} O\left(n^{2v-i} p^{-i \cdot m(G')}\right).$$

Finally, applying the second moment method gives

$$\mathbf{P}[N_{G'} = 0] \leq \frac{\mathrm{Var}(N_{G'})}{\mathbb{E}[N_{G'}]^2} = \frac{\sum_{i=2}^{v} O\left(n^{2v-i} p^{-i \cdot m(G')}\right)}{\Omega(n^{2v})} = \sum_{i=2}^{v} O\left(n^{-i} p^{-i \cdot m(G')}\right).$$

Since $p \gg n^{-1/m(G')}$, we conclude that $np^{m(G')} \to \infty$, and therefore, $\mathbf{P}[N_{G'} = 0] \to 0$. It follows that $\mathbf{P}\left[G' \overset{ind}{\subset} G(n,p)\right] \to 1$. □

We now turn to the proof of proposition 2.4.1.

Proof of proposition 2.4.1. Recall that X_m is the complex from theorem 2.3.1, and let H_m be its underlying graph. Moreover, the underlying graph of $\Delta(n,p)$ is the Erdős-Rényi random graph $G(n,p)$. Since a flag complex is uniquely determined by its underlying graph, it suffices to show that $\mathbf{P}\left[H_m \overset{ind}{\subset} G(n,p)\right] \to 1$.

Since $\text{maxdeg}(H_m) \leq 12$, every subgraph has average degree at most 12. Thus, the essential density $m(H_m)$ satisfies $m(H_m) \leq 6$. Since $p \gg n^{-1/6}$, we have $p \gg n^{-1/m(H_m)}$. Applying proposition 2.4.3 gives $\mathbf{P}\left[H_m \overset{ind}{\subset} G(n,p)\right] \to 1$; thus, $\mathbf{P}\left[X_m \overset{ind}{\subset} \Delta(n,p)\right] \to 1$. □

Remark 2.4.4. Explicitly computing the essential density $m(H_m)$ seems difficult in general, and our chosen bound $m(H_m) \leq 6$, which is determined by the fact that $6 = \frac{1}{2}\,\text{maxdeg}(X_m)$, is likely too coarse. It would be interesting to see a sharper result on $m(H_m)$, as this could potentially provide a heuristic for decreasing the bound on r in Conjecture 1. Might it even be the case that $m(H_m)$ is half the average degree, $\frac{1}{2}\,\text{avgdeg}(H_m)$?

In any case, $\frac{1}{2}\,\text{avgdeg}(H_m)$ at least provides a lower bound on $m(H_m)$. Due to the detailed nature of the constructions in section 2.3, we can estimate this value. Let $k \geq 13$ and $m \gg 0$ so that $n_k = \lfloor \log_2(m) \rfloor$ will be much larger than δ. By table 2.4, the number of vertices will be approximately $\frac{5}{2}k + 12n_k$ and the number of edges will be approximately $\frac{29}{2}k + 40n_k$. The smallest the ratio of edges to vertices can be is when $n_k \gg k$, in which case the ratio will be approximately $3\frac{1}{3}$. A similar computation holds for $k \leq 12$ and for $m \gg 0$. We can conclude that $m(H_m) \geq 3\frac{1}{3} - \epsilon$, where ϵ is a positive constant that goes to 0 as $m \to \infty$.

2.5 A detailed analysis of 2-torsion

The goal of this section is to provide a more detailed analysis of what happens in the case of 2-torsion (when $m = 2$ in proposition 2.4.1). In [CFH15], Costa, Farber, and Horak analyze

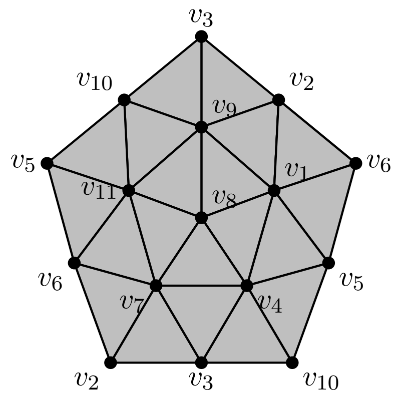

Figure 2.6: A minimal flag triangulation of $\mathbb{R}P^2$.

the 2-torsion of the fundamental group of $\Delta(n, p)$. Their results, specifically Theorem 7.2, give that if $n^{-11/30} \ll p \ll n^{-1/3-\epsilon}$ where $0 < \epsilon < \frac{1}{30}$ is fixed, then $H_1(\Delta(n, p))$ has 2-torsion with high probability as $n \to \infty$. Since our aim is to show that there is 2-torsion with high probability in the homology of an induced subcomplex of $\Delta(n, p)$, rather than in the global homology, we are able to extend their threshold to $n^{-11/30} \ll p \leq 1 - \epsilon$ where $\epsilon > 0$. We use the same techniques as in section 2.4, but instead of using X_2 from theorem 2.3.1, we use a known flag triangulation of $\mathbb{R}P^2$ that minimizes the number of vertices and where we can easily compute its essential density. This gives the less restrictive threshold of $p \gg n^{-11/30}$ in the 2-torsion case as opposed to $p \gg n^{-1/6}$ in the general case. In [Bib+19, Figure 1], the authors found two (nonisomorphic) minimal flag triangulations of $\mathbb{R}P^2$, each of which have 11 vertices and 30 edges and differ by a single bistellar 0-move; one of these is used in [CFH15], and the other, which we use in this section, is depicted in fig. 2.6.

For the remainder of this section, let G denote the underlying graph of this flag triangulation of $\mathbb{R}P^2$, which we denote by $\Delta(G)$. To understand the probability that $\Delta(G)$ appears as an induced subcomplex of $\Delta(n, p)$, we need to compute the essential density $m(G)$.

Lemma 2.5.1. *For the graph G underlying the flag triangulation of $\mathbb{R}P^2$ exhibited in fig. 2.6, the essential density $m(G)$ is* $30/11$.

Proof. This amounts to an exhaustive computation, which is summarized in table 2.5. In particular, table 2.5 identifies the maximal number of edges that a subgraph $H \subset G$ on $|V(H)|$ vertices can have, for each $|V(H)| \leq 11$. One can see from the table that $m(G)$ is maximized by the entire graph, and thus $m(G) = |E(G)|/|V(G)| = 30/11$. □

Table 2.5: With G as the underlying graph of the complex in fig. 2.6, this table computes the maximal number of edges of subgraphs $H \subset G$ with varying number of vertices.

$\|V(H)\|$	$\max\{\|E(H)\|\}$	$V(H)$	$\max\left\{\frac{\|E(H)\|}{\|V(H)\|}\right\}$
1	0	$\{v_1\}$	0
2	1	$\{v_1, v_2\}$	$\frac{1}{2}$
3	3	$\{v_1, v_2, v_6\}$	1
4	5	$\{v_1, v_2, v_5, v_6\}$	$\frac{5}{4}$
5	7	$\{v_1, v_2, v_4, v_5, v_6\}$	$\frac{7}{5}$
6	10	$\{v_1, v_4, v_7, v_8, v_9, v_{11}\}$	$\frac{5}{3}$
7	13	$\{v_1, v_2, v_4, v_7, v_8, v_9, v_{11}\}$	$\frac{13}{7}$
8	17	$\{v_1, v_2, v_4, v_6, v_7, v_8, v_9, v_{11}\}$	$\frac{17}{8}$
9	21	$\{v_1, v_2, v_3, v_4, v_6, v_7, v_8, v_9, v_{11}\}$	$\frac{7}{3}$
10	25	$\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{11}\}$	$\frac{5}{2}$
11	30	$\{v_1, \ldots, v_{11}\}$	$\frac{30}{11}$

Lemma 2.5.1 shows that the graph G is strongly balanced in the sense of definition 2.2.2. While we expect the essential density of our complexes X_m to be lower than the coarse bound of $\frac{1}{2}\operatorname{maxdeg}(X_m)$ (see remark 2.4.4), we note that in the case of the graph G, this difference is not very large. In fact, we have $\frac{1}{2}\operatorname{maxdeg}(G) = 3$ and $m(G) = 30/11 \approx 2.72$.

Combining lemma 2.5.1 and theorem 2.4.2 we obtain an analogue of proposition 2.4.1.

Proposition 2.5.2. *If $\Delta \sim \Delta(n,p)$ is a random flag complex with $n^{-11/30} \ll p \leq 1-\epsilon$ for some $\epsilon > 0$, then $\mathbf{P}\left[\Delta(G) \overset{ind}{\subset} \Delta(n,p)\right] \to 1$ as $n \to \infty$.*

Proof. The proof is nearly identical to that of proposition 2.4.1, so we omit the details. □

Question 2.5.3. *It would be interesting to know whether $p \gg n^{-11/30}$ is a sharp threshold for the appearance of 2-torsion in the homology of any induced subcomplex of $\Delta(n,p)$. While*

[CFH15, Theorem 7.1] shows that the global homology has no torsion if $p \ll n^{-11/30}$*, it is possible that some induced subcomplex of* $\Delta(n,p)$ *has* 2*-torsion. A closely related question is whether there exists a flag complex* X *with 2-torsion homology and a smaller essential density than* $30/11$.

2.6 Torsion in the Betti tables associated to Δ

We now prove theorem 2.1.3. The hard work was done in the previous sections.

Proof of theorem 2.1.3. Assume $n^{-1/6} \ll p \leq 1-\epsilon$ and let $\Delta \sim \Delta(n,p)$. Let X_m be as constructed in the proof of theorem 2.3.1. By proposition 2.4.1, Δ contains X_m as an induced subcomplex with high probability as $n \to \infty$. Since $H_1(X_m)$ has m-torsion, Hochster's formula (see fact 2.2.3) gives that the Betti table of the Stanley–Reisner ideal of Δ has ℓ-torsion for every prime ℓ dividing m. □

We can also apply the more detailed study of 2-torsion from section 2.5 to obtain a result on the appearance of 2-torsion in the Betti tables of random flag complexes.

Proposition 2.6.1. *Let* $\Delta \sim \Delta(n,p)$ *be a random flag complex with* $n^{-11/30} \ll p \leq 1-\epsilon$ *for some* $\epsilon > 0$*. With high probability as* $n \to \infty$*, the Betti table of the Stanley–Reisner ideal of* Δ *has* 2*-torsion.*

Proof. The proof is the same as the proof of theorem 2.1.3, but utilizing proposition 2.5.2 in place of proposition 2.4.1. □

As a generalization of question 2.5.3, it would be interesting to understand a precise threshold on the attaching probability p such that the Betti table of the Stanley–Reisner ideal of Δ does not depend on the characteristic. A related question is posed in question 2.7.3.

Remark 2.6.2. Our constructions are based entirely on torsion in the H_1-groups, and thus we obtain Betti tables where the entries in the second row of the Betti table (the row of entries of the form $\beta_{i,i+2}$) depend on the characteristic. Since Newman's work also produces small simplicial complexes where the H_i-groups have torsion for any $i \geq 1$ [New18, Theorem 1], one could likely apply the methods of section 2.3 to produce thresholds for where the other rows of the Betti table would depend on the characteristic, and it might be interesting to explore the resulting thresholds.

2.7 Further Questions

In this final section, we discuss some further questions about torsion for flag complexes and for the asymptotic syzygies of geometric examples.

Question 2.7.1. *Can one find new examples of Veronese embeddings of $\mathbb{P}^r$, or of any other reasonably simple variety (Grassmanian, toric variety, etc.), whose Betti tables depend on the characteristic? For a given ℓ, can one produce a specific example of a variety whose Betti table has ℓ-torsion?*

We find it especially surprising that there are no known examples of 2-torsion for d-uple embeddings of $\mathbb{P}^r$. Focusing on the case of projective space, the following question is open:

Question 2.7.2. *What is the minimal value of r such that the Betti table of the d-uple embedding of $\mathbb{P}^r$ depends on the characteristic for some d? (It is known that $2 \leq r \leq 6$.)*

An analogous question, in the context of random monomial ideals, would be as follows:

Question 2.7.3. *Let $m \geq 2$. For a random flag complex $\Delta \sim \Delta(n, p)$, what is the threshold on p such that the Betti table of the Stanley–Reisner ideal of Δ has m-torsion with high probability as $n \to \infty$?*

A closely related result is [CFH15, Theorem 8.1], which implies that for any given odd prime ℓ, the Betti table of the Stanley–Reisner ideal of Δ (with high probability as $n \to \infty$) has no ℓ-torsion when $p \ll n^{-1/3-\epsilon}$ where $\epsilon > 0$ is fixed.

Remark 2.7.4. We know of two natural ways that one could improve the threshold for p in theorem 2.1.3. First, one could perform a more detailed study of the essential density $m(H_m)$, as that value is surely lower than our chosen bound $\frac{1}{2}\operatorname{maxdeg}(X_m)$. Second, one could aim to produce flag complexes X'_m with torsion homology (not necessarily in H_1) whose underlying graphs have a lower essential density than H_m. Of course, following the heuristic discussed in the introduction, any such improvement of the threshold for p in theorem 2.1.3 would suggest a corresponding improvement of the bound on r in Conjectures 1 and 2.

In a different direction, one might ask about how large n needs to be before we expect to see that the Betti table associated to Δ has ℓ-torsion.

Question 2.7.5. *Fix a prime ℓ and $\epsilon > 0$. Let $\Delta \sim \Delta(n, p)$ be a random flag complex with $n^{-1/6} \ll p \ll 1 - \epsilon$. For a constant $0 < \delta < 1$, approximately how large does n need to be to guarantee that*

$$\mathbb{P}[\text{ Betti table associated to } \Delta \text{ has } \ell\text{-torsion }] \geq 1 - \delta?$$

It would be interesting to even answer this question for 2-torsion, where the thresholds from [CFH15, Theorems 7.1 and 7.2] make the question seemingly quite tractable. An analogous question for Veronese embeddings of projective space would be the following:

Question 2.7.6. *Fix a prime ℓ and integer $r \geq 2$. Can one provide lower/upper bounds on the minimal value of d such that the Betti table of the d-uple embedding of $\mathbb{P}^r$ has ℓ-torsion?*

Of course, one could ask similar questions, replacing $\mathbb{P}^r$ by other varieties. We could also turn to even more quantitative questions related to Conjecture 2 as well.

Question 2.7.7. *Fix a prime ℓ and an integer $r \geq 2$. Can one describe the set of $d \in \mathbb{Z}$ such that the Betti table of the d-uple embedding of $\mathbb{P}^r$ has ℓ-torsion? Can one bound or estimate the density of this set?*

Chapter 3

Virtual criterion for generalized Eagon–Northcott complexes

Joint work with John Cobb [BC22]

3.1 Introduction

The Eagon-Northcott complex of a matrix has been an object of interest since its introduction in [EN62], where the authors showed that it is a minimal free resolution of the ideal of maximal minors of the matrix if the depth of this ideal is the greatest possible value. In 1975, Buchsbaum and Eisenbud described a family of generalized Eagon-Northcott complexes associated to a map of free modules, which are free resolutions if the depth of the ideal of maximal minors of the matrix is as large as possible [BE75; Eis04]. Our main result provides an analogue in the setting of virtual resolutions.

Let us recall the setup of virtual resolutions. Let X be a smooth projective toric variety over an algebraically closed field k, and let S be its $\mathrm{Pic}(X)$-graded Cox ring, which is a multigraded polynomial ring with irrelevant ideal B, as defined in [Cox95]. In 2017, Berkesch, Erman, and Smith formalized the notion of *virtual resolutions* as a natural analogue to minimal graded free resolutions for smooth projective toric varieties [BES20].[1]

[1] Even without a formal definition, this notion appeared in the literature prior [MS04; EL18].

In exchange for allowing some higher homology supported on the irrelevant ideal B, virtual resolutions tend to be shorter and to better capture geometrically meaningful properties of S-modules, such as unmixedness, well-behavedness of deformation theory, and regularity of tensor products.

Despite their utility, there continues to be a lack of families of examples of virtual resolutions. Our work adds a new method for explicitly constructing virtual resolutions, and it adds to the growing literature on developing virtual analogues of classical homological results [Ber+21; Yan21; HNV22; DM19; Lop21; Ken+20; Gao+21; BHS21]. In particular, we show when the Eagon-Northcott and Buchsbaum-Rim complexes (and the other complexes in this generalized family) are virtual resolutions.

Let X be a smooth projective toric variety with $S = \mathrm{Cox}(X)$ and irrelevant ideal B. Our main theorem is as follows:

Theorem 3.1.1. *Let $\varphi : F \to G$ be a $\mathrm{Pic}(X)$-graded map of free S-modules of ranks $f \geq g$ where $I_m(\varphi)$ denotes the ideal of $m \times m$ minors of φ, and let*

$$\mathcal{C}^i : 0 \longrightarrow F_e \xrightarrow{\varphi_e} F_{e-1} \xrightarrow{\varphi_{e-1}} \cdots \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0$$

be one of the generalized Eagon-Northcott complexes of φ with $i \geq -1$ (see §2.2). Then $\mathcal{C}^i$ is a virtual resolution whenever $\mathrm{depth}(I_m(\varphi) : B^\infty) \geq f - m + 1$ for $f - e + 1 \leq m \leq g$.

Notably, if $\mathrm{depth}(I_g(\varphi) : B^\infty) \geq f - g + 1$, then each of the complexes $\mathcal{C}^i$ with $-1 \leq i \leq f - g + 1$ (including the Eagon-Northcott and Buchsbaum-Rim) is a virtual resolution of length $f-g+1$ by Theorem 3.1.1. The proof of Theorem 3.1.1 involves a combination of the corresponding methods from [Eis04], Loper's criterion for when a complex is a virtual resolution [Lop21], and several concrete arguments to address how these auxiliary results interact with saturation.

To illustrate its use, consider the graph of the twisted cubic in the following example.

Example 3.1.2 (Graph of the twisted cubic)**.** Let $X = \mathbb{P}^1 \times \mathbb{P}^3$ with $S = k[x_0, x_1, y_0, y_1, y_2, y_3]$. This is $\mathrm{Pic}(X) = \mathbb{Z}^2$-graded where $\deg(x_i) = (1, 0)$ and $\deg(y_i) = (0, 1)$. The irrelevant ideal B is $\langle x_0, x_1\rangle \cap \langle y_0, y_1, y_2, y_3\rangle$. Consider the following map of $\mathrm{Pic}(X)$-graded S-modules:

$$\varphi := \begin{bmatrix} x_0^3 & x_0^2x_1 & x_0x_1^2 & x_1^3 \\ y_0 & y_1 & y_2 & y_3 \end{bmatrix} : S(-3,-1)^4 \longrightarrow \begin{matrix} S(0,-1) \\ \oplus \\ S(-3,0) \end{matrix}$$

The variety in X defined by $I_2(\varphi)$, the ideal of 2×2 minors of φ, is precisely the graph of the embedding of the twisted cubic curve defined by $[s:t] \mapsto ([s:t],[s^3 : s^2t : st^2 : t^3])$. The Eagon-Northcott complex $\mathcal{C}^0$ is computed to be

$$\mathcal{C}^0 : 0 \longrightarrow \begin{matrix} S(-9,-1) \\ \oplus \\ S(-6,-2) \\ \oplus \\ S(-3,-3) \end{matrix} \longrightarrow \begin{matrix} S(-6,-1)^4 \\ \oplus \\ S(-3,-2)^4 \end{matrix} \longrightarrow S(-3,-1)^6 \xrightarrow{I_2(\varphi)} S$$

If depth $I_2(\varphi) = 3$, then Theorem A2.10c in [Eis04] would ensure that $\mathcal{C}^0$ is a free resolution of $S/I_2(\varphi)$. However, since $\langle x_0, x_1\rangle$ is a minimal prime of $I_2(\varphi)$, we have depth $I_2(\varphi) \leq 2$, and one can in fact check that $H_1(\mathcal{C}^0) \neq 0$ (see Example 3.4.3). A similar computation shows that the Buchsbaum-Rim complex $\mathcal{C}^1$ is not a free resolution of $\operatorname{coker} \varphi$. However, since $\operatorname{depth}(I_2(\varphi) : B^\infty) = 3$, Theorem 3.1.1 implies that both $\mathcal{C}^0$ and $\mathcal{C}^1$ are virtual resolutions.

This paper is organized as follows: in §2, we provide notation and necessary background about virtual resolutions and generalized Eagon-Northcott complexes, in §3, we give a proof of Theorem 3.1.1, and in §4, we explore more examples of the utility of our result.

Acknowledgements

The authors thank Daniel Erman for his valuable insight and guidance and Mahrud Sayrafi for helpful conversations. We also thank our anonymous referees for their valuable suggestions. The computer algebra system Macaulay2 [M2] was used extensively, in particular the `VirtualResolutions` package [Alm+20].

3.2 Background and Notation

3.2.1 Virtual Resolutions

On $\mathbb{P}^n$, minimal free resolutions capture geometric properties well because the maximal ideal, which plays a key role in the definition of such resolutions, is the irrelevant ideal. However, on toric varieties such as $\mathbb{P}^1 \times \mathbb{P}^1$, the irrelevant ideal $B = \langle x_0, x_1 \rangle \cap \langle y_0, y_1 \rangle$ is strictly contained in the maximal ideal. This inequality results in the minimal free resolution containing algebraic structure that is geometrically irrelevant. By focusing on the irrelevant ideal, virtual resolutions are better able to represent important geometric information.

From now on, let X be a smooth projective toric variety with $S = \mathrm{Cox}(X)$ and irrelevant ideal B.[2] Given that there is a correspondence between $\mathrm{Pic}(X)$-graded B-saturated S-modules M and sheaves $\widetilde{M}$ on X (for more details see §5.2, [CLS11]; a generalization can be found in [Mus02]), allowing for some "irrelevant homology" in our complexes stands to make them shorter and closer linked to the geometric situation. This motivates the following definition.

Definition 3.2.1 ([BES20])**.** A complex $\mathcal{C} : \cdots \xrightarrow{\varphi_3} F_2 \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0$ of $\mathrm{Pic}(X)$-graded free S-modules is called a **virtual resolution** of a $\mathrm{Pic}(X)$-graded S-module M if the corresponding complex $\widetilde{\mathcal{C}}$ of vector bundles on X is a locally free resolution of the sheaf $\widetilde{M}$.

Algebraically, $\mathcal{C}$ is a virtual resolution if all of the higher homology groups are supported on the irrelevant ideal, i.e. for each $i \geq 1$, $B^n H_i(\mathcal{C}) = 0$ for some n. Note that all exact complexes are virtual resolutions, but not all virtual resolutions are exact, since they allow for a specific type of homology.

Our proof of Theorem 3.1.1 utilizes a result from [Lop21] which provides a virtual analogue of Buchsbaum and Eisenbud's famous criterion for checking the exactness of a complex without directly computing the homology [BE73]. Given a $\mathrm{Pic}(X)$-graded map

[2] S and B are generalizations of the homogeneous coordinate ring and its maximal ideal, respectively. For more precise definitions, see [Cox95, §5.1], where $\mathrm{Cox}(X)$ is called the total coordinate ring.

of free S-modules $\varphi : F \to G$, we can choose a matrix representation for φ. Let $I_m(\varphi) \subseteq S$ be the ideal generated by the $m \times m$ minors of this matrix, where, by convention, $I_m(\varphi) = S$ for $m \leq 0$. Note that these ideals of minors are Fitting invariants of $\operatorname{coker} \varphi$, independent of the choice of matrix representation [Eis04, §20.2]. We define the rank of φ to be $\operatorname{rank}(\varphi) := \max\{m \mid I_m(\varphi) \neq 0\}$, and we set $I(\varphi) := I_{\operatorname{rank}(\varphi)}(\varphi)$ to be the corresponding ideal of minors, which will play a key role in our study of φ. Loper's criterion for a complex to be a virtual resolution is as follows.

Theorem 3.2.2 ([Lop21]). *Suppose*

$$\mathcal{C} : 0 \longrightarrow F_e \xrightarrow{\varphi_e} F_{e-1} \longrightarrow \cdots \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0$$

is a $\operatorname{Pic}(X)$*-graded complex of free* S*-modules. Then* $\mathcal{C}$ *is a virtual resolution if and only if both of the following conditions are satisfied for each* $j = 1, \ldots, e$*:*

(i) $\operatorname{rank} \varphi_j + \operatorname{rank} \varphi_{j+1} = \operatorname{rank} F_j$ *(taking* $\varphi_{e+1} = 0$*),*

(ii) $\operatorname{depth}(I(\varphi_j) : B^\infty) \geq j$

Therefore, in order for a complex in the toric setting to be a virtual resolution, we only need to consider the depth of the B-saturation of the maximal ideals of minors of the complex's differentials. By convention, the depth of the unit ideal is infinity, so that condition (ii) above is satisfied if $(I(\varphi_j) : B^\infty) = S$.

3.2.2 Generalized Eagon-Northcott Complexes

Let $\varphi : F \to G$ be any map of free R-modules where $f = \operatorname{rank} F \geq g = \operatorname{rank} G$. Associated to φ is a family $\{\mathcal{C}^i\}_{i \in \mathbb{Z}}$ of *generalized Eagon-Northcott complexes* defined in [Eis04, §A2.6] by splicing together linear strands of particular Koszul complexes. Two of the most important of these complexes are $\mathcal{C}^0$, the Eagon-Northcott complex, and $\mathcal{C}^1$, the Buchsbaum-Rim complex, which are shown below. Here, $\bigwedge^d F$ denotes the d^{th} exterior power of F and $(\operatorname{Sym}_d(G))^*$ denotes the dual of the d^{th} symmetric power of G.

$$\mathcal{C}^0 : 0 \to \bigwedge^f F \otimes (\mathrm{Sym}_{f-g}(G))^* \longrightarrow \cdots \longrightarrow \bigwedge^{g+1} F \otimes (\mathrm{Sym}_1(G))^* \longrightarrow \bigwedge^g F \otimes (\mathrm{Sym}_0(G))^* \xrightarrow{\wedge^g \varphi} R \longrightarrow 0$$

$$\mathcal{C}^1 : 0 \to \bigwedge^f F \otimes (\mathrm{Sym}_{f-g-1}(G))^* \to \cdots \to \bigwedge^{g+2} F \otimes (\mathrm{Sym}_1(G))^* \to \bigwedge^{g+1} F \otimes (\mathrm{Sym}_0(G))^* \xrightarrow{\epsilon} F \xrightarrow{\varphi} G \to 0$$

This family of complexes has unique properties, leading to their use in a wide variety of situations [Sch86; GLP83; Zam+13]. By their construction, each complex is dual to another: $\mathcal{C}^i$ is dual to $\mathcal{C}^{f-g-i}$. The most interesting complexes in this family are those $\mathcal{C}^i$ with $-1 \leq i \leq f-g+1$. These complexes are of length $f-g+1$ and are generically free resolutions [Eis04, Theorem A2.10c]. The $\mathcal{C}^i$ can be described explicitly, and they preserve homogeneity in the case that R is a (multi)graded ring.

Example 3.2.3. Consider the homogeneous map

$$\varphi := \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \end{bmatrix} : S(-1,-1)^4 \longrightarrow \begin{matrix} S(0,-1) \\ \oplus \\ S(-1,0) \end{matrix}$$

where $S = k[x_{ij}] = \mathrm{Cox}(\mathbb{P}^3 \times \mathbb{P}^3)$ is bigraded with $\deg(x_{1j}) = (1,0)$ and $\deg(x_{2j}) = (0,1)$. Let m_{ij} be the 2×2 minor of φ involving columns i and j, so that $I_2(\varphi) = \langle m_{ij} \rangle$. The Eagon-Northcott complex $\mathcal{C}^0$ and Buchsbaum-Rim complex $\mathcal{C}^1$ are given by

$$\mathcal{C}^0 : \begin{matrix} S(-3,-1) \\ \oplus \\ S(-2,-2) \\ \oplus \\ S(-1,-3) \end{matrix} \xrightarrow{\begin{bmatrix} -x_{14} & -x_{24} & 0 \\ x_{13} & x_{23} & 0 \\ -x_{12} & -x_{22} & 0 \\ x_{11} & x_{21} & 0 \\ 0 & -x_{14} & -x_{24} \\ 0 & x_{13} & x_{23} \\ 0 & -x_{12} & -x_{22} \\ 0 & x_{11} & x_{21} \end{bmatrix}} \begin{matrix} S(-2,-1)^4 \\ \oplus \\ S(-1,-2)^4 \end{matrix} \xrightarrow{\begin{bmatrix} x_{13} & x_{14} & 0 & 0 & x_{23} & x_{24} & 0 & 0 \\ -x_{12} & 0 & x_{14} & 0 & -x_{22} & 0 & x_{24} & 0 \\ x_{11} & 0 & 0 & x_{14} & x_{21} & 0 & 0 & x_{24} \\ 0 & -x_{12} & -x_{13} & 0 & 0 & -x_{22} & -x_{23} & 0 \\ 0 & x_{11} & 0 & -x_{13} & 0 & x_{21} & 0 & -x_{23} \\ 0 & 0 & x_{11} & x_{12} & 0 & 0 & x_{21} & x_{22} \end{bmatrix}} S(-1,-1)^6 \xrightarrow{I_2(\varphi)} S$$

$$\mathcal{C}^1 : \begin{matrix} S(-3,-2) \\ \oplus \\ S(-2,-3) \end{matrix} \xrightarrow{\begin{bmatrix} -x_{14} & -x_{24} \\ x_{13} & x_{23} \\ -x_{12} & -x_{22} \\ x_{11} & x_{21} \end{bmatrix}} S(-2,-2)^4 \xrightarrow{\begin{bmatrix} m_{23} & m_{24} & m_{34} & 0 \\ -m_{13} & -m_{14} & 0 & m_{34} \\ m_{12} & 0 & -m_{14} & -m_{24} \\ 0 & m_{12} & m_{13} & m_{23} \end{bmatrix}} S(-1,-1)^4 \xrightarrow{\varphi} \begin{matrix} S(0,-1) \\ \oplus \\ S(-1,0) \end{matrix}$$

These are minimal free resolutions of $S/I_2(\varphi)$ and $\operatorname{coker} \varphi$, respectively.

3.3 Main Result and Proof

Our main result gives a sufficient criterion for the generalized Eagon-Northcott complexes $\{\mathcal{C}^i\}_{i\geq -1}$ of a $\mathrm{Pic}(X)$-graded map of free S-modules $\varphi : F \to G$ with ranks $f \geq g$ to be virtual resolutions. This is a virtual analogue of [Eis04, Theorem A2.10c] (see section 3.2.2) and will require Theorem 3.2.2 to prove. Note that while one could apply Theorem 3.2.2 directly to a given $\mathcal{C}^i$ to determine if it is a virtual resolution, this would require checking $\mathrm{depth}(I(\varphi_j) : B^\infty)$ for each differential in $\mathcal{C}^i$. The main utility of Theorem 3.1.1 is that it enables one to determine that the entire family $\{\mathcal{C}^i\}_{i\geq -1}$ consists of virtual resolutions by only checking $\mathrm{depth}(I_m(\varphi) : B^\infty)$ for particular m and the single map φ. The proof of Theorem 3.1.1 will require the following lemma, a virtual analogue of [Eis04, Theorem A2.10b].

Lemma 3.3.1. *For $i \geq -1$, let*

$$\mathcal{C}^i : 0 \longrightarrow F_e \xrightarrow{\varphi_e} F_{e-1} \xrightarrow{\varphi_{e-1}} \cdots \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0$$

be one of the generalized Eagon-Northcott complexes of φ, and let $r(j) = \sum_{\ell=j}^{e} (-1)^{\ell-j} \operatorname{rank} F_\ell$. Then for each $1 \leq j \leq e$, we have $\operatorname{rank} \varphi_j \leq r(j)$ and $(I_{r(j)}(\varphi_j) : B^\infty)$ is contained in and has the same radical as the ideal $(I_{s(j)}(\varphi) : B^\infty)$, where $s(j) = \min(g, f - j + 1)$.

Proof. This will follow readily from [Eis04, Theorem A2.10b] once we understand how taking radicals plays with saturation. We claim that for ideals I, J in a Noetherian ring R, $\sqrt{(I : J^\infty)} = (\sqrt{I} : J)$. (This is likely known to experts, but we provide a proof for completeness.) For the first containment, let $r \in \sqrt{(I : J^\infty)}$. Then there exists a such that $r^a \in (I : J^\infty)$ so there exists b such that $r^a J^b \subseteq I$. For $c = \max\{a, b\}$ and any $j \in J$, we have $(rj)^c = r^{c-a}(r^a j^c) \in r^a J^c \subseteq I$. Thus, $rJ \subseteq \sqrt{I}$, so $r \in (\sqrt{I} : J)$. Conversely, let $r \in (\sqrt{I} : J)$, and suppose $J = \langle j_1, \ldots, j_s \rangle$. Then $rj_i \in \sqrt{I}$ and we can choose $a \gg 0$ such that $r^a j_i^a \in I$ for each i. Since $J^{sa} \subseteq \langle j_i^a, \ldots, j_s^a \rangle$, we have $r^a J^{sa} \subseteq I$. Thus, $r^a \in (I : J^\infty)$ and $r \in \sqrt{(I : J^\infty)}$.

By [Eis04, Theorem A2.10b], $\operatorname{rank} \varphi_j \leq r(j)$, $I_{r(j)}(\varphi_j) \subseteq I_{s(j)}(\varphi)$, and $\sqrt{I_{r(j)}(\varphi_j)} =$

$\sqrt{I_{s(j)}(\varphi)}$. Therefore, $(I_{r(j)}(\varphi_j) : B^\infty) \subseteq (I_{s(j)}(\varphi) : B^\infty)$, and the above argument gives that $\sqrt{(I_{r(j)}(\varphi_j) : B^\infty)} = (\sqrt{I_{r(j)}(\varphi_j)} : B) = (\sqrt{I_{s(j)}(\varphi)} : B) = \sqrt{(I_{s(j)}(\varphi) : B^\infty)}$. □

Proof of Theorem 3.1.1. First, note that by the construction of the $\mathcal{C}^i$ the length satisfies $f - g + 1 \leq e \leq f$. To show that $\mathcal{C}^i$ is a virtual resolution, we show that (i) and (ii) from Theorem 3.2.2 hold for all $j = 1, \ldots, e$. The first property to show is that $\operatorname{rank} \varphi_j + \operatorname{rank} \varphi_{j+1} = \operatorname{rank} F_j$. Since

$$\begin{aligned} r(j+1) + r(j) &= \Big(\operatorname{rank} F_{j+1} - \operatorname{rank} F_{j+2} + \cdots + (-1)^{e-(j+1)} \operatorname{rank} F_e\Big) + \cdots \\ &\qquad \cdots + \Big(\operatorname{rank} F_j - \operatorname{rank} F_{j+1} + \cdots + (-1)^{(e-j)} \operatorname{rank} F_e\Big) \\ &= \operatorname{rank} F_j, \end{aligned}$$

condition (i) will follow if $\operatorname{rank} \varphi_j = r(j)$. By Lemma 3.3.1, we have $\operatorname{rank} \varphi_j \leq r(j)$. Suppose instead that $\operatorname{rank} \varphi_j < r(j)$. Then, since $\operatorname{rank} \varphi_j$ is the largest ideal of minors of φ_j that is nonzero, we have that $I_{r(j)}(\varphi_j) = 0$. Note that $\operatorname{depth}(I_{r(j)}(\varphi_j) : B^\infty) = \operatorname{depth}(I_{s(j)}(\varphi) : B^\infty)$ since the depth of ideals is preserved under taking radicals ([Eis04, Cor. 17.8b] and these ideals have the same radical by Lemma 3.3.1. Thus, $\operatorname{depth}(I_{s(j)}(\varphi) : B^\infty) = 0$. However, since $f - e + 1 \leq s(j) \leq g$ for each j, this contradicts our assumption that $\operatorname{depth}(I_{s(j)}(\varphi) : B^\infty) \geq f - s(j) + 1 \geq 1$. Thus, $\operatorname{rank} \varphi_j = r(j)$ for each $j = 1, \ldots, e$.

For (ii), we need to show that $\operatorname{depth}(I(\varphi_j) : B^\infty) \geq j$ for each $j = 1, \ldots, e$. Since $\operatorname{rank} \varphi_j = r(j)$ by the above argument, we have $I_{r(j)}(\varphi_j) = I_{\operatorname{rank} \varphi_j}(\varphi_j) = I(\varphi_j)$. Saturating then gives $(I_{r(j)}(\varphi_j) : B^\infty) = (I(\varphi_j) : B^\infty)$. Since $\sqrt{(I_{r(j)}(\varphi_j) : B^\infty)} = \sqrt{(I_{s(j)}(\varphi) : B^\infty)}$ by Lemma 3.3.1, we have $\operatorname{depth}(I(\varphi_j) : B^\infty) = \operatorname{depth}(I_{r(j)}(\varphi_j) : B^\infty) = \operatorname{depth}(I_{s(j)}(\varphi) : B^\infty)$. In the case that $s(j) = g$, our assumption gives that $\operatorname{depth}(I(\varphi_j) : B^\infty) = \operatorname{depth}(I_{s(j)}(\varphi) : B^\infty) \geq f - g + 1 \geq j$ since $g \leq f - j + 1$. Otherwise, if $s(j) = f - j + 1$, then $\operatorname{depth}(I(\varphi_j) : B^\infty) \geq f - (f - j + 1) + 1 = j$ since $f - e + 1 \leq s(j) \leq g$. In any case, $\operatorname{depth}(I(\varphi_j) : B^\infty) \geq j$. Thus, $\mathcal{C}^i$ is a virtual resolution by Theorem 3.2.2. □

Remark 3.3.2. If $\operatorname{depth}(I_g(\varphi) : B^\infty) \geq f - g + 1$, then the complexes $\mathcal{C}^i$ with $-1 \leq i \leq f - g + 1$ are virtual resolutions of length $f - g + 1$ by Theorem 3.1.1. In particular, $\mathcal{C}^{-1}$

is a virtual resolution of $\bigwedge^{f-g+1}(\operatorname{coker}\varphi^*)$, the Eagon-Northcott complex $\mathcal{C}^0$ is a virtual resolution of $S/I_g(\varphi)$, the Buchsbaum-Rim complex $\mathcal{C}^1$ is a virtual resolution of $\operatorname{coker}\varphi$, and $\mathcal{C}^i$ is a virtual resolution of $\operatorname{Sym}_i(\operatorname{coker}\varphi)$ when $1 < i \leq f-g+1$.

3.4 Examples of Virtual Resolutions

Theorem 3.1.1 gives a new way of producing virtual resolutions, especially those which are not themselves free resolutions. One aspect that is new is that our virtual resolutions are not constructed by paring down a minimal free resolution, as is the case in the construction of a virtual resolution of a pair in [BES20]. In particular, Theorem 3.1.1 tells us that we can restrict our search to finding $\operatorname{Pic}(X)$-graded maps φ of free S-modules where depth $I_g(\varphi) < f-g+1$ but $\operatorname{depth}(I_g(\varphi) : B^\infty) \geq f-g+1$. Note that while the depth of $I_g(\varphi)$ is bounded above by $f-g+1$, saturating allows for the depth to increase, potentially to infinity if $(I_g(\varphi) : B^\infty) = S$. Under these conditions, we know that the Eagon-Northcott and Buchsbaum-Rim, along with the other complexes $\mathcal{C}^i$ with $-1 \leq i \leq f-g+1$, are virtual resolutions which may not be free resolutions. If, in addition, $\operatorname{depth}(I_m(\varphi) : B^\infty) \geq f-m+1$ for all $1 \leq m \leq g$, then Theorem 3.1.1 ensures that the remaining $\mathcal{C}^i$ with $i > f-g+1$ are virtual resolutions as well.

Example 3.4.1 (Graph of a Hirzebruch)**.** Consider the Hirzebruch surface $\mathscr{H}_1 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$ with projective coordinates $[x_0 : x_1 : x_2 : x_3]$ and Cox ring $C = k[x_0, x_1, x_2, x_3]$ with $\deg(x_0) = \deg(x_2) = (1,0)$, $\deg(x_3) = (0,1)$, and $\deg(x_1) = (-1,1)$ (see [CLS11, p. 112]). The smallest ample line bundle on the Hirzebruch is $\mathcal{O}_{\mathscr{H}_1}(1,1)$, and its global sections are given by

$$H^0(\mathscr{H}_1, \mathcal{O}_{\mathscr{H}_1}(1,1)) = C_{(1,1)} = k\langle x_0x_3, x_2x_3, x_0^2x_1, x_0x_1x_2, x_1x_2^2\rangle.$$

That is, its global sections are spanned by all combinations of generators whose total bi-degree is $(1,1)$. Since $\mathcal{O}_{\mathscr{H}_1}(1,1)$ is actually globally generated by these sections, we get a map $\varphi : \mathscr{H}_1 \to \operatorname{Proj}(C_{(1,1)}) = \mathbb{P}^4$ defined by $\begin{bmatrix} x_0x_3 & x_2x_3 & x_0^2x_1 & x_0x_1x_2 & x_1x_2^2 \end{bmatrix}$.

Let $X = \mathscr{H}_1 \times \mathbb{P}^4$, so that our Cox ring is $S = C \otimes k[z_0, z_1, z_2, z_3, z_4]$ with grading $\deg(x_0) = \deg(x_2) = (1, 0, 0)$, $\deg(x_3) = (0, 1, 0)$, and $\deg(x_1) = (-1, 1, 0)$ inherited from C and $\deg(z_i) = (0, 0, 1)$. This has irrelevant ideal $B = \langle x_0, x_2\rangle \cap \langle x_1, x_3\rangle \cap \langle z_0, z_1, z_2, z_3, z_4\rangle$. Consider the Pic$(X)$-graded map

$$\psi := \begin{bmatrix} x_0x_3 & x_2x_3 & x_0^2x_1 & x_0x_1x_2 & x_1x_2^2 \\ z_0 & z_1 & z_2 & z_3 & z_4 \end{bmatrix} : S(-1,-1,-1)^5 \longrightarrow \begin{matrix} S(0,0,-1) \\ \oplus \\ S(-1,-1,0) \end{matrix}$$

The variety in X defined by $I_2(\psi)$ is precisely the graph of φ constructed above. Then $\operatorname{depth} I_2(\psi) = 2$ and $\operatorname{depth}(I_2(\psi) : B^\infty) = 4$ so by Remark 3.3.2, the Eagon-Northcott complex $\mathcal{C}^0$ and the Buchsbaum-Rim complex $\mathcal{C}^1$ are virtual resolutions of $S/I_2(\psi)$ and $\operatorname{coker} \psi$, respectively:

$$\mathcal{C}^0 : 0 \longrightarrow \begin{matrix} S(-4,-4,-1) \\ \oplus \\ S(-3,-3,-2) \\ \oplus \\ S(-2,-2,-3) \\ \oplus \\ S(-1,-1,-4) \end{matrix} \longrightarrow \begin{matrix} S(-3,-3,-1)^5 \\ \oplus \\ S(-2,-2,-2)^5 \\ \oplus \\ S(-1,-1,-3)^5 \end{matrix} \longrightarrow \begin{matrix} S(-2,-2,-1)^{10} \\ \oplus \\ S(-1,-1,-2)^{10} \end{matrix} \longrightarrow S(-1,-1,-1)^{10} \xrightarrow{I_2(\psi)} S$$

$$\mathcal{C}^1 : 0 \longrightarrow \begin{matrix} S(-4,-4,-2) \\ \oplus \\ S(-3,-3,-3) \\ \oplus \\ S(-2,-2,-4) \end{matrix} \longrightarrow \begin{matrix} S(-3,-3,-2)^5 \\ \oplus \\ S(-2,-2,-3)^5 \end{matrix} \longrightarrow S(-3,-3,-3)^5 \longrightarrow S(-1,-1,-1)^5 \xrightarrow{\psi} \begin{matrix} S(0,0,-1) \\ \oplus \\ S(-1,-1,0) \end{matrix}$$

However, one can compute nonzero elements of $H_1(\mathcal{C}^0)$ and $H_1(\mathcal{C}^1)$, so these are virtual resolutions which are not exact.

Example 3.4.2. Let $X = \mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^2$ with $S = k[x_0, x_1, y_0, y_1, y_2, z_0, z_1, z_2]$, which is $\operatorname{Pic}(X) = \mathbb{Z}^3$-graded with $\deg(x_i) = (1, 0, 0), \deg(y_i) = (0, 1, 0)$, and $\deg(z_i) = (0, 0, 1)$ and has $B = \langle x_0, x_1\rangle \cap \langle y_0, y_1, y_2\rangle \cap \langle z_0, z_1, z_2\rangle$. Consider the Pic$(X)$-graded map:

$$\varphi \coloneqq \begin{bmatrix} x_0^4 & x_0^3x_1 & x_0^2x_1^2 & x_0x_1^3 & x_1^4 \\ 0 & y_0^2 & y_1^2 & y_2^2 & 0 \\ z_0 & z_1 & z_2 & z_1 & z_0 \end{bmatrix} : S(-4,-2,-1)^5 \longrightarrow \begin{matrix} S(0,-2,-1) \\ \oplus \\ S(-4,0,-1) \\ \oplus \\ S(-4,-2,0) \end{matrix}$$

Here, we have that $\operatorname{depth} I_3(\varphi) = 2$ and $\operatorname{depth}(I_3(\varphi) : B^\infty) = 3$, so Remark 3.3.2 gives that each $\mathcal{C}^i$ with $-1 \leq i \leq 3$ is a virtual resolution, and one can check that these are not exact.

Example 3.4.3 (Graph of the degree d rational normal curve)**.** Let $X = \mathbb{P}^1 \times \mathbb{P}^d$ with $S = k[x_0, x_1, y_0, \ldots, y_d]$ and $d \geq 3$. Then S is $\operatorname{Pic}(X) = \mathbb{Z}^2$-graded with $\deg(x_i) = (1,0)$ and $\deg(y_i) = (0,1)$, and the irrelevant ideal B is $\langle x_0, x_1\rangle \cap \langle y_0, \ldots, y_d\rangle$. Consider the following map of $\operatorname{Pic}(X)$-graded S-modules:

$$\varphi \coloneqq \begin{bmatrix} x_0^d & x_0^{d-1}x_1 & \cdots & x_0x_1^{d-1} & x_1^d \\ y_0 & y_1 & \cdots & y_{d-1} & y_d \end{bmatrix} : S(-d,-1)^{d+1} \longrightarrow \begin{matrix} S(0,-1) \\ \oplus \\ S(-d,0) \end{matrix}$$

The variety in X defined by $I_2(\varphi)$ is the graph of the embedding of the degree d rational normal curve into projective space. Let $f_{i,j}$ denote the 2×2 minor of φ involving columns i and j (starting at 0) so that $I_2(\varphi) = \langle f_{i,j} \,|\, 0 \leq i < j \leq d\rangle$. Note that $\operatorname{depth} I_2(\varphi) \leq 2$ since $\langle x_0, x_1\rangle$ is a codimension 2 associated prime of $I_2(\varphi)$ (so Theorem A2.10c from [Eis04] does not apply). The Eagon-Northcott complex of φ is given by

$$\mathcal{C}^0 : 0 \longrightarrow \bigoplus_{j=1}^{d} S(-jd, -(d+1-j))^{\binom{d+1}{d+1}} \longrightarrow \cdots \longrightarrow \bigoplus_{j=1}^{i} S(-jd, -(i+1-j))^{\binom{d+1}{i+1}} \longrightarrow \cdots \longrightarrow S(-d,-1)^{\binom{d+1}{2}} \xrightarrow{I_2(\varphi)} S$$

and we wish to show that this is a virtual resolution of $S/I_2(\varphi)$ which is not a free resolution.

Let $J \subseteq S$ be the ideal

$$J = I_2\left(\begin{bmatrix} x_0 & y_0 & \cdots & y_{d-1} \\ x_1 & y_1 & \cdots & y_d \end{bmatrix}\right)$$

which also defines the graph of the degree d rational normal curve, and let $g_{i,j}$ for $0 \leq i < j \leq d$ be the 2×2 minors defining J. The following relations show that $J \subseteq (I_2(\varphi) : \langle x_0, x_1\rangle^\infty) \subseteq (I_2(\varphi) : B^\infty)$.

$$x_0^{d+j-2} \cdot g_{0,j} = x_0^{j-i} f_{0,j} - x_0^{j-2} x_1 f_{0,j-1} \qquad \text{for } 1 \leq j \leq d$$

$$x_0^{d-i+2} x_1^{i-1} \cdot g_{i,j} = x_0 y_{i-1} f_{i-1,j} - x_1 y_{i-1} f_{i-1,j-1} - x_0 y_{j-1} f_{i-1,i} \qquad \text{for } 1 \leq i < j \leq d$$

Since J is an ideal defining a rational normal scroll, we know that S/J is Cohen-Macaulay, so depth $J = d$. Therefore, $\operatorname{depth}(I_2(\varphi) : B^\infty) \geq d$, and thus, Remark 3.3.2 implies that each of the generalized Eagon-Northcott complexes $\mathcal{C}^i$ for $-1 \leq i \leq d$ is a virtual resolution. In particular, the Eagon-Northcott complex $\mathcal{C}^0$ is a virtual resolution of $S/I_2(\varphi)$ that is not exact since the relation $x_0^2 f_{1,2} - x_0 x_1 f_{0,2} + x_1^2 f_{0,1} = 0$ gives a nonzero element of $H_1(\mathcal{C}^0)$.

Chapter 4

Hilbert–Burch virtual resolutions for points in $\mathbb{P}^1 \times \mathbb{P}^1$

4.1 Introduction

Virtual resolutions, which were defined by Berkesch, Erman, and Smith [BES20] as natural analogues to minimal free resolutions in the setting of smooth projective toric varieties, have been a topic of much recent research [DM19; Alm+20; DS20; Ken+20; Gao+21; Lop21; Ber+21; Yan21; HNV22; BC22; BE23]. Since one of their most notable properties is being shorter than minimal free resolutions while still capturing important geometric information, many investigations have sought out short virtual resolutions. Specifically, given a smooth projective toric variety X with Cox ring S, a virtual analog of Hilbert's Syzygy Theorem would ensure that S-modules have virtual resolutions of length at most $\dim X$ (whereas their minimal free resolutions could have length up to $\dim S = \dim X + \operatorname{rank} \operatorname{Pic}(X)$). Such a result was shown for particular cases of X and conjectured for more general X in [BES20; Yan21; BS22]; recently, work of Hanlon-Hicks-Lazarev resolved these conjectures by proving that short virtual resolutions of length at most $\dim X$ exist when X is any smooth projective toric variety [HHL23] (see also [BE23]). However, even though short virtual resolutions are known to exist, there are still more precise questions about the structure of such resolutions, including understanding which ones best capture geometric or algebraic data.

This work focuses on exploring virtual resolutions in the simplest nontrivial geometric setting: we let X be a finite set of points in $\mathbb{P}^1 \times \mathbb{P}^1$, and we analyze virtual resolutions of S/I_X, where $S = \mathrm{Cox}(\mathbb{P}^1 \times \mathbb{P}^1)$ and I_X is the defining ideal of the points. In the classical case of points in $\mathbb{P}^2$, every minimal free resolution is of **Hilbert–Burch type**, i.e. the resolution has the shape $S \leftarrow S^{n+1} \leftarrow S^n \leftarrow 0$ and the defining ideal is given by the maximal minors of the syzygy matrix [BH93, Theorem 1.4.17] In our case, we want to understand the relationships among the minimal free resolution of S/I_X, the multigraded regularity of S/I_X, and the virtual resolutions that are of Hilbert–Burch type. An example will help illustrate the main ideas.

Example 4.1.1. Let X be the following set of four points in $\mathbb{P}^1 \times \mathbb{P}^1$:

$$X = \{([1:0],[0:1]),([0:1],[1:0]),([1:1],[1:1]),([1:2],[3:1])\}.$$

The minimal free resolution of S/I_X is [1]

$$\mathcal{F}: \qquad 0 \leftarrow S \leftarrow \begin{array}{c} S(0,-4) \\ \oplus \\ S(-1,-2)^2 \\ \oplus \\ S(-2,-1)^2 \\ \oplus \\ S(-4,0) \end{array} \leftarrow \begin{array}{c} S(-1,-4)^2 \\ \oplus \\ S(-2,-2)^3 \\ \oplus \\ S(-4,-1)^2 \end{array} \leftarrow \begin{array}{c} S(-2,-4) \\ \oplus \\ S(-4,-2) \end{array} \leftarrow 0.$$

The multigraded regularity of S/I_X turns out to be the region in $\mathbb{Z}^2$ consisting of points (i,i') such that $(i+1)(i'+1) \geq 4$ (see Proposition 4.2.6). The three minimal elements of the regularity—$(0,3)$, $(1,1)$, and $(3,0)$—each give a virtual resolution of a pair for S/I_X (see definition 4.2.3). For example, the virtual resolution of the pair $(S/I_X,(0,3))$, which is the subcomplex of $\mathcal{F}$ consisting of all summands generated in degree up to $(1,4)$, is given

[1]This is the same as the minimal free resolution of four sufficiently general points (as defined in section 4.2), and for this example, we are assuming that the characteristic of the underlying field is not 2 or 3.

by

$$(S/I_X,(0,3)):\qquad 0 \leftarrow S \xleftarrow{A} \begin{matrix} S(0,-4) \\ \oplus \\ S(-1,-2)^2 \end{matrix} \xleftarrow{B} S(-1,-4)^2 \leftarrow 0$$

where $A = \begin{bmatrix} y_0^3y_1 - 4y_0^2y_1^2 + 3y_0y_1^3 & 4x_0y_0^2 - 7x_1y_0y_1 + 3x_1y_1^2 & 4x_0y_0y_1 - x_1y_0y_1 - 3x_1y_1^2 \end{bmatrix}$

$$\text{and } B = \begin{bmatrix} 4x_0 & x_1 \\ -y_0y_1 - 3y_1^2 & -y_0y_1 \\ 7y_0y_1 - 3y_1^2 & y_0^2 \end{bmatrix}.$$

Notice that this is a Hilbert–Burch resolution where the 2×2 minors of B generate the same ideal as the entries in A. Similarly, one can check that the virtual resolutions of a pair $(S/I_X,(1,1))$ and $(S/I_X,(3,0))$ are also length two and of Hilbert–Burch type.

In Example 4.1.1, each minimal element of the multigraded regularity of S/I_X yielded a short virtual resolution of a pair that was of Hilbert–Burch type. The primary purpose of this paper is to analyze such virtual resolutions of a pair $(S/I_X,(i,i'))$ (see definition 4.2.3) for minimal elements of the multigraded regularity (see Proposition 4.2.6). Motivated by the previous example and recent work of Harada, Nowroozi, and Van Tuyl [HNV22], we pose the following question.

Question 4.1.2. *If X is a finite set of points in sufficiently general position in $\mathbb{P}^1 \times \mathbb{P}^1$ (see section 4.2) and (i,i') is a minimal element of the multigraded regularity of S/I_X, then is the virtual resolution of a pair $(S/I_X,(i,i'))$ of Hilbert–Burch type?*

Initial evidence pointed towards a positive answer. Specifically, Theorem 3.1 in [HNV22] gives an affirmative answer for minimal elements of regularity (i,i') satisfying $(i+1)(i'+1) = |X|$. Furthermore, in [HNV22, Remark 3.8] the authors say that computer experimentation suggests that the answer to Question 4.1.2 may be yes for more, or perhaps all, minimal elements of regularity. Our own experimentation confirmed this observation for small sets of points as well.

Our main results show that Question 4.1.2 is a bit nuanced. In the positive direction, Theorem 4.1.3 shows that most such virtual resolutions are of Hilbert–Burch type; more specifically, away from some particular numerical inequalities, this is the case. But, in the negative direction, Theorem 4.1.4 gives that when certain numerical criteria are achieved, Question 4.1.2 can and does have a negative answer.

Theorem 4.1.3 (Theorem 4.3.3)**.** *Let X be a set of $n \geq 2$ points in sufficiently general position in $\mathbb{P}^1 \times \mathbb{P}^1$, and let (i, i') be a minimal element of* $\operatorname{reg}(S/I_X)$*. By symmetry, without loss of generality, assume that $i \leq i'$. Then the virtual resolution of a pair $(S/I_X, (i, i'))$ has length two if either of the following holds:*

(a) $i(i'+2) \leq n$, *or*

(b) $i(i'+2) > n$, $-3n + 3ii' + 4i + i' \leq 0$, *and* $3n - 3ii' - 2i - 2i' \geq 0$.

A fuller statement is found in Theorem 4.3.3 and includes how these Hilbert–Burch virtual resolutions are determined by the Hilbert function of X. For $n \leq 10,000$ points, Macaulay2 [M2] gives that either condition (a) or (b) in Theorem 4.1.3 is satisfied by nearly 89.1% of the minimal elements of regularity. Hence, this theorem shows that the vast majority of virtual resolutions of a pair for minimal elements of regularity for generic points in $\mathbb{P}^1 \times \mathbb{P}^1$ are of Hilbert–Burch type. Since elements (i, i') such that $(i+1)(i'+1) = n$ satisfy condition (a), Theorem 4.1.3 greatly extends the work of Harada, Nowroozi, and Van Tuyl (see remark 4.3.4), as demonstrated below in Example 4.1.5.

Our second result gives a condition that guarantees that the virtual resolution of a pair is *not* length two, providing a partial converse of Theorem 4.1.3.

Theorem 4.1.4. *Assume the hypotheses of Theorem 4.1.3. If $i(i'+2) > n$ and $3n - 3ii' - 2i - 2i' < 0$, then the virtual resolution of a pair $(S/I_X, (i, i'))$ has length three.*

For $n \leq 10,000$ points, Macaulay2 [M2] indicates that just under 5.3% of the minimal elements of regularity satisfy Theorem 4.1.4. This provides the first evidence of negative answers to Question 4.1.2. In particular, combining the conditions from Theorems 4.1.3 and 4.1.4 (and directly checking the three cases that they don't cover) shows that for $n \leq 20$ points, *all* of the minimal elements of regularity yield Hilbert–Burch virtual resolutions of a pair. See Example 4.3.6 for the first instance where this fails when $n = 21$. Together, Theorems 4.1.3 and 4.1.4 indicate that the answer to Question 4.1.2 is yes in most cases but not all, hinting at the subtlety in answering this question fully. The only case not covered by Theorems 4.1.3 and 4.1.4 is when $i(i'+2) > n$, $-3n + 3ii' + 4i + i' > 0$, and $3n - 3ii' - 2i - 2i' \geq 0$. For $n \leq 10,000$ points, these conditions are only met for roughly 5.6% of the minimal elements of regularity, and based on computational evidence we conjecture that these remaining virtual resolutions are also length two.

Our methods build on those in [HNV22], which in turn are based on work of Giuffrida, Maggioni, and Ragusa in [GMR92; GMR94; GMR96], and use second difference functions of the bigraded Hilbert function of S/I_X to predict the minimal free resolution of a generic set of points (see section 4.2). While [GMR96, Theorem 4.3](stated below as Theorem 4.2.8) proves that the Hilbert function determines the minimal generators of I_X, the technical challenge for us amounts to determining when it correctly predicts the syzygies and second syzygies. Specifically, we need to rule out the possibility of Betti numbers which would not be forced by the Hilbert function; this amounts to understanding the Minimal Resolution Conjecture for points in $\mathbb{P}^1 \times \mathbb{P}^1$.

Originally stated for general sets of points in $\mathbb{P}^n$ by Lorenzini [Lor93], the Minimal Resolution Conjecture predicts that there are no redundant (or ghost) Betti numbers. This conjecture has been intensely studied for various n (see [BG86; Wal95; Eis+00] for details on when it holds and fails) and has been generalized by Mustată to points lying on arbitrary projective varieties [Mus98]. This has led to several studies of the Minimal Resolution Conjecture on curves [FMP03; FL22] and different surfaces [Cas06; MP11; MP12b; MP12a]. See Remark 4.3.5 for more details about how our work is related to the Minimal Resolution Conjecture for points in $\mathbb{P}^1 \times \mathbb{P}^1$ and how a proof of this conjecture in our setting would close the gap between Theorems 4.1.3 and 4.1.4, providing a full answer to Question 4.1.2.

To prove our theorems, we utilize two lemmas. In Lemma 4.3.1, we perform an in-depth analysis of the second difference functions of the Hilbert function of S/I_X to understand all possible cases that can occur for generic sets of points. Then, our key novelty, which enables us to extend [HNV22, Theorem 3.1] to the majority of the possible cases, is Lemma 4.3.2, which proves that the Hilbert function determines certain first syzygies. Finally, in the proof of Theorem 4.1.3, we essentially show that if either of the given conditions holds, then the Minimal Resolution Conjecture is true for the degrees in question. Once we know that "consecutive cancellations" of Betti numbers do not occur, we are able to use Lemma 4.3.1 to explicitly describe the virtual resolution of the pair $(S/I_X, (i,i'))$. In the one case not covered by Theorems 4.1.3 and 4.1.4, the obstacle in the proof is that our techniques are not sufficient for showing that the Minimal Resolution Conjecture holds.

Example 4.1.5. This example illustrates how Theorems 4.1.3 and 4.1.4 can be used to understand the structure of the virtual resolutions of a pair for minimal elements of regularity. Let X be a set of $n = 502$ points in sufficiently general position in $\mathbb{P}^1 \times \mathbb{P}^1$, and let $I_X \subseteq S$ be its defining ideal. Then $\operatorname{reg}(S/I_X) = \{(i,i') \in \mathbb{Z}^2 | (i+1)(i'+1) \geq 502\}$ has 22 minimal elements with $i \leq i'$, which are listed below and shown in Figure 4.1.

$$\{(0,501),(1,250),(2,167),(3,125),(4,100),(5,83),(6,71),(7,62),(8,55),(9,50),(10,45),$$
$$(11,41),(12,38),(13,35),(14,33),(15,31),(16,29),(17,27),(18,26),(19,25),(20,23),(21,22)\}.$$

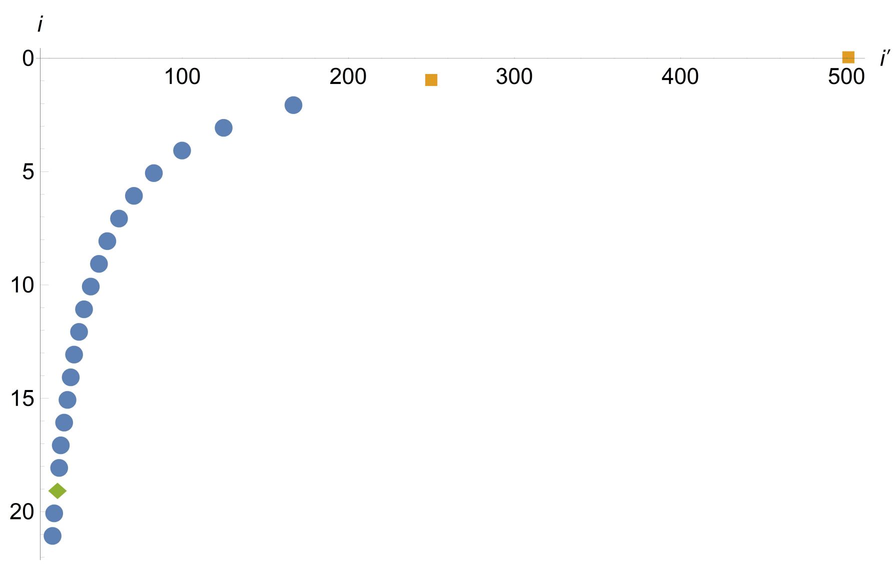

Figure 4.1: Minimal elements (i,i') of $\operatorname{reg}(S/I_X)$ for 502 points with $i \leq i'$.

The orange squares $(0, 501)$ and $(1, 250)$ satisfy $(i+1)(i'+1) = 502$, so by [HNV22, Theorem 3.1] (and Theorem 4.1.3), these elements yield Hilbert–Burch virtual resolutions of a pair. The 19 blue circles are ones where $(i+1)(i'+1) > 502$ (so they are not covered by Harada, Nowroozi, and Van Tuyl's work) that satisfy either (a) or (b) in Theorem 4.1.3 and thus give Hilbert–Burch virtual resolutions of a pair. The green diamond $(19, 25)$ does not satisfy condition (b) since $-3n+3ii'+4i+i' = 20$ and $3n-3ii'-2i-2i' = -7$; instead, Theorem 4.1.4 gives that the virtual resolution of a pair $(S/I_X, (19, 25))$ has length three. Therefore, Theorems 4.1.3 and 4.1.4 provide a complete answer to Question 4.1.2 for 502 points.

Acknowledgements

The author would like to thank Daniel Erman for his valuable guidance throughout this project. They also thank Adam Van Tuyl, John Cobb, and Mahrud Sayrafi for their helpful conversations. The computer algebra system Macaulay2 [M2] was used extensively for experimentation, especially the `VirtualResolutions` package [Alm+20].

4.2 Background

We now review the necessary background on virtual resolutions and multigraded regularity in the specific setting of $\mathbb{P}^1 \times \mathbb{P}^1$. The Cox ring of $\mathbb{P}^1 \times \mathbb{P}^1$ is the $\mathbb{Z}^2$-graded polynomial ring $S = k[x_0, x_1, y_0, y_1]$ over an algebraically closed field k of arbitrary characteristic, where $\deg(x_i) = (1, 0)$ and $\deg(y_i) = (0, 1)$. The irrelevant ideal of S is $B = \langle x_0, x_1\rangle \cap \langle y_0, y_1\rangle$, and an ideal $I \subseteq S$ is homogeneous if its generators are homogeneous elements with respect to the $\mathbb{Z}^2$-grading. We will use the component-wise partial order on $\mathbb{Z}^2$ denoted by $\preceq$, where $(i, i') \preceq (j, j')$ if and only if $i \leq j$ and $i' \leq j'$, and $(i, i') \prec (j, j')$ if and only if $(i, i') \preceq (j, j')$ and either $i < j$ or $i' < j'$. We can then define virtual resolutions in this setting as follows.

Definition 4.2.1 ([BES20])**.** A complex $\mathcal{C} : F_0 \leftarrow F_1 \leftarrow F_2 \leftarrow \cdots$ of $\mathbb{Z}^2$-graded free S-modules is called a **virtual resolution** of a $\mathbb{Z}^2$-graded S-module M if the corresponding complex $\widetilde{\mathcal{C}}$ of vector bundles on $\mathbb{P}^1 \times \mathbb{P}^1$ is a locally free resolution of the sheaf $\widetilde{M}$.

Algebraically, $\mathcal{C}$ is a virtual resolution if all of the higher homology groups are annihilated by some power of the irrelevant ideal, i.e. for each $i \geq 1$, $B^n H_i(\mathcal{C}) = 0$ for some n. Note that all exact complexes are virtual resolutions, but not all virtual resolutions are exact, since they allow for "irrelevant" homology.

In this paper, we will focus our attention on a specific type of virtual resolution called the virtual resolution of a pair, which was introduced in [BES20]. These virtual resolutions are determined by a pair of a module M and an element of its multigraded regularity, $\operatorname{reg}(M)$, which is defined below for $\mathbb{P}^1 \times \mathbb{P}^1$ and involves the vanishing of various local cohomology groups.

Definition 4.2.2. [MS04, Definition 1.1] For $\boldsymbol{r} \in \mathbb{Z}^2$, we say that a $\mathbb{Z}^2$-graded S-module M is $\boldsymbol{r}$-regular if the following conditions are satisfied:

1. $H^i_B(M)_{\boldsymbol{p}} = 0$ for all $i \geq 1$ and all $\boldsymbol{p} \in \bigcup(\boldsymbol{r} - \boldsymbol{\lambda} + \mathbb{N}^2)$ where the union is over all $\boldsymbol{\lambda} = (\lambda, \lambda') \in \mathbb{N}^2$ such that $\lambda + \lambda' = i - 1$.

2. $H^0_B(M)_{\boldsymbol{p}} = 0$ for all $\boldsymbol{p} \in (\boldsymbol{r} + (1,0) + \mathbb{N}^2) \cup (\boldsymbol{r} + (0,1) + \mathbb{N}^2)$.

We set $\operatorname{reg}(M) := \{\boldsymbol{r} \in \mathbb{Z}^2 \mid M \text{ is } \boldsymbol{r}\text{-regular}\}$.

Once we know the elements $\boldsymbol{r} \in \mathbb{Z}^2$ in the multigraded regularity region of M, we can compute the virtual resolution of the pair $(M, \boldsymbol{r})$. This is done by using $\boldsymbol{r}$ to "trim" the minimal free resolution of M in a specific way. In the case of $\mathbb{P}^1 \times \mathbb{P}^1$, the definition is as follows. See Example 4.1.1 for an example of how to find a virtual resolution of a pair.

Definition 4.2.3. [BES20, Theorem 1.3] Let M be a finitely generated $\mathbb{Z}^2$-graded, B-saturated S-module that is $\boldsymbol{r}$-regular. The free subcomplex of the minimal free resolution of M consisting of all summands generated in degree at most $\boldsymbol{r}+(1,1)$ is a virtual resolution of M called the **virtual resolution of the pair** $(M, \boldsymbol{r})$.

We are interested in understanding virtual resolutions of a pair when $M = S/I_X$, where I_X is the ideal defining a finite set of points $X \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ and $\boldsymbol{r}$ is a *minimal* element of $\operatorname{reg}(S/I_X)$ with respect to $\preceq$. Note that since $\operatorname{reg}(S/I_X)$ is a region in $\mathbb{Z}^2$, there may be several minimal elements. Although computing $\operatorname{reg}(M)$ is generally challenging, it turns out that $\operatorname{reg}(S/I_X)$ reduces to a simpler definition when X has a generic Hilbert function. To state this definition, we first recall the definition of the Hilbert function of S/I, as well as its first and second difference functions, which will play a key role in Section 4.3.

Definition 4.2.4. For any homogeneous ideal $I \subseteq S$, the **Hilbert function** of S/I is the function $H_{S/I} : \mathbb{N}^2 \to \mathbb{N}$ given in degree (i, i') by

$$H_{S/I}(i,i') := \dim_k (S/I)_{(i,i')} = \dim_k S_{(i,i')} - \dim_k I_{(i,i')}.$$

To simplify notation, when $I = I_X$ is the defining ideal of a subscheme $X \subseteq \mathbb{P}^1 \times \mathbb{P}^1$, we will denote H_{S/I_X} by H_X. Observe that we can view H_X as an infinite matrix $(m_{i,i'})$ for $(i,i') \in \mathbb{N}^2$ by setting $m_{i,i'} := H_X(i,i')$. Then the **first difference function** of H_X, $\Delta H_X : \mathbb{N}^2 \to \mathbb{N}$, is another infinite matrix $\Delta H_X = (c_{i,i'})$ defined by

$$\Delta H_X(i,i') := c_{i,i'} = m_{i,i'} + m_{i-1,i'-1} - m_{i,i'-1} - m_{i-1,i'},$$

where, by convention, $m_{i,i'} = 0$ if $i < 0$ or $i' < 0$. Repeating this operation, we define the **second difference function** of H_X, $\Delta^2 H_X : \mathbb{N}^2 \to \mathbb{N}$, to be the infinite matrix $\Delta^2 H_X = (d_{i,i'})$ where

$$\Delta^2 H_X(i,i') = \Delta(\Delta H_X(i,i')) := d_{i,i'} = c_{i,i'} + c_{i-1,i'-1} - c_{i,i'-1} - c_{i-1,i'}.$$

Example 4.2.5. Let X be as in Example 4.1.1. Then H_X and its difference functions are given by the following infinite matrices, where the rows and columns are indexed by $0, 1, 2, \ldots$:

$$H_X = \begin{bmatrix} 1 & 2 & 3 & 4 & 4 & 4 & \\ 2 & 4 & 4 & 4 & 4 & 4 & \\ 3 & 4 & 4 & 4 & 4 & 4 & \\ 4 & 4 & 4 & 4 & 4 & 4 & \cdots \\ 4 & 4 & 4 & 4 & 4 & 4 & \\ 4 & 4 & 4 & 4 & 4 & 4 & \\ & & & \vdots & & & \end{bmatrix} \Delta H_X = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & \\ 1 & 1 & -1 & -1 & 0 & 0 & \\ 1 & -1 & 0 & 0 & 0 & 0 & \\ 1 & -1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & \\ & & & \vdots & & & \end{bmatrix} \Delta^2 H_X = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 & \\ 0 & 0 & -2 & 0 & 2 & 0 & \\ 0 & -2 & 3 & 0 & -1 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ -1 & 2 & -1 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & \\ & & & \vdots & & & \end{bmatrix}$$

As explored in [GMR92; GMR94; GMR96; HNV22], when X is a finite set of points in $\mathbb{P}^1 \times \mathbb{P}^1$, the functions $H_X, \Delta H_X$, and $\Delta^2 H_X$ reveal various algebraic and geometric properties of X. One particularly interesting case is when X has a **generic Hilbert function**, which means that

$$H_X(i, i') = \min\{|X|, (i+1)(i'+1)\} \text{ for all } (i, i') \in \mathbb{N}^2.$$

Note that if X has a generic Hilbert function, then H_X and its difference functions are all symmetric matrices, so it suffices to study entries with $i \leq i'$. Observe that this is the case in Example 4.2.5. Therefore, switching the roles of i and i' in Theorems 4.1.4 and 4.3.3 gives the corresponding statements when $i > i'$. Also, I_X is a homogeneous ideal, and since the ideal of each point is B-saturated, I_X is a **B-saturated ideal**, i.e. $I_X = \bigcup_{n=1}^{\infty}(I_X : B^n)$. This implies that condition (2) of Definition 4.2.2 is satisfied for $M = S/I_X$.

Furthermore, if X has a generic Hilbert function, then we can utilize [MS04, Proposition 6.7], which says that $\boldsymbol{r} \in \operatorname{reg}(S/I_X)$ if and only if the space of forms vanishing on X has codimension $|X|$ in the space of forms of degree $\boldsymbol{r}$. Since this happens precisely when the Hilbert function H_X agrees with the Hilbert polynomial of S/I_X, we see that Definition 4.2.2 simplifies to the following.

Proposition 4.2.6. *Let $X \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ be a finite set of points with a generic Hilbert function. Then*

$$\operatorname{reg}(S/I_X) = \{(i, i') \mid H_X(i, i') = |X|\} = \{(i, i') \mid (i+1)(i'+1) \geq |X|\}.$$

Now that we know how to easily compute the multigraded regularity of S/I_X, we will discuss the minimal free resolution of S/I_X, which will be "trimmed" to construct the virtual resolutions of a pair. As in [GMR92] and [HNV22], the minimal free resolution $\mathcal{F}$

of S/I_X is given by

$$\mathcal{F}: \qquad 0 \longleftarrow S \longleftarrow \bigoplus_{\ell=1}^{m} S(-\boldsymbol{a}_{1,\ell}) \longleftarrow \bigoplus_{\ell=1}^{n} S(-\boldsymbol{a}_{2,\ell}) \longleftarrow \bigoplus_{\ell=1}^{p} S(-\boldsymbol{a}_{3,\ell}) \longleftarrow 0 \tag{4.1}$$

where $\boldsymbol{a}_{i,\ell} = (a_{i,\ell}, a'_{i,\ell})$. Then the bigraded Betti numbers of S/I_X are

$$\beta_{0,(0,0)} = 1,\ \beta_{1,\boldsymbol{r}} = \#\{\boldsymbol{a}_{1,\ell} = \boldsymbol{r}\},\ \beta_{2,\boldsymbol{r}} = \#\{\boldsymbol{a}_{2,\ell} = \boldsymbol{r}\},\ \text{and}\ \ \beta_{3,\boldsymbol{r}} = \#\{\boldsymbol{a}_{3,\ell} = \boldsymbol{r}\}.$$

Note that our notation differs slightly from that in [HNV22] and [GMR92]; in their notation, we have $\beta_{1,\boldsymbol{r}} = \alpha_{r,r'}, \beta_{2,\boldsymbol{r}} = \beta_{r,r'}$, and $\beta_{3,\boldsymbol{r}} = \gamma_{r,r'}$. In [GMR92], the authors explore several combinatorial properties of these Betti numbers. We will use the following property which relates the Betti numbers to the entries in $\Delta^2 H_X$ many times in our arguments.

Proposition 4.2.7. *[GMR92, Proposition 3.3(vi)] Let X be a set of points in $\mathbb{P}^1 \times \mathbb{P}^1$ with $\Delta^2 H_X = (d_{i,i'})$. For all $\boldsymbol{r} = (r, r') \succ (0,0)$, we have $d_{r,r'} = -\beta_{1,\boldsymbol{r}} + \beta_{2,\boldsymbol{r}} - \beta_{3,\boldsymbol{r}}$.*

Our results require that the points in X are in *sufficiently general position*, as defined first by Giuffrida, Maggioni, and Ragusa and used more recently by Harada, Nowroozi, and Van Tuyl. This condition on X ensures that the points not only have a generic Hilbert function but are "random" enough to ensure that the minimal generators of I_X are determined by H_X. Specifically, a set of n points $X = \{P_1, \dots, P_n\}$ in $\mathbb{P}^1 \times \mathbb{P}^1$ is in **sufficiently general position** if $(P_1, \dots, P_n)$ is in the open set U in the following theorem.

Theorem 4.2.8. *[GMR96, Theorem 4.3],[HNV22, Theorem 2.13] Let $n \geq 1$ be an integer. There exists a dense open subset $U \subseteq (\mathbb{P}^1 \times \mathbb{P}^1)^n$ such that for every $(P_1, \dots, P_n) \in U$, the set of points $X = \{P_1, \dots, P_n\}$ satisfies:*

1. *X has a generic Hilbert function, and*

2. *the nonzero $\beta_{1,\boldsymbol{r}}$ are precisely given by $\beta_{1,\boldsymbol{r}} = -d_{r,r'}$ for the entries $d_{r,r'} < 0$ such that either $d_{r,i'} > 0$ for some $i' > r'$ or $d_{i,r'} > 0$ for some $i > r$.*

Example 4.2.9. Let X be as in Examples 4.1.1 and 4.2.5. Since X has a generic Hilbert function, Proposition 4.2.6 gives that $\operatorname{reg}(S/I_X) = \{(i,i') \mid (i+1)(i'+1) \geq 4\}$. Observe that the minimal free resolution of S/I_X shown in Example 4.1.1 has the same structure as $\mathcal{F}$ in eq. (4.1), and the Betti numbers in the first homological degree are given by

$$\beta_{1,\boldsymbol{r}} = \begin{cases} 1, & \boldsymbol{r} = (0,4) \text{ or } (4,0) \\ 2, & \boldsymbol{r} = (1,2) \text{ or } (2,1) \\ 0, & \text{otherwise.} \end{cases}$$

We can verify directly that these points are in sufficiently general position by checking that condition (2) from Theorem 4.2.8 holds. Notice that the entries $d_{r,r'} \in \Delta^2 H_X$ that are negative and have a positive entry either to the right of or below them are $d_{0,4} = -1, d_{1,2} = -2$, $d_{2,1} = -2$, and $d_{4,0} = -1$, and the negations of these entries give the nonzero $\beta_{1,\boldsymbol{r}}$.

Furthermore, observe that the nonzero Betti numbers in the second homological degree are given by

$$\beta_{2,\boldsymbol{r}} = \begin{cases} 2, & \boldsymbol{r} = (1,4) \text{ or } (4,1) \\ 3, & \boldsymbol{r} = (2,2) \\ 0, & \text{otherwise} \end{cases}$$

and these match the positive entries $d_{1,4} = 2, d_{2,2} = 3$, and $d_{4,2} = 2$ in $\Delta^2 H_X$. Lemma 4.3.2 proves that this follows because these are each the first positive entry in their row.

We refer the reader to Examples 2.14 and 2.15 in [HNV22] to get more familiar with computing the Hilbert function and its difference functions, to see the connection between the entries in $\Delta^2 H_X$ and the Betti numbers $\beta_{1,\boldsymbol{r}}$, and for an illustrative example of a set of points with a generic Hilbert function that doesn't satisfy condition (2) in Theorem 4.2.8.

Remark 4.2.10. When $X \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ is a set of points in sufficiently general position, we see that much is known about the minimal generators of I_X. Most importantly, Theorem 4.2.8 gives that minimal generators of I_X correspond precisely to the negative entries of $\Delta^2 H_X$ that have a positive entry either to the right of or below them. In addition, the sentence following [GMR94, Lemma 4.2] says that this lemma implies that all of the minimal generators of I_X must occur in consecutive degrees, i.e. in degrees (i, i') and $(i, i'+1)$ or in degrees (i, i') and $(i+1, i')$. Furthermore, the paragraph on the top of page 202 in [GMR94] describes that if we ignore the entry $d_{0,0}$, then the first nonzero entry (if it exists) of every row (resp. column) of $\Delta^2 H_X$ is negative and corresponds to the number of minimal generators in that degree. These properties can be seen below in Example 4.3.6.

Remark 4.2.11. In [GMR96, Definition 2.1] Giuffrida, Maggioni, and Ragusa introduce the notion of a good rectangle of H_X. When X has a generic Hilbert function, it follows from [GMR96, Proposition 2.3] and Proposition 4.2.6 that degree (i, i') gives a good rectangle of H_X iff $(i-1, i'-1) \in \operatorname{reg}(S/I_X)$. Therefore, [GMR96, Proposition 2.7] implies that every virtual resolution of a pair $(S/I_X, (i, i'))$ for an element $(i, i') \in \operatorname{reg}(S/I_X)$ is actually acyclic. Note that this would follow from the Acyclicity Lemma [Eis04, Lemma 20.11] if the resolution is length two, but [GMR96, Proposition 2.7] gives acyclicity for those that are length three as well. This means that every virtual resolution of a pair considered in this paper is acyclic, and the ones that are length two are thus of Hilbert–Burch type. In Example 4.3.6, observe that degree $(4, 6)$ gives a (minimal) good rectangle of H_X since $(3, 5)$ is a (minimal) element of $\operatorname{reg}(S/I_X)$. Moreover, each of the virtual resolutions of a pair in this example is acyclic.

4.3 Results

To prove Theorems 4.1.4 and 4.3.3, we need to determine the relationships between the Betti numbers of S/I_X and the nonzero entries in the second difference function $\Delta^2 H_X$. If (i, i') is a minimal element of the multigraded regularity of S/I_X, then the virtual resolution of a pair $(S/I_X, (i, i'))$ is determined by the Betti numbers up to degree $(i+1, i'+1)$ with respect to the partial order $\preceq$. Therefore, we need to understand the nonzero entries in $\Delta^2 H_X$ up to degree $(i+1, i'+1)$. Remark 4.2.10 describes what is known about the Betti numbers in the first homological degree, and we aim to describe what happens in the second and third homological degrees. In particular, we will prove that in most cases if $d_{i+1,i'+1}$ is non-negative, then $(S/I_X, (i,i'))$ is length two (Theorem 4.3.3), and if $d_{i+1,i'+1}$ is negative, then $(S/I_X, (i,i'))$ is length three (Theorem 4.1.4) We do so by utilizing the following two lemmas. Lemma 4.3.1 gives numerical conditions that determine exactly when this corner entry is non-negative. In addition, we work out all possible cases for the form of $\Delta^2 H_X$ to determine the structure of the Hilbert–Burch virtual resolutions coming from Theorem 4.3.3 and given in Appendix A. Then, Lemma 4.3.2 will show that, assuming the numerical conditions in Theorem 4.3.3, the positive entries in $\Delta^2 H_X$ correspond to Betti numbers in the second homological degree. From here, we will prove that since $d_{i+1,i'+1} \geq 0$ under these hypotheses, the virtual resolution of a pair is length two. Furthermore, if $d_{i+1,i'+1} < 0$, then we show that $\beta_{3,(i+1,i'+1)} > 0$, and so $(S/I_X, (i+1, i'+1))$ is length three.

Lemma 4.3.1. *Let X be a set of $n \geq 2$ points in sufficiently general position in $\mathbb{P}^1 \times \mathbb{P}^1$, and let (i,i') be a minimal element of $\operatorname{reg}(S/I_X)$. By symmetry of H_X, without loss of generality, assume that $i \leq i'$. Then $d_{i+1,i'+1} \geq 0$ if and only if one of the following holds:*

(a) $i(i'+2) \leq n$, *or*

(b) $i(i'+2) > n$ *and* $3n - 3ii' - 2i - 2i' \geq 0$.

Furthermore, one can compute the entries in $\Delta^2 H_X$ up to degree $(i+1, i'+1)$ in all possible cases.

Proof. Consider the Hilbert function H_X near position (i,i'):

$$H_X = \begin{array}{c} \\ \vdots \\ i-1 \\ i \\ i+1 \\ \vdots \end{array} \begin{array}{c} \begin{array}{ccccc} \cdots & i'-1 & i' & i'+1 & \cdots \end{array} \\ \begin{bmatrix} \ddots & & \vdots & & \\ & ii' & i(i'+1) & \min\{i(i'+2), n\} & \\ & (i+1)i' & n & n & \\ & \min\{(i+2)i', n\} & n & n & \\ & & \vdots & & \ddots \end{bmatrix} \end{array}$$

Since (i,i') is a minimal element of $\operatorname{reg}(S/I_X)$, we know that $m_{i,i'} = n$ (so $(i+1)(i'+1) \geq n$), $m_{i-1,i'} = i(i'+1) < n$, $m_{i,i'-1} = (i+1)i' < n$, and $m_{i-1,i'-1} = ii' < n$. We also know that $m_{r,r'} = n$ for every $(r,r') \succeq (i,i')$. However, since the entries $m_{i-1,i'+1}$ and $m_{i+1,i'-1}$ are dependent on the specific values of i, i', and n, in order to determine the sign of $d_{i+1,i'+1}$, we need to consider three cases corresponding to the possible values of these two entries (note that since $i \leq i'$, $i(i'+2) \leq (i+2)i'$, so it is not possible for $i(i'+2) > n$ and $(i+2)i' \leq n$). Cases 1 and 2 will show that if (a) holds, then $d_{i+1,i'+1} > 0$, and Case 3 will show that if (b) holds, then $d_{i+1,i'+1} \geq 0$. Moreover, if neither (a) nor (b) holds, then we must be in Case 3, and we must have that $d_{i+1,i'+1} < 0$. To compute $d_{i+1,i'+1}$ in each case, we will use that by the definition of $\Delta^2 H_X$ (see Section 4.2) we have

$$d_{i+1,i'+1} = n + ii' - 2i(i'+1) - 2(i+1)i' + \min\{i(i'+2), n\} + \min\{(i+2)i', n\}. \quad (4.2)$$

Furthermore, since they will be used in the proof of Theorem 4.3.3, in each case we will compute all of the entries in $\Delta^2 H_X$ up to degree $(i+1, i'+1)$, and we will determine what is known about the sign of each entry. Note that in each case, $d_{0,0} = 1$ and $d_{r,r'} = 0$ in all degrees $(0,0) \prec (r,r') \preceq (i+1, i'+1)$ that are not explicitly shown in $\Delta^2 H_X$.

Case 1: If $i(i'+2) \leq n$ and $(i+2)i' \leq n$, then simplifying eq. (4.2) gives $d_{i+1,i'+1} = n - ii' > 0$ since $ii' < n$. In this case, we have

$$\Delta H_X = \begin{array}{c} \\ \vdots \\ i-1 \\ i \\ \\ i+1 \\ \vdots \end{array} \begin{array}{c} \cdots \quad i'-1 \qquad i' \qquad i'+1 \quad \cdots \\ \begin{bmatrix} \ddots & & \vdots & & \\ & 1 & 1 & 1 & \\ & 1 & \begin{matrix} n-ii' \\ -i-i' \end{matrix} & -i & \\ & 1 & -i' & 0 & \\ & & \vdots & & \ddots \end{bmatrix} \end{array} \qquad \Delta^2 H_X = \begin{array}{c} \\ \vdots \\ i-1 \\ i \\ \\ i+1 \\ \vdots \end{array} \begin{array}{c} \cdots \quad i'-1 \qquad i' \qquad i'+1 \quad \cdots \\ \begin{bmatrix} \ddots & & \vdots & & \\ & 0 & 0 & 0 & \\ & 0 & \begin{matrix} n-ii' \\ -i-i'-1 \end{matrix} & -n+ii'+i' & \\ & 0 & -n+ii'+i & n-ii' & \\ & & \vdots & & \ddots \end{bmatrix} \end{array}$$

Observe also that $d_{i,i'} \leq 0$ since $(i'+1)(i+1) \geq n$, $d_{i,i'+1} < 0$ since $(i+1)i' < n$, and $d_{i+1,i'} < 0$ since $i(i'+1) < n$.

Case 2: If $i(i'+2) \leq n$ and $(i+2)i' > n$, then eq. (4.2) becomes $d_{i+1,i'+1} = 2(n-ii'-i') > 0$ since $(i+1)i' = ii'+i' < n$. In this case, to determine $\Delta^2 H_X$ we need to consider the entry $m_{i+1,j'} \in H_X$ which is the first one in that row that is equal to n. In other words, $(i+1, j')$ is also a minimal element of $\operatorname{reg}(S/I_X)$. We know that $j' \leq i'-1$ since $m_{i+1,i'-1} = n$, and we can compute that

$$\Delta H_X = \begin{array}{c} \\ \vdots \\ i-1 \\ i \\ i+1 \\ \vdots \\ \\ \end{array} \begin{array}{c} \begin{array}{ccccccccc} \cdots & j'-1 & j' & j'+1 & \cdots & i'-1 & i' & i'+1 & \cdots \end{array} \\ \begin{bmatrix} \ddots & & \vdots & & & & \vdots & & \\ & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & \\ & 1 & 1 & 1 & \cdots & 1 & n-ii'-i-i' & -i & \\ & 1 & \begin{matrix} n-ij' \\ -i-2j'-1 \end{matrix} & -i-1 & \cdots & -i-1 & -n+ii'+i' & 0 & \\ & & \vdots & & & & \vdots & & \ddots \end{bmatrix} \end{array}$$

where if $j' = i'-1$, then we only take the column labeled j' and delete the columns labeled $j'+1$ through $i'-1$. We then compute $\Delta^2 H_X$ in two cases depending on the value of j'.

Case 2.1: If $j' < i'-1$, then we have

$$\Delta^2 H_X = \begin{array}{c} \\ \vdots \\ i-1 \\ i \\ i+1 \\ \vdots \\ \\ \end{array} \begin{array}{c} \begin{array}{ccccccccc} \cdots & j'-1 & j' & j'+1 & \cdots & i'-1 & i' & i'+1 & \cdots \end{array} \\ \begin{bmatrix} \ddots & & \vdots & & & & \vdots & & \\ & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \\ & 0 & 0 & 0 & \cdots & 0 & \begin{matrix} n-ii' \\ -i-i'-1 \end{matrix} & -n+ii'+i' & \\ & 0 & \begin{matrix} n-ij' \\ -i-2j'-2 \end{matrix} & \begin{matrix} -n+ij' \\ +2j' \end{matrix} & \cdots & 0 & \begin{matrix} -2n+2ii' \\ +2i+2i'+2 \end{matrix} & 2(n-ii'-i') & \\ & & \vdots & & & & \vdots & & \ddots \end{bmatrix} \end{array}$$

Observe that $d_{i,i'} \leq 0$, $d_{i,i'+1} < 0$, $d_{i+1,j'} \leq 0$ since $(i+1, j') \in \operatorname{reg}(S/I_X)$ implies that $(i+2)(j'+1) \geq n$, $d_{i+1,j'+1} < 0$ since $(i+1, j'-1) \notin \operatorname{reg}(S/I_X)$ implies $(i+2)j' < n$, and $d_{i+1,i'} = -2d_{i,i'} \geq 0$.

Case 2.2: If $j' = i' - 1$, then we have

$$\Delta^2 H_X = \begin{array}{c} \\ \vdots \\ i-1 \\ \\ i \\ \\ \\ i+1 \\ \vdots \\ \\ \end{array} \begin{array}{c} \cdots \quad i'-2 \quad\quad i'-1 \quad\quad\quad i' \quad\quad\quad i'+1 \quad\quad \cdots \\ \left[\begin{array}{ccccccc} \ddots & & & \vdots & & & \\ & 0 & 0 & 0 & 0 & & \\ & & & & & & \\ & 0 & 0 & n - ii' & -n + ii' + i' & & \\ & & & -i - i' - 1 & & & \\ & & & & & & \\ & 0 & n - ii' - 2i' & -3n + 3ii' & 2(n - ii' - i') & & \\ & & & +i + 4i' & & & \\ & & & \vdots & & & \ddots \end{array}\right] \end{array}$$

Here, $d_{i,i'} \leq 0$, $d_{i,i'+1} < 0$, $d_{i+1,i'-1} \leq 0$, and $d_{i+1,i'}$ could be negative or non-negative.

Case 3: If $i(i' + 2) > n$ and $(i + 2)i' > n$, then eq. (4.2) becomes $d_{i+1,i'+1} = 3n - 3ii' - 2i - 2i'$. If (b) holds, then $d_{i+1,i'+1} \geq 0$. Otherwise, $d_{i+1,i'+1} < 0$.

To compute $\Delta^2 H_X$, we again let j' be the column corresponding to the first entry equal to n in the $(i+1)$st row of H_X, and similarly, let j be the row corresponding to the first entry equal to n in the $(i'+1)$st column. In other words, $(i+1, j')$ and $(j, i'+1)$ are also minimal elements of $\text{reg}(S/I_X)$. Then, a priori, $j' \leq i' - 1$ and $j \leq i - 1$. However, since $i \leq i'$, we must have $j = i - 1$. Indeed, since $m_{i-1,i'+1} = n$ in this case, it is enough to show that $m_{i-2,i'+1} = (i-1)(i'+2) < n$. We have $(i-1)(i'+2) = i(i'+1) + (i-i') - 2 < n-2 < n$ since $i(i' + 1) < n$ and $i - i' \leq 0$. Thus, $(i - 1, i' + 1)$ must be a minimal element of $\text{reg}(S/I_X)$. Furthermore, $j' \in \{i - 2, i - 1\}$. This is because $m_{i+1,i'-3} = (i + 2)(i' - 2) = (i + 1)i' + (i' - 2i) - 4 < n - 4 < n$, where we have used that $i' < 2i$, which is necessary in order to be in Case 3 since $(i + 1)i' < n$, $i(i' + 2) > n$, and $i' \geq 2i$ is an inconsistent system of inequalities for $i, i', n \in \mathbb{N}$. So, we really only have two cases to consider. First, we can compute that

$$\Delta H_X = \begin{array}{c} \\ \vdots \\ i-2 \\ i-1 \\ i \\ i+1 \\ \vdots \\ \\ \end{array} \begin{array}{c} \begin{array}{ccccccc} \cdots & j'-1 & j' & i'-1 & i' & i'+1 & \cdots \end{array} \\ \begin{bmatrix} \ddots & & & \vdots & & & \\ & 1 & 1 & 1 & 1 & 1 & \\ & 1 & 1 & 1 & 1 & n-ii'-2i+1 & \\ & 1 & 1 & 1 & n-ii'-i-i' & -n+ii'+i & \\ & 1 & n-ij' & -i-1 & -n+ii'+i' & 0 & \\ & & -i-2j'-1 & & & & \\ & & & \vdots & & & \ddots \end{bmatrix} \end{array}$$

where if $j' = i'-1$, then we only take the column labeled j' and delete the column labeled $i'-1$. We now use ΔH_X to compute $\Delta^2 H_X$ in two cases depending on the value of j'.

Case 3.1: If $j' = i'-2$, then we have

$$\Delta^2 H_X = \begin{array}{c} \\ \vdots \\ i-2 \\ i-1 \\ i \\ \\ i+1 \\ \vdots \\ \\ \end{array} \begin{array}{c} \begin{array}{ccccccc} \cdots & i'-3 & i'-2 & i'-1 & i' & i'+1 & \cdots \end{array} \\ \begin{bmatrix} \ddots & & & \vdots & & & \\ & 0 & 0 & 0 & 0 & 0 & \\ & 0 & 0 & 0 & 0 & n-ii'-2i & \\ & 0 & 0 & 0 & n-ii' & -3n+3ii' & \\ & & & & -i-i'-1 & +4i+i' & \\ & 0 & n-ii' & -n+ii' & -2n+2ii' & 3n-3ii' & \\ & & +i-2i'+2 & -2i+2i'-4 & +2i+2i'+2 & -2i-2i' & \\ & & & \vdots & & & \ddots \end{bmatrix} \end{array}$$

Here, $d_{i-1,i'+1} < 0$, $d_{i,i'} < 0$, $d_{i,i'+1}$ could be negative or non-negative, $d_{i+1,i'-2} \leq 0$, $d_{i+1,i'-1} < 0$, $d_{i+1,i'} > 0$, and $d_{i+1,i'+1} \geq 0$ if and only if (b) holds.

Case 3.2: If $j' = i' - 1$, then we have

$$\Delta^2 H_X = \begin{array}{c} \\ \vdots \\ i-2 \\ i-1 \\ i \\ \\ i+1 \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccccc} \cdots & i'-2 & i'-1 & i' & i'+1 & \cdots \end{array} \\ \begin{bmatrix} \ddots & & & \vdots & & \\ & 0 & 0 & 0 & 0 & \\ & 0 & 0 & 0 & n-ii'-2i & \\ & 0 & 0 & \begin{array}{c} n-ii' \\ -i-i'-1 \end{array} & \begin{array}{c} -3n+3ii' \\ +4i+i' \end{array} & \\ & 0 & n-ii'-2i' & \begin{array}{c} -3n+3ii' \\ +i+4i' \end{array} & \begin{array}{c} 3n-3ii' \\ -2i-2i' \end{array} & \\ & & & \vdots & & \ddots \end{bmatrix} \end{array}$$

Here, $d_{i-1,i'+1} < 0$, $d_{i,i'} < 0$, $d_{i+1,i'-1} < 0$, $d_{i,i'+1}$ and $d_{i+1,i'}$ could be negative or non-negative (but note that $d_{i,i'+1} \leq d_{i+1,i'}$), and $d_{i+1,i'+1} \geq 0$ if and only if (b) holds.

By examining the three possible cases above, we see that $d_{i+1,i'+1} \geq 0$ if and only if (a) holds (Case 1 or 2) or (b) holds (covered by Case 3). □

The following is our second major lemma. Recall that one of the main challenges in using $\Delta^2 H_X$ to analyze virtual resolutions of a pair comes from the fact that the Minimal Resolution Conjecture is not known in this setting; see Remark 4.3.5. Lemma 4.3.2 is a partial result in that vein. Giuffrida-Maggioni-Ragusa showed that the first negative entry in a given row or column of $\Delta^2 H_X$ *always* corresponds to the number of minimal generators of I_X of that degree (see Remark 4.2.10); Lemma 4.3.2 shows that, in a similar way, the first positive entry always corresponds to the number of minimal first syzygies of I_X of that degree. See Example 4.2.9 for an illustration of this in the case of four points.

Lemma 4.3.2. *Let X be a set of $n \geq 2$ points in sufficiently general position in $\mathbb{P}^1 \times \mathbb{P}^1$, and let $\mathcal{F}$ be the minimal free resolution of S/I_X (see eq. (4.1)). If $d_{r,r'}$ is the first positive entry in the rth row (resp. r'th column) of $\Delta^2 H_X$ (excluding the 0th and 1st row and column),* *then $\beta_{2,\boldsymbol{r}} = d_{r,r'}$, i.e. that positive entry corresponds exactly to the number of first syzygies of degree $\boldsymbol{r} = (r, r')$. Furthermore, $\beta_{2,(r,u')} = 0$ for all $u' < r'$ (resp. $\beta_{2,(u,r')} = 0$ for all $u < r$), i.e. there are no first syzygies of smaller degree coming from that row (resp. column).*

Proof. We consider the following subcomplexes of the minimal free resolution $\mathcal{F}$: for $r \geq 0$, let $\mathcal{C}^{\leq r}$ be the subcomplex of $\mathcal{F}$ consisting of all summands $S(-\boldsymbol{a}_{i,\ell})$ where $\boldsymbol{a}_{i,\ell} = (a_{i,\ell}, a'_{i,\ell})$ satisfies $a_{i,\ell} \leq r$. So, $\mathcal{C}^{\leq r}$ consists of all of the terms whose first coordinate has degree at most r, with no restriction on the degree of the second coordinate. Then $\mathcal{F}$ has a filtration by the $\mathcal{C}^{\leq r}$ since there are natural inclusions $\mathcal{C}^{\leq r-1} \hookrightarrow \mathcal{C}^{\leq r}$ for each $r \geq 1$. Let $\mathcal{C}^r$ denote the cokernel of this inclusion for each $r \geq 1$. Note that $\mathcal{C}^r$ is a free complex of S-modules whose summands each have degree r in the first coordinate.

We next aim to show that for $r \geq 2$, $\mathcal{C}^r$ is the minimal free resolution of a finite length S-module. Observe that $\mathcal{C}^{\leq r}$ for $r \geq 1$ is actually the virtual resolution of the pair $(S/I_X, (r-1, N-1))$ for some sufficiently large $N \geq n$ (see definition 4.2.3). Indeed, since X is a set of n points in sufficiently general position, $(0, n-1) \in \operatorname{reg}(S/I_X)$ (see Proposition 4.2.6), which implies that $(r-1, N-1) \in \operatorname{reg}(S/I_X)$ for all $r \geq 1$ and $N \geq n$. Thus, we can choose N to be the largest degree in the second coordinate appearing in $\mathcal{C}^{\leq r}$ to ensure that the virtual resolution of the pair $(S/I_X, (r-1, N-1))$ is precisely $\mathcal{C}^{\leq r}$. This means that in the short exact sequence of complexes $0 \longrightarrow \mathcal{C}^{\leq r-1} \longrightarrow \mathcal{C}^{\leq r} \longrightarrow \mathcal{C}^r \longrightarrow 0$ the left and middle complex are both virtual resolutions of S/I_X for each $r \geq 2$. Therefore, the long exact sequence in homology gives that $\mathcal{C}^r$ must have purely irrelevant homology for each $r \geq 2$, i.e. $H_i(\mathcal{C}^r)$ is annihilated by some power of B for each $i \geq 0$. In addition, each $\mathcal{C}^r$ with $r \geq 2$ has projective dimension at most two since its summands solely come from homological degrees one, two, and three in $\mathcal{F}$, and each $\mathcal{C}^r$ is actually a complex over $k[y_0, y_1]$ since the maps only involve the second coordinate variables. This means we must actually have that each $H_i(\mathcal{C}^r)$ is annihilated by some power of $\langle y_0, y_1 \rangle$ and thus has depth zero over $k[y_0, y_1]$. Observe that $\mathcal{C}^r$ satisfies the hypotheses of the Acyclicity Lemma [Eis04, Lemma 20.11] since it has projective dimension at most two and each module in the complex is a free $k[y_0, y_1]$-module and so has depth two. Applying the Acyclicity Lemma gives that each $H_i(\mathcal{C}^r) = 0$ for $i > 0$ since otherwise the depth of some homology group would have to be at least one. Thus, each $\mathcal{C}^r$ with $r \geq 2$ is a minimal free resolution of a finite length S-module since $H_i(\mathcal{C}^r) = 0$ for $i > 0$ and some power of B annihilates $H_0(\mathcal{C}^r)$.

Now, to prove the lemma, suppose $d_{r,r'}$ is the first positive entry in the rth row of $\Delta^2 H_X$, where $r \geq 2$, and write $\mathcal{C}^r : 0 \leftarrow C_1^r \leftarrow C_2^r \leftarrow C_3^r \leftarrow 0$ since there are no terms in homological degree zero from $\mathcal{F}$. Since this is a minimal free resolution, there are no unit entries in the maps, so the minimal degrees of the generators of each module must strictly increase. By Remark 4.2.10, we know precisely the Betti numbers that appear in C_1^r: they come from the first and second (if it exists) negative entries in the rth row of $\Delta^2 H_X$. Let $d_{r,s'}$ be the first negative entry (so, by our assumption that $d_{r,r'}$ is the first positive entry, $s' < r'$).

Let (r, t') be the minimal degree of a generator of C_2^r. Then $\beta_{2,(r,t')} > 0$ and $\beta_{2,(r,u')} = 0$ for $u' < t'$, and we want to show that $t' = r'$ and $d_{r,r'} = \beta_{2,(r,r')}$. Since the minimal degrees of generators of the C_i^r must increase, we know that $s' < t'$.

Let us first suppose that $r' = s' + 1$. Then by Theorem 4.2.8, $\beta_{1,(r,r')} = 0$, so Proposition 4.2.7 gives that $d_{r,r'} = \beta_{2,(r,r')} - \beta_{3,(r,r')}$. Since the minimal degrees of the generators of C_2^r and C_3^r must increase and $d_{r,r'} > 0$ by assumption, we must have that $\beta_{3,(r,r')} = 0$ and $d_{r,r'} = \beta_{2,(r,r')}$. Thus, $t' = r'$ as desired.

Next, we consider when $r' > s' + 1$, and let's suppose towards a contradiction that $s' < t' < r'$. If $d_{r,s'+1}$ is negative, then by Proposition 4.2.7, $\beta_{1,(r,s'+1)} = -d_{r,s'+1} = \beta_{1,(r,s'+1)} - \beta_{2,(r,s'+1)} + \beta_{3,(r,s'+1)}$, which implies that $\beta_{2,(r,s'+1)} = \beta_{3,(r,s'+1)}$. But then these must both be zero since the minimal degrees of generators of C_2^r and C_3^r are not the same. This shows that $t' \neq s' + 1$ if $d_{r,s'+1}$ is negative. If $d_{r,s'+1}$ is not negative, then it must be zero (since it's not the first positive entry) and $\beta_{1,(r,s'+1)} = 0$ by Theorem 4.2.8. So, $d_{r,(s'+1)} = 0 = \beta_{2,(r,s'+1)} - \beta_{3,(r,s'+1)}$. Again, since degrees must increase, $\beta_{2,(r,s'+1)} = \beta_{3,(r,s'+1)} = 0$. Thus, if $r' > s'+1$, then $t' \neq s'+1$ no matter the sign of $d_{r,s'+1}$. This means that $s'+1 < t' < r'$. Then $\beta_{1,(r,t')} = 0$ by Remark 4.2.10, so we have $d_{r,t'} = \beta_{2,(r,t')} - \beta_{3,(r,t')}$.

Since $\beta_{2,(r,t')} > 0$ and $d_{r,t'}$ is not positive by our assumption that $t' < r'$, this implies that $\beta_{3,(r,t')} > 0$. But this contradicts that the minimal degree of a generator of C_3^r must be larger than (r, t'). Therefore, we have shown that $t' \geq r'$. Finally, by similar arguments $d_{r,r'} = \beta_{2,(r,r')} - \beta_{3,(r,r')} > 0$ implies that $\beta_{2,(r,r')} > 0$ and $\beta_{3,(r,r')} = 0$. Thus, $t' = r'$ and $d_{r,r'} = \beta_{2,(r,r')}$.

By symmetry of $\Delta^2 H_X$, the same argument works for the first positive entry in a given column. □

Theorem 4.3.3. *Let X be a set of $n \geq 2$ points in sufficiently general position in $\mathbb{P}^1 \times \mathbb{P}^1$, and let (i, i') be a minimal element of $\operatorname{reg}(S/I_X)$. By symmetry, without loss of generality, assume that $i \leq i'$. Then the virtual resolution of a pair $(S/I_X, (i, i'))$ has length two if either of the following holds:*

(a) $i(i' + 2) \leq n$, or

(b) $i(i' + 2) > n$, $-3n + 3ii' + 4i + i' \leq 0$, and $3n - 3ii' - 2i - 2i' \geq 0$.

Moreover, if either (a) or (b) holds, then for all degrees $\boldsymbol{r} = (r, r') \preceq (i + 1, i' + 1)$, the bigraded Betti numbers of S/I_X in degree $\boldsymbol{r}$ can be read from $\Delta^2 H_X$ in the following way:

(i) $\beta_{0,\boldsymbol{r}} = d_{r,r'}$ iff $\boldsymbol{r} = (0, 0)$;

(ii) $\beta_{1,\boldsymbol{r}} = -d_{r,r'}$ iff $d_{r,r'} < 0$ and either $d_{r,s'} > 0$ for some $s' > r'$ or $d_{s,r'} > 0$ for some $s > r$;

(iii) $\beta_{2,\boldsymbol{r}} = d_{r,r'}$ iff $d_{r,r'} > 0$ and $\boldsymbol{r} \succ (0, 0)$.

Furthermore, minimal elements of regularity satisfying either (a) or (b) give rise to seven types of Hilbert–Burch virtual resolutions of a pair, which are described in Appendix A as eqs. (A.1) to (A.7).

Remark 4.3.4. Note that in the specific case where $n = (i + 1)(i' + 1)$, condition (a) is satisfied and Theorem 4.3.3 exactly reduces to [HNV22, Theorem 3.1]. The virtual resolution of a pair that they give agrees with ours in the following way. If $i' \in \{i, i + 1\}$, then $(S/I_X,(i,i'))$ is given by eq. (A.1), which is the same as the complex given in [HNV22, Theorem 3.1] by taking $j=i'$, $q=i'$, and $r=0$. Otherwise, if $i'>i+1$, then $(S/I_X,(i,i'))$ is given by eq. (A.2), which is the same as the complex given in [HNV22, Theorem 3.1] by taking $j=i'$, $q=j'$ and $r=n-(i+2)j'$.

Proof of Theorem 4.3.3. The main idea of the proof is to show that if (a) or (b) holds, then $(S/I_X,(i,i'))$ is determined by the nonzero entries in $\Delta^2 H_X$ up to degree $(i+1,i'+1)$ in the sense of $(i),(ii)$, and (iii). Furthermore, using some technical arguments from Lemmas 4.3.1 and 4.3.2, we will see that under either of these conditions, $(S/I_X,(i,i'))$ has length two and is given by one of the seven Hilbert–Burch complexes in Appendix A.

Let $\mathcal{F}$ be the minimal free resolution of S/I_X as in eq. (4.1). Observe that (i) follows from $\mathcal{F}$ and (ii) is a restatement of Theorem 4.2.8. So, (i) and (ii) are true for all degrees $\boldsymbol{r}$, not just for $\boldsymbol{r}\preceq(i+1,i'+1)$. (See remark 4.2.10 for further discussion about the minimal generators of I_X.) Therefore, we just need to show that (iii) holds if either (a) or (b) is satisfied.

First, assume (a). Then Lemma 4.3.1 gives that $d_{i+1,i'+1}\geq 0$, and Cases 1 and 2 from the proof of this lemma describe the possibilities for $\Delta^2 H_X$. We will use these descriptions to prove that (iii) holds, which will indicate that $(S/I_X,(i,i'))$ has length two and is given by one of eqs. (A.1) to (A.4).

We first note that if $i=0$, then $i'=n-1$ and $n=(i+1)(i'+1)$. Then, as described in Remark 4.3.4, [HNV22, Theorem 3.1] gives that $(S/I_X,(i,i'))$ has length two and is given by either eq. (A.1) or eq. (A.2), from which we can see that (iii) holds by comparing the complex to $\Delta^2 H_X$ in Case 1 or Case 2.1, respectively.

Since we have covered the $i=0$ case when (a) holds, we now suppose that $i\geq 1$. Then we have two possibilities: either $(i+2)i'\leq n$ or $(i+2)i'>n$. If the former holds, then $\Delta^2 H_X$ is given in Case 1 of the proof of Lemma 4.3.1. Since $d_{i+1,i'+1}$ is the only positive entry in $\Delta^2 H_X$ of degree $(0,0)\prec\boldsymbol{r}\preceq(i+1,i'+1)$, we see that (iii) is true if and only if $\beta_{2,\boldsymbol{r}}=0$ for all $\boldsymbol{r}\prec(i+1,i'+1)$ and $\beta_{2,(i+1,i'+1)}=d_{i+1,i'+1}$. Since this is the first positive entry in the $(i+1)$st row and the $(i'+1)$st column, Lemma 4.3.2 gives that $\beta_{2,(i+1,i'+1)} = d_{i+1,i'+1}$ and $\beta_{2,\boldsymbol{r}} = 0$ for all degrees $\boldsymbol{r} = (i+1, j')$ with $j' < i'+1$ and $\boldsymbol{r} = (j, i'+1)$ with $j < i+1$. Furthermore, $\beta_{2,\boldsymbol{r}} = 0$ for the remaining degrees $\boldsymbol{r} \preceq (i,i')$ since $d_{r,r'} = -\beta_{1,\boldsymbol{r}} + \beta_{2,\boldsymbol{r}} - \beta_{3,\boldsymbol{r}}$ (Proposition 4.2.7), $\beta_{1,\boldsymbol{r}} = 0$ for all $\boldsymbol{r} \prec (i,i')$, $\beta_{1,(i,i')} = -d_{i,i'}$, and the Betti numbers must increase in degree, i.e. each first syzygy has to have degree larger than some minimal generator and each second syzygy has to have degree larger than some first syzygy. Thus, (iii) holds, and since $(S/I_X, (i,i'))$ is determined precisely by the Betti numbers of degree at most $(i+1, i'+1)$, this complex has length two and in this situation is given by eq. (A.1).

If instead $(i+2)i' > n$, then $\Delta^2 H_X$ is given by one of the matrices in Case 2. If it is given by Case 2.1, then we know that the only (potentially) positive entries are $d_{i+1,i'}$ and $d_{i+1,i'+1}$. Since each of these is the first positive entry in a row or column of $\Delta^2 H_X$, Lemma 4.3.2 gives that $\beta_{2,(i+1,i')} = d_{i+1,i'}$, $\beta_{2,(i+1,i'+1)} = d_{i+1,i'+1}$, and there are no other first syzygies coming from the $(i+1)$st row or the $(i'+1)$st column. Then we can use the same argument as in the previous paragraph to show that $\beta_{2,\boldsymbol{r}} = 0$ for $\boldsymbol{r} \preceq (i,i')$ to establish (iii), which shows that $(S/I_X, (i,i'))$ has length two and is given by eq. (A.2). If $\Delta^2 H_X$ is given by Case 2.2, then we know that $d_{i+1,i'+1} > 0$, but we do not know the sign of $d_{i+1,i'}$. If $d_{i+1,i'} < 0$, then by (ii), $\beta_{1,(i+1,i')} = -d_{i+1,i'}$. By Lemma 4.3.2, $\beta_{2,(i+1,i'+1)} = d_{i+1,i'+1}$ and there are no syzygies coming from the $(i+1)$st row or $(i'+1)$st column. Then, by the same argument as before, there are no syzygies of smaller degree, so (iii) holds. If $d_{i+1,i'} = 0$, then $\beta_{1,(i+1,i')} = 0$, so $\beta_{2,(i+1,i')} = \beta_{3,(i+1,i')} = 0$ by Lemma 4.3.2, and again, we see that (iii) is true. Lastly, if $d_{i+1,i'} > 0$, then we apply Lemma 4.3.2 to both $d_{i+1,i'}$ and $d_{i+1,i'+1}$, and by the same arguments (iii) holds. Therefore, in this situation, no matter the sign of $d_{i+1,i'}$, $(S/I_X, (i,i'))$ has length two and is given by either eq. (A.3) if $d_{i+1,i'} \leq 0$ or eq. (A.4) if $d_{i+1,i'} > 0$.

We have now shown that if (a) holds, then $(S/I_X, (i,i'))$ has length two and is given by one of eqs. (A.1) to (A.4), which are determined by the nonzero entries in $\Delta^2 H_X$ up to degree $(i+1, i'+1)$ in the sense of $(i), (ii)$, and (iii).

Next, assume (b). Then Lemma 4.3.1 gives that $d_{i+1,i'+1} = 3n - 2ii' - 2i - 2i' \geq 0$

and $d_{i,i'+1} = -3n + 3ii' + 4i + i' \leq 0$, and Case 3 from the proof of this lemma describes the two possibilities for $\Delta^2 H_X$. Note that because $i' < 2i$ in Case 3, we must actually have that $d_{i+1,i'+1} > 0$ since if it was equal to zero, then that would force $d_{i,i'+1} > 0$. We will implore very similar arguments as those used for Cases 1 and 2 to show that (iii) is true and $(S/I_X, (i, i'))$ has length two and is given by one of eqs. (A.5) to (A.7). If $\Delta^2 H_X$ is given by Case 3.1, then $d_{i,i'+1} \leq 0$ ensures that the only positive entries are $d_{i+1,i'}$ and $d_{i+1,i'+1}$. Then (iii) holds since Lemma 4.3.2 gives that these entries correspond to minimal first syzygies, and the virtual resolution of a pair is given by eq. (A.5). If $\Delta^2 H_X$ is given by Case 3.2, then the only entry with undetermined sign is $d_{i+1,i'}$. If this entry is negative, then it corresponds to a minimal generator by Theorem 4.2.8, and if it is positive, then it corresponds to a first syzygy by Lemma 4.3.2. Either way, since $d_{i+1,i'+1}$ is the first positive entry in the $(i'+1)$st column, we have $\beta_{2,(i+1,i'+1)} = d_{i+1,i'+1}$ by Lemma 4.3.2. Thus, (iii) holds and $(S/I_X, (i, i'))$ is given by eq. (A.6) if $d_{i+1,i'} \leq 0$ and by eq. (A.7) if $d_{i+1,i'} > 0$. □

Proof of Theorem 4.1.4. If $i(i'+2) > n$ and $3n - 3ii' - 2i - 2i' < 0$, then $\Delta^2 H_X$ is given by one of the matrices in Case 3 in the proof of Lemma 4.3.1 and has $d_{i+1,i'+1} < 0$. In this case, we can show that $d_{i,i'+1} > 0$. Indeed, since $i' < 2i$ in Case 3, we have $2i' - i' < 4i - 2i$, which gives $2i + 2i' < 4i + i'$. Therefore, $3n - 3ii' < 2i + 2i' < 4i + i'$, so $d_{i,i'+1} = -3n + 3ii' + 4i + i' > 0$. As in the proof of Theorem 4.3.3, we can use Lemma 4.3.2 to conclude that $\beta_{2,(i,i'+1)} = d_{i,i'+1}$ and $\beta_{2,(i+1,i')} = d_{i+1,i'}$. Then by Proposition 4.2.7, we have that $d_{i+1,i'+1} = \beta_{2,(i+1,i'+1)} - \beta_{3,(i+1,i'+1)}$. Since this entry is negative, we must have that $\beta_{3,(i+1,i'+1)} > 0$, and thus, $(S/I_X, (i, i'))$ has length three. □

Remark 4.3.5. As stated in [Cas06; MP11; MP12a; Boi+19], Giuffrida, Maggioni, and Ragusa's work in [GMR96] proves that the Minimal Resolution Conjecture is true for general sets of points lying on a smooth quadric in $\mathbb{P}^3$. Since any smooth quadric is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, one might initially think that this means that the Minimal Resolution Conjecture holds for general sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$, the objects of study in this paper. However, we are considering the bigraded Betti numbers of the bihomogeneous ideal of a set of general points, which lives in the bigraded Cox ring S of $\mathbb{P}^1 \times \mathbb{P}^1$. So, although the Minimal Resolution Conjecture holds for the Betti numbers of the ideal when viewed as living in the standard graded coordinate ring of $\mathbb{P}^3$, it does not, a priori, hold in the setting that we study here. It seems, though, that experts in the field believe the conjecture does hold; however, no one has formally proven this. In the proofs of Theorems 4.1.4 and 4.3.3, we are able to show that the Minimal Resolution Conjecture is true up to degree $(i+1, i'+1)$ under certain hypotheses which enable us to rule out the possibility of having overlapping Betti numbers.

For example, in the proof of Theorem 4.1.4, we know only that $d_{i+1,i'+1} = \beta_{2,(i+1,i'+1)} - \beta_{3,(i+1,i'+1)}$. If the Minimal Resolution Conjecture holds, then at most one of these Betti numbers is nonzero. Since we are assuming that $d_{i+1,i'+1} < 0$, this would imply that $d_{i+1,i'+1} = -\beta_{3,(i+1,i'+1)}$, and we would then know the entire virtual resolution of a pair $(S/I_X, (i, i'))$, not just that it has length three.

Furthermore, if we assume that the Minimal Resolution Conjecture for sufficiently general sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$ is true, then the nonzero entries $d_{r,r'} \in \Delta^2 H_X$ would precisely determine the minimal free resolution of S/I_X in the following sense: $(i), (ii)$, and (iii) from Theorem 4.3.3 would hold for all degrees, and we'd have

(iv) $\beta_{3,\boldsymbol{r}} = -d_{r,r'}$ iff $d_{r,r'} < 0$ and $d_{s,s'} > 0$ for some nonzero $(s, s') \prec (r, r')$.

This follows from Proposition 4.2.7 since the Minimal Resolution Conjecture implies that at most one of the Betti numbers $\beta_{1,(r,r')}, \beta_{2,(r,r')}, \beta_{3,(r,r')}$ is nonzero. Therefore, this would indicate that every virtual resolution of a pair $(S/I_X, (i, i'))$ with $(i, i') \in \text{reg}(S/I_X)$ (not necessarily minimal) is also determined by the nonzero entries in $\Delta^2 H_X$. In particular, this would mean that Theorem 4.3.3 would become an if and only if statement after removing the hypothesis that $-3n + 3ii' + 4i + i' \leq 0$ from condition (b). In other words, for minimal elements of regularity, the sign of $d_{i+1,i'+1}$ would determine the length of $(S/I_X, (i, i'))$: it would have length three if and only if $d_{i+1,i'+1} < 0$ and length two otherwise.

Example 4.3.6. Let X be a set of $n = 21$ points in sufficiently general position in $\mathbb{P}^1 \times \mathbb{P}^1$, and let $I_X \subseteq S$ be its defining ideal. Then $\text{reg}(S/I_X) = \{(i, i') \in \mathbb{Z}^2 | (i+1)(i'+1) \geq 21\}$ has

5 minimal elements with $i \leq i'$: $\{(0,20),(1,10),(2,6),(3,5),(4,4)\}$. Since X has a generic Hilbert function, H_X and $\Delta^2 H_X = (d_{i,i'})$ are given by the following infinite matrices. Because of symmetry, we only show a representative portion of each matrix, and the rows and columns are indexed by i and i' starting at 0, respectively.

$$H_X = \left[\begin{array}{ccccccc:ccccccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 21 \\
2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 & 18 & 20 & 21 & 21 & 21 & 21 & 21 & 21 & 21 & 21 & 21 & 21 & 21 & 21 \\
3 & 6 & 9 & 12 & 15 & 18 & 21 & 21 & 21 & 21 & 21 & 21 & 21 & 21 & 21 & 21 & 21 & 21 & 21 & 21 & 21 & 21 \\
4 & 8 & 12 & 16 & 20 & 21 & 21 & 21 & 21 & 21 & 21 & 21 & 21 & 21 & 21 & 21 & 21 & 21 & 21 & 21 & 21 & 21 \\
5 & 10 & 15 & 20 & 21 & 21 & 21 & 21 & 21 & 21 & 21 & 21 & 21 & 21 & 21 & 21 & 21 & 21 & 21 & 21 & 21 & 21 \\
\hdashline
6 & 12 & 18 & 21 & 21 & 21 & 21 & 21 & 21 & 21 & 21 & 21 & 21 & 21 & 21 & 21 & 21 & 21 & 21 & 21 & 21 & 21
\end{array}\right]$$

$$\Delta^2 H_X = \left[\begin{array}{ccccccc:ccccccccccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & -3 & -1 & 6 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -4 & 5 & 2 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hdashline
0 & 0 & 0 & -3 & 5 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]$$

Let's consider the virtual resolution of a pair $(S/I_X,(i,i'))$ for each minimal element of regularity. We will first show how to compute $(S/I_X,(3,5))$ using the nonzero entries in $\Delta^2 H_X$ up to degree $(3,5)+(1,1)=(4,6)$. For $(i,i')=(3,5)$, $i(i'+2)=21$, so condition (a) in Theorem 4.3.3 is satisfied. Therefore, the 1 in degree $(0,0)$ gives $\beta_{0,(0,0)}=1$, the negative entries in degrees $(3,5),(3,6)$, and $(4,4)$ give $\beta_{1,(3,5)}=3$, $\beta_{1,(3,6)}=1$, and $\beta_{1,(4,4)}=4$, and the positive entries in degrees $(4,5)$ and $(4,6)$ give $\beta_{2,(4,5)}=5$ and $\beta_{2,(4,6)}=2$. Thus, the virtual resolution of a pair is as follows (see eq. (A.4)):

$$(S/I_X,(3,5)): \qquad 0 \leftarrow S \leftarrow \begin{array}{c} S(-3,-5)^3 \\ \oplus \\ S(-3,-6) \\ \oplus \\ S(-4,-4)^4 \end{array} \leftarrow \begin{array}{c} S(-4,-5)^5 \\ \oplus \\ S(-4,-6)^2 \end{array} \leftarrow 0.$$

For $(i,i')=(2,6)$, $i(i'+2)=16$, so (a) is satisfied, and we have the following (see eq. (A.3)):

$$(S/I_X,(2,6)): \qquad 0 \leftarrow S \leftarrow \begin{matrix} S(-2,-7)^3 \\ \oplus \\ S(-3,-5)^3 \\ \oplus \\ S(-3,-6) \end{matrix} \leftarrow S(-3,-7)^6 \leftarrow 0.$$

For $(i,i') = (1,10)$, $i(i'+2) = 12$, so (a) is satisfied, and we have the following complex (see eq. (A.2) with $j' = 6$):

$$(S/I_X,(1,10)): \qquad 0 \leftarrow S \leftarrow \begin{matrix} S(-1,-10) \\ \oplus \\ S(-1,-11) \\ \oplus \\ S(-2,-7)^3 \end{matrix} \leftarrow \begin{matrix} S(-2,-10)^2 \\ \oplus \\ S(-2,-11)^2 \end{matrix} \leftarrow 0.$$

For $(i,i') = (0,20)$, $i(i'+2) = 0$, so (a) is satisfied, and we have the following complex (see eq. (A.2) with $j' = 10$):

$$(S/I_X,(0,20)): \qquad 0 \leftarrow S \leftarrow \begin{matrix} S(0,-21) \\ \oplus \\ S(-1,-10) \\ \oplus \\ S(-1,-11) \end{matrix} \leftarrow S(-1,-21)^2 \leftarrow 0.$$

Finally, for $(i,i') = (4,4)$, $i(i'+2) = 24 > 21$ and $3n - 3ii' - 2i - 2i' = d_{5,5} = -1$, so Theorem 4.1.4 gives that the virtual resolution of a pair has length three. This is the first situation where the virtual resolution of a pair for a minimal element of regularity is *not* Hilbert–Burch; for sets of $n \leq 20$ points, they are all length two! The complex is as follows, and we used [M2] to confirm that $\beta_{2,(5,5)} = 0$ and $\beta_{3,(5,5)} = -d_{5,5} = 1$.

$$(S/I_X,(4,4)): \qquad 0 \leftarrow S \leftarrow \begin{matrix} S(-3,-5)^3 \\ \oplus \\ S(-4,-4)^4 \\ \oplus \\ S(-5,-3)^3 \end{matrix} \leftarrow \begin{matrix} S(-4,-5)^5 \\ \oplus \\ S(-5,-4)^5 \end{matrix} \leftarrow S(-5,-5) \leftarrow 0.$$

Appendix A

Classification of Known Hilbert–Burch $(S/I_X,(i,i'))$

Let X be a set of $n \geq 2$ points in sufficiently general position in $\mathbb{P}^1 \times \mathbb{P}^1$, and let (i, i') be a minimal element of $\operatorname{reg}(S/I_X)$ with $i \leq i'$. Then Theorem 4.3.3 gives sufficient conditions for when the virtual resolution of a pair $(S/I_X,(i,i'))$ is length two. Furthermore, it implies that these virtual resolutions are determined by the nonzero entries in $\Delta^2 H_X$ up to degree $(i+1, i'+1)$ in the sense that the negative entries give the Betti numbers in the first module (the minimal generators) and the positive entries (excluding degree (0,0)) give the Betti numbers in the second module (the first syzygies). Here, we use the $\Delta^2 H_X$ matrices computed in Lemma 4.3.1 to give explicit descriptions of the Hilbert–Burch type $(S/I_X,(i,i'))$ that come from Theorem 4.3.3. In what follows, let $j' \leq i'-1$ be the degree such that $(i+1, j')$ is also a minimal element of $\operatorname{reg}(S/I_X)$, if such a degree exists.

- If $i(i'+2) \leq n$ and $(i+2)i' \leq n$ (see Case 1 in Lemma 4.3.1), then

$$(S/I_X,(i,i')): \quad 0 \leftarrow S \leftarrow \begin{matrix} S(-i,-i')^{-n+ii'+i+i'+1} \\ \oplus \\ S(-i,-i'-1)^{n-ii'-i'} \\ \oplus \\ S(-i-1,-i')^{n-ii'-i} \end{matrix} \leftarrow S(-i-1,-i'-1)^{n-ii'} \leftarrow 0. \tag{A.1}$$

- If $i(i'+2) \le n$, $(i+2)i' > n$, and $j' < i'-1$ (see Case 2.1 in Lemma 4.3.1), then

$$(S/I_X,(i,i')):\quad 0 \leftarrow S \leftarrow \begin{array}{c} S(-i,-i')^{-n+ii'+i+i'+1} \\ \oplus \\ S(-i,-i'-1)^{n-ii'-i'} \\ \oplus \\ S(-i-1,-j')^{-n+ij'+i+2j'+2} \\ \oplus \\ S(-i-1,-j'-1)^{n-ij'-2j'} \end{array} \leftarrow \begin{array}{c} S(-i-1,-i')^{2(-n+ii'+i+i'+1)} \\ \oplus \\ S(-i-1,-i'-1)^{2(n-ii'-i')} \end{array} \leftarrow 0. \tag{A.2}$$

- If $i(i'+2) \le n$, $(i+2)i' > n$, $j' = i'-1$, and $-3n+3ii'+i+4i' \le 0$ (see Case 2.2 in Lemma 4.3.1), then

$$(S/I_X,(i,i')):\quad 0 \leftarrow S \leftarrow \begin{array}{c} S(-i,-i')^{-n+ii'+i+i'+1} \\ \oplus \\ S(-i,-i'-1)^{n-ii'-i'} \\ \oplus \\ S(-i-1,-i'+1)^{-n+ii'+2i'} \\ \oplus \\ S(-i-1,-i')^{3n-3ii'-i-4i'} \end{array} \leftarrow S(-i-1,-i'-1)^{2(n-ii'-i')} \leftarrow 0. \tag{A.3}$$

- If $i(i'+2) \le n$, $(i+2)i' > n$, $j' = i'-1$, and $-3n+3ii'+i+4i' > 0$ (see Case 2.2 in Lemma 4.3.1), then

$$(S/I_X,(i,i')):\quad 0 \leftarrow S \leftarrow \begin{array}{c} S(-i,-i')^{-n+ii'+i+i'+1} \\ \oplus \\ S(-i,-i'-1)^{n-ii'-i'} \\ \oplus \\ S(-i-1,-i'+1)^{-n+ii'+2i'} \end{array} \leftarrow \begin{array}{c} S(-i-1,-i')^{-3n+3ii'+i+4i'} \\ \oplus \\ S(-i-1,-i'-1)^{2(n-ii'-i')} \end{array} \leftarrow 0. \tag{A.4}$$

- If $i(i'+2) > n$, $3n-3ii'-2i-2i' \ge 0$, $j' = i'-2$, and $-3n+3ii'+4i+i' \le 0$ (see

Case 3.1 in Lemma 4.3.1), then

$$(S/I_X,(i,i')):\quad 0\leftarrow S\leftarrow \begin{array}{c} S(-i+1,-i'-1)^{-n+ii'+2i}\\ \oplus\\ S(-i,-i')^{-n+ii'+i+i'+1}\\ \oplus\\ S(-i,-i'-1)^{3n-3ii'-4i-i'}\\ \oplus\\ S(-i-1,-i'+2)^{-n+ii'-i+2i'-2}\\ \oplus\\ S(-i-1,-i'+1)^{n-ii'+2i-2i'+4}\end{array} \leftarrow \begin{array}{c} S(-i-1,-i')^{2(-n+ii'+i+i'+1)}\\ \oplus\\ S(-i-1,-i'-1)^{3n-3ii'-2i-2i'}\end{array}\leftarrow 0. \tag{A.5}$$

- If $i(i'+2)>n$, $3n-3ii'-2i-2i'\geq 0$, $j'=i'-1$, $-3n+3ii'+4i+i'\leq 0$, and $-3n+3ii'+i+4i'\leq 0$ (see Case 3.2 in Lemma 4.3.1), then

$$(S/I_X,(i,i')):\quad 0\leftarrow S\leftarrow \begin{array}{c} S(-i+1,-i'-1)^{-n+ii'+2i}\\ \oplus\\ S(-i,-i')^{-n+ii'+i+i'+1}\\ \oplus\\ S(-i,-i'-1)^{3n-3ii'-4i-i'}\\ \oplus\\ S(-i-1,-i'+1)^{-n+ii'+2i'}\\ \oplus\\ S(-i-1,-i')^{3n-3ii'-i-4i'}\end{array} \leftarrow S(-i-1,-i'-1)^{3n-3ii'-2i-2i'}\leftarrow 0. \tag{A.6}$$

- If $i(i'+2)>n$, $3n-3ii'-2i-2i'\geq 0$, $j'=i'-1$, $-3n+3ii'+4i+i'\leq 0$, and $-3n+3ii'+i+4i'>0$ (see Case 3.2 in Lemma 4.3.1), then

$$(S/I_X,(i,i')):\quad 0\leftarrow S\leftarrow \begin{array}{c} S(-i+1,-i'-1)^{-n+ii'+2i}\\ \oplus\\ S(-i,-i')^{-n+ii'+i+i'+1}\\ \oplus\\ S(-i,-i'-1)^{3n-3ii'-4i-i'}\\ \oplus\\ S(-i-1,-i'+1)^{-n+ii'+2i'}\end{array} \leftarrow \begin{array}{c} S(-i-1,-i')^{-3n+3ii'+i+4i'}\\ \oplus\\ S(-i-1,-i'-1)^{3n-3ii'-2i-2i'}\end{array}\leftarrow 0. \tag{A.7}$$

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